UC Davis

Physics 7B

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TABLE OF CONTENTS

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Agenda

DLs 1 and 2:

DLs 3 and 4:

DLs 5, 6, and 7:

DLs 8 and 9:

DLs 10 and 11:

DLs 12, 13, and 14:

DLs 15, 16, 17, and 18:

CHAPTER OVERVIEW

5: Flow, Transport and Exponential

This chapter focuses on phenomena involving real fluids with viscosity and electric circuits with resistance. The effect of resistance in both kinds of flow means that energy will be reduced in the fluid system while thermal energy systems will increase. Frequently the flow is described as being *dissipative*. We call the model/approach we use to make sense of dissipative flow the *Steady-State Energy Density Model*, where the flow is constant over time. In the second part of the chapter we generalize the underlying ideas about flow to flow phenomena in which changes in energy are not of paramount importance. Rather, the focus is simply on the "fluid" and medium properties and the "driving force" that keeps the flow going. The "thing" that flows can be a real fluid, electric charge, energy, or other things that diffuse – in short, any phenomenon in which the flow of something becomes constant can be understood with this approach/model, which we call the *Linear Transport Model*. Toward the end of the chapter we look at examples where the flow is no longer constant, but displays exponential decay behavior.

5.0: Overview of Flow, Transport and Exponential

In Physics 7A, Chapters 1 and 2, we explored a very general and universally applicable model based on conservation of energy: the *Energy-Interaction Model*. The approach of focusing on initial and final values was extended in Chapter 4 about thermodynamics, to include all state variables. We now turn back to energy conservation, but with a significant difference. Up to now the changes in energy and other variables occurred over time. We focused on an initial time and a final time and looked at how indicators of various energies changed between those two times. We begin Chapter 5 by looking at phenomena that occur in a *steady-state* fashion. *Steady-state* refers to systems that do not change with time. In particular, we will study the steady-state flow of fluids and electric charge in electrical circuits. In these phenomena the change in energy occurs over *position*, but is constant in time. We will *not* be starting from scratch, however, since we will be able to use many of the ideas and constructs we previously developed when working with the Energy-Interaction Model.

We begin by focusing on phenomena that involve flow of fluids in fluid systems and charge in electric circuits. The effect of resistance in both kinds of flow means that energy will be reduced in the fluid system while thermal energy will increase. Frequently the flow is described as being *dissipative*. We call the model/approach we use to make sense of dissipative flow, the *Steady-State Energy Density Model*.

Later in the chapter we generalize the underlying ideas about flow to flow phenomena in which changes in energy are not of paramount importance. Rather, the focus is simply on the “fluid” and medium properties and the “driving force” that keeps the flow going. The “thing” that flows can be a real fluid, electric charge, energy, or other things that diffuse – in short, any phenomenon in which the flow of something becomes constant can be understood with this approach/model, which we call the *Linear Transport Model*.

Frequently in science the same basic relationship is rediscovered by practitioners in different disciplines. Usually a different name is given to the relationship and different symbols are used to express it in each different discipline. These historical differences manage to last through multiple generations of textbooks. This is certainly true of the linear transport equation we develop in this chapter. As an example, some of you will study environmental science or soil and water science. You may have come across Darcy’s law, describing the flow of water in soils. After studying this chapter, Darcy’s law should seem pretty familiar to you, even if some of the symbols that are used are different. Practitioners in other branches of science, technology, medicine and engineering, will use different laws relating to transport, each with its different name and specialized symbols, but all referring to the same basic underlying transport model. Hopefully, you will develop expertise in “reading past the particular symbols” and recognize the fundamental content expressed by the relationship, which, after all, is independent of which letters of the alphabet we choose to use.

The kinds of phenomena we can make sense of using the *Steady-State Energy Density Model* or the *Linear Transport Model* do not change in time. That is, the transport phenomena remain constant for the time interval of interest. For example, once a resistor network has been arranged and a battery connected, the current very quickly comes to its steady-state values in the various parts of the circuit and then remains constant. Or, when there is a constant temperature difference from one end of a bar of metal to the other, the rate of heat flow is constant along the bar. These two powerful models, though broadly applicable to many real situations, exclude any situation where the amount of something transported changes with time. There are certainly many physical systems where the rate of change is not constant.

A very common “non-steady state” phenomenon in nature is exponential growth or decay. You might have seen discussions of exponential growth in biology classes, or perhaps in a business or economics class when discussing compound interest. You may also have seen *exponential decay* in chemistry when exploring radioactivity and nuclear decay. Many of the physical systems that exhibit steady-state flow also exhibit exponential behavior as they evolve to a steady-state condition. The “charging up” of an electrical circuit, the heat flow from a hot cup of coffee or tea, the draining of a container of liquid through a small hole all exhibit exponential behavior. In the last part of this chapter we develop a model that that provides the foundation for understanding this kind of exponential-change phenomena. We include exponential change in this chapter, because all of the physical systems that exhibit steady-state behavior and to which the *Steady-State Energy Density Model* and the *Linear Transport Model* apply, also exhibit exponential change under different circumstances.

5.1: Steady-State Energy-Density Model

Most traditional physics textbooks discuss the flow of fluids and the flow of electric charge in electric circuits as completely separate topics. Our goal in this chapter is to understand both kinds of transport phenomena using the *Steady-State Energy Density Model*. Historically, different words and symbols have been used for the description of each of these phenomena, making their similarity even more difficult to see. We will generally use conventional notation and vocabulary, because this is the notation you will see and use in your other science courses and in research. While becoming comfortable with the specific notation and vocabulary of each of these different types of steady-state transport phenomena, you need to simultaneously become conscious of the more universal nature of the underlying model and approach. In this way, you can use your understanding in one area to help develop understanding in other less familiar areas.

The principle of conservation of energy applied to fluid phenomena is expressed in a relation historically known as *Bernoulli equation*. When dissipation effects and sources of energy input are included, the term *extended Bernoulli equation* is sometimes used. We will generally use the terms "energy-density equation" and "complete energy-density equation" for both fluid flow and electric circuit phenomena. Changes in the total energy-density of a small element of fluid include changes in its kinetic energy-density, its gravitational potential energy-density, and its pressure. The first two terms are familiar, except the energy is not divided by volume to yield energy density. The pressure term, however, is more challenging. The energy density represented by the pressure is related to whatever it is that confines the liquid and gives rise to the pressure. For the common occurrence of static incompressible fluids in open containers, the pressure is ultimately due to gravitational forces acting on the fluid.

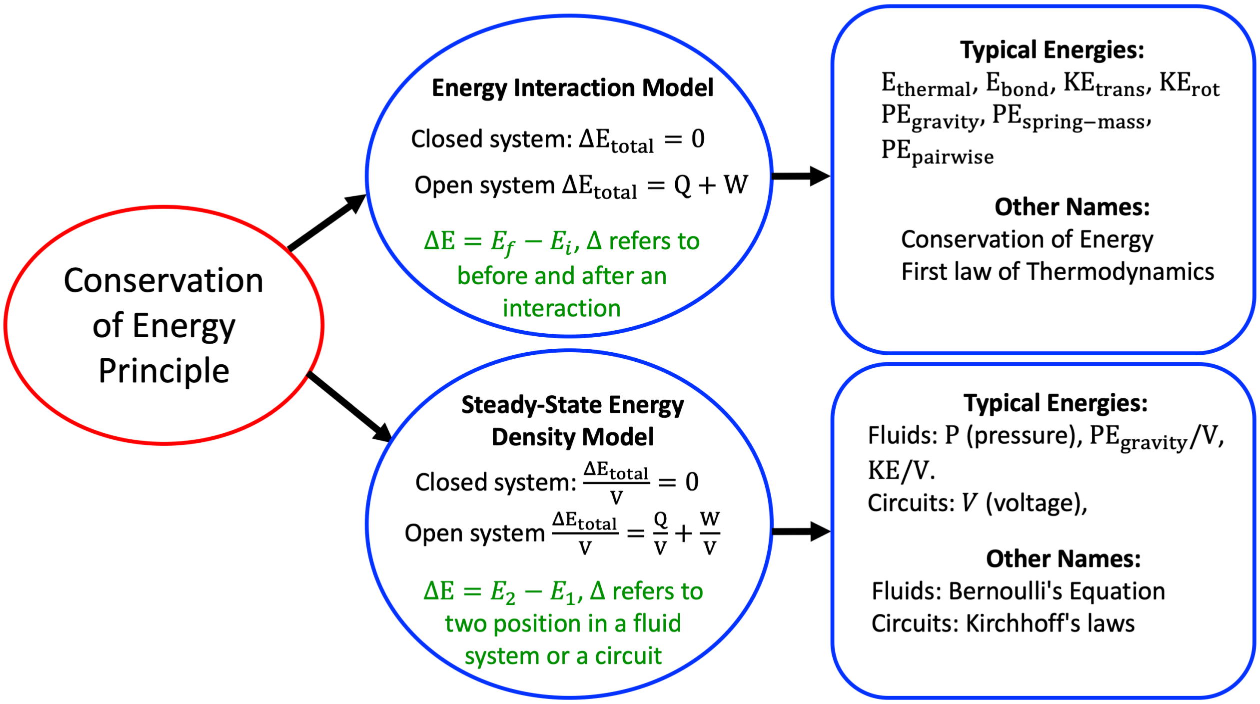
Relationships of exactly the same form and meaning as used to make sense of fluid phenomena are used to express energy conservation in electric circuits. Instead of pressure, the energy density is the voltage. This is the same voltage you are familiar with when installing 1.5 or 9 Volt batteries in your electronic gadgets. We will develop and use the *Steady-State Energy Density Model* in terms of both real fluid flow and electric circuits. Our goal is to make sense of both kinds of phenomena using a common model, and thus take advantage of the understanding we have in one domain to help us make sense of the other. Both static-fluid and flowing-fluid phenomena are very common, of course, in living organisms.

Differences in the Two Basic Energy Models

The most general and universal approach to getting answers to questions about phenomena in the physical world is to use conservation principles. The most universal of these principles is conservation of energy. However, to actually use this very general principle of conservation of energy, we need to cast it in a form that is useful for analyzing particular types of phenomena.

Here is a comparison of the energy-interaction model from Chapters 1 and 2 of 7A and the steady-state energy density model for fluids and electricity that we are developing in this chapter of 7B. Figure 5.1.1 shows both the similarities and differences in our two energy conservation models.

Figure 5.1.1: Two Energy Model Comparison



To use either of these models, we follow these basic steps in carrying out the approach:

1. Determine useful initial and final states (two useful positions) of the physical system based on the questions we are asking.
2. Identify the observable parameters (indicators) that significantly changes in time (or as we move from one position to another) for the two states (position) selected, and calculate the corresponding change in energy (or energy-density).
3. Write down an equation expressing conservation of energy (or energy density). One way to write this equation is as a sum of the changes in energy (or energy-density). If no energy is added or removed from outside systems, i.e., a closed system, the sum is set equal to zero. If there are outside sources of energy, i.e., an open system, then the sum is set equal to those energy sources.

5.2: Static Fluids

Static Fluid

A *fluid* is a state of matter that include gases and liquids. The distinguishing feature of fluids is that the individual molecules do not have fixed positions relative to one another, unlike in a solid. We will introduce several parameters that characterize fluids, some of which apply to solids as well. One important characteristic of fluids is pressure, which is where we begin.

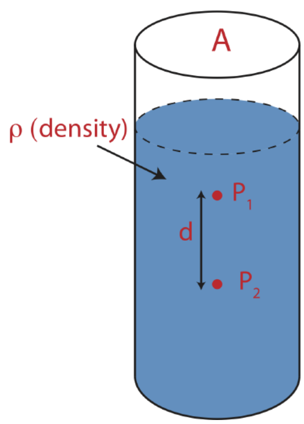
What is *pressure*? You are familiar with this word from both everyday usage and within a scientific context. Fundamentally, pressure is an *intensive* property of fluids. An intensive property does not scale with the volume of the substance as energies does. Similarly temperature and density are intensive properties. While solids are characterized by temperature and density, only fluids (liquids and gases) have a characteristic pressure. For fluids, then, pressure is another state variable, telling us something about the state of the fluid system. The common use of the term "pressure" in a phrase such as "I can really feel the pressure building up inside me as finals approach; I think I'm about to explode!" somewhat captures the spirit of the meaning of pressure in a fluid. The pressure in a fluid can be changed by external means, and an increase in pressure represents an increase in energy.

Energy Perspective

Perhaps one of the most useful ways to think of pressure is as an *energy density*. More precisely, the pressure at a particular point in a fluid is the energy per unit volume that must be transferred from another system into the fluid system in order to create a unit volume of fluid at that point.

Figure 5.2.1 illustrates an a example of a static fluid in a container which is open to the atmosphere. We would like to use an energy argument to figure out how the pressure at point 1, P_1, is related to pressure at point 2, P_2.

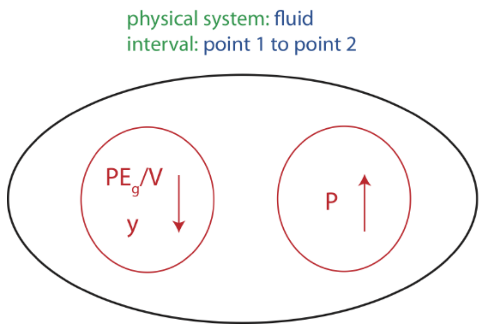
Figure 5.2.1 – Static Fluid.



In 7A we learned that energy of a system is conserved when no heat enters or leaves the system or no work is done by or on the system. The fluid in Figure 5.2.1 is in thermal equilibrium and no work is being done on the fluid. In 7A we were concerned with changes of energy as a system evolved in time from one state to another, such as ice melting on a hot day or a ball rolling down a ramp. In fluid systems at equilibrium, we are interested in looking at how energy changes in space within the fluid rather than in time.

From point 1 to point 2 in the liquid in Figure 5.2.1, there is a change of height which is an indicator of gravitational potential energy as described in Chapter 2.4. By applying the principle of conservation of energy to this closed system from position 1 to 2, we know that another energy has to increase since gravitational potential energy decreases. That energy is precisely related to pressure. In order to write down the conservation of energy equation we either have to convert gravitational potential energy into units of energy density or pressure into units of energy, since pressure has units of energy density. It is easier to work with fluid density rather than fluid mass and volume. Thus, by convention we will stick with energy densities when analyzing fluid systems. To convert an energy to an energy density we need to divide energy by volume, V. The interval of going from point 1 to point 2 is illustrated in the Energy-Interaction diagram below. This will be most likely the first and the last time we will use these diagram in 7B, but it is drawn here to help draw the connection between energy conservation in 7A to energy density conservation in 7B.

Figure 5.2.2: Energy-Interaction Diagram for a Static Fluid.



Energy conservation tells us that as we move vertically downward in a fluid at rest the gravitational potential energy density decreases so the pressure must increase by the same amount. Since the indicator of gravitational potential energies is height, it is only the change in vertical distance that matters. If we move in a horizontal direction in a fluid, the pressure remains constant.

Gravitational potential energy is given by $PE_g = mgy$. When converted into energy density by dividing by volume V, we can further simplify the equation by using *mass density*, $\rho = m/V$, mass per unit volume.

Writing down an energy density conservation equation for the interval depicted in Figure 5.2.2:

$$\frac{\Delta PE_g}{V} + \Delta P = 0 \tag{5.2.1}$$

$$\rho g \Delta y + \Delta P = 0 \tag{5.2.2}$$

$$\rho g (y_2 - y_1) + P_2 - P_1 = 0 \tag{5.2.3}$$

Since y_2 is at a lower height than y_1, the difference between the two values will be negative or $y_2 - y_1 = -d$, where d is the distance between the two points as marked in Figure 5.2.1. Equation 5.2.3 becomes:

$$-\rho g d + P_2 - P_1 = 0 \tag{5.2.4}$$

Solving for pressure at point 2:

$$P_2 = P_1 + \rho g d \tag{5.2.5}$$

Example 5.2.1

In this example we want to determine the difference in pressure in air and water for the same change in height. This calculation will demonstrate why when you move from a second to the first floor (about 4 meters) in a building you do not feel a change in air pressure. However, when you dive 4 meters in water the increase in pressure is very apparent.

Calculate the change in air pressure when you descent 4 meters in air and calculate the change in water pressure when you dive down 4 meters. Use $\rho_{\text{air}} \sim 1.2\ kg/m^3$ and $\rho_{\text{water}} \sim 1100\ kg/m^3$. You can approximate $g \sim 10\ m/s^2$.

Solution

For air:

$$\Delta P = -\rho_{\text{air}} g \Delta y = -1.2 kg/m^3 \times 10 m/s^2 \times (-4.0 m) = -48 Pa = 4.7 \times 10^{-4} atm$$

For water:

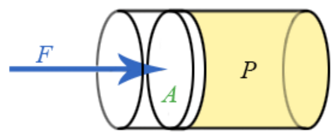
$$\Delta P = -\rho_{water} g \Delta y = -1000 kg/m^3 \times 10 m/s^2 \times (-4.0 m) = 4 \times 10^4 Pa = 0.4 atm$$

Since the question asks about the change in pressure when you go down 4 meters in air or water, the change in height is negative, resulting in an increase in pressure or a positive change in pressure. The result shows that the change in water pressure is close to 1000 times greater than the change in air pressure for the same change in height.

Force Perspective

Everything we have done up to this point has treated pressure as an energy density. Pressure is also related to force, just as energy is related to force through the concept of work. Consider a piston pushing on the fluid as see in the animation below.

Figure 5.2.3: Piston Doing Work on a Fluid.



The work done by an outside agent in moving the piston a distance dx down is given by:

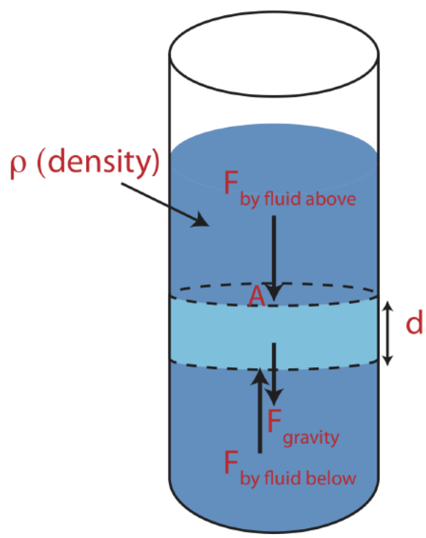
$$dW = Fdx = -PdV \tag{5.2.6}$$

Refer back to Chapter 4.2 on the relationship of work and force and pressure and volume. If a piston of area A pushes down on the fluid a distance dx, the change is volume will be $dV = -Adx$, resulting in $Fdx = PAdx$ or:

$$F = PA \tag{5.2.7}$$

That is for a fluid system the force acting perpendicular on an area is directly proportional to the area and the fluid pressure. We can imagine that the fluid exerts this force on any area, whether it is a wall of a container or just an imaginary area separating two parts of fluid. Let us explore how we can use these ideas to determine the change in pressure with depth. Consider the forces acting on a small elements of a fluid, colored light blue (outlined by dashed lines) in the figure below.

Figure 5.2.4: Forces on an Element of Fluid.



We will study forces in much more detail later in Chapter 6. The main idea to focus on here is that when a system is in equilibrium, such as a static fluid, all the forces acting on that system must be balanced. The light blue element has a force acting on it by the fluid above, $F_{\text{by fluid above}}$, a force by the fluid below it, $F_{\text{by fluid below}}$, and the force of gravity pulling it down, F_{gravity} as illustrated in the figure above. For the forces to be balanced, the sum of forces pushing down has to add up to the sum of the forces pushing up:

$$F_{\text{by fluid above}} + F_{\text{gravity}} = F_{\text{by fluid below}} \tag{5.2.8}$$

Plugging in $F = PA$ and $F_{\text{gravity}} = mg$ we get:

$$P_{\text{above}}A + mg = P_{\text{below}}A \tag{5.2.9}$$

Using $m=\rho V=\rho A d$ we arrive at:

$$P_{\text{below}}=P_{\text{above}}+\rho g d \tag{5.2.10}$$

Comparing Equations 5.2.5 to 5.2.10 we find that the two approaches lead to an identical result. This result states that pressure in a fluid only depends on its density and the depth of the fluid and is completely independent of the amount of fluid present.

Applications

The result in Equation 5.2.10 tells us that the pressure in a fluid is the same at all points of equal height regardless of the volume of fluid above that height, such as along the dashed line in Figure 5.2.5. We can think about this by realizing that pressure does not have a direction, but represents an average forced exerted on a system per unit area. That pressure on the system, or element of a fluid, is equal in all directions. If the pressure, for example, along the dashed line in Figure 5.2.5 was greater on the right side of the container compared to the left, that would imply that the force pushing from the left would be greater than the force pushing from the right, resulting in a fluid which is not static.

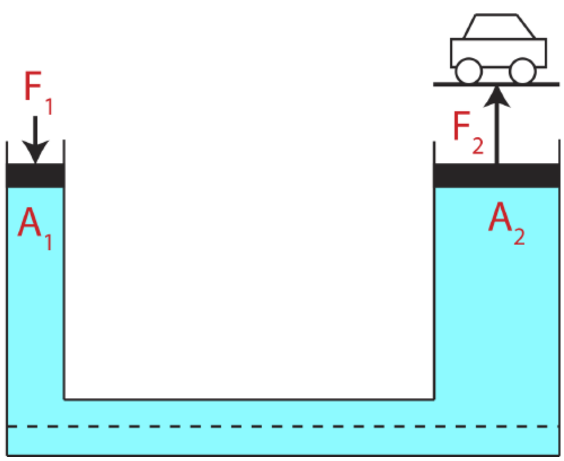
This relationship between fluid pressure and depth is true as long as the fluid is:

1. *static*: not moving. We will see in the next section that when the fluid is moving its pressure may change even if the fluid stays at the same height.

2. *continuous*: the fluid density must the uniform everywhere throughout the fluid. There are no physical barriers separating parts of the fluid or two fluids of different densities in contact.

3. *incompressible*: there are no variations of densities within the same fluid. Thus, this relationship is most accurate when applied to liquids rather than gasses which are compressible.

Figure 5.2.5: Pascal's Principle Demonstrated.



Suppose we add a constant pressure to the entire fluid, by compressing it with a piston. Then the pressure increases uniformly throughout the fluid by this same amount. Otherwise, the change in pressure due to a change in height would not be independent of the total pressure. This result is often referred to *Pascal's Principle.* One common application of Pascal's Principle is hydraulic machines. Figure 5.2.5 shows an enclosed fluid with two pistons of different areas. If the pistons are at the same elevation, then the pressure in the fluid is the same, $P=P_1=P_2$. The force on each piston is simply the pressure times the area of the piston. If the small piston is pressed down with a force F_1, the pressure created in the fluid is $P=F_1/A_1$. The force this pressure exerts on the larger piston is then:

$$F_2=PA_2=F_1\frac{A_2}{A_1} \tag{5.2.11}$$

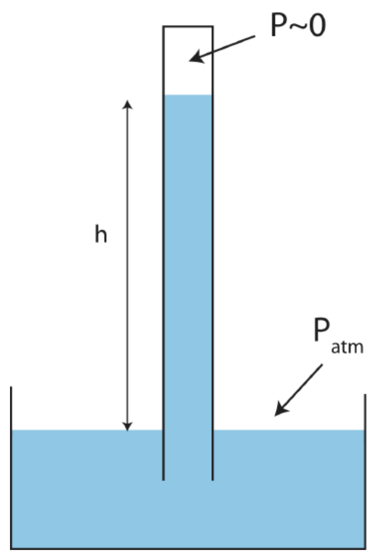
Tremendous increases in force are readily obtained in this manner, by varying the area of the pistons appropriately. For example, this gives the piston with a larger area the ability to lift a car as shown in Figure 5.2.5. The bigger the ratio $\dfrac{A_2}{A_1}$, the bigger will be the force exerted by the larger piston. Of course, the energy or power output of a hydraulic machine, as with any machine, cannot be greater than the energy or power put in. The distance the small piston has to move for a corresponding movement of the large piston is in exactly the inverse ratio of the forces, making the magnitudes of the work done by each piston the same.

A common example of a hydraulic system is the braking system of an automobile. In an “old fashioned” braking system (without power assist), pushing in on the brake pedal increased the pressure throughout the system, and uniformly applied the same force to the brake shoes on each of the four wheels. This would be tricky to do with mechanical linkages, but is “automatic” using a hydraulic system.

The fact that the pressure is the same at the same height also tells us that when a fluid meets another fluid at a boundary, the pressure of both fluids at the location of the boundary has to be the same. In Figure 5.2.1 in the open container water meets air, which means that the pressure of the water at the top is the same as air pressure, known as *atmospheric pressure*. It is frequently the case that the *absolute pressure* of a system is most easily regarded as being the sum of atmospheric pressure plus an additional pressure caused by some process. The pressure in your bicycle tires, for example, is given in lbs per square inch above atmospheric pressure, caused by the process of “pumping up your tire” which increases the pressure above atmospheric pressure. When you use a *gauge* to measure your tire pressure you are measuring only the increase in pressure above one atmosphere. This is called *gauge pressure*. A "flat tire" occurs when the air pressure inside the tire is equal to the air outside due to a leak. Thus, the tire pressure become atmospheric pressure, or zero gauge pressure. The absolute pressure is equal to the gauge pressure plus the atmospheric pressure. A good rule of thumb is when you see ΔP it refers to gauge pressure or $\Delta P = P_{\text{fluid}} - P_{\text{atm}}$, and when you see just P this refers to absolute pressure or $P = \Delta P + P_{\text{atm}}$.

There are many ways that pressure is measured. Generally, a device for measuring pressure is known as *manometer*. There are many different kinds of manometers, such as a *barometer* used for measuring atmospheric pressure and a *sphygmomanometer* used for measuring blood pressure. An example of a barometer in shown below.

Figure 5.2.6: Barometer.



A thin long container initially filled with fluid is submerged in a large container of fluid which is open to the atmosphere as seen in Figure 5.2.6. The fluid in the thin container will flow down until pressure equilibrium is reached. We assume that the pressure at the top of the thin container is nearly zero, since air cannot enter at that end. The height of the fluid in the thin container will allow us to determine atmospheric pressure. Applying Equation 5.2.5 from the top of the thin container where the pressure is zero down a distance h at the height where the fluid in the larger container is exposed to atmosphere we get:

$$P_{\text{atm}} = \rho g h \tag{5.2.12}$$

Let us see what the height would have to be to measure atmospheric pressure of $P_{\text{atm}} = 1atm = 1.01 \times 10^5 Pa$ if water was used. Using the density of water is $\rho_{\text{water}} \sim 1000kg/m^3$ and solving Equation 5.2.12 for height, h, we get:

$$h = \frac{P_{\text{atm}}}{\rho g} = \frac{1.01 \times 10^5 Pa}{1000kg/m^3 \times 9.8m/s^2} \sim 10m \tag{5.2.13}$$

The height of 10m is about a three-story building! Although, we do not "feel" air pressure since our bodies are in equilibrium with atmospheric pressure, this results demonstrates the immense magnitude of atmospheric pressure if it is able to push water three stories high. It also tells us that making a water barometer is not very practical since it would require a 10 m long container and a lot of water. Using a fluid with a much larger density would allow us to use a shorter barometer. Mercury is an example of liquid with a very large density $\rho_{\text{mercury}} = 13600 kg/m^3$, more than 10 times larger than the density of water. Using Equation 5.2.13 and replacing the density of water with the the density of mercury, we find that the height will be approximately 760 mm, less than 1 meter. Historically using mercury, Hg, in manometers has been so common that a unit of pressure was created based on measurements of pressure with mercury, $1atm = 760mmHg$. A unit of *Torr* is almost identical to 1 mmHg. Blood pressure is also measured in units of mmHg since mercury sphygmomanometers were commonly used before the health dangers of mercury were known.

Example 5.2.2

Intravenous infusions are usually made with the help of the gravitational force. At what height should the IV bag be placed above the entry point so that the fluid just enters the vein if the blood pressure in the vein is 18 mmHg above atmospheric pressure? Assume that the density of the fluid being administered is about the density of water 1.00 g/ml, and that the IV bag is always filled with the fluid.

Solution

For the fluid to just enter the vein its pressure at entry must exceed the blood pressure in the vein (18 mm Hg above atmospheric pressure). We therefore need to find the height of fluid that corresponds to this gauge pressure, $\Delta P = \rho g h$. We need to convert the pressure and density into SI units, using $1.0 mmHg = 133 Pa$ and $1.0 g/ml = 1000 kg/m^3$. Solving for height:

$$h = \frac{\Delta P}{\rho g} = \frac{2394 Pa}{1000 kg/m^3 \times 10 m/s^2} = 0.24 m = 24 cm$$

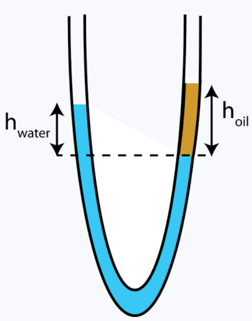
The IV bag must be placed at 24 cm above the entry point into the arm for the fluid to just enter the arm. Generally, IV bags are placed higher than this. You may have noticed that the bags used for blood collection are placed below the donor to allow blood to flow easily from the arm to the bag, which is the opposite direction of flow than required in the example presented here.

Example 5.2.3

A U-shaped container with both ends open to the atmosphere, is initially fill with water. Then, some oil is slowly poured on the right side of the U-tube. Assume the oil does not mix with water and settles at levels at shown in the picture below, where $h_{\text{water}} = 18cm$ cm and $h_{\text{oil}} = 20cm$ at equilibrium. Use $\rho_{\text{water}} \sim 1000 kg/m^3$.

a) Find the density of oil.

b) If the ends of the tube were no longer open to the atmosphere, which side would need to have more pressure in order for the top of the water on the left side to settle at the same height as the top of the oil on the right side? Calculate this pressure difference between the two sides.



Solution

a) The pressure at the dashed line in the figure is the same, since it is where water is located in one fluid system at the same height. Applying energy-density conservation equation from the open ends to the dashed line on each side:

$$\text{right side of tube: } \Delta P+\rho_{\text{oil}} g h_{\text{oil}}=0$$

$$\text{left side of tube: } \Delta P+\rho_{\text{water}} g h_{\text{water}}=0$$

Since the pressure difference, $\Delta P=P_{\text{atm}}-P_{\text{dashed line}}$ *, is the same for each side, the gravitational potential energy density terms have to equal:*

$$\rho_{\text{water}} g h_{\text{water}}=\rho_{\text{oil}} g h_{\text{oil}}$$

resulting in:

$$\rho_{\text{oil}}=\rho_{\text{water}} \times \frac{h_{\text{water}}}{h_{\text{oil}}}=1000 \frac{kg}{m^3} \times \frac{18 cm}{20 cm}=900 \frac{kg}{m^3} \quad .$$

b) In this case, the heights of each fluid from a line from where the oil touches the water to the top of the fluids on each side of the tube are the same height $h=20 cm$*, since the amount of oil is the same. Let us define the pressure at the line where the two fluids touch* P_{down}*, and the pressures at the top of the left and right sides of the U-tube* P_{left} *and* P_{right} *, respectively. The equations become:*

$$\text{right side of tube: } P_{\text{right}}-P_{\text{down}}+\rho_{\text{oil}} g h=0$$

$$\text{left side of tube: } P_{\text{left}}-P_{\text{down}}+\rho_{\text{water}} g h=0$$

Subtracting the two equations:

$$P_{\text{right}}-P_{\text{left}}+\left(\rho_{\text{oil}}-\rho_{\text{water}}\right) g h=0$$

$$P_{\text{right}}-P_{\text{left}}=\left(\rho_{\text{water}}-\rho_{\text{oil}}\right) g h=(1000-900) \frac{kg}{m^3} \times 10 \frac{m}{s^2} \times 0.2 m=200 Pa$$

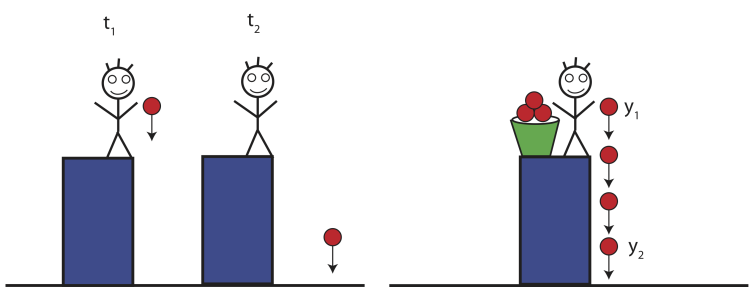
5.3: Fluid Flow

Steady-State Systems

In the Energy-Interaction model, the change in energy (or other state variables) was always from an initial to final state; that is, from a state earlier in time to a state later in time. The interaction of interest occurred during the time between the initial and final states. Now we will look at a *steady-state* energy model where the states are not distinguished in time; rather they are distinguished by spatial location in the physical system.

When we studied systems in 7A we analyzed how energy changed as a function of time during some interaction. In Figure 5.3.1 shown below, a person drops a ball at some initial time, t_1. We can apply conservation of energy to figure out how the energy of the ball at some later time t_2 changed. In a different scenario, instead of dropping one ball the person drops multiple balls at equal time intervals, such as in the drawing on the right in Figure 5.3.1. In this case the system of falling balls looks identical as a function of time. If you took a picture of the left scenario at t_1 and then later at t_2, you would see the ball higher and then lower, respectively. This this system is changing with time. On the other hand, if you took two pictures of the right scenario at two different times, the pictures would look identical, if the balls are indistinguishable, even though the balls are moving. We call this scenario a *steady-state* system. For a steady-state system instead of analyzing energy differences between two time periods, such as t_1 and t_2 in the left diagram, we will analyze energy differences between two spatial locations, such as y_1 and y_2 in the right diagram below.

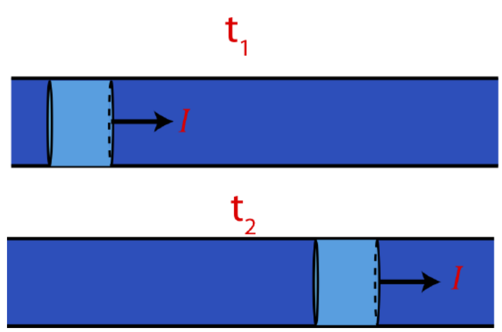
Figure 5.3.1: Time Varying versus a Steady-State System.



Flow Rate and Continuity

A flowing fluid at equilibrium is an example of a steady-state system. If you are observing a steady-state fluid system flowing past you, the system looks identical with passage of time. For the fluid to be in a steady-state, it cannot pile up or leak out of anywhere (what goes in at one end of the pipe must come out at the other end). In other words, the amount of time that it takes for a volume of fluid to flow past one point must equal the amount of time it takes for that same volume to flow past another point at a later time. As depicted in Figure 5.3.2 this means that the light blue fluid element at some initial time t_1 will have the same flow rate as at a late time t_2.

Figure 5.3.2: Steady-State Fluid Flow System.



We define the rate at which the fluid flows, the volume of fluid passing through the pipe at a particular location along the pipe per second, the *volumetric flow rate*, I, sometimes referred to as *current*:

$$I = \frac{dV}{dt} \tag{5.3.1}$$

with standard SI units of m^3/s. The letter "I" is a standard symbol for electric current, so we will use it here in our discussion of fluids as the symbol for the volumetric flow rate or current since our next discussion will be on electric flow systems. In everyday language we do often refer to volumetric flow rate as "current", such as "this river has a fast current". (Advanced texts in other disciplines might use other symbols for flow rate, such as Q, for example.)

When the fluid is in a steady-state and is incompressible (uniform density throughout), it cannot pile up or leak out. This means the current is conserved and remains constant in space throughout the entire fluid system. This is known as the *continuity equation* or conservation of mass:

$$I = \text{constant} \tag{5.3.2}$$

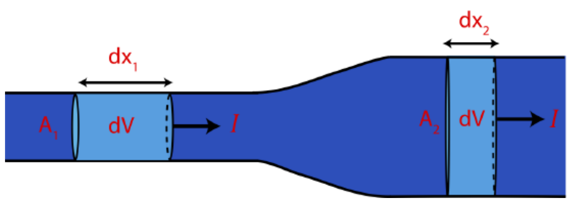
For this fluid dynamic model to work the flow has to be *laminar*. This means that neighboring fluid particles are flowing nearly parallel to each other, known as a *streamline*. When the flow becomes too fast it the streamline becomes *turbulent*, resulting in swirls and non-linear and non-laminar flow. Thus, in our model we will confine ourselves to slower and linear flow rates.

For static fluids in equilibrium all locations in the connected fluid system with a constant density must have the same total energy-density. For flowing fluids, once steady-state has been reached, all locations in the connected fluid system must also have the same total energy-density if there are not external sources that either add or remove energy. We will look at those cases shortly. We make the restrictions that the fluid is incompressible and that all elements of the fluid move uniformly at the same speed at any particular cross-section. We will now proceed to write down the energy-densities that are relevant for a steady-steady flowing fluid.

Kinetic energy-density

Let us think what happens in a fluid flowing through a pipe system where the pipe either narrows or widens. If the system is in a steady-state, the continuity equation tells us that current must be constant throughout the pipe system. Because we are dealing with an incompressible fluid that has fixed density, there is a simple relation between the velocity of the fluid and the cross-sectional area of the pipe. Figure 5.3.3 depicts a segment of a pipe where it becomes wider.

Figure 5.3.3: Fluid Flow in a Non-uniform Pipe.



The same volume element, dV, in the narrow part of pipe has to pass through the wider section of the pipe at the same rate. We can rewrite the volume element, dV, as the product of its cross-sectional area, A and the horizontal distance, dx: $dV = Adx$. The rate at which the fluid moves perpendicular to the pipe's cross-sectional area (or horizontally in Figure 5.3.3) is defined as the fluid velocity, $v = dx/dt$. Thus, we can write Equation 5.3.1 for flow rate as:

$$I = \frac{dV}{dt} = A\frac{dx}{dt} = Av \tag{5.3.3}$$

Applying the continuity equation, which states that current must remain the same in a steady-state fluid system, between the narrow and wide regions of the pipe system in Figure 5.3.3, we find that:

$$A_1 v_1 = A_2 v_2 \tag{5.3.4}$$

The above result tells us that if the cross-sectional area changes, then the velocity of the fluid must change to keep the flow rate constant. The fluid speeds up when entering a narrower section and slows down when entering a wider segment of the pipe.

Alert

Although, the volumetric flow rate, or current, and the fluid velocity are both related to the rate at which the fluid moves, these quantities describe different fluid properties. Flow rate is the rate of volume per unit time and has units of, m^3/s. Fluid velocity is the rate at which the fluid moves on average in the direction perpendicular to the cross-sectional area of the container confining the fluid and has units of speed, m/s. Fluid velocity describes average motion since each fluid particle is moving in random directions, but when averaged over all the particles there is a net "ordered" motion in the direction of the fluid velocity. Current and velocity are related to each other, since the greater current the greater the velocity. However, once the fluid is in a steady-state the current which is determined by the properties of the entire system will stay constant throughout the fluid system from one location to another. The fluid velocity, on the other hand, will change if the cross-sectional area of the pipe changes.

The total energy E_{total} of the fluid element of volume dV will consist of the internal energy U (thermal plus all bond systems) plus any macroscopic energies. At first, we will neglect friction and assume that internal energy stays constant, which describes *non-dissipative* flow. As established in Equation 5.3.4 the speed of the fluid will change when the pipe's cross-sectional area changes. Since speed is an indicator of kinetic energy this implies that a change in area will result in a change of translational *kinetic energy-density*, "density" since we describe fluids in terms of their "energy-density". We will omit in this treatment rotational motions, like the vortices that sometimes form when water goes down the drain of your bathtub, since in our model the flow is laminar.

Using the expression for translational kinetic energy $KE = \frac{1}{2}mv^2$, dividing by volume to convert to an energy-density, and then using $\rho = m/V$, we find that change in kinetic energy-density between two locations is given by:

$$\frac{\Delta KE}{V} = \frac{1}{2}\rho\Delta(v^2) \tag{5.3.5}$$

Based on Equation 5.3.4 the change in kinetic energy-density will be non-zero only when the pipe's cross-sectional area will change between the two points being analyzed. It is only the difference in the areas between the initial and final locations that will matter. For example, if a pipe gets wider in the middle of the interval analyzed but then returns to its original width at the end, the speed of the fluid will also return to its original value, and the kinetic energy will not change between the start and the end of the pipe.

In Section 5.2 we established how gravitational potential energy-density and pressure change with depth. For a dynamic fluid system we will typically ignore any height variations within a horizontal pipe since pipes are typically too narrow to result in any significant pressure changes even within a more dense liquid. However, if we are considering a pipe which is not positioned horizontally, such as water flowing downward from a reservoir at high elevation or from a water tower, or water flowing upward to the second floor of a house from a water tank on the ground floor, changes in gravitational energy-density need to be considered.

Combining pressure and gravitational potential energy-density from Section 5.2 and kinetic energy-density terms, the conservation of energy-density equation becomes:

$$\Delta P+\rho g\Delta y+\frac{1}{2}\rho\Delta(v^2)=0 \tag{5.3.6}$$

The equation above is often referred to as the *Bernoulli equation*.

Thermal energy-density

In many physical systems although friction is never completely absent, it is often negligible due to other dominating effects. For example, you can determine that a ball falling from a few meters conserves its mechanical energy to a great precision. Thus, in the case of the ball we can neglect the effects of air friction. This is no longer true for a piece of paper where its small weight and larger surface area make air friction a much greater effect.

Likewise, in some cases friction in fluid flow is negligible compared to other transfers of energy. The traditional Bernoulli Equation 5.3.6 does not include friction or any internal energy-density changes. Whether frictional effects need to be taken into account in a flowing fluid depends on many factors, some having to do with the properties of the fluid itself, others having to do with the geometrical properties of the pipe or channel confining the fluid, and still others relate to the rate of fluid flow and the type of flow.

We treat frictional effects in fluid flow the same way we did in the energy-interaction model, by including a thermal energy term or defining an open system which loses energy as work due to friction. We can extend Bernoulli's equation to include frictional effects by adding a *thermal energy-density* term:

$$\Delta P+\rho g\Delta y+\frac{1}{2}\rho\Delta(v^2)+\frac{\Delta E_{th}}{V}=0 \tag{5.3.7}$$

Although we now have a general energy conservation equation to use with many common fluid systems, we can make it much more useful by representing the rate of energy transfer to the thermal system in terms of two variables: the first is the fluid flow rate and the second is what is called the *resistance* of the particular section of the channel we are analyzing. We first give a plausibility argument for why this works.

Let us consider how friction comes into play in a fluid flowing through a pipe. Molecular attractions exist between the molecules of the fluid and the walls of the pipe. This will cause the molecules closest to the pipe to be essentially stationary. So molecules a little further away from the wall of the pipe will have to slide past the molecules nearer the wall. But this sliding involves the momentary making and breaking of bonds as the molecules slide past each other, which leads to the creation of additional random molecular motion. Random molecular motion, of course, is precisely what thermal energy is.

The amount of thermal energy generated by molecules sliding past one another should be less if the average fluid velocity is less. This will occur if the rate of flow is reduced. It will also occur if the diameter of the pipe is increased, even if the overall amount of fluid flowing through the pipe remains the same. The amount of thermal energy generated should also depend on the fluid itself. Molasses molecules do not slide past each other as readily as water molecules, for example. Viscosity is the fluid property of interest here. It turns out that it is possible to incorporate these various factors into the two parameters volumetric flow rate and resistance, which incorporates the fluid properties and the properties of the medium in which the fluid is flowing. In other words, resistance is the parameter incorporating all factors that contribute to energy dissipation, or friction, other than current. When resistance is multiplied by current this results in the energy transferred to the thermal system per unit volume of fluid:

$$\frac{\Delta E_{th}}{V}\equiv IR \tag{5.3.8}$$

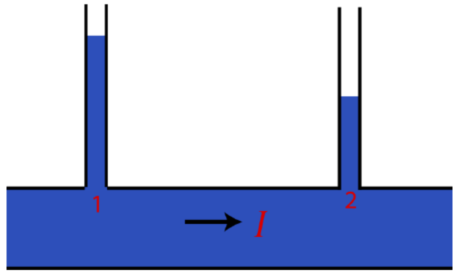
It is convenient to rewrite Equation 5.3.7 in terms of an open system with the energy-density of the system is transferred to thermal energy:

$$\Delta P+\rho g\Delta y+\frac{1}{2}\rho\Delta(v^2)=-IR \tag{5.3.9}$$

Our definition of the resistance, R, does not require it to be independent of the current, I. For most fluids, however, the resistance is independent of current for flow rates up to a certain critical value. Then it jumps up to a higher (usually non-constant) value as the flow becomes turbulent. Both current and resistance are positive quantities, so the term $-IR$ is always negative. However, when frictional effects are included it is now important that the energy-density terms are analyzed in the direction of current, since friction increases in the direction of motion. In other words, when using Equation 5.3.9, "Δ" in the left-hand side of the equation has to represent the difference between two locations, "p_2-p_1", where p_2 is downstream compared to p_1.

Figure 5.3.4 demonstrates a system where *dissipative flow* is apparent. In this case the steady-state fluid is flowing horizontally in a pipe with uniform area. Thus, there is no change in gravitational or kinetic energy-density from point 1 to point 2 in the figure. Equation 5.3.9 simplified to $\Delta P=-IR$. The decrease in pressure from points 1 and 2 is seen here by introducing *standpipes* into the fluid system. A standpipe is connected to a fluid system with the top of the standpipe exposed to the atmosphere. Initially, as fluid flows past a standpipe the higher pressure of the flowing fluid will push against the air which is at a lower atmospheric pressure. The fluid will continue to rise in the standpipe until the gravitational potential energy-density difference compensates the pressure difference between between the top and the bottom of the fluid in the standpipe, and equilibrium is established.

Figure 5.3.4: Dissipative Fluid Flow.

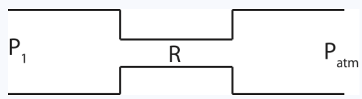


Once the system is in steady state, the fluid in a standpipe is not flowing. Although the flowing fluid system and the fluid in the standpipe indicate two different fluid systems, since one has a zero current and the other is flowing, the pressure at their boundary has to be equivalent. Thus, the height of the fluid in standpipe indirectly gives us the gauge pressure of the flowing system. For example, the difference between pressure right below standpipe marked 1 in Figure 5.3.4, P_1 and atmospheric pressure is $\Delta P = P_1 - P_{atm} = \rho g h$. The larger the height in the standpipe the greater is the pressure of the flowing fluid below, since the greater pressure is able to push more fluid upward against atmospheric pressure.

Figure 5.3.4 shows that as the fluid flows to the right the pressure drops as seen by the drop of height of the fluid in the standpipes. It turns out that for steady-state laminar flow resistance scales linearly with distance. Thus, the pressure drop between equidistance points would be the same as long as the properties of the pipe do not change over that segment. The longer the fluid flows the more its molecules experience frictional forces with other molecules and pipe surfaces, the greater will be the loss of its energy-density, or pressure in this case, to thermal energy.

Example 5.3.1

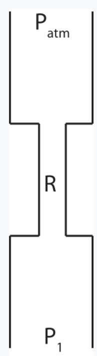
Shown below is a segment of a pipe system in which water flows in a steady-state to the right. The wider pipes have equal areas and negligible resistance. The narrower pipe has resistance of $2500 Js/m^6$. For all three questions below, assume that the pressure P_1 on the left is kept constant, and the pipe is open to the atmosphere on the right.



a) You replace the narrower pipe with another pipe that has half the area and a different resistance. The water now moves 1.5 times slower in the new narrower pipe. Calculate the resistance of the new narrow pipe.

b) Now you equally increase the areas of the wider pipes (their resistance is still negligible). Everything else is as in the **original set-up** (not with the modified pipe from part a). Does the speed in the wider pipes increase, decrease, or stay the same? What about the speed in the narrower pipe?

c) **The original set-up** is now vertical as shown below. Does the current increase, decrease, or stay the same compared to the **original set-up** on the top of the page?



Solution

a) The pressure difference is kept constant across this segment, so the different resistance when a new narrow pipe is placed will result in a different current. Applying Bernoulli's equation across the system:

$$\text{Before}: P_{\text{atm}} - P_1 = -I_{\text{old}} R_{\text{old}}$$

$$\text{Before}: P_{\text{atm}} - P_1 = -I_{\text{new}} R_{\text{new}}$$

Since the left-side of each equation are equal, we can set the right-side of each equation equal to each other:

$$I_{\text{old}} R_{\text{old}} = I_{\text{new}} R_{\text{new}}$$

Using $I = Av$, we can re-write the above equation as:

$$A_{\text{old}} v_{\text{old}} R_{\text{old}} = A_{\text{new}} v_{\text{new}} R_{\text{new}}$$

and solve for R_{new}:

$$R_{\text{new}} = \frac{A_{\text{old}}}{A_{\text{new}}}\frac{v_{\text{old}}}{v_{\text{new}}}R_{\text{old}} = 2 \times 1.5 \times 2500\frac{Js}{m^6} = 7500\frac{Js}{m^6}$$

b) If we apply Bernoulli's equation across the system, $P_{\text{atm}} - P_1 = -IR$, increasing the area of the wider pipes does not change the current, since R is only non-zero in the narrower portion of the fluid system. Thus, the current will be the same but since $I = Av$, the increase in area will result in a slower speed in the wider pipes. For the narrower pipe, there was no change in area, so the speed will stay the same.

c) In this case when we apply the Bernoulli's equation across the system:

$$P_{\text{atm}} - P_1 + \rho g h = -IR$$

There is an additional gravitational potential energy-density term which is positive, when we subtract top minus bottom. Since ΔP is kept constant, this implies that the current must decrease. Conceptually, the fluid now has to flow vertically upward, so it will flow slower (smaller flow rate), given the same pressure difference (assuming it's sufficient to make the fluid flow uphill).

Adding Energy from Outside Systems: Pumps

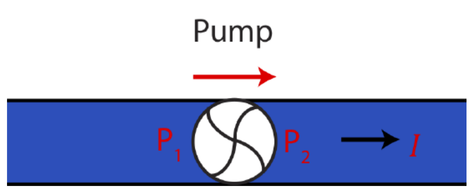
When we are analyzing dissipative fluid-flow phenomena, we still restrict ourselves to steady-state phenomena. But without an outside source of energy, the fluid energy-density systems would gradually transfer all of their energy to the thermal energy-density system and the flow would stop. Also, how is it ever possible to make a fluid flow uphill without providing some energy source? This outside energy comes from a *pump*. To include pumps we add a term to the right-hand side of the extended Bernoulli equation that represents the amount of energy per unit volume transferred into the fluid energy-densities. That is, we have created an open system in which energy added from outside sources. The fully extended *complete Bernoulli equation* becomes:

$$\Delta P + \rho g \Delta y + \frac{1}{2}\rho\Delta(v^2) = \frac{E_{\text{pump}}}{V} - IR \tag{5.3.10}$$

In biological systems an example of such as pump is the heart in our circulatory system which allows blood to flow around the body and up to the brain.

Figure 5.3.5 below shows a pump which is pumping fluid to the right (a pump must have a direction). Once the pump is turned on, and the system reaches an equilibrium steady-state, the current has to stay constant as it passes the pump. Since the pipe is horizontal and does not change area, the pump creates a greater pressure after the pump, $P_2 > P_1$, since Equation 5.3.10 simplifies to, $\Delta P = \frac{E_{\text{pump}}}{V}$, assuming dissipation is negligible between points 1 and 2. Note, the term $\frac{E_{\text{pump}}}{V}$ is positive as long as Bernouilli equation is analyzed in the direction of the pump.

Figure 5.3.5: Fluid Flow with a Pump.



Alert

We have stressed that for steady-state system the current remains constant. However, intuitively introducing friction to your system should slow down fluid flow or introducing a pump should speed it up. How can these seemingly contradictory ideas be resolved? A constant current in steady-state systems implies that the current is the same everywhere in that particular fluid system. But it does not imply that currents must be equal in all fluid systems. Thus, if you have a pump moving your fluid, the strength of that pump establishes the magnitude of the current which is the same before and after the pump. If you suddenly turn up the strength of the pump, you create a new fluid system with a different pump energy resulting in a faster current. This larger value of current will the same throughout this new fluid system once steady-state is reached. Likewise, if a segment of a pipe suddenly gets partially blocked, increasing the overall resistance, the fluid will slow down until it reaches a steady-state. Once the steady-state is reached and the physical properties of the system remain unchanged, the flow rate will remain the same throughout the system, even in the section with with little resistance compared to one with high resistance.

We will write the fully extended Bernoulli Equation 5.3.10 in yet one more way for several reasons. The first reason is simply to connect with experts who deal a lot with liquids, such as soil and water scientists and civil engineers, who use the term *head* with each of the specific terms for the fluid energy-density. These terms are *pressure head*, *gravity head*, and *velocity head*. Together, all three terms are called the *total head*. Also, rewriting Equation 5.3.10 in terms of total head, makes its similarity to electric circuits, which we discuss in the next section, more obvious. In terms of total head, the fully extended Bernoulli Equation 5.3.10 can be simply written as:

$$\Delta(\text{total head}) = \frac{E_{pump}}{V} - IR \tag{5.3.11}$$

We have finally arrived at the basic fluid transport equation. It can be applied along any continuous current path between whichever two points we specify. The change in the fluid energy-density (encompassed in the total head) depends explicitly, of course, on the location of the two points along the pipe. The pump term will be present if there is in fact a pump between the two chosen points. The thermal energy-density term depends on the resistance between the two points.

Alert

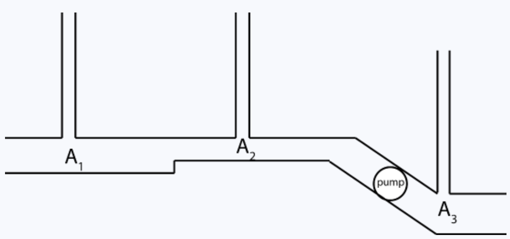
Intuitively, one might think that friction slows down fluid flow and pumps speed it up. In fact, from previous analysis of conservation of energy in 7A we analyzed systems where objects slow down due to effects of friction and speed up when external forces do work on them. Therefore, when first encountering fluid dynamics it is tempting to associate effects of dissipation with a decrease in kinetic energy-density and a pump doing work on the fluid with an increase in kinetic energy-density. However, in fluids kinetic energy-density is related with the amount fluid that flows per unit time, or the flow rate. As described in this section, the continuity equation for a steady-state system ensures that flow rate stays constant throughout the system, and kinetic energy-density can only change if the cross-sectional area of a pipe changes. This kinetic energy-density is associated with motion which is perpendicular to the cross sectional area of the pipe. However, there is also kinetic energy-density associated with random particle motion, and we call this pressure. When a fluid enters a narrower pipe more of its motion is directed parallel to the pipe and less motion is in random directions, thus kinetic energy-density increases and pressure drops. Likewise, friction decreases the overall random motion, i.e. pressure, but the kinetic energy-density stays the same since it is fixed at a steady-state. Likewise, pumps increase pressure or the average random kinetic energy of the fluid particles, while kinetic energy-density stays the same as a consequence of conservation of mass.

Example 5.3.2

A segment of a fluid system is shown below with three equally spaced standpipes. The top of each standpipe is open to the atmosphere. The pump is pumping water downhill. The areas in regions 1 and 3 are equal, and the area in region 2 is smaller: $A_2 < A_1 = A_3$. Region 3 is lower in height than regions 1 and 2.

a) Assuming there is no resistance in the pipe, rank the water levels in each stand pipe.

b) The pipe system becomes eroded, so resistance becomes significant and is uniform throughout the pipe. Assume speed of the fluid in region 1 is $0.5m/s$, $A_1 = 4cm^2$, and $A_2 = 2cm^2$. The water level decrease from standpipe 1 to 2 by $0.3m$, and then increase from standpipe 2 to 3 by $1.2m$. The height difference between regions 2 and 3 is $70cm$. Assume water is flowing through the pipe with density, $\rho_{water} \sim 1000kg/m^3$. Find the resistance of the horizontal pipe between regions 1 and 3, and find the energy of the pump per unit volume. Show your work.



Solution

a) Writing down only the non-zero terms of the full Bernoulli's equation between regions 1 and 2:

$$\Delta P + \Delta KE = 0$$

Since the area decreases from 1 to 2, the speed will increase, $A_1 v_1 = A_2 v_2$ *, resulting in an increase of kinetic energy-density. Therefore, the pressure must decrease,* $P_2 < P_1$ *.*

Between regions 2 and 3, the Bernoulli's equation is:

$$\Delta P + \Delta KE + \Delta PE_g = \frac{E_{pump}}{V}$$

Since the area increases from 2 to 3, the speed will decrease resulting in a decrease of kinetic energy-density. The height is lower in region 3, so the gravitational potential energy will also decrease. The pump adds energy to the system, thus, the left side of the equation must be positive. Since the other two terms are both negative, the change in pressure must be positive. Therefore, the pressure must increase, $P_3 > P_2$ *.*

Between regions 1 and 3, the Bernoulli's equation is:

$$\Delta P + \Delta PE_g = \frac{E_{pump}}{V}$$

The argument is the similar to above, except there is no change in area. Thus, since the change in gravitational potential energy-density is negative, and the pump adds energy to the system, the pressure must increase, $P_3 > P_1$ *.*

Putting the three results together we find, $P_3 > P_1 > P_2$ *. The higher the pressure, the higher will be the height in each standpipe, so* $h_3 > h_1 > h_2$ *.*

b) Use the given information for region 1 to find the flow rate:

$$I = A_1 v_1 = 4 \times 10^{-4} m^2 \times 0.5 m/s = 2 \times 10^{-4} m^3/s \quad .$$

The speed in region 2 can be found using the continuity equation, $A_1 v_1 = A_2 v_2$ *:*

$$v_2 = v_1 \frac{A_1}{A_2} = 0.5 m/s \times 2 = 1.0 m/s .$$

The pressure difference between the standpipes can be determined from the height difference information. Below each standpipe the pressure is, $P_{below} = P_{atm} + \rho g h$ *, where h is the height of the fluid level in the stand pipe. The difference between any two standipes is related to the difference in the heights of the fluids that fill the standpipes,* $\Delta P = \rho g \Delta h$*.*

There are two unknowns here, the resistance and the pump energy. Thus, first solve for the resistance between regions 1 and 2, since there is no pump present in this region:

$$\Delta P + \Delta KE = -IR$$

$$R_{12} = -\frac{\Delta P_{12} + \Delta KE_{12}}{I} = -\frac{\rho g (h_2 - h_1) + \frac{1}{2}\rho (v_2^2 - v_1^2)}{I} = -\frac{1000 kg/m^3 \times 10 m/s^2 \times (-0.3 m) + \frac{1}{2} \times 1000 kg/m^3 \times (1.0^2 - 0.5^2) m^2/s^2}{2 \times 10^{-4} m^3/s}$$

$$= 1.3 \times 10^7 \frac{Js}{m^6}$$

.

Since the standpipes are equally spaces and resistance is uniform, the total resistance between standpipes 1 and 3 is double the resistance between regions 1 and 2. So, $R_{13} = 2.6 \times 10^7 Js/m^6$ *. Applying Bernoulli's equation between regions 1 and 3 to find the pump energy-density:*

$$\Delta P_{13} + \Delta PE_{g,13} = \frac{E_{pump}}{V} - IR_{13}$$

$$\frac{E_{pump}}{V} = \rho g (h_3 - h_1) + \rho g (y_3 - y_1) + IR_{13} = 1000 kg/m^3 \times 10 m/s^2 \times 0.9 m + 1000 kg/m^3 \times 10 m/s^2 \times (-0.7 m) + 2 \times 10^{-4} m^3/s \times 2.6$$
$$\times 10^7 Js/m^6 = 7200 \frac{J}{m^3}$$

In the equation above $\Delta h = h_3 - h_1$ *is the height difference between the water levels in standpipes 3 and 1. Since the water levels drops from 1 to 2 by 0.3m and then increases from standpipe 2 to 3 by 1.2m,* $h_3 - h_1 = 0.9 m$*. The height difference between regions 2 and 3 is* $\Delta y = y_3 - y_1 = -70 cm = -0.7 m$ *.*

Power in Relation to Fluid Flow

In general, power is simply the rate of energy transfer. Each term in our fluid transport equation represents either a change in an energy-density ΔP, $\Delta PE_g/V$, and $\Delta KE/V$) or a transfer of energy per unit volume of fluid IR and E_{pump}/V). If we want to determine the amount of energy change that occurs in the fluid as it passes between two points along a pipe per time, we need to multiply the energy change per volume by the volume of fluid passing through the pipe and divide by the time. But volume divided by time is just the current. So the power associated with each energy-density term, is simply the the change in energy-density multiplied by the current. In the case of transfer terms, power is simply the energy transferred per volume multiplied by the current:

$$Power = \frac{\Delta E}{t} = \frac{\Delta E}{V} \times \frac{V}{t} = \frac{\Delta E}{V} I \tag{5.3.12}$$

It is almost universal to use the symbol P for power, and we will follow this custom. You will need to be sensitive to the context of the equation to know whether "P" means power or pressure. This will not be confusing, if you always think about the meaning of the equation in which "P" appears.

Exactly how we algebraically express power will depend, of course, on the particular context, i.e., exactly which section of the pipe or channel we are focusing on and which energy-density system we are focusing on. There is not one single expression. Rather, we have to use the basic meaning of power and apply it appropriately to each context. Several of these are listed below.

Rate of change of all fluid energy densities:

$$P = |\Delta(\text{total head})| I \tag{5.3.13}$$

Rate at which energy is transferred into the fluid by a pump:

$$P = \frac{E_{pump}}{V} I \tag{5.3.14}$$

Rate energy is transferred into the thermal energy-density from the fluid energy densities:

$$P = I^2 R \tag{5.3.15}$$

Contributors

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5.4: Electric Circuits

From Fluid to Electric Charge Flow

Another type of steady-state system is the flow of electric charge in an electric circuit. Although, they are seemingly very different system as we will shortly see, the application of energy conservation to electric flow results in an analogous model to that of fluid flow. Instead of a fluid flowing, current electricity (as opposed to static electricity) involves the flow of electric charge. To create intensive energy systems we divide energy by electric charge, rather than by volume as we did for fluids.

Fluid Circuits

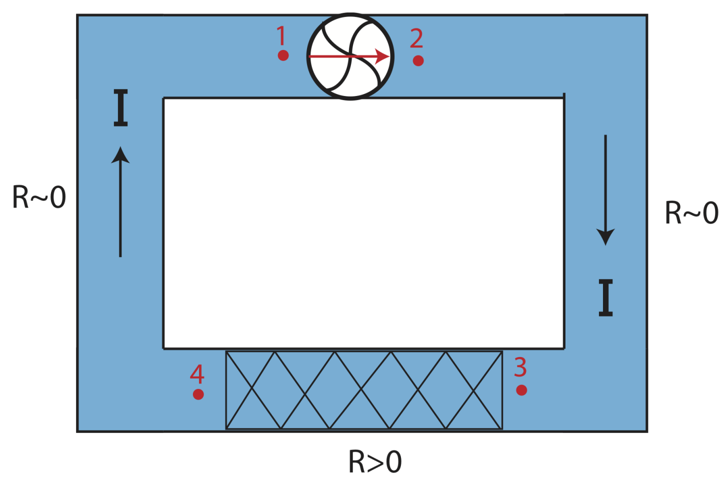
The examples we worked through in Section 5.3 have shown how we can use the steady-state energy-density model to calculate various fluid flow parameters given sufficient details about the physical situation. We have mostly focused on segments of a fluid system where there are changes in the physical properties of the system that result in changes in energy-density or addition of energy with a pump or dissipation of energy due to resistance. Now we extend this analysis to *circuits*. We begin by summarizing the components of the steady-state energy-density model we developed in the context of fluids and which we will now generalize to the flow of electric charge. The complete energy-density Equation 5.3.11 as applied to fluid phenomena is given by:

$$\Delta(\text{total head}) = \frac{E_{pump}}{V} - IR \tag{5.4.1}$$

The above equation states that the change in the total fluid energy-density as we move from one point to another point in the stead-state flow will increase due to energy added by a pump and will decrease due to the transfer of fluid energy-density to thermal energy-density.

Let us define a *fluid circuit* to represent a system where the fluid flows in a circular manner or in a loop as shown in Figure 5.4.1 below.

Figure 5.4.1: Fluid Circuit.



The fluid circuit above has a pump between points marked 1 and 2. The pump pushes the fluid to the right causing a current, I, in the clockwise direction. We will assume that only one section of the pipe system between points marked 3 and 4 has significant resistance, $R > 0$, and the pipe in the rest of the circuit has negligible resistance, $R \sim 0$. (We do this to make a clear analogy to electric circuits as you will see below.) The view of the circuit above is a top view so that the pipe is horizontal throughout the entire circuit. Thus, there is no change in gravitational potential energy-density anywhere along the circuit. The pipe also has uniform area throughout, so the Equation 5.4.1 for this circuit simplifies to:

$$\Delta P = \frac{E_{pump}}{V} - IR \tag{5.4.2}$$

The pressure difference in the equation above depends on which particular part of the circuit being analyzed. Let us analyze the energy-density changes for the specific locations, 1-4, shown in the circuit in Figure 5.4.1.

Across the pump:

$$P_2 - P_1 = \frac{E_{pump}}{V} \tag{5.4.3}$$

Across the pipe with negligible resistance:

$$P_3 - P_2 = 0; \quad P_1 - P_4 = 0 \tag{5.4.4}$$

Across the pipe with non-zero resistance:

$$P_4 - P_3 = -IR \tag{5.4.5}$$

If we were to add the four equations above that would analyze the entire circuit going from 1 and back to 1 clockwise in the direction of the current. Adding up the left-hand-sides of the four equations, we find they add up to zero. This is because we went around the entire circuit and returned back to the original location "1". Since energy-density is conserved in a steady-state system, the pressure at location 1 is fixed, energy cannot be created or destroyed. In other words, $\Delta P = 0$ when going around the circuit. Adding the right-hand side of the 4 equations above we arrive at:

$$0 = \frac{E_{pump}}{V} - IR \tag{5.4.6}$$

Rearranging we find that:

$$I = \frac{E_{pump}/V}{R} \tag{5.4.7}$$

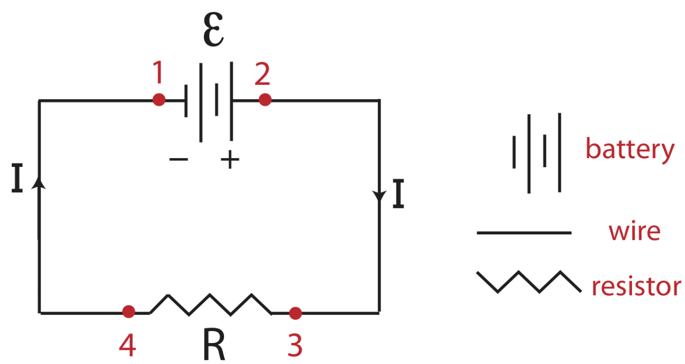
As was stressed in Section 5.3 although the current in a steady-state system is the same everywhere within that system, the value of that constant current depends on the strength of the pump and the amount of resistance present. Equation 5.4.7 demonstrates exactly that. The larger the strength of the pump the larger the current, while the more resistance the system has the smaller the overall current. In Section 5.3, we have treated the current as an independent variable, as we looked at drops in total head for fluid systems. However, frequently, we deal with complete circuits. The current is no longer an independent variable, but rather the resistance(s) and pump(s) determine the current that exists in the circuit.

Electric Circuits

The electric circuit depicted in Figure 5.4.2 below is analogous to the fluid circuit in Figure 5.4.1. Instead of fluid flowing through pipes, electric charge is flowing through wires. Wires are depicted as a straight lines making right angles as charges move through them in a circuit. The "mechanism" that adds energy to the system is a battery or a power source labeled as $\mathcal{E}$ in the circuit below, which stands for emf or *electromotive force* of that power source. The analogous mechanism in the fluid circuit is the pump which adds energy to the system and allows the fluid to flow. In a fluid circuit where the pipes are horizontal, the fluid would not flow without a pump. Likewise, the charges would not flow without a battery. Like the pump which has a direction, the batteries positive and negative terminal determines the direction of electric current. The symbol for battery shown in Figure 5.4.2 has a long line which indicates the positive side and a short line which indicates a negative sides. The double lines in the symbol arise from the traditional two-cell batteries, each long and short line pair representing one cell. Charge will flow around the circuit since charge is attracted to the opposite charge.

In addition, as in fluid flow pipe properties introduce resistance to flow, *resistors* introduce resistance to charge flow. We separate resistors from wires by indicating them by a zigzag line as shown below. Wires typically have negligible resistance, thus we treat them as having zero resistance relative to the resistance of resistors.

Figure 5.4.2: Electric Circuit.



There is an important point we need to get very clear about right from the start. When charge flows in an electric circuit it is electrons, the negative charges, that flow from the negative terminal of the battery to the positive side in a counterclockwise direction in circuit in Figure 5.4.2. However, *electric current* is defined by convention as the flow of positive charge from the positive to the negative terminal of the battery, which is in the clockwise direction in Figure 5.4.2. Historically, positive charge was defined in a way that makes the charge on an electron negative. Now, when we speak of current as being in a particular direction, we mean positive charge flow. So, if that charge flow is due to the motion of electrons, then those electrons are in fact moving in the opposite direction. We will always emphasize *charge flow*, not the flow of the charge carriers, such as electrons, when using the steady-state energy-density model with electrical phenomena.

Electric current, I, is the amount of charge that flows past a particular point per unit of time. Electric charge has units of *coulombs*, abbreviated C. The unit of electric current is the *ampere* or *amp* for short, abbreviated with a uppercase letter "A", such that $A \equiv C/s$. This is analogous to the volumetric flow rate which has units of volume per second, so it describes the amount of fluid flowing per unit time rather than the amount of charge flowing per unit time for charge flow.

The Complete Energy-Density Equation for Electric Circuits

In one way, current electricity is simpler than dissipative fluid flow. With fluids we have three energy-density systems that all contribute to the total head. In current electricity, there is only one energy system: the *electric potential energy per charge*. Since the mass of charge carriers and velocities are so small, both the gravitational potential energy and the kinetic energy changes are totally negligible compared to the changes in the electric potential energy. When dividing energy by electric charge, we turn the extensive electric potential energy, which depends on the amount of charge, into an intensive quantity. The electric potential energy per charge is given the name *electric potential*. Another common way to call the electric potential is *voltage*, which is what we will do from now on. Voltage has SI units of *volts*, abbreviated with uppercase "V". Since electric potential has units of energy per charge, a volt is a joule per coulomb, $V = J/C$.

Batteries convert chemical energy (bond energy) into electric potential energy. Electromotive force, $\mathcal{E}$ is an energy per charge and has units of volts, just like for electric potential. Common practice today is to speak of voltage instead of emf when referring to batteries and generators. Thus one commonly hears phrases such as, “The voltage of a ‘D' battery is 1.5 volts”.

Resistance to the flow of a fluid causes a transfer of energy from the fluid energy-density to thermal energy-density. Likewise, in electric circuits, resistance to the flow of charge causes transfer of electric potential energy to thermal energy-density. In both cases the amount of energy-density transferred is equal to the product of the current and the resistance. That is, $\Delta E_{th}/C = IR$. The unit of *electrical resistance* is the *ohm* with abbreviation Ω. Using the four electric components just discussed, voltage, emf, current, and resistance, the complete energy-density equation for electric charge becomes:

$$\Delta V = \mathcal{E} - IR \tag{5.4.8}$$

The meaning of Equation 5.4.8 is completely analogous to the meaning of the complete energy-density Equation 5.4.1 used for fluid flow phenomena. The arguments we made in developing the fluid version of the energy-density Equation 5.4.1 apply to current electricity as well. If there are no sources or energy transfer into or out of the electric charge system, then the electric potential does not change. But if we attach batteries or generators, we put energy into the system. If there is a current and charge flows through conductors that have resistance, then electric potential energy per charge will be converted to thermal energy, which decreases the electric potential.

As with fluid circuits, we must always remember that the complete energy-density Equation 5.4.8 applies to two specific points along the current path. The algebraic sign of the "IR" terms also works the same way. If we move in the direction of positive charge flow, i.e., in the direction of the current, then "IR" is positive, and the minus sign insures that voltage decreases in as we move in that direction. This is often referred to as a *voltage drop*, indicating that the value of ΔV across a resistor in the direction of current is negative.

As we did with the fluid circuit, let us apply Equation 5.4.8 across various points, 1-4, marked in Figure 5.4.2.

Across the battery:

$$V_2 - V_1 = \mathcal{E} \tag{5.4.9}$$

Across the wires:

$$V_3 - V_2 = 0; \quad V_1 - V_4 = 0 \tag{5.4.10}$$

Across the resistor:

$$V_4 - V_3 = -IR \tag{5.4.11}$$

When we add the four equation we find as we did for a fluid circuit, we find that the voltage around the circuit adds up to zero. If we add the right-hand sides of the 4 question and solve for current, we will get an equation analogous to Equation 5.4.7:

$$I = \frac{\mathcal{E}}{R} \tag{5.4.12}$$

The current in an electric circuit remains constant throughout the circuit since the flow is steady-state. But the magnitude of the current depends on the energy provided to the circuit by the battery and the amount of resistance present in the circuit. In the next section we will analyze circuits with more complex sets of resistors and batteries, but we will discover that all circuits can be reduced to the simplest circuit shown in Figure 5.4.2, and Equation 5.4.12 can be used to find the total current coming out of a battery in any circuit.

Power Relationships

The power relationships for current electricity are completely analogous to those for fluids shown in Section 5.3.

Rate of change of the electric potential energy:

$$P = |\Delta V| I \tag{5.4.13}$$

Rate energy is transferred into the electric potential by a battery or generator:

$$P = \mathcal{E} I \tag{5.4.14}$$

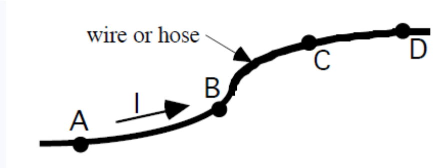
Rate energy is transferred into the thermal system from electric potential energy or voltage:

$$P = I^2 R = \frac{(\Delta V)^2}{R} \tag{5.4.15}$$

Using dimensional analysis we can see that these equations give us units of power, $W = J/s$. Voltage has units of energy per charge, $V = J/C$, and current is charge per using time, $A = C/s$. Multiplying units of voltage and current we find the units of power, J/s.

Example 5.4.1

Suppose a wire or a hose is carrying a steady-state current. The wire must be attached to batteries or other sources of emf, and be part of a larger closed circuit, but we will focus only on one segment of the circuit. Likewise, if this was a hose it would represent a section of a larger fluid system. We will not distinguish wires from resistors and will just assume that the wire has some internal resistance. (The shape of *section* of wire or hose in the figure is meant to represent any general section of wire or hose. It continues in both directions.)



a) Suppose the current in the wire is $10A$ and the resistance per meter of wire length is $0.01\Omega/m$. Find the voltage drop between points ***A*** and ***B***, if the length of wire between points ***A*** and ***B*** is $200m$.

b) A voltmeter measures a voltage drop from ***B*** to ***C*** of -16 V. Find the resistance in segment BC.

c) Let us consider the analogous fluid example. Imagine that we now have a section of constant diameter fire hose carrying a current of $1.0\times10^{-3}m^3/s$. The resistance per meter of length of hose is $1.0\times10^{6}Js/m^7$. Find the drop in total head in going from ***A*** to ***B*** which is 200m long.

d) Is the change in pressure the same, greater, or smaller than the change in total head you found in part c)?

e) Find the power loss from ***A*** to ***B*** for both the wire and hose scenarios.

Solution

a) The resistance of the wire between points A and B is

$$R_{AB}=0.01\frac{\Omega}{m}\times200m=2\Omega$$

The voltage drop, $\Delta V=-IR$ *is:*

$$\Delta V_{AB}=V_B-V_A=-IR_{AB}=-10A\times2\Omega=-20V$$

b) The current is constant throughout the wire, so a smaller voltage drop implies a smaller resistance in the BC segment compared to AB segment.

$$R_{BC}=-\frac{\Delta V_{BC}}{I}=-\frac{-16V}{10A}=1.6\Omega$$

c) The resistance of the hose between points A and B is

$$R_{AB}=1.0\times10^{6}\frac{Js}{m^7}\times200m=2.0\times10^{8}\frac{Js}{m^6}$$

The total head change, $\Delta(\text{total head})=-IR$ *is:*

$$\Delta(\text{total head})_{AB}=-1.0\times10^{-3}\frac{m^3}{s}\times2.0\times10^{8}\frac{Js}{m^6}=-2.0\times10^{5}Pa=-2.0atm$$

d) Total head represents the sum of changes in energy-densities. The hose has uniform area, so there is no change in kinetic energy-density between A and B. However, point B is higher than point A, so there is an increase in gravitational potential energy:

$$\Delta(\text{total head})_{AB}=\Delta P+\Delta PE_g=-2.0atm$$

Since ΔPE_g is positive, to get a negative change in total head the change in pressure must be negative as well and with a great magnitude than the change in total head. Thus, the change in pressure is greater than the change in total head.

e) The power loss of the electric charge or the fluid system from ***A*** *to* ***B*** *is simply the product of the energy drop (voltage or total head) and the current:*

electric: $P=|\Delta V|I=20V\times10A=200W$

fluid: $P=|\Delta(\text{total head})|I=2.0\times10^{5}Pa\times1.0\times10^{-3}m^3/s=200W$

Household Electricity

Utility companies such as PG&E and SMUD supply our workplaces and houses with *alternating current* (AC) electricity. The current varies sinusoidally, at 60 Hz, switching directions 120 times each second. The fundamental ideas we have developed to understand *direct current* (DC) circuits (as from a battery) can also be applied to AC sources. In fact, the values of the voltages and currents used when dealing with 60 Hz AC electricity are typically the root mean square values, which makes all the algebraic relationships we have developed applicable. Therefore, unless stated otherwise, values of voltages and currents for 60 Hz AC can be treated as DC values.

The primary reason AC is used in power distribution systems is the ease with which voltages can be changed. The large round "cans" hanging on power poles are *transformers*. In a typical residential power distribution system, the wires at the top of the pole are often at 12,000 to 22,000 V. The transformer steps this voltage down to 120 V and 240 V, the voltage of wires entering most apartments and houses. We will study how transformers work in Physics 7C when we get into the fascinating world of the interaction of electricity and magnetism.

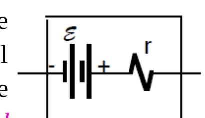
Household appliance	Typical power consumption [Watts]
Wall clock	4
Boom box	10
Light bulb	100
TV	100
Fan	200
Washer	550
Vacuum	600
Refrigerator	800
Toaster	1100
Iron	1100
Dishwasher	1200-1500
Hair dryer	1000-1300
Oven	4000-8000
Clothes dryer	5000-8000
Range	12,000-16,000

You probably use several small appliances everyday that make use of a "step-down" transformer. Many small computer peripherals have a small black box with two prongs sticking out that plugs into the 120 V power-outlet strip. The 120 V from the power strip is dropped down to 6 to 12 V in the *isolation t*ransformer. This is an excellent safety feature. Voltages in the 10 to 20 V range are relatively harmless to humans, if contact is limited to skin and not to internal organs. The 120 V in the wall outlet can cause sufficient currents through a person's body when contact is made through skin and is definitely considered dangerous. The major health risk from 120 V shocks are currents in the chest region, which can cause the heart to go into fibrillation. Much larger currents cause burns, and ironically, can also be effective in stopping fibrillation of the heart. This is exactly what an AED, automated external defibrillator, does.

Every outlet in our homes can be considered a source of constant voltage of about 120 Volts. Each appliance has a characteristic resistance R, and this determines how much power is used by this appliance when it is turned, using $P = \dfrac{\Delta V^2}{R}$. The table on the rights gives some typical power values. When you pay your PG&E or SMUD electrical bill, they charge you for energy, not power. The unit they use, however, sounds like a power unit: kilowatt-hour. Can you explain why this is an energy unit?

Internal Resistance of Power Sources

Most wires used in house wiring, for appliances, and in lab, have very low resistance. In particular, the voltage drop across wire are small compared to other voltage changes in the circuits. Therefore, we typically model them as having zero resistance. Similarly, new batteries have resistance that are also small compared to the resistance of other components in the circuit to which the battery is attached. However, the *internal resistance* of batteries, labeled "r'" in the figure here, does increase over time, as the reactant chemicals inside turn into by-products that impede the flow of electrons through the battery. A 1.5V battery that is almost "used up" still provides charges that pass through it with almost 1.5 joules per coulomb. That is, it might still have an emf of nearly 1.5 V. But due to its high internal resistance *r*, most of this voltage is converted to thermal energy inside the battery itself when the battery is connected into a circuit. A battery that is very warm when it is being used is probably almost completely depleted.



"real" battery with internal resistance r

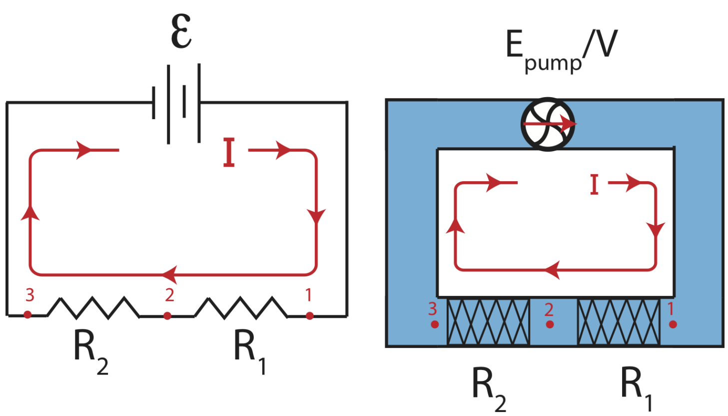
5.5: Resistors in Parallel and Series

Up to this points we have mostly treated resistance as some general value in a given circuit or segment of a flow system, whether fluid or electric. Sometimes, however, it is useful to look deeper at individual elements in a flow system which have high resistance. In fluid circuits you might have section of a pipe with different resistance values. In an electric circuit, for example, we often have more than one light bulb which act as resistors. These light bulbs might have difference resistance and they might be wired in various way that we will shortly see. In addition, we have only considered a flow system, where there is only one possible path for the current. Here we will consider more complex system where the current has multiple paths such as a fluid channel that splits into several channels, or circuit wiring that allows current to flow along different wires.

Resistors in Series

When two or more resistors (whether they are electric elements or different section of a pipe) have the same amount of current flowing through them, the resistors are in *series*. In other words, the current that flows through the first resistor follows the same path to flow through the remaining resistors that are all in series. An example in Figure 5.5.1 below shows an electric circuit with two resistors and an analogous fluid circuit where two sections of a pipe have different resistances.

Figure 5.5.1: Resistors in Series.



Our goal is to figure out how the individual resistances, R_1 and R_2, are related to the overall resistance of the entire circuit. This combined resistance is known as the *equivalent resistance*. It implies that if we were to replace all the resistors with one equivalent resistor it would have the same effect, or result in the same current, as the individual resistors. We assume that for both circuits the resistance is zero elsewhere in the circuit, the wires or other sections of the pipe. Let us analyzing the electric circuit first between points 1 and 2 and then between points 1 and 3:

$$V_2 - V_1 = -IR_1 \tag{5.5.1}$$

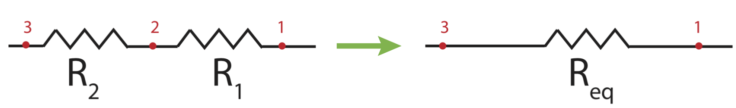
$$V_3 - V_2 = -IR_2 \tag{5.5.2}$$

Adding the two equation gives us the voltage drop across both resistors:

$$V_3 - V_1 = -I(R_1 + R_2) \tag{5.5.3}$$

Let us replace the two resistors by an equivalent resistor, R_{eq} as shown in the Figure 5.5.2 below.

Figure 5.5.2: Equivalent Resistance in Series.



Finding the voltage drop across the equivalent resistor:

$$V_3 - V_1 = -IR_{eq} \tag{5.5.4}$$

and combining Equations 5.5.3 and 5.5.4 we get the following expression for combing resistors in series:

$$R_{eq} = R_1 + R_2 \tag{5.5.5}$$

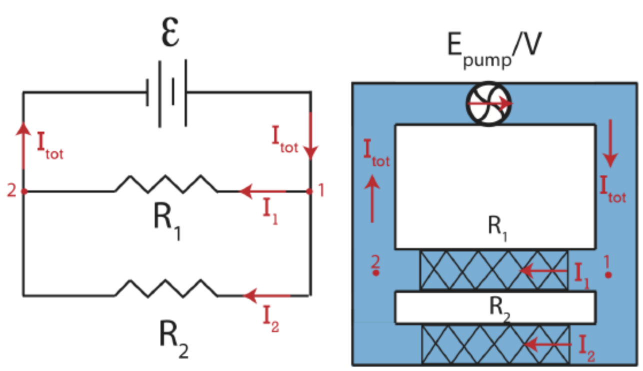
More generally, for an n number of resistors in series, the equivalent resistance is:

$$R_{eq} = \sum_i^n R_i \tag{5.5.6}$$

Resistors in Parallel

There is another way in which resistors can be arranged in a circuit, known as *parallel resistors* as depicted in Figure 5.5.3 below. Once we understand how the current flows when resistors are in parallel, we will see advantages of wiring resistors in this manner.

Figure 5.5.3: Resistors in Parallel.



In the electric circuit in Figure 5.5.3 we see that a total current, I_{tot}, reaching a point 1 after which there are two path for the current to travel, either through resistor R_1 or resistor R_2. This location in a circuit where the current is able to split is known as a *junction*. This circuit still describes steady-state electric flow, thus the current has to be constant. This implies that when the current splits into two paths the sums of the currents in the individual paths have to add to the current coming in:

$$I_{tot} = I_1 + I_2 \tag{5.5.7}$$

Likewise, the current coming out of the junction as points 2 has to equal to the current, I_{tot} that entered the junction at 1. In other words, since current is conserved, the amount of current leaving the battery has to equal to the amount of current coming back to the battery. This is analogously true to the fluid circuit where the current splits into two separate pipes at a junction 1 and recombines at junction 2.

Another concept that could be applied here is the fact that when you apply the transport equation around the circuit as we did in Section 5.4 the change in energy-density has to be zero around the entire circuit. If we applied this concept to the upper path in Figure 5.5.3, following the current that goes through R_1, we find that going this way, say from point 1, through R_1, through the battery, and back to point 1 give us:

$$-\Delta V_1 + \mathcal{E} = 0 \tag{5.5.8}$$

Similarly, if we follow the lower path in Figure 5.5.3, from point 1, through R_2, through the battery, and back to point 1 we get:

$$-\Delta V_2 + \mathcal{E} = 0 \tag{5.5.9}$$

Combining Equations 5.5.8 and 5.5.9 we get:

$$\Delta V_1 = \Delta V_2 \tag{5.5.10}$$

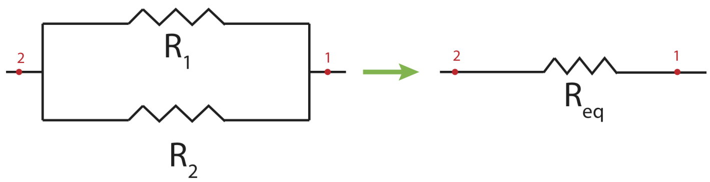
This bring us to the definition of *parallel resistors*:

1. The sum of the currents through each parallel resistor adds to the total current going into a junction where the current splits.

2. The voltage drop across resistors that are in parallel is the same.

Our next goal is to figure out how to determine an equivalent resistance for parallel resistors. In other words, if we were to replace two parallel resistors R_1 and R_2 as shown in Figure 5.5.4 below by an equivalent resistor R_{eq} how would their resistances be related?

Figure 5.5.4: Equivalent Resistance in Parallel.



Rewriting Equation 5.5.10 in terms of currents and resistances we find:

$$I_1R_1 = I_2R_2 \tag{5.5.11}$$

This results tells us that the path with larger resistance will have smaller current and vice versa. Intuitively, it is logical that the path that has least resistance will get more current. Think about the fluid circuit in Figure 5.5.3. When the fluid reaches the junction a point 1, there are two paths to go. If one of the paths contains a narrow and mostly clogged pipe with high resistance, naturally, majority of the fluid will choose the other path. This is consistent with the result in Equation 5.5.11.

Figure 5.5.4 shows the replacement of parallel resistors R_1 and R_2 with R_{eq}. For resistors in series, the equivalent resistor has the same current as the two original resistors. Here, for the same argument that led us to Equation 5.5.10, the voltage across the equivalent resistors is the same as the voltage across the original parallel resistors.

The current that goes through R_{eq} is the same as the current that enters the junction in the original circuit. Using $\Delta V = -IR$ to rewrite current conservation, $I_{eq} = I_1 + I_2$, in terms of voltage drops we get:

$$\frac{\Delta V_{eq}}{R_{eq}} = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2} \tag{5.5.12}$$

Since the voltage drops are the same for both resistors and the equivalent resistor the above equation simplifies to:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{5.5.13}$$

More generally, for an n number of resistors in parallel, the equivalent resistance is:

$$\frac{1}{R_{eq}} = \sum_i^n \frac{1}{R_i} \tag{5.5.14}$$

Generalized Rules

Kirchhoff Loop Rule: for any complete loop in a circuit (no matter how complicated the path appears, and how many batteries and resistors are in the loop), the total increase in potential caused by the emf of batteries or generators must equal the total voltage drops caused by all resistors in the loop. This *loop rule* in electricity is a formal statement of energy-density conservation. Mathematically the *loop rule* is written as:

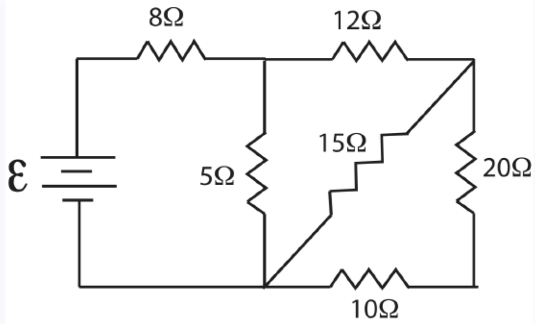
$$\Delta V_{\text{loop}} = \sum \mathcal{E} - \sum IR = 0 \tag{5.5.15}$$

Kirchhoff Junction Rule: the sum of currents going into a junction has to equal to the sum of currents coming out of a junction. This *junction rule* in electricity is a formal statement of conservation of current. Mathematically the *loop rule* is written as:

$$\sum I_{in} = \sum I_{out} \tag{5.5.16}$$

Example 5.5.1

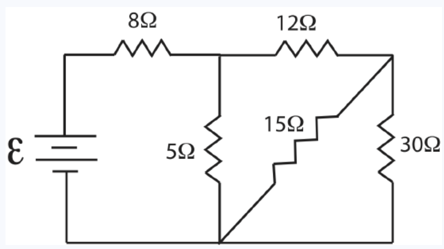
Shown below is a circuit with multiple resistors.



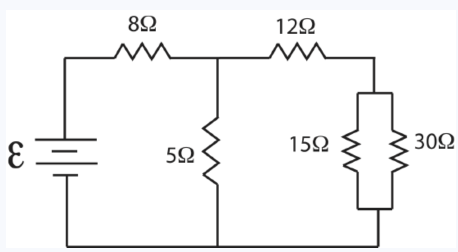
Find the equivalent resistance, R_{eq}, for this circuit.

Solution

Let's think about the path the current takes. After leaving the battery the first junction is after the 8Ω resistor. The current splits and part of it goes through the 12Ω resistor and part through the 5Ω resistor and back to the battery. The fraction of current that went through the 12Ω resistor meets another junction, such that part of that current goes through 15Ω resistor. Another part goes through the 20Ω, which also goes through the 10Ω resistor. It is not always obvious in which order to combine resistors, and sometimes there are several ways of doing it. One rule of thumb is if there is a resistor parallel to multiple resistors in series, you need to combine the series resistors first. Thus, let us start with combining the 20Ω and the 10Ω resistors, calling the combination $R_{1,eq} = 20\Omega + 10\Omega = 30\Omega$. After this step the circuit simplified to the following diagram.



Now recall the current split after the 12Ω resistor, so the 15Ω resistor is parallel to the now combined 30Ω resistor. To make it simpler to visualize since the 15Ω resistor is drawn in a diagonal manner, we can redraw the above circuit like this.

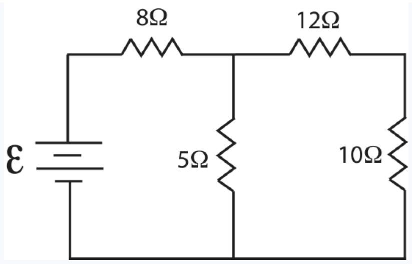


Now it is more clear that 15Ω and 30Ω resistors are in parallel. Let us call their combined resistance, $R_{2,eq}$:

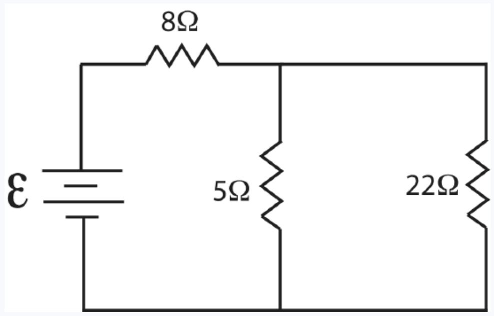
$$\frac{1}{R_{2,eq}} = \frac{1}{15\Omega} + \frac{1}{30\Omega} = \frac{1}{10\Omega}$$

$$R_{2,eq} = 10\Omega$$

The further simplified circuit looks like the following diagram.



The 12Ω *and* 10Ω *resistors are in series,* $R_{3,eq} = 12\Omega + 10\Omega = 22\Omega$. *The simplified circuit now looks like this:*

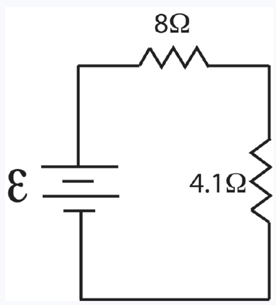


Next, we see that 5Ω *and* 22Ω *resistors are in parallel:*

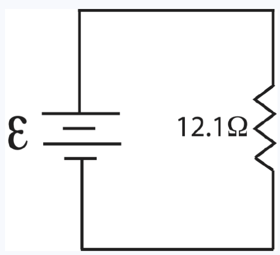
$$\frac{1}{R_{4,eq}} = \frac{1}{5\Omega} + \frac{1}{22\Omega} = 0.245\frac{1}{\Omega}$$

$$R_{4,eq} = 4.1\Omega$$

The diagram is now:



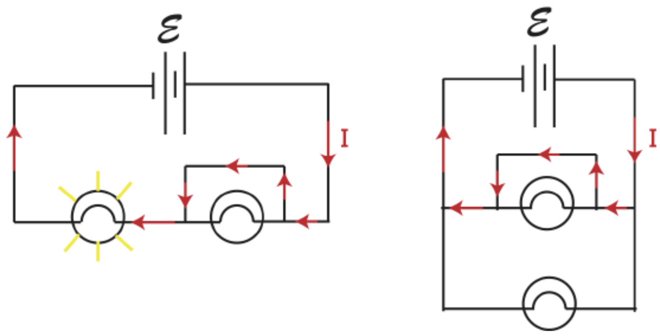
And finally combing the two remaining resistor in series, we obtain the equivalent resistance of this circuit, $R_{eq} = 8\Omega + 4.1\Omega = 12.1\Omega$.



Special Cases

You must have heard the expression "short circuit" before, which you most likely associate with something going wrong with electronics, possible leading to overheating and electric sparks. We will now describe what a "short circuit" is using our developed circuit analysis tools. Imagine that you initially built a circuit with a battery and either two light bulbs in series as in the left diagram in Figure 5.5.5 or with two light bulbs in parallel as on the right diagram below (the circle with the curved line inside is a standard symbol for a light bulb). While wiring these circuits by mistake an extra wire is added across a resistor as drawn below.

Figure 5.5.5: Shorted Resistor.



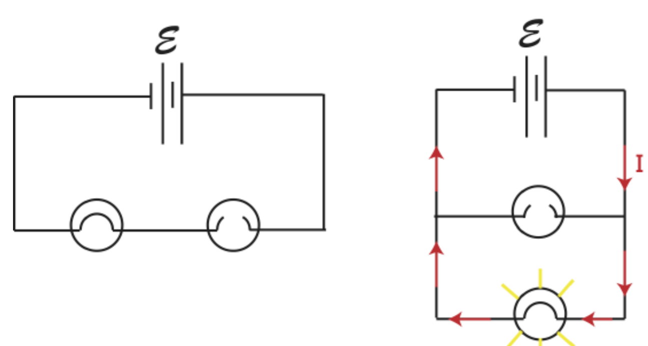
Let us carefully what what happens in this case. We know that wire have nearly zero resistance, and as a result there is no voltage drop across a wire. We also know that when things are wired in parallel, the voltage drops across each element are the same to conserve energy. If the voltage drop across the wire is zero, that means that the voltage drop across the light bulb parallel to the wire has to be zero as well. The only way for the voltage across a resistor to be zero is if there is no current going through that resistor due to $\Delta V = -IR$. Another way to look at this is to refer back to Equation \label{paths} which tells us the relative currents when they are split at a junction. In this case, there is one path that has zero resistance, implying that all the current will go through that path and none through the path with resistance.

This path of the current with the extra wire is depicted with arrows in each circuit in Figure 5.5.5. In the left circuit one of the light bulbs with the wire across it will not be lit since all the current goes through the added wire. But all the current still goes through the second light bulb to return to the battery. It is as if the added wire removed that light bulbs completely, leaving a simple circuit with one lit light bulb. Thus, the first light bulb was *shorted out*. Also, an important thing to note is that when the wire is added the value of the total current coming out of the battery changes since the equivalent resistance of the circuit decreases from, $R_1 + R_2$ to just R_1. A decrease in resistance means that the total current increased, since $I = \mathcal{E}/R_{eq}$. Since the current increased more current will flow through the one lit light bulb, and it will be brighter compared to the original circuit since, $P = I^2R$. The more power dissipated through the light bulb, the brighter it will shine.

For the circuit on the right in Figure 5.5.5 the wire is again placed across one of the light bulbs. In this case the current has three paths to chose from, the wire, and the two paths with a light bulb. Since current chooses the path with least resistance and the wire has zero resistance, all of the current will go through the wire, and both light bulbs will not be lit. What we are left with is a battery connected directly to a wire with extremely low resistance implying that the current is extremely high. This large current may result in the wire heating up or even catching on fire, and is clearly not a safe situation. This is the scenario which is known as *short circuit*.

The two scenarios below depict another situation where one of the light bulbs *burns out*.

Figure 5.5.6: Burnt Resistor.



When a light bulb burns out, the connection is broken, and current can no longer flow along that path. In the circuit with two light bulbs in series, the current only has one path, and since there is no longer a closed circuit, both light bulbs will not be lit. This is reason that in old-fashioned Christmas lights the entire row of lights would not be lit if even just one light burnt out, making it impossible to know easily which light to change. Now there are more clever mechanisms to deal with this problem.

In the circuit with parallel light bulbs, on the other hand, the current will still have a closed path with through the other light bulb as shown in the figure. Thus, the unburned light bulb will stay lit. In addition, the burnt out light bulb will not change the brightness, or power dissipated, of the second light bulb in the circuit with two parallel light bulbs. Why would this be since the total resistance of the circuit changes from $R/2$ (if both light bulbs have equal resistance) to R. In the original circuit, the total current was $I = 2\mathcal{E}/R$ but it split in half, each light bulb getting a current of $\mathcal{E}/R$. That is exactly the amount of current the lit light bulb receives when the one of the light bulbs burns out, since the circuit is reduced to a simple circuit with a battery and a resistor. Or if you simply apply the loop rule, you can see that the burning of one light bulb does not change the bottom loop, resulting in, $\mathcal{E} - \Delta V = 0$.

If batteries are hooked up in series, their internal resistance (if any) will add, just as do their emfs. For example, four "AA" cells of 1.5 volts each used in series used in some electronic device would have a total emf 6 volts, but will also have four times the internal resistance of one cell. Because of its larger size, a “D” cell, while still having an emf of 1.5 V, will have considerable less internal resistance than will a “AA” cell.

When similar size batteries are wired in parallel in a circuit, the voltage across each battery will be the same (since they are hooked together with low resistance wire. The potential drop across each is the same. But their internal resistance and actual values of $\mathcal{E}$, will determine how much current gets into the rest of the circuit and how much is “wasted” “going around in circles” among the batteries. In general, it is not a good idea to hook up batteries in parallel.

Contributors

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5.6: Circuit Problem Solving

Solving Electric Circuit Problems

When tackling a circuit problem you may need to figure out the equivalent resistance of the circuit, voltage drops across resistors, total current coming out of the battery or current through specific resistors, power dissipated by resistors or provided by the battery, relative brightness of light bulbs in a circuit, the effect of a shorted resistor, or a burnt one, and more.

Below are a few useful steps to follow. Even though these basics steps are provided, it is never a good idea to follow a procedure verbatim. This procedure is a good starting guide of tacking some circuit problems, but might not apply to all of them in the order provided. As you start working with more advanced scenarios, think of ways of how you can work backwards, follow the steps in different order, or only use some of the steps to solve your particular problem most efficiently.

Circuit problem solving procedure:

1) Calculate the **equivalent resistance** of the circuit. First combine all the series resistors and then calculate the parallel ones. Use the following equations:

$$\text{series}: \quad R_{eq} = \sum_i^n R_i \tag{5.6.1}$$

$$\text{parallel}: \quad \frac{1}{R_{eq}} = \sum_i^n \frac{1}{R_i} \tag{5.6.2}$$

2) Use your result of equivalent resistance to find the **total current** coming out of the battery:

$$I_{tot} = \frac{\mathcal{E}}{R_{eq}} \tag{5.6.3}$$

3) Apply the **loop rule** to all the loops present in the circuit in order to find the relationship between voltage drops and emf of battery. Make sure you are consistent with the direction of the loop you choose. If your loop takes you from a negative to the positive terminal of the battery, you will get, $+\mathcal{E}$. If you loop takes you across a resistor in the direction of the current, then the correct sign of the voltage difference should be negative, $\Delta V = -IR$. In general for any given loop, the following must be true:

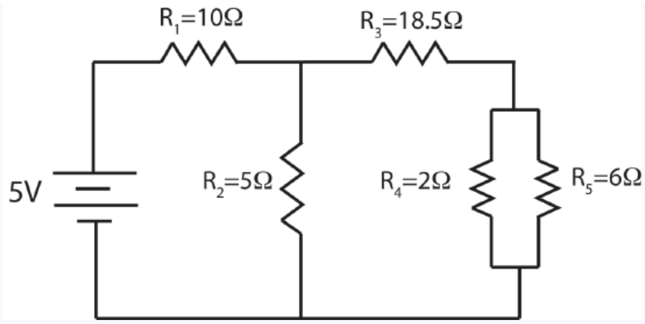
$$\sum \mathcal{E} + \sum \Delta V = 0 \tag{5.6.4}$$

4) Apply the **junction rule** at all the junctions to find the relationship between the current going into the junction and the individual currents in each of n paths:

$$I_{in} = \sum_i^n I_i \tag{5.6.5}$$

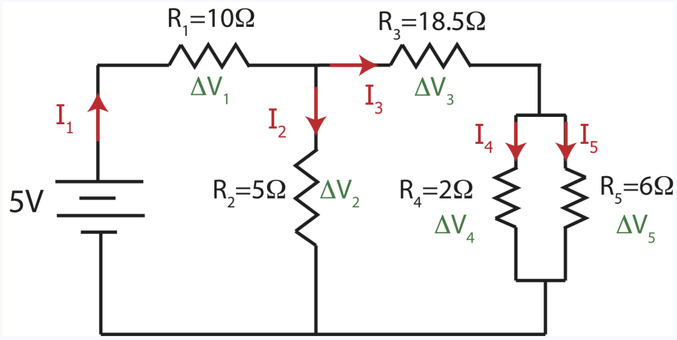
Example 5.6.1

For the circuit show below find the current and voltage for each of the five resistors.



Solution

It is a good idea to start by labeling all the currents and voltage as shown below.



Since all the resistances are known, the first natural step is to find the equivalent resistance. For this particular circuit the best way to combine resistors is: resistors 4 and 5 are in parallel, their combination R_{45} is in series with R_3. The combination of 3,4, and 5, R_{345} is in parallel with 2, and their combination, R_{2345} is in series with R_1.

First, calculating R_4 and R_5 in parallel:

$$\frac{1}{R_{45}} = \frac{1}{R_4} + \frac{1}{R_5} = \frac{1}{2\Omega} + \frac{1}{6\Omega} = \frac{2}{3\Omega}$$

$$R_{45} = \frac{3}{2}\Omega$$

Combining R_3 in series with R_{45}:

$$R_{345} = R_3 + R_{45} = 18.5\Omega + 1.5\Omega = 20\Omega$$

Combining R_3 in parallel with R_{345}:

$$\frac{1}{R_{2345}} = \frac{1}{R_2} + \frac{1}{R_{345}} = \frac{1}{5\Omega} + \frac{1}{20\Omega} = \frac{1}{4\Omega}$$

$$R_{2345} = 4\Omega$$

And finally combining R_1 in series with R_{2345} to find the equivalent resistance of this circuit:

$$R_{eq} = R_1 + R_{2345} = 10\Omega + 4\Omega = 14\Omega$$

Resistor 1 is the only resistors which is in series with the battery, so the current through resistor 1, I_1, will be equal to the total current coming out of the battery:

$$I_1 = I_{tot} = \frac{\mathcal{E}}{R_{eq}} = \frac{5V}{14\Omega} = 0.357A$$

Once we know the current through 1, we can find the voltage across resistor 1:

$$\Delta V_1 = -I_1 R_1 = 0.357A \times 10\Omega = -3.57V$$

Now we can apply the loop rule to the first loop on the left that goes through the battery, resistor 1, and resistor 2 to find the voltage across resistor 2:

$$\mathcal{E}+\Delta V_1+\Delta V_2=0$$

$$\Delta V_2=-\mathcal{E}-\Delta V_1=-5V+3.57V=-1.43V$$

Since we know the voltage across resistors 2, we can figure out the current through that resistor, I_2:

$$I_2=\frac{-\Delta V_2}{R_2}=\frac{1.43V}{5\Omega}=0.286A$$

Next, we can use the junction rule to find the amount of current that goes to the other branch and through resistors 3, I_3:

$$I_1=I_2+I_3$$

$$I_3=I_1-I_2=0.357A-0.286A=0.071A$$

Knowing the current, allows us to find the voltage drop across R_3:

$$\Delta V_3=-I_3R_3=-0.071A\times 18.5\Omega=-1.3135V$$

Applying the loop rule to the outermost loop we can find the voltage drop across the parallel combination of R_4 and R_5:

$$\mathcal{E}+\Delta V_1+\Delta V_3+\Delta V_{45}=0$$

$$\Delta V_{45}=-\mathcal{E}-\Delta V_1-\Delta V_3=-5V+3.57V+1.3135V=-0.1165V$$

Lastly, since we know the voltage drop across the parallel set of 4 and 5, it must equal to the voltage drops across each one of the resistors, $\Delta V_{45}=\Delta V_4=\Delta V_5$. Using this, we can find the currents I_4 and I_5:

$$I_4=\frac{-\Delta V_4}{R_4}=\frac{0.1165V}{2\Omega}=0.0583A$$

$$I_4=\frac{-\Delta V_5}{R_5}=\frac{0.1165V}{6\Omega}=0.0194A$$

Example 5.6.2

While playing with electronics in class you build a circuit with one battery and five resistors. You calculate the following equivalent resistance:

$$R_{eq}=\left[\frac{1}{R_1+\left(\frac{1}{R_2}+\frac{1}{R_3}\right)^{-1}}+\frac{1}{R_4}\right]^{-1}+R_5$$

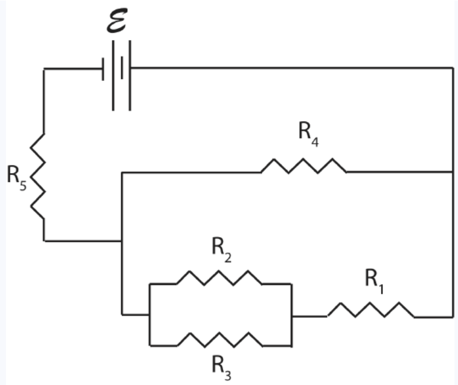
a) Draw a possible circuit connected to a battery that has the above equivalent resistance. Clearly mark each resistor with R_1, R_2, R_3, R_4, and R_5.

b) Assume that $R_{eq}=25\Omega$, $R_4=R_1+\left(\frac{1}{R_2}+\frac{1}{R_3}\right)^{-1}$, and $R_2=2R_3$. If the circuit is connected to a 10V battery, how much current will flow through R_2?

c) If R_2 was shorted out, what would be the new equivalent resistance in terms of the resistor numbers? Would the total current coming out of the battery increase, decrease, or stay the same?

Solution

a) The equation above states that R_5 is in series with a parallel branch which contains R_4 in one path and $R_1+\left(\frac{1}{R_2}+\frac{1}{R_3}\right)^{-1}$ in the other. The other path has R_1 in series with a parallel branch containing R_2 and R_3 as shown.



b) The total current coming out of the battery is

$$I = \frac{\mathcal{E}}{R_{eq}} = \frac{10V}{25\Omega} = 0.4A$$

The two paths of the main parallel branch have equal resistance, so the current will split equally at the junction. This means that half the total current will flow through R_1, which is $0.2A$ The current will split further between R_2 and R_3. Since $R_2 = 2R_3$, R_3 will get double the current of R_2 since their voltage drops have to be the same.

$$\Delta V_2 = \Delta V_3$$

$$I_2 R_3 = I_3 R_3$$

$$I_2 = \frac{I_3}{2}$$

Using the fact that $I_2 + I_3 = 0.2A$ and the above result we get:

$$I_2 = \frac{I_3}{2} = \frac{0.2 - I_2}{2}$$

Resulting in $I_2 = \frac{1}{15}A$.

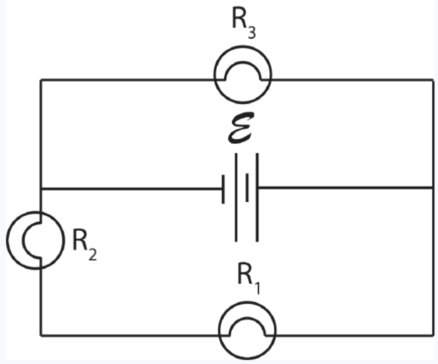
c) If R_2 was shorted out, all the current going through R_1 would go though the wire, so R_2 and R_3 would not get any current and are no longer part of the active circuit. The new equivalent resistance becomes:

$$R_{eq} = R_5 + \left(\frac{1}{R_4} + \frac{1}{R_1}\right)^{-1}$$

Since the lower branch now has reduced resistance, the combined resistance between R_4 and the lower branch will decrease, thus the total equivalent resistance of the entire circuit will be decreased. Smaller total resistance means that the total current will increase since, $I = \frac{\mathcal{E}}{R_{eq}}$.

Example 5.6.3

You build the circuit shown here and find that the brightness of light bulb 1 twice as bright (double power) as 3, and the brightness of light bulb 1 is half as bright as 2 (half power).



a) You know that light bulb 3 has resistance of 36Ω. Find the resistance of light bulbs 1 and 2.

b) You add a wire across light bulb 1. Describe what happens to the brightness of each light bulb (gets brighter, gets dimmer, stays the same brightness, or is not lit) compared to the original circuit.

c) If light bulb 1 in the original circuit burnt out instead, what happens to the brightness of each light bulb (gets brighter, gets dimmer, stays the same brightness, or is not lit) compared to the original circuit?

Solution

a) Resistors R_1 and R_2 have the same current since they are in series, $I_1 = I_2$. Since $P = I^2R$ and $P_2 = 2P_1$ we conclude that:

$$P_2 = I^2R_2 = 2P_1 = 2I^2R_1$$

$$R_2 = 2R_1$$

When it comes to resistors in parallel, it is often simpler to think in terms of voltage drops rather than current. The voltage drop across R_3 has to equal to the voltage drop across the other branch which includes the sum of voltage drop across R_1 and R_2:

$$\Delta V_3 = \Delta V_1 + \Delta V_2$$

Since $R_2 = 2R_1$ and $\Delta V = -IR$, $\Delta V_2 = 2\Delta V_1$, and the above equation becomes:

$$\Delta V_3 = 3\Delta V_1$$

We also know that $P_1 = 2P_3$. Using $P = \dfrac{\Delta V^2}{R}$ we find that:

$$\frac{\Delta V_1^2}{R_1} = 2\frac{\Delta V_3^2}{R_3}$$

Using the result $\Delta V_3 = 3\Delta V_1$ we find that:

$$\frac{\Delta V_1^2}{R_1} = 18\frac{\Delta V_1^2}{R_3}$$

$$R_3 = 18R_1$$

Using $R_3 = 36\Omega$, we find that $R_1 = 2\Omega$, and using $R_2 = 2R_1$, we find that $R_2 = 4\Omega$.

b) Adding a wire across R_1 shorts out R_1, since all the current will go through the wire. So R_1 will not be lit. The voltage drop across R_2 has to increase, since due to the loop rule applied to the bottom loop, all the voltage now drops across rather R_2, $\mathcal{E} = -\Delta V_2$, rather than being split between R_1 and R_2 in the original circuit. Since power is proportional to voltage drop, 2 gets brighter. The voltage drop across R_3 doesn't change due to loop rule for top loop, $\mathcal{E} = -\Delta V_3$, so 3 stays the same brightness.

c) If light bulb 1 burns out, the path on the lower loop is broken (there is no place for current to go), so both R_1 and R_2 will not be lit. The voltage drop across R_3 doesn't change due to loop rule for top loop as in the argument for b), so 3 stays the same brightness.

5.7: The Linear Transport Model

Steady-State Transport Phenomena

We have extensively described two different systems whose behavior can be described with a steady-state transport model, flowing fluid and moving electric charge systems. In both systems, we saw that to have a current there has to be a energy-density difference between two locations in space, such as a voltage or pressure difference. In this section we will generalize this phenomena to other systems that behave in this matter. In the next section we will look at a non steady-state transport model, where current is no longer constant as a function of time.

We generalize the difference in pressure or voltage, or a driving potential that causes current to a *gradient*, which means change in the quantity with change in position. For fluid flow there must be a gradient in the total head. In order to have electric charge flow, there must be a gradient in the electric potential. In each case, the flow is proportional to the inverse of the resistance.

The transport equations for fluids and current electricity (without pumps or batteries) and using the subscripts “F” to denote fluids and “E” to denote electric charge become:

$$\Delta(\text{total head}) = -I_F R_F \tag{5.7.1}$$

$$\Delta V = -I_E R_E \tag{5.7.2}$$

And solving for I results in:

$$I_F = -\frac{\Delta(\text{total head})}{R_F} \tag{5.7.3}$$

$$I_E = -\frac{\Delta V}{R_E} \tag{5.7.4}$$

For any system that abided by this behavior, we can define a general potential, ϕ. Then, the current, I due to the gradient, $\Delta\phi$ is defined as:

$$I = -\frac{\Delta\phi}{R} \tag{5.7.5}$$

There are many phenomena that involve the motion or transport of some quantity that behave similarly to the way fluids and electric charge flow in those sections of a circuit without sources of energy density (pumps or batteries). In steady-state, after all transients have settled out, we can model these phenomena the same way we model fluids and electric charge flow. These are such common and general phenomena that it is useful to collect them together under their own specially named model, *the linear transport model*.

This model has the following main components:

- a current describing flow as a function of time.

- a gradient causing current.

- resistance impeding the flow.

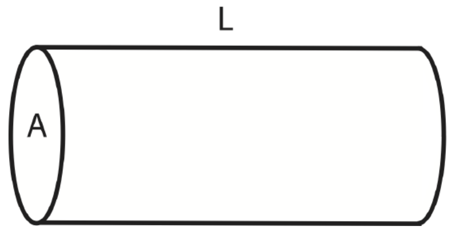
- the gradient is proportional to current and inversely propositional to resistance.

Resistance to flow is typically depends on the material and the geometry of the medium. It can be written in terms of *conductivity*, k, which characterizes the specific material such as a steel or plastic pipe, the cross sectional area of the pipe, the wire, or whatever is constraining the flow, A, and the length of the circuit under consideration, L.

$$R = \frac{1}{k}\left(\frac{L}{A}\right) \tag{5.7.6}$$

This relation says that the resistance increases with the length of the pipe or wire, is inversely proportional to the cross-sectional area, and is inversely proportional to the conductivity of the pipe or wire.

Figure 5.7.1: *Flow through a pipe of length L and cross section A.*



Plugging this into the expression for current in Equation 5.7.5 we get:

$$I = -k\frac{A}{L}\Delta\phi \tag{5.7.7}$$

We can generalize these relations further by defining the *flux* j, whihc is the amount of what is flowing that passes a unit area per unit time.

$$j = \frac{I}{A} \tag{5.7.8}$$

Then the expression for j becomes:

$$j = -\frac{k}{L}\Delta\phi \tag{5.7.9}$$

Our final generalization is to imagine the length, L, shrinking down to infinitesimal size, so that $\Delta\phi$/L becomes simply the derivative of the potential d$\phi/$dx in the direction of the flow, called the gradient of the potential. Hence

$$\lim_{L\to 0}\frac{\Delta\phi}{L} = \frac{d\phi}{dx} \tag{5.7.10}$$

Thus Equation 5.7.9 turns into a *generalized linear transport equation*:

$$j = -k\frac{d\phi}{dx} \tag{5.7.11}$$

In words this relation says that the flux of whatever it is that flows is proportional to a constant that depends on the characteristics of the physical system, which is the conductivity for fluids and electric current and to the gradient of the potential. The faster the potential changes in the direction of the flow, the higher the flow. The gradient of the potential is the driving force that causes the fluid or electric charge to flow in the presence of the resistance. Thinking back to our energy-density model, the gradient of the potential is simply the change (loss) in fluid or electric energy density as you move along with the flow. The minus sign is consistent with the convention that the flux is positive in the direction of flow. Because the gradient of the potential is negative in the direction of flow, the extra minus sign in the relation, makes the flux positive.

The units of the flux and current, conductivity, and potential depend on whatever it is that is flowing. We now look at several specific cases. To get an expression in terms of resistance, we use the relation in Equation 5.7.6.

Fluid Flow

We consider a current flowing through a round pipe of radius r, cross sectional area A and length L. If the flow is in the low-velocity regime (streamline or laminar flow) the expression for resistance is simply the product of two factors. One factor incorporates pipe geometry and the other incorporates fluid properties. We can intuitively see how geometry, the length and area of the pipe (or other similar structure) influences resistance. If fluid flows through the pipes that differ only in length, the longer one will have more loss to the thermal energy-density. If two pipes differ in cross-sectional area, the pipe with the smaller area will have a greater proportion of fluid interacting with the pipe walls, and thus will lose more energy to the thermal energy-density.

Incorporating pipe geometry properties with a fluid property, viscosity, η, we have the following relations:

$$R = (\text{fluid properties}) \times (\text{geometry properties}) \tag{5.7.12}$$

$$R = 8\eta \times \frac{L}{Ar^2} \tag{5.7.13}$$

Note the very strong dependence of flow resistance on the radius of the pipe or tube: inversely proportional to the fourth power of the radius. This has interesting consequences in many common situations, including partially clogged water pipes and arteries. Viscosity has units of Js/m^3. A table of viscosities for several fluids is given below.

Table 5.7.1: Viscosities of selected fluids at selected temperatures.

Fluid at specific temperature	viscosity, η (Js/m^3)
air (20° C)	1.8×10^{-5}
water (20° C)	1.8×10^{-3}
water (20° C)	1.0×10^{-3}
water (90° C)	0.32×10^{-3}
blood (37° C)	4×10^{-3}
light motor oil (20° C)	0.03
glycerin (20° C)	1.5

When the velocity of the fluid is high, the flow is turbulent, a chaotic and irregular flow. The value of the resistance when flow is turbulent is always greater than for smooth streamline (laminar) flow. Because it is complicated to model and not independent of velocity, we won't discuss turbulent flow quantitatively in this course. It is useful to know, however, in case you encounter fluid flow in your future studies, that it is possible to calculate a particular parameter, which is a function of current, fluid properties as well as dimensions and shape of the pipe, that allows one to roughly predict whether laminar or turbulent flow will occur. This dimensionless parameter is called the Reynolds number. The flow is laminar for values of the Reynolds number less than about 2000 and turbulent for values greater than about 3000. The flow is unstable for intermediate values. For now, the important point to remember is simply that as flow velocity increases, the flow will eventually become turbulent and the resistance will increase significantly. This phenomenon of increasing resistance occurs in many common fluid systems including air moving in the ductwork in forced-air heating and air conditioning systems, water flowing in typical household water distribution systems, and blood flowing in arteries.

Electric Charge Flow

As with the resistance to fluid flow through pipes, the resistance to the movement of electric charge can be separated into a factor dependent solely on the geometry of the material through which charge moves and another factor independent of geometry, but dependent on the details of the charge carriers and how they interact with the material through which they move.

Focusing first on the geometry factors, the resistance will increase proportionally with the length, L, and the inverse of the cross sectional area, A. In fluid flow, the geometry independent factor is a property of the fluid itself, viscosity; in charge flow it is a property of the material through which the charge flows. This geometry independent factor is called *resistivity*, ρ, which is the reciprocal of the *conductivity*, $k = 1/\rho$. From Equation 5.7.6 we can determine that the units of conductivity are $1/\Omega m$, thus, the units of resistivity are Ωm. Combining the two factors, the resistance is:

$$R = \frac{1}{k}\frac{L}{A} = \rho\frac{L}{A} \tag{5.7.14}$$

Shown below is a table of resistivities for various materials. Substances that have high conductivities (low resistivities) are good electrical conductors. Metals are usually good electrical conductors. Substances that have low conductivities (high resistivities) are said to be good insulators. Glass and most plastics are good insulators. Note the extremely large range of resistivities for common materials. In most materials there is a small dependence of resistivity on temperature. In semiconductors the dependence of resistivity on temperature is typically larger than for insulators or metals. A strong current dependence arises if the thermal heating, due to the resistance, causes the temperature of the material to increase substantially. This is the case with ordinary tungsten light bulbs, the kind that get hot. The resistance of the filament of these light bulbs increases many times as they go from room temperature to their typical operating temperature, which is several thousand degrees.

Table 5.7.2: Electric resistivity of certain materials.

Material	**Electric Resistivity, ρ (Ωm)**
glass	$10^{10} - 10^{14}$
diamond	2.7
silicon	2.5×10^3
mercury	96×10^{-8}
iron	10×10^{-8}
turngsten	5.5×10^{-8}
aluminum	2.8×10^{-8}
copper	1.7×10^{-8}
silver	1.6×10^{-8}
fat	25
blood	1.5
saturated NaCl solution	0.044

Heat Flow

In heat conduction, the quantity that flows is heat which transfers thermal energy. We are imagining a non-equilibrium situation in which the temperature is hotter at one end of a rod, for example, than at the other end. The driving potential is the gradient of the temperature in the material. The "current" is the heat per time, which is power. Power in heat flow is analogous to the current in fluid or charge flow. Here, power is the amount of thermal energy that is transferred per second past a plane which is oriented perpendicular to the gradient of the temperature. The thermal flux is the power per unit area. Heat flows due to a temperature gradient within a solid, liquid, or gas.

For heat flow in the generalized transform Equation 5.7.11, k is the thermal conductivity in units of Watts per meter Kelvin, W/mK, and $d\phi/dx$ becomes the temperature gradient in units of Kelvin per meter, K/m. Flux is power per unit area resulting in:

$$\frac{P}{A} =- k\frac{dT}{dx} \tag{5.7.15}$$

For an object (window pane perhaps) with cross sectional area A, thickness L, and temperature difference across the surfaces, ΔT, the total heat transported across the object would be:

$$P =- kA\frac{\Delta T}{L} \tag{5.7.16}$$

Metals that are good electrical conductors are usually also good thermal conductors. This is because the movement of "free" electrons in the metal is primarily responsible for both electrical and thermal conduction. However, the vibrations of atoms can also transport thermal energy with little resistance, provided the material is very pure. Diamond is an electrical insulator; its electrons are strongly bound to the carbon atoms and are not free to transport electric charge. However, diamond has a thermal conductivity considerably larger (thermal resistivity smaller) than any metal at room temperature! List of materials and their thermal conductivities is given in the table below

Table 5.7.3: Thermal conductivity of certain materials.

Material	Thermal Conductivity, k (W/mK)
down	0.02
air	0.026
Styrofoam	0.03
fiberglass	0.04
water	0.0597
human tissue (no blood)	0.21
fat	0.17
glass	$0.7-1.0$
wood	0.1
lead	35
iron	74
aluminum	235
copper	400
silver	427
diamond	990

We see from the table that air is an extremely poor conductor of heat (as long as there is no convection; i.e., macroscopic movement of the air), which makes it an excellent insulator. When you bundle up on those ski trips in the mountains, it is most effective to trap air between you and your outerwear with a wool sweater and/or long johns, and wear a down or Holofil™ jacket with many microscopic pockets of trapped air. The walls and roofs of our houses are layered with fiberglass mats or similar material that are designed to trap air in order to be effective insulators. Double-pane windows also trap "dead air" between the outside environment and the inner window.

Diffusion

There are many phenomena in various branches of science that are generally referred to as diffusion. In its basic form, diffusion is simply the net motion (net transport) of a particular species of particles that is due to the presence of random thermal motion that naturally exists at all temperatures above $0K$ and due to to the presence of a concentration gradient of that species of particle. Diffusion of particles occurs in solids, liquids, and gases. Another common example occurs when permeable or semipermeable membranes are present. If a concentration difference of some particle species exists across a membrane that is permeable to those particles, then there will be a flow of those particles across the membrane from the region of high concentration to the region of low concentration. If the concentration gradient is removed, transport ceases. The greater the concentration gradient, the greater the number of particles transported per unit time.

Steady-state diffusion is described by the linear transport model. We re-write the generalized linear transport equation as:

$$j=-D\frac{dc}{dx} \tag{5.7.17}$$

This is historically known as Fick's Law. The letter, D is the diffusion constant. In SI units of square meters per second, \dfrac{m^2}{s} \), c is the particle concentration in SI units of number per cubic meter, $1/m^{-3}$, and $\dfrac{dc}{dx}$ is the particle concentration gradient in SI units of $1/m^{-4}$. The particle flux, j, has units of number per square meter second, $1/m^2 s$. The units of j and D are often expressed in terms of moles rather than number of particles.

Values of diffusion constants range over many orders of magnitude. The diffusion of nuclear magnetization (due to the very weak magnetic moments of nuclei) in pure insulating crystals can be as small as $1.0\times10^{-17} m^2 s^{-1}$ while the diffusion of hydrogen in air at STP is nearly $1.0 \times 10^{-4} m^2 s^{-1}$. The table below lists some representative values of solute-fluid combinations at 20°C and 1 atm pressure.

Table 5.7.4: Diffusion constant for various solutes in various media.

Solute	Medium	Thermal Conductivity, D (m^2/s)
oxygen	water	1.0×10^{-9}
oxygen	air	1.8×10^{-5}
oxygen	tissue	2.0×10^{-11}
hydrogen	air	6.4×10^{-5}
glucose	water	6.7×10^{-10}
sucrose	water	5.0×10^{-10}
hemoglobin	water	6.9×10^{-11}
DNA	water	1.3×10^{-8}
pollen grain	water	1.0×10^{-12}

In steady state conditions, the linear transport model using the constructs of concentration and concentration gradient works well. However, in transient conditions, a better approach is to use the mobility, rather than concentration, and the free energy gradient, rather than the concentration gradient in the linear transport equation. This latter approach avoids having to let the diffusion constant be a function of the concentration.

Other Examples of Linear Transport

As previously indicated, there are many transport phenomena that can be understood with the linear transport model. Many of these phenomena have been given special names (laws). It is easy to not realize that they are all understandable from the general perspective of the linear transport model. For example, in studying subsurface water flow Henry Darcy established in the 1850's with a series of sand column experiments that, for a given type of sand, the volume flow rate, i.e., the current I, was proportional to the head difference and cross-sectional area of the column and inversely proportional to the column length. From what you know about linear transport, you should be able to predict the law that bears his name, Darcy's law, simply by assuming that there is a soil-dependent conductivity (known today as the hydraulic conductivity of the soil) and writing down the general linear transport equation:

$$\frac{I}{A} = k_{soil} \frac{d(head)}{dx}. \tag{5.7.18}$$

Summary

A summary of the algebraic relationships and relationships in *Linear Transport Model* as it applies to fluid flow, current electricity, heat conduction and diffusion is given here.

Flux:

$$j = \frac{I}{A}$$

Generalized linear transport equation:

$$j = -k \frac{d\phi}{dx}$$

Relationship of resistance, R, conductivity, k, and resistivity, ρ :

$$R = \rho \frac{L}{A} \quad k = \frac{1}{\rho}$$

Table 5.7.5: Summary and Comparison of Several Steady-State Transport Systems

Parameter	Laminar Fluid Flow	Electric Flow	Heat Conduction	Diffusion
Transported quantity	Fluid Volume: V $[m^3]$	Electric charge, q $[C]$	Heat, Q $[J]$	Particles, n [number or mol]
Current	Volume per unit time: flow rate I $[m^3/s]$	Charge per unit time: electric current I $[C/s]$, or $[A]$	Heat per unit time: power P $[W]$	Particle number per unit time: particle current I $[1/s]$
Flux: $j=\dfrac{I}{A}$	$[m/s]$	$[A/m^2]$	$[W/m^2]$	$[1/m^2 s]$
Potential, ϕ	Total head: energy per unit volume $[J/m^3]$ or $[N/m^2]$ or $[Pa]$	Voltage, V: energy per unit charge $[J/C]$ or [volts, V]	Temperature, T: [Kelvin, K]	Particle concentration, c: $[1/m^3]$
Potential gradient: $\dfrac{d\phi}{dx}$	$\dfrac{d(\text{head})}{dx}$ $[J/m^4]$	$\dfrac{dV}{dx}$ $[V/m]$	$\dfrac{dT}{dx}$ $[K/m]$	$\dfrac{dc}{dx}$ $[1/m^4]$
Conductivity: (inverse of resistivity)	Fluid conductivity, k $[m^5/Js]$	Electric conductivity, k $[1/\Omega m]$	Thermal conductivity, k $[W/Km]$	Diffusion constant, D $[m^2/s]$
Transport Equation	$j=-k\dfrac{d(\text{head})}{dx}$	$j=-k\dfrac{dV}{dx}$	$j=-k\dfrac{dT}{dx}$	$j=-D\dfrac{dc}{dx}$

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5.8: Exponential Change Model

We have dealt mainly with steady-state conditions in the discussion up to this point. Among other things, this means that the gradient of the potential remains constant in time. This will typically be the case when there is a large reservoir of whatever it is that is being transported on both sides of the region through which the transport is occurring. Then, if the reservoirs remain at constant potential, one higher than the other, the potential gradient remains constant resulting in a constant flow rate. For example, heat will flow from a building kept at a fixed temperature, by an air conditioner, to the outside environment whose temperature is also fixed. Since the difference between the temperatures of the inside and outside remains the same, the rate of heat flow will be constant and the system will remain in a steady-state.

In more general cases, the potential (and the potential gradient) is a function of both time and spatial dimensions. We know that a cold drink will eventually warm up when left outside on hot day, so the potential gradient, i.e. the temperature difference between the drink and the outside will change with time, resulting in a *non steady-state* flow rate or power. By combining the transport equation with the condition of continuity and conservation of energy-density, a differential equation is obtained which will result in an *exponential change* of flow rate.

Exponential Decay

Before we go into physical examples that demonstrate exponential change, we will describe the mathematics of the exponential function. Things grow (or decay) exponentially when the time rate of change in a quantity is proportional to the amount already present. Bacteria grow exponentially because the more you have, the more they reproduce (until they run out of food, or their waste starts to poison the environment). The decay of radioactive elements is exponential because as the number of radioactive atoms decreases, there are fewer available to decay, so less decay occurs. The key idea is that the rate of change depends on the amount present. Mathematically, the previous statement it looks like this:

$$\frac{dy}{dt} = \pm\lambda y \tag{5.8.1}$$

In words the above equation states that the rate of change in y is proportional to y , and the constant of proportionality is λ. If the sign in front of λ is positive, then y is getting bigger (e.g., growth of organisms or compound interest). If the sign is negative, then y is getting smaller (e.g., decay of voltage across a cell membrane or nuclear decay.) Again, this is the key to having exponential change. The rate of change is proportional to the amount present at that time. There is only one function which results in itself with you take it derivative, and it is an exponential: $\frac{d}{dx}e^x = e^x$. In this section we will focus on things that decay exponential, thus we will only see a negative sign in the exponent for *exponential decay* phenomena.

Solving Equation 5.8.1 for $y(t)$ required some calculus. We will focus on the importance of the result and interpretation of exponential decay phenomena. Thus, you can skip directly to the result shown below in Equation 5.8.2 below. But if you are interested in seeing the mathematics of obtaining the result, you can read through the "derivation" box shown below.

Derivation

We start with equation for general exponential decay:

$$\frac{dy}{dt} = -\lambda y$$

The equation above can be rearranged such as the y-dependent terms are on one side and the time term is on the other side:

$$\frac{dy}{y} = -\lambda dt$$

Now we can take the integral of each side of the equation and integrate from initial time, $t = 0$, to same later time t'. Let us define the value of y at $t = 0$ to be y_0. So we integrate y from y_0 to some arbitrary value y'. The "prime" is there to distinguish t' from t and y' from y. Integrating both sides we get:

$$\int_{y_0}^{y'} \frac{dy}{y} = -\int_0^{t'} \lambda dt$$

This results in:

$$\ln(y') - \ln(y_0) = -\lambda t'$$

Using the property of log, $\ln\left(\frac{a}{b}\right) \equiv \ln(a) - \ln(b)$, we get:

$$\ln\left(\frac{y}{y_0}\right) = -\lambda t$$

Using the following property that related the exponential with the natural log:

$$e^{\ln(x)} = x,$$

and taking the exponential of each side of the equation and dropping the "prime", we get:

$$\frac{y}{y_0} = e^{-\lambda t}.$$

Finally solving to y gives us the expression of how y changes as a function of time:

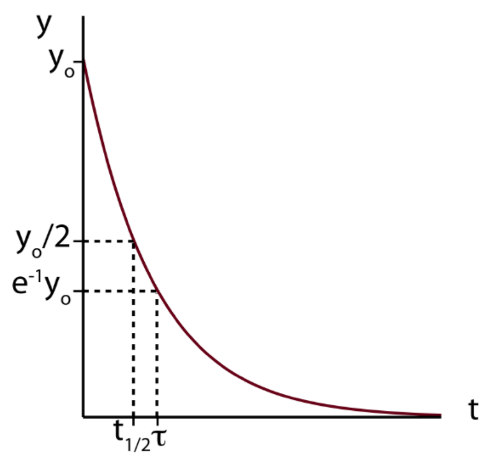
$$y(t) = y_0 e^{-\lambda t}$$

When solving the differential equation 5.8.1 as outlined in the derivation above we arrive at the following expression for the time dependence of the variable y:

$$y(t) = y_0 e^{-\lambda t} \tag{5.8.2}$$

Whenever you get a result like this with an exponential function it is always helpful to calculate the extreme values, in other words, the value of y at $t = 0$ and at $t \gg 0$ to figure out the value which the variable approaches in the case of exponential decay. Using the exponential property, $e^0 = 1$ and evaluating Equation 5.8.2 at $t = 0$ we conclude that $y(t = 0) = y_0$. This is exactly the definition of y_0, the value of y at $t = 0$ (see Derivation). To find the value of y when $t \gg 0$ we can take the limit of $y(t)$ when t goes to infinity. Using the following property, $\lim_{x\to\infty} e^{-x} = 0$, we find that y decays to zero. A plot of y as a function of t is shown below.

Figure 5.8.1: Function Demonstrating Exponential Decay.



Another question we want to address is how fast the decay occurs. A very useful way to measure the rate of this decay is by using a concept of *half-life*. Half-life, $t_{1/2}$ tells us how long it takes for an amount which is changing exponentially to get to the value which is halfway between the original value at $t = 0$ and the final equilibrium value at $t \gg 0$. Half-life has units of time, such as seconds, minutes, hours, or years. The definition of half-life is depicted in Figure 5.8.1 above, the time at which when the amount y has reached a value of $y_0/2$.

Alert

It does not necessarily mean that it took one half-life for the amount to get to half of its original amount, since the final equilibrium value does not have to be zero for exponential decay to occur. For example, if the system starts at a value of 10 and goes to 0, then indeed half-life will mean the time it takes for the amount to get to half of the original value of 10, i.e. 5. However, if the system starts at a value of 10 but approaches 2 as its equilibrium final value, then half-life will be the time it takes for the system to get to 6, which is the amount halfway between 10 and 2.

Now let us see how to calculate half-life from Equation 5.8.2. Since in this case y decays to zero in this case, half-life, $t_{1/2}$, is the time when y reaches half of its original value, $y(t_{1/2}) = \frac{y_0}{2}$. Plugging this into Equation 5.8.2 we find that:

$$y(t = t_{1/2}) = \frac{y_0}{2} = y_0 e^{-\lambda t_{1/2}} \tag{5.8.3}$$

Cancelling y_0 from both sides of the equation and taking the natural log of both sides we get:

$$\ln\left(\frac{1}{2}\right) = -\lambda t_{1/2}$$

Finally, using the natural log property, $\ln\left(\frac{1}{x}\right) = -\ln(x)$ we find that:

$$t_{1/2} = \frac{\ln 2}{\lambda} \tag{5.8.4}$$

The above equation tells us that the larger the value of λ the smaller will be the half-life, the faster the decay and vice versa. (Try plotting the exponential Equation 5.8.2 for different values of λ and observe how it effects the decay rate). Half-life can be easily estimated if you have data or a plot the exponential decay. Another useful measure of decay rate is the *time constant*, which also has units of time and can readily obtained if you know the value of λ. By definition time constant, τ, is the inverse of λ:

$$\tau \equiv \frac{1}{\lambda} \tag{5.8.5}$$

To find the value of y when $t = \tau$ we can plug the definition of τ into Equation 5.8.2:

$$y(t = \tau) = y_0 e^{-\lambda\tau} = y_0 e^{-1} \sim 0.368 y_0 \tag{5.8.6}$$

The value of τ is depicted in Figure 5.8.1. The time constant is slightly larger than half-life since it is the time it take for the system to get to $0.368 y_0$ of its original value which is less than $0.5 y_0$ for half-life. So the system has to decay for a longer time to get to a smaller amount. Half-time and time constant can be related to each other by using Equations 5.8.4 and 5.8.5:

$$t_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2 \tag{5.8.7}$$

Example 5.8.1

A radioactive material decays exponentially from its initial amount of N_0. It take the material 15 years to get to 75% of its initial amount. Find the half life of this material.

Solution

For radioactive decay:

$$N = N_0 e^{-\lambda t}$$

It takes 15 years to get to 75% of N_0, such that $N(t = 15 \text{ years}) = \frac{3}{4} N_0$:

$$y(t = 15) = \frac{3}{4} N_0 = N_0 e^{-15\lambda}$$

resulting in,

$$\frac{3}{4} = e^{-15\lambda}$$

Taking the natural log of both sides:

$$\ln\left(\frac{3}{4}\right) = -15\lambda$$

Using $\ln \frac{1}{x} = -\ln x$ and solving for λ:

$$\lambda=\frac{\ln\left(\frac{4}{3}\right)}{15}=0.01918\,\frac{1}{\text{years}}$$

We can find half-life using Equation 5.8.7:

$$t_{1/2}=\frac{\ln 2}{\lambda}=\frac{\ln 2}{0.01918}=36.1\text{ years}$$

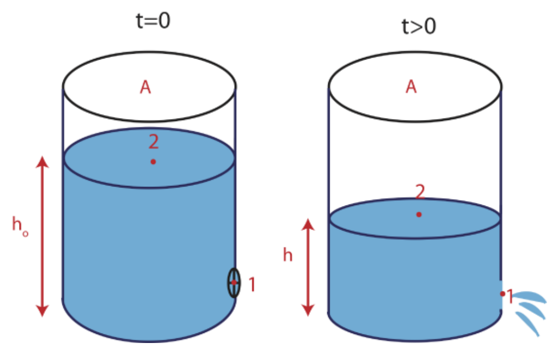
We now have introduced all the important components of exponential decay and are ready to apply them to real physical situations.

5.9: Exponential Fluid Flow

A Leaking Container

The scenario shown below in Figure 5.9.1 is a cylinder that is initially filled with a liquid. There is an opening toward the bottom of the container which is initially closed with a valve. When the valve is open at some later time, the liquid becomes exposed to the atmosphere and starts to leak out of the container. The rate at which the fluid is flowing, the current, depends on the total head gradient between the bottom and the top of the container. We will assume that there is an overall resistance to the flow. Once the valve is open the pressure difference between locations 1 and 2 is zero since both are at atmospheric pressure. The gravitation potential energy-density gradient is not zero, on the other hand, and is proportional to the height of the fluid inside the container. However, this height decreases as the liquid flows out of the cylinder, resulting in a decrease of current. Thus, we have an example of a fluid flow system where the flow is no longer steady-state. In fact, the current, which is rate of change of volume, is proportional to the amount of volume present. This is exactly how we described a system which behaves exponentially in Section 5.8.

Figure 5.9.1: Non Steady-State Fluid System: leaking container



When the valve is open to the atmosphere the pressure difference between points 1 and 2 is zero, but there is a change in gravitational potential energy-density. Applying Bernoulli equation between these two points in the direction of current we get:

$$-\rho g h=-I R \tag{5.9.1}$$

This equation may seem very familiar from examples in previous sections, but the current is no longer a constant here, since height, h, depends on time. At t=0 the height is h_0 resulting in an initial current I_0 of:

$$I_0=\frac{\rho g h_0}{R} \tag{5.9.2}$$

Current is the rate of change of volume. In the Bernoulli equation current must be a positive quantity in order for the " $-IR$ " term to be negative to represent a loss of mechanical energy to thermal energy. In this case volume is decreasing as a function of time, so the quantity $\frac{dV}{dt}$ is negative, thus the negative sign needs to be added in order to make the current positive:

$$I=-\frac{dV}{dt} \tag{5.9.3}$$

We can represent height in terms of volume by using $V=Ah$, where A is the area of the cylinder as shown in Figure 5.9.1. Plugging in Equation 5.9.3 into Equation 5.9.1 and solving for the rate of change of volume we arrive at:

$$\frac{dV}{dt}=-\frac{\rho g}{AR} V \tag{5.9.4}$$

If you refer back to Section 5.8 you will see that the equation above has the exact form as the equation in the derivation of exponential decay, where $\lambda=\frac{\rho g}{AR}$. Using the result from the derivation we find that height changes with time as:

$$V(t)=V_0 \exp \left(-\frac{\rho g}{AR} t\right), \tag{5.9.5}$$

where $\exp(x) \equiv e^x$ and $V_0 = Ah_0$. Using the Equation 5.9.3 we can solve for current by taking the derivative of Equation ??? and multiplying by a minus sign:

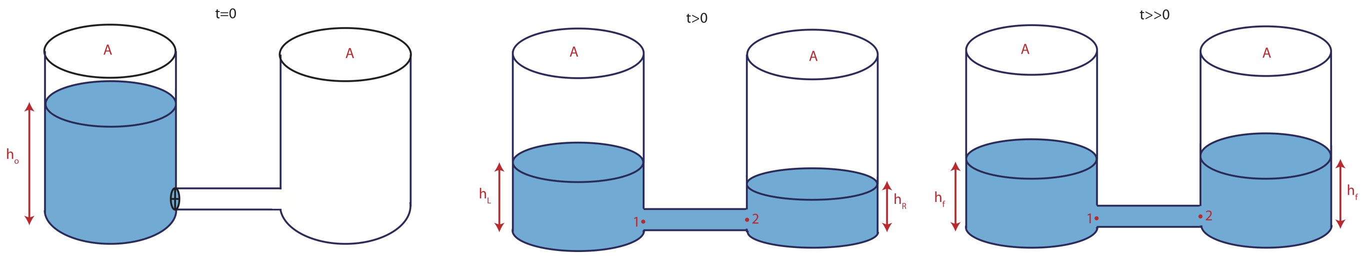
$$I(t) = \frac{\rho g h_0}{R} \exp\left(-\frac{\rho g}{AR} t\right) = I_0 \exp\left(-\frac{\rho g}{AR} t\right) \tag{5.9.6}$$

We see that we get the exact value of initial current we found in Equation 5.9.2. The current and the volume both decay exponentially to zero.

Leaking Between Two Containers

Let us look at another similar but slightly modified situation shown below in Figure 5.9.2. Here there are two cylinder connected by a thin pipe. We start with all the fluid in the left cylinder held in place by a closed valve. Once the valve is open the liquid is allowed to flow. The current is determined by the pressure gradient between points 1 and 2 which is proportional to the height difference of the liquid in the two containers. As more fluid flows to the right cylinder the pressure difference decreases, and so does the current. Eventually the system approaches equilibrium when the water levels in the cylinders approach the same value resulting in equal pressures and zero current.

Figure 5.9.2: Non Steady-State Fluid System: two cylinder system

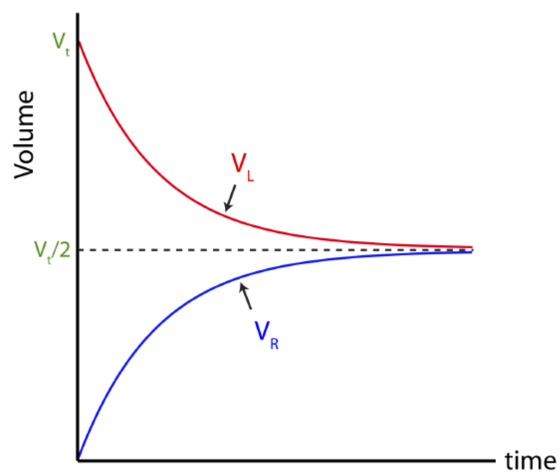


Since rate of charge of volume is proportional to the volume that is present, which means that the behavior will be exponential once again. You can arrive at these conclusions without doing all the math that we previously showed. Here we see an example where the quantity decaying exponential does not approach zero. The volume in the right cylinder starts at zero and approaches half the total volume exponentially. In the left cylinder the volume starts at the total volume and approaches half the total volume, again exponentially with the same rate. The volume in both cylinders displays exponential decay, even though the volume is increasing in one cylinder and decreasing in the other. The plots for volume as a function of time are shown in Figure 5.9.3 below.

Alert

Exponential decay does not mean that the value decreases from some initial value to zero or a smaller value. The word "exponential decay" implies that the parameter approaches another value in an exponential way. That values can be larger than its initial values.

Figure 5.9.3: Volume as a function of time: two cylinder system



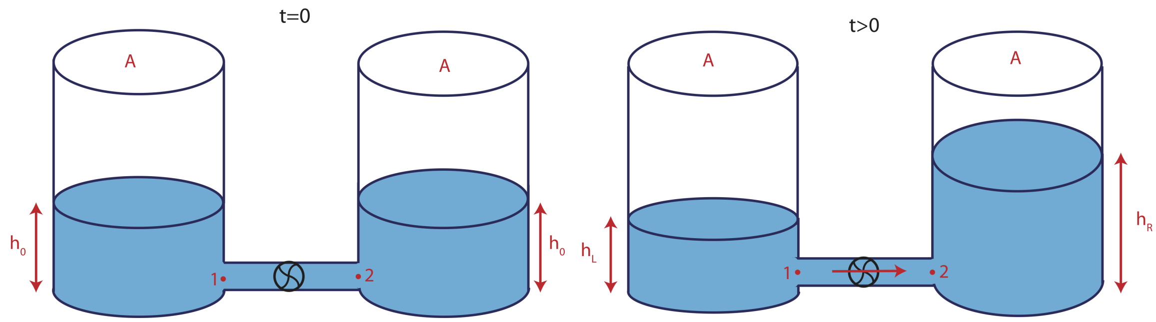
You can follow the following basics procedures to conceptually arrive at the exponential change of the system without going through all the mathematics:

1. Establish that the system will change exponentially by arguing that the rate of change of some parameter is proportion to the value of that parameter.
2. Determine the value of the parameter at initial time.
3. Determine the value parameter as the system approaches steady-state or equilibrium.
4. Connect the initial and final values with an exponential function using any information you have about decay rate, such as time constant or half-life.

A Pump Moving Liquid Between Two Containers

Imagine another example depicted below in Figure 5.9.4. There are two cylinders that initially contain the same amount of water. Thus, the heights of the water levels in each cylinder are equal. This mean that the pressures at points marked 1 and 2 are equal as well. The pump shown in the figure is initially turned off. There is no other source of potential gradient, the change in total head between points 1 and 2 is zero, resulting in zero current at t=0. Suddenly, the pump is turned on pumping fluid to the right, which results in more fluid accumulating in the right cylinder and fluid draining from the left cylinder. There is now a pressure gradient between points 1 and 2, resulting in a current which is proportional to this pressure difference. However, as the relative water level continue to change, so does the pressure gradient, resulting in a changing non steady-state current.

Figure 5.9.4: Non Steady-State Fluid System: filling up a container with a pump



We can use similar reasoning described above in red to figure out how the current and the volume change as a function of time. But we do require slightly more mathematical thinking here to found out the initial value of current, the final height and volume in each cylinder, and the decay rate. We will do through this exercise here. Initially, at t=0, the instant the pump is turned on and water starts to flow to the right. We can assume that the instant the pump is turned on the water levels are still equal, resulting in $\Delta(\text{total head}) = 0$ between points 1 and 2. The complete Bernoulli equation at t=0 becomes:

$$0 = \tilde{E}_{\text{pump}} - I_0 R, \tag{5.9.7}$$

where we defined $\tilde{E}_{\text{pump}} \equiv \dfrac{E_{\text{pump}}}{V}$ for simplicity. Solving for current we find:

$$I_0 = \frac{\tilde{E}_{\text{pump}}}{R} \tag{5.9.8}$$

At some time later as the pump continues to work at constant strength, the two heights are no longer equal as shown in Figure 5.9.4 at $t > 0$. Assuming both cylinders are open to the atmosphere, the pressures at points 1 and 2 are given by:

$$\text{point 1: } P_1 = P_{atm} + \rho g h_L \tag{5.9.9}$$

$$\text{point 2: } P_2 = P_{atm} + \rho g h_R \tag{5.9.10}$$

Subtracting the two equations we can find the pressure difference between the two points and write down the Bernoulli equation between points 1 and 2:

$$P_2 - P_1 = \rho g(h_R - h_L) = \tilde{E}_{\text{pump}} - IR \tag{5.9.11}$$

Solving for I and expressing heights in terms of volumes we get:

$$I=\frac{\tilde{E}_{\text{pump}}}{R}-\frac{\rho g}{AR}(V_R-V_L) \tag{5.9.12}$$

The current depends on time since the volume difference, V_R-V_L , changes with time. The current decreases from its initial value at t=0 in Equation 5.9.8, since the volume on the right is greater than the volum on the left, which means that the second term on the right-side of Equation 5.9.12 is negative. As the volume difference increases, the current continues to decrease until the pressure difference is too large for the pump to be able to continue pumping the water upward. Mathematically, we can see that at $t\gg 0$, the current will go to zero in Equation 5.9.12 when:

$$\tilde{E}_{\text{pump}}=\frac{\rho g}{A}(V_R-V_L) \tag{5.9.13}$$

We would like to find a general expression for current as a function of time, so we can describe the behavior of the current at all three times we analyzed, t=0, intermediate time t, and $t\gg 0$ in one equation. This will require a little calculus, as it did in the general derivation in Section 5.8. First, let us represent Equation 5.9.12 in a form which is similar to Equation 5.9.4.

The total volume, which is the sum of the volumes in the right and left cylinders, $V_t=V_R+V_L$, is conserved. No water flows in from the outside or leaks out anywhere, the volume only redistributes itself from the left to the right cylinder. Defining the current as the rate of charge of volume in the right cylinder, equation 5.9.12 for the current becomes:

$$\frac{dV_R}{dt}=\frac{\tilde{E}_{\text{pump}}}{R}-\frac{\rho g}{AR}(V_R-V_L) \tag{5.9.14}$$

Since both volumes in the right and left cylinders are changing with time, we would want to express everything in terms of one of the volume which is changing. Let us choose the volume in the right cylinder, V_r. Using the definition of total volume and substituting $V_L=V_t-V_R$ for the volume in the left cylinder, the above equation becomes:

$$\frac{dV_R}{dt}=\frac{\tilde{E}_{\text{pump}}}{R}-\frac{\rho g}{AR}(2V_R-V_t) \tag{5.9.15}$$

Comparing the equation above with Equation 5.9.4, a similar structure appears up to some additive constants. The rate of change of volume is proportional to volume itself, thus, we expect an exponential result for volume as a function of time. The exam mathematics of solving for volume are cumbersome, so we leave is as an aside derivation below. You can move directly to the result below in Equation 5.9.16 since our main focus should be on interpreting the physical behavior rather then the mathematics of obtaining the result.

Derivation

This derivation to follow is very similar to the one we did at the beginning of this section, except with addition of more constants. To simplify the expression in Equation 5.9.15, let us separate the terms on the right-hand side that depend on V_R from terms that do not, and define some variables:

$$\frac{dV_R}{dt}=a-bV_R$$

where,

$$a\equiv\frac{\tilde{E}_{\text{pump}}}{R}+\frac{\rho g}{AR}V_t$$

and,

$$b\equiv\frac{2\rho g}{AR}$$

Our next step is to separate the volume and time variables and integrating each side from the initial value to some arbitrary volume V_R' and time t'. The initial volume is $\frac{V_t}{2}$ since the volumes are equal on both side.

$$\int_{V_t/2}^{V_R'}\frac{V_R}{a-bV_R}=\int_0^{t'}dt$$

Taking an integral of both sides and dropping the "prime", we get:

$$\ln(a-bV_R)-\ln\left(a-b\frac{V_t}{2}\right)=t$$

$$\ln\left[\frac{a-bV_R}{a-bV_t/2}\right]=-bt$$

Taking the exponential of each side and using the relationship $e^{ln(x)}=x$, and then solving for V_R we get:

$$V_R=\frac{a}{b}+\left(\frac{V_t}{2}-\frac{a}{b}\right)e^{-bt}$$

Plugging back expressions for a and b and rearranging, we finally arrive at:

$$V_R(t)=\frac{V_t}{2}+\frac{A\tilde{E}_{\text{pump}}}{2\rho g}\left[1-\exp\left(-\frac{2\rho g}{AR}t\right)\right] \quad .$$

The derivation above gives us the following results for volume as a function of time:

$$V_R(t)=\frac{V_t}{2}+\frac{A\tilde{E}_{\text{pump}}}{2\rho g}\left[1-\exp\left(-\frac{2\rho g}{AR}t\right)\right] \tag{5.9.16}$$

We can check this result by using what we know about what happens at $t=0$ and $t\gg 0$. Initially, each side has half the total volume. Plugging in $t=0$ into Equation 5.9.16 we get exactly this result, $V_R(0)=\frac{V_t}{2}$. When $t\gg 0$, we get:

$$V_R(t\gg 0)=\frac{V_t}{2}+\frac{A\tilde{E}_{\text{pump}}}{2\rho g}, \tag{5.9.17}$$

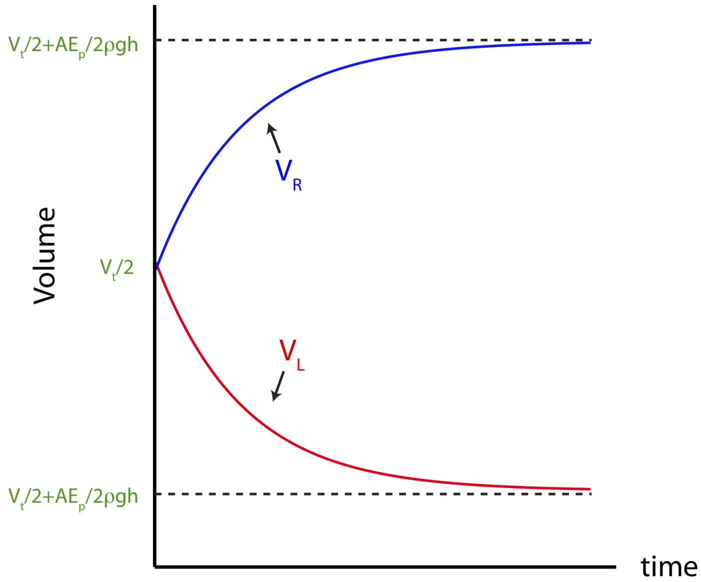
which is the result you can obtain from Equation 5.9.13 which states the relationship between the pump strength and height as the system approaches an equilibrium steady state.

To solve for current we can take the derivative of the above equation and arrive at:

$$I(t)=\frac{dV_R}{dt}=\frac{\tilde{E}_{\text{pump}}}{R}\exp\left(-\frac{2\rho g}{AR}t\right) \tag{5.9.18}$$

The current starts at its initial maximum values of $\frac{\tilde{E}_{\text{pump}}}{R}$ and decays to zero exponentially. Using Equation 5.9.16 we can plot volume as a function of time for the right cylinder as it fills up w and for left cylinder as the water depletes. To find the equation for V_L we just need to use the relationship $V_L=V_t-V_R$. Figure 5.9.5 below displays the change of volume as a function of time. In both cylinders the volume starts out at half the total volume. In the right cylinder the volume increase to its maximum value, while in the left cylinder the volume decreases by the same volume, always keeping the combined volume in the two cylinders constant.

Figure 5.9.5: Non Steady-State Fluid System.



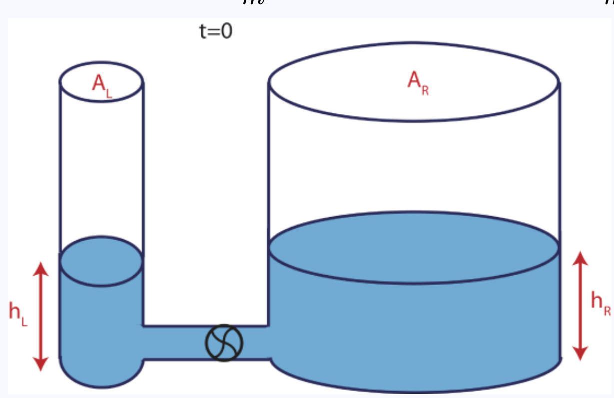
Also, since the two plots represent change the same system both cylinders will have the same decay rate which can be obtained from the time constant in Equation 5.9.16:

$$\tau = \frac{AR}{2\rho g} \tag{5.9.19}$$

Recall, the large the time constant the slower the rate of decay. It is reasonable that the time constant is proportional to A and R. A larger area would indicate it would take longer for the cylinder to fill up to the maximum height. A larger resistance would slow down the current, and thus the time it would take the reach its equilibrium value. The time constant is inversely proportional to density, because with the same pump the maximum height at equilibrium will be reduced with a higher density since the gravitational energy-density is increased with density. All three scenarios described in this section decay with an analogous rate.

Example 5.9.2

Two cylinders below have different cross sectional areas, such that the area of cylinder on the right is nine times greater than the area of the cylinder on the left. Initially the water levels are at 2m in both cylinders and the pump is turned off, such that $h_L = h_R = 2m$ at $t = 0$ in the figure below. The pump in this system can support a maximum height difference of 10 meters. When the pump is turned on, it pumps water from the right to the left cylinder. The main source of resistance is in the thin connecting pipe, which has a resistance of $1.6 \times 10^4 \, \frac{Js}{m^6}$. The density of water is $1000 \frac{kg}{m^3}$.



a) Make a plot of the water levels, h_R and h_L, as a function of time from the moment the pump is turned on until the system reaches equilibrium. Make sure to mark numerical values for initial and final heights. (You do not need to derive any equations here.)

b) Likewise, make a plot of current as a function of time. Make sure to mark numerical values for initial and final currents.

c) What will be the height in the left and the right cylinders after 2 half-lives?

Solution

The flow rate I is related to rate of change of height since, $I = \frac{dV}{dt} = A\frac{dh}{dt}$ since $V = Ah$. In this system the rate of change of the height is proportional to the height, the change is exponential, and approaching a total height difference of 10m as stated in the problem. The magnitude of the current flowing out of the right standpipe is equal to the current flowing into the left standpipe, except the volume is increasing in the left cylinder and decreasing in the right one. This is accounted by the minus sign shown here:

$$I = \frac{dV_L}{dt} = -\frac{dV_R}{dt}$$

$$I = A_L\frac{dh_L}{dt} = -A_R\frac{dh_R}{dt}$$

Plugging in $A_R = 9A_L$:

$$\frac{dh_L}{dt} = -9\frac{dh_R}{dt}$$

We can rewrite the above equation in terms of change in height, starting at initial height and going to a maximum value:

$$h_{L,max} - h_0 = -9(h_{R,min} - h_0)$$

Thus, the height in the left cylinder will increase nine times more than the height in the right one will decrease. Also, once the system reaches equilibrium:

$$h_{L,max} - h_{R,min} = 10m$$

Using the two previous results and setting $h_0 = 2m$ we get:

$$(10 + h_{R,min}) - 2 = 9(2 - h_{R,min})$$

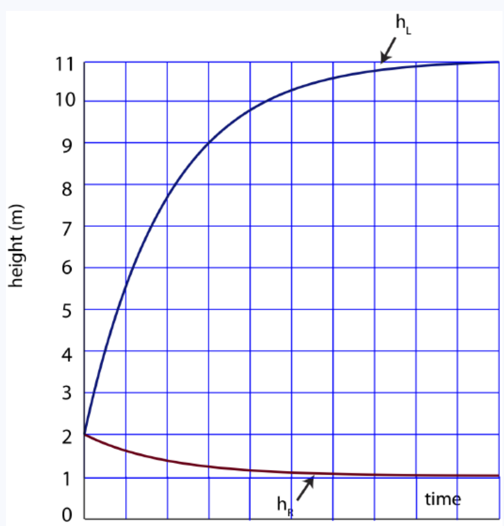
$$h_{R,min} + 8 = 18 - 9h_{R,min}$$

$$h_{R,min} = 1m$$

and

$$h_{L,max} = h_{R,min} + 10m = 11m$$

The plot for h_L and h_R is shown below.



b) When the pump is first turned on, this sets the initial current. Initially, there is no change in head across the system since the water levels are the same in both cylinders. Applying the Bernoulli equation across the thin pipe we get:

$$0 = \tilde{E}_{\text{pump}} - I_0 R$$

so

$$I_0 = \frac{\tilde{E}_{\text{pump}}}{R}$$

We can also solve for $\tilde{E}_{\text{pump}}$, *since when the system reaches its equilibrium state the current goes to zero, and the height difference between the two pipes approaches 10m. Applying Bernoulli equation at equilibrium we get:*

$$\tilde{E}_{\text{pump}} = \rho g(h_L - h_R) = 1000\frac{kg}{m^3} \times 10\frac{m}{s^3} \times 10m = 1.0 \times 10^5 \frac{J}{m^3}$$

Solving for initial current we find:

$$I_0 = \frac{\tilde{E}_{\text{pump}}}{R} = \frac{1.0 \times 10^5 \frac{J}{m^3}}{1.6 \times 10^4 \frac{Js}{m^6}} = 6.25\frac{m^3}{s}$$

Thus the current starts at its initial values of $6.25\frac{m^3}{s}$ *and approaches zero exponentially as shown below.*

c) After one half-life the heights will be halfway between its initial and final values. The left cylinder starts at 2m and ends at 11m. The height at one half-life is:

$$h_L(t = t_{1/2}) = h_0 + \frac{h_{max} - h_0}{2} = 2 + \frac{11 - 2}{2} = 6.5m \quad .$$

For the right cylinder which starts at 2m and ends at 1m, after one half-life it will be at 2-1/2=1.5m.

$$h_R(t = t_{1/2}) = h_0 - \frac{h_0 - h_{min}}{2} = 2 - \frac{2 - 1}{2} = 1.5m \quad .$$

After another half-life the height for the left cylinder has to be halfway between 6.5m and 11m:

$$h_L(t = 2t_{1/2}) = 6.5 + \frac{11 - 6.5}{2} = 8.75m\ .$$

For the right cylinder the height is halfway between 1.5m and 1m which is:

$$h_R(t = 2t_{1/2}) = 1.5 - \frac{1.5 - 1}{2} = 1.25m\ .$$

We saw two specific examples above of fluid flow exhibiting exponential change behavior. In one situation a pump was used to store water in one cylinder and in another the stored water flowed from one cylinder to another. We will now introduced analogous situations for electric change flow and make connections to these two fluid examples.

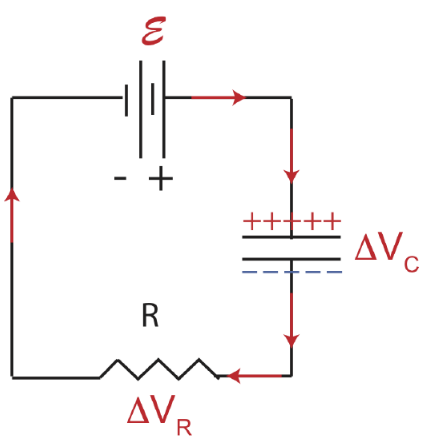
5.10: Exponential Charge Flow

Charging Capacitor

When we discussed electric circuits earlier in this chapter we limited ourselves to circuits with batteries, wires, and resistors resulting in steady-state charge flow. We will now consider another circuit component, the *capacitor*. Capacitors are useful because they can store electric energy and release that stored energy quickly. Batteries store energy too, they just let it trickle out over a relatively long time. When you take a photograph with a flash, you may have noticed a high-pitched whine as the camera charged a capacitor. The capacitor then discharges a large burst of energy to light the flashbulb. Capacitors store energy by accumulating charge on two conducting plates, a net positive charge on one plate and a net negative charge on the other. Like charges repel each other, so it makes sense that as the charge builds up on each plate, it becomes increasingly difficult to add more charge. Or if you think about a capacitor that is already charged, at first there will be a large accumulation of charge pushing charges off the plates, and as the charges move the “pressure” pushing them will decrease. Here is another situation where the change in an amount is related to the amount already present.

Figure 5.10.1 shows a typical *RC circuit* where a battery, a capacitor, and a resistor are all connected in series. The two parallel lines used to symbolize a capacitor represent the two conducting parallel plates with the space in between filled with an insulator. Charge cannot move across the capacitor since the insulating material does not allow charge to move across it. When the circuit is initially connected, electrons from the plate closest to the positive terminal of the battery get pulled to the positive terminal. This leaves behind a depletion of electrons on that plate making the net charge positive, as shown below. The red arrows represent the direction of current, which is the motion of positive charge carriers in the opposite direction of the motion of electrons. An analogous situation is occurring with the other other plate where electrons move from the negative terminal of the battery to the plate causing an accumulation of negative charge there. Thus, current flows toward the negative terminal at the same rate as it flows away from the positive terminal of the battery, *charging the capacitor*. We can consider this a closed circuit the same way we did for circuits without a capacitor. Although, charge is not moving across the capacitor, there is a uniform direction of charge flow in this circuit. Current does not technically flow through the battery either, there is a chemical reaction that occurs in the battery which keeps it at a fixed emf.

Figure 5.10.1: Charging Capacitor.



Let us think move deeply about the behavior of current as a function of time. Initially, the capacitor is not charged, and the two plates easily become charged. However, as the charges build up on each plate, the like charges repel each other on each plate, and it becomes harder to add more charge. So the current per unit time decreases until the force that pushing the charges onto the plate balances the force repelling those charges, resulting in zero net charge movement or current. Analogously, think back to the scenario in Figure 5.9.4. As the pump pushed water to the right cylinder, gravity was pulling the fluid back down, which made it harder for the pump to push more water upward, until the two effects were balanced. This is analogous to this RC circuit scenario, as the battery pushes charge onto the capacitor, the accumulated charge pushes those charges back, until the two effects become balanced, the emf of the battery will be equal to the voltage across the capacitor.

You can also think about this RC circuit in terms of the loop rule which still applies there:

$$\mathcal{E}+\Delta V_C+\Delta V_R=0 \tag{5.10.1}$$

Initially the capacitor is not charged, $\Delta V_C = 0$, so all the voltage drops across the resistor, $\Delta V_R = -I_0 R = -\mathcal{E}$, exactly how a simple circuit without a capacitor would behave. The initial current is then $I_0 = \frac{\mathcal{E}}{R}$. At equilibrium the voltage across the capacitor will equal to the emf of the battery, $\mathcal{E} = -\Delta V_C$. Since no voltage will drop across the resistor, the current will go to zero.

How much charge exactly can accumulate on a capacitor? Not all capacitors are made equally, some are able to hold more charge than others. The property that determines how much charge a capacitor can hold when charged with some battery is known as *capacitance*, C, which is given by:

$$C = \frac{Q}{\Delta V} \tag{5.10.2}$$

The unit of capacitance is called a *farad*, which is abbreviated as "F", where $F = \frac{C}{V}$. A capacitor with a large capacitance is able to store more charge per voltage difference. Capacitance is proportional to the area of the capacitor plate, the larger the area the more charges can spread out without repelling each other. This is analogous to the area of the cylinder, the larger the area the more volume can be stored in the cylinder. In addition, capacitance is inversely proportional to the distance between the two plates. As the plates are moved closer together, there is an additional attractive force between the two plates since they have opposite charge. This attraction allows more charge to be added.

Using the known expressions for the voltage drops across the capacitor and resistor and rewriting Equation 5.10.1, we get:

$$\mathcal{E} - \frac{Q}{C} - IR = 0 \tag{5.10.3}$$

Expressing current as the rate of change of charge, $I = \frac{dQ}{dt}$ and solving for I we arrive at:

$$I(t) = \frac{dQ}{dt} = \frac{\mathcal{E}}{R} - \frac{Q}{RC} \tag{5.10.4}$$

We once again have an expression that shows the dependence the rate of charge of some amount, here the rate of charge, $\frac{dQ}{dt}$ on the amount of charge, Q. The equation above has a similar form to Equation 5.9.15 for the rate of volume change in the two cylinder system. Applying a similar procedure to solve the differential Equation 5.10.4 as we did for the cylinder system, we arrive at the following expression for charge as a function of time:

$$Q(t) = \mathcal{E}C\left[1 - \exp\left(-\frac{t}{RC}\right)\right] \tag{5.10.5}$$

Using the definition of current and taking the derivative of Equation 5.10.5 we find that current has the following expression as a function of time:

$$I(t) = \frac{\mathcal{E}}{R}\exp\left(-\frac{t}{RC}\right) \tag{5.10.6}$$

The time constant is given by $\tau = RC$ resulting in a half-life for the RC circuit:

$$t_{1/2} = RC\ln 2 \tag{5.10.7}$$

Note the similarity between the way current behaves when a pump is used to store water in a cylinder (Equation 5.9.18) and when a battery is used to charge a capacitor (Equation 5.10.6). In both cases the current starts with an initial maximum value which is proportional to the strength of the pump or battery and inversely proportional to the amount of resistance present that impedes the flow. Also, in both situations the rate of charge of current is proportional to the amount of current is present at a given time, which leads to exponential decay of the current to zero. An equilibrium state of zero current is reached when the strength of the pump or battery is balanced by an opposing force, gravity in the case of the fluid system and electric force in the case of an RC circuit. Both situations have a half-life which is determined by the properties of the system. A longer half-life for the water storing system is determined by a larger area allowing for a greater volume to be stored which takes more time and larger resistance making the flow slower. For the RC circuit the half-life is increased by a larger capacitance allowing more storage of charge which take more time, and resistance which slows down the current causing slower decay. It is fascinating that these two seeming different situations have extremely similar physical behavior.

Conceptually, we can argue that the voltage across the capacitor starts and zero and approaches $-\mathcal{E}$ exponentially while the voltage across the resistor starts at $-\mathcal{E}$ and approaches zero exponentially as shown below in Figure 5.10.2. Mathematically, we can use the above results to get an expression for voltage as a function of time. Using Equations 5.10.2 and 5.10.5 we can find the voltage across the capacitor as a function of time:

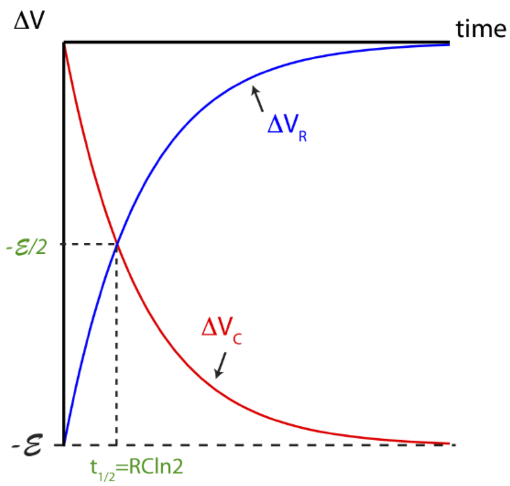
$$\Delta V_C(t) = -\frac{Q(t)}{C} = -\mathcal{E}\left[1 - \exp\left(-\frac{t}{RC}\right)\right] \tag{5.10.8}$$

And using, $\Delta V_R = -IR$ and Equation 5.10.6 we find the following expression of the voltage drop across the resistor as a function of time:

$$\Delta V_R(t) = -\mathcal{E}\exp\left(-\frac{t}{RC}\right) \tag{5.10.9}$$

When we add the two equations above we find that they add up to $-\mathcal{E}$. This is because energy is conserved during the entire process and the loop rule given in Equation 5.10.1 applies at all times. You can see this in Figure 5.10.2 below. In the figure the half-life is also labeled at the time when the voltage for both the resistor and capacitor reaches $-\mathcal{E}/2$.

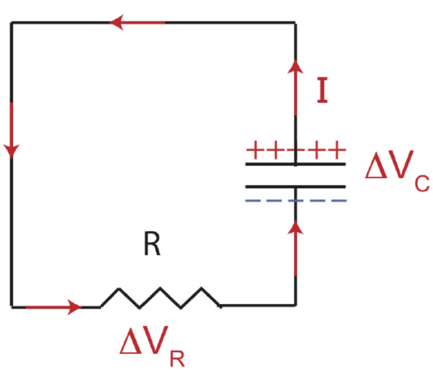
Figure 5.10.2: Voltages when Capacitor is Charging.



Discharging Capacitor

Now suppose we take the capacitor that was charged in a circuit in Figure 5.10.1, disconnected from a battery, and connected to just to a resistor as shown in Figure 5.10.3 below. In this case electrons from the negatively charged plate will be attracted to the positive plate and flow accordingly. Since current is the opposite direction of electrons, current will flow in the counterclockwise direction in the circuit below. The system will come to equilibrium when there is no longer a net charge on the two plates, resulting in no flow of electric charge, *discharging the capacitor*. Remember, a current flows when there is a attractive electric force present, such as a terminal of a battery or a charged plate in this case of a discharging capacitor.

Figure 5.10.3: Discharging Capacitor.



For this circuit the loop rule is:

$$\Delta V_C + \Delta V_R = 0 \tag{5.10.10}$$

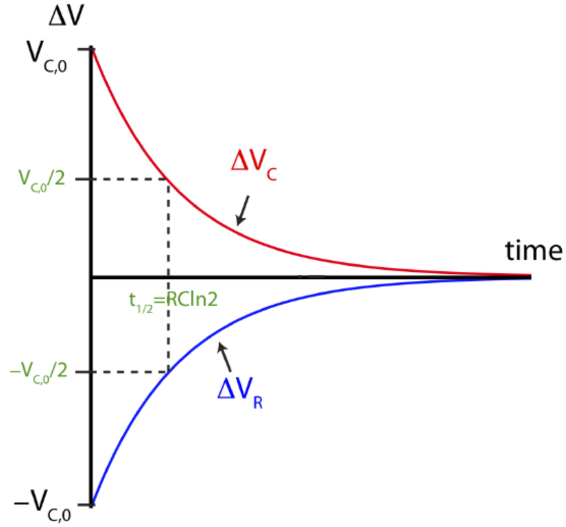
Since the voltage across a resistor in the direction of current is always negative, the voltage across the capacitor has to be positive. If you follow the direction of the current in Figure 5.10.3 it goes from the negative plate to the positive plate, the same way the current in Figure 5.10.1 flows from the negative to the positive terminal of a battery resulting in a positive emf with the loop rule is applied. In Figure 5.10.1 the current "flows" from the positive to the negative plate of the capacitor resulting in a negative change in the voltage of the capacitor in that case.

Alert

The voltage across a capacitor is always negative when it is charging and is positive when it is discharging when following the direction of current.

The voltage across the capacitor for the circuit in Figure 5.10.3 starts at some initial value, $V_{C,0}$, decreases exponential with a time constant of $\tau = RC$, and reaches zero when the capacitor is fully discharged. For the resistor, the voltage is initially $-V_{C,0}$ and approaches zero as the capacitor discharges, always following the loop rule so the two voltages add up to zero. This behavior is depicted in Figure 5.10.4 below. The half-life is also indicated when the voltages reach half of their initial value for both the resistor and the capacitor.

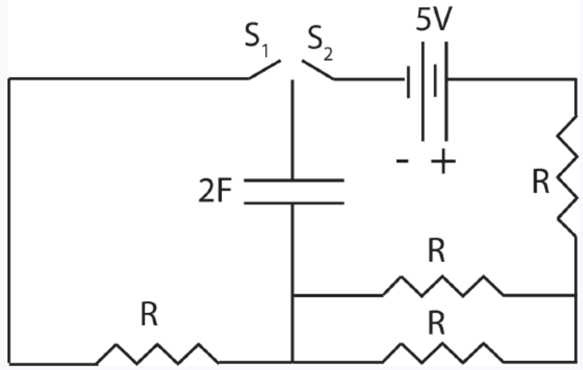
Figure 5.10.4: Voltages when Capacitor is Discharging.



If you are more keen on showing it mathematically, start with Equation 5.10.10, and follow the method outlined in the derivations shown in this section, to obtain mathematical exponential decay equations for charge across the capacitor, voltages across the capacitor and resistance, and the current. In this case a capacitor discharging is analogous to a cylinder with stored water flowing out to reach equilibrium as described in Figure 5.9.2.

Example 5.10.3

Shown here is a circuit that contains a $5V$ battery, a $2F$ capacitor, several resistors with the same resistance R, and two switches. Assume the capacitor is initially discharged.



a) Initially the switch, S2, is closed while S1 remains open. It takes 9 seconds for the capacitor to charge to 2 volts in this case. Once the capacitor is fully charged, S2 is open and S1 is closed. How long will it take the capacitor to reach 2.5 volts after S1 is closed?

b) On the same plot, make a graph of the magnitude of the voltage across the capacitor as it charges and as it discharges in this circuit. Assume both processes start at t=0. Mark at least one half-life with a numerical value.

Solution

a) To solve this problem, we first need to use the information given about the charging RC circuit to find the resistance R, since we have some information about the time it takes to discharge. Once we know R, we can find the half-life of the discharging circuit.

The magnitude of voltage across a capacitor as it charges is:

$$|\Delta V_C| = \mathcal{E}\left[1 - \exp\left(-\frac{t}{R_{eq}C}\right)\right]$$

We are given that at t=9sec, $|\Delta V_C(9s)| = 2V$. *Also, the equivalent resistance for the circuit when only S2 is closed is* $R_{eq} = R + \frac{R}{2} = \frac{3}{2}R$ *. Plugging these values into the equation above we get:*

$$2V = 5V\left[1 - \exp\left(-\frac{9s}{3R/2 \times 2F}\right)\right] = 5V\left[1 - \exp\left(-\frac{3}{R}\frac{s}{F}\right)\right]$$

Solving for R:

$$\exp\left(-\frac{3}{R}\frac{s}{F}\right) = 1 - \frac{2}{5} = \frac{3}{5}$$
$$-\frac{3}{R}\frac{s}{F} = \ln\left(\frac{3}{5}\right) = -0.51$$
$$R = 5.87\Omega$$

For the discharging circuit, there is only one resistor, so:

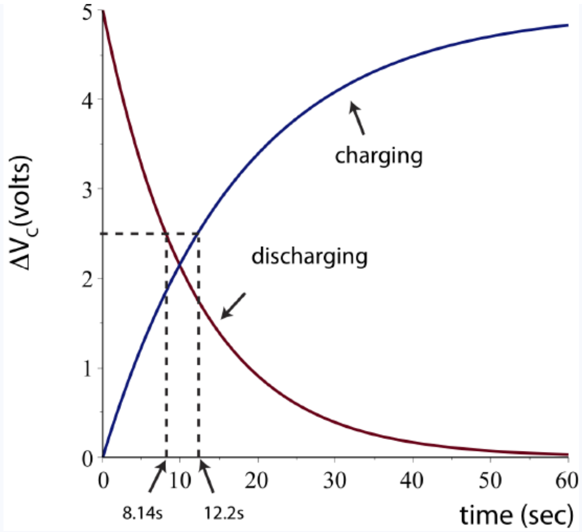
$$t_{1/2} = \ln 2RC = \ln 2 \times 5.87\Omega \times 2F = 8.14s$$

Thus, it will take 8.14 seconds for the capacitor to discharge to half of time maximum voltage of 5V, which is 2.5V.

b) For the charging circuit the half life is:

$$t_{1/2} = \ln 2R_{eq}C = \ln 2\frac{3}{2}RC = \ln 2 \times 5.87\frac{3}{2}\Omega \times 2F = 12.2s$$

The plots with the half-lives marked are shown below.



Other Systems

In each of these phenomena we can understand the change by applying the basic ideas of the exponential change model. The fact that each version of the equation looks a bit different can easily hide that fact that the ideas underlying how the system changes are the same. The advantage of understanding the underlying behavior makes it possible for you to recognize the general pattern, even though the symbols are different or the equation is written differently.

Another example that displays exponential change is the the cooling of objects. Most of us have observed that an unfinished cup of hot coffee or tea will cool down to room temperature eventually. What might not be so obvious without taking some data is that the rate of cooling depends on the temperature difference between the hot object and its environment. So the hotter the cup of coffee, and the colder the room, the faster heat will move from the coffee to the room. This is known as Newton's Law of Cooling given by:

$$\Delta T(t) = \Delta T_0 e^{-t/\tau} \tag{5.10.11}$$

where ΔT is the temperature difference between the object and its environment. So as the hot object approaches the temperature of its environment, the rate of cooling decreases and asymptotically approaches zero. The temperature difference behaves exactly like the example of nuclear decay, fluid flow examples described in this section, and RC circuits.

5.11: Wrap up

We began this chapter by developing an approach that enabled us to use conservation of energy to understand and make sense of fluid flow and electric current. In many respects the approach is similar to the approach we developed previously to use energy conservation for understanding all kinds of phenomena by looking at the changes that occurred from before to after an interaction. We called this the energy-interaction model. For steady-state flow phenomena involving fluids and electricity, we needed to make two important modifications to our model. First, instead of focusing on energy, we focused on energy-density. Second, instead of looking at changes from before to after an interaction, we looked at changes in the energy-density as we moved along the flow path. But the general approach and methods are similar in both models. These are very powerful and general models. Energy conservation can be applied to all phenomena. The energy interaction model and the steady-state energy-density model provide us the tools we need to apply conservation of energy.

In the second part of this chapter, we saw how the approach we used in the stead-state energy density model could also be used to make sense of other transport phenomena that didn't even involve energy conservation. The linear transport model applies to any kind of transport that is linearly proportional to a gradient of something. Some of the more interesting cases of transport involve more than one potential gradient. For example, the diffusion of charged particles (ions) across a cell membrane. A cell wall will typically have a voltage, i.e., an electrical potential gradient of the order of 100 mV. In addition there will typically be concentration gradients across the cell wall as well. And there can be active "pumps" that act selectively on certain ions or particles. You should be able to readily make sense of these phenomena when you encounter them in your more advanced courses and lab work by focusing on the underlying transport mechanisms in terms of the general linear transport model.

In the last part of this chapter we looked at a general and universal way systems approach steady state: the exponential change model. We saw that in some cases the flow rate that we discussed so much in this chapter was no longer constant. Instead it depended on the amount of "stuff" that was flowing, whether it was the volume of a liquid or the amount of charge. Mathematically, we saw that this relationship between current and what is flowing resulted in exponential decay phenomena.

One of the difficult things for all of us to do when we are first learning a new subject that is expressed in *quantitative relationships involving symbols* is to get past the symbols themselves and focus on the meaning of the relationships represented in the mathematical relationships. The mathematical relationships must, by necessity, be represented with symbols. The symbols are typically different for the different physical phenomena to which the relationship is applied, even though it is the same underlying relationship. Each of the three models treated in this chapter, the steady-state energy density model, the linear transport model and the exponential change model, apply to multiple kinds physical phenomena. The symbols used to express the relationship, however, are typically different for each different application. In the case of exponential change, the equations for exponential growth look very different from the equation expressing the charge on a capacitor, because some of the symbols are different. Likewise, the expression for diffusion or osmosis "looks different" from the expression for heat flow, because many of the symbols are different. Working through this chapter gives you many opportunities to practice getting past the symbols to the meaning of the relationship behind the symbols. Take advantage of this opportunity.

CHAPTER OVERVIEW

6: Newton's Laws of Motion

This chapter introduces two new models: The Galilean Space-Time Model, which is the basis for developing a useful way of representing variables that are based on spatial dimensions and time. The second is a model of how "things" interact in our physical universe with *forces* acting as the agents of interactions in this model.

Topic hierarchy

6.0: Overview

This chapter is somewhat different from the other chapters in this text, in that much of the material serves as reference for the following two chapters. We start this Chapter with introducing the concept of vectors, which is a mathematical tool that we will need to understanding physical concepts and to solve problems for the remainder of this course. We will introduce specific vectors that we frequency use in physics, especially those use to describe the dynamics of objects.

For the remainder of the chapter we will focus on one specific type of vector, a force. Forces are extremely important in physics since they will help us explain why objects move the way they do or why they stay stationary. These ideas are explained by the famous Newton's Laws of nature.

6.1: Overview of Vectors

Basic Vector Definition

What we are really interested in now is describing how objects move. A very useful tool that will help us achieve this goal is the concept of a *vector*. We can give a definition of a *vector* as something that exhibits both *direction* and *magnitude*. This concept might appear abstract at first, but as we begin to use the vectors in different contexts, we will develop a deeper and richer understanding. Remember how hard it was to define *energy*. Our understanding of energy continually increased as we applied the idea to multiple physical phenomena. Similarly with vectors, even though we currently give it a short one-sentence definition, understanding comes as we use the concept and work with it over the remainder of this and the next two chapters.

In Physics 7A we were only concerned with *magnitudes* of quantities, such as speed and mass of an object which was sufficient information to calculate its kinetic energy. However, if you want to know precisely where in space a moving object will be after some time, in addition to speed, we also need to know its direction. The *velocity* of an object is a vector quantity which describes both its speed (magnitude of the velocity vector) and its direction. One example where the velocity is important to know is when two objects are interacting. Two moving objects colliding head-on compared to two moving objects colliding while moving in the same direction will yield very different outcomes, even if their speeds are the same in both scenarios. We will explore such interaction when we discuss conservation of momentum in the next chapter. *Force* is another quantity where both the magnitude (how strong is the push or pull) and then direction (in which direction is the push or pull) will matter to explain how an object will react under the influence of forces. For example, two people pushing the table with the same strength in the same direction will result in different dynamics (the table will move most likely) compared with the same strength is applied in opposite directions (the table will not budge). *Acceleration* is another useful vector quantity which describes the rate of change of velocity and is directly related to the forces acting on an object. We will explore the effects of forces in much greater details in this and the following chapters.

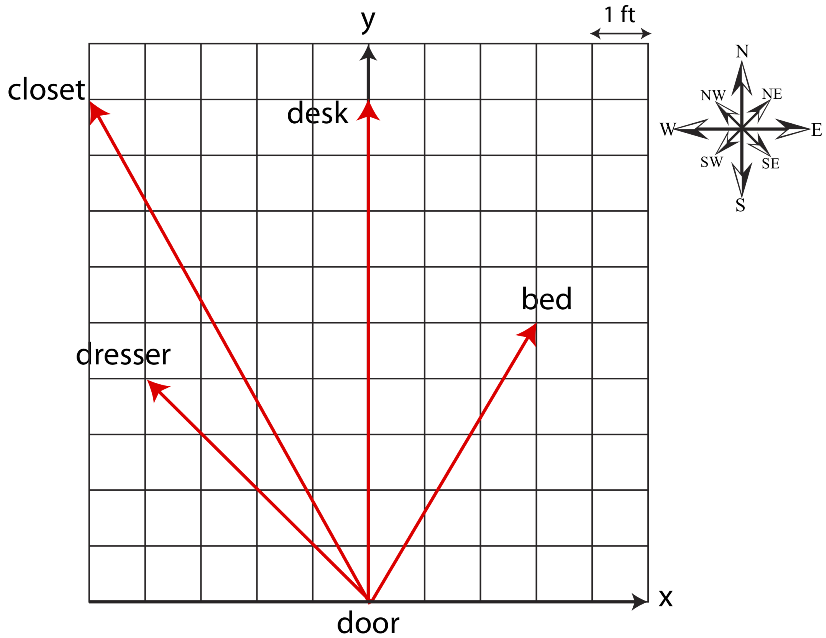
Vector Representation

One way to represent vectors is with arrows pointing in the direction of the vector, where the length of the arrow represents the magnitude of the vector. It is only the relative lengths of various arrows that matter, there is not rule that says that a specific length represents a specific magnitude. For example, if we choose a length of 1 cm to represent a velocity vector with a speed of 10 m/s, then another velocity vector whose length is double, 2 cm, will represent a speed of 20 m/s. We refer to the arrowhead as the *head* of the vector, and the other end as the *tail*.

Let us explore vector representation in the context of a *position vector*, since it is perhaps the simplest vector quantity to apply to every day experience. A *position vector* describes the location of an object relative to some *point of reference*. If you are trying to describe the layout of your bedroom to your friend, you can use the door as the reference point, or the origin, of the position vectors which will represent the locations of objects in your room. It is not sufficient to say that your desk is 9 feet away from the door. Your friend will not know on which side of the room the desk located. Thus, to describe the position of your desk exactly, you need a vector quantity to specify that your desk is 9 feet straight from the door. Rather then using "straight", "right" or "left", it is more convenient to use the x-y coordinate system. Figure 6.1.1 below shows a set of position vectors to the center of objects in a bedroom, where each grid square is one foot in distance. In the figure the positive x-direction is to the right of the door, the negative x-direction is to the rest, the positive y-direction is straight ahead, and the negative y-direction is straight behind. We will often use compass coordinates as well to describe direction: in Figure 6.1.1 "north" represents straight from the door, east to the right, and west to the left.

These vectors are two-dimensional since we are only concerned with where the various objects are located on the floor. If we also represented the height of each object, then we would have to describe the vector in three-dimensions. For example, the bed is 5 feet straight, 3 feet to the right, and 2 feet up from the floor. For now, we will mostly focus on two-dimensional vectors.

Figure 6.1.1: Position Vectors



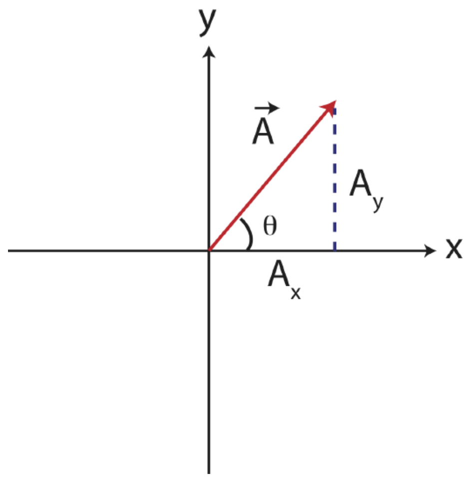
In print, a small arrow is put over the symbol to represent a vector. For example, force, position, velocity, and acceleration vectors are represented as $\vec{F}, \vec{r}, \vec{v}$, and $\vec{a}$. In this book we will stick to this notation, but in other text you might encounter a bold symbol for vector notation (such as ***F***, ***r***, ***v***, ***a***).

Vectors are represents by their *components*, which are the values of the vectors in each of the spacial direction. For example, the position vector of the desk in Figure 6.1.1 has a component of 0 ft in the x-direction and of 9 ft in the positive y-direction. To write down the values of the vectors we will use the following notation: $\vec{A} = (A_x, A_y, A_z)$, where the values in the parenthesis are the vector components. Note, the vector components can be either zero, positive, or negative depending in which quadrant the vector lies in. The position vector for the desk is then written as, $\vec{r}_{\text{desk}} = (0, 9)$ ft since it is precisely on 9 feet in the positive y-direction. The dresser's position vector is $\vec{r}_{\text{dresser}} = (-4, 4)$ ft since it is four feet from the door in the negative x-direction and four feet from the door in the +y direction. Note, we have dropped the the z-component all together from the position vectors in this room, since for this particular situation we are only concerned with the x-y position of the objects and not their heights. Thus, the z-component is always zero in this particular description of position of objects.

This notation of the position vector, $\vec{r}$, fully describes the location of objects from the door, which is defined as the origin for all position vectors. But this is not the only way to represent the position vector. One can also do it by using the magnitude (the distance form the door to the object) and the direction. For example, the desk is a distance of 9 feet in the positive y-direction (or north) from the door. The notation for magnitude of a vector $\vec{A}$ is $|\vec{A}|$, or sometimes you will encounter simply, A, without the arrow over it to specify magnitude. Thus, either writing $\vec{r}_{\text{desk}} = (0, 9)$ or $|\vec{r}_{\text{desk}}| = 9$ ft pointing north are equivalent ways of describing the position vector of the desk.

One way to represent the position vector of the bed is by components, $\vec{r}_{\text{bed}} = (3, 5)$ ft. But how do we describe it by using magnitude and direction? We need to know how far the bed is from the door, we cannot simply count the squares in Figure 6.1.1 and get an exact answer. Using a ruler is one way to do it, but there is a more accurate calculation that can be done as described shortly. Also, simply stating its direction as northeast or above the +x axis is not accurate, since it only specifies the first quadrant but not the bed's exact location. Thus, we need to both calculate the distance and to specify the exact angle the vector makes with the +x axis. To do so we need to turn to trigonometry. Consider the figure below which depicts some vector $\vec{A}$ with components A_x and A_y, $\vec{A} = (A_x, A_y)$.

Figure 6.1.2: Vector Components



The magnitude (length) of the vector makes a right triangle with the two components of the vector, where the magnitude is the hypotenuse. Thus, to find the magnitude of the vector $\vec{A} = (A_x, A_y)$ we can use the Pythagorean theorem:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2} \tag{6.1.1}$$

To find the angle, θ shown in the figure above, we can use the property of a tangent:

$$\theta = \arctan\left(|\frac{A_y}{A_x}|\right) \tag{6.1.2}$$

We use the absolute value symbol in Equation 6.1.2 since we want to describe the direction by stating the angle between 0° and 90°, and then by specifying the quadrant either by stating, for example, the vector points "down from the negative x-axis" or using the compass coordinate "southwest". Thus, even if the x- or y- components are negative you should use the absolute values of these components when calculating the angle.

Returning to the position vector of the bed, $\vec{r}_{\text{bed}} = (3, 5)$ ft, we can now use the trigonometric properties, described above, to represent the vector with its magnitude $|\vec{r}_{\text{bed}}| = \sqrt{3^2 + 5^2} = 5.83$ ft and angle of $\theta = \arctan\left(\frac{5}{3}\right) = 59^\circ$ above the positive x-axis or pointing northeast. The position vector of the closet, $\vec{r}_{\text{closet}} = (-5, 9)$ ft has magnitude of $|\vec{r}_{\text{bed}}| = \sqrt{5^2 + 9^2} = 10.3$ ft and direction of $\theta = \arctan\left(\frac{9}{5}\right) = 60.9^\circ$ above the negative x-axis or pointing northwest.

You might be wondering why one should bother to express the vector in terms of magnitude and direction direction since expressing the vectors in Figure 6.1.1 directly in terms of components is simpler, you can read the values of the x- and y- components directly from the figure using the provided grid and scale. However, sometime information for a vector quantity is provided or measured directly in terms of magnitude and direction. And it is also more intuitive to picture the location of a bed if one gave you the information in terms distance and angle from the door rather than in terms of components. In other words, instead of measuring directly that the closet is 5 feet east and 9 feet north from the door, you may have measured with a ruler the distance of 10.3 feet from the door and with a protractor an angle of 60.9° above the negative x-axis. Having this information does describe the vector completely, but it is still often very useful to represent in terms of its components, as we will see shortly below when discussing adding and subtracting vectors. We can find the components from the magnitude $|\vec{A}|$ and direction θ plus quadrant, by using trigonometry once again. From Figure 6.1.2 using the definition of sine and cosine the x- and y- components of the vector in terms of magnitude and angle are given by:

$$A_x = |\vec{A}| \cos\theta \tag{6.1.3}$$

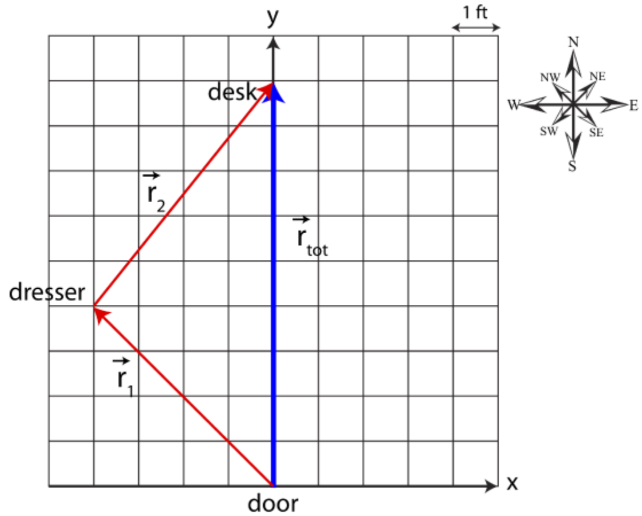
$$A_y = |\vec{A}| \sin\theta \tag{6.1.4}$$

Let us check that these equations give us the correct components for the closet's position vector, if you were just told that the closet is 10.3 feet away from the door and at an angle of 60.9° northwest. Using the equation above and noting that "northwest" implies that the the x-component will be negative and the y-component positive we find that, $r_x = -(10.3 \text{ ft}) \cos 60.9^\circ = -5 \text{ ft}$ and $r_y = (10.3 \text{ ft}) \sin 60.9^\circ = 9 \text{ ft}$, resulting $\vec{r}_{\text{closet}} = (-5, 9)$ ft as expected.

Adding and Subtracting Vectors

As we introduce new models in the following chapters, we will see multiple applications where we will need to add or subtract vectors. Adding vectors is not as trivial as adding scalar quantities. We know that the combined mass of a 2kg and a 3kg objects is 5kg. But we cannot simply add magnitudes of vectors together to obtain the magnitude of combined vectors. This method would only work if the two vectors point in the same direction. For example, if you walk 2 feet north and then walk 3 more feet also north, the combined vector (which represents walking directly from the start to end) is indeed the sum of the magnitudes pointing in the same direction, 5 feet north. However, let us look at another example as depicted in the figure below.

Figure 6.1.2: Vector Addition



In this scenario, you walk to your dresser first, and then to your desk. This is equivalent to walking to your desk directly, 9 feet north. But 9 feet is clearly a shorter distance than the distance to get to the dresser plus the distance to walk to the desk. Thus, we cannot simply add magnitudes of the position vectors $\vec{r}_1$ and $\vec{r}_2$ in Figure 6.1.2.

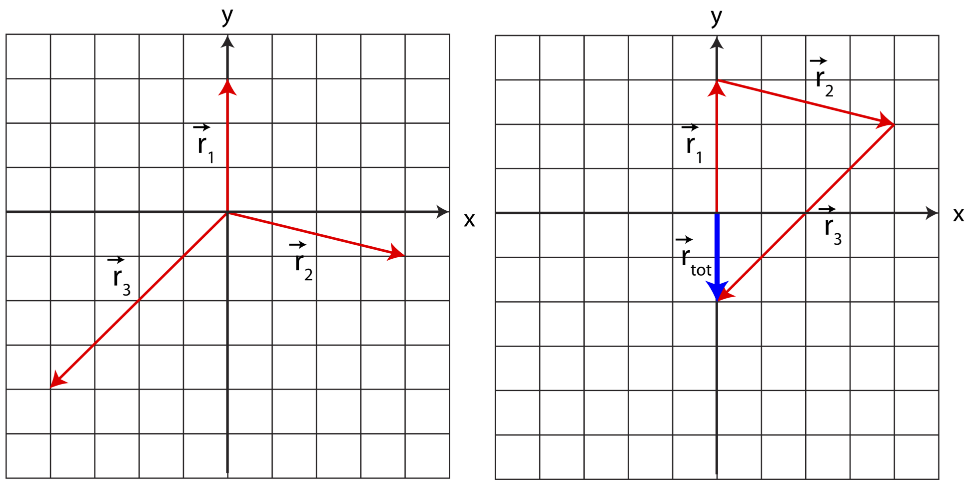
Our goal is to come up with a method such that adding the position vector to the dresser, $\vec{r}_1$, with the position vector from the dresser to the desk, $\vec{r}_1$, gives us the position vector directly to the desk, here defined as $\vec{r}_{tot}$ and represented by the bolder blue arrow in Figure 6.1.2: $\vec{r}_{tot} = \vec{r}_1 + \vec{r}_2$. Let us represent the vectors by components. The position vector to the dresser is $\vec{r}_1 = (-4, 4)$. When representing $\vec{r}_2$ we have to remember that all position vectors are defined relative to the origin (0,0) defined at the location of the door. Thus, although the tail of the vector is at the location of the dresser, it must be respresented relative to the origin. You can always take the vector $\vec{r}_2$ and move it to the location of the door (origin) preserving the length and direction of the vector. From figure 6.1.2, you can simply count the number of squares in the x- and y- directions starting from the tail of $\vec{r}_2$, resulting in $\vec{r}_2 = (4, 5)$. The total vector, directly to the desk is $\vec{r}_{tot} = (0, 9)$. You can see from the components of $\vec{r}_1$ and $\vec{r}_2$ that to obtain the components of the total vector, you need to sum the x-components, $\vec{r}_{tot,x} = -4 + 4 = 0$, to get the x-component of the total vector, and likewise you need to sum the y-components, $\vec{r}_{tot,y} = 4 + 5 = 9$. Mathematically, the sum of two vectors is written as:

$$\vec{A}_{tot} = \vec{A}_1 + \vec{A}_2 = (A_{1,x} + A_{2,x}, A_{1,y} + A_{2,y}) \tag{6.1.5}$$

To obtain the magnitude and direction of the total vector, you can use Equations 6.1.1 and 6.1.2.

You can also see from Figure 6.1.2 that the tail of $\vec{r}_2$ is positioned at the head of $\vec{r}_1$, and the sum of the two vectors, $\vec{r}_{tot}$, is positioned such that its tail is at the tail of $\vec{r}_1$ and its head is at the head of $\vec{r}_2$. This describes the geometric method of vector addition, known as the *head-to-tail* method. Take three vectors shown in Figure 6.1.3 and move them such that the head of one vector touches the tail of the next one as shown on the right figure. The resultant vector, $\vec{r}_{tot}$ is drawn from the tail of the first vector to the head of the last vector as shown.

Figure 6.1.3: Head-to-Tail Method for Vector Addition



To check this method, let us sum the three vectors by components:

$$\vec{r}_{tot} = \vec{r}_1 + \vec{r}_2 + \vec{r}_3 = (0, 3) + (4, -1) + (-4, -4) = (0 + 4 - 4, 3 - 1 - 4) = (0, -2), \tag{6.1.6}$$

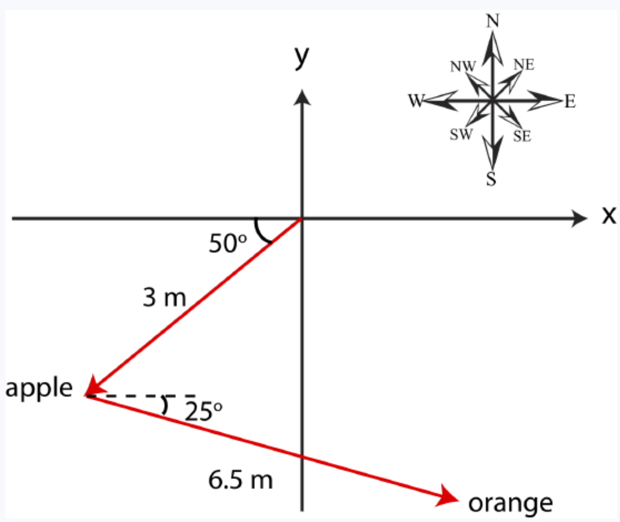
which is exactly the vector $\vec{r}_{tot}$ drawn in Figure 6.1.3.

Example 6.1.1

An apple tree is positioned 3 meters away from you pointing 50° southwest. And orange tree is is positioned 6.5 meters away from the apple tree, in the direction of 25° pointing southeast from the apple tree. If you were to walk directly from your location to the orange tree, how far should you walk an in which direction (angle and compass direction)?

Solution

It is often much easier to solve problems that involve vector algebra if you first draw then out, if they are not drawn for you already. The figure below depicts the information provided in the problem.



Now you can see that the vector from the origin (your position) directly to the orange tree will point southeast and will have a magnitude somewhere between 3 and 6.5 meters. In order to find the vector to the orange tree, we need to add the vector from the origin to the apple tree with the vector from the apple to the orange tree. To do so, we first need to express both vectors in terms of their components, then add them by components, and finally calculate the magnitude and angle of the vector resulting from this addition.

The apple tree's position vector $\vec{r}_a$ *in terms of components is:*

$$\vec{r}_a = |\vec{r}_a|(-\cos\theta, -\sin\theta) = 3m(-\cos 50^{\circ}, -\sin 50^{\circ}) = (-1.93, -2.30)m \tag{6.1.7}$$

The orange tree's position vector $\vec{r}_o$ *is:*

$$\vec{r}_o = |\vec{r}_o|(\cos\theta, -\sin\theta) = 6.5m(\cos 25^\circ, -\sin 25^\circ) = (5.89, -2.75)m \tag{6.1.8}$$

The sum of the two vectors $\vec{r}_{tot}$ *is:*

$$\vec{r}_{tot} = \vec{r}_a + \vec{r}_o = (-1.93 + 5.89, -2.30 - 2.75) = (3.96, -5.05) \tag{6.1.9}$$

Magnitude of $\vec{r}_{tot}$ *is:*

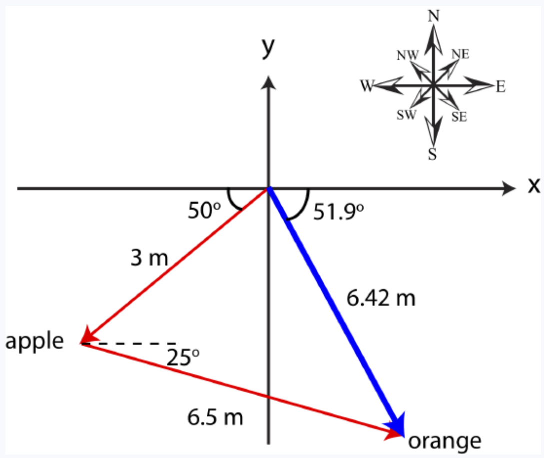
$$|\vec{r}_{tot}| = \sqrt{3.96^2 + 5.05^2} = 6.42m \tag{6.1.10}$$

Angle is:

$$\theta = \arctan\left(\frac{5.05}{3.96}\right) = 51.9^\circ \tag{6.1.11}$$

in the southeast direction.

You can see the vector $\vec{r}_{tot}$ *represented by the thick blue arrow in the figure below.*



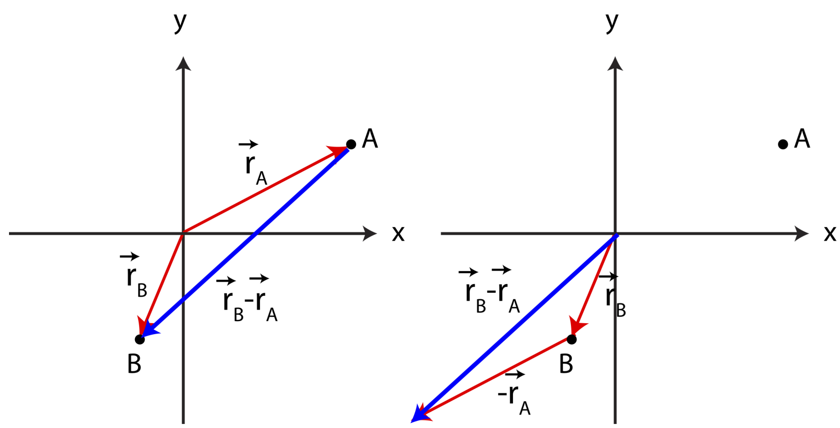
To subtract two or more vectors, we use a very similar method as adding vectors. As for scalar addition, the same principle applies: the difference between two numbers is just the sum of one number plus the negative of another one: $a - b = a + (-b)$. Likewise, for vectors, $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$. To negate a vector $\vec{A}$ you need to negative each component of that vector: $-\vec{A} = (-A_x, -A_y)$. This means that the direction of the vector flips by 180° when you negate a vector. For example a vector $\vec{A} = (4, 3)$ has a magnitude of 5 and points northeast at 53.1° from the positive x-axis. The negative of this vector $-\vec{A} = (-4, -3)$ still has the same magnitude of 5, but now points southwest with an angle of 53.1° below the negative x-axis, so $-\vec{A}$ is just $\vec{A}$ rotated by 180°. Therefore, to subtract vectors geometrically, you need to flip the vector(s) being subtracted by 180°, and then apply the usual head-to-tail method for addition. To subtract vectors by components, you simply need to subtract each component individually:

$$\vec{A}_1 - \vec{A}_2 = (A_{1,x} - A_{2,x}, A_{1,y} - A_{2,y}) \tag{6.1.12}$$

In the next few chapters we will focus on analyzing motion of objects. One important property of a moving object is its *displacement*, a vector quantity which describes both the distance by which an object moved in some interval and the direction in which it moved. If the initial position of an object is some location A and its final position is another location B, *displacement vector* points from A to B.

Figure 6.1.4 below shows two locations A and B, described by position vectors $\vec{r}_A$ and $\vec{r}_B$, respectively. If an object changed its position from point A to B, then the displacement vector is $\Delta\vec{r}_{AB} \equiv \vec{r}_B - \vec{r}_A$, as drawn pointing from A to B. The right side of the figure depicts the head-to-tail method for obtaining $\Delta\vec{r}_{AB}$ from the two position vectors. Vector $\vec{r}_A$ is flipped 180°, and its tail is placed at the head of $\vec{r}_B$. The summation vector pointing from the tail of $\vec{r}_B$ to the head of $-\vec{r}_A$ represents the displacement vector $\vec{r}_B - \vec{r}_A$. Note, two vectors are equivalent as long as their magnitude and direction is the same, so you can see, by eye, that the displacement vectors in both figures look the same.

Figure 6.1.4: Displacement Vector



Generally, the displacement vector represents a vector which points from some initial position, i, to a final position f:

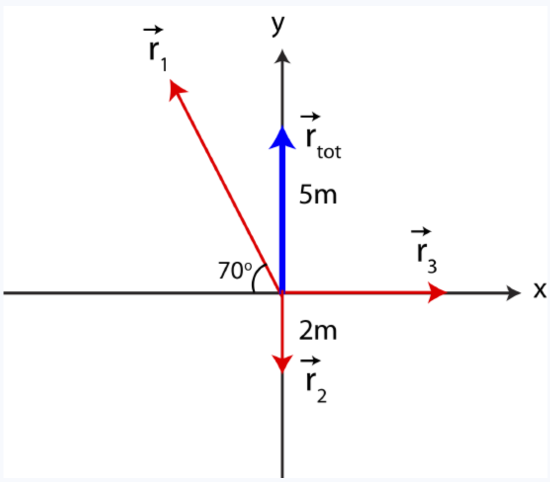
$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i \tag{6.1.13}$$

Example 6.1.2

Three position vectors are added resulting in a net vector pointing north with a magnitude of 5 meters. The first vector points northwest at 70°, the second vector points south and has a magnitude of 2m, and the third vector points east. Find the magnitude of the third vector.

Solution

The graph below depicts the provided information.



The three vectors add up to the total vector:

$$\vec{r}_{tot} = \vec{r}_1 + \vec{r}_2 + \vec{r}_3 = (-|\vec{r}_1|\cos 70^{\circ} + r_{3,x}, |\vec{r}_1|\sin 70^{\circ} - 2) = (0, 5) \tag{6.1.14}$$

Let us split the equation above in terms of components:

$$r_{tot,x} = -|\vec{r}_1|\cos 70^{\circ} + r_{3,x} = 0 \tag{6.1.15}$$

$$r_{tot,y} = |\vec{r}_1|\sin 70^{\circ} - 2 = 5 \tag{6.1.16}$$

Looking at the equation above, we can first solve for the magnitude of $\vec{r}_1$ using the y-components since that's the only unknown in the y-direction, and then use the result to solve for the magnitude of $\vec{r}_3$.

Solving for the magnitude of $\vec{r}_1$:

$$|\vec{r}_1| = \frac{3}{\sin 70^{\circ}} = 3.19m \tag{6.1.17}$$

Now plugging in this result in the equation in the x-direction and solving for r_{3x}:

$$r_{3x} = 3.19 \cos 70^{\circ} = 1.09m \tag{6.1.18}$$

Since the vector $\vec{r}_3$ only has a component in the x-direction, its magnitude is just its x-component: $|\vec{r}_3| = r_{3x} = 1.09m$.

Velocity and Acceleration Vectors

We just defined the displacement vector which is a vector representing the change in position as an object moves from an initial to a final location. The *velocity* vector will give us additional information, specifically how fast the object is moving in the direction of displacement. Since the velocity of the object does not need to be constant, we define the *average velocity*, $\vec{v}_{avg}$, which is the average rate of displacement, $\Delta\vec{r}$, over time, Δt, that it took to move from the initial to final state:

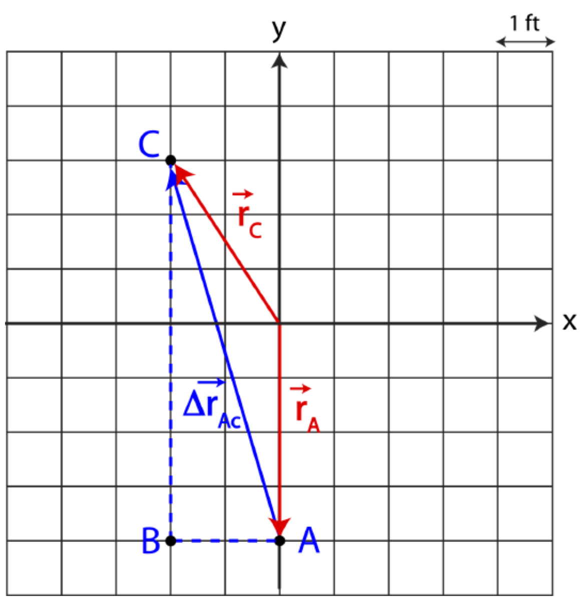
$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} \tag{6.1.19}$$

A given motion may cover a lot of distant and takes a long time, there could be a lot of variation in both speed and direction of motion between the initial and final locations, so a lot of information might be missing from knowing only the average velocity. Therefore, it is also useful to define *instantaneous velocity*, $\vec{v}$, over some small time period dt:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \tag{6.1.20}$$

Figure 6.1.5 below shows an example of a path you take marked by a dashed line to move from location A to B to C. It is important to realize that displacement vector does not describe a particular path an object takes to move between two locations, but it is a vector which points from some initial location to a final location, regardless of the path taken to move between the two locations.

Figure 6.1.5: Displacement Example



In the example depicted in Figure 6.1.5, even though the path is described by the dashed lines, the displacement vector, $\Delta\vec{r}_{AC} = \vec{r}_C - \vec{r}_A$, points directly from A to C. Reading from the grid we find by subtracting the two position vectors that $\Delta\vec{r}_{AC} = (-2, 3) - (0, -4) = (-2, 7)$ ft, which is the same result you could obtain by simply reading the components of the displacement vector directly from the graph. If the entire path took 60 seconds to complete, then the average velocity would be:

$$\vec{v}_{avg} = \frac{\Delta \vec{r}_{AC}}{\Delta t} = \frac{(-2,7)}{60}\frac{\text{ft}}{\text{sec}} \tag{6.1.21}$$

Note, when you divide or multiple a vector by a constant, you simply need to divide or multiple each component by that constant, so $\vec{v}_{avg} = (-1/30, 7/60)$ ft/sec. The average speed is the magnitude of the velocity vector resulting in $v_{avg} = 0.121$ ft/sec, and the angle is calculated to be $\theta = 74.1^\circ$ in the northwest direction. Although, you never take the path directly in the northwest direction, this is just the average velocity over the entire path.

Knowing the average velocity as calculated above does not give us any information about the specific path that you took to get from location A to C. Let us assume that on your journey between locations A and B you were walking with a constant speed and it took 15 seconds to cover the distance between A and B. This information allows us to compute your instantaneous velocity between points A and B, $\vec{v} = (-2/15, 0)$ ft/sec, which tells us that you moved at a speed of 0.133 ft/sec in the west direction during that portion of the path. If the profile of your motion is not described by a simple constant speed and a straight line direction, but rather given by some curve, you would need to use the derivative definition of instantaneous velocity given in Equation 6.1.20.

Another important vector quantity that we will discuss in greater detail in later chapters is *acceleration* which describes the rate of change of velocity. If the motion is at a constant speed and in a straight line (no change in direction), the the acceleration is zero. However, if the speed and/or direction are changing then the object will experience non-zero acceleration which we will relate to forces acting on the object in the next section. Like for velocity, we can define an *average acceleration* as the rate of change of velocity from some initial velocity, $\vec{v}_i$, to a final velocity, $\vec{v}_f$, over a time period, Δt:

$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t} \tag{6.1.22}$$

The average velocity does not tell us about variations of changes in velocity between the intial and final times. So *instantaneous acceleration* is sometimes a more useful quantity to define:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \tag{6.1.23}$$

Alert

When an object is accelerating, in every day language, it is interpreted as the object is speeding up. In physics, the proper definition of acceleration is the rate of change of velocity. This means that when acceleration is not zero, the object can be either speeding up, slowing down, or changing direction. Often the word "deceleration" is used to describe the act of slowing down, but in physics we will always refer to any kind of velocity change as "acceleration".

6.2: The Force model

The Definition of Force

Force is another type of vector which we will study extensively in this course. Forces will give us information about *dynamics* of objects: they will explain why some objects remain stationary and why other objects will move they way they do. A *force is a push or a pull* which describes an interaction between two objects. The word *object* is used here in a general sense to include any identifiable mass, which might be a particular volume of a fluid or a single electron, as well as ordinary objects such as a table, a book, or a car. The essential idea here is that forces do not exist in the absence of interacting objects. When you exert a "push" with your hand, there are two objects involved, your hand the object that you are pushing. A force is a vector quantity which describes both the strength of the push or pull (the magnitude of the force vector) and the direction of push and pull. As we will see in the next few chapters both the strength and the direction of the force is essential to describing the dynamics of objects. Our understanding of just what a force is how how to use it to predict or explain motion will become sharper and more refined as we progress through the next couple of chapters.

Types of Forces

An interaction can occur between objects that are in direct physical contact, as well as between objects that are widely separated in space. An example of the former is the force you apply on an object when you push it, or the force your bat exerts on a ball as you hit it into left field. An example of the latter is the force of gravity the Earth exerts on an orbiting satellite 200 miles above the surface of the Earth. The gravitational interaction between the Earth and other objects which may or may not be touching the Earth is said to be *long range*. The electric force between charged particles is another example of a long range force. The force resulting in an interaction continue to exist, even when there is no direct contact. This is distinctly different from the example of the bat and ball, once the ball is in the air after the bat has hit it, the bat no longer exerts a force on the ball. The type of forces such as that of the bat on the ball is often referred to as a *contact forces*, since there is no interaction or force if the bat and ball are not in physical contact (in a macroscopic sense).

Long Range Forces:

The electric and gravitational forces are said to be long-range forces, also known as *fundamental* forces. They are called *fundamental* since all other forces between macroscopic objects which will be describe shortly can be described in terms of fundamental forces (except for gravity) at a microscopic scale. Although the strength of the long-range forces decreases with distance, the interactions still exist and have profound effects even at large separation distances. The magnitude of the electric and gravitational force between two particles decrease as $1/r^2$, thus, these two forces are often known as *inverse-square law* forces. In both cases, electric and gravitational, the force acts along the direction of the vector from one particle to the other. In addition, each force depends on a fundamental property of matter: electric charge and gravitational mass, respectively. These two forces are the manifestations of two of the four fundamental interactions: gravitational, electromagnetic, weak nuclear, and strong nuclear. We will cover the electromagnetic force is greater detail in Physics 7C.

We directly experience one of the fundamental forces all the time, the *gravitational force*. We know this force as the *weight* of an object. The weight of an object on the surface of the Earth is proportional to the mass of the object times acceleration due to gravity, *g*, having the value 9.80 m/s^2, which we often rounded to 10 m/s^2. We will mostly use the symbol, F_g, for gravitational force which at the surface of the Earth is equal to:

$$F_g = mg \tag{6.2.1}$$

Contact Forces:

Recall in Physics 7A what happens when molecules get pushed very close together. The electrons then exert very large repulsive forces on each other. This is why substances resist compression. When we push on an object, it is the electrical forces between the electrons in the molecules of our skin and the electrons on the atoms at the surface of what we are pushing that are really doing the pushing. And it is the electrical forces holding molecules together that allow us to establish a tension in a stretched wire or cord.

So, we actually do experience electrical forces all the time. But because they are for the most part due to electrically neutral molecules interacting with each other, the net forces are “short range” and are not given by our simple formula. We often describe these electric forces as *contact forces*. In other words, the forces that electrical neutral objects exert on each other when they are brought into close proximity really are electric forces, but they are very short range, and are not described by the inverse-square law. We often categorize these laws based on the type of interaction that is occurring macroscopically.

Here are basic description of some of these forces:

- *normal force:* a repulsive "push" force due to physical contact of an object with a surface which always acts perpendicular (which is the meaning of the word "normal") to that surface. We often use the symbol, F_N, for this force.

- *tension force:* an attractive "pull" force which is transmitted through a string or rope. We often use the symbol, T or F_t for this force.

- *friction:* a force which acts between two objects parallel to their surfaces, and acts in the direction opposite their motion (known as *kinetic friction*) or in the direction opposite their intended motion (known as *static friction*). Kinetic friction explains why a toy car when pushed on a rough surface will eventually come to a stop. The expression for kinetic friction which is determined experimentally is given by:

$$F_k = \mu_k F_N \tag{6.2.2}$$

where μ_k is the *coefficient of kinetic friction* and F_N is the normal force. The coefficient of kinetic friction depends of the nature of the materials that are rubbing against each other. The more slippery the materials are, the less is the effect of friction, resulting in a smaller coefficient of kinetic friction. The normal force also plays a role on how much the two materials are rubbing against each other. The harder you push perpendicular to the two surfaces, the harder it will be for them to slide relative to each other. A heavier object, resulting in a larger normal force, will stop faster when moving on a surface than a lighter object.

Static friction explains why a car parked on a slopped hill does not simply roll down, or why a heavy object doesn't budge when pushed. For static friction the applied force need to reach a critical magnitude before the friction is overcome, and the object starts to move. The expression for static friction is:

$$F_s \leq \mu_s F_N \tag{6.2.3}$$

where mu_s is the *coefficient of static friction* and F_N is the normal force. When an object is stationary, all forces need to be balanced. If you lightly push on a heavy object and it does not move, then the force is static friction is equal to your push. If you now push a little harder but the object still does not move, then the force of static friction is now larger than before to balance with the stronger push. The maximum value that static friction can have is $F_s = \mu_s F_N$. The coefficient of static friction is typically larger than the coefficient of kinetic friction. It is more difficult to get the motion started than to keep the motion going.

- *air friction or drag force*: when an object moves through a gas or liquid there are there are *drag* or *viscous forces* retarding the motion of the object. These forces act in the direction opposite to the velocity of an object and are zero is the object is stationary. When a parachute is falling down or a car moves on a highway at a high speed, air exerts a drag force, sometimes called *air friction*.

- *spring force:* Robert Hooke (a contemporary of Isaac Newton) discovered in 1676 that the force exerted by many stretched springs is proportional to the elongation or compression of the spring and in the opposite direction to the elongation or compression. The constant of proportionality depends on the way the particular spring is made (its material, size, number of coils, etc.). We used this model for springs extensively in Physics 7A. In equation form we express the force exerted by a spring, known as *Hooke's Law*, as

$$\vec{F}_s = -k\Delta\vec{x} \tag{6.2.4}$$

where k is the spring constant and $\Delta\vec{x}$ is the displacement away from equilibrium. We write Hooke's law in vector form order to stress the relationship between the direction of force and displacement. Whether the spring is stretched or compressed the vector $\Delta\vec{x}$ points away from equilibrium, while the spring force always points in the opposite direction, toward equilibrium, due to the minus sign in the equation. Note the "Δ" in the expression for displacement does not correspond to the more common "before and after" application of the symbol. This "Δ" stresses that it is the distance from equilibrium that is measured. If the equilibrium position is at some distance x_o, then $\Delta\vec{x} = \vec{x} - \vec{x}_o$, where $\vec{x}$ is position of the spring after it have been stretched or compressed.

Newton's First Law

Before Isaac Newton, arguably the greatest scientist of all time, came along and changed the view of nature forever, the belief established by thinkers such as Aristotle was that motion required forces. In other words, ancient (pre-Newtonian) understanding of motion was that for an object to move a force needs to be acting on it and that eventually everything comes to a stop. Newton claimed the opposite, that for an object to stay at rest or to remain moving with a constant velocity (constant speed moving in a straight line) no total force can be acting on it. This statement is known as *Newton's First Law*.

Newton's 1st Law of Motion

Objects at rest stay at rest and objects moving in a straight line at a constant speed continue moving this way unless acted upon by an external total force.

Let us think carefully about the implications of Newton's First Law by thinking about a specific scenario. When you are standing on the floor in an airport terminal and then one hour later when you are standing in the aisle of a jet plane going 500 miles per hour with no turbulence, you experience the same sensations on your body. This illustrates a fundamental aspect of the 1st Law. If the total net force is zero on you when standing on the floor of the airport terminal, and the net force on you is also zero when on the jet traveling at a constant speed with respect to the ground, the effect is the same: no change in motion. From your perspective you are standing still on the ground and standing still on the airplane. But from another person's perspective at the airport, when you are on the airplane you moving at a constant 500 miles per hour. Evidently, being motionless and moving with a constant velocity are the same thing with respect to how forces work. Because forces work this way, we can turn moving at a constant velocity to being motionless by switching to reference frame that is moving with the same velocity with respect to the original reference frame.

The word *total force* (also called *net force*) in the statement above has very important implications. Newton's First Law does not claim that no forces can be acting on the object for it to remain at rest or moving at constant velocity, but rather the *total* force acting on the object must be zero. A book which is stationary on a table has a force of gravity acting on it. Thus, Newton's First Law implies that there must be another force acting on the book that must *balance* the force of gravity. This force is the normal force of the table pushing back on the book. In other words, the fundamental aspect of Newton's First Law is that if all the forces acting on an object, both contact and long-range forces, are balanced and the total force is zero, then all of those forces acting together have no effect on the object. It is as if there are no forces acting on the object. The effect of all forces acting on a particular object can be represented by a single vector called the *total force* or the *net force* which is defined as the sum of all forces acting on that object:

$$\vec{F}_{net} = \Sigma \vec{F} \tag{6.2.5}$$

Mathematically, Newton's First Law can be stated as:

$$\text{if } \vec{F}_{net} = 0, \text{then } \Delta \vec{v} = 0 \tag{6.2.6}$$

The quantity $\Sigma \vec{F}$ is a vector sum, since the individual forces acting on the system are vector quantities. The net force is not a physical force due to a particular interaction with another object acting on the system, but rather it is the *effect of all the forces* acting on our system. Because of this lack of connection to a particular interaction, net force is a rather abstract concept, but one that turns out to be very useful.

Newton's First Law describes situations when all the forces acting on an object are balanced. There are two complementary ways of applying the 1st Law. If we know that there is *no change* in motion, then we know that the forces acting on the object must be balanced. If we know all but one of the forces, we can solve for the the magnitude and direction of the unknown force. The second way to use the 1st Law is to add up the known forces to see if they balance. If they do, than there cannot be a change in motion of the object.

The sum over all forces $\Sigma \vec{F} = 0$ is a vector quantity which can re-written in terms of its components, assuming forces act in only two-dimensions, as:

$$\Sigma \vec{F} = (\Sigma F_x, \Sigma F_y) = (0, 0) \tag{6.2.7}$$

The equation above implies that for the net force to be zero, the sum components of all the forces in the x-direction must be zero, as well as in the y-direction. In other words, when balancing forces, you are really trying to solve two independent equations (if you are working in 2D), on the x-direction and another in the y-direction. Technically, the equation above can be written as a set of two independent equations:

$$\Sigma F_x = 0 \tag{6.2.8}$$

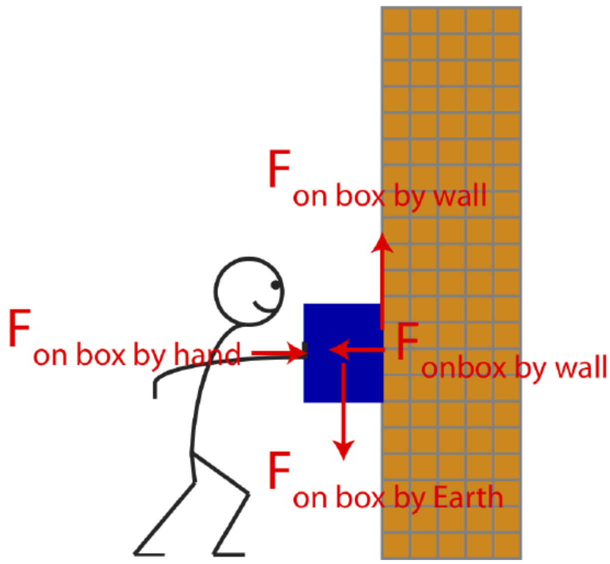
$$\Sigma F_y = 0 \tag{6.2.9}$$

Let us look at an example shown below in Figure 6.2.1, where we would like to analyze a box which is being pushed and kept stationary against the wall by a person in terms of Newton's First Law. All forces acting on a box are labeled.

Force notation

In order to stress that two objects are interacting when there is a force present an "on-by" notation is often used when labeling forces. So if a force is acting on object A and is being exerted by object B we write this as: $F_{\text{on object A by object B}}$.

Figure 6.2.1: Balanced Forces



Once you define a system, in this case a box, in order to figure out which forces are acting on the system you need to identify all the objects that the system is in direct physical contact with and if any long-range forces involved. If the system is in a gravitational field (on Earth, for example) and the mass of the system is not negligible, then we always must include the long-range gravitational force acting directly down toward the surface of the Earth. In Figure 6.2.1, the gravitational force is labeled as $F_{\text{on box by Earth}}$ to stress the fact that it represents an interaction between two objects, the box and Earth. The "on box" is due to our system being the box, and "by Earth" is the object which is exerting a force on our system. As you get more comfortable with working with forces, you can abbreviate the gravitational forces acting on a system as simply F_g. Since the box is stationary (not slipping down), Newton's First Law tells us that there has to be an equal an opposite force pointing up. The box is in physical contact with the wall, which is exactly what provides the force holding the box. It is a horizontal force between two surfaces (the box and the wall), which we defined as the force of friction. The notation $F_{\text{on box by wall}}$ is again to stress that it is a contact force due to the interaction between the box and the wall, but it can be abbreviated as the frictional force F_f. Using this notation, we balance the two forces by summing them to zero. If we defining up as the positive direction then the net force in the y-direction is:

$$F_f - F_g = 0, \tag{6.2.10}$$

resulting in $F_f = F_g$. The frictional force generated by wall on the the box when in contact must point up and have the same magnitude as the gravitation force on the box by the Earth.

In the x-direction, the person is pushing the box with their hand, so there is a force on the box by the hand pointing to the right. To balance this force, there must be an force pointing in the opposite direction. Since the box is touching the wall, this force is by the wall and is perpendicular to the surface, which we defined as the normal force. The notation for this normal force can be abbreviated as F_N, but again the notation $F_{\text{on box by wall}}$ is to stress that this force is due to the interaction of our system with the wall. Defining the positive x-direction as the one pointing to the right and balancing forces in the x-direction we get:

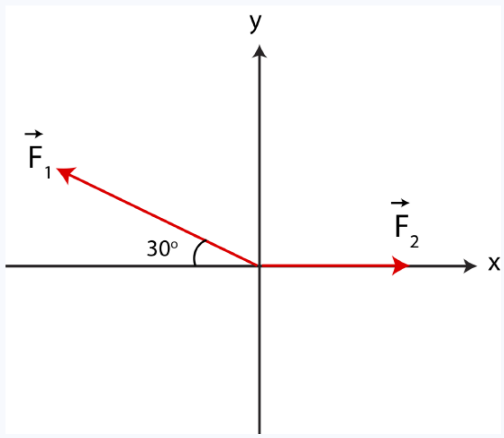
$$F_{\text{on box by hand}} - F_N = 0, \tag{6.2.11}$$

resulting in $F_{\text{on box by hand}} = F_N$, so the normal force needs to be equal and opposite to the push force by the person.

As seen in this example, both the perpendicular (the normal force) and parallel (friction) forces arise from the same interaction, the box with the wall. Thus, these seemingly two separate forces are just the parallel and the perpendicular complements of one force, the force exerted by the wall on the box. Often, using the words "normal" and "friction" are abstract since they do not directly tell you about the particular interaction. You should always start your force analysis by first considering the physical objects which are interacting with your system, either by direct contact (like the wall) or at a distance (like the Earth).

Example 6.2.1

An object is moving with a speed of 5m/s in the positive y-direction while three forces are acting on it. Two of the forces are shown in the figure below. The magnitude of $\vec{F}_1$ is 16 N and the magnitude of $\vec{F}_2$ is 10 N. Find the magnitude and direction of $\vec{F}_3$.



Solution

Since the object is moving at a constant velocity the three forces must add up to zero:

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 \tag{6.2.12}$$

This implies that:

$$\vec{F}_3 = -\vec{F}_1 - \vec{F}_2 \tag{6.2.13}$$

Expressing each force by components we get:

$$\vec{F}_1 = 16N(-\cos 30^\circ, \sin 30^\circ) = (-13.86, 8)N \tag{6.2.14}$$

$$\vec{F}_2 = (10, 0)N \tag{6.2.15}$$

Plugging into the expression for $\vec{F}_3$:

$$\vec{F}_3 = -(-13.86 + 10, 8 + 0) = (3.86, -8) \tag{6.2.16}$$

The magnitude of the vector $\vec{F}_3$ is:

$$|\vec{F}_3 = \sqrt{3.86^2 + 8^2} = 8.88N \tag{6.2.17}$$

And the angle:

$$\theta = \arctan\left(\frac{8}{3.86}\right) = 64.2^\circ \tag{6.2.18}$$

in the southeast direction.

Newton's Second Law

Newton's First Law is a special case of *Newton's Second Law* which describes what happens when the net force acting on the system is not zero. It states that when the net force acting on an object is not zero, then it will experience an acceleration in the direction of the net force.

Newton's 2nd Law of Motion

The net force acting on an object is equal to its mass times its acceleration.

Mathematically, Newton's Second Law is written as:

$$\vec{F}_{net} = m\vec{a} \tag{6.2.19}$$

As we described in Section 6.1 acceleration is a vector which describes the rate of change of velocity, or change of motion. Some examples of the influence of this non-zero net force could be an stationary object starting to move, a moving object speeding up, slowing down, or coming to a rest, of a moving object changing its direction even while its speed is staying constant. If there is no net force acting on the system, Equation 6.2.19 states that acceleration is zero, which directly implies that $\Delta\vec{v} = 0$ and we recover Newton's First Law as given in Equation 6.2.6.

Although Equation 6.2.19 is rather simple mathematically, its physical meaning is often misinterpreted. The equation is often read as: "an object which is accelerating is carrying a force which is equal to the object's mass times its acceleration, and then will exert this force if it interacts with another object". This way to interpreting Newton's 2nd Law is completely wrong. To help resolve this interpretation it more physical to think about Equation 6.2.19 in reverse: "an object which has a net external force acting on it will experience an acceleration which is equal to this net force divided by its mass". In other words, the common misinterpretation is due to cause and effect being reversed: it is not that an accelerating object produces a net force, but rather a net forces results in an acceleration. Thus, it is better to re-write Newton's Second Law as:

$$\text{if } \vec{F}_{net} \neq 0, \text{then } \vec{a} = \frac{\vec{F}_{net}}{m} \tag{6.2.20}$$

Having the 2nd Law written as above also makes it more obvious to see that the 1st Law, as given in Equation 6.2.6, is just a special case of the 2nd law when the net force is zero. We will going into more details of applying Newton's 2nd Law in later chapters.

Newton's Third Law

As we start to understand forces in greater details, it is important to remind ourselves that the starting point is a very important aspect that forces come in pairs, which is a direct consequence of the fact that forces describe interactions between two objects. To have an interaction, we must have *two* objects that are interacting. Suppose *object A* is interacting with *object B*. Focusing on *object B*, we would say that *object A* exerts a force on *object B*. Alternatively, focusing on *object A*, we would say *object B* exerts a force on *object A*. These *two* forces, associated with the same interaction between the same two objects must be related to each other. But how are they related? It turns out these forces are equal and opposite. This is known as *Newton's Third Law* of motion. Mathematically, the two forces between two interacting objects, *object A* and *object B* point in *opposite* directions and the magnitude of the force that *object A* exerts on *object B* has the same magnitude as the force that *object B* exerts on *object A*.

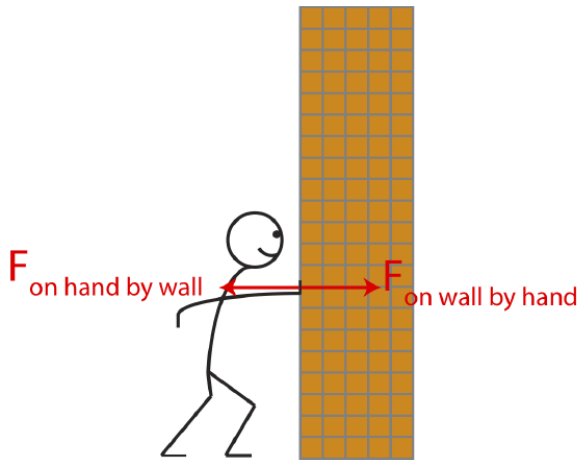
$$\vec{F}_{\text{on B by A}} = -\vec{F}_{\text{on A by B}} \tag{6.2.21}$$

In words, this relationship can be expressed as, "If object A exerts a force on object B, then object B exerts a force equal in magnitude and opposite in direction on object A".

The effect of the interaction on each of the objects, however, does not have to be the same, and usually it is not the same. The *effect* of the interaction depends on other properties of the specific object, such as its mass and its motion prior to the interaction. For example, when a bug collides with the windshield of a moving car, Newton's Third Law tells us that the car and the bug experience the same magnitude of force in opposite direction, but the effect on the motion of the two objects is obviously very different. This is an important distinction: the interaction is always the same with respect to the two objects, but the effect that the interaction has on each of the two objects does not have to be the same and usually is not. It is very important to keep this distinction in mind when thinking about forces. Newton's Second Law is what gives us an answer about why the effect is different even though the forces are the same. The car which has a much greater mass compared to a bug, will experience a much smaller acceleration for the same magnitude of force. Thus, the change of motion of the car will be negligible, while the bug will experience a much greater acceleration due to its tiny mass.

Figure 6.2.2 below shows an example of a *Third Law pair*, which describes the two forces involved in one interaction. In this example, a person exerts a force on the wall by pushing on it with their hand, labeled as $F_{\text{on wall by hand}}$ and drawn with an arrow pointing to the right, toward the wall. Newton's Third Law tells us that the wall must exert a force on the hand with the same magnitude as the push by the hand by in the opposition direction. This force is labeled in the figure as $F_{\text{on hand by wall}}$ and drawn with an arrow which has the same length as $F_{\text{on wall by hand}}$ but in the opposite direction, toward the hand.

Figure 6.2.2: Third Law Pair



A Third Law pair can always be recognized by looking at the "on by" notation, with the objects in this notation are flipped, a 3rd law pair is recognized. This is another reason to stick with the "on by" notation when analyzing forces, since it is an clear way to track which object is experiencing the force and which object is exerting it.

Here is another common misinterpretation that often occurs when thinking about Newton's Third Law this time, "since for every force there is an equal and opposite force which is present, how can anything ever move since all the forces always balance?". It is important to distinguish between forces exerted by a particular object on other objects, and the forces those other objects exert on the original object. Only forces acting *on* an object affect that object. Forces that an object itself exerts on other objects do not affect itself. Therefore, when applying Newton's 1st and 2nd laws we only consider forces on the object whose motion we are analyzing, and ignore all the forces that object exerts on other objects. It is sometimes easy to confuse the 1st and 3rd laws, especially when there are two forces acting on an object. The pairs of forces that balance each other in the First Law act on the same object, so they cannot possibly be Third Law force pairs, even though they are equal and point in opposite directions.

The wall in Figure 6.2.2 does not move not because the two forces shown in the figure which describe the 3rd law pair balance, but because the wall is attached to the floor, and the floor exerts an force on the wall (static friction) in the opposite direction of the push on the wall. These two forces (on the wall by the hand and on the wall by the floor) balance and prevent the wall from moving due to Newton's First Law.

- Authors of Phys7A (UC Davis Physics Department)

6.3: Applying the Force Model

Free-Body Diagrams

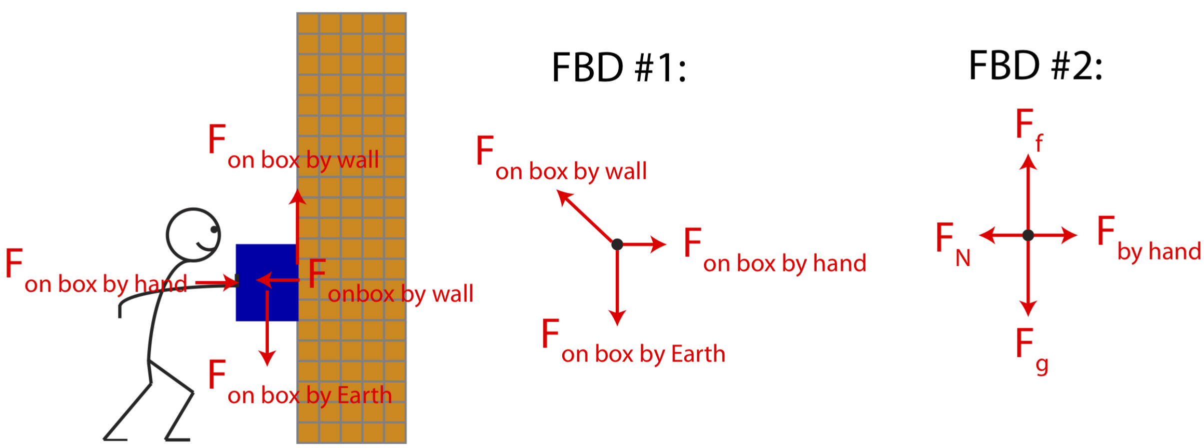
When we want to analyze a physical situation in terms of forces, it is necessary to focus on forces acting on a particular object or a collection of objects. We will always be interested in the net force on a particular system. To help in identifying forces that act on a particular object it is helpful to pictorially represent the forces as clearly labeled arrows on a diagram. There is a standard convention for representing forces like this, called a *free-body diagram or FBD*. We will also sometimes refer to this diagram as a *force diagram*.

The following is a list of common conventions that we will follow when making a force diagram. It is important to be familiar with these and strictly follow them; it will make things clearer in the long run:

- A force diagram refers only to *one system*. A system might consist of one object or multiple objects, and the forces that act on that system must come from objects or systems which are not part of the defined system.
- The system is shown as an enlarged dot in a force diagram. The dot is clearly labeled to indicate what system it refers to.
- All the forces acting *on* the chosen system are shown on the force diagram. Forces that the system exerts on *other* systems are *not* shown on its own force diagram. You should never find forces that represent a third-law pair on the same FBD.
- Forces like friction and weight (that act over multiple points) are modeled as a *single* force acting at a *single* point.
- To avoid confusion, we usually don't show velocity or other vector quantities on the force diagram.
- If a particular force has components in multiple directions, it is useful to draw separate arrows for each component, since it will help with finding the net force in each spacial direction.
- A net force vector can be added to the force diagram, but there should be a clear indication that the force is the sum of all the forcing acting in the system, rather than a new force.

Figure 6.3.1 shows a free-body diagram of the example we analyzed in the previous section. The picture on the left is a diagram of the physical situation showing all the forces that act on the box as a person pushes it against the wall while keeping it in place. There are two FDBs of this scenario shown. In both, the dot in the center represents our selected physical system, in this case the box. All the forces shown act on the box. By convention, the tails of the forces start at the dot representing the system.

Figure 6.3.1: Example of Free-Body Diagrams with Zero Net Force

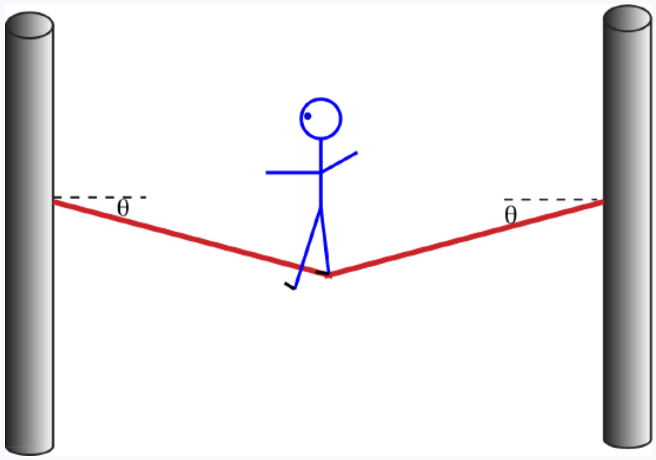


The central free-body diagram labeled FBD #1, shows three forces which represent the three objects with which the box is interacting: the hand, the wall, and the Earth. The "on by" notation in the central diagram stresses the fact that the forces present act on the box by these three objects. The force by the box on the wall points in the northwest direction because it shows the combined force by the wall, which is the sum of the vertical and horizontal components of this force shown in the left picture. The diagram labeled FBD #2 splits $F_{\text{on box by wall}}$ into its vertical component, which represents the force of friction with the wall, F_f, and its horizontal component which here represents the normal force, F_N, which is perpendicular to the surface of the wall. Also, FBD #2 abbreviates the gravitational forces as F_g and drops all the "on box" notations. It is often more useful to start with labeling forces like in FBD #1 which assures that you are not missing any objects with which your system is interacting. Once you get more comfortable with forces with lots of practice, you may go straight to the notation used in FBD #2.

The arrows drawn in horizontal direction have the same length, so do the arrows in the vertical direction. This is to indicate that the forces are balanced. Since the box is stationary, the forces in both the x- and the y- directions must cancel. The fact that F_g arrow is drawn longer than the $F_{\text{by hand}}$ arrow shows that these forces do not need to have the same magnitude. Of course, the reverse is possible, the force of the push could be greater than the weight of the box. In that case, the magnitude of the normal force would have to be greater than the magnitude of the frictional force.

Example 6.3.1

Shown below is a tightrope walker who is balancing on a rope tied between two poles. The weight of the person is 600 N. The rope makes an angle of 5° with the horizontal.



a) Draw a free-body diagram for the tightrope walker.

b) Explain why the tension in the rope to the right of the person's position must equal to the tension of the rope to the left of her position.

c) Calculate the tension in the rope.

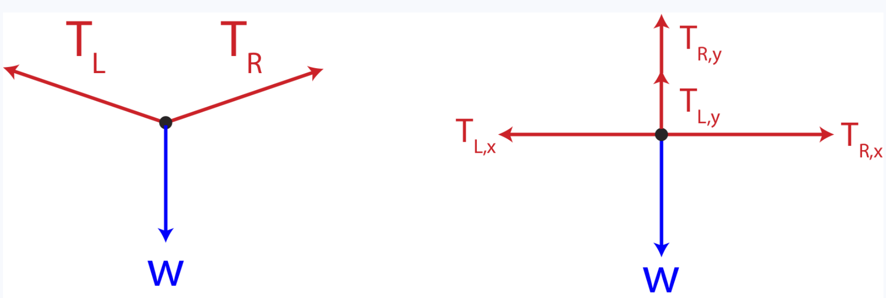
Solution

a) Since the person is stationary on the rope, there is no net force. The forces must be balanced in both the x- and y- directions:

$$\sum F_x = 0$$

$$\sum F_y = 0$$

Shown below is a free-body diagram for the tightrope walker. The left figure shows the three forces present: the weight, and the tension from each side of the rope relative to the position of the person. The right figure shows the forces by components, since we need to split the tension forces by components in order to balance them.



b) In the x-direction there are only two forces:

$$\sum F_x = T_{R,x} - T_{L,x} = 0$$

Thus, the x-components must be equal:

$$T_{R,x} = T_{L,x}$$

In terms of angle and the magnitudes of the tension forces we get:

$$T_R \cos\theta = T_L \cos\theta$$

Resulting in:

$$T_R = T_L$$

Thus, the tensions on each side of the person are equal.

b) In the y-direction the forces are:

$$\sum F_y = T_{R,y} + T_{L,y} - w = 0$$

In terms of angle and magnitude and using the result from a), $T = T_R = T_L$ *, we get:*

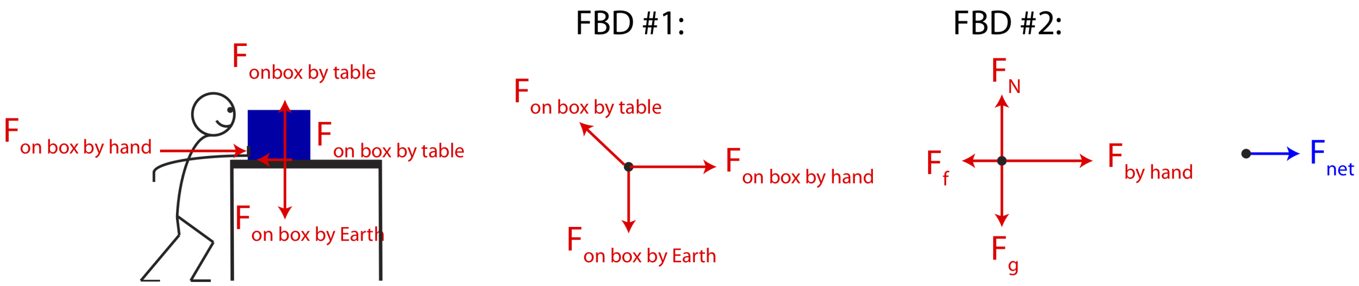
$$2T\sin\theta - w = 0$$

Solving for T:

$$T = \frac{w}{2\sin\theta} = \frac{600N}{2\sin 5^\circ} = 3442N$$

In another example below, Figure 6.3.2 shows a student pushing a box on a surface of a table. Let us assume that the box is initially stationary and begins to move as the student pushes it. Based on Newton's 2nd Law this implies that the net force is no longer zero in the horizontal direction since the box experiences a change in velocity in that direction.

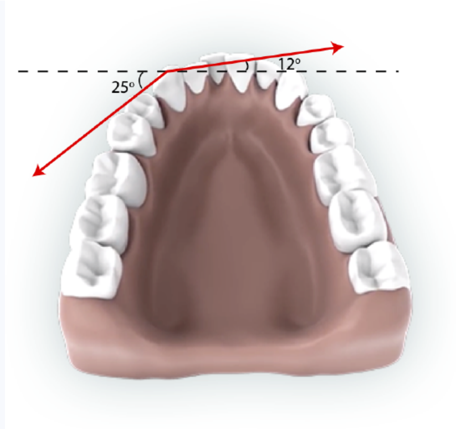
Figure 6.3.2: Example of Free-Body Diagrams with Non-zero Net Force



The force-diagrams drawn in the figure above indicate that the net force is not zero in this case. The box does not move in the vertical direction, so the force of gravity on the box must still be balanced with the vertical force exerted on the box by the table. In this case, the normal force is vertical, since it is perpendicular to the surface of the table which is horizontal. In the horizontal direction, however, the box changes its velocity from zero to non-zero, which implies an acceleration in the positive x-direction based on Newton's 2nd Law. This means that the force on the box by the pushing hand is greater than the frictional force on the box by the table which opposes motion and, and thus, points to the left. This is indicated in FBD #2 by drawing the $F_{\text{by hand}}$ arrow longer than the F_f arrow. The diagram shown on the very right shows the net force, F_{net}, which is the sum of all the forces in the FBDs. Note, it is better to draw the net force on a separate diagram, so it is not confused with all the physical forces which describe interactions, but rather its a force which is the summation of all of the physical forces present.

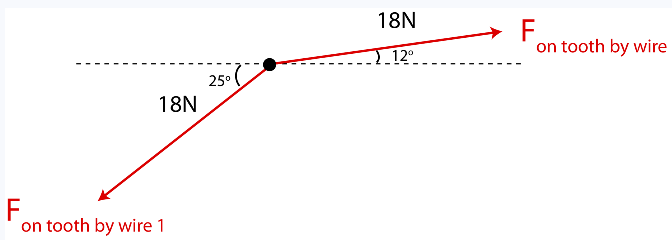
Example 6.3.2

Braces are used to apply forces to teeth to realign them. Shown in this figure below are the tensions applied by the wire to a protruding tooth. The angles relative to the horizontal are marked in the figure. Calculate the total force (magnitude and direction) exerted on the tooth if the tension in wire is 18 N.



Solution

First, let's draw a FBD with the took being the system. The only forces applied to the tooth are the two tension forces as shown below.



In the x-direction the net force is:

$$F_{net,x} = 18N\cos 12^\circ - 18N\cos 25^\circ = 1.29N \tag{6.3.1}$$

In the y-direction:

$$F_{net,y} = 18N\sin 12^\circ - 18N\sin 25^\circ = -3.86N \tag{6.3.2}$$

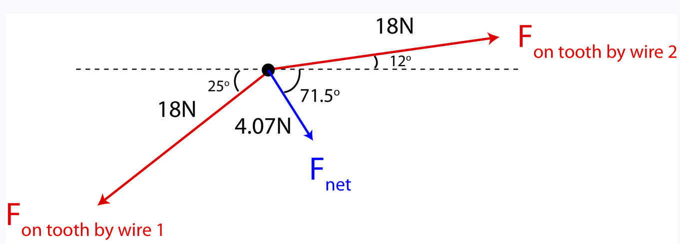
The magnitude is:

$$|\vec{F}_{net}| = \sqrt{F_{net,x}^2 + F_{net,y}^2} = \sqrt{1.29^2 + 3.86^2} = 4.07N \tag{6.3.3}$$

The direction is:

$$\arctan\left(\frac{3.86}{1.29}\right) = 71.5^\circ \tag{6.3.4}$$

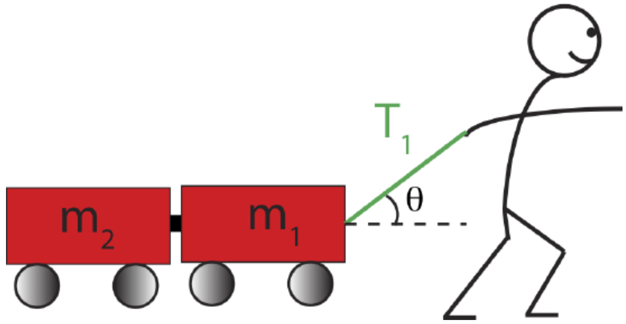
pointing southeast. The net force is added to the FBD below, pointing toward the center of the mouth such that the protruding tooth aligns with its neighboring teeth.



One Dimensional Acceleration

Let us look at an example where we can use information about the system's acceleration to determine forces involved. In the figure below a child pulls two carts with a rope, at an angle of 30° relative to the horizontal, along a frictionless surface. As a result the carts accelerate at the rate of 2 m/s^2. Assume $m_1 = 0.5 \text{ kg}$ and $m_2 = 0.8 \text{ kg}$.

Figure 6.3.3: Two Carts Pulled by a Rope



Below we outline some general problem-solving steps that can be applied to many types of problems that describe physical phenomemon explained by Newton's Laws. We apply these steps to answer multiple questions about the system in Figure 6.3.3. Given some angle and masses of the two carts, we aim to calculate the tension in the rope, the normal force by the surface on both cars and on each individual cart, and the force of one cart pushing on another.

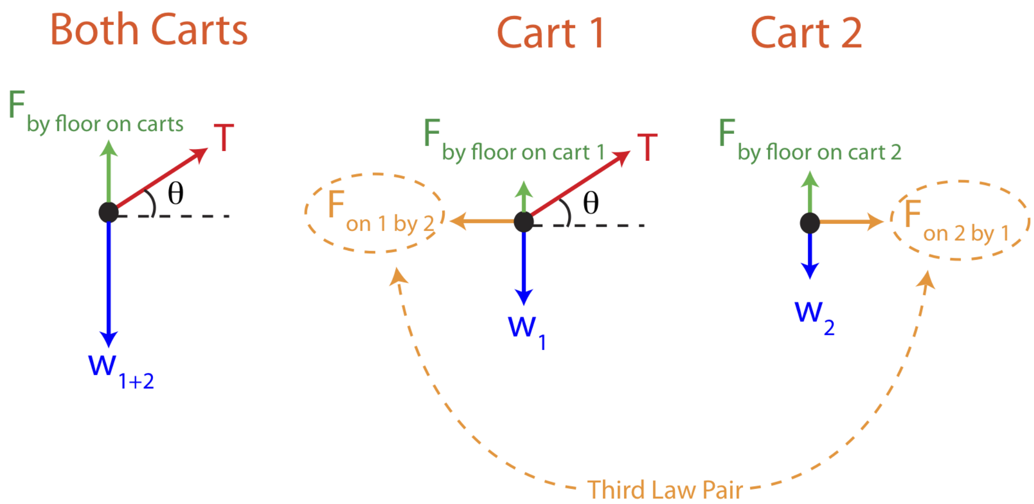
Step 1: Define Physical System

The first thing we would like to determine is how to define our physical system to answer these questions. Since the questions all ask about forces on the cart(s), we want to include the cart(s) in the system, rather than the child, for example, since we are not trying to calculate any forces on the child at this point. But should our physical system include one of both carts? The forces between the carts and the normal force on each cart are forces on the individual carts. If we define our system as the two carts combined, then all the internal forces cancel, and we would not be able to answer those questions. But to find the tension in the rope it is sufficient to think of the system as the two carts combined, although it can still be done by thinking about each cart individually. Thus, we often choose our physical system based on the types of questions we are trying to answer and the simplicity of answering those questions.

Step 2: Draw a Free-Body Diagram

We start with definition the system as both carts combined. The free-body diagrams on this system is the first diagram from the left in Figure 6.3.4 below. Once we define our system to be composed of both carts, it no longer matters that two carts are present, and only the forces acting on the two carts are present in the FBD. These forces are either contact or long-range. Contact forces in this case are the force on the carts by the rope labeled as tension, T, the force by the floor on the carts, $F_{\text{by floor on carts}}$, also known as the normal force and can be abbreviated as $F_{N,\text{carts}}$. The long-range force is the force on the carts by the Earth (the force of gravity) or the weight of the two carts, $w_{1+2} = (m_1 + m_2)g$.

Figure 6.3.4: Free-Body Diagrams For the Carts Scenario



Step 3: Split forces by Components

In order to apply Newton's Laws we first need to split all the forces into their components to solve for forces depending on acceleration in each spatial direction. In this example there is no acceleration in the y-direction (the motion is purely horizontal), but there is a non-zero acceleration in the x-direction. The only force that has both components is the tension force vector, $\vec{T}$. By components the tension force is:

$$T_x = T\cos\theta \tag{6.3.5}$$

$$T_y = T\sin\theta \tag{6.3.6}$$

The angle θ is given in this problem, so we aim to determine the magnitude of the tension force, T, based on the information provided.

Step 4: Apply Newton's Laws in Each Spatial Direction and Solve for Unknowns

Once all the forces are expressed in terms of their components, we are ready to apply Newton's Laws. In the y-direction the acceleration is zero, so the forces must be balanced. The equation is the y-direction is:

$$\sum F_y = T\sin\theta + F_{\text{N,carts}} - (m_1+m_2)g = 0 \tag{6.3.7}$$

We can't yet solve for any unknown forces using the above equation since there are two unknowns, the tension and the normal force. So, let us move on to writing down the Newton's 2nd Law for the horizontal x-direction:

$$\sum F_x = T\cos\theta = (m_1+m_2)a \tag{6.3.8}$$

Since only the magnitude of the tension is unknown in the above equation, we can solve for it:

$$T = \frac{(m_1+m_2)a}{\cos\theta} = \frac{(0.5kg+0.8kg)(2m/s^2)}{\cos 30^\circ} = 3.0N \tag{6.3.9}$$

Now we can return to Equation 6.3.7 for the y-direction and solve for the normal force:

$$F_{\text{N,carts}} = (m_1+m_2)g - T\sin\theta = (0.5kg+0.8kg)(9.8m/s^2) - 3N\sin 30^\circ = 11.24N \tag{6.3.10}$$

Alert

The normal force does not always equal to the weight of the object. This is only true when the weight of the system and the normal force are the only two forces present in the vertical direction, the surface which generates a normal force is horizontal, and the object is not accelerating.

Equation 6.3.10 demonstrates an example of a situation when the normal forces does not equal to the weight of the system, $w_{1+2} = 12.74N$. This is because there is another force present in the vertical direction, the vertical component of the tension. Imagine holding an object by a vertical rope. Even if the object is still (just barely) in contact with the floor, the floor does not exert a normal force on the object, whose weight is fully supported by the rope.

Let us now return to step 1 in the problem solving procedure in order to calculate the forces between the two carts and the individual normal forces. Since the forces between the carts are internal to the two-cart system, they did not play a role on the two cart system. Thus, we need to redefine our system as consisting of just one of the carts. The middle illustration in Figure 6.3.4 shows the FBD for cart 1, where all the forces labeled act on cart one only. Similarly, the last figure from the left shows all the forces acting on cart 2. Note, the tension is no longer present in the FBD for cart 2 since the rope is not in physical contact with the second cart. Your intuition might tell you that cart 2 accelerates because of the child pulling on the rope. This is only an indirect effect. The force that acts on cart 2 that causes acceleration is the force due to cart 1 pulling on cart 2, not the tension force. But cart 1 accelerates due to the rope pulling on it. Using the middle FBD the forces in each spatial direction on cart 1 are:

$$\sum F_y = T\sin\theta + F_{\text{N,1}} - m_1 g = 0 \tag{6.3.11}$$

$$\sum F_x = T\cos\theta - F_{\text{on 1 by 2}} = m_1 a \tag{6.3.12}$$

We can use the result for the tension we found in Equation 6.3.9 to first solve for two unknown forces in the equation above, but we would like to demonstrate that we can also calculate the tension force by analyzing the two carts individually. Below are the two equations for cart 2 using the last FBD in Figure 6.3.4:

$$\sum F_y = F_{\text{N,2}} - m_2 g = 0 \tag{6.3.13}$$

$$\sum F_x = F_{\text{on 2 by 1}} = m_2 a \tag{6.3.14}$$

Solving for the normal force on cart 2 we find:

$$F_{\text{N,2}} = m_2 g = (0.8kg)(9.8m/s^2) = 7.84N \tag{6.3.15}$$

We can see that we can solve for the force on cart 2 by car 1 immediately since there is not tension acting directly on cart 2:

$$F_{\text{on 2 by 1}} = m_2 a = (0.8kg)(2m/s^2) = 1.6N \tag{6.3.16}$$

Next we can use Newton's Third Law to obtain the force on cart 1 by cart 2. The Third Law pair is marked in Figure 6.3.4, identified because the "on by" notation is switched in the two forces which must be equal in magnitude and point in opposite directions. Thus, we can use the 3rd law and solve for the tension force using Equation 6.3.12:

$$T = \frac{F_{\text{on 1 by 2}} + m_1 a}{\cos\theta} = \frac{1.6N + (0.5kg)(2m/s^2)}{\cos 30^\circ} = 3.0N \tag{6.3.17}$$

We obtain the same results as we did in Equation 6.3.9 using the two-cart system. if our only goal was to calculate the tension in the rope, doing the method outlined above of treating each cart individually would be unnecessary work, since you need more equation and calculation to solve for the tension. But since we are also interested in calculating the forces between the two carts, we see that this method produces the same result.

Lastly, solving for the normal force on cart 1 using Equation 6.3.11 we find:

$$F_{\text{N,1}} = m_1 g - T\sin\theta = (0.5kg)(9.8m/s^2) - 3N\sin 30^\circ = 3.4N \tag{6.3.18}$$

We can see that the total normal force obtained in Equation 6.3.10 is the sum on the individual normal forces:

$$F_{\text{N,carts}} = F_{\text{N,1}} + F_{\text{N,2}} = 3.4N + 7.84N = 11.24N \tag{6.3.19}$$

This result can be seen by looking at the three FBDs in Figure 6.3.4. The combined FBD shows that the total normal force plus the y-component of the tension force equals to the total weight of the two carts, which is the result you can get by combining the two FBDs for each cart.

Two Dimensional Acceleration

The example we worked out above involved acceleration in one dimension, although there were forces acting in both direction. We will not look at an example when the acceleration is two-dimensional. And we do so by analyzing a special phenomemon of an object moving in a horizontal circle at constant speed. Imagine holding a ball attached by a string and twirling it around in a horizontal circle. As first guess you might think that the ball has zero acceleration since it is moving with a constant speed, but we need to be careful here. Acceleration is the change and velocity, so we need to think about both speed and direction before making such conclusions.

6.3.5: Circular Motion

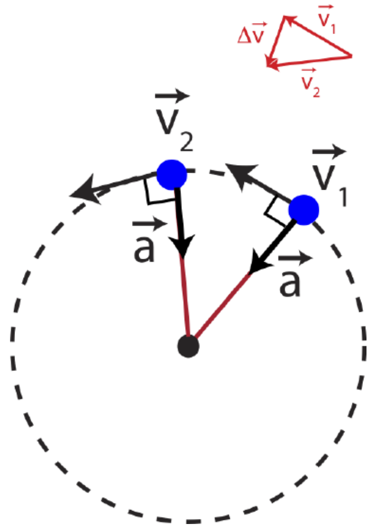


Figure 6.3.5 above shows the top view of a ball (blue dot) moving a circle. The vectors tangent to the circle represent the velocity of the ball at different locations of its motion. Although, the length of the arrows is the same, representing constant speed, the direction is changing around the circle. Thus, there has to be non-zero acceleration to change the direction of velocity. If the ball was speeding up, then the acceleration would have to be in the same direction as velocity. If the ball was slowing down, the acceleration would be in the opposite direction of velocity. But since the speed is staying constant, we conclude that the only possible direction of acceleration is perpendicular to the velocity.

We can also see this by thinking about the direction of the change is velocity which is proportional to the direction of acceleration since acceleration is the rate of change of velocity. The change of velocity between the two marked locations, $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$, can be seen by using a geometric addition of vectors in Figure 6.3.5. We can see that $\Delta\vec{v}$ points toward the center of the circle as we just argued for acceleration.

In terms of forces acting on the ball, there is only the tension of the rope and gravity present. The ball is not moving vertically since it is a horizontal circle, so the vertical component of tension equals to the weight of the ball. In the horizontal direction there is only the horizontal component of tension which points toward the center of the circle, which is exactly in the direction of the acceleration in Figure 6.3.5. This is consistent with Newton's 2nd Law that states that acceleration must in the same direction as the net force, in this case toward the center of the circle.

This type of acceleration that describes a change of direction while speed is constant is known as known as *centripetal acceleration*. To experience centripetal acceleration the object does not necessarily need to be moving in a circle, but simply changing direction at a constant speed. Centripetal acceleration has a special expression in terms of speed, v and radius r :

$$a_c = \frac{v^2}{r} \tag{6.3.20}$$

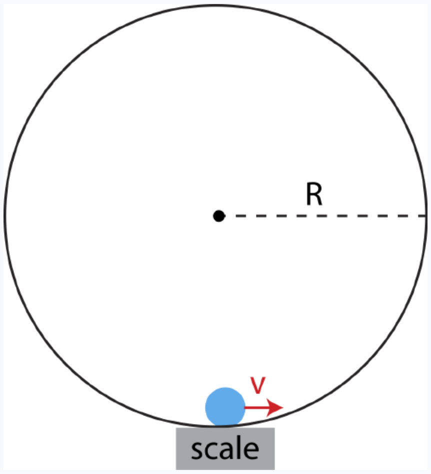
The example of a ball moving in a circle on a string, is the same reason why satellites orbit around the Earth. Of course, there is no string that is attached to the Earth and the satellite, but there is a force of gravity which always points toward the Earth. Thus, as the satellite orbit the Earth at a constant speed, the force of gravity is always perpendicular to the satellite's velocity, changing its direction and keep it in a circular motion around the Earth.

Alert

When an object experiences centripetal acceleration it is often stated that a "centripetal force" is present. This statement is often misinterpreted as a presence of a special type of force that cause objects to move in a circular way. In reality, there is no additional magical force that makes objects move in a circle, but it is simply due to a net force that happens to point perpendicular to the motion of the object in some particular situation. This force is due to an interaction of the system with other object(s), and centripetal acceleration could arise from forces such as tension, gravity, or the normal force. Thus, it is best to avoid using the word "centripetal force", but instead stick with "centripetal acceleration due to a net force which is perpendicular to the velocity of the object".

Example 6.3.3

Shown below is a ball moving around on a vertical frictionless track of radius R. At the bottom of the track, there is a scale. Assume the track is massless.

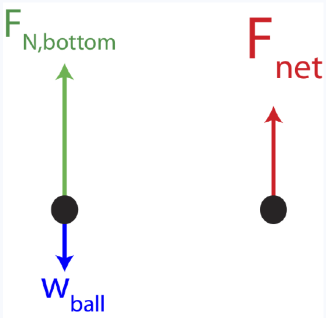


a) Determine the reading on the scale in Newtons if the mass of the ball is 0.4 kg, the radius is 0.8 m, and the speed of the ball at the bottom is 1.5 m/s.

b) Does the ball move at a constant speed? Why or why not?

Solution

a) The force on the scale is the force by the ball on the scale, which is Newton's 3rd Law pair with the force by the scale on the ball, which is the normal force. Since the ball is moving in a circle it experiences centripetal acceleration that points toward the center of the circle. There are two forces present, gravity and the normal force by the track on the ball. At the bottom of the track, gravity points down and the normal force points up. Since the net force is up in the direction of centripetal acceleration, the normal force is greater than the weight of the ball. The force diagram at the bottom of the track is shown below, along with the net force.



Using equation for centripetal acceleration we find the net force at the bottom of the track is:

$$\sum F_y = F_N - mg = ma_c = m\frac{v^2}{R}$$

Solving for the normal force we get:

$$F_N = mg + m\frac{v^2}{R} = (0.4kg)\left(9.8m/s^2 + \frac{1.5^2 m^2/s^2}{0.8m}\right) = 5.0N$$

Thus, the scale reads 5.0 N.

b) In this case since the circle is vertical and gravity always points down, the net force is not always toward the center. The normal force always points toward the center and contributes to centripetal acceleration, but the force of gravity acts along *the direction of motion except at the top and at the bottom. As the ball goes from the bottom to the top, the ball slows down, and as it returns to the bottom it speeds us again. Thus, the speed is not constant.*

Contributors

6.4 : Wrap Up

In this Chapter we introduced a mathematical concept of vectors which is needed to understand and apply Newton's Laws of Motion. We also introduced Newton's three laws and some some examples of how they are applied to real physical situations. Specifically, we saw how to calculate the net forces acting on a system to figure out whether the velocity of the system will remain unchanged when the net force is zero or will change in the case when the total force is non-zero.

In the remaining two chapter we will see multiple physical situations and analyze them by applying Newton's Laws of Motion. In the next chapter we will introduce conservation of momentum, which is a direct consequence of Newton's Laws. We will apply this model to both linear and angular motion. In the last chapter we will study in more detail how Newton's Second Law can predict motion of a object experiencing acceleration due to a net force. As you continue to read this textbook, refer back to this chapter as you begin using momentum conservation and Newton’s second law in greater detail.

CHAPTER OVERVIEW

7: Momentum

The conservation of momentum is a fundamental concept of physics along with the conservation of energy and the conservation of mass. Momentum is defined to be the mass of an object multiplied by the velocity of the object. The conservation of momentum states that the amount of momentum remains constant; momentum is neither created nor destroyed, but only changed through the action of forces as described by Newton's laws of motion. Dealing with momentum is more difficult than dealing with mass and energy because momentum is a vector quantity having both a magnitude and a direction. Momentum is conserved in all three physical directions at the same time.

Topic hierarchy

7.0: Overview

We began this course by focusing on energy and transfers of energy between different physical systems. In 7A we tried to answer was what happened to a physical system from a time before to a time after the system interacted with other systems. We tried to avoid needing to understand the details of the interaction. We discovered that changes of energy of a physical system is a very useful measure of the interaction, although it is not the only measure. We couldn't completely avoid the details of interactions, however, since force was involved in the amount of energy transferred during an interaction.

We saw how we could apply this energy formalism to more traditional thermodynamic systems (gases, heat engines) as well as to mechanical systems. We also developed a simple particulate model of matter in 7A that involved modeling the bonds between atoms and molecules as analogous to masses hanging on springs, the masses being in continuous random oscillation. This simple model allowed us to explain and predict many of the thermal properties of matter in its various states. Again, we avoided the details of oscillations and focused only on changes in energies.

In this chapter, we continue our focus on the results of interactions. We are still trying to address what happens to a physical system from a time before to a time after the system interacted with other systems. We will analyze two new physical quantities (momentum and angular momentum) that round out our understanding of the results of an interaction. We still cannot completely avoid the details of interactions, since force will determines whether momentum is transferred during an interaction.

The first model in this chapter gets us into the meaning of linear momentum and how changes in momentum are related to forces. Then in the second model of this chapter, we explore the fascinating world of rotating objects. We extend the ideas of force, impulse, and momentum to their analogous rotational or angular counterparts: torque, angular impulse, and angular momentum.

Contributors

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7.1: Linear Momentum

Overview of Model

In the previous chapter we discussed how the forces acting on an object effect the motion of that object. In this chapter some of these concepts will lead us directly to a new conservation law: the *conservation of momentum*. The goal of the remaining chapters in this book is to understand the effects of forces on motion in greater detail, and we begin to do this in this chapter through a discussion of momentum and transfers of momentum.

We introduce two new concepts: *momentum and impulse*. Although these terms are new, it is often helpful to make connections to things we already know. So we start the discussion of momentum and impulse by pointing out their similarities to energy and work. When work is done on a system it changes the energy of the physical system. When there is no work that is done on the system, *energy is conserved*. Similarly, a net *impulse* done on a defined system changes the *momentum* of a system. When the net impulse acting on the system is zero, *momentum is conserved*.

Effect of Force

We have learned from Physics 7A that objects experience changes in energy when other objects exert forces on them and do work on them. We recall that the amount of energy change caused by a force is the integral of the force over a displacement or path of the object. This integral is called the *work* done on a system. The general algebraic expression for work done along a path s from point A to B was defined in Section 2.3 is given by:

$$W \equiv \int_A^B \overrightarrow{F}_{net} \cdot \overrightarrow{ds} = \Delta E = E_f - E_i \tag{7.1.1}$$

The *dot product*, \(\overrightarrow F_{net} \cdot \overrightarrow {ds}\), in the equation above means that only the component of the force parallel to the displacement contributes to work. In other words, the parallel component of force integrated over the path of the motion is the work. This work equals the amount of energy transferred to the system due to the application of the force by an object outside the system. It is always important to stress the the change in energy on the right-side of Equation 7.1.1 is the change of energy of some defined system, while the work on the left-side of the equation refers to *external* work done on the system by objects outside the system.

Equation 7.1.1 tells us that the path over which the force will act will determine how the energy of the system will change. There is another factor that will determine how the system will change, the time over which the force will act. We know that the resulting motion of a toy car will be different depending if you briefly give it a push or keep pushing it for a prolonged time before letting it go. In the latter case, the car will move for a longer distance, after you stop pushing it, until it reaches an obstacle or friction will cause it to stop. Thus, we want to introduce a new concept that depends on force and the time over which the force acts, known as the *impulse* of the force. If the force is not constant during some range of time, then instead of simply multiplying force and time, we would have to integrate it. Impulse which is a vector (force) times a scalar (time), is a vector. We represent impulse with the symbol $\vec{J}$:

$$\vec{J} \equiv \int_{t_i}^{t_f} \vec{F} dt \tag{7.1.2}$$

If the force is constant over some time, $\Delta t = t_f - t_i$, then the impulse equation simplifies to, $\vec{J} = \vec{F}\Delta t$. Let us now use Newton's second law described in Section 6.2 to help us determine how impulse effects the motion of a system:

$$\begin{aligned}\vec{J} &= \int_{t_i}^{t_f} \vec{F}dt \\ &= \int_{t_i}^{t_f} (m\vec{a})dt \\ &= \int_{t_i}^{t_f} \left(m\frac{d\vec{v}}{dt}\right)dt \\ &= m\Delta\vec{v}\Big|_{t_i}^{t_f} \\ &= m\vec{v}_f - m\vec{v}_i\end{aligned} \tag{7.1.3}$$

We arrive at a definition of a vector quantity we call *momentum, $\vec{p}$*:

$$\vec{p} \equiv m\vec{v} \tag{7.1.4}$$

From Equation 7.1.3, this results in a relationship between impulse and momentum, known as the *impulse-momentum model*:

$$\vec{J} = \Delta\vec{p} \tag{7.1.5}$$

To summarize the impulse-momentum model in words: impulse, $\vec{J}$, is a vector quantity, defined as an external force acting on a system over some time period, which causes a change in a vector property of a system, specifically, a change in momentum, $\Delta\vec{p}$. This is analogous to the result in Equation 7.1.1: work, W, is a scalar quantity, defined as an external force acting on a system over some path, which causes a change in a scalar property of a system, specifically, a change in energy, ΔE. Although, both work and impulse give us a quantity which is the effect of forces, there are some fundamental differences. Work is a scalar, while impulse is a vector. Also, to find work you need to know the specific path over which the system moves, while only the time over which the force acts determine impulse.

The momentum introduced here, $\vec{p}$, is *linear momentum*, the word "linear" implies that the motion is described by displacement (distance changing) with time. In the following section we will introduce *angular momentum*, which describes motion in terms of a change of an angle with time. In this section, we will mostly refer to $\vec{p}$ as just "momentum", for simplicity. In general, we will refer to this type of momentum as linear momentum when we want to distinguish it from angular momentum.

Note on units: force has SI units of Newtons, N. Impulse must therefore have units of Newton seconds, $\mathrm{N \cdot s}$. Momentum, the product of mass and velocity, must have SI units of kilogram meter per second, $\mathrm{kg \cdot m/s}$. Since these two quantities are equated, these units must be equivalent, as you can show using the relationship that force equals to mass times accelation: $\mathrm{N = kg \cdot m/s^2}$.

Until we consider rotation of objects in the next mode, when we talk about angular momentum, we will consider phenomena in which extended objects act only like point particles. A useful concept that will become much more meaningful when we consider rotation, is *center of mass*. Right now we can simply consider that any extended object acts like a single particle whose mass is equal to the mass of the object, located at the special point, the center of mass.

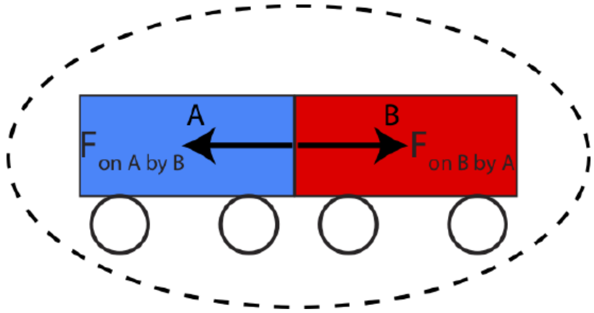
Momentum of a System with Multiple Particles

It is often useful to evaluate interaction between multiple particles in terms of momentum. Specifically, the impulse-momentum model will help us understand the motion of particles after they exert forces on each other, such as two billiard balls colliding. The momentum of a single object is simply the product of its mass and velocity. Suppose we define a physical system that contains several particles which may have different masses and move with different velocities. The total linear momentum of this physical system consisting of N particles is the vector sum of the individual linear momenta:

$$\vec{p}_{\text{system}} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots + \vec{p}_N = \sum_{i=1}^{N} \vec{p}_i \tag{7.1.6}$$

If the particles in our system interact with each other, they exert forces on each other, and there will be an impulse associated with each of these forces. Figure 7.1.1 below depicts two carts that come in contact with each other, thus exert a force on each other. Newton's 3rd law, described in Section 6.2, tells us that the force that cart A exerts on cart B is equal in magnitude and opposite in direction to the force that cart B exerts on cart A, as shown in the picture below.

Figure 7.1.1: A system of Two Objects



Since impulse is the force integrated over time during which the carts are in contact, the impulses on A and on B will also be equal and opposite:

$$\vec{J}_{\text{on A by B}} = -\vec{J}_{\text{on B by A}} \tag{7.1.7}$$

The impulse on each separate object within the system equals to the change in that object's momentum:

$$\begin{aligned} \vec{J}_{\text{on A by B}} &= \Delta\vec{p}_A \\ \vec{J}_{\text{on B by A}} &= \Delta\vec{p}_B \end{aligned} \tag{7.1.8}$$

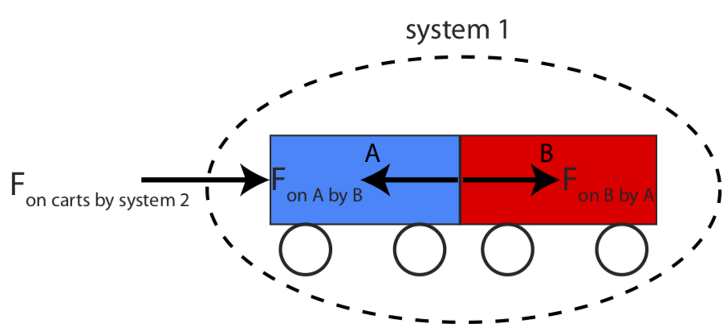
Using Equation 7.1.6 for the total momentum of a system and the two equations above, we then find that the total change of momentum of a system consisting of two objects A and B is:

$$\Delta\vec{p}_{\text{system}} = \Delta\vec{p}_A + \Delta\vec{p}_B = \vec{J}_{\text{on A by B}} + \vec{J}_{\text{on B by A}} = 0 \tag{7.1.9}$$

Generalizing the above argument to interactions *between* any of the particles *within* the system, we see that if the momentum of one particle changes a certain amount, another particle's momentum changes the same amount in the opposite direction. Thus, when we sum over all the momenta of the system, the *total momentum* of the system does not change in response to interactions among the particles *within the system*.

However, if the particles of our define system interact with particles (objects) *outside* the system, then the total momentum of the system might change. Figure 7.1.2 below shows a force that acts on the two-cart system. This force is from another system, for example, as a third cart or a hand pushing the two carts. As argued above the internal forces between the two carts do not change the total momentum of the system. However, the external force labeled *ext* (for external) does change the momentum of the system based on Equation 7.1.5. There is no other force present to cancel the force on system 1 by system 2.

Figure 7.1.2: A system of Two Objects with External Force



Also, it is important to note that when we derived Equation 7.1.3 using Newton's 2nd Law, it is the *net* force acting on the system which determines its acceleration, so the impulse which changes momentum is the impulse due to the net force acting on the system.

Conservation of Momentum

To summarize the above discussion, for a system of particles or objects it is useful to write the impulse-momentum model in a way that emphasizes the external net interaction:

$$\text{Net external impulse} = \vec{J}_{net,ext} = \int_{t_i}^{t_f} \sum \vec{F}_{ext}\, dt = \Delta \vec{p}_{\text{system}} \tag{7.1.10}$$

We will often drop the "net" and "ext" subscripts, for simplicity, but please always keep in that that this is what is implied in the the impulse-momentum model. A system acted on by external forces undergoes a change in total linear momentum equal to the net impulse (total impulse) of the external forces. We can rephrase the relationship stated above as a conservation principle for the total linear momentum of a system of particles:

Conservation of Momentum

If the net external impulse acting on a system is zero, then there is no change in the total linear momentum of that system. Otherwise, the change in momentum is equal to the net external impulse.

This statement is an expression of *conservation of linear momentum*. The total linear momentum of a system of objects remains constant as long as there is no net impulse due to forces that arise from interactions with objects outside the system. It does not matter that the objects of the system interact with each other and exert impulses on each other. These internal impulses cause changes in the individual momenta of the objects, but not the sum or total momentum of the *system* of objects.

In the light of this discussion, it is useful to categorize systems as either closed or open:

1. *Closed system* - a closed system does not interact with its environment so there is no net external impulse acting on it. The total momentum of a closed system is conserved. That is, the total momentum of the system remains constant. A closed system can consist of one or multiple objects.
2. *Open system* - an open system interacts with its environment, so that it can exchange both energy and momentum with the environment. The change in the total momentum of an open system is equal to the net impulse from the environment outside the system. An open system can consist of one or multiple objects.

7.2: Applications of Momentum Conservation

Collisions

Types of interaction where analysis of momentum is applied are known as *collisions*. But the word "collision" should not be taken literally. From a physics perspective it means an interaction between objects. When you throw a ball, we can study the change of momentum of you and the ball since an interaction occurred, but you would not call this event a "collision" in daily language. But in physics we clump all these interactions as "collisions" when we want to analyze momentum. A collision can involve one object which defines a system colliding with some external object from the environment or it may involve collisions of two or more objects.

We want to analyze momentum conservation in the interval before and after the collision, similarly to the way we studied conservation of energy over some interval by comparing the initial and final energies. Once a physical system is defined one can determine whether the momentum will be conserved after a collision by observing whether external forces act on the system during that interval. If there are no net force or a negligible net force, then momentum will be conserved. Otherwise, the momentum of the system will change. The reverse is also true, if information about external forces is not known but momentum before and after the collision is measured, one can determine whether an external net force was present by comparing initial and final momentum of the system.

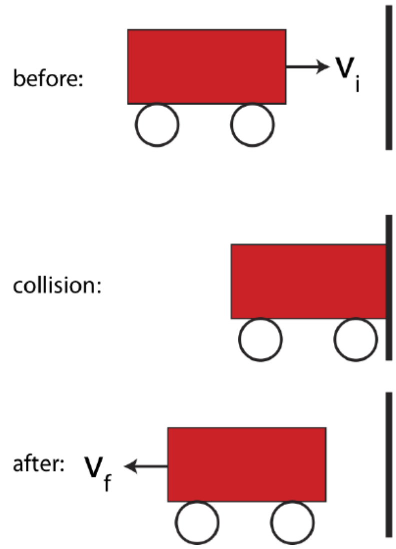
The next question we would like to ask is what happens to energy during a collision, specifically the kinetic energy of the system. We learned from 7A that mechanical energy can be converted to thermal energy. For example, when a object falls to the ground and comes to rest, its initial potential energy at some height is converted to kinetic energy as it falls, and then to thermal energy when it hits the ground. However, a very elastic ball could bounce back up with kinetic energy almost exactly the same as the kinetic energy it had before hitting the ground. In this case kinetic energy is conserved, since no mechanical energy is converted to thermal.

Thus, another piece of information about collisions which is important is the conservation of kinetic energy. If the collision is between two protons or two billiard balls, kinetic energy might be exactly or almost conserved. Collisions in which kinetic energy is conserved are called *elastic collisions*. In most types of collisions, however, the interaction between two objects results in some sort of deformation or friction between them converting some of kinetic energy to internal energy, which could be either thermal energy resulting in a rise of temperature or bond energy in a case when bonds are broken. These types of collisions are known as *inelastic collisions*. Below we will look at multiple example of different types of collisions. We will analyze system that consist of one object and multiple object. At first we will focus on one-dimensional collisions, and then we will expand our analysis to two-dimensional events.

One-Body Systems

A example of a system which consists of one object is shown below in Figure 7.2.1. In this scenario a cart moving to the right with initial velocity, v_i, collides with the wall after which it is moving to the left with some final velocity, $-v_f$. Note, in one-dimensional problems when vector quantities are involved, typically the arrow above the vector quantity is dropped, and only the signs in front of the vector quantity, plus or minus, are used to specify the direction for a given convention. In this case for velocity of a cart whose motion is in one-dimension, it can only move to the right or to left, and our convention states that velocity to the right is positive, and to the left it is negative.

Figure 7.2.1: One Cart Collision



We define the physical system here as the cart. Since the direction of its motion changed, regardless of the magnitude of the speed after the collision, the cart's momentum is not conserved. Momentum is a vector quantity which is proportional to velocity, so even if the cart bounced back with the same speed, the direction changed, so did the momentum. Another reason why we know that momentum changed is that there was an external force during this interval which was exerted on the cart by the wall during the collision. The force is external since the wall is not included in our physical system. Mathematically, applying the impulse-momentum model from the previous section to this situation, we get:

$$J_{ext} = m\Delta v = -mv_f - mv_i \tag{7.2.1}$$

To analyze conservation of kinetic energy, we need to compare the difference between initial and final energies:

$$\Delta KE = KE_f - KE_i = \frac{1}{2}m\Delta(v^2) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \tag{7.2.2}$$

Let us now examine different cases of this type of collision.

Case 1: Elastic Collision

Assume that the cart from Figure 7.2.1 bounced back to the left with the same speed as its initial speed, which is define as speed, v:

$$v \equiv |v_i| = |v_f| \tag{7.2.3}$$

Perhaps this happened because there is a spring attached to the cart which compressed during the collision, converting kinetic energy to spring energy, and then converting back to kinetic energy as the spring returns to equilibrium when the cart separates from the wall. Thus, we are assuming negligible conversion of mechanical energy to thermal energy. Thus, for this case, equation 7.2.1 becomes:

$$J_{ext} = m\Delta v = m(-v) - mv = -2mv \tag{7.2.4}$$

The result above tells us that the momentum of our system is not conserved and is equal to the external impulse which points to the left and has a magnitude of $2mv$. We often depict collision problems with a *momentum chart*. Momentum charts are a visualization tool which demonstrates the collision geometrically using arrows to represent momentum vectors. This tool will be especially useful when dealing with systems which consist of two or more objects, when algebra become more cumbersome and the momentum chart can act as a good tool to check for mathematical errors. Figure 7.2.2 below depicts a momentum chart for our case 1.

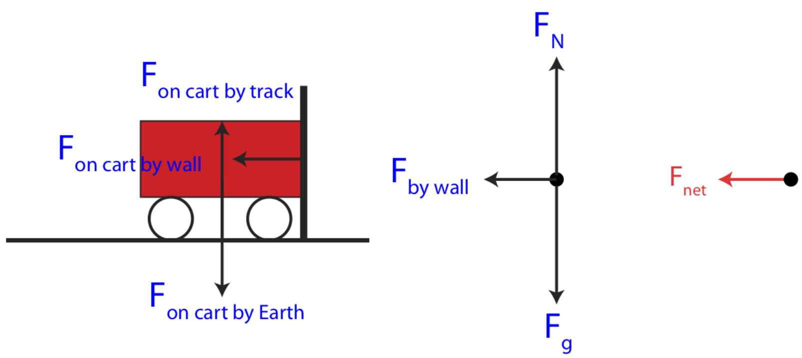
Figure 7.2.2: Momentum Chart One Cart Elastic Collision

System	$\vec{p}_i$ +	$\Delta\vec{p}$	= $\vec{p}_f$
Cart	mv	-2mv	-mv

The momentum chart shows a connection between the initial and final momenta by converting difference to two vectors, $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$, into a sum, $\vec{p}_i + \Delta\vec{p} = \vec{p}_f$. The arrows depict both the direction (right or left), and the magnitude, since the arrow for $\Delta\vec{p}$ is drawn twice as long. Below the arrows the algebraic expressions are written. Depending on the purpose of using a momentum chart algebraic expressions are not always shown.

Equation 7.2.4 shows that the external impulse which equals to the change in momentum has twice the magnitude and points in the opposite direction of initial momentum. A force diagram (or FBD) is also a useful tool for explaining the conservation of momentum for a given situation. Figure 7.2.3 below shows the forces that act on the cart during the collision. The leftmost figure below shows the cart all the forces that act on the cart at different locations. The forces are labeled with the "on by" notation. The middle figure is a FBD, where the central dot represents the cart, and the force labels are abbreviated with their standard notations. The leftmost diagram shows the net force, which is the sum of all the forces. The forces in the y-direction cancel since there is no change in momentum in that direction. Thus, the only remaining force is the force by the wall, which is the net force.

Figure 7.2.3: FBD for a One Cart Collision



Let us now check what type of collision this is by looking at conservation of energy. Kinetic energy is a scalar quantity independent of direction of motion, so only the magnitude of velocity, the speed, of the system before and after the collision will matter. Using Equation 7.2.2 and plugging in values of velocity for case 1 we get:

$$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mv^2 = 0 \tag{7.2.5}$$

Since kinetic energy is conserved, case 1 describes an *elastic collision*.

Case 2: Inelastic Collision

For the next scenario, let us assume that after the collision the cart is traveling with half of its initial speed:

$$v \equiv |v_i| = 2|v_f| \tag{7.2.6}$$

For this case equation 7.2.1 becomes:

$$J_{ext} = m\Delta v = m\left(-\frac{v}{2}\right) - mv = -\frac{3}{2}mv \tag{7.2.7}$$

The momentum chart for this scenario will have the same directions but different magnitudes of momentum, as shown below.

Figure 7.2.4: Momentum Chart One Cart Inelastic Collision

System	$\vec{p}_i$ +	$\Delta\vec{p}$	= $\vec{p}_f$
Cart	→ mv	← -3/2mv	← -mv/2

In this case the change in kinetic energy is no longer zero. Plugging velocity values into Equation 7.2.2 we get:

$$\Delta KE=\frac{1}{2}m\left(\frac{v}{2}\right)^2-\frac{1}{2}mv^2=-\frac{3}{8}mv^2 \tag{7.2.8}$$

Since kinetic energy is not conserved, case 2 describes an *inelastic collision*. The loss of kinetic energy is converted to thermal energy which represents the increase in the random motion of particles in the wall and the cart due to the collision.

Example 7.2.1

A 0.4 kg ball is moving down at 5 m/s before it hits the floor. It bounces back up with kinetic energy of 0.8 J. Answer the following questions:

a) Is this collision elastic or inelastic? If kinetic energy is not conserved, what is the change in thermal energy during the collision?

b) Is momentum conserved? If not, calculate the total impulse on the ball during the collision with the floor.

Solution

To check whether the collision is elastic or inelastic, we need to calculate the change in kinetic energy.The initial kinetic energy is:

$$KE_i=\frac{1}{2}mv_i^2=\frac{1}{2}(0.4kg)(5m/s)^2=5J$$

Since the final kinetic energy is less than 5 J, it is not conserved and the collision is inelastic. The decrease in kinetic energy is equal to the increase in thermal energy, since total energy is conserved:

$$\Delta KE+\Delta E_{th}=0$$

$$\Delta KE=KE_f-KE_i=0.8J-5J=-4.2J$$

,

so the increase in thermal energy is 4.2 J.

b) Momentum is not conserved since the ball changes its velocity, both the speed and direction are different after the collision. From kinetic energy we can find the final speed:

$$v_f=\sqrt{\frac{2KE_f}{m}}=\sqrt{\frac{2\times 0.8J}{0.4kg}}=2m/s$$

The change in momentum will equal to the impulse on the ball:

$$J=\Delta p=mv_f-mv_i=0.4kg\Big[2m/s-(-5m/s)\Big]=2.8Ns$$

In the equation above, we defined up as the positive direction and down as negative, such that the initial velocity is - 5 m/s and the final velocity is + 2 m/s. The net impulse is in the positive direction, since it is due to the force by the floor pushing on the ball in the positive upward direction.

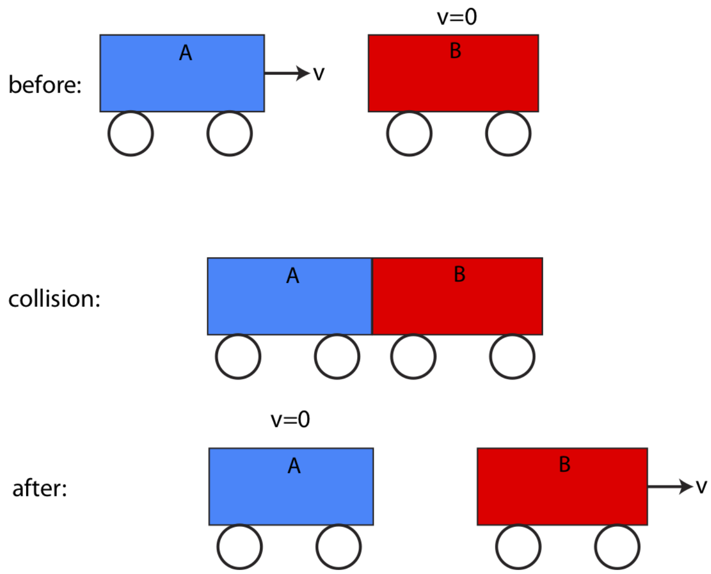
Multiple-Body Systems

Another very common category of momentum conservation problems is studying collisions between multiple objects, such as billiard balls, asteroids in space, or even automobile crashes. Often there are external forces acting on these multiple-body system, such as friction with the road or track, or gravity. However, often the effects of the collision are dominant during a short time scale immediately after the collision, and the effects of forces such as friction can be neglected over a short time-frame. Thus, we often idealize these situation and assume the momentum of the system is conserved. Of course, if we are discussing asteroids colliding in empty outer space this is already an ideal situation where no external forces are present.

Case 1: Two-body Elastic Collision

Figure 7.2.5 below shows example of a two-body cart elastic collision. We assume that no external forces act on the two carts, and both carts have the same mass, m. Cart A is initial moving to the right with speed v while cart B is stationary. After the collision, cart A stops and transfers all of its momentum to cart B which moves to the right with the same speed v.

Figure 7.2.5: Two-Body Elastic Collision



Momentum is conserved because the initial momentum of the system equals to the final momentum of the two cart system:

$$p_{i,tot} = mv_{i,A} + mv_{i,B} = mv + 0$$
$$p_{f,tot} = mv_{f,A} + mv_{f,B} = 0 + mv \tag{7.2.9}$$

Thus,

$$p_{i,tot} = p_{f,tot} \tag{7.2.10}$$

The collision is elastic since kinetic energy is also conserved:

$$KE_{i,tot} = KE_{i,A} + KE_{i,B} = \frac{1}{2}mv^2 + 0$$
$$KE_{f,tot} = KE_{f,A} + KE_{f,B} = 0 + \frac{1}{2}mv^2 \tag{7.2.11}$$

Resulting in

$$KE_{i,tot} = KE_{f,tot} \tag{7.2.12}$$

.

The momentum chart below depicts this scenario using vector notation.

Figure 7.2.6: Two-Body Elastic Collision Momentum Chart

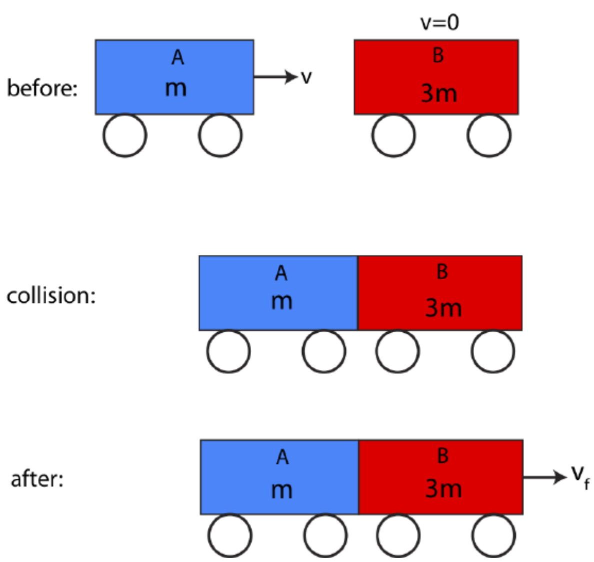
System	p_i +	Δp	= p_f
Cart A	→ mv	← -mv	0
Cart B	0	→ mv	→ mv
Total	→ mv	0	→ mv

This momentum chart now has three rows, one for each object in the system and the third for the total momentum of the system. For multiple-object momentum charts, not only the first two entries in each row must add up to the third entry in a given row, the same is true for columns. When two objects describe the system, the first two entries in each column add up to the last entry in column, labeled "total". The quantity Δp in the last row represents the total change in momentum of the system. When it is zero, as in this example, it shows that the system's total momentum is conserved implying that no net external impulse acts on the system, even though the momentum of individual objects that define the system can change.

Case 2: Two-body Inelastic Collision

In an inelastic collision between two objects kinetic energy is not conserved, so we can not equate initial and final kinetic energies. A standard inelastic collision between two objects involves them colliding and changing their momentum after the collision which results in some loss of kinetic energy. An interesting case occurs when the collision is when the objects stick together during the collision, and thus both have the same final velocity after the collision. This is known as a *perfectly inelastic* collision. An example is shown in Figure 7.2.7 below. In this scenario we also look at a case of having different masses. Cart B is three times heavier than cart A. Initially, cart A is moving to the right with speed v and collides with a stationary cart B. The two carts stick together and move to the right with some final speed v_f.

Figure 7.2.7: Two-Body Inelastic Collision



Assuming that there are no net external force acting on the system, we can figure out the final speed of the two stuck carts using momentum conservation. Let us use a momentum chart to guide us through this problem. The momentum chart can be filled out in the following order:

- initial momentum is given: $p_{i,tot} = p_{i,A} + p_{i,B} = mv + 0$. This fills out the first column.
- we set $\Delta p_{tot} = 0$ since there are no external forces acting on the two carts, $J_{net,ext} = 0$.
- this implies that $p_{f,tot} = p_{i,tot} = mv$, filling in the last entry in the bottom row.
- the final momentum of cart A is $p_{f,A} = mv_f$, third entry in first row.

- cart B has the same speed as cart A but triple the momentum since it has triple the mass of cart A: $p_{f,B} = 3mv_f$, third entry in second row.
- the individual values of Δp_A and Δp_B are calculated using $\Delta p = p_f - p_i$

Figure 7.2.8: Two-Body Inelastic Collision Momentum Chart

System	p_i +	Δp	= p_f
Cart A	→ mv	← mv_f-mv	→ mv_f
Cart B	0	→ $3mv_f$	→ $3mv_f$
Total	→ mv	0	→ mv

Once we have the momentum chart filled out we can solve for v_f by other using the third column:

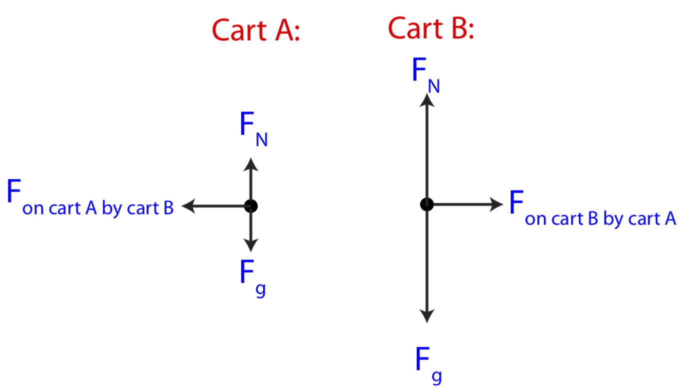
$$mv_f + 3mv_f = mv \tag{7.2.13}$$

Resulting in:

$$v_f = \frac{v}{4} \tag{7.2.14}$$

Or you can use the second column which states the the sum of the individual momentum changes must be equal to zero. The middle column is also consistent with Newton's 3rd law of motion since each cart experiences an equal and opposite force (and thus impulse) from the other cart. Below are force diagrams for each cart. The FBDs illustrate the 3rd law pair during this collision, since each cart is experiences a force by the other cart with equal magnitude by in the opposite direction. The force due to gravity is triple on cart B since it has triple the mass.

Figure 7.2.9: Two-Body Inelastic Collision Free-Body Diagrams



To show that the collision is inelastic, we need to compare the final and initial kinetic energies:

$$KE_{i,tot} = KE_{i,A} + KE_{i,B} = \frac{1}{2}mv^2 + 0$$

$$KE_{f,tot} = KE_{f,A} + KE_{f,B} = \frac{1}{2}(m+3m)\left(\frac{v}{4}\right)^2 = \frac{1}{8}mv^2$$

$$\Delta KE = KE_{f,tot} + KE_{i,tot} = \frac{1}{8}mv^2 - \frac{1}{2}mv^2 = -\frac{3}{8}mv^2 \tag{7.2.15}$$

If you think deeper about the examples we have see so far, one-body systems where momentum is not conserved and two-body where momentum is conversed, you might ask whether momentum conservation depends on a particular physical situation. This question goes back to the idea of defining a "system", which is an arbitrarily chosen particle or object or a collection of particles or objects. As in 7A an open system defined an object or a collection of objects which interacted with the environment allowing heat and/or work to enter and leave the system, thus, changing the energy of the system. But we could always redefine the system to include everything, i.e. the environment, in this case the system becomes closed, and energy of the system is always conserved.

Similarly, when calculating momentum conservation, we saw that an open system where momentum changed was one where there was a net force exerted by the environment on the system. As for energy analysis, we can always include the environment in the system, in which case it would become closed, with no external forces, and the only forces present being internal third-law pairs. In this case momentum would be always conserved.

This idea at first thought has rather mind-blowing consequences. Imagine that a rock falls on the ground and we analyzed the rock as an open system whose momentum changed when it fell and stopped on the ground, due to the force by the Earth on the rock. However, if we included the Earth in the system, there would be no longer external forces acting on the rock-Earth system, implying that the total momentum of this system has to be conserved. But this means that the collision of the rock "sticking" to the ground would make the Earth move, as in the scenario described in Figure 7.2.7, where the rock would represent the initially moving cart and the Earth is the stationary cart. This, of course, does not seem correct, we do not cause the Earth to move back and forth as we bounce balls and throw objects on the ground. Let us take a closer look at what is happening here. Applying Equation 7.2.13 (which helps us calculate the final speed due to a perfectly inelastic collision between two objects) to this scenario we need to replace the mass of the initially moving cart with the mass of the rock, m_R, and the mass of the stationary cart becomes the mass of the Earth, m_E. Solving for final velocity, we find that:

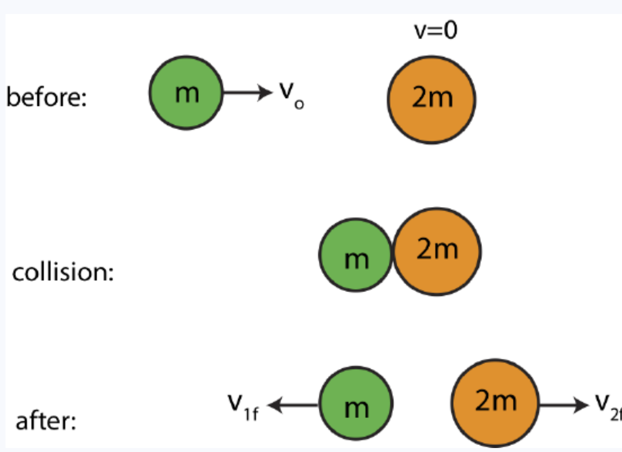
$$v_f = \frac{m_R}{m_R + m_E} v_i \tag{7.2.16}$$

Since the mass of the Earth is so much bigger than the mass of the rock, the ratio of masses in the above equation is basically zero, resulting in zero final velocity. So we saved the Earth from moving off into outer space by dropping a ball on the ground. The main message to get from this fun digression is that it is momentum which is conserved, and not velocity. So even though by including the Earth in the system we are seeing that the momentum of the Earth must changed due to this collision, its change of velocity is negligible since its mass so much larger than the mass of the object with which it interacts. This is directly related to the truck and bug example from the previous section, although the collision of the bug with the truck changes the momentum of the truck the same way the ball changes the momentum of the Earth by landing on the ground, the truck keeps on going with the same velocity as before the collision, since it is much heavier than the bug.

A good rule of thumb when choosing a system of objects which are interacting, is to include objects of similar size only. If you include everything and some of those objects are much more massive than others, you will not learn anything interesting about the massive objects since their motion will not change due to the interaction with the much smaller objects. Thus, it is more informative to leave them out, and define an open system where the massive object become a source of an external force on our selected system.

Example 7.2.2

Below is an illustration of a two balls colliding. Ball 1 on the left is initially moving to the right with speed, $v_o = 4m/s$ while ball 2 is stationary. After the collision both balls are moving in opposite directions with some speeds as shown below. Ball 2 is twice the mass of ball 1. Assume no external forces are acting on the two-balls, and the collision is elastic.



a) Calculate v_{1f} and v_{2f}.

b) Find the impulse on ball 1 by ball 2 and the impulse on ball 2 by ball 1, if the mass of ball 1 is 1.5 kg.

c) if in another scenario the two ball stick together instead, find the final velocity. Assume the initial conditions are the same. Is the collision elastic or inelastic? If inelastic, find the change in kinetic energy.

Solution

a) Since there are no external forces acting on the two balls, the momentum of the two-ball system is conserved:

$$mv_o = 2mv_{2f} - mv_{1f}$$

The equation above has 2 unknowns, both final speeds. But we are also given that the collision is elastic, so kinetic energy is conserved:

$$KE_i = KE_f$$

$$KE_{1i} + KE_{2i} = KE_{1f} + KE_{2f}$$

$$\frac{1}{2}mv_o^2 + 0 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}(2m)v_{2f}^2$$

Now we have 2 equations with 2 unknowns, v_{1f} and v_{2f}. The initial velocity, v_o is given but for now we will leave it as a known variable. The conservation of momentum and kinetic energy conservation equations can be simplified by canceling out common factors:

$$v_o = 2v_{2f} - v_{1f}$$

$$v_o^2 = v_{1f}^2 + 2v_{2f}^2$$

The remaining steps in the solution are algebraic, let's use the first equation and plug into the second equation for v_o:

$$(2v_{2f} - v_{1f})^2 = v_{1f}^2 + 2v_{2f}^2$$

Squaring the left-hand side:

$$4v_{2f}^2 - 4v_{2f}v_{1f} + v_{1f}^2 = v_{1f}^2 + 2v_{2f}^2$$

Canceling common terms on both sides and moving all the terms to the left-hand side we get:

$$2v_{2f}^2 - 4v_{2f}v_{1f} = 0$$

Factoring common terms:

$$2v_{2f}(v_{2f} - 2v_{1f}) = 0$$

The above equation has two solutions, one states that $v_{2f} = 0$. This cannot be true in this case since you are told that both ball are moving after the collision. The other solution states that:

$$v_{2f} = 2v_{1f}$$

So ball 2 is moving with twice the speed of ball one. To get a numerical result we need to relate the above result to initial speed v_o:

$$v_o = 2v_{2f} - v_{1f} = 4v_{1f} - v_{1f} = 3v_{1f}$$

Solving for v_{1f}:

$$v_{1f} = \frac{v_o}{3} = \frac{4}{3}m/s$$

Which means that:

$$v_{2f} = 2v_{1f} = \frac{8}{3}m/s$$

b) The impulse on ball 1 by ball 2 is the change of momentum of ball 1:

$$J_{\text{on 1 by 2}} = \Delta p_1 = mv_{1f} - mv_0 = 1.5kg\left(-\frac{4}{3} - 4\right)m/s = -8Ns$$

The impulse on ball 2 by ball 1 is the change of momentum of ball 2:

$$J_{\text{on 2 by 1}} = \Delta p_2 = 2mv_{2f} - 0 = 2 \times 1.5kg \times \frac{8}{3}m/s = 8Ns$$

You can see that the two impulses are equal in magnitude and opposite in direction, which is consistent with Newton's 3rd Law. It also shows that internal impulses cancel, so although the momentum of each individual object in a system consisting of multiple objects changes, the total momentum is conserved, since there is no net impulse acting on the system. All internal forces do not contribute to the total change of momentum of the entire system, in this case the two balls.

c) If the two ball stick together instead, then their final velocity has to be the same since they are moving together while attached. Let us call this velocity v_f. Momentum is still conserved, since no external forces act on the two balls. Conservation of momentum for this case becomes:

$$mv_o = mv_f + 2mv_f = 3v_f$$

This results in:

$$v_{1f} = \frac{v_o}{3} = \frac{4}{3}m/s$$

So both balls are moving to the right with speed of 4/3 m/s.

In this case since the balls stick together the collision is perfectly inelastic. The change in KE is:

$$\Delta KE = KE_{f,tot} - KE_{i,tot}$$

$$\Delta KE = \frac{1}{2}(m+2m)v_f^2 - \frac{1}{2}mv_o^2$$

$$\Delta KE = \frac{1}{2}(3m)\left(\frac{v_o}{3}\right)^2 - \frac{1}{2}mv_o^2 = -\frac{1}{3}mv_o^2 = -\frac{1}{3}(1.5kg)(4m/s)^2 = -8J$$

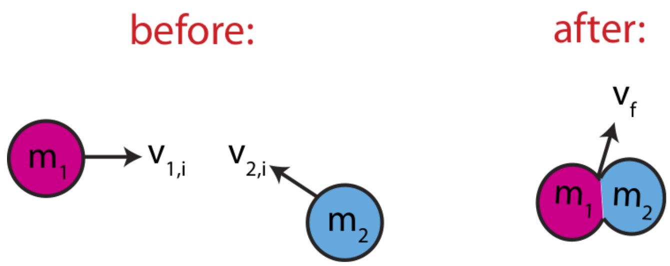
Two-Dimensional Collisions

We use the same impulse-momentum model to solve two-dimensional momentum conservation problems, except all the vectors are no longer described with just a plus or minus sign specifying the direction of motion, but now have two components for each spatial direction. Thus, although the physics is the same, the algebra become more cumbersome. The impulse-momentum model, $\vec{J}_{ext} = \Delta\vec{p}$, written by components in 2D is:

$$J_{ext,x} = \Delta p_x$$
$$J_{ext,y} = \Delta p_y \tag{7.2.17}$$

So one physical collision can be treated independently in each spatial direction, as if you are solving two separate 1D collisions. The only difference is the each component of given velocity describe the motion of the same object. Figure 7.2.10 is an example of a perfectly inelastic 2D collision, where two balls with different masses moving in two-dimensions collide and stick together. We assume no external forces are acting on the two-ball systems, thus the total momentum is conserved.

Figure 7.2.10: Two-Body Collision in 2D

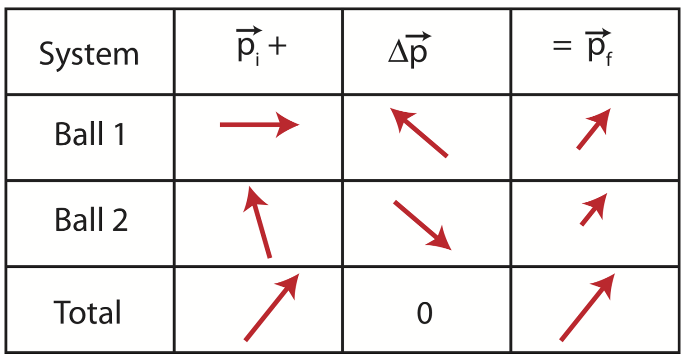


The momentum chart in Figure 7.2.11 illustrates an example of a momentum chart for the scenario. With the information given we cannot exactly draw a momentum chart, since we do not know the relative masses and velocities of the two objects.

But this momentum chart demonstrates the basic ideas of a 2D perfectly inelastic collision:

- the sum of initial momenta of each object equals to the total initial momentum.
- the total initial momentum is equal to the total final momentum, since the change in total momentum is zero when there is no net external impulse acting on the system in either spatial direction
- the individual final momenta for each object point in the same direction since they move together, but do not have the same magnitude when their masses are different.
- the individual changes of momenta for each object are equal in magnitude and opposite in direction.

Figure 7.2.11: Two-Body Collision in 2D



To solve for the final velocity of the two balls algebraically, we equate the initial and final total momentum, as we did for one-dimensional examples:

$$\vec{p}_{i,tot} = \vec{p}_{f,tot} \tag{7.2.18}$$

The total momentum is a vector sum of the individual momenta of each ball:

$$\vec{p}_{i,1} + \vec{p}_{i,2} = \vec{p}_{f,1} + \vec{p}_{f,2} \tag{7.2.19}$$

Alert

When doing momentum conservation problem it is really important to label your masses and velocities. Otherwise, it become very easy to loose track of variables and make mistakes. The masses should have subscripts specifying which objects you are referring to, such as m_1 and m_2. The velocities often have three labels, indicating the object, time period (before the collision or after the collision), and the spatial direction. For example, the final velocity of object 2 in the y-direction can be labeled as $v_{2,f,y}$.

To sum the vectors we need to express each momentum by components. In the x-direction Figure 7.2.10 shows m_1 moving to the right and m_2 moving to the left initially. After the collision, both masses have a component of momentum in the positive x-direction. Thus, Equation 7.2.19 in the x-direction is:

$$m_1 v_{i,1} + m_2 v_{i,2,x} = (m_1 + m_2) v_{f,x} \tag{7.2.20}$$

Note, the subscripts in Equation 7.2.20: the initial velocity of m_1 does not have an "x" subscript since all of its velocity is in the x-direction so there is no reason to keep the x-component notation. When we were working with 1D examples at the start of this section, we didn't carry the notation of direction along which the system was moving since it was understood that motion was happening in that one direction. The only distinction which was necessary is to specify right/left or up/down with a plus or minus sign. Since $\vec{v}_{i,2}$ has both x- and y-components we need to specify that the momentum conservation equation in the x-direction includes the x-component of this velocity, which is negative since m_2 is moving in the northwest direction (negative x- and positive y-). For the final momentum, the velocity is the same for both masses, so there is no meaning to the subscripts "1" and "2", thus it is left simply as $\vec{v}_f$ with the x-component included in this equation for the x-components.

In the y-direction Equation 7.2.19 becomes:

$$m_2 v_{i,2,y} = (m_1 + m_2) v_{f,y} \tag{7.2.21}$$

If your goal for to calculate the final velocity, assuming that you were given values of initial velocity and masses, you would solve for each component of the final velocity using Equations 7.2.20 and 7.2.21:

$$v_{f,x} = \frac{m_1 v_{i,1} + m_2 v_{i,2,x}}{m_1 + m_2}$$
$$v_{f,y} = \frac{m_2 v_{i,2,y}}{m_1 + m_2} \tag{7.2.22}$$

After calculating the components you can use Pythagorean theorem and trigonometry to solve for the speed and the angle of the final velocity.

To figure out the change in kinetic energy the concepts are the same as they were in 1D, since it is only the speeds of the objects that matter, not the direction in which they travel. The change is kinetic energy for this example is:

$$\Delta KE = KE_{f,tot} - KE_{i,tot} = \left[\frac{1}{2}(m_1 + m_2) v_f^2\right] - \left[\frac{1}{2} m_1 v_{i,1}^2 + \frac{1}{2} m_2 v_{i,2}^2\right] \tag{7.2.23}$$

In this particular example, as in appears in Figure 7.2.10 the combined mass system is moving in the northeast direction. However, they can be other scenarios where the masses move in a different direction after the collision. The final velocity depends on the initial velocities and masses. Let us think of an extreme example where both objects move with the same speed but ball 1 is much heavier than ball 2. In this case the momentum of ball 1 will be much greater and the initial momentum of ball 2 will have very little effect, so the final velocity of the two objects will be in the east direction. Think back to our truck and bug example. If m_1 represented a moving truck and m_2 is a bug that landed on the truck and got stuck to its windshield, the truck would continue moving with its initial momentum.

As another example, let us examine a situation where both masses are moving north (in the positive y-direction) after the collision. If both have the same initial speed, and ball 2 is moving at a angle of 60° from the negative x-axis, let us figure out their ratio of their masses in order for this to happen. Applying Equation 7.2.20 to this scenario we get:

$$m_1 v_i - m_2 v_i \cos 60^\circ = 0 \tag{7.2.24}$$

The right-hand side of the equation is zero since the final velocity does not have an x-component in this example, and we drop the "1" and "2" subscripts from the initial velocity since the speed is the same for both balls. Solving for the ratio of the masses we get:

$$\frac{m_1}{m_2} = \cos 60^\circ = \frac{1}{2} \tag{7.2.25}$$

So ball 2 has to have weight double of ball 1, for them to stick and travel north. This result is independent of the initial speed. To find the final speed we need to set up the equation for the y-direction. Equation 7.2.21 for this situation becomes:

$$m_2 v_i \sin 60^\circ = (m_1 + m_2) v_f \tag{7.2.26}$$

So the final speed depends on the initial speed:

$$v_f = \frac{m_2}{m_1 + m_2} v_i \sin 60^\circ = \frac{2}{3} v_i \sin 60^\circ \tag{7.2.27}$$

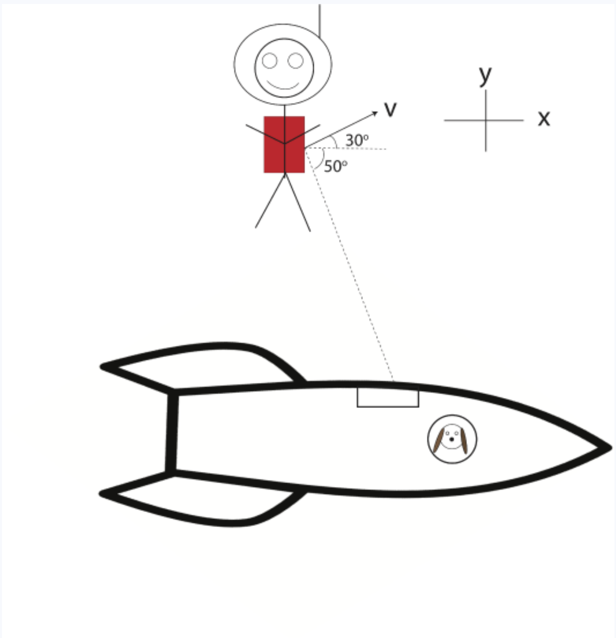
This result shows that the final speed will depend on the initial speed of the two masses.

Example 7.2.3

An astronaut is drifting in space away from her spaceship with speed of 3 m/s and an angle of 30° from the positive x-axis as shown below. She needs to get back to her spaceship in order to feed her dog, Cupcake. Using her physics knowledge, she takes off her heavy 20 kg backpack and throws it. After the throw the astronaut is traveling toward the spaceship at an angle of 50° below the positive x-axis, and the backpack is moving straight in the y-direction. The astronaut weighs 70 kg. Assume no external forces are present.

a) Find the speed of the backpack after the astronaut threw it and the speed of the astronaut as she approaches the spaceship, as seen by Cupcake.

b) Find the impulse with which the astronaut threw the backpack. Express your answer in terms of magnitude and direction.



Solution

a) We want to define the system to include the astronaut and the backpack, since they initially travel together but after the interaction (the throw), they move with different velocities. The total momentum of the system is conserved since no external forces act on the astronaut-backpack system. This means that:

$$\Delta \vec{p}_{tot} = 0$$

$$\vec{p}_{i,tot} = \vec{p}_{f,tot}$$

The initial momentum is just the combined mass of the astronaut and backpack, since they are attached, multiplied by the initial velocity:

$$\vec{p}_{i,tot} = (m_A + m_B)\vec{v}_i$$

where m_A is the mass of the astronaut, and m_B is the mass of the backpack. By component the initial momentum is written as:

$$p_{i,x} = (m_A + m_B)v_i \cos 30^\circ$$

$$p_{i,y} = (m_A + m_B)v_i \sin 30^\circ$$

Both components are positive since the initial velocity is in the northeast direction.The total final momentum includes the individual momentum of the astronaut and backpack since they are no longer attached:

$$\vec{p}_{f,tot} = \vec{p}_{f,A} + \vec{p}_{f,B} = m_A\vec{v}_{f,A} + m_B\vec{v}_{f,B}$$

The backpack's velocity only has a positive y-component after the throw. Simplifying the above equation and using the information about the direction of astronaut's motion after the throw we get:

$$p_{f,tot,x} = m_A v_{f,A} \cos 50^\circ + 0$$

$$p_{f,tot,y} = -m_A v_{f,A} \sin 50^\circ + m_B v_{f,B}$$

Equating initial and final momentum in each direction we get:

$$(m_A + m_B) v_i \cos 30^\circ = m_A v_{f,A} \cos 50^\circ$$

$$(m_A + m_B) v_i \sin 30^\circ = -m_A v_{f,A} \sin 50^\circ + m_B v_{f,B}$$

First we use the equation for the x-direction to calculate the final speed of the astronaut, since the equation in the y-direction has two unknowns:

$$v_{f,A} = \frac{(m_A + m_B) v_i \cos 30^\circ}{m_A \cos 50^\circ} = \frac{(90kg)(3m/s)\cos 30^\circ}{(70kg)\cos 50^\circ} = 5.20m/s$$

Next we use the equation for the y-direction to calculate the final speed of the backpack:

$$v_{f,B} = \frac{(m_A + m_B) v_i \sin 30^\circ + m_A v_{f,A} \sin 50^\circ}{m_B} = \frac{(90kg)(3m/s) sin 30^\circ + (70kg)(5.20m/s)\sin 50^\circ}{20kg} = 20.7m/s$$

b) The impulse on the backpack by the astronaut is equal to the change in the backpack's momentum:

$$\vec{J}_{\text{on B by A}} = \Delta \vec{p}_B = \vec{p}_{f,B} - \vec{p}_{i,B}$$

Using the results from a) and solving by component:

$$J_{\text{on B by A},x} = p_{f,x,B} - p_{i,x,B} = 0 - m_B v_{i,x,B} = -(20kg)(3m/s)\cos 30^\circ = -52kgm/s$$

$$J_{\text{on B by A},y} = p_{f,y,B} - p_{i,y,B} = m_B v_{f,y,B} - m_B v_{i,y,B} = (20kg)(20.7m/s) - (20kg)(3m/s)\sin 30^\circ = 384kgm/s$$

Combining results the magnitude of the impulse is:

$$|\vec{J}_{\text{on B by A}}| = \sqrt{52^2 + 384^2} = 387.5kgm/s \tag{7.2.28}$$

And the angle is the 2nd quadrant (northwest):

$$\theta = \arctan\left(\frac{384}{52}\right) = 82.3^\circ \tag{7.2.29}$$

Contributors

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7.3: Angular Motion

Overview

In the next few sections we will see that the ideas we have developed for linear momentum and impulse apply to rotational motion as well. To describe *rotational or angular motion* we will need to develop rotational analogs of the various variables and concepts we have been using. Force, acceleration, momentum, velocity, and impulse all have rotational analogs. Newton's Laws, the concept that impulse equals change in linear momentum, the principle of conservation of momentum all have analogs in rotational motion.

In the impulse-momentum model, we focused both on the properties of forces and the momentum transfers governing the connection of force to motion. Now we extend the formalism to enable us to analyze and make sense of the motion of extended objects that can *rotate* as well as *translate*. When objects *translate*, we say that they have linear momentum. When objects *rotate*, we will describe their motion with *angular momentum* (which could also be called "rotational momentum"). We will introduce a couple of additional concepts: *torque* (rotational analog of force) and *rotational inertia* (rotational analog of mass).

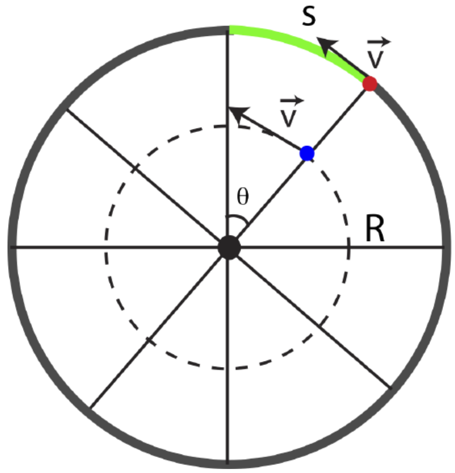
Although torque arises directly from the presence of a force, it is a rather different quantity. We will see that not all forces result in torques, and that torque also depends on the exact location of where the force is applied. Even the units of torque are different from the units of force. Angular momentum is analogous to momentum (translational or linear momentum) but the two types of momentum describe different physical situations. The impulse-momentum model stated that the momentum of an object is conserved if there is no net external force acting on it. We will find that angular momentum of an object is conserved if there is no net external *torque* acting on it (even if there is a net force). Similarly, a transfer of angular momentum is called *angular impulse*.

Translational or linear and angular motion describe different physical situations, although there is a connection between them. The difference is obvious when you see a physical situation but, when discussing abstract ideas without a physical picture in mind, it is easy to confuse the two quantities. For instance, a ball may be spinning (i.e. have angular momentum) and flying through the air in a straight line (i.e. have linear momentum). Or it may be spinning (have any angular momentum) and not be flying through the air. Or it may be flying through the air but not spinning at all. So, you see that the amount of angular momentum the ball has is completely independent of its linear momentum. The moral of this little story is the same as with all physics problems: try to keep a concrete physical picture in your head as you learn new abstract ideas.

Rotational Velocity

We begin by developing some useful relationships to describe the motion of a point object. Figure 7.3.1 shows a wheel, with radius R which rotates around its center. The question that we want to ask is how do we describe the motion of the wheel. We can attempt to use our developed ideas of linear velocity. Let us focus on two arbitrary points on the wheel (marked with red and blue points in the figure). The direction of the velocity of each point is tangent to the circle since the wheel rotates about its center. The magnitude of the two velocities will be different. The red dot on the outer rim moves the entire circumference of the wheel, $2\pi r$, over the same period of time as the blue dot moves a much smaller distance, since it covers a smaller circumference being closer to the center of the wheel.

Figure 7.3.1: Rotational Analog of Velocity



Thus, if we use these two different points to describe the motion of the wheel, we would obtain different answer. And even more extreme, if we choose to focus on the center of the wheel, it does not move at all, so we conclude the velocity is zero. This is exactly what leads us to the distinction between angular and linear motion. When we discussed linear motion, we always stated that we can "collapse" an extended object to a point and only think about the motion of the point. Likewise, all the forces acting on the object were treated as if they were all applied at one location, the point representing the extended object. If we redefine this wheel as just a point object at the center, we conclude that it has zero linear motion. This makes sense since the wheel is not translating in space, its center remains in the same location as it rotates. In fact, we can attempt to add up all the velocities of all the points on the wheel, and we would find that they all cancel. For example, there is an analog of a red dot on the opposition side of the wheel that moves in the opposite direction with the same speed, so the sum of those two velocities is zero. Equivalently, the same analog exists for the blue dot and all other points on the wheel.

So we have concluded that the wheel's *linear velocity*, $\vec{v}$, is zero. But the wheel is clearly not stationary, it is rotating, so there must be more information that we can provide to describe the motion. Let us think about what is the same for the red and the blue dots. Even though their velocities are different, they cover the same angle, θ, during the same amount of time. Thus, instead of describing a *linear displacement*, Δx, per unit time, we will use an an *angular displacement*, $\Delta\theta$, per using time. In Figure 7.3.1 the angle is measured relative to some arbitrarily chosen axis which is frequently measured from the positive x axis, but it could be measured from any reference line. Recall, we define linear speed as the rate of change of a small displacement over time. Analogously, we define *angular speed, ω*, as the rate of change of a small change in angle over time:

$$v=\frac{dx}{dt} \iff \omega=\frac{d\theta}{dt} \tag{7.3.1}$$

When the wheel in Figure 7.3.1 rotates by some angle θ, the red point move by a distance of arc length, s, as illustrated by a green color in the figure. Since the change in angle is describing the same motion as the change in distance, we can connect the two using a known geometric relationship between angle, θ, radius, R, and arc length, s:

$$s=R\theta \tag{7.3.2}$$

Combining Equations 7.3.1 and 7.3.2, and substituting displacement dx with change in arc length, ds, we obtain a relationship between angular and linear speed:

$$v=\frac{ds}{dt}=R\frac{d\theta}{dt}=R\omega \tag{7.3.3}$$

The result above is consistent with our original discussion, the red dot on the outer rim of the wheel has a larger speed since further from the center compared to the green dot. Both points has the same angular speed, so the red dot with have a greater linear speed since the distance R is greater for the red dot.

The units of θ and ω are, respectively, an angle unit and an angle unit divided by time. We can use any units we want and that are useful for a particular application of rotational motion. Typical units for angular speed are degrees/second, revolutions/second, revolutions/hour, etc. The SI units, are, however, *radians* for angle and *radians per second* for angular speed. We must use radians and radians per second when we use the relationship connecting v to ω. Note that a “radian” is a rather unique kind of unit. For instance, radians multiplied by meters is just meters, not rad·meters. A wheel which makes 5 revolutions per second, in radians will move at an angular speed of $10\pi\ \mathrm{rad/sec}$, since each full revolution is 2π in radians.

Note that so far we have been discussing objects constrained to move in a circle. We defined that motion in terms of *polar coordinates*, using the radial distance from the origin, r, and the angle relative to the positive x-axis, θ. We can also describe the kinematics of any extended object (e.g. a baseball bat) that is rotating about a fixed origin (where we grip it) by θ and ω , as long as we define the coordinates about the fixed axis of rotation.

Extending the connection of linear and angular speed to acceleration, we can define an *angular acceleration, α*, which describes the rate of change of angular velocity:

$$\alpha = \frac{d\omega}{dt} \tag{7.3.4}$$

Similarly, we can connect linear and angular accelerations:

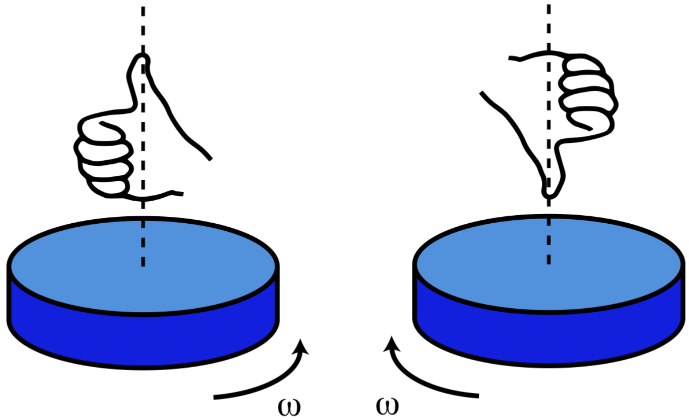
$$a = \frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha \tag{7.3.5}$$

When forces act on extended objects, in addition to causing changes in *translational* motion, but can also cause changes in *rotational* motion. That is, these forces can cause translational acceleration and/or angular acceleration. It turns out that it is not just the magnitude and direction of the force that is important in causing angular accelerations, but also *where* the force is applied on an extended object. *Torque* is the angular analog of force that incorporates both the force vector as well as where it is applied to an object. We will define torque in greater detail in the coming sections.

So far we have conveniently ignored the direction aspect of angular velocity and purely focused on speed. Just as the translational velocity vector, $\vec{v}$, describes both direction and magnitude, the angular velocity, $\vec{\omega}$, is also a vector. What direction does angular velocity have? Looking back at the wheel in Figure 7.3.1, we can summarize its motion as counter-clockwise. But this description or rotation (clockwise or counter-clockwise) is not a direction in a particular spatial coordinate. In addition, there is another problem. If another observer is standing on the other side of the wheel, then they would describe the motion as clockwise, and there there would be a disagreement. And how would you distinguish its rotational direction is the wheel was horizontally positioned instead? The only unique direction in space associated with a rotation is along the *axis of rotation*, which is an axis perpendicular to the plane of rotation. For the wheel in Figure 7.3.1 the axis of rotation would be a line which is perpendicular to the screen, going in and out of the screen.

So, if the axis of rotation gives the direction, the only thing left to specify which way along the axis corresponds to a particular direction of rotation. For the wheel in Figure 7.3.1, we want to know whether the counter-clockwise rotation corresponds to a direction which is in or out of the page. By convention, the direction is specified by the *right-hand-rule (RHR)*. If you curl the fingers of your right hand in the direction of positive θ or in the direction rotation is occurring, your thumb points in the direction (along the axis of rotation) of θ or ω. For the wheel in Figure 7.3.1 the RHR (fingers curl to the left so thumb stick out of the page) would result in angular velocity pointing out of the page. Figure 7.3.2 below shows several examples of the right-hand-rule. On the left, a cylinder is rotation counter-clockwise as viewed from above. A right hard is shown with the fingers curled in the direction of rotation, resulting in the thumb pointing up. So, we conclude that the direction of rotation is in the positive vertical direction. On the contrary, the cylinder on the right is rotating clockwise as seen from above. In this case the RHR gives us the direction of angular velocity to be in the negative vertical direction.

Figure 7.3.2: Right Hand Rule for Rotational Motion



Alert

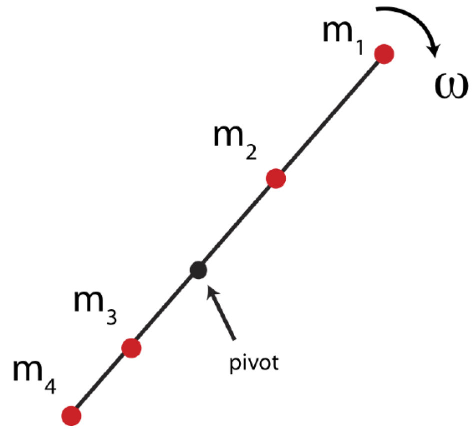
The direction of angular velocity does not specify the direction in which the object is physically rotating. In fact, it is always the direction in which no motion is occurring since it is the only unique direction for the entire span of rotation.

7.4: Rotational Inertia

Rotational Kinetic Energy

Recall that kinetic energy is described by the mass of the object and its speed. For rotating bodies there must be an analog to kinetic energy as well. We already have a relationship between linear and angular speed, which we can use to redefine kinetic energy for rotational motion. Let us simplify our wheel example by observing a few point masses rotating on a massless rod as shown in Figure 7.4.1 below. The *pivot* shown in the figure defines a fixed point about which the object rotates. In some cases it is an obvious choice, such as in this example since we are assuming rod is help in place at a pilot. In other examples a rotating object might not be fixed at a particular point, but will still rotate about the center-of-mass of the object, which we will discuss later.

Figure 7.4.1: Rotational Kinetic Energy



In this scenario the total kinetic energy of the rotating object is the sum of the kinetic energies of the four masses shown. For mass 1, m_1, we can write down the kinetic energy in terms of linear speed, and then use the relationship between angular and linear speeds, $v_1 = r_1\omega$, where r_1 is the distance from the pivot to the location of m_1:

$$KE = \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1(r_1\omega)^2 = \frac{1}{2}m_1r_1^2\omega^2 \tag{7.4.1}$$

The expression is similar for all the other masses, using the appropriate distances from the pivot and recognizing that all masses will rotate with the same angular velocity, ω. The total kinetic energy for this rotating object would be the sum of all the individual kinetic energies:

$$KE = \sum_{i=1}^{4}\frac{1}{2}m_iv_i^2 = \frac{1}{2}(m_1r_1^2+m_2r_2^2+m_3r_3^2+m_4r_4^2)\omega^2 \tag{7.4.2}$$

In general, for a set of N masses, the kinetic energy of a rotating object about a fixed pivot becomes:

$$KE = \sum_{i=1}^{N}\frac{1}{2}m_iv_i^2 = \frac{1}{2}\left(\sum_{i=1}^{N}m_ir_i^2\right)\omega^2 \tag{7.4.3}$$

Rotational Inertia

Let us now use the result in Equation 7.4.3 to write down the rotational analog of kinetic energy:

$$KE_{rot} \equiv \frac{1}{2}I\omega^2 \tag{7.4.4}$$

where *I*, is the *rotational inertia* of a object consisting of point masses:

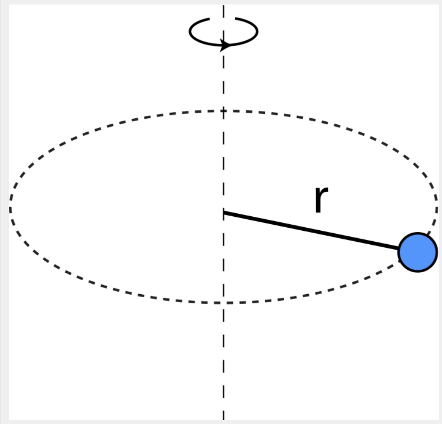
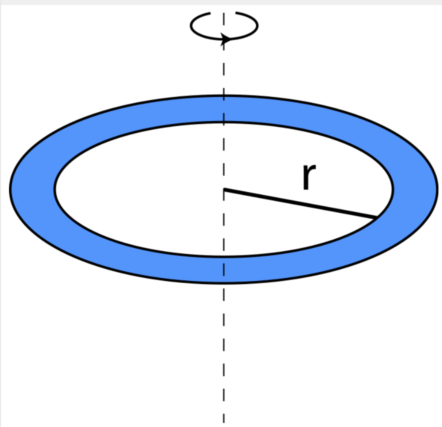
$$I \equiv \sum_{i=1}^{N}m_ir_i^2 \tag{7.4.5}$$

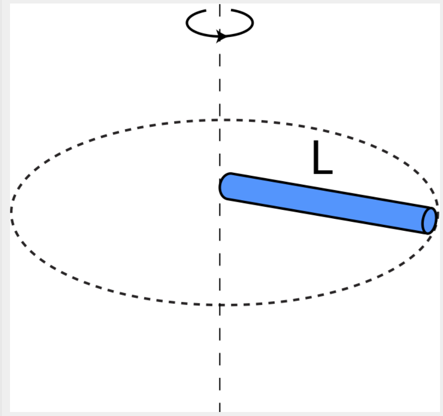
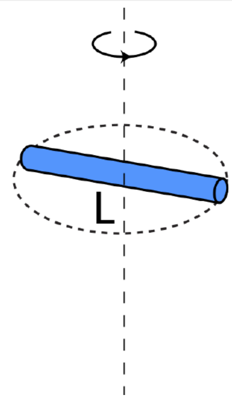
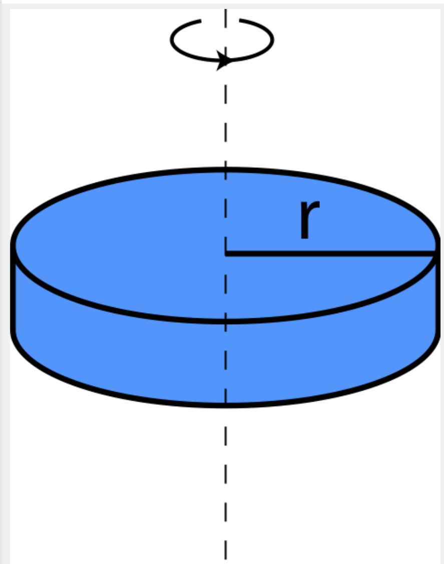
The SI units of rotational inertia are $kg \cdot m^2$. Comparing the expression for linear and angular kinetic energies, we see that rotationalinertia is the rotational analog of mass. The rotational inertia of an object does not depend solely on the amount of mass in the object, but on how this mass is distributed relative to the axis of rotation. If the pivot in Figure 7.4.1 changed to a different location, the rotational inertia of the object would change as well, even though its total mass would stay the same.

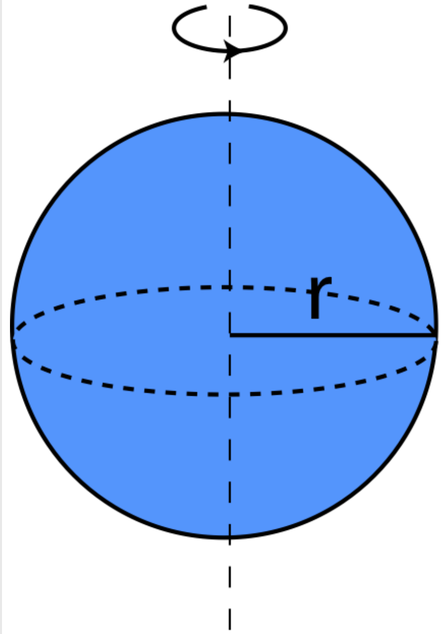
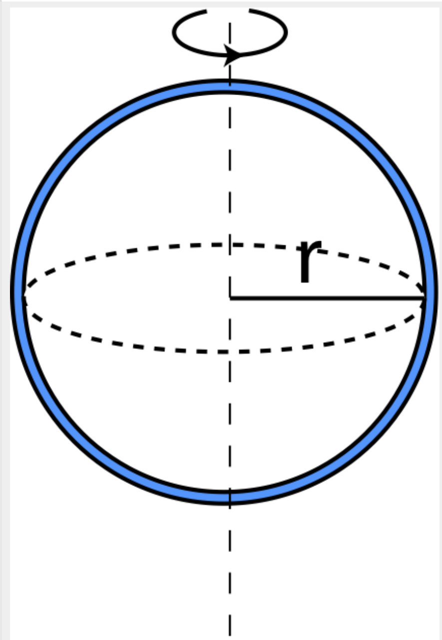
In general, most objects are not made-up of point masses but are continuous mass. Equation 7.4.5 is a sum which can be turned into an integral in the limit of infinitesimally small mass increments when the mass becomes continuous:

$$I = \int r^2 dm \tag{7.4.6}$$

We will not go into deriving rotational inertia for different objects, but the table below gives the rotational inertia of several simple geometric shapes, as calculated in the limit of infinitesimal increments of mass using this equation. All of there geometric shapes have uniform mass.

Object	Rotational Inertia	Illustration
Point mass m moving in radius r	$I = mr^2$	
Thin ring of mass m and radius r rotating about center	$I = mr^2$	

Thin rod of mass m and length L rotating about one end perpendicular to the rod	$I = \frac{1}{3}mL^2$	
Thin rod of mass m and length L rotating about its center perpendicular to the rod	$I = \frac{1}{12}mL^2$	
Disk of mass m radius r rotating about an axis perpendicular to disk though the center	$I = \frac{1}{2}mr^2$	

Sphere of mass m and radius r rotating about an axis thorough the center	$I = \frac{2}{5}mr^2$	
Thin hollow spherical shell of mass m and radius r rotating about an axis through the center	$I = \frac{2}{3}mr^2$	

As seen from the formulas in the table, objects with the same mass can have very different rotational inertias, depending on how the mass is distributed with respect to the axis of rotation. For example, when the mass of a sphere is concentrated at its radius (as for the spherical shell), its rotational inertia is greater than for the sphere of the same radius and mass but with the mass uniformly distributed from the center (solid sphere). This is consistent with Equation 7.4.5 which shows that rotational inertia increases as mass gets further from the rotational axis.

The rotational inertia of a composite object is the sum of the rotational inertias of each component, all calculated about the same axis.

$$I_{total} = I_1 + I_2 + I_3 + \dots \tag{7.4.7}$$

So for a ring and a disk stacked upon each other and rotating about the symmetry axis of both, the rotational inertia is:

$$I_{total} = I_{ring} + I_{disk} \tag{7.4.8}$$

Rotational Analog to Force

We have developed multiple analogs between linear and rotational motion so far: velocity, acceleration, kinetic energy, and mass. We have seen Newton's 2nd Law which relates force to mass and acceleration, which tells us how the linear motion of an object changes due to external forces. Now we want to develop the analog of the 2nd Law with will relate "rotational force", known as *torque*, $\vec{\tau}$, which will determine how angular motion will change. There is a fundamental difference between force and torque as we will see shortly, specifically, not all forces will result in torques, and not all torques will results in net forces. Since rotational inertia is analogous to mass, and angular acceleration is analogous to linear acceleration, we can write down the analog of Newton's 2nd Law:

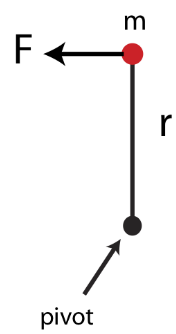
$$\vec{a}=\frac{\vec{F}}{m} \Longleftrightarrow \vec{\alpha}=\frac{\vec{\tau}}{I} \tag{7.5.1}$$

Torque is a vector since angular acceleration is a vector, and rotational inertia is a scalar. Let us examine which variables torque depends on by thinking about its units:

$$\tau=I\alpha=\left[kg\cdot m^2\right]\left[\frac{\text{rad}}{s^2}\right]=\left[\frac{kg\cdot m^2}{s^2}\right]=[N\cdot m] \tag{7.5.2}$$

The last equality in the above equation comes from definition of Newtons: $F=ma=[kg\cdot m/s^2]\equiv[N]$. So we find that torque will have units of force times length. Let's examine a simple example of a point mass attached to a massless rod of length r as shown in Figure 7.5.1. The rod is attached at a pivot on the opposite end of the mass.

Figure 7.5.1: Torque on a Point Mass



Focusing on magnitude only for now and applying Newton's 2nd Law for linear motion to the mass and using the connection between linear and angular acceleration, $a=r\alpha$, we find that:

$$F=ma=mr\alpha \tag{7.5.3}$$

For a point mass the rotational inertia from the pivot is $I=mr^2$. Multiplying both sides of the equation above by r we get:

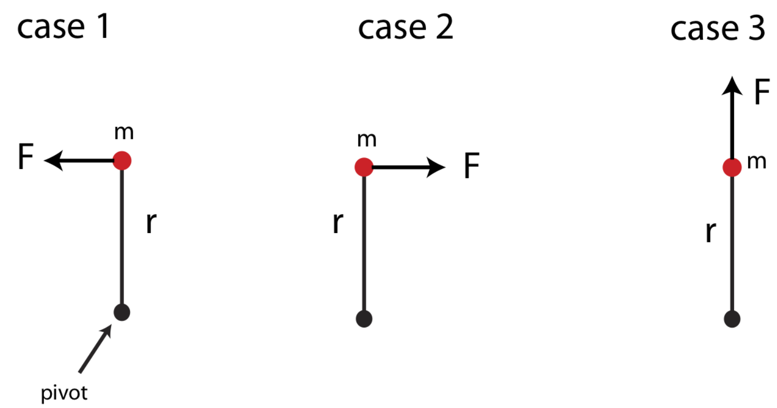
$$Fr=mr^2\alpha=I\alpha \tag{7.5.4}$$

Comparing the above result with Equation 7.5.1 we conclude that for a point mass:

$$\tau=rF \tag{7.5.5}$$

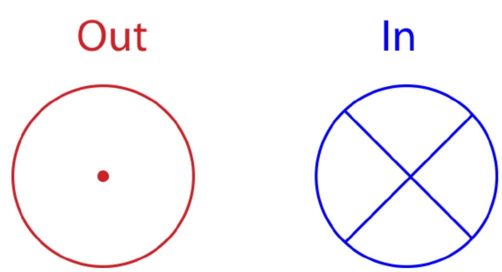
This result is consistent with the argument in Equation 7.5.2 that torque has units of $N\cdot m$. Although we demonstrated the result in Equation 7.5.5 specifically for a point mass, it turns out to be general for any set of masses or for an object with continuous mass, where r represents a distance from the pivot to the location where the force is applied. The result of torque being proportional to distance from pivot implies that the further the force is from the pivot, the bigger will be the torque, and the greater the angular acceleration. We have experienced this phenomena in many instances. For example, the pivot can represent a hinge of a door. The door swings back and forth about the hinge, and we apply forces to the door when we push it in order to open it. The further away from the door you push, the easier it is to open it since you are applying a greater torque simply by applying a force further from the hinge (pivot). We have all faced challenges of trying to open a heavy door by accidentally pushing it very close to the hinge.Our discussion on torque is not yet complete since we also need to consider the direction of torque and its dependence on the direction of force. Figure 7.5.2 below shows three different cases of a force acting on a point mass attached to a massless rod, which is fixed in place at the pivot on other end of the bar. In case one the force will result in an angular acceleration counterclockwise. In case 2 the acceleration will be in the clockwise direction. While in case 3 the rod will not rotate even if there is a force of the same magnitude acting on the mass.

Figure 7.5.2: Direction of Torque on a Point Mass



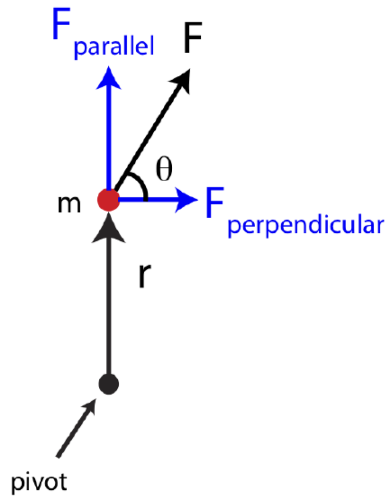
Thus, cases 1 and 2 generate a torque but in the opposite directions based on Equation 7.5.1, while case 3 results in zero torque. For cases 1 and 2 we can use the same right hand rule as we did for angular velocity to determine the direction of torque. To use the RHR you point the fingers of your right hand from the pivot toward the force and then curl them in the direction of the force, which is the direction it which the object would rotate due to that force. Your thumb will point in the direction of torque. In case 1 this direction is out of the page, and for case 2 it is into the page. Since very often when discussing angular motion vectors will point either in or out of the page, we need symbols to represent these vectors, since we can no longer draw arrows in the two-dimensional space of the page. These symbols are shown below in Figure 7.5.4. The "out" vector show the head of an arrow pointing toward you, and the "in" vector shows the tail of the arrow which is sticking into the page. Unlike arrows drawn on the plane of the page which have a length and a direction, these vector symbols only demonstrate direction, since you cannot draw them as longer or shorter arrows.

Figure 7.5.3: Vector Direction In and Out of the Page



What we can conclude from the three cases in Figure 7.5.2 that whether a force generates a torques depends of its orientation relative to the pivot. We can define a force position vector, $\vec{r}$, which points from the pivot or reference point to the location of the force. In Figure 7.5.2 we see that forces that are perpendicular to the position vector generate torque. On the contrary, forces that are parallel to the position vector, as in case 3, general zero torque. If a force was at an angle relative to the position vector, as shown in Figure 7.5.4, then it can be split into components parallel, $F_{||}$, and perpendicular, $F_{\perp}$ to $\vec{r}$. Based on the argument of the three cases in Figure 7.5.2, only the perpendicular component of this force would contribute to torque.

Figure 7.5.4: Direction of Force and Torque



Equation 7.5.5 for a more general case of a force that has both a parallel and perpendicular component relative to the position vector is:

$$\tau = rF_{\perp} \tag{7.5.6}$$

The equation above only give the magnitude of the torque, while the right-hand rule determines the direction. Mathematically, a *cross product* contains both the magnitude and direction of torque:

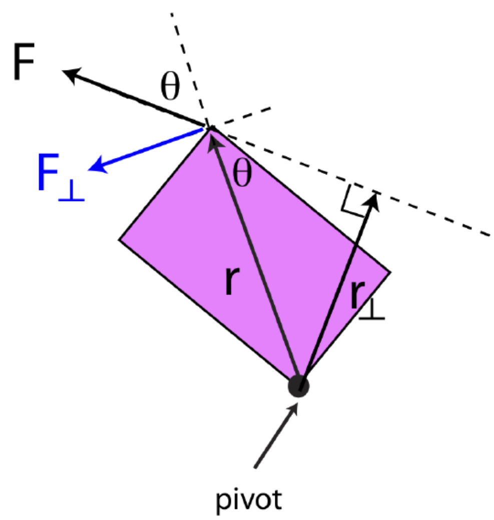
$$\vec{\tau} = \vec{r} \times \vec{F} \tag{7.5.7}$$

But in this class, we will typically use equation 7.5.6 to calculate the torque, and the RHR to determine its direction.

Figure 7.5.5 shows the position vector $\vec{r}$ drawn from the pivot to the location of the force, and then the perpendicular component of the force is depicted as $F_{\perp}$. If we define the angle, θ as the angle between the force vector and the position vector, then $F_{\perp} = F\sin\theta$ and Equation 7.5.6 becomes:

$$\tau = rF\sin\theta \tag{7.5.8}$$

Figure 7.5.5: Position Vector and Moment arm



However, there are cases where using the position vector to calculate the magnitude of torque is not the simplest method geometrically. Thus, another method with the *moment arm*, $r_{\perp}$, is sometimes used as shown in Figure 7.5.5. The vector $r_{\perp}$ is drawn from the pivot point to the location where it is perpendicular to the force. Even if the force arrow is not physically there, we can extend it (dashed line extending the force vector in the figure) such that $r_\perp$ is perpendicular to the line extending from the force. The two distances can be related by the same angle, θ as:

$$r_\perp = r\sin\theta \tag{7.5.9}$$

Combing the above equation and Equation 7.5.8, the magnitude of torque can be written as:

$$\tau = r_\perp F \tag{7.5.10}$$

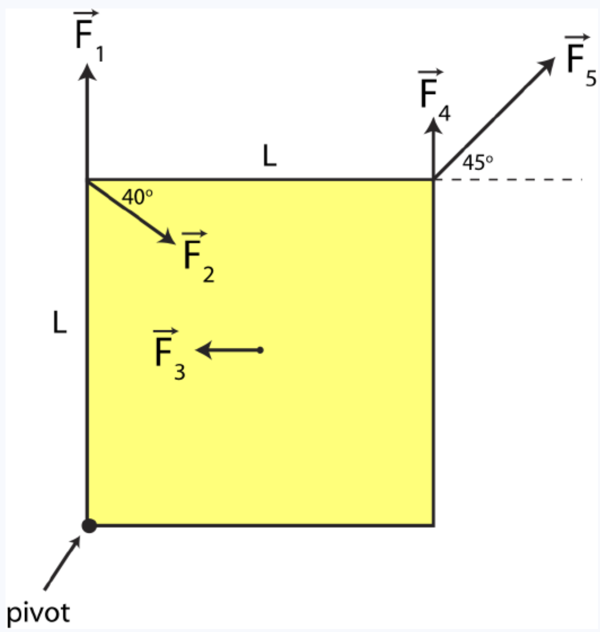
The two methods described by Equations 7.5.6 and 7.5.10 give equivalent results. With practice you will find situations where one method is easier or quicker to use than another, depending on what is given and what needs to be calculated.

What we have learned by describing torque in this section so far is that the location of the force is essential in order to determine torque. Thus, when drawing a force diagram on a selected system, we can no longer collapse the extended object(s) into a point, which we did when calculating forces. For torque we need to draw the entire object and draw forces at the locations of their application. This is known as the *extended force diagram*, which needs to contain all the force vectors at their locations, the pivot or the point of reference from which torques are calculated, and all the distances marked from the pivot to each force. The extended force diagram contains all the information that a "collapsed" free-body diagram contains, so it can be used for calculating the net force as well as the net torque.

Example 7.5.1

Shown below is a top view of a square shape anchored at the lower left corner with 5 forces acting on it. The length of each side of the square is 0.4 m. Forces 1-4 at at one of the corners of the square, while force 3 acts right at the center, as shown below. The magnitudes of the forces are given as:

$F_1 = F_2 = 5N, F_3 = 3N, F_4 = 2N,$ and $F_5 = 7N$.

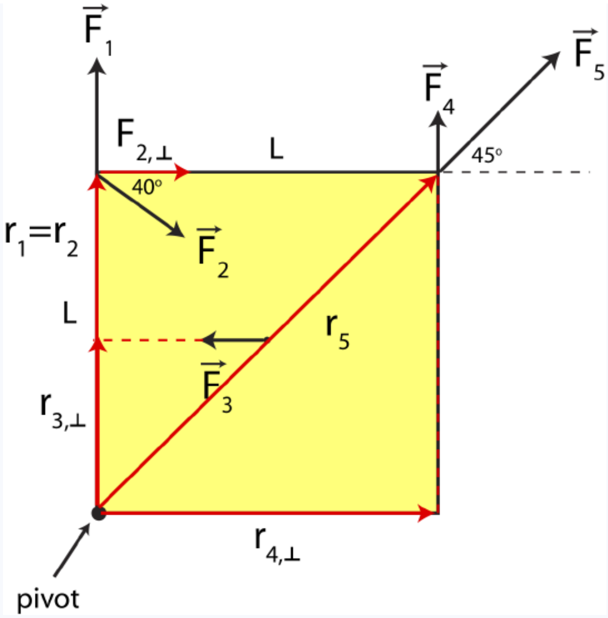


a) Calculate the net torque on the square. Express your answer in terms of magnitude and direction.

b) Calculate the net force on the square. Express your answer in terms of magnitude and direction.

Solution

a) The following diagram shows a convenient choice of using either position vector, $\vec{r}$ or the moment arm $r_\perp$ to solve for the torque due to each force.



Force $\vec{F}_1$ is parallel to the position vector $\vec{r}_1$. Thus, there is no perpendicular component which results in zero torque due to this force:

$$\tau_1 = 0$$

Force $\vec{F}_2$ has the same position vector as $\vec{F}_1$ ($\vec{r}_2 = \vec{r}_1$), but there perpendicular component of this force to the position vector, $F_{2,\perp}$:

$$F_{2,\perp} = F_2 \cos 40^\circ = 5N \cos 40^\circ = 3.83 \text{ N}$$

The magnitude of the torque due to force 2 is then:

$$\tau_2 = r_2 F_{2,\perp} = (0.4m)(3.83N) = 1.53 \text{ Nm}$$

Using the right hand-rule, fingers point up and curl toward $F_{2,\perp}$ in the clockwise direction resulting in the thumb pointing into the page. Thus, the torque due to $\vec{F}_1$ points into the page.

For $\vec{F}_3$ it is simpler to use the moment arm method since you do not need to calculate any angles, as labeled in the figure $r_{3,\perp}$. If you were to use the position vector, you would need to draw a vector pointing from the pivot to the location of $\vec{F}_3$, and then calculate component of $\vec{F}_3$ which is perpendicular to $\vec{r}_3$. Also, you would need to calculate the length of $\vec{r}_3$, which is half of the diagonal of the square. But using the moment arm gives a simple result to the magnitude of the torque due to force 3:

$$\tau_3 = r_{3,\perp} F_3 = \left(\frac{0.4m}{2}\right)(3N) = 0.6 \text{ Nm}$$

Using RHR, we find that $\vec{F}_3$ results in a clockwise rotation, so the torque point out of the page.

For $\vec{F}_4$ it is again simpler to use the moment arm to calculate torque. Using $r_{4,\perp}$ as marked in the figure we find:

$$\tau_4 = r_{4,\perp} F_4 = (0.4)(2N) = 0.8 \text{ Nm}$$

Using RHR, we find that force 4 results in a clockwise rotation, so the torque point out of the page.

Force $\vec{F}_5$ is parallel to the position vector, since $\vec{r}_5$ makes exactly a 45° angle with the corner of the square, as does $\vec{F}_5$. Since the two vectors are parallel there is no perpendicular component of the force to the position vector which results in zero torque:

$$\tau_5 = 0$$

Let us define a convention that vectors pointing out of the page are positive, and those pointing into the page are negative. Putting everything together the net torque is:

$$\vec{\tau}_{net} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \vec{\tau}_4 + \vec{\tau}_5 = 0 - 1.53 + 0.6 + 0.8 + 0 = -0.13 \text{ Nm}$$

The magnitude of net torque is 0.13 Nm*and it point into the page.*

b) To find the net force we need to break down the forces by components. In the x-direction:

$$F_{net,x} = F_{1,x} + F_{2,x} + F_{3,x} + F_{4,x} + F_{5,x}$$

$$F_{net,x} = 0 + 5N\cos 40^\circ - 3N + 0 + 7\cos 45^\circ = 5.78 \text{ N}$$

In the y-direction:

$$F_{net,y} = F_{1,y} + F_{2,y} + F_{3,y} + F_{4,y} + F_{5,y}$$

$$F_{net,y} = 5N - 5N\sin 40^\circ + 0 + 2N + 7\sin 45^\circ = 8.74 \text{ N}$$

The magnitude is:

$$F_{net} = \sqrt{F_{net,x}^2 + F_{net,y}^2} = \sqrt{5.78^2 + 8.74^2} = 10.5 \text{ N}$$

And the angle is:

$$\theta = \arctan\left(\frac{F_{net,y}}{F_{net,x}}\right) = \arctan\left(\frac{8.74}{5.78}\right) = 56.5^\circ$$

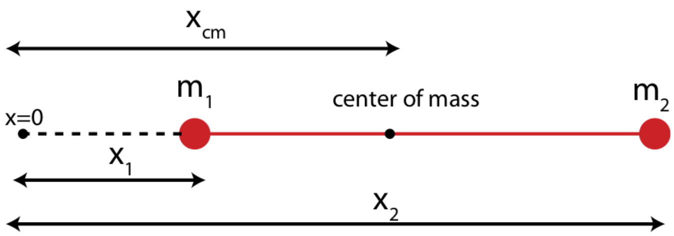
pointing in the northeast direction.

The Center of Mass

You may have realized by now that modeling objects as point particles is a rather drastic oversimplification, even though it is often very useful. We have now seen instances when the extended geometry of a non-point object becomes important. Focusing on just one point of an object can describe perfectly adequately the translational motion of that object, but it does not tell us anything about the object's rotation. It turns out that we can consider all of the forces acting on the object as if they acted at one point, the *center of mass*, as far as translation is concerned. That is, if we are concerned only about an object's translation, it doesn't matter where the forces act on the object. We can consider them all to act at a single point! This is truly a great simplification. The special point where we consider the forces to act is called the *center of mass*.

The center of mass is defined as a geometric average position of all the masses. For example, the center of mass of two point particles of equal mass is exactly at the midpoint between the two masses. If the masses are unequal, the center of mass would be closer to the heavier mass. Mathematically the center of mass can be described with an equation which expresses the average based on the relative weights of the masses. Figure 7.5.6 below shows a generalized system of two points masses, m_1 and m_2. The center of mass is the location measured relative to some arbitrary origin (x=0) along the line of the two masses. The distances marked x_1 and x_2 are the distances from the origin to each of the masses.

Figure 7.5.6: Center of Mass



For the two masses shown in Figure 7.5.6, the center of mass is written as:

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{7.5.11}$$

From the equation we see that when $m_1 = m_2$ the center of mass becomes $x_{cm} = (x_1 + x_2)/2$ which is exactly the midpoint between the two masses. If $m_1 \gg m_2$, then the center of mass is very close to the location of m_1, $x_{cm} \sim x_1$. Extending this to a system of multiple number of point masses the center of mass equation becomes:

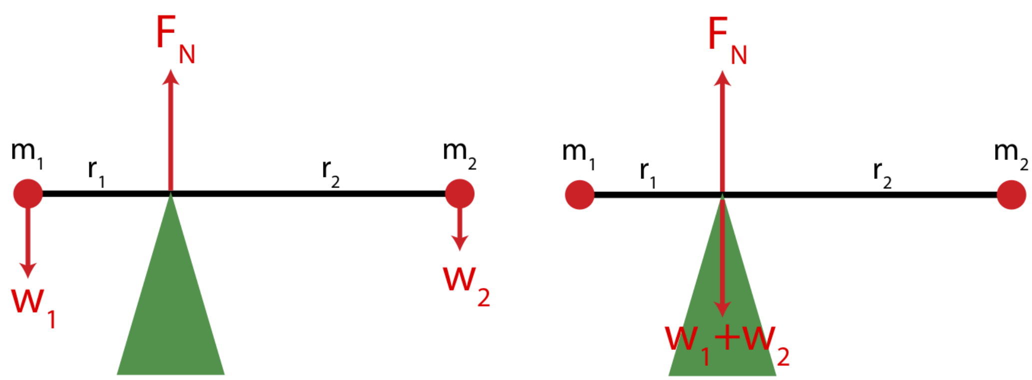
$$x_{cm} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M_{tot}} \sum_{i=1}^{N} m_i x_i \tag{7.5.12}$$

In nature most objects are not made up of a collection of point masses, but rather of a continuous mass. Mathematically, this means that we need to turn the summation to an integral over small mass elements, which can become rather complicated for an object in three-dimension with mass which is not uniform.

Torque due to Gravity

As we have just established in order to calculate torque on an object due to a force, you need to know the exact location where that force act. Gravity is a special case since it acts everywhere on a given object. And gravity can certainly cause torque, if you attempt hold a long and heavy stick at one end, it will want to rotate down due to torque cause by gravity. Thus, we want to figure out where to place the force of gravity on an extended object so we can calculate the amount of torque generated by gravity relative to some reference point.Consider the example in Figure 7.5.7 below, where a massless rod with two point masses at each end is balanced by a *fulcrum*, the pivot point where an object is balanced. The left picture shows an extended force diagram. Since the two masses are point masses, the force due to gravity on each mass is located exactly at the location of each mass.

Figure 7.5.7: Balanced Rod with Two Point Masses



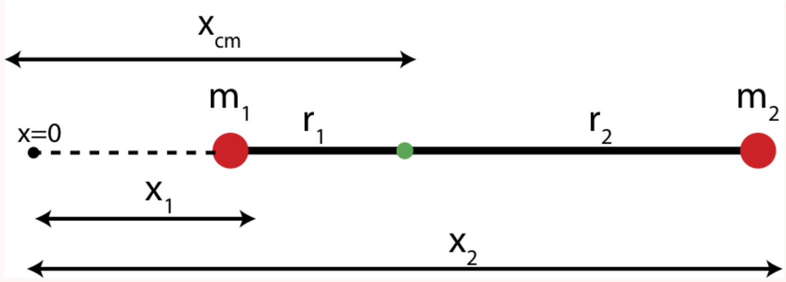
Relative to the fulcrum about which the rod pivots, the two weights generate torque in opposite directions. The right mass causes a torque into the page, while the left mass generates a torque out of the page. To balance torque, in addition to the direction being opposite, their magnitudes need to be the equal, $\tau_1 = \tau_2$. Using Equation 7.5.6 we find that:

$$r_1 m_1 = r_2 m_2 \tag{7.5.13}$$

Imagine that we did not have knowledge of the two point masses at the end of this rod, but instead just moved the fulcrum along the rod until it became balanced in this particular location. Since we now view this object as having some total mass (rather than a sum of two point masses) but with unknown distribution of that mass and there is no net torque once the fulcrum balances the object, the logical location of the weight of the object is at balance point. At the fulcrum which is the pivot, the position vector to the weight force is zero, so there is zero torque due to the weight. If we placed the weight vector anywhere other than the fulcrum, then it would generate a torque relative to the pivot, which is not consistent with the object being balanced. In other words, we can replace the two weights at the two point masses in the left picture with just one weight at the fulcrum of magnitude, $w_1 + w_2$, as shown in the right illustration. It turns out that this location is exactly at the center of mass of the object. You can see derivation of this fact below.

Derivation

We want to show that the result in Equation 7.5.13 is exactly the center of mass of a two point particle system. Earlier in this section defined the center of mass of an object relative to some origin, as shown in the figure below.



The center of mass relative to x=0 is given by:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

We can rewrite r_1 and r_2 in terms of x_{cm} and x_1 and x_2. From the figure we see that:

$$r_1 = x_{cm} - x_1$$

and

$$r_2 = x_2 - x_{cm}$$

Plugging in the expression for x_{cm} into each equation we get:

$$r_1 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} - x_1 = \frac{m_2}{m_1 + m_2}(x_2 - x_1)$$

and

$$r_2 = x_2 - \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}(x_2 - x_1)$$

Now let us multiple the equation for r_1 by m_1 and the equation for r_2 by m_2, and we see that the right hand sides of both equations becomes identical. So we can conclude that:

$$r_1 m_1 = r_2 m_2$$

which is the exact equation that we obtained from balancing torque. Thus, the fulcrum was balanced at the center of mass of the two point object.

It turns out that the center of mass is the same as the *center of gravity* (where you can support the object and it will not rotate) as long as the gravitational force is uniform. Near the surface of the Earth, for all objects of ordinary size, the gravitational force can certainly be considered uniform. Thus, for all problems we consider, the center of mass and center of gravity are the same point.

If objects are constrained to rotate about a particular axis, such as a rod mounted by a hinge to a vertical wall the object is fixed at a specific location, the torques are typically computed about that axis. If there is no constraint, torques should be computed about the center of mass, the point about which the object will rotate. To see this, imagine you exert two forces of equal magnitude and opposite direction at different locations along the object. The different locations of force will result in a non-zero net torque, so the object will start rotating. Equal and opposite forces result in zero net force, so the center of mass cannot accelerate. If the object is rotating about the center of mass, it means that the center of mass is stationary, which is consistent with zero net force. If the object was rotating about any other reference point, that would imply that the center of mass is rotating about that point. In that case the velocity of the center of mass would be changing since its direction is changing, which can only happen if there is net force. Thus, the object must rotate about its center of mass.

Work

We are familiar with the concept of work as a way that the energy of a system is changed. In terms of force and distance, work is done to translate a system from position x_1 to a new position x_2 is given by:

$$W = \int_{x_1}^{x_2} \vec{F} \cdot d\vec{s} \tag{7.5.14}$$

where the dot product of the two vectors reminds us that it is only the components of force and displacement in the same direction that contribute to the integral.

A similar expression holds for the work done by a torque to rotate a system from angle θ_1 to a new angle θ_2 is given by:

$$W = \int_{\theta_1}^{\theta_2} \vec{\tau} \cdot d\vec{\theta} \tag{7.5.15}$$

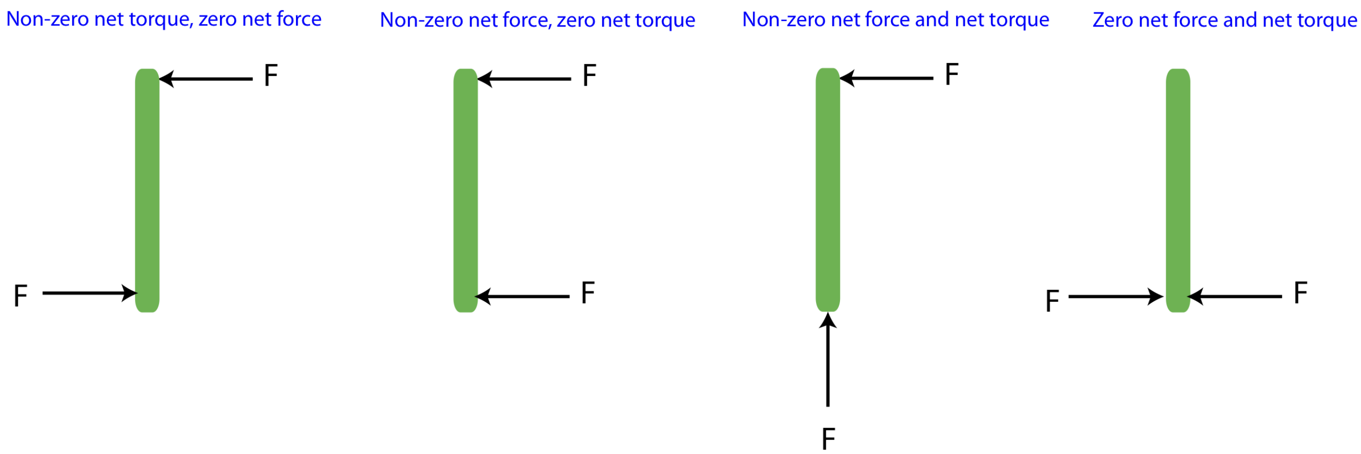
The energy of a particular system can be changed by the process of a force exerted by an outside object doing work on an object in the system and/or by a torque exerted by an outside object doing work on an object within the system. In either case, the work can be positive (increases the energy of the system) or negative (decreases the energy of the system).

7.6: Static Equilibirium

Net Force and Torque

Although torque is derived from force, the two concepts are fundamentally different. Forces cause changes in linear motion, while torques cause changes in angular motion. It is possible for forces to cause a net force but no net torque, as it is possible for forces to create a net torque but no net force. Figure 7.6.1 demonstrates a few scenarios using two forces with equal magnitude but varying directions, that result in different combinations of net force and torque.

Figure 7.6.1: Net force and torque



We assume that the bar in the figure is initially stationary, its mass is uniform, and it will rotate about its center. In the first scenario from the left, measuring torque relative to the center, both forces result in a counterclockwise rotation, so the bar will experience angular acceleration since the torques do not cancel. But the forces are equal and opposite, the net force is zero implying that the center of the mass of the bar will remain stationary at the same location. In the second scenario from the left, the two force result in torques of equal magnitude but opposite direction, the upper force result in torque pointing out of the page, while the lower one give a torque pointing into the page, so the two torques will cancel. The net force is to the left since the two forces add in this case, thus the bar will (linearly) accelerate to the left without any rotations. In the third scenario from the left, the lower force does not result in a torque, since it's parallel to the position vector of the force. The upper force will cause a counterclockwise rotation, so there is a net torque pointing out of the page. The forces do not cancel either, and the combination points in northwest direction. Thus, the bar will acceleration linearly in the northwest direction and will rotate counterclockwise with angular acceleration pointing out of the page. In the the last scenario from the last, the forces are again equal and opposite, so the net force is zero. Relative to the center, these forces also result in equal and opposite torques, so the net torque is also zero. This, this bar will remain stationary, neither translating or rotating.

It is often of interest to find situations when that all the forces that act on a system result in zero net force and zero net torque. This ensures that even under the influence of the forces, if the system is initially stationary, it will remain in *static equilibrium*, meaning that it will not move translationally and will not rotate. The conditions that describe a system in static equilibrium are:

$$\vec{\tau}_{net} = \sum \vec{\tau} = 0$$

$$\vec{F}_{net} = \sum \vec{F} = 0$$

Often you are faced with a problem where you are trying to determine some unknown forces with the assumption that the system is in static equilibrium. For example, you want to hang a heavy object using several ropes at various locations and angles, and you want to make sure the the ropes you use will be able to withstand the type of forces that will be present when the object is hanging in one place. You might find that adjusting the location of the ropes will still result in static equilibrium but will decrease the tension the ropes will feel. Below is a general outline for standard static equilibrium problems, followed by several examples.

Static Equilibrium Problems

Problem-Solving Strategies:

1. Isolate and define the physical system which is in static equilibrium.
2. Establish a pivot point from which you will measure all distances. Note, a pivot point does not have to be a physical pivot or even on the object you are trying to analyze. If there is a physical pivot present such as a hinge, then a good choice of pivot is at a location of an unknown force. Then the force at the chosen pivot is eliminated in the torque balancing calculation. For a system in static equilibrium, the net torque is zero about any chosen pivot point.
3. Draw an extended force diagram on the system, labeling all the forces at the locations where they act and the relevant distances (using position vector or moment arm) from the pivot to each force.
4. Determine if any of the forces result in zero torque.
5. Using the right hand rule to determine the direction of each torque.
6. Since not all forces contribute to torque, it is typically best to start with balancing torques since you can eliminate one of the unknown forces to solve for the second unknown force. Balance torques using the chosen pivot. Instead of choosing a sign convention, you can just equate the sum over torques in one direction to the sum in the opposite direction. For example, the sum of all torques that cause clockwise (CW) rotations is equal to the sum of all torques that cause counterclockwise (CCW) rotations:

$$\sum \tau_{CW} = \sum \tau_{CCW} \tag{7.6.1}$$

7. Calculate the remaining unknown force by balancing forces by component:

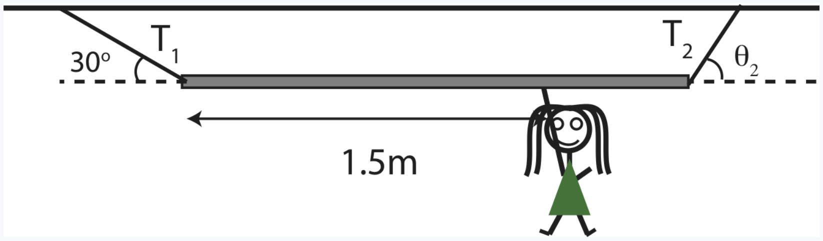
$$\sum F_{net,x} = 0 \tag{7.6.2}$$

$$\sum F_{net,y} = 0 \tag{7.6.3}$$

8. Calculate the magnitude and direction of the unknown forces.

Example 7.6.1

A child is hanging on a bar attached by two ropes. The rope on the left makes 30° with the horizontal, as shown. The girl weighs 200N and is hanging 1.5 m from the left end of the bar. The bar is uniform, 1.8 m long, and weighs 60N. (Angles and lengths may not be drawn to scale.)

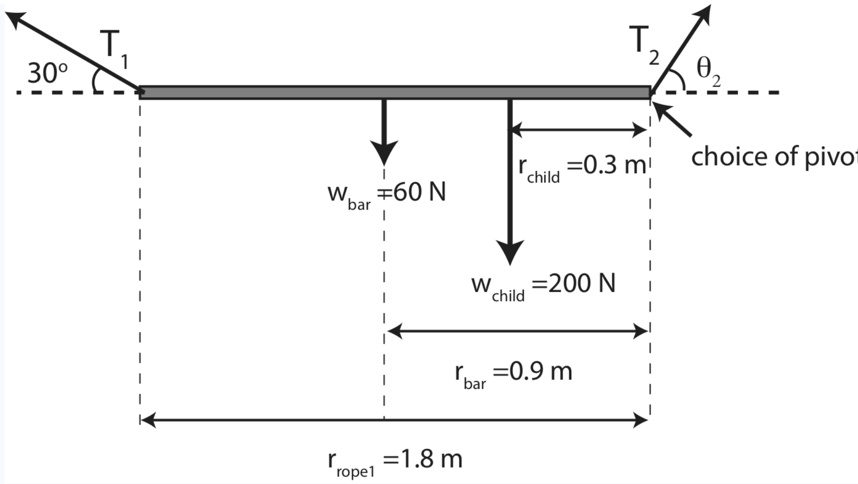


a) Find the tensions in the two ropes, T_1 and T_2, and the angle, θ_2, that the right rope makes with the horizontal.

b) If the length of the bar was longer while its weight stayed the same, and the girl was still hanging a distance of 5/6 of the bar from the left edge, would the tensions in the ropes increase, decrease, or stay the same?

Solution

a) Choose the location of the right rope as the pivot point, since both the magnitude and direction T_2 is unknown and only the magnitude of T_1 is unknown. The distance to the girl from the pivot is 1.8-1.5=0.3 m. Since the bar is uniform, the center of mass is at L/2=0.9m. The extended force diagram is shown below.



Relative to the chosen pivot point, the weight of the bar and girl will result in counterclockwise rotations, while the tension of rope 1 gives a clockwise rotation. Balancing torques we get:

$$\tau_1 = \tau_{\text{bar}} + \tau_{\text{child}}$$

$$r_{\text{rope 1}} T_1 \sin 30^\circ = r_{\text{bar}} w_{\text{bar}} + r_{\text{child}} w_{\text{child}}$$

Solving for tension we get:

$$T_1 = \frac{r_{\text{bar}} w_{\text{bar}} + r_{\text{child}} w_{\text{child}}}{r_{\text{rope 1}} \sin 30^\circ}$$

$$T_1 = \frac{(0.9m)(60N) + (0.3m)(200N)}{(1.8m)(\sin 30^\circ)} = 126.7 \text{ N}$$

Now we can solve for the components of T_2 by balancing forces. The x-components are:

$$T_{2,x} - T_{1,x} = 0$$

$$T_{2,x} = T_{1,x} = 126.7 \cos 30^\circ = 109.7 \text{ N}$$

The y-components are:

$$T_{2,y} + T_{1,y} - w_{\text{bar}} - w_{\text{child}} = 0$$

$$T_{2,y} = w_{\text{bar}} + w_{\text{child}} - T_{1,y} = 200 + 60 - 126.7 \sin 30^\circ = 196.7 \text{ N}$$

The magnitude of T_2 is:

$$T_2 = \sqrt{109.7^2 + 196.7^2} = 225.2 \text{ N}$$

The angle is:

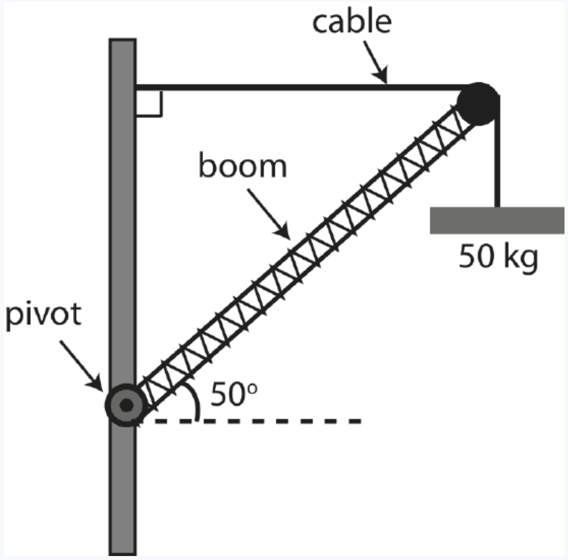
$$\theta = \arctan\left(\frac{196.7}{109.7}\right) = 60.9^\circ$$

pointing in the northeast direction.

b) When balancing torques in this problem, the tensions don't depend on the total length of the system, but rather where all the forces act relative to each other which is kept the same here. Thus, the answer doesn't depend on L, and the tensions are the same. And "balancing forces equations" do not depend on the length of the bar either.

Example 7.6.2

Shown here is a boom of a crane supporting a 50 kg weight with a cable. The boom has a uniform mass of 150 kg and makes a 50° angle with the horizontal.



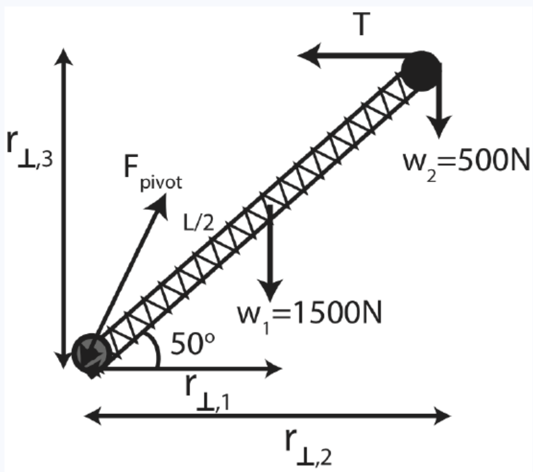
a) Draw an extended force diagram on the boom. Mark all the forces and relevant distances. You may leave forces and distances in terms of variables.

b) Find the magnitude of the tension of the supporting cable (marked on diagram).

c) Calculate the force exerted by the pivot on the boom in terms of magnitude and direction.

Solution

a) Below is an extended force diagram on the boom.



For all the forces here, we choose to use the moment arm method. These distances are calculated in part b). The position vector would be along the boom.

b) We need to balance torques first to find the tension, since the force of the pivot is unknown as well. The two weights give clockwise torques, τ_1 and τ_2, and the tension gives counterclockwise torque, τ_T relative to the pivot point. Thus, equating the clockwise to counterclockwise torques we get:

$$\tau_1 + \tau_2 = \tau_T$$

Using the $r_\perp$s drawn in the extended FBD we get:

$$r_{1,\perp} w_1 + r_{2,\perp} w_1 = r_{3,\perp} T$$

Let the length of the boom be L which results in:

$$\frac{L}{2} \cos 50^\circ \times 1500 N + L \cos 50^\circ \times 500 N = L \sin 50^\circ \times T$$

The distance L cancels. Solving for T we get:

$$T = \frac{\cos 50^\circ}{\sin 50^\circ} \left(\frac{1500}{2} + 500 \right) = 1049 N$$

b) Balance forces to calculate the pivot force, $\vec{F}_p$. In the x-direction:

$$F_{p,x} - T = 0$$

Resulting in:

$$F_{p,x} = T = 1049 \text{ N}$$

Forces in the y-direction:

$$F_{p,y} - w_1 - w_2 = 0$$

$$F_{p,y} = w_1 + w_2 = 1500 + 500 = 2000 \text{ N}$$

The magnitude of the pivot force is:

$$F_p = \sqrt{1049^2 + 2000^2} = 2258 \text{ N}$$

And the angle is:

$$\theta = \arctan\left(\frac{2000}{1049} \right) = 62.3^\circ$$

pointing in the northeast direction.

7.7: Angular Momentum

Angular Impulse and Momentum

Up to this point, we have developed analogs between linear and angular velocity and acceleration, mass and rotational inertia, and force and torque. Thus, it is only natural to consider the angular analog to linear momentum, known as *angular momentum*. Since linear momentum is equal to mass times linear velocity, we can make an educated guess that angular momentum is equal to rotational inertia times angular velocity. But let us develop a mathematical definition for angular momentum. To do that, first let's define *angular impulse*, $\text{ang }\vec{J}$, which is analogous to linear impulse:

$$\text{ang }\vec{J} \equiv \int_{t_i}^{t_f} \vec{\tau} dt \tag{7.7.1}$$

Following a similar procedure to Equation 7.1.3 and using Equation 7.5.1 we find that:

$$\begin{aligned} \text{ang }\vec{J} &= \int_{t_i}^{t_f} \vec{\tau} dt \\ &= \int_{t_i}^{t_f} (I\vec{\alpha}) dt \\ &= \int_{t_i}^{t_f} \left(I\frac{d\vec{\omega}}{dt}\right) dt \\ &= I\Delta\vec{\omega}\Big|_{t_i}^{t_f} \\ &= I\vec{\omega}_f - I\vec{\omega}_i \end{aligned} \tag{7.7.2}$$

Defining *angular momentum*, $\vec{L}$:

$$\vec{L} \equiv I\vec{\omega} \tag{7.7.3}$$

We see that the direction of angular momentum is the same as the direction of angular velocity which can be obtained using the right-hand rule. The units of angular momentum are $N \cdot m \cdot sec$, which is equivalent to $kg \cdot m^2/sec$. Combining Equations 7.7.2 and 7.7.3 we get the angular equivalent of the impulse- momentum model:

$$\text{ang }\vec{J} = \Delta\vec{L} \tag{7.7.4}$$

Conservation of Angular Momentum

If a system has many parts, its total angular momentum is the vector sum of the angular momenta of all the parts:

$$L = L_1 + L_2 + L_3 \dots = \sum L_i \tag{7.7.5}$$

For a system of particles or objects we can write a model for angular momentum:

$$\text{Net external angular impulse} = \text{ang }\vec{J}_{net,ext} = \int_{t_i}^{t_f} \sum \vec{\tau}_{ext} dt = \Delta\vec{L}_{\text{system}} \tag{7.7.6}$$

A system acted on by external torque undergoes a change in total angular momentum equal to the net angular impulse (total angular impulse) of the external torques. We can rephrase the relationship stated above as a conservation principle for the total angular momentum of a system of particles:

Conservation of Angular Momentum

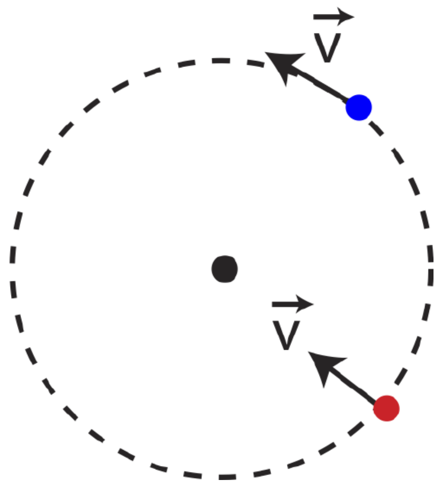
If the net external angular impulse acting on a system is zero, then there is no change in the total angular momentum of that system. Otherwise, the change in angular momentum is equal to the net external angular impulse.

This statement is an expression of *conservation of angular momentum*. As for linear momentum, total angular momentum of a system of objects remains constant as long as there is no net angular impulse due to torques that arise from interactions with objects outside the system.

Analog of Linear and Angular Momentum

We will consider the angular momentum of both a point object as well as the angular momentum of extended objects. In either case, we need to be clear about the axis (or reference point) about which we are calculating the angular momentum. Let us start with a point particle moving in a circle, as the blue particle in Figure 7.7.1.

Fig 7.7.1: Angular and Linear Momentum



If the particle with mass, m and speed, v, its linear momentum is, $\vec{p} = m\vec{v}$. The angular momentum of the particle is then:

$$\vec{L} = I\vec{\omega} = mr^2\frac{\vec{v}}{r} = mr\vec{v} = r\vec{p} \tag{7.7.7}$$

However, ff we consider the red particle instead, which is moving with the same speed but toward the center of the circle, it has no angular momentum relative to the center of the circle. Thus, even though the red particle has the same linear momentum as the blue one, it has zero angular momentum relative to the center of the circle, since there is no change in angle relative to the center. Generally, what matters when defining angular momentum is the component of the momentum which is perpendicular to the line that connects some reference point (such as the center of the circle):

$$L = r\vec{p}_\perp \tag{7.7.8}$$

Since the velocity of the red particle is parallel to the line connecting the center and the particle, its angular momentum is zero. This discussion is starting to seem very similar to the one on torque in Section 7.5. In fact Equation 7.7.8 looks analogous to Equation 7.5.6, when we were developing the connection between torque and force. Thus, it is no surprise that the general connection between linear and angular momentum has the form analogous to Equation 7.5.7:

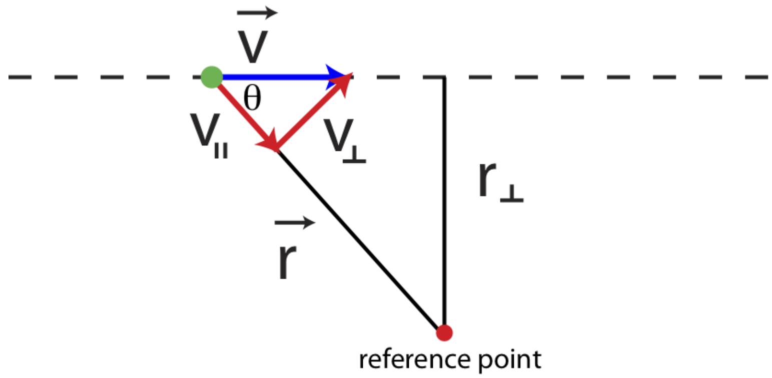
$$\vec{L} = \vec{r} \times \vec{p} \tag{7.7.9}$$

Alert

Angular motion should not be interpreted as circular motion. Objects do not need to be rotating or moving in a circle in order to have angular velocity and momentum. Angular velocity is defined as a rate of change of angle which is defined relative to some chosen reference point. If there is a change of angle relative to this reference point as the object moves, then it have angular velocity and momentum, even if it is moving in a straight line.

An interesting consequence of this is that a particle that moves in a straight line can have non-zero angular momentum. Figure 7.7.2 shows a particle moving in a straight line with a reference point defined. As the particle moves the angle θ will change, so the particle has angular velocity, and thus angular momentum relative to this reference point.

Fig 7.7.2: Angular and Linear Momentum



As we just argued in Equation 7.7.8, only the perpendicular component of velocity to vector $\vec{r}$, $v_{\perp}$, (depicted in Figure 7.7.2) will contribute to angular momentum. As the particle moves and the angle changes, $v_{\perp}$ will change as well since $v_{\perp} = v \sin\theta$. Rewriting Equation 7.7.8 we get:

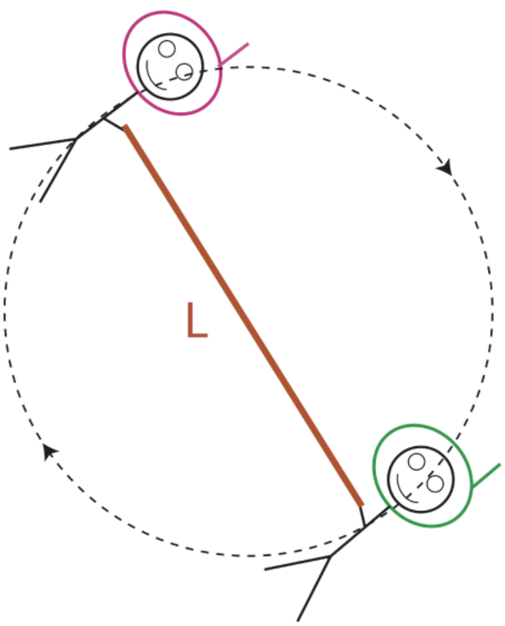
$$L = r\vec{p}_{\perp} = rmv\sin\theta \tag{7.7.10}$$

The distance r changes with angle, making Equation 7.7.10 difficult to use, but there is a useful simplification. Notice that the distance marked $r_{\perp}$ in Figure 7.7.2 does not change as the particle moves, and it can related to distance r as $r = r_{\perp}\sin\theta$. Equation 7.7.10 can the be simplified as:

$$L = rmv\sin\theta = mvr_{\perp} \tag{7.7.11}$$

Let us explore some of these connections between linear and angular motion using a concrete example shown below in Figure 7.7.3. In this scenario two astronauts are holding on to a rope of length L while rotating in space in a clockwise direction, as shown. We want to analyze what happens to angular and linear momentum and velocities when the two astronauts get closer to each other by pulling in the rope, such that the length of the rope between them decreases by half. For simplicity, let us assume that the rope is nearly massless, and that the two astronauts have the same mass, m. We also assume that the astronauts are in empty space with no other external forces present. No forces also implies not external torques are present.

Fig 7.7.3: Angular and Linear Momentum Example



When the two astronauts change the length of the rope, both linear and angular momentum of the two-astronaut system is conserved since there is no net force or torque. For linear momentum, since the two astronauts are across from each other, their velocities will be always in opposite direction, and the combined momentum is zero. In other words, the center of mass of system remains stationary. Thus, for linear momentum:

$$\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 = m|\vec{v}| - m|\vec{v}| = 0 = \vec{p}_f \tag{7.7.12}$$

Angular momentum is conserved as well:

$$\vec{L}_i = \vec{L}_f \tag{7.7.13}$$

We can treat the two-astronaut system as two point masses rotating about their center of mass. The rotational inertia of this system is then:

$$I = \sum mr^2 = 2mr^2 \tag{7.7.14}$$

where r is the distance from the center to each astronaut. Initially when the rope is length L the rotational inertia is:

$$I_i = 2m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{2} \tag{7.7.15}$$

When the astronauts get closer to each other shortening the rope by half, the distance from the center becomes $L/4$ and the rotational inertia changes to:

$$I_f = 2m\left(\frac{L}{4}\right)^2 = \frac{mL^2}{8} \tag{7.7.16}$$

We can rewrite the conservation of angular momentum Equation 7.7.13 as:

$$I_i\omega_i = I_f\omega_f \tag{7.7.17}$$

Inserting the results for initial and final rotational inertia, we get:

$$\frac{mL^2}{2}\omega_i = \frac{mL^2}{8}\omega_f \tag{7.7.18}$$

Solving for final angular speed:

$$\omega_f = 4\omega_i \tag{7.7.19}$$

The astronauts are spinning four times as fast because they changed the rotational inertia of the system by bringing their masses closer to the center of rotation. This is a very interesting results which does not have a direct analogy for linear momentum. We have a situation where angular momentum is conserved, the system did not change into separate objects, yet the speed changed because the system changed its rotational inertia.

This is exactly the reason why figure skaters spin faster when they pull their arms toward their bodies, decreasing rotational inertial while increasing rotational speed in order to conserve angular momentum. When an ice skater begins to spin with a leg extended, there is only a small torque exerted on the skater by the ice. Thus, angular momentum diminishes rather slowly (she can spin for a long time). Now, if she pulls in her leg, her rotational inertia is reduced considerably, and her rotational velocity (spin velocity) increases considerably. This is most easily seen by writing the angular momentum as $L = I\omega$ and noting that if L remains almost constant, then the product $I\omega$ must remain constant.

Since angular speed changes, so does linear speed. Relating angular and linear velocity:

$$\omega_i = \frac{v_i}{L/2} \tag{7.7.20}$$

$$\omega_f = \frac{v_f}{L/4} \tag{7.7.21}$$

Using the above equation and Equation 7.7.19 we find the relationship between initial and speeds for each astronaut:

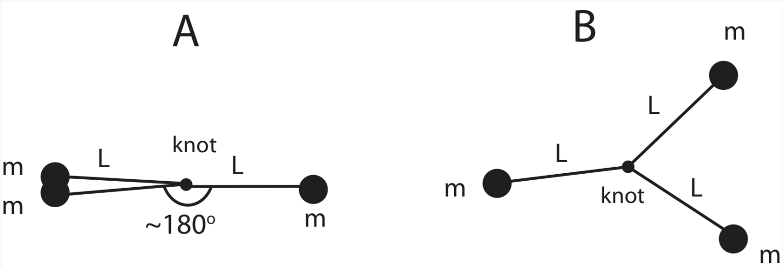
$$v_f = 2v_i \tag{7.7.22}$$

To maintain the same angular momentum the linear momentum according to Equation 7.7.8 had to increase by half since the distance r decreased by half.

Example 7.7.1

Pictured is a gaucho, an Argentinian "cowboy". In his hand he is holding *bolas* (aka boleadoras), a commonly used throwing weapon among the gauchos. A bolas consists of three equal masses, m, each tied to a rope with length, L, and the three ropes are tied in a knot. You may ignore the mass of the three ropes and the knot in this problem.

A detailed illustration of bolas and the two different configurations you will analyze is shown here:



a) Imagine two gauchos Juan and Carlos are each swinging a bolas above their heads. Juan is holding it by one of the masses such that the bolas form a shape as in picture in above **A** (assume the three ropes form a straight line). Carlos is holding it by the knot. If both gauchos start swinging their bolas at the same time and apply the same torque for 5 seconds, whose bolas will be rotating faster or will they rotate at the same rate?

b) When used as a throwing weapon, a gaucho holds the bolas by one of the balls. He swings the bolas above his head and releases it. Once released the bolas rotate about their center of mass located at the knot. Calculate the final angular velocity given that once released the bolas are rotating at 3 rev/sec. You may neglect the effects of gravity in this interval.

Solution

a) The change in angular momentum is the same for both situations since angular impulse is the same (equal torque applied for the same duration of time). Since the bolas start at rest, both Juan and Carlos's bolas will have the same final angular momentum magnitude of $L_f = I\omega$. Thus, the one with smaller rotational inertia will be rotating faster. To find rotational inertia use:

$$I = \sum_i (m_i r_i^2)$$

For Juan's bolas, the axis of rotation is about one of the masses, scenario A pictured above. So the reference point for the rotational inertia calculation is one of the masses, with the two other masses being a distance 2L away:

$$I_J = (2m)(2L)^2 = 8mL^2$$

For Carlos's bolas, the axis of rotation is about the knot, scenario B pictured above. Each mass is a distance L away from the axis of rotation:

$$I_C = 3mL^2$$

Since Carlos's bolas have a smaller rotational inertia, they will be rotating faster after 5 seconds.

b) We assume that there are no external torques, so the initial angular momentum must equal the final angular momentum:

$$\vec{L}_i = \vec{L}_f$$

$$I_i \omega_i = I_f \omega_f$$

The initial rotational inertia is about one of the balls and the final is about the knot which we calculated in part a):

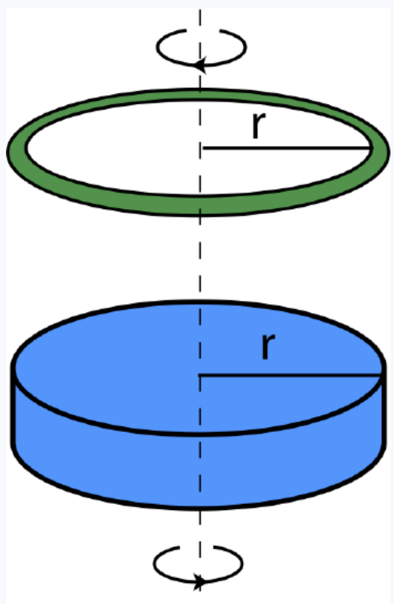
$$(8mL^2)\omega_i = (3mL^2)\omega_f$$

Solving for the final angular speed we get:

$$\omega_f = \frac{8}{3}\omega_i = \frac{8}{3}(3 \text{ rev/s}) = 8 \text{ rev/s} = 16\pi \text{ rad/s}$$

Example 7.7.2

Shown below is a disk which is rotating counterclockwise, as viewed from above, with angular speed of 6 rad/sec, and a ring which is rotating clockwise, as viewed from above, with angular speed of 4 rad/sec. Both are rotating about their center. The ring and the disk have the same radius, r. The ring then drops onto the disk, after which they stick together and continue spinning together counterclockwise, as viewed from the top, with angular speed of 2 rad/sec. Assume that no external torques act on the disk/ring system.



a) Determine which object, the disk or the ring, has a greater magnitude of angular momentum before the collision.

b) Determine the ratio of the mass of the ring over the mass of the disk, m_r/m_d.

c) You would like to stop the rotating disk/ring system. You do think by spinning another ring which is identical to the first one and dropping it onto the system. How fast should you spin the ring and in what direction?

Solution

a) The figure below illustrates the initial and final configurations of this system. Using RHR, we see that the angular momentum points upward for the disk and downward for the ring. The initial total angular momentum has to equal to the final total angular momentum, since there are no external torques present. Since the final momentum point upward, this means that the sum of the initial angular momenta has to point upward. For this to be true, the magnitude of the ring's initial angular momentum has to be smaller than the magnitude of the disk's initial angular momentum.

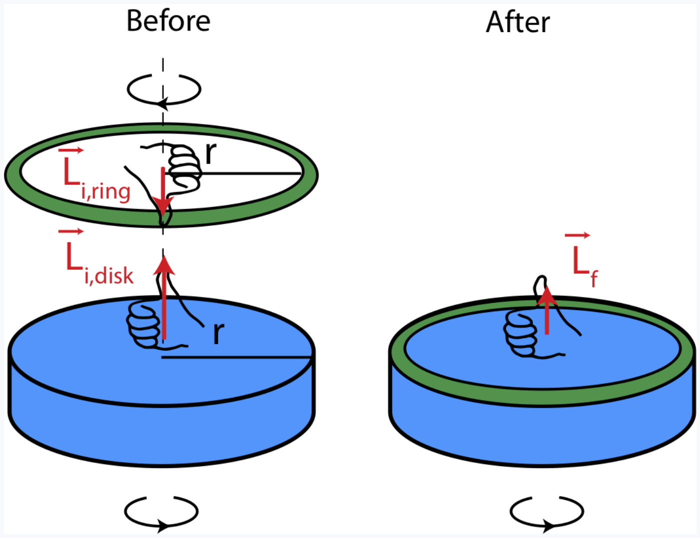
$$\vec{L}_{tot,i} = \vec{L}_{tot,f}$$

Let us assume that angular momentum is positive when it points up:

$$|\vec{L}_{i,d}| - |\vec{L}_{i,r}| = |\vec{L}_{tot,f}|$$

Resulting in:

$$|\vec{L}_{i,d}| > |\vec{L}_{i,r}|$$



a) Following up from equation in a) we can write the conservation of angular momentum equation as:

$$I_d\omega_{i,d} - I_r\omega_{i,r} = (I_d + I_r)\omega_f$$

Looking up the rotational inertia in Section 7.4 for the disk and the ring rotating about their center, and setting the radii equal to each other, $r_d = r_r = r$, we get:

$$\frac{1}{2}m_d r^2\omega_{i,d} - m_r r^2\omega_{i,r} = \left(\frac{1}{2}m_d r^2 + m_r r^2\right)\omega_f$$

Dividing each side of the equation by m_d to get the ratio of masses and canceling the factor of r^2 we get:

$$\frac{1}{2}\omega_{i,d} - \frac{m_r}{m_d}\omega_{i,r} = \left(\frac{1}{2} + \frac{m_r}{m_d}\right)\omega_f$$

The rest of the calculation involves algebraically solving for the ratio. Let's start by moving all the terms containing the mass ratio to the left-side and the remaining terms to the right side:

$$-\frac{m_r}{m_d}(\omega_{i,r} + \omega_f) = \frac{1}{2}(\omega_f - \omega_{i,d})$$

Moving the minus sign and solving for the ratio we get:

$$\frac{m_r}{m_d} = \frac{1}{2}\left[\frac{\omega_{i,d} - \omega_f}{\omega_{i,r} + \omega_f}\right] = \frac{1}{2}\left[\frac{6-2}{4+2}\right] = \frac{1}{3}$$

c) In order the system to spot the final angular momentum must be zero, so the sum of the new ring's initial momentum and the combined disk/ring system must add up to zero:

$$\vec{L}_{tot,i} = vecL_{i,r} + \vec{L}_{i,rd} = \vec{L}_{tot,f} = 0$$

For this to be true the direction of the new ring's momentum has to be in the opposite direction, so the ring must be rotating clockwise as seen from above. And the magnitudes must be equal:

$$|\vec{L}_{i,r}| = |\vec{L}_{i,rd}|$$

Rewriting angular momentum in terms of rotational inertia and angular speed we get:

$$m_r r^2 \omega_{i,r} = \left(\frac{1}{2} m_d r^2 + m_r r^2\right) \omega_{i,rd}$$

Solving for $\omega_{i,r}$ and using the result from b) of the ratio of masses, we get:

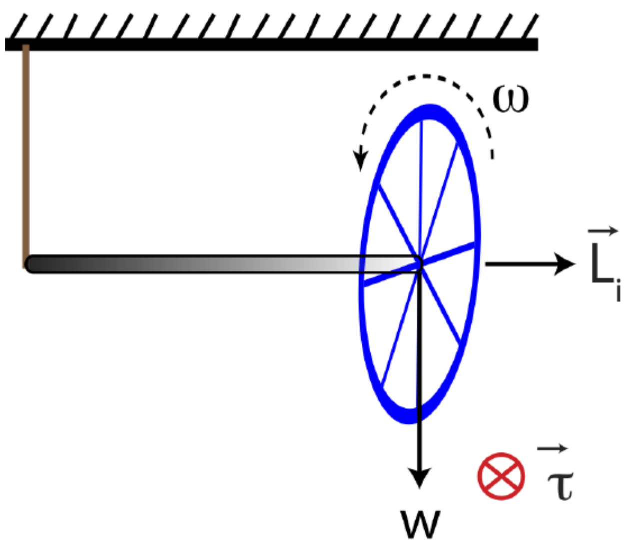
$$\omega_{i,r} = \left(\frac{1}{2}\frac{m_d}{m_r} + 1\right)\omega_{i,rd} = \left(\frac{1}{2} \times 3 + 1\right)(2 \text{ rad/s}) = 5 \text{ rad/s}$$

A Special System

There are some fascinating aspects of angular motion. One that we have already encountered is that when rotational inertia is changed by either changing the shape of the rotating object or the rotational axis, it can effect the rotational speed. For example, a diver spins faster when she pulls her body in. Another interesting example is somewhat analogous to a ball moving in a horizontal circle at a constant speed. In that case, linear momentum of the ball, which is tangent to the circle, is always perpendicular to the force by the string pulling toward the center of the circle. As a result, the force changes the direction of momentum while keeping its magnitude constant. For rotation motion, we will look at a situation where the torque is perpendicular to angular momentum.

Figure 7.7.4 shows a bicycle wheel supported by a rope attached to the ceiling at the left end of a short axle. The torque caused by the gravitational force acting down at the center of gravity of the wheel produces a torque that points into page, by the right-hand rule. If the wheel is not spinning, it will just fall rotating about the pivot point at the location of the rope, since this is the only point of support. However, if the wheel is initially spinning before being hung by the rope, the situation is much different. Figure 7.7.4 below show a wheel spinning counterclockwise as seen from the right, resulting in angular momentum, labeled $\vec{L}_i$, pointing to the right.

Figure 7.7.4: Wheel on a Rope



Since torque points into the board, it's easier to visualize all the vectors from the top view. Figure 7.7.5 is a top view of the rotating wheel, where initially the wheel's axle is horizontal resulting in torque pointing up in this view. Our goal is to figure out how this torque will change the wheel's angular momentum as some later time. When torque acts for a time Δt, the angular impulse equation gives us the change in angular momentum,

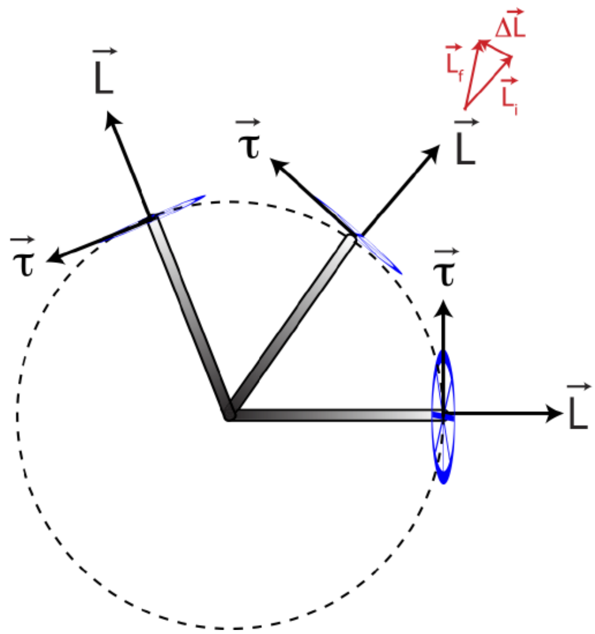
$$\vec{\tau}\Delta t = \Delta\vec{L} = \vec{L}_f - \vec{L}_i \tag{7.7.23}$$

To see what the wheel will do at some later time, it is more convenient to solve the final momentum:

$$\vec{L}_f = \vec{L}_i + \vec{\tau}\Delta t \tag{7.7.24}$$

Since torque is perpendicular to the initial angular momentum, instead of slowing down or speeding up the rotational speed it acts to change its direction only, as seen by the vector addition in the figure below. That is, the direction of the initial angular momentum, $\vec{L}_i$, is changed by the presence of the angular impulse, and is moved to the direction shown by $\vec{L}_f$. This turning motion of the orientation of the wheel is called *precession*. Instead of falling, the wheel *precesses*. Once the angular momentum (and the wheel) point in a new direction, the torque comes into play again, causing the wheel to precess still farther. The wheel continues to rotate in a horizontal circle with torque always pointing tangent to the circle and angular momentum pointing outward from the center, in this example of counterclockwise rotation as viewed from the side. In this fashion, the wheel is caused to precess in a horizontal circle about the pivot point. This is the weird behavior exhibited by a spinning top or gyroscope.

Figure 7.7.5: Top View of a Spinning Wheel



Precession is analogous to the situation of a ball being twirled around in a circle on the end of a string. Why doesn't the tension in the string pull the ball in toward the center of the circle? The answer is that it does, but the large tangential velocity also moves the ball in a direction tangent to the circle. The net result is that the ball travels in a circular path. If there were no large tangential velocity, the ball would indeed be pulled directly toward the center of the circle due to the tension in the string. A similar thing happens with the bike wheel. The torque causes a change in the direction of the large angular momentum of the spinning wheel. If the wheel did not have this large angular momentum, the torque would cause the wheel to tip over or fall down. Keep in mind, we did this analysis assuming friction is negligible. Presence of friction would slow down the rotational speed, and the wheel would eventually fall.

Putting it all together

The chart on below shows all of the linear motion and dynamic variables along with their rotational counterparts. Keep this chart out and handy for ready reference to help you from getting "lost" in all the symbols. You should make sure that you recognize the meaning behind the symbols when you see on of these relationships.

Summary Listing Fundamental Concepts Used in Mechanics Emphasizing Translational and Rotational Counterparts

Category	Concept	Translation	Rotation	Relation
Kinematic Variables	Position Velocity Acceleration	x $v = \frac{dx}{dt}$ $a = \frac{dv}{dt}$	θ $\omega = \frac{d\theta}{dt}$ $\alpha = \frac{d\omega}{dt}$	$\theta = \frac{s}{r}$ $\omega = \frac{v}{r}$ $\alpha = \frac{a}{r}$
Fundamental Dynamic Variables	Force/Torque Mass/Inertia Momentum Impulse Momentum-Impulse	F m $p = mv$ $J = \int F dt$ $J_{ext} = \Delta p$	τ I $L = I\omega$ ang $J = \int \tau dt$ ang $J_{ext} = \Delta L$	$\tau = rF_{\perp}$ $I = \sum mr^2$ $L = rp_{\perp}$
Newton's Laws	First Law Second Law Third Law	if $F_{net} = 0$, then $\Delta p = 0$ $F_{net} = ma$ or $F_{net} = \frac{dp}{dt}$ $F_{1\,on\,2} = -F_{2\,on\,1}$ $J_{1\,on\,2} = -J_{2\,on\,1}$	if $\tau_{net} = 0$, then $\Delta L = 0$ $\tau_{net} = I\alpha$ or $\tau_{net} = \frac{dL}{dt}$ $\tau_{1\,on\,2} = -\tau_{2\,on\,1}$ ang $\boldsymbol{J_{1\,on\,2}} = -$ang $\boldsymbol{J_{2\,on\,1}}$	
Energy	Kinetic Energy Work	$KE = \frac{1}{2}mv^2$ $W = \int_{x_1}^{x_2} \vec{F} \cdot d\vec{s}$	$KE = \frac{1}{2}I\omega^2$ $W = \int_{\theta_1}^{\theta_2} \vec{\tau} \cdot d\vec{\theta}$	

Contributors

7.9: Wrap-up

We have now developed models to enable us to use the three fundamental conservation laws of all of science: energy, momentum and angular momentum. The "before and after the interaction" approach, which now includes momentum and angular momentum, as well as energy, is extremely general and universally applicable. It allows us to get answers to most questions we ask regarding the behavior of interacting systems, as long as we don't need the time dependence of the dynamical variables.

What are the limitations to the approaches we have developed these past two chapters? We have mentioned some of these before, but it is good to emphasize them again. We know from our prior studies in chemistry and from some of what we have done in this course, that strange things begin to happen when the systems we are studying get very small, the size of molecules and atoms. Energies become quantized. Atoms and molecules can absorb and emit only certain amounts of energy, not a continuous range. We saw how specific heat modes became frozen out at low temperatures in solids. Things also get weird when speeds become large. In this case, large means moving at speeds that begin to approach the speed of light. Both at very small scales and when things go fast, some of our models break down and must be replaced by more complicated theories. But the primary variables in both quantum mechanics and in special relativity turn out to be energy, momentum, and angular momentum. There is something very special about these quantities. They apparently represent some of the most basic aspects of the universe. The fundamental ideas of conservation of energy, momentum, and angular momentum carry through all of the models we use to describe our universe.

The concepts of energy, momentum, and angular momentum (and the conservation of energy and momentum) remain as we delve into the details of the microscopic and the realm of very high speeds, but we do have to make changes in our understanding of these concepts. Energy, momentum and angular momentum take on discreet values; i.e., they become quantized. When we go to high speeds momentum and energy become intertwined. Even the separate idea of mass conservation gets pulled into a unified mass-energy conservation principle. We will explore the quantum world a little further in Physics 7C, but you will need to explore the fascinating world of special and general relativity on your own or in more advanced courses.

CHAPTER OVERVIEW

8: Force and Motion

In this chapter we take a different approach. We focus on the *details* of what happens during interactions, not just at net changes that occur as a result of interactions. We explicitly look at the *time-dependence* of the change of various parameters. We have already spent considerable time in the last two chapters learning how to work with forces to determine work and impulse and both translational and rotational motion. Here, we augment this knowledge to develop an approach to predict the detailed time dependence of motion from a knowledge of the net force and vice versa.

Topic hierarchy

8.0: Overview

Up to this point in this text we have concentrated on analyzing interacting physical systems with a “before and after the interaction” approach. The defining characteristic of this approach is that we need to know only *net changes* that occur in various quantities as the result of interactions. We do not need to know the details of what took place between the initial and final times. What makes this such a powerful approach is that there exist very general conservation laws for energy, momentum, and angular momentum. If all interactions are included in the definition of the physical system (making it a closed system), then the total energy, momentum, and angular momentum of this total system remain constant, regardless of how the energy, momentum, and angular momentum of various parts of the system change. On the other hand, if we don’t include all interactions in our definition of the physical system (making it an open system), then the net change in these quantities is due to the energy, momentum, and angular momentum transferred into the system from outside. In the case of energy this transfer can occur as heat or work. For momentum, the transfer occurs as an impulse. For angular momentum, the transfer occurs as an angular impulse.

In this chapter we take a different approach. We focus on the details of what happens during interactions, not just at net changes that occur as a result of interactions. We explicitly look at the time-dependence of the change of various parameters. Work, impulse, and angular impulse all involve the net force that acts on the physical system. Consequently, we have already spent considerable time in the last two chapters learning how to work with forces to determine work and impulse. In the last chapter we also developed greater skill at describing motion. We will need to augment this knowledge of force and motion only slightly in this chapter to develop an approach that lets us predict the detailed time dependence of motion from a knowledge of the net force and vice versa.

8.1: Graphing Motion

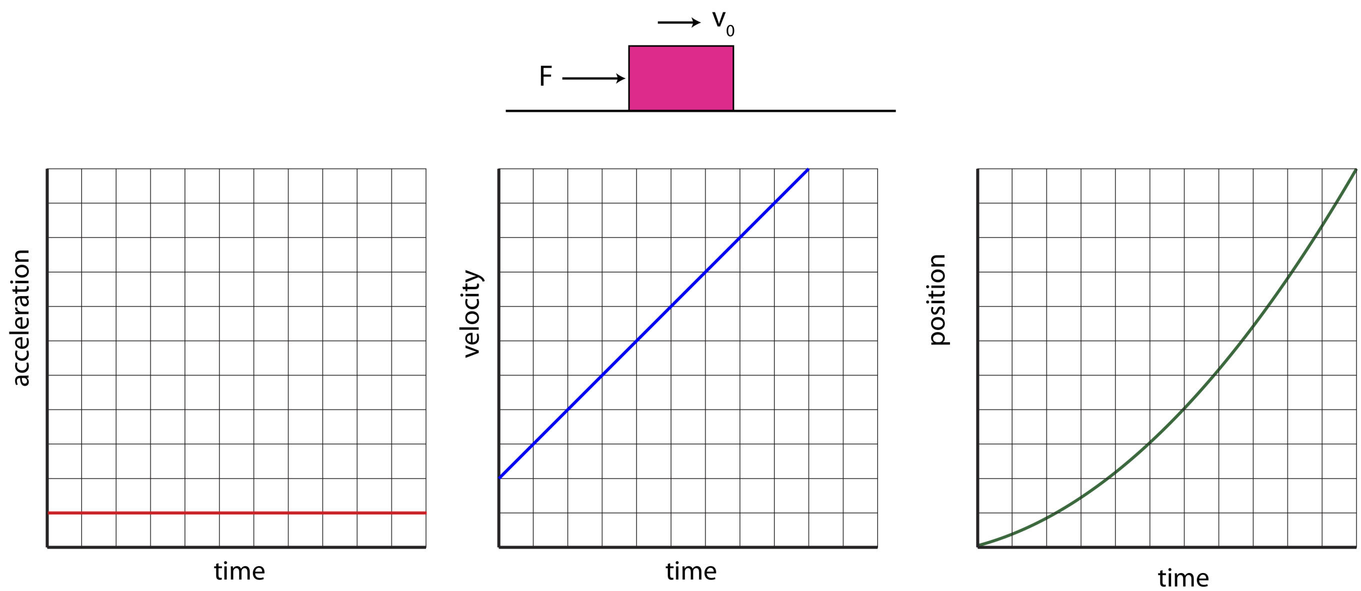
Force and Kinematics

Our focus so far has been on the details of force, and comparing the motion of an object before and after the force acted on the object, typically at two time instances. We will now look at the motion of an object for a continuous duration of time while a net force acts on the system or when the net force is zero. We first do this by graphically representing the time dependence of motion by analyzing acceleration, velocity, and position as a function of time. These three vectors are connected by the following equations that we have introduced in the earlier chapters:

$$\vec{v}=\frac{d\vec{x}}{dt};\quad \vec{a}=\frac{d\vec{v}}{dt} \tag{8.1.1}$$

We will see how to make sense of these equations graphically by looking at a few specific examples. Below are plots demonstrating motion of a box which is initially moving to the right with a net force also pointing to the right.

Figure 8.1.1: Force and Initial Velocity in the Same Direction



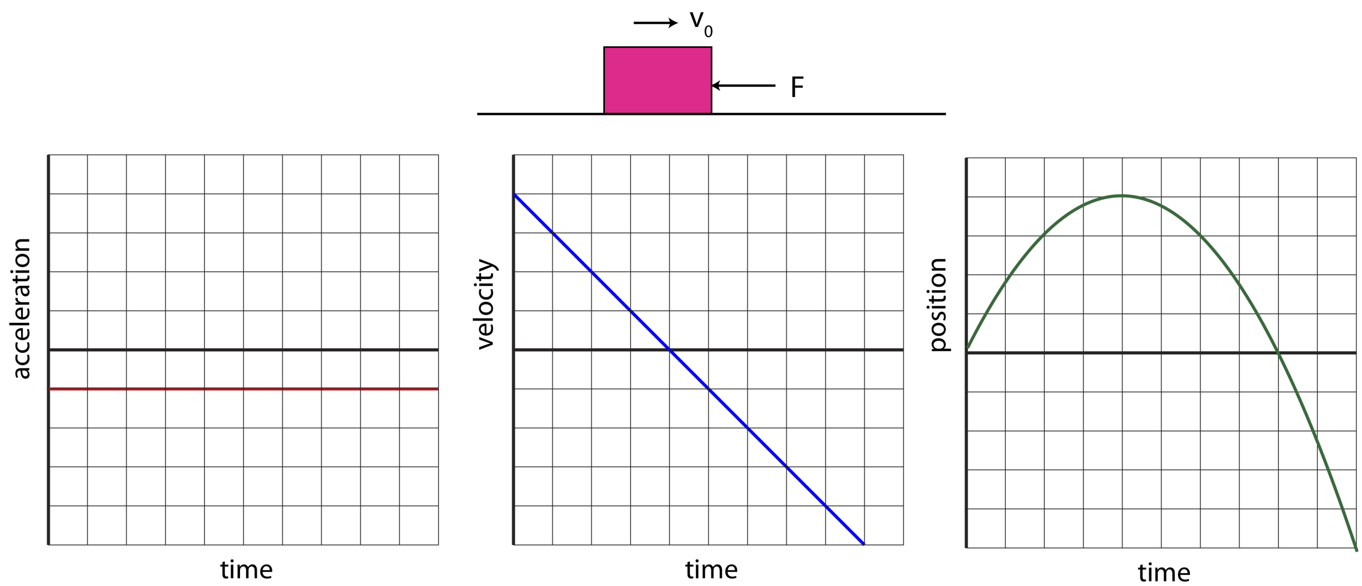
By convention we define to the right as positive. Then, acceleration will be positive as well according to Newton's second law since the net force is pointing to the right. We assume that the force is constant over this time range, resulting in acceleration being constant as well. Thus, acceleration plotted as a function of time is a horizontal line, as shown in Figure 8.1.1. The acceleration is arbitrarily chosen as 1 grid unit on this scale.

Equation 8.1.1 states that the slope of the velocity plot will give acceleration. This means that if acceleration is a constant, velocity must be linear as a function of time with a slope of 1 unit. This is consistent with Newton's second law which states that when there is a net force acting on the system, acceleration is non-zero, which implies that velocity is changing. There are infinite ways to have velocity changing with time with a slope of 1, since the line can intersect anywhere on the y-axis and have the same slope. Thus, another piece of information that is required to make the velocity versus time plot is the initial velocity at $t=0$. In the example in Figure 8.1.1 the initial velocity, v_o, is arbitrarily set to 2 units. From the velocity plot we can see that the box is speeding up, since its speed is increasing with time.

Lastly, Equation 8.1.1 states that the slope of position plot is the velocity plot. In this case since velocity is linear and increasing, this means that the slope of position plot increases with time, and it does so in a quadratic manner. Thus, the position versus time plot in Figure 8.1.1 has a parabolic shape. As for the velocity plot, there are infinite ways to have the have shape, since the plot can be moved vertically up or down while retaining its slope. So we need to define the origin in order to determine where the position crosses the y-axis. It is convenient to define that origin at initial time, so in this case the position is zero at $t=0$. The position versus time tells us that the object is moving to the right and speeding up since the slope is positive and increasing with time.

Let us now turn to another example where the force and the initial velocity point in opposite direction as shown in Figure 8.1.2 below. The box is still moving to the right initially, but the force now points to the left. We will again assume that the force is constant over the time range that we want to analyze the motion of the box.

Figure 8.1.2: Force and Initial Velocity in the Opposite Directions



In this example acceleration is negative since the net force is to the left. Again, we choose a magnitude of 1 unit on this scale to represent constant acceleration over this time range. This means that velocity will have a slope of negative 1 units. Again, we need to choose initial velocity, and in this case we choose 4 units as shown in Figure 8.1.2. The initial velocity is positive since the box is moving to the right, but we see from the plot that when the slope is negative, the velocity plot will cross the x-axis and will start increasing in the negative direction. This means that initially the box is slowing down as the magnitude of velocity decreases, since the force is in the opposite direction of position. But at the moment when the velocity plot crosses the x-axis (at 4 units of time), the box temporarily stops, after which it turns around and starts moving in the negative direction. Recall, since velocity is a vector, negative velocity means that its it moving in the negative direction. And since the velocity is getting more negative after the box turns around, the speed, which is the magnitude of velocity, is increasing, so the box is now speeding up. The box starts to speed up after 4 units of time, the motion of the box is now in the same direction as the force.

For the position plot, we set the origin at the initial time again as seen in Figure 8.1.2. The shape of the plot is again parabolic since its slope has to be linear based of the velocity plot. Initially, the slope is positive and decreasing, corresponding to the box moving to the right and slowing down. At time of 4 units the slope of the position plot is zero. This is the exact time when the velocity goes to zero before the box turns around. After this, the slope of the parabolic shape negative and increasing since the box is now moving to the left and speeding up. Note, that the position is not negative when the box starts moving to the left, since negative position just means that the box is to the left of the origin. Initially, the box moved to the right of the origin, as it was slowing down. When it first turned around and started moving to the left, it was still to the right of the origin until it returned back to its starting position, exactly at 8 units of time on the position plot. After this, the box is located to the left of the origin, thus, position is negative.

As you work with analysis of motion for different physical situations, here are a list of a few key points to keep in mind when making acceleration, velocity and position plots:

General:

- Acceleration, velocity, and position are vectors which can have positive, negative, or zero values depending on their direction and location of origin.
- If you know one of the graphs, you can obtain the other two as long as you know initial conditions.

Acceleration:

- Acceleration points in the same direction as the net force.

- Acceleration is zero when the net force is zero.
- Acceleration plot alone does not contain any information whether the object is speeding up or slowing down, since it does not tell you which way the object is moving, only in which direction its velocity is changing.

Velocity:

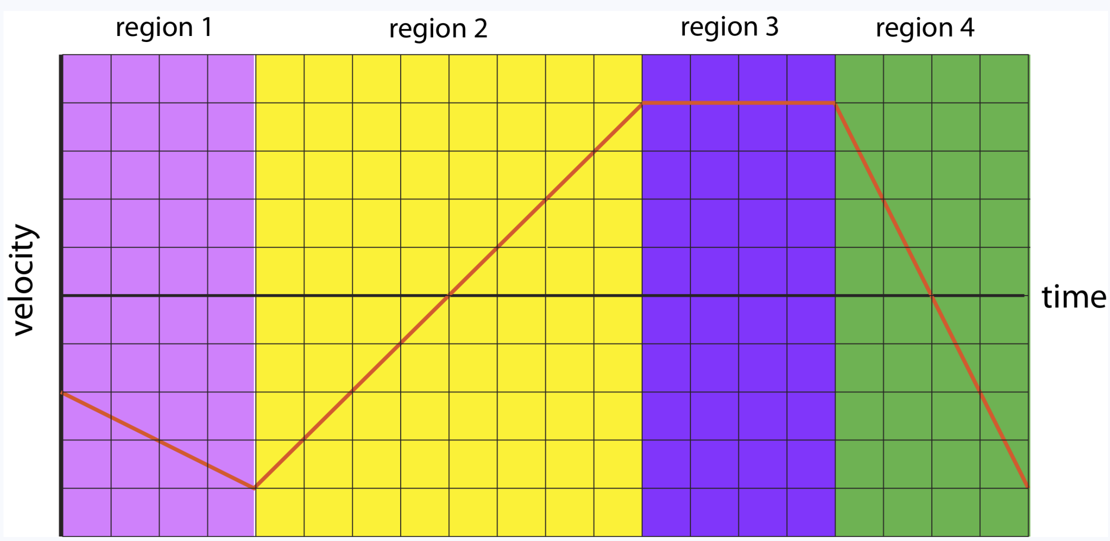
- The slope of the velocity plot is the acceleration.
- Initial velocity has to be known to make the velocity versus time plot.
- When acceleration and velocity have the same sign, the system is speeding up.
- When acceleration and velocity have opposite signs, the system is slowing down.
- When velocity plot crosses the x-axis (changes sign), the system is turning around.
- When the slope of velocity plot is zero, acceleration is zero, implying zero net force.

Position:

- The slope of the position plot is the velocity.
- Initial position has to be known to make the position versus time plot.
- Positive slope means the object is moving to the right.
- Negative slope means the object is moving to the left.
- Increasing magnitude of slope (either negative or positive) means the object is speeding up.
- Decreasing magnitude of slope (either negative or positive) means the object is slowing down.
- Zero slope means the object is stationary.
- When the sign of the slope of the position plot changes, this means the object has changed direction of motion.
- The sign of the position plot does not tell us about the direction of motion, but an indication of whether the object is located on the positive or negative side of the origin.

Example 8.1.1

Below is a velocity versus time plot.



a) For each marked region specify the direction of motion (right, left, or turning around) and describe the speed (zero, speeding up, slowing down, or constant speed).

b) Make an acceleration versus time plot for the entire time shown.

Solution

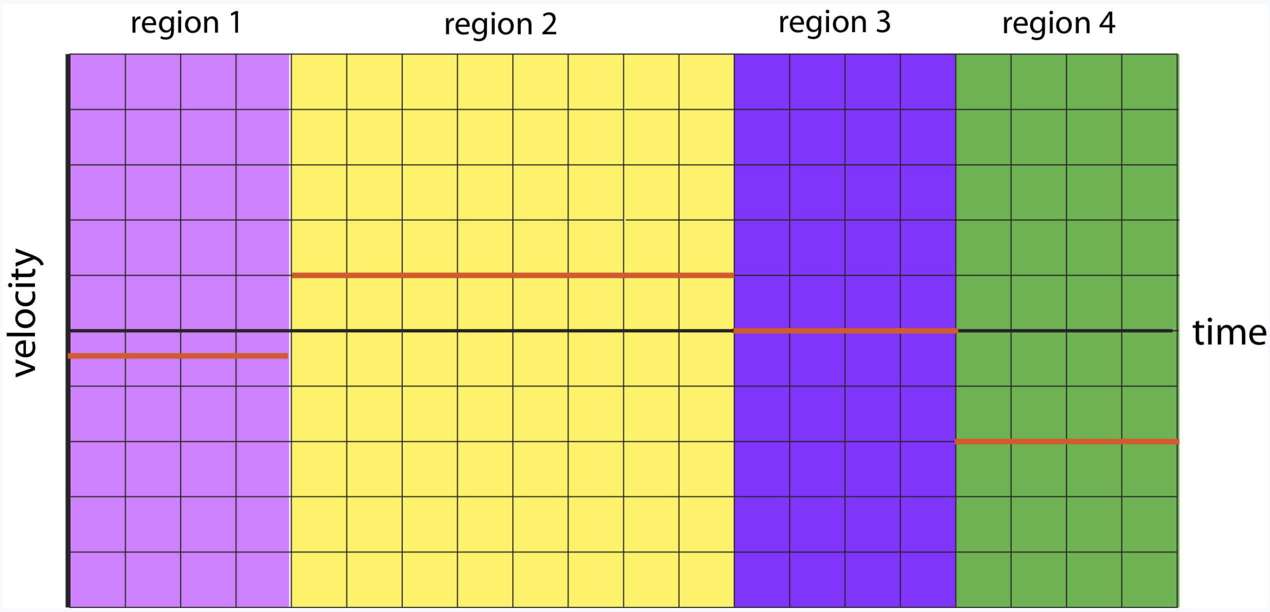
a) Region 1: object starts with non-zero velocity, is moving to the left since velocity is negative and speeding up since the magnitude of velocity is increasing.

Region 2: object is moving to the left and slowing down. After 4 units of time in this region, the object turns around, after which it's moving to the right and speeding up.

Region 3: object is moving to the right with constant speed.

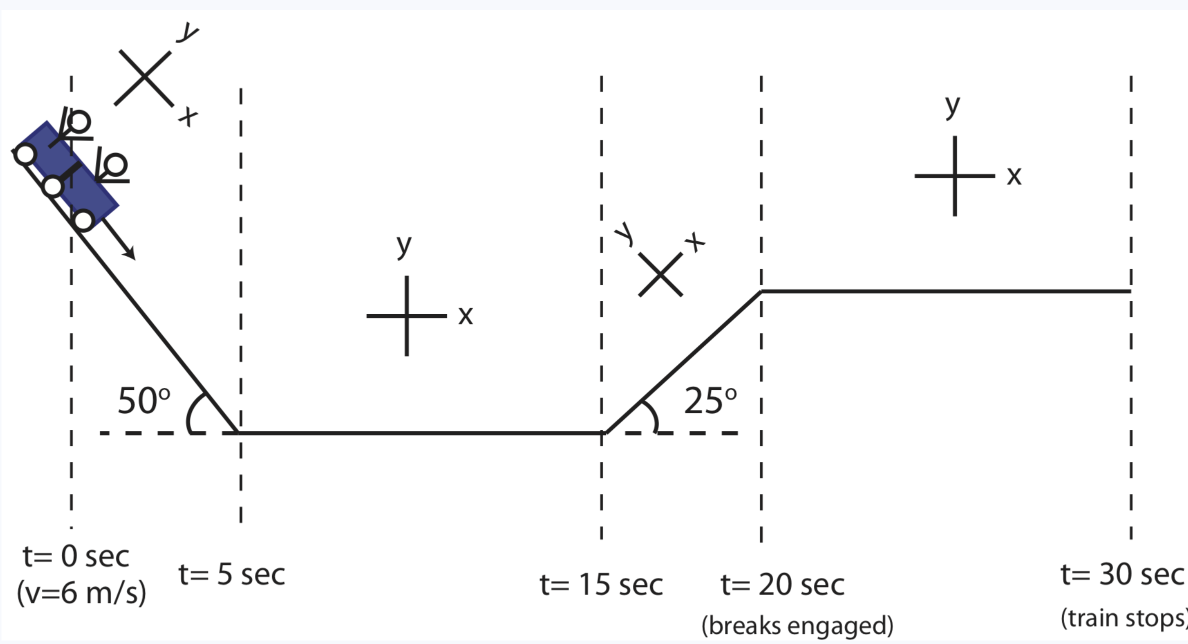
Region 4: object is moving to the right and slowing down, after 2 units of time it turns around, then moving to the left and speeds up.

b) Below is the acceleration versus time plot. The plots in each region are obtained from the slope of velocity versus time plot for each marked region: region 1 the slope is -1/2, in region 2 the slope is 1, in region 3 the slope is zero, and in region 4 the slope is-2.



Example 8.1.2

Shown below is a rollercoaster ride. At the start of a drop, defined as t=0 sec, the train is moving with a speed of 6 m/s. The rest of the motion is depicted in the picture below. Assume the track is frictionless from t=0 to t=20sec. Between t=20sec and 30sec, the breaks create friction with the tracks. For this problem assume that the x-axis always points horizontal to the track (in the direction of motion), and the y-axis points perpendicular to the track, as depicted in the picture.



a) Draw four force diagrams for the train at t=2 sec, t=12 sec, t=18 sec, and t=25 sec. For each force diagram, split the forces into x-components (horizontal to the track) and y-component (perpendicular to the track).

b) Make a plot for component of velocity and acceleration horizontal the track (x-direction as shown in the figure) as a function of time from t=0 to t=30 sec.

Solution

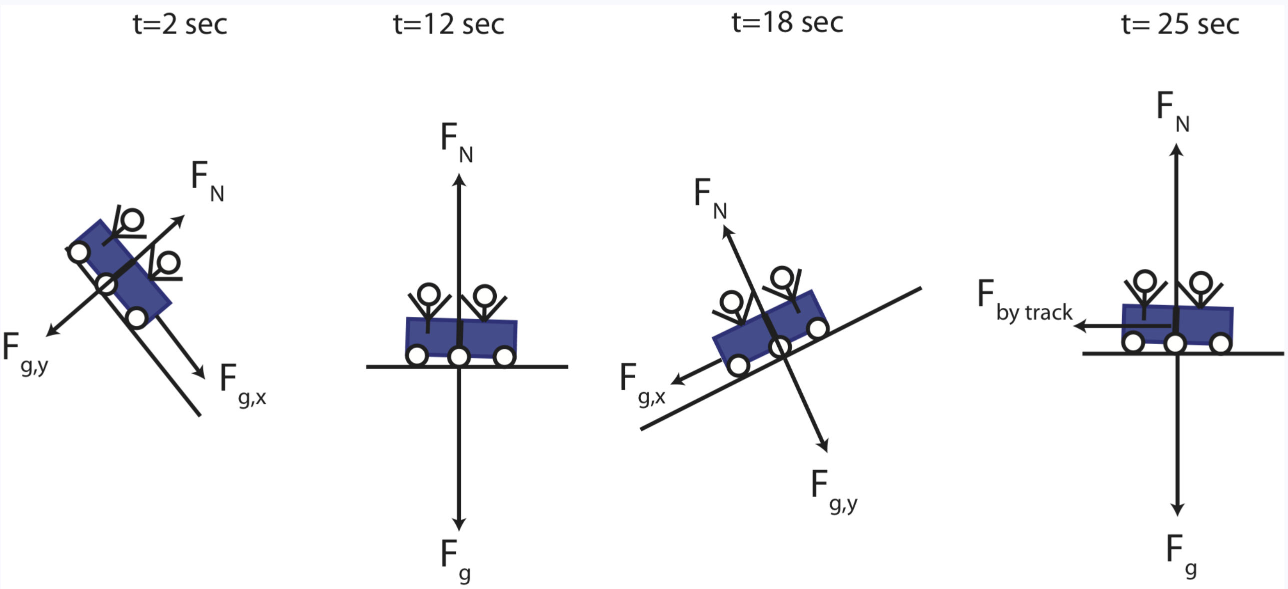
a) The free-body diagrams are shown below.

t=2 sec: in the first 5 seconds the track is frictionless, so there is only the force of gravity and the normal force. Gravity always points down, so it has a component along the track (x-direction in the tilted axis) and perpendicular to the track (y-direction in the tilted axis). The normal force is always perpendicular to the surface, thus it points in the y-direction. The y-component of gravity and the normal force must cancel since there is no motion in the y-direction. The net force is due to the force of gravity in the x-direction.

t=12 sec: the train is moving horizontally with zero net force, since there are only vertical forces present (friction is still zero in this region). The normal force and gravity are equal and opposite.

t=18 sec: the component of the gravitation force points in the negative x-direction. Also, since the slope of the track is less steep than at 2 sec, the component of gravity along the track is smaller, so acceleration will have a smaller magnitude compared to when the train was moving down during the first 5 seconds.

t=25 sec: the motion is also horizontal, but the breaks are engaged creating friction with the track pointing to the left as shown below.



b) For the first 5 seconds, to find acceleration you need to calculate the x-component of gravity using the angle provided in the figure:

$$a=\frac{F_{gx}}{m}=g\sin\theta=(9.8m/s^2)(\sin 50^\circ)=7.51m/s^2$$

The velocity is related to acceleration as:

$$a=\frac{\Delta v}{\Delta t}=\frac{v_f-v_i}{t_f-t_i}$$

The initial time is 0 seconds. Solving for v_f at the bottom of the ramp, when $t_f=5sec$:

$$v_f=v_i+at_f=(6m/s)+(7.51m/s^2)(5sec)=43.5m/s$$

Between 5 and 15 seconds acceleration is zero since there is no net force, so velocity remains constant at 43.5 m/s. Between 15 and 20 seconds the net forces points in the negative x-direction:

$$a=\frac{F_{gx}}{m}=-g\sin\theta=-(9.8m/s^2)(\sin 25^\circ)=-4.14m/s^2$$

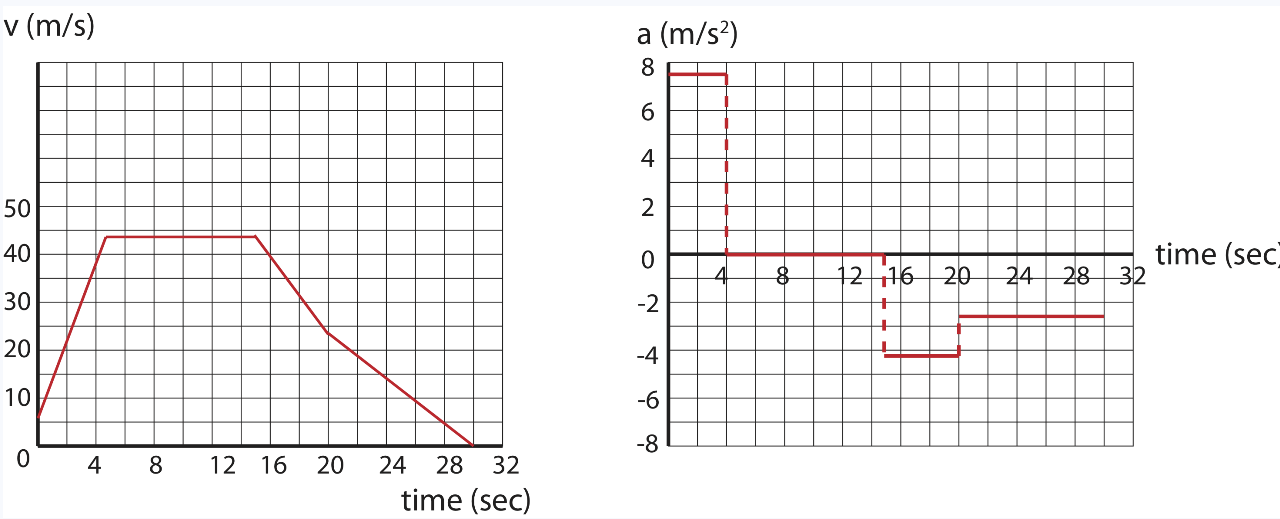
Solving for v_f at 20 seconds at the top of the ramp:

$$v_f=v_i+a(t_f-t_i)=(43.5m/s)+(-4.14m/s^2)(20sec-15sec)=22.8m/s$$

Between 20 and 30 seconds, we don't know the magnitude of the force, but we know that the train stops at 30 seconds:

$$a=\frac{v_f-v_i}{t_f-t_i}=\frac{0-22.8m/s}{30sec-20sec}=-2.28m/s^2$$

All of these calculations are summarized in the graphs below.



In this section we focused on depicting and interpreting motion graphically. In the next section, we will develop mathematical equations which will equivalently describe the motion and allow for more exact calculations of velocity and position at specific instances of time.

Contributors

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8.2: Kinematics

Constant Acceleration

We have mostly focused so far on the source of acceleration, the forces, and what happens after the force acted on a system. Now we will look at the details of what happens to the motion of a system during the interaction, knows as *kinematics*. We will look at equations of motion for a specific case of *constant acceleration*. For example, an object in free fall near the surface of the Earth experiences constant acceleration due to gravity. Many other physical situations can be modeled by assuming the acceleration is constant. In this model, there are simple algebraic expressions relating the motion variables that make it convenient to get quantitative answers. Specifically, we are interested to get expression of velocity and position as a function of time. Even when acceleration is not constant over the entire range of motion, it is often possible to model the motion by considering acceleration to be constant over select intervals of time.

The algebraic equations for constant acceleration can be obtained by starting with the definition of acceleration:

$$a=\frac{dv}{dt} \tag{8.2.1}$$

To obtain the dependence of velocity on time, we need to change the above equation into integral form, and then set a= constant. You can check the results by differentiating the resulting expression. Integrating from some initial time, t_o, to some later time, t, to solve for change in velocity over time:

$$\Delta v=\int_{t_o}^{t} adt \tag{8.2.2}$$

In general, when acceleration is not constant over time, the above integral can become complex. When acceleration is constant over time:

$$a(t)=a \tag{8.2.3}$$

then it simply comes out of the integral in Equation 8.2.2. Defining the change in velocity as, $\Delta v=v_f-v_o$, where v_o is the velocity at some initial time $t_o=0$, and v_f is velocity as time later time t. Equation 8.2.2 becomes:

$$v_f=v_o+at \tag{8.2.4}$$

We can follow a similar procedure to determine how position varies with time for the case of constant acceleration. Starting with definition of velocity:

$$v=\frac{dx}{dt} \tag{8.2.5}$$

and integrating to solve for change in position over time:

$$\Delta x=x_f-x_o=\int_{0}^{t} vdt \tag{8.2.6}$$

Plugging in Equation 8.2.4 into the above equation we get:

$$x_f=x_o+\int_{0}^{t}(v_o+at)dt \tag{8.2.7}$$

Integrating we obtain the following expression for position as a function of time:

$$x_f=x_o+v_ot+\frac{1}{2}at^2 \tag{8.2.8}$$

Equations 8.2.4 and 8.2.8 for velocity and position as a function of time can provide all the needed information about the motion of an object at constant acceleration. These *equations of motion* enable us to determine the position and velocity of an object at all times.

When we derived the equations of motion, we assume that the motion is one-dimensional. At the end of this section, you will see how to us equations of motion when the motion is in two-dimensions. But even with one-dimensional motion velocity have a direction which means that the numerical value of v can be either positive or negative depending on the direction of motion. The position, x, can also have an either positive or negative value, which will depend of the location of the object relative to the defined origin. Thus, it is important before starting to solve any kinematics problem to define the origin and assign a sign to each direction. By convention, typically to the right and upward from the origin is positive and to the left and downward from the origin is negative, but you might choose a different convention for convenience. There are many variables to keep track of, so let us categorize them.

The following variables are **constants**:

- x_o: the initial position of the system at $t=0$ which is based on the choice of origin. Often, it is convenient to define the origin at the location of the object at $t=0$, such that $x_o=0$.
- v_o: the initial velocity of the system at $t=0$, which can be zero, positive, or negative.
- a: the acceleration which is a constant by definition, since these equations of motion are only valid when acceleration is constant. It can also be zero, positive, or negative.

The following are **independent variables**:

- x_f: the position of the system at some later time t.
- v_f: the velocity of the system at some later time t.

The following is a **dependent variable**:

- t: some later time, $t>t_o$ during which one you aim to calculate x_f and v_f.

Although Equations 8.2.4 and 8.2.8 are sufficient to provide all the information about the kinematics of an object as constant acceleration, another equation is often used for convenience, since it does not contain the time variable. This equation has the following form:

$$v_f^2 - v_o^2 = 2a(x_f - x_o) \tag{8.2.9}$$

It it left up to you to show that the above equation comes from the two equations of motion in the following example.

Example 8.2.1

Show that Equation 8.2.9 comes directly from Equations 8.2.4 and 8.2.8.

Solution

In order to eliminate time from the combination of the two equations, let us solve for time using Equation 8.2.4and plug the resulting for t into Equation 8.2.8 We choose to do so in this order since Equation 8.2.4 depends only linearly on time. Solving to t using using Equation 8.2.4we get:

$$t = \frac{v_f - v_o}{a}$$

Plugging in the above result into Equation 8.2.8we get:

$$x_f = x_o + v_o\left(\frac{v_f - v_o}{a}\right) + \frac{1}{2}a\left(\frac{v_f - v_o}{a}\right)^2$$

Separating x and v on each side on the equation:

$$x_f - x_o = \frac{v_f v_o - v_o^2}{a} + \frac{1}{2a}(v_f^2 - 2v_f v_o + v_o^2)$$

Simplifying we arrive at:

$$x_f - x_o = \frac{v_f^2 - v_o^2}{2a}$$

And finally rearranging the above equation to obtain the form of Equation 8.2.9we arrive at:

$$v_f^2 - v_o^2 = 2a(x_f - x_o)$$

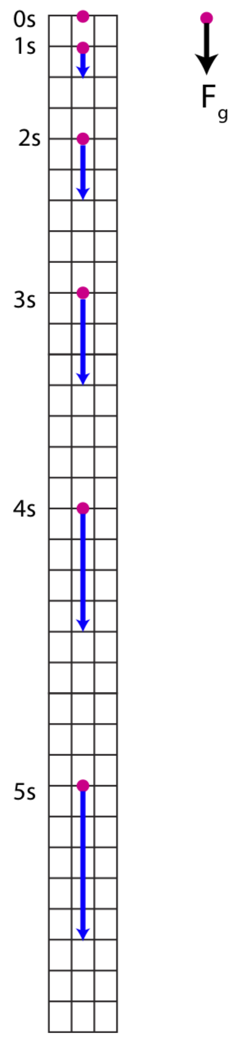
It is important to remember that these equations are valid when acceleration is constant which occurs when the net force is constant. A common example is free fall since the gravitational force on an object near the surface of the Earth is approximately constant. But even a freely falling object does not always experience constant acceleration when the effects of air friction become significant, such as a falling parachute. In that case, it is even possible to have zero acceleration, when air friction balances gravity, and the system reaches at *terminal velocity*. However, there are many situations where acceleration is never constant. A mass hanging on a spring oscillating up and down represents a common situation where the acceleration is certainly not constant. The force changes as the mass oscillates, changing direction to pull the system toward equilibrium and increasing in magnitude as the mass goes further from equilibrium.

One-Dimensional Motion

We will now look at an example of an object falling near the surface of the Earth, in the case when the motion is in one-dimension, vertical motion only. If air friction on the object is sufficiently small compared to the pull of the Earth, we can model the situation as if only the force of gravity acts on the object. The object is said to be in *free fall*. For any object in free fall on the Earth, regardless its mass, its acceleration is equal to the gravitational constant: $a = -g = -9.8 m/s^2$. The minus sign implies that acceleration is downward toward the center of the Earth, which by convention is a negative direction. In general, the object can have velocity of any direction, up, down, sideways, or something in between, but acceleration is always pointing down.

Figure 8.2.1 shows the free-body diagram for an object in free fall moving down. It also illustrates a diagram of position and velocity of the object at equal intervals of time.

Figure 8.2.1: An object in Free Fall



We assume that the object is stationary at $t=0$, right before it is dropped. The blue arrows pointing down in the diagram specify the objects. Based on Equation 8.2.4 the velocity of an object in free fall changes according to:

$$v=-gt \tag{8.2.10}$$

where we set $v_o=0$ for this example. Thus the velocity of the object increases linearly with time:

$$\begin{gathered} t=1s: \;\; v=-g \\ t=2s: \;\; v=-2g \\ t=3s: \;\; v=-3g \\ t=4s: \;\; v=-4g \\ t=5s: \;\; v=-5g \\ \ldots \end{gathered} \tag{8.2.11}$$

The linear increase in velocity is indicated by the lengths of the blue arrows in Figure 8.2.1, the length of the first arrow is 1 unit, the second one is 2 units, the third is 3 units, and so on.

Based on Equation 8.2.8 position changes quadratically with time:

$$x=-\frac{1}{2}gt^2 \tag{8.2.12}$$

where we set $x_o=0$ at the location of where the object is dropped, and $v_o=0$ as before. The position of the object will be negative since it's moving down below the origin. Writing out the quadratic behavior for the first 5 seconds we get:

$$\begin{gathered} t=1s: \;\; x=-\tfrac{1}{2}g \\ t=2s: \;\; x=-\tfrac{4}{2}g \\ t=3s: \;\; x=-\tfrac{9}{2}g \\ t=4s: \;\; x=-\tfrac{16}{2}g \\ t=5s: \;\; x=-\tfrac{25}{2}g \\ \ldots \end{gathered} \tag{8.2.13}$$

You can see this quadratic increase in position in Figure 8.2.1, since the dots representing the location of the object at equal intervals of time get further and further apart.

Since position depends quadratically on time, the quadratic equation is often needed to solve for time. Rewriting Equation 8.2.8 in the form of a quadratic equation we get:

$$\frac{1}{2}gt^2+v_ot+(x_o-x_f)=0 \tag{8.2.14}$$

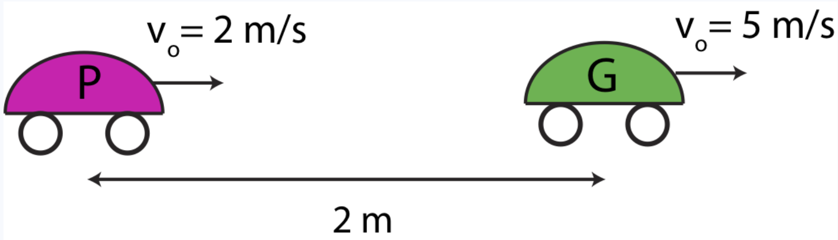
Applying the quadratic equation to solve for time:

$$t=\frac{-v_o\pm\sqrt{v_o^2-2g(x_o-x_f)}}{g} \tag{8.2.15}$$

Although, the quadratic equation has two solutions, both of them are not always valid. If one of the solutions results in negative time, then that solution is not physical since time is always positive. However, sometimes two solutions are possible. Imagine you throw an object upward, and you want to know how long it takes it to get to a specific height, which is less than its maximum height. In this case there are two possible answers: the time it takes for the object to get to that height on the way up, and then the other solution is time it takes for the object to go all the way up and back down to that specific height.

Example 8.2.2

The starting conditions of two cars at $t=0$ are shown in the figure below. The green car, G, starts out 2 m ahead of the pink car, P. At $t=0$ the pink car is moving to the right at 2 m/s, while the green car is moving to the right at 5 m/s. The pink car is accelerating to the right with magnitude of 6 m/s^2, while the green car is accelerating to the left with with magnitude of 4 m/s^2. Assume acceleration is constant for both cars. Define the origin at the location of the pink car at $t = 0$.



a) Calculate the time when the pink car catches up to the green one.

b) At which positions do the two cars meet?

Solution

a) We will use subscripts "P" and "G" to refer to the variables of the pink and green car, respectively. The pink car catches up to the green one when they are at the same location:

$$x_{f,P} = x_{f,G}$$

Using Equation 8.2.8for each car and setting them equal to each other we get:

$$x_{o,P} + v_{o,P}t + \frac{1}{2}a_P t^2 = x_{o,G} + v_{o,G}t + \frac{1}{2}a_G t^2$$

Moving all terms to the left-hand side of the equation and combining terms based on their dependence on t we get:

$$\frac{1}{2}(a_P - a_G)t^2 + (v_{o,P} - v_{o,G})t + (x_{o,P} - x_{o,G}) = 0$$

Let us now plug in all the constants for each car. By definition the pink car is at initially the origin,$x_{o,P} = 0$. If we define to the right as positive, then $a_P = 6\ \mathrm{m/s^2}$ and $a_G = -4\ \mathrm{m/s^2}$. Substituting these values and the initial velocities given into the last equation we get:

$$\frac{1}{2}(-6-4)t^2 + (2-5)t + (0-2) = 0$$

The above equation has the form of a quadratic equation:

$$0 = 5t^2 - 3t - 2$$

Solving the quadratic equation:

$$t = \frac{3 \pm \sqrt{3^2 - (4)(5)(-2)}}{(2)(5)} = \frac{3 \pm 7}{10} = 1, -\frac{2}{5}$$

Since time cannot be negative, the two cars meet at $t = 1s$.

b) Now we simply need to solve to the position of either car at $t = 1s$. Solving for the pink car we get:

$$x_{f,P} = x_{o,P} + v_{o,P}t + \frac{1}{2}a_P t^2 = (2m/s)(1s) + \frac{1}{2}(6m/s^2)(1^2 s^2) = 5m$$

To check that we get the same position for the green car at $t = 1s$:

$$x_{f,G} = x_{o,G} + v_{o,G}t + \frac{1}{2}a_G t^2 = 2m + (5m/s)(1s) + \frac{1}{2}(-4m/s^2)(1^2 s^2) = 5m$$

Thus, the two cars meet 5 m to the right of the pink car's initial position.

Two-Dimensional Motion

When the object is moving in two-dimensions, we still use the same kinematics equations developed above, treating each direction independently. These kinematic equations for each spatial direction are summarized in the table below.

Table 8.1.1: Kinematics Equations in 2D

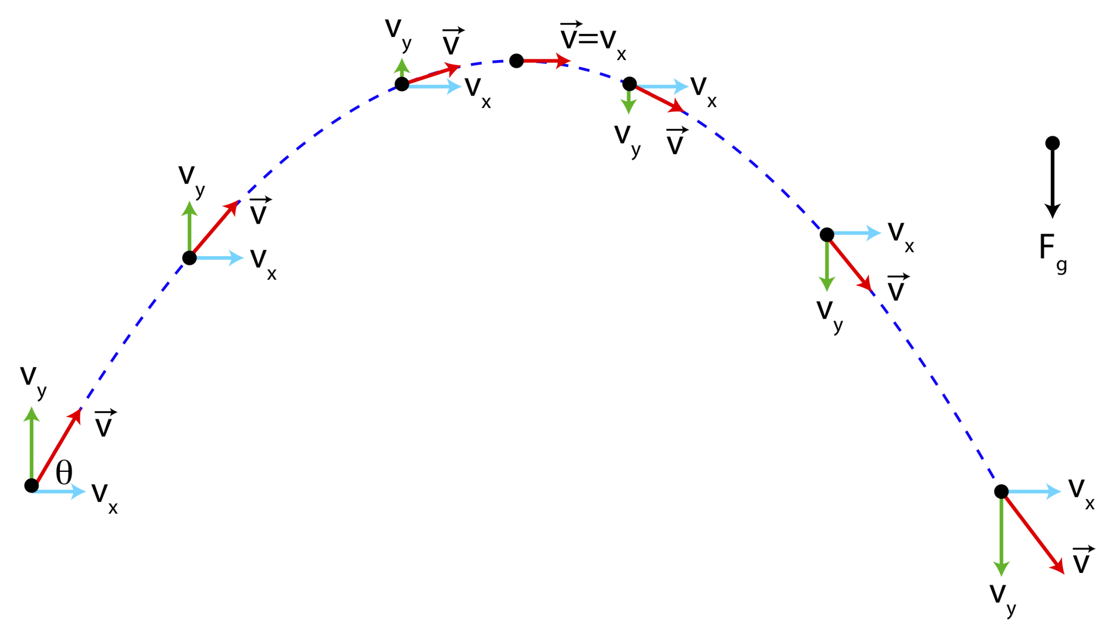
Description	Horizontal Direction, x	Vertical Direction, y
Constant Acceleration	$a_x(t) = a_x$	$a_y(t) = a_y$
Velocity as a function of time	$v_{f,x} = v_{o,x} + a_x t$	$v_{f,y} = v_{o,y} + a_y t$
Position as a function of time	$x_f = x_o + v_{o,x} t + \frac{1}{2} a_x t^2$	$y_f = y_o + v_{o,y} t + \frac{1}{2} a_y t^2$
Time independent equation	$v_{f,x}^2 - v_{o,x}^2 = 2a_x(x_f - x_o)$	$v_{f,y}^2 - v_{o,y}^2 = 2a_y(y_f - y_o)$

One common example of 2D motion is projectile motion, a object in free-fall whose initial velocity has a component in the horizontal direction. Since the object is in free-fall, it is still implies that the only force acting on it is the force of gravity pointing down. Thus, the equations in the table above can be simplified for projectile motion in the following way:

$$\begin{array}{ll} a_x = 0; & a_y = -g \\ v_{f,x} = v_{o,x}; & v_{f,y} = v_{o,y} - gt \\ x_f = x_o + v_{o,x} t; & y_f = y_o + v_{o,y} t - \frac{1}{2} g t^2 \\ v_{f,x} = v_{o,x}; & v_{f,y}^2 = v_{o,y}^2 - 2g(y_f - y_o) \end{array} \tag{8.2.16}$$

The equations above state the motion in the x-direction remains constant since there is no net force in the x-direction: the x-component of velocity is constant and position in the x-direction changes linearly with time. In the y-direction the object follows the standard free-fall motion, as depicted in Figure 8.2.1. Projectile motion is illustrated in Figure 8.1.2 below, where initially the object has velocity pointing northeast, with some angle θ above the positive x-axis.

Figure 8.1.2: Projectile Motion



Since there is no force in the x-direction, which means that the component of velocity, v_x, will be constant in this trajectory (light blue arrow marked v_x stays the same length throughout the motion). In the y-direction there is acceleration due to the force of gravity. Since the vertical component of initial velocity points up, the y-component of velocity will first decrease, then reach zero, and increase downward as the object starts moving down. Note, that this motion in the y-direction is identical to one-dimensional motion of an object being thrown directly upward. The fact that the object is also moving horizontally does not influence its vertical motion, since the two spatial direction are independent.

You can calculate the maximum height of the object in projectile motion using the last row in Equation 8.2.16. At maximum height the speed in the vertical direction is zero, $v_{f,y}=0$. Setting the initial height to zero, $y_o=0$ and solving for the final height we get:

$$y_{max}=\frac{v_{o,y}^2}{2g} \tag{8.2.17}$$

Recall from 7A, that this is the same result you would obtain from energy conservation:

$$\begin{gathered}\Delta KE+\Delta PE=0 \\ \tfrac{1}{2}m(v_{max}^2-v_o^2)+mg(y_{max}-y_o)=0 \\ -\tfrac{1}{2}v_o^2+gy_{max}=0\end{gathered} \tag{8.2.18}$$

The last line in the last equation is identical to the result in Equation 8.2.17. When it comes to calculating positions and speeds at different times, conservation of energy can give us the same results. However, conservation of energy model is limited since it cannot provide the details of the dynamics, such as the time it took for a certain motion or the direction of motion. Thus, we need to turn to Newton's Laws and kinematics to answer those questions.

For example in order to calculate the amount of time it takes for the object to return to its initial height, $y_f=y_o$, you need to use kinematics:

$$0=v_{o,y}t-\frac{1}{2}gt^2 \tag{8.2.19}$$

This equation have two solutions:

$$t=0, t=\frac{2v_{o,y}}{g}=\frac{2v_o\sin\theta}{g} \tag{8.2.20}$$

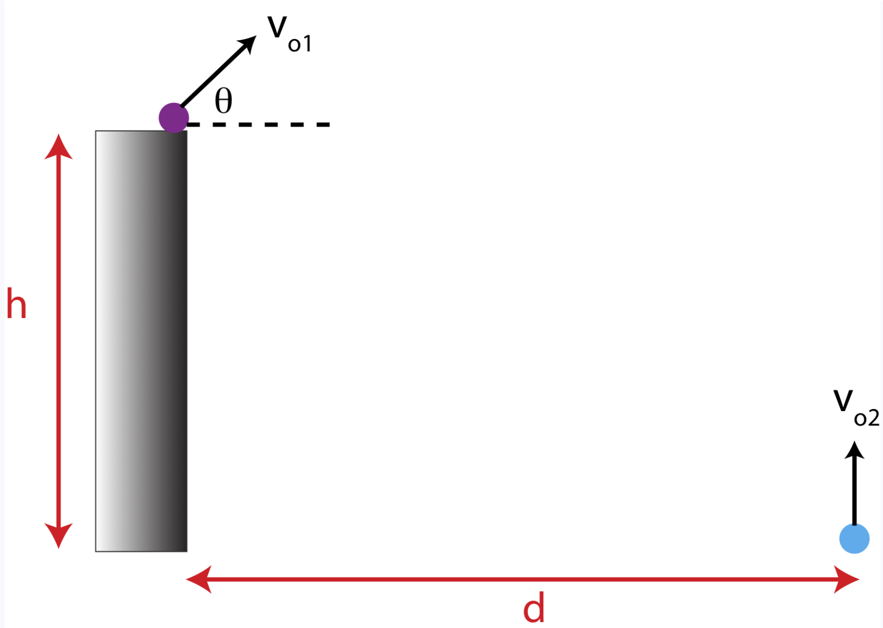
Of course, the $t=0$ solution just specifies the initial time, so the second solution tells us how long it would take to return to the same height. To obtain this time, we did not use any information about the motion in the x-direction. So if you throw an object straight up and another object at an angle with the same initial vertical speed, the two objects would land back on the ground at the same time even though one of the objects traveling in the horizontal direction as well. To calculate how far it would land from its initial position we can use the expression for position in Equation 8.2.16 in the x-direction and the time obtained in Equation 8.2.20:

$$x_f=x_o+v_{o,x}t=(v_o\cos\theta)\left(\frac{2v_o\sin\theta}{g}\right)=\frac{2v_o^2\cos\theta\sin\theta}{g}=\frac{v_o^2\sin(2\theta)}{g} \tag{8.2.21}$$

where in the last step we used a trigonometric identity. The equation above tells that that the faster the object is moving the further it will travel horizontally. If the angle is 90°, the motion is purely vertical, so the horizontal displacement will be zero. When the angle is 0°, again we get a displacement of zero since we are looking for "an arc" when we calculated the time for the object to come back to the same height. Of course, you can throw an object purely horizontally from above the ground, and it would land some distance away. But it would never be at the same height again, since that's equivalent to an object being dropped down.

Example 8.2.3

Below is a diagram ball 1 being thrown at an angle $\theta=40^\circ$ above the horizontal at a height of $h=1.5m$ above the ground. The initial speed of ball 1 is 10 m/s. Ball 2 is thrown from the ground directly upward, a distance $d=2m$ to the right of ball 1.



a) The two balls collide as ball 2 travels up. Find the time when the two balls collide, and the speed at which ball 2 is thrown upward.

b) If the two balls did not collide and ball 1 continued on its trajectory, calculate the velocity of ball 1 right before it reaches the ground.

Solution

The time at which the balls collide is the time that it takes the first ball to move by distance d horizontally. Setting $x=0$ *at the initial location of ball 1 we get:*

$$d=v_{o1,x}t=(v_{o1}\cos\theta)(t)$$

Solving for time:

$$t=\frac{d}{v_{o1}\cos\theta}=\frac{2m}{(10m/s)(\cos 40^{\circ})}=0.261s$$

The height of both balls at this time is:

$$\text{ball 1}:\quad y_1=h+v_{o1}\sin\theta t-\frac{1}{2}gt^2$$

$$\text{ball 2}:\quad y_2=v_{o2}t-\frac{1}{2}gt^2$$

Setting them equal to each other:

$$h+v_{o1}\sin\theta t-\frac{1}{2}gt^2=v_{o2}t-\frac{1}{2}gt^2$$

$$v_{o2}=\frac{h}{t}+v_{o1}\sin\theta=\frac{1.5m}{0.261s}+(10m/s)(\sin 40^{\circ})=12.2m/s$$

b) Since there is no force in the x-direction, the component of velocity in the x-direction will remain the same:

$$v_{fx}=v_{ox}=v_{o1}\cos\theta=(10m/s)(\cos 40^{\circ})=7.66m/s$$

We don't need to solve for time directly, thus, it is convenient to use the kinematics equation which does not depend on time to directly calculate the y-component of velocity when the ball reaches the ground, $y_f=0$*:*

$$v_{f,y}^2=v_{o,y}^2+2a_y(y_f-y_o)$$

Solving for $v_{f,y}$:

$$|v_{f,y}| = \sqrt{[(10m/s)(\sin 40^{\circ})]^2 + 2(-9.8m/s^2)(0-1.5m)} = 8.41m/s$$

This is the absolute value of the velocity, but the componet is negative since the ball is moving downward before it hits the ground:

$$\vec{v}_f = (7.66, -8.41)m/s$$

Solving for magnitude:

$$|\vec{v}_f| = \sqrt{7.66^2 + 8.41^2} = 11.4m/s$$

And angle:

$$\theta_f = \arctan\left(\frac{8.41}{7.66}\right) = 47.7^{\circ}$$

moving down in the southeast direction.

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8.3: Wrap-Up

In this chapter we have learned how to find the details, and particularly, the time dependence of interacting systems: we use Newton's second law. We know how motion is connected to forces. But as we see, things can get pretty "messy" very quickly, when we look at the details.

We previously mentioned some of the limitations of our models. As we go to atomic sized systems Newton's laws totally break down. Energy and momentum become quantized, but still exist and the idea of conservation exists. However, at atomic sizes we have to get a totally new model to replace the Newtonian model. As we get to systems the size of atoms and molecules, a different fundamental law of nature takes over. The various aspects of this very different law of nature are collectively known as quantum mechanics. One way to represent what is going on in quantum mechanics is with what are called wave packets (probability waves). Particles cease to have meaning in terms of what we normally think of as a particle. An electron is modeled not as a particle circling the nucleus, but is represented by a probability wave.

So, should we replace Newton's laws with wave mechanics (or any other representation of quantum mechanics)? No, not for large scale objects. The very peculiar aspects of quantum mechanics become totally insignificant as the size of the objects approach macroscopic size. In the limit of large scale objects, Newtonian mechanics is a particularly useful approximation (model) of how matter interacts with other matter. Is it a correct model? On a macroscopic level, the very weird quantum effects that the individual objects experience are simply not large enough to be observed when we look at the behavior of the macroscopic objects themselves (at least most of the time). Remember when we addressed the issue of models at the beginning of this course. We emphasized that we shouldn't think of models as being right or wrong, but rather as useful or not useful. This is probably a good way to think about Newton's laws. They are a very useful model of the interactions of (most) macroscopic objects as a whole.

Why do we insert the parenthetical "most" in the previous sentence? Because, even on a macroscopic level, sometimes quantum mechanics takes charge. Consider the phenomenon of superconductivity. This is truly a quantum mechanical effect on a macroscopic scale. The electrons going around the thousands of turns of wire in a superconducting magnet act in some ways as if they were one giant electron. And they go around the wire as if there were no electrical resistance! The currents in superconducting magnets can last months or even years. The magnetic field surrounding the space in a MRI machine (magnetic resonance imaging) is created by super currents in the superconducting magnet around the outside. The electrons act sort of like the entire magnet is one giant atom. But this only happens if the thermal energy is sufficiently low. Thermal energy is the "enemy" of macroscopic quantum effects. At the present time, these superconducting magnets must be kept at "helium temperatures", so called, because liquid helium is used as the refrigerant. At atmospheric pressure the boiling point of helium is 4.2 K, and this is the temperature of the magnet coils.

There is another limit to the applicability of Newton's laws that has nothing to do with size, but rather with speed. When objects move relative to each other with speeds that are not significantly less than the speed of light, very strange behavior begins to be observed. Clocks run slow. Distances shrink. We enter the strange world of special relativity. One thing Einstein's theory of special relativity, developed a hundred years ago, is telling us, is that our concepts of space and time are not "correct". That is, there really isn't a three dimensional space and an independent time. Rather, space and time get all tangled up. This, of course, plays havoc with Newton's laws! Every time you watch television, you are a witness to an example of one of the consequences of special relativity: namely, that nature has an ultimate speed limit of the speed of light: $3 \times 10^8 m/s$. The electrons that are "fired" out of the back end of the picture tube travel toward the phosphors on the screen at almost the speed of light. As energy is transferred to the electrons by an electric field, their energy increases, but pretty soon their velocity approaches the speed of light. Does this mean their kinetic energy can't increase further. No, it is as if their velocity tops out at just under the speed of light, but their *mass* increases. Pretty weird and fascinating stuff.

Index

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    - 5.3: Fluid Flow - *CC BY 4.0*
    - 5.4: Electric Circuits - *CC BY 4.0*
    - 5.5: Resistors in Parallel and Series - *CC BY 4.0*
    - 5.6: Circuit Problem Solving - *CC BY 4.0*
    - 5.7: The Linear Transport Model - *CC BY 4.0*
    - 5.8: Exponential Change Model - *CC BY 4.0*
    - 5.9: Exponential Fluid Flow - *CC BY 4.0*
    - 5.10: Exponential Charge Flow - *CC BY 4.0*
    - 5.11: Wrap up - *CC BY 4.0*
  - 6: Newton's Laws of Motion - *CC BY 4.0*
    - 6.0: Overview - *CC BY 4.0*
    - 6.1: Overview of Vectors - *CC BY 4.0*
    - 6.2: The Force model - *CC BY 4.0*
    - 6.3: Applying the Force Model - *CC BY 4.0*
    - 6.4 : Wrap Up - *CC BY 4.0*
  - 7: Momentum - *CC BY 4.0*
    - 7.0: Overview - *CC BY 4.0*
    - 7.1: Linear Momentum - *CC BY 4.0*
    - 7.2: Applications of Momentum Conservation - *CC BY 4.0*
    - 7.3: Angular Motion - *CC BY 4.0*
    - 7.4: Rotational Inertia - *CC BY 4.0*
    - 7.5: Torque - *CC BY 4.0*
    - 7.6: Static Equilibirium - *CC BY 4.0*
    - 7.7: Angular Momentum - *CC BY 4.0*
    - 7.8: Summary of Linear and Angular Analogs - *CC BY 4.0*
    - 7.9: Wrap-up - *CC BY 4.0*
  - 8: Force and Motion - *CC BY 4.0*
    - 8.0: Overview - *CC BY 4.0*
    - 8.1: Graphing Motion - *CC BY 4.0*
    - 8.2: Kinematics - *CC BY 4.0*
    - 8.3: Wrap-Up - *CC BY 4.0*
  - Agenda - *CC BY 4.0*
  - Back Matter - *CC BY 4.0*
    - Index - *CC BY 4.0*
    - Glossary - *Undeclared*
    - Detailed Licensing - *Undeclared*