3.4: Momentum 4-Vector

Construction of the Momentum 4-Vector

Back when we first discussed momentum, the form of the relativistic momentum was just given and shown to work. Here we will demonstrate the power of 4-vector thinking by deriving the momentum 4-vector, of which the momentum 3-vector is a part. The whole "derivation" consists of one trivial step. We get the momentum 4-vector from the velocity 4-vector by multiplying it by the mass. The mass is an invariant, so this will ensure that the 4-vector we construct has the necessary property – it has a magnitude that is measured to be the same by all observers. We see that making this definition results in momentum components in the spatial dimensions are exactly the "new" momentum we defined previously:

$$P \equiv mV \quad \leftrightarrow \quad \begin{pmatrix} \gamma_u mc \\ \gamma_u mu_x \\ \gamma_u mu_y \\ \gamma_u mu_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc \\ \gamma_u m \overrightarrow{u} \end{pmatrix} \tag{3.4.1}$$

The ct component of this 4-vector looks familiar. It is in fact common to write the components of the momentum 4-vector as:

$$P \quad \leftrightarrow \quad \begin{pmatrix} \frac{E}{c} \\ \overrightarrow{p} \end{pmatrix} \tag{3.4.2}$$

One compelling reason to write the matrix this way is that it works equally-well for photons as massive objects. But it also assists us in seeing the physical interpretation of momentum and energy according to relativity. Just as space and time need to be treated as different parts of the same whole we now call spacetime, so too are energy and momentum linked. We no longer see conservation of momentum and conservation of total energy as separate physical principals – they are just two parts of the single principle of conservation of 4-vector momentum.

Let's compute the magnitude-squared of this 4-vector, and let's do it the "easy way" – by doing it in the rest frame. After all, the magnitude computed in that frame will be the same in every other frame. Setting $\rightarrow u=0$ and $\gamma_u=1$ in the components of the momentum 4-vector above leaves only the ct component, making its square the magnitude-squared:

$$P \cdot P = m^2c^2 \tag{3.4.3}$$

If we now also compute the magnitude-squared the "hard way" by doing it in an arbitrary frame, we get:

$$P \cdot P = \left(\frac{E}{c}\right)^2 - \overrightarrow{p} \cdot \overrightarrow{p} = \frac{E^2}{c^2} - p^2 \tag{3.4.4}$$

Setting the "easy" and "hard" calculations equal gives us a familiar relation:

$$E^2 = p^2c^2 + m^2c^4 \tag{3.4.5}$$

Using the Momentum 4-Vector

We use the momentum 4-vector in precisely the same way that we use the momentum 3-vector (and in fact three quarters of this use is exactly this). That is, we add up all of the momenta in a system before a collision (which can include photons as well as massive particles), add it all up afterward, and set them equal to invoke the conservation principle. Of course, the presence of the γ_u's complicates things somewhat, as particles that collide will naturally change frames going from "before" to "after." On the other hand, the method of choosing a simpler frame to work in such as the system's overall rest frame (just as we occasionally employed the center-of-mass frame in the case of non-relativistic collisions) can be quite powerful in such problems. As with the position and velocity 4-vectors, the Lorentz transformation correctly translates the components of the momentum 4-vector between frames.