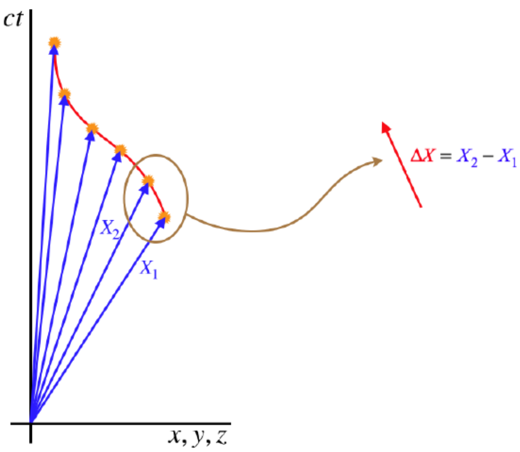
3.3: Velocity and Acceleration 4-Vectors

Calculus of 4-Vectors

There is not a lot we can do with just the position 4-vector. Just as we did in 9HA, we need to use the position vector to construct a velocity vector. We do it pretty much the same way, though as we will see, there is one additional detail that arises in relativity. An object that moves through spacetime is tracked by its position vector at each moment (spacetime event):

Figure 3.3.1 – Position 4-Vector Tracks an Object's World Line



The change in the position 4-vector is defined in the same way as it is for position 3-vectors – using tail-to-head addition. As before, we want the instantaneous rate of change of the position vector, so we choose a change that is infinitesimally-small. We then need to divide by an infinitesimally-small time interval, which is where relativity complicates things – which time do we use?

Different observers will have different spacetime diagrams, and will measure different coordinate time intervals between events. The choice is therefore clear – if we want to construct a velocity 4-vector that is universal (the vector, not its components!), we have to use the proper time between the two events. We don't have to worry about there being several proper time intervals between events, because these events are infinitesimally-close, making the change a straight line, even if the world line path is non-inertial. We therefore define the velocity 4-vector as:

$$V \equiv \frac{dX}{d\tau} \tag{3.3.1}$$

This process of constructing new 4-vectors from others by incorporating invariants is our go-to tactic. We can construct the acceleration 4-vector this way, and we will use this method to construct the momentum 4-vector in the next section.

Properties of Velocity 4-Vectors

Looking at how we constructed the velocity 4-vector, we see that the magnitude of the tiny displacement along the world line also happens to be the spacetime interval between the two nearby events. We therefore find that the magnitude of the velocity 4-vector is:

$$|V| = \frac{ds}{d\tau} = c \tag{3.3.2}$$

Thanks to the invariance of the interval and the proper time, every observer agrees on this magnitude. At first this must seem like a very strange result – every object's velocity 4-vector has the same magnitude, no matter what its world line looks like, and that magnitude is the speed of light?! This is only confusing until one gets accustomed to thinking about velocity 4-vectors differently from velocity 3-vectors. To demonstrate this difference and show this property more clearly, let's express the velocity 4-vector in a specific reference frame:

$$V = \frac{dX}{d\tau} \quad \leftrightarrow \quad \frac{d}{d\tau}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{pmatrix} \tag{3.3.3}$$

We want to express this 4-vector in terms of quantities measured in the frame, namely the 3-vector velocity $\overrightarrow{u}$ of the object being observed. The derivatives of the positions with respect to the proper time are not the components of the 3-vector velocity – for that we need derivatives with respect to coordinate time. To this end, we use the chain rule and the relation between proper time and coordinate time (time dilation):

$$\frac{dt}{d\tau} = \gamma_u \quad \Rightarrow \quad \frac{dx}{d\tau} = \frac{dx}{dt}\cdot\frac{dt}{d\tau} = u_x \cdot \gamma_u \tag{3.3.4}$$

Putting this into all the components of the velocity 4-vector matrix gives:

$$V \;\leftrightarrow\; \gamma_u \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} = \gamma_u \begin{pmatrix} c \\ \overrightarrow{u} \end{pmatrix} \tag{3.3.5}$$

Example 3.3.1

Use the velocity 4-vector matrix to show that its magnitude-squared is c^2 two different ways:

a. *Do it for an arbitrary reference frame.*
b. *Pick a convenient reference frame and argue why you can do this.*

Solution

a. The magnitude-squared of any vector is the dot product of that vector with itself, but for 4-vectors we have to be careful to incorporate the Minkowski metric, so following the matrix multiplication shown in Equation 3.2.3:

$$V\cdot V = \left(\gamma_u c \;\; \gamma_u u_x \;\; \gamma_u u_y \;\; \gamma_u u_z\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u z \end{pmatrix} = \gamma_u^2 \left(c^2 - u_x^2 - u_y^2 - u_z^2\right) = \gamma_u^2 \left(1 - \frac{u^2}{c^2}\right) c^2 = c^2 \tag{3.3.6}$$

b. If we choose the rest frame of the object in question, then its 3-velocity is zero, making $\gamma_u = 1$, and leaving the velocity 4-vector with only one component – the time component, equal to c, giving a magnitude squared of c^2. The magnitude of this 4-vector is an invariant, which means that all reference frames will get this same result.

Acceleration 4-Vectors

Knowing that every object's velocity 4-vector has the same magnitude, and that this magnitude remains the same for all time, may inspire us to ask about acceleration. On this count, there are two important considerations: First, objects can clearly change the magnitudes of their velocity 3-vectors (i.e. they can accelerate in the sense that we are used to) – it's just that the time component of their velocity 4-vectors will compensate for these changes such that the 4-vector magnitude remains fixed. Second, just because the magnitude of a velocity 4-vector doesn't change, it doesn't mean that its *direction* (in spacetime) doesn't. This can happen two distinct ways – the velocity 3-vector can change direction, or it can speed-up/slow down (or both, obviously). The first is an rotation in space, and the second is a "rotation" between the space and time components.

We can construct the acceleration 4-vector in the same manner that we constructed the velocity 4-vector – by taking a derivative with respect to proper time. To write this vector in terms of a column matrix gets significantly messier than it was for the velocity, because the derivative will now act on factors of γ_u present in the components of the velocity 4-vector that were not present in the position 4-vector. If we consider acceleration that is only along the direction of motion x (so the object is only speeding-up or slowing-down), then the magnitude of the (3-vector) acceleration is just the derivative of the magnitude of the (3-vector) velocity:

$$\overrightarrow{a}\,||\,\overrightarrow{u} \quad\Rightarrow\quad a = \frac{du}{dt} \tag{3.3.7}$$

This makes the derivative of γ_u equal to:

$$\frac{d\gamma_u}{dt} = \frac{d}{dt}\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{3}{2}} \frac{u}{c^2}\frac{du}{dt} = \frac{u}{c^2}\,\gamma_u^3\, a \tag{3.3.8}$$

Before proceeding, let's note the following useful identity:

$$1 + \frac{u^2}{c^2}\,\gamma_u^2 = \gamma_u^2 \tag{3.3.9}$$

Now we apply all of this to the construction of the acceleration 4-vector matrix:

$$A = \frac{dV}{d\tau} = \frac{dt}{d\tau}\cdot\frac{dV}{dt} \;\leftrightarrow\; \frac{dt}{d\tau}\cdot\frac{d}{dt}\begin{pmatrix} \gamma_u c \\ \gamma_u u \\ 0 \\ 0 \end{pmatrix} = \gamma_u \begin{pmatrix} \frac{u}{c}\,\gamma_u^3\, a \\ \frac{u^2}{c^2}\,\gamma_u^3\, a + \gamma_u a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{u}{c}\,\gamma_u^4\, a \\ \gamma_u^2\left(\frac{u^2}{c^2}\,\gamma_u^2 + 1\right) a \\ 0 \\ 0 \end{pmatrix} = \gamma_u^4\, a \begin{pmatrix} \frac{u}{c} \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.3.10}$$

Example 3.3.2

Derive the 4-vector acceleration components in terms of the 3-vector velocity and 3-vector acceleration for the more general case when these two 3-vectors are not parallel. [Note: You will need to write the u^2 that appears in γ_u as a dot product of the 3-vector velocity with itself, and then make use of the product rule on the dot product.]

Solution

Start with the definition of the acceleration 4-vector as the derivative of the 4-vector velocity with respect to proper time:

$$A = \frac{dV}{d\tau} = \frac{dt}{d\tau}\frac{dV}{dt} = \gamma_u \frac{dV}{dt}$$

Now we need to perform the derivative. The 4-vector velocity includes a factor of γ_u, which includes the quantity $u^2 = \vec{u}\cdot\vec{u}$, so let's do this little pieces at a time. The derivative of u^2 gives us the derivative of γ_u:

$$\frac{d}{dt}u^2 = \frac{d}{dt}\vec{u}\cdot\vec{u} = 2\vec{u}\cdot\frac{d\vec{u}}{dt} = 2\vec{u}\cdot\vec{a} \quad\Rightarrow\quad \frac{d}{dt}\gamma_u = \frac{d}{dt}\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}} = \frac{1}{2c^2}\left(1-\frac{u^2}{c^2}\right)^{-\frac{3}{2}}[2\vec{u}\cdot\vec{a}] = \gamma_u^3\frac{\vec{u}\cdot\vec{a}}{c^2}$$

Now we put this result into the full derivative of V:

$$A = \gamma_u\frac{d}{dt}\left[\gamma_u\begin{pmatrix} c\\ \vec{u}\end{pmatrix}\right] = \gamma_u\left[\frac{d\gamma_u}{dt}\begin{pmatrix} c\\ \vec{u}\end{pmatrix} + \gamma_u\frac{d}{dt}\begin{pmatrix} c\\ \vec{u}\end{pmatrix}\right] = \gamma_u^4\frac{\vec{u}\cdot\vec{a}}{c^2}\begin{pmatrix} c\\ \vec{u}\end{pmatrix} + \gamma_u^2\begin{pmatrix} 0\\ \vec{a}\end{pmatrix} = \begin{pmatrix} \gamma_u^4\frac{(\vec{u}\cdot\vec{a})}{c} \\ \frac{1}{c^2}\gamma_u^4\left(\vec{u}\cdot\vec{a}\right)\vec{u} + \gamma_u^2\vec{a}\end{pmatrix}$$

It is left as an exercise for the reader to show that this general result reduces to the result above when $\vec{u}$ is parallel to $\vec{a}$. [Note that this means $\vec{u}\cdot\vec{a} = ua$.]

Example 3.3.3

Show that the 4-vector acceleration is always perpendicular to the 4-vector velocity.

Solution

*There are no fewer than **three** good ways to solve this. Two of them require clever and powerful relativistic arguments, while the third method consists of brute force algebra. All three of these are satisfying in their own way...*

Method 1: using the invariant magnitude of V

The magnitude-squared of the 4-vector velocity is c^2, so:

$$0 = \frac{d}{d\tau}\left(c^2\right) = \frac{d}{d\tau}(V\cdot V) = 2\frac{dV}{d\tau}\cdot V = A\cdot V$$

Naturally, two vectors with a zero dot product are orthogonal.

Method 2: using the invariance of the dot product and a convenient frame

The dot product is invariant with respect to choice of frame, so if it is zero in one frame, it vanishes in all frames. Choose the rest frame, $\vec{u} = 0$, $\gamma_u = 1$:

$$V\cdot A = \begin{pmatrix} c\\ \vec{0}\end{pmatrix}\cdot\begin{pmatrix} 0\\ \vec{a}\end{pmatrix} = 0$$

Method 3: brute force

It is left as an exercise for the reader to perform the dot product between the 4-vector velocity given in Equation 3.3.5 and the 4-vector acceleration given in the result of Example 3.3.2. It is important to remember, however, that the Minkowski metric must be used in this dot product. That is, the product of the time components are multiplied by +1, and the products of the spatial components are multiplied by -1 before these are added together in the dot product.

Let's step back for a moment and consider the case of two observers: One is accelerating at a constant rate along the x-axis, while the other remains in an inertial frame. The person who is accelerating *knows* that they are doing so (they can do a test to see that they are not in an inertial frame), so how does the measurement of acceleration made by one observer relate to the measurement made by the other?

At first we might expect both observers to measure the same (3-vector) acceleration, but there is a major problem with this. There is no physical law that says that the observer in the accelerated frame can't keep accelerating indefinitely at the same rate – they just need to keep the rocket thrusters set at the same level for as long as they like. But the other observer cannot see this occur, or after a long enough period of time the speed of the other frame relative to theirs will exceed c, since $u(t) = at + u_o$. So the observer in the inertial frame must witness gradually *decreasing* acceleration while the accelerated observer observes constant acceleration. We can show this in a couple of ways.

First, we're not really equipped to talk about non-inertial frames, so let's change the situation to two inertial observers, each looking at the same object that is accelerating through space. One of these observers (we'll call it the primed frame), at the moment of observation, happens to be (momentarily) at rest relative to the object, while the other observer (the unprimed frame) sees the primed frame (and therefore momentarily the object) moving in the $+x$-direction at a speed u. The claim above is that these two observers cannot agree upon the 3-vector acceleration. In the primed frame, we have $u = 0$ and $\gamma_u = 1$, which gives the following components of 4-vector acceleration:

$$A \overset{u=0}{\leftrightarrow} \begin{pmatrix} 0 \\ a' \\ 0 \\ 0 \end{pmatrix} \tag{3.3.11}$$

Now use the fact that the magnitude-squared of this 4-vector is an invariant to compare the (3-vector) accelerations measured in the two frames. In the rest frame this is easy to compute:

$$A \cdot A = -a'^2 \tag{3.3.12}$$

In the unprimed frame we have:

$$A \cdot A = \gamma_u^8 \, a^2 \left(\frac{u^2}{c^2} - 1\right) = -\gamma_u^6 \, a^2 \tag{3.3.13}$$

Applying the invariance of the magnitude of 4-vectors means we can set these equal from the two frames, giving the simple result:

$$a' = \gamma_u^3 \, a \tag{3.3.14}$$

As expected, the magnitude of the acceleration measured in the inertial frame (a) is less than what is measured in the rest frame (a').

A second way to check this is to perform the Lorentz transformation on the 4-vector acceleration in one frame to get the 4-vector acceleration in the other, and then check the 3-vector transformation:

$$\gamma_u^4 \, a \begin{pmatrix} \frac{u}{c} \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_u & \frac{u}{c}\gamma_u & 0 & 0 \\ \frac{u}{c}\gamma_u & \gamma_u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ a' \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \gamma_u^3 \, a = a' \tag{3.3.15}$$

Example 3.3.4

Bob moves at a constant speed in a circle around Ann, who is in an inertial frame. Use the result of Example 3.3.2 to derive the relationship between the magnitude of the acceleration Bob that measures in his frame to the magnitude measured by Ann.

Solution

The answer comes out immediately by setting $\overrightarrow{u} \cdot \overrightarrow{a} = 0$, *and setting the two acceleration 4-vector magnitudes equal. With Bob being the primed frame and Ann the unprimed frame, we get:*

$$a' = \gamma_u^2 \, a$$

A Classic Example

With what we understand about acceleration 4-vectors, we can now work out a classic problem solved in most classes in special relativity. It involves the twin paradox, where one twin gets into a spaceship that is constructed to simulate close to the Earth's gravity by accelerating at a constant rate only slightly less than that of earth's gravity: $a = \frac{1.0 \; light-year}{year^2} = 9.5 \frac{m}{s^2} \approx g$. This twin takes a round-trip to the nearest star (approximately 4 light years distant) in this ship, increasing its speed for the first half of the trip to the star, decreasing it for the second half so that the ship stops at the star. Then the process is repeated for the return trip, speeding up then slowing down. Our goal is to determine how long this trip takes for the twin in the ship, and how long it takes for the twin on the Earth. [*Notice that this is a much more reasonable set of circumstances than what we have used for the twin paradox before now, where the acceleration was instantaneous.*]

We start by noting that all 4 legs of the trip (speeding up toward the star, slowing down toward the star, speeding up toward the Earth, and slowing down toward the Earth) are all going to give identical results for the times measured in each frame, as they are completely symmetric and the relative directions of the acceleration and velocity 3-vectors are irrelevant as long as they remain parallel. So we only need to do the calculation for the first leg and multiply the result by 4.

The acceleration in the frame of the spaceship is simply g, which means that at the moment during the trip when the spaceship is moving at speed u relative to the Earth, the twin on Earth measures an acceleration of:

$$a = \frac{g}{\gamma_u^3} = g\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}} \tag{3.3.16}$$

This acceleration is the rate at which the speed of the ship is changing, as measured by the earth:

$$\frac{du}{dt} = g\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}} \quad \Rightarrow \quad \int \left(1 - \frac{u^2}{c^2}\right)^{-\frac{3}{2}} du = \int g \, dt \tag{3.3.17}$$

These are not difficult integrals to perform. Noting that $u = 0$ at $t = 0$, we have:

$$\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}=gt \tag{3.3.18}$$

Solving for u gives:

$$u(t)=\frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}} \tag{3.3.19}$$

Integrating the velocity over the time of the trip gives the distance (halfway to the star):

$$\Delta x=\int_0^T \frac{gt\;dt}{\sqrt{1+\frac{g^2t^2}{c^2}}}=\frac{c^2}{g}\left(\sqrt{1+\frac{g^2T^2}{c^2}}-1\right) \quad\Rightarrow\quad T=\sqrt{\frac{2\Delta x}{g}+\frac{\Delta x^2}{c^2}} \tag{3.3.20}$$

Notice that if not for the effect of "slowing acceleration" seen from the earth frame, the time this leg of the trip takes would be found to be simply:

$$\Delta x=\frac{1}{2}gt^2 \quad\Rightarrow\quad t=\frac{2\Delta x}{g} \tag{3.3.21}$$

This only includes the first term in the square root given above. Plugging-in our values of $\Delta x=2\;ly$, $g\approx 1\frac{ly}{year^2}$, and of course $c=1\frac{ly}{year}$, and multiplying by 4 to get the full time, we get the time of the trip measured on the Earth:

$$T=16\sqrt{2}\;years\approx 23\;years \tag{3.3.22}$$

Now we need to calculate the time measured aboard the spaceship. The spaceship measures the proper time, which we can obtain from Equation 1.2.9, now that we know the speed of the ship as a function of time. Again, we compute one of the 4 intervals, and multiply by 4 to get the total time:

$$\Delta\tau=\frac{\Delta s}{c}=4\int_0^{\frac{T}{4}}\sqrt{1-\frac{u(t)^2}{c^2}}\;dt=4\int_0^{\frac{T}{4}}\sqrt{1-\frac{g^2t^2}{c^2+g^2t^2}}\;dt=4\int_0^{\frac{T}{4}}\frac{dt}{\sqrt{1+\frac{g^2}{c^2}\;t^2}} \tag{3.3.23}$$

At last we come upon an integral that is not a single simple substitution away from being solved, so we take the coward's way out and look it up in an integral table:

$$\int\frac{dx}{\sqrt{1+x^2}}=\ln\left[x+\sqrt{1+x^2}\right] \quad\Rightarrow\quad \Delta\tau=4\frac{c}{g}\;\ln\left[\frac{gT}{4c}+\sqrt{1+\frac{g^2T^2}{16c^2}}\right] \tag{3.3.24}$$

[*Note: Most folks use the identity:* $\sinh^{-1}x=\ln\left[x+\sqrt{1+x^2}\right]$ *to reduce the space required to write this solution.*]

Plugging in for T, g and c and noting that $\frac{c}{g}\approx 1\;year$, gives:

$$\Delta\tau=(4\;years)\ln\left[4\sqrt{2}+\sqrt{1+32}\right]=9.7\;years \tag{3.3.25}$$

It takes less than half as much time to make the round trip for the twin aboard the spacetime as elapses on Earth.