# 6.1: Force

As we saw in the previous chapter, when an interaction can be described by a potential energy function, it is possible to use this to get a full solution for the motion of the objects involved, at least in one dimension. In fact, energy-based methods (known as the Lagrangian and Hamiltonian methods) can be also generalized to deal with problems in three dimensions, and they also provide the most direct pathway to quantum mechanics and quantum field theory. It might be possible to write an advanced textbook on classical mechanics without mentioning the concept of force at all.

On the other hand, as you may have also gathered from the example I worked out at the end of the previous chapter (section 5.7), solving for the equation of motion using energy-based methods may involve somewhat advanced math, even in just one dimension, and it only gets more complicated in higher dimensions. There is also the question of how to deal with interactions that are not conservative (at the macroscopic level) and therefore cannot be described by a potential energy function of the macroscopic coordinates. And, finally, there are specialized problem areas (such as the entire field of statics) where you actually *want* to know the forces acting on the various objects involved. For all these reasons, the concept of force will be introduced here, and the next few chapters will illustrate how it may be used to solve a variety of elementary problems in classical mechanics. This does not mean, however, that we are going to forget about energy from now on: as we will see, energy methods will continue to provide useful shortcuts in a variety of situations as well.

We start, as usual, by considering two objects that form an isolated system, so they interact with each other and with nothing else. As we have seen, under these circumstances their individual momenta change, but the total momentum remains constant. We are going to take the *rate of change* of each object's momentum as a measure of the *force* exerted on it by the other object. Mathematically, this means we will write for the *average force exerted by 1 on 2* over the time interval $\Delta t$ the expression

$$\left(F_{12}\right)_{av}=\frac{\Delta p_{2}}{\Delta t}. \tag{6.1.1}$$

Please observe the notation we are going to use: the subscripts on the symbol $F$ are in the order "by,on", as in "force exerted by" (object identified by first subscript) "on" (object identified by second subscript). (The comma is more or less optional.)

You can also see from Equation (6.1.1) that the SI units of force are kg·m/s$^2$. This combination of units has the special name "newton," and it's abbreviated by an uppercase N.

In the same way as above, we can write the average force exerted by object 2 on object 1:

$$\left(F_{21}\right)_{av}=\frac{\Delta p_{1}}{\Delta t} \tag{6.1.2}$$

and we know, by conservation of momentum, that we must have $\Delta p_1=-\Delta p_2$, so we get our first important result,

$$\left(F_{12}\right)_{av}=-\left(F_{21}\right)_{av}. \tag{6.1.3}$$

That is, *whenever two objects interact, they always exert equal* (in magnitude) *and opposite* (in direction) *forces on each other*. This is most often called **Newton's third law** of motion, or informally "the law of action and reaction."

We might as well now proceed along familiar lines and take the limit of Eqs. (6.1.1) and (6.1.2) above, as $\Delta t$ goes to zero, in order to introduce the more general concept of the instantaneous force (or just the "force," without any further qualifiers). We then get

$$\begin{aligned}F_{12}&=\frac{dp_{2}}{dt}\\ F_{21}&=\frac{dp_{1}}{dt}\end{aligned} \tag{6.1.4}$$

and, since Equation (6.1.3) should hold for a time interval of any size,

$$F_{12}=-F_{21}. \tag{6.1.5}$$

Now, under most circumstances the mass of, say, object 2 will not change during the interaction, so we can write

$$F_{12}=\frac{d}{dt}\left(m_{2}v_{2}\right)=m_{2}\frac{dv_{2}}{dt}=m_{2}a_{2}. \tag{6.1.6}$$

This is the result that we often refer to as "$F=ma$", also known as **Newton's second law** of motion: *the (net) force acting on an object is equal to the product of its inertial mass and its acceleration*. The formulation in terms of the rate of change of momentum, as in Eqs. (6.1.4), is, however, somewhat more general, so it is technically preferred, even though this semester we will directly use $F=ma$ throughout.

If you want an example of a physical situation where $F=dp/dt$ is *not* equivalent to $F=ma$, consider a system where object 1 is a rocket, including its fuel, and "object" 2 are the gases ejected by the rocket. In this case, the mass of both "objects" is constantly changing, as the fuel is burned and more gases are ejected, and so the more general form $F=dp/dt$ needs to be used to calculate the force on the rocket (the thrust) at any given time.

At this point you may be wondering just what is Newton's *first law*? It is just the law of inertia: an object on which no force acts will stay at rest if it is initially at rest, or will move with constant velocity.

## Forces and Systems of Particles

What if you had, say, three objects (let us make them "particles," for simplicity), all interacting with one another? In physics we find that all our interactions are pairwise additive, that is, we can write the total potential energy of the system as the sum of the potential energies associated with each pair of particles separately. As we will see in a moment, this means that the corresponding forces are additive too, so that, for instance, the total force on particle 1 could be written as

$$F_{all,1}=F_{21}+F_{31}=\frac{dp_1}{dt}. \tag{6.1.7}$$

Consider now the most general case of a system that has an arbitrary number of particles, and is *not* isolated; that is, there are other objects, outside the system, that exert forces on some or all of the particles that make up the system. We will call these *external forces*. The sum of all the forces (both internal and external) acting on all the particles will take a form like this:

$$F_{\text{total}}=F_{\text{ext},1}+F_{21}+F_{31}+\ldots+F_{\text{ext},2}+F_{12}+F_{32}+\ldots+\ldots=\frac{dp_1}{dt}+\frac{dp_2}{dt}+\ldots \tag{6.1.8}$$

where $F_{ext,1}$ is the sum of all the external forces acting on particle 1, and so on. But now, observe that because of Newton's third law, Equation (6.1.5), for every term of the form $F_{ij}$ appearing in the sum (6.1.8), there is a corresponding term $F_{ji}=-F_{ij}$ (you can see this explicitly already in Equation (6.1.8) with $F_{12}$ and $F_{21}$), so all those terms (which represent all the internal forces) are going to cancel out, and we will be left only with the sum of the external forces:

$$F_{ext,1}+F_{ext,2}+\ldots=\frac{dp_1}{dt}+\frac{dp_2}{dt}+\ldots \tag{6.1.9}$$

The left-hand side of this equation is the sum of all the external forces; the right-hand side is the rate of change of the total momentum of the system. But the total momentum of the system is just equal to $Mv_{cm}$ (compare Equation (3.3.4), in the "Momentum" chapter). So we have

$$F_{ext,all}=\frac{dp_{sys}}{dt}=\frac{d}{dt}(Mv_{cm}). \tag{6.1.10}$$

This extends a previous result. We already knew that in the absence of external forces, the momentum of a system remained constant. Now we see that the system's momentum responds to the net external force as if the whole system was a single particle of mass equal to the total mass $M$ and moving at the center of mass velocity $v_{cm}$. In fact, assuming that $M$ does not change we can rewrite Equation (6.1.10) in the form

$$F_{\text{ext },all}=Ma_{cm}. \tag{6.1.11}$$

where $a_{cm}$ is the acceleration of the center of mass. This is the key result that allows us to treat extended objects as if they were particles: as far as the motion of the center of mass is concerned, all the internal forces cancel out (as we already saw in our study of collisions), and the point representing the center of mass responds to the sum of the external forces as if it were just a particle of mass $M$ subject to Newton's second law, $F=ma$. The result (6.1.11) applies equally well to an extended solid object that we choose to mentally break up into a collection of particles, as to an actual collection of separate particles, or even to a collection of separate extended objects; in the latter case, we would just have each object's motion represented by the motion of its own center of mass.

Finally, note that all the results above generalize to more than one dimension. In fact, forces are *vectors* (just like velocity, acceleration and momentum), and all of the above equations, in 3 dimensions, apply separately to each vector component. In one dimension, we just need to be aware of the sign of the forces, whenever we add several of them together.