9.1: Rotational Kinetic Energy, and Moment of Inertia

If a particle of mass m is moving on a circle of radius R, with instantaneous speed v, then its kinetic energy is

$$K_{rot} = \frac{1}{2}mv^2 = \frac{1}{2}mR^2\omega^2 \tag{9.1.1}$$

using $|\vec{v}| = R|\omega|$, Equation (8.4.12). Note that, at this stage, there is no real reason for the subscript "rot": equation (9.1.1) is all of the particle's kinetic energy. The distinction will only become important later in the chapter, when we consider extended objects whose motion is a combination of translation (of the center of mass) and rotation (around the center of mass).

Now, consider the kinetic energy of an extended object that is rotating around some axis. We may treat the object as being made up of many "particles" (small parts) of masses m_1, m_2 .... If the object is rigid, all the particles move together, in the sense that they all rotate through the same angle in the same time, which means they all have the same angular velocity. However, the particles that are farther away from the axis of rotation are actually moving faster—they have a larger v, according to Equation (8.4.12). So the expression for the total kinetic energy in terms of all the particles' speeds is complicated, but in terms of the (common) angular velocity is simple:

$$\begin{aligned} K_{rot} &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \dots \\ &= \frac{1}{2}\left(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots\right)\omega^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned} \tag{9.1.2}$$

where r_1, r_2,... represent the distance of the 1st, 2nd... particle to the axis of rotation, and on the last line I have introduced the quantity

$$I = \sum_{\text{all particles}} mr^2 \tag{9.1.3}$$

which is usually called the *moment of inertia* of the object about the axis considered. In general, the expression (9.1.3) is evaluated as an integral, which can be written symbolically as $I = \int r^2 dm$; the "mass element" dm can be expressed in terms of the local density as ρdV, where V is a volume element. The integral is a multidimensional integral that may require somewhat sophisticated calculus skills, so we will not be calculating any of these this semester; rather, we will rely on the tabulated values for I for objects of different, simple, shapes. For instance, for a homogeneous cylinder of total mass M and radius R, rotating around its central axis, $I = \frac{1}{2}MR^2$; for a hollow sphere rotating through an axis through its center, $I = \frac{2}{3}MR^2$, and so on.

As you can see, the expression (9.1.2) for the kinetic energy of a rotating body, $\frac{1}{2}I\omega^2$, parallels the expression $\frac{1}{2}mv^2$ for a moving particle, with the replacement of v by ω, and m by I. This suggests that I is some sort of measure of a solid object's rotational inertia, by which we mean the resistance it offers to being set into rotation about the axis being considered. We will see later on, when we introduce the torque, that this interpretation for I is indeed correct.

It should be stressed that the moment of inertia depends, in general, not just on the shape and mass distribution of the object, but also on the axis of rotation. In general, the formula (9.1.3) shows that, the more mass you put farther away from the axis of rotation, the larger I will be. Thus, for instance, a thin rod of length l has a moment of inertia $I = \frac{1}{12}Ml^2$ when rotating around a perpendicular axis through its midpoint, whereas it has the larger $I = \frac{1}{3}Ml^2$ when rotating around a perpendicular axis through one of its endpoints.