11.6: Advanced Topics

Mass on a Spring Damped By Friction with a Surface

Consider the system depicted in Figure 11.2.1 in the presence of friction between the block and the surface. Let the coefficient of kinetic friction be μ_k and the coefficient of static friction be μ_s. As usual, we will assume that $\mu_s \geq \mu_k$.

As the mass oscillates, it will experience a kinetic friction force of magnitude $F^k = \mu_k mg$, in the direction opposite the direction of motion; that is to say, a force that changes direction every half period. As shown in section 11.2, this force does not change the frequency of the motion, but it displaces the equilibrium position in the direction of the force, which is to say, closer to the starting point for each half-swing. As a result of that, the amplitude for each half-swing is reduced from the previous one.

Let the original equilibrium position (in the absence of friction) be $x_0 = 0$. Suppose we displace the mass a distance A to the right (call this position, the starting point for the first half-swing, $x_1 = A$), and let go. In the presence of friction, the equilibrium position for this first half-swing becomes the point $x_0' = F^k/k = \mu_k mg$, so the real amplitude of this first half-oscillation will be $A_1 = x_1 - x_0' = A - x_0'$, and the resulting motion will be

$$x(t) = x_0' + A_1 \cos(\omega t) \quad \left(\text{first half-period, } 0 \leq t \leq \pi/\omega\right). \tag{11.6.1}$$

Continuing the process, we see that $A_1 = A - x_0'$, $A_2 = A - 3x_0'$, $A_3 = A - 5x_0' \ldots A_n = A - (2n-1)x_0'$. Of course, this can't go on forever, since we require the amplitude to be a positive quantity; so the motion will consist of only n half-periods, where n is an integer such that $A - (2n-1)x_0' > 0$ but $A - (2n+1)x_0' < 0$. (That is to say, n is equal to the integer part of $\left(A/x_0' + 1\right)/2.$).)

The figure shows an example of how this would go, for the following choice of parameters: period T = 1 s, μ_k = 0.1, and A = 0.18 m. Note that, since x_0' depends only on the ratio $m/k = 1/\omega^2$, there is no need to specify m and k separately. We get $\omega = 2\pi/T = 2\pi$ rad/s, $x_0' = \mu_k g/\omega^2$ = 0.0248 m, and $\left(A/x_0' + 1\right)/2 = 4.13$, which means that the motion will go on for 4 half-periods before stopping.

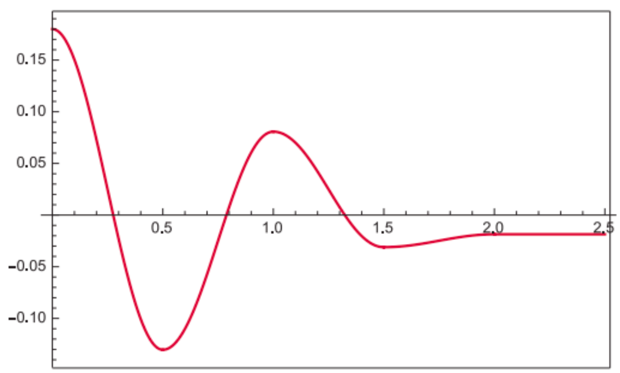

Figure 11.6.1: Damped oscillations.

Note that, in general, the oscillator does not stop at the equilibrium position. Rather, its final position will be at the end of the last half-swing, which is either $x_0' - A_n$ (if the number n of half-periods is odd), or $-x_0' + A_n$, if the number n is even. Either way, at that point the spring will be exerting a force of magnitude

$$F^{spr} = k\left|x_0' - A_n\right| = k\left|x_0' - A + (2n-1)x_0'\right| = k\left|A - 2nx_0'\right| < kx_0' = F^k. \tag{11.6.2}$$

Since we expect the force of static friction, F^s, to be greater than F^k, this tells us that at this point the spring is not exerting enough force to get the mass to move again.

Note: Just for the record, this is *not* the way dissipation in simple harmonic motion is usually handled. The conventional thing is to assume a damping force that is proportional to the oscillator's velocity. You will almost certainly see this more standard approach (which leads to a relatively simple differential equation) in some later course.

The Cavendish Experiment- How to Measure G with a Torsion Balance

Suppose that you want to try and duplicate Cavendish's experiment to measure directly the gravitational force between two masses (and hence, indirectly, the value of G). You take two relatively small, identical objects, each of mass m, and attach them to the ends of a rod of length l (let us say the mass of the rod is negligible, for simplicity), making a sort of dumbbell; then you suspend this from the ceiling, by the midpoint, using a nylon line.

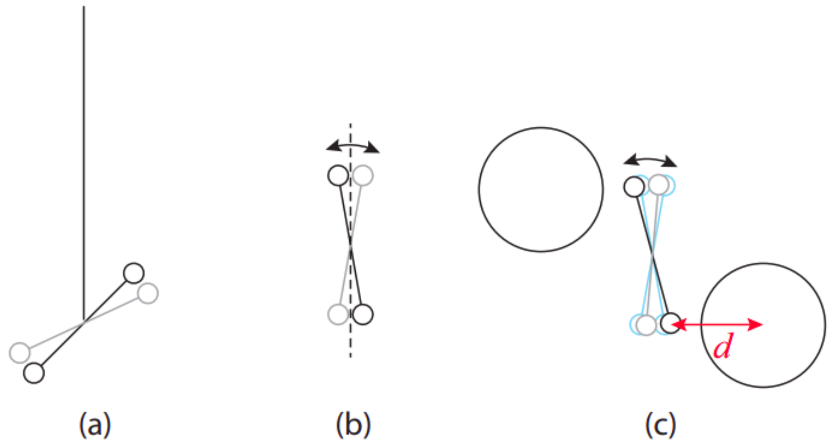

Figure 11.6.2: : (a) Torsion balance. The extremes of the oscillation are drawn in black and gray, respectively. (b) The view from the top. The dashed line indicates the equilibrium position. (c) In the presence of two nearby large masses, the equilibrium position is tilted very slightly; the light blue lines in the background show the oscillation in the absence of the masses, for reference.

You have now made a *torsion balance* similar to the one Cavendish used. You will probably find out that it it is very hard to keep it motionless: the slightest displacement causes it to oscillate around an equilibrium position. The way it works is that an angular displacement θ from equilibrium puts a small twist on the line, which results in a restoring torque $\tau = -\kappa\theta$, where κ is the *torsion constant* for your setup. If your dumbbell has moment of inertia I, then the equation of motion $\tau = I\alpha$ gives you

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta. \tag{11.6.3}$$

If you compare this to Equation (11.3.3), and follow the derivation there, you can see that the period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{11.6.4}$$

so if you measure T you can get κ, since $I = 2m(l/2)^2 = ml^2/2$ for the dumbbell.

Now suppose you bring two large masses, a distance d each from each end of the dumbbell, perpendicular to the dumbbell axis, and one on either side, as in the figure. The gravitational force $F^G = GmM/d^2$ between the large and small mass results in a net "external" torque of magnitude

$$\tau_{ext} = 2F^G \times \frac{l}{2} = F^G l. \tag{11.6.5}$$

This torque will cause a very small displacement, so small that the change in d will be practically negligible, so you can treat F^G, and hence τ_{ext}, as a constant. Then the situation is analogous to that of an oscillator subjected to a constant external force (section 11.2): the frequency of the oscillations will not change, but the equilibrium position will. In Equation (11.2.14) we found that $y_0' - y_0 = F_{ext}/k$ for a spring of spring constant k, where y_0 was the old and y_0' the new equilibrium position (the force was equal to $-mg$; the displacement of the equilibrium position will be in the direction of the force). For the torsion balance, the equivalent result is

$$\theta_0' - \theta_0 = \frac{\tau_{ext}}{\kappa} = \frac{F^G l}{\kappa}. \tag{11.6.6}$$

So, if you measure the angular displacement of the equilibrium position, you can get F^G. This displacement is going to be very small, but you can try to monitor the position of the dumbbell by, for instance, reflecting a laser from it (or, one or both of your small masses could be a small laser). Tracking the oscillations of the point of laser light on the wall, you might be able to detect the very small shift predicted by Equation (11.6.6).