13.10: Chapter- 10

Problem (10.1).

(a) Use Stokes' theorem to show that the Maxwell equation curl $\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}$ can be written in the form

$$\oint_C \mathbf{E} \cdot d\mathbf{L}=-\frac{\partial}{\partial t} \quad \int_{\text{Surface S}} \mathbf{B} \cdot \mathbf{dS} \qquad (1)$$

where the surface S is bounded by the closed curve c.

(b) Apply the above equation to a loop which straddles the boundary between two materials to show that the tangential component of **E** must be continuous across the boundary.

Answer (10.1).

(a) curl $\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}$

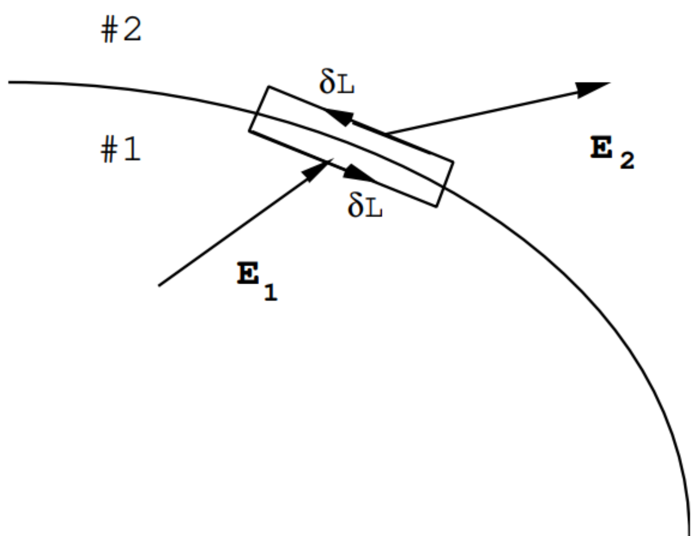
Integrate over a surface S bounded by a curve c:

$\int_S \text{curl} \quad \mathbf{E} \cdot d\mathbf{S}=-\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$

But from Stokes' theorem

$\int_S \text{Curl}\, \mathbf{E} \cdot \mathrm{d}\mathbf{S}=\oint_c \mathbf{E} \cdot \mathrm{d}\mathbf{L}$, and the result follows.

(b) Apply the above to a loop δL long and of negligible width, δd.



Then $\oint_c \mathbf{E} \cdot \mathrm{d}\mathbf{L}=\left.\mathrm{E}_2\right)_{\text{tang}} \delta \mathrm{L}-\left.\mathrm{E}_1\right)_{\text{tang}} \delta \mathrm{L}$

$$=-\frac{\partial}{\partial t}\left(\mathrm{B}_{\text{perp}} \cdot \delta \mathrm{L} \delta \mathrm{d}\right) \Rightarrow 0$$

therefore

$\left.\mathrm{E}_2\right)_{\text{tangential}}=\left.\mathrm{E}_1\right)_{\text{tangential}}$

Problem (10.2).

(a) Use Stokes' theorem to transform the Maxwell equation

$$\text{curl}\, \mathbf{H}=\mathbf{J}_{\mathrm{f}}+\frac{\partial \mathbf{D}}{\partial \mathrm{t}}$$

into

$$\oint_C \mathbf{H}\cdot d\mathbf{L} = \int_S \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{S},$$

where the surface S is bounded by the closed curve, c.

(b) Use the above equation to show that at the surface of discontinuity between two materials the tangential component of **H** must be continuous.

Answer (10.2).

(a) $\operatorname{Curl} \mathbf{H} \quad = \quad \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right)$

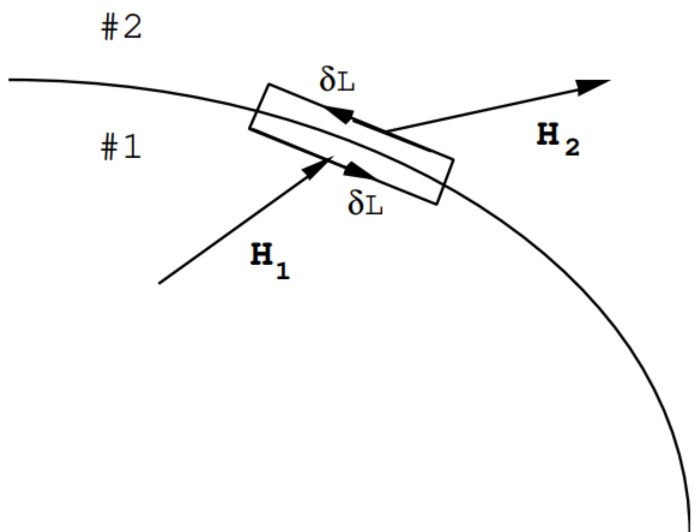
$\therefore \quad \int_S \operatorname{Curl} \mathbf{H}\cdot d\mathbf{S} = \int_S \mathbf{J}_f \cdot d\mathbf{S} + \frac{\partial}{\partial t}\int_S \mathbf{D}\cdot d\mathbf{S}.$

But by Stokes' theorem:

$$\int_S \operatorname{Curl} \mathbf{H}\cdot d\mathbf{S} = \oint_c \mathbf{H}\cdot d\mathbf{L}$$

from which the result follows.

(b) Apply the above theorem to a loop straddling the boundary. The loop is δL long and δd wide.



$\oint_c \delta \mathbf{H}\cdot d\mathbf{L} = \left.H_2\right)_{\text{tang}} \delta L - \left.H_1\right)_{\text{tang}} \delta L + \text{ terms 2nd order in } \delta d$

$\int_S \left(J_f + \frac{\partial D}{\partial t}\right)\cdot ds = \left(J_f + \frac{\partial D}{\partial t}\right)_{\text{normal}} \delta L \delta d \Rightarrow 0 \text{ as } \delta d \to 0$

$\therefore \quad \left.H_2\right)_{\text{tang}} = \left.H_1\right)_{\text{tang}}$

Problem (10.3).

a. From div **B** = 0 show that the normal component of **B** is continuous across the boundary between two different materials.
b. From div **D** = ρ_f show that there will be a surface charge density on the surface of discontinuity between two materials. Show that the magnitude of this surface charge density is given by

$$\rho_f = \left.D_2\right)_{\text{normal}} - \left.D_1\right)_{\text{normal}}$$

where $\left.D_2\right)_{\text{normal}}$ and $\left.D_1\right)_{\text{normal}}$ are the normal components of the vector **D**.

Answer (10.3).

(a) $\operatorname{div} \mathbf{B} = 0$

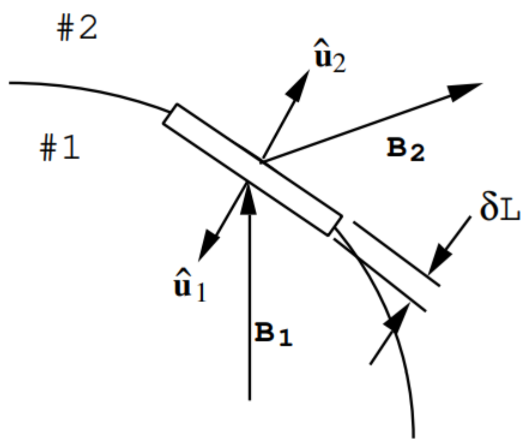
$\therefore \int_V \operatorname{div}\mathbf{B}\, d\tau = 0$

But by Gauss' theorem $\int_V \operatorname{div}\mathbf{B}\, d\tau = \int_S \mathbf{B}\cdot d\mathbf{S}$

where S is the surface bounding the closed volume V.

Therefore $\int_S \mathbf{B}\cdot d\mathbf{S} = 0$

Apply this to a pill box of area δA and thickness δL which straddles the boundary between material (1) and material (2)



$\int_{\text{Pill Box}} \underset{\text{box}}{\mathbf{B}\cdot}\mathrm{d}\mathbf{S} = \left[\mathrm{B}_2)_{\text{normal}} - \mathrm{B}_1\Big)_{\text{normal}}\right]\delta A + \text{ terms of 2nd order in } \delta L$

(As shown, $\mathbf{B}_2\cdot\hat{\mathbf{u}}_2$ makes a positive contribution and $\mathbf{B}_1\cdot\hat{\mathbf{u}}_1$ makes a negative contribution).

Therefore $\left[\mathrm{B}_2)_{\text{normal}} - \mathrm{B}_1)_{\text{normal}}\right]\delta\mathrm{A} = 0$ for arbitrary δA and

$$\therefore \mathrm{B}_2)_{\text{normal}} = \mathrm{B}_1)_{\text{normal}}$$

(b) div **D** = ρ_f

$\therefore$ for any closed volume V bounded by a surface S

$\int_V \operatorname{div}\mathbf{D}\, d\tau = \int_V \rho_f d\tau$

But by Gauss' theorem:

$\int_V (div\mathbf{D})d\tau = \int_S \mathbf{D}\cdot\mathbf{ds}$

Apply this to a pill-box which straddles material (1) and material (2):

Then $\int_S \mathrm{D}\cdot\mathrm{ds} = [\mathrm{D}_2)_{\text{normal}}\,\delta\mathrm{A} - \mathrm{D}_1)_{\text{normal}}\,\delta\mathrm{A}]$

+ higher order corrections of order δLδA.

$\therefore \left[\mathrm{D}_2)_{\text{normal}} - \mathrm{D}_1)_{\text{normal}}\right]\delta\mathrm{A} = \rho_f\delta\mathrm{A}\delta\mathrm{L}$

So $\left[\mathrm{D}_2)_{\text{normal}} - \mathrm{D}_1)_{\text{normal}}\right] = \rho_f\delta\mathrm{L} = \rho_s$,

where (ρ_f δ_L) does not depend upon the length δ_L and therefore represents a surface charge ρ_s. A discontinuity in the normal component of **D** means that there exists a surface charge density.

Problem (10.4).

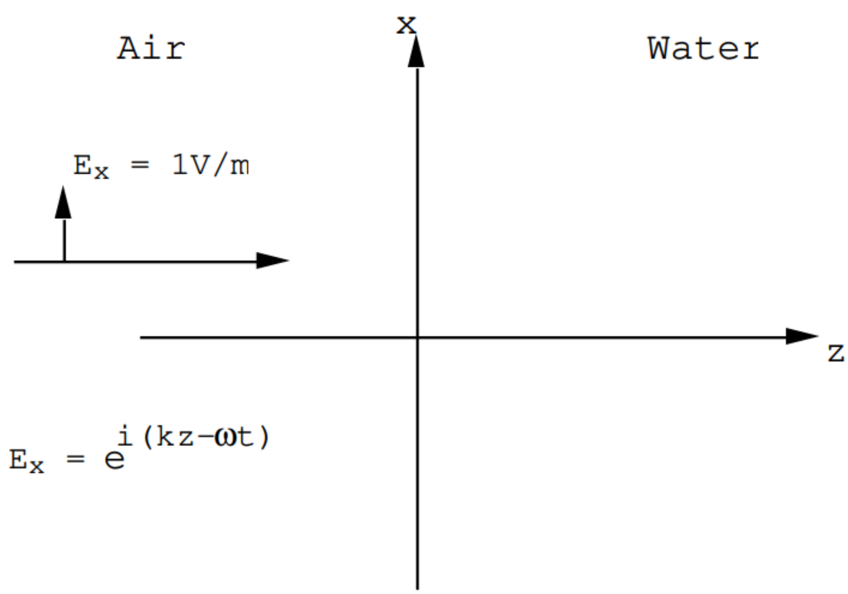
A plane wave falls at normal incidence on the plane surface of a large, deep, body of water. The real and imaginary parts of the index of refraction for water are n = 4/3 and κ = 10^{-8} corresponding to a time dependence ~ $e^{-i\omega t}$. The amplitude of the electric field in the incident wave is 1 V/m. Let the z-axis be directed into the water, and let the x, y axes lie in the surface of the water. Let the electric field be polarized along x. The index of refraction of air is n = 1, κ = 0.

a. Write an equation for the space and time variation of the electric field in the incident wave.

b. Write an equation for the space and time variations of **B**,**H** in the incident wave. What is the amplitude, H_o, of the **H** field?
c. Write expressions for the space and time variation of the reflected wave. Let the reflected electric field amplitude be E_R. Write the reflected magnetic field amplitude in terms of E_R.
d. Write expressions for the space and time variations of the electric and magnetic field waves (**H** field) transmitted into the water. Let the electric field amplitude at the water surface, at z = 0, be E_T. Write the magnetic field amplitude in terms of E_T.
e. State the boundary conditions which **E**, **H** must satisfy at the surface of the water.
f. Apply the boundary conditions of part (e) to obtain the reflected electric field amplitude, E_R, and the transmitted wave electric field amplitude, E_T.
g. What is the intensity of the incident wave? i.e. At what rate, in Watts/m^2, is energy transported to the water surface?
h. At what rate is energy absorbed by the water?
i. What will be the electric field amplitude at a depth of 2 m if the wavelength of the light is 1/2 micron?

Answer (10.4).

(a)



(b) $B_y = \frac{E_x}{c} = \frac{1}{c} e^{i(kz-\omega t)}$

$$\mathrm{H_y} = \frac{\mathrm{B_Y}}{\mu_\mathrm{O}} = \frac{1}{\mathrm{c}\mu_\mathrm{O}} e^{\mathrm{i(kz-\omega t)}} = \frac{1}{120\pi} \mathrm{e}^{\mathrm{i(kz-\omega t)}} \quad \text{Amps/m} \ .$$

$$\text{Amplitude } = \frac{1}{120\pi} = \frac{1}{377} \text{ Amps /m } .$$

c) Let the reflected electric field be

$$E_x = E_R e^{-i(kz+\omega t)}$$

(note change in sign of k).

Then $\mathrm{H_y} = -\frac{\mathrm{E_R}}{120\pi} \mathrm{e}^{-\mathrm{i(kz+\omega t)}}$

(d) In the water the propagation vector is given by $\mathrm{k} = \frac{\omega}{\mathrm{c}}(\mathrm{n}+\mathrm{i}\kappa)$

$$\therefore \mathrm{E_X} = \mathrm{E_T} e^{-\kappa \frac{\omega}{c} z} e^{i\left(\frac{n\omega z}{c} - \omega t\right)}$$

Now curl $\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial \mathrm{t}} = \mathrm{i}\omega \mathbf{B} = \mathrm{i}\omega\mu_\mathrm{o}\mathbf{H}$

$$\mathrm{i}\omega\mu_\mathrm{O}\mathbf{H} = \begin{vmatrix} \hat{\mathbf{u}}_\mathrm{x} & \hat{\mathbf{u}}_\mathrm{y} & \hat{\mathbf{u}}_\mathrm{z} \\ 0 & 0 & \frac{\partial}{\partial \mathrm{z}} \\ \mathrm{E_x} & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{\partial \mathrm{E_x}}{\partial \mathrm{z}} \\ 0 \end{vmatrix}$$

$\therefore \mathrm{H_Y} = \frac{1}{\mathrm{i}\omega\mu_o}\frac{\partial \mathrm{E_x}}{\partial \mathrm{z}} = \frac{\mathrm{i}\left(\frac{\omega}{\mathrm{c}}\right)(\mathrm{n}+\mathrm{i}\kappa)\mathrm{E_X}}{\mathrm{i}\omega\mathrm{u_O}} = \frac{(\mathrm{n}+\mathrm{i}\kappa)}{\mu_\mathrm{O}\mathrm{c}}\mathrm{E_X}$

and

$\mathrm{H_y} = \left(\frac{\mathrm{n}+\mathrm{i}\kappa}{\mu_\mathrm{O}\mathrm{c}}\right)\mathrm{E_T} e^{-\kappa\frac{\omega z}{\mathrm{c}}} e^{\mathrm{i}\left(\mathrm{n}\frac{\omega z}{\mathrm{c}}-\omega t\right)}.$

(e) At the interface the required boundary conditions are

(1) Tangential components of **E** must be continuous.

(2) Tangential components of **H** must be continuous.

(f) At z = 0

Incident Wave $E_x = (1)e^{-i\omega t}$

$\mathrm{H_y} = \frac{1}{\mathrm{c}\mu_0}\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}$

Reflected Wave $E_x = E_R e^{-i\omega t}$

$\mathrm{H_Y} = -\frac{\mathrm{E_R}}{\mathrm{c}\mu_\mathrm{O}} e^{-\mathrm{i}\omega\mathrm{t}}$

Transmitted Wave $E_x = E_T e^{-i\omega t}$

$\mathrm{H_Y} = \frac{(\mathrm{n}+\mathrm{i}\kappa)}{\mathrm{c}\mu_\mathrm{O}}\mathrm{E_T} e^{-\mathrm{i}\omega\mathrm{t}}$

Continuity of $\mathrm{E_x}$: $1+\mathrm{E_R} = \mathrm{E_T}$ (1)

Continuity of $\mathrm{H_y}$: $\frac{1}{\mathrm{c}\mu_0} - \frac{\mathrm{E_R}}{\mathrm{c}\mu_0} = \frac{(\mathrm{n}+\mathrm{i}\kappa)}{\mathrm{c}\mu_0}\mathrm{E_T}$

or $1-\mathrm{E_R} = (\mathrm{n}+\mathrm{i}\kappa)\mathrm{E_T}$ (2)

Solve eqns. (1) and (2) to obtain:

$E_T = \frac{2}{(1+n)+i\kappa} = \frac{2[(n+1)-i\kappa]}{(n+1)^2+\kappa^2}$

But $\kappa \simeq 0$ so $E_T = \frac{14/3}{(7/3)^2} = \frac{6}{7} = \underline{0.86\ Volts/m}.$

Also $E_T \simeq \left(\frac{2}{n+1}\right)$

and $\mathrm{E_R} = \mathrm{E_T} - 1 = \underline{-0.143\ Volts/m}.$

(NOTICE THE PHASE CHANGE IN THE ELECTRIC FIELD!!)

(g) Rate of transport of energy to the water surface is

$\mathrm{S_z} = \mathrm{E_x H_y}$

$\langle S_z\rangle = \left(\frac{1}{2}\right)(1)\left(\frac{1}{c\mu_0}\right) = \frac{1}{754}\ \mathrm{Watts}/\mathrm{m}^2$

$= \underline{1.33\ mW/m^2}.$

(h) The rate of energy reflected from the surface is

$< S_z >_R = \frac{1}{2}(E_R)\frac{(E_R)}{c\mu_0} = \frac{(0.143)^2}{754} = \underline{0.027\ \mathrm{mW/m^2}} = \underline{27\ \mu\mathrm{W/m^2}}.$

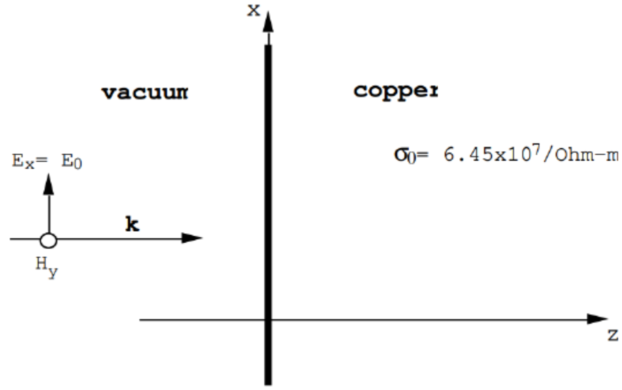
$\therefore$ Energy absorbed in H_2O = $\underline{1.30\ \mathrm{mW/m^2}}$.

(i) At z = 2m

$|\mathrm{E_X}| = \mathrm{E_T e}^{-\kappa 4\pi/\lambda} = \mathrm{E_T e}^{-0.251} = 0.78\ \mathrm{E_T}$

$\therefore$ @ 2m the electric field strength = $\underline{0.67\ \mathrm{V/m}}$.

Problem (10.5).



A wave having an electric field amplitude E_o = 1 V/m falls at normal incidence on a plane copper surface as shown in the above sketch. Its frequency is 10^6 Hz.

a. Write expressions for the electric and magnetic fields in the incident wave. How big is H_y?
b. Calculate the magnitude of the vacuum wave-vector.
c. Calculate the wave-vector in the metal (k_m) in the expressions:

$$E_x = E_T e^{i(k_m z - \omega t)}$$

$$H_Y = H_T e^{i(k_m z - \omega t)}$$

d. Calculate the amplitude of the electric field at the surface of the metal i.e. E_T.
e. Calculate the magnetic field amplitude at the surface of the metal i.e. H_T.
f. Calculate the time average Poynting vector for the incident wave i.e. $\langle S_o \rangle$
g. Calculate the time average Poynting vector for the energy flow into the metal i.e. $\langle S_m \rangle$
h. From (f) and (g) calculate the absorption coefficient $\alpha = \langle S_m \rangle / \langle S_o \rangle$.
i. Calculate the average rate of energy dissipation as Joule heat in the metal. Show that the integral of this quantity from z = o to ∞ is just equal to $\langle S_m \rangle$ from (g) above.

Answer (10.5).

(a) $k = \frac{\omega}{c} = \frac{2\pi \times 10^6}{3 \times 10^8} = \mathbf{2.094 \times 10^{-2}\ m^{-1}}$

$E_x = E_0 e^{i(kz - \omega t)} = e^{i(kz-\omega t)}$ since E_o = 1 V/m.

$H_Y = \frac{E_0}{Z_0} e^{i(kz - \omega t)} = \left(\mathbf{2.653 \times 10^{-3}}\right) e^{i(kz-\omega t)}$

since Z_o = 377 Ohms.

(b) See above. k = 2.094 x 10^{-2} /meter.

(c) In the metal:

$$\operatorname{curl} \mathbf{E} = \begin{vmatrix} \hat{\mathbf{u}}_x & \hat{\mathbf{u}}_y & \hat{\mathbf{u}}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{\partial E_x}{\partial z} \\ 0 \end{vmatrix} = i\omega\mu_0 \mathbf{H},$$

$$\operatorname{curl} \mathbf{H} = \begin{vmatrix} \hat{\mathbf{u}}_x & \hat{\mathbf{u}}_y & \hat{\mathbf{u}}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = \begin{vmatrix} -\frac{\partial H_y}{\partial z} \\ 0 \\ 0 \end{vmatrix} = \sigma \mathbf{E}.$$

$$\therefore \frac{\partial E_x}{\partial z} = i\omega\mu_0 H_y = ik_m E_x$$

or $\mathbf{H_y} = \left(\frac{\mathbf{k_m}}{\omega\mu_0}\right)\mathbf{E_x}$

and $\frac{\partial H_Y}{\partial z} = -\sigma E_X$

$\therefore \mathrm{ik_m H_y} = -\sigma \mathrm{E_x}$

or $\mathbf{H_y} = \left(\frac{\mathrm{i}\sigma}{\mathbf{k_m}}\right) \mathbf{E_x}$.

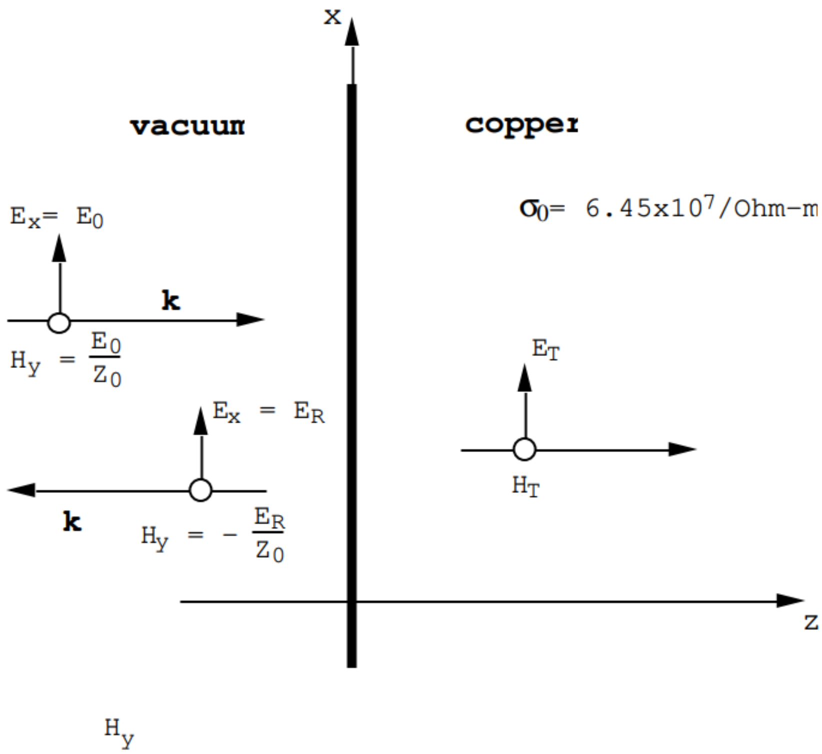
So $\frac{\mathrm{i}\sigma}{\mathrm{k_m}} = \frac{\mathrm{k_m}}{\omega\mu_o}$ or $k_m^2 = i\omega\mu_0\sigma$

$k_m^2 = i\left(2\pi \times 10^6\right)\left(4\pi \times 10^{-7}\right)\left(6.45 \times 10^7\right)$

= i (5.093 x 10^8)

$k_m = \left(\frac{1+i}{\sqrt{2}}\right)\left(2.257 \times 10^4\right) = \left(\mathbf{1.596} \times \mathbf{10^4}\right)(\mathbf{1}+\mathbf{i})$.

N.B. km is very large c.f. k = ω/c. Approx. 10^6 larger!!



At z = 0: a) Continuity of $\mathrm{E_x}$: $\mathrm{E_O + E_R = E_T}$

b) Continuity of $\mathrm{H_y}$: $\frac{E_0}{Z_0} - \frac{E_R}{Z_0} = H_T$

or $\mathrm{E_O - E_R = Z_O H_T}$

$\therefore 2\mathrm{E_O} = (\mathrm{E_T + Z_O H_T}) = \left[1 + \frac{\mathrm{i}\sigma \mathrm{z_O}}{\mathrm{k_m}}\right] \mathrm{E_T}$

$2\mathrm{E_R} = (\mathrm{E_T - Z_O H_T}) = \left[1 - \frac{\mathrm{i}\sigma \mathrm{Z_O}}{\mathrm{k_m}}\right] \mathrm{E_T}$

But $\mathrm{H_T} = \left(\frac{\mathrm{i}\sigma}{\mathrm{k_m}}\right) \mathrm{E_T}$

$\therefore \frac{E_R}{E_0} = \frac{1-\frac{i\sigma Z_0}{k_m}}{1+\frac{i\sigma Z_0}{k_m}} \quad \frac{E_T}{E_0} = \frac{2}{1+\frac{i\sigma Z_0}{k_m}}$.

$\frac{1}{k_m} = \left(\frac{10}{1.596}\right) \times 10^{-5}\left(\frac{1}{1+i}\right) = \left(\frac{5 \times 10^{-5}}{1.596}\right)(1-i)$

$= 3.133(1-i) \times 10^{-5}$

$\therefore \left(\frac{\mathrm{i}}{\mathrm{k_m}}\right) = \left(\mathbf{3.133} \times \mathbf{10^{-5}}\right)(\mathbf{1}+\mathbf{i})$.

So $\frac{i\sigma Z_0}{k_m} = (3.133)\left(10^{-5}\right)\left(6.45\times 10^7\right)(377)(1+i)$

$= \left(\mathbf{7.618\times 10^5}\right)\mathbf{(1+i)}$.

This is much larger than 1.

$\therefore \frac{E_T}{E_0} \cong \frac{2}{\frac{i\sigma Z_0}{k_m}} = \frac{-i2k_m}{\sigma Z_0} = \mathbf{1.313\times 10^{-6}(1-i).}$

$\frac{E_R}{E_0} = \frac{-\left[1+\frac{ik_m}{\sigma Z_0}\right]}{\left[1-\frac{ik_m}{\sigma Z_0}\right]} \simeq -\left[1+\frac{2ik_m}{\sigma Z_0}\right] \cong -1$

to approximately 1 part in 10^6!

e) From part (c) $\mathrm{H_Y} = \left(\frac{\mathrm{i}\sigma}{\mathrm{k_m}}\right)\mathrm{E_X}\ \therefore \mathrm{H_T} = \left(\frac{\mathrm{i}\sigma}{\mathrm{k_m}}\right)\quad \mathrm{E_T}$

and $\mathrm{H_T} \cong \left(\frac{\mathrm{i}\sigma}{\mathrm{k_m}}\left(-\mathrm{i}\frac{2}{\sigma}\frac{\mathrm{k_m}}{\mathrm{Z_0}}\right)\right) = \frac{2}{\mathrm{z_0}}$

N.B. To first order in $\left(\frac{2}{\sigma}\frac{\mathrm{k_m}}{\mathrm{Z_0}}\right)$ the magnetic field amplitude in the metal is INDEPENDENT of σ,ω !!

The factor 2 comes from the sum $H_T = H_o + H_R$, where $\mathrm{H_0} = \frac{\mathrm{E_0}}{\mathrm{Z_0}}$ & $\mathrm{H_R} = \frac{|\mathrm{E_R}|}{\mathrm{Z_0}}$

But E_o = 1 V/m & E_R = -1 v/m (to 1 part in 10^6)

$\therefore \mathrm{H_T} = \frac{2}{\mathrm{Z_0}} = \mathbf{5.305\times 10^{-3}\ Amps/m.}$

(f) For the incident wave $\langle \mathrm{S_0}\rangle = \frac{1}{2}$ Real $\{\mathrm{E_X H_Y^*}\}$

$= \frac{\mathrm{E_0^2}}{2\mathrm{z_0}} = \frac{1}{2\mathrm{z_0}} = \mathbf{1.326\times 10^{-3}\ watts/m^2.}$

(g) At the metal surface (z = 0)

$$\begin{aligned}\langle S_m\rangle &= \frac{1}{2}\text{ Real }\left\{E_T H_T^*\right\}\\ &= \frac{1}{2}\text{Real}\left\{\left(1.313\times 10^{-6}\right)(1-i)\frac{(2)}{Z_0}\right\}\\ &= \left(\frac{1.313}{Z_0}\times 10^{-6}\right) = \mathbf{3.48\times 10^{-9}\ Watts/m^2.}\end{aligned}$$

(h) $\alpha = \langle S_m\rangle / \langle S_0\rangle = \frac{3.48}{1.326}\times 10^{-6} = \mathbf{2.627\times 10^{-6}.}$

(i) In the metal the current density is given by

$\mathrm{J_x} = \sigma \mathrm{E_x} = \sigma \mathrm{E_T} e^{\mathrm{i}(\mathrm{k_m z}-\omega \mathrm{t})}$

The Joule heat/volume (time averaged) is

$$\begin{aligned}\frac{dQ}{dt} &= \frac{1}{2}\text{Real}\{J_x E_x^*\}\\ &= \frac{1}{2}\text{Real}\left\{\sigma E_T e^{i(k_m z-\omega t)}\cdot E_T^* e^{-i(k_m^* z-\omega t)}\right\}\\ &= \frac{1}{2}\text{Real}\left\{\sigma |E_T|^2 e^{i(k_m-k_m^*)z}\right\}\end{aligned}$$

But $\mathrm{k_m} = \gamma(1+\mathrm{i})$ and $\mathrm{k_m^*} = \gamma(1-\mathrm{i})$ $\quad\therefore \mathrm{k_m}-\mathrm{k_m^*} = 2\mathrm{i}\gamma$

and γ = 1.596 x 10^4 from part (c)

& $\mathrm{i}\left(\mathrm{k_m}-\mathrm{k_m^*}\right) = -2\gamma$

$\therefore \frac{dQ}{dt} = \frac{\sigma}{\mathbf{2}}|\mathbf{E_T}|^{\mathbf{2}}\mathbf{e^{-2\gamma z}.}$

Total rate of heat production $= \frac{\sigma|E_T|^2}{2}\int_0^\infty e^{-2\gamma z}dz = \frac{\sigma|E_T|^2}{4\gamma}.$

$$\therefore Q_{\text{Total}} = \frac{(6.45\times10^7)}{(4)(1.596\times10^4)}(1.313)^2\times10^{-12})(2) = \mathbf{3.48\times10^{-9}\ Watts/m^2.}$$

$$=<\mathrm{S_m}>(\text{ from (g)}) \; .$$

Problem (10.6).

Light having a wavelength of 5145 Å (0.5145 μm) falls upon a plane copper surface at normal incidence. The intensity of the light is 10^5 Watts/m^2 (i.e. 100 mW in a laser beam 1x1 mm in cross-section). The complex index of refraction for copper at 5145 Å is $\sqrt{\varepsilon_r} = (1.19+2.60i)$ for a time dependence of $e^{-i\omega t}$.

(a) Calculate the amplitudes of the electric and magnetic fields in the incident wave.

(b) Calculate the amplitudes of the electric and magnetic fields in the reflected wave.

(c) Calculate the intensity of the reflected wave;i.e. calculate the time-averaged value of the Poynting vector.

(d) Calculate the wave-vector of the light in the copper. What is the phase velocity associated with the wave in the copper?

(e) Calculate the amplitudes of the electric and magnetic fields in the copper but near the surface at z=0.

(f) Calculate the time averaged value of the Poynting vector inside the copper but near the surface at z=0.

(g) How far into the copper does the light penetrate before its intensity has decreased to 1% of its intensity at the surface?

(h) Calculate the time averaged energy density, <W> , stored in the electric and magnetic fields in the copper but at the surface z=0. Show that $\langle \mathrm{S_z}\rangle = \frac{\mathrm{c}}{\mathrm{n}} < \mathrm{W} >$ Watts/m^2.

Answer (10.6).

(a) Incident wave:

$$E_x = E_0 e^{i(kz-\omega t)}$$

$$\mathrm{H_y} = \frac{\mathrm{E_0}}{\mathrm{Z_0}} e^{\mathrm{i(kz-\omega t)}},$$

where k= ω/c and Z_0= μ_0c = 377 Ohms.

$$<\mathrm{S_z}> = \frac{1}{2}\mathrm{Real}\left(\mathrm{E_x H_y^*}\right) = \frac{\mathrm{E_0^2}}{2\mathrm{Z_0}} = \mathrm{I_0} = 10^5 \text{ Watts }/\mathrm{m}^2.$$

Therefore, $\mathrm{E_0^2} = 75.4\times10^6$, and E_0= 8.683x10^3 Volts/m, and H_y= 23.03 Amps/m.

(b) From the boundary value problem

$$\frac{E_R}{E_0} = \frac{1-\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}} = \frac{(1-n)-i\kappa}{1+n+i\kappa} = r.$$

For this problem n=1.19 and κ=2.60;

r = -0.621- 0.45i, and therefore r= - $Re^{i\phi}$ where R=0.767, and Tanϕ= 0.725 so that ϕ= 35.93° = 0.627 radians. The minus sign means that the direction of the reflected wave amplitude is reversed relative to the amplitude in the incident wave.

$$|E_R| = R|E_0| = 6.66\times10^3\ \mathrm{V/m},$$

and

$$|\mathrm{H_R}| = \mathrm{R}\,|\mathrm{H_y}| = 17.66 \text{ Amps/m}.$$

(c) The intensity of the reflected wave is given by

$$\mathrm{I_R} = \mathrm{R^2 I_0} = 0.588\times10^5 \text{ Watts }/\mathrm{m}^2.$$

(d) In the copper $\mathrm{k_m^2} = \varepsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2$

$$\mathrm{k} = \frac{\omega}{\mathrm{c}} = \frac{2\pi}{\lambda} = 1.221\times10^7\mathrm{m}^{-1}.$$

$$\mathrm{k_m} = (\mathrm{n}+\mathrm{ik})\frac{\omega}{\mathrm{c}} = (1.453+\mathrm{i}3.175)\times10^7\mathrm{m}^{-1}.$$

In the copper the fields are proportional to

$$e^{-\kappa\left(\frac{\omega}{c}\right)z}e^{i\left(n\frac{\omega}{c}z-\omega t\right)}.$$

The phase velocity is $\frac{c}{n}=2.52\times 10^{8}\ \mathrm{m/sec}$.

(e) $\frac{E_T}{E_0}=Te^{i\theta}=\frac{2}{1+\sqrt{\varepsilon}}=\frac{2}{(n+1)+i\kappa}$.

$\mathrm{Te^{i\theta}}=(0.379-\mathrm{i}0.450)$,

and T= 0.588 and θ= - 49.9° = - 0.871 radians.

$$|\mathrm{E_T}|=\mathrm{TE_0}=5.11\times 10^{3}\ \mathrm{V/m}.$$

$$\mathrm{H_Y}=\frac{(\mathrm{n}+\mathrm{i}\kappa)}{\mathrm{Z_0}}\mathrm{E_T}=(37.33+10.36\mathrm{i});$$

$|H_y|=\frac{\sqrt{n^2+K^2}}{Z_0}E_T=\frac{2.859}{377}E_T=38.75\ \text{Amps}/\mathrm{m}.$

phase=0.271 rad= 15.51°.

(f) In the metal

$$E_x=E_Te^{-\kappa\left(\frac{\omega}{c}\right)z}e^{i\left(n\frac{\omega}{c}z-\omega t\right)}$$

$$\mathrm{H_y}=\frac{(\mathrm{n}+\mathrm{i}\kappa)}{\mathrm{Z_0}}\mathrm{E_T}e^{-\kappa\left(\frac{\omega}{c}\right)\mathrm{z}}e^{\mathrm{i}\left(\mathrm{n}\frac{\omega}{c}\mathrm{z}-\omega\mathrm{t}\right)},$$

so at z=0 these become

$$E_x=E_Te^{-i\omega t}$$

and

$$\mathrm{H_y}=\frac{(\mathrm{n}+\mathrm{i}\kappa)}{\mathrm{Z_0}}\mathrm{E_T}e^{-\mathrm{i}\omega\mathrm{t}}.$$

$$<S_z>=\frac{1}{2}\mathrm{Real}(E_xH_y^*)=<S_z>=\frac{1}{2}\mathrm{Real}\left(E_T\frac{(n-i\kappa)}{Z_0}E_T^*\right),$$

$$<\mathrm{S_z}>=\frac{\mathrm{nE_T^2}}{2\mathrm{Z_0}}=0.4119\times 10^{5}\ \mathrm{Watts/m^2}.$$

$$\langle\mathrm{S_z}\rangle|_{\text{Reflected}}+\langle\mathrm{S_z}\rangle|_{\text{Transmitted}}=1.0\times 10^{5}\ \text{Watts}/\mathrm{m}^2.$$

(g) The electric and magnetic field amplitudes are multiplied by $e^{-\kappa\left(\frac{\omega}{c}\right)z}$ and therefore the intensity is multiplied by

$$e^{-2\kappa\left(\frac{\omega}{c}\right)z}.$$

If $e^{-2\kappa\left(\frac{\omega}{c}\right)z}=0.01$ then $2\kappa\frac{\omega}{c}z=4.605$.

But ω/c= 1.221x10. m^{-1} , therefore z= 0.725x10^{-7} meters, or z= 72.5 nm, or z= 0.0725 µm.

The free space wavelength of the light is 0.5145 µm, so that the light penetrates $\sim\left(\frac{\lambda}{7.1}\right)$, approximately 1/10 of a free space wavelength.

(h) At the surface of the copper the electric and magnetic field amplitudes are given by

$$E_x=E_Te^{-i\omega t},$$

$$\mathrm{H_y}=\frac{(\mathrm{n}+\mathrm{i}\kappa)}{\mathrm{Z_0}}\mathrm{E_T}e^{-\mathrm{i}\omega\mathrm{t}}.$$

$$\langle W_E\rangle=\frac{1}{4}\mathrm{Real}(E_xD_x)=\langle S_Z\rangle=\frac{1}{4}\mathrm{Real}\left(\varepsilon_0E_T^2\left((n^2-\kappa^2)-2n\kappa\right)\right),$$

$$< W_E >= \frac{\varepsilon_0}{4}\left(n^2-\kappa^2\right) E_T^2.$$

$$< W_B >= \frac{\mu_0}{4} \text{ Real } \left(H_y H_y^*\right) = \frac{\mu_0}{4} \frac{\left(n^2+\kappa^2\right)}{Z_0^2} E_T^2.$$

But $z_0^2 = \mu_0^2 c^2 = \frac{\mu_0}{\varepsilon_0}$, and

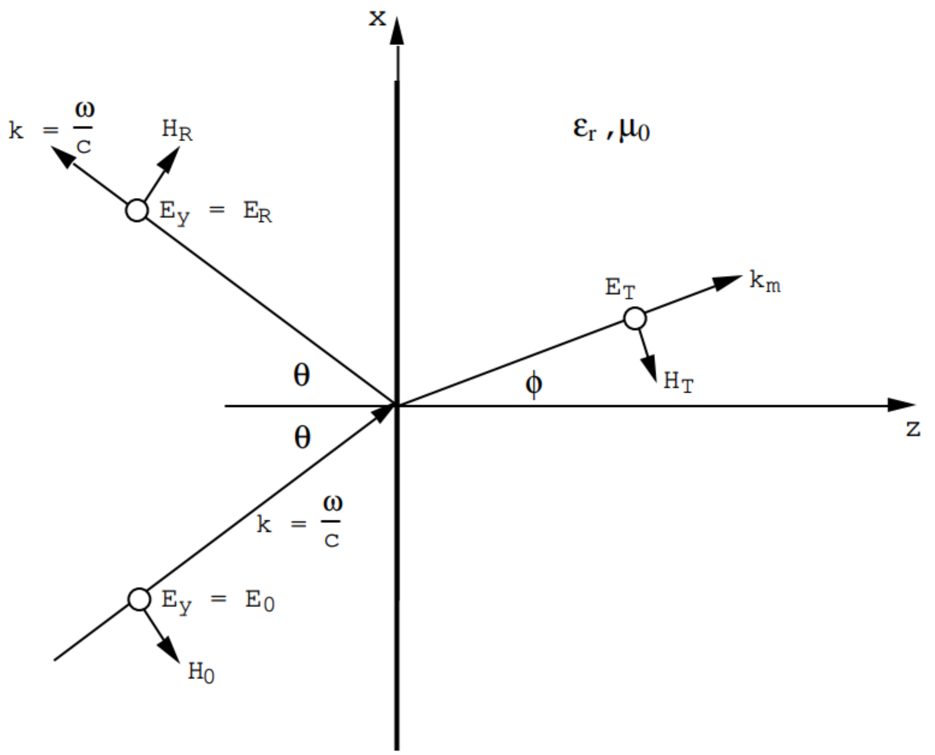
$$< W_B >= \frac{\varepsilon_0}{4}\left(n^2+\kappa^2\right) E_T^2.$$

$$< \mathrm{W} >=< \mathrm{W_E} > + < \mathrm{W_B} >= \frac{\varepsilon_0}{2} \mathrm{n}^2 \mathrm{E_T^2} = 1.63 \times 10^{-4} \text{ Joules }/\mathrm{m}^3.$$

$$\langle \mathrm{S_z} \rangle = \frac{\mathrm{n}}{2\mu_0 \mathrm{c}} \mathrm{E_T^2} = \frac{\mathrm{c}\varepsilon_0 \mathrm{n}}{2} \mathrm{E_T^2} = \left(\frac{\mathrm{c}}{\mathrm{n}}\right) \frac{\mathrm{n}^2 \varepsilon_0}{2} \mathrm{E_T^2} = \left(\frac{\mathrm{c}}{\mathrm{n}}\right) < \mathrm{W} >,$$

where for this case c/n= 2.52×10^8 m/sec.

Problem (10.7).



An s-polarized electromagnetic wave is incident on a plane interface at the angle θ (see the sketch). The amplitude of the incident electric field is E_o, that of the reflected electric field is E_R, and the transmitted electric field is E_T. The material for z > 0 is characterized by a relative dielectric constant, ε_r, which is real (no losses in the medium). The material is characterized by the magnetic permeability of free space.

(a) Write expressions for the components of **E** and **H** in the incident wave e.g.

$$\mathrm{E_y} = \mathrm{E_0} \mathrm{e}^{\mathrm{i}[(\mathrm{k}\sin\theta)\mathrm{x}+(\mathrm{k}\cos\theta)\mathrm{z}-\omega\mathrm{t}]}$$

$$\text{etc. where } \mathrm{k} = \omega/\mathrm{c}.$$

(b) Write expressions for the components of **E**, **H** in the reflected wave.

(c) Write expressions for the components of **E**, **H** in the transmitted wave.

(d) Show that $\frac{E_R}{E_0}=\left[\frac{\cos\theta-n\cos\phi}{\cos\theta+n\cos\phi}\right]$

where n = $\sqrt{\varepsilon_r}$ and $\sin\phi=\frac{\sin\theta}{n}$

and $\frac{E_T}{E_0}=\left[\frac{2\cos\theta}{\cos\theta+n\cos\phi}\right]$.

(e) Show that the normal component of **B**, B_z, is continuous across the boundary at z = 0.

(f) Construct a graph of $\left(\frac{E_R}{E_0}\right)$ vs the angle of incidence, θ, for $\varepsilon_r = 4$.

Answer (10.7).

(a) Incident Wave:

$$\mathrm{E_Y=E_0e^{i[(ksin\theta)x+(kcos\theta)z-\omega t]}}$$

$$\mathrm{H_x=\frac{-E_0}{Z_0}\cos\theta e^{i[(ksin\theta)x+(kcos\theta)z-\omega t]}}$$

$$\mathrm{H_z=\frac{E_O}{Z_O}\sin\theta e^{i[(ksin\theta)x+(k\cos\theta)z-\omega t]}}$$

where $Z_o = 377\ \Omega = c\mu_o$.

(b) Reflected Wave:

$$E_y=E_Re^{i[(k\sin\theta)x-(k\cos\theta)z-\omega t]}$$

$$\mathrm{H_X=\frac{E_R}{Z_0}\cos\theta e^{i[(ksin\theta)x-(k\cos\theta)z-\omega t]}}$$

$$\mathrm{H_z=\frac{E_R}{Z_0}\sin\theta e^{i[(ksin\theta)x-(k\cos\theta)z-\omega t]}}$$

(c) Transmitted Wave:

$$E_y=E_Te^{i[(k\sin\theta)x+(k_m\cos\phi)z-\omega t]}$$

$$\mathrm{H_X\frac{-n\cos\phi}{Z_0}E_Te^{i[(ksin\theta)x+(k_m\cos\phi)z-\omega t]}}$$

$$\mathrm{H_z=\frac{\sin\theta}{Z_0}E_Te^{i[(ksin\theta)x+(k_m\cos\phi)z-\omega t]}}$$

Since curl $\mathbf{E}=i\omega\mu_0\mathbf{H}$ or $\frac{\partial E_y}{\partial z}=-i\omega\mu_0H_x$

and $\mathrm{\frac{\partial E_y}{\partial x}=i\omega\mu_0H_z}$

and $\mathrm{Curl}\ \mathbf{H}=-\mathrm{i}\omega\varepsilon_r\varepsilon\mathbf{E}$ or $\mathrm{\frac{\partial H_X}{\partial z}-\frac{\partial H_z}{\partial x}=i\omega\varepsilon_r\varepsilon_0E_y}$

$\therefore\ \mathrm{\frac{\partial^2E_y}{\partial x^2}+\frac{\partial^2E_y}{\partial z^2}=-\varepsilon_r\left(\frac{\omega}{c}\right)^2E_Y}$

or $\mathrm{k^2\sin^2\theta+k_m^2=E_r\left(\frac{w}{c}\right)^2}$

or $k_m^2=\varepsilon_r\left(\frac{\omega}{c}\right)^2\qquad\therefore\qquad k_m=\sqrt{\varepsilon_r}\left(\frac{\omega}{c}\right)=n\left(\frac{\omega}{c}\right)$

$$\mathrm{k_m\sin\phi=k\sin\theta=\left(\frac{\omega}{c}\right)\sin\theta}$$

$$\therefore\sin\phi=\sin\theta/n$$

At z = 0 $E_o + E_R = E_T$ (1)

$$\mathrm{-\frac{E_0\cos\theta}{Z_O}+\frac{E_R\cos\theta}{Z_O}=-\frac{n\cos\phi}{Z_O}E_T}$$

or $-E_0+E_R=-\frac{n\cos\phi}{\cos\theta}E_T$ (2)

$\therefore \frac{2E_R}{E_T}=\left(1-\frac{n\cos\phi}{\cos\theta}\right)$

$2E_O=\left(1+\frac{n\cos\phi}{\cos\theta}\right)E_T$

$\therefore \frac{\mathbf{E_R}}{\mathbf{E_0}}=\left[\frac{\cos\boldsymbol{\theta}-\mathbf{n}\cos\boldsymbol{\phi}}{\cos\boldsymbol{\theta}+\mathbf{n}\cos\boldsymbol{\phi}}\right]$ where $n=\sqrt{\varepsilon_r}$

(d) $\frac{E_T}{E_O}\left[\frac{2\cos\theta}{\cos\theta+n\cos\phi}\right]$, where $\cos\phi=\sqrt{1-\frac{\sin^2\theta}{\varepsilon_r}}$

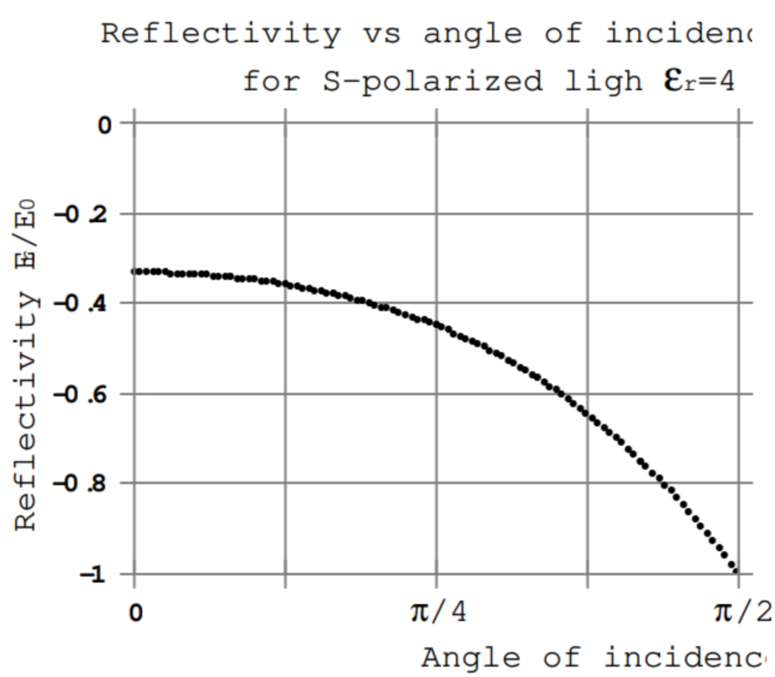
(e) At z = 0

on the left: $H_z=(E_O+E_R)\frac{\sin\theta}{Z_O}$

on the right: $H_z=E_T\frac{\sin\theta}{Z_O}$

Therefore, because of eqn (1), the normal component of $B_z = \mu_o H_z$ is continuous across the interface.

(f)



The ratio $\frac{E_R}{E_0}$ is plotted in the figure. Notice that

(1) The phase of the electric field is reversed in the reflected wave i.e. the total electric field at the interface is smaller than the incident electric field amplitude;

(2) The reflectivity approaches 1 at large angles of incidence i.e. as the beam becomes parallel with the interface plane. It is a common experience that surfaces appear more reflecting at shallow angles.

Problem (10.8).

Let p-polarized radiation, λ= 0.50 μm, be incident from vacuum on glass at an angle of incidence of 45°. The index of refraction of the glass is 1.5 and the glass is lossless. Let the plane of incidence be the x-z plane, and let the surface of the glass be parallel with the x-y plane and located at z=0.

(a) Write expressions for the incident fields (**E**,**H**) assuming a time dependence $e^{-i\omega t}$. Let the incident electric field amplitude be E_0= 1 V/m.

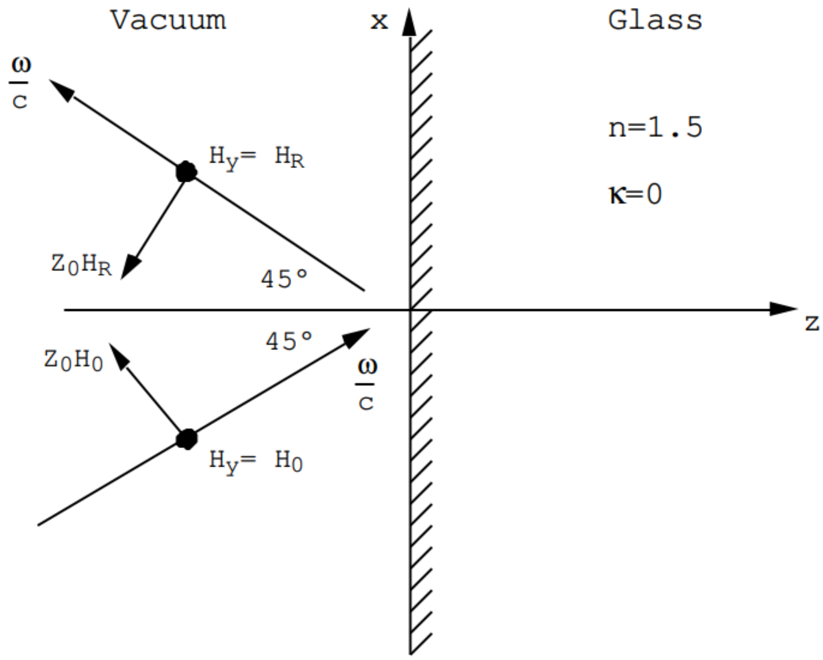
(b) Write expressions for the reflected fields. Let the reflected electric field amplitude be E_R.

(c) Write expressions for the transmitted fields. Let the transmitted electric field amplitude be E_T.

(d) Solve the appropriate boundary value problem to obtain the complex ratios E_R/E_0 and E_T/E_0.

(e) Calculate all of the components of the time averaged Poynting vectors for each of the incident, reflected, and transmitted waves.

Answer (10.8).



$\frac{\omega}{c} = \frac{2\pi}{\lambda} = 4\pi \times 10^6 \text{ rad/sec } = 1.2566 \times 10^7 \mathrm{m}^{-1}$;

the component along x is $q = \frac{1}{\sqrt{2}}\frac{\omega}{c} = 0.889 \times 10^7 \mathrm{m}^{-1}$; the component along z is k= q= $0.889\mathrm{x}10^7$ m^{-1}.

(a) Incident Wave:

$$H_0 = \frac{E_0}{Z_0}$$

$$E_x = \frac{E_0}{\sqrt{2}} e^{iqx} e^{iqz} e^{-i\omega t}$$

$$E_z = -\frac{E_0}{\sqrt{2}} e^{iqx} e^{iqz} e^{-i\omega t}$$

$$H_y = \frac{E_0}{Z_0} e^{iqx} e^{iqz} e^{-i\omega t}$$

(b) Reflected Wave:

$$H_R = \frac{H_R}{Z_0}$$

$$E_x = -\frac{E_R}{\sqrt{2}} e^{iqx} e^{-iqz} e^{-i\omega t}$$

$$E_z = -\frac{E_R}{\sqrt{2}} e^{iqx} e^{-iqz} e^{-i\omega t}$$

$$H_y = \frac{E_R}{Z_0} e^{iqx} e^{-iqz} e^{-i\omega t}.$$

(c) In the glass $q^2+k_m^2=n^2\left(\frac{\omega}{c}\right)^2$,

therefore $\mathrm{k_m^2}=\left(\mathrm{n}^2-\frac{1}{2}\right)\left(\frac{\omega}{\mathrm{c}}\right)^2$, since $\mathrm{q}^2=\frac{1}{2}(\omega/\mathrm{c})^2$,

and $\mathrm{k_m}=1.3229\left(\frac{\omega}{\mathrm{c}}\right)=1.6624\times10^7\ \mathrm{m}^{-1}$.

The angle of refraction is such that $\tan\phi=\frac{q}{k_m}=0.534$, ϕ= 28.13°.

In the glass $\mathrm{E_T}=\frac{\mathrm{Z_0H_T}}{\mathrm{n}}$:

$$\mathrm{E_X}=\left(\frac{\mathrm{k_m}}{\mathrm{n}\frac{\omega}{\mathrm{c}}}\right)\mathrm{E_T}e^{\mathrm{i}q\mathrm{x}}e^{\mathrm{ik_mz}}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{E_z}=\left(\frac{-\mathrm{q}}{\mathrm{n}\frac{\omega}{\mathrm{c}}}\right)\mathrm{E_T}e^{\mathrm{i}q\mathrm{x}}e^{\mathrm{ik_mz}}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{H_Y}=\frac{\mathrm{nE_T}}{\mathrm{Z_0}}e^{\mathrm{iqx}}\mathrm{e}^{\mathrm{ik_mz}}e^{-\mathrm{i}\omega\mathrm{t}},$$

where $\frac{\mathrm{k_m}}{\mathrm{n}\omega/\mathrm{c}}=0.882$ and $\frac{q}{n\omega/c}=0.4714$.

(d) Boundary Value Problem.

(i) Continuity of $\mathrm{H_y}$:

$$\frac{\mathrm{E_0}}{\mathrm{Z_0}}+\frac{\mathrm{E_R}}{\mathrm{Z_0}}=\frac{\mathrm{nE_T}}{\mathrm{Z_0}}$$

(ii) Continuity of $\mathrm{E_x}$:

$$\frac{E_0}{\sqrt{2}}-\frac{E_R}{\sqrt{2}}=(0.882)E_T.$$

Therefore $\mathrm{E_0}+\mathrm{E_R}=1.5\ \mathrm{E_T}$

$\mathrm{E_0}-\mathrm{E_R}=1.247\ \mathrm{E_T}$

from which $\mathbf{\frac{E_R}{E_0}=0.0920}$ and $\mathbf{\frac{E_T}{E_0}=0.7280}$.

(e) Time averaged Poynting Vectors.

(i) Incident Wave.

$$<\mathrm{S_X}>=-\frac{1}{2}\mathrm{Real}(\mathrm{E_Z H_Y^*})$$

$$\mathbf{<S_x>=\frac{E_0^2}{z_0 2\sqrt{2}}=9.38\times10^{-4}\ Watts\ /m^2}.$$

$$\mathbf{<S_z>=\frac{E_0^2}{z_0 2\sqrt{2}}=9.38\times10^{-4}\ Watts\ /m^2}.$$

(ii) Reflected Wave.

$$\mathbf{<S_x>=\frac{E_R^2}{z_0 2\sqrt{2}}=7.94\times10^{-6}\ Watts\ /m^2}$$

$$\mathbf{<S_z>=-\frac{E_R^2}{z_0 2\sqrt{2}}=7.94\times10^{-6}\ Watts\ /m^2}.$$

(iii) Transmitted Wave.

$$<\mathrm{S_x}>=\frac{1}{2\mathrm{n}}\frac{\mathrm{q}}{\mathrm{w/c}}\frac{\mathrm{n}}{\mathrm{Z_0}}\mathrm{E_T^2}=\frac{\mathrm{E_T^2}}{\mathrm{Z_0}2\sqrt{2}}$$

$$< \mathbf{S_x} >= \mathbf{4.97 \times 10^{-4} Watts\ /m^2.}$$

$$< S_z >= \frac{1}{2n}\frac{k_m}{w/c}\frac{n}{Z_0}E_T^2 = 1.323\frac{E_T^2}{2Z_0}$$

$$< \mathbf{S_z} >= \mathbf{9.30 \times 10^{-4}\ Watts\ /m^2.}$$

Problem (10.9).

Reverse the configuration of Problem (10.8); i.e. let p-polarized radiation be incident on a glass-vacuum interface from inside the glass. The interface is parallel with the x-y plane and it is located at z=0: the glass is on the left in the half-space z<0. Let the index of the glass be n=1.5 (the imaginary part of the index may be set equal to zero, κ=0). The vacuum wavelength of the light is λ= 0.50 μm, and the angle of incidence is 45°. The magnetic vector of the incident light is polarized along the y-direction.

(a) Calculate the z-component of the Poynting vector in the vacuum at z=0.

(b) Calculate the amplitude of the vacuum wave at z=0 if the incident wave electric field amplitude is E_0= 1 V/m

Answer (10.9).

(a) The wave-vector in the glass is given by

$$k^2 = n^2\left(\frac{\omega}{c}\right)^2,$$

or

$$k = n\left(\frac{\omega}{c}\right).$$

For this problem $\left(\frac{\omega}{c}\right) = 1.2566 \times 10^7\ m^{-1}$ and k= 1.8849x10^7 m^{-1}.

The wave-vector component along the interface (along x) is

$$q = k\sin 45^\circ = \frac{k}{\sqrt{2}} = 1.3328 \times 10^7\ m^{-1}.$$

On the vacuum side of the interface the fields are proportional to

$$e^{iqx}e^{ik_vz}e^{-i\omega t}$$

where $q^2 + k_v^2 = \left(\frac{\omega}{c}\right)^2$,

therefore

$$k_V^2 = \left(\frac{\omega}{c}\right)^2 - q^2 = -0.1974 \times 10^{14}\ m^{-2}.$$

Notice that k_v^2 is negative. This means that the square root is pure imaginary.

$$k_v = \left(4.443 \times 10^6\right) i\ m^{-1} = i\alpha = i\frac{\omega}{c}\sqrt{(n^2/2) - 1}.$$

The wave in the vacuum is a pure exponential, it does not oscillate in space. The fields are confined to a distance of the order of 1/α near the interface, i.e ~ 1 λ. In the vacuum

$$H_Y = H_T e^{iqx} e^{-\alpha z} e^{-i\omega t}$$

where α= 4.443x10^6 m^{-1}. In the vacuum curl**H** = -iωε$_0$ **E**, therefore

$$-i\omega\varepsilon_0 E_x = -\frac{\partial H_y}{\partial z} = \alpha H_y$$

$$-i\omega\varepsilon_0 E_z = -\frac{\partial H_y}{\partial x} = iqH_y,$$

or

$$E_x=\left(\frac{i\alpha}{\omega\varepsilon_0}\right)H_Y$$

$$\mathrm{E_z}=-\left(\frac{\mathrm{q}}{\omega\varepsilon_0}\right)\mathrm{H_Y}.$$

The time averaged Poynting vector at the interface is

$$<\mathrm{S_z}>=\frac{1}{2}\mathrm{Real}(\mathrm{E_X H_Y^*})$$

so

$$<S_z>=\frac{1}{2}\mathrm{Real}\left(\frac{i\alpha}{\omega\varepsilon_0}|H_T|^2\right)\equiv 0.$$

There is no energy flow from the glass to the vacuum. The light is totally reflected.

(b) From the continuity of the tangential components of **E** and **H** one finds

$$\mathrm{H_0+H_R=H_T}$$

$$\frac{\mathrm{nZ_0H_0}}{\sqrt{2}}-\frac{\mathrm{nZ_0H_R}}{\sqrt{2}}=\left(\frac{\mathrm{i}\alpha}{\omega\varepsilon_0}\right)\mathrm{H_T}=\mathrm{iZ_0}\left(\frac{\alpha}{\omega/\mathrm{c}}\right)\mathrm{H_T}$$

or $\mathrm{H_0-H_R}=\frac{\mathrm{i}}{2\mathrm{n}}\mathrm{H_T}$, since $\frac{\alpha}{\omega/c}=\frac{1}{2\sqrt{2}}$.

Consequently, $\mathrm{H_0+H_R=H_T}$

$\mathrm{H_0-H_R}=\frac{\mathrm{i}}{3}\mathrm{H_T}$,

from which

$$\frac{\mathrm{H_T}}{\mathrm{H_O}}=\frac{6}{(3+\mathrm{i})}=(1.80-0.60\mathrm{i})=1.897\mathrm{e}^{-\mathrm{i}\phi}.$$

where ϕ= 18.43°,

and

$$\frac{\mathrm{H_R}}{\mathrm{H_0}}=\frac{1}{6}(3-\mathrm{i})\frac{\mathrm{H_T}}{\mathrm{H_0}}=(0.8-0.6\mathrm{i})=\mathrm{e}^{-\mathrm{i}\theta},$$

where θ= 36.87°.

NB. $\left|\frac{\mathrm{H_R}}{\mathrm{H_0}}\right|^2\equiv 1$ as expected.

The electric field amplitude in the glass is given by

$$\mathrm{E_0}=\left(\frac{\mathrm{Z_0}}{\mathrm{n}}\right)\mathrm{H_0},$$

so if E_0= 1 V/m, then H_0= 3.98x10^{-3} Amps/m. The vacuum wave amplitude is given by

$$\mathrm{H_T}=\left(1.897\mathrm{e}^{-\mathrm{i}\phi}\right)\mathrm{H_0}=7.548\times10^{-3}\mathrm{e}^{-\mathrm{i}\phi}\quad \mathrm{Amps/m},$$

and

$$\mathrm{E_T}=\mathrm{Z_0H_T}=2.846\mathrm{e}^{-\mathrm{i}\phi}\quad \mathrm{V/m},\qquad \text{where } \phi=18.43^{\circ}.$$

Problem (10.10).

Light of wavelength λ= 0.50 μm falls from vacuum on a plane glass interface; the angle of incidence is 60°. Let the plane of incidence be the x-z plane, and let z be directed into the glass; the interface is located at z=0. The complex index of refraction of the glass, n+iκ, has components n=1.5, κ=0. The incident light is plane polarized but the electric vector has equal amplitudes, E_0, for the component perpendicular to the plane of incidence (the s-polarized component), and for the component parallel with the plane of incidence (the p-polarized component). Calculate the reflected electric field amplitudes and show that the electric field in the reflected light is plane polarized, but that the plane of polarization has been rotated relative to that of the incident light.

Answer (10.10).

From Snell's law

$$\sin\theta = n\sin\phi$$

where θ= 60°.Thus

$$\begin{aligned}&\sin\phi = 0.5774\\&\phi = 35.26^{\circ}\\&\cos\phi = 0.8165\\&\cos\theta = 1/2.\end{aligned}$$

For the s-polarized component

$$\mathrm{R_S} = \frac{\mathrm{E_R}}{\mathrm{E_0}} = \frac{\cos\theta - \mathrm{n}\cos\phi}{\cos\theta + \mathrm{n}\cos\phi} = -0.4202.$$

For the p-polarized component

$$\mathrm{R_p} = \frac{\mathrm{H_R}}{\mathrm{H_0}} = \frac{\mathrm{E_R}}{\mathrm{E_0}} = \frac{\mathrm{n}\cos\theta - \cos\phi}{\mathrm{n}\cos\theta + \cos\phi} = -0.0425.$$

The reflected light is polarized almost perpendicular to the plane of incidence. The angle which the electric vector makes with the plane of incidence is α, where

$\tan\alpha = \frac{0.4202}{0.0425}$, so that α = 84.2°.

The amplitude of the electric vector is 0.422 E_0.

Problem (10.11).

Light of wavelength λ= 0.5145 μm falls on a plane copper interface; the complex index of refraction for copper, $\sqrt{\varepsilon_r} = (n + i\kappa)$, has components n=1.19, and κ= 2.60, for a time dependence $e^{-i\omega t}$. Let the copper-vacuum interface lie in the x-y plane at z=0. The plane of incidence is the x-z plane, and the angle of incidence is 60°. The incident wave is plane polarized and its electric vector is oriented at 45° with respect to the plane of incidence. Take the amplitudes of the s-polarized and p-polarized components to be equal to E_0. Calculate the reflected wave electric field amplitudes and show that the reflected light is elliptically polarized.

Answer (10.11).

In the copper one has a spatial variation of the form

$$e^{iqx}e^{ikz}$$

where $\mathrm{q}^2 + \mathrm{k}^2 = \varepsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2$

and $q = \left(\frac{\omega}{c}\right)\sin 60^{\circ} = 0.8660\left(\frac{\omega}{c}\right)$.

Therefore $\mathrm{k}^2 = (\varepsilon_{\mathrm{r}} - 0.75)\left(\frac{\omega}{\mathrm{c}}\right)^2$.

For copper $\varepsilon_r = (n + i\kappa)^2 = \left(n^2 - \kappa^2\right) + 2ni\kappa$,

or $\varepsilon_r = -5.34 + i6.19$,

and $\mathrm{k}^2 = (-6.09 + \mathrm{i}6.19)\left(\frac{\omega}{\mathrm{c}}\right)^2$

$\mathrm{k}^2 = 8.686\mathrm{e}^{\mathrm{i}134.5^{\circ}}\left(\frac{\omega}{\mathrm{c}}\right)^2$,

so that $\mathrm{k} = 2.947\mathrm{e}^{\mathrm{i}\phi}\left(\frac{\omega}{\mathrm{c}}\right)$ where $\phi = 67.27^{\circ}$.

This can be written $\mathrm{k} = (\mathrm{n}_\theta + \mathrm{i}\kappa_\theta)\left(\frac{\omega}{\mathrm{c}}\right)$

where n_θ= 1.139 κ_θ= 2.718.

For the s-polarized wave

$$\frac{E_R}{E_0}=\frac{\cos\theta-(n_\theta+i\kappa_\theta)}{\cos\theta+(n_\theta+i\kappa_\theta)}=0.880e^{-i162.14^\circ}.$$

For the p-polarized wave

$$\frac{\mathrm{H_R}}{\mathrm{H_0}}=\frac{\varepsilon_\mathrm{r}\cos\theta-(\mathrm{n}_\theta+\mathrm{iK}_\theta)}{\varepsilon_\mathrm{r}\cos\theta+(\mathrm{n}_\theta+\mathrm{i}\kappa_\theta)}=0.637\mathrm{e}^{\mathrm{i}69.59^\circ}.$$

The reflected waves can be described at z=0 by

$$E_y=0.880E_0e^{-i(\omega t+162.14^\circ)}$$

and $E_{x'}=-0.637E_0e^{-i(\omega t-69.59^\circ)}$,

or $E_{x'}=0.637\ \mathrm{E}\ _0e^{-i(\omega t+110.41^\circ)}$

where the x' refers to a co-ordinate system in which the x'-axis lies in the plane perpendicular to the reflected wave wave-vector.

These expressions mean

$$\mathrm{E_{x'}}=0.637\mathrm{E_0}\cos(\omega\mathrm{t}+110.41^\circ)$$

$$\mathrm{E_Y}=0.880\mathrm{E_0}\cos(\omega\mathrm{t}+162.14^\circ).$$

The phase shift between these two components is

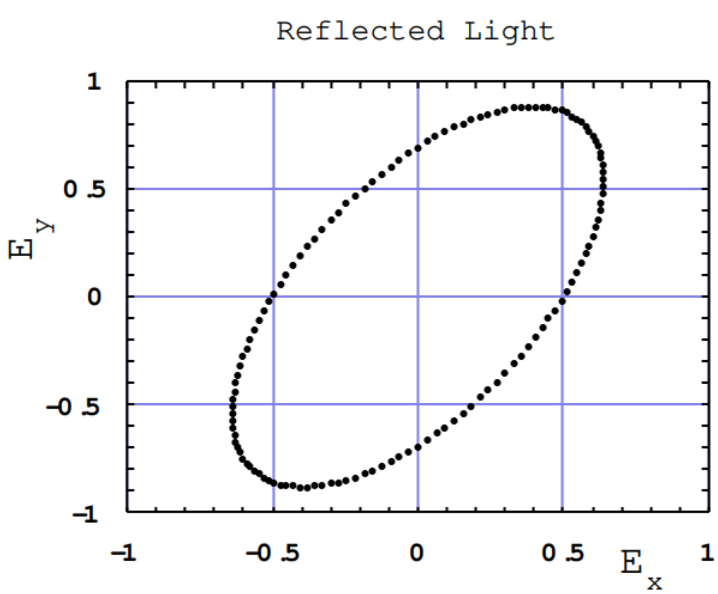
ϕ= 162.14 - 110.41 = 51.73° .

Shift the zero of time so as to make the component E_x vary as Cosωt:

$$E_{x'}=a\cos\omega t$$

$$E_y=b\cos(\omega t+51.73^\circ),$$

where a= $0.637E_0$ and b= $0.880E_0$. These relations are plotted below for E_0= 1 V/m.



This ellipse can be put in standard form by a co-ordinate rotation through the angle θ:

$$\mathrm{E}_\xi=\mathrm{E_X}\cos\theta+\mathrm{E_Y}\sin\theta$$

$$E_\eta = -E_X \operatorname{Sin}\theta + E_Y \cos\theta.$$

Using these relations the electric field components in the rotated frame can be written:

$$E_\xi = 0.637\ \cos\theta \cos\omega t + 0.5451\ \sin\theta \cos\omega t - 0.6909\ \sin\theta \sin\omega t$$

$$E_\eta = -0.637\ \sin\theta\ \cos\omega t + 0.5451\ \cos\theta \cos\omega t - 0.6909\ \cos\theta \sin\omega t$$

These have the form

$$E_\xi = A\cos(\omega t + \alpha) = A\ \cos\alpha \cos\omega t - A\ \sin\alpha\ \sin\omega t$$

where

$$\begin{aligned} A\cos\alpha &= 0.637\cos\theta + 0.5451\sin\theta \\ A\sin\alpha &= 0.6909\sin\theta \end{aligned}$$

and

$$E_\eta = B\sin(\omega t + \alpha) = \cos\alpha \sin\omega t + \sin\alpha \cos\omega t$$

where

$$\begin{aligned} B\cos\alpha &= -0.6909\cos\theta \\ B\sin\alpha &= 0.5451\cos\theta - 0.637\sin\theta. \end{aligned}$$

These give two expressions for Tan α which when equated provide an equation for the angle of rotation θ.

$$\frac{0.637\sin\theta - 0.5451\cos\theta}{0.6909\cos\theta} = \frac{0.6909\sin\theta}{0.637\cos\theta + 0.5451\sin\theta}.$$

Solutions are θ= -31.01° and θ= 58.98°. Use θ= -31.01°

so that Cosθ= 0.857, Sinθ= - 0.515.

These can be used to write

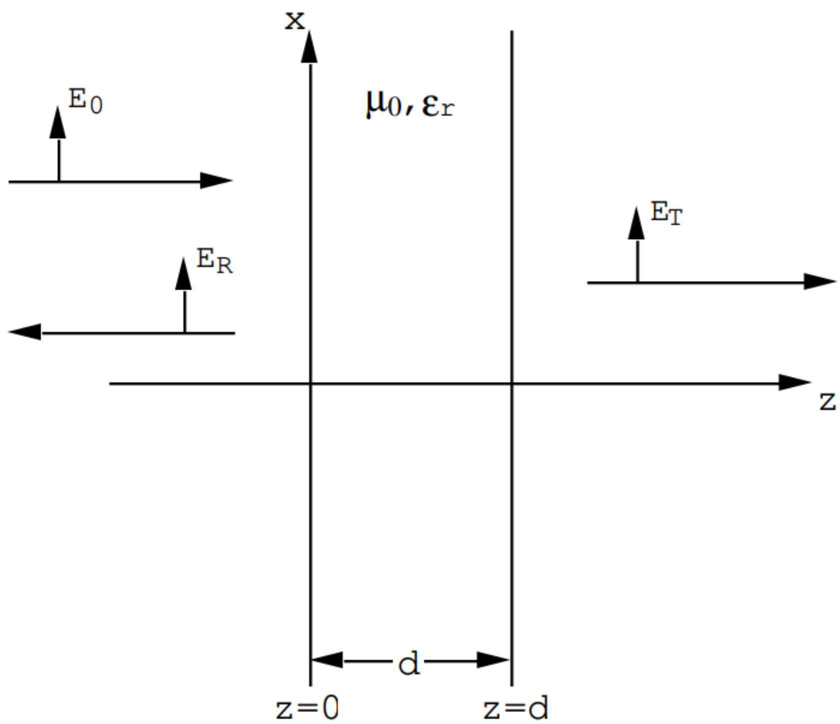
$E\xi = 0.265\cos\omega t + 0.356\sin\omega t = \mathbf{0.444}\ \cos(\omega \mathbf{t} - \mathbf{53.3}^\circ)$

$E_\eta = 0.796\cos\omega t - 0.592\sin\omega t = \mathbf{-0.992}\ \sin(\omega t - \mathbf{53.30})$.

The light is elliptically polarized. The ratio of the major to the minor axes of the ellipse is 2.23, and one of the principle axes of the ellipse is rotated 31° from the plane of incidence of the light. The electric vector is rotating counter-clockwise when viewed looking into the reflected beam along the +z direction.

Problem (10.12).

Consider a block of dielectric material of thickness d immersed in vacuum. A wave having an amplitude E_o is incident on the block as shown: the angle of incidence is θ = 0.



Calculate the amplitudes of the reflected and transmitted waves E_R, E_T.

HINT: Inside the dielectric block there is both a forward and a backward moving wave: ie. in the block

$$E_x = ae^{i[kmz-\omega t]} + be^{-i[kmz+\omega t]}$$

One must satisfy boundary conditions at both z = 0 and at z = d.

Answer (10.12).

In the dielectric block $\mathrm{k_m^2} = \varepsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2$

$\therefore \mathrm{k_m} = (\mathrm{n}+\mathrm{ik})\left(\frac{\omega}{\mathrm{c}}\right)$ if ε_r is complex.

We require curl **E** = $i\omega\mu_o$**H**

∴ since there is only an x-component of **E**

$$\mathrm{i}\omega\mu_\circ \mathrm{H_Y} = \frac{\partial \mathrm{E_X}}{\partial \mathrm{z}} = \mathrm{ik_m}\left[\mathrm{ae^{i[kmz-\omega t]}} - \mathrm{be^{-i[kmz+\omega t]}}\right]$$

Incident Wave:

$$E_x = E_0 e^{i[\omega/c\, z-\omega t]}$$

$$\mathrm{H_y} = \frac{\mathrm{E_0}}{\mathrm{Z_0}} e^{\mathrm{i}[\omega/\mathrm{c\,z}-\omega \mathrm{t}]}$$

Reflected Wave:

$$E_x = E_R e^{-i[\omega/c\, z+\omega t]}$$

$$\mathrm{H_Y} = -\frac{\mathrm{E_R}}{\mathrm{Z_0}} e^{-\mathrm{i}[\omega/\mathrm{c\,z}+\omega \mathrm{t}]}$$

Boundary Conditions at z = 0

(1) Continuity of E_x $\mathbf{E_0} + \mathbf{E_R} = \mathbf{a} + \mathbf{b}$ (1)

(2) Continuity of H_y $\frac{E_0}{Z_0}-\frac{E_R}{Z_0}=\frac{ck_m}{\omega Z_0}[a-b]$

or $\mathbf{E}_0-\mathbf{E}_R=\frac{\mathbf{ck}_m}{\omega}[\mathbf{a}-\mathbf{b}]$ (2) (2)

Now at z = d one can write the transmitted fields as

$$E_x=E_T e^{i[\omega/c\,(z-d)}-\omega t]$$

$$\mathrm{H_y}=\frac{\mathrm{E_T}}{\mathrm{Z_0}}e^{\mathrm{i}[\omega/\mathrm{c}\,(\mathrm{z-d})-\omega\mathrm{t}]}$$

∴ at z = d $E_x = E_T$ and $H_y = E_T/Z_o$

But in the dielectric at z = d one has

$$E_x=\left(ae^{ik_md}+be^{-ik_md}\right)e^{-i\omega t}$$

$$\mathrm{H_Y}=\frac{\mathrm{ck_m}}{\omega\mathrm{Z_0}}\left(a_{\mathrm{b}}e^{\mathrm{ik_md}}-\mathrm{be}^{-\mathrm{ik_md}}\right)\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}$$

Therefore from continuity of E_x one obtains

$\mathbf{ae^{ik_md}+be^{-ik_md}=E_T}$ (1)

and from continuity of H_y

$$\frac{ck_m}{\omega Z_0}\left(ae^{ik_md}-be^{-ik_md}\right)=\frac{E_T}{Z_0}$$

or

$\mathbf{a}e^{\mathbf{ik_md}}-\mathbf{b}e^{-\mathbf{ik_md}}=\left(\frac{\omega}{\mathbf{ck_m}}\right)\mathbf{E_T}$ (4)

From (3) and (4) one has

$\mathrm{ae^{ik_md}-be^{-ik_md}}=\left(\frac{\omega}{\mathrm{ck_m}}\right)\left[\mathrm{ae^{ik_md}+be^{-ik_md}}\right]$

$\therefore\quad \frac{\mathrm{b}}{\mathrm{a}}=\mathrm{e}^{2\mathrm{ik_md}}\left[\frac{1-\left(\frac{\omega}{\mathrm{ck_m}}\right)}{1+\left(\frac{\omega}{\mathrm{ck_m}}\right)}\right]$

and from (1) and (2)

$2\mathrm{E_0}=\mathrm{a}\left\{\left[1+\left(\frac{\mathrm{ck_m}}{\omega}\right)\right]+\left[1-\left(\frac{\mathrm{ck_m}}{\omega}\right)\right]\left(\frac{\mathrm{b}}{\mathrm{a}}\right)\right\}$

or

$$\frac{\mathrm{a}}{\mathrm{E_o}}=\frac{2\left(1+\frac{\omega}{\mathrm{ck_m}}\right)}{\left[\left(2+\frac{\omega}{\mathrm{ck_m}}+\frac{\mathrm{ck_m}}{\omega}\right)+\left(2-\frac{\omega}{\mathrm{ck_m}}-\frac{\mathrm{ck_m}}{\omega}\right)\mathrm{e}^{2\mathrm{ik_md}}\right]}$$

$$\frac{\mathbf{b}}{\mathbf{E_O}}=\frac{2\left(1-\frac{\omega}{\mathbf{ck_m}}\right)}{\left[\left(2+\frac{\omega}{\mathbf{ck_m}}+\frac{\mathbf{ck_m}}{\omega}\right)+\left(2-\frac{\omega}{\mathbf{ck_m}}-\frac{\mathbf{ck_m}}{\omega}\right)\mathrm{e}^{2\mathbf{ik_m}\mathrm{d}}\right]}$$

$$\frac{E_R}{E_0}=\frac{\left[\left(\frac{\omega}{ck_m}\right)-\left(\frac{ck_m}{\omega}\right)\right]\left[1-e^{2ik_md}\right]}{\left[\left(2+\frac{\omega}{ck_m}+\frac{ck_m}{\omega}\right)+\left(2-\frac{\omega}{ck_m}-\frac{ck_m}{\omega}\right)e^{2ik_md}\right]}$$

$$\frac{E_T}{E_0}=\frac{4e^{ik_md}}{\left[\left(2+\frac{\omega}{ck_m}+\frac{ck_m}{\omega}\right)+\left(2-\frac{\omega}{ck_m}-\frac{ck_m}{\omega}\right)e^{2ik_md}\right]}$$

If ε_r is real $k_m = n\omega/c$ and

$$\left(\frac{E_R}{E_0}\right) = \frac{(1-n^2)\left[1-e^{2ik_m d}\right]}{\left[(n+1)^2-(n-1)2e^{2ik_m d}\right]}$$

$$\left(\frac{E_T}{E_0}\right) = \frac{4ne^{ik_m d}}{[(n+1)2-(n-1)2e^{2ik_m d}]}$$

The above two equations are oscillatory functions of the wavelength.

if $2k_m d = 2\pi, 4\pi, 6\pi$, etc.

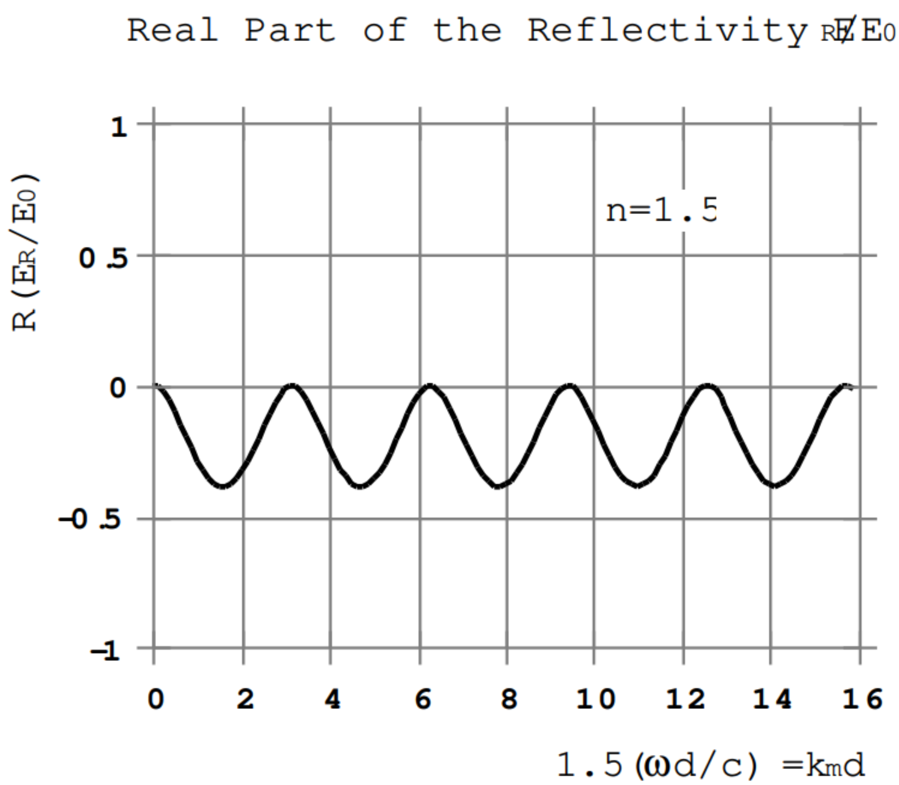
then $\frac{E_R}{E_0} = 0 \ \frac{E_T}{E_0} = \pm 1$

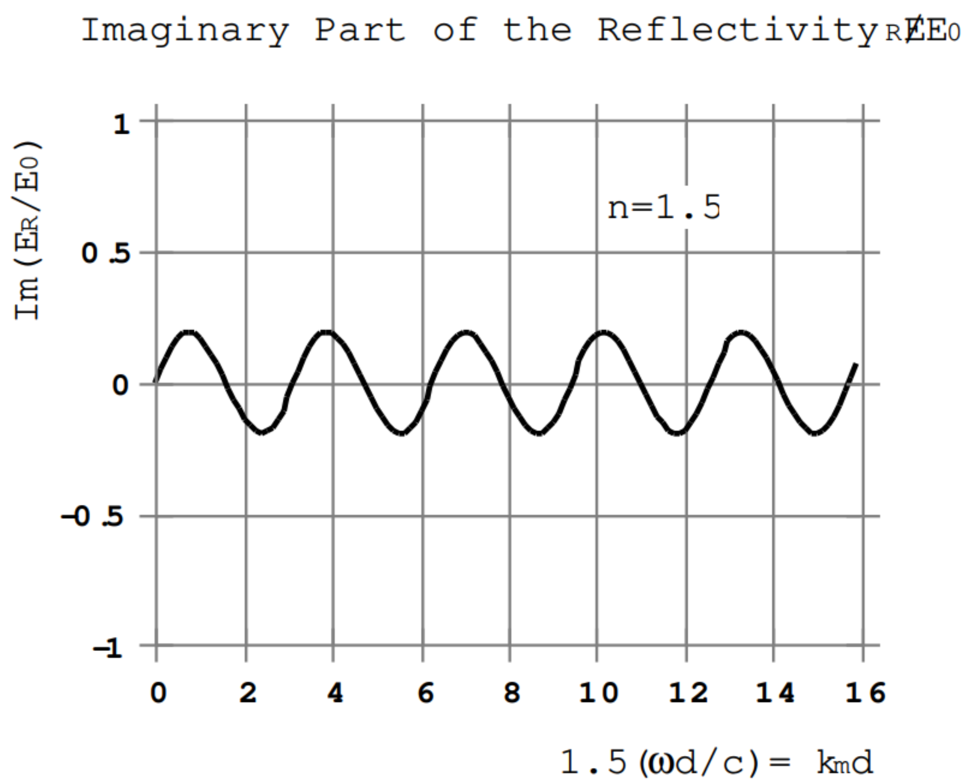
If $2k_m d = \pi, 3\pi, 5\pi$, etc.

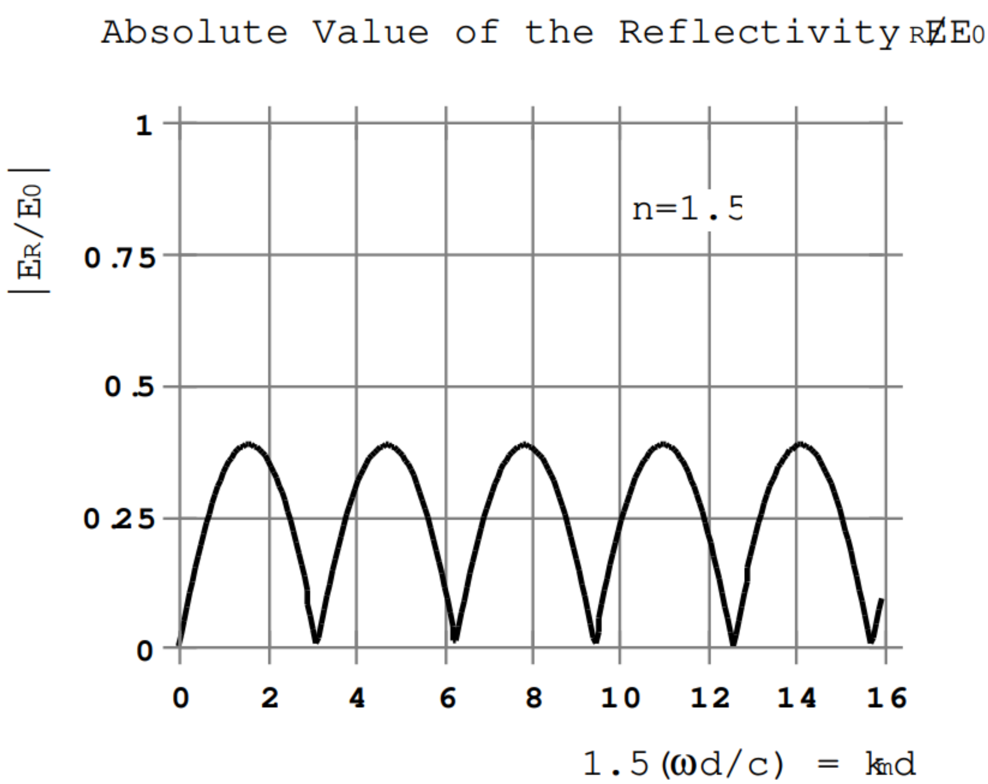
then $\frac{E_R}{E_0} = \frac{(1-n^2)}{(1+n^2)}$ i.e. a maximum

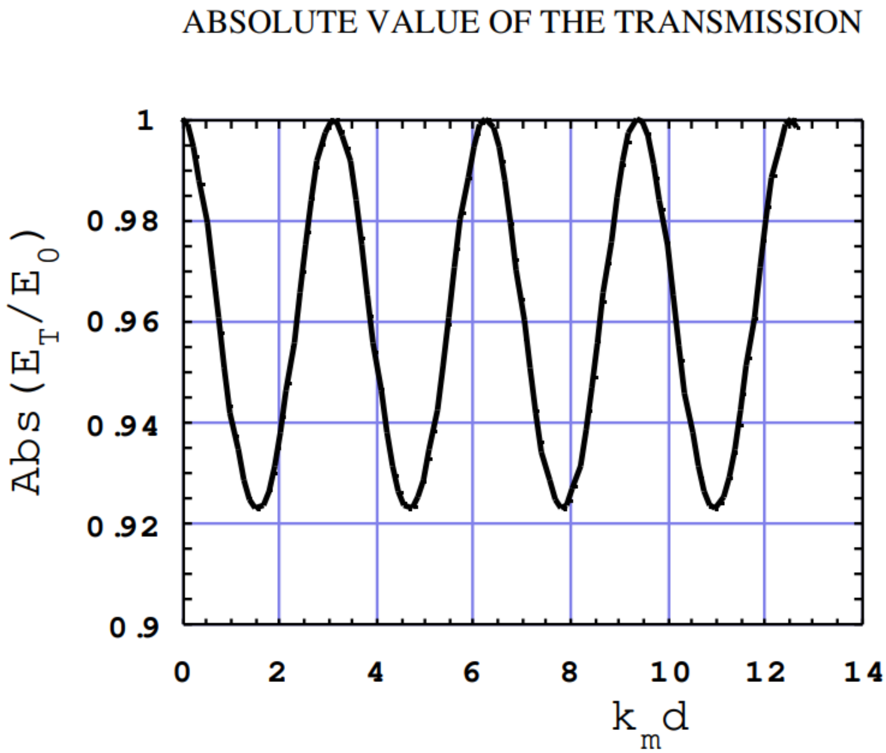
$$\frac{E_T}{E_0} = \frac{\pm 2ni}{(n^2+1)}$$

The variation with frequency of the reflectivity and the transmission coefficient are plotted below for a real dielectric constant ε_r= 2.25 (n= 1.5).




Imaginary Part of the Reflectivity E_R/E_0
Im(E_R/E_0)
n=1.5
1
0.5
0
-0.5
-1
0 2 4 6 8 10 12 14 16
1.5($\omega d/c$) = $k_m d$


Absolute Value of the Reflectivity E_R/E_0
$|E_R/E_0|$
n=1.5
1
0.75
0.5
0.25
0
0 2 4 6 8 10 12 14 16
1.5($\omega d/c$) = $k_m d$



Problem (10.13).

Let a material be described by electric and magnetic linear response: i.e.

$$\mathbf{D}=\varepsilon(\omega)\mathbf{E},$$

and

$$\mathbf{B}=\boldsymbol{\mu}(\boldsymbol{\omega})\mathbf{H},$$

where both ε(ω) and μ(ω) are complex numbers. These are usually written

$$\varepsilon(\omega)=\varepsilon_0\varepsilon_r=\varepsilon_1+i\varepsilon_2$$

and

$$\mu(\omega)=\mu_0\mu_r=\mu_1+\mu_2.$$

For a time dependence $e^{-i\omega t}$ the imaginary parts of the response functions, $\varepsilon_2(\omega)$ and $\mu_2(\omega)$, are greater than zero.

(a) According to Poynting's theorem the rate of increase of energy stored in the fields is given by

$$\frac{dW}{dt}=\mathbf{E}\cdot\frac{d\mathbf{D}}{dt}+\mathbf{H}\cdot\frac{d\mathbf{B}}{dt}.$$

Show that for a time dependence $e^{-i\omega t}$ the imaginary parts of ε and μ must be greater than zero for any finite frequency. This conclusion follows from the restriction that the time average of $\frac{dW}{dt}$ must be greater than or equal to zero according to the second law of thermodynamics.

(b) Show that for a time dependence $e^{-i\omega t}$ a plane wave solution of Maxwell's equations can be found in the form

$$E_x=E_0e^{i(k_mz-\omega t)} \qquad (1)$$

$$\mathrm{H_Y}=\frac{\mathrm{k_m}}{\omega\mu}\mathrm{E_0e^{i(k_mz-\omega t)}}, \qquad (2)$$

where $k_m^2 = \varepsilon_r \mu_r \left(\frac{\omega}{c}\right)^2$,

and $k_m = \sqrt{\varepsilon_r \mu_r}\left(\frac{\omega}{c}\right) \equiv (N + i\kappa)\left(\frac{\omega}{c}\right)$, where K>0

for a wave damped towards the interior of a semi-infinite slab.

(c) Calculate the time averaged value of the Poynting vector corresponding to the fields of eqns.(1) and (2). Show that

$\langle S_z \rangle = \frac{1}{2C} \frac{(N\mu_1 + K\mu_2)}{(\mu_1^2 + \mu_2^2)} |E_0|^2 e^{-2\kappa\left(\frac{\omega}{c}\right)z}$. (3)

Notice that for a passive medium $<S_z>$ must be greater than, or equal, to zero; this means that $(N\mu_1 + K\mu_2) \geq 0$. For a nonmagnetic material $\mu_1 = \mu_0$ and $\mu_2=0$; thus for a non-magnetic material eqn.(3) states that $n \geq 0$ (for this case N=n).

(d) Calculate the time averaged energy densities corresponding to the waves of eqns.(1) and (2). Show that

$< W_E >= \frac{1}{2}\mathrm{Real}\left(\frac{\varepsilon E E^*}{2}\right) = \frac{\varepsilon_1 |E_0|^2}{4} e^{-2K\left(\frac{\omega}{c}\right)z}$, (4)

and

$< W_B >= \frac{1}{2}\mathrm{Real}\left(\frac{\mu H H^*}{2}\right) = \frac{\varepsilon_0\mu_0}{4}\left(\frac{\mu_1}{\mu_1^2 + \mu_2^2}\right)\left(N^2 + K^2\right)|E_0|^2 e^{-2K\left(\frac{\omega}{c}\right)z}$ (5)

Expressions (4) and (5) do not appear to have much in common except the factor $|E_0|^2 e^{-2K\left(\frac{\omega}{c}\right)z}$. However, from the definition

$$(\varepsilon_1 + i\varepsilon_2)(\mu_1 + i\mu_2) \equiv (N + iK)^2 \varepsilon_0 \mu_0$$

plus some tedious algebra, it can be shown that

$\varepsilon_1 = \left(\frac{(N^2 - K^2)\mu_1 + 2NK\mu_2}{(\mu_1^2 + \mu_2^2)}\right)\varepsilon_0\mu_0$, (6)

and

$\varepsilon_2 = \left(\frac{2NK\mu_1 - (N^2 - K^2)\mu_2}{(\mu_1^2 + \mu_2^2)}\right)\varepsilon_0\mu_0$. (7)

These can be used to write

$< W_E >= \frac{\varepsilon_0\mu_0}{4}\left(\frac{(N^2 - K^2)\mu_1 + 2NK\mu_2}{(\mu_1^2 + \mu_2^2)}\right)|E_0|^2 e^{-2K\left(\frac{\omega}{c}\right)z}$. (8)

(e) Calculate the total time averaged energy density associated with the electric and magnetic fields of eqns.(1) and (2). Show that since $< W >= \langle W_E \rangle + \langle W_B \rangle$ it follows that

$< W >= \frac{\varepsilon_0\mu_0}{2} \frac{N(N\mu_1 + K\mu_2)}{(\mu_1^2 + \mu_2^2)} |E_0|^2 e^{-2K\left(\frac{\omega}{c}\right)z}$. (9)

If this energy density is to be non-negative, it follows from eqn.(3) for $<S_z>$ which must be greater than or equal to zero, that $N \geq 0$. By comparison of eqns.(3) and (9) one finds also that

$$< S_z >= \left(\frac{c}{N}\right) < W >.$$

I know of no fundamental microscopic reason why the real part of the index of refraction should be confined to positive values. It is true, however, that for the metals that I have checked, Fe,Co,Ni,Cu,Ag,Au, and Al, the real part of the index of refraction, n, is greater than zero over the energy range 0.1 to 100 eV. For example,

(i) Cu: n is a minimum at 1.80 eV where n=0.21 and κ=4.25; the index then increases with energy but becomes less than 1 for energies greater than 9.0 eV.

(ii) Ag: n is a minimum at 3.5 eV where n=0.21 and κ=1.42; the index then increases with energy and becomes again less than 1 for energies greater than 25 eV.

(iii) Au: n is a minimum at 1.40 eV where n=0.08 and κ=5.44; the index then increases with energy but becomes less than 1 for energies greater than 22 eV.

(iv) Al: n is a minimum at 12.0 eV where n=0.033 and κ=5.44; the index then increases with energy but drops below 1 for energies greater than 95 eV.

Answer (10.13).

(a) Let $E_x = E_0 e^{-i\omega t} = E_0 \cos\omega t$

then $D_{\mathrm{X}} = (\varepsilon_1 + i\varepsilon_2) E_0 e^{-i\omega t}$

or

$$\mathrm{D_x} = \varepsilon_1 \mathrm{E_0 Cos}\omega \mathrm{t} + \varepsilon_2 \mathrm{E_0}\,\mathrm{Sin}\,\omega \mathrm{t}.$$

$$\frac{dW_E}{dt} = E_x \frac{dD_x}{dt} = E_0 \cos\omega t\left(-\varepsilon_1 \omega E_0\, \mathrm{Sin}\,\omega t + \varepsilon_2 \omega E_0 \cos\omega t\right),$$

$$\frac{\mathrm{d}W_{\mathrm{E}}}{\mathrm{dt}} = -\omega\varepsilon_1 \mathrm{E}_0^2 \sin\omega \mathrm{t}\cos\omega \mathrm{t} + \omega\varepsilon_2 \mathrm{E}_0^2 \cos^2\omega \mathrm{t}.$$

Therefore $< \frac{dW_E}{dt} >= \omega\varepsilon_2 \frac{E_0^2}{2}$.

It follows that if $< \frac{dW_E}{dt} > \ \geq 0$ then $\varepsilon_2 \geq 0$ for any finite frequency.

Similarly, $\mathrm{H_y} = \mathrm{H_0 e^{-i\omega t}} = \mathrm{H_0}\cos\omega \mathrm{t}$,

and $\mathrm{B_y} = \mu \mathrm{H_y} = \mu_1 \mathrm{H_0}\,\cos\omega \mathrm{t} + \mu_2 \mathrm{H_0}\sin\omega t$.

$$\frac{\mathrm{dB_Y}}{\mathrm{dt}} = \omega \mathrm{H_0}\left(-\mu_1 \sin\omega \mathrm{t} + \mu_2 \cos\omega \mathrm{t}\right),$$

therefore

$$\frac{dW_B}{dt} = \mathbf{H}\cdot\frac{d\mathbf{B}}{dt}$$

$$\frac{\mathrm{d}W_B}{dt} = \omega H_0^2\left(-\mu_1 \sin\omega t\cos\omega t + \mu_2 \cos^2\omega t\right),$$

and

$$< \frac{dW_B}{dt} >= \omega\mu_2 \frac{H_0^2}{2}.$$

It follows that if $< \frac{dW_B}{dt} > \ \geq 0$ then $\mu_2 \geq 0$ for any finite frequency.

(b) Maxwell's equations for a time dependence $e^{-i\omega t}$ can be written

$$\mathrm{curl}\,\mathbf{E} = \mathrm{i}\,\omega\mu\mathbf{H} = i\omega\mu_r\mu_0\mathbf{H} \qquad (i)$$

$$\mathrm{curl}\,\mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E} = -\mathrm{i}\omega\varepsilon_{\mathrm{r}}\varepsilon_0\mathbf{E} \qquad (ii)$$

where from (i) div**H**=0 and from (ii) div**E**=0 because the divergence of any curl must vanish. The fields **E**,**H** therefore satisfy

$$\nabla^2\mathbf{E} = -\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2\mathbf{E},$$

$$\nabla^2\mathbf{H} = -\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2\mathrm{H}.$$

Let **E** be polarized along x and **H** be polarized along y. Then plane wave solutions of the above equations are

$$E_x = E_0 e^{i(k_m z - \omega t)}$$

and

$$\mathrm{H_y} = \mathrm{H_0 e^{i(k_m z - \omega t)}}$$

or

$$\mathrm{H_Y} = \frac{\mathrm{k_m}}{\omega\mu}\mathrm{E_0}\mathrm{e}^{\mathrm{i}(\mathrm{k_m}z-\omega t)},\ \text{from eqn. (i)},$$

where $\mathrm{k}_{\mathrm{m}}^2 = \varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2$

or $\mathrm{k_m} = (\mathrm{N}+\mathrm{iK})\left(\frac{\omega}{\mathrm{c}}\right)$,

where $\mathrm{N}+\mathrm{iK} = \sqrt{\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}}$.

It is necessary to use the branch of the square root for which K≥0, since this branch corresponds to a disturbance which dies away with increasing z.

$$<\mathrm{S_z}> = \frac{1}{2}\mathrm{Real}\left(\mathrm{E_x}\mathrm{H}_{\mathrm{y}}^{*}\right),$$

$$<\mathrm{S_z}> = \frac{1}{2}\mathrm{Real}\left(\mathrm{E_0}\mathrm{e}^{\mathrm{ik_m}z}\frac{\mathrm{k}_{\mathrm{m}}^{*}}{\omega\mu^{*}}\mathrm{E}_0^{*}\mathrm{e}^{-\mathrm{ik}_{\mathrm{m}}^{*}z}\right)$$

$$<\mathrm{S_z}> = \frac{1}{2\mathrm{c}}\mathrm{Real}\left(\frac{(\mathrm{N}-\mathrm{iK})(\mu_1+\mathrm{i}\mu_2)}{\mu\mu^{*}}|\mathrm{E_0}|^2 e^{-2\mathrm{K}\left(\frac{\omega}{\mathrm{c}}\right)\mathrm{z}}\right),$$

$$\langle \mathrm{S_z}\rangle = \frac{1}{2\mathrm{c}}\frac{(\mathrm{N}\mu_1+\mathrm{K}\mu_2)}{(\mu_1^2+\mu_2^2)}|\mathrm{E_0}|^2\mathrm{e}^{-2\mathrm{K}\left(\frac{\omega}{\mathrm{c}}\right)\mathrm{z}}.$$

(d) $\langle W_E\rangle = \left\langle\frac{\mathbf{E}\cdot\mathbf{D}}{2}\right\rangle = \frac{1}{4}\mathrm{Real}(E_xD_x^{*})$, or

$$\langle W_E\rangle = \frac{1}{4}\mathrm{Real}\left(E_0e^{i(k_mz-\omega t)}(\varepsilon_1-i\varepsilon_2)E_0^{*}e^{-i(k_m^{*}z-\omega t)}\right),$$

$$<W_E> = \frac{1}{4}\mathrm{Real}\left((\varepsilon_1-i\varepsilon_2)|E_0|^2e^{i(k_m-k_m^{*})z}\right).$$

But $(\mathrm{k_m}-\mathrm{k}_{\mathrm{m}}^{*}) = 2\mathrm{iK}\left(\frac{\omega}{\mathrm{c}}\right)$, therefore

$$<W_E> = \frac{\varepsilon_1}{4}|E_0|^2e^{-2K\left(\frac{\omega}{c}\right)z}.$$

$$<W_B> = <\frac{\mu H^2}{2}>,$$

$$<W_B> = \frac{1}{4}\mathrm{Real}\left(\frac{(\mu_1+i\mu_2)}{\omega^2}\frac{k_mk_m^{*}}{\mu\mu^{*}}|E_0|^2e^{-2K\left(\frac{\omega}{c}\right)z}\right).$$

But $\mathrm{k_m}\mathrm{k}_{\mathrm{m}}^{*} = \left(\mathrm{N}^2+\mathrm{K}^2\right)\left(\frac{\omega}{\mathrm{c}}\right)^2$

and $\mu\mu^{*} = \left(\mu_1^2+\mu_2^2\right)$,

so that

$$<W_B> = \frac{1}{4c^2}\frac{\mu_1\left(N^2+K^2\right)}{\left(\mu_1^2+\mu_2^2\right)}|E_0|^2e^{-2K\left(\frac{\omega}{c}\right)z},$$

where $c^2 = 1/\varepsilon_0\mu_0$.

(e) Just add together $<W_E>$ and $<W_B>$ and use eqn.(6) above to get

$$<\mathrm{W}> = \frac{\varepsilon_0\mu_0}{2}\frac{\mathrm{N}(\mathrm{N}\mu_1+\mathrm{K}\mu_2)}{(\mu_1^2+\mu_2^2)}|\mathrm{E_0}|^2\mathrm{e}^{-2\mathrm{K}\left(\frac{\omega}{\mathrm{c}}\right)\mathrm{z}}. \qquad (9)$$

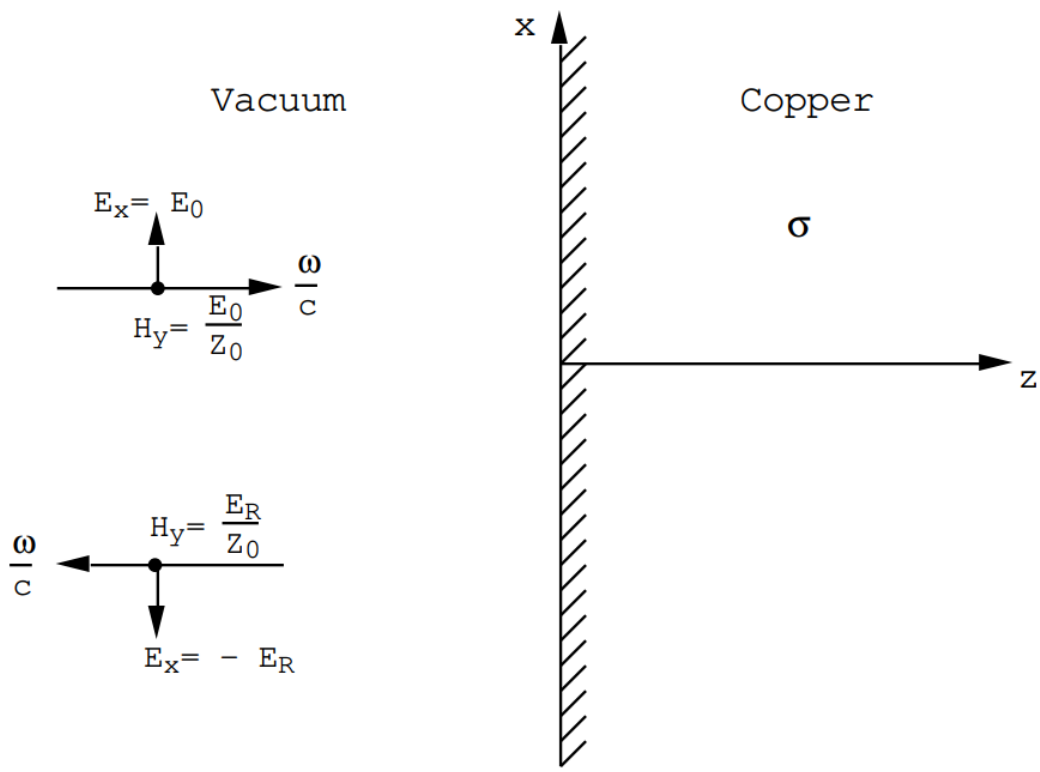
Problem (10.14).

Radiation having a frequency of 1 MHz falls at normal incidence from vacuum upon a thick copper sheet. The copper sheet is parallel with the x-y plane and the surface of the sheet lies at z=0. The resistivity of copper is ρ= $2.0\mathrm{x}10^{-8}$ Ohm-meters at room temperature.

(a) How much energy is absorbed per square meter by the copper sheet if the electric field strength in the incident wave is 1 V/m?

(b) What will be the energy absorbed per m^2 if the incident radiation falls on the surface at an angle of incidence of 45°? Let the incident radiation be p-polarized.

Answer (10.14).



In the metal

$$\begin{aligned}\operatorname{curl}\mathbf{E} &= i\omega\mu_0\mathbf{H}\\ \operatorname{curl}\mathbf{H} &= \sigma\mathbf{E}\\ \operatorname{div}\mathbf{E} &= 0\\ \operatorname{div}\mathbf{H} &= 0\end{aligned}$$

therefore

$$\operatorname{curl}\operatorname{curl}\mathbf{H} = \mathbf{i}\omega\sigma\mu_0\mathbf{H},$$

and $\operatorname{curl}\operatorname{curl}\mathbf{E} = \mathbf{i}\omega\sigma\mu_0\mathbf{E},$

where

$$\mathrm{k}_\mathrm{m}^2 = \mathrm{i}\omega\sigma\mu_0,$$

or

$$\mathrm{k}_\mathrm{m} = \sqrt{\frac{\omega\mu_0\sigma}{2}}(1+\mathrm{i}) = \frac{(1+\mathrm{i})}{\delta}.$$

Also $\frac{\partial \mathrm{E}_\mathrm{x}}{\partial z} = \mathrm{i}\omega\mu_0\mathrm{H}_\mathrm{Y}$

or

$$\mathrm{E}_\mathrm{x} = \left(\frac{\omega\mu_0}{\mathrm{k}_\mathrm{m}}\right)\mathrm{H}_\mathrm{y} = \left(\frac{\delta\omega}{\mathrm{c}}\right)\mu_0\mathrm{c}\frac{(1-\mathrm{i})}{2}\mathrm{H}_\mathrm{Y}.$$

For this problem $\omega = 2\pi \mathrm{x} 10^6$ radians/sec

$\frac{\omega}{c}=0.0209/\mathrm{m}$

$\delta=\sqrt{\frac{2}{\omega\mu_0\sigma}}=0.71\times10^{-4}\ \mathrm{m}=71\mu\mathrm{m}$

and $\frac{\delta\omega}{\mathrm{c}}=1.49\times10^{-6}$.

In the incident wave E_0= 1 V/m, and $\mathrm{H}_0=\frac{\mathrm{E}_0}{\mathrm{Z}_0}=2.65\times10^{-3}$ Amps/m.

Just inside the metal surface $\mathrm{H_T}\cong 2\mathrm{H}_0=5.31\times10^{-3}$ Amps/m.

Therefore $E_x=\left(1.49\times10^{-6}\right)(377)\frac{(1-i)}{2}\left(5.31\times10^{-3}\right)$

$$E_x=\left(1.49\times10^{-6}\right)(1-i)\quad V/m\,.$$

$$\langle S_x\rangle=\frac{1}{2}\mathrm{Real}\left(H_yE_x^*\right)$$

$$\langle S_x\rangle=\mathrm{Real}\left(\left(2\cdot65\times10^{-3}\right)\left(1\cdot49\times10^{-6}\right)(1+i)\right)$$

$$\langle S_x\rangle=3.95\times10^{-9}\mathrm{Watts}\,/\mathrm{m}^2\,.$$

(b) The incident wave is given by

$$\mathrm{H_Y}=\mathrm{H_0e^{iqx}e^{ikz}}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{E_x}=\frac{\mathrm{Z_0H_0}}{\sqrt{2}}\mathrm{e^{iqx}e^{ikz}e^{-i\omega t}}$$

$$\mathrm{E_z}=-\frac{\mathrm{Z_0H_0}}{\sqrt{2}}e^{\mathrm{iqx}}e^{\mathrm{ikz}}e^{-\mathrm{i}\omega\mathrm{t}}.$$

The reflected wave is given by

$$\mathrm{H_y}=\mathrm{H_Re^{-ikz}}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{E_x}=-\frac{\mathrm{Z_0H_R}}{\sqrt{2}}e^{iqx}e^{-\mathrm{ik}z}e^{-\mathrm{i}\omega\mathrm{t}}$$

$$\mathrm{E_z}=-\frac{\mathrm{Z_0H_R}}{\sqrt{2}}e^{iqx}e^{-\mathrm{ik}z}e^{-\mathrm{i}\omega t}.$$

In the metal $-\nabla^2\mathbf{H}=i\omega\sigma\mu_0\mathbf{H}$

therefore

$$q^2+k_m^2=i\omega\sigma\mu_0$$

$$q^2+k_m^2=\frac{i\left(4\pi\times10^{-7}\right)\left(2\pi\times10^6\right)}{2\times10^{-8}}=3.95i\times10^8/\mathrm{m}^2$$

and

$$q=\frac{\omega}{\mathrm{c}\sqrt{2}}=0.0148/\mathrm{m},\quad \mathrm{i.e}\cdot\mathrm{q}^2=2.18\times10^{-4}/\mathrm{m}^2.$$

In other words, q^2 is completely negligible compared with k_m^2. This is, for all intents and purposes, the same problem as part (a). The energy absorbed from the incident wave will be **3.95x10^{-9} Watts/m^2**. For completeness, if

$\mathrm{H_Y}=\mathrm{H_T}e^{iqx}e^{ik_mz}$, then $\mathrm{E_x}=-\frac{\mathrm{ik_m}}{\sigma}\mathrm{H_Te^{iqx}e^{ik_mz}}$, where $\mathrm{H_T}\cong 2/\mathrm{Z}_0$, and Z_0= $c\mu_0$.