

## 13.8: Chapter 8

### Problem (8.1)

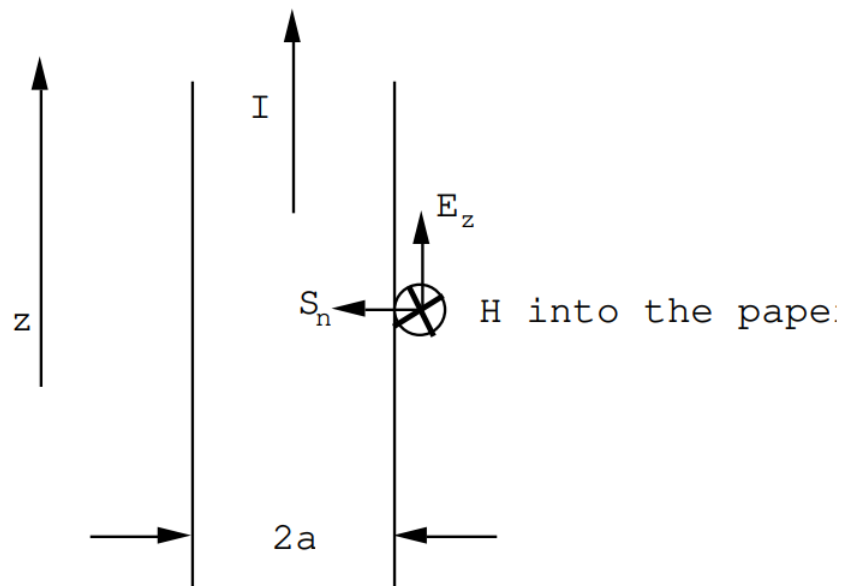
A long straight non-magnetic wire carries a steady current of  $I$  Amps. The resistance of the wire is  $R$  Ohms/meter. Use Poynting's theorem to show that the energy flow into the wire is  $I^2 R$ /meter.

### Answer (8.1)

At the surface of the wire the magnetic field is tangential. From  $\text{Curl} \mathbf{B} = \mu_0 \mathbf{J}_f$  (No  $\mathbf{M}$ , no time variation) one has, using Stokes' theorem,

$$2\pi a B_\theta = \mu_0 I$$

$$\text{So } B_\theta = \frac{\mu_0 I}{2\pi a} \text{ or } H_\theta = \frac{I}{2\pi a}.$$



The electric field is also tangential  $E_z = IR$  Volts/m.

$$\therefore \mathbf{S} = \mathbf{E} \times \mathbf{H} \text{ is normal to the wire surface and } S_n = \frac{I^2 R}{2\pi a}. \text{ Energy flow/m} = (S_n)(2\pi a) = I^2 R \text{ Watts/m.}$$

### Problem (8.2).

One meter of wire is bent into a circular form to make a magnetic dipole antenna. The wire carries a current  $I(t) = I_0 \sin \omega t$  where  $I_0 = 2$  Amps,  $\omega = 2\pi f$ , and  $f = 50$  MHz. At what rate does this loop radiate energy?

### Answer (8.2).

For a magnetic dipole at the origin,  $m_z$ , one has

$$B_r = \left(\frac{\mu_0}{4\pi}\right) 2 \cos \theta \left[ \frac{m_z}{R^3} + \frac{\dot{m}_z}{cR^2} \right]$$

$$B_\theta = \frac{\mu_0}{4\pi} \sin \theta \left[ \frac{m_z}{R^3} + \frac{\dot{m}_z}{cR^2} + \frac{\ddot{m}_z}{c^2 R} \right]$$

$$E_\phi = -cB_\theta = -\left(\frac{\mu_0}{4\pi}\right) c \sin \theta \left[ \frac{\dot{m}_z}{cR^2} + \frac{\ddot{m}_z}{c^2 R} \right]$$

(part of  $B_\theta$  that is proportional to the time derivatives).

Very far from the magnetic dipole one has only the radiation field terms  $\sim 1/R$ :

$$E_\phi = -\left(\frac{\mu_0}{4\pi}\right) \sin \theta \frac{\ddot{m}_z}{cR}$$

$$B_{\theta} = \left( \frac{\mu_0}{4\pi} \right) \sin \theta \frac{\ddot{m}_z}{c^2 R}$$

$$\text{or } H_{\theta} = \frac{1}{4\pi} \sin \theta \frac{\ddot{m}_z}{c^2 R}$$

$\mathbf{S} = \mathbf{E} \times \mathbf{H}$  has only a radial component

$$S_r = \frac{\mu_0}{(4\pi)^2} \sin^2 \theta \frac{(\ddot{m}_z)^2}{c^3 R^2}.$$

If  $I = I_0 \sin \omega t$  then  $m_z = (\pi a^2) I_0 \sin \omega t$

$$\ddot{m}_z = -\omega^2 (\pi a^2 I_0) \sin \omega t.$$

Let  $m_0 = \pi a^2 I_0$ . In this problem  $2\pi a = 1$  m so  $m_0 = 0.159$  Amp m<sup>2</sup>.

At  $R$  and at the angle  $\theta$  the time averaged Poynting vector is given by

$$\langle S_r \rangle = \frac{\mu_0}{(4\pi)^2} \sin^2 \theta \frac{\omega^4 m_0^2}{2c^3 R^2} = 0.0363 \frac{\sin^2 \theta}{R^2} \text{ Watts } / m^2$$

Integrate this over a sphere of radius  $R$ . The element of surface area is  $dA = 2\pi R^2 \sin \theta d\theta$

$$\therefore \text{radiated power } P = (0.0363)(2\pi) \int_0^\pi \sin^3 \theta d\theta \text{ Watts}$$

$$\text{but } \int_0^\pi \sin^3 \theta d\theta = 4/3$$

$$\therefore P = \mathbf{0.305 \text{ Watts}}$$

### Problem (8.3)

An electric dipole whose strength is  $p_0 = 10^{-7}$  Coulomb-meters oscillates at a frequency  $f = 50$  MHz. Let the dipole be oriented along  $z$ .

(a) Estimate the electric field amplitude measured by an observer on the  $x$ -axis at a mean distance of 5 meters from the dipole. Compare the near field terms with the far-field, or radiation term. Note that the  $\dot{p}$  term is in quadrature with the other two terms so that it contributes only about 2% to the electric field amplitude.

(b) How big is the phase shift between the time variation of the dipole and the electric field measured by the observer?

(c) What intensity would the observer measure at the coordinates (5,0,0); i.e. what is the value of  $\langle S_x \rangle$ ?

(d) What would be the intensity of the radiation measured by an observer on the  $z$ -axis 5 meters from the oscillating dipole?

(e) At what total rate does this dipole radiate energy?

(f) This dipole can be modelled by two spheres each having a radius of 0.1 meters and separated by 0.5 meters center to center. One sphere carries an initial charge of  $Q = +2 \times 10^{-7}$  Coulombs, the other sphere carries an initial charge of  $-2 \times 10^{-7}$  Coulombs. The two spheres are suddenly connected by a conducting wire and the two charges oscillate back and forth. Estimate how long is required for this system to radiate away  $e^{-1}$  of its initial energy. (The stored energy is proportional to  $Q^2$ ; the rate at which energy is radiated away is proportional to  $Q^2$  because  $p_z = Qd$ . It follows that the energy stored on the two spheres will decay exponentially in time).

### Answer (8.3).

$$(a) p_z = p_0 e^{-i\omega t}$$

$$\text{where } \omega = 2\pi f = 3.14 \times 10^8 \text{ radians/sec, and } \frac{e}{c} = 1.047 \text{ m}^{-1}.$$

$$E_r = \frac{1}{4\pi\epsilon_0} 2 \cos \theta \left( \frac{p_z}{R^3} + \frac{\dot{p}_z}{cR^2} \right)$$

$$E_{\theta} = \frac{1}{4\pi\epsilon_0} \sin \theta \left( \frac{p_z}{R^3} + \frac{\dot{p}_z}{cR^2} + \frac{\ddot{p}_z}{c^2 R} \right)$$

$$E_{\phi} = 0$$

$$cB_{\phi} = \frac{1}{4\pi\epsilon_0} \sin \theta \left( \frac{\dot{p}_z}{cR^2} + \frac{\ddot{p}_z}{c^2 R} \right)$$

$$B_r = B_{\theta} = 0.$$

On the x-axis  $\theta = \pi/2$ , and  $\cos\theta=0$ ,  $\sin\theta=1$ . Consequently, one has only to worry about the electric field component,  $E_\theta$ . Taking out the common factors one has

$$\text{first term: } \frac{1}{R^3} = \frac{1}{125} = 0.008$$

$$\text{second term: } -\frac{i\omega}{cR^2} = -0.042i$$

$$\text{third term: } \frac{-\omega^2}{c^2 R} = -0.219$$

The total field is proportional to  $-0.211 - 0.042i$ . The quadrature term makes only a  $\sim 2\%$  correction to the field. For an observer along x and 5 meters from the dipole the electric field is polarized along z: it is given by

$$|E_z| = 193 \text{ Volts/meter.}$$

(b) The phase shift between the time variation of the dipole and the electric field at the observer is

$$\Delta\phi = \frac{\omega R}{c} = 5.235 \text{ radians} = 300^\circ$$

(c) For an observer at (5,0,0)

$$E_\theta = -90(2.11 + 0.419i)$$

$$H_\phi = -0.2387(2.193 + 0.419i)$$

Therefore

$$\langle S_X \rangle = \frac{1}{2} \text{Real}(E_\theta H_\phi^*) = 10.74(4.627 + 0.176 + 0.035i) \text{ Watts/m}^2,$$

$$\langle S_X \rangle = 51.6 \text{ Watts/m}^2$$

(d) For an observer at (0,0,5) the angle  $\theta$  is zero, and consequently  $E_\theta=0$  and  $B_\phi = H_\phi = 0$ ; thus  $\langle S_z \rangle = 0$ .

$$(e) \langle P \rangle = \frac{1}{3} \frac{C}{4\pi\epsilon_0} p_0^2 \left(\frac{\omega}{c}\right)^4 = 10.8 \text{ kWatts}$$

#### Problem (8.4)

A 10 turn circular coil of wire is centered on the origin and the plane of the coil lies parallel with the xy plane. The coil has a mean radius of 5 cm and it carries a current  $I(t) = I_0 \sin\omega t$  where  $I_0 = 100$  Amps, and  $\omega = 2\pi f$  where  $f = 20$  MHz. An observer in the xy plane, and 1 km distant from the coil, measures the emf induced in a piece of straight wire 1 meter long due to the radiation field produced by the oscillating current in the coil.

(a) In what direction should the observer orient the wire in order to obtain the maximum signal?

(b) What maximum receiver power would you expect the observer to measure using a matched receiver? The radiation resistance of a short wire of length  $L$  meters ( $L/\lambda \ll 1$ ) is given by  $R = 80\pi^2 \left(\frac{L}{\lambda}\right)$  Ohms.

(c) Calculate the total average rate at which energy is radiated by the oscillating magnetic dipole formed by the coil.

#### Answer (8.4).

(a) The wire should be oriented parallel with the xy plane and perpendicular to the line joining the observer to the coil. The electric field has only an  $E_\phi$  component.

(b) For this problem one can ignore the near field terms and calculate only the radiation field terms. These are

$$B_\theta = \frac{\mu_0}{4\pi} \sin\theta \frac{\ddot{m}_z}{c^2 R} \text{ Teslas,}$$

$$E_\phi = -cB_\theta \text{ Volts/meter.}$$

In this problem  $m_z = m_0 e^{-i\omega t}$  where  $m_0 = 7.85$  Amp-meters<sup>2</sup>,  $\omega = 1.257 \times 10^8$  radians/sec.,  $\frac{\omega}{c} = 0.419 \text{ m}^{-1}$  and  $\lambda = 15$  m. For an observer in the x-y plane the angle  $\theta = \pi/2$  so that

$$B_\theta = -\frac{\mu_0}{4\pi} \frac{\omega^2}{c^2} \frac{m_0}{R} = 1.378 \times 10^{-10} \text{ Teslas}$$

The electric field strength at the observer will be  $E_\phi = -cB\theta = 41.3 \times 10^{-3}$  Volts/m. The current induced in the wire will have the spatial variation  $I(z) = I_0 \sin\left(\frac{2\pi}{\lambda} \left[\frac{L}{2} - z\right]\right)$  for  $z > 0$  with a similar variation for  $z < 0$ . The average power delivered to the antenna will be

$$\langle P_i \rangle = \frac{1}{2} (2) E_0 I_0 \int_0^L dz \sin\left(\frac{2\pi}{\lambda} \left[\frac{L}{2} - z\right]\right) = E_0 I_0 \frac{\lambda}{2\pi} \left(1 - \cos\left(\frac{\pi L}{\lambda}\right)\right),$$

or for small  $L/\lambda$   $\langle P_i \rangle \cong E_0 I_0 \left(\frac{\pi}{4}\right) \left(\frac{L^2}{\lambda}\right)$ . Half this power is delivered to the load:

$$E_0 I_0 \left(\frac{\pi}{4}\right) \left(\frac{L^2}{\lambda}\right) = I_0^2 R, \text{ where } R = 20\pi^2 \left(\frac{L}{\lambda}\right)^2 = 0.877 \text{ Ohms.}$$

$I_0 = 2.468 \times 10^{-3}$  Amps. The power delivered to the matched load is  $\langle P_L \rangle = \frac{R}{2} I_0^2 = 2.67 \times 10^{-6}$  Watts.

(c) The total rate at which a magnetic dipole radiates energy is given by

$$P_M = \frac{c}{3} \frac{\mu_0}{4\pi} \left(\frac{\omega}{c}\right)^4 m_0^2 \text{ Watts} = 19.0 \text{ Watts.}$$

This contributes an amount  $Z$  Ohms to the coil resistance where  $\frac{I_0^2 Z}{2} = 19.0$  Watts. This gives  $Z = 3.8 \times 10^{-3}$  Ohms since the current amplitude was assumed to be 100 Amps. This means that very heavy wire should be used for the oscillator tank coil if one wishes most of the input power to be radiated as electromagnetic energy rather than dissipated in the coil as heat.

### Problem (8.5).

Two identical electric dipoles are driven by the same oscillator at a frequency of 20 MHz but there is a phase shift of  $\beta$  radians between them. The dipoles are both oriented along the  $z$ -axis, but one dipole is located at (5,0,0), the other is located at (-5,0,0). Describe the angular variation of the maximum radiation field intensity produced by these two dipoles as measured by an observer confined to the  $x$ - $y$  plane and located a constant distance of 1 km from the origin; i.e. make a plot of the time-averaged Poynting vector as a function of the angle  $\phi$  measured from the  $x$ -axis for (a)  $\beta = 0$  radians, and (b)  $\beta = \frac{\pi}{2}$  radians.

### Answer (8.5).

One has only to worry about the far field terms generated by the oscillating dipoles, however the phase at the observer is very important in this problem.

Dipole #1:

$$E_z^1 = \frac{1}{4\pi\epsilon_0} \left(\frac{\omega}{c}\right)^2 \frac{p_0}{R} e^{-i\omega\left(t - \left[\frac{R-d\cos\phi}{c}\right]\right)} \exp(i\beta)$$

Dipole #2:

$$E_z^2 = \frac{1}{4\pi\epsilon_0} \left(\frac{\omega}{c}\right)^2 \frac{p_0}{R} e^{-i\omega\left(t - \left[\frac{R+d\cos\phi}{c}\right]\right)}.$$

In writing these expressions explicit account has been taken of the fact that the distance from the observer to each dipole is slightly different. But slight as it may be compared with  $R$ , the difference in distance is a large fraction of a wavelength ( $\lambda = 15$  meters). The total electric field measured by the observer in the  $x$ - $y$  plane is given by

$$E_z = \frac{1}{4\pi\epsilon_0} \left(\frac{\omega}{c}\right)^2 \frac{p_0}{R} e^{i\omega(t-R/c)} e^{i\beta/2} \left( e^{i\frac{\omega}{c} d \cos\phi + i\beta/2} + e^{-i\frac{\omega}{c} d \cos\phi - i\beta/2} \right),$$

or

$$E_S = 2E_0 e^{i\omega(t-R/c)} e^{i\beta/2} \cos\left(\frac{\omega}{c} d \cos\phi + \beta/2\right),$$

where  $E_0 = \frac{1}{4\pi\epsilon_0} \left(\frac{\omega}{c}\right)^2 \frac{p_0}{R}$ .

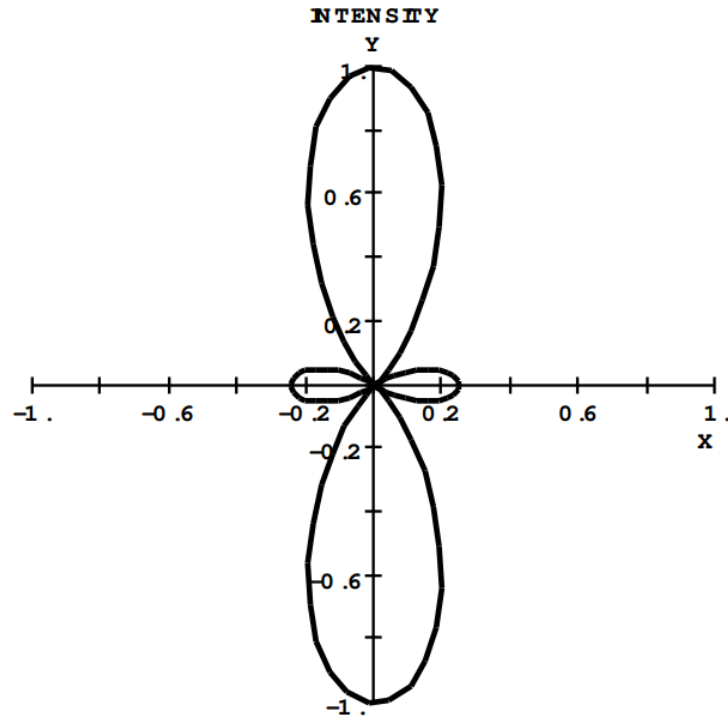
The time averaged Poynting vector is proportional to the square of the electric field strength; for one oscillator  $\langle S_0 \rangle = \frac{E_0^2}{2Z_0}$ , where  $Z_0 = 377$  Ohms. We may therefore write

$$\langle S \rangle = 4 \langle S_0 \rangle \cos^2 \left( \frac{\omega}{c} d \cos \phi + \beta/2 \right).$$

(a)  $\beta=0$ , and  $d= 5.0$  m.  $(\omega/c)= 0.419 \text{ cm}^{-1}$  so

$$\langle S \rangle = 4 \langle S_0 \rangle \cos^2 (2.094 \cos \phi).$$

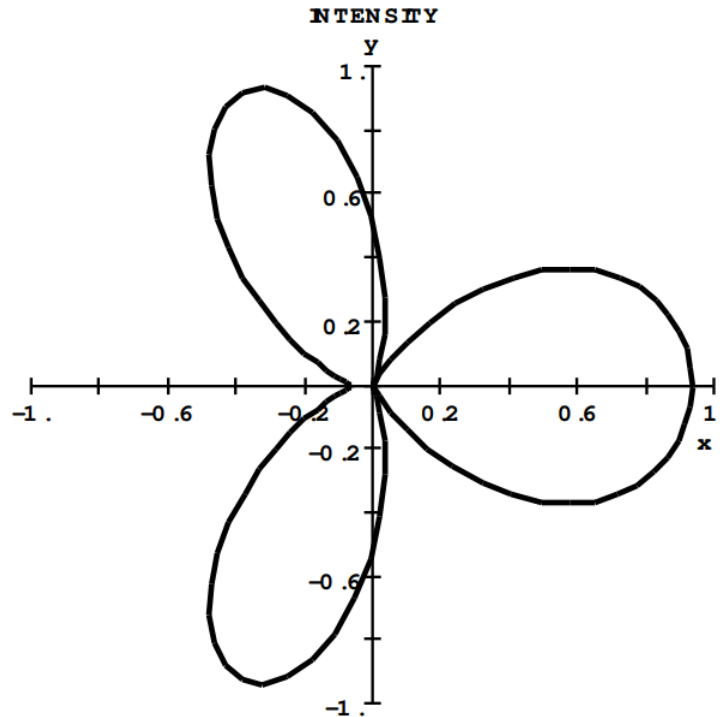
A polar plot of  $\cos^2(2.094 \cos \phi)$  is shown below.



(b) For a phase shift of  $\pi/2$  between the oscillators one finds

$$\langle S \rangle = 4S_0 \cos^2 \left( \frac{\omega}{c} d \cos \phi + \frac{\pi}{4} \right) = 4S_0 \cos^2 \left( 2.094 \cos \phi + \frac{\pi}{4} \right).$$

A plot of the  $\cos^2 \left( 2.094 \cos \phi + \frac{\pi}{4} \right)$  pattern is shown below.



**Problem (8.6).**

100 Watts/m<sup>2</sup> of monochromatic laser light having a free space wavelength of  $\lambda = 0.5145 \mu\text{m}$  is used to illuminate a stationary hydrogen atom. For this problem atomic hydrogen can be modelled by an oscillator having a single resonant frequency given by the  $n=1$  to  $n=2$  transition at 10.18 eV. the oscillator strength may be taken to be unity

- Estimate the total power removed from the incident beam by the hydrogen atom.
- How much energy would be scattered per second into a photomultiplier tube having an aperture of 3 cm and located 10 cm from the atom? The axis of the photomultiplier tube is oriented perpendicular to the incident beam in such a direction as to intercept the maximum scattered power.

**Answer (8.6).**

- Calculate the resonant frequency associated with the  $n=1$  to  $n=2$  transition in the hydrogen atom:

$$10.18\text{eV} = 16.31 \times 10^{-19} \text{ Joules.}$$

$$hf_0 = 16.31 \times 10^{-19}, \quad \text{so}$$

$$f_0 = 2.461 \times 10^{15} \text{ Hz and } \lambda = 0.1219 \mu\text{m.}$$

$$\omega_0 = 2\pi f_0 = 1.546 \times 10^{16} \text{ radians/sec.}$$

The equation of motion for the bound electron in the hydrogen atom can be written

$$m \frac{d^2 z}{dt^2} + kz = -eE_0 e^{-i\omega t},$$

where  $k/m = \omega_0^2$ . Under the influence of a driving field at circular frequency  $\omega$  the electron amplitude is given by

$$z_0 = -\frac{(e/m)E_0}{[\omega_0^2 - \omega^2]}.$$

Now calculate the electric field amplitude in the incident beam. The frequency of the incident light is  $f = c/\lambda = 5.831 \times 10^{14}$  Hz, and  $\omega = 3.664 \times 10^{15}$  radians/sec. The power in the incident beam is

$$\langle S \rangle = \frac{E_0 H_0}{2} = \frac{E_0^2}{2Z_0} = 10^2 \text{ Watts/m}^2,$$

where  $Z_0 = 377 \text{ Ohms}$ . From this expression  $E_0 = 275 \text{ Volts/m}$ . For this field amplitude the electron amplitude is

$$z_0 = -2.14 \times 10^{-19} \text{ meters, corresponding to}$$

an induced dipole moment amplitude  $p_0 = -ez_0 = 3.43 \times 10^{-38} \text{ Coulomb-meters}$ . The averaged power radiated by the oscillating atomic electron is

$$P_E = \frac{c}{3} \frac{1}{4\pi\epsilon_0} \left(\frac{\omega}{c}\right)^4 p_0^2 = 23.53 \times 10^{-30} \text{ Watts}$$

A photon at  $0.5145 \mu\text{m}$  carries an energy of  $38.64 \times 10^{-20} \text{ Joules}$ . The hydrogen atom scatters the equivalent of  $6.1 \times 10^{-11}$  photons per second. One would have to wait approximately 3200 years in order to get only 6 photons! (1 year =  $3.15 \times 10^7$  seconds).

(b) The area of the photomultiplier tube is  $\pi r^2 = 7.07 \text{ cm}^2$ , therefore  $\frac{\text{Area}}{R^2} = d\Omega = \frac{7.07}{100} = 0.0707 \text{ steradians}$ . The power intercepted by the tube will be given by

$$P_T = \frac{1}{8\pi} \left(\frac{c}{4\pi\epsilon_0}\right) \left(\frac{\omega}{c}\right)^4 p_0^2 d\Omega, \text{ since } \sin\theta = 1$$

Therefore  $P_T = 1.99 \times 10^{-31} \text{ Watts}$ . In order to get one count per second one would need  $N = 19.5 \times 10^{11}$  hydrogen atoms. This corresponds to  $7.2 \times 10^{-11} \text{ liters}$  at NTP or  $7.2 \times 10^{-8} \text{ cc}$  of gas at NTP. Such an experiment would be feasible using 1 mm cubed of gas at NTP ( $10^{-3} \text{ cc}$ ); the dark count for a good tube is of order 1.0 counts per second at room temperatures.

### Problem (8.7).

A very thin plane, uniform, sheet of dipoles is located in the y-z plane. The dipole moments are aligned along the z-axis and they vary in time like  $e^{-i\omega t}$ . The polarization density for such a sheet can be written

$$P_z(x, y, z, t) = P_0 \delta(x) e^{-i\omega t}.$$

(a) Show that this time-varying polarization density generates a vector potential which can be written in the form

$$\underline{X} > 0: A_z(X, Y, Z, t) = \frac{\mu_0 P_0 C}{2} e^{i(kX - \omega t)};$$

$$\underline{X} < 0: A_z(X, Y, Z, t) = \frac{\mu_0 P_0 C}{2} e^{-i(kX + \omega t)}.$$

(Hint: one can use the particular solution for the vector potential in terms of the current density. Note also that  $\mathbf{A}$  cannot depend upon either Y or Z from the symmetry of the problem; this means that one can carry out the calculation for  $Y=Z=0$ . You will run into an indeterminate constant upon carrying out the integral; this does not matter because any constant, no matter how large, can be added to the potential without altering the fields calculated from it.)

(b) Use the above vector potential to calculate the scalar potential from the Lorentz condition  $\text{div } \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0$ . Use the potentials to calculate the corresponding electric and magnetic fields.

(c) What is the rate at which energy is radiated away from this sheet of dipoles?

### Answer (8.7).

The vector potential is generated by the total current density  $\mathbf{J}_{\text{tot}} = \mathbf{J}_f + \text{curl } \mathbf{M} + \frac{\partial \mathbf{p}}{\partial t}$ . For this problem the current density is entirely due to the term  $\frac{\partial \mathbf{p}}{\partial t}$ ; this term has only a z-component, so the vector potential which it generates will have only a z-component.

$$J_z = \frac{\partial P_z}{\partial t} = -i\omega P_0 \delta(x) e^{-i\omega t},$$

therefore

$$A_z(X, Y, Z, t) = \frac{\mu_0}{4\pi} \int_{\text{Space}} \frac{dx dy dz (-i\omega P_0) \delta(x) e^{-i\omega(t-R/c)}}{R}$$

where

$$R = ((X-x)^2 + (Y-y)^2 + (Z-z)^2)^{1/2}.$$

From the plane symmetry the vector potential  $A_z$  cannot depend upon the co-ordinates  $Y, Z$ ; it is convenient to take  $Y=Z=0$ . One can use polar co-ordinates in the plane and write

$$R = \sqrt{r^2 + X^2},$$

and  $dydz = 2\pi r dr$ . The integral over  $x$  can be carried out with no effort because of the  $\delta$ -function. One obtains

$$A_z(X, t) = \frac{\mu_0}{4\pi} (-i\omega P_0) \int_0^\infty \frac{2\pi r dr e^{-i\omega t} e^{i\frac{\omega\sqrt{r^2+X^2}}{c}}}{\sqrt{r^2+X^2}}.$$

The integral can be carried out with the help of the substitutions

$$u = \sqrt{r^2 + X^2} \text{ and } u du = r dr.$$

$$A_z(X, t) = -\frac{i\omega\mu_0 P_0}{2} e^{-i\omega t} \int_{|X|}^\infty du e^{i\omega u/c}.$$

The constant is ill-defined; ignore it because a constant has no effect on the fields calculated from the vector potential.

$$A_z(X, t) = \left(\frac{\mu_0 c P_0}{2}\right) e^{i(k|X| - \omega t)}, \text{ where } k = \omega/c.$$

For  $X > 0$  this represents a wave propagating to the right.

For  $X < 0$  this represents a wave propagating to the left.

(b) The scalar potential can be calculated from the Lorentz condition

$$\text{div } \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0.$$

For this example,  $\text{div } \mathbf{A} = 0$  so that the scalar potential is also zero.

The electric field has only a  $z$ -component which is given by  $E_z = -\frac{\partial A_z}{\partial t}$ .

The magnetic field has only a  $y$ -component which is given by  $B_Y = -\frac{\partial A_z}{\partial X}$ .

$$\text{For } X > 0: E_z = \frac{i\omega\mu_0 c P_0}{2} e^{i(kX - \omega t)} \text{ Volts/m}$$

$$B_Y = -\frac{ik\mu_0 P_0 c}{2} e^{i(kX - \omega t)} = -\frac{i\omega\mu_0 P_0}{2} e^{i(kX - \omega t)} \text{ Teslas}.$$

$$\text{For } X < 0: E_z = \frac{i\omega\mu_0 c P_0}{2} e^{-i(kX + \omega t)} \text{ Volts/m,}$$

$$B_Y = \frac{ik\mu_0 P_0 c}{2} e^{-i(kX + \omega t)} = \frac{i\omega\mu_0 P_0}{2} e^{-i(kX + \omega t)} \text{ Teslas}.$$

(c) The plane sheet of polarization radiates a plane wave in each direction which is shifted in phase by  $90^\circ$  with respect to the polarization sheet. The average rate at which energy is radiated in either direction is

$$\langle |S_X| \rangle = \frac{C}{8} \mu_0 \omega^2 P_0^2 \text{ Watts/m}^2.$$

### Problem (8.8).

A very thin plane, uniform, sheet of dipoles is located in the  $y$ - $z$  plane. The dipole moments are aligned along the  $x$ -axis and they vary in time like  $e^{-i\omega t}$ . The polarization density for such a sheet can be written

$$P_X(X, Y, z, t) = P_0 \delta(x) e^{-i\omega t}.$$



This problem differs from Prob.(8.7) in the orientation of the dipoles. It is still true that  $\mathbf{A}$  cannot depend upon either  $Y$  or  $Z$  because of the planar symmetry.

(a) Show that this time-varying polarization density generates a vector potential which can be written in the form

$$X>0: A_X(X, Y, Z, t) = \frac{\mu_0 P_0 c}{2} e^{i(kX - \omega t)},$$

$$X<0: A_X(X, Y, Z, t) = \frac{\mu_0 P_0 c}{2} e^{-i(kX + \omega t)}.$$

(Hint: one can use the particular solution for the vector potential in terms of the current density. You will run into an indeterminate constant upon carrying out the integral; this does not matter because any constant, no matter how large, can be added to the potential without altering the fields calculated from it.)

(b) Show that the electric and magnetic fields corresponding to the above potentials are zero.

**Answer (8.8).**

This problem can be solved by direct integration after the manner of Problem(8.7). A more elegant and usefull method starts from the differential equation for the vector potential:

$$\nabla^2 A_x - \frac{1}{c^2} \frac{\partial A_x}{\partial t^2} = -\mu_0 J_x = i\omega\mu_0 P_0 e^{-i\omega t} \delta(x)$$

The free space solutions of this differential equation are:

$$(i) \text{ on the right, } x>0; A_X = A_0 e^{i(kx - \omega t)},$$

$$(ii) \text{ on the left, } x<0; A_X = A_0 e^{-i(kx + \omega t)},$$

where  $k = \omega/c$ . These plane waves satisfy the homogeneous wave equation

$$\nabla^2 A_X - \frac{1}{c^2} \frac{\partial A_X}{\partial t^2} = 0.$$

The amplitudes of the two waves must be the same by symmetry. Near  $x=0$  one requires

$$\frac{\partial^2 A_x}{\partial x^2} + \left(\frac{\omega}{c}\right)^2 A_x = i\omega\mu_0 P_0 \delta(x).$$

Integrate this equation over a vanishingly small interval around  $x=0$ , from  $x=-\epsilon$  to  $x=\epsilon$ . The result is

$$\left. \frac{\partial A_x}{\partial x} \right|_{\epsilon} - \left. \frac{\partial A_x}{\partial x} \right|_{-\epsilon} + \left(\frac{\omega}{c}\right)^2 O(\epsilon) = i\omega\mu_0 P_0.$$

$$\lim_{\epsilon \rightarrow 0} \left. \frac{\partial A_x}{\partial x} \right|_{\epsilon} = ikA_0$$

$$\lim_{\epsilon \rightarrow 0} \left. \frac{\partial A_x}{\partial x} \right|_{-\epsilon} = -ikA_0,$$

and the term of order  $\epsilon$  goes to zero in the limit as  $\epsilon \rightarrow 0$ . Therefore,

$$2ikA_0 = i\omega\mu_0 P_0$$

and

$$A_0 = \frac{\mu_0 c P_0}{2}.$$

(b) The scalar potential can be calculated from the Lorentz condition

$$\text{div } \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0.$$

Consider the potentials for  $x>0$ ; the potentials for  $x<0$  can be handled in the same way.

$$\text{div } \mathbf{A} = \frac{i\omega\mu_0 P_0}{2} e^{i(kx - \omega t)},$$

consequently

$$V = \frac{c^2 \mu_0 P_0}{2} e^{i(kx - \omega t)} = \frac{P_0}{2\epsilon_0} e^{i(kx - \omega t)}.$$

$$\mathbf{E} = -\text{grad } V - \frac{\partial \mathbf{A}}{\partial t},$$

so that

$$E_X = -\frac{ikP_0}{2\epsilon_0} e^{i(kx - \omega t)} + \frac{i\omega\mu_0 P_0 c}{2} e^{i(kx - \omega t)}.$$

$$\text{But } \frac{i\omega\mu_0 P_0 c}{2} = \frac{i\omega\mu_0 P_0 c^2}{2c} = \frac{ikP_0}{2\epsilon_0}$$

so that  $E_X \equiv 0$

Furthermore,  $\text{curl} \mathbf{A} \equiv 0$  so that there is also no magnetic field component.

### Problem (8.9).

very thin plane, uniform, sheet of dipoles is located in the y-z plane. The dipole moments are aligned along the x-axis and their time variation can be described by

$$e^{i(qy - \omega t)}.$$

The polarization density for such a sheet can be written

$$P_X(x, y, z, t) = P_0 \delta(x) e^{i(qy - \omega t)}.$$

Calculate the electric and magnetic fields generated by this space and time varying polarization density.

Hint: It is very difficult to calculate the vector potential by direct integration of the expression

$$\mathbf{A}(\mathbf{R}, t) = \frac{\mu_0}{4\pi} \int \frac{d\tau \frac{\partial \mathbf{p}}{\partial t} \Big|_{t_R}}{|\mathbf{R} - \mathbf{r}|}.$$

It is better to work directly with the differential equation

$$\nabla^2 A_x - \frac{1}{c^2} \frac{\partial^2 A_x}{\partial t^2} = \mu_0 i\omega P_0 \delta(x) e^{i(qy - \omega t)}.$$

For  $x > 0$  let  $A_X = A_0 e^{i(kx + qy - \omega t)}$

For  $x < 0$  let  $A_X = A_0 e^{i(-kx + qy - \omega t)}$ .

These functions satisfy the homogeneous equation and represent plane waves propagating away from the dipole plane. The left and right propagating plane waves have the same amplitude by symmetry; this can be deduced directly from the integral for  $\mathbf{A}(\mathbf{R}, t)$ . Near  $x=0$  one finds

$$\frac{\partial^2 A_x}{\partial x^2} + \left[ \left( \frac{\omega}{c} \right)^2 - q^2 \right] A_x = i\omega\mu_0 P_0 \delta(x),$$

where the factor  $e^{i(qy - \omega t)}$  has been cancelled out on both sides. Integrate this equation from  $-\epsilon$  to  $+\epsilon$  where  $\epsilon$  is allowed to go to zero. This gives

$$\frac{\partial A_x}{\partial x} \Big|_{\epsilon} - \frac{\partial A_x}{\partial x} \Big|_{-\epsilon} + O(A_0 \epsilon) \left[ \left( \frac{\omega}{c} \right)^2 - q^2 \right] = i\omega\mu_0 P_0,$$

which may be used to find  $A_0 = \frac{\mu_0 P_0}{2} \left( \frac{\omega}{k} \right)$ , where  $\left( \frac{\omega}{k} \right)^2 = k^2 + q^2$  (the term proportional to  $\epsilon$  goes to zero with  $\epsilon$ ).

### Answer (8.9).

Following the procedure outlined in the problem, one obtains

for  $x > 0$ :  $A_X = \left( \frac{\omega \mu_0 P_0}{2k} \right) e^{i(kx+qy-\omega t)}$

for  $x < 0$ :  $A_X = \left( \frac{\omega \mu_0 P_0}{2k} \right) e^{i(-kx+qy-\omega t)}$ .

For  $x > 0$  one finds  $\text{div } \mathbf{A} = \frac{\partial A_x}{\partial x} = \left( \frac{i\omega \mu_0 P_0}{2} \right) e^{i(kx+qy-\omega t)}$  ;

from  $\text{div } \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0$  this gives

for  $x > 0$ :  $V = \frac{P_0}{2\epsilon_0} e^{i(kx+qy-\omega t)}$ .

For  $x < 0$  a similar calculation gives

$$V = -\frac{P_0}{2\epsilon_0} e^{i(-kx+qy-\omega t)}.$$

$\mathbf{E} = -\text{grad } V - \frac{\partial \mathbf{A}}{\partial t}$ , from which

for  $x > 0$   $E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} = \frac{iP_0}{2\epsilon_0} \left( \frac{q}{k} \right) e^{i(kx+qy-\omega t)}$ ,

$$E_y = -\frac{\partial V}{\partial y} = -\frac{iqP_0}{2\epsilon_0} e^{i(kx+qy-\omega t)},$$

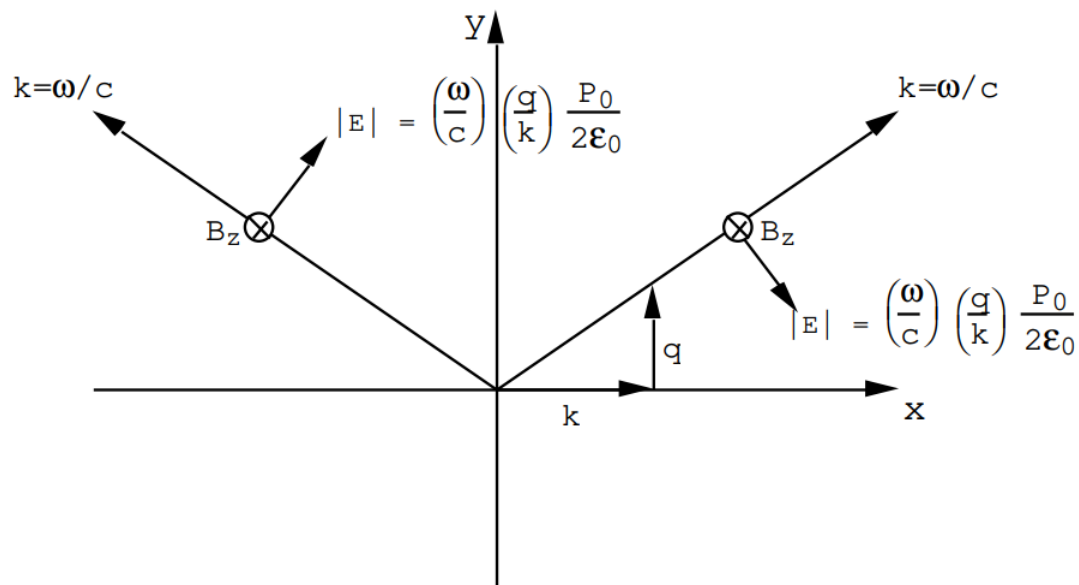
and  $\mathbf{B} = \text{curl } \mathbf{A}$ , so that

$$B_z = -\frac{\partial A_x}{\partial y} = -\frac{i\omega \mu_0 P_0}{2} \left( \frac{q}{k} \right) e^{i(kx+qy-\omega t)}.$$

The amplitude of the electric field is

$$|E| = \sqrt{E_x^2 + E_y^2} = \left( \frac{\omega}{c} \right) \left( \frac{q}{k} \right) \frac{P_0}{2\epsilon_0},$$

therefore  $cB_z = |E|$  which is the correct ratio for a plane wave propagating through empty space. A similar calculation shows that a plane wave propagates to the left; its  $\mathbf{B}$  field is the same as the wave propagating to the right. See the diagram below.



### Problem (8.10).

The electron on a hydrogen atom is characterized by the resonant frequency  $f_0 = 15 \times 10^{15}$  Hz. The dipole moment induced on each hydrogen atom by an electric field can be written  $p = \alpha E$  where  $\alpha$  is the polarizability.

- (a) Estimate the polarizability of a hydrogen atom for an electric field oscillating at a frequency of  $10^{18}$  Hz.
- (b) Consider a hydrogen atom at the origin. A plane wave is incident on the atom where  $E_z = E_0 e^{i(kx - \omega t)}$  where  $E_0 = 1$  Volt/meter and  $\omega = 2\pi \times 10^{18}$  rad/sec. How large are the electric field components measured by an observer 1 meter distant and located in the x-y plane?

**Answer (8.10).**

- (a) The equation of motion of the electron on the H atom can be written

$$m \frac{d^2 z}{dt^2} + kz = |e| E_0 e^{-i\omega t}$$

or

$$\frac{d^2 z}{dt^2} + \frac{k}{m} z = -\frac{|e|}{m} E_0 e^{-i\omega t}.$$

The resonant frequency is  $\frac{3}{2} \times 10^{16}$  Hz therefore  $\frac{k}{m} = (3\pi \times 10^{16})^2 (\text{rad/sec})^2 = \Omega^2$ .

If  $z = z_0 e^{-i\omega t}$  then  $z_0 = \frac{-(|e|/m)E_0}{(\Omega^2 - \omega^2)}$ .

The dipole moment is  $P_z = -|e|z = \frac{(e^2/m)}{(\Omega^2 - \omega^2)} E_0 e^{-i\omega t}$ ,

$$\therefore \alpha = \frac{e^2/m}{(\Omega^2 - \omega^2)} \simeq -\frac{e^2}{m\omega^2} \text{ since } \omega^2 \gg \Omega^2.$$

At  $f = 10^{18}$  Hz  $\omega = 6.28 \times 10^{18}$  and  $\alpha = -0.714 \times 10^{-45}$  Coulomb meters.

The induced dipole moment is  $P_z = \alpha E_z$ . For an observer in the x-y plane

$$\theta = \frac{\pi}{2} \quad \therefore \cos \theta = 0 \\ \sin \theta = 1$$

There is no radial component of electric field!

$$E_\theta = \frac{1}{4\pi\epsilon_0} \left[ \frac{P_z}{R^3} + \frac{\dot{P}_z}{cR^2} + \frac{\ddot{P}_z}{c^2 R} \right]$$

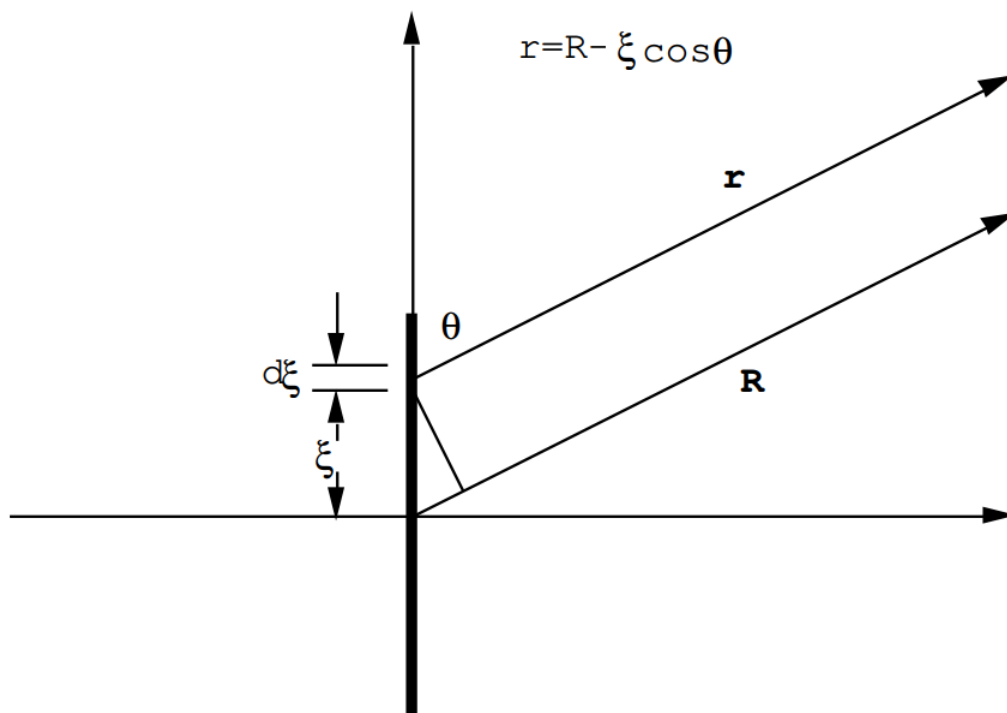
Now  $\frac{\omega}{c} = 2.09 \times 10^{10}$   $\left(\frac{\omega}{c}\right)^2 = 4.39 \times 10^{20}$

So the quantities 1 &  $\frac{\omega}{c}$  are negligible c.f.  $\left(\frac{\omega}{c}\right)^2$

$$E_\theta = -\frac{(0.71)(10^{-45})(1)}{4\pi\epsilon_0} (4.39 \times 10^{20}) = -2.82 \times 10^{-15} \text{ Volts/meter.}$$

**Problem (8.11).**

A short thin-wire center-fed dipole antenna of length L meters is oriented along the z-axis with its center at the origin as shown in the sketch.



The current distribution on the antenna is given by:

$$\begin{aligned} z > 0 : \quad I(\xi) &= I_0 \left( 1 - \frac{2\xi}{L} \right) e^{-i\omega t} \\ z < 0 : \quad I(\xi) &= I_0 \left( 1 + \frac{2\xi}{L} \right) e^{-i\omega t}. \end{aligned}$$

Show that the radiation resistance of the antenna is given by

$$R = 20\pi^2 \left( \frac{L}{\lambda} \right)^2 \text{ Ohms.}$$

Hint: The distance from the element  $d\xi$  to the observer is given approximately by

$$r = R - \xi \cos \theta = R \left( 1 - \frac{\xi \cos \theta}{R} \right);$$

here  $\xi/R$  is a very small quantity. Expand all relevant terms as a power series in  $(\xi/R)$  and keep only terms proportional to  $(\xi/R)$ . Also use the approximation that  $e^{-i\omega\xi \cos \theta/c}$  can be set equal to  $\left( 1 - i \frac{\omega\xi}{c} \cos \theta \right)$ .

**Answer (8.11).**

$$\begin{aligned} dA_z &= \frac{\mu_0}{4\pi} \frac{d\xi I(\xi)}{(R - \xi \cos \theta)} e^{-i\omega \left( t - \frac{R}{c} + \frac{\xi \cos \theta}{c} \right)}, \\ A_z &\cong \left\{ \frac{\mu_0}{4\pi R} \int_{-L/2}^{L/2} d\xi \left( 1 + \frac{\xi \cos \theta}{R} \right) I(\xi) e^{\frac{-i\omega\xi}{c} \cos \theta} \right\} e^{-i\omega(t - R/c)}. \end{aligned}$$

Drop the term  $e^{-i\omega(t - R/c)}$ ; it is just a factor throughout.

Let  $e^{-i\omega\xi \cos \theta/c} = \left( 1 - i\omega\xi \cos \theta/c \right)$ . Then setting  $k = \omega/c$ , one finds

$$A_z \cong \frac{\mu_0 I_0}{4\pi R} \int_0^{L/2} d\xi \left(1 - \frac{2\xi}{L}\right) \left(1 + \frac{\xi \cos \theta}{R}\right) (1 - ik\xi \cos \theta) + \frac{\mu_0 I_0}{4\pi R} \int_{-L/2}^0 d\xi \left(1 + \frac{2\xi}{L}\right) \left(1 + \frac{\xi \cos \theta}{R}\right) (1 - ik\xi \cos \theta)$$

where  $\frac{\xi}{R} \ll 1$  and it is assumed that  $k\xi \ll 1$ . With these approximations

$$A_z = \frac{\mu_0 I_0}{4\pi R} \left(\frac{L}{2}\right) e^{-i\omega[t-R/c]}.$$

In spherical polar co-ordinates

$$A_R = \frac{\mu_0 I_0}{4\pi} \left(\frac{L}{2}\right) \frac{\cos \theta}{R} e^{-i\omega[t-R/c]},$$

$$A_\theta = -\frac{\mu_0 I_0}{4\pi} \left(\frac{L}{2}\right) \frac{\sin \theta}{R} e^{-i\omega[t-R/c]}.$$

$\mathbf{B} = \text{curl} \mathbf{A}$ . In this case  $\mathbf{B}$  has only the component  $B_\phi$ . The radiation component of this field is proportional to  $1/R$  and is

$$B_\phi = -i \frac{\mu_0 I_0}{4\pi} \left(\frac{L}{2}\right) \left(\frac{\omega}{c}\right) \frac{\sin \theta}{R} e^{-i\omega[t-R/c]}.$$

The electric field is

$$E_\theta = cB_\phi = -i \frac{\mu_0 I_0}{4\pi} \left(\frac{L}{2}\right) (\omega) \frac{\sin \theta}{R} e^{-i\omega[t-R/c]}.$$

$$H_\phi = B_\phi / \mu_0 = -i \frac{I_0}{4\pi} \left(\frac{L}{2}\right) \left(\frac{\omega}{c}\right) \frac{\sin \theta}{R} e^{-i\omega[t-R/c]}.$$

The time average of the Poynting vector,  $\langle S_R \rangle$  is

$$\langle S_R \rangle = \frac{1}{2} \text{Real} \left( E_\theta H_\phi^* \right) = \frac{1}{2} \frac{\mu_0 I_0^2}{16\pi^2} \frac{L^2}{4} \frac{\omega^2}{c} \frac{\sin^2 \theta}{R^2}.$$

Integrate this expression over the sphere of radius  $R$  to get the total radiated power:

$$\langle P \rangle = \left(\frac{\pi}{12}\right) (CH_0) \left(\frac{I_0^2 L^2}{\lambda^2}\right) \text{ Watts}.$$

But  $c\mu_0 = 120\pi$ , and by definition the radiation resistance  $R_R$  is such that  $\langle P \rangle = \frac{I_0^2 R_R}{2}$ , therefore

$$R_R = 20\pi^2 \left(\frac{L}{\lambda}\right)^2 \text{ Ohms}.$$