10.5: Oblique Incidence

When a plane wave falls upon the plane interface between two media the incident and reflected wave-vectors define the plane of incidence, see Figures (10.4.6) and (10.5.7). The direction of the electric field vector in the incident wave may make an arbitrary angle with the plane of incidence. The general case may be treated as the sum of two special cases: an electric vector perpendicular to the plane of incidence (called s-polarized light from the German word for perpendicular, ”senkrecht”), and an electric vector which lies in the plane of incidence (p-polarized light).

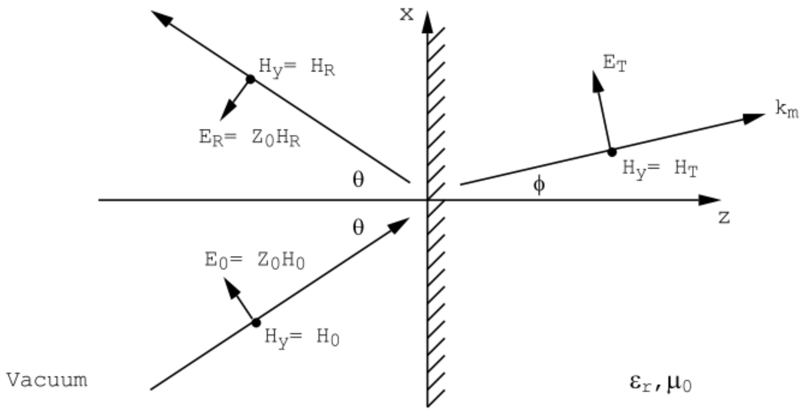

Figure 10.5.7: A P-polarized plane wave incident at the angle θ on the plane interface between vacuum and an isotropic medium characterized by material parameters ϵ_r and μ_0. The electric vector in the incident wave is parallel with the plane of incidence.

10.5.1 S-polarized Waves.

Consider first S-polarized waves, Figure (10.4.6). The incident wave electric vector can be described by the equation

$$\mathrm{E_y} = \mathrm{E_0} \exp(i[\mathrm{x}(\mathrm{k}\sin\theta) + \mathrm{z}(\mathrm{k}\cos\theta) - \omega\mathrm{t}]), \tag{10.5.1}$$

where k = ω/c because this wave is incident on the interface from vacuum. Eventually one is going to have to ensure that the tangential components of the electric and magnetic fields are continuous across the interface, and these boundary conditions must hold at any particular time at all points on the interface. This requirement means that all the waves in this problem, both inside the material and on the vacuum side of the interface, must have the same spatial dependence on the co-ordinates which lie in the interface plane. For the present example, Figure (10.6), the incident wave varies with the in-plane co-ordinate like

$$\exp(i\mathrm{xk}\sin\theta) = \exp(i\mathrm{x}\omega\sin\theta/\mathrm{c}),$$

therefore this same factor must appear both in the reflected wave and in the transmitted wave that is generated in the region z>0. Since the reflected wave-vector has the same magnitude as the incident wave-vector, k = ω/c as determined by Maxwell’s equations, and since its x-component of the wavevector must be the same as for the incident wave, it follows that **the angle of reflection must be the same as the angle of incidence** as is shown in Figure (10.4.6). The electric vector of the reflected wave is given by

$$\mathrm{E_y} = \mathrm{E_R} \exp(i[\mathrm{xk}\sin\theta - \mathrm{zk}\cos\theta - \omega\mathrm{t}]). \tag{10.5.2}$$

(Note the change in the sign of the z-component of k). The magnetic field vector in the incident wave must be perpendicular both to the electric field vector and to the wave-vector:

$$\begin{aligned} \mathrm{H_x^{(i)}} &= -\mathrm{H_0}\cos\theta\exp(i[\mathrm{xk}\sin\theta + \mathrm{zk}\cos\theta - \omega\mathrm{t}]) \\ \mathrm{H_z^{(i)}} &= \mathrm{H_0}\sin\theta\exp(i[\mathrm{x},\mathrm{k}\sin\theta + \mathrm{zk}\cos\theta - \omega\mathrm{t}]) \end{aligned} \tag{10.5.3}$$

where $\mathrm{H_0} = \mathrm{E_0}/\mathrm{Z_0}$, and $\mathrm{Z_0} = \mathrm{c}\mu_0 = \sqrt{\mu_0/\epsilon_0} = 377$ Ohms. The magnetic field vector in the reflected wave must simultaneously be orthogonal to the reflected wave electric vector and also to the wave-vector:

$$\begin{aligned}
\mathrm{H}_{\mathrm{x}}^{(\mathrm{R})} &= \mathrm{H}_{\mathrm{R}} \cos\theta \exp(i[\mathrm{xk}\sin\theta - \mathrm{zk}\cos\theta - \omega t]) \\
\mathrm{H}_{z}^{(\mathrm{R})} &= \mathrm{H}_{\mathrm{R}} \sin\theta \exp(i[\mathrm{xk}\sin\theta - \mathrm{zk}\cos\theta - \omega t])
\end{aligned} \tag{10.5.4}$$

where $H_R = E_R/Z_0$. Eqns.(10.5.3 and 10.5.4) satisfy Maxwell's equations for the vacuum.

The electric field in the transmitted wave will be polarized along y because the material in the region z≥0 is assumed to be linear and isotropic so that a y-directed incident electric field will generate a y-directed transmitted electric field:

$$\mathrm{E}_{\mathrm{y}} = \mathrm{A}\exp(i[\mathrm{xk}\sin\theta])\exp(i\,[\mathrm{zk}_{\mathrm{z}} - \omega \mathrm{t}]). \tag{10.5.5}$$

The Maxwell equation $\operatorname{curl}(\vec{\mathrm{H}}) = -\frac{\partial\vec{\mathrm{B}}}{\partial \mathrm{t}} = i\omega\mu_0\vec{\mathrm{H}}$ becomes

$$\begin{aligned}
\frac{\partial \mathrm{E}_{\mathrm{y}}}{\partial \mathrm{z}} &= -i\omega\mu_0\mathrm{H}_{\mathrm{x}}, \\
\frac{\partial \mathrm{E}_{\mathrm{y}}}{\partial \mathrm{x}} &= i\omega\mu_0\mathrm{H}_{\mathrm{z}}.
\end{aligned} \tag{10.5.6}$$

The Maxwell equation $\operatorname{curl}(\vec{\mathrm{H}}) = \frac{\partial\vec{\mathrm{D}}}{\partial \mathrm{t}} = -i\epsilon_{\mathrm{r}}\epsilon_0\omega\vec{\mathrm{E}}$ becomes

$$\frac{\partial \mathrm{H}_{\mathrm{x}}}{\partial \mathrm{z}} - \frac{\partial \mathrm{H}_{\mathrm{z}}}{\partial \mathrm{x}} = -i\omega\epsilon_{\mathrm{r}}\epsilon_0\mathrm{E}_{\mathrm{y}}. \tag{10.5.7}$$

Combine Equations (10.5.6) and (10.5.7) to obtain

$$\frac{\partial^2 \mathrm{E}_{\mathrm{y}}}{\partial \mathrm{z}^2} + \frac{\partial^2 \mathrm{E}_{\mathrm{y}}}{\partial \mathrm{x}^2} = -\left(\frac{\omega}{\mathrm{c}}\right)^2 \epsilon_{\mathrm{r}}\mathrm{E}_{\mathrm{y}}. \tag{10.5.8}$$

This equation requires that

$$\mathrm{k}_{\mathrm{z}}^2 + \mathrm{k}^2\sin^2\theta = \epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^2,$$

or, since k = ω/c,

$$\mathrm{k}_{\mathrm{z}}^2 = \left[\epsilon_{\mathrm{r}} - \sin^2\theta\right]\left(\frac{\omega}{\mathrm{c}}\right)^2.$$

The z-component of the transmitted wave-vector must therefore be calculated from

$$\mathrm{k}_{\mathrm{z}} = \sqrt{\left[\epsilon_{\mathrm{r}} - \sin^2\theta\right]}\left(\frac{\omega}{\mathrm{c}}\right), \tag{10.5.9}$$

where the imaginary part of k_z must be chosen to be positive in order that the wave (10.5.5) be damped out as the wave travels along the z -direction. The wave-vector component k_z will in general be a complex number corresponding to the fact that the relative dielectric constant, $\epsilon_{\mathrm{r}} = \epsilon_{\mathrm{R}} + i\epsilon_{\mathrm{I}}$, is a complex number: here ϵ_{R} and ϵ_{I} I are both real numbers. A complex index of refraction can be defined for the case of oblique incidence by setting

$$\mathrm{k}_{\mathrm{z}} = (\mathrm{n}_\theta + i\kappa_\theta)\left(\frac{\omega}{\mathrm{c}}\right). \tag{10.5.10}$$

the parameters n_θ and κ_θ are explicit functions of the angle of incidence. The electric field transmitted into the material on the right of z=0 will be given by

$$\mathrm{E}_{\mathrm{y}} = \mathrm{A}\exp(i[\mathrm{xk}\sin\theta])\exp(-\kappa_\theta\omega\mathrm{z}/\mathrm{c})\exp\left(i\left[\frac{\mathrm{n}_\theta\omega\mathrm{z}}{\mathrm{c}} - \omega\mathrm{t}\right]\right), \tag{10.5.11}$$

and from Equations (10.5.6) the magnetic field components are given by

$$\begin{aligned}
\mathrm{H}_{\mathrm{x}} &= -\frac{(\mathrm{n}_\theta + i\kappa_\theta)}{\mathrm{Z}_0}. \\
&\cdot \mathrm{A}\exp(i[\mathrm{xk}\sin\theta])\exp(-\kappa_\theta\omega\mathrm{z}/\mathrm{c})\exp\left(i\left[\frac{\mathrm{n}_\theta\omega\mathrm{z}}{\mathrm{c}} - \omega\mathrm{t}\right]\right), \\
\mathrm{H}_{\mathrm{z}} &= \frac{\sin\theta}{\mathrm{Z}_0}\mathrm{A}\exp(i[\mathrm{xk}\sin\theta])\exp(-\kappa_\theta\omega\mathrm{z}/\mathrm{c})\exp\left(i\left[\frac{\mathrm{n}_\theta\omega\mathrm{z}}{\mathrm{c}} - \omega\mathrm{t}\right]\right),
\end{aligned} \tag{10.5.12}$$

where $Z_0 = c\mu_0 = 377$ Ohms. The planes of **constant amplitude** are parallel with the plane interface. The planes of constant phase are tilted at an angle ϕ with respect to the interface plane. The wave-vector in the material, which is perpendicular to the planes of **constant phase**, has components that are given by

$$k_x = \left(\frac{\omega}{c}\right)\sin\theta,$$

and

$$\mathrm{Real}(k_z) = n_\theta\left(\frac{\omega}{c}\right),$$

therefore the tilt angle ϕ illustrated in Figure (10.4.6) can be calculated from

$$\tan\phi = \frac{\sin\theta}{n_\theta}. \tag{10.5.13}$$

In cases for which the dielectric constant can be taken to be real, i.e. negligible losses, one has

$$k_m = \sqrt{\left[k^2\sin^2\theta + k_z^2\right]} = \sqrt{\epsilon_r}\left(\frac{\omega}{c}\right).$$

Then

$$\sin\phi = \frac{k\sin\theta}{\sqrt{\epsilon_r}(\omega/c)} = \frac{\sin\theta}{\sqrt{\epsilon r}}.$$

This is just Snell's law:

$$\sin\theta = \sqrt{\epsilon_r}\sin\phi. \tag{10.5.14}$$

For this case a real index of refraction can be defined for the medium, $n = \sqrt{\epsilon_r}$, and the phase velocity of the wave in the medium is c/n; the refracted wave propagates in the direction specified by the angle ϕ obtained from Snell's law. In the more general case of a lossy medium the angle between the surfaces of constant phase and the boundary surface must be calculated from Equation (10.5.13).

At z=0 the tangential components of $\vec{E}$ and $\vec{H}$ must be continuous across the interface and this condition determines the amplitudes of the reflected and transmitted waves. One finds

$$E_0 + E_R = A, \tag{10.5.15}$$

and

$$-H_0\cos\theta + H_R\cos\theta = -\frac{(n_\theta + i\kappa_\theta)}{Z_0}A,$$

or, since $H_0 = E_0/Z_0$ and $H_R = E_R/Z_0$

$$-E_0 + E_R = -\frac{(n_\theta + i\kappa_\theta)}{\cos\theta}A. \tag{10.5.16}$$

The parameters n_θ and κ_θ are defined by equations (10.5.9) and (10.5.10). The two equations, (10.5.15) and (10.5.16), can be solved for the amplitudes E_R and A in terms of the incident wave amplitude E_0.

$$\begin{aligned}\frac{A}{E_0} &= \frac{2\cos\theta}{[\cos\theta + (n_\theta + i\kappa_\theta)]},\\ \frac{E_R}{E_0} &= \left(\frac{\cos\theta - (n_\theta + i\kappa_\theta)}{\cos\theta + (n_\theta + i\kappa_\theta)}\right),\end{aligned} \tag{10.5.17}$$

where, it will be recalled,

$$(n_\theta + i\kappa_\theta) = \sqrt{\epsilon_r - \sin^2\theta},$$

and the sign must be chosen so that $\kappa_\theta > 0$.

10.5.2 P-polarized Waves.

Arguments for P-polarized light are similar to those for S-polarized light. However, for P-polarized radiation the magnetic field is polarized perpendicular to the plane of incidence, Figure (10.5.7). The incident wave can be written

$$\begin{gathered}\mathrm{H}_{\mathrm{y}}^{\mathrm{inc}}=\mathrm{H}_{0} \exp (i[\mathrm{xk} \sin \theta+\mathrm{zk} \cos \theta-\omega \mathrm{t}]), \\ \mathrm{E}_{\mathrm{x}}^{\mathrm{inc}}=\mathrm{Z}_{0} \mathrm{H}_{0} \cos \theta \exp (i[\mathrm{xk} \sin \theta+\mathrm{zk} \cos \theta-\omega \mathrm{t}]), \\ \mathrm{E}_{\mathrm{z}}^{\mathrm{R}}=-\mathrm{Z}_{0} \mathrm{H}_{\mathrm{R}} \sin \theta \exp (i[\mathrm{xk} \sin \theta-\mathrm{zk} \cos \theta-\omega \mathrm{t}]),\end{gathered} \tag{10.5.18}$$

and for the reflected wave:

$$\begin{gathered}\mathrm{H}_{y}^{\mathrm{R}}=\mathrm{H}_{\mathrm{R}} \exp (i[\mathrm{xk} \sin \theta-\mathrm{zk} \cos \theta-\omega \mathrm{t}]), \\ \mathrm{E}_{\mathrm{x}}^{\mathrm{R}}=-\mathrm{Z}_{0} \mathrm{H}_{\mathrm{R}} \cos \theta \exp (i[\mathrm{xk} \sin \theta-\mathrm{zk} \cos \theta-\omega \mathrm{t}]), \\ \mathrm{E}_{\mathrm{z}}^{\mathrm{R}}=-\mathrm{Z}_{0} \mathrm{H}_{\mathrm{R}} \sin \theta \exp (i[\mathrm{xk} \sin \theta-\mathrm{zk} \cos \theta-\omega \mathrm{t}]).\end{gathered} \tag{10.5.19}$$

Inside the material, z≥0, which is assumed to be characterized by a complex relative dielectric constant ϵ_r, one finds from

$$\begin{gathered}\operatorname{curl}(\vec{\mathrm{H}})=-i \omega \epsilon_{\mathrm{r}} \epsilon_{0} \vec{\mathrm{E}}, \\ \frac{\partial \mathrm{H}_{\mathrm{y}}}{\partial \mathrm{z}}=i \omega \epsilon_{\mathrm{r}} \epsilon_{0} \mathrm{E}_{\mathrm{x}}, \\ \frac{\partial \mathrm{H}_{\mathrm{y}}}{\partial \mathrm{x}}=-i \omega \epsilon_{\mathrm{r}} \epsilon_{0} \mathrm{E}_{\mathrm{z}},\end{gathered} \tag{10.5.20}$$

and from

$$\begin{gathered}\operatorname{curl}(\vec{\mathrm{E}})=-\frac{\partial \vec{\mathrm{B}}}{\partial \mathrm{t}}=i \omega \mu_{0} \vec{\mathrm{H}}, \\ \frac{\partial \mathrm{E}_{\mathrm{x}}}{\partial \mathrm{z}}-\frac{\partial \mathrm{E}_{\mathrm{z}}}{\partial \mathrm{x}}=i \omega \mu_{0} \mathrm{H}_{\mathrm{y}}.\end{gathered} \tag{10.5.21}$$

Equations (10.5.20) and (10.5.21) can be combined to give

$$\frac{\partial^{2} \mathrm{H}_{y}}{\partial \mathrm{x}^{2}}+\frac{\partial^{2} \mathrm{H}_{y}}{\partial \mathrm{z}^{2}}=-\epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^{2} \mathrm{H}_{\mathrm{y}}. \tag{10.5.22}$$

The solution of Equation (10.5.22) can be written

$$\mathrm{H}_{y}=\mathrm{H}_{\mathrm{T}} \exp \left(i\left[\mathrm{xk} \sin \theta+\mathrm{zk}_{\mathrm{z}}-\omega \mathrm{t}\right]\right), \tag{10.5.23}$$

where

$$\mathrm{k}^{2} \sin ^{2} \theta+\mathrm{k}_{\mathrm{z}}^{2}=\epsilon_{\mathrm{r}}\left(\frac{\omega}{\mathrm{c}}\right)^{2}. \tag{10.5.24}$$

In these Equations $\mathbf{k}=(\omega / \mathbf{c})$. Equation (10.5.24) for k_z is the same as that which was obtained for the case of an incident S-polarized wave. Solving for k_z one obtains:

$$\mathrm{k}_{z}=\sqrt{\left[\epsilon_{r}-\sin ^{2} \theta\right]}\left(\frac{\omega}{\mathrm{c}}\right),$$

or

$$\mathrm{k}_{\mathrm{z}}=\left(\mathrm{n}_{\theta}+i \kappa_{\theta}\right)\left(\frac{\omega}{\mathrm{c}}\right),$$

where

$$\mathrm{n}_{\theta}+i \kappa_{\theta}=\sqrt{\left[\epsilon_{\mathrm{r}}-\sin ^{2} \theta\right]},$$

and the sign of the square root must be chosen so as to make the imaginary part of k_z positive in order to describe an optical disturbance that is attenuated as z increases.

From the form of the magnetic field, Equation (10.5.23), and from the Maxwell Equations (10.5.20), it follows that

$$\begin{aligned} \mathrm{E_x} &= \frac{(\mathrm{n}_\theta + i\kappa_\theta)}{\epsilon_\mathrm{r}} \mathrm{Z_0 H_T} \exp(i\mathrm{xk}\sin\theta)\exp(-\kappa_\theta \omega \mathrm{z/c})\exp\left(i\left[\frac{\mathrm{n}_\theta \omega \mathrm{z}}{\mathrm{c}} - \omega \mathrm{t}\right]\right), \\ \mathrm{E}_z &= -\frac{\sin\theta}{\epsilon_\mathrm{r}} \mathrm{Z_0 H_T} \exp(i\mathrm{xk}\sin\theta)\exp(-\kappa_\theta \omega \mathrm{z/c})\exp\left(i\left[\frac{\mathrm{n}_\theta \omega \mathrm{z}}{\mathrm{c}} - \omega \mathrm{t}\right]\right). \end{aligned} \tag{10.5.25}$$

And from the boundary conditions at z=0 (continuity of the tangential components of $\vec{E}$ and $\vec{H}$) one finds:

$$\mathrm{H_0 + H_R = H_T},$$

$$\mathrm{Z_0 H_0}\cos\theta - \mathrm{Z_0 H_R}\cos\theta = \frac{(\mathrm{n}_\theta + i\kappa_\theta)}{\epsilon_\mathrm{r}} \mathrm{Z_0 H_T},$$

or

$$\mathrm{H_0 - H_R} = \frac{(\mathrm{n}_\theta + i\kappa_\theta)}{\epsilon_\mathrm{r}\cos\theta} \mathrm{H_T}.$$

These two equations can be solved to obtain

$$\begin{aligned} \frac{\mathrm{H_T}}{\mathrm{H_0}} &= \frac{2\epsilon_\mathrm{r}\cos\theta}{(\epsilon_\mathrm{r}\cos\theta + (\mathrm{n}_\theta + i\kappa_\theta))}, \\ \frac{\mathrm{H_R}}{\mathrm{H_0}} &= \left(\frac{\epsilon_\mathrm{r}\cos\theta - (\mathrm{n}_\theta + i\kappa_\theta)}{\epsilon_\mathrm{r}\cos\theta + (\mathrm{n}_\theta + i\kappa_\theta)}\right), \end{aligned} \tag{10.5.26}$$

where $(\mathrm{n}_\theta + i\kappa_\theta) = \sqrt{[\epsilon_\mathrm{r} - \sin^2\theta]}$ and $\kappa_\theta > 0$.

Notice that $\mathrm{div}(\vec{\mathrm{D}}) = 0$ for both the S- and P-polarized waves. This is obvious for the S-polarized light because the electric field has only a y-component and this component does not depend upon the y co-ordinate, Equation (10.5.11). For P-polarized radiation, from Equations (10.5.25),

$$\frac{\partial \mathrm{E_x}}{\partial \mathrm{x}} + \frac{\partial \mathrm{E_z}}{\partial \mathrm{z}} = 0,$$

so that $\mathrm{div}(\vec{\mathrm{E}}) = 0$ and, since $\vec{\mathrm{D}} = \epsilon_\mathrm{r}\epsilon_0\vec{\mathrm{E}}$, so also $\mathrm{div}(\vec{\mathrm{D}}) = 0$. There are no free charges set up in the material for either S- or P-polarized radiation. The condition $\mathrm{div}(\vec{\mathrm{D}}) = 0$ can also be deduced directly from the Maxwell's equation

$$\mathrm{curl}(\vec{\mathrm{H}}) = \frac{\partial \vec{\mathrm{D}}}{\partial \mathrm{t}} = -i\omega\vec{\mathrm{D}},$$

because the divergence of any curl is zero. It is easy to show by direct calculation that the normal component of the magnetic field $\vec{B}$ is continuous across the surface of the dielectric material for both S- and P-polarized radiation.

10.5.3 Oblique Incidence on a Lossless Material.

For a material in which the losses are very small so that the imaginary part of the dielectric constant can be neglected, a real index of refraction can be defined by

$$\mathrm{n} = \sqrt{\epsilon_\mathrm{r}}.$$

For S-polarized radiation the reflection and transmission coefficients, Equations (10.5.17), become

$$\begin{aligned} \mathrm{R_S} &= \frac{\mathrm{E_R}}{\mathrm{E_0}} = \left(\frac{\cos\theta - \mathrm{n}\cos\phi}{\cos\theta + \mathrm{n}\cos\phi}\right), \\ \mathrm{T_S} &= \frac{\mathrm{E_T}}{\mathrm{E_0}} = \left(\frac{2\cos\theta}{\cos\theta + \mathrm{n}\cos\phi}\right), \end{aligned} \tag{10.5.27}$$

where $\sin\phi = \sin\theta/\mathrm{n}$.

For P-polarized radiation, and $\mathbf{n} = \sqrt{\epsilon_\mathbf{r}}$ a real number, the reflection and transmission coefficients (10.5.26) become

$$\mathrm{R_P} = \frac{\mathrm{H_R}}{\mathrm{H_0}} = \left(\frac{\mathrm{n}\cos\theta - \cos\phi}{\mathrm{n}\cos\theta + \cos\phi}\right), \tag{10.5.28}$$

$$\mathrm{T_P} = \frac{\mathrm{H_T}}{\mathrm{H_0}} = \left(\frac{2\mathrm{n}\cos\theta}{\mathrm{n}\cos\theta + \cos\phi}\right),$$

where, as above, sin ϕ = sin θ/n and $\mathrm{n} = \sqrt{\epsilon_\mathrm{r}}$. The relation

$$\mathrm{n}_\theta = \sqrt{\mathrm{n}^2 - \sin^2\theta} = \mathrm{n}\cos\phi$$

has also been used.