12.5: Circular Waveguides

The details are different, but the modes sustained by a circular wave-guide have much in common with the rectangular wave-guide modes. They may, for example, be classified as **transverse electric modes** (TE modes) in which there is no component of electric field along the guide axis, or as transverse magnetic modes (TM modes) in which there is no component of the magnetic field along the guide axis.

12.5.1 TM Modes.

A vector potential whose transverse components are zero but for which A_z is not zero will generate transverse magnetic modes because H_z is necessarily zero since curl($\vec{A}$) has a zero z-component. A_z must satisfy the wave equation (12.2.6) in order that the fields generated by A_z satisfy Maxwell's equations:

$$\nabla^2 \mathrm{A_z} + \epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2 \mathrm{A_z} = 0. \tag{12.5.1}$$

It is convenient to use cylindrical polar co-rdinates (r,θ,z) because of the cylindrical symmetry implied by the shape of a cylindrical wave-guide. For a wave travelling along the z-direction, one can write

$$\mathrm{A_z} = \mathrm{A}(\mathrm{r}, \theta)\exp(i\left[\mathrm{k_g z} - \omega \mathrm{t}\right]).$$

From now on the factor $\exp(i[k_g z - \omega t])$ **will be understood and not written out explicitly**. Using cylindrical polar co-ordinates Equation (12.5.1) becomes

$$\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial \mathrm{A}}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 \mathrm{A}}{\partial \theta^2} + \left[\epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2 - \mathrm{k_g^2}\right]\mathrm{A} = 0.$$

or setting

$$\mathrm{k_c^2} = \epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2 - \mathrm{k_g^2}, \tag{12.5.2}$$

and multiplying through by r^2

$$\mathrm{r}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial \mathrm{A}}{\partial \mathrm{r}}\right) + \frac{\partial^2 \mathrm{A}}{\partial \theta^2} + \mathrm{k_c^2 r^2 A} = 0. \tag{12.5.3}$$

Now let the amplitude $A(r, \theta)$ be written as the product of a function F(r) that depends only on the radius r and the function cos (mθ), where m is an integer. The constant m must be an integer so that $A(r, \theta)$ will be single valued in angle: ie. A(r,0) must be equal to A(r,2πm). The use of the function cos(mθ) is arbitrary. We could just as well use sin (mθ) or a function of the form $f(\theta) = a\cos(m\theta) + b\sin(m\theta)$, where a and b are constants. All of these choices have in common that $d^2f/d\theta^2 = -m^2 f$. The various choices of a,b simply amount to a choice of the orientation of the wave-guide mode pattern with respect to the axis θ = 0.

The equation for the radial function, $F(r)$, becomes

$$\mathrm{r}\frac{\mathrm{d}}{\mathrm{dr}}\left(\mathrm{r}\frac{\mathrm{dF}}{\mathrm{dr}}\right) + \left(\mathrm{k_c^2 r^2} - \mathrm{m^2}\right)\mathrm{F} = 0. \tag{12.5.4}$$

This equation for F(r) can be put in the standard form of Bessel's equation by the introduction of a change of variable:

$$\mathrm{x} = \mathrm{k_c r}, \tag{12.5.5}$$

then Equation (12.5.4) becomes

$$\mathrm{x}\frac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{x}\frac{\mathrm{dF}}{\mathrm{dx}}\right) + \left(\mathrm{x^2} - \mathrm{m^2}\right)\mathrm{F} = 0.$$

The solutions of this equation that remain finite at r=0 are

$$\mathrm{F(x)} = \mathrm{J_m(x)},$$

where the $J_m(x)$ are Bessel's functions of integer order because m is an integer. See "Schaum's Outline Series, Mathematical Handbook" by Murray R. Spiegel, McGraw-Hill, New York, 1968, Chapter 24. The required form of the vector potential is

$$A(r,\theta) = A_0 J_m(k_c r)\cos(m\theta), \tag{12.5.6}$$

where A_0 is a constant, and k_c is given by Equation (12.5.2).

The magnetic field components are obtained from $\vec{H}$ = curl($\vec{A}$):

$$\begin{aligned} H_r &= \frac{1}{r}\left(\frac{\partial A_z}{\partial\theta}\right) = \frac{-m}{r} A_0 J_m(k_c r)\sin(m\theta), \\ H_\theta &= -\left(\frac{\partial A_z}{\partial r}\right) = -k_c A_0 J_m(k_c r)\cos(m\theta), \\ H_z &= 0, \end{aligned} \tag{12.5.7}$$

(these are all multiplied by the factor exp ($i[k_g z - \omega t]$), of course). The notation $\dot{J}_m(x)$ means the derivative of the Bessel function with respect to the argument x.

The electric field components can be calculated from $\operatorname{curl}(\overrightarrow{H}) = -i\omega\epsilon\overrightarrow{E}$:

$$\begin{aligned} E_r &= \frac{-i}{\epsilon\omega}\left(\frac{\partial H_\theta}{\partial z}\right) = -\frac{k_g}{\epsilon\omega} k_c A_0 \dot{J}_m(k_c r)\cos(m\theta), \\ E_\theta &= \frac{i}{\epsilon\omega}\left(\frac{\partial H_r}{\partial z}\right) = \frac{k_g}{\epsilon\omega}\frac{mA_0}{r} J_m(k_c r)\sin(m\theta), \\ E_z &= \frac{i}{\epsilon\omega}\frac{1}{r}\left(\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right) \\ &= \frac{-i}{\epsilon\omega}\frac{A_0}{r^2}\left(k_c^2 r^2 \ddot{J}_m + k_c r\dot{J}_m - m^2 J_m\right)\cos(m\theta). \end{aligned} \tag{12.5.8}$$

The expression for E_z can be simplified because $J_m(k_c r)$ must satisfy the differential equation(12.5.4), therefore

$$\left(k_c^2 r^2 \ddot{J}_m + k_c r\dot{J}_m + k_c^2 r^2 J_m - m^2 J_m\right) = 0.$$

Using this expression E_z becomes

$$E_z = i\frac{k_c^2}{\epsilon\omega} A_0 J_m(k_c r)\cos(m\theta). \tag{12.5.9}$$

The fields of Equations (12.5.7) and (12.5.8) satisfy Maxwell's equations. They must also satisfy the boundary conditions E_θ=0, E_z=0, and H_r=0 at the walls of the wave-guide. Let the inner radius of the wave-guide be R meters. The boundary conditions can be met if $J_m(k_c R)$=0. This condition fixes allowable values for k_c and therefore fixes k_g through Equation (12.5.2)

$$k_g^2 = \epsilon_r\left(\frac{\omega}{c}\right)^2 - k_c^2. \tag{12.5.10}$$

Table(12.5.2) lists the four lowest roots of the equation $J_m(x)$=0 for Bessel's functions with m=0,1,2 and 3. These roots determine the wave-vector, k_g. In particular, they determine the minimum frequency for which energy can be propagated down the wave-guide. The cut-off frequencies correspond to k_g=0, and are given by

$$\frac{\omega}{c} = \frac{k_c}{\sqrt{\epsilon_r}}. \tag{12.5.11}$$

To take a concrete example, suppose that R=1cm =0.01m. The lowest TM mode corresponds to m=0 and to the first root of the Bessel's function J_0: this is called the TM_{01} mode. For this case

$$\begin{aligned} \mathrm{E_r} &= -\frac{\mathrm{k_g}}{\epsilon\omega}\mathrm{k_c A_0 \dot{J}_0(k_c r)}, \\ \mathrm{E_\theta} &= 0, \\ \mathrm{E_z} &= i\frac{\mathrm{k_c^2}}{\epsilon\omega}\mathrm{A_0 J_0(k_c r)}, \\ \mathrm{H_r} &= 0, \\ \mathrm{H_\theta} &= -\mathrm{k_c A_0 \dot{J}_0(k_c r)}, \\ \mathrm{H_z} &= 0, \end{aligned} \tag{12.5.12}$$

Here k_c = 2.4048/R = 240.48 per meter. This corresponds to a cut-off frequency of 11.48 GHz for ϵ_r =1. The TM_{01} mode pattern is shown in Figure (12.5.12(b)).

	$\mathrm{J_m}$	$\mathrm{\dot{J}_m}$
m=0	2.4048	3.8317
	5.5200	7.0156
	8.6537	10.1735
	11.7915	13.3237
m=1	3.8317	1.8412
	7.0156	5.3314
	10.1735	8.5363
	13.3237	11.7060
m=2	5.1356	3.0542
	8.4172	6.7061
	11.6198	9.9695
	14.7960	13.1704
m=3	6.3802	4.2012
	9.7610	8.0152
	13.0152	11.3459
	16.2235	14.5858

Table 12.5.2: The values of x corresponding to the roots of the equations $J_m(x)=0$ and $\mathrm{\dot{J}_m(x)} = 0$ for the first four Bessel's functions.

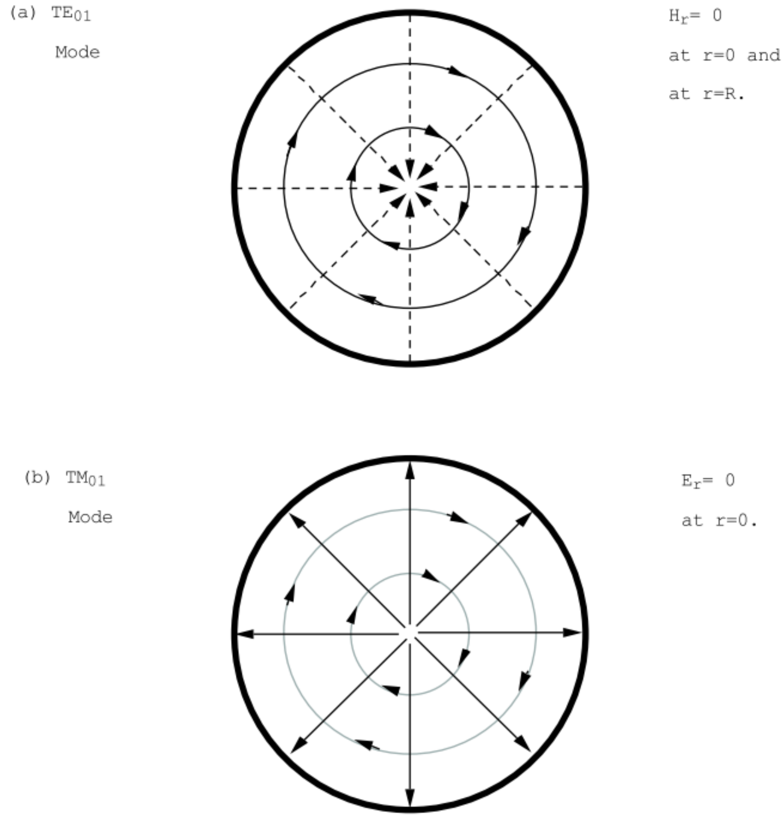

Figure 12.5.12: Electric and magnetic field distributions for the TE_{01} and TM_{01} modes in a circular wave-guide. The dashed lines represent the magnetic field, the full lines the electric field.

12.5.2 TE Modes.

Using the symmetry relations (12.2.3) one can write down the electric field components corresponding to transverse electric modes directly from Equations (12.5.7):

$$\begin{aligned} \mathrm{E_r} &= \frac{\mathrm{mE_0}}{\mathrm{r}} \mathrm{J_m}\left(\mathrm{k_c r}\right) \sin(\mathrm{m}\theta), \\ \mathrm{E}_\theta &= \mathrm{k_c E_0 J_m}\left(\mathrm{k_c r}\right) \cos(\mathrm{m}\theta), \\ \mathrm{E_z} &= 0, \end{aligned} \tag{12.5.13}$$

where, as before,

$$\mathrm{k_c^2} = \epsilon_\mathrm{r}\left(\frac{\omega}{\mathrm{c}}\right)^2 - \mathrm{k_g^2}.$$

The magnetic field components can be calculated from $\operatorname{curl}(\overrightarrow{\mathrm{E}}) = i\omega\mu_0 \overrightarrow{\mathrm{H}}$:

$$\begin{aligned} \mathrm{H_r} &= \frac{i}{\omega\mu_0}\frac{\partial \mathrm{E}_\theta}{\partial \mathrm{z}} = -\frac{\mathrm{k_g k_c}}{\omega\mu_0}\mathrm{E_0}\dot{\mathrm{J}}_\mathrm{m}\left(\mathrm{k_c r}\right)\cos(\mathrm{m}\theta), \\ \mathrm{H}_\theta &= \frac{-i}{\omega\mu_0}\frac{\partial \mathrm{E_r}}{\partial \mathrm{z}} = \frac{\mathrm{k_g}}{\omega\mu_0}\frac{\mathrm{mE_0}}{\mathrm{r}}\mathrm{J_m}\left(\mathrm{k_c r}\right)\sin(\mathrm{m}\theta), \\ \mathrm{H_z} &= \frac{-i}{\omega\mu_0 \mathrm{r}}\left[\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{rE}_\theta\right) - \frac{\partial \mathrm{E_r}}{\partial \theta}\right] \\ &= \frac{i\mathrm{k_c^2}}{\omega\mu_0}\mathrm{E_0 J_m}\left(\mathrm{k_c r}\right)\cos(\mathrm{m}\theta). \end{aligned} \tag{12.5.14}$$

In Equations (12.5.14) the factor exp $(i[k_g z - \omega t])$ has been suppressed. The simple form for H_z has been obtained using the fact that $J_m(k_c r)$ must satisfy Equation (12.5.4), the differential equation for the radial function F(r). In order to satisfy the boundary conditions E_θ=0 and H_r=0 at the wave-guide walls k_cR must be set equal to one of the roots of the equation $\dot{J}_m(k_c r) = 0$, where R is the inner radius of the wave-guide. The lowest four roots of $\dot{J}_m(x) = 0$ have been listed in Table(12.5.2) for the first four Bessel's functions. The lowest cut-off frequency occurs for the first root of $\dot{J}_1(x)$: this mode is called the TE_{11} mode. The cut-off frequency for the TE_{11} mode is 8.79 GHz for ϵ_r=1 and R= 1cm. Compare this with the cut-off frequency for the TM_{01} mode, 11.48 GHz. Thus, over the frequency interval 8.79 to 11.48 GHz an air-filled circular pipe having an inner radius of R=1cm can support only a single mode, the TE_{11} mode. The TE_{11} mode pattern is shown in Figure (12.5.13).

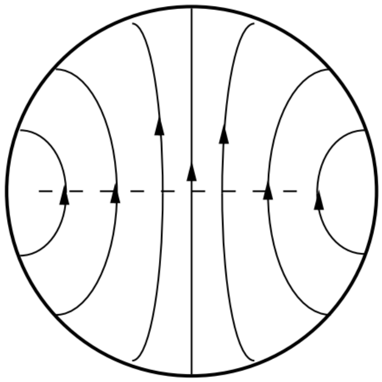

Figure 12.5.13: Electric field lines for the TE_{11} mode in a cylindrical Waveguide. The electric field lines must be normal to the walls at r=R, where R is the inner radius of the wave-guide. The magnetic field lines are orthogonal to the electric field lines, and H_r=0 at r=R.

The TE_{01} mode is of particular interest; the mode pattern is shown in Figure (12.5.12(a)). This mode is very useful for the construction of high-Q cavities of variable frequency. The length of the cavity can be altered by means of a sliding piston. No currents need flow across the gap between the piston and the walls of the cylinder for the TE_{01} mode: the current lines on the face of the piston are similar to the electric field lines shown in Figure (12.5.12(a)) and are concentric circles. Even if the piston does not make good electrical contact with the cavity walls the field lines in the TE_{01} mode remain unperturbed by any small gap between the piston and the cylinder walls. This mode is often used to construct microwave frequency meters.

Wave-guide modes are discussed in detail in the book "Electron Spin Resonance" by Charles P. Poole, 2cd Edition, John Wiley and Sons, New York, 1983.