12.2: Higher Order Modes

The wave-guide modes discussed above are very simple ones because they presumed that there was no spatial variation of the fields along the y-direction. There exist wave-guide solutions of Maxwell's equations that involve spatial variations along all three axes: these higher order modes correspond to the co-ordinated propagation of plane waves whose wave-vectors make an oblique angle with the guide axis so that they are repeatedly reflected from all four walls. These modes divide naturally into two classes:

a. Transverse Electric (TE) Modes;
b. Transverse Magnetic (TM) Modes.

A transverse electric mode is one in which there is no component of the electric field parallel to the direction of propagation. A transverse magnetic mode is one in which there is no component of the magnetic field parallel to the direction of propagation. For both classes of modes one seeks solutions of Maxwell's equations that correspond to waves travelling down the waveguide; i.e. all of the field components are required to be proportional to the phasor

$$\exp(i\left[\mathrm{k_g z}-\omega \mathrm{t}\right]).$$

Furthermore, it is convenient at this point to change the description of the wave-guide co-ordinate system so that the origin is located at one corner of the hollow rectangular pipe as shown in Figure (12.2.4): in the new system the walls of the guide are formed by the intersection of the planes x=0,a and y=0,b. For a time variation of the form exp (−iωt) Maxwell's equations become

$$\begin{aligned} \operatorname{curl}(\overrightarrow{\mathrm{E}}) &= i\omega\mu_0 \overrightarrow{\mathrm{H}}, \\ \operatorname{curl}(\overrightarrow{\mathrm{H}}) &= -i\omega\epsilon \overrightarrow{\mathrm{E}}. \end{aligned} \tag{12.2.1}$$

The divergence of any curl is zero, and therefore the electric and the magnetic fields satisfy the conditions

$$\begin{aligned} div(\overrightarrow{\mathrm{E}}) &= 0, \\ \operatorname{div}(\overrightarrow{\mathrm{H}}) &= 0. \end{aligned} \tag{12.2.2}$$

Note that the equations for $\vec{E}$ and $\vec{H}$ are very similar. This symmetry between the equations for $\vec{E}$ and $\vec{H}$ can be exploited to generate a second set of solutions to Maxwell's equations from a primary set of fields that satisfy Maxwell's equations. This works as follows: suppose that one has found the fields $\vec{E}_1$ and $\vec{H}_1$ that satisfy Equations (12.2.1). Now consider a second set of fields

$$\begin{aligned} \overrightarrow{\mathrm{E}}_2 &= \mathrm{Z}\overrightarrow{\mathrm{H}}_1, \\ \overrightarrow{\mathrm{H}}_2 &= -\overrightarrow{\mathrm{E}}_1/\mathrm{Z}. \end{aligned} \tag{12.2.3}$$

where $\mathrm{Z}=\sqrt{\mu_0/\epsilon}$ is the wave impedance for a medium characterized by a permeability μ_0 and a dielectric constant ϵ. Substitute these new fields into Equations (12.2.1) to obtain

$$\operatorname{curl}\left(\overrightarrow{\mathrm{E}}_2\right) = i\omega\mu_0 \overrightarrow{\mathrm{H}}_2.$$

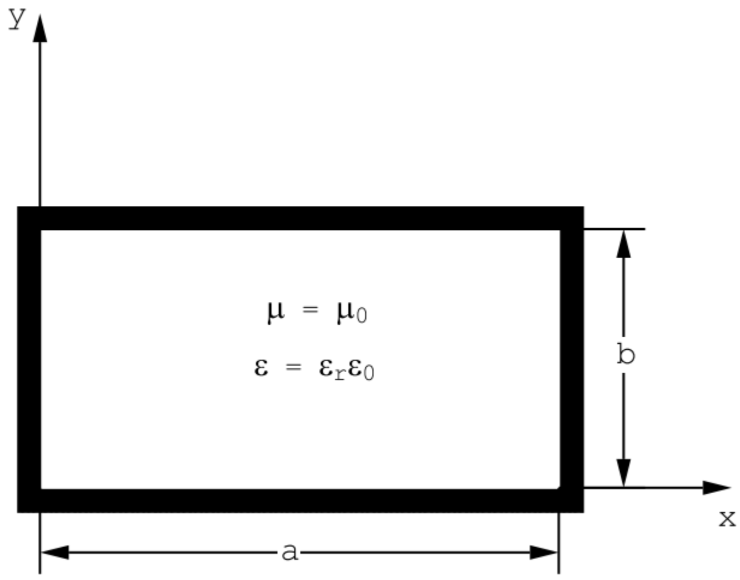

Figure 12.2.4: A rectangular wave-guide formed by conducting walls at x=0, x=a, y=0, and y=b. The lossless material inside the guide is characterized by a real dielectric constant ϵ, and a permeability μ_0.

Upon the substitution (12.2.3) this becomes

$$\operatorname{curl}\left(\overrightarrow{\mathrm{H}}_1\right) = -\frac{i\omega\mu_0}{\mathrm{Z}^2}\overrightarrow{\mathrm{E}}_1 = -i\omega\epsilon\overrightarrow{\mathrm{E}}_1,$$

and this by hypothesis satisfies Maxwell's Equations (12.2.1). Similarly, from (12.2.1) one has

$$\operatorname{curl}\left(\overrightarrow{\mathrm{H}}_2\right) = -i\omega\epsilon\overrightarrow{\mathrm{E}}_2.$$

Upon substitution of Equations (12.2.3) one finds

$$\operatorname{curl}\left(\overrightarrow{\mathrm{E}}_1\right) = i\omega\mu_0\overrightarrow{\mathrm{H}}_1,$$

so that the new fields, $\vec{E}_2$ and $\vec{H}_2$ satisfy both of Equations (12.2.1). Clearly Equations (12.2.2) are satisfied since $\vec{E}_2$ and $\vec{H}_2$ are proportional to $\vec{E}_1$ and $\vec{H}_1$. It follows that the prescription of Equation (12.2.3) can be used to generate a second, different, set of solutions for Maxwell's equations from a primary set of solutions. This procedure can often be used to avoid a great deal of computational tedium.

12.2.1 TM Modes.

In order to proceed with the rectangular wave-guide problem it is convenient to use the vector potential $\vec{A}$, and the scalar potential, V, where

$$\begin{aligned}\overrightarrow{\mathrm{H}} &= \operatorname{curl}(\overrightarrow{\mathrm{A}}),\\ \overrightarrow{\mathrm{E}} &= -\operatorname{grad}(\mathrm{V}) - \mu_0\frac{\partial\overrightarrow{\mathrm{A}}}{\partial \mathrm{t}}.\end{aligned} \tag{12.2.4}$$

The choice

$$\mathrm{A}_z = \mathrm{A}(\mathrm{x},\mathrm{y})\exp(i\left[\mathrm{k_g z} - \omega \mathrm{t}\right]), \tag{12.2.5}$$

plus $A_x = A_y = 0$ will guarantee that the z-component of the magnetic field, $\vec{H}$, is zero: in other words, this choice of vector potential will generate only TM modes, and

$$H_x = \frac{\partial A_z}{\partial y}$$
$$H_y = -\frac{\partial A_z}{\partial x}$$

For a time dependence exp (−iωt), and using (12.2.4) in Maxwell's equations (12.2.1), one finds

$$\mathrm{curlcurl}(\vec{A}) = \epsilon_r \left(\frac{\omega}{c}\right)^2 \vec{A} + i\omega\epsilon\,\mathrm{grad}(V).$$

But in cartesian co-ordinates

$$\mathrm{curlcurl}(\vec{A}) = -\nabla^2 \vec{A} + \mathrm{graddiv}(\vec{A}),$$

so that

$$\nabla^2 \vec{A} + \epsilon_r \left(\frac{\omega}{c}\right)^2 \vec{A} = \mathrm{grad}(\mathrm{div}(\vec{A}) - i\omega\epsilon V).$$

As explained in Chapter(7), one can set

$$i\omega\epsilon V = \mathrm{div}(\vec{A}),$$

so that for this problem where there are no driving charges or currents one finds

$$\nabla^2 \vec{A} + \epsilon_r \left(\frac{\omega}{c}\right)^2 \vec{A} = 0.$$

In particular, if $\vec{A}$ has only a z-component one finds

$$\nabla^2 A_z + \epsilon_r \left(\frac{\omega}{c}\right)^2 A_z = 0, \quad \text{and} \quad i\omega\epsilon V = \frac{\partial A_z}{\partial z}. \tag{12.2.6}$$

We require solutions that propagate along z: ie solutions that are proportional to exp $(ik_g z)$. Thus write

$$A_z(x, y, z, t) = A(x, y) \exp(i\,[k_g z - \omega t]),$$

for which A(x, y) must satisfy

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial Y^2} - k_g^2 A + \epsilon_r \left(\frac{\omega}{c}\right)^2 A = 0. \tag{12.2.7}$$

This equation is solved by products of sines and cosines:

$$A(x, y) = \text{ constant } (\sin(px) \text{ or } \cos(px))(\sin(qy) \text{ or } \cos(qy)),$$

where

$$p^2 + q^2 + k_g^2 = \epsilon_r \left(\frac{\omega}{c}\right)^2. \tag{12.2.8}$$

The particular combination of sines and cosines required must be chosen so that $\vec{H}$ satisfies the boundary condition that the normal component of $\vec{H}$ vanishes at the wave-guide walls. Using the magnetic field components calculated from Equation (12.2.4) and the co-ordinate system of Figure (12.2.4), it can be readily concluded that we require

$$A(x, y) = A_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \tag{12.2.9}$$

so that

$$\begin{aligned} \mathrm{H_x} &= \frac{\partial \mathrm{A}}{\partial \mathrm{y}} = \left(\frac{\mathrm{n}\pi}{\mathrm{b}}\right) \mathrm{A}_0 \sin\left(\frac{\mathrm{m}\pi \mathrm{x}}{\mathrm{a}}\right) \cos\left(\frac{\mathrm{n}\pi \mathrm{y}}{\mathrm{b}}\right), \\ \mathrm{H_y} &= -\frac{\partial \mathrm{A}}{\partial \mathrm{x}} = -\left(\frac{\mathrm{m}\pi}{\mathrm{a}}\right) \mathrm{A}_0 \cos\left(\frac{\mathrm{m}\pi \mathrm{x}}{\mathrm{a}}\right) \sin\left(\frac{\mathrm{n}\pi \mathrm{y}}{\mathrm{b}}\right), \end{aligned} \tag{12.2.10}$$

where m,n are integers, and Equation (12.2.8) becomes

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_g^2 = \epsilon_r \left(\frac{\omega}{c}\right)^2. \tag{12.2.11}$$

Notice that A_z=0 at the walls of the wave-guide. E_z is proportional to A_z so that if A_z=0 on the walls of the guide then the tangential component E_z will also vanish on the walls of the wave-guide as is required by the boundary conditions on the tangential components of E.

The electric field components can be most easily calculated from the second of Equations (12.2.1),

$$\begin{aligned} \mathrm{curl}(\overrightarrow{\mathrm{H}}) &= -i\epsilon\omega \overrightarrow{\mathrm{E}}. \\ \mathrm{E_x} &= \frac{-i}{\epsilon\omega}\left(\frac{\partial \mathrm{H_y}}{\partial \mathrm{z}}\right), \\ \mathrm{E_y} &= \frac{+i}{\epsilon\omega}\left(\frac{\partial \mathrm{H_x}}{\partial \mathrm{z}}\right), \\ \mathrm{E_z} &= \frac{i}{\epsilon\omega}\left[\frac{\partial \mathrm{H_y}}{\partial \mathrm{x}} - \frac{\partial \mathrm{H_x}}{\partial \mathrm{y}}\right]. \end{aligned}$$

The resulting electric field components are (dropping the factor exp $(i[k_g z - \omega t])$):

$$\begin{aligned} \mathrm{E_x} &= -\left(\frac{\mathrm{m}\pi}{\mathrm{a}}\right)\left(\frac{\mathrm{k_g}}{\epsilon\omega}\right) \mathrm{A}_0 \cos\left(\frac{\mathrm{m}\pi \mathrm{x}}{\mathrm{a}}\right) \sin\left(\frac{\mathrm{n}\pi \mathrm{y}}{\mathrm{b}}\right), \\ \mathrm{E_y} &= -\left(\frac{\mathrm{n}\pi}{\mathrm{b}}\right)\left(\frac{\mathrm{k_g}}{\epsilon\omega}\right) \mathrm{A}_0 \sin\left(\frac{\mathrm{m}\pi \mathrm{x}}{\mathrm{a}}\right) \cos\left(\frac{\mathrm{n}\pi \mathrm{y}}{\mathrm{b}}\right), \\ \mathrm{E_z} &= \left(\frac{i}{\epsilon\omega}\right)\left[\left(\frac{\mathrm{m}\pi}{\mathrm{a}}\right)^2 + \left(\frac{\mathrm{n}\pi}{\mathrm{b}}\right)^2\right] \mathrm{A}_0 \sin\left(\frac{\mathrm{m}\pi \mathrm{x}}{\mathrm{a}}\right) \sin\left(\frac{\mathrm{n}\pi \mathrm{y}}{\mathrm{b}}\right). \end{aligned} \tag{12.2.12}$$

Notice that these electric field components satisfy the requirement that the tangential components of E must vanish at the walls of the wave-guide. The field components Equations (12.2.10) and (12.2.12) correspond to the TM_{mn} mode.

For a propagating wave the value of k_g^2 calculated from (12.2.11) must be positive. This introduces a cut-off frequency, ω_m, such that k_g =0. This cut-off frequency is given by

$$\epsilon_r \left(\frac{\omega_m}{\mathrm{c}}\right)^2 = \left(\frac{\mathrm{m}\pi}{\mathrm{a}}\right)^2 + \left(\frac{\mathrm{n}\pi}{\mathrm{b}}\right)^2. \tag{12.2.13}$$

For given interior dimensions of the wave-guide there is a lower limit to the frequency for which a particular mode may be propagated along the waveguide. For example, a popular X-band wave-guide has interior dimensions a= 2.286 cm and b= 1.016 cm. For this guide the TM_{11} mode can be propagated only for frequencies greater than 16.15 GHz if ϵ_r= 1. There are no TM modes corresponding to m=0 or n=0 since the fields are zero if m=0 or if n=0 because A(x,y)=0 from Equation (12.2.9). Thus the lowest TM frequency that can be propagated down the above guide is 16.15 GHz.

Non-propagating TM modes do exist for frequencies less than the cutoff frequency. If $\omega < \omega_m$ then k_g calculated from (12.2.8) is negative. This means that kg is a purely imaginary number, $k_g = i\beta$ say. The phasor exp $(i[k_g z - \omega t])$ becomes exp $(-\beta z)$ exp $(-i\omega t)$ corresponding to a disturbance that decays to a small amplitude over a distance $z \sim (1/\beta)$.

12.2.2 TE Modes.

There exists another group of modes for which $E_z = 0$; these are the TE modes. Using the symmetry relations Equations (12.2.3) and the magnetic fields (12.2.10) one might guess that the TE mode electric fields ought to be given by (the factor exp $(i[k_g z - \omega t])$ is suppressed)

$$\begin{aligned} E_x &= Z\left(\frac{n\pi}{b}\right) A_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \\ E_y &= -Z\left(\frac{m\pi}{a}\right) A_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right). \end{aligned}$$

These electric fields satisfy Maxwell's equations but they do not satisfy the boundary condition that the tangential components of $\vec{E}$ must vanish at the wave-guide walls: ie. E_x=0 at y=0,b and E_y=0 at x=0,a (see Figure (12.2.4)). However, the following equations for the electric field components do satisfy the required boundary conditions:

$$\begin{aligned} E_x &= E_1 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \\ E_y &= E_2 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \\ E_z &= 0. \end{aligned}$$

These field components vanish on the wave-guide walls. The electric field must also satisfy the Maxwell equation div($\vec{E}$) = 0. This condition requires

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{y} = 0.$$

It follows that

$$\left(\frac{m\pi}{a}\right) E_1 = -\left(\frac{n\pi}{b}\right) E_2.$$

Using this relation between E_1 and E_2 the electric field components corresponding to the TE modes in a rectangular wave-guide have the form:

$$\begin{aligned} E_x &= \left(\frac{n\pi}{b}\right) E_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \\ E_y &= -\left(\frac{m\pi}{a}\right) E_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \\ E_z &= 0, \end{aligned} \tag{12.2.14}$$

where E_0 is a constant, and the factor exp (i[k_gz − ωt]) has again been suppressed. The magnetic field components corresponding to Equations (12.2.14) can be calculated from Faraday's law: iωμ0 $\vec{H}$ = curl($\vec{E}$). The resulting field components are

$$\begin{aligned} H_x &= \frac{+i}{\omega\mu_0}\frac{\partial E_y}{\partial z} \\ &= \frac{k_g}{\omega\mu_0}\left(\frac{m\pi}{a}\right) E_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \\ H_y &= \frac{-i}{\omega\mu_0}\frac{\partial E_x}{\partial z} \\ &= \frac{k_g}{\omega\mu_0}\left(\frac{n\pi}{b}\right) E_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \\ H_z &= \frac{-i}{\omega\mu_0}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \\ &= \frac{i}{\omega\mu_0} E_0 \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right). \end{aligned} \tag{12.2.15}$$

Eqns.(12.2.14) and (12.2.15) satisfy Maxwell's equations and also the boundary conditions that the tangential components of $\vec{E}$ and the normal components of $\vec{H}$ vanish on the wave-guide walls. The TE_{10} mode discussed in section(12.1) corresponds to m=1,n=0: for this mode E_x = 0 and H_y = 0. Referring to the co-ordinate system of Figure (12.2.4) the field components for the TE_{10} mode are:

$$\begin{aligned} E_y &= A\sin\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z-\omega t]), \\ H_x &= -\frac{k_g}{\omega\mu_0}A\sin\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z-\omega t]), \\ H_z &= -\frac{i}{\omega\mu_0}\left(\frac{\pi}{a}\right)A\cos\left(\frac{\pi x}{a}\right)\exp(i\,[k_g z-\omega t]), \end{aligned} \tag{12.2.16}$$

where A is a constant and

$$k_g^2=\epsilon_r\left(\frac{\omega}{c}\right)^2-\left(\frac{\pi}{a}\right)^2. \tag{12.2.17}$$

The cut-off frequency for this mode, corresponding to k_g=0, is given by

$$\epsilon_r\left(\frac{\omega}{c}\right)^2=\left(\frac{\pi}{a}\right)^2.$$

For the popular X-band wave-guide used above for illustrative purposes one has a=2.286 cm and b=1.016 cm. For this guide, and $\epsilon_r = 1$, the cut-of

m	n	TE_{mn}(GHz)	TM_{mn}(GHz)
1	0	6.557	*No* TM *Mode*
0	1	14.753	*No* TM *Mode*
1	1	16.145	16.145
2	0	13.114	*No* TM *Mode*
2	1	19.740	19.740
0	2	29.507	*No* TM *Mode*
1	2	30.227	30.227
2	2	32.290	32.290
3	0	19.671	*No* TM *Mode*
3	1	24.589	24.589
3	2	35.463	35.463
3	3	48.435	48.435

Table 12.2.1: Cut-off frequencies for the lowest transverse electric (TE) modes and the lowest transverse magnetic (TM) modes in X-band waveguides (RG52/U or WR90 brass guides). The internal dimensions of X-band waveguides are a= 0.900 inches = 2.286 cm, and b= 0.400 inches = 1.016 cm. The external dimensions of the guide are 1.00 x 0.50 inches. The cut-off frequencies were calculated for ϵ_r = 1 using $\left(\frac{\omega_{mn}}{c}\right)^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$.

frequency for the TE_{10} mode is 6.56 GHz. Cut-off frequencies for various modes in this X-band wave-guide are listed in Table(12.2.1). Notice that only one mode, the TE_{10} mode, can be propagated along this wave-guide for frequencies between 6.6 and 13.1 GHz.