13.3: Chapter 3

Problem (3.1)

It is desired to construct a 100 pF capacitor using a mylar spacer 10^{-4} m. thick, ε_r= 3.0, placed between two metal plates. How large an area is required for the metal electrodes?

Answer (3.1)

$C=\frac{AE_r\varepsilon_0}{D}=\frac{3A}{10^{-4}}\left(8.84\times10^{-12}\right)\ \text{Farads }=10^{-10}\text{ Farads;}$

$A=3.77\times10^{-4}\ \text{m}^2=3.8\ \text{cm}^2$,

i.e. one requires electrodes approx. 2x2 cm square.

Problem (3.2)

A small drop of oil is characterized by a relative dielectric constant ε_r= 1.5 and a density of 800 kg/m^3; its radius is R= 10^{-4} m. It is placed between condenser plates which are parallel and which are separated by 1 cm. The oil drop is uncharged. A potential difference of 100 Volts is placed across the capacitor plates. The relative dielectric constant of air may be taken to be ε_r= 1.00.

(a) Estimate the dipole moment induced on the drop by the electric field.

(b) How large a field gradient would be required to suspend the drop in the gravitational field?

Answer (3.2)

(a) The potential function outside the drop has the form

$$V_2=-E_0r\cos\theta+\frac{b\cos\theta}{r^2};$$

The potential function inside the drop has the form

$$\mathrm{V}_1=-\mathrm{a\,r}\cos\theta.$$

At r= R these potentials must satisfy the two boundary conditions

(1) The potential function must be continuous;

and (2) The normal component of **D** must be continuous.

These boundary conditions require

$$a+\frac{b}{R^3}=E_0$$
$$a-\frac{2}{\varepsilon_r}\frac{b}{R^3}=\frac{E_0}{\varepsilon_r}.$$

These equations have the solutions

$a=\left(\frac{3E_0}{\varepsilon_r+2}\right)\qquad b=\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right)R^3E_0$

The dipole moment on the sphere is therefore given by

$$\mathrm{p_z}=4\pi\varepsilon_0\mathrm{R}^3\left(\frac{\varepsilon_\mathrm{r}-1}{\varepsilon_\mathrm{r}+2}\right)\mathrm{E}_0$$

For the present case

$$p_z=\left(\frac{10^{-12}}{9\times10^9}\right)\left(\frac{0.5}{3.5}\right)\left(\frac{100}{10^{-2}}\right)=1.59\times10^{-19}\text{ Coulomb-meters.}$$

(b) The gravitational force on the drop is

$$F_g=mg=\frac{4\pi R^3}{3}(800)(9.8)=3.28\times10^{-8}\text{Newtons.}$$

In order to suspend the drop one would require

$$p_z \frac{dE_z}{dz} = 3.28 \times 10^{-8} \mathrm{N}.$$

This implies a field gradient of $\mathbf{\frac{dE_z}{dz} = 2.07 \times 10^{11}\ Volts/meter.}$ This is an enormous field gradient!

Problem (3.3)

This problem concerns the calculation of the dielectric constant for a material composed of a lattice of atoms each of which carries a permanent electric dipole moment which is free to rotate. The calculation follows the article by L.Onsager, J.Amer.Chem.Soc.58,1486-1493(1936). According to the Onsager model, each electric dipole, of strength p, is located at the center of a sphere of radius R: inside the sphere the relative dielectric constant is 1, outside the sphere the relative dielectric constant is ε_r. The spherical hole is supposed to represent the volume of the atom which carries the dipole. The average electric field in the material far from the dipole under examination is uniform, it has the value E, and it is directed along z.

(a) Calculate the field in the cavity in the absence of the dipole moment; let this field be E_c.

(b) Calculate the field inside the cavity for the case when the dipole is present in the cavity but the average applied field strength is zero, i.e. E=0. Let the field in the cavity due to the presence of the dipole be the reaction field **R**. Notice that the reaction field is always parallel with the direction of the dipole. The field outside the cavity is a dipole field; what is the corresponding effective dipole moment?

(c) The total field in the cavity due both to the presence of the dipole and due to the applied field E can be obtained by superposition. The result is the vector sum of the cavity field, $\mathbf{E}_c$, and the reaction field, **R**. However, **R** exerts no torque on the dipole because it is parallel with it. The potential energy of the electric dipole because of the presence of the cavity field is given by

$$\mathrm{U_d} = -\mathbf{p} \cdot \mathbf{E}_\mathrm{C} = -\mathrm{pE_C} \operatorname{Cos}\theta$$

If pE_c is small compared with kT it can be shown, using standard statistical mechanics, that the average value of Cosθ due to thermal agitation is

$$<\cos\theta>=\left(\frac{\mathrm{pE_C}}{3\mathrm{kT}}\right)$$

Use this result to calculate the mean value of the polarization per unit volume. Let the number density of dipoles be N per m^3.

(d) Use the results of part (c) to show that the dielectric constant of the medium is related to the individual atomic dipole moment, p, through the expression

$$\varepsilon_\mathrm{r} - 1 = \left(\frac{\varepsilon_\mathrm{r}}{1+2\varepsilon_\mathrm{r}}\right)\left(\frac{\mathrm{Np}^2}{\varepsilon_0 \mathrm{kT}}\right)$$

This relation can be used to deduce the dipole moment of polar molecules from the measured values of the static dielectric constant.

(e) A certain material contains a density of molecules $\mathrm{N} = \frac{1}{3} \times 10^{29}$ per meter^3, and each molecule carries an electric dipole moment $p = \frac{1}{2} x 10^{-29}$ Coulomb-meters. Calculate the relative dielectric constant, ε_r, at 300K.

Answer (3.3)

(a) The Cavity Field.

Outside the cavity $\mathrm{V_2} = -\mathrm{Ercos}\,\theta + \frac{\mathrm{b}\cos\theta}{\mathrm{r}^2}$

Inside the cavity $V_1 = -E_C r\cos\theta$

At r=R: $V_1 = V_2$

$$\varepsilon_0 \frac{\partial V_1}{\partial r} = \varepsilon \frac{\partial V_2}{\partial r},$$

from which $\mathbf{E}_\mathrm{c} = \frac{3\varepsilon_\mathrm{r}\mathrm{E}}{(2\varepsilon_\mathrm{r}+1)}$.

(b) The Reaction Field.

Outside the cavity the potential function is that of a dipole; for this part of the problem there is no external field, so that $V_2 = \frac{b\cos\theta}{r^2}$.

Inside the cavity the potential function, V_1, must include the singular dipole field due to the point dipole plus a reaction field due to the polarization of the medium:

$$V_1 = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} + \mathrm{ar}\cos\theta$$

At r=R: $V_1 = V_2$

$$\varepsilon_0\frac{\partial V_1}{\partial r} = \varepsilon\frac{\partial V_2}{\partial r}$$

from which

$$\mathrm{V}_1 = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{p}\cos\theta}{\mathrm{r}^2} - \frac{2\mathrm{p}}{4\pi\varepsilon_0\mathrm{R}^3}\left(\frac{\varepsilon_\mathrm{r}-1}{2\varepsilon_\mathrm{r}+1}\right)\mathrm{r}\cos\theta$$

and

$$\mathrm{V}_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\varepsilon_{\mathrm{rp}}}{2\varepsilon_\mathrm{r}+1}\right)\frac{\cos\theta}{\mathrm{r}^2}$$

From these expressions one obtains

$$|\mathbf{R}| = \frac{2p}{4\pi\varepsilon_0 R^3}\left(\frac{\varepsilon_\mathbf{r}-1}{2\varepsilon_\mathbf{r}+1}\right)$$

and the effective dipole moment for the region external to the cavity is given by

$$p^* = \frac{3\varepsilon_r^2 p}{(2\varepsilon_r+1)},$$

where $\mathrm{V}_2 = \frac{\mathrm{p}^*}{4\pi\varepsilon_\mathrm{r}\varepsilon_0}\frac{\cos\theta}{\mathrm{r}^2}$.

(c) The mean polarization per unit volume is parallel with the field and is given by

$$P = Np < \cos\theta >$$

Consequently,

$$P = \frac{Np^2}{3kT}\left(\frac{3\varepsilon_r}{2\varepsilon_r+1}\right)E = \frac{Np^2}{kT}\left(\frac{\varepsilon_r}{2\varepsilon_x+1}\right)E \text{ coulombs }/m^2$$

(d) D = εE = ε0E + P,

or $(\varepsilon_r-1)\,E = P/\varepsilon_0$.

Therefore

$$(\varepsilon_\mathrm{r}-1) = \frac{\mathrm{Np}^2}{\varepsilon_0\mathrm{kT}}\left(\frac{\varepsilon_\mathrm{r}}{2\varepsilon_\mathrm{r}+1}\right),$$

or

$$\frac{\mathrm{Np}^2}{\varepsilon_0\mathrm{kT}} = \frac{(\varepsilon_\mathrm{r}-1)(2\varepsilon_\mathrm{r}+1)}{\varepsilon_\mathrm{r}}$$

(e)

$$\frac{\mathrm{Np}^2}{\varepsilon_0\mathrm{kT}} = \frac{(1/3)\left(10^{29}\right)(1/4)\left(10^{-29}\right)\left(10^{-29}\right)}{\left(8.84\times10^{-12}\right)\left(1.38\times10^{-23}\right)(300)} = 22.8$$

From this $\varepsilon_\mathrm{r}^2 - 11.9\varepsilon_\mathrm{r} - (1/2) = 0$

and

$\varepsilon_r = 11.93$

Problem (3.4)

The radius of the sun is 6.98x10^5 km. and its surface temperature is 6000°C, corresponding to an energy kT= 0.34 electron Volts. Treat the sun as a conducting sphere isolated in space and calculate the net positive charge required to produce a potential of 0.34 Volts relative to zero potential at infinity.

Answer (3.4)

At the surface of a conducting sphere the potential is given by

$V = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}$

Therefore, Q= $(4\pi\varepsilon_0)(0.34)(6.98\text{x}10^8)$ = 2.64x10^{-2} Coulombs. This is a surprisingly small amount of charge. It corresponds to a deficit of 1.65x10^{17} electrons.

Problem (3.5)

A capacitor is constructed of two concentric metal cylinders. The relevant radius of the inner electrode is a, the relevant radius of the outer electrode is b. The space between the electrodes is filled with air for which ε_r= 1.00.

(a) What is the capacitance per unit length of this device?

(b) The above capacitor, whose length is L= 10 cm, is placed upright in a dish of oil, ε_r= 3.00, so that the space between the cylindrical electrodes is filled with oil to a depth of 5 cm. What is the capacitance of this configuration if the radii are a= 5 cm and b= 6 cm?

(c) The capacitor of part (b) is charged to a potential difference of 1000 Volts. How high will the oil rise between the capacitor electrodes if the density of the oil is 800 kg/m^3?

Answer (3.5)

(a) Let the charge on the inner electrode be Q Coulombs/meter, that on the outer electrode -Q Coulombs/meter. The field is radial, so from Gauss' law

$2\pi r\, E_r = Q/\varepsilon_0$,

and

$E_r = \frac{Q}{2\pi\varepsilon_0 r}$.

The potential difference between the electrodes is

$$\Delta V = \int_a^b E_r dr = \frac{Q}{2\pi\varepsilon_0}\ln(b/a)$$

But Q= CΔV, therefore

$$c = \frac{2\pi\varepsilon_0}{\ln(b/a)} \text{ Farads/meter.}$$

(b) If oil is placed betweeen the electrodes the capacitance per unit length becomes

$C_{oil} = \varepsilon_r C$ Farads/meter.

For a system having a length of L= 5cm= 0.05 meters the capacitance is

$$c = \frac{(2\pi\varepsilon_0)(0.05)}{\ln(6/5)} = 15.2 \times 10^{-12}\text{ F} = 15.2\text{ pF}$$

The oil filled part has a capacitance which is 3 times this value: C_{oil}= 45.7 pF. The total capacitance is the sum of the above figures:

$$C_{tot} = C_{oil} + C = 60.9\text{pF}$$

(c) The electrostatic field energy per unit length of capacitor is given by

$$\mathrm{U_E}=\int_{\mathrm{a}}^{\mathrm{b}}(2\pi\mathrm{rdr})\varepsilon_{\mathrm{r}}\varepsilon_0\frac{\mathrm{E}^2}{2},$$

where $E=\frac{Q}{2\pi\varepsilon_0\varepsilon_r}$ and $\Delta\mathrm{V}=\frac{\mathrm{Q}}{2\pi\varepsilon_0\varepsilon_{\mathrm{r}}}\ln(\mathrm{b}/\mathrm{a})$.

That is $E=\frac{\Delta V}{\ln(b/a)}\frac{1}{r}$, so that

$$\mathrm{U_E}=\pi\varepsilon_0\varepsilon_{\mathrm{r}}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{rdr}\frac{(\Delta\mathrm{V})^2}{(\ln(\mathrm{b}/\mathrm{a}))^2}\frac{1}{\mathrm{r}^2}=\frac{\pi\varepsilon_0\varepsilon_{\mathrm{r}}(\Delta\mathrm{V})^2}{\ln(\mathrm{b}/\mathrm{a})}$$

If the oil level between the capacitor electrodes rises by dz, the increase in electrostatic field energy will be given by

$$\mathrm{dU_E}=\frac{\pi\varepsilon_0\left(\varepsilon_{\mathrm{r}}-1\right)(\Delta\mathrm{V})^2}{\ln(\mathrm{b}/\mathrm{a})}\mathrm{dz}$$

since a slice dz thick of air (ε_r=1) is replaced by oil (ε_r= 3.0). But this change in energy must be equal to the work done by the electrostatic forces: Fdz= dU_E, so that

$$\mathrm{F}=\frac{\pi\varepsilon_0\left(\varepsilon_{\mathrm{r}}-1\right)(\Delta\mathrm{V})^2}{\ln(\mathrm{b}/\mathrm{a})}\text{ Newtons}.$$

This force will support a column of oil whose height is H meters, where

$$\mathrm{F}=\pi\left(\mathrm{b}^2-\mathrm{a}^2\right)\mathrm{H}\rho\mathrm{g}$$

For our problem F = $\pi(1.1\mathrm{x}10^{-3})(800)(9.8)$H,

or F = 27.09H Newtons

Therefore $\mathbf{H}=\mathbf{1.12}\times\mathbf{10^{-5}meters}=\mathbf{11.2\mu m}$.

Problem (3.6)

An electron is located a distance d in front of the plane interface with a material characterized by a relative dielectric constant ε_r= 3.00.

(a) Calculate the force on the electron.

(b) How much work must be done on the electron to bring it from infinity to a distance a= 10^{-10} m from the surface?

Answer (3.6)

(a) In the vacuum at the interface

$$V=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r}+\frac{Q'}{r}\right)$$

and

$\mathrm{D_n}\sim\left(\mathrm{Q}-\mathrm{Q}'\right)$.

In the slab at the interface

$$V=\frac{1}{4\pi\varepsilon_0}\frac{1}{\varepsilon_r}\frac{Q''}{r}$$

and

$$\mathrm{D_n}\sim Q''.$$

Therefore $Q-Q'=Q''$

$$Q+Q'=\frac{Q''}{\varepsilon_r},$$

so that

$$Q'' = \frac{2\varepsilon_r Q}{(\varepsilon_r + 1)}$$

and

$$Q' = Q\left(\frac{1-\varepsilon_r}{1+\varepsilon_r}\right).$$

For the present problem Q"= 3Q/2 and Q'= -Q/2.

The force on the charge Q is given by (z is measured towards the interface)

$$\mathrm{F_z} = \mathrm{Q}\left(\frac{\mathrm{Q}/2}{4\pi\varepsilon_0(2\mathrm{d})^2}\right) = \frac{\mathrm{Q}^2}{32\pi\varepsilon_0\mathrm{d}^2}.$$

This is an attractive force.

(b) The electron will be attracted to the interface, consequently the work done to bring it up from infinity is negative. The binding energy is given by

$$U = -\int_{-\infty}^{-a} F_z dz = \frac{Q^2}{32\pi\varepsilon_0 a},$$

or

$$\mathrm{U} = \frac{\left((1.6)^2 \times 10^{-38}\right)\left(9 \times 10^9\right)}{\left(8 \times 10^{-10}\right)} = \mathbf{2.88 \times 10^{-19}\,Joules\ = 1.8eV}$$

Problem (3.7)

(W.Shockley, J.Appl.Phys.9 ,635(1938)).

A capacitor is made using conducting concentric cylinders with a vacuum in the space between the electrodes. The radii of the relevant surfaces are a,b where b>a. Place a charge q at position r between the electrodes. What is the charge induced on each of the electrodes?

The solution of this problem is related to the calculation of the noise spectrum in vacuum tubes. The current through such a tube is carried by discrete charges, electrons, and as each electron leaves one electrode it induces a characteristic current spike in an external circuit. The time variation of the current pulse depends upon the electron transit time. The Fourier transform of the time variation of the current pulse gives the noise spectrum.

Hint for the solution.

(1) Use the linearity between charge and Voltage to write three equations involving generalized capacitance coefficients (see Equations (10.111)). One can think of the test charge q as being located on a very tiny spherical electrode.

(2) Construct two thought experiments:

(a) Put a charge Q on the inner electrode(#1), a charge -Q on the outer electrode(#2) which is grounded, and put no charge on the tiny sphere (electrode #3); i.e. Q_1= Q, Q_2= -Q, and Q_3= 0. The corresponding potentials are V_1, which can be calculated, V_2= 0 (grounded electrode), and V_3 which can also be calculated assuming that electrode 3 is so small that it makes a negligible perturbation of the field between the electrodes.

(b) Put V_1= V_2= 0 and let the charge on electrode #3 be q.

The results of these two experiments enables one to deduce that the induced charge on the inner electrode is given by

$$Q_1 = -q\frac{\ln(b/r)}{\ln(b/a)}.$$

Similarly, one can show that $Q_2 = -q\frac{\ln(r/a)}{\ln(b/a)}$. Thus Q_1 + Q_2 = -q corresponding to charge conservation.

Answer (3.7)

Put a charge Q on the inner electrode and ground the outer electrode so that V_2=0. In the space between the electrodes the potential is given by

$$V(r)=\frac{Q}{2\pi\varepsilon_0}\ln(b/r),$$

corresponding to the electric field $E_r=\frac{Q}{2\pi\varepsilon_0}$. The potential at the position of the uncharged electrode #3 is just V(r). One has

$$Q_1=Q,\quad V_1=\frac{Q}{2\pi\varepsilon_0}\ln(b/a);$$

$$Q_2=-Q,\quad V_2=0;$$

$$Q_3=0,\quad V_3=\frac{Q}{2\pi\varepsilon_0}\ln(b/r).$$

But

$$\begin{aligned}Q_1&=C_{11}V_1+C_{12}V_2+C_{13}V_3\\ Q_2&=C_{12}V_1+C_{22}V_2+C_{23}V_3\\ Q_3&=C_{13}V_1+C_{23}V_2+C_{33}V_3\end{aligned}$$

Therefore

$$Q=C_{11}\frac{Q}{2\pi\varepsilon_0}\ln(b/a)+C_{13}\frac{Q}{2\pi\varepsilon_0}\ln(b/r)\qquad(1)$$

$$-Q=C_{12}\frac{Q}{2\pi\varepsilon_0}\ln(b/a)+C_{23}\frac{Q}{2\pi\varepsilon_0}\ln(b/r)\qquad(2)$$

$$0=C_{13}\frac{Q}{2\pi\varepsilon_0}\ln(\mathrm{b/a})+\mathrm{C}_{33}\frac{\mathrm{Q}}{2\pi\varepsilon_0}\ln(\mathrm{b/r})\qquad(3).$$

From (3) $\frac{C_{13}}{C_{33}}=-\frac{\ln(b/r)}{\ln(b/a)}$.

Now let V_1=0, V_2=0, and Q_3=q. Then

q = $C_{33}V_3$ and Q_1= $C_{13}V_3$,

from which

$$\mathbf{Q_1}=\left(\frac{\mathbf{c_{13}}}{\mathbf{c_{33}}}\right)\mathbf{q}=-\mathbf{q}\frac{\ln(\mathbf{b/r})}{\ln(\mathbf{b/a})}.$$

When r=b Q_1=0 as it should; no charge is induced on the inner electrode, but there is a charge -q induced on the outer electrode. When r=a the full charge -q is induced on the inner electrode. The induced charge -q is transferred from the outer to the inner electrode through the external circuit during the time required for the charge q to move from one electrode to the other.

Problem (3.8)

Let an air-filled capacitor (ε_r= 1.00) be constructed of two square shaped metal plates of length L on a side separated by a space D. The edges of the two plates are parallel. Now let one of the plates be rotated slightly around one of its edges so that the two electrodes make an angle θ with respect to one another; along one edge the spacing is D and along the other edge the spacing is D+Lθ. Estimate the capacitance of this wedged capacitor. This can be done by equating $\frac{\mathrm{CV}^2}{2}$ with the electrostatic field energy $\frac{\varepsilon_0}{2}\int E^2d\tau$, and by making a plausible assumption about the electric field distribution between the wedged conductors. The electrostatic field energy is an extremum (a minimum) for the correct field distribution and therefore its value is insensitive to small deviations of the field from its correct distribution.

I assumed that

$$E_\theta = \frac{V}{(R+x)\theta},$$

where Rθ= D, and where V is the potential difference between the electrodes. This assumption makes E perpendicular to the electrode surfaces and it preserves a constant potential difference between the plates as x goes from 0, corresponding to one edge of the plates, to x=L corresponding to the other edge. Unfortunately, div **E** is not zero so that the its potential function does not satisfy LaPlace's equation. Nevertheless, this calculation will give an upper bound for the change in capacitance with wedge angle.

Answer (3.8)

Let x be the distance from the narrow edge of the wedge between the two conductors. At any point x one can use the volume element

$$d\tau = L(R+x)\theta dx;$$

this expression is based upon a cylindrical co-ordinate system in which the z-axis lies at the apex of the wedge. If

$$E_\theta = \frac{V}{(R+x)\theta},$$

then the field energy is given by (neglecting edge effects)

$$U_E = L\int_0^L \left(\frac{\varepsilon_0 E^2}{2}\right)(R+x)\theta dx = \frac{\varepsilon_0 L V^2}{2\theta}\int_0^L \frac{dx}{(R+x)},$$

or

$$U_E = \frac{\varepsilon_0 L V^2}{2\theta}\int_R^{R+L}\frac{du}{u} = \frac{\varepsilon_0 L V^2}{2\theta}\ln\left(1+\frac{L}{R}\right),$$

from which $\mathbf{c} \cong \frac{\varepsilon_0 \mathbf{L}}{\theta}\ln\left(\mathbf{1}+\frac{\mathbf{L}}{\mathbf{R}}\right)$.

If $\frac{L}{R} << 1$, (small wedge angle), this gives

$$C = C_0 = \frac{\varepsilon_0 L^2}{R\theta} = \frac{\varepsilon_0 L^2}{D},$$

the correct expression for a parallel plate capacitor. From the expansion $\ln(1+x) = x - \frac{x^2}{2} + \ldots$ the correction to the parallel plate value C_0 is given by

$$C \cong C_0\left(1-\frac{L}{2R}\right) = C_0\left(1-\frac{L\theta}{2D}\right).$$

The effect of tilting the plates is to reduce the capacitance by an amount corresponding to the average increase in spacing between the plates.