13.1: Chapter 1

Problem (1.1).

Two charges, each q=+1.6x10^{-19} Coulombs, are located at (0,0,a) and at (0,0,-a) where a=1.0x10^{-9} meters.

(a) Calculate the electric field at the origin (0,0,0).

(Answ: the field is zero.)

(b) Calculate the electric field at (a,a,a).

(Answ: **E**= (6.07,6.07,1.96)x10^{8} Volts/m.)

(c) An electron, q=-1.6x10^{-19} Coulombs, flies through the point (a,a,a) with the velocity **v**= v_0(1,2,3) where v_0= 10^5 m/sec. What forces are exerted on the electron due to the two stationary charges?

(Answ: **F**= q**E**= (-9.71,-9.71,-3.14)x10^{-11} Newtons. There is no magnetic force.)

Problem (1.2).

At a certain moment a moving proton, q=+1.6x10^{-19} Coulombs, is located at (0,0,a) with velocity components v_0(1,1,0) where a=10^{-9} m. and v_0=10^5 m/sec. At the same moment a moving electron, q=-1.6x10^{-19} Coulombs, is located at (a,a,a) with velocity components (0,10^6,0) m/sec.

(a) Calculate the electric and magnetic fields at the position of the electron due to the proton.

(Answ: **E**= (E_0,E_0,0) where E_0= 5.09x10^8 V/m. and **B**= (0,0,0) because $\mathbf{v_p}$x**E** =0.)

(b) Calculate the force on the electron due to the electric field of the proton.

(Answ: **F**=(-F_0,-F_0,0) where F_0= |q|E_0 =8.14x10^{-11} N.)

(c) Calculate the force on the electron due to the magnetic field of the proton.

(Answ: **F**= $\mathbf{v}_{\text{electron}}$x**B** = 0 N.)

(d) Calculate the electric and magnetic forces on the proton due to the fields generated by the electron.

Answ: The electric field at the position of the proton, **R**=(0,0,a), due to the electron at **r**=(a,a,a) is given by

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\quad\left(-1.6\times10^{-19}\right)\frac{\rho}{\rho^3},$$

where ρ= **R**-**r** = (-a,-a,0)= - a(1,1,0), where a= 10^{-9} m.

Therefore

$$\mathbf{E}=\left(5\cdot09\times10^{8}\right)(1,1,0).$$

The magnetic field at the position of the proton due to the motion of the electron is given by c^2**B**= **v**x**E**, where the velocity of the electron is **v**= 10^6(0,1,0) m/sec. c^2**B**= (5.09x10^{14})(0,0,-1) so **B**= (0.566x10^{-2})(0,0,-1) Teslas.

The force on the proton due to the electric field is $\mathbf{F_E}$= 8.15x10^{-11}(1,1,0) N. The force on the proton due to the magnetic field is $\mathbf{F_M}$= q($\mathbf{v_p}$x**B**) = 0.906x10^{-16}(-1,1,0) N.

Problem (1.3).

A particle having a velocity **V**=$v_1\mathbf{u_x}$ carries a charge q_1 C and is located at the origin. A second particle, charge q_2, is located at **r**= a$\mathbf{u_x}$ + b$\mathbf{u_y}$ + c$\mathbf{u_z}$, and it has a velocity $\mathbf{V_2}$=$v_y\mathbf{u_y}$.

(a) Show that the force on charge #2 due to the magnetic field generated by charge #1 is $\mathbf{F}_{21}=\frac{\mu_0}{4\pi}\frac{q_1q_2}{r^3}bv_1v_y\mathbf{u}_x$.

(b) Show that the force on charge #1 due to the magnetic field generated by charge #2 is $\mathbf{F}_{12}=-\frac{\mu_0}{4\pi}\frac{q_1q_2}{r^3}av_1V_y\mathbf{u}_y$. Notice that $\mathbf{F_{21}}$ does not equal -$\mathbf{F_{12}}$ so that Newton's law of the equality of forces of action and reaction is not obeyed in this case.

Answer (1.3).

(a) The electric field at the position of particle #2 due to particle #1 is

$$\mathbf{E_{21}}=\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}_1}{\mathrm{r}^3}(\mathrm{a},\mathrm{b},\mathrm{c}).$$

The magnetic field at the position of particle #2 due to the motion of particle #1 is given by

$$c^2\mathbf{B_{21}}=\mathbf{v}_1\times\mathbf{E}_{21}=\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}_1\mathrm{v}_1}{\mathrm{r}^3}(0,-\mathrm{c},\mathrm{b}),$$

or

$$\mathbf{B_{21}}=\frac{\mu_0}{4\pi}\frac{\mathrm{q}_1\mathrm{v}_1}{\mathrm{r}^3}(0,-\mathrm{c},\mathrm{b}).$$

The magnetic force on particle #2 due to its motion is

$$\mathbf{F_{2M}}=q_2\left(\mathbf{v_2}\times\mathbf{B_{21}}\right)=\frac{\mu_0}{4\pi}\frac{q_1q_2v_1v_y}{r^3}\quad(b,0,0).$$

(b) The electric field at the position of particle #1 due to particle #2 is

$$\mathbf{E_{12}}=-\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q2}}{\mathrm{r}^3}(\mathrm{a},\mathrm{b},\mathrm{c}).$$

The magnetic field at the position of particle #1 due to the motion of particle #2 is given by

$$c^2\mathbf{B_{12}}=\mathbf{v}_2\times\mathbf{E_{12}}=-\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}_2\mathrm{v}_\mathrm{y}}{\mathrm{r}^3}(\mathrm{c},0,-\mathrm{a}),$$

or

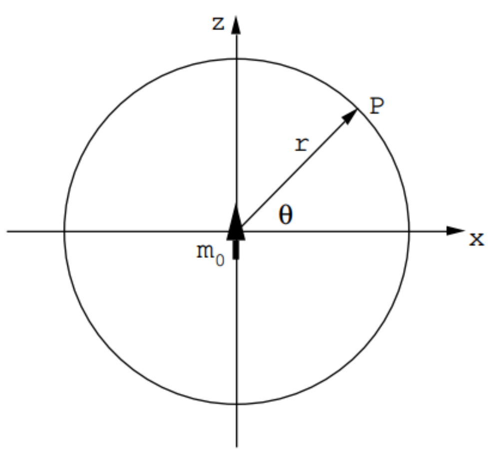
$$\mathbf{B_{12}}=\frac{\mu_0}{4\pi}\frac{\mathrm{q}_2\mathrm{v}_\mathrm{y}}{\mathrm{r}^3}(\mathrm{c},0,-\mathrm{a}).$$

The magnetic force on particle #1 due to its motion is

$$\mathbf{F_{1M}}=\mathrm{q}_2\left(\mathbf{v_1}\times\mathbf{B_{12}}\right)=-\frac{\mu_0}{4\pi}\frac{\mathrm{q}_1\mathrm{q}_2\mathrm{v}_1\mathrm{v}_\mathrm{y}}{\mathrm{r}^3}(0,\mathrm{a},0).$$

Problem (1.4).

An electron carries a magnetic moment of $|\mathbf{m}_0|$=9.27x10^{-24} Joules/Tesla= 1 Bohr magneton. Suppose that this magnetic moment is oriented along the z-axis as shown in the figure.



(a) At what angle θ is the field measured by an observer at P a maximum?

(Answ: θ= ±π/2.)

(b) If r= 1 micron (10^{-6}m.) what is the magnitude and direction of this maximum field?

(Answ: $|\mathbf{B}_{max}|$= 18.54x10^{-13} Teslas directed along +z).

(c) What is the minimum magnetic field? At what angle θ does it occur, and what is the direction of the field?

(Answ: |$\mathbf{B}_{\min}$|= 9.27x10-13 Teslas directed along -z. The observer is at θ=0 or π.)

Answer

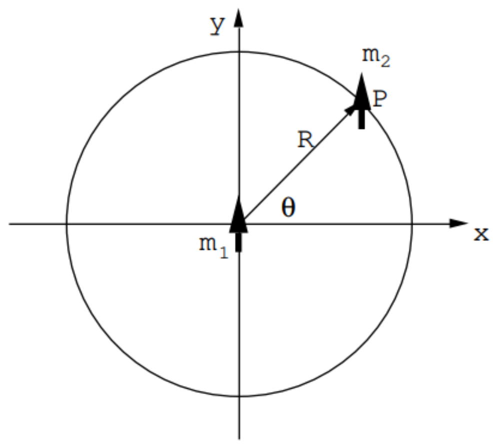
$$\mathbf{B}=\frac{\mu_0}{4\pi}\left(3\frac{m_0 z\mathbf{r}}{r^5}-\frac{m_0\mathbf{u_z}}{r^3}\right) \text{ therefore } \mathrm{B_x}=\frac{\mu_0 \mathrm{m_0}}{4\pi}\frac{3\mathrm{xz}}{\mathrm{r}^5}, \quad \mathrm{B_y}=\frac{\mu_0\mathrm{m_0}}{4\pi}\frac{3\mathrm{yz}}{\mathrm{r}^5}, \quad \mathrm{B}_z=\frac{\mu_0\mathrm{m_0}}{4\pi}\left(\frac{3\mathrm{z}^2}{\mathrm{r}^5}-\frac{1}{\mathrm{r}^3}\right).$$

$\mathrm{B}^2=\left(\frac{\mu_0\mathrm{m_0}}{4\pi\mathrm{r}^3}\quad\left(1+\frac{3\mathrm{z}^2}{\mathrm{r}^2}\right)\right)$ so B^2 is a maximum at x=0, y=0, z=r. B^2 is a minimum at z=0. $B_{\min}=\frac{\mu_0 m_0}{4\pi}\frac{1}{r^3}$. $B_{\max}=2B_{\min}$.

Problem (1.5).

The energy of interaction between two magnetic dipoles is given by - $\mathbf{m_1}$•$\mathbf{B_2}$ or by - $\mathbf{B_1}$•$\mathbf{m_2}$ where $\mathbf{B_1}$ is the field generated at the position of dipole #2 by dipole #1, and $\mathbf{B_2}$ is the field at dipole #1 generated by dipole #2. Let these two magnetic dipoles be separated by a constant distance R= 10^{-6}m (1 µm).

(a) Assume that the two dipoles are forced to remain parallel as shown in the figure. At what angle θ is the interaction energy a minimum? What is this minimum energy?



(Answ: θ= ± π/2, $\mathrm{U_{min}}=-2\frac{\mu_0\mathrm{m_1m_2}}{4\pi\mathrm{R}^3}$.

(b) Assume that θ=0 in the figure, but that the two dipoles are free to rotate in the x-y plane. Let $m_1|_x$ = $m_1\cos\alpha_1$ and $m_1|_y$ = $m_1\sin\alpha_1$. Similarly let $m_2|_x$ = $m_2\cos\alpha_2$ and $m_2|_y$ = $m_2\sin\alpha_2$. What will be the minimum energy configuration, and what will be the minimum energy?

(Answ: α_1 = α_2 = 0 or π. $\mathrm{U_{min}}=-2\frac{\mu_0\mathrm{m_1m_2}}{4\pi\mathrm{R}^3}$.)

Answer

(a) $\mathbf{B_1}=\frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m_1}\bullet\mathbf{r})\mathbf{r}}{\mathrm{r}^5}-\frac{\mathbf{m_1}}{\mathrm{r}^3}\right), \quad \mathrm{x}=\mathrm{R}\cos\theta, \quad \mathrm{y}=\mathrm{R}\sin\theta, \quad \mathrm{z}=0$

$\mathrm{B_{1x}}=\frac{\mu_0}{4\pi}\frac{\mathrm{m_1}}{\mathrm{R}^3}3\sin\theta\cos\theta, \quad \mathrm{B_{1y}}=\frac{\mu_0}{4\pi}\frac{\mathrm{m_1}}{\mathrm{R}^3}\left(3\sin^2\theta-1\right)$,

therefore

$$U=-\mathbf{B_1}\bullet\mathbf{m_2}=-\frac{\mu_0}{4\pi}\frac{m_1m_2}{R^3}\quad\left(3\sin^2\theta-1\right).$$

This expression is clearly a minimum when sinθ= ±1.

(b) When θ=0 and **r**=(R,0,0) one finds

$\mathrm{B_{1x}}=\frac{\mu_0}{4\pi\mathrm{R}^3}2\mathrm{m_1}\cos\alpha_1, \quad \mathrm{B_{1y}}=-\frac{\mu_0}{4\pi\mathrm{R}^3}\mathrm{m_1}\sin\alpha_1$,

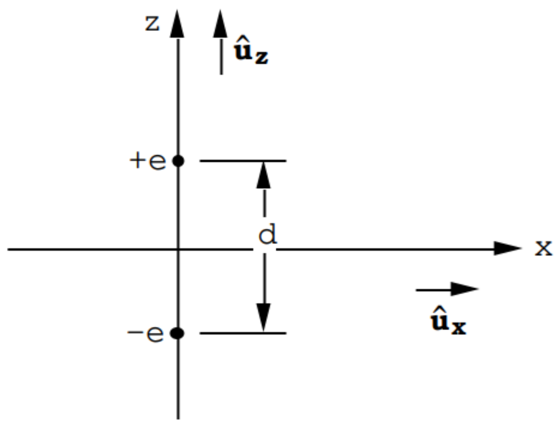
therefore

$\mathrm{U}=-\mathbf{m_2}\bullet\mathbf{B_1}=-\frac{\mu_0\mathrm{m_1m_2}}{4\pi\mathrm{R}^3}\quad\left(2\cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2\right)$.

This expression clearly has a minimum when $\cos\alpha_1$=$\cos\alpha_2$=1 and $\sin\alpha_1$=$\sin\alpha_2$=0, ie. when α_1=α_2= 0 or π.

Problem (1.6)

A proton and an electron are separated by 10^{-12} m = d as shown in the sketch.



(a) Calculate the strength of the electric field 1 micron (= 10^{-6} m) distant from a proton.

(b) Calculate the strength of the electric field a = 1 micron from the above pt dipole at $\mathbf{r}=\mathrm{a}\hat{\mathbf{u}}_z$. What is the direction of this electric field?

(c) Calculate the strength of the electric field a distance a = 1 micron from the dipole at the point $\mathbf{r}=-\mathrm{a}\hat{\mathbf{u}}_z$. What is the direction of this electric field?

(d) Calculate the strength and direction of the electric field at $\mathbf{r}=\mathrm{a}\hat{\mathbf{u}}_x$ where a = 1 micron.

(e) Calculate the strength and direction of the electric field for the above dipole at $\mathbf{r}=\frac{1}{\sqrt{2}}a\left(\hat{\mathbf{u}}_x+\hat{\mathbf{u}}_z\right)$ and a= 1 micron.

N.B. $\hat{\mathbf{u}}_x, \hat{\mathbf{u}}_y, \hat{\mathbf{u}}_z$ are unit vectors along x,y,z.

Answer (1.6)

a) $|\mathbf{E}|=\frac{1}{4\pi\varepsilon_0}\frac{e}{r^2}$

$$\frac{1}{4\pi\varepsilon_0}=9\times10^9 \mathrm{e}=1.6\times10^{-19}\text{Coulombs}$$

$$\mathrm{r}=10^{-6}\mathrm{m}$$

$\therefore|\mathbf{E}|=\frac{(9\times109)\left(1.6\times10^{-19}\right)}{10^{-12}}=\underline{1440}\mathrm{Volts/m}$.

b) For a point dipole $\mathbf{E}_{\mathrm{d}}=\frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{\mathrm{r}^5}-\frac{\mathbf{p}}{\mathrm{r}^3}\right]$

In this case **p** and **r** are both along z and hence parallel

$$\begin{aligned}\therefore\left|\mathbf{E}_{\mathrm{d}}\right| &= \frac{1}{4\pi\varepsilon_0}\quad\left(\frac{2\mathrm{p}}{\mathrm{r}^3}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\quad\left(\frac{\mathrm{e}}{\mathrm{r}^2}\right)\quad\left(\frac{2\mathrm{d}}{\mathrm{r}}\right)\\ &= \frac{(2)\left(10^{-12}\right)}{10^{-6}}=\left(2\times10^{-6}\right)\times\text{part (a)}\end{aligned}$$

So $|\mathbf{E}_{\mathrm{d}}|$ = 2.88 x 10^{-3} Volts/m and directed along z.

c) For this part **p.r** = -e d a

since p = e d $\hat{\mathbf{u}}_z$

and $\mathbf{r}=-\mathrm{a}\hat{\mathbf{u}}_z$

therefore $(\mathbf{p}\cdot\mathbf{r})\mathbf{r}=ea^2d\hat{\mathbf{u}}_z$

So $\frac{(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5}=\frac{ed}{a^3}\hat{\mathbf{u}}_z$

and $|\mathbf{E}\mathrm{d}|=\frac{1}{4\pi\varepsilon_0}\left|\left(\frac{3\mathrm{ed}}{\mathrm{a}^3}\hat{\mathbf{u}}_z-\frac{\mathrm{ed}}{\mathrm{a}^3}\hat{\mathbf{u}}_z\right)\right|=\frac{1}{4\pi\varepsilon_0}\frac{2\mathrm{ed}}{\mathrm{a}^3}$

or exactly the same as part (b). The electric field is also directed along z, just as in part (b).

(d) Here **p.r**=0 because **p** is directed along z whereas **r** is directed along x.

Therefore

$$\begin{aligned}\mathbf{E}_{\mathrm{d}}&=\frac{-1}{4\pi\varepsilon_0}\quad\mathbf{p}/\mathrm{r}^3\\&=-\left(\frac{\mathrm{e}}{4\pi\varepsilon_0}\right),\frac{1}{\mathrm{a}^2}(\mathrm{d}/\mathrm{a})\hat{\mathbf{u}}_{\mathrm{z}}\end{aligned}$$

i.e. directed along –z and half as large as the electric field for a point along the dipole axis and a meters from the dipole.

∴|**E**d| = 1.44 x 10^{-3} Volts/m

e) $\mathbf{p}=\mathrm{ed}\hat{\mathbf{u}}_z\quad\therefore\mathbf{p}\cdot\mathbf{r}=\frac{\mathrm{eda}}{\sqrt{2}}$

$\mathbf{r}=\frac{a}{\sqrt{2}}\quad(\hat{\mathbf{u}}_x+\hat{\mathbf{u}}_z)$

$\therefore\quad\mathbf{E}\mathrm{d}=\frac{1}{4\pi\varepsilon_0}\left[\frac{3\mathrm{ed}}{(\sqrt{2})(\sqrt{2})}\frac{\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_z}{\mathrm{a}^3}-\frac{\mathrm{ed}\hat{\mathbf{u}}_z}{\mathrm{a}^3}\right]$

$\mathbf{E}_{\mathrm{d}}=\frac{e}{4\pi\varepsilon_0}\frac{\mathrm{d}}{\mathrm{a}^3}\left[\frac{3}{2}\quad(\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_z)-\hat{\mathbf{u}}_z\right]$
$=\frac{\mathrm{e}}{4\pi\varepsilon_0}\left(\frac{1}{\mathrm{a}^2}\right)\left(\frac{\mathrm{d}}{2\mathrm{a}}\right)(3\hat{\mathbf{u}}_{\mathrm{x}}+\hat{\mathbf{u}}_z)$

So **E**d is directed 18.4° from the xy plane and has the magnitude |**E**d| = (1.58 x 10^{-6}) (1440) = 2.28 x 10^{-3} Volts/m.

Problem (1.7).

Show that the magnetic field at the center of a uniformly magnetized sphere containing a small hole at the center is zero. Uniform magnetization means **M** is constant. Without loss of generality, one can take the magnetization to be directed along the z-axis, ie **M**= $M_0\mathbf{u_z}$.

(Hint: Add up all the contributions to the field at the center due to volume elements at a distance r from the center. In polar co-ordinates $d\tau = r^2 dr \sin\theta d\theta\, d\phi$, and d**m** = $M_0 d\tau \mathbf{u}_z$.)

Answer (1.7).

If **r**= -x$\mathbf{u_x}$ - y$\mathbf{u_y}$ - z$\mathbf{u_z}$ then **m•r** = - $M_0 d\tau z$ (remember that **r** is the vector drawn from the magnetic moment to the point of observation).

$\mathbf{B}=\frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5}-\frac{\mathbf{m}}{r^3}\right)$, so that

$$B_x=\frac{\mu_0}{4\pi}\frac{M_0 d\tau}{r^5}\quad(3xz),\quad B_y=\frac{\mu_0}{4\pi}\frac{M_0 d\tau}{r^5}\quad(3yz),$$

$$B_z=\frac{\mu_0}{4\pi}\frac{M_0 d\tau}{r^5}\quad\left(2z^2-x^2-y^2\right),$$

Convert to polar co-ordinates and integrate over θ from 0 to π, and over ϕ from 0 to 2π. All field components integrate to zero.

Problem (1.8)

The fields generated at the position **r** from a slowly moving, spinless, point charge are given by $\mathbf{E}=\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}$ and $c\mathbf{B}=\frac{\mathbf{v}}{c}\times\mathbf{E}$. Consider a particle moving in a circular orbit whose position at time t is given by $\mathbf{a}=a\cos\omega t\hat{\mathbf{u}}_{\mathbf{x}}+\sin\omega t\quad\hat{\mathbf{u}}_y$.

(a) Show that the time averaged electric field seen by an observer at $\mathbf{R}=\mathrm{x}\hat{\mathbf{u}}_{\mathbf{x}}+\mathrm{Y}\hat{\mathbf{u}}_{\mathbf{y}}+\mathrm{z}\hat{\mathbf{u}}_{\mathbf{z}}$ is given by $<\mathbf{E}_p>=\frac{q}{4\pi\varepsilon_0}\left(\frac{\mathbf{R}}{R^3}\right)$ to terms of order $(a/R)^2$.

(b) Show that to lowest order in (a/R) the magnetic field observed at **R** is given by

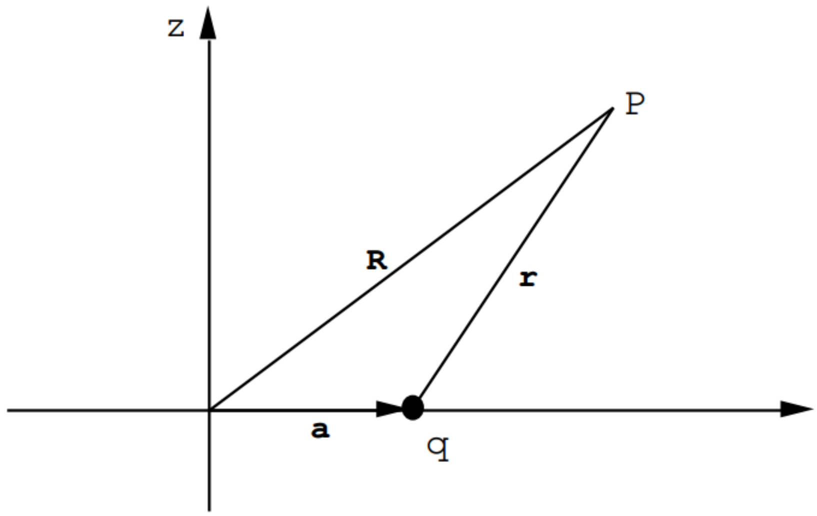
$$< \mathbf{B}_\mathrm{p} >= \frac{1}{\mathrm{c}^2} \frac{1}{4\pi\varepsilon_0}\left[\frac{3z\mathbf{R}}{\mathrm{R}^5} - \frac{\hat{\mathbf{u}}_\mathrm{z}}{\mathrm{R}^3}\right] \quad \left(\frac{\mathrm{qa}^2\omega}{2}\right)$$

or since $\mathbf{m} = \left(\frac{\mathrm{qa}^2\omega}{2}\right) \hat{\mathbf{u}}_\mathrm{z}$ is the magnetic moment (|**m**|= Iπa^2 where I is the current in Amps)

$$\langle \mathbf{B}_\mathrm{p} \rangle = \frac{\mu_0}{4\pi} \quad \left[\quad 3\left(\frac{\mathbf{m}\cdot\mathbf{R}}{\mathrm{R}^5}\right)\mathbf{R} - \frac{\mathbf{m}}{\mathrm{R}^3}\right]$$

and $c^2 = \frac{1}{\varepsilon_0\mu_0}$.

Answer (1.8)



We have **r** + **a** = **R**

∴ **r** = **R** - **a**

where $\mathbf{R} = X\hat{\mathbf{u}}_\mathrm{x} + \mathrm{Y}\hat{\mathbf{u}}_\mathrm{y} + \mathrm{z}\hat{\mathbf{u}}_\mathrm{z}$

and $\mathbf{a} = \mathrm{a}\cos\omega\mathrm{t} \quad \hat{\mathbf{u}}_\mathrm{x} + \mathrm{a}\sin\omega\mathrm{t}\hat{\mathbf{u}}_\mathrm{y}$

$\therefore \quad r^2 = (X - a\cos\omega t)^2 + (Y - a\sin\omega t)^2 + Z^2$

$r^2 = X^2 + Y^2 + Z^2 + a^2 - 2aX\cos\omega t - 2aY\sin\omega t$

or $r^2 = R^2\left[1 + \left(\frac{a}{R}\right)^2 - \frac{2aX}{R^2}\cos\omega t - \frac{2aY}{R^2}\sin\omega t\right]$.

Keep only the lowest terms in $\left(\frac{a}{R}\right)$:

$r \cong R\left[1 - \frac{2aX}{R^2}\cos\omega t - \frac{2aY}{R^2}\sin\omega t\right]^{1/2}$

So $\frac{1}{r^3} \cong \frac{1}{R^3}\left[1 - \frac{2aX}{R^2}\cos\omega t - \frac{2aY}{R^2}\sin\omega t\right]^{-3/2}$

$\frac{1}{r^3} \cong \frac{1}{R^3}\left[1 + \frac{3aX}{R^2}\cos\omega t + \frac{3aY}{R^2}\sin\omega t\right]$

$\mathbf{E}_\mathrm{p} = \frac{q}{4\pi\varepsilon_0}\left(\frac{\mathbf{r}}{r^3}\right) = \frac{q}{4\pi\varepsilon_0}\frac{(\mathbf{R}-\mathbf{a})}{\mathrm{R}^3}\left[1 + \frac{3aX}{R^2}\cos\omega t + \frac{3aY}{R^2}\sin\omega t\right]$

Now multiply out and take time averages.

$< \cos\omega t >=< \sin\omega t >=< \sin\omega t \quad \cos\omega t >= 0$

$< \cos^2\omega t >=< \sin^2\omega t >= 1/2$

Notice that all terms proportional to a average to zero.

(a) $\therefore \quad \langle\mathbf{E}_{\mathrm{p}}\rangle \cong \frac{q}{4\pi\varepsilon_0} \quad \left(\frac{\mathbf{R}}{R^3}\right)$. The correction terms are of order $\left(\frac{a}{R}\right)^2$.

(b) $\mathbf{B}_{\mathrm{p}} = \frac{1}{\mathrm{c}^2} \quad (\mathbf{V} \quad \times\mathbf{E})$

$\mathbf{v} = \frac{\mathbf{da}}{dt} = -a\omega\sin\omega t \quad \hat{\mathbf{u}}_x + a\omega\cos\omega t \quad \hat{\mathbf{u}}_y$

$\therefore \quad \mathbf{B}_p \cong \frac{1}{c^2}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{[(\mathbf{v}\times\mathbf{R})-(\mathbf{v}\times\mathbf{a})]}{\mathbb{R}^3}\left\{1+\frac{3aX}{R^2}\cos\omega t+\frac{3aY}{R^2}\sin\omega t\right\}$

$\mathbf{v}\times\mathbf{R} = (\,a\omega Z \quad \cos\omega t\,) \quad \hat{\mathbf{u}}_x + (a\omega z\sin\omega t)\hat{\mathbf{u}}_y - [\mathrm{a}\omega\mathrm{Y}\sin\omega\mathrm{t} + \mathrm{a}\omega\mathrm{X}\cos\omega\mathrm{t}]\hat{\mathbf{u}}_{\mathrm{z}}$

$\mathbf{v}\times\mathbf{a} = -\mathrm{a}^2\omega\hat{\mathbf{u}}_z$

Multiply out the terms in $\mathbf{B}_{\mathrm{p}}$ and take time averages. The result is

$\langle\mathbf{B}_{\mathrm{p}}\rangle = \frac{1}{\mathrm{c}^2}\left(\frac{\mathrm{q}}{4\pi\varepsilon_{\mathrm{O}}}\right)\frac{1}{\mathrm{R}^3}\left\{\left(\frac{3\mathrm{a}^2\omega\mathrm{XZ}}{2\mathrm{R}^2}\right)\hat{\mathbf{u}}_{\mathrm{x}} + \left(\frac{3\mathrm{a}^2\omega\mathrm{YZ}}{2\mathrm{R}^2}\right)\hat{\mathbf{u}}_{\mathrm{y}} - -\left(\frac{3a^2\omega}{2R^2}\right) \quad (X^2+Y^2) \quad \hat{\mathbf{u}}_z + a^2\omega\hat{\mathbf{u}}_z\right\}$

add and subtract $\frac{3a^2\omega}{2R^2}z^2\hat{\mathbf{u}}_z$ to obtain

$<\mathbf{B}_p> \frac{1}{c^2}\left(\frac{q}{4\pi\varepsilon_0}\right)\frac{1}{R^3}\left\{\left(\frac{3a^2\omega}{2R^2}\right) \quad Z\mathbf{R} - \left(\frac{a^2\omega}{2}\right)\hat{\mathbf{u}}_z\right\}$.

Now $\frac{1}{c^2} = \varepsilon_0\mu_0$ and $\mathbf{m} = \frac{\mathrm{qa}^2\omega}{2}\hat{\mathbf{u}}_{\mathrm{z}}$

Therefore $\langle\mathbf{B}_p\rangle = \left(\frac{\mu_0}{4\pi}\right)\left[\frac{3(\mathbf{m}\cdot\mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}}{R^3}\right]$.

Problem (1.9)

Given the following scalar functions, V, expressed in cylindrical polar co-ordinates. For each function calculate

(1) the components of grad V

(2) ∇^2 V

(a) V = r Cosθ

(b) V = ln r

(c) $V = \frac{\operatorname{Cos}\theta}{r}$

(d) $V = \frac{\cos n\theta}{r^n}$, where n is an integer either positive or negative.

Answer (1.9)

$\operatorname{grad} V = \left(\frac{\partial V}{\partial r}\right)\hat{\mathbf{u}}_r + \frac{1}{r}\left(\frac{2V}{\partial\theta}\right)\hat{\mathbf{u}}_\theta + \left(\frac{\partial V}{\partial z}\right)\hat{\mathbf{u}}_z$

$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2}$

(a) V= rCosθ.

$\operatorname{grad} V|_r = \cos\theta$

$\operatorname{grad} V|_\theta = -\sin\theta$

These correspond to a unit vector along x!

∇^2V = 0

(b) V= lnr.

$\operatorname{grad} V|_r = \frac{1}{r}$

$\operatorname{grad} V|_\theta = 0$

∇^2V = 0

(c) $V = \frac{\cos\theta}{r}$

$\operatorname{grad} V|_r = -\frac{\cos\theta}{r^2}$

$\mathrm{grad}\, V|_{\theta} = -\frac{\sin\theta}{r^2}$

∇^2V = 0

(d) $\mathrm{V} = \frac{\cos \mathrm{n}\theta}{\mathrm{r}^{\mathrm{n}}}$.

$\mathrm{gradv}|_{r} = -\frac{n\cos n\theta}{r^{n+1}}$

$\mathrm{grad}\, V|_{\theta} = -\frac{\mathrm{nsin}(n\theta)}{r^{n+1}}$

∇^2V = 0 for any n.

Problem (1.10)

Given the following scalar functions V expressed in spherical polar co-ordinates. For each function calculate

(1) the components of grad V

(2) ∇^2V

(a) V = r Cosθ

(b) $V = \frac{\mathrm{Cos}\,\theta}{r^2}$

(c) V = $r^2(3\mathrm{Cos}^2\theta - 1)$

(d) $V = \frac{(3\cos^2\theta - 1)}{r^3}$

(e) $V = \frac{\cos n\theta}{r^n}$ where n is a positive integer.

Answer (1.10)

$$\nabla \mathrm{V} \quad = \left(\frac{\partial \mathrm{V}}{\partial \mathrm{r}}\right)\hat{\mathbf{u}}_{\mathrm{r}} + \frac{1}{\mathrm{r}}\left(\frac{\partial \mathrm{V}}{\partial \theta}\right)\hat{\mathbf{u}}_{\theta} + \frac{1}{\mathrm{r}\sin\theta}\left(\frac{\partial \mathrm{V}}{\partial \phi}\right)\hat{\mathbf{u}}_{\phi}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

(a) V = rCosθ

$\mathrm{grad}\, V|_{r} = \cos\theta$

$\mathrm{gradv}|_{\theta} = -\sin\theta$

$\mathrm{grad}\, V|_{\phi} = 0$

These correspond to a constant field $\hat{\mathbf{u}}_z$.

∇^2V = 0.

(b) $V = \frac{\mathrm{Cos}\,\theta}{r^2}$

$\mathrm{grad}\, V|_{r} = -\frac{2\cos\theta}{r^3}$

$\mathrm{grad}\, V|_{\theta} = -\frac{\sin\theta}{r^3}$

$\mathrm{grad}\, V|_{\phi} = 0$

Corresponds to a dipole field.

∇^2V = 0.

(c) $\mathrm{V} = \mathrm{r}^2 \quad (3\cos^2\theta - 1) \quad \mathrm{grad}\,\mathrm{V}|_{\mathrm{r}} = 2\mathrm{r}\,(3\cos^2\theta - 1)$

$\mathrm{gradv}|_{\theta} = -6r\sin\theta\cos\theta$

$\mathrm{grad}\, V|_{\phi} = 0$

∇^2V = 0

(d) $V = \frac{3\cos^2\theta - 1}{r^3}$

$\mathrm{gradv}|_{r} = -\frac{3}{r^4}(3\cos^2\theta - 1)$

$\operatorname{grad} V|_{\theta} = -\frac{6}{r^4}\sin\theta\cos\theta$

$\operatorname{grad} V|_{\phi} = 0$

$\nabla^2 V = 0$

(e) $V = \frac{\cos n\theta}{r^n}$

$\operatorname{grad} V|_{r} = -\frac{n\cos(n\theta)}{r^{n+1}}$

$\operatorname{grad} V|_{\theta} = -\frac{n\sin(n\theta)}{r^{n+1}}$

$\operatorname{grad} V|_{\phi} = 0$

$\nabla^2 V = -\frac{n}{\sin\theta r^{n+2}}(\sin\theta\cos(n\theta) + \cos\theta\sin(n\theta))$.

Problem (1.11)

Calculate the vector field **B** = curl**A** for the following fields, **A**.

(a) In cylindrical polar co-ordinates

$$\begin{aligned} A_T &= 0 \\ A\theta &= 0 \\ A_z &= -\frac{\mu_0 I}{2\pi}\ln r \end{aligned}$$

(b) In cylindrical polar co-ordinates

$$\begin{aligned} A_r &= 0 \\ A\theta &= \frac{B_O r}{2} \\ A_Z &= 0 \end{aligned}$$

(c) $\mathbf{A} = \frac{\mu_0}{4\pi}\left(\frac{\mathbf{m}\times\mathbf{r}}{r^3}\right)$, where $\mathbf{m} = m_0\hat{\mathbf{u}}_z$.

Show that in spherical polar co-ordinates if $\mathbf{m} = m_0\hat{\mathbf{u}}_z$ then $A_r = A_\theta = 0$ and $A_\phi = \frac{\mu_0}{4\pi}\frac{m_0\sin\theta}{r^2}$. This can be used to calculate curl **A**.

Answer (1.11)

$$\text{(a) } \mathbf{B} = \operatorname{curl}\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = \begin{vmatrix} \frac{1}{r}\frac{\partial A_z}{\partial\theta} \\ -\frac{\partial A_z}{\partial r} \\ 0 \end{vmatrix}$$

But $\frac{A_Z}{\partial\theta} = 0 \;\therefore\quad B_r = 0$

$B_\theta = \frac{\mu_0 I}{2\pi r}$

$B_z = 0$

The field due to a current I Amps flowing along a long wire oriented along z.

$$\text{(b) } \mathbf{B} = \operatorname{curl}\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & rA_\theta & 0 \end{vmatrix} = \begin{vmatrix} -\frac{\partial A_\theta}{\partial z} \\ \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} \end{vmatrix}$$

But $\frac{\partial A_\theta}{\partial z} = 0$ and $A_\theta = \frac{B_0 r}{2}$

$$\begin{aligned} \therefore\quad B_r &= 0 \\ B_\theta &= 0 \\ B_z &= B_0 \end{aligned}$$

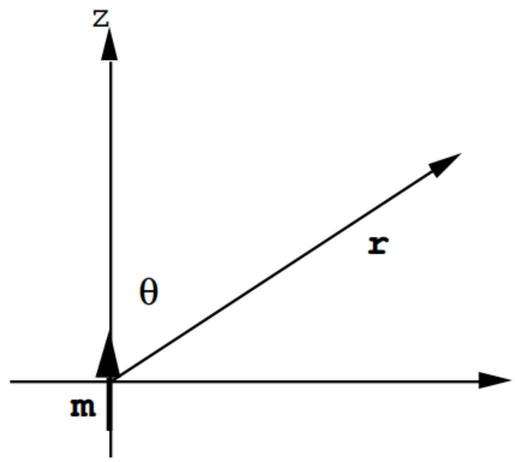
This is the field inside an infinitely long solenoid.

(c) $\mathbf{A}=\frac{\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{r})}{r^3}$

If $\mathbf{m}=\mathrm{m_O}\hat{\mathbf{u}}_\mathrm{z}$ this generates the field due to a magnetic dipole.

For $\mathbf{m}=m_0\hat{\mathbf{u}}_z$ $(\mathbf{m}\times\mathbf{r})$ is a vector in the ϕ direction having the magnitude $m_o r\mathrm{Sin}\theta$.

$$\mathrm{A_r}=0$$
$$\mathrm{A}_\theta=0$$
$$\mathrm{A}_\phi=\frac{\mu_0}{4\pi}\frac{\mathrm{m}_0}{\mathrm{r}^2}\sin\theta,\qquad \text{(see the figure)}$$

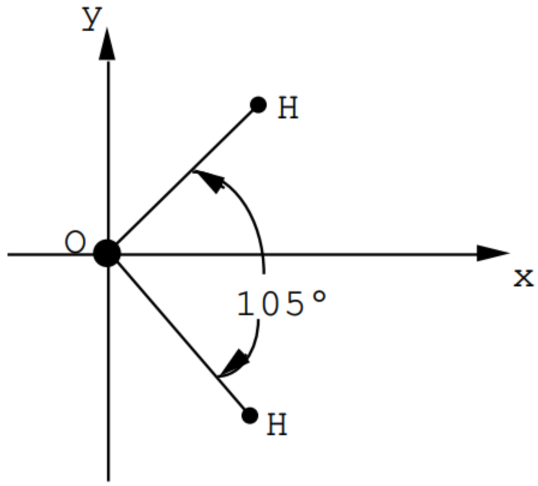


$$\mathbf{B}=\operatorname{curl}\mathbf{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & r\sin\theta\hat{\mathbf{u}}\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ 0 & 0 & r\sin\theta A_\phi\end{vmatrix}=\begin{vmatrix}\frac{1}{r^2\sin\theta} & \frac{\partial(r\sin\theta A_\phi)}{\partial\theta}\\ -\frac{1}{\mathrm{r}\sin\theta} & \partial(\sin\theta A_\phi)\\ & 0\end{vmatrix}$$

$$\therefore \mathrm{B_r}=\frac{\mu_0}{4\pi}\frac{2\mathrm{m}_0\cos\theta}{\mathrm{r}^3},\quad \mathrm{B}_\theta=\frac{\mu_0}{4\pi}\frac{\mathrm{m}_0\sin\theta}{\mathrm{r}^3},\quad \mathrm{B}_\phi=0\ .$$

Problem (1.12)

A water molecule is planar but the angle between the two oxygen-hydrogen bonds is 105˚ as shown in the sketch.



(a) If the charge on the oxygen is twice the electronic charge i.e. -2|e| and the charge on each hydrogen is q_H = +|e|, calculate the dipole moment of the molecule assuming an O-H bond length of 5 x 10^{-10} m. [The measured dipole moment is p = 6.17 x 10^{-30} Coulomb-m].

(b) If all of the dipoles in a cubic meter of water were aligned what would be the resulting density of electric dipoles |P|?

Use p = 6.17 x 10^{-30} cm.

Answer (1.12)

(a) The dipole moment is p = qd. In the H_2O molecule q = 2|e| = (2)(1.60 x 10^{-19}) Coulombs or q = 3.2 x 10^{-19} C

The distance $d = b\cos\left(\frac{105}{2}\right)$ where b = 5 x 10^{-10} m is the bond length; d = 3.04 x 10^{-10} m

∴ p = (3.2)(3.04) x 10^{-29} Coulomb m = 9.74 x 10^{-29} Cm

Compared with experiment this is too large by ~ 15.7 times.

(b) The molar volume of H_2O is 18 c.c.

∴ No. of moles in 1 m^3 = 10^6/18 = 5.56 x 10^4 moles.

∴ No. of molecules in 1 m^3 = (6.02 x 10^{23})(5.56 x 10^4) = 3.34 x 10^{28} molecules.

∴ |**P**| = (3.34 x 10^{28})(6.17 x 10^{-30}) = 0.21 Coulombs/m^2.

(This is very large--in fact H_2O has no permanent dipole moment because the molecules are oriented at random).

Problem (1.13)

An iron atom in metallic iron carries a magnetic moment of 2.2 Bohr magnetons. (1 Bohr magneton, μ_B, is μ_B = 9.27 x 10^{-24} Amp m^2 (= Joules/Tesla)). The density of iron is 7.87 gms/cc and its molecular weight is 55.85 gms. If all of the atomic moments were aligned parallel what would be the magnetization per unit volume of iron? Compare this value with the observed internal magnetic field of saturated iron at room temperature |**B**| = μ_o|**M**| = 2.15 Teslas = 2.15 Webers/m^2.

Answer (1.13)

The molar volume of iron is $\frac{55.85}{7.87} = 7.10\mathrm{cc}$.

The number of atoms in /m^3 is

$\mathrm{N} = \left(6.02 \times 10^{23}\right)\left(\frac{10^6}{7.10}\right) = 0.848 \times 10^{29} \mathrm{atoms/m^3}$.

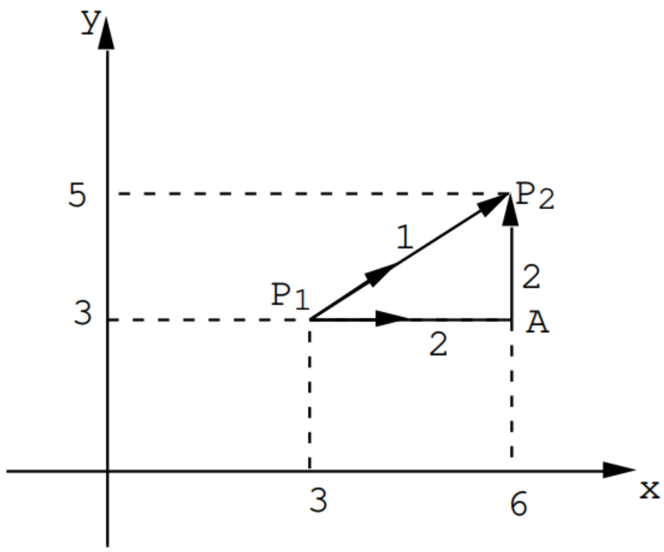
The magnetization/m^3 |**M**| = (N)(2.2) μ_B

|**M**| = 0.173 x 10^7 Amps/m.

This would give an internal field |**B**| = μ_o |**M**| of |**B**| = (4π x 10^{-7})(0.173 x 10^7) = 2.17 Teslas.

This means that at room temperature the fraction of aligned spins in iron is $\frac{2\cdot 15}{2\cdot 17} = 0.989$ i.e. Very nearly completely aligned!

Problem (1.14)



Given a vector function $\mathbf{F} = \mathrm{xy}\hat{\mathbf{u}}_\mathrm{x} + \left(3\mathrm{x} - \mathrm{y}^2\right)\hat{\mathbf{u}}_\mathrm{y}$ evaluate the line integral from P_1 to P_2 along

a) the direct path (1).

b) the indirect path $P_1 \rightarrow A \rightarrow P_2$ (path (2)).

Answer (1.14)

The line P_1P_2 can be written $\mathbf{s} = (3\hat{\mathbf{u}}_x + 3\hat{\mathbf{u}}_y) + (3\hat{\mathbf{u}}_x + 2\hat{\mathbf{u}}_y)\,\mathrm{L}$ where L varies from L = 0 to L = 1. L= 0 corresponds to $P_1\ (3\hat{\mathbf{u}}_x + 3\hat{\mathbf{u}}_y)$ whereas L=1 corresponds to $P_2\quad (6\hat{\mathbf{u}}_x + 5\hat{\mathbf{u}}_y)$.

So $\mathbf{ds} = (3\hat{\mathbf{u}}_x + 2\hat{\mathbf{u}}_y)\,\mathrm{dL}$ or dx= 3dL and dy= 2dL.

(a) Now $\mathbf{F} \cdot \mathbf{ds}$ = 3xy dL + 2(3x - y^2) dL

$$\therefore \quad \int_{P_2}^{R} \mathbf{F}\cdot\mathbf{ds} = \int_{P_2(L=0)}^{P_1(L=1)} \left[3xy + 6x - 2y^2\right] \mathrm{dL}$$

But x = (3 + 3L) y = 3 + 2L along the line (components of **S**)

$$\therefore \int_{P_2}^{P_1} \mathbf{F}\cdot\mathbf{ds} = \int_0^1 \mathrm{dL}\{3(3+3L)(3+2L) + 6(3+3L)\,2(3+2L)^2\}$$

$$\therefore \quad \int_{P_2}^{P} \mathbf{F}\cdot\mathbf{ds} = \int_0^1 \mathrm{dL}\left[27 + 39L + 10L^2\right] \\ = 27 + (137/6) = \frac{299}{6}$$

(b) Along path (2)

$$\begin{aligned}\int_2 \mathbf{F}\cdot\mathbf{ds} &= \int_3^6 F_{\mathbf{X}}(y=3)\mathrm{dx} + \int_3^5 \mathrm{F_Y}(x=6)\mathrm{dy} \\ &= 3\int_3^6 \mathrm{xdx} + \int_3^5 \left(18 - y^2\right)\mathrm{dy} \\ &= \left(\frac{3}{2}\right)(27) + 36 - \frac{98}{3} = 36 + \left(\frac{47}{6}\right) = \frac{263}{6}\end{aligned}$$

The line integral is different for the two paths.

Therefore **F** is not a conservative field. Indeed, curl $\mathbf{F} = \begin{vmatrix} 0 \\ 0 \\ 3x \end{vmatrix}$ and therefore curl **F** does not vanish everywhere.

Problem (1.15)

Given the vector function $\mathbf{E} = y\hat{\mathbf{u}}_x + x\hat{\mathbf{u}}_y$. Evaluate the line integral $\int_1^2 \mathbf{E}\cdot\mathbf{dL}$ from P_1 (2,1,-1) to P_2 (8,2,-1)

a) along the parabola x = $2y^2$,

b) along the straight line joining the two points.

c) Is **E** a conservative vector field?

Answer (1.15)

$$\operatorname{curl}\mathbf{E} = \begin{vmatrix} \hat{\mathbf{u}}_{\mathbf{x}} & \hat{\mathbf{u}}_y & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ (1-1)=0 \end{vmatrix} \equiv 0\ .$$

Therefore **E** is a conservative vector field.

$$\text{(a)}\quad \begin{aligned}\int_1^2 \mathbf{E}\cdot\mathbf{dL} &= \int_1^2 E_{\mathbf{X}}\mathrm{dx} + \int_1^2 E_Y \mathrm{dy} \\ &= \int_2^8 \mathrm{ydx} + \int_1^2 \mathrm{xdy}\end{aligned}$$

But $y = \sqrt{x/2} \quad x = 2y^2$ along the parabola

$$\therefore \int_1^2 \mathbf{E}\cdot\mathbf{dL} = \int_2^8 \frac{x^{1/2}dx}{\sqrt{2}} + 2\left[\int_1^2 y\hat{}\{2\}dy = \frac{\sqrt{2x^3}}{3}\right|_2^8 + \left.\frac{2y^3}{3}\right|_1^2$$

$$= \frac{1}{3}\left[2^5 - 2^2 + 14\right] = 42/3 = 14$$

(b) Since curl **E** ≡ 0 the line integral along the second path must also be equal to 14.

<u>Check</u>

Let $\mathbf{r}_1 = 2\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_y - \hat{\mathbf{u}}_z$ (the vector to P_1)

Let $\mathbf{r}_2 = 8\hat{\mathbf{u}}_x + 2\hat{\mathbf{u}}_y - \hat{\mathbf{u}}_z$ (the vector to P2).

Then any point on the straight line from P_1 to P_2 can be specified by $\mathbf{L} = \mathbf{r}_1 + L(\mathbf{r}_2 - \mathbf{r}_1)$ where L runs from L = 0 (P_1) to L = 1 (P_2)

$\therefore \mathbf{L} = (2\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_y - \hat{\mathbf{u}}_z) + (6\hat{\mathbf{u}}_x + \hat{\mathbf{u}}_y)$

$\mathbf{dL} = (6\hat{\mathbf{u}}_\mathbf{x} + \hat{\mathbf{u}}_\mathbf{y})\,dL$

$\therefore \quad \mathbf{E}\cdot\mathbf{dL} = 6ydL + xdL$

However, along the st. line **L** x = 2 + 6L y = 1 + L

$\therefore \quad \mathbf{E}\cdot\mathbf{dL} = 6(1+L) \quad dL + (2+6L) \quad dL = (8+12L) \quad dL$

$\therefore \quad \int_{P_1}^{P_2} \mathbf{E}\cdot\mathbf{dL} = \int_0^1 dL \quad (8+12L) = 8+6 = 14$ Q.E.D.