3.3: Electrostatic Field Energy

It will be shown in Chapter(8) that it costs energy to set up an electric field. As the electric field increases from zero the energy density stored in the electrostatic field, $\mathrm{W_E}$, increases according to

$$\frac{\partial \mathrm{W_E}}{\partial \mathrm{t}} = \vec{\mathrm{E}} \cdot \frac{\partial \vec{\mathrm{D}}}{\partial \mathrm{t}}.$$

For the particular case in which the electric field is set up in a dielectric medium that can be described by a dielectric constant so that $\vec{\mathrm{D}} = \epsilon \vec{\mathrm{E}}$, this expression can be written

$$\frac{\partial \mathrm{W_E}}{\partial \mathrm{t}} = \epsilon \vec{\mathrm{E}} \cdot \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}} = \frac{\epsilon}{2} \frac{\partial \mathrm{E}^2}{\partial \mathrm{t}}. \tag{3.3.1}$$

Eqn.(3.3.1) can be integrated immediately to obtain

$$\mathrm{W_E} = \frac{\epsilon \mathrm{E}^2}{2} = \frac{1}{2} \vec{\mathrm{E}} \cdot \vec{\mathrm{D}} \quad \text{Joules } / m^3. \tag{3.3.2}$$

In the above expressions the zero of energy has been chosen to be zero when the electrostatic field is everywhere zero. The total energy stored in the electrostatic field is obtained as an integral of $\mathrm{W_E}$ over all space. This total energy, $\mathrm{U_E}$, can be expressed in terms of the potentials and charges on the electrodes that created the electric field. This can be shown by starting from the vector identity

$$\operatorname{div}(\mathrm{V}\vec{\mathrm{D}}) = \mathrm{V}\operatorname{div}(\vec{\mathrm{D}}) + \vec{\mathrm{D}} \cdot \operatorname{grad}(\mathrm{V}), \tag{3.3.3}$$

where $\vec{D}$ is any vector field and V is a scalar function. This identity can be proved by writing out the divergence in cartesian co-ordinates and by carrying out the differentiations. But from Maxwell's equations $\operatorname{div}(\vec{\mathrm{D}}) = \rho_f$, and by definition $\vec{\mathrm{E}} = -\operatorname{grad}(\mathrm{V})$, so that

$$\int\int\int_{Volume} \operatorname{div}(\mathrm{VD}) d(\mathrm{Vol}) = \int\int\int_{Volume} \left(\rho_f \mathrm{V} - \vec{\mathrm{E}} \cdot \vec{\mathrm{D}}\right) d(\mathrm{Vol}). \tag{3.3.4}$$

The volume integral on the left can be replaced by a surface integral by using Gauss' theorem:

$$\int\int\int_{Volume} \operatorname{div}(\mathrm{V}\vec{\mathrm{D}}) d(\mathrm{Vol}) = \int\int_{Surface} \mathrm{VD} \cdot \overrightarrow{dS}.$$

As the volume becomes very large and the surface S recedes to infinity, the surface integral becomes very small. Very far from all charges the potential V must decrease at least as fast as 1/R (the potential due to a point charge) and $|\vec{D}|$ must decrease at least as fast as $1/\mathrm{R}^2$ (again a point charge) whereas the surface area increases like R^2 . It follows that the surface integral must decrease at least as fast as 1/R in the limit as the dimensions of the surface become infinitely large. It follows from Equation (3.3.4) that

$$\int\int\int_{Volume} \rho_f \mathrm{V} d(\mathrm{Vol}\,) = \int\int\int_{Volume} (\vec{\mathrm{E}} \cdot \vec{\mathrm{D}}) d(\mathrm{Vol}\,),$$

and therefore

$$\mathrm{U_E} = \int\int\int_{Space} \mathrm{W_E} d(\mathrm{Vol}) = \frac{1}{2} \int\int\int_{Space} \rho_f \mathrm{V} d(\mathrm{Vol}). \tag{3.3.5}$$

For a collection of conductors embedded in a non-conducting dielectric medium all of the charges are on the conductor surfaces and the charges on a given conductor are all at the same potential. In that case the integrals in Equation (3.3.5) simply give the product of electrode potential and the total charge on the electrode:

$$\mathrm{U_E} = \frac{1}{2} \sum_{\mathrm{n}} \mathrm{Q_n} \mathrm{V_n}. \tag{3.3.6}$$

3.3.1 Generalized Capacitance Coefficients

Maxwell's equations are linear, therefore the potentials associated with electrodes embedded in a material that obeys linear response must obey the principle of superposition. The potential distribution that is generated by a particular charge is proportional to the quantity of that charge. It follows from superposition that for any collection of charges the potential at any point must be a linear function of the charge strengths. The converse must also be true. Given a collection of conducting electrodes embedded in a linear dielectric medium the charge on each of the electrodes must be a linear function of the electrode potentials: if the potentials are doubled then so must the charge on each electrode be doubled and vice versa.

This linear dependence of the charge on potentials can be expressed as follows (see Figure (3.3.13):

$$\begin{aligned} Q_1 &= C_{11}V_1 + C_{12}V_2 + \ldots + C_{1n}V_n \\ Q_2 &= C_{21}V_1 + C_{22}V_2 + \ldots + C_{2n}V_n \\ &\vdots \quad \vdots \\ Q_n &= C_{n1}V_1 + C_{n2}V_2 + \ldots + C_{nn}V_n \end{aligned} \tag{3.3.7}$$

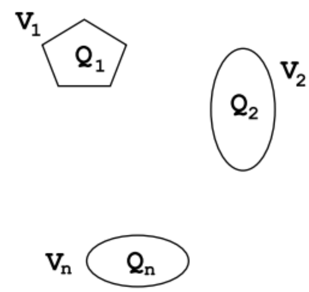

Figure 3.3.13: Charged conductors embedded in a linear dielectric medium. The charges are a linear function of the potentials, see Equation (3.3.8) in the text.

The factors of proportionality, C_{mn}, are called **capacitance coefficients**; they have the units of Farads. These equations express the observation that a change in the potential of one electrode causes a change in the amount of charge stored on every electrode, not just on the electrode whose potential was altered. The energy stored in the electric field, which can be calculated from Equation (3.3.6), must be independent of how the charging process was carried out. It must not matter, for example, whether electrode (1) is first charged, then electrode (2), then electrode (3), and so on, or whether (3) is charged first, then (2), then (1), then (4), and so on. The energy contained in the final state of the system must be independent of the way in which that final state was reached: in order that this be so, it can be shown that

$$C_{mn} = C_{nm}.$$

Instead of N^2 independent capacitance coefficients there are only N(N+1)/2 of them. Notice that these capacitance coefficients are geometry dependent. Any change in the shape of any electrode, or a change in the position of any electrode, will result in a change in all of the capacitance coefficients. It follows also that the energy stored in the electric field must change. This change in field energy can, in principle, be used to calculate the electrostatic forces on the conductors or on the dielectric medium.

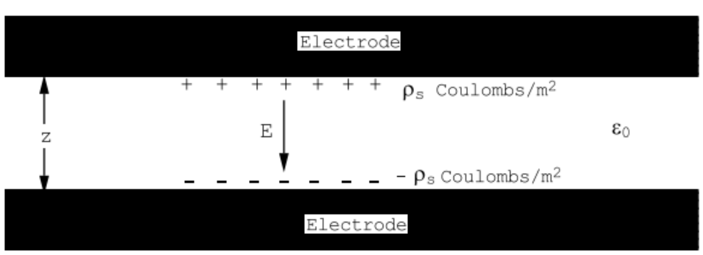

Figure 3.3.14: A parallel plate capacitor. The two plates have an area A and are separated by a distance z. The charge density on each plate is ρ_s Coulombs/m^2 .

3.3.2 Electrostatic Forces.

Case(1) The Charges are Fixed.

The charges on each conductor are held fixed, and one of the conductors is allowed to undergo a slight displacement $\overrightarrow{\delta \mathrm{r}}$. During this displacement the electric forces will do an amount of work

$$\delta \mathrm{w}=\vec{\mathrm{F}} \cdot \overrightarrow{\delta \mathrm{r}}.$$

This work can only be done at the expense of the energy stored in the electric field since there are no other energy sources. Consequently

$$\overrightarrow{\mathrm{F}_{\mathrm{E}}} \cdot \overrightarrow{\delta \mathrm{r}}=-\delta \mathrm{U}_{\mathrm{E}}.$$

The energy stored in the electric field acts like a potential function for the electrical forces. As an example, consider the parallel plate capacitor of Figure (3.3.14). It is convenient in this case to work with a unit area of electrode surface, and to take metal plates that are so large that edge effects can be neglected. For a fixed surface charge density on each electrode the electric field strength between the plates is independent of the electrode spacing, z. The energy stored in the electric field per unit area of electrode can be calculated from the energy density Equation (3.3.2); the result of the calculation is

$$\mathrm{U}_{\mathrm{E}}=\left(\frac{\rho_{s}^{2}}{2 \epsilon_{0}}\right) \mathrm{z}$$

since the electric field strength is given by $\mathrm{E}=\rho_{s} / \epsilon_{0}$. Let the plates be moved apart by a small increment dz. The work done on the displaced plate by the electrical force per unit area is given by Fdz. This work must be done at the cost of the stored electrical energy, therefore

$$\mathrm{F} \mathrm{dz}=-\left(\frac{\rho_{s}^{2}}{2 \epsilon_{0}}\right) \mathrm{dz},$$

or

$$\mathrm{F}=-\left(\frac{\rho_{s}^{2}}{2 \epsilon_{0}}\right)=-\frac{1}{2} \rho_{s} \mathrm{E} \quad \text{newtons} / \mathrm{m}^{2}. \tag{3.3.8}$$

The electric forces act in such a way as to pull the electrodes together. This is the expected result because one plate carries a positive charge and the other plate carries a negative charge. As a guess, one might have thought that the force per unit area on a given electrode would just be given by the charge density multiplied by the electric field at the surface of the electrode, i.e. ρ_sE. The result Equation (3.3.8) shows that the average field acting on the charges must be used to calculate the force (remember that E=0 inside the conductor).

Although the above result for the force on a conductor has been derived for a plane parallel plate, it turns out to be valid for the electric force per unit area acting on the surface of any conductor facing vacuum. There is a negative pressure acting on the conductor surface that depends only upon the local values of the field strength and the surface charge density. This negative pressure, or tension t_{E}, is given by

$$\mathrm{t}_{\mathrm{E}}=\frac{\rho_{s} \mathrm{E}}{2}=\frac{\epsilon_{0}}{2} \mathrm{E}^{2} \quad \text{Newtons} / \mathrm{m}^{2}.$$

For a conducting surface immersed in a fluid characterized by a dielectric constant ϵ it is easy to show that this tension becomes

$$\mathrm{t}_{\mathrm{E}}=\frac{1}{2} \vec{\mathrm{E}} \cdot \vec{\mathrm{D}}=\frac{\epsilon}{2} \mathrm{E}^{2} \quad \text{Newtons} / \mathrm{m}^{2}. \tag{3.3.9}$$

Case(2) The Potentials are Fixed.

In many instances it is convenient to investigate the electrical force distribution under circumstances in which the electrode potentials are held fixed. Any change in the electrode configuration at fixed potentials that results in a change in the capacitance coefficients will also lead to a change in the amount of charge carried by each conductor. If the change in the charge carried by a particular electrode is $\delta \mathrm{Q}_{\mathrm{M}}$, the work required to add this charge to the conductor is $\delta \mathrm{W}_{\mathrm{B}}=\mathrm{V}_{\mathrm{M}} \delta \mathrm{Q}_{\mathrm{M}}$ and this energy is provided by the source of emf that is attached to conductor M, i.e. by the battery which is used to maintain the constant potential. The change in energy stored in the electric field can be calculated from Equation (3.3.6); the result for constant potentials is

$$\delta \mathrm{U_E} = \frac{1}{2} \sum_N \mathrm{V_N} \delta \mathrm{Q_N}. \tag{3.3.10}$$

The energy provided by the batteries that hold the potentials $\mathrm{V_N}$ constant is given by

$$\delta \mathrm{W_B} = \sum_N \mathrm{V_N} \delta \mathrm{Q_N}. \tag{3.3.11}$$

The energy supplied from the batteries is exactly twice the increase in the energy stored in the electric field. The work done by the electrical forces in moving an electrode is $\overrightarrow{\mathrm{F}}_\mathrm{E} \cdot \overrightarrow{\mathrm{dr}}$. Conservation of energy now gives

$$\overrightarrow{\mathrm{F}}_\mathrm{E} \cdot \overrightarrow{\mathrm{dr}} + \delta \mathrm{U_E} = \delta \mathrm{W_B},$$

or

$$\overrightarrow{\mathrm{F}}_\mathrm{E} \cdot \overrightarrow{\mathrm{dr}} = \delta \mathrm{U_E} = \frac{1}{2} \sum_N \mathrm{V_N} \delta \mathrm{Q_N}, \tag{3.3.12}$$

since $\delta \mathrm{W_B} = 2 \delta \mathrm{U_E}$. For this case the increase in electrical energy stored in the field is exactly equal to the external work done by the electrical forces in changing the electrode geometry.

As an example, consider the configuration shown in Figure (3.3.15). A slab of dielectric material characterized by a dielectric constant ϵ, lies with one end near the center of a plane parallel capacitor and the other end lies well outside the capacitor. The slab has a thickness d meters and a width w meters. The specimen is so long that the electric field at the end that lies

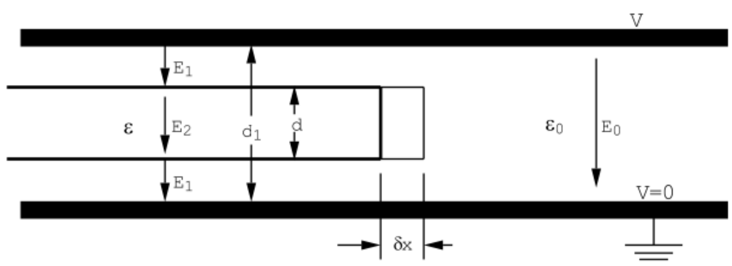

Figure 3.3.15: A plane parallel capacitor partially containing a slab of dielectric material d thick and w wide. The dielectric constant of the slab is ϵ. The objective is to calculate the electric forces acting on the slab.

outside the capacitor is nearly zero and may be neglected. An uncomplicated but tedious calculation gives (refer to Figure (3.3.15)):

$$\mathrm{E_1} = \frac{\mathrm{V}}{\left[\mathrm{d_1} + \left(\frac{\epsilon_0}{\epsilon} - 1\right)\mathrm{d}\right]} \quad \mathrm{Volts/m};$$

$$\mathrm{E_2} = \frac{\left(\frac{\epsilon_0}{\epsilon}\right)\mathrm{V}}{\left[\mathrm{d_1} + \left(\frac{\epsilon_0}{\epsilon} - 1\right)\mathrm{d}\right]} \quad \mathrm{Volts\ /m};$$

and

$$\mathrm{E_0} = \frac{\mathrm{V}}{d_1} \quad \mathrm{Volts/m}.$$

$\mathrm{E_1}$ is the field in the vacuum in a region occupied by the dielectric slab, but far enough from the end of the slab so that inhomogeneities in the field can be neglected: in practice, this means that one is considering a position several slab thicknesses, d, from the end. The quantity $\mathrm{E_2}$ is the electric field strength in the dielectric slab, but at a position several d removed from its end. $\mathrm{E_0}$ is the electric field strength in the region of the capacitor where there is no slab, and far enough from the end of the slab so that fringing fields can be neglected. Now let the slab be inserted δx farther between the capacitor plates. The change in energy stored in the electric field will just be that corresponding to removing a volume $(d_1 w)\,\delta x$ of dielectric-free space where the field is $\mathrm{E_0}$ Volts/m and replacing it with the volume (wd) δx of dielectric material subject to the field $\mathrm{E_2}$ plus the vacuum volume

$w\left(d_1-d\right)\delta x$ subject to the field E_1. This change in energy will be independent of the exact shape of the end of the slab providing that the extent of the non-uniform field region around the end of the slab is very small compared with the lateral dimensions, D, of the capacitor plates, i.e. providing that d/D ≪ 1. The change in stored electrostatic energy for a small displacement δx is given by

$$\delta \mathrm{U}_{\mathrm{E}}=\mathrm{wd}\,\delta \mathrm{x}\left(\frac{\epsilon \mathrm{E}_2^2}{2}+\frac{\epsilon_0 \mathrm{E}_1^2}{2}\left[\frac{\mathrm{d}_1}{\mathrm{d}}-1\right]\right)-\left(\mathrm{wd}_1 \delta \mathrm{x}\right) \frac{\epsilon_0 \mathrm{E}_0^2}{2}.$$

After some algebra this may be written

$$\delta \mathrm{U}_{\mathrm{E}}=(\mathrm{w} \delta \mathrm{x})\left(\frac{\epsilon_0 \mathrm{V}^2}{2 \mathrm{d}_1}\right)\left(\frac{1-\frac{\epsilon_0}{\epsilon}}{\frac{\epsilon_0}{\epsilon}-1+\frac{\mathrm{d}_1}{\mathrm{d}}}\right).$$

In general the dielectric constant ϵ is greater than ϵ_0 so that the electrostatic energy stored in the field increases if the dielectric slab moves farther into the capacitor. For constant applied voltage this means that the electric forces are such as to pull the slab further between the capacitor plates: at constant applied potential the geometry tends to change so as to maximize the energy stored in the field. The force on the slab is given by

$$\mathrm{F}_{\mathrm{x}}=\mathrm{w}\left(\frac{\epsilon_0 \mathrm{V}^2}{2 \mathrm{d}_1}\right)\left(\frac{1-\frac{\epsilon_0}{\epsilon}}{\frac{\epsilon_0}{\epsilon}-1+\frac{\mathrm{d}_1}{\mathrm{d}}}\right) \quad \text{Newtons.} \tag{3.3.13}$$

The force on a dielectric slab may be measured and used to obtain the dielectric constant for the slab material, ϵ. A variant of this method is often used to measure the dielectric constant of a fluid, see Figure (3.3.16). Eqn.(3.3.13) becomes much simpler if the thickness of the dielectric slab is the same, or nearly the same, as the spacing between the capacitor plates. When d = d_1 one finds

$$\mathrm{F}_{\mathrm{x}}=\mathrm{w}\left(\frac{\epsilon_0 \mathrm{V}^2}{2 \mathrm{d}}\right)\left(\frac{\epsilon}{\epsilon_0}-1\right)=\mathrm{w}\left(\frac{\epsilon_0 \mathrm{V}^2}{2 \mathrm{d}}\right) \chi_e, \tag{3.3.14}$$

where χ_e is the electrical susceptibility defined by $\epsilon=\epsilon_0\left(1+\chi_e\right)$. In the opposite limit, d/d_1 ≪ 1, the force is given by

$$\mathrm{F}_{\mathrm{x}}=\mathrm{wd}\left(\frac{\epsilon_0 \mathrm{V}^2}{2 \mathrm{d}_1^2}\right)\left(\frac{\epsilon-\epsilon_0}{\epsilon}\right)=\mathrm{wd}\left(\frac{\epsilon_0 \mathrm{V}^2}{2 \mathrm{d}_1^2}\right)\left(\frac{\chi_e}{1+\chi_e}\right) \quad \text{Newtons.} \tag{3.3.15}$$

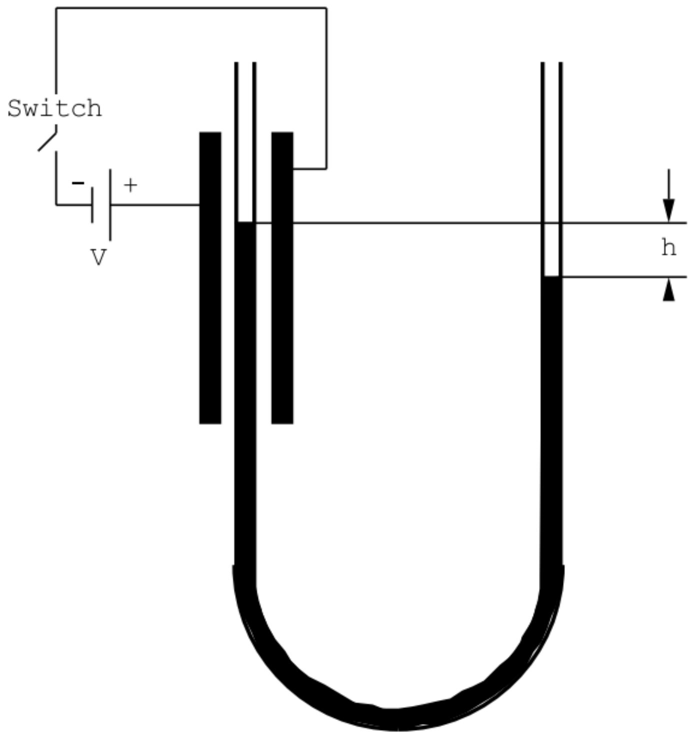

Figure 3.3.16: Quincke's method for measuring the dielectric constant of a fluid. Upon the application of the voltage V to the capacitor plates the dielectric fluid is sucked up between the plates. In equilibrium the electrical force on the fluid just balances the gravitational force. The gravitational force is proportional to the level difference h. The electrical force per unit area, t, is given by t = ρgh where ρ is the fluid density.

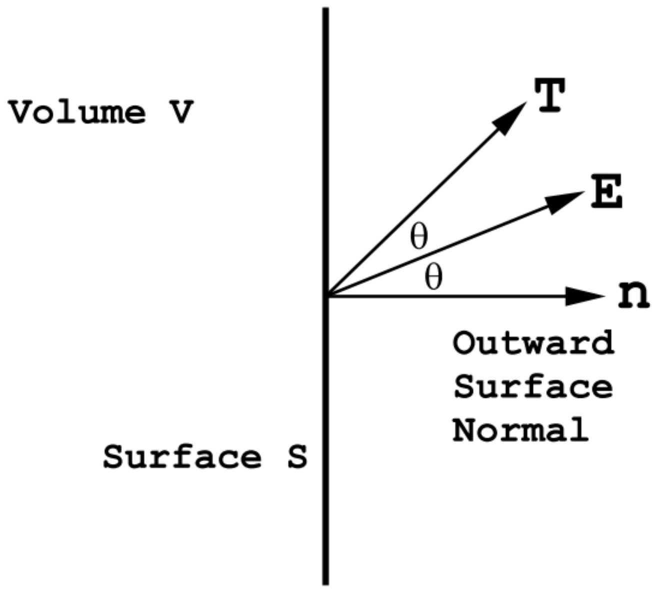

Figure 3.3.17: The force acting on the matter contained within a volume V can be obtained as the surface integral of a vector $\vec{T}$ over a surface S that encloses V. It is assumed that $\vec{D}$ is everywhere inside S proportional to the electric field, $\vec{E}$. It is further assumed that the surface S is immersed in a fluid that can support no shear stresses, and that $\vec{D}$ and $\vec{E}$ are parallel on S. The force per unit area is given by $|\mathrm{T}| = \vec{\mathrm{E}} \cdot \vec{\mathrm{D}}/2$ and the direction of the force per unit area is such that the angle between $\vec{T}$ and the surface normal is bisected by the direction of the electric field.

3.3.3 The Maxwell Stress Tensor

The forces acting on a static charge distribution located in a linear isotropic dielectric medium can be obtained as the divergence of an object called the **Maxwell stress tensor**. It can be shown that there exists a vector $\vec{T}$ associated with the elements of the stress tensor such that the surface integral of $\vec{T}$ over a closed surface S enclosing a volume V gives the net force acting on the charges within V: see, for example, Electromagnetic Theory by J.A.Stratton, section 2.5, (McGraw-Hill, N.Y., 1941). One can write

$$\overrightarrow{\mathrm{F}_{\mathrm{E}}}=\int\int_{\mathrm{S}}\vec{\mathrm{T}}\mathrm{dS}. \tag{3.3.16}$$

In this integral $\vec{T}$ is a vector whose magnitude is given by $|\vec{\mathrm{T}}|=(\vec{\mathrm{E}}\cdot\vec{\mathrm{D}})/2$ and whose direction is given by the construction shown in Figure (3.3.17). Note that the element of area, dS, in Equation (3.3.16) is not represented by a vector; it is simply a scalar quantity. When the electric field, E, is directed parallel with the outward normal to the surface element the force contribution is a tension, but when E lies in the surface the contribution to the force is a pressure. It is an interesting exercise to show that the force between two charges in vacuum is given by $\left(\mathrm{q}_1\mathrm{q}_2/4\pi\epsilon_0\mathrm{R}^2\right)$ if one integrates the vector $\vec{T}$ over a suitably chosen surface that completely surrounds one of the charges.