7.3: A Simple Radio Antenna

See (Electromagnetic Theory by J.A.Stratton, McGraw-Hill, N.Y., 1941. Section 8.7 and the following).

Consider a center fed linear antenna such as that depicted in Figure (7.3.2). In order to apply Equation (7.2.18) to an antenna of finite length it is necessary to know the current distribution along the wire. An exact solution of this problem is very difficult. A useful approximation assumes that the current distribution along the antenna is sinusoidal if the time variation of the current is sinusoidal. For a thin wire the current must be zero at the ends of the wire since there is no place to store charge. At other places along the wire charge may be stored on the wire surfaces and so the current need not be the same at every cross-section. The antenna is supposed to be driven by a sinusoidal generator at the circular frequency ω. A wave of current propagates along the wire, which can be regarded as a transmission line, and is reflected from the open ends of the wire. The resulting current distribution is a sinusoidal

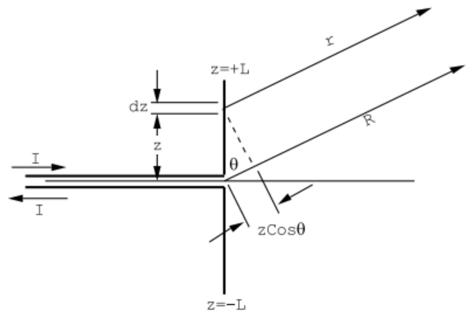

Figure 7.3.2: A center fed linear radio antenna. The current forms a standing wave with nodes at the wire ends.

standing wave along the wire having zero current at the ends of the wire at z = ±L. Such a current distribution can be described by

$$\begin{aligned} \mathrm{I}(\mathrm{z}, \mathrm{t}) &= \mathrm{I}_0 \exp(-i\omega t) \sin \tfrac{\omega}{\mathrm{c}}(\mathrm{L}-\mathrm{z}), \quad \mathrm{z} \geq 0 \\ \mathrm{I}(\mathrm{z}, \mathrm{t}) &= \mathrm{I}_0 \exp(-i\omega t) \sin \tfrac{\omega}{\mathrm{c}}(\mathrm{L}+\mathrm{z}), \quad \mathrm{z} \leq 0 \end{aligned} \tag{7.3.1}$$

see Figure (7.3.2). Use this current distribution in Equation (7.2.18) to calculate the vector potential. The currents flow only in the z-direction so the vector potential will only have a z-component. Further, it is assumed that we will be interested only in those fields that are far removed from the antenna: this means that for an observer at distance R from the antenna we shall assume that R ≫ L,λ where the antenna length is 2L and λ is the wavelength of the electric and magnetic fields produced by the antenna where λ = c/f, and ω = 2πf where f is the frequency. The antenna wire is assumed to be very thin so that every point on a cross-section at z can be considered to be at the same distance from the observer. This means that in the integral of (7.2.18) the integral of current density over x,y simply gives the total current at that place along the wire. Eqn.(7.2.18) can be written

$$\mathrm{A}_{\mathrm{z}}(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{t}) = \frac{\mu_0}{4\pi} \int_{-\mathrm{L}}^{\mathrm{L}} \mathrm{dz} \frac{\mathrm{I}_{\mathrm{z}}(\mathrm{z}, \mathrm{t}_{\mathrm{R}})}{\mathrm{r}},$$

where I_z is given by (7.3.1), t_R= t - r/c, and

$$r^2 = X^2 + Y^2 + (Z-z)^2.$$

Since | z |< L and R ≫ L one can use the binomial theorem to expand r in powers of (z/R):

$$\mathrm{r} = \mathrm{R}\left[1 - \frac{2\mathrm{Zz}}{\mathrm{R}^2} + \left(\frac{\mathrm{z}}{\mathrm{R}}\right)^2\right]^{1/2},$$

or

$$\mathrm{r} \cong \mathrm{R} - \frac{\mathrm{Zz}}{\mathrm{R}},$$

neglecting terms of order $(z/R)^2$ or higher. Thus the distance to the observer from a point z on the antenna is given by

$$\mathrm{r} = \mathrm{R} - \mathrm{z}\cos\theta,$$

where θ is the angle between the direction of $\vec{R}$ and the antenna, see Figure (7.3.2). The vector potential can therefore be written

$$A_z(R,\theta,t)=\frac{\mu_0}{4\pi}\int_{-L}^{0}dz\frac{I_0\sin\left[\frac{\omega}{c}(L+z)\right]}{[R-z\cos\theta]}\exp(-i\omega t)\exp\left(\frac{i\omega}{c}[R-z\cos\theta]\right)+\frac{\mu_0}{4\pi}\int_0^L dz\frac{I_0\sin\left[\frac{\omega}{c}(L-z)\right]}{[R-z\cos\theta]}\exp(-i\omega t)\exp\left(\frac{i\omega}{c}[R-z\cos\theta]\right). \tag{7.3.2}$$

This rather formidable appearing expression can be simplified if one notices that z cos θ in the denominator can be neglected compared with the distance R since R≫L. On the other hand, the term (ωz cos θ/c) in the exponentials cannot be neglected because L is usually comparable with λ and (ωz/c) = 2πz/λ. With the above simplification we have

$$A_z(R,\theta,t)=\frac{\mu_0}{4\pi}\frac{I_0\exp(-i\omega[t-R/c])}{R}[I_1+I_2], \tag{7.3.3}$$

where

$$I_1=\int_{-L}^{0}dz\sin\left[\frac{\omega}{c}(L+z)\right]\exp\left(-i\frac{\omega}{c}z\cos\theta\right)$$

and

$$I_2=\int_0^L dz\sin\left[\frac{\omega}{c}(L-z)\right]\exp\left(-i\frac{\omega}{c}z\cos\theta\right).$$

The integrals are messy but can be easily carried out. The result is

$$F(\theta)=I_1+I_2=\frac{2}{\frac{\omega}{c}\sin^2\theta}\left[\cos\left(\frac{\omega L\cos\theta}{c}\right)-\cos\left(\frac{\omega L}{c}\right)\right]. \tag{7.3.4}$$

The next step uses (7.3.3) to calculate the B-field from $\vec{B}=\text{curl}(\vec{A})$. For this purpose it is convenient to work in spherical polar co-ordinates:

$$A_R=A_z\cos\theta$$
$$A_\theta=-A_z\sin\theta$$
$$A_\phi=0$$

But since A_ϕ = 0 and there is no angular dependence on the angle ϕ, it follows that

$$B_R=B_\theta=0,$$

and

$$B_\phi=\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta)-\frac{\partial A_R}{\partial\theta}\right],$$

or

$$B_\phi=-i\frac{\mu_0}{4\pi}I_0\frac{\omega}{c}\frac{\exp(-i\omega[t-R/c])}{R}\sin\theta F(\theta)-\frac{\mu_0}{4\pi}I_0\frac{\omega}{c}\frac{\exp(-i\omega[t-R/c])}{R^2}\left[-\sin\theta F(\theta)+\cos\theta\frac{dF}{d\theta}\right] \tag{7.3.5}$$

This field contains two terms. The first term decreases with distance to the observer like $(\lambda R)^{-1}$. The second term decreases with distance like R^{-2} . This means that for the condition R≫ λ one can ignore the second term because it becomes very small relative to the first term. So in the far field of the antenna (R ≫ λ) one finds

$$B_\phi=-i\frac{\mu_0}{4\pi}I_0\frac{\omega}{c}\frac{\exp(-i\omega[t-R/c])}{R}\sin\theta F(\theta), \tag{7.3.6}$$

and

$$B_\theta=B_R=0.$$

The electric field can be most easily obtained from B by means of the third Maxwell equation (7.2.11.). In free space $\vec{J}_T=0$ so that

$$\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\text{curl}(\vec{B}),$$

or, since the time variation is proportional to exp (−iωt),

$$-i\frac{\omega}{c^2}\vec{E} = \mathrm{curl}(\vec{B}). \tag{7.3.7}$$

The electric field components calculated from Equation (7.3.7) are:

$$\begin{aligned} -i\frac{\omega}{c^2}E_R &= \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\phi) \\ -i\frac{\omega}{c^2}E_\theta &= -\frac{1}{R}\frac{\partial}{\partial R}(RB_\phi) \\ E_\phi &= 0 \end{aligned} \tag{7.3.8}$$

Notice that the component E_R varies with distance like $1/R^2$, whereas the component E_θ varies with distance like 1/R. For distances such that $R \gg \lambda$ the component E_R becomes very small compared with E_θ and can be ignored. Thus in the **far field limit** $\vec{E}$ has only the component E_θ and $\vec{B}$ has only the component B_ϕ. Notice that $\vec{E}$ and $\vec{B}$ are orthogonal to each other and both are orthogonal to the line joining the observer to the center of the antenna. Also note that from Equation (7.3.8)

$$E_\theta = cB_\phi, \tag{7.3.9}$$

independent of the angle of observation.

In the limit of small angles, θ, the factor sin θF(θ) in Equation (7.3.6) simplifies to $2L\theta\sin\left(\frac{\omega L}{c}\right)$. This means that in the limit θ → 0 the field amplitudes fall off to zero as the observer becomes aligned with the antenna (θ is defined in Figure (7.3.2)). On the other hand, for an observer in the X-Y plane θ = ϕ/2 and

$$\sin\theta F(\theta) = \left(\frac{2c}{\omega}\right)\left[1-\cos\left(\frac{\omega L}{c}\right)\right].$$

The radiation fields in the equatorial plane are non-zero and become particularly large when cos(ωL/c)= 0 or -1. Such an antenna is said to be resonant. The condition ωL/c = π/2 corresponds to the commonly used half-wave antenna for which L = λ/4, where λ is the free space wavelength 2πc/ω. For such a half-wave antenna the angular dependence of the radiation fields becomes

$$\sin\theta F(\theta) = \frac{(2c/\omega)\cos(\pi\cos\theta/2)}{\sin\theta}. \tag{7.3.10}$$

Despite its more complicated appearance, this function is very similar to the sin θ angular variation that characterizes a point electric dipole radiator, as we shall see in the next section.

The present section has demonstrated how one can calculate the strength of a radio signal generated by a typical linear antenna. It also demonstrates that relatively complex fields are a consequence of the presence of the retarded time in the relatively simple formula for the vector potential, Equation (7.2.18).