5.2: Calculation of off-axis Fields

It is relatively easy to calculate the magnetic field along the symmetry axis of an axially symmetric coil system using the law of Biot-Savart, Equation (5.1.8). The calculation can be easily carried out because the magnetic field has only one component, an axial component, and the cylindrical symmetry makes the integration over the current distribution relatively simple. M.W.Garrett has pointed out that off-axis fields can be readily calculated from the magnetic scalar potential (M.W.Garrett, J.Appl.Phys.22,1091-1107(1951); ”Axially Symmetric Systems for Generating and Measuring Magnetic Fields. Part I”). In any current-free region

$$\operatorname{curl}(\overrightarrow{\mathrm{H}})=0 \tag{5.2.1}$$

and therefore one can write

$$\overrightarrow{\mathrm{H}}=-\operatorname{grad}(\mathrm{V_m}), \tag{5.2.2}$$

where $\mathrm{V_m}$ is a scalar function of position. In a uniform medium for which $\overrightarrow{\mathrm{B}}=\mu\overrightarrow{\mathrm{H}}$ the equation $\operatorname{div}(\overrightarrow{\mathrm{B}})=0$ can be re-written as

$$\operatorname{div}(\overrightarrow{\mathrm{H}})=0. \tag{5.2.3}$$

It follows from Equation (5.2.2) that the magnetic scalar potential must satisfy LaPlace’s equation in any region free of currents. In spherical polar coordinates LaPlace’s equation is written

$$\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2\frac{\partial \mathrm{V_m}}{\partial \mathrm{r}}\right)+\frac{1}{\mathrm{r}^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \mathrm{V_m}}{\partial\theta}\right)+\frac{1}{\mathrm{r}^2\sin^2\theta}\frac{\partial^2\mathrm{V_m}}{\partial\phi^2}=0. \tag{5.2.4}$$

The magnetic scalar potential cannot depend upon the azimuthal angle ϕ for an axially symmetric coil system, so that $\nabla^2\mathrm{V_m}=0$ reduces to

$$\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2\frac{\partial \mathrm{V_m}}{\partial \mathrm{r}}\right)+\frac{1}{\mathrm{r}^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \mathrm{V_m}}{\partial\theta}\right)=0. \tag{5.2.5}$$

The general solution of this equation can be written as a series expansion in Legendre polynomials:

$$\mathrm{V_m}=\sum_{n=0}^{\infty}\left(\mathrm{a_n r^n}+\frac{\mathrm{b_n}}{\mathrm{r^{n+1}}}\right)\mathrm{P_n}(\cos\theta). \tag{5.2.6}$$

This is the same expansion as was used for the electrostatic potential in Chpt.(3),section(3.2.1(d)) to treat the problem of a dielectric sphere in a uniform applied electric field. The functions $\mathrm{P_n(x)}$ are Legendre polynomials, the first five of which are listed in Table (3.2.2). The terms proportional to $1/\mathrm{r^{n+1}}$ in the expansion (5.2.6) are not acceptable for describing the magnetic potential function for a system of axially symmetric coils because they blow up at r=0; there are no singularities in the magnetic field along the axis of the coil system. This means that the magnetic potential must be describable by the series

$$\mathrm{V_m}=\sum_{n=1}^{\infty}\mathrm{a_n r^n P_n}(\cos\theta). \tag{5.2.7}$$

(The n=0 term corresponds to a constant; it is not important and may be set equal to zero because any constant may be added to $\mathrm{V_m}$ without changing the magnetic field). Along the axis of the coil system, the z-axis of the spherical polar co-ordinate system, the angle θ is fixed; cos θ=+1 for the region z > 0 and cos θ=-1 for the region z < 0. Moreover,along the axis of the coil system r =| z |, so that along the axis Equation (5.2.7) becomes a power series in z. The magnetic field calculated from this power series may be compared term by term with the power series for the magnetic field calculated

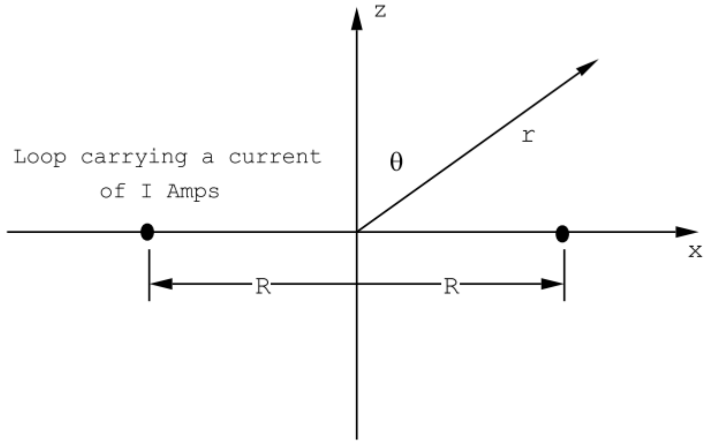

Figure 5.2.1: A circular loop of radius R carrying a current of I Amps and lying in the x-y plane.

directly from the law of Biot-Savart. The comparison of the two series yields values for the coefficients an that appear in the expansion for the magnetic potential, Equation (5.2.7). Once the coefficients an have been determined the magnetic field at any point within the coil system can be readily calculated from $\vec{H} = -\text{grad}(V_m)$.

This procedure can be illustrated for a single loop of wire lying in the xy plane, Figure (5.1.1). The magnetic field along the axis of such a loop is given by

$$\mathrm{B_z} = \frac{\mu \mathrm{IR}^2}{2} \frac{1}{\left(\mathrm{z}^2 + \mathrm{R}^2\right)^{3/2}}, \tag{5.2.8}$$

see Equation (4.3.4) of Chpt.(4). This expression can be expanded in a Taylor seies in the variable z:

$$\mathrm{B_z(z)} = \mathrm{B_z(0)} + \left(\frac{d\mathrm{B_z}}{dz}\right)_{\mathrm{z}=0} \mathrm{z} + \left(\frac{d^2\mathrm{B_z}}{dz^2}\right)_{\mathrm{z}=0} \left(\frac{\mathrm{z}^2}{2}\right) + + \left(\frac{d^3\mathrm{B}_z}{dz^3}\right)_{z=0} \left(\frac{z^3}{6}\right) + \cdots + \left(\frac{d^n B_z}{dz^n}\right)_{z=0} \left(\frac{z^n}{n!}\right) + \cdots \tag{5.2.9}$$

For the single current loop of Figure (5.1.1) this Taylor's series becomes

$$\mathrm{B_z(z)} = \left(\frac{\mu \mathrm{I}}{2\mathrm{R}}\right) \left(1 - \frac{3}{2}\left(\frac{\mathrm{z}}{\mathrm{R}}\right)^2 + \frac{45}{24}\left(\frac{\mathrm{z}}{\mathrm{R}}\right)^4 + \cdots\right). \tag{5.2.10}$$

Notice that this series contains only even powers of (z/R) because the magnetic field is symmetric with respect to the plane of the coil, i.e. $B_z(-z) = Bz(z)$. Now $B_z(z)$ is derived from the magnetic potential function through a differentiation with respect to z:

$$\mathrm{B_z(z)} = -\frac{\partial \mathrm{V_m}}{\partial \mathrm{z}}, \tag{5.2.11}$$

The series (5.2.11) must be compared with the general series Equation (5.2.7) using r=z and cos θ = 1, i.e. with

$$\mathrm{V_m(z, 0)} = \mathrm{a_1 z P_1(1)} + \mathrm{a_2 z^2 P_2(1)} + \mathrm{a_3 z^3 P_3(1)} + \mathrm{a_4 z^4 P_4(1)} + \mathrm{a_5 z^5 P_5(1)} + \cdots \tag{5.2.12}$$

It is clear from this comparison that the coefficients of all the even terms must be zero. The Legendre polynomials are normalized so that $P_n(1) = 1$ (see Table (3.2.2), section(3.2.1(d))). It follows from a comparison of (5.2.12) with (5.2.11) that

$$\begin{aligned} \mathrm{a_1} &= -\tfrac{\mu \mathrm{I}}{2\mathrm{R}} \\ \mathrm{a_3} &= \tfrac{\mu \mathrm{I}}{2}\left(\tfrac{1}{2\mathrm{R}^3}\right) \\ \mathrm{a_5} &= -\tfrac{\mu \mathrm{I}}{2}\left(\tfrac{9}{24\mathrm{R}^5}\right), \quad \text{etc.} \end{aligned}$$

The first three terms in the expansion for the potential function, valid for any point in space such that (r/R) < 1, are given by

$$\mathrm{V_m(r, \theta)} = -\frac{\mu \mathrm{I}}{2}\left[\left(\frac{\mathrm{r}}{\mathrm{R}}\right) \mathrm{P_1}(\cos\theta) - \frac{1}{2}\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^3 \mathrm{P_3}(\cos\theta) + \frac{9}{24}\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^5 \mathrm{P_5}(\cos\theta) + \cdots\right], \tag{5.2.13}$$

where

$$
\begin{aligned}
\mathrm{P}_1(x) &= x \\
\mathrm{P}_3(x) &= \frac{1}{2}\left(5\mathrm{x}^3 - 3\mathrm{x}\right)
\end{aligned}
$$

and

$$\mathrm{P}_5(x) = \frac{1}{8}\left(63\mathrm{x}^5 - 70\mathrm{x}^3 + 15\mathrm{x}\right);$$

(see Schaum's Outline Series: Mathematical Handbook, by Murray R. Spiegel, McGraw-Hill,N.Y., 1968)). The components of the magnetic field can be calculated from

$$\mathrm{B}_\mathrm{r} = -\frac{\partial \mathrm{V}_\mathrm{m}}{\partial \mathrm{r}},$$

and

$$\mathrm{B}_\theta = -\frac{1}{\mathrm{r}}\frac{\partial \mathrm{V}_\mathrm{m}}{\partial \theta}.$$

These fields can be calculated very readily for particular values of r, θ by means of a modern digital computer; programs for calculating Legendre polynomials and their derivatives are readily available.