1.5: The Force Density and Torque Density in Matter

The presence of an electric field, $\vec{E}$, and a magnetic field,$\vec{B}$, in matter results in a force density if the matter is charged and in a torque density if the matter carries electric and magnetic dipole densities. In addition, if the electric field varies in space (the usual case) then a force density is created that is proportional to the electric dipole density and to the electric field gradients. Similarly, if the magnetic field varies in space then a force density is exerted on the matter that is proportional to the magnetic dipole density and to the magnetic field gradients. These force and torque densities are stated below; their proof is left for the problem sets.

1.5.1 The Force Density in Charged and Polarized Matter.

There is a force density that is the direct analogue of Equation (1.1.8), the force acting on a charged particle moving with the velocity $\vec{v}$ in electric and magnetic fields, ie

$$\overrightarrow{\mathrm{f}}=q(\overrightarrow{\mathrm{E}}+[\overrightarrow{\mathrm{v}}\times\overrightarrow{\mathrm{B}}]).$$

If this force acting on each charged particle is averaged in time over periods longer than characteristic atomic or molecular orbital times and summed over the particles contained in a volume, $\Delta\mathrm{V}$, where $\Delta\mathrm{V}$ is large compared with atomic or molecular dimensions, then one can divide this total averaged force by $\Delta\mathrm{V}$ to obtain the force density

$$\overrightarrow{\mathbf{F}}=\rho_f\overrightarrow{\mathbf{E}}+\left(\overrightarrow{\mathbf{J}}_f\times\overrightarrow{\mathbf{B}}\right)\quad\text{Newtons}\,/m^3.\tag{1.5.1}$$

If the electric field in matter varies from place to place there is generated a force density proportional to the dipole moment per unit volume, $\vec{P}$, given by

$$\overrightarrow{\mathrm{F}}_E=\left(\overrightarrow{\mathrm{P}}\cdot\nabla E_x\right)\hat{\mathbf{u}}_x+\left(\overrightarrow{\mathrm{P}}\cdot\nabla E_y\right)\hat{\mathbf{u}}_y+\left(\overrightarrow{\mathrm{P}}\cdot\nabla E_z\right)\hat{\mathbf{u}}_z\quad\text{Newtons}\,/m^3.\tag{1.5.2}$$

In addition, if the magnetic field, $\vec{B}$, varies from place to place there will be generated a force density proportional to the magnetic dipole density, $\vec{M}$, given by

$$\overrightarrow{\mathrm{F}}_B=\left(\overrightarrow{\mathrm{M}}\cdot\nabla B_x\right)\hat{\mathbf{u}}_x+\left(\overrightarrow{\mathrm{M}}\cdot\nabla B_y\right)\hat{\mathbf{u}}_y+\left(\overrightarrow{\mathrm{M}}\cdot\nabla B_z\right)\hat{\mathbf{u}}_z\quad\text{Newtons}\,/m^3.\tag{1.5.3}$$

The nabla operator denotes the operation of calculating the gradient of a scalar function $\phi(\overrightarrow{\mathrm{r}})$. In cartesian co-ordinates

$$\nabla\phi=\frac{\partial\phi}{\partial x}\hat{\mathbf{u}}_x+\frac{\partial\phi}{\partial y}\hat{\mathbf{u}}_y+\frac{\partial\phi}{\partial z}\hat{\mathbf{u}}_z.$$

1.5.2 The Torque Densities in Polarized Matter.

It can be shown that an electric field exerts a torque on polarized matter. The torque density is given by

$$\overrightarrow{\mathrm{T}}_E=\overrightarrow{\mathrm{P}}\times\overrightarrow{\mathrm{E}}\quad\text{Newtons}\,/\mathrm{m}^2.\tag{1.5.4}$$

The magnetic field also exerts a torque on magnetized matter. This torque density is given by

$$\overrightarrow{\mathrm{T}}_B=\overrightarrow{\mathrm{M}}\times\overrightarrow{\mathrm{B}}\quad\text{Newtons}\,/\mathrm{m}^2.\tag{1.5.5}$$