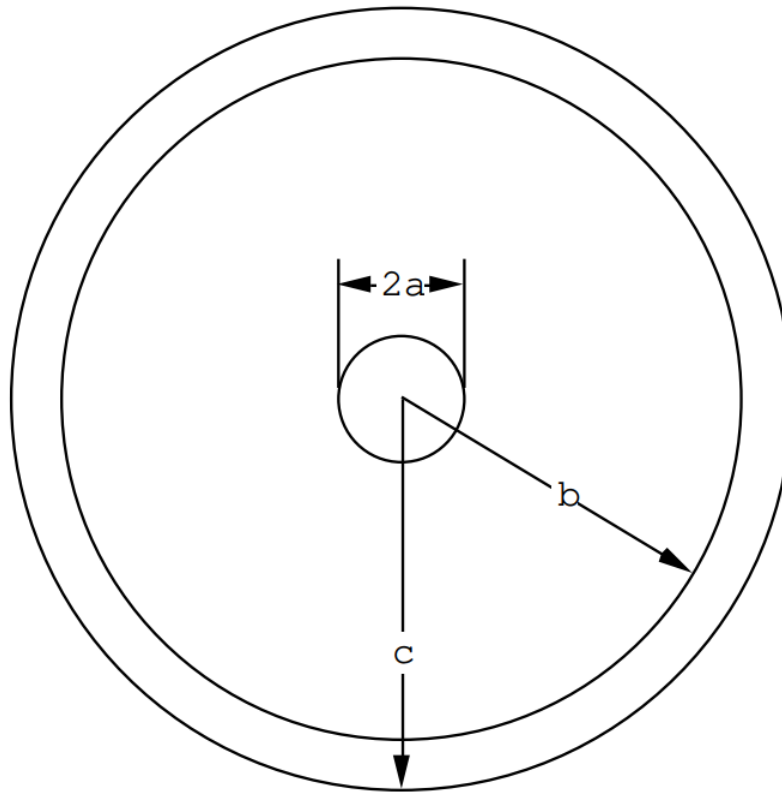


## 13.4: Chapter 4

### Problem (4.1)

A current  $I$  amps flows in the inner conductor of an infinitely long co-axial line and returns via the outer conductor. The radius of the inner conductor is  $a$ , and  $b$  and  $c$  are the inner and outer radii of the outer conductor (see the sketch). The current density is uniform in the two conductors. Calculate the magnetic flux density in all regions. The magnetization density can be set equal to zero everywhere.



### Answer (4.1)

This problem exhibits cylindrical symmetry so that it is ideal for an application of Stokes' theorem. Let  $z$  be the direction along the cable. Then there is only a component  $A_z$  of the vector potential  $\left(\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{L}}{r}\right)$ . Moreover, by symmetry  $A_z$  cannot depend upon the angle  $\theta$ , nor can it depend upon  $z$  (infinite wire).

$$\therefore A_z = A_z(r).$$

In cylindrical co-ordinates

$$\text{curl } \mathbf{A} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}, \therefore \mathbf{B} \text{ has only a } \theta \text{ component}$$

$$B_\theta = -\frac{\partial A_z}{\partial r}.$$

But since there is no magnetization and no time dependence

$$\text{Curl } \mathbf{B} = \mu_0 \mathbf{J}_f$$

$$\therefore \int_{\text{surface}} \text{curl } \mathbf{B} \cdot d\mathbf{s} = \mu_0 \int_{\text{surface}} \mathbf{J}_f \cdot d\mathbf{s}$$

or

$$\oint_C \mathbf{B} \cdot d\mathbf{L} = \mu_0 \int_{\text{surface}} \mathbf{J}_f \cdot d\mathbf{s}$$

Apply this to a circle of radius r:

Case (1)  $r \leq a$   $J_f = \frac{I}{\pi a^2}$

$$\therefore 2\pi r B_\theta = \mu_0 \left( \frac{I}{\pi a^2} \right) (\pi r^2) = \mu_0 I \left( \frac{r}{a} \right)^2$$

$$\therefore B_\theta = \left( \frac{\mu_0 I}{2\pi a^2} \right) r$$

So when  $r = 0$   $B_\theta = 0$

when  $r = a$   $B_\theta = \frac{\mu_0 I}{2\pi a}$ .

Case (2)  $a \leq r \leq b$

In this case  $\int_{\text{surface}} \mathbf{J}_f \cdot d\mathbf{s} \equiv I$

$$\therefore 2\pi r B_\theta = \mu_0 I$$

$$B_\theta = \frac{\mu_0 I}{2\pi r}$$

When  $r=a$   $B_\theta = \frac{\mu_0 I}{2\pi a}$

When  $r=b$   $B_\theta = \frac{\mu_0 I}{2\pi b}$

Case (3)  $b \leq r \leq c$

In the outer conductor

$$|J_f| = \frac{I}{\pi(c^2 - b^2)}$$

and the current flow is negative. Therefore this time one has

$$2\pi r B_\theta = \mu_0 \left[ I - \frac{I\pi(r^2 - b^2)}{\pi(c^2 - b^2)} \right]$$

$$B_\theta = \frac{\mu_0 I}{2\pi r} \left[ 1 - \left( \frac{r^2 - b^2}{c^2 - b^2} \right) \right]$$

So when  $r = b$   $B_\theta = \frac{\mu_0 I}{2\pi b}$

When  $r = c$   $B_\theta = 0$

Case (4)  $R \geq C$

Here  $2\pi r B_\theta = \mu_0 (I - I) \equiv 0 \therefore B_\theta = 0$ .

There is no field outside this co-axial cable. Notice that the tangential component of  $\mathbf{B}$  is continuous across the boundaries.

#### Problem (4.2)

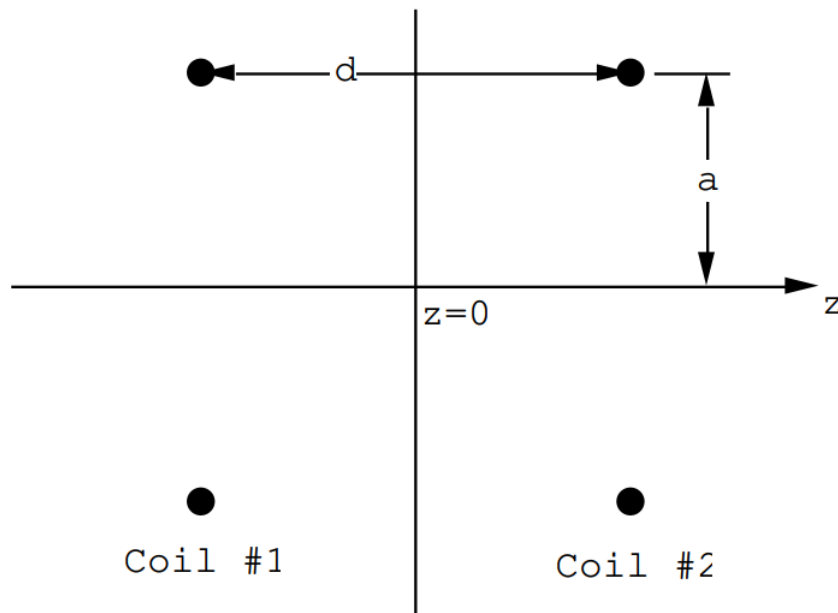
Two identical coaxial coils, each of N turns and radius a, are separated by a distance d as shown in the sketch. A current flows through each coil so that the fields of the two coils add at the origin.

(a) Calculate  $B_z$  at the origin

(b) Show that  $\frac{dB_z}{dz} = 0$  at  $z = 0$ .

(c) Find d such that  $\frac{d^2 B_z}{dz^2} = 0$  at  $z = 0$ .

Such a configuration is the simplest system for generating a uniform magnetic field. It is known as a Helmholtz pair.



#### Answer (4.2)

The field of a single coil along its axis is

$$B_z = \frac{\mu_0 NI}{2} \frac{a^2}{(a^2 + z^2)^{3/2}}$$

where  $z$  is measured from the center of the coil. For the above pair of coils

$$B_z = \frac{\mu_0 NI a^2}{2} \left\{ \frac{1}{\left[ \left( z - \frac{d}{2} \right)^2 + a^2 \right]^{3/2}} + \frac{1}{\left[ \left( z + \frac{d}{2} \right)^2 + a^2 \right]^{3/2}} \right\}$$

(a) At  $z = 0$   $B_z = \mu_0 NI \left( \frac{a^2}{\left[ \frac{d^2}{4} + a^2 \right]^{3/2}} \right)$

(b)

$$\frac{dB_z}{dz} \propto \frac{-3 \left( z - \frac{d}{2} \right)}{\left[ \left( z - \frac{d}{2} \right)^2 + a^2 \right]^{5/2}} - \frac{3 \left( z + \frac{d}{2} \right)}{\left[ \left( z + \frac{d}{2} \right)^2 + a^2 \right]^{5/2}}$$

Thus at  $z = 0$   $\frac{dB_z}{dz} = 0$ .

(c)

$$\frac{d^2 B_z}{dz^2} \propto \frac{-3}{\left[ \left( z - \frac{d}{2} \right)^2 + a^2 \right]^{5/2}} - \frac{3}{\left[ \left( z + \frac{d}{2} \right)^2 + a^2 \right]^{5/2}} + \frac{15 \left( z - \frac{d}{2} \right)^2}{\left[ \left( z - \frac{d}{2} \right)^2 + a^2 \right]^{7/2}} + \frac{15 \left( z + \frac{d}{2} \right)^2}{\left[ \left( z + \frac{d}{2} \right)^2 + a^2 \right]^{7/2}}$$

$\therefore$  at  $z = 0$

$$\begin{aligned} \frac{d^2 B_z}{dz^2} &\propto \frac{-3 \left( \frac{d^2}{4} + a^2 \right) - 3 \left( \frac{d^2}{4} + a^2 \right) + 15 \left( \frac{d^2}{4} \right) + 15 \left( \frac{d^2}{4} \right)}{\left[ \frac{d^2}{4} + a^2 \right]^{7/2}} \\ &\propto 6d^2 - 6a^2 = 6(d^2 - a^2) \end{aligned}$$

So  $\frac{d^2 B_z}{dz^2} = 0$  if  $d = a$ .

Thus for a Helmholtz pair  $d = a$ .

The magnetic field strength at the center of the Helmholtz pair is given by

$$B_z(0) = \frac{\mu_0 NI}{a} \left(\frac{4}{5}\right)^{3/2} = 0.716 \frac{\mu_0 NI}{a}.$$

#### Problem (4.3)

A solenoid is 1 meter long and it carries  $10^4$  turns of wire. The average radius of the coil is 0.1 meters. The coil carries a current of 10 Ampères.

(a) Calculate the field at the center of the solenoid.

(b) If the wire of the coil has a cross-sectional area of  $10^{-6}$  meters<sup>2</sup> calculate the resistance of the coil.  $R = \rho L/A$  and for copper  $\rho = 2 \times 10^{-8}$  ohm meters.

(c) How much power is required to produce the magnetic field of part (a)?

This calculation explains why iron core magnets are used to generate fields of  $\sim 1$  Tesla.

#### Answer (4.3)

$$B_z(z) = \frac{\mu_0 NI}{2} \left\{ \frac{(z + L/2)}{\sqrt{R^2 + (z + L/2)^2}} - \frac{(z - L/2)}{\sqrt{R^2 + (z - L/2)^2}} \right\}$$

$N$  is the number of turns/m,  $L$  the length of the coil.

$$\text{At } z = 0 \quad B_z(0) = \frac{\mu_0 NI}{2} \frac{L}{\sqrt{R^2 + (L/2)^2}}$$

Here  $N = 10^4/\text{m}$ ,  $I = 10$  Amps,  $\frac{L}{2} = \frac{1}{2} \text{ m}$ , and  $R = 0.1$  m

$$(a) \therefore B_z(0) = \frac{(2\pi \times 10^{-2})}{\sqrt{.01 + .25}} = 0.123 \text{ Tesla} \quad \text{i.e. } \sim 10^3 \text{ x earth's field!}$$

$$(b) L = (2\pi R)(10^4) = 6.283 \times 10^3 \text{ m} \therefore R = 125.7 \text{ Ohms}$$

$$(c) \text{ For 10 Amps one would require 1257 Volts and a power } = VI = 12,570 \text{ Watts!!} = 12.57 \text{ kWatts!}$$

#### Problem (4.4)

A square loop of wire 1 cm on a side carries a current of 2 Ampères.

(a) Estimate the magnitude of the magnetic field on the axis of the current loop and 1 meter from its center. The loop may be treated like a point dipole.

(b) Estimate the magnitude and direction of the magnetic field one meter from the center of the loop but at a point in the plane of the loop.

#### Answer (4.4)

The magnetic moment of the loop is  $M_0 = IA = (2)(10^{-4})$  Amp m<sup>2</sup>.

$$\text{Now } \mathbf{B} = \frac{\mu_0}{4\pi} \left[ \frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3} \right]$$

(a) On the axis of the dipole  $\mathbf{m} \cdot \mathbf{r} = M_0 r$

$$\text{So } B_z = \left(\frac{\mu_0}{4\pi}\right) \left(\frac{2M_0}{r^3}\right) = 4 \times 10^{-11} \text{ Tesla}$$

(The earth's field is  $\sim 10^{-4}$  Tesla so this is very weak).

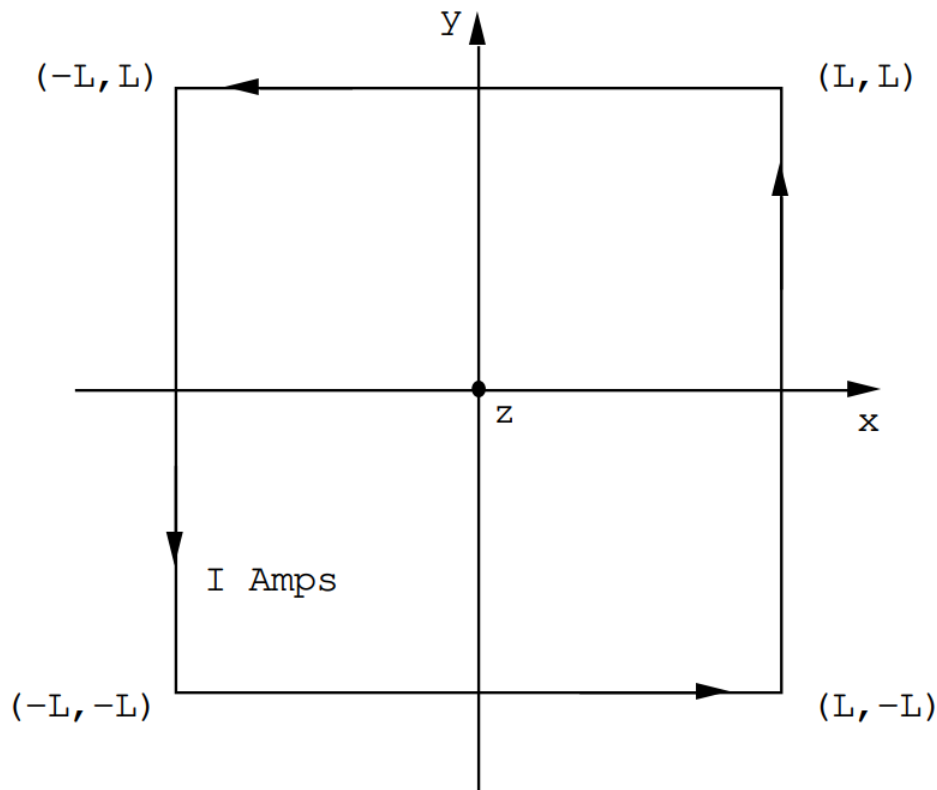
(b) On the equatorial plane  $\mathbf{m} \cdot \mathbf{r} = 0$

$$\therefore B_z = -\frac{\mu_0}{4\pi} \frac{M_0}{r^3} = -2 \times 10^{-11} \text{ Tesla}$$

Directed opposite to the dipole moment.

**Problem (4.5).**

Calculate the magnetic field along the z-axis of a square coil which carries a current of I Amps (see the sketch).



Each side of the square is  $2L$  meters long.

**Answer (4.5).**

Along the axis of the coil there will be only a z-component of magnetic field by symmetry. In order to get the total field it is only necessary to calculate the z-component of the field generated by one side of the coil and then multiply by four. Consider the right hand side.

Let  $d\mathbf{L} = dy\hat{\mathbf{u}}_y$  at  $(L, Y)$

The position of the element of length,  $d\mathbf{L}$ , is specified by  $\mathbf{r}$  where  $\mathbf{r} = L\hat{\mathbf{u}}_x + y\hat{\mathbf{u}}_y$ . The position of the point of observation along the z-axis is specified by  $\mathbf{R} = z\hat{\mathbf{u}}_z$ .

Therefore,

$$(\mathbf{R} - \mathbf{r}) = -L\hat{\mathbf{u}}_x - y\hat{\mathbf{u}}_y + z\hat{\mathbf{u}}_z$$

and

$$|\mathbf{R} - \mathbf{r}| = \sqrt{L^2 + y^2 + z^2}.$$

From the law of Biot-Savard one obtains

$$d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{d\mathbf{L} \times (\mathbf{R} - \mathbf{r})}{|\mathbf{R} - \mathbf{r}|^3};$$

from which

$$dB_z = \frac{\mu_0}{4\pi} I \frac{L dy}{(L^2 + y^2 + z^2)^{3/2}},$$

and

$$B_z = \frac{\mu_0}{4\pi} I \frac{2L^2}{(z^2 + L^2) \sqrt{z^2 + 2L^2}}.$$

This must be multiplied by 4x because the coil has four sides:

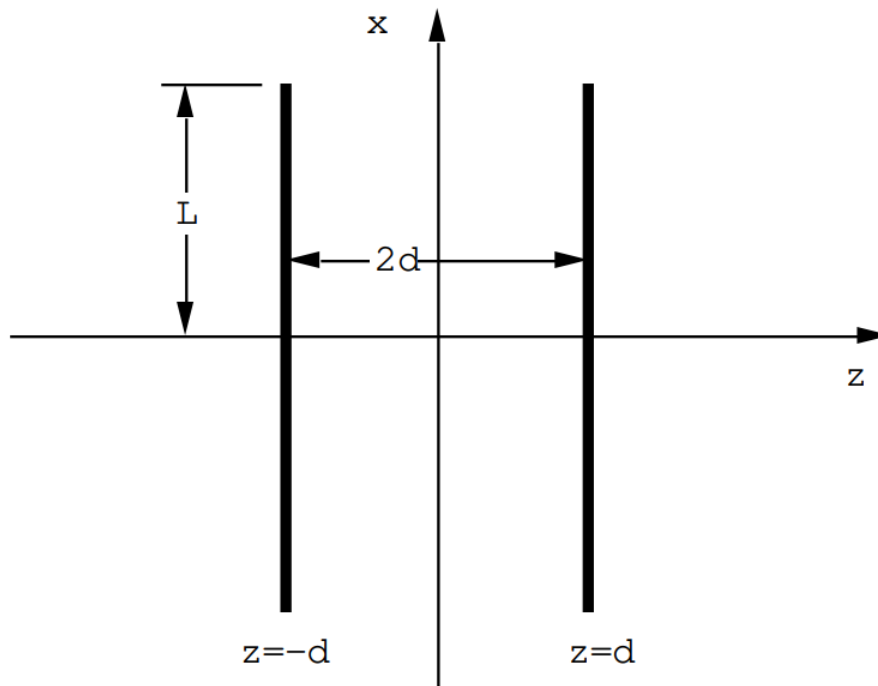
$$B_z(z) = \frac{2\mu_0 I}{\pi} \frac{L^2}{(z^2 + L^2) \sqrt{z^2 + 2L^2}}.$$

$$\text{At } z=0 \quad B_z(0) = \frac{2\sqrt{2}}{\pi} \frac{\mu_0 I}{2L} = 0.9003 \frac{\mu_0 I}{2L}$$

This value can be compared with  $B_z(0) = \frac{\mu_0 I}{2R}$  for a circular coil.

#### Problem (4.6).

(a) How far apart should two square coils be mounted in order to obtain as homogeneous as possible a magnetic field? See the sketch.



(One wants  $\frac{d^2 B_z}{dz^2} = 0$  at the center of the coil system. With a little thought one can convince oneself that at  $z=0$  the quantity  $\frac{d^2 B_z}{dz^2}$  is exactly the same for each coil, so that the work of differentiation can be reduced by a factor two.)

(b) Over what distance along the z-axis will the field deviate by less than 1% from the field at the center of the coil system if  $L = 1$  meter?

Such square coils are often more convenient to build than circular coils if the earth's magnetic field is to be cancelled over a large volume.

#### Answer (4.6)

(a) From the results of Problem (3.5) one can write

$$B_z(z) = \frac{2\mu_0 I L^2}{\pi} (\Psi_1 + \Psi_2), \text{ where}$$

$$\Psi_1 = \frac{1}{((z+d)^2 + L^2)((z+d)^2 + 2L^2)^{1/2}},$$

$$\text{and } \Psi_2 = \frac{1}{((z-d)^2 + L^2)((z-d)^2 + 2L^2)^{1/2}}.$$

$$\frac{d\psi_1}{dz} = \frac{-(z+d)(3(z+d)^2+5L^2)}{((z+d)^2+L^2)^2((z+d)^2+2L^2)^{3/2}}, \text{ and}$$

$$\frac{d\psi_2}{dz} = \frac{-(z-d)(3(z-d)^2+5L^2)}{((z-d)^2+L^2)^2((z-d)^2+2L^2)^{3/2}}.$$

Note that at  $z=0$   $\frac{d}{dz}(\psi_1 + \psi_2) \equiv 0$ ; the field gradient vanishes by symmetry.

$$\frac{d^2\psi_1}{dz^2} = \frac{N}{D}, \text{ where}$$

$$N = - (9(z+d)^2+5L^2)((z+d)^2+L^2)((z+d)^2+2L^2) \\ + (12(z+d)^4+20L^2(z+d)^2)((z+d)^2+2L^2) \\ + (9(z+d)^4+15L^2(z+d)^2)((z+d)^2+L^2)$$

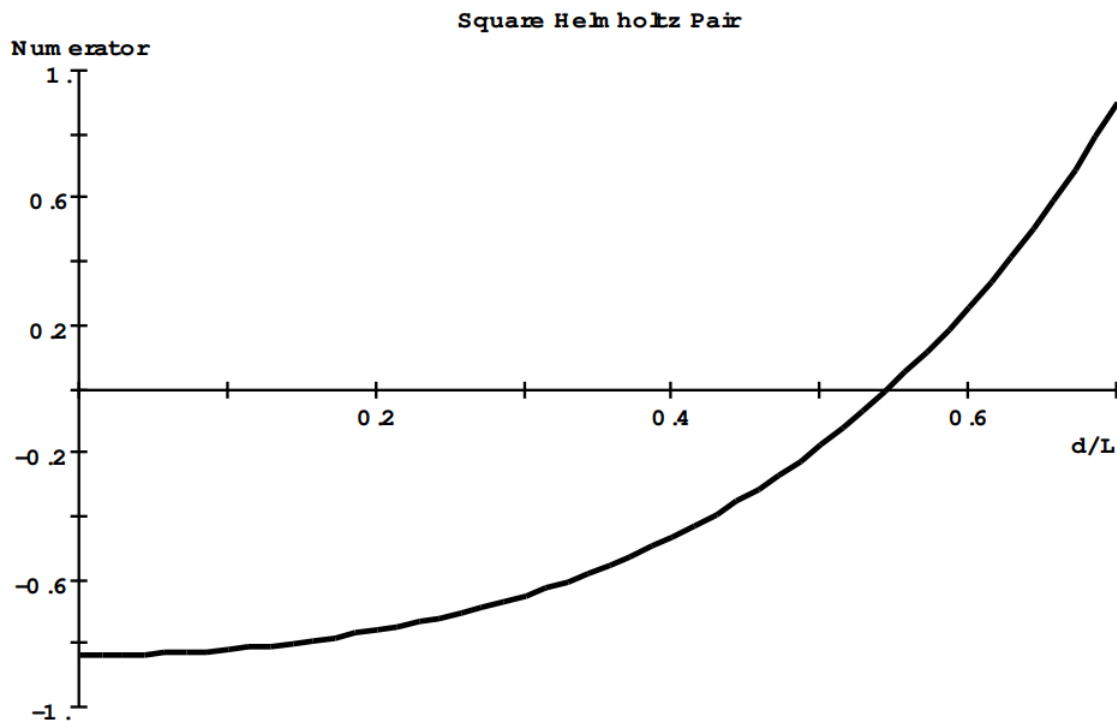
and

$$D = ((z+d)^2+L^2)^3((z+d)^2+2L^2)^{5/2}.$$

At  $z=0$   $\frac{d^2\psi_1}{dz^2} = \frac{d^2\psi_2}{dz^2}$  so that for optimum uniformity We require the numerator in the second derivative to vanish at  $z=0$ . This condition gives

$$\eta^6 + 3\eta^4 + \left(\frac{11}{6}\right)\eta^2 - \left(\frac{5}{6}\right) = 0 \quad (1)$$

where  $\eta = (d/L)$ . The solution is  $\frac{d}{L} = 0.5445057$  (see the figure below). The coils should be placed  $2d = 1.0890L$  apart.

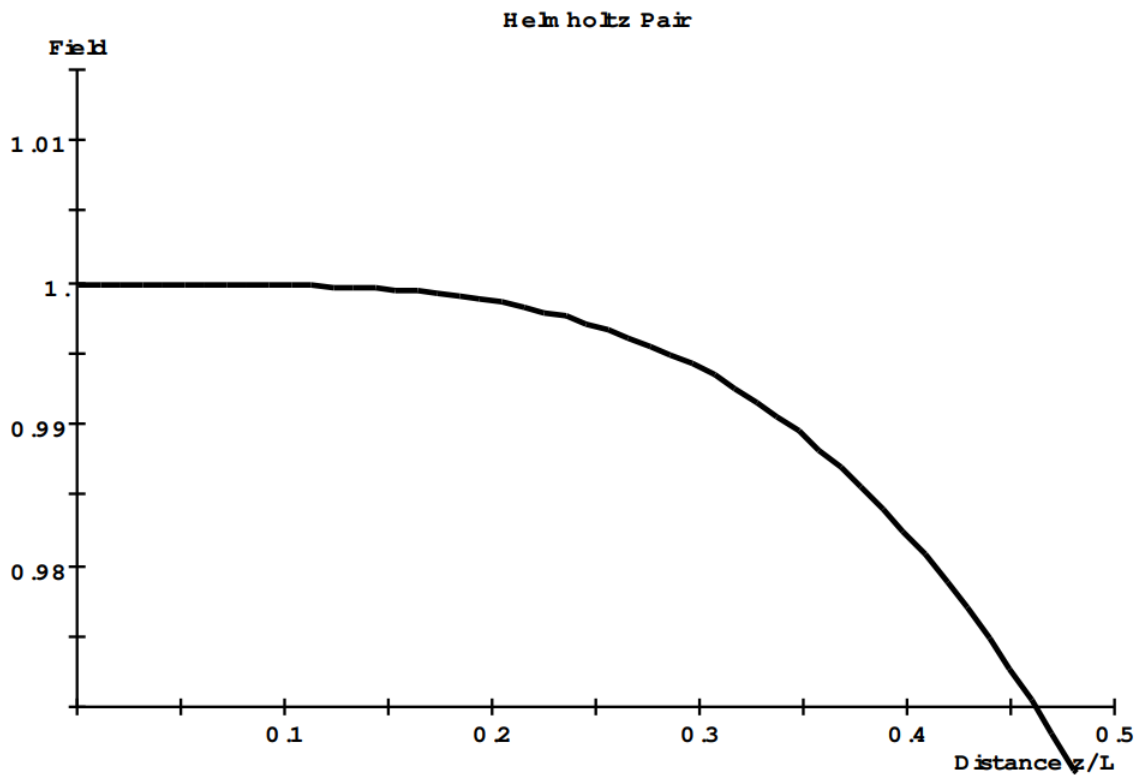


(b) The simplest way to examine the homogeneity is to plot the field function:

$$B_z(\zeta) = \frac{2\mu_0 I}{\pi L} \left( \frac{1}{((\zeta+\eta)^2+1)\sqrt{(\zeta+\eta)^2+2}} + \frac{1}{((\zeta-\eta)^2+1)\sqrt{(\zeta-\eta)^2+2}} \right),$$

where  $\zeta = (z/L)$  and where  $\eta = (d/L) = 0.5445057$ . At the center of the coil system

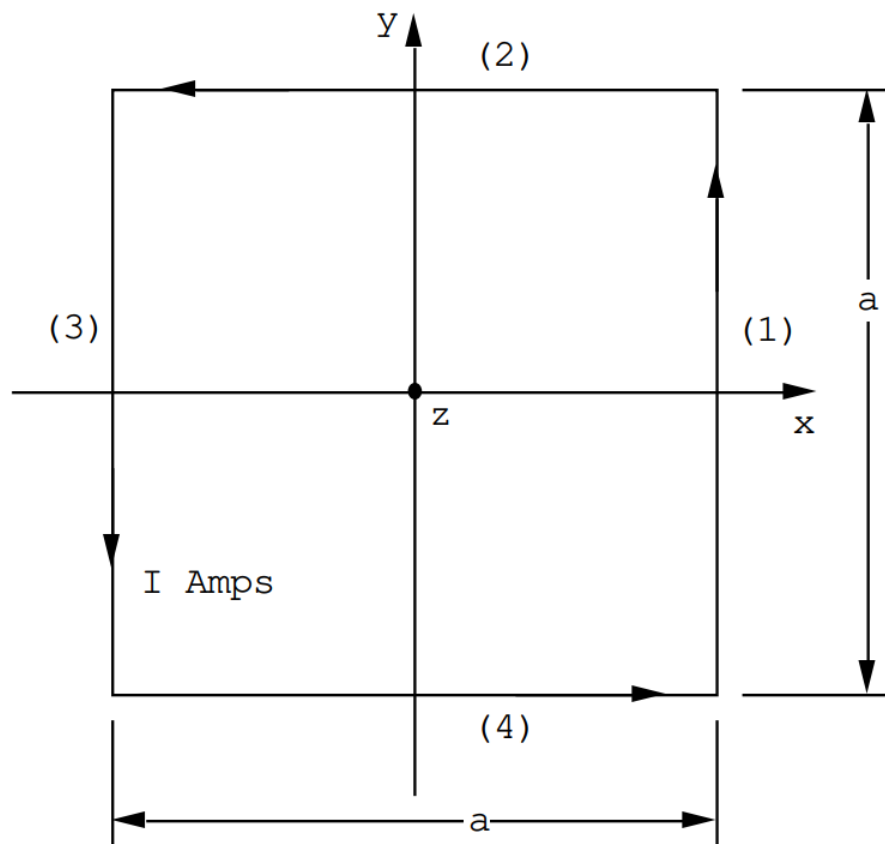
$$B_z(0) = \frac{2\mu_0 I}{\pi L} (1.017958) \text{ Teslas.}$$



From this graph one finds that the field has decreased by 1% when  $\left(\frac{z}{L}\right) = 0.344$ . This means that the field varies by less than 1% over a central region whose length is  $0.688L$ . It turns out that the field is homogeneous to within 1% within a **volume** whose diameter is  $0.688L$ : i.e. within the sphere whose diameter is  $\sim 68$  cm if  $L=1$  meter.

#### Problem (4.7)

Consider a square loop of wire lying in the  $xy$ -plane as shown in the sketch. The loop carries a current of  $I$  amps and is centered on the origin.



(a) Show that the contribution to the vector potential at a point  $P(X,Y,Z)$  from side (1) has only a y component and that this component is given by

$$A_{Y1} = \frac{\mu_0 I}{4\pi} \ln \left( \frac{Y - (a/2) - \sqrt{(X - a/2)^2 + (Y - a/2)^2 + Z^2}}{Y + (a/2) - \sqrt{(X - a/2)^2 + (Y + a/2)^2 + Z^2}} \right).$$

(b) Show that the contribution to the vector potential at a point  $P(X,Y,Z)$  from side (3) has only a y component and that this component is given by

$$A_{Y3} = -\frac{\mu_0 I}{4\pi} \ln \left( \frac{Y - (a/2) - \sqrt{(X + a/2)^2 + (Y - a/2)^2 + Z^2}}{Y + (a/2) - \sqrt{(X + a/2)^2 + (Y + a/2)^2 + Z^2}} \right).$$

(c) Show that the contribution to the vector potential at a point  $P(X,Y,Z)$  from side (2) has only an x component and that this component is given by

$$A_{X2} = -\frac{\mu_0 I}{4\pi} \ln \left( \frac{X - (a/2) - \sqrt{(X - a/2)^2 + (Y - a/2)^2 + Z^2}}{X + (a/2) - \sqrt{(X + a/2)^2 + (Y - a/2)^2 + Z^2}} \right).$$

(d) Show that the contribution to the vector potential at a point  $P(X,Y,Z)$  from side (4) has only an x component and that this component is given by

$$A_{X4} = \frac{\mu_0 I}{4\pi} \ln \left( \frac{X - (a/2) - \sqrt{(X - a/2)^2 + (Y + a/2)^2 + Z^2}}{X + (a/2) - \sqrt{(X + a/2)^2 + (Y + a/2)^2 + Z^2}} \right).$$

(e) Now consider the point  $P(X,0,Z)$  which is specified by the vector  $\mathbf{R} = X\hat{\mathbf{u}}_X + Z\hat{\mathbf{u}}_Z$ . Show that

$$A_x = 0,$$

$$A_y = \frac{\mu_0 I}{4\pi} \ln \left( \left( \frac{1 + \frac{a}{2\sqrt{(x-a/2)^2 + \frac{a^2}{4} + z^2}}}{1 - \frac{a}{2\sqrt{(x-a/2)^2 + \frac{a^2}{4} + z^2}}} \right) \left( \frac{1 - \frac{a}{2\sqrt{(x+a/2)^2 + \frac{a^2}{4} + z^2}}}{1 + \frac{a}{2\sqrt{(x+a/2)^2 + \frac{a^2}{4} + z^2}}} \right) \right).$$

In the limit as  $a/R \rightarrow 0$ , where  $R = \sqrt{X^2 + Z^2}$ , the expression for  $A_y$  can be shown to go to the limit

$$A_y = \frac{\mu_0 I a^2}{4\pi} \left( \frac{X}{R^3} \right) = \frac{\mu_0}{4\pi} \left( \frac{m_z x}{R^3} \right),$$

where  $m_z = I a^2$  Amp-meters<sup>2</sup>. This is just the x-component of the expression  $\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{R}}{R^3}$ , the dipole vector potential.

#### Answer (4.7)

We shall show the calculation for side (1). The procedure for the other three sides is very similar. For side (1) the element of length is given by

$$d\mathbf{L} = dy \hat{\mathbf{u}}_y.$$

This element is located at  $\mathbf{r} = \frac{a}{2} \hat{\mathbf{u}}_x + y \hat{\mathbf{u}}_y$ . The point of observation is located at  $\mathbf{R} = X \hat{\mathbf{u}}_x + Y \hat{\mathbf{u}}_y + Z \hat{\mathbf{u}}_z$ , therefore

$$\mathbf{R} - \mathbf{r} = \left( X - \frac{a}{2} \right) \hat{\mathbf{u}}_x + (Y - y) \hat{\mathbf{u}}_y + Z \hat{\mathbf{u}}_z.$$

The length of this line is given by

$$|\mathbf{R} - \mathbf{r}| = \sqrt{\left( X - \frac{a}{2} \right)^2 + (Y - y)^2 + Z^2}.$$

The contribution to the vector potential at P has only a y-component because the current element has only a y component:

$$dA_y = \frac{\mu_0 I}{4\pi} \frac{dy}{\sqrt{\left( X - \frac{a}{2} \right)^2 + (Y - y)^2 + Z^2}}, \quad \text{and}$$

$$A_y = \frac{\mu_0 I}{4\pi} \int_{-a/2}^{a/2} \frac{dy}{\sqrt{\left( X - \frac{a}{2} \right)^2 + (Y - y)^2 + Z^2}}.$$

This is a standard integral; it can be written

$$A_{y1} = \frac{\mu_0 I}{4\pi} \ln \left( \frac{Y - (a/2) - \sqrt{(X - a/2)^2 + (Y - a/2)^2 + Z^2}}{Y + (a/2) - \sqrt{(X - a/2)^2 + (Y + a/2)^2 + Z^2}} \right).$$

(e) The expansion for  $A_y$  in the limit of  $(a/R) \rightarrow 0$  can be carried out as follows: ( it is convenient to use the notation

$$\sqrt{(x - a/2)^2 + \frac{a^2}{4} + z^2} = \sqrt{x^2 + z^2 - ax + \frac{a^2}{2}} = R_-,$$

and

$$\sqrt{(x + a/2)^2 + \frac{a^2}{4} + z^2} = \sqrt{x^2 + z^2 + ax + \frac{a^2}{2}} = R_+.$$

$$\ln \left( \frac{\left( 1 + \frac{a}{2R_-} \right) \left( 1 - \frac{a}{2R_+} \right)}{\left( 1 - \frac{a}{2R_-} \right) \left( 1 + \frac{a}{2R_+} \right)} \right) = \ln \left( 1 + \frac{a}{2R_-} \right) + \ln \left( 1 - \frac{a}{2R_+} \right) - \ln \left( 1 - \frac{a}{2R_-} \right) - \ln \left( 1 + \frac{a}{2R_+} \right).$$

Expand to first order in small quantities:

$$\cong \frac{a}{R_-} - \frac{a}{R_+},$$

since  $(a/R_-)$  and  $(a/R_+)$  are very small. One can finally write

$$(a/R_-) = \frac{a}{R} \left( 1 - \frac{aX}{R^2} + \frac{a^2}{2R^2} \right)^{-1/2} \cong \frac{a}{R} \left( 1 + \frac{aX}{2R^2} - \frac{a^2}{4R^2} \right),$$

$$(a/R_+) = \frac{a}{R} \left( 1 + \frac{aX}{R^2} + \frac{a^2}{2R^2} \right)^{-1/2} \cong \frac{a}{R} \left( 1 - \frac{aX}{2R^2} - \frac{a^2}{4R^2} \right),$$

where  $R = \sqrt{X^2 + Z^2}$ .

It follows from this that to first order in small quantities

$$A_Y = \frac{\mu_0 I}{4\pi} \frac{a^2 X}{R^3}.$$

#### Problem (4.8)

A short cylindrical solenoid has a radius of  $R = 5 \times 10^{-2}$  meters and a length of  $L = 5 \times 10^{-2}$  meters. It is wound with  $N = 8 \times 10^4$  turns/meter, and the windings carry a current of  $I = 10$  Amps.

(a) What is the magnetic field at the center of the solenoid?

(b) What is the magnetic field strength on the axis of the solenoid but at the end face ( $z = L/2$ )?

#### Answer (4.8)

The magnetic field along the axis of a short solenoid is given by ( $z$  is measured from the solenoid center)

$$B_z(z) = \frac{\mu_0 NI}{2} \left( \frac{(z + L/2)}{\sqrt{(z + L/2)^2 + R^2}} + \frac{(L/2 - z)}{\sqrt{(z - L/2)^2 + R^2}} \right).$$

$$(a) \text{ At } z=0 \quad B_z(0) = \frac{\mu_0 NI}{2} \frac{L}{\sqrt{(L/2)^2 + R^2}}$$

For this problem  $\frac{\mu_0 NI}{2} = 0.503$  Teslas. .

Therefore  $B_z(0) = 0.450$  Teslas.

(b) At  $z = L/2 = 2.5 \times 10^{-2}$  meters:

$$B_z(L/2) = \frac{\mu_0 NI}{2} \left( \frac{L}{\sqrt{L^2 + R^2}} \right) = 0.707 \left( \frac{\mu_0 NI}{2} \right) = 0.356 \text{ Teslas}.$$

#### Problem (4.9)

A short cylindrical disc has a radius of  $R = 5 \times 10^{-2}$  meters and a length of  $L = 5 \times 10^{-2}$  meters. It is uniformly magnetized; the magnetization density is parallel with the axis of the disc, the  $z$ -axis, and the magnetization has the value  $M_0 = 0.955 \times 10^6$  Amps/meter.

(a) What is the magnetic field at the center of the disc?

(b) What is the magnetic field strength on the axis of the disc but at the end face ( $z = L/2$ )?

#### Answer (4.9)

The magnetic field distribution generated by a uniformly magnetized disc is the same as that generated by the windings of a short solenoid. The magnetic field along the axis of a short solenoid is given by

$$B_z(z) = \frac{\mu_0 NI}{2} \left( \frac{(z + L/2)}{\sqrt{(z + L/2)^2 + R^2}} + \frac{(L/2 - z)}{\sqrt{(z - L/2)^2 + R^2}} \right).$$

It is only necessary to replace the product  $NI$  in this formula by the magnetization  $M_0$ .

$$(a) \text{ At } z=0 \ B_z(0) = \frac{\mu_0 M_0}{2} \frac{L}{\sqrt{(L/2)^2 + R^2}} .$$

For this problem  $\frac{\mu_0 M_0}{2} = 0.600$  Teslas.

Therefore  $B_z(0) = 0.537$  Teslas.

(b) At  $z = L/2 = 2.5 \times 10^{-2}$  meters:

$$B_z(L/2) = \frac{\mu_0 M_0}{2} \left( \frac{L}{\sqrt{L^2 + R^2}} \right) = 0.707 \left( \frac{\mu_0 M_0}{2} \right) = 0.424 \text{ Teslas} .$$

#### Problem (4.10).

Given a sphere which is uniformly polarized along the z direction i.e.  $M_z = M_0$  Amps/meter.

- What is  $\mathbf{H}$  inside the sphere?
- What is  $\mathbf{B}$  inside the sphere?
- What is the value of  $B_z$  on the axis of the sphere but just outside the surface of the sphere?
- What is the value of  $\mathbf{H}$  just outside the equator of the sphere?
- A neutron star is typically an object  $10^4$  meters in diameter having the density of nuclear matter ( $\sim 10^{21}$  kg/m<sup>3</sup>). The maximum magnetic field at its surface is estimated to be  $10^8$  Tesla. What is its average magnetization density,  $M_0$ ?
- A neutron has a mass of  $1.68 \times 10^{-27}$  kg. From (e) what is the average magnetic moment of a neutron in a neutron star?

#### Answer (4.10).

(a) The demagnetizing factor for a sphere is  $1/3$ . Therefore  $H_z = -\frac{M_0}{3}$  .

$$(b) B_z = \mu_0 (H_z + M_z) = \frac{2}{3} \mu_0 M_0 .$$

(c) From  $\text{div } \mathbf{B} = 0$  The normal component of  $\mathbf{B}$  must be continuous  $\therefore B_z = \frac{2}{3} \mu_0 M_0$  .

(d) From  $\text{curl } \mathbf{H} = 0$  (there are no currents) the tangential component of  $\mathbf{H}$  must be continuous across the surface of the sphere. It follows that  $H_z = -\frac{M_0}{3}$  Amps/meter at the equator just outside the sphere. From the fact that  $\mathbf{M}$  has no component normal to the surface of the sphere at the equator it follows that the normal component of  $\mathbf{H}$  must be continuous across the surface of the sphere at its equator and therefore  $\mathbf{H}$  has only a z-component just outside the sphere on the equator. Also on the equator just outside the sphere  $B_z = -\frac{\mu_0 M_0}{3}$  . The tangential component of  $\mathbf{B}$  is discontinuous.

(e)  $\frac{2}{3} \mu_0 M_0 = 10^8$  Teslas.

$$\therefore M_0 = 1.19 \times 10^{14} \text{ Amps/m (i.e. Large!!)}$$

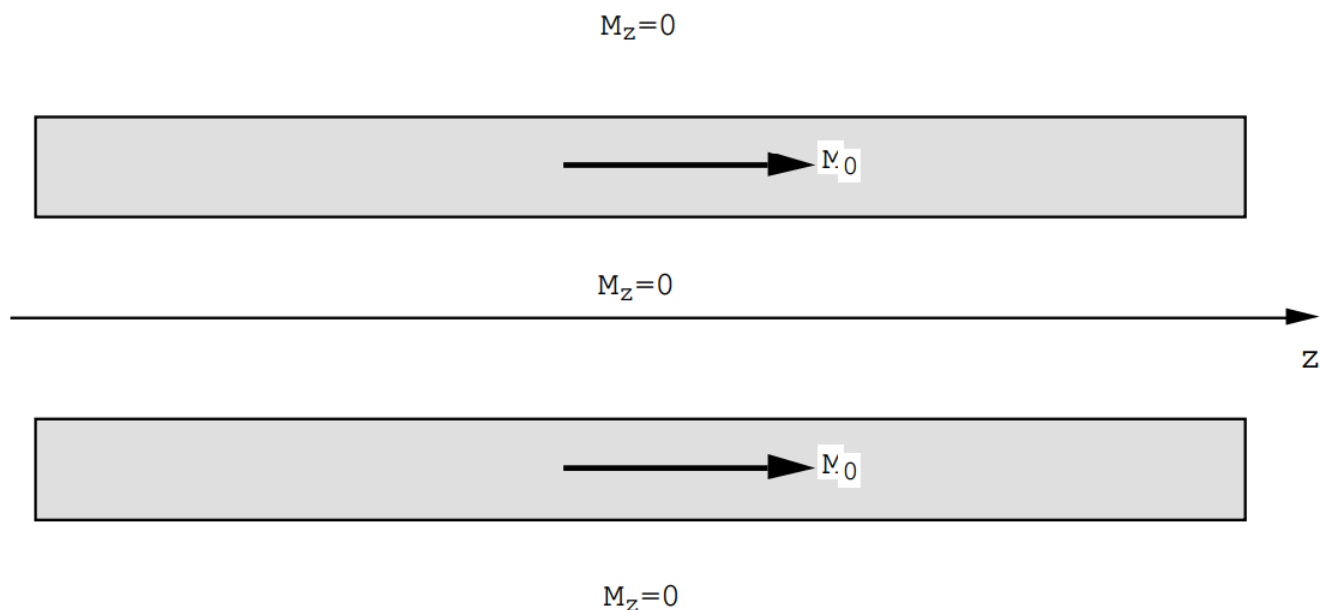
(f) The number of neutrons/m<sup>3</sup> =  $\frac{10^{21}}{1.68 \times 10^{-27}} = 5.95 \times 10^{47}$

$$\therefore \langle \mu_N \rangle = \frac{11.9 \times 10^{13}}{5.95 \times 10^{47}} = 2.0 \times 10^{-34} \text{ Amp m}^2 .$$

The neutron magnetic moment is  $9.7 \times 10^{-27}$  Amp m<sup>2</sup> so that on average only  $2 \times 10^{-8}$  of a neutron is aligned.

#### Problem (4.11)

The material of a very long, hollow, rod is uniformly magnetized as shown in the sketch. (Although the rod is shown as having a finite length in the sketch, it is supposed to be infinitely long).



- (a) What is the value of the magnetic field  $\mathbf{B}$  outside the rod?
- (b) What is the value of the magnetic fields  $\mathbf{H}, \mathbf{B}$  in the central hollow region where  $M_z=0$ ?
- (c) What are the values of  $\mathbf{B}, \mathbf{H}$  in the material of the rod where the magnetization is  $M_0$ ?

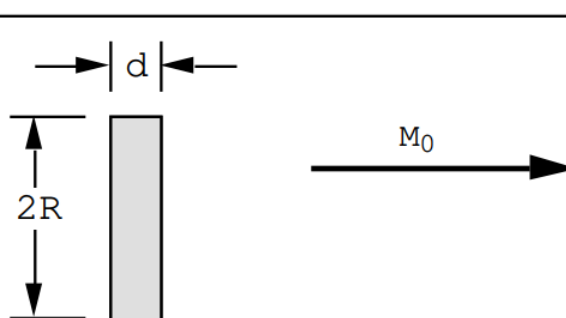
**Answer (4.11)**

By superposition this problem can be reduced to the problem of nested solenoids. The outer surface discontinuity in the tangential component of  $\mathbf{M}$  is equivalent to a solenoid for which  $NI = M_0$ . This current sheet produces a field  $B_1 = \mu_0 M_0$ . The inner surface discontinuity in the tangential component of  $\mathbf{M}$  is equivalent to a solenoid for which  $NI = -M_0$ .

- (a) Outside the rod the fields  $\mathbf{B}, \mathbf{H}$  are both zero.
- (b) In the hollow region the fields due to the two current sheets cancel so that  $\mathbf{B} = \mathbf{H} = 0$ .
- (c) In the region between the two current sheets the  $\mathbf{B}$  field is that due to the outer current sheet;  $B_z = \mu_0 M_0$ . But by definition,  $B_z = \mu_0 (H_z + M_0)$ , and therefore  $H_z = 0$ . Thus  $\mathbf{H} = 0$  everywhere because there are no real currents and no magnetic charge density to generate an  $\mathbf{H}$ -field.

**Problem (4.12)**

An infinitely long rod is uniformly magnetized except for a disc shaped cavity shown shaded in the figure. Inside the cavity the magnetization is zero. What is the magnetic field strength at the center of the cavity?



**Answer (4.12)**

This problem can be worked as the superposition of a uniformly magnetized, infinitely long rod plus a uniformly magnetized disc, but for the disc  $M_z = -M_0$ . For the uniform rod  $B_z = \mu_0 M_0$ . Along the axis of the disc

$$B_z(z) = -\frac{\mu_0 M_0}{2} \left( \frac{(z + d/2)}{\sqrt{(z + d/2)^2 + R^2}} + \frac{(d/2 - z)}{\sqrt{(z - d/2)^2 + R^2}} \right),$$

and at  $z=0$

$$B_z(0) = -\frac{\mu_0 M_0}{2} \frac{d}{\sqrt{(d/2)^2 + R^2}}.$$

The total field at the center of the disc will be

$$B_z(0) = \mu_0 M_0 \left( 1 - \frac{d}{2\sqrt{(d/2)^2 + R^2}} \right).$$

In the limit  $(d/R) \rightarrow 0$  the field at the center of the cavity is just  $B_0 = \mu_0 M_0$ .

#### Problem (4.13)

A uniformly magnetized ellipsoid possesses magnetization components

$$M_X = 2 \times 10^5 \text{ Amps/meter},$$

$$M_Y = 2 \times 10^5 \text{ Amps/meter}$$

$$M_Z = 4 \times 10^5 \text{ Amps/meter}$$

when referred to the principle axes of the ellipsoid. Demagnetizing coefficients for the ellipsoid are

$$N_X = 0.2,$$

$$N_Y = 0.3.$$

(a) Calculate the components of **H** inside the ellipsoid.

(b) Calculate the components of **B** inside the ellipsoid.

(c) Calculate the angle between **B** and **M**.

#### Answer (4.13)

The demagnetizing coefficients obey the sum rule

$$N_x + N_y + N_z = 1.$$

For this problem

$$N_x = 0.20,$$

$$N_Y = 0.30,$$

$$N_z = 0.50.$$

$$H_X = -N_X M_X = -0.40 \times 10^5 \text{ Amps/meter}.$$

$$(a) H_Y = -N_Y M_Y = -0.60 \times 10^5 \text{ Amps/meter}.$$

$$H_Z = -N_Z M_Z = -2.00 \times 10^5 \text{ Amps/meter}.$$

(b) **B** =  $\mu_0(\mathbf{H} + \mathbf{M})$ , therefore

$$B_X = \mu_0 (H_X + M_X) = 0.201 \text{ Teslas}.$$

$$B_Y = \mu_0 (H_Y + M_Y) = 0.176 \text{ Teslas}.$$

$$B_z = \mu_0 (H_Z + M_Z) = 0.251 \text{ Teslas}.$$

(c) **B** · **M** =  $|\mathbf{B}||\mathbf{M}| \cos \theta$ ;

$$|\mathbf{M}| = 4.899 \times 10^5 \text{ Amps/meter},$$

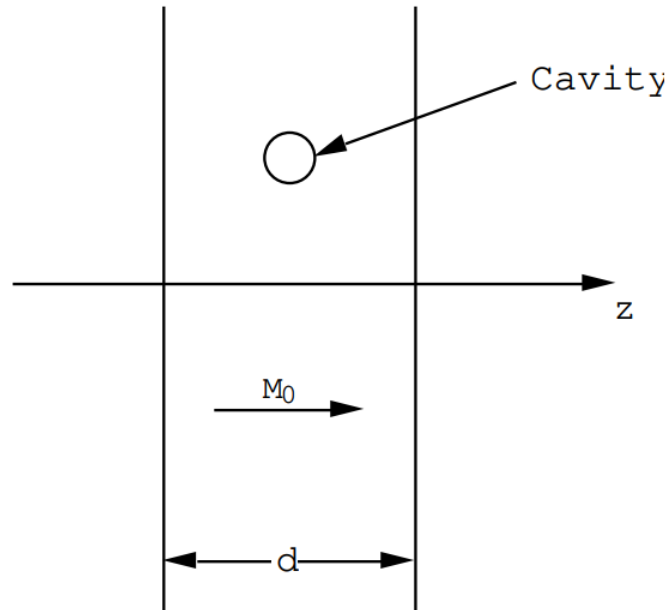
$$|\mathbf{B}| = 0.367 \text{ Teslas},$$

$$\cos \theta = \frac{M_x B_x + M_y B_y + M_z B_z}{|\mathbf{M}||\mathbf{B}|} = \frac{1.758}{(4.899)(.3667)} = 0.9786.$$

So  $\theta = 11.9^\circ$ .

**Problem (4.14)**

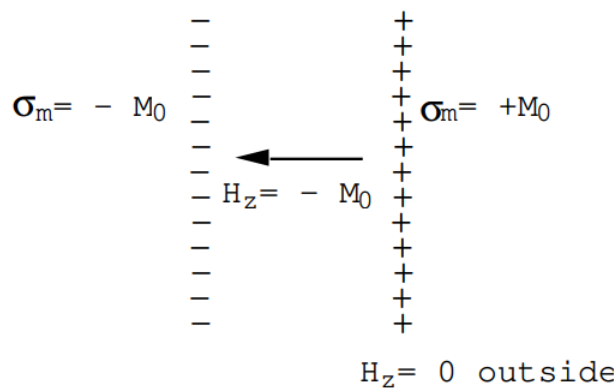
A very large disc whose radius is infinite is magnetized along its normal as shown in the figure.



- What is  $\mathbf{H}$  in the disc?
- What is  $\mathbf{H}$  outside the disc?
- What is  $\mathbf{B}$  inside the slab?
- A spherical cavity is cut out of the material of the disc. Use the principle of superposition to calculate the magnetic field  $\mathbf{B}$  in the cavity.

**Answer (4.14)**

- The demagnetizing factor for the direction along the disc normal is  $N_z = 1$ . Therefore  $H_z = -M_0$ .
- Outside the disc the field is zero by analogy with the equivalent electrostatic problem i.e. two infinite charge sheets



- $B_z = \mu_0 (H_z + M_z) \equiv 0$ .
- Inside a uniformly polarized sphere  $B_z = \frac{2}{3}\mu_0 M_0$ . Therefore in the cavity one must have  $B_z = -\frac{2}{3}\mu_0 M_0$  so that the sum of the two fields gives zero when the sphere is put into the hole.

**Problem (4.15)**

A very long cylinder of magnetic material has a radius  $R$ . The axis of the cylinder lies along the  $z$ -axis. The magnetization depends upon the distance from the cylinder axis:

$$M_z = M_0(r/R) \quad \text{Amps / meter.}$$

- (a) Calculate the effective current density curl  $\mathbf{M}$  both inside and outside the cylinder.
- (b) Note that there is an effective surface current density on the surface of the cylinder due to the discontinuity in the tangential component of the magnetization. Calculate this surface current density,  $J_s$ .
- (c) Calculate the radial dependence of the magnetic field in the cylinder.

**Answer (4.15)**

$$\text{curl } \mathbf{M} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{u}}_r & r\hat{\mathbf{u}}_\theta & \hat{\mathbf{u}}_z \\ \frac{\partial}{\partial r} & 0 & 0 \\ 0 & 0 & M_z \end{vmatrix} = \begin{vmatrix} 0 \\ -\frac{\partial M_z}{\partial r} \\ 0 \end{vmatrix},$$

(There is no angular or  $z$  dependence).

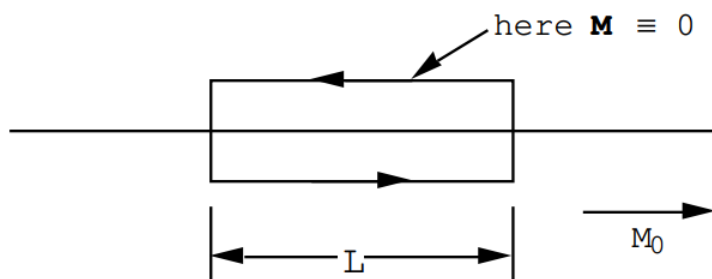
(a)  $\therefore J_\theta = -\frac{\partial M_z}{\partial r} = -M_0/R$  ie. independent of position.

(b) At the outer surface there is a discontinuity in the tangential component of  $\mathbf{M}$ . Use Stokes' theorem to obtain the effective surface current density:

$$\mathbf{J}_b = \text{curl } \mathbf{M}$$

$$\therefore \int_{\text{surface}} \mathbf{J}_b \cdot d\mathbf{s} = \int_{\text{surface}} \text{curl } \mathbf{M} \cdot d\mathbf{s} = \oint_C \mathbf{M} \cdot d\mathbf{L}$$

Apply this to the loop shown below:



$$\text{Current through the loop } \int \mathbf{J}_b \cdot d\mathbf{s} = J_s L$$

$J_s$  is the effective surface current density.

$$\oint_C \mathbf{M} \cdot d\mathbf{L} = M_0 L$$

$$\therefore J_s L = M_0 L$$

$$\text{or } J_s = M_0 \text{ Amps/m.}$$

(c) Calculate the field along the axis of the cylinder. By symmetry there is only a  $z$ -component which is independent of  $z$ . The uniform effective current density,  $-\frac{M_0}{R}$ , can be treated like a nested solenoid problem in order to calculate the magnetic field along the cylinder axis.

$$\text{The effective current sheet strength is } NI = -\left(\frac{M_0}{R}\right) dr \quad \text{Amps/m.}$$

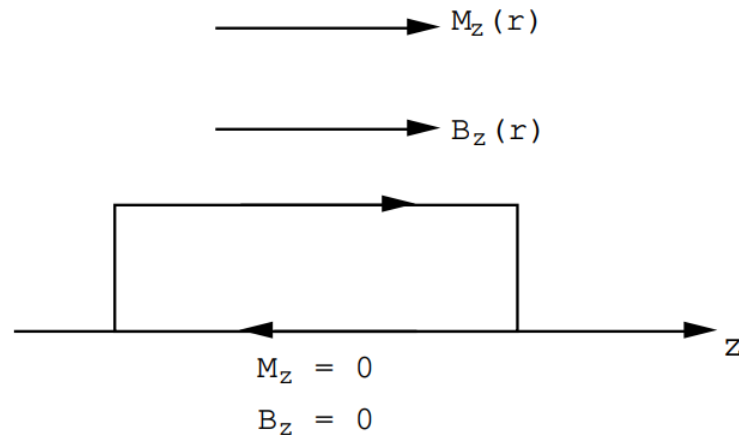
$$\text{This produces the solenoid field contribution } dB_z = -\mu_0 \left(\frac{M_0}{R}\right) dr.$$

Integrate from  $r = 0$  to  $r = R$ :  $B_z = -\mu_0 M_0$  Tesla. However, this is just cancelled by the surface current sheet which produces  $B_z = \mu_0 M_0$  Tesla.

$$\therefore \text{On the axis } B_z = 0.$$

Now use  $\text{curl } \mathbf{B} = \mu_0 \text{curl } \mathbf{M}$

or  $\oint_C \mathbf{B} \cdot d\mathbf{L} = \mu_0 \oint_C \mathbf{M} \cdot d\mathbf{L}$  and integrate around the loop shown in the figure:



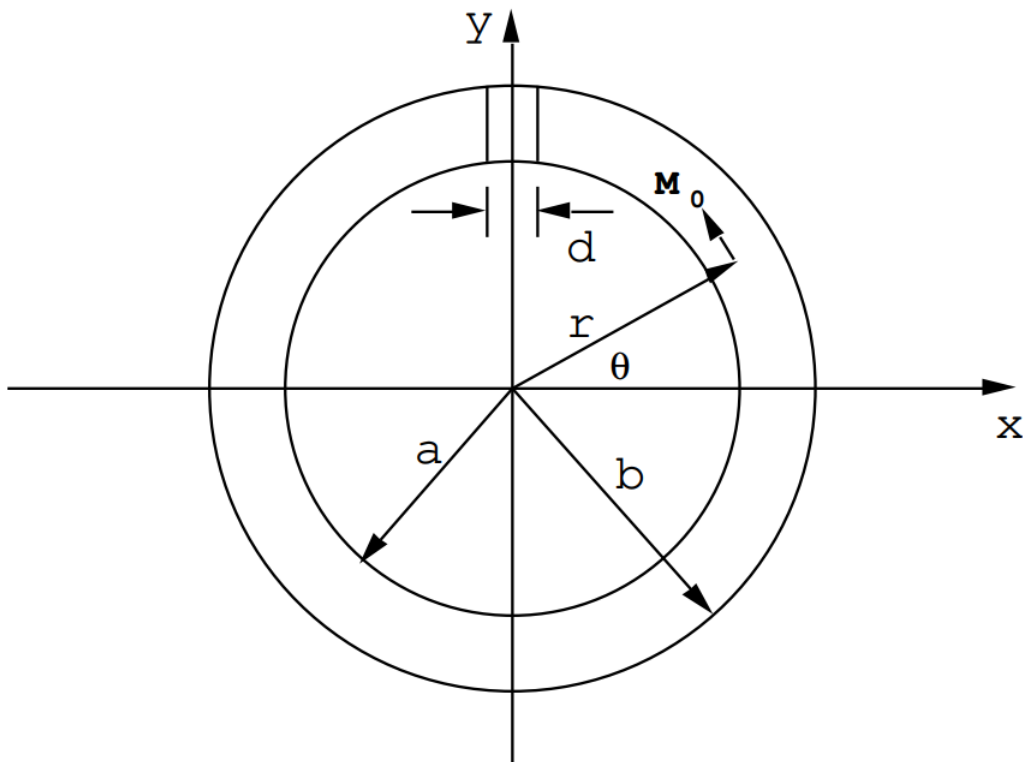
from which  $B_z = \mu_0 M_z$

$\therefore B_z(r) = \mu_0 M_0(r/R)$

and  $H_z = 0$  everywhere.

#### Problem (4.16)

A permanent magnet is formed in the shape of a dough-nut having an inner radius  $a$  meters and an outer radius of  $b$  meters (see the figure). The magnetization density has the components  $M_r=0$ ,  $M_\theta = M_0$ ,  $M_z=0$  in cylindrical polar coordinates, where  $M_0$  is constant.



(a) Calculate the field  $\mathbf{H}$  everywhere.

(Answ:  $\text{div } \mathbf{M}=0$  everywhere, and there are no free currents. Therefore there are no sources for  $\mathbf{H}$  and consequently  $\mathbf{H}=0$  everywhere.)

(b) Suppose that a gap  $d$  meters wide is opened in the ring as shown in the figure. Calculate the field  $\mathbf{B}$  at the center of the gap.

$$(\text{Answ: } B_0 = \mu_0 M_0 \left( 1 - \frac{d}{\sqrt{4(b-a)^2 + d^2}} \right) \text{ Teslas. } )$$

#### Answer (4.16)

A uniform magnetic charge density will appear on the faces of the cut due to the discontinuity in  $\mathbf{M}$ . The surface charge density on the left hand face is  $+M_0/m^2$ ; the surface charge density on the right hand face is  $-M_0/m^2$ . These charge distributions produce a field at the gap center given by

$$H_0 = \frac{M_0 d}{4} \int_0^R 2r dr \frac{1}{(r^2 + (d/2)^2)^{3/2}}$$

where  $R = (b-a)/2$ .

$$H_0 = M_0 \left( 1 - \frac{d}{\sqrt{(b-a)^2 + d^2}} \right),$$

$B_0 = \mu_0 H_0$  directed along  $M_0$ , ie along  $-x$  in the above figure. This problem can also be solved by treating the magnetized plug removed from the gap as a short solenoid: for a short solenoid of radius  $R = (b-a)/2$  and of length  $d$  the field at its center is given by

$$B_x = -\frac{\mu_0 M_0}{2} \frac{d}{\sqrt{R^2 + (d/2)^2}}.$$

This field plus the gap field,  $B_x^G$ , must equal the field in the gapless ring,  $-\mu_0 M_0$ , by superposition. Therefore

$$B_x^G = -\mu_0 M_0 \left( 1 - \frac{d/2}{\sqrt{(d/2)^2 + R^2}} \right) = -\mu_0 M_0 \left( 1 - \frac{d}{\sqrt{d^2 + (b-a)^2}} \right),$$

the same answer as above.

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