3.2: Soluble Problems

3.2.1 (1) Orthogonal Systems.

The only problems that can be solved analytically are those for which the conducting electrodes can be described by u1=const., or u2=const., or u3=const. where u1, u2, u3 form a system of orthogonal co-ordinates. Stratton discusses eight such orthogonal systems (Electromagnetic Theory by Julius Adams Stratton, McGraw-Hill, New York, 1941). We shall be interested only in the three most commonly used systems (1) cartesian co-ordinates, (2) cylindrical polar co-ordinates, and (3) spherical polar co-ordinates.

(a) Cartesian Co-ordinates.

In the Cartesian co-ordinate system LaPlace's equation becomes

$$\frac{\partial^2 \mathrm{V}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{V}}{\partial \mathrm{y}^2}+\frac{\partial^2 \mathrm{V}}{\partial \mathrm{z}^2}=0.$$

Let the surfaces of two semi-infinite electrodes lie at z=0 and at z=D, Figure (3.2.2). In this case the potential function must be independent of x and y by symmetry: the potential on any plane z=constant must be featureless because there are no edges with which to locate oneself in the plane. In other words, any shift of the electrodes in the x-y plane does not change the geometry of the problem. Thus LaPlace's equation is reduced to

$$\frac{\partial^2 \mathrm{V}}{\partial \mathrm{z}^2}=0.$$

This simple equation has the general solution

$$\mathrm{V}(\mathrm{z})=A+Bz, \tag{3.2.1}$$

where A,B are constants that must be determined from the boundary conditions. For z=0 the potential is required to be V_1 and therefore A = V_1. For z=D the potential must equal V_2 and therefore B = $(V_2 - V_1)/D$. Thus the required potential function for this problem is given by

$$\mathrm{V}(\mathrm{z})=\mathrm{V}_1+\frac{(\mathrm{V}_2-\mathrm{V}_1)\, z}{\mathrm{D}},$$

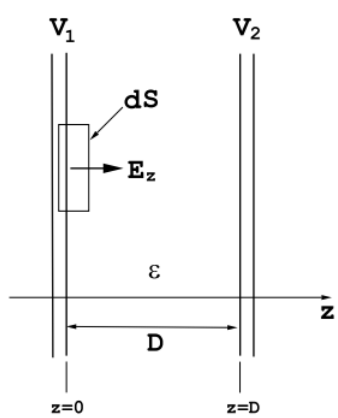

Figure 3.2.2: Two plane parallel, semi-infinite metal electrodes separated by a distance D. The electrode potentials are V_1 and V_2. The space between the electrodes is filled with a material having a dielectric constant ϵ.

and this solution is unique. It corresponds to an electric field whose components are:

$$\begin{gathered}\mathrm{E}_{\mathrm{x}}=0\\ \mathrm{E}_{y}=0\\ \text{and}\quad \mathrm{E}_{\mathrm{z}}=\frac{(\mathrm{V}_1-\mathrm{V}_2)}{\mathrm{D}}\quad \text{Volts}/\mathrm{m}.\end{gathered}$$

The electric field is forced to be uniform simply because the potential function has no spatial variation along x or y.

The surface charge density on each electrode must also be independent of x and y, and the charge density on the two electrodes are equal in magnitude but opposite in sign. They may be calculated by means of Gauss' theorem. Consider a small pillbox that spans an electrode surface such as that shown in Figure (3.2.2). According to Gauss' Theorem the surface integral of the normal component of D over the pillbox is equal to the total free charge contained within the pillbox:

$$\mathrm{Q}=\rho_s dS=\int\int_{Pillbox}\overrightarrow{\mathrm{dS}}\cdot\vec{\mathrm{D}}.$$

But $\vec{\mathrm{D}}=\epsilon\vec{\mathrm{E}}$ so that the surface integral of $\vec{D}$ can be written as a surface integral of $\vec{E}$. Since $\vec{E}$ is zero inside the metal electrode it follows that the only contribution to the surface integral comes from the surface of the pillbox that lies in the dielectric; the surface integral of $\vec{E}$ gives dSE_z. Thus one finds

$$\mathrm{Q}=\mathrm{dS}\rho_s=\epsilon\mathrm{dSE}_z$$

and therefore

$$\rho_s=\epsilon\frac{(\mathrm{V}_1-\mathrm{V}_2)}{\mathrm{D}}.$$

This expression can be used to estimate the relation between total charge and voltage difference on a parallel plate capacitor. Consider two parallel plate electrodes each having an area of A meters^2 , and let the charge on one plate be Q Coulombs and on the other plate be -Q Coulombs. If edge effects are neglected, and if it is assumed that the charge density is uniform, one can write ρ_s = Q/A. It follows that

$$\mathrm{Q}=\left(\frac{\epsilon\mathrm{A}}{\mathrm{D}}\right)(\mathrm{V}_1-\mathrm{V}_2),$$

or writing $(V_1 - V_2) = \Delta V$,

$$\mathrm{C}=\frac{\mathrm{Q}}{\Delta\mathrm{V}}=\left(\frac{\epsilon\mathrm{A}}{\mathrm{D}}\right)\quad\text{Farads. .}$$

C is the capacitance of the parallel plate system.

Variations of this problem involve two regions having different dielectric constants, see Figure (3.2.3).

The potential function in each region must satisfy Laplace's equation (3.1.11), and therefore in the infinite plate approximation

$$\frac{\partial^2\mathrm{V}}{\partial\mathrm{z}^2}=0,$$

since the potential function can not depend upon the co-ordinates x and y. It follows from $E_z = -dV/dz$ that the electric field in each region must be

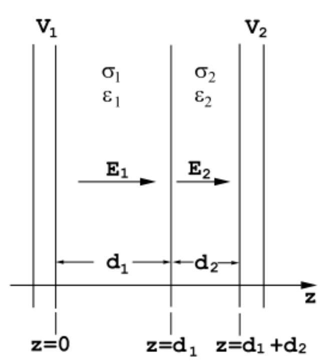

Figure 3.2.3: The parallel plate capacitor problem with two different dielectric materials. The electric field in each region is independent of position.

independent of position. The magnitude of the electric field in each region depends upon whether or not the dielectric material is, or is not, conducting. There are two main limiting cases: (1) the conductivity, σ in each region is zero; and (2) each region is conducting with a current density given by $\vec{J} = \sigma\vec{E}$.

Case(1). If the conductivities are zero there can be no free charges anywhere in the dielectric materials. As a consequence it follows from Maxwell's equation(2.3.2) that $\vec{D}$ must be divergence free, ie div($\vec{D}$) = 0. This in turn means that the normal component of $\vec{D}$ must be continuous across the interface between the two dielectrics, or $D_1 = D_2$. This implies that

$$\epsilon_1 E_1 = \epsilon_2 E_2. \tag{3.2.2}$$

But also

$$d_1 E_1 + d_2 E_2 = (V_1 - V_2) = \Delta V. \tag{3.2.3}$$

These two equations, Equations (3.2.2 and 3.2.3) can be solved to obtain the electric field strengths in each region of the dielectric insulators. The charge density on each electrode has the magnitude $\rho_s = D_1 = D_2$.

Case(2). If the dielectric materials are conducting the current density in each region must be the same in the steady state in order to prevent a time dependent build up of charge at the interface between the two dielectric slabs. But $J_1 = \sigma_1 E_1$ and $J_2 = \sigma_2 E_2$ so that

$$\sigma_1 E_1 = \sigma_2 E_2. \tag{3.2.4}$$

Equation(3.2.4) replaces Equation (3.2.2) which is only valid providing that there is no charge flow through the dielectric slabs. Eqns.(3.2.3 and 3.2.4) form a system of two equations that may be solved for the two unknowns E_1 and E_2. Notice that for this case of conducting materials the displacement vector will have a different value in each of the two regions:

$$D_1 = \epsilon_1 E_1 = \epsilon_1 \left(\frac{\sigma_2}{\sigma_1}\right) E_2,$$

and

$$D_2 = \epsilon_2 E_2.$$

Notice that the free surface charge density on each electrode will be different in magnitude because $\rho_s = D_1$ for the positive electrode and $\rho_s = -D_2$ for the negative electrode. The surface free charge density at the interface between the two dielectric slabs is given by $\rho_s = (D_2 - D_1)$.

(b) A Leaky Capacitor.

The potential function for a leaky capacitor is the same as the potential function for a non-leaky capacitor because in both cases the potential must satisfy Equation (3.1.11), $\nabla^2 V = 0$, and in both cases the potential must satisfy the same boundary conditions. In the infinite electrode approximation in which edge effects are neglected the plane symmetry requires that the potential function have the form $V = a + bz$, Equation (3.2.1), where a and b are constants. This means that $E_z = -(\partial V/\partial z)$ must be independent of position. If the potential difference between the electrodes is ΔV, see Figure (3.2.4), the electric field strength is $E = \Delta V/d$, and the corresponding strength of the displacement vector is $D = \epsilon E = \epsilon \Delta V/d$. But from Gauss' Theorem

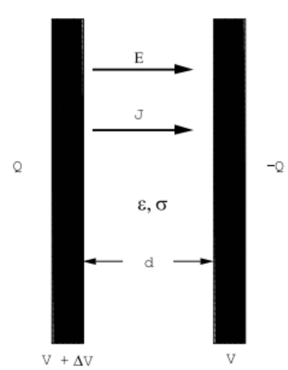

Figure 3.2.4: Charge decay through a leaky capacitor. $\epsilon = \epsilon_r \epsilon_0$ is the dielectric constant for the spacer material. σ is the conductivity of the spacer material.

applied to div($\vec{D}$) = ρ_f it follows that the surface charge density on the positive electrode is given by $\sigma_f = D = \epsilon \Delta V/d = Q/A$, where Q is the total charge on the electrode and A is the electrode area. (Do not confuse the free charge density, σ_f, with the conductivity, σ). The capacitance is defined by $C = Q/\Delta V$ so

$$C = \frac{\epsilon A}{d} \quad \text{Farads},$$

exactly the same formula as for a non-leaky capacitor! However, there is a flow of charge between the two electrodes of a leaky capacitor. The current density is given by

$$J = \sigma E = \frac{\sigma \Delta V}{d} = \frac{\sigma Q}{\epsilon A}.$$

The total current is

$$I = JA = \frac{\sigma Q}{\epsilon} \quad Amps.$$

Unless the current is maintained by some external source such as a battery this current flow must deplete the electrodes of charge. For an isolated capacitor the charge on the positive electrode must change with time according to the equation of charge conservation:

$$\frac{dQ}{dt} = -I = -\frac{\sigma Q}{\epsilon}.$$

The solution of this differential equation is

$$Q(t) = Q_0 \exp -\sigma t/\epsilon. \tag{3.2.5}$$

Thus the charge on a leaky capacitor dies away exponentially with a time constant, τ, given by

$$\tau = \frac{\epsilon}{\sigma} = \rho\epsilon \quad \text{seconds}, \tag{3.2.6}$$

where ρ is the resistivity of the material between the conducting electrodes (not the charge density!), see Figure (3.2.4). Whether or not a capacitor should be treated as leaky depends entirely upon the time scale associated with the problem. For most materials the relative dielectric constant, ϵ_r, lies between 1 and 10, so that differences in the intrinsic time constant, τ, from one material to another are determined primarily by the resistivity. Resistivities for some selected materials are listed in Table (3.2.1). The most striking feature of this Table is the wide range of resistivities exhibited by these solid materials. It is clear that the best candidates for an insulating dielectric material listed in the Table are yellow sulphur and paraffin wax.

(c) Cylindrical Co-ordinates.

Consider a problem that exhibits cylindrical symmetry so that the potential function does not depend upon the z co-ordinate. LaPlace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0. \tag{3.2.7}$$

The general solution of this equation can be written

$$V(r,\theta) = a + b\ln r + \sum_{n=1}^{\infty}\left(a_n r^n + \frac{b_n}{r^n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(c_n r^n + \frac{d_n}{r^n}\right)\sin n\theta, \tag{3.2.8}$$

where a_n, b_n, c_n, and d_n are arbitrary constants. The series (3.2.8) satisfies the equation $\nabla^2 V = 0$ term by term as can be verified by direct differentiation.

Table 3.2.1: Resistivities and time constants for some selected materials at a temperature of 20C. (Handbook of Chemistry and Physics, 53rd Ed., CRC Press (1972). The dielectric constant has been taken to be ε_0 for convenience.

Material ρ (Ohm m)	**Material ρ (Ohm m)**	$\tau_0 = \epsilon_0\rho(\text{ seconds })$
Copper	1.67×10^{-8}	1.48×10^{-19}
Intrinsic Ge	0.46	4.1×10^{-12}
Boron	1.8×10^{4}	1.6×10^{-7}
Yellow Sulpher	2×10^{15}	1.8×10^{4}= 4.9 hours

Material ρ (Ohm m)	Material ρ (Ohm m)	$\tau_0 = \epsilon_0 \rho($ seconds $)$
Pyrex 7060	1.3×10^5	1.2×10^{-6}
Pyrex 1710	2.5×10^7	2×10^{-4}
Fused Silica	$\sim 10^8$	$\sim 9 \times 10^{-4}$
Beeswax	$\sim 10^{13}$	~ 89
Paraffin	$10^{13} - 10^{17}$	$89 - 8.9 \times 10^5$ (up to 247 hours)
Wet Ground	$10^2 - 10^3$	$10^{-9} - 10^{-8}$

The constants in (3.2.8) have to be chosen so as to satisfy the boundary conditions for a particular problem. The term *b ln r* corresponds to the potential generated by an infinite line charge, see Section (2.7.3) of Chpt.(2). A line charge of strength ρ_L Coulombs/meter in free space generates the potential

$$\mathrm{V(r)} = -\left(\frac{\rho_L}{2\pi\epsilon_0}\right)\ln \mathrm{r}. \tag{3.2.9}$$

If the line charge is immersed in a medium of dielectric constant ϵ then ϵ_0 must be replaced by ϵ in Equation (3.2.9).

The term $V_1 = (b_1/r)\cos\theta$ corresponds to a line of dipoles in which the dipole moment is oriented along the x-axis, see Chpt.(2), Section (2.7.3). The potential generated by a line of dipoles in free space and having a strength of P_x Coulombs is given by

$$\mathrm{V_x(r,\theta)} = \frac{\mathrm{P_x}\cos\theta}{2\pi\epsilon_0 \mathrm{r}}.$$

If the dipole moments are oriented along the y-axis the potential is given by

$$V_y(r,\theta) = \frac{P_y \sin\theta}{2\pi\epsilon_0 r}:$$

this is one of the terms proportional to sin θ in Equation (3.2.8).

The terms $a_1 r\cos(\theta)$ and $c_1 r\sin(\theta)$ in (3.2.8) correspond to uniform fields along x and y. This can be seen by using the substitutions

$$x = r\cos\theta,$$

and

$$y = r\sin\theta,$$

to obtain $V_x = a_1 x$, corresponding to an electric field $E_x = -a_1$, and $V_y = c_1 y$, corresponding to the electric field $E_y = -c_1$. Such terms are appropriate for discussing the problem of a uniform dielectric cylinder immersed in a uniform applied electric field, Figure (3.2.5). If the applied electric field, E_0, is taken to lie along the x-direction it is clear from the symmetry of the problem that the potential function must be symmetric in y: a reflection of the system through the xz plane gives one exactly the same problem. This implies that the potential for θ and for −θ must be the same. This being the case, the amplitudes of all the sin nθ terms in the expansion (3.2.8) must be zero for

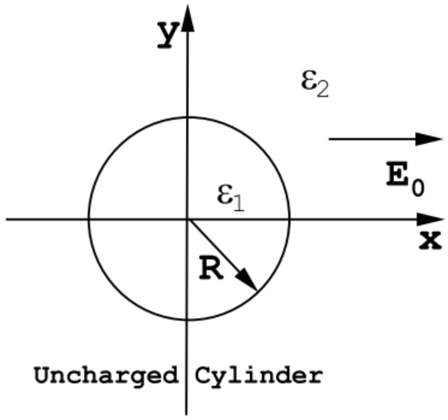

Figure 3.2.5: A uniform cylinder, infinitely long in the z-direction, and immersed in a uniform electric field $E_x = E_0$. The cylinder is characterized by a dielectric constant ϵ_1. It is situated in a medium whose dielectric constant is $\epsilon 2$.

each n because $\sin(n\theta)$ is an odd function of its argument. Furthermore, the cylinder is uncharged and therefore the amplitude, b, of the ln(r) term in the expansion must be zero because b is proportional to the strength of the line charge, ρ_L, which generates this term in the potential. A second argument requires this term to be absent in the potential function for this problem: the potential function inside the cylinder is required to remain finite at r=0, and ln(r) diverges at r=0. In addition, the potential function in the region outside the cylinder is required to approach the value corresponding to a uniform electric field, $V(r,\theta) = -E_0 r \cos(\theta)$, for very large distances r, whereas ln(r) diverges at large r much more slowly. On the basis of these arguments one may conclude that the potential function required for the problem of a cylinder immersed in a uniform electric field must have the form

$$\mathrm{V}(\mathrm{r},\theta) = a + a_1 \mathrm{r}\cos\theta + \frac{b_1\cos\theta}{\mathrm{r}} + a_2\mathrm{r}^2\cos 2\theta + \frac{b_2\cos 2\theta}{\mathrm{r}^2} + \cdots \tag{3.2.10}$$

In the limit as $r \to \infty$ the potential function outside the cylinder, V_0, must converge to the potential corresponding to a uniform electric field, E_0, along the x direction. That is

$$\lim_{r\to\infty} \mathrm{V}_0 \to -\mathrm{E}_0\mathrm{r}\cos\theta.$$

This condition requires all the terms an to vanish for n > 1. It also requires $a_1 = -E_0$.

Inside the cylinder the potential function, V_i , must remain finite in the limit as $r \to 0$: there are no charges inside the cylinder to produce any singularity in the potential. Thus inside the cylinder all the terms proportional to $1/r^n$ must vanish for all n. These considerations now leave the following possibilities for the potential functions inside and outside the cylinder:

Inside(r ≤ R)

$$\mathrm{V}_\mathrm{i}(\mathrm{r},\theta) = a + a_1\mathrm{r}\cos\theta + a_2\mathrm{r}^2\cos 2\theta + a_3\mathrm{r}^3\cos 3\theta\cdots$$

Outside(r ≥ R)

$$\mathrm{V}_0(\mathrm{r},\theta) = A - \mathrm{E}_0\mathrm{r}\cos\theta + \frac{b_1\cos\theta}{\mathrm{r}} + \frac{b_2\cos 2\theta}{\mathrm{r}^2} + \frac{b_3\cos 3\theta}{\mathrm{r}^3} + \cdots$$

These two series for the potential functions inside and outside the cylinder must be matched on the surface of the cylinder in order to satisfy two conditions: (1) the tangential component of $\vec{E}$ must be continuous across the interface (from curl$\vec{E}$ = 0); and (2) the normal component of $\vec{D}$ must be continuous across the interface at r=R because there is no free charge density (from div($\vec{D}$) = ρ_f). Condition (1) will obviously be satisfied if the potential function is forced to be continuous at r=R,

$$\mathrm{V}_\mathrm{i}(\mathrm{R},\theta) = \mathrm{V}_0(\mathrm{R},\theta).$$

Condition (2) requires that

$$\epsilon_1\left(\frac{\partial V_i}{\partial r}\right)_{r=R} = \epsilon_2\left(\frac{\partial V_0}{\partial r}\right)_{r=R}.$$

These two conditions must be satisfied for every angle θ, and this means that they must be separately satisfied for each term cos (nθ) in the above two series. For example:

n=0

$$a = A$$

n=1

$$a_1 \mathrm{R} \cos\theta = \left(-\mathrm{E}_0 \mathrm{R} + \frac{b_1}{\mathrm{R}}\right) \cos\theta,$$

and

$$\epsilon_1 a_1 \cos\theta = -\epsilon_2 \left(\mathrm{E}_0 + \frac{b_1}{\mathrm{R}^2}\right) \cos\theta.$$

These last two equations can be solved to give

$$a_1 = \frac{-2\epsilon_2 \mathrm{E}_0}{(\epsilon_1 + \epsilon_2)}, \tag{3.2.11}$$

$$\frac{b_1}{\mathrm{R}^2} = \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) \mathrm{E}_0. \tag{3.2.12}$$

n=2

$$a_2 \mathrm{R}^2 \cos 2\theta = \frac{b_2}{\mathrm{R}^2} \cos 2\theta,$$

and

$$2\epsilon_1 a_2 R \cos 2\theta = \frac{-2\epsilon_2 b_2}{R^3} \cos 2\theta.$$

The latter two equations have only the solution $a_2 = b_2 = 0$. These procedures can be continued for all n with the result that all coefficients for n ≥ 2 are zero. The potential function for the problem of an infinite cylinder subjected to a uniform applied field turns out to be rather simple:

Inside the cylinder (r ≤ R)

$$V_i(r, \theta) = a - \frac{2\epsilon_2 E_0 r \cos\theta}{(\epsilon_1 + \epsilon_2)}. \tag{3.2.13}$$

Outside the cylinder (r ≥ R)

$$\mathrm{V}_0(\mathrm{r}, \theta) = a - \mathrm{E}_0 \mathrm{r} \cos\theta + \mathrm{R}^2 \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) \frac{\mathrm{E}_0 \cos\theta}{\mathrm{r}}. \tag{3.2.14}$$

The constant a has no physical significance and could just as well have been set equal to zero. The potential function inside the cylinder corresponds to a uniform electric field along the x-direction:

$$\mathrm{E}_{\mathrm{x}} = \frac{2\epsilon_2}{(\epsilon_1 + \epsilon_2)} \mathrm{E}_0. \tag{3.2.15}$$

This means that the material inside the cylinder is uniformly polarized along the x-direction. This is an example of the depolarizing coefficients discussed in Section (2.7.4) of Chpt.(2). In order to make contact with the treatment of Chpt.(2), consider the problem of a cylinder characterized by a dielectric constant ϵ, surrounded by free space $(\epsilon_2 = \epsilon_0)$, and located in a uniform external field $E_x = E_0$. A uniform polarization density transverse to the cylinder axis, P_x, produces an internal field given by

$$\mathrm{E}_{\mathrm{x}} = -\frac{\mathrm{P}_{\mathrm{x}}}{2\epsilon_0},$$

because the depolarization coefficient for this geometry is 1/2. When this field is added to the applied field the total electric field along the x-direction inside the cylinder is given by

$$E_x = E_0 - \frac{P_x}{2\epsilon_0}. \tag{3.2.16}$$

It is this total field that polarizes the material of the cylinder. By definition

$$D_x = \epsilon E_x = \epsilon_0 E_x + P_x;$$

therefore

$$P_x = (\epsilon - \epsilon_0) E_x.$$

Now substitute Equation (3.2.16) for the electric field to obtain

$$P_x = (\epsilon - \epsilon_0)\left(E_0 - \frac{P_x}{2\epsilon_0}\right).$$

Solving for P_x this gives

$$P_x = 2\epsilon_0 E_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right). \tag{3.2.17}$$

From Equation (3.2.16) the total electric field inside the cylinder is

$$E_x = \frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0,$$

in agreement with Equation (3.2.15) deduced from the potential function for the case $\epsilon_2 = \epsilon_0$ and $\epsilon_1 = \epsilon$.

The potential function outside the cylinder corresponds to the uniform applied electric field, E_0, plus the potential due to a line of dipoles whose dipole moment per unit length is given by

$$P_{Lx} = \left(\pi R^2\right) 2\epsilon_0 E_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right)$$

(by comparison of Equation (3.2.14) with the expression for the potential function for a line of dipoles given in Section(2.7.3)). This is equivalent to a dipole moment per unit volume

$$P_x = \frac{P_{Lx}}{\pi R^2} = 2\epsilon_0 E_0 \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\right),$$

in agreement with Equation (3.2.17).

This problem has been treated in detail because it is the prototype for all problems in cylindrical polar co-ordinates that involve boundaries describable by the form r= constant. At each surface of discontinuity one must require the potential function to be continuous through the surface. In addition the normal component of the displacement vector, $vecD$, is required to be continuous through the surface if that surface contains no surface free charge density. These conditions, together with the requirement that the potential function behave properly in the limits as r approaches zero and as r approaches infinity, serve to determine the coefficients in the expansions Equation (3.2.8). The solution so found is guaranteed to be **the solution** apart from an additive constant.

(d) Spherical Polar Co-ordinates.

LaPlace's equation written in spherical polar co-ordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 V}{\partial \phi^2}\right) = 0. \tag{3.2.18}$$

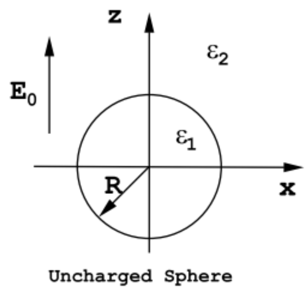

Figure 3.2.6: An uncharged dielectric sphere, dielectric constant ϵ_1, situated in a medium characterized by a dielectric constant, $\epsilon 2$, in the presence of a uniform electric field $E_z = E_0$.

For simplicity consider problems that are symmetric around the z-axis so that the potential function does not depend upon the angle variable ϕ. The general solution of LaPlace's equation for that case is

$$\mathrm{V}(\mathrm{r}, \theta)=\sum_{n=0}^{n=\infty}\left(a_n \mathrm{r}^{\mathrm{n}}+\frac{b_n}{\mathrm{r}^{\mathrm{n}+1}}\right) \mathrm{P}_{\mathrm{n}}(\cos \theta). \tag{3.2.19}$$

The angular functions $P_n(\cos\theta)$ are called Legendre polynomials: the first few of them are listed in Table (3.2.2). The coefficients a_n, b_n must be chosen to satisfy the boundary conditions for a particular problem. As an example, consider a dielectric sphere having a dielectric constant ϵ_1 surrounded by a medium characterized by a dielectric constant $\epsilon 2$ and immersed in a uniform applied field, $E_z = E_0$, see Figure (3.2.6). The electric field is directed along the z-axis, and is supposed to be produced by sources that are very far removed from the position of the sphere. Far from the sphere the potential function must have the form

$$\mathrm{V}(\mathrm{r}, \theta) \rightarrow-\mathrm{E}_0 \mathrm{r} \cos \theta,$$

corresponding to a uniform field E_0. This suggests that the potential both inside and outside the sphere should be proportional to the Legendre polynomial $P_1 = \cos(\theta)$. One is therefore led to try

Table 3.2.2: The first five Legendre polynomials $P_n(x)$: see Schaum's Outline Series "Mathematical Handbook" by Murray R. Spiegel, McGraw-Hill, N.Y.,1968. The multiplicative constant in front of each polynomial has been chosen so that the polynomials satisfy the condition $\int_{-1}^{1} \mathrm{dxP}_{\mathrm{m}} \mathrm{P}_{\mathrm{n}}=\left(\frac{2}{2 n+1}\right) \delta_{m n}$, where $\delta_{\mathrm{mn}}=1$ 1 if m=n, and zero otherwise.

$P_0=1$
$P_1=x$
$\mathrm{P}_2=\frac{1}{2}\left(3 \mathrm{x}^2-1\right)$
$P_3=\frac{1}{2}\left(5 x^3-3 x\right)$
$P_4=\frac{1}{8}\left(35 x^4-30 x^2+3\right)$
$\mathrm{P}_5=\frac{1}{8}\left(63 \mathrm{x}^5-70 \mathrm{x}^3+15 \mathrm{x}\right)$

Inside.

$$\mathrm{V}_{\mathrm{i}}=A \mathrm{r} \cos \theta.$$

There is no term b_1/r^2 term because there is no charge at r=0 that would cause the potential function to be singular at the origin.

Outside.

$$\mathrm{V}_{\mathrm{o}}=\left(a \mathrm{r}+\frac{b}{\mathrm{r}^2}\right) \cos \theta,$$

where $a = -E_0$ in order that the potential reduce to that corresponding to a uniform field of strength E_0 at distances far from the sphere.

On the surface of the sphere the potential must be continuous on passing from the inside to the outside the sphere; this continuity of the potential function guarantees that the tangential component of E will be continuous across the surface of the sphere as is required by the Maxwell equation curl$(\vec{E}) = 0$. One finds, for r=R,

$$A\mathrm{R} = -\mathrm{E}_0\mathrm{R} + \frac{b}{\mathrm{R}^2},$$

or

$$A = -\mathrm{E}_0 + \frac{b}{\mathrm{R}^3}. \tag{3.2.20}$$

On the surface of the sphere the normal component of $\vec{D}$ must be continuous. This condition gives

$$\epsilon_1 A = -\epsilon_2\left(\mathrm{E}_0 + \frac{2b}{\mathrm{R}^3}\right). \tag{3.2.21}$$

Equations (3.2.20) and (3.2.21) can be solved for A and b. The result of the calculation is

$$A = -\left(\frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2}\right)\mathrm{E}_0,$$

and

$$b = \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}\right)\mathrm{R}^3\mathrm{E}_0.$$

The function

$$\mathrm{V_i}(\mathrm{r}, \theta) = -\frac{3\epsilon_2\mathrm{E}_0\mathrm{r}\cos\theta}{(\epsilon_1 + 2\epsilon_2)}.$$

satisfies $\nabla^2\mathrm{V} = 0$ everywhere in the region inside the sphere. The function

$$\mathrm{V}_o(\mathrm{r}, \theta) = -\mathrm{E}_0\mathrm{r}\cos\theta + \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}\right)\frac{\mathrm{R}^3\mathrm{E}_0\cos\theta}{\mathrm{r}^2}$$

satisfies $\nabla^2\mathrm{V} = 0$ everywhere in the region outside the sphere. Moreover, these two functions satisfy all of the boundary conditions for this problem. The uniqueness theorem guarantees that this is **the solution** of the problem of an uncharged dielectric sphere subject to a uniform applied electrostatic field.

For the particular case in which an uncharged dielectric sphere characterized by a dielectric constant ϵ is located in free space, dielectric constant ϵ_0, the above result reduces to

$$\mathrm{V_i}(\mathrm{r}, \theta) = -\left(\frac{3\mathrm{E}_0}{2 + \epsilon_r}\right)\mathrm{r}\cos\theta, \tag{3.2.22}$$

and

$$\mathrm{V_o}(\mathrm{r}, \theta) = -\mathrm{E}_0\mathrm{r}\cos\theta + \left(\frac{\epsilon_r - 1}{2 + \epsilon_r}\right)\frac{\mathrm{R}^3\mathrm{E}_0\cos\theta}{\mathrm{r}^2}, \tag{3.2.23}$$

where the relative dielectric constant is $\epsilon_r = (\epsilon_2/\epsilon_0)$. These expressions are consistent with the results of Chpt.(2), Section(2.7.4) in which it was stated that the depolarization factor for a sphere is 1/3. The second term in Equation (3.2.23) corresponds to the potential generated by a point dipole at the center of the sphere having the strength

$$\mathrm{p}_z = 4\pi\epsilon_0\left(\frac{\epsilon_r - 1}{2 + \epsilon_r}\right)\mathrm{R}^3\mathrm{E}_0.$$

This moment corresponds to a polarization per unit volume directed along z and having the value

$$\mathrm{P} = \mathrm{p}_z / \frac{4\pi\mathrm{R}^3}{3} = 3\epsilon_0\left(\frac{\epsilon_r - 1}{2 + \epsilon_r}\right)\mathrm{E}_0 \quad \text{Coulombs}/\mathrm{m}^2. \tag{3.2.24}$$

This uniform polarization would produce a depolarizing field within the sphere given by

$$\mathrm{E_z} = -\left(\frac{\epsilon_r - 1}{2 + \epsilon_r}\right)\mathrm{E}_0 \quad \text{Volts/m}.$$

When this is combined with the applied field, E_0, the total field within the dielectric sphere becomes

$$\mathrm{E_i} = \frac{3\mathrm{E}_0}{2+\epsilon_r} \quad \text{Volts}/\mathrm{m}$$

in agreement with the inner field calculated from the potential function Equation (3.2.22).

3.2.2 The Method of Images.

(a) A Charge Located Near a Plane Interface.

This is a very specialized technique for solving electrostatic problems that involves setting up a distribution of non-existent charges in such a way that the boundary conditions on the real problem are satisfied. For example, consider a point charge, q, located in vacuum and at a distance d in front of an infinite conducting plane, Figure (3.2.7). In the region to the left of the interface the potential function must satisfy $\nabla^2 V = 0$. The boundary conditions are:

(1) Very near the position of the charge the potential function must have the form required for a point charge q, i.e.

$$\mathrm{V(r)} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathrm{r}}\right).$$

(2) The conducting surface must be an equipotential surface, i.e. V=const.

These two boundary conditions are satisfied by the system of two charges shown in the bottom diagram of Figure (3.2.7). The real problem involving a conducting surface has been replaced by an image problem which just involves two charges in free space. The potential at any point in space for the image problem is

$$\mathrm{V} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathrm{r}_1} - \frac{q}{\mathrm{r}_2}\right).$$

On the symmetry plane $r_1 = r_2$ and therefore V=0, and is constant, everywhere on the symmetry plane. Moreover, this potential function satisfies $\nabla^2 V = 0$ everywhere, except right at the two charges, because it is the sum of two point charge potentials each of which separately satisfies LaPlace's

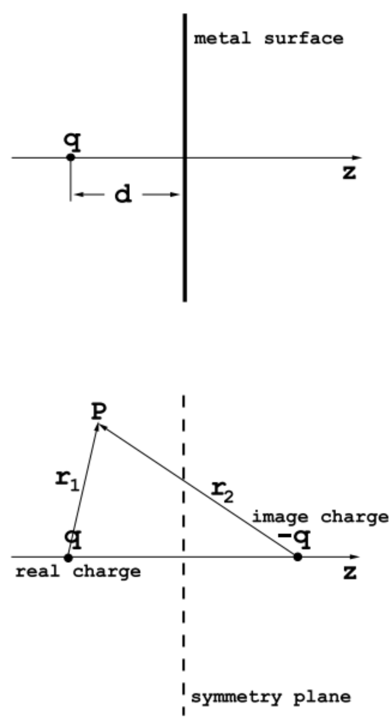

Figure 3.2.7: Top figure: a point charge located a distance d in front of an infinite conducting metal plane. Bottom figure: The system of charges whose electrostatic potential satisfies $\nabla^2 V = 0$ as well as the boundary conditions for the problem posed in the top figure. This solution is only valid in the vacuum region: in the metal V = 0.

equation. This potential function obviously approaches the limit $\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}\right)$ as $r_1 \rightarrow 0$. Therefore the potential function for the image problem of Figure (3.2.7) is also the potential function which satisfies all the requirements for the problem shown in the top diagram of Figure (3.2.7) in the region outside the conductor, ($z \leq 0$). According to the uniqueness theorem, this is therefore the required solution. Of course this solution is only valid for the region outside the conductor: inside the conductor the potential is constant and equal to zero. This image problem can be easily generalized to the problem in which the space outside the conducting plane is filled with a dielectric material ϵ simply by replacing ϵ_0 with ϵ.

(b) A Charged Particle Located Near an Interface between Two Dielectric Materials.

The problem of a point charge outside a plane interface of discontinuity in the dielectric constant can also be solved by the method of images, although in this case the required image charge distribution is not so obvious. Refer to Figure (3.2.8). Let the potential function in Region(1) be that due to the real charge q plus an image charge q_1 symmetrically placed with respect to the interface, see Figure (3.2.8). If space were homogeneously filled with material characterized by a dielectric constant ϵ_1 the resulting potential would be given by

$$\mathrm{V_L} = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{\mathrm{r_1}} + \frac{q_1}{\mathrm{r_2}}\right). \tag{3.2.25}$$

Let the potential to the right of the interface be the same as that due to an image charge q_2 located at the position of the real charge, but a charge that is immersed in a homogeneous dielectric material characterized by a dielectric constant ϵ_2:

$$\mathrm{V_R} = \frac{1}{4\pi\epsilon_2}\left(\frac{q_2}{\mathrm{r_1}}\right). \tag{3.2.26}$$

Clearly both V_L and V_R satisfy LaPlace's equation. The trick now is to choose the image charges q_1,q_2 so as to satisfy the boundary conditions

(1) $V_L = V_R$ on the interface between the two dielectrics; and

(2) the normal component of $\vec{D}$ must be continuous across the interface

between the two dielectrics, ie.

$$\epsilon_1\left(\frac{\partial \mathrm{V_L}}{\partial \mathrm{n}}\right) = \epsilon_2\left(\frac{\partial \mathrm{V_R}}{\partial \mathrm{n}}\right).$$

Boundary condition (1) gives

$$\frac{1}{4\pi\epsilon_1}\left(\frac{q}{\mathrm{r_0}} + \frac{q_1}{\mathrm{r_0}}\right) = \frac{1}{4\pi\epsilon_2}\left(\frac{q_2}{\mathrm{r_0}}\right), \tag{3.2.27}$$

where $r_1 = r_2 = r_0$ on the boundary between the two dielectrics. Boundary condition (2) gives

$$-\frac{q\mathrm{d}}{\mathrm{r_0^3}} + \frac{q_1\mathrm{d}}{\mathrm{r_0^3}} = -\frac{q_2\mathrm{d}}{\mathrm{r_0^3}}. \tag{3.2.28}$$

Equation(3.2.28) is the consequence of the relation

$$\frac{\partial}{\partial \mathrm{n}}\left(\frac{1}{\mathrm{r_1}}\right) = \frac{\partial}{\partial \mathrm{z}}\left(\frac{1}{\sqrt{\mathrm{x^2+y^2+(z+d)^2}}}\right) = -\frac{\mathrm{(z+d)}}{\mathrm{(x^2+y^2+(z+d)^2)}^{3/2}}.$$

When this expression is evaluated on the interface, ie. at z = 0, the result is

$$\frac{\partial}{\partial \mathrm{z}}\left(\frac{1}{\mathrm{r_1}}\right) = \frac{-d}{\mathrm{r_0^3}}.$$

Similarly

$$\frac{\partial}{\partial \mathrm{z}}\left(\frac{1}{\mathrm{r_2}}\right) = \frac{+d}{\mathrm{r_0^3}}.$$

The boundary condition Equation (3.2.28) requires

$$q - q_1 = q_2.$$

When this is combined with the first boundary condition, Equation (3.2.27), one obtains

$$q_1 = \frac{\left(\frac{\epsilon_1}{\epsilon_2} - 1\right)}{\left(\frac{\epsilon_1}{\epsilon_2} + 1\right)} q \tag{3.2.29}$$

and

$$q_2 = \frac{2q}{\left(\frac{\epsilon_1}{\epsilon_2} + 1\right)}. \tag{3.2.30}$$

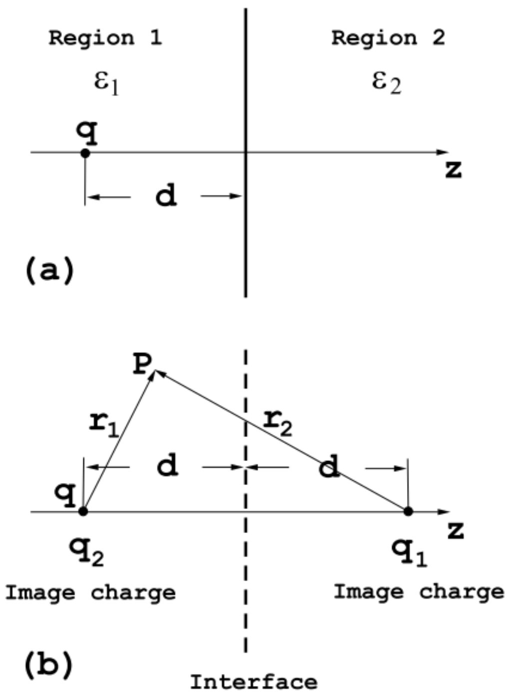

Figure 3.2.8: (a) The real problem: a charge q located a distance d from the interface between two uncharged dielectric media. (b) The configuration of image charges that produce a potential that satisfies $\nabla^2 V = 0$ and that can be used to satisfy the required boundary conditions.

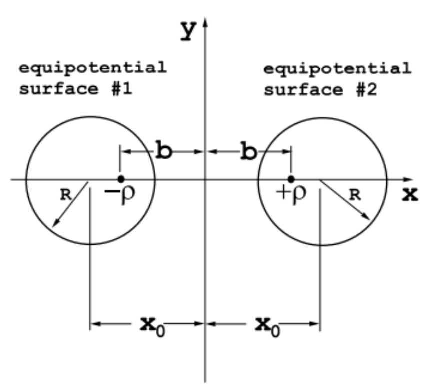

Figure 3.2.9: Equipotential surfaces for two line charges of equal strength but of opposite sign.

The solution of the original problem illustrated in part (a) of Figure (3.2.8) is given by Equation (3.2.25), valid for region(1) characterized by ϵ_1, and by Equation (3.2.26) in region(2) characterized by ϵ_2, where q_1 and q_2 are given by Equations (3.2.29 and

3.2.30). The force acting on the real charge q is just q multiplied by the electric field generated at the position of q by the image charge q1: ie. the electric field $\vec{E} = -\mathrm{grad}(V_L)$. It follows that if $\epsilon_2 > \epsilon_1$ the charge q is attracted to the interface, but if $\epsilon_2 < \epsilon_1$ then the charge q is repelled by the interface.

(c) Parallel Conducting Cylinders

Let two line charges of strengths −ρ and +ρ Coulombs/m be separated by the distance 2b along the x-axis as shown in Figure (3.2.9). In the first instance let these line charges be immersed in vacuum. The potential generated by a line charge of strength ρ is given by

$$\mathrm{V}(\mathrm{r}) = -\left(\frac{\rho}{2\pi\epsilon_0}\right)\ln(\mathrm{r}),$$

(see Section(2.7.3)). If r_1 is the distance from −ρ to an observer at P , and if r_2 is the distance from the line charge +ρ to the observer at P, then the potential at P is given by

$$\mathrm{V_P} = \frac{\rho}{2\pi\epsilon_0}\ln(\mathrm{r_1}/\mathrm{r_2}).$$

Let $(r_1/r_2) = k$, a constant, so that

$$\mathrm{V} = \frac{\rho}{2\pi\epsilon_0}\ln(k) \quad \text{Volts}.$$

This is the potential on all points that satisfy the condition $r_1 = kr_2$, or$r_1^2 = k^2 r_2^2$. This last condition can be written out explicitly in cartesian co-ordinates:

$$(x+b)^2 + y^2 = k^2\left((x-b)^2 + y^2\right).$$

With the application of some tedious algebra this last expression may be put in the form:

$$\left(x + b\left(\frac{1+k^2}{1-k^2}\right)\right)^2 + y^2 = \frac{4b^2k^2}{(1-k^2)^2}. \tag{3.2.31}$$

Equation (3.2.31) describes a circle centered at

$$\mathrm{x_0} = -\mathrm{b}\left(\frac{1+\mathrm{k}^2}{1-\mathrm{k}^2}\right), \tag{3.2.32}$$

with a radius

$$\mathrm{R} = \frac{2\mathrm{bk}}{|1-\mathrm{k}^2|}. \tag{3.2.33}$$

Notice that k′ = 1/k corresponds to an equipotential surface centered at

$$\mathrm{x}_0' = \mathrm{b}\left(\frac{1+\mathrm{k}^2}{1-\mathrm{k}^2}\right) = -\mathrm{x}_0,$$

and having the same radius R as the equipotential surface corresponding to k and centered at x_0. Equipotential surfaces for k and 1/k are illustrated in Figure (3.2.9). The two equipotential surfaces shown correspond to different potentials. The cylinder on the left corresponds to $r_1/r_2 = k$; the cylinder on the right corresponds to $r_1/r_2 = 1/k$. It follows that the potential of the cylinder on the right is equal in magnitude but opposite in sign to the potential of the cylinder on the left.

These families of displaced equipotential cylindrical surfaces can be used to solve a number of parallel conducting cylinder problems. The same treatment works if the cylinders are immersed in a dielectric medium; one has only to replace $epsilon_0$ by the dielectric constant for the medium, ϵ.

(d) A Point Charge Outside a Conducting Sphere.

The problem of the potential function generated by a point charge located outside a conducting sphere can also be solved using the method of images. Consider the geometry shown in Figure (3.2.10). The potential everywhere outside the conducting sphere is the same as the potential generated by two point charges: (1) the original point charge q; and (2) an image charge $-q_1$. (The minus sign has been introduced for convenience; it makes sense that the charges induced on the sphere should have a sign that is opposite to that of the charge q.) The assertion is that if the position of the charge q_1 as well as its magnitude are properly chosen then these two charges will create a potential that is constant on the surface of the sphere. This assertion is by no means obvious, but let us see how this comes about. The potential generated at the point of observation, P, is given by

$$V_p = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} - \frac{q_1}{r_1}\right),$$

where

$$r = \sqrt{x^2 + y^2 + (z-d)^2}$$

and

$$r_1 = \sqrt{x^2 + y^2 + (z-e)^2}\ .$$

It is now obvious from the form of the above potential that the potential will be zero on all points such that

$$\frac{q}{r} = \frac{q_1}{r_1}.$$

So let $qr_1 = q_1 r$, or more conveniently let $q^2 r_1^2 = q_1^2 r^2$. Write out this last equation explicitly in cartesian co-ordinates:

$$x^2 + y^2 + (z-e)^2 = \left(\frac{q_1}{q}\right)^2 \left[x^2 + y^2 + (z-d)^2\right],$$

or

$$x^2 + y^2 + z^2 - 2ez + e^2 = \left(\frac{q_1}{q}\right)^2 \left[x^2 + y^2 + z^2 - 2zd + d^2\right].$$

Gather terms to obtain

$$\left[1 - \left(\frac{q_1}{q}\right)^2\right](x^2 + y^2 + z^2) - 2z\left[e - \left(\frac{q_1}{q}\right)^2 d\right] = \left(\frac{q_1}{q}\right)^2 (d^2) - e^2.$$

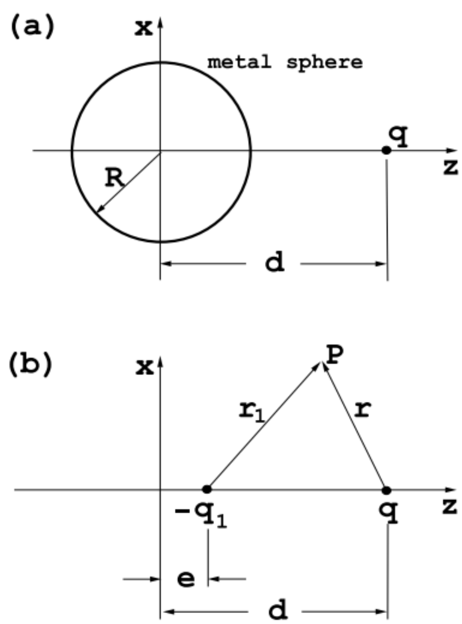

Figure 3.2.10: The real problem of a point charge located a distance d from the center of a metal sphere of radius R is shown in (a). In part (b) the real problem has been replaced by two point charges as shown in the figure.

Note that if

$$e=\left(\frac{q_1}{q}\right)^2$$

then the potential V = 0 at all points such that

$$x^2+y^2+z^2=\frac{\left(\frac{a_1}{q}\right)^2 d^2-e^2}{\left[1-\left(\frac{q_1}{q}\right)^2\right]},$$

or using

$$e=\left(\frac{q_1}{q}\right)^2$$

$$x^2+y^2+z^2=\left(\frac{q_1 d}{q}\right)^2. \tag{3.2.34}$$

This means that the spherical equipotential surface corresponding to V = 0 will coincide with the surface of the metal sphere if

$$\mathrm{q}_1=\mathrm{q}\left(\frac{\mathrm{R}}{\mathrm{d}}\right), \tag{3.2.35}$$

and

$$\mathrm{e}=\left(\frac{\mathrm{R}^2}{\mathrm{d}}\right). \tag{3.2.36}$$

So if Equations (3.2.35 and 3.2.36) are satisfied then the potential everywhere outside the metal sphere will satisfy $\nabla^2 V = 0$ because the potential is the sum of two point charge potentials each of which satisfies the LaPlace equation. Moreover, this potential function satisfies the boundary condition that the surface of the sphere be an equipotential surface. This solution corresponds to the special case in which charge is allowed to flow onto the sphere as the driving charge q is brought up from infinity. It can be shown using Gauss' theorem that the charge induced on the sphere is just $-q_1$. The problem of an isolated sphere such that the net charge on it is zero can be solved by adding a third charge of strength $+q_1$ to the position of the center of the sphere in the image problem of Figure (3.2.10(b)).The potential in the region outside the sphere is now given by

$$\mathrm{V}(\mathrm{x}, \mathrm{y}, \mathrm{z})=\frac{1}{4 \pi \epsilon_0}\left[\frac{\mathrm{q}_1}{\mathrm{r}_2}+\frac{\mathrm{q}}{\mathrm{r}}-\frac{\mathrm{q}_1}{\mathrm{r}_1}\right], \tag{3.2.37}$$

where

$$r_2=\sqrt{x^2+y^2+z^2}.$$

The potential function Equation (3.2.37) satisfies the LaPlace equation, $\nabla^2 V = 0$, the surface of the sphere is an equipotential surface, and it corresponds to a net charge of zero on the sphere. The potential of the sphere is just

$$\mathrm{V}_{\mathrm{s}}=\frac{1}{4 \pi \epsilon_0}\left(\frac{\mathrm{q}_1}{\mathrm{R}}\right)=\frac{1}{4 \pi \epsilon_0}\left(\frac{q}{\mathrm{d}}\right),$$

because the last two terms in Equation (3.2.37) cancel each other, by construction, on the surface of the sphere.

3.2.3 Two-dimensional Problems.

(a) The Theory of Complex Variables.

The theory of complex variables may be useful for solving problems in which the potential function does not depend upon one co-ordinate- the z coordinate, say. Let z=x+iy represent a complex number, and let F(z)= U(x,y)+iV(x,y) be an analytic function of the complex variable z. In order for F(z) to exhibit a well-defined derivative it can be shown that the Cauchy-Riemann equations must be satisfied:

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y},$$

and

$$\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}.$$

See, for example, Schaum's Outline Series: Complex Variables by Murray R. Spiegel, McGraw-Hill, N.Y.,1964. It follows from the Cauchy-Riemann relations by direct differentiation that

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0,$$

and

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0.$$

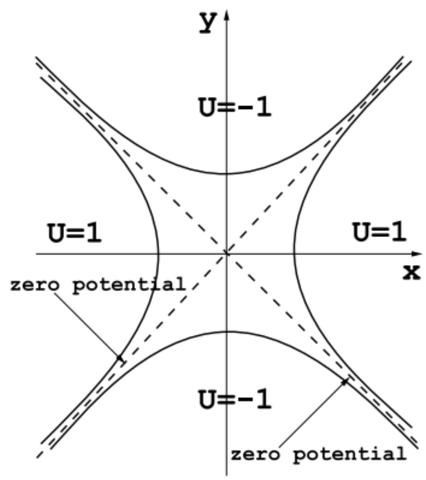

Figure 3.2.11: Electrodes in the form of cylindrical hyperboloids. One pair of electrodes is held at a potential U=-1 Volt; the other pair of electrodes is held at a potential U=+1 Volt. The potential function in the space between the electrodes is given by $U = x^2 - y^2$. The electric field components in the space between the electrodes are given by $E_x = -2x$ and $E_y = +2y$.

That is, both of the functions U(x,y) and V(x,y) satisfy LaPlace's equation. Both U and V are therefore candidates for the solution of some problem in electrostatics. Consider, for example, the analytic function

$$F(z)=z^2=\left(x^2-y^2\right)+2ixy. \tag{3.2.38}$$

In this case

$$U(x,y)=x^2-y^2$$

and

$$V(x,y)=2xy.$$

The families of curves U= const. and V= const. are orthogonal to each other. If the equipotential surfaces are represented by U(x,y)= const. (see Figure (3.2.11)) then the curves V(x,y)= const represent the electric field lines: electric field lines are constructed so that their tangent at each point is parallel with the direction of the electric field. Conversely, if the equipotential surfaces are described by the curves V(x,y)= const. then the curves U(x,y)= const. represent the field lines. Other examples are described in the Feynman Lectures on Physics,Vol.(II), section 7-2. In principle, the technique of conformal mapping (described in Schaum's Outline Series: Complex Variables, loc.cit.) can be used to determine the potential distribution around electrodes whose shape can be represented by a polygon.

(b) Analogue Solution Using Conducting Paper.

Two dimensional problems can also be solved by means of a kind of analogue computer. The desired electrode configuration is painted on conducting paper using a metallic conducting paint, eg. silver dag, see Figure (3.2.12). The silver paint electrodes portrayed in this figure would represent an infinitely long metal, circular, cylinder placed between two infinitely long parallel metal plates. The electrodes are held at fixed potentials V_1, V_2, and V_3. The currents which flow in the conducting paper must be such that there is no charge build-up anywhere; they must, therefore, satisfy the equation

$$\operatorname{div}(\mathrm{J})=0.$$

But in a conducting medium one has

$$\mathrm{J}=\sigma \mathrm{E}$$

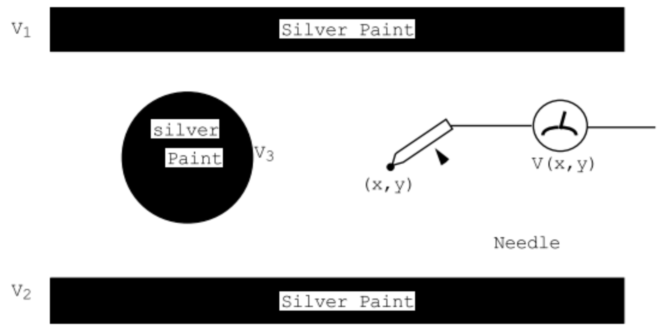

Figure 3.2.12: Electrodes of silver paint drawn on a sheet of conducting paper. The resistivity of the paper is much larger than that of the silver paint electrodes.The equipotential lines can be mapped out by means of a voltmeter connected to a pointed probe.

so that

$$\operatorname{div}(\mathrm{E})=0.$$

From this last equation it follows that the potential distribution in the conducting paper must satisfy $\nabla^2 V = 0$ because $E = -\text{grad}V$. The equipotential lines corresponding to a desired potential value can be traced out on the conducting paper by means of a high input impedance voltmeter connected to a sharply pointed probe. The electric field can, of course, be obtained from the known potential distribution via a numerical differentiation.