

### 3.1: Introduction

Chapter(2) demonstrated that the electrostatic field could be calculated everywhere in space from a knowledge of the spatial distribution of free charges and a knowledge of the spatial distribution of the electric dipole moment density. However, in most cases of interest one does not initially know the free charge and dipole moment distributions. In a typical problem one is given two or more metal electrodes embedded in a material medium, in which the electrode potentials are specified. In this kind of problem the free charge and dipole moment distributions must be determined as part of the problem solution. In order to solve such problems it is necessary to know the relation between the electric field in a material and the dipole density that is induced in that material by the electric field. In general such problems are extremely difficult unless the dipole density,  $\vec{P}(\vec{r})$ , at a point  $\vec{r}$  in the material is linearly related to the electric field at that same point,  $\vec{E}(\vec{r})$ . In this chapter we shall assume that we have to do with **linear, isotropic, media** such that

$$\vec{P}(\vec{r}) = \chi \epsilon_0 \vec{E}(\vec{r}), \quad (3.1.1)$$

where  $\chi$  is a pure number called the static dielectric susceptibility.  $\chi$  is supposed to be independent of position within a given material; it will exhibit discontinuous jumps at the boundary between two different materials. For a dielectric material characterized by  $\chi$  Maxwell's equations for the electrostatic field become (no variation with time)

$$\begin{aligned} \text{curl}(\vec{E}) &= 0, \\ \text{div}(\vec{E}) &= \frac{1}{\epsilon_0} [\rho_f - \text{div}(\vec{P})] = \frac{1}{\epsilon_0} [\rho_f - \chi \epsilon_0 \text{div}(\vec{E})]. \end{aligned} \quad (3.1.2)$$

This last equation can be written  $\epsilon_0(1 + \chi) \text{div}(\vec{E}) = \rho_f$ . But  $\epsilon_0(1 + \chi)$  is independent of position within a given dielectric material so that

$$\text{div}[(1 + \chi)\epsilon_0 \vec{E}] = \rho_f = \text{div}(\vec{D}), \quad (3.1.3)$$

since

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = (1 + \chi)\epsilon_0 \vec{E}. \quad (3.1.4)$$

The number  $(1 + \chi)$  is called the **relative dielectric constant**,  $\epsilon_r$ . Thus for a linear isotropic material

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E} = \epsilon \vec{E}, \quad (3.1.5)$$

and

$$\text{div}(\vec{D}) = \rho_f \quad \text{or} \quad \text{div}(\vec{E}) = \rho_f / \epsilon. \quad (3.1.6)$$

Equation(3.5) defines the dielectric constant  $\epsilon$  which has the same units as  $\epsilon_0$ , namely Farads/meter or Coulombs/Volt meter.

As in Chapter(2) one can introduce a potential function,  $V(\vec{r})$ , such that

$$\vec{E}(\vec{r}) = -\text{grad } V(\vec{r}). \quad (3.1.7)$$

This definition guarantees that the Maxwell Equation (3.1.2) will be satisfied because the curl of any gradient is zero. Using Equation (3.1.7) in Equation (3.1.6) gives

$$-\text{div grad } V(\vec{r}) = \rho_f(\vec{r}) / \epsilon$$

or

$$\nabla^2 V(\vec{r}) = -\frac{1}{\epsilon} \rho_f(\vec{r}) \quad (3.1.8)$$

The differential equation, Equation (3.1.8), is called Poisson's equation. It is similar in form to Equation (2.2.5) of Chapter(2) for  $\vec{P} = 0$  except that  $\epsilon_0$  is replaced by  $\epsilon$ . It follows by analogy with Equation (2.2.6) of Chapter(2) that the particular solution of Equation (3.1.8) is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon} \iiint_{Space} dV_{vol} \frac{\rho_f(\vec{r})}{|\vec{R} - \vec{r}|} \quad (3.1.9)$$

Unfortunately, Equation (3.1.9) is seldom helpful because one does not usually know a priori the free charge distribution,  $\rho_f(\vec{r})$ . The usual problem involves a number of conducting electrodes embedded in a dielectric medium: either the potential or the total charge on each electrode is specified as a boundary condition. Generally the dielectric medium is taken to be either charge free,  $\rho_f = 0$ , or else slightly conducting where the current density at any point in the medium is proportional to the electric field strength at that point

$$\vec{J}_f = \sigma \vec{E}. \quad (3.1.10)$$

For the charge-free case Poisson's equation, Equation (3.1.8), becomes

$$\nabla^2 V = 0. \quad (3.1.11)$$

This is called LaPlace's equation. For a conducting medium the charge density will not generally be zero. However, any distribution of charges must be time independent because, by hypothesis, we are dealing with static field distributions which are independent of the time. It is a consequence of charge conservation that the current density,  $\vec{J}_f$ , and the charge density,  $\rho_f$ , must satisfy the equation

$$\text{div}(\vec{J}_f(\vec{r})) + \frac{\partial \rho_f(\vec{r})}{\partial t} = 0, \quad (3.1.12)$$

so that if the charge distribution does not depend upon time the current density must be divergence free;

$$\text{div}(\vec{J}_f) = 0. \quad (3.1.13)$$

But if the current density is proportional to the electric field, Equation (3.1.10), it follows that a divergence free current density must be paired with a divergence free electric field,

$$\text{div}(\vec{E}) = 0, \quad (3.1.14)$$

and therefore that the potential function must satisfy LaPlace's equation, Equation (3.1.11),  $\nabla^2 V = 0$ . The time-independent problem associated with conductors having a specified potential embedded in a linear, isotropic, chargefree dielectric medium has exactly the same potential distribution as the

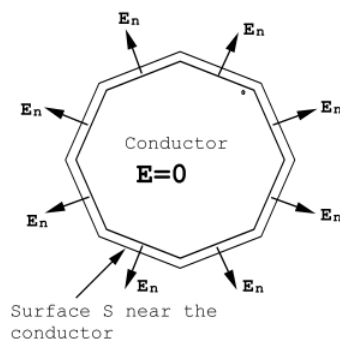


Figure 3.1.1: A charged conductor. The free charge on the conductor is given by  $Q = \int_S \vec{D} \cdot d\vec{S}$ .

problem of the same conductors embedded in an isotropic, conducting, linear dielectric medium. In either case one requires a potential distribution that satisfies LaPlace's equation, Equation (3.1.11), and for which the potential on each conducting electrode either reduces to a specified value, or else the total charge on each conductor calculated from the potential distribution gives a specified value. A conductor must have the same potential throughout by definition because the electric field within a conductor is zero; a zero electric field means that the potential function has no dependence upon position. The electric field within a conductor must be zero because otherwise mobile charges within the conductor would flow to its surface and build up time dependent electric fields in contradiction with the original assumption that the electric fields were time independent. It is a general rule that the tangential components of the electrostatic field must be continuous across any surface (see sections 2.4 and 2.5). Therefore, if  $\vec{E} =$

0 inside the conducting body it follows that the tangential components of the electrostatic field must be zero just outside the conductor surface. In other words, **the electrostatic field must be normal to the surface of a conductor**.

The total charge on a conductor, which must be located entirely on the surface of the conductor, can be calculated by means of an application of Gauss' theorem to Equation (3.1.6); see Figure (3.1.1).

$$Q = \int \int_{\text{Surface}} \vec{dS} \cdot \vec{D} = \epsilon \int \int_{\text{Surface}} \vec{dS} \cdot \vec{E},$$

where the integral is evaluated on a surface that lies just outside the actual conductor surface. The electric field just outside the conductor is normal to the surface and has the magnitude  $E_n = -\partial V / \partial n$ , where  $\partial V / \partial n$  is the gradient of the potential at the surface of the conductor. The total charge contained on the conductor is therefore related to the normal gradient of the potential function at the conductor

$$Q = -\epsilon \int \int_{\text{Surface}} \vec{dS} \cdot \left( \frac{\partial V}{\partial n} \right). \quad (3.1.15)$$

It can be proved mathematically, and it makes sense physically, that there is only one solution of LaPlace's equation,  $\nabla^2 V = 0$ , which satisfies the given boundary conditions. In other words, a solution of LaPlace's equation that satisfies the boundary conditions is unique; it is **the solution**. The boundary conditions may be of three different types: (1) the potential on each conductor is specified; (2) the total charge on each conductor is specified; (3) the potential on some of the conductors is specified and the total charge on the remaining conductors is specified. In addition, the fields very far from any charges must fall to zero at least as fast as  $1/R^2$ . This requirement follows from the fact that any collection of charges when viewed from very far away must look very nearly like a point charge, and hence the potential function must fall off like  $1/R$  where  $R$  is the mean distance to the group of charges.

In general it is very difficult to find a solution of LaPlace's equation that satisfies the boundary conditions imposed by a particular problem. Usually it is necessary to resort to numerical or approximate techniques in order to find a suitable solution. In the following sections a number of standard problems are posed and their solutions are discussed. The solutions of these standard problems enable one to build up, by analogy, a picture of how the electrostatic field should behave given a problem whose parameters lie outside those corresponding to one of the categories discussed below.

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