13.2: Chapter 2

Problem (2.1)

Given an electric field of the form $\mathbf{E}=100\mathrm{x}\ \hat{\mathbf{u}}_{\mathrm{x}}\mathrm{V}/\mathrm{m}$ find the total charge contained in the following volumes:

1) A cubical volume 1 cm on a side centered on the origin. The cube edges are parallel with the x, y, and z axes.

2) A cylindrical volume having a radius of 1 cm and a height of 2 cm centered at the origin. The axis of the cylinder is parallel with the z-axis.

Answer (2.1)

Ex = 100x

Ey = 0

Ez = 0

$\therefore \operatorname{div}\mathbf{E}=100=\rho_{\mathrm{f}}/\varepsilon_{\mathrm{o}}$

This electric field corresponds to a uniform charge distribution $\rho_f=100\varepsilon_0$ Coulombs/m^3

∴ The total charge in

(1) The cube

$$\begin{aligned}Q&=(100\varepsilon_0)\left(10^{-6}\right)=10^{-4}\varepsilon_0\\&=\frac{10^{-13}}{36\pi}=8.84\times10^{-16}\text{ Coulombs.}\end{aligned}$$

(2) The cylinder

$$\begin{aligned}Q&=(100\varepsilon_0)\left(2\pi\times10^{-6}\right)=2\pi\varepsilon_0\times10^{-4}\\&=\frac{10^{-13}}{18}=5.56\times10^{-15}\text{Coulombs}\end{aligned}$$

since $\varepsilon_0=\frac{1}{\mu_0C^2}=\frac{10^{-9}}{36\pi}$

Note that the above charge distribution though uniform must have planar symmetry (because Ey = Ez = 0).

Problem (2.2)

A free charge distribution is given by ρ_f = ar Coulombs/m^3 for $0\le r\le R$ and $\rho_f=0$ for $r>R$. (The electric polarization P is everywhere zero).

a) Calculate the components of the electric field in and around this charge distribution. The problem has spherical symmetry so one can use Gauss' theorem (the divergence theorem).

b) Calculate the potential function corresponding to the electric field of part (a). Choose the arbitrary constants so that (1) V → 0 as r → ∞.

(2) V is continuous at r= R.

In this way show that the potential at r = 0 is given by $V(0)=\frac{aR^3}{3\varepsilon_0}$ Volts.

Answer (2.2)

(a) $\operatorname{div}\mathbf{E}=\frac{\rho_{\mathrm{f}}}{\varepsilon_{\mathrm{o}}}$ since P ≡ 0

and therefore div P = 0.

So $\int_{\mathrm{V}}\operatorname{div}\mathbf{E}d\tau=\int_{\mathrm{s}}\mathbf{E}\cdot\mathbf{ds}=\frac{Q(\mathrm{r})}{\varepsilon_0}$

where Q(r) is the charge contained within a sphere of radius r.

$Q(r)=a\int_0^r(r)4\pi r^2dr=\pi ar^4$ Coulombs for r ≤ R.

But $\mathbf{E}$ has only a radial component by symmetry. Therefore for a spherical surface of radius r , $\int_S\mathbf{E}\cdot\mathbf{ds}=4\pi r^2E_r$

So for r≤R $4\pi r^2 E_r = \frac{\pi a r^4}{\varepsilon_0}$

or $\mathrm{E_r} = \frac{\mathrm{ar^2}}{4\varepsilon_O}\mathrm{Volts/m}$.

for r>R the charge is independent of r: Q = πaR^4

$$\therefore \quad E_r = \frac{\pi a R^4}{4\pi\varepsilon_O r^2}\ \mathrm{Volts/m\ (Coulomb's\ Law)}\ .$$

(b) Since **E** has only a radial component, the potential function will depend only upon r:

$$E_r = -\left(\frac{\partial V}{\partial r}\right)$$

∴ in the region r ≤ R $V = \frac{-ar^3}{12\varepsilon_0} + V_0$

in the region r > R $V = \frac{\pi a R^4}{4\pi\varepsilon_0 r}$

(The constant is zero so that V → 0 as r → ∞).

At r = R we require V to be continuous. Therefore

$$V_0 - \frac{aR^3}{12\varepsilon_0} = \frac{aR^3}{4\varepsilon_0} = \frac{3aR^3}{12\varepsilon_0}$$

So $V_0 = \frac{4aR^3}{12\varepsilon_0} = aR^3/3\varepsilon_0$

The potential at the center of the charge distribution is therefore ($aR^3/3\ \varepsilon_o$) Volts

Problem (2.3)

A cube of side length L m is centered on the origin and its edges are parallel with the x, y, and z axes. The electric dipole vector per unit volume, **P**, is given by $\mathbf{P} = \mathrm{P}_0\left(\mathrm{x}\hat{\mathbf{u}}_\mathrm{x} + \mathrm{y}\hat{\mathbf{u}}_\mathrm{y} + \mathrm{z}\hat{\mathbf{u}}_\mathrm{z}\right)\ \ \mathrm{Coulombs\ /m^2}$

a) Calculate the bound charge density ρ_b = - div **P**.

b) Calculate the surface bound charge density on each face of the cube.

c) Show that the total bound charge on the cube is zero.

Answer (2.3)

(a) $\mathbf{P} = \ \mathrm{Po}\ \left(\mathrm{x}\hat{\mathbf{u}}_\mathbf{x} + \mathrm{y}\hat{\mathbf{u}}_\mathrm{y} + \mathrm{z}\hat{\mathbf{u}}_\mathrm{z}\right)$ inside cube

P ≡ 0 Outside cube

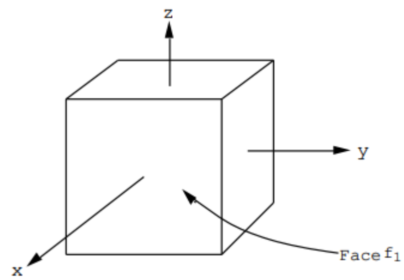
∴ ρ_b = - div **P** = - $3P_o$ inside the cube

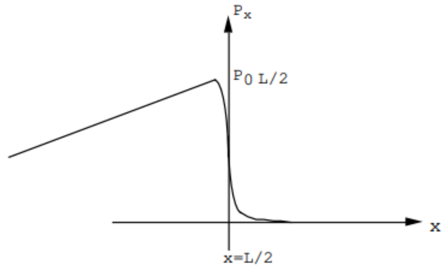
= 0 outside the cube.

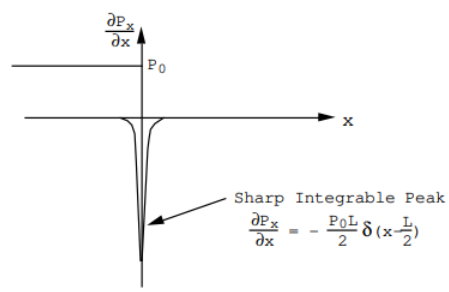
The total bound charge inside the cube is therefore Q_v = - $3P_oL^3$ Coulombs

(b) The discontinuity in the normal component of **P** gives an effective surface charge density on each face of the cube.

For example, on the face f_1 there is a discontinuity in P_x which is illustrated in the sketch below.



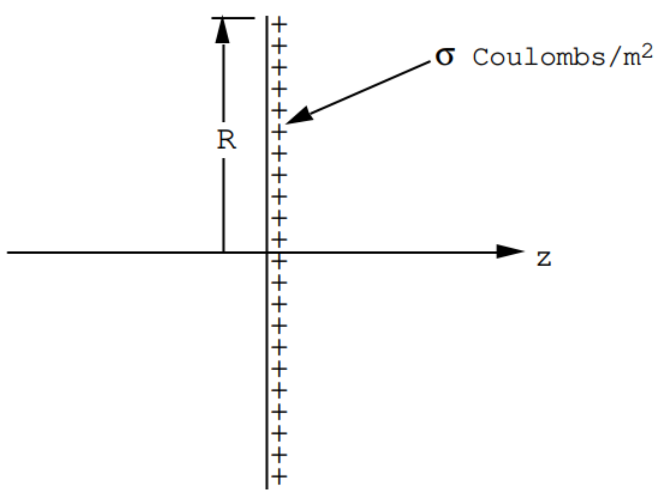




∴ The surface bound charge density is $\sigma_b = +\frac{P_0 L}{2}$ Coulombs/m^2 The total surface charge on f_1 is $Q_s = \sigma_b L^2 = P_o L^3/2$ Coulombs. There is a similar charge on each of the other faces. Therefore the total surface charge = $6Qs = 3P_oL^3$. The same as the volume charge.

Problem (2.4)

A disc of charge whose diameter is R meters is centered on the origin with its plane normal to the z-axis as shown in the sketch.



(a) Calculate the potential function V(z) on the axis of the disc. Sketch V(z).

(b) Make a sketch of $E_z(z)$. Show that as z → 0 $\mathbf{E}_z = \frac{\sigma}{2\varepsilon_0}$ for z > 0 and $E_z = -\frac{\sigma}{2\varepsilon_0}$ for z < 0.

(This problem is not as trivial as it looks. Remember V(z) must be an even function of z. It must also go to zero as | z | → ∞, and it must be continuous at z = 0. The answer is $V(z) = \frac{\sigma}{2\varepsilon_0}[\sqrt{R^2+z^2} - |z|]$.

Answer (2.4)

$$V(z) = \frac{1}{4\pi\varepsilon_0} \quad \int_0^R \frac{2\pi r dr \sigma}{\sqrt{r^2+z^2}} = \frac{\sigma}{4\varepsilon_0}\int_{z^2}^{R+z^2} \frac{du}{\sqrt{u}}$$

where u = r^2 + z^2 du = 2r dr

$$\therefore V(z) = \frac{\sigma}{2\varepsilon_0}[\sqrt{R^2+z^2} - \sqrt{z^2}]$$

(There is a temptation to write $\sqrt{z^2} = z$ but this would be wrong because one must use the +'ve root of z^2 even when z is negative. Hence $\sqrt{z^2} = |z|$.)

For z > 0 but z small $V \rightarrow -\frac{\sigma_z}{2\varepsilon_0} + \frac{\sigma_R}{2\varepsilon_0}$

For z < 0 but z small $V \rightarrow \frac{\sigma_z}{2\varepsilon_0} + \frac{\sigma R}{2\varepsilon_0}$

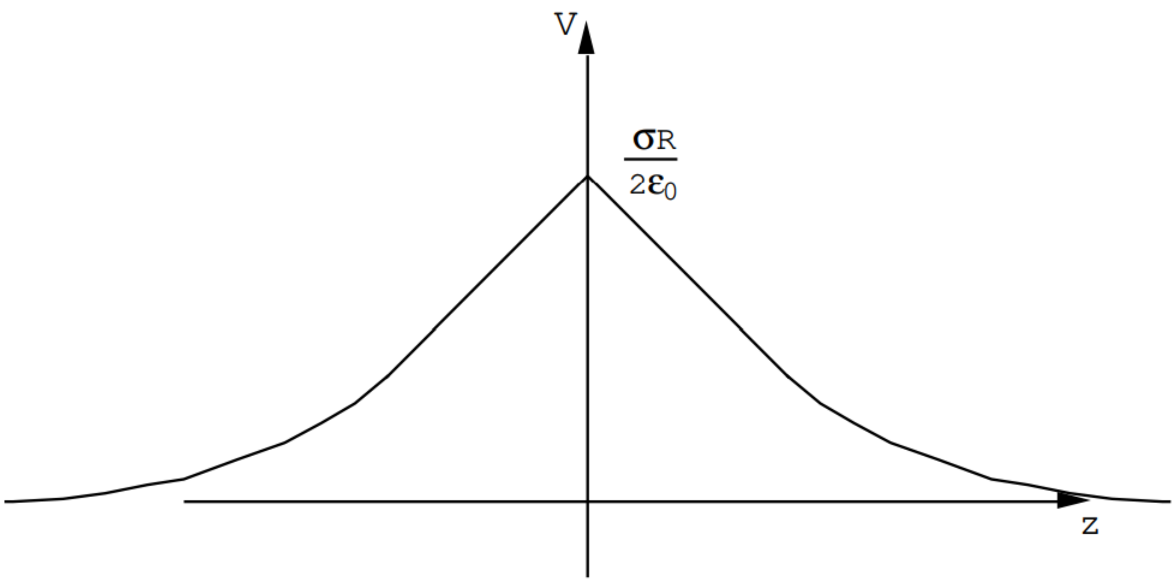
Therefore for z > 0 $\lim_{z\to 0} -\frac{\partial V}{\partial z} = E_z = \frac{\sigma}{2\varepsilon_0}$.

for z < 0 $\lim_{z\to 0} -\frac{\partial V}{\partial z} = E_z = -\frac{\sigma}{2\varepsilon_0}$.

for z > 0 but |z| >> R $V \rightarrow \frac{\pi\sigma R^2}{4\pi\varepsilon_0}\left(\frac{1}{z}\right)$,

for z < 0 but |z| >> R $\mathrm{V} \rightarrow -\frac{\pi\sigma \mathrm{R}^2}{4\pi\varepsilon_0}\left(\frac{1}{\mathrm{z}}\right)$,

ie. the potential looks like a point charge q = $\pi R^2 \sigma$ Coulombs.



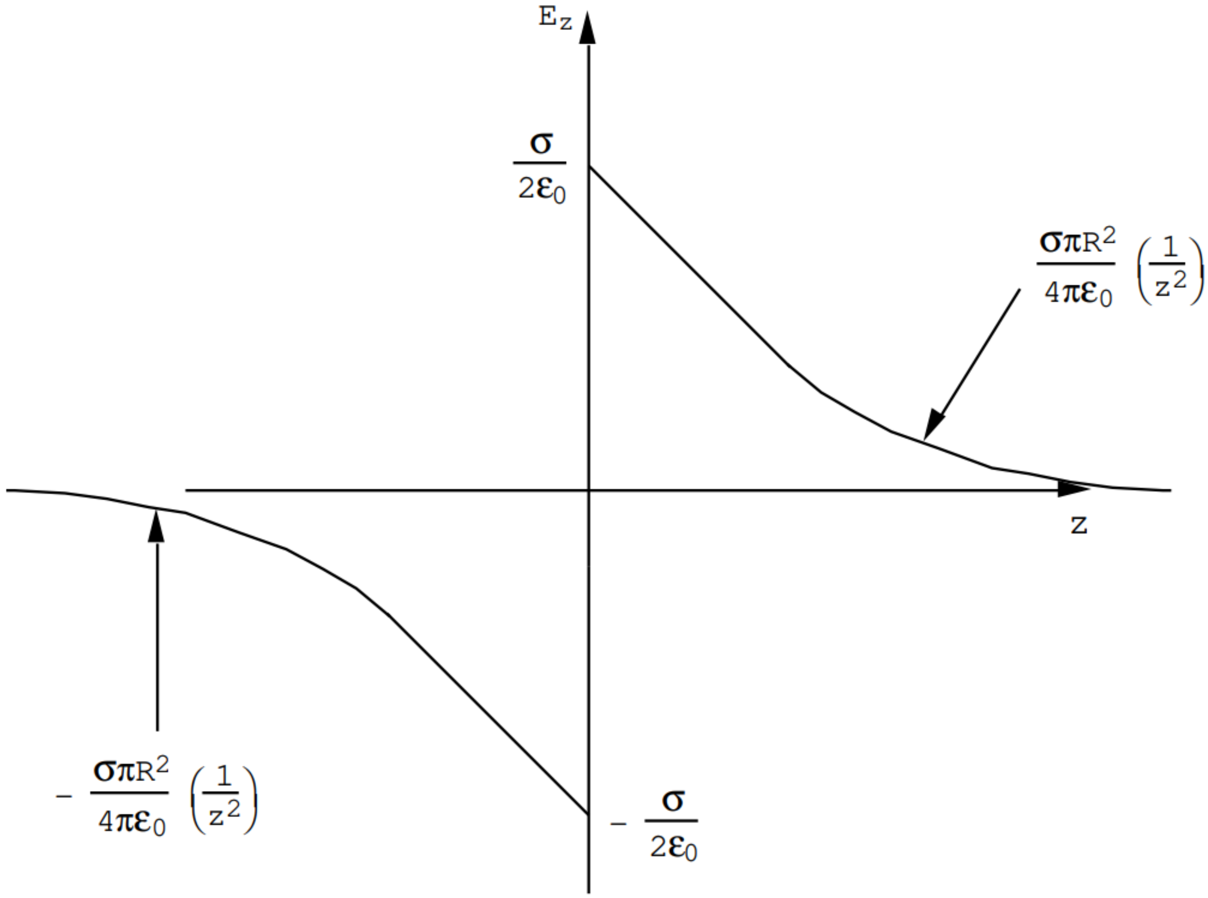
(b) $E_z = -\frac{\partial V}{\partial z}$

for z > 0 $E_z = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)$

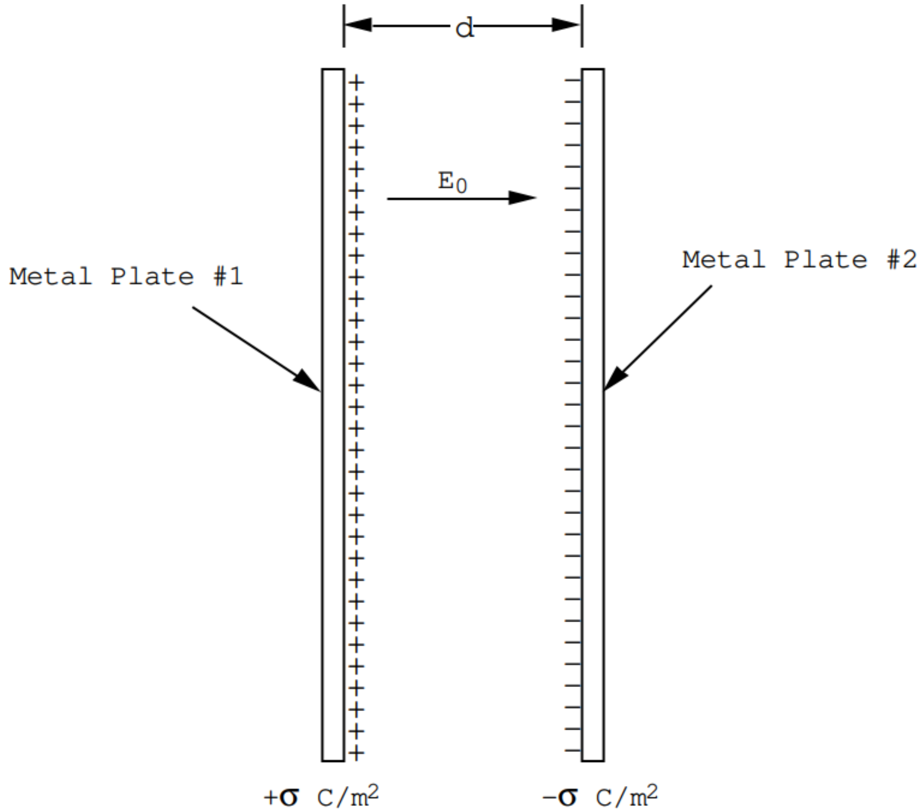
∴ @ z = 0 $E_z = +\frac{\sigma}{2\varepsilon_0}$.

for z < 0 $\mathrm{E_Z} = \frac{\sigma}{2\varepsilon_0}\left(-1 + \frac{\mathrm{z}}{\sqrt{\mathrm{R}^2+\mathrm{z}^2}}\right)$

$\therefore$ @ z = 0 $E_z = -\frac{\sigma}{2\varepsilon_0}$



Problem (2.5)



Two thin infinite plane metal plates are parallel and separated by a gap d meters as shown in the sketch. Plate #1 carries a surface charge density of +**σ** Coulombs/m^2. Plate #2 carries a surface charge of -**σ** Coulombs/m^2. In the metal **E** = 0, otherwise the charges in the metal would move and one would not have an electrostatic problem. Let the direction normal to the plates be the z direction.

(a) Use Gauss' theorem to calculate the electric field strength in the gap between the plates. Let this value be E_o.

(b) What is the value of D_z between the plates?

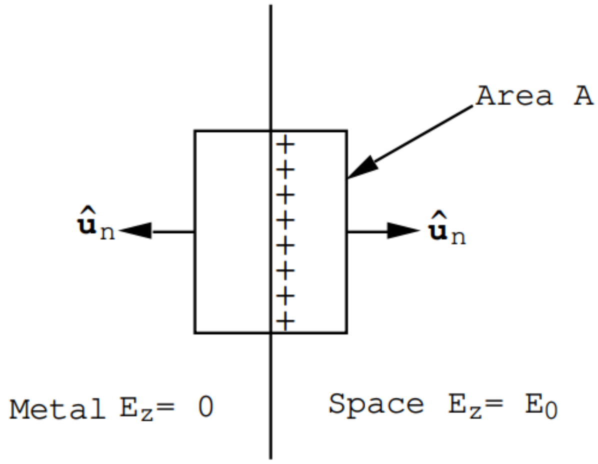
(c) What is the potential difference between the two metal plates?

(d) Suppose that a slab of matter whose thickness was (d/2) meters was slipped between the two metal plates. Suppose further that this slab were polarized such that $P_z = P_o$. What would now be the potential difference between the two plates?

(e) Show that D_z is continuous across the faces of the polarized slab.

Answer (2.5)

(a) Since we have two infinite sheets of charge the electric field is uniform and parallel with z (normal to the plates). Use a pill box which penetrates the metal surface on the left



$$\int_S (\mathbf{E}\cdot\mathbf{n})\mathrm{d}S = Q/\varepsilon_0 \backslash) \qquad \backslash(\therefore \mathrm{E_o A} = \frac{\sigma_A}{\varepsilon_o}$$

or $E_o = \sigma/\varepsilon_o$ Volts/meter.

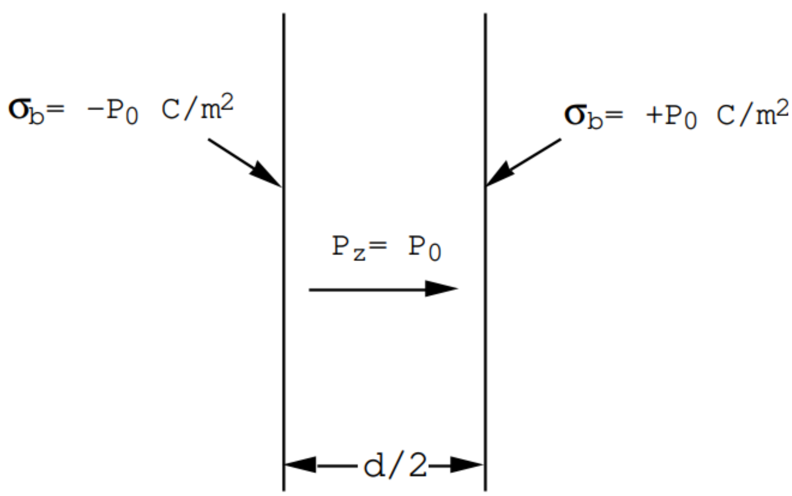
(b) $D_z = \varepsilon_0 E_z + P_z$ Here $P_z = 0$

$\therefore D_z = \varepsilon_0 E_o = \sigma$ Coulombs/m^2.

Notice that for $\boldsymbol{\sigma} = 1$ Coulomb/m^2 the electric field would be 1.13 x 10^{11} Volts/meter. This is huge: air breaks down in a field of ~ 3 x 10^6 Volts/meter. Therefore 1 Coulomb/m^2 is a huge charge density.

(c) $\Delta V = E_o d = \sigma d/\varepsilon_0$ Volts.

(d)



Outside the slab $E_z = 0$

Inside the slab $E_z = -\frac{P_0}{\varepsilon_0}$

When placed between the two metal plates the field distributions add. Therefore in the gap $E_z = E_o$ but in the slab $E_z = E_O - \frac{P_O}{\varepsilon_0}$.

The total potential drop between the plates will be

$$\Delta V = E_O\left(\frac{d}{2}\right) + \left[E_O - \frac{P_O}{\varepsilon_0}\right]\left(\frac{d}{2}\right)$$

$$\therefore \Delta V = E_O d - \frac{P_o d}{2\varepsilon_0} = \frac{\sigma d}{\varepsilon_0} - \frac{P_O d}{2\varepsilon_0}$$

The potential drop is decreased by the presence of the slab.

(e) In the gap between the slab and the plates $D_z = \varepsilon_0 E_o = \sigma$ Coulombs/m^2.

In the slab

$$\begin{aligned} D_z &= \varepsilon_0 E_z + P_0 \\ &= \varepsilon_0 E_0 - P_0 + P_0 \\ &= \varepsilon_0 \mathrm{E}_0 = \sigma \text{ Coulombs } / \mathrm{m}^2 \end{aligned}$$

D_z is continuous across the slab faces!

Problem (2.6)

An ellipsoid of revolution has the shape of a cigar with its axis oriented along z. The length of the cigar is 1 cm and its diameter is $\frac{1}{2}$ cm. The cigar is uniformly polarized: The polarization is given by

$$\mathbf{P} = P_X \hat{\mathbf{u}}_X + P_Y \hat{\mathbf{u}}_Y + P_Z \hat{\mathbf{u}}_Z$$

where

P_x = 0.1 Coulombs/m^2

P_y = 0.2 Coulombs/m^2

P_z = 0.3 Coulombs/m^2.

Calculate the electric field components in the ellipsoid. (They turn out to be huge ~ 10^{10} V/m. Air breaks down in a field of ~ 10^6 V/m).

For the cigar whose length is 2d and whose diameter is 2R the depolarizing coefficient is given by (where $\frac{R}{d} < 1$)

$N_z = \left(\frac{1-\varepsilon^2}{\varepsilon^3}\right)\left(\frac{1}{2}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right) - \varepsilon\right) \quad$ where $\varepsilon = \sqrt{1-\left(\frac{R}{d}\right)^2}$.

Answer (2.6)

For this problem the ratio $\frac{R}{d} = \frac{1}{2}$ and therefore $\varepsilon = \sqrt{3/4}$.

according to my calculations, N_z = 0.1736.

But the sum rule states that $N_x + N_y + N_z = 1$ and therefore the sum $N_x + N_y = 0.826$. By symmetry $N_x = N_y$, therefore $N_x = N_y = 0.413$.

$\mathbf{E} = -\frac{N_X P_X}{\varepsilon_0}\hat{\mathbf{u}}_x - \frac{N_Y P_Y}{\varepsilon_0}\hat{\mathbf{u}}_Y - \frac{N_Z P_z}{\varepsilon_0}\hat{\mathbf{u}}_z,$ where $\varepsilon_0 = 8.85 \times 10^{-12}$MKS ,

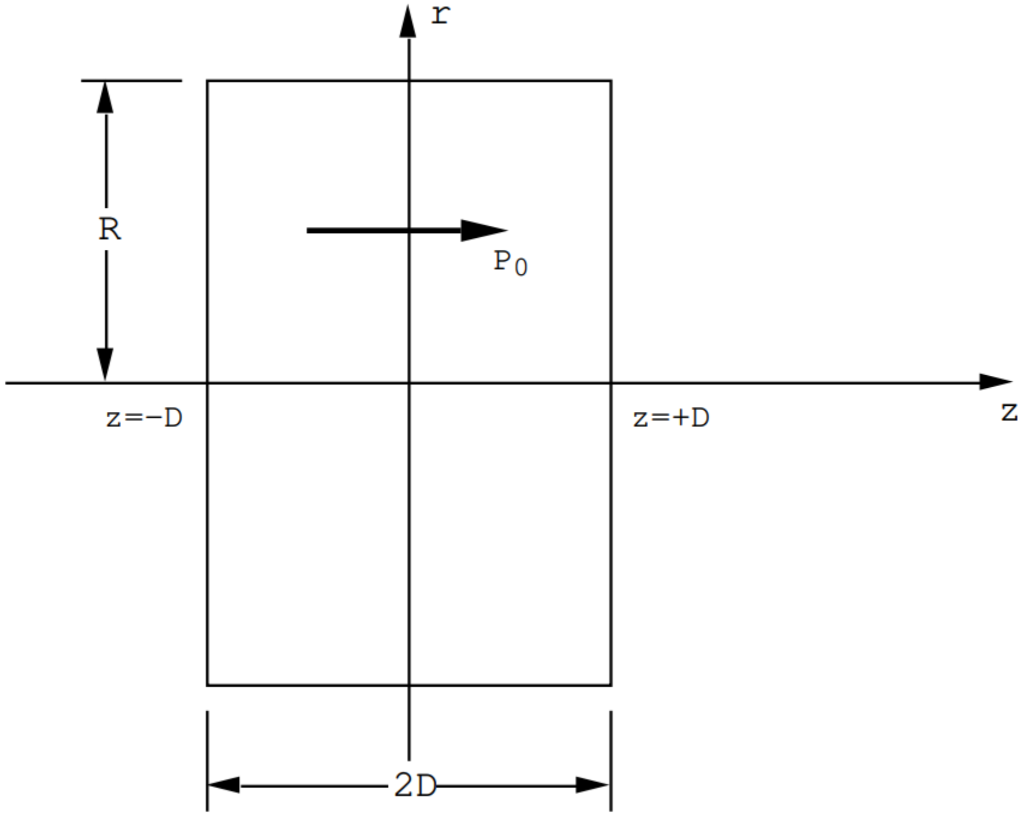
therefore E_x = -0.47x10^{10} Volts/meter,

E_y = -0.94x10^{10} Volts/meter

E_z = -0.59x10^{10} Volts/meter.

Problem (2.7)

An uncharged uniformly polarized disc of radius R meters and thickness 2D meters is shown in the figure. The polarization, P_0 Coulombs/m^2, is directed along the axis of the disc.



(a) Calculate the effective bound charge density, $\boldsymbol{\rho}_b$= - div **P**, everywhere.

(b) Use the bound charge density of part (a) to calculate the potential function along the axis of the disc.

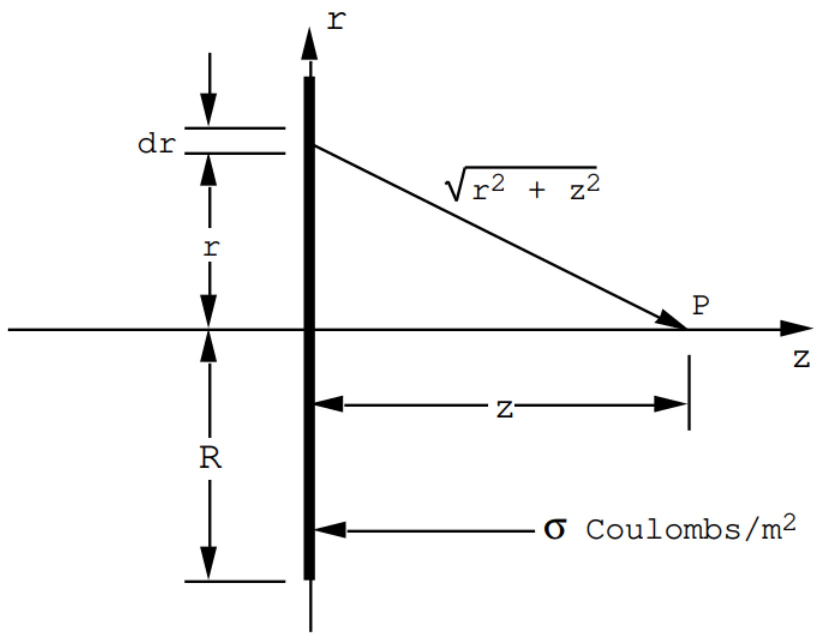
(c) Calculate the electric field along the axis of the disc. Check your answer by looking at three limits: (1) the limit (D/R)<<1; (2) the limit z>0 and (z/R)>>1; and (3) the limit z<0 and (|z|/R)>>1.

(d) Calculate the displacement vector **D**= ε_0**E** + **P** for all points along the axis of the disc.

Answer(2.7)

(a) **P**= 0 everywhere outside the disc and therefore $\boldsymbol{\rho}_b$= - div**P** = 0 everywhere outside the disc. Everywhere inside the disc **P** is constant and so its divergence is zero; $\boldsymbol{\rho}_b$= 0 inside the disc. The discontinuity in the normal component of **P** on the surfaces at z=-D and at z=+D produces surface bound charge densities. The surface charge density carried by the surface at z= -D is $\boldsymbol{\sigma}_b$= -P_0 Coulombs/m^2; the surface charge density carried by the surface at z= +D is $\boldsymbol{\sigma}_b$= +P_0 Coulombs/m^2.

(b) In order to calculate the potential function along the axis of the disc that is generated by the two surface charge distributions, it is useful to begin by considering just one plane surface charge distribution; see the sketch below.



$$\mathrm{dV_p} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi \mathrm{rdr}}{\sqrt{\mathrm{r}^2+\mathrm{z}^2}},$$

and

$$V_p = \frac{\sigma}{4\varepsilon_0}\int_0^R \frac{2rdr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}(\sqrt{R^2+z^2}-\sqrt{z^2}).$$

Notice that the potential function must be symmetric in z. There is a temptation to replace $\sqrt{z^2}$ in Equation (1) by z but that would be quite wrong because z is an odd function. One must replace $\sqrt{z^2}$ by |z|.

For z>0, but (z/R)<<1, $V \rightarrow \frac{\sigma_R}{2\varepsilon_0} - \frac{\sigma_z}{2\varepsilon_0}$.

For z<0, but (|z|/R)<<1, $V \rightarrow \frac{\sigma_R}{2\varepsilon_0} + \frac{\sigma_z}{2\varepsilon_0}$.

Therefore near the charged disc the electric field has the value E_z= +σ/2ε_0 on the right, and E_z= -σ/2ε_0 on the left; this is the expected result based upon an analysis of an infinite uniformly charged plane. Far from the charged disc, $\frac{R}{|z|} << 1$, one finds

For z>0, $V \rightarrow \frac{\pi R^2\sigma}{4\pi\varepsilon_0}\left(\frac{1}{z}\right)$.

For z<0, $V \rightarrow -\frac{\pi R^2\sigma}{4\pi\varepsilon_0}\left(\frac{1}{z}\right)$.

From a great distance the disc of charge looks like a point charge, where Q= πR^2σ Coulombs.

Returning to the problem of the polarized disc, the potential function along the axis of the disc can be written by inspection using Equation (1).

For z≥ D:

$$V(z) = \frac{P_0}{2\varepsilon_0}\quad(\sqrt{R^2+(z-D)^2}-\sqrt{R^2+(z+D)^2}+2D).$$

For -D≤ z ≤ +D:

$$V(z) = \frac{P_0}{2\varepsilon_0}(\sqrt{R^2+(z-D)^2}-\sqrt{R^2+(z+D)^2}+2z).$$

For z≤ -D:

$$V(z) = \frac{P_0}{2\varepsilon_0}(\sqrt{R^2 + (z-D)^2} - \sqrt{R^2 + (z+D)^2} - 2D).$$

(c) The electric field along the axis is given by $E_z = -\frac{\partial V}{\partial z}$.

For z≥ +D:

$$E_z(z) = \frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2 + (z+D)^2}} - \frac{(z-D)}{\sqrt{R^2 + (z-D)^2}}\right).$$

For -D ≤ z ≤ +D:

$$E_z(z) = \frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2 + (z+D)^2}} - \frac{(z-D)}{\sqrt{R^2 + (z-D)^2}} - 2\right).$$

For z ≤ -D:

$$E_z(z) = \frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2 + (z+D)^2}} - \frac{(z-D)}{\sqrt{R^2 + (z-D)^2}}\right).$$

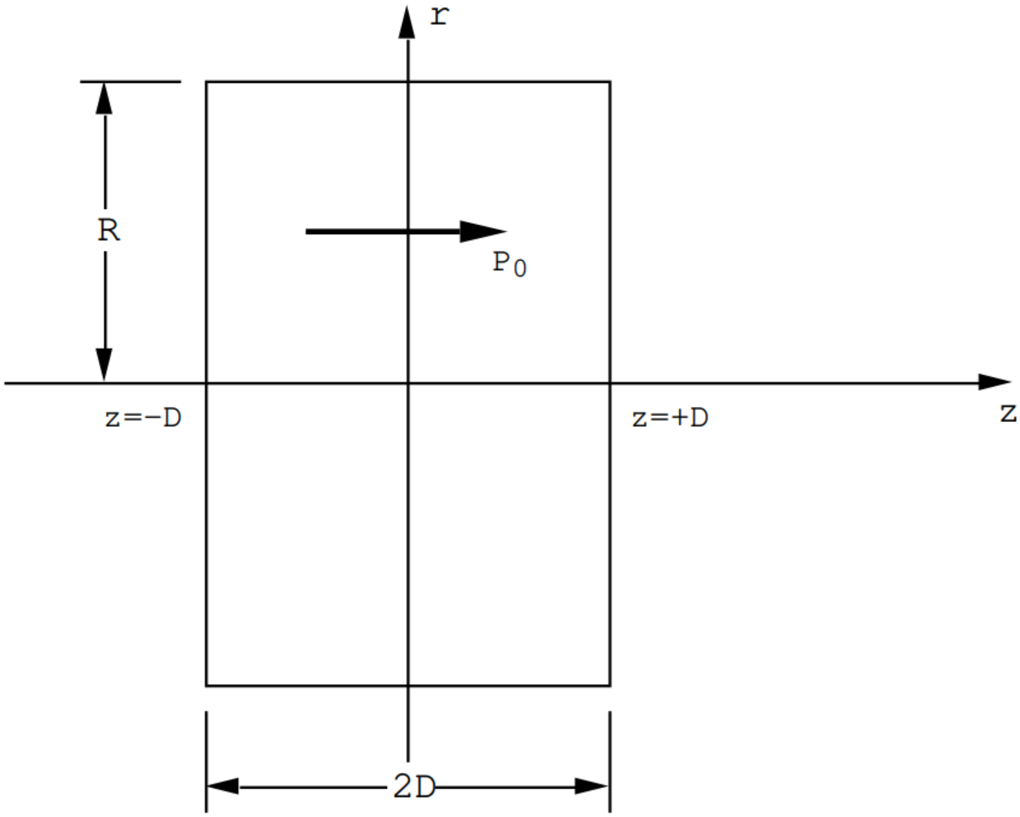
In the limit as $(D/R) \to 0$ the field outside the disc goes to zero like (2D/R); notice that the electric field is symmetric in z. The field inside the disc approaches the value $E_z = \frac{2D}{R} - \frac{P_0}{\varepsilon_0}$; i.e. the field approaches the value expected for an infinite pair of oppositely charged planes in the limit $D \to 0$.

In the limit $|z|/R \to \infty$, the electric field approaches the limit $E_z = \frac{1}{4\pi\varepsilon_0} 4\pi R^2 D P_0 \frac{1}{|z|^3}$; i.e. the field due to a point dipole of moment $p = 2D\pi R^2 P_0$.

(d) By definition $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$. Outside the slab $\mathbf{D} = \varepsilon_0\mathbf{E}$ since $\mathbf{P}=0$. Inside the slab the term P_0 just cancels the constant term in E_z. The displacement vector is continuous through the surfaces of the slab. It is given for all points along the axis by $D_z(z) = \frac{P_0}{2}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}} - \frac{(z-D)}{\sqrt{R^2+(z-D)^2}}\right)$.

Problem (2.8)

An uncharged uniformly polarized disc of radius R meters and thickness 2D meters is shown in the figure. The polarization, P_0 Coulombs/m^2, is directed along the axis of the disc.



Calculate the potential function along the axis of the disc for z≥ D by summing the potential contributions from a collection of point dipoles. Show that

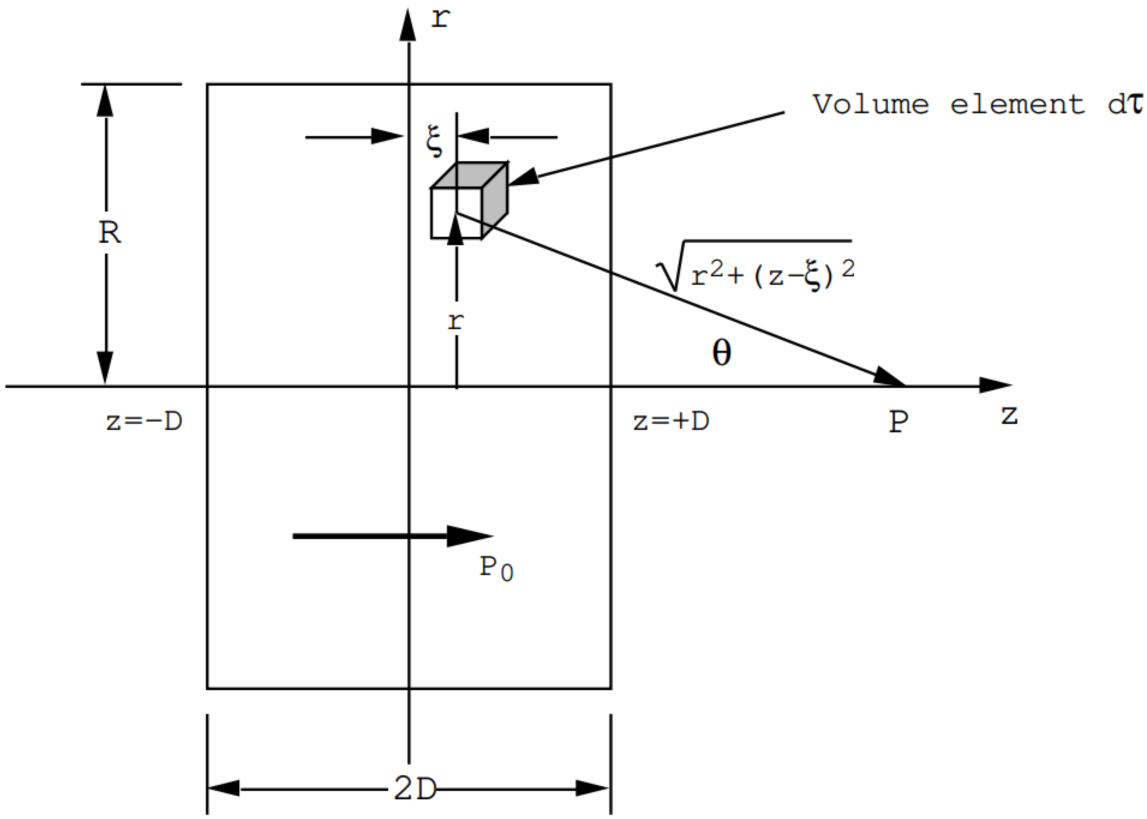
$$V(z)=\frac{P_0}{2\varepsilon_0}(\sqrt{R^2+(z-D)^2}-\sqrt{R^2+(z+D)^2}+2D),$$

and therefore that the field for z≥D is given by

$$E_z(z)=\frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}}-\frac{(z-D)}{\sqrt{R^2+(z-D)^2}}\right).$$

(Compare with the results of Problem(2.7)).

Answer (2.8)



The contribution from the illustrated volume element to the potential at P can be written

$$\mathrm{dV}=\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{P}_0\mathrm{d}\tau\cos\theta}{(r^2+(z-\xi)^2)}.$$

The total potential at z is given by

$$V(z)=\frac{1}{4\pi\varepsilon_0}\int_0^R\int_{-D}^D\frac{P_0(z-\xi)d\tau}{(r^2+(z-\xi)^2)^{3/2}},$$

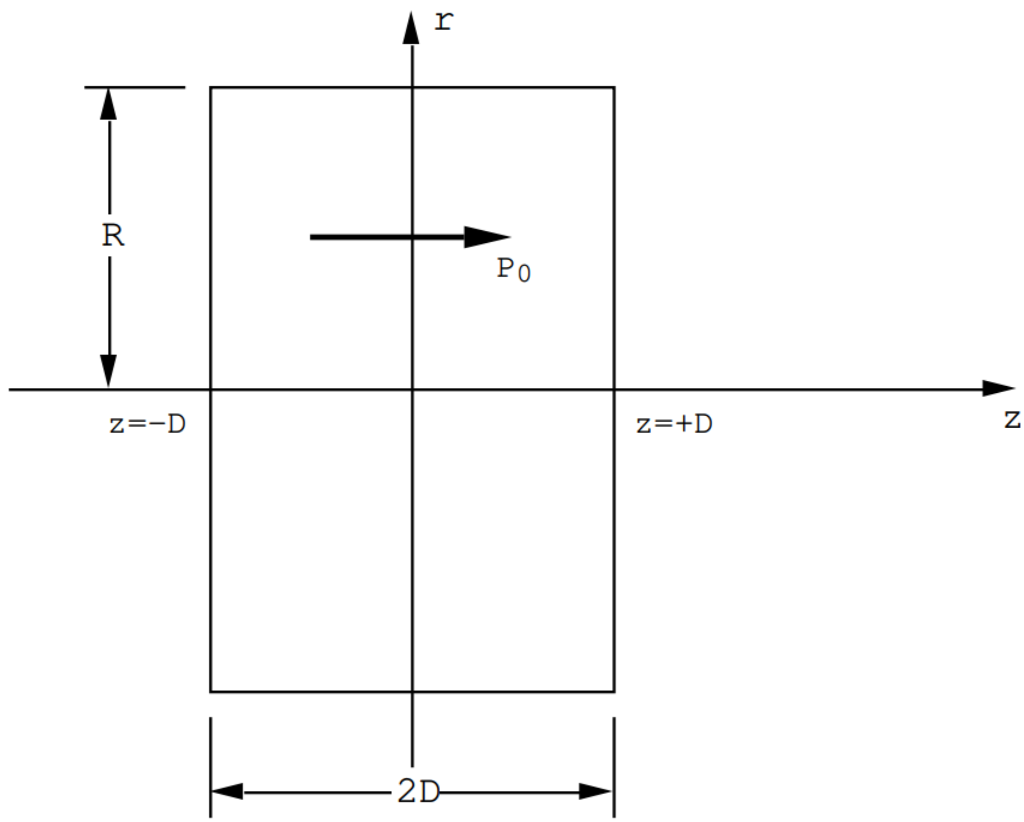
where $d\tau=2\pi r dr d\xi$. The integrations can be readily carried out. The result is

$$V(z)=\frac{P_0}{2\varepsilon_0}(\sqrt{R^2+(z-D)^2}-\sqrt{R^2+(z+D)^2}+2D)\ \text{Volts.}\ .$$

$$E_z(z)=-\frac{\partial V}{\partial z}=\frac{P_0}{2\varepsilon_0}\left(\frac{(z+D)}{\sqrt{R^2+(z+D)^2}}-\frac{(z-D)}{\sqrt{R^2+(z-D)^2}}\right).$$

Problem (2.9)

An uncharged uniformly polarized disc of radius R meters and thickness 2D meters is shown in the figure. The polarization, P_0 Coulombs/m^2, is directed along the axis of the disc.



The electric field at the center of the disc is, by direct calculation (see Problem(2.7)),

$$E_Z(0) = -\frac{P_0}{\varepsilon_0}\left(1 - \frac{D}{\sqrt{R^2 + D^2}}\right).$$

Compare this value for the electric field with that obtained using the depolarizing coefficient for an ellipsoid of revolution having the same ratio of (D/R) as the disc. Carry out the calculation for (a) $(D/R) = \frac{1}{10}$, and for (b) $(D/R) = \frac{1}{100}$.

Answer (2.9)

The appropriate depolarizing coefficient for a pancake shaped ellipsoid is stated in the E&M notes, Figure (2.15):

$$N_z = \frac{R^2 D}{(R^2 - D^2)^{3/2}}\left(\frac{\sqrt{R^2 - D^2}}{D} - \mathrm{Tan}^{-1}\left(\frac{\sqrt{R^2 - D^2}}{D}\right)\right) \text{ for } (D/R) < 1.$$

If (D/R)<<1, $N_z \cong 1 - \frac{\pi}{2}\left(\frac{D}{R}\right)$. Using the exact expression for N_z

(a) For R= 10D one finds N_z= 0.861, and this gives

$$E_z(0) = -0.861\left(\frac{P_0}{\varepsilon_0}\right).$$

By direct calculation, the exact value is $-0.8995\left(\frac{P_0}{\varepsilon_0}\right)$.

(b) For R= 100D one finds N_z= 0.9845, and therefore

$$E_z(0) = -0.9845\left(\frac{P_0}{\varepsilon_0}\right).$$

By direct calculation, the value is $-0.9900\left(\frac{P_0}{\varepsilon_0}\right)$.

Cylindrical discs are often approximated as ellipsoids of revolution, especially in magnetic problems, for purposes of estimating the first order correction to the field at the center of a disc having an infinite radius, $E_z(0) = -\frac{P_0}{\varepsilon_0}$.

Problem (2.10)

An uncharged sphere of radius R is polarized in such a way that the polarization vector **P** is radial, and its magnitude is given by $\mathrm{P_r(r) = P_0\left(\frac{r}{R}\right)^2}$.

(a) Calculate the electric field at all points inside the sphere.

(b) Calculate the electric field at all points outside the sphere.

Answer (2.10)

The polarization vector possesses only a radial component, therefore $\operatorname{div}\mathbf{P} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right)$. The bound charge density is $\rho_b = -\operatorname{div}\mathbf{P} = -\left(\frac{4P_0}{R^2}\right) r$.

(a) The electric field inside the sphere can be calculated from Gauss' theorem because the field must be radial by symmetry. Thus

$$4\pi r^2 E_r = \frac{1}{\varepsilon_0}\int_0^r \rho_b 4\pi r^2 dr = -\frac{4\pi P_0 r^4}{\varepsilon_0 R^2}$$

or

$$\mathrm{E_r} = -\frac{\mathrm{P_0}}{\varepsilon_0}\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2.$$

(b) There is a surface charge density on the sphere, $\rho_s = P_0$ Coulombs/m^2 because of the discontinuity in the normal component of the polarization vector. The total charge contained within a sphere whose radius is slightly larger than the radius R is zero. Therefore the electric field is zero everywhere outside the sphere.

Problem (2.11)

Consider an uncharged sphere having a very large radius R which is uniformly polarized along the z direction. The polarization is $P_z = P_0$.

(a) What is the direction and strength of the electric field inside the sphere? How does this field depend upon the radius of the sphere?

(b) A tiny spherical cavity of radius b, b/R<<1, is cut out of the sphere at some point not too far from its center. The polarization in the remainder of the big sphere remains unchanged. Use the principle of superposition to calculate the electric field strength inside the small cavity of radius b.

Answer (2.11)

(a) The depolarizing factors obey the sum rule $N_x + N_y + N_z = 1$. But for a sphere $N_x=N_y=N_z$, therefore each is equal to (1/3). Inside the sphere $E_z = -\frac{P_0}{3\varepsilon_0}$. This fieldstrength does not depend at all on the radius of the sphere.

(b) The field inside the tiny sphere of polarized material which has been cut out of the big sphere is $\mathrm{E_z} = -\frac{\mathrm{P_0}}{3\varepsilon_0}$. When this is added to the field in the cavity of radius b it must give a total field equal to the field strength before the tiny sphere was removed. It can therefore be concluded that the field in the cavity is zero!

Problem (2.12)

Consider an uncharged cylinder of radius R and length L. The axis of the cylinder lies along the zaxis. Let both R and L become infinitely large, but in such a way that the ratio (R/L)→0.

(a) Let the material of the cylinder be polarized along its length, i.e. $P_z = P_0$. What is the direction and strength of the electric field inside the cylinder?

(b) Let the material of the cylinder be polarized transverse to its axis, along the x-axis say; i.e. $P_x = P_0$. What is the direction and magnitude of the electric field inside the cylinder?

Answer (2.12)

(a) The depolarizing coefficient for a very long needle in the direction of its length is zero. Therefore, when the cylinder is polarized along its axis there is no electric field inside it.

(b) By symmetry the transverse depolarizing coefficients must be equal: $N_x=N_y$. But from the sum rule $N_x+N_y= 1$, since $N_z=0$. It follows that $N_x=N_y=(1/2)$. The electric field inside the cylinder is given by $E_x=-\frac{P_0}{2\varepsilon_0}$.

This problem demonstrates that the dipole field has such a long range that the electric field inside an infinitely large body depends upon its shape.

Problem (2.13)

Consider two charges q1=Q and q2=- αQ. The charge q1 is located at (-b,0,0); the charge q2 is located at (b,0,0).

(a) Let α=2. Show that the equipotential V=0 is a sphere of radius $R=\frac{4b}{3}$ centered at $x_0=-\frac{5b}{3}$.

(b) Let α= 1/2. Show that the equipotential V=0 is again a sphere of radius $R=\frac{4b}{3}$ but centered at $x_0=+\frac{5b}{3}$.

The equipotential V=0 can be replaced by a grounded metal sphere without disturbing the potential distribution outside the sphere. This construction therefore provides the solution of the problem of a point charge brought up to a grounded conducting sphere.

Answer (2.13)

$V_p=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1}+\frac{q_2}{r_2}\right)$, where $q_1=Q$ and $q_2=-\alpha_Q$

$r_1=\sqrt{(x+b)^2+y^2+z^2}$, and $r_2=\sqrt{(x-b)^2+y^2+z^2}$. Therefore V=0 when $r_2=\alpha_{r_1}$, or $(r_2)^2=\alpha^2(r_1)^2$. A bit of algebra gives

$$x^2+y^2+z^2+2bx\left(\frac{\alpha^2+1}{\alpha^2-1}\right)+b^2=0 \qquad (1)$$

By adding $b^2\left(\frac{\alpha^2+1}{\alpha^2-1}\right)^2$ to both sides of (1) this equation can be written

$$\left(x+b\left(\frac{\alpha^2+1}{\alpha^2-1}\right)\right)^2+y^2+z^2=\left(\frac{2\alpha b}{|\alpha^2-1|}\right)^2 \qquad (2)$$

This is the equation of a sphere centered at $x_0=-b\frac{\alpha^2+1}{\alpha^2-1}$, and having a radius $\mathrm{R}=\frac{2\alpha\mathrm{b}}{|\alpha^2-1|}$.

Problem (2.14)

Consider two charges q_1=Q and q_2= -Q. The charge q_1 is located at (-b,0,0); the charge q_2 is located at (b,0,0).

(a) Show that V=0 on the plane x=0. The region to the right of x=0 can be replaced by a conducting metal without disturbing the potential in the region x<0. This construction provides the solution of the problem of a point charge brought up to a grounded conducting plane.

(b) Show that the charge q_1=Q is attracted to a grounded metal plane with a force

$$\mathrm{F_x}=\frac{\mathrm{Q}^2}{4\pi\varepsilon_0}\frac{1}{4\mathrm{b}^2}\ \text{Newtons}.$$

Answer (2.14)

The electric field at x=-b is just that due to a point charge -Q located at x=+b. Therefore $E_x=\frac{Q}{4\pi\varepsilon_0}\frac{1}{4b^2}$, and E_y= E_z= 0. Thus the force on the charge pulling it towards the metal surface is just

$$\mathrm{F_x}=\frac{\mathrm{Q}^2}{4\pi\varepsilon_0}\frac{1}{4\mathrm{b}^2}\ \text{Newtons.}\ .$$