9.2: Phasors

It is very convenient to represent sinusoidal functions i.e. sines and cosines, by complex exponential functions when dealing with linear differential equations such as Maxwell's equations. For example

$$\mathrm{y} = \mathrm{A} \exp[i(\mathrm{kx} - \omega \mathrm{t})]$$

means

$$\mathrm{y} = \text{Real Part}(\mathrm{A} \exp[i(\mathrm{kx} - \omega \mathrm{t})]) = \mathrm{A} \cos(\mathrm{kx} - \omega \mathrm{t})$$

if A is a real number, or if A= a+ib is a complex number

$$\begin{aligned} \mathrm{y} &= \text{Real Part}((\mathrm{a}+\mathrm{ib}) \exp[i(\mathrm{kx} - \omega \mathrm{t})]), \\ &= \mathrm{a} \cos(\mathrm{kx} - \omega \mathrm{t}) - \mathrm{b} \sin(\mathrm{kx} - \omega \mathrm{t}). \end{aligned} \tag{9.2.1}$$

A complex amplitude represents a phase shift. Since

$$\cos(\alpha + \beta) = \cos\beta \cos\alpha - \sin\beta \sin\alpha, \tag{9.2.2}$$

Equation (9.2.1) can be written

$$y = \sqrt{a^2 + b^2} \cos(kx - \omega t + \beta),$$

where

$$\sin\beta = \frac{b}{\sqrt{a^2 + b^2}}$$

and

$$\cos\beta = \frac{\mathrm{a}}{\sqrt{\mathrm{a}^2 + \mathrm{b}^2}}, \tag{9.2.3}$$

or

$$\tan\beta = \frac{b}{a}. \tag{9.2.4}$$

In phasor notation

$$\mathrm{y} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2} \exp[i(\mathrm{kx} - \omega \mathrm{t} + \beta)] = (\sqrt{\mathrm{a}^2 + \mathrm{b}^2} \exp i\beta) \exp[i(\mathrm{kx} - \omega \mathrm{t})].$$

The prefactor $(\sqrt{a^2 + b^2} \exp i\beta)$ is just the polar representation of the complex number (a+ib).

Derivatives are particularly convenient in the complex phasor notation because the derivative of an exponential function gives back the same exponential function multiplied by a constant (usually a complex number).

One must be careful when calculating energy densities or when calculating the Poynting vector using the phasor notation because the Real Part of the product of two complex exponentials is not the same as the product of the two Real sinusoidal functions that appear in the product. There is, however, a trick which is useful. Consider a plane wave propagating along z and which can be described by

$$\begin{aligned} \mathrm{E}_{\mathrm{x}} &= \mathrm{E}_0 \mathrm{e}^{i(\mathrm{kz} - \omega t + \phi_1)} \\ \mathrm{H}_y &= \mathrm{H}_0 \mathrm{e}^{i(\mathrm{k}z - \omega t + \phi_2)} \end{aligned} \tag{9.2.5}$$

These electric and magnetic fields are not in phase because ϕ_1 and ϕ_2 are different, and therefore this plane wave is not propagating in free space. It corresponds to a wave propagating in a medium characterized by a complex dielectric constant as will be discussed in a later chapter. Now calculate the time average of the Poynting vector, $\vec{\mathrm{S}} = \vec{\mathrm{E}} \times \vec{\mathrm{H}}$, using Equations (9.2.5). It is asserted that **the time average of the product of two phasors can be obtained as one-half of the real part of the product of one phasor with the complex conjugate of the other phasor.**

Thus

$$< S_z >= \frac{1}{2} \mathrm{Real}\left(E_x H_y^*\right) = \frac{1}{2} \mathrm{Real}(E_x^* H_y), \tag{9.2.6}$$

where E_x^* means the complex conjugate of E_x, and H_y^* means the complex conjugate of H_y. Using Equation (9.2.5) in Equation (9.2.6) one obtains

$$< S_z >= \frac{1}{2} \mathrm{Real}(E_0 H_0 \exp i\,(\phi_1 - \phi_2)) = \frac{E_0 H_0}{2} \cos(\phi_1 - \phi_2), \tag{9.2.7}$$

since E_0, H_0 are taken to be real amplitudes. Eqn.(9.2.6) can be checked by writing the fields (9.2.5) in real form:

$$S_z = E_0 H_0 \cos(kz - \omega t + \phi_1) \cos(kz - \omega t + \phi_2),$$

or, using Equation (9.2.2),

$$\begin{aligned} S_z = E_0 H_0 &(\cos\phi_1 \cos(kz - \omega t) - \sin\phi_1 \sin(kz - \omega t)) \\ &\times (\cos\phi_2 \cos(kz - \omega t) - \sin\phi_2 \sin(kz - \omega t))\,, \end{aligned}$$

or upon an explicit multiplication

$$\begin{aligned} S_z = &\ E_0 H_0 \left(\cos\phi_1 \cos\phi_2 \cos^2(kz - \omega t) - \cos\phi_2 \sin\phi_1 \sin(kz - \omega t) \cos(kz - \omega t)\right) \\ &- E_0 H_0 \left(\sin\phi_2 \cos\phi_1 \sin(kz - \omega t) \cos(kz - \omega t) - \sin\phi_1 \sin\phi_2 \sin^2(kz - \omega t)\right) \end{aligned}$$

Upon taking the time averages one obtains

$$< S_z >= \frac{E_0 H_0}{2} (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)\,.$$

This equation can be written compactly as

$$< S_z >= \frac{E_0 H_0}{2} \cos(\phi_1 - \phi_2),$$

in agreement with the result Equation (9.2.7) obtained using the prescription (9.2.6).

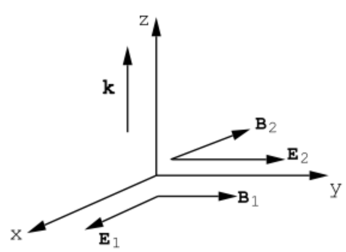

Figure 9.2.3: Two coherent plane waves having orthogonal polarizations, and propagating along the z-direction. Each wave is characterized by the same circular frequency, ω, and the same wave-vector, $\vec{k}$, where $k_z = |\vec{k}| = \omega/c$. Let the fields in wave number (2) be shifted in phase by ϕ radians relative to the fields in wave number (1).