

## 5.4: The Magnetostatic Field Energy

Energy is required to establish a magnetic field. The energy density stored in a magnetostatic field established in a linear isotropic material is given by

$$W_B = \frac{\mu}{2} H^2 = \frac{\vec{H} \cdot \vec{B}}{2} \quad \text{Joules / m}^3. \quad (5.4.1)$$

The total energy stored in the magnetostatic field is obtained by integrating the energy density,  $W_B$ , over all space (the element of volume is  $d\tau$ ):

$$U_B = \iiint_{\text{Space}} d\tau \left( \frac{\vec{H} \cdot \vec{B}}{2} \right). \quad (5.4.2)$$

This expression for the total energy,  $U_B$ , can be transformed into an integral over the sources of the magnetostatic field. The transformation can be carried out by means of the vector identity

$$\text{div}(\vec{A} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{H}). \quad (5.4.3)$$

(There is a nice discussion of this identity in The Feynman Lectures on Physics, Vol.II, section 27.3, by R.P.Feynman, R.B.Leighton, and M.Sands, Addison-Wesley, Reading, Mass.,1964). Proceed by integrating Equation (5.4.3) over all space, then use Gauss' theorem to transform the left hand side into a surface integral. The result is

$$\iiint_{\text{Surface}} (\vec{A} \times \vec{H}) \cdot d\vec{S} = \iiint_{\text{Volume}} d\tau (\vec{H} \cdot \vec{B} - \vec{J}_f \cdot \vec{A}), \quad (5.4.4)$$

where  $d\vec{S}$  is the element of surface area,  $\vec{B} = \vec{\nabla} \times \vec{A} = \text{curl}(\vec{A})$ , and  $\vec{\nabla} \times \vec{H} = \text{curl}(\vec{H}) = \vec{J}_f$ . Here  $\vec{A}$  is the vector potential and  $\vec{J}_f$  is the current density. When the integrals in Equation (5.4.4) are extended over all space the surface integral goes to zero: the surface area of a sphere of large radius  $R$  is proportional to  $R^2$  but for currents confined to a finite region of space  $|\vec{A}|$  must decrease at least as fast as a dipole source, i.e.  $\propto 1/R^2$ , and  $|\vec{H}|$  must decrease at least as fast as  $1/R^3$ . It follows that in the large  $R$  limit the surface integral must go to zero like  $1/R^3$ . This requires the two terms on the right hand side of (5.4.4) to be equal, and this result can be used to rewrite the expression (5.4.2) in terms of the vector potential and the source current density:

$$U_B = \frac{1}{2} \iiint_{\text{Space}} d\tau (\vec{H} \cdot \vec{B}) = \frac{1}{2} \iiint_{\text{Space}} d\tau (\vec{J}_f \cdot \vec{A}). \quad (5.4.5)$$

In many problems the current density is confined to a wire whose dimensions are small compared with other lengths in the problem. For such a circuit the contribution to the second volume integral in (5.4.5) vanishes except for points within the wire, and therefore the volume integral can be replaced by a line integral along the wire providing that the variation of the vector potential,  $\text{vec } A$ , over the cross-section of the wire can be neglected. For a wire of negligible thickness

$$\iiint_{\text{Space}} d\tau (\vec{J}_f \cdot \vec{A}) \rightarrow I \oint_C \vec{A} \cdot d\vec{L}, \quad (5.4.6)$$

where  $I$  is the current through the wire; the current must be the same, of course, at all points along the circuit. The line integral of the vector potential around a closed circuit is equal to the magnetic flux,  $\Phi$ , through the circuit. This equivalence can be seen by using the definition  $\vec{B} = \text{curl}(\vec{A})$  along with Stokes' theorem to transform the integral for the flux:

$$\Phi = \iint_S \vec{B} \cdot d\vec{S} = \iint_S \text{curl}(\vec{A}) \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{L}, \quad (5.4.7)$$

where the curve  $C$  bounds the surface  $S$ . Combining Equations (5.4.7) and (5.4.5), the magnetic energy associated with a single circuit can be written

$$U_B = \frac{1}{2} \iiint_{\text{Space}} d\tau (\vec{J}_f \cdot \vec{A}) = \frac{1}{2} I \Phi, \quad (5.4.8)$$

and for a number of circuits,  $N$ ,

$$U_B = \frac{1}{2} \sum_{k=1}^N I_k \Phi_k. \quad (5.4.9)$$

The latter expression is similar to Equation (3.3.6) for the electrostatic energy associated with a collection of charged conductors: currents in the magnetostatic case play a role similar to that of charges in the electrostatic case, and flux plays a role that is similar to the role played by the potentials.

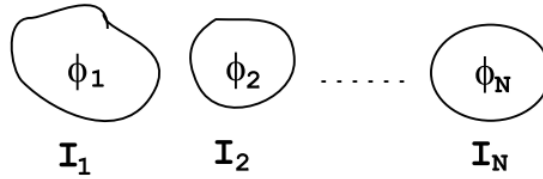


Figure 5.4.7:  $N$  circuits embedded in a linear, isotropic medium. The magnetic fluxes,  $\Phi_k$ , are linear functions of the currents  $I_j$ . The inductance coefficients satisfy the symmetry relations  $L_{MN} = L_{NM}$ .

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