2.8: Appendix 2A

2.8 Appendix 2A.

It was pointed out in sections 2.2.1 and 2.2.2 above that the potential function, V($\vec{R}$) generated by a distribution of electric dipoles, P($\vec{r}$), can be calculated in two ways:

$$(1)\quad \mathrm{V}(\overrightarrow{\mathrm{R}})=\frac{1}{4\pi\epsilon_0}\int\int\int_{S_{\text{pace}}\, dV_{\text{vol}}}\frac{(-\operatorname{div}(\overrightarrow{\mathrm{P}}))}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}. \tag{2.8.1}$$

This equation for the potential function is calculated from the distribution of bound charges, $\rho_b = -\mathrm{div}(\vec{P})$. The second equation for the potential function can be written as the potential due to point dipoles $\vec{P}\mathrm{dV}_{\mathrm{vol}}$ summed over the entire distribution of dipoles:

$$(2)\quad \mathrm{V}(\overrightarrow{\mathrm{R}})=\frac{1}{4\pi\epsilon_0}\int\int\int_{S_{pace}} dV_{\text{vol}}\,\frac{\overrightarrow{\mathrm{P}}\cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}. \tag{2.8.2}$$

These two formulae, Equations (2.8.1 and 2.8.2), give the same potential function apart from a possible constant that has no effect on the resulting electric field. This statement can be proved by applying Gauss' Theorem to the function

$$\operatorname{div}\left(\frac{\overrightarrow{\mathrm{P}}}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right)=\operatorname{div}\left(\frac{\overrightarrow{\mathrm{P}}}{\sqrt{[X-x]^2+[Y-y]^2+[Z-z]^2}}\right).$$

The divergence is calculated with respect to the co-ordinates of the source point, (x,y,z):

$$\operatorname{div}\left(\frac{\overrightarrow{\mathrm{P}}}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right)=\frac{\partial}{\partial x}\left(\frac{\mathrm{P}_x}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right)+\frac{\partial}{\partial y}\left(\frac{\mathrm{P}_y}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right)+\frac{\partial}{\partial z}\left(\frac{\mathrm{P}_z}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right).$$

By direct differentiation one can readily show that

$$\operatorname{div}\left(\frac{\overrightarrow{\mathrm{P}}}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}\right)=\frac{\operatorname{div}(\overrightarrow{\mathrm{P}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}+\frac{\overrightarrow{\mathrm{P}}\cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}.$$

Remember that the differentiations are with respect to the co-ordinates of $\vec{r}$, (x,y,z), and not with respect to the observer co-ordinates $\vec{R}$, (X,Y,Z). Integrate the above equation over a volume, V_{vol}, bounded by a surface S and apply Gauss' Theorem, section 1.3.3, to the term on the left. The result is

$$\int\int_S\frac{dS(\overrightarrow{\mathrm{P}}\cdot\hat{\mathbf{n}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}=\int\int\int_{V_{\text{rot}}}\frac{dV_{\text{vol}}\,div(\overrightarrow{\mathrm{P}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}+\int\int\int_{V_{\text{rot}}}\frac{dV_{\text{vol}}\,\overrightarrow{\mathrm{P}}\cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}.$$

Now let the volume V_{vol} become very large so that the surface S recedes to infinity. If the polarization distribution is limited to a finite region of space, as we shall assume, the surface integral must vanish because the polarization density on the surface, S, is zero. We are left with the identity

$$-\int\int\int_{V_{\text{vol}}}\frac{dV_{\text{vol}}\,div(\overrightarrow{\mathrm{P}})}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|}=\int\int\int_{V_{\text{red}}}\frac{dV_{\text{vol}}}{|\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}|^3}\cdot(\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}}) \tag{2.8.3}$$

Upon multiplying both sides of Equation (2.8.3) by $1/(4\pi\epsilon_0)$ one obtains the integral of Equation (2.8.1) on the left and the integral of Equation (2.8.2) on the right. It follows that the same value for the potential will be obtained, aside from a possible unimportant constant, whether one uses the formulation based upon the potential for a point charge or the formulation based upon the potential function for a point dipole.