# 10.8: Metals at Radio Frequencies

We are interested in the practical case of metals at room temperature and frequencies less than 1000 GHz so that the metallic response to an electric field may be characterized by its dc conductivity, $\sigma_0$. We are also interested in the general case of radiation at oblique incidence. In this relatively low frequency regime the conduction current density in a metal is much larger than the displacement current density; i.e. for a time dependence $\sim \exp(-i\omega t)$ one finds that $\sigma_0 \gg \omega\epsilon_r\epsilon_0$ in the Maxwell equation

$$\operatorname{curl}(\vec{H}) = \sigma_0\vec{E} - i\omega\epsilon_r\epsilon_0\vec{E}.$$

The relevant Maxwell's equations for low frequency fields in a non-magnetic metal, $\mu \cong \mu_0$, become

$$\operatorname{curl}(\vec{E}) = -\mu_0\frac{\partial\vec{H}}{\partial t} \tag{10.8.1}$$

and

$$\operatorname{curl}(\vec{H}) = \sigma_0\vec{E}. \tag{10.8.2}$$

Take the curl of (10.68) and use a time variation $\sim \exp(-i\omega t)$ to obtain

$$\operatorname{curlcurl}(\vec{E}) = i\omega\mu_0\sigma_0\vec{E},$$

or

$$-\nabla^2\vec{E} + graddiv(\vec{E}) = i\omega\mu_0\sigma_0\vec{E}. \tag{10.8.3}$$

However, the divergence of any curl of a vector is equal to zero, and consequently $\operatorname{div}(\vec{E}) = 0$ from Equation (10.8.2). It follows that for a metal at low frequencies the electric field components must satisfy the equation

$$\nabla^2 E_\alpha = -i\omega\mu_0\sigma_0 E_\alpha, \tag{10.8.4}$$

where $\alpha$ stands for each of the three cartesian components x,y, or z.

The solution of the problem of a plane wave incident at an oblique angle on a plane metallic surface proceeds just as for the general case of oblique incidence discussed in section(10.5). Two cases are of interest: (1) S-polarization in which the electric vector of the incident wave is parallel with the plane interface, see Figure (10.4.6), and (2) P-polarization in which the electric vector of the incident wave lies in the plane of incidence and the magnetic vector therefore lies parallel with the interface, see Figure (10.5.7).

## 10.8.1 S-polarization.

Using the co-ordinate system of Figure (10.4.6) the fields in the metal can be written

$$\begin{aligned} E_y &= E_T \exp(i[xk\sin\theta + zk_z - \omega t]), \\ H_x &= -\frac{k_z}{\omega\mu_0}E_T \exp(i[xk\sin\theta + zk_z - \omega t]), \\ H_z &= \frac{\sin\theta}{Z_0}E_T \exp(i[xk\sin\theta + zk_z - \omega t]), \end{aligned} \tag{10.8.5}$$

where $Z_0 = c\mu_0$= 377 Ohms, and $k = \omega/c$. The wave-vector component $k_z$ in the metal must be chosen so that $E_y$ satisfies Equation (10.8.4), i.e.

$$k_z^2 = i\omega\mu_0\sigma_0 - \left(\frac{\omega\sin\theta}{c}\right)^2.$$

Apparently the wave-vector component $k_z$ depends upon the angle of incidence of the driving incident plane wave. This dependence is illusory because $\mu_0\sigma_0$ is much larger than $w/c^2$ : for copper at 100 GHz $w/c^2 = 7\times10^{-6}$ whereas $\mu_0\sigma_0$= 81. For the range of frequencies and conductivities that are of interest here the term in $\sin^2\theta$ is negligible compared with the term proportional to the conductivity, and for any angle of incidence one may use

$$k_2^2 = i\omega\mu_0\sigma_0,$$

and

$$\mathrm{k_z} = \sqrt{\frac{\omega\mu_0\sigma_0}{2}}(1+i) = \frac{(1+i)}{\delta}, \tag{10.8.6}$$

where $\delta$ is a length that is inversely proportional to the square root of the frequency. It is handy to remember that $\delta$= 2μm for copper at 1 GHz and at room temperature.

At the metal-vacuum interface the tangential components of $\vec{E}$ and $\vec{H}$ must be continuous through the surface. These boundary conditions at z=0 result in two equations for the two unknown electric field amplitudes $E_R$ and $E_T$; $E_R$ is the amplitude of the wave reflected from the metal surface, and $E_T$ is the amplitude of the electric field transmitted into the metal. The solutions of these equations are

$$\begin{aligned}\frac{E_R}{E_0} &= \left(\frac{(\omega/c)\cos\theta - k_z}{(\omega/c)\cos\theta + k_z}\right), \\ \frac{\mathrm{E_T}}{\mathrm{E_0}} &= \left(\frac{2(\omega/\mathrm{c})\cos\theta}{(\omega/\mathrm{c})\cos\theta + \mathrm{k}_z}\right).\end{aligned} \tag{10.8.7}$$

The wave-vector $k_z$ is very large compared with (ω/c) cos θ so that if one divides the equations in (10.74) by $k_z$ top and bottom the reflection and transmission coefficients can be expressed as a power series expansion in the small parameter ω cos θ/($ck_z$): for example

$$\frac{\mathrm{E_R}}{\mathrm{E_0}} = \frac{-1+\frac{\omega\cos\theta}{\mathrm{ck_z}}}{1+\frac{\omega\cos\theta}{\mathrm{ck_z}}} \cong -1+\left(\frac{2\omega\cos\theta}{\mathrm{ck_z}}\right).$$

In terms of $(1/\mathrm{k_z}) = \frac{\delta}{2}(1-i)$ one finds

$$\frac{\mathrm{E_R}}{\mathrm{E_0}} \cong -1+\left(\frac{\delta\omega\cos\theta}{\mathrm{c}}\right)(1-i), \tag{10.8.8}$$

$$\frac{\mathrm{E_T}}{\mathrm{E_0}} \cong \left(\frac{\delta\omega\cos\theta}{\mathrm{c}}\right)(1-i). \tag{10.8.9}$$

The rate at which energy is carried through the surface per meter squared to be dissipated as Joule heat in the metal is given by the time average of the Poynting vector at z=0.

$$<\mathrm{S_z}> = \frac{1}{2}\mathrm{Real}(-\mathrm{E_y}\mathrm{H_x^*})$$

so that

$$<\mathrm{S_z}> = \frac{1}{2}\mathrm{Real}\left(\mathrm{E_T}\left(\frac{\mathrm{k_z^*}}{\omega\mu_0}\right)\mathrm{E_T^*}\right) = \frac{1}{2\delta\omega\mu_0}|\mathrm{E_T}|^2,$$

or

$$<\mathrm{S_z}> = \frac{\delta\omega}{\mathrm{c}}\cos^2\theta\left(\frac{\mathrm{E_0^2}}{\mathrm{Z_0}}\right). \tag{10.8.10}$$

The time-averaged rate at which the incident wave transports energy in the z-direction is given by the z-component of the incident wave Poynting vector:

$$<\mathrm{S_0}> = \frac{1}{2}\mathrm{Real}(-\mathrm{E_y}\mathrm{H_x^*}) = \frac{\mathrm{E_0^2}}{2\mathrm{Z_0}}\cos\theta. \tag{10.8.11}$$

The absorption coefficient associated with the metal surface is given by

$$\alpha = \frac{<S_z>}{<S_0>} = 2\left(\frac{\delta\omega}{c}\right)\cos\theta, \tag{10.8.12}$$

where $\delta=\sqrt{2/(\omega\mu_0\sigma_0)}$. The absorption coefficient is very small and increases with frequency like $\sqrt{\omega}$, and decreases in proportion with the increase of the square root of the conductivity. Notice that at the surface of the metal the magnetic field components $H_x$ in the incident and reflected waves add in phase so that at z=0

$$\mathrm{H_x}=-\mathrm{H_0}\cos\theta-\mathrm{H_R}\cos\theta,$$

or

$$\mathrm{H_x}=\frac{\mathrm{E_0}\cos\theta}{\mathrm{Z_0}}\left(-2+\frac{\delta\omega\cos\theta}{\mathrm{c}}(1-i)\right). \tag{10.8.13}$$

Since $\delta\omega/\mathrm{c}=2\pi(\delta/\lambda)$ is very small one makes very little error by taking the parallel component of the magnetic field at the metal surface to be just **twice the parallel magnetic field component of the incident wave**. In the limit of infinite conductivity the parameter $\delta\to 0$, the electric field in the metal becomes zero, and the component $H_x$ at the metal surface has twice the amplitude of $H_x$ in the incident wave. The component $H_z$ also becomes zero at the metal surface in the limit of infinite conductivity, so that the normal component of $\vec{B}$, $B_z = \mu_0 H_z$, is continuous across the vacuum-metal interface as is required by the Equation div($\vec{B}$) = 0.

### 10.8.2 P-polarization.

The magnetic vector of the incident wave is parallel with the metal surface, Figure (10.5.7). For this case the waves in the metal are described by

$$\begin{aligned}
\mathrm{H_y} &=\mathrm{H_T}\exp(i\,[\mathrm{xk}\sin\theta+\mathrm{zk_z}-\omega\mathrm{t}]),\\
\mathrm{E_x} &=-\frac{i\mathrm{k_z}}{\sigma_0}\mathrm{H_T}\exp(i\,[\mathrm{xk}\sin\theta+\mathrm{zk_z}-\omega\mathrm{t}]),\\
\mathrm{E_z} &=\frac{i\omega\sin\theta}{\mathrm{c}\sigma_0}\mathrm{H_T}\exp(i\,[\mathrm{xk}\sin\theta+\mathrm{zk_z}-\omega\mathrm{t}]),
\end{aligned} \tag{10.8.14}$$

where

$$\mathrm{k_z^2}=i\omega\mu_0\sigma_0-\left(\frac{\omega}{\mathrm{c}}\right)^2\sin^2\theta\cong i\omega\mu_0\sigma_0$$

and therefore

$$\mathrm{k_z}=\sqrt{\frac{\omega\mu_0\sigma_0}{2}}(1+i)=\frac{(1+i)}{\delta}. \tag{10.8.15}$$

The boundary conditions on $H_y$ and on $E_x$ at the interface z=0 (continuity of the tangential components of $\vec{E}$ and $\vec{H}$), plus a bit of algebra, readily gives the results

$$\begin{aligned}
\frac{\mathrm{H_R}}{\mathrm{H_0}} &=\frac{\left(\cos\theta-\left(\frac{\omega\delta}{2\mathrm{c}}\right)(1-i)\right)}{\left(\cos\theta+\left(\frac{\omega\delta}{2\mathrm{c}}\right)(1-i)\right)},\\
\frac{\mathrm{H_T}}{\mathrm{H_0}} &=\frac{2\cos\theta}{\left(\cos\theta+\left(\frac{\omega\delta}{2\mathrm{c}}\right)(1-i)\right)},\\
\frac{\mathrm{E_x}}{\mathrm{H_0}} &=\frac{\mathrm{Z_0}\left(\frac{\omega\delta}{\mathrm{c}}\right)\cos\theta(1-i)}{\left(\cos\theta+\left(\frac{\omega\delta}{2\mathrm{c}}\right)(1-i)\right)}.
\end{aligned} \tag{10.8.16}$$

In the above expressions $Z_0$= 377 Ohms, the impedance of free space. The ratio $\omega\delta/\mathrm{c}=2\pi\delta/\lambda$ is very small, approximately $4\times 10^{-5}$ for copper at 1 GHz and 300K. It therefore follows that

$$\frac{\mathrm{H_R}}{\mathrm{H_0}}\cong 1$$

and

$$\frac{\mathrm{H_T}}{\mathrm{H_0}} \cong 2,$$

and

$$\frac{\mathrm{E_x}}{\mathrm{H_0}} \cong \mathrm{Z_0}\left(\frac{\omega\delta}{\mathrm{c}}\right)(1-i) \sim 0.$$

In fact, for a perfect metal, one for which the conductivity becomes infinitely large, the length parameter, $\delta$, goes to zero and the electric field does not penetrate into the metal.

The rate at which energy is carried into the metal surface at z=0 is given by

$$<\mathrm{S_z}>=\frac{1}{2}\mathrm{Real}\left(\mathrm{E_x H_y^*}\right)=\left(\frac{2\omega\delta}{\mathrm{c}}\right)\frac{\mathrm{Z_0}}{2}|\mathrm{H_0}|^2 \quad \text{Watts}\,/m^2.$$

The rate at which energy is carried to the surface by the incident wave is given by

$$<\mathrm{S_0}>=\frac{1}{2}\mathrm{Real}\left(\mathrm{E_x H_y^*}\right)=\frac{\mathrm{Z_0}}{2}\cos\theta|\mathrm{H_0}|^2 \quad \text{Watts}\,/m^2.$$

It follows that the absorption coefficient associated with the metal surface is

$$\alpha=\frac{<\mathrm{S_z}>}{<\mathrm{S_0}>}=\frac{2\omega\delta}{\mathrm{c}\cos\theta}. \tag{10.8.17}$$

Equation (10.83) is only valid if cos θ ≫ (ω$\delta$/c). In the opposite limit, for angles very near to π/2 so that cos θ ≪ (ω$\delta$/c), it can be shown that

$$\alpha \rightarrow 4\cos\theta/(\omega\delta/\mathrm{c}),$$

so that the absorption coefficient goes to zero as the angle of incidence approaches $\pi$/2.