11.3: Co-axial Cables

Cylindrical co-ordinates are appropriate for the problem of a co-axial cable, Figure (11.2.3). The relevant Maxwell's equations become

$$\operatorname{curl}(\vec{\mathrm{E}})=-\mu_0\frac{\partial\vec{\mathrm{H}}}{\partial \mathrm{t}},$$

and

$$\operatorname{curl}(\vec{\mathrm{H}})=\epsilon\frac{\partial\vec{\mathrm{E}}}{\partial \mathrm{t}},$$

where ϵ is a real number for a lossless line. Look for solutions of these equations in which, by analogy with a strip-line curved around on itself, the electric field has only a radial component, E_r , that is independent of angle, and the magnetic field has only an angularly independent component H_θ:

$$\frac{\partial \mathrm{E_r}}{\partial \mathrm{z}}=-\mu_0\frac{\partial \mathrm{H}_\theta}{\partial \mathrm{t}}, \tag{11.3.1}$$

$$\frac{\partial \mathrm{H}_\theta}{\partial \mathrm{z}}=-\epsilon\frac{\partial \mathrm{E_r}}{\partial \mathrm{t}}. \tag{11.3.2}$$

In addition, take $E_z = 0$ because the tangential components of the electric field must be zero at the perfectly conducting walls of the co-axial cable. But if $E_z = 0$ it follows from Maxwell's equations that

$$\operatorname{curl}(\vec{\mathrm{H}})_z=0=\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{rH}_\theta).$$

This implies that

$$\mathrm{H}_\theta=\frac{\mathrm{a(z,t)}}{\mathrm{r}}, \tag{11.3.3}$$

where a(z,t) is a function of time and of position along the cable. Similarly, from div($\vec{E}$) = 0 one has

$$\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{rE_r})=0,$$

and this is satisfied by

$$\mathrm{E_r}=\frac{\mathrm{b(z,t)}}{\mathrm{r}}. \tag{11.3.4}$$

By combining the Maxwell Equations (11.3.1) the electric and magnetic fields, Equations (11.3.3) and (11.3.4), must satisfy

$$\frac{\partial^2 \mathrm{E_r}}{\partial \mathrm{z}^2}=-\mu_0\frac{\partial^2 \mathrm{H}_\theta}{\partial \mathrm{z}\partial \mathrm{t}}=\epsilon\mu_0\frac{\partial^2 \mathrm{E_r}}{\partial \mathrm{t}^2}, \tag{11.3.5}$$

$$\frac{\partial^2 \mathrm{H}_\theta}{\partial \mathrm{z}^2}=-\epsilon\frac{\partial^2 \mathrm{E_r}}{\partial \mathrm{z}\partial \mathrm{t}}=\epsilon\mu_0\frac{\partial^2 \mathrm{H}_\theta}{\partial \mathrm{t}^2}. \tag{11.3.6}$$

These have the same form as the strip-line equations (11.2.3). It follows from these equations,and from the requirements (11.3.3) and (11.3.4), that the general solution for the electric field can be written

$$\mathrm{E_r(z,t)}=\frac{\mathrm{F(z-vt)}}{\mathrm{r}}+\frac{\mathrm{G(z+vt)}}{\mathrm{r}}, \tag{11.3.7}$$

where F(u) and G(u) are arbitrary functions of their arguments, and where

$$\mathrm{v}=\frac{1}{\sqrt{\epsilon\mu_0}}.$$

The corresponding general solution for the magnetic field is

$$\mathrm{H}_\theta(\mathrm{z},\mathrm{t}) = \epsilon \mathrm{v}\left(\frac{\mathrm{F}(\mathrm{z}-\mathrm{vt})}{\mathrm{r}} - \frac{\mathrm{G}(\mathrm{z}+\mathrm{vt})}{\mathrm{r}}\right). \tag{11.3.8}$$

The above electric and magnetic fields satisfy the wave equations (11.3.5), they satisfy Equations (11.3.1), and they have the form required by Equations (11.3.3 and 11.3.4).

Instead of the electric field strength, the state of the electric field in the cable can be specified by the potential difference between the inner and outer conductors:

$$\mathrm{V} = \int_{\mathrm{R}_1}^{\mathrm{R}_2} \mathrm{E}_\mathrm{r} \mathrm{dr} = \mathrm{F}(\mathrm{z}-\mathrm{vt}) \int_{\mathrm{R}_1}^{\mathrm{R}_2} \frac{\mathrm{dr}}{\mathrm{r}} = \mathrm{F}(\mathrm{z}-\mathrm{vt}) \ln\left(\frac{\mathrm{R}_2}{\mathrm{R}_1}\right)$$

for a forward propagating wave. Note that the inner conductor is positive with respect to the outer conductor. The corresponding current on the inner conductor is given by

$$\mathrm{I} = \mathrm{J}_\mathrm{z}\left(2\pi \mathrm{R}_1\right) = \mathrm{H}_\theta\left(\mathrm{R}_1\right)\left(2\pi \mathrm{R}_1\right) = \epsilon \mathrm{v}\left(2\pi \mathrm{R}_1\right) \frac{\mathrm{F}(\mathrm{z}-\mathrm{vt})}{\mathrm{R}_1},$$

so that

$$\mathrm{I} = 2\pi\epsilon \mathrm{v} \mathrm{F}(\mathrm{z}-\mathrm{vt}).$$

The current flows flows towards +z for the current on the inner conductor; the current flows towards minus z on the outer conductor. That is, on the outer conductor

$$\mathrm{I} = -2\pi \mathrm{R}_2 \mathrm{H}_\theta\left(\mathrm{R}_2\right) = -2\pi\epsilon \mathrm{v} \mathrm{F}(\mathrm{z}-\mathrm{vt}).$$

so that the net current flow through a section of the cable is zero. The characteristic impedance of the cable is given by

$$\mathrm{Z}_0 = \frac{\mathrm{V}}{\mathrm{I}} = \frac{1}{2\pi\epsilon \mathrm{v}} \ln\left(\frac{\mathrm{R}_2}{\mathrm{R}_1}\right)$$

or

$$\mathrm{Z}_0 = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon}} \ln\left(\frac{\mathrm{R}_2}{\mathrm{R}_1}\right). \tag{11.3.9}$$

The potential difference, V, is proportional to the electric field, E_r , and the current, I, is proportional to the magnetic field, H_θ, therefore from Equations (11.3.5) the voltage and current satisfy the wave equations

$$\frac{\partial^2 \mathrm{V}}{\partial \mathrm{z}^2} = \frac{1}{\mathrm{v}^2} \frac{\partial^2 \mathrm{V}}{\partial \mathrm{t}^2}, \tag{11.3.10}$$

$$\frac{\partial^2 I}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 I}{\partial t^2}, \tag{11.3.11}$$

where $v^2 = 1/(\epsilon\mu_0)$. For a forward propagating pulse having the form

$$V(z,t) = F(z-vt)$$

the corresponding current pulse is described by

$$\mathrm{I}(\mathrm{z},\mathrm{t}) = \frac{1}{\mathrm{Z}_0} \mathrm{F}(\mathrm{z}-\mathrm{vt}) = \frac{\mathrm{V}(\mathrm{z},\mathrm{t})}{\mathrm{Z}_0}, \tag{11.3.12}$$

where the characteristic impedance for a co-axial cable is given by Equation (11.18). For a backward propagating potential pulse of the form

$$\mathrm{V}(\mathrm{z},\mathrm{t}) = \mathrm{G}(\mathrm{z}+\mathrm{vt})$$

the corresponding current pulse is described by

$$\mathrm{I}(\mathrm{z},\mathrm{t}) = -\frac{1}{\mathrm{Z}_0} \mathrm{V}(\mathrm{z},\mathrm{t}) = -\frac{\mathrm{G}(\mathrm{z}+\mathrm{vt})}{\mathrm{Z}_0}. \tag{11.3.13}$$

In the above equations F(z-vt) and G(z+vt) are arbitrary functions of their arguments.
