

10.5: Mathematical Formulas - Vector Operators

This section contains a summary of vector operators expressed in each of the three major coordinate systems:

- Cartesian (x, y, z)
- cylindrical (ρ, ϕ, z)
- spherical (r, θ, ϕ)

Associated basis vectors are identified using a caret (^) over the symbol. The vector operand \mathbf{A} is expressed in terms of components in the basis directions as follows:

- Cartesian: $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$
- cylindrical: $\mathbf{A} = \hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{\mathbf{z}}A_z$
- spherical: $\mathbf{A} = \hat{\mathbf{r}}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi$

Gradient

Gradient in Cartesian coordinates:

$$\nabla f = \hat{\mathbf{x}} \frac{\partial f}{\partial x} + \hat{\mathbf{y}} \frac{\partial f}{\partial y} + \hat{\mathbf{z}} \frac{\partial f}{\partial z}$$

Gradient in cylindrical coordinates:

$$\nabla f = \hat{\rho} \frac{\partial f}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial f}{\partial z}$$

Gradient in spherical coordinates:

$$\nabla f = \hat{\mathbf{r}} \frac{\partial f}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}$$

Divergence

Divergence in Cartesian coordinates:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence in cylindrical coordinates:

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Divergence in spherical coordinates:

$$\begin{aligned} \nabla \cdot \mathbf{A} = & \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) \\ & + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) \\ & + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \end{aligned}$$

Curl

Curl in Cartesian coordinates:

$$\begin{aligned}\nabla \times \mathbf{A} = & \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \\ & + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\ & + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)\end{aligned}$$

Curl in cylindrical coordinates:

$$\begin{aligned}\nabla \times \mathbf{A} = & \hat{\rho} \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \\ & + \hat{\phi} \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \\ & + \hat{\mathbf{z}} \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right]\end{aligned}$$

Curl in spherical coordinates:

$$\begin{aligned}\nabla \times \mathbf{A} = & \hat{\mathbf{r}} \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] \\ & + \hat{\theta} \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \\ & + \hat{\phi} \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right]\end{aligned}$$

Laplacian

Laplacian in Cartesian coordinates:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian in cylindrical coordinates:

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian in spherical coordinates:

$$\begin{aligned}\nabla^2 f = & \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) \\ & + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial \theta} \sin \theta \right) \\ & + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}\end{aligned}$$

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