5.5: Gauss' Law - Integral Form

Gauss' Law is one of the four fundamental laws of classical electromagnetics, collectively known as *Maxwell's Equations*. Before diving in, the reader is strongly encouraged to review Section 2.4. In that section, Gauss' Law emerges from the interpretation of the electric field as a flux density. Section 2.4 does not actually identify Gauss' Law, but here it is:

> *Gauss' Law* (Equation 5.5.1) states that the flux of the electric field through a closed surface is equal to the enclosed charge.

Gauss' Law is expressed mathematically as follows:

$$\boxed{\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s} = Q_{encl}} \tag{5.5.1}$$

where $\mathbf{D}$ is the electric flux density $\epsilon\mathbf{E}$, $\mathcal{S}$ is a closed surface with differential surface normal $d\mathbf{s}$, and Q_{encl} is the enclosed charge. We can see the law is dimensionally correct; $\mathbf{D}$ has units of C/m^2, thus integrating $\mathbf{D}$ over a surface gives a quantity with units of C/m^2 $\cdot$ m^2 = C, which are the units of charge.

Gauss' Law has a number of applications in electromagnetic theory. One of them, as explored below, is as a method to compute the electric field in response to a distribution of electric charge. Note that a method to do this, based on Coulomb's Law, is described in Sections 5.1, 5.2, and 5.4. Gauss' Law provides an alternative method that is easier or more useful in certain applications.

Example 5.5.1: Electric field associated with a charged particle, using Gauss' Law.

In this example, we demonstrate the ability of Gauss' Law to predict the field associated with a charge distribution. Let us do this for the simplest possible charge distribution. A particle of charge q located at the origin, for which we already have the answer (Section 5.1).

Solution

Gauss' Law applies to *any* surface that encloses the charge, so for simplicity we chose a sphere of radius r centered at the origin. Note that Q_{encl} on the right hand side is just q for any surface having $r > 0$. Gauss' Law in this case becomes

$$\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \mathbf{D} \cdot \left(\hat{\mathbf{r}} r^2 \sin\theta \, d\theta \, d\phi\right) = q$$

If we can solve for $\mathbf{D}$, we can get $\mathbf{E}$ using $\mathbf{D} = \epsilon\mathbf{E}$. The simplest way to solve for $\mathbf{D}$ is to use a symmetry argument, which proceeds as follows. In this problem, the *magnitude* of $\mathbf{D}$ can depend only on r, and not θ or ϕ. This is because the charge has no particular orientation, and the sphere is centered on the charge. Similarly, it is clear that $\mathbf{D}$ must point either directly toward or directly away from the charge. In other words, $\mathbf{D} = \hat{\mathbf{r}} D(r)$. Substituting this in the above equation, we encounter the dot product $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}$, which is simply 1. Since $D(r)$ and r^2 are constants with respect to the integration, we obtain:

$$r^2 D(r) \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta d\phi = q$$

The remaining integral is simply 4π, thus we obtain:

$$D(r) = \frac{q}{4\pi r^2}$$

Bringing the known vector orientation of $\mathbf{D}$ back into the equation, we obtain

$$\mathbf{D} = \hat{\mathbf{r}} \frac{q}{4\pi r^2}$$

and finally using $\mathbf{D} = \epsilon\mathbf{E}$ we obtain the expected result

$$\mathbf{E} = \hat{\mathbf{r}} \frac{q}{4\pi\epsilon r^2}$$

Here's the point you should take away from the above example:

Gauss’ Law combined with a symmetry argument may be sufficient to determine the electric field due to a charge distribution. Thus, Gauss’ Law may be an easier alternative to Coulomb’s Law in some applications.
