

## 4.10: The Laplacian Operator

The Laplacian  $\nabla^2 f$  of a field  $f(\mathbf{r})$  is the divergence of the gradient of that field:

$$\nabla^2 f \triangleq \nabla \cdot (\nabla f) \quad (4.10.1)$$

Note that the Laplacian is essentially a definition of the second derivative with respect to the three spatial dimensions. For example, in Cartesian coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (4.10.2)$$

as can be readily verified by applying the definitions of gradient and divergence in Cartesian coordinates to Equation 4.10.1.

The Laplacian relates the electric potential (i.e.,  $V$ , units of V) to electric charge density (i.e.,  $\rho_v$ , units of C/m<sup>3</sup>). This relationship is known as *Poisson's Equation*:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

where  $\epsilon$  is the permittivity of the medium. The fact that  $V$  is related to  $\rho_v$  in this way should not be surprising, since electric field intensity ( $\mathbf{E}$ , units of V/m) is proportional to the derivative of  $V$  with respect to distance (via the gradient) and  $\rho_v$  is proportional to the derivative of  $\mathbf{E}$  with respect to distance (via the divergence).

The Laplacian operator can also be applied to *vector* fields; for example, Equation 4.10.2 is valid even if the scalar field “ $f$ ” is replaced with a vector field. In the Cartesian coordinate system, the Laplacian of the vector field  $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$  is

$$\nabla^2 \mathbf{A} = \hat{\mathbf{x}}\nabla^2 A_x + \hat{\mathbf{y}}\nabla^2 A_y + \hat{\mathbf{z}}\nabla^2 A_z$$

An important application of the Laplacian operator of vector fields is the *wave equation*; e.g., the wave equation for  $\mathbf{E}$  in a lossless and source-free region is

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0$$

where  $\beta$  is the **phase propagation constant**.

It is sometimes useful to know that the Laplacian of a vector field can be expressed in terms of the gradient, divergence, and curl as follows:

$$\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

The Laplacian operator in the cylindrical and spherical coordinate systems is given in Appendix B2.

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