

5.15: Poisson's and Laplace's Equations

The electric scalar potential field $V(\mathbf{r})$, defined in Section 5.12, is useful for a number of reasons including the ability to conveniently compute potential differences (i.e., $V_{21} = V(\mathbf{r}_2) - V(\mathbf{r}_1)$) and the ability to conveniently determine the electric field by taking the gradient (i.e., $\mathbf{E} = -\nabla V$). One way to obtain $V(\mathbf{r})$ is by integration over the source charge distribution, as described in Section 5.13. This method is awkward in the presence of material interfaces, which impose boundary conditions on the solutions that must be satisfied simultaneously. For example, the electric potential on a perfectly conducting surface is constant¹ – a constraint which is not taken into account in any of the expressions in Section 5.13 (this fact is probably already known to the reader from past study of elementary circuit theory; however, this is established in the context of electromagnetics in Section 5.19.)

In this section, we develop an alternative approach to calculating $V(\mathbf{r})$ that accommodates these boundary conditions, and thereby facilitates the analysis of the scalar potential field in the vicinity of structures and spatially-varying material properties. This alternative approach is based on *Poisson's Equation*, which we now derive.

We begin with the differential form of Gauss' Law (Section 5.7):

$$\nabla \cdot \mathbf{D} = \rho_v$$

Using the relationship $\mathbf{D} = \epsilon \mathbf{E}$ (and keeping in mind our standard assumptions about material properties, summarized in Section 2.8) we obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon}$$

Next, we apply the relationship (Section 5.14):

$$\mathbf{E} = -\nabla V$$

yielding

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon}$$

This is Poisson's Equation, but it is not in the form in which it is commonly employed. To obtain the alternative form, consider the operator $\nabla \cdot \nabla$ in Cartesian coordinates:

$$\begin{aligned} \nabla \cdot \nabla &= \left[\frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right] \cdot \left[\frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \right] \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \nabla^2 \end{aligned}$$

i.e., the operator $\nabla \cdot \nabla$ is identically the Laplacian operator ∇^2 (Section 4.10). Furthermore, this is true regardless of the coordinate system employed. Thus, we obtain the following form of Poisson's Equation:

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}} \quad (5.15.1)$$

Poisson's Equation (Equation 5.15.1) states that the Laplacian of the electric potential field is equal to the volume charge density divided by the permittivity, with a change of sign.

Note that Poisson's Equation is a partial differential equation, and therefore can be solved using well-known techniques already established for such equations. In fact, Poisson's Equation is an *inhomogeneous differential equation*, with the inhomogeneous part $-\rho_v/\epsilon$ representing the source of the field. In the presence of material structure, we identify the relevant boundary conditions at the interfaces between materials, and the task of finding $V(\mathbf{r})$ is reduced to the purely mathematical task of solving the associated boundary value problem (see "Additional Reading" at the end of this section). This approach is particularly effective when one of the materials is a perfect conductor or can be modeled as such a material. This is because – as noted at the beginning of this section

– the electric potential at all points on the surface of a perfect conductor must be equal, resulting in a particularly simple boundary condition.

In many other applications, the charge responsible for the electric field lies outside the domain of the problem; i.e., we have non-zero electric field (hence, potentially non-zero electric potential) in a region that is free of charge. In this case, Poisson's Equation simplifies to *Laplace's Equation*:

$$\nabla^2 V = 0 \quad (\text{source-free region}) \quad (5.15.2)$$

Laplace's Equation (Equation 5.15.2) states that the Laplacian of the electric potential field is zero in a source-free region.

Like Poisson's Equation, Laplace's Equation, combined with the relevant boundary conditions, can be used to solve for $V(\mathbf{r})$, but only in regions that contain no charge.

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