25.4: Energy Diagram, Effective Potential Energy, and Orbits

The energy (Equation (25.3.8)) of the single body moving in two dimensions can be reinterpreted as the energy of a single body moving in one dimension, the radial direction r, in an effective potential energy given by two terms,

$$U_{\text{eff}}=\frac{L^2}{2\mu r^2}-\frac{Gm_1m_2}{r}$$

The energy is still the same, but our interpretation has changed,

$$E=K_{\text{eff}}+U_{\text{eff}}=\frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2+\frac{L^2}{2\mu r^2}-\frac{Gm_1m_2}{r}$$

where the effective kinetic energy K_{eff} associated with the one-dimensional motion is

$$K_{\text{eff}}=\frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2$$

The graph of U_{eff} as a function of $u=r/r_0$ where r_0 as given in Equation (25.3.13), is shown in Figure 25.4. The upper red curve is proportional to $L^2/\left(2\mu r^2\right)\sim 1/2r^2$ The lower blue curve is proportional to $-Gm_1m_2/r\sim -1/r$. The sum U_{eff} is represented by the middle green curve. The minimum value of U_{eff} is at $r=r_0$, as will be shown analytically below. The vertical scale is in units of $-U_{\text{eff}}\left(r_0\right)$. Whenever the one-dimensional kinetic energy is zero, $K_{\text{eff}}=0$, the energy is equal to the effective potential energy,

$$E=U_{\text{eff}}=\frac{L^2}{2\mu r^2}-\frac{Gm_1m_2}{r}$$

Recall that the potential energy is defined to be the negative integral of the work done by the force. For our reduction to a one-body problem, using the effective potential, we will introduce an effective force such that

$$U_{\text{eff},B}-U_{\text{eff},A}=-\int_A^B \overrightarrow{\mathbf{F}}^{\text{eff}}\cdot d\overrightarrow{\mathbf{r}}=-\int_A^B F_r^{\text{eff}}dr$$

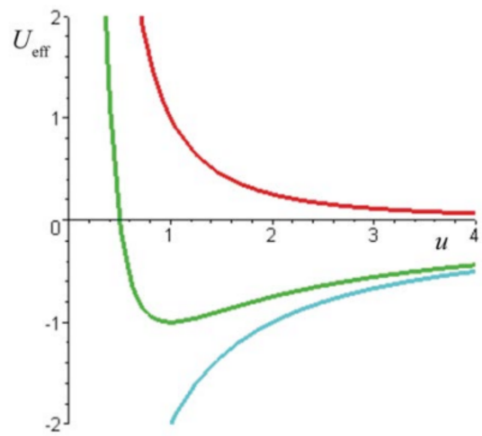

Figure 25.4 Graph of effective potential energy

The fundamental theorem of calculus (for one variable) then states that the integral of the derivative of the effective potential energy function between two points is the effective potential energy difference between those two points,

$$U_{\text{eff},B}-U_{\text{eff},A}=\int_A^B\frac{dU_{\text{eff}}}{dr}dr$$

Comparing Equation (25.4.6) to Equation (25.4.5) shows that the radial component of the effective force is the negative of the derivative of the effective potential energy,

$$F_r^{\text{eff}} = -\frac{dU_{\text{eff}}}{dr}$$

The effective potential energy describes the potential energy for a reduced body moving in one dimension. (Note that the effective potential energy is only a function of the variable r and is independent of the variable θ). There are two contributions to the effective potential energy, and the radial component of the force is then

$$F_r^{\text{eff}} = -\frac{d}{dr} U_{\text{eff}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}\right)$$

Thus there are two "forces" acting on the reduced body,

$$F_r^{\text{eff}} = F_{r,\text{ centrifugal}} + F_{r,\text{ gravity}}$$

with an effective centrifugal force given by

$$F_{r,\text{ centrifugal}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) = \frac{L^2}{\mu r^3}$$

and the centripetal gravitational force given by

$$F_{r,\text{ gravily}} = -\frac{G m_1 m_2}{r^2}$$

With this nomenclature, let's review the four cases presented in Section 25.3.

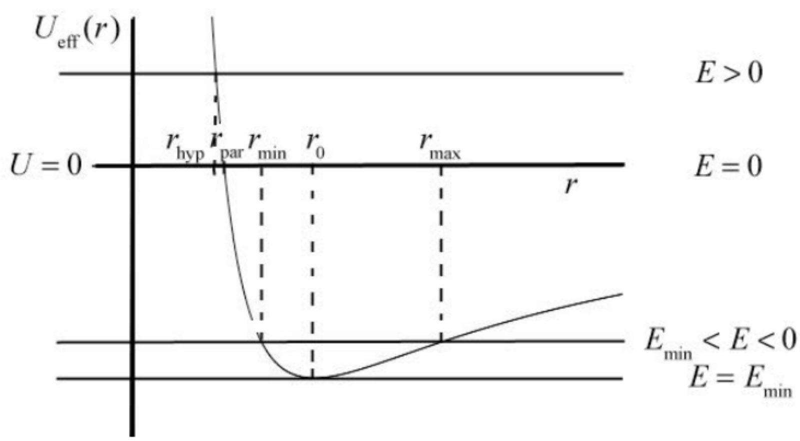

Figure 25.5 Plot of $U_{\text{eff}}(r)$ vs. r with four energies corresponding to circular, elliptic, parabolic, and hyperbolic orbits

Circular Orbit $E = E_{\text{min}}$

The lowest energy state, E_{min}, corresponds to the minimum of the effective potential energy, $E_{\text{min}} = (U_{\text{eff}})_{\text{min}}$. We can minimize the effective potential energy

$$0 = \left.\frac{dU_{\text{eff}}}{dr}\right|_{r=r_0} = -\frac{L^2}{\mu r_0^3} + \frac{G m_1 m_2}{r_0^2}$$

and solve Equation (25.4.12) for r_0,

$$r_0 = \frac{L^2}{\mu G m_1 m_2}$$

reproducing Equation (25.3.13). For $E = E_{\text{min}}, \quad r = r_0$ which corresponds to a circular orbit.

Elliptic Orbit $E_{\text{min}} < E < 0$

For $E_{\text{min}} < E < 0$, there are two points r_{min} and r_{max} such that $E = U_{\text{eff}}(r_{\text{min}}) = U_{\text{eff}}(r_{\text{max}})$. At these points $K_{\text{eff}} = 0$, therefore $dr/dt = 0$ which corresponds to a point of closest or furthest approach (Figure 25.6). This condition corresponds to the minimum and maximum values of r for an elliptic orbit.

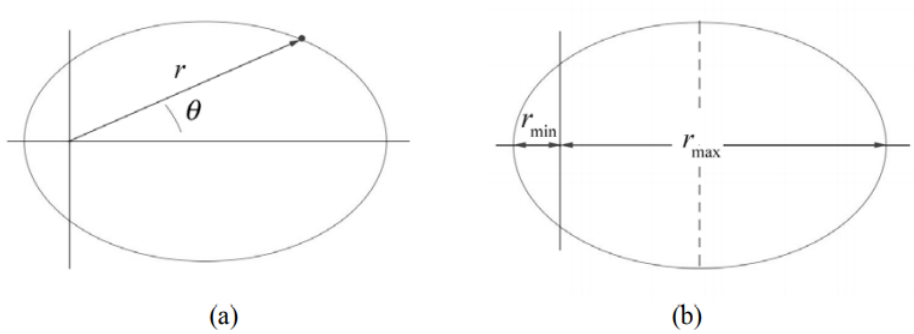

Figure 25.6 (a) elliptic orbit, (b) closest and furthest approach

The energy condition at these two points

$$E=\frac{L^{2}}{2 \mu r^{2}}-\frac{G m_{1} m_{2}}{r}, \quad r=r_{\min }=r_{\max }$$

is a quadratic equation for the distance r ,

$$r^{2}+\frac{G m_{1} m_{2}}{E} r-\frac{L^{2}}{2 \mu E}=0$$

There are two roots

$$r=-\frac{G m_{1} m_{2}}{2 E} \pm\left(\left(\frac{G m_{1} m_{2}}{2 E}\right)^{2}+\frac{L^{2}}{2 \mu E}\right)^{1 / 2}$$

Equation (25.4.16) may be simplified somewhat as

$$r=-\frac{G m_{1} m_{2}}{2 E}\left(1 \pm\left(1+\frac{2 L^{2} E}{\mu\left(G m_{1} m_{2}\right)^{2}}\right)^{1 / 2}\right)$$

From Equation (25.3.14), the square root is the eccentricity ε

$$\varepsilon=\left(1+\frac{2 E L^{2}}{\mu\left(G m_{1} m_{2}\right)^{2}}\right)^{\frac{1}{2}}$$

and Equation (25.4.17) becomes

$$r=-\frac{G m_{1} m_{2}}{2 E}(1 \pm \varepsilon)$$

A little algebra shows that

$$\begin{aligned}
\frac{r_{0}}{1-\varepsilon^{2}} &=\frac{L^{2} / \mu G m_{1} m_{2}}{1-\left(1+\frac{2 L^{2} E}{\mu\left(G m_{1} m_{2}\right)^{2}}\right)} \\
&=\frac{L^{2} / \mu G m_{1} m_{2}}{-2 L^{2} E / \mu\left(G m_{1} m_{2}\right)^{2}} \\
&=-\frac{G m_{1} m_{2}}{2 E}
\end{aligned}$$

Substituting the last expression in (25.4.20) into Equation (25.4.19) gives an expression for the points of closest and furthest approach,

$$r=\frac{r_{0}}{1-\varepsilon^{2}}(1 \pm \varepsilon)=\frac{r_{0}}{1 \mp \varepsilon}$$

The minus sign corresponds to the distance of closest approach,

$$r \equiv r_{\min} = \frac{r_0}{1+\varepsilon}$$

and the plus sign corresponds to the distance of furthest approach,

$$r \equiv r_{\max} = \frac{r_0}{1-\varepsilon}$$

Parabolic Orbit E = 0

The effective potential energy, as given in Equation (25.4.1), approaches zero $(U_{\text{eff}} \rightarrow 0)$ when the distance r approaches infinity $(r \rightarrow \infty)$. When $E=0$, as $r \rightarrow \infty$, the kinetic energy also approaches zero, $K_{\text{eff}} \rightarrow 0$. This corresponds to a parabolic orbit (see Equation (25.3.23)). Recall that in order for a body to escape from a planet, the body must have an energy $E=0$ (we set $U_{\text{eff}}=0$ at infinity) This escape velocity condition corresponds to a parabolic orbit. For a parabolic orbit, the body also has a distance of closest approach. This distance r_{par} can be found from the condition

$$E = U_{\text{eff}}\left(r_{\text{par}}\right) = \frac{L^2}{2\mu r_{\text{par}}^2} - \frac{G m_1 m_2}{r_{\text{par}}} = 0$$

Solving Equation (25.4.24) for r_{par} yields

$$r_{\text{par}} = \frac{L^2}{2\mu G m_1 m_2} = \frac{1}{2} r_0$$

the fact that the minimum distance to the origin (the focus of a parabola) is half the semilatus rectum is a well-known property of a parabola (Figure 25.5).

Hyperbolic Orbit E > 0

When $E>0$, in the limit as $r \rightarrow \infty$ the kinetic energy is positive, $K_{\text{eff}}>0$. This corresponds to a hyperbolic orbit (see Equation (25.3.24)). The condition for closest approach is similar to Equation (25.4.14) except that the energy is now positive. This implies that there is only one positive solution to the quadratic Equation (25.4.15), the distance of closest approach for the hyperbolic orbit

$$r_{\text{hyp}} = \frac{r_0}{1+\varepsilon}$$

The constant r_0 is independent of the energy and from Equation (25.3.14) as the energy of the single body increases, the eccentricity increases, and hence from Equation (25.4.26), the distance of closest approach gets smaller (Figure 25.5).
