13.3: Kinematics and Kinetic Energy in One Dimension

Constant Accelerated Motion

Let's consider a constant accelerated motion of a rigid body in one dimension in which we treat the rigid body as a point mass. Suppose at t = 0 the body has an initial x - component of the velocity given by $\mathcal{V}_{x,i}$. If the acceleration is in the direction of the displacement of the body then the body will increase its speed. If the acceleration is opposite the direction of the displacement then the acceleration will decrease the body's speed. The displacement of the body is given by

$$\Delta x=v_{x,i} t+\frac{1}{2} a_{x} t^{2}$$

The product of acceleration and the displacement is

$$a_{x} \Delta x=a_{x}\left(v_{x,i} t+\frac{1}{2} a_{x} t^{2}\right)$$

The acceleration is given by

$$a_{x}=\frac{\Delta v_{x}}{\Delta t}=\frac{\left(v_{x,f}-v_{x,i}\right)}{t}$$

Therefore

$$a_{x} \Delta x=\frac{\left(v_{x,f}-v_{x,i}\right)}{t}\left(v_{x,i} t+\frac{1}{2} \frac{\left(v_{x,f}-v_{x,i}\right)}{t} t^{2}\right)$$

Equation (13.3.4) becomes

$$a_{x} \Delta x=\left(v_{x,f}-v_{x,i}\right)\left(v_{x,i}\right)+\frac{1}{2}\left(v_{x,f}-v_{x,i}\right)\left(v_{x,f}-v_{x,i}\right)=\frac{1}{2} v_{x,f}^{2}-\frac{1}{2} v_{x,i}^{2}$$

If we multiply each side of Equation (13.3.5) by the mass m of the object this kinematical result takes on an interesting interpretation for the motion of the object. We have

$$m a_{x} \Delta x=\frac{1}{2} m v_{x,f}^{2}-m \frac{1}{2} v_{x,i}^{2}=K_{f}-K_{i}$$

Recall that for one-dimensional motion, Newton's Second Law is $F_{x}=m a_{x}$, for the motion considered here, Equation (13.3.6) becomes

$$F_{x} \Delta x=K_{f}-K_{i}$$

Non-constant Accelerated Motion

If the acceleration is not constant, then we can divide the displacement into N intervals indexed by j = 1 to N . It will be convenient to denote the displacement intervals by Δx_{j} the corresponding time intervals by Δt_{j} and the x -components of the velocities at the beginning and end of each interval as $\mathcal{V}_{x,j-1}$ and $\mathcal{V}_{x,j}$. Note that the x -component of the velocity at the beginning and end of the first interval j =1is then $v_{x,1}=v_{x,i}$ and the velocity at the end of the last interval, $j=N$ is $v_{x,N}=v_{x,j}$. Consider the sum of the , products of the average acceleration $\left(a_{x,j}\right)_{\text{ave}}$ and displacement Δx_{j} in each interval,

$$\sum_{j=1}^{j=N}\left(a_{x,j}\right)_{\text{ave}} \Delta x_{j}$$

The average acceleration over each interval is equal to

$$\left(a_{x,j}\right)_{\text{ave}}=\frac{\Delta v_{x,j}}{\Delta t_{j}}=\frac{\left(v_{x,j+1}-v_{x,j}\right)}{\Delta t_{j}}$$

and so the contribution in each integral can be calculated as above and we have that

$$\left(a_{x,j}\right)_{\text{ave}} \Delta x_{j}=\frac{1}{2} v_{x,j}^{2}-\frac{1}{2} v_{x,j-1}^{2}$$

When we sum over all the terms only the last and first terms survive, all the other terms cancel in pairs, and we have that

$$\sum_{j=1}^{j=N}\left(a_{x,j}\right)_{\text{ave}} \Delta x_j=\frac{1}{2} v_{x,f}^2-\frac{1}{2} v_{x,i}^2$$

In the limit as $N \rightarrow \infty$ and $\Delta x_j \rightarrow 0$ for all j (both conditions must be met!), the limit of the sum is the definition of the definite integral of the acceleration with respect to the position,

$$\lim_{\substack{N \rightarrow \infty \\ \Delta x_j \rightarrow 0}} \sum_{j=1}^{j=N}\left(a_{x,j}\right)_{\text{ave}} \Delta x_j \equiv \int_{x=x_i}^{x=x_f} a_x(x) dx$$

Therefore In the limit as $N \rightarrow \infty$ and $\Delta x_j \rightarrow 0$ for all j , with $v_{x,N} \rightarrow v_{x,f}$ Equation (13.3.11) becomes

$$\int_{x=x_i}^{x=x_f} a_x(x) dx=\frac{1}{2}\left(v_{x,f}^2-v_{x,i}^2\right)$$

This integral result is consequence of the definition that $a_x \equiv dv_x / dt$. The integral in Equation (13.3.13) is an integral with respect to space, while our previous integral

$$\int_{t=t_i}^{t=t_f} a_x(t) dt=v_{x,f}-v_{x,i}$$

requires integrating acceleration with respect to time. Multiplying both sides of Equation (13.3.13) by the mass m yields

$$\int_{x=x_i}^{x=x_f} m a_x(x) dx=\frac{1}{2} m\left(v_{x,f}^2-v_{x,i}^2\right)=K_f-K_i$$

When we introduce Newton’s Second Law in the form $F_x=m a_x$, then Equation (13.3.15) becomes

$$\int_{x=x_i}^{x=x_f} F_x(x) dx=K_f-K_i$$

The integral of the x -component of the force with respect to displacement in Equation (13.3.16) applies to the motion of a point-like object. For extended bodies, Equation (13.3.16) applies to the center of mass motion because the external force on a rigid body causes the center of mass to accelerate.