23.7: Small Oscillations

Any object moving subject to a force associated with a potential energy function that is quadratic will undergo simple harmonic motion,

$$U(x)=U_0+\frac{1}{2}k\left(x-x_{eq}\right)^2$$

where k is a “spring constant”, x_{eq} is the equilibrium position, and the constant U_0 just depends on the choice of reference point x_{ref} for zero potential energy, $U\left(x_{ref}\right)=0$,

$$0=U\left(x_{ref}\right)=U_0+\frac{1}{2}k\left(x_{ref}-x_{eq}\right)^2$$

Therefore the constant is

$$U_0=-\frac{1}{2}k\left(x_{ref}-x_{eq}\right)^2$$

The minimum of the potential x_0 corresponds to the point where the x -component of the force is zero,

$$\left.\frac{dU}{dx}\right|_{x=x_0}=2k\left(x_0-x_{eq}\right)=0 \Rightarrow x_0=x_{eq}$$

corresponding to the equilibrium position. Therefore the constant is $U\left(x_0\right)=U_0$ and we rewrite our potential function as

$$U(x)=U\left(x_0\right)+\frac{1}{2}k\left(x-x_0\right)^2$$

Now suppose that a potential energy function is not quadratic but still has a minimum at x_0. For example, consider the potential energy function

$$U(x)=-U_1\left(\left(\frac{x}{x_1}\right)^3-\left(\frac{x}{x_1}\right)^2\right)$$

(Figure 23.22), which has a stable minimum at x_0,

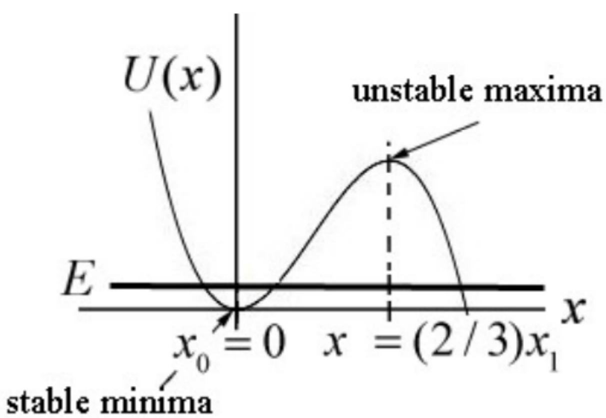

Figure 23.22 Potential energy function with stable minima and unstable maxima

When the energy of the system is very close to the value of the potential energy at the minimum $U\left(x_0\right)$, we shall show that the system will undergo small oscillations about the minimum value x_0. We shall use the Taylor formula to approximate the potential function as a polynomial. We shall show that near the minimum x_0 we can approximate the potential function by a quadratic function similar to Equation (23.7.5) and show that the system undergoes simple harmonic motion for small oscillations about the minimum x_0.

We begin by expanding the potential energy function about the minimum point using the Taylor formula

$$U(x)=U\left(x_0\right)+\left.\frac{dU}{dx}\right|_{x=x_0}\left(x-x_0\right)+\left.\frac{1}{2!}\frac{d^2U}{dx^2}\right|_{x=x_0}\left(x-x_0\right)^2+\left.\frac{1}{3!}\frac{d^3U}{dx^3}\right|_{x=x_0}\left(x-x_0\right)^3+\cdots$$

where $\left.\frac{1}{3!}\frac{d^3U}{dx^3}\right|_{x=x_0}\left(x-x_0\right)^3$ is a third order term in that it is proportional to $\left(x-x_0\right)^3$, and $\left.\frac{d^3U}{dx^3}\right|_{x=x_0}$, $\left.\frac{d^2U}{dx^2}\right|_{x=x_0}$, and $\left.\frac{dU}{dx}\right|_{x=x_0}$ are constants. If x_0 is the minimum of the potential energy, then the linear term is zero, because

$$\left.\frac{dU}{dx}\right|_{x=x_0}=0$$

and so Equation ((23.7.7)) becomes

$$U(x)\simeq U\left(x_0\right)+\left.\frac{1}{2}\frac{d^2U}{dx^2}\right|_{x=x_0}\left(x-x_0\right)^2+\left.\frac{1}{3!}\frac{d^3U}{dx^3}\right|_{x=x_0}\left(x-x_0\right)^3+\cdots$$

For small displacements from the equilibrium point such that $\left|x-x_0\right|$ is sufficiently small, the third order term and higher order terms are very small and can be ignored. Then the potential energy function is approximately a quadratic function,

$$U(x)\simeq U\left(x_0\right)+\left.\frac{1}{2}\frac{d^2U}{dx^2}\right|_{x=x_0}\left(x-x_0\right)^2=U\left(x_0\right)+\frac{1}{2}k_{eff}\left(x-x_0\right)^2$$

where we define k_{eff}, the effective spring constant, by

$$k_{eff}\equiv\left.\frac{d^2U}{dx^2}\right|_{x=x_0}$$

Because the potential energy function is now approximated by a quadratic function, the system will undergo simple harmonic motion for small displacements from the minimum with a force given by

$$F_x=-\frac{dU}{dx}=-k_{eff}\left(x-x_0\right)$$

At $x=x_0$, the force is zero

$$F_x\left(x_0\right)=\frac{dU}{dx}\left(x_0\right)=0$$

We can determine the period of oscillation by substituting Equation (23.7.12) into Newton's Second Law

$$-k_{eff}\left(x-x_0\right)=m_{eff}\frac{d^2x}{dt^2}$$

where m_{eff} is the effective mass. For a two-particle system, the effective mass is the reduced mass of the system.

$$m_{eff}=\frac{m_1m_2}{m_1+m_2}\equiv\mu_{red}$$

Equation (23.7.14) has the same form as the spring-object ideal oscillator. Therefore the angular frequency of small oscillations is given by

$$\omega_0=\sqrt{\frac{k_{eff}}{m_{eff}}}=\sqrt{\left(\left.\frac{d^2U}{dx^2}\right|_{x=x_0}\right)/m_{eff}}$$

Example 23.6: Quartic Potential

A system with effective mass m has a potential energy given by

$$U(x)=U_0\left(-2\left(\frac{x}{x_0}\right)^2+\left(\frac{x}{x_0}\right)^4\right)$$

where U_0 and x_0 are positive constants and $U(0)=0$ (a) Find the points where the force on the particle is zero. Classify these points as stable or unstable. Calculate the value of $U(x)/U_0$ at these equilibrium points. (b) If the particle is given a small displacement from an equilibrium point, find the angular frequency of small oscillation.

Solution: (a) A plot of $U(x)/U_0$ as a function of x/x_0 is shown in Figure 23.23.

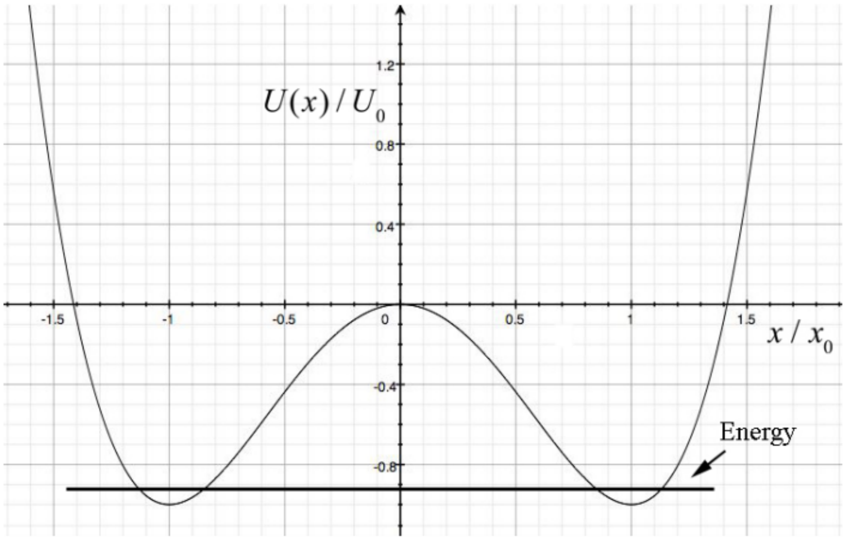

Figure 22.23 Plot of $U(x)/U_0$ as a function of x/x_0

The force on the particle is zero at the minimum of the potential energy,

$$0=\frac{dU}{dx}=U_0\left(-4\left(\frac{1}{x_0}\right)^2 x+4\left(\frac{1}{x_0}\right)^4 x^3\right)$$
$$=-4U_0 x\left(\frac{1}{x_0}\right)^2\left(1-\left(\frac{x}{x_0}\right)^2\right)\Rightarrow x^2=x_0^2 \text{ and } x=0$$

The equilibrium points are at $x=\pm x_0$ which are stable and x = 0 which is unstable. The second derivative of the potential energy is given by

$$\frac{d^2U}{dx^2}=U_0\left(-4\left(\frac{1}{x_0}\right)^2+12\left(\frac{1}{x_0}\right)^4 x^2\right)$$

If the particle is given a small displacement from $x=x_0$ then

$$\left.\frac{d^2U}{dx^2}\right|_{x=x_0}=U_0\left(-4\left(\frac{1}{x_0}\right)^2+12\left(\frac{1}{x_0}\right)^4 x_0^2\right)=U_0\frac{8}{x_0^2}$$

(b) The angular frequency of small oscillations is given by

$$\omega_0=\sqrt{\left.\frac{d^2U}{dx^2}\right|_{x=x_0}/m}=\sqrt{\frac{8U_0}{mx_0^2}}$$

Example 23.7: Lennard-Jones 6-12 Potential

A commonly used potential energy function to describe the interaction between two atoms is the Lennard-Jones 6-12 potential

$$U(r)=U_0\left[(r_0/r)^{12}-2(r_0/r)^6\right];r>0$$

where r is the distance between the atoms. Find the angular frequency of small oscillations about the stable equilibrium position for two identical atoms bound to each other by the LennardJones interaction. Let m denote the effective mass of the system of two atoms.

Solution: The equilibrium points are found by setting the first derivative of the potential energy equal to zero,

$$0=\frac{dU}{dr}=U_0\left[-12r_0^{12}r^{-13}+12r_0^6 r^{-7}\right]=U_0 12r_0^6 r^{-7}\left[-\left(\frac{r_0}{r}\right)^6+1\right]$$

The equilibrium point occurs when $r = r_0$ The second derivative of the potential energy function is

$$\frac{d^2U}{dr^2} = U_0 \left[+(12)(13) r_0^{12} r^{-14} - (12)(7) r_0^6 r^{-8} \right]$$

Evaluating this at $r = r_0$ yields

$$\left.\frac{d^2U}{dr^2}\right|_{r=r_0} = 72 U_0 r_0^{-2}$$

The angular frequency of small oscillation is therefore

$$\begin{aligned} \omega_0 &= \sqrt{\left.\frac{d^2U}{dr^2}\right|_{r=r_0} / m} \\ &= \sqrt{72 U_0 / m r_0^2} \end{aligned}$$