

4.5: Constant Acceleration

When the x -component of the velocity is a linear function (Figure 4.5.1a), the average acceleration, $\Delta v / \Delta t$, is a constant and hence is equal to the instantaneous acceleration (Figure 4.5.1b).

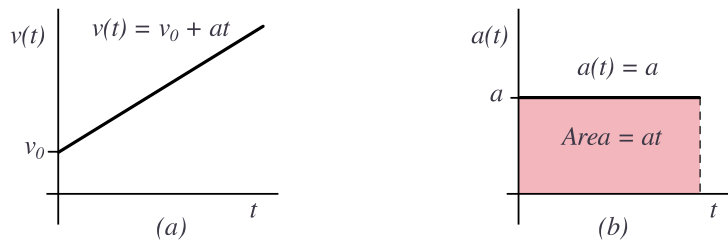


Figure 4.5.1: Constant acceleration: (a) velocity, (b) acceleration. (CC BY-NC; Ümit Kaya)

Let's consider a body undergoing constant acceleration for a time interval $[0, t]$, where $\Delta t = t$. Denote the x -component of the velocity at time $t = 0$ by v_0 . Therefore the x -component of the acceleration is given by

$$a(t) = \frac{\Delta v}{\Delta t} = \frac{v(t) - v_0}{t}$$

Thus, the x -component of the velocity is a linear function of time given by

$$v(t) = v_0 + at$$

Velocity: Area Under the Acceleration vs. Time Graph

In Figure 4.8(b), the area under the acceleration vs. time graph, for the time interval $\Delta t = t - 0 = t$, is

$$\text{Area}(a(t), t) = at$$

From Equation (4.5.2), the area is the change in the x -component of the velocity for the interval $[0, t]$:

$$\text{Area}(a(t), t) = at = v(t) - v_0 = \Delta v$$

Displacement: Area Under the Velocity vs. Time Graph

In Figure 4.5.2 shows a graph of the x -component of the velocity vs. time for the case of constant acceleration (Equation (4.5.2)).

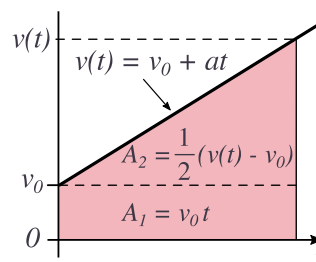


Figure 4.5.2: Graph of velocity as a function of time for a constant. (CC BY-NC; Ümit Kaya)

The region under the velocity vs. time curve is a trapezoid, formed from a rectangle with area $A_1 = v_0 t$, and a triangle with area $A_2 = (1/2)(v(t) - v_0)$. The total area of the trapezoid is given by

$$\text{Area}(v(t), t) = A_1 + A_2 = v_0 t + \frac{1}{2}(v(t) - v_0)$$

Substituting for the velocity (Equation (4.5.2)) yields

$$\text{Area}(v(t), t) = v_0 t + \frac{1}{2}at^2$$

Recall that from Example 4.2 (setting $b = a$ and $\Delta t = t$),

$$v_{ave} = v_0 + \frac{1}{2}at = \Delta x / t$$

therefore Equation (4.5.6) can be rewritten as

$$Area(v(t), t) = \left(v_0 + \frac{1}{2}at \right) t = v_{ave}t = \Delta x$$

The displacement is equal to the area under the graph of the x -component of the velocity vs. time. The position as a function of time can now be found by rewriting Equation (4.5.8) as

$$x(t) = x_0 + v_0t + \frac{1}{2}at^2$$

Figure 4.5.3 shows a graph of this equation. Notice that at $t = 0$ the slope is non-zero, corresponding to the initial velocity component v_0

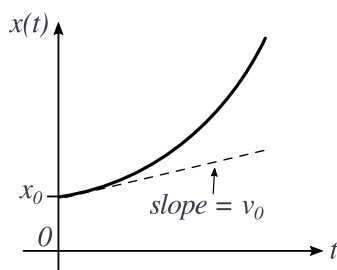


Figure 4.5.3: Graph of position vs. time for constant acceleration. (CC BY-NC; Ümit Kaya)

Example 4.4 Accelerating Car

A car, starting at rest at $t = 0$, accelerates in a straight line for 100 m with an unknown constant acceleration. It reaches a speed of $20 \text{ m} \cdot \text{s}^{-1}$ and then continues at this speed for another 10 s.

- Write down the equations for the position and velocity of the car as a function of time.
- How long was the car accelerating?
- What was the magnitude of the acceleration?
- Plot speed vs. time, acceleration vs. time, and position vs. time for the entire motion.
- What was the average velocity for the entire trip?

Solution

(a) For the acceleration a , the position $x(t)$ and velocity $v(t)$ as a function of time t for a car starting from rest are

$$\begin{aligned} x(t) &= (1/2)at^2 \\ v_x(t) &= at \end{aligned}$$

b) Denote the time interval during which the car accelerated by t_1 . We know that the position $x(t_1) = 100\text{m}$ and $v(t_1) = 20\text{m} \cdot \text{s}^{-1}$. Note that we can eliminate the acceleration a between the Equations (4.4.10) to obtain

$$x(t) = (1/2)v(t)t$$

We can solve this equation for time as a function of the distance and the final speed giving

$$t = 2 \frac{x(t)}{v(t)}$$

We can now substitute our known values for the position $x(t_1) = 100\text{m}$ and $v(t_1) = 20\text{m} \cdot \text{s}^{-1}$ and solve for the time interval that the car has accelerated

$$t_1 = 2 \frac{x(t_1)}{v(t_1)} = 2 \frac{100\text{m}}{20\text{m} \cdot \text{s}^{-1}} = 10\text{s}$$

c) We can substitute into either of the expressions in Equation (4.4.10); the second is slightly easier to use,

$$a = \frac{v(t_1)}{t_1} = \frac{20\text{m} \cdot \text{s}^{-1}}{10\text{s}} = 2.0\text{m} \cdot \text{s}^{-2}$$

d) The x -component of acceleration vs. time, x -component of the velocity vs. time, and the position vs. time are piece-wise functions given by

$$a(t) = \begin{cases} 2\text{m} \cdot \text{s}^{-2}; & 0 < t \leq 10\text{s} \\ 0; & 10\text{s} < t < 20\text{s} \end{cases}$$

$$v(t) = \begin{cases} (2\text{m} \cdot \text{s}^{-2}) t; & 0 < t \leq 10\text{s} \\ 20\text{m} \cdot \text{s}^{-1}; & 10\text{s} \leq t \leq 20\text{s} \end{cases}$$

$$x(t) = \begin{cases} (1/2) (2\text{m} \cdot \text{s}^{-2}) t^2; & 0 < t \leq 10\text{s} \\ 100\text{m} + (20\text{m} \cdot \text{s}^{-1}) (t - 10\text{s}); & 10\text{s} \leq t \leq 20\text{s} \end{cases}$$

The graphs of the x -component of acceleration vs. time, x -component of the velocity vs. time, and the position vs. time are shown in Figure 4.11.

(e) After accelerating, the car travels for an additional ten seconds at constant speed and during this interval the car travels an additional distance $\Delta x = v(t_1) \times 10\text{s} = 200\text{m}$ (note that this is twice the distance traveled during the 10 s of acceleration), so the total distance traveled is 300 m and the total time is 20 s, for an average velocity of

$$v_{\text{ave}} = \frac{300\text{m}}{20\text{s}} = 15\text{m} \cdot \text{s}^{-1}$$

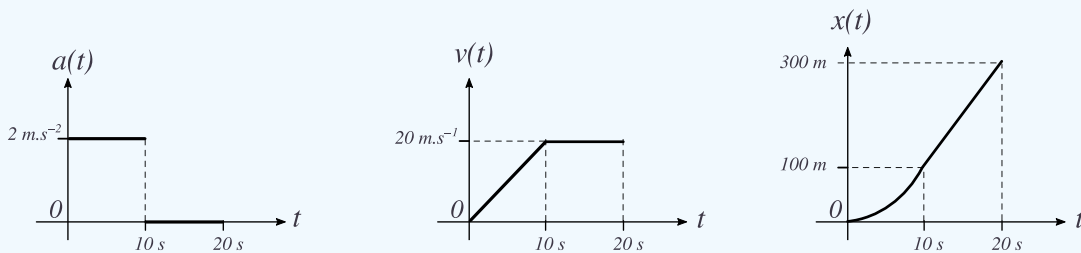


Figure 4.5.4: Graphs of the x-components of acceleration, velocity and position as piecewise functions. (CC BY-NC; Ümit Kaya)

Example 4.5.1: Catching a Bus

At the instant a traffic light turns green, a car starts from rest with a given constant acceleration, $3.0\text{m} \cdot \text{s}^{-2}$. Just as the light turns green, a bus, traveling with a given constant velocity, $1.6 \times 10^1 \text{m} \cdot \text{s}^{-1}$, passes the car. The car speeds up and passes the bus sometime later. How far down the road has the car traveled, when the car passes the bus?

Solution

There are two moving objects, bus and the car. Each object undergoes one stage of one-dimensional motion. We are given the acceleration of the car, the velocity of the bus, and infer that the position of the car and the bus are equal when the bus just passes the car. Figure 4.5.5 shows a qualitative sketch of the position of the car and bus as a function of time.

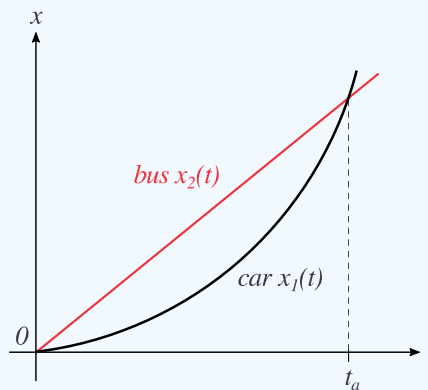


Figure 4.5.5: Position vs. time of the car and bus. (CC BY-NC; Ümit Kaya)

Choose a coordinate system with the origin at the traffic light and the positive x-direction such that car and bus are traveling in the positive x-direction. Set time $t = 0$ as the instant the car and bus pass each other at the origin when the light turns green. Figure 4.5.6 shows the position of the car and bus at time t .

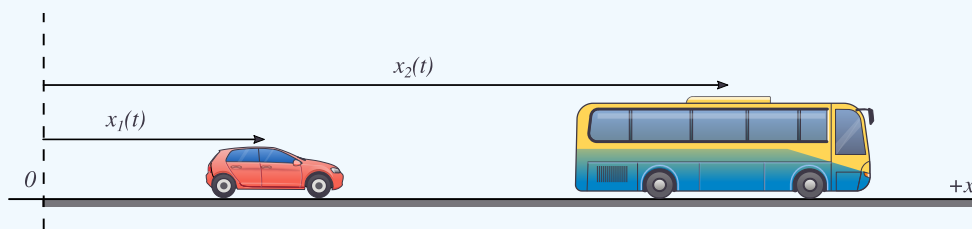


Figure 4.5.6: Coordinate system for car and bus. (CC BY-NC; Ümit Kaya)

Let $x_1(t)$ denote the position function of the car, and $x_2(t)$ the position function for the bus. The initial position and initial velocity of the car are both zero, $x_{1,0} = 0$ and $v_{1,0} = 0$, and the acceleration of the car is non-zero $a_1 \neq 0$. Therefore the position and velocity functions of the car are given by

$$x_1(t) = \frac{1}{2}a_1t^2$$

$$v_1(t) = a_1t$$

The initial position of the bus is zero, $x_{2,0}(t) = 0$, the initial velocity of the bus is non-zero, $v_{2,0} \neq 0$, and the acceleration of the bus is zero, $(a_2) = 0$. Therefore the velocity is constant $v_2(t) = v_{2,0}$, and the position function for the bus is given by $x_2(t) = v_{2,0}t$.

Let $t = t_a$ correspond to the time that the car passes the bus. Then at that instant, the position functions of the bus and car are equal, $x_1(t_a) = x_2(t_a)$. We can use this condition to solve for t_a :

$$(1/2)a_1t_a^2 = v_{2,0}t_a \Rightarrow t_a = \frac{2v_{2,0}}{a_1} = \frac{(2)(1.6 \times 10^1 \text{ m} \cdot \text{s}^{-1})}{(3.0 \text{ m} \cdot \text{s}^{-2})} = 1.1 \times 10^1 \text{ s}$$

Therefore the position of the car at t_a is

$$x_1(t_a) = \frac{1}{2}a_1t_a^2 = \frac{2v_{2,0}^2}{a_1} = \frac{(2)(1.6 \times 10^1 \text{ m} \cdot \text{s}^{-1})^2}{(3.0 \text{ m} \cdot \text{s}^{-2})} = 1.7 \times 10^2 \text{ m}$$