23.10: Solution to the Underdamped Simple Harmonic Oscillator

Consider the underdamped simple harmonic oscillator equation),

$$\frac{d^2x}{dt^2}+\frac{b}{m}\frac{dx}{dt}+\frac{k}{m}x=0$$

When $(b/m)^2<4k/m$ we show that the equation has a solution of the form

$$x(t)=x_{\mathrm{m}}e^{-\alpha t}\cos(\gamma t+\phi)$$

Solution: Let’s suppose the function x(t) has the form

$$x(t)=A\,\mathrm{Re}\left(e^{zt}\right)$$

where z is a number (possibly complex) and A is a real number. Then

$$\frac{dx}{dt}=zAe^{zt}$$

$$\frac{d^2x}{dt^2}=z^2Ae^{zt}$$

We now substitute Equations (23.C.3), (23.C.4), and (23.C.5), into Equation (23.C.1) resulting in

$$z^2Ae^{zt}+\frac{b}{m}zAe^{zt}+\frac{k}{m}Ae^{zt}=0$$

Collecting terms in Equation (23.C.6) yields

$$\left(z^2+\frac{b}{m}z+\frac{k}{m}\right)Ae^{zt}=0$$

The condition for the solution is that

$$z^2+\frac{b}{m}z+\frac{k}{m}=0$$

This quadratic equation has solutions

$$z=\frac{-(b/m)\pm\left((b/m)^2-4k/m\right)^{1/2}}{2}$$

When $(b/m)^2<4k/m$, the oscillator is called underdamped, and we have two solutions for z , however the solutions are complex numbers. Let

$$\gamma=\left(k/m-(b/2m)^2\right)^{1/2}$$

and

$$\alpha=b/2m$$

Recall that the imaginary number $i=\sqrt{-1}$. The two solutions are then $z_1=-\alpha+i\gamma t$ and $z_2=-\alpha-i\gamma t$. Because our system is linear, our general solution is a linear combination of these two solutions,

$$x(t)=A_1e^{-\alpha+i\gamma t}+A_2e^{-\alpha-i\gamma t}=\left(A_1e^{i\gamma t}+A_2e^{-i\gamma t}\right)e^{-\alpha t}$$

where A_1 and A_2 are constants. We shall transform this expression into a more familiar equation involving sine and cosine functions with help from the Euler formula,

$$e^{\pm i\gamma t}=\cos(\gamma t)\pm i\sin(\gamma t)$$

Therefore we can rewrite our solution as

$$x(t)=\left(A_1(\cos(\gamma t)+i\sin(\gamma t))+A_2(\cos(\gamma t)-i\sin(\gamma t))\right)e^{-\alpha t}$$

A little rearrangement yields

$$x(t)=\left(\left(A_1+A_2\right)\cos(\gamma t)+i\left(A_1-A_2\right)\sin(\gamma t)\right)e^{-\alpha t}$$

Define two new constants $C=A_1+A_2$ and $D=i\left(A_1-A_2\right)$. Then our solution looks like

$$x(t)=(C\cos(\gamma t)+D\sin(\gamma t))e^{-\alpha t}$$

Recall from Example 23.5 that we can rewrite

$$C\cos(\gamma t)+D\sin(\gamma t)=x_{\mathrm{m}}\cos(\gamma t+\phi)$$

where

$$x_{\mathrm{m}}=\left(C^2+D^2\right)^{1/2},\ \text{and}\ \phi=\tan^{-1}(D/C)$$

Then our general solution for the underdamped case (Equation (23.C.16)) can be written as

$$x(t)=x_{\mathrm{m}}e^{-\alpha t}\cos(\gamma t+\phi)$$

There are two other possible cases which we shall not analyze: when $(b/m)^2>4k/m$, a case referred to as overdamped, and when $(b/m)^2=4k/m$, a case referred to as critically damped.
