19.3: Torque and the Time Derivative of Angular Momentum about a Point for a Particle

We will now show that the torque about a point S is equal to the time derivative of the angular momentum about S,

$$\vec{\tau}_S = \frac{d\overrightarrow{\mathbf{L}}_S}{dt}$$

Take the time derivative of the angular momentum about S,

$$\frac{d\overrightarrow{\mathbf{L}}_S}{dt} = \frac{d}{dt}\left(\overrightarrow{\mathbf{r}}_S \times \overrightarrow{\mathbf{p}}\right)$$

In this equation we are taking the time derivative of a vector product of two vectors. There are two important facts that will help us simplify this expression. First, the time derivative of the vector product of two vectors satisfies the product rule,

$$\frac{d\overrightarrow{\mathbf{L}}_S}{dt} = \frac{d}{dt}\left(\overrightarrow{\mathbf{r}}_S \times \overrightarrow{\mathbf{p}}\right) = \left(\left(\frac{d\overrightarrow{\mathbf{r}}_S}{dt}_S\right) \times \overrightarrow{\mathbf{p}}\right) + \left(\overrightarrow{\mathbf{r}}_S \times \left(\frac{d\overrightarrow{\mathbf{p}}}{dt}\right)\right)$$

Second, the first term on the right hand side vanishes,

$$\frac{d\mathbf{r}_S}{dt} \times \overrightarrow{\mathbf{p}} = \overrightarrow{\mathbf{v}} \times m\overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{0}}$$

The rate of angular momentum change about the point S is then

$$\frac{d\overrightarrow{\mathbf{L}}_S}{dt} = \overrightarrow{\mathbf{r}}_S \times \frac{d\overrightarrow{\mathbf{p}}}{dt}$$

From Newton's Second Law, the force on the particle is equal to the derivative of the linear momentum,

$$\overrightarrow{\mathbf{F}} = \frac{d\overrightarrow{\mathbf{p}}}{dt}$$

Therefore the rate of change in time of angular momentum about the point S is

$$\frac{d\overrightarrow{\mathbf{L}}_S}{dt} = \overrightarrow{\mathbf{r}}_S \times \overrightarrow{\mathbf{F}} \tag{19.3.1}$$

Recall that the torque about the point S due to the force $\overrightarrow{\mathbf{F}}$ acting on the particle is

$$\vec{\tau}_S = \overrightarrow{\mathbf{r}}_S \times \overrightarrow{\mathbf{F}} \tag{19.3.2}$$

Combining the expressions in Equation 19.3.1 and 19.3.2, it is readily seen that the torque about the point S is equal to the rate of change of angular momentum about the point S,

$$\vec{\tau}_S = \frac{d\overrightarrow{\mathbf{L}}_S}{dt}$$