20.4: Kinetic Energy of a System of Particles

Consider a system of particles. The i^{th} particle has mass m_i and velocity $\vec{\mathbf{v}}_i$ with respect to a reference frame O . The kinetic energy of the system of particles is given by

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i \vec{\mathbf{v}}_i \cdot \vec{\mathbf{v}}_i \\ &= \frac{1}{2} \sum_i m_i \left(\vec{\mathbf{v}}_{\text{cm},i} + \vec{\mathbf{V}}_{\text{cm}} \right) \cdot \left(\vec{\mathbf{v}}_{\text{cm},i} + \vec{\mathbf{V}}_{\text{cm}} \right) \end{aligned}$$

where Equation 15.2.6 has been used to express $\vec{\mathbf{v}}_i$ in terms of $\vec{\mathbf{v}}_{cm,i}$ and $\vec{\mathbf{V}}_{cm}$. Expanding the last dot product in Equation (20.4.1),

$$\begin{aligned} K &= \frac{1}{2} \sum_i m_i \left(\vec{\mathbf{v}}_{\text{cm},i} \cdot \vec{\mathbf{v}}_{cm,i} + \vec{\mathbf{V}}_{\text{cm}} \cdot \vec{\mathbf{V}}_{\text{cm}} + 2 \vec{\mathbf{v}}_{\text{cm},i} \cdot \vec{\mathbf{V}}_{\text{cm}} \right) \\ &= \frac{1}{2} \sum_i m_i \left(\vec{\mathbf{v}}_{\text{cm},i} \cdot \vec{\mathbf{v}}_{cm,i} \right) + \frac{1}{2} \sum_i m_i \left(\vec{\mathbf{V}}_{\text{cm}} \cdot \vec{\mathbf{V}}_{\text{cm}} \right) + \sum_i m_i \vec{\mathbf{v}}_{\text{cm},i} \cdot \vec{\mathbf{V}}_{\text{cm}} \\ &= \sum_i \frac{1}{2} m_i v_{\text{cm},i}^2 + \frac{1}{2} \sum_i m_i V_{\text{cm}}^2 + \left(\sum_i m \vec{\mathbf{v}}_{\text{cm},i} \right) \cdot \vec{\mathbf{V}}_{\text{cm}} \end{aligned}$$

The last term in the third equation in (20.4.2) vanishes as we showed in Equation (20.3.7). Then Equation (20.4.2) reduces to

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i v_{\text{cm},i}^2 + \frac{1}{2} \sum_i m_i V_{\text{cm}}^2 \\ &= \sum_i \frac{1}{2} m_i v_{\text{cm},i}^2 + \frac{1}{2} m^{\text{total}} V_{\text{cm}}^2 \end{aligned}$$

We interpret the first term as the sum of the individual kinetic energies of the particles of the system in the center of mass reference frame O_{cm} and the second term as the kinetic energy of the center of mass motion in reference frame O.

At this point, it's important to note that no assumption was made regarding the mass elements being constituents of a rigid body. Equation (20.4.3) is valid for a rigid body, a gas, a firecracker (but K is certainly not the same before and after detonation), and the sixteen pool balls after the break, or any collection of objects for which the center of mass can be determined.