3.2: Coordinate Systems

Physics involve the study of phenomena that we observe in the world. In order to connect the phenomena to mathematics we begin by introducing the concept of a coordinate system. A coordinate system consists of four basic elements:

1. Choice of origin
2. Choice of axes
3. Choice of positive direction for each axis
4. Choice of unit vectors at every point in space

There are three commonly used coordinate systems: Cartesian, cylindrical and spherical. In this chapter, we will describe a Cartesian coordinate system and a cylindrical coordinate system.

Cartesian Coordinate System

Cartesian coordinates consist of a set of mutually perpendicular axes, which intersect at a common point, the origin O. We live in a three-dimensional spatial world; for that reason, the most common system we will use has three axes.

Choice of Origin

Choose an origin O at any point that is most convenient.

Choice of Axes

The simplest set of axes is known as the Cartesian axes, x-axis, y-axis, and the z-axis, that are at right angles with respect to each other. Then each point P in space can be assigned a triplet of values (x_P, y_P, z_P). The ranges of these values are: $-\infty < x_P < +\infty$, $-\infty < y_P < +\infty$, $-\infty < z_P < +\infty$.

Choice of Positive Direction

Our third choice is an assignment of positive direction for each coordinate axis. We shall denote this choice by the symbol $+$ along the positive axis. In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem. Problems that are very difficult using the 6 conventional choices may turn out to be much easier to solve by making a thoughtful choice of axes.

Choice of Unit Vectors

We now associate to each point P in space, a set of three unit vectors $\left(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P, \hat{\mathbf{k}}_P\right)$. A unit vector has magnitude one: $\left|\hat{\mathbf{i}}_P\right| = 1, \left|\hat{\mathbf{j}}_P\right| = 1, \text{ and } \left|\hat{\mathbf{k}}_P\right| = 1$. We assign the direction of $\hat{\mathbf{i}}_P$ to point in the direction of the increasing x-coordinate at the point P. We define the directions for $\hat{\mathbf{j}}_P$ and $\hat{\mathbf{k}}_P$ P in the direction of the increasing y-coordinate and z-coordinate respectively, (Figure 3.10). If we choose a different point S, and define a similar set of unit vectors $\left(\hat{\mathbf{i}}_S, \hat{\mathbf{j}}_S, \hat{\mathbf{k}}_S\right)$, the unit vectors at S and P satisfy the equalities

$$\hat{\mathbf{i}}_S = \hat{\mathbf{i}}_P, \hat{\mathbf{j}}_S = \hat{\mathbf{j}}_P, \text{ and } \hat{\mathbf{k}}_S = \hat{\mathbf{k}}_P,$$

because vectors are equal if they have the same direction and magnitude regardless of where they are located in space.

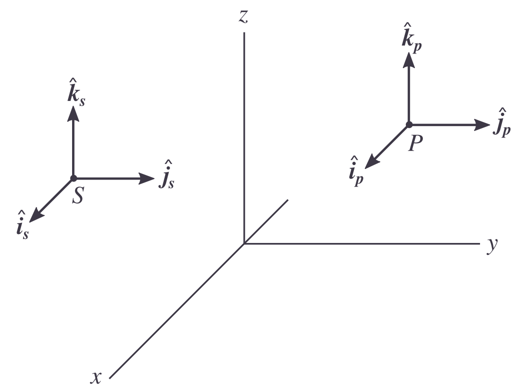

Figure 3.10 Choice of unit vectors at points P and S . (CC BY-NC; Ümit Kaya)

A Cartesian coordinate system is the only coordinate system in which Equation (3.2.1) holds for all pair of points. We therefore drop the reference to the point P and use $\left(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P, \hat{\mathbf{k}}_P\right)$ to represent the unit vectors in a Cartesian coordinate system (Figure 3.11).

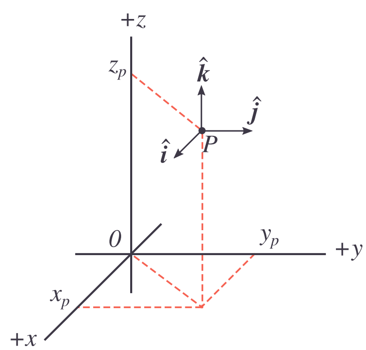

Figure 3.11 Unit vectors in a Cartesian coordinate system. (CC BY-NC; Ümit Kaya)

Cylindrical Coordinate System

Many physical objects demonstrate some type of symmetry. For example, if you rotate a uniform cylinder about the longitudinal axis (symmetry axis), the cylinder appears unchanged. The operation of rotating the cylinder is called a symmetry operation, and the object undergoing the operation, the cylinder, is exactly the same as before the operation was performed. This symmetry property of cylinders suggests a coordinate system, called a cylindrical coordinate system, that makes the symmetrical property under rotations transparent.

First choose an origin O and axis through O, which we call the z-axis. The **cylindrical coordinates** for a point P are the three numbers (r, θ, z) (Figure 3.12). The number z represents the familiar coordinate of the point P along the z-axis. The nonnegative number r represents the distance from the z-axis to the point P. The points in space corresponding to a constant positive value of r lie on a circular cylinder. The locus of points corresponding to $r=0$ is the z-axis. In the plane $z=0$, define a reference ray through O, which we shall refer to as the positive x-axis. Draw a line through the point P that is parallel to the z-axis. Let D denote the point of intersection between that line PD and the plane $z=0$. Draw a ray OD from the origin to the point D. Let θ denote the directed angle from the reference ray to the ray OD. The angle θ is positive when measured counterclockwise and negative when measured clockwise.

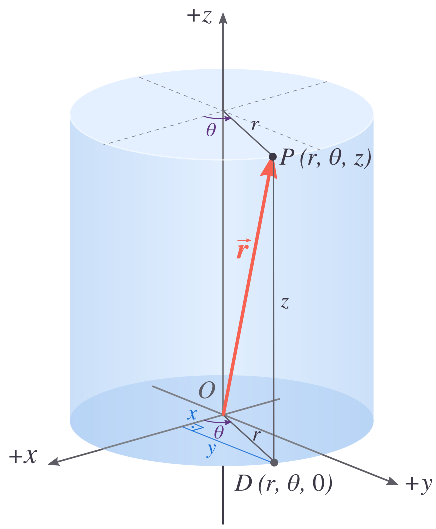

Figure 3.12 Cylindrical Coordinates. (CC BY-NC; Ümit Kaya)

The coordinates (r, θ) are called **polar coordinates**. The coordinate transformations between (r, θ) and the Cartesian coordinates (x, y) are given by

$$x = r\cos\theta,$$

$$y = r\sin\theta.$$

Conversely, if we are given the Cartesian coordinates (x, y), the coordinates (r, θ) can be determined from the coordinate transformations

$$r = +\left(x^2 + y^2\right)^{1/2}$$

$$\theta = \tan^{-1}(y/x)$$

We choose a set of unit vectors $\left(\hat{\mathbf{r}}_P, \hat{\boldsymbol{\theta}}_P, \hat{\mathbf{k}}_P\right)$ at the point P as follows. We choose $\hat{\mathbf{k}}_P$ to point in the direction of increasing z. We choose $\hat{\mathbf{r}}_P$ to point in the direction of increasing r, directed radially away from the z-axis. We choose $\hat{\boldsymbol{\theta}}_P$ to point in the direction of increasing θ. This unit vector points in the counterclockwise direction, tangent to the circle (Figure 3.13a). One crucial difference between cylindrical coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point S in the plane. The unit vectors $\left(\hat{\mathbf{r}}_s, \hat{\boldsymbol{\theta}}_s, \hat{\mathbf{k}}_s\right)$ at the point S are also shown in Figure 3.13. Note that $\hat{\mathbf{r}}_P \neq \hat{\mathbf{r}}_S$ and $\hat{\boldsymbol{\theta}}_p \neq \hat{\boldsymbol{\theta}}_S$ because their direction differ. We shall drop the subscripts denoting the points at which the unit vectors are defined at and simple refer to the set of unit vectors at a point as $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{k}})$, with the understanding that the directions of the set $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ depend on the location of the point in question.

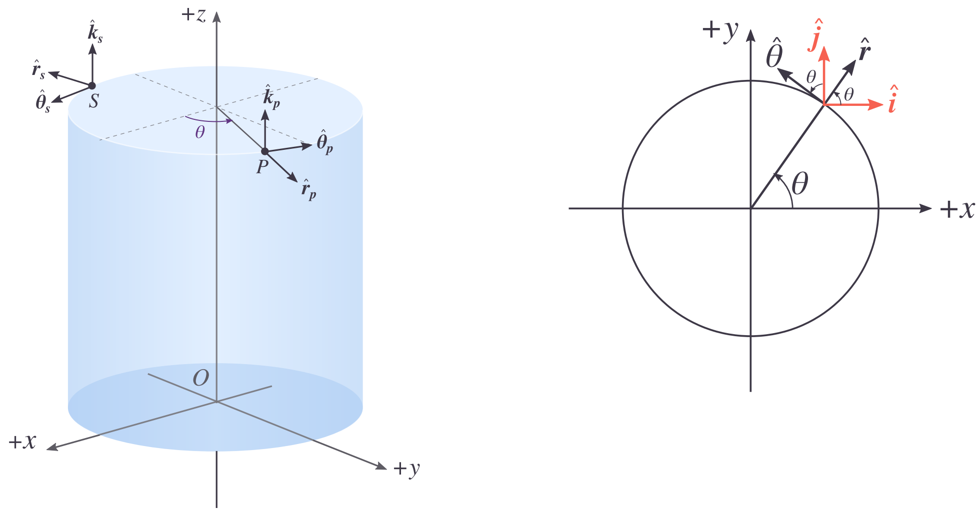

Figure 3.13: (a) Unit vectors at two different points in cylindrical coordinates. (b) Unit vectors in polar coordinates and Cartesian coordinates. (CC BY-NC; Ümit Kaya)

The unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ at the point P also are related to the Cartesian unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$ by the transformations

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$$

Similarly the inverse transformations are given by

$$\hat{\mathbf{i}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{j}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}$$

A cylindrical coordinate system is also a useful choice to describe the motion of an object moving in a circle about a central point. Consider a vertical axis passing perpendicular to the plane of motion passing through that central point. Then any rotation about this vertical axis leaves circles unchanged.