3.3: Vectors

The Use of Vectors in Physics

From the last section we have three important ideas about vectors:

1. vectors can exist at any point P in space,
2. vectors have direction and magnitude, and
3. any two vectors that have the same direction and magnitude are equal no matter where in space they are located.

When we apply vectors to physical quantities it's nice to keep in the back of our minds all these formal properties. However, from the physicist's point of view, we are interested in representing physical quantities such as displacement, velocity, acceleration, force, impulse, and momentum as vectors. We can't add force to velocity or subtract momentum from force. We must always understand the physical context for the vector quantity. Thus, instead of approaching vectors as formal mathematical objects we shall instead consider the following essential properties that enable us to represent physical quantities as vectors.

Vectors in Cartesian Coordinates

Vector Decomposition

Choose a coordinate system with an origin, axes, and unit vectors. We can decompose a vector into component vectors along each coordinate axis (Figure 3.14).

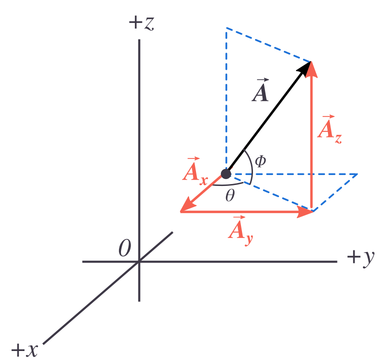

Figure 3.14 Component vectors in Cartesian coordinates. (CC BY-NC; Ümit Kaya)

A vector $\overrightarrow{\mathbf{A}}$ at P can be decomposed into the vector sum,

$$\overrightarrow{\mathbf{A}}=\overrightarrow{\mathbf{A}}_x+\overrightarrow{\mathbf{A}}_y+\overrightarrow{\mathbf{A}}_z$$

where $\overrightarrow{\mathbf{A}}_x$ is the x-component vector pointing in the positive or negative x-direction, $\overrightarrow{\mathbf{A}}_y$ is the y-component vector pointing in the positive or negative y-direction, and $\overrightarrow{\mathbf{A}}_z$ is the z-component vector pointing in the positive or negative z-direction.

Vector Components

Once we have defined unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, we then define the **components** of a vector. Recall our vector decomposition, $\overrightarrow{\mathbf{A}}=\overrightarrow{\mathbf{A}}_x+\overrightarrow{\mathbf{A}}_y+\overrightarrow{\mathbf{A}}_z$. We define the x-component vector, $\overrightarrow{\mathbf{A}}_x$, as

$$\overrightarrow{\mathbf{A}}_x=A_x\,\hat{\mathbf{i}}.$$

In this expression the term A_x (without the arrow above) is called the x-component of the vector $\overrightarrow{\mathbf{A}}_x$. The x-component A_x can be positive, zero, or negative. It is not the magnitude of $\overrightarrow{\mathbf{A}}_x$ which is given by $\left(A_x^2\right)^{1/2}$. The x-component A_x is a scalar quantity and the x-component vector, $\overrightarrow{\mathbf{A}}_x$ is a vector. In a similar fashion we define the y-component, A_y , and the z-component, A_z , of the vector $\overrightarrow{\mathbf{A}}$ according to

$$\overrightarrow{\mathbf{A}}_y = A_y\,\hat{\mathbf{j}}, \quad \overrightarrow{\mathbf{A}}_z = A_z\,\hat{\mathbf{k}}.$$

A vector $\overrightarrow{\mathbf{A}}$ is represented by its three components (A_x, A_y, A_z). Thus we need three numbers to describe a vector in three-dimensional space. We write the vector $\overrightarrow{\mathbf{A}}$ as

$$\overrightarrow{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$$

Magnitude

Using the Pythagorean theorem, the magnitude of $\overrightarrow{\mathbf{A}}$ is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Direction

Let's consider a vector $\overrightarrow{\mathbf{A}} = (A_x, A_y, 0)$. Because the z-component is zero, the vector $\overrightarrow{\mathbf{A}}$ lies in the $x-y$ plane. Let θ denote the angle that the vector $\overrightarrow{\mathbf{A}}$ makes in the counterclockwise direction with the positive x-axis (Figure 3.15).

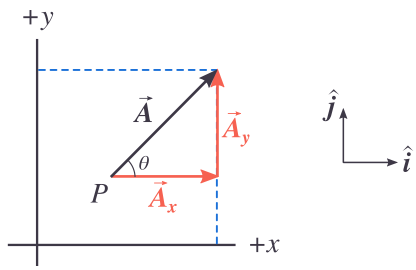

Figure 3.15 Components of a vector in the *xy* -plane. (CC BY-NC; Ümit Kaya)

Then the x -component and y -component are

$$A_r = A\cos(\theta), \quad A_y = A\sin(\theta)$$

We now write a vector in the xy -plane as

$$\overrightarrow{\mathbf{A}} = A\cos(\theta)\hat{\mathbf{i}} + A\sin(\theta)\hat{\mathbf{j}}$$

Once the components of a vector are known, the tangent of the angle θ can be determined by

$$\frac{A_y}{A_x} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta)$$

and hence the angle θ is given by

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

Clearly, the direction of the vector depends on the sign of A_x and A_y. For example, if both $A_x > 0$ and $A_y > 0$, then $0 < \theta < \pi/2$. If $A_x < 0$ and $A_y > 0$ then $\pi/2 < \theta < \pi$. If $A_x < 0$ and $A_y < 0$ then $\pi < \theta < 3\pi/2$. If $A_x > 0$ and $Ay < 0$, then $3\pi/2 < \theta < 2$. Note that tanθ is a double valued function because

$$\frac{-A_y}{-A_x} = \frac{A_y}{A_x}, \text{ and } \frac{A_y}{-A_x} = \frac{-A_y}{A_x}$$

Unit Vectors

Unit vector in the direction of $\overrightarrow{\mathbf{A}}$: Let $\overrightarrow{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_v\hat{\mathbf{j}} + A_7\hat{\mathbf{k}}$. Let $\overrightarrow{\mathbf{A}}$ denote a unit vector in the direction of $\overrightarrow{\mathbf{A}}$. Then,

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{|\vec{\mathbf{A}}|} = \frac{A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}}{\left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}}$$

Vector Addition

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors in the $x-y$ plane. Let θ_A and θ_B denote the angles that the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ make (in the counterclockwise direction) with the positive x -axis. Then

$$\vec{\mathbf{A}} = A\cos(\theta_A)\hat{\mathbf{i}} + A\sin(\theta_A)\hat{\mathbf{j}}$$

$$\vec{\mathbf{B}} = B\cos(\theta_B)\hat{\mathbf{i}} + B\sin(\theta_B)\hat{\mathbf{j}}$$

In Figure 3.16, the vector addition $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is shown. Let θ_C denote the angle that the vector $\vec{\mathbf{C}}$ makes with the positive x -axis.

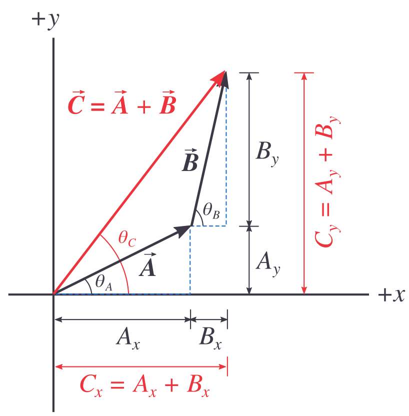

Figure 3.16 Vector addition using components. (CC BY-NC; Ümit Kaya)

From Figure 3.16, the components of $\vec{\mathbf{C}}$ are

$$C_x = A_x + B_x, \quad C_y = A_y + B_y$$

In terms of magnitudes and angles, we have

$$C_x = C\cos(\theta_C) = A\cos(\theta_A) + B\cos(\theta_B)$$
$$C_y = C\sin(\theta_C) = A\sin(\theta_A) + B\sin(\theta_B)$$

We can write the vector $\vec{\mathbf{C}}$ as

$$\vec{\mathbf{C}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} = C\cos(\theta_C)\hat{\mathbf{i}} + C\sin(\theta_C)\hat{\mathbf{j}}$$

Example 3.1: Vector Addition

Given two vectors, $\vec{\mathbf{A}} = 2\hat{\mathbf{i}} + -3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, find: (a) $|\vec{\mathbf{A}}|$; (b) $|\vec{\mathbf{B}}|$; (c) $\vec{\mathbf{A}} + \vec{\mathbf{B}}$; (d) $\vec{\mathbf{A}} - \vec{\mathbf{B}}$; (e) a unit vector $\hat{\mathbf{A}}$ pointing in the direction of $\vec{\mathbf{A}}$; (f) a unit vector $\hat{\mathbf{B}}$ pointing in the direction of $\vec{\mathbf{B}}$;

Solution

(a)

$$|\vec{\mathbf{A}}| = \left(2^2 + (-3)^2 + 7^2\right)^{1/2} = \sqrt{62} = 7.87\,.$$

(b)

$$|\vec{\mathbf{B}}| = \left(5^2 + 1^2 + 2^2\right)^{1/2} = \sqrt{30} = 5.48\,.$$

(c)

$$\begin{aligned}\vec{\mathbf{A}}+\vec{\mathbf{B}} &= (A_x+B_x)\,\hat{\mathbf{i}}+(A_y+B_y)\,\hat{\mathbf{j}}+(A_z+B_z)\,\hat{\mathbf{k}}\\ &=(2+5)\hat{\mathbf{i}}+(-3+1)\hat{\mathbf{j}}+(7+2)\hat{\mathbf{k}}\\ &=7\hat{\mathbf{i}}-2\hat{\mathbf{j}}+9\hat{\mathbf{k}}\end{aligned}$$

(d)

$$\begin{aligned}\vec{\mathbf{A}}-\vec{\mathbf{B}} &= (A_x-B_x)\,\hat{\mathbf{i}}+(A_y-B_y)\,\hat{\mathbf{j}}+(A_z-B_z)\,\hat{\mathbf{k}}\\ &=(2-5)\hat{\mathbf{i}}+(-3-1)\hat{\mathbf{j}}+(7-2)\hat{\mathbf{k}}\\ &=-3\hat{\mathbf{i}}-4\hat{\mathbf{j}}+5\hat{\mathbf{k}}\end{aligned}$$

(e)

A unit vector $\hat{\mathbf{A}}$ in the direction of $\vec{\mathbf{A}}$ can be found by dividing the vector $\vec{\mathbf{A}}$ by the magnitude of $\vec{\mathbf{A}}$. Therfore

$$\hat{\mathbf{A}}=\vec{\mathbf{A}}/|\vec{\mathbf{A}}|=(2\hat{\mathbf{i}}+-3\hat{\mathbf{j}}+7\hat{\mathbf{k}})/\sqrt{62}$$

(f)

In a similar fashion, $\hat{\mathbf{B}}=\vec{\mathbf{B}}/|\vec{\mathbf{B}}|=(5\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}})/\sqrt{30}$

Example 3.2 Sinking Sailboat

A Coast Guard ship is located 35 km away from a checkpoint in a direction 52° north of west. A distressed sailboat located in still water 24 km from the same checkpoint in a direction 18° south of east is about to sink. Draw a diagram indicating the position of both ships. In what direction and how far must the Coast Guard ship travel to reach the sailboat?

Solution

The diagram of the set-up is Figure 3.17.

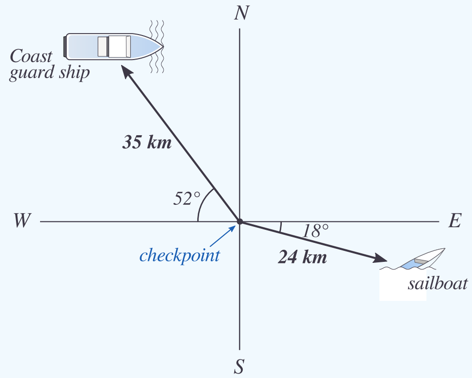

Figure 3.17 Example 3.2. (CC BY-NC; Ümit Kaya)

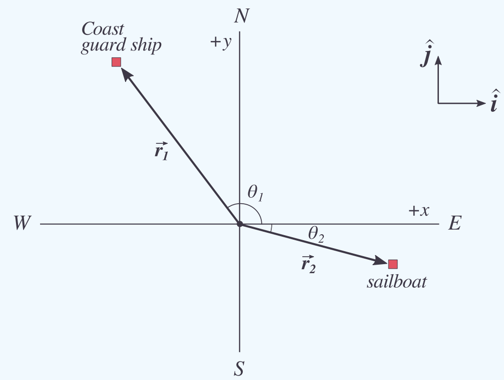

Figure 3.18 Coordinate system for sailboat and ship. (CC BY-NC; Ümit Kaya)

Choose the checkpoint as the origin of a Cartesian coordinate system with the positive x -axis in the East direction and the positive y –axis in the North direction. Choose the corresponding unit vectors $\hat{\mathbf{i}}$ and$\hat{\mathbf{j}}$ as shown in Figure 3.18. The Coast Guard ship is then a distance $r = 35$ km at an angle $\theta_1 = 180^\circ - 52^\circ = 128^\circ$ from the positive x -axis, The position of the Coast Guard ship is then

$$\overrightarrow{\mathbf{r}}_1 = r_1 \left(\cos\theta_1 \hat{\mathbf{i}} + \sin\theta_1 \hat{\mathbf{j}} \right)$$

$$\overrightarrow{\mathbf{r}} = -21.5 \mathrm{km} \hat{\mathbf{i}} + 27.6 \mathrm{km} \hat{\mathbf{j}}$$

and the position of the sailboat is

$$\overrightarrow{\mathbf{r}}_2 = r_2 \left(\cos\theta_2 \hat{\mathbf{i}} + \sin\theta_2 \hat{\mathbf{j}} \right)$$

$$\overrightarrow{\mathbf{r}}_2 = 22.8 \mathrm{km} \hat{\mathbf{i}} - 7.4 \mathrm{km} \hat{\mathbf{j}}$$

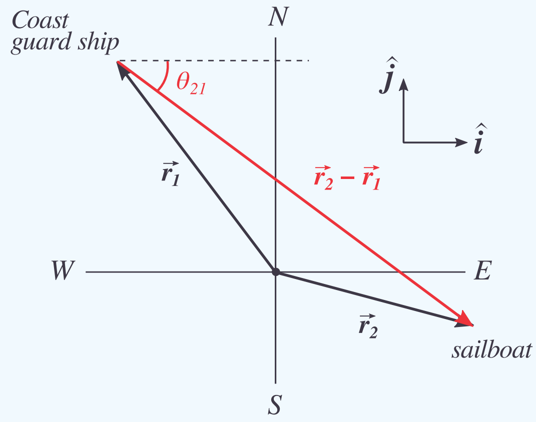

Figure 3.19 Relative position vector from ship to sailboat. (CC BY-NC; Ümit Kaya)

The relative position vector from the Coast Guard ship to the sailboat is (Figure 3.19)

$$\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = (22.8\mathrm{km}\hat{\mathbf{i}} - 7.4\mathrm{km}\hat{\mathbf{j}}) - (-21.5\mathrm{km}\hat{\mathbf{i}} + 27.6\mathrm{km}\hat{\mathbf{j}})$$
$$\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = 44.4\mathrm{km}\hat{\mathbf{i}} - 35.0\mathrm{km}\hat{\mathbf{j}}$$

The distance between the ship and the sailboat is

$$\left|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1\right| = \left((44.4\mathrm{km})^2 + (-35.0\mathrm{km})^2\right)^{1/2} = 56.5\mathrm{km}$$

The rescue ship's heading would be the inverse tangent of the ratio of the *y* - and *x* - components of the relative position vector,

$$\theta_{21} = \tan^{-1}(-35.0\mathrm{km}/44.4\mathrm{km}) = -38.3^\circ$$

or 38.3° South of East.

Example 3.3: Vector Addition

Two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, such that $|\vec{\mathbf{B}}| = 2|\vec{\mathbf{A}}|$, have a resultant $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ of magnitude 26.5. The vector $\vec{\mathbf{C}}$ makes an angle $\theta_c = 41^\circ$ with respect to vector $\vec{\mathbf{A}}$. Find the magnitude of each vector and the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Solution: We begin by making a sketch of the three vectors, choosing $\vec{\mathbf{A}}$ to point in the positive *x*-direction (Figure 3.20).

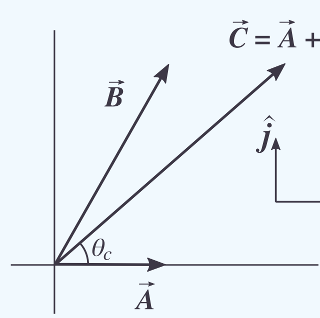

Figure 3.20: Choice of coordinates system for Example 3.3. (CC BY-NC; Ümit Kaya)

Denote the magnitude of $\vec{\mathbf{C}}$ by $C \equiv |\vec{\mathbf{C}}| = \sqrt{(C_x)^2 + (C_y)^2} = 26.5$. The components of $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ are given by

$$C_x = A_x + B_x = C\cos\theta_C = (26.5)\cos(41^\circ) = 20$$

$$C_y = B_y = C\sin\theta_C = (26.5)\sin(41^\circ) = 17.4.$$

From the condition that $|\vec{\mathbf{B}}| = 2|\vec{\mathbf{A}}|$, the square of their magnitudes satisfy

$$(B_x)^2 + (B_y)^2 = 4(A_x)^2.$$

Using Equations (3.3.17) and (3.3.18), Equation (3.3.19) becomes

$$(C_x - A_x)^2 + (C_y)^2 = 4(A_x)^2$$

$$(C_x)^2 - 2C_xA_x + (A_x)^2 + (C_y)^2 = 4(A_x)^2$$

This is a quadratic equation

$$0 = 3(A_x)^2 + 2C_xA_x - C^2$$

which we solve for the component A_x:

$$A_x = \frac{-2C_x \pm \sqrt{(2C_x)^2 + (4)(3)\left(C^2\right)}}{6} = \frac{-2(20) \pm \sqrt{(40))^2 + (4)(3)(26.5)^2}}{6} = 10.0$$

where we choose the positive square root because we originally chose $A_x > 0$. The components of $\vec{\mathbf{B}}$ are then given by Equations (3.3.17) and (3.3.18):

$$B_x = C_x - A_x = 20.0 - 10.0 = 10.0$$

$$B_{.} = 17.4$$

The magnitude of $|\vec{\mathbf{B}}| = \sqrt{(B_x)^2 + (B_y)^2} = 20.0$ which is equal to two times the magnitude of $|\vec{\mathbf{A}}| = 10.0$. The angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is given by

$$\theta = \sin^{-1}\left(B_y/|\vec{\mathbf{B}}|\right) = \sin^{-1}(17.4/20.0\mathrm{N}) = 60^\circ$$

Example 3.4 Vector Description of a Point on a Line

Consider two points, P_1 with coordinates (x_1, y_1) and P_2 with coordinates (x_1, y_1) are separated by distance d . Find a vector $\vec{\mathbf{A}}$ from the origin to the point on the line connecting P_1 and P_2 that is located a distance a from the point P_2 (Figure 3.21).

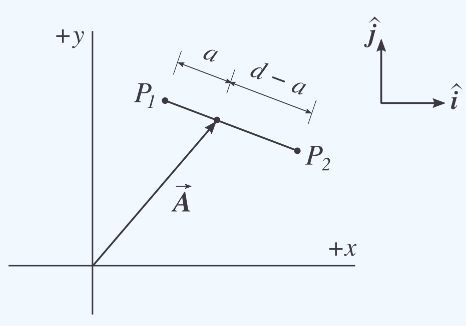

Figure 3.3.21: Example 3.4. (CC BY-NC; Ümit Kaya)

Solution

Let $\vec{\mathbf{r}}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$ be the position vector of P_1 and $\vec{\mathbf{r}}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$ the position vector of P_2. Let $\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$ be the vector from P_2 to P_1 (Figure 3.22a). The unit vector pointing from P_2 to P_1 is given by

$\hat{\mathbf{r}}_{21} = \left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)/\left|\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right| = \left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)/d$, where $d = \left((x_2 - x_1)^2 + (y_2 - y_1)^2\right)^{1/2}$

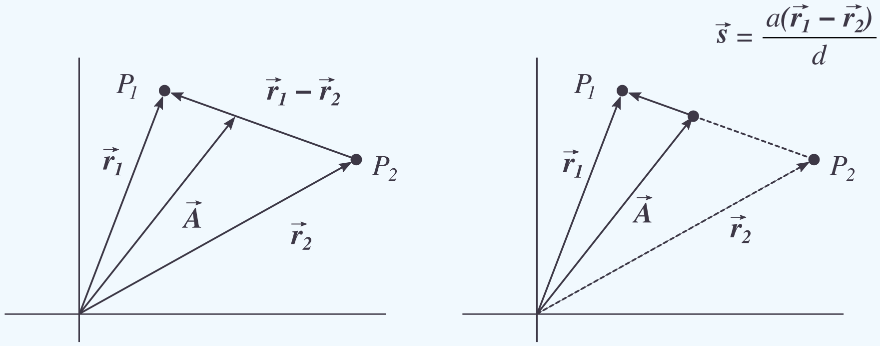

Figure 3.22a: Relative position vector Figure 3.22b: Relative position vector

The vector $\overrightarrow{\mathbf{s}}$ in Figure 3.22b connects $\overrightarrow{\mathbf{A}}$ to the point at $\overrightarrow{\mathbf{r_1}}$, points in the direction of $\overrightarrow{\mathbf{r_{12}}}$ and has length a. Therefore $\overrightarrow{\mathbf{s}} = a\hat{\mathbf{r}}_{21} = a\left(\overrightarrow{\mathbf{r}}_1 - \overrightarrow{\mathbf{r}}_2\right)/d$. The vector $\overrightarrow{\mathbf{r}}_1 = \overrightarrow{\mathbf{A}} + \overrightarrow{\mathbf{s}}$. Therefore

$$\overrightarrow{\mathbf{A}} = \overrightarrow{\mathbf{r}}_1 - \overrightarrow{\mathbf{s}} = \overrightarrow{\mathbf{r}}_1 - a\left(\overrightarrow{\mathbf{r}}_1 - \overrightarrow{\mathbf{r}}_2\right)/d = (1 - a/d)\overrightarrow{\mathbf{r}}_1 + (a/d)\overrightarrow{\mathbf{r}}_2$$

$$\overrightarrow{\mathbf{A}} = (1 - a/d)\left(x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}\right) + (a/d)\left(x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}\right)$$

$$\overrightarrow{\mathbf{A}} = \left(x_1 + \frac{a\left(x_2 - x_1\right)}{\left(\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2\right)^{1/2}}\right)\hat{\mathbf{i}} + \left(y_1 + \frac{a\left(y_2 - y_1\right)}{\left(\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2\right)^{1/2}}\right)\hat{\mathbf{j}}$$

Transformation of Vectors in Rotated Coordinate Systems

Consider two Cartesian coordinate systems S and S' such that the (x', y') coordinate axes in S' are rotated by an angle θ with respect to the (x, y) coordinate axes in S, (Figure 3.23).

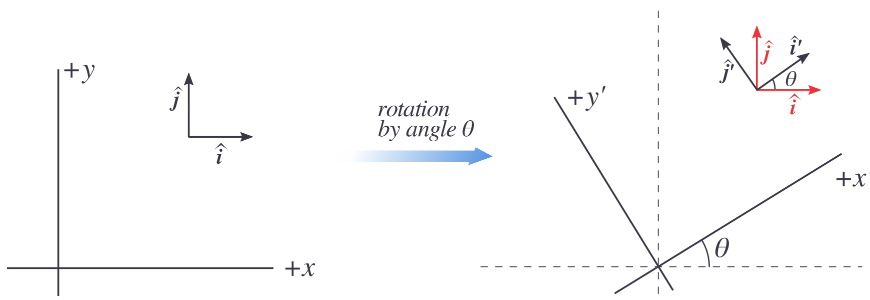

Figure 3.23: Rotated coordinate systems. (CC BY-NC; Ümit Kaya)

The components of the unit vector $\hat{\mathbf{i}}'$ in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ direction are given by

$$i'_x = \left|\hat{\mathbf{i}}'\right|\cos\theta = \cos\theta$$

and

$$i'_y = \left|\hat{\mathbf{i}}'\right|\sin\theta = \sin\theta.$$

Therefore

$$\hat{\mathbf{i}}' = i'_x\hat{\mathbf{i}} + i'_y\hat{\mathbf{j}} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta$$

A similar argument holds for the components of the unit vector $\hat{\mathbf{j}}'$. The components of $\hat{\mathbf{j}}'$ in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ direction are given by

$$j'_x = -\left|\hat{\mathbf{j}}'\right| \sin\theta = -\sin\theta$$

and

$$j'_y = \left|\hat{\mathbf{j}}'\right| \cos\theta = \cos\theta.$$

Therefore

$$\hat{\mathbf{j}}' = j'_x \hat{\mathbf{i}} + j'_y \hat{\mathbf{j}} = \hat{\mathbf{j}} \cos\theta - \hat{\mathbf{i}} \sin\theta$$

Conversely, from Figure 3.23 and similar vector decomposition arguments, the components of (\hat{\mathbf{i}}\) and (\hat{\mathbf{j}}\) in S' are given by

$$\hat{\mathbf{i}} = \hat{\mathbf{i}}' \cos\theta - \hat{\mathbf{j}}' \sin\theta$$

$$\hat{\mathbf{j}} = \hat{\mathbf{i}}' \sin\theta + \hat{\mathbf{j}}' \cos\theta$$

Consider a fixed vector $\overrightarrow{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ with components (x, y) in coordinate system S. In coordinate system S', the vector is given by $\overrightarrow{\mathbf{r}} = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}'$, where (x', y') , where (x', y') are the components in S', (Figure 3.24).

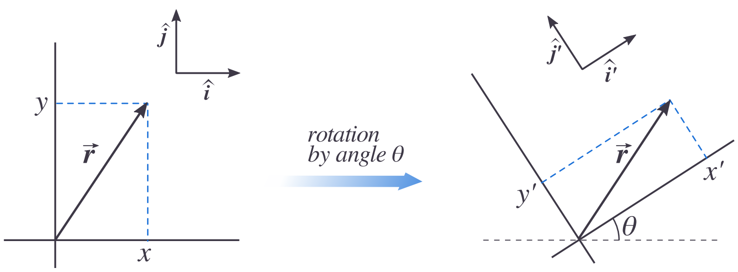

Figure 3.24: Transformation of vector components. (CC BY-NC; Ümit Kaya)

Using the Equations (3.3.20) and (3.3.21), we have that

$$\overrightarrow{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = x\left(\hat{\mathbf{i}}' \cos\theta - \hat{\mathbf{j}}' \sin\theta\right) + y\left(\hat{\mathbf{j}}' \cos\theta + \hat{\mathbf{i}}' \sin\theta\right)$$

$$\overrightarrow{\mathbf{r}} = (x\cos\theta + y\sin\theta)\hat{\mathbf{i}}' + (x\sin\theta - y\cos\theta)\hat{\mathbf{j}}'$$

Therefore the components of the vector transform according to

$$x' = x\cos\theta + y\sin\theta$$

$$y' = x\sin\theta - y\cos\theta$$

We now consider an alternate approach to understanding the transformation laws for the components of the position vector of a fixed point in space. In coordinate system S, suppose the position vector $\overrightarrow{\mathbf{r}}$ has length $r = |\overrightarrow{\mathbf{r}}|$ and makes an angle ϕ with respect to the positive x-axis (Figure 3.25).

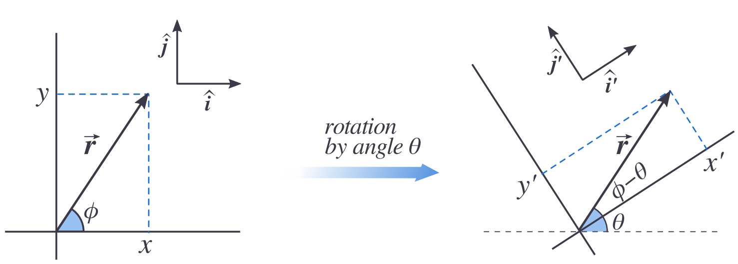

Figure 3.25: Transformation of vector components of the position vector. (CC BY-NC; Ümit Kaya)

Then the components of $\overrightarrow{\mathbf{r}}$ in S are given by

$$x = r\cos\phi$$

$$y = r\sin\phi$$

. In coordinate system S', the components of $\overrightarrow{\mathbf{r}}$ are given by

$$x' = r\cos(\phi - \theta)$$

$$y' = r\sin(\phi - \theta)$$

Apply the addition of angle trigonometric identities to Equations (3.3.29) and (3.3.30) yielding

$$x' = r\cos(\phi - \theta) = r\cos\phi\cos\theta + r\sin\phi\sin\theta = x\cos\theta + y\sin\theta$$

$$y' = r\sin(\phi - \theta) = r\sin\phi\cos\theta - r\cos\phi\sin\theta = y\cos\theta - x\sin\theta$$

in agreement with Equations (3.3.25) and (3.3.26).

Example 3.5 Vector Decomposition in Rotated Coordinate Systems

With respect to a given Cartesian coordinate system S, a vector $\overrightarrow{\mathbf{A}}$ has components $A_x = 5$, $A_y = -3$, $A_z = 0$. Consider a second coordinate system S' such that the $(x', y'$ coordinate axes in S' are rotated by an angle $\theta = 60^\circ$ with respect to the (x, y) coordinate axes in S, (Figure 3.26).

a. What are the components A_x and A_y of vector $\overrightarrow{\mathbf{A}}$ in coordinate system S'?
b. Calculate the magnitude of the vector using the (A_x, A_y) components and using the (A_x, A_y) components. Does your result agree with what you expect?

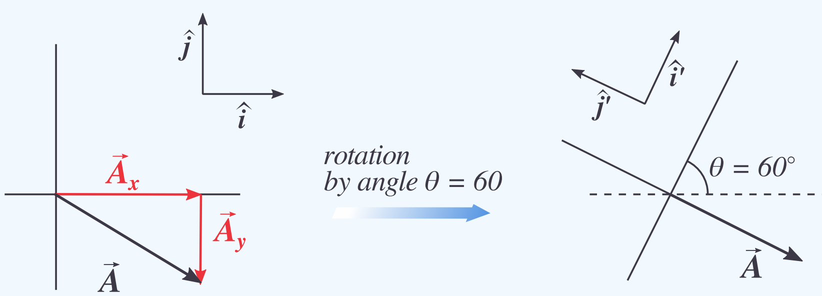

Figure 3.26 Example 3.4. (CC BY-NC; Ümit Kaya)

Solution:

We begin by considering the vector decomposition of $\overrightarrow{\mathbf{A}}$ with respect to the coordinate system S,

$$\overrightarrow{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$

Now we can use our results for the transformation of unit vectors $\overrightarrow{\mathbf{i}}$ and $\overrightarrow{\mathbf{j}}$ in terms of $\overrightarrow{\mathbf{i}'}$ and $\overrightarrow{\mathbf{j}'}$, (Equations (3.3.22) and (3.3.23)) in order decompose the vector $\overrightarrow{\mathbf{A}}$ in coordinate system S'

$$\begin{aligned}\overrightarrow{\mathbf{A}} &= A\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} = A_x\left(\cos\theta\hat{\mathbf{i}}' - \sin\theta\hat{\mathbf{j}}'\right) + A_y\left(\sin\theta\hat{\mathbf{i}}' + \cos\theta\hat{\mathbf{j}}'\right) \\ &= (A_x\cos\theta + A_y\sin\theta)\,\hat{\mathbf{i}}' + (-A_x\sin\theta + A_y\cos\theta)\,\hat{\mathbf{j}}' \\ &= A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\end{aligned}$$

where

$$A_{x'} = A_x\cos\theta + A_y\sin\theta$$

$$A_{y'} = -A_x\sin\theta + A_y\cos\theta$$

We now use the given information that $A_x = 5$, $A_y = -3$, and $\theta = 60^\circ$ to solve for the components of $\overrightarrow{\mathbf{A}}$ in coordinate system S'

$$A_{x'} = A_x \cos\theta + A_y \sin\theta = (1/2)(5 - 3\sqrt{3})$$

$$A_{y'} = -A_x \sin\theta + A_y \cos\theta = (1/2)(-5\sqrt{3} - 3)$$

b) The magnitude can be calculated in either coordinate system

$$|\overrightarrow{\mathbf{A}}| = \sqrt{(A_x)^2 + (A_y)^2} = \sqrt{(5)^2 + (-3)^2} = \sqrt{34}$$

$$|\overrightarrow{\mathbf{A}}| = \sqrt{(A_{x'})^2 + (A_{y'})^2} = \sqrt{((1/2)(5 - 3\sqrt{3}))^2 + ((1/2)(-5\sqrt{3} - 3))^2} = \sqrt{34}$$

This result agrees with what I expect because the length of vector $\overrightarrow{\mathbf{A}}$ independent of the choice of coordinate system.