![LibreTexts]

25.2: Planetary Orbits

We now commence a study of the Kepler Problem. We shall determine the equation of motion for the motions of two bodies interacting via a gravitational force (two-body problem) using both force methods and conservation laws.

Reducing the Two-Body Problem into a One-Body Problem

We shall begin by showing how the motion of two bodies interacting via a gravitational force (two-body problem) is mathematically equivalent to the motion of a single body acted on by an external central gravitational force, where the mass of the single body is the reduced mass μ,

$$\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2} \Rightarrow \mu=\frac{m_1 m_2}{m_1+m_2}$$

Once we solve for the motion of the reduced body in this equivalent one-body problem, we can then return to the real two-body problem and solve for the actual motion of the two original bodies. The reduced mass was introduced in Chapter 13 Appendix A of these notes. That appendix used similar but slightly different notation from that used in this chapter.

Consider the gravitational interaction between two bodies with masses m and as 1 m2 shown in Figure 25.1.

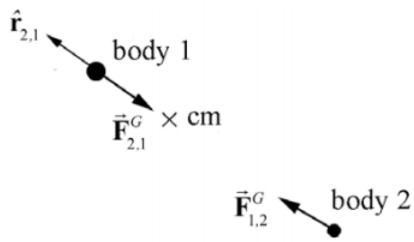

Figure 25.1 Gravitational force between two bodies

Choose a coordinate system with a choice of origin such that body 1 has position $\overrightarrow{\mathbf{r}}_1$ and body 2 has position $\overrightarrow{\mathbf{r}}_2$ (Figure 25.2). The relative position vector $\overrightarrow{\mathbf{r}}$ pointing from body 2 to body 1 is $\overrightarrow{\mathbf{r}}=\overrightarrow{\mathbf{r}}_1-\overrightarrow{\mathbf{r}}_2$ We denote the magnitude of $\overrightarrow{\mathbf{r}}$ by $|\overrightarrow{\mathbf{r}}|=r$, where r is the distance between the bodies, and $\hat{\mathbf{r}}$ is the unit vector pointing from body 2 to body 1, so that $\overrightarrow{\mathbf{r}}=r\hat{\mathbf{r}}$.

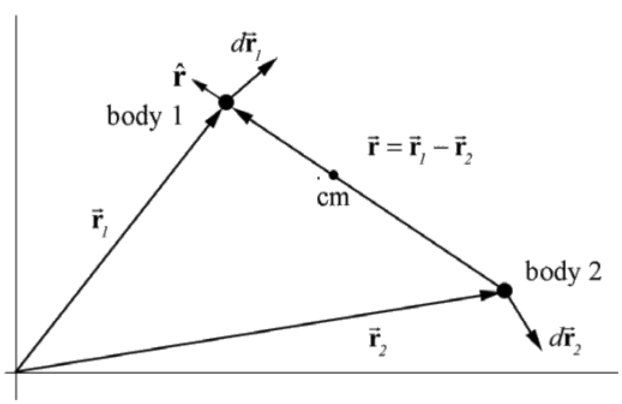

Figure 25.2 Coordinate system for the two-body problem

The force on body 1 (due to the interaction of the two bodies) can be described by Newton's Universal Law of Gravitation

$$\overrightarrow{\mathbf{F}}_{2,1}=-F_{2,1}\hat{\mathbf{r}}=-G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

Recall that Newton's Third Law requires that the force on body 2 is equal in magnitude and opposite in direction to the force on body 1,

$$\overrightarrow{\mathbf{F}}_{1,2}=-\overrightarrow{\mathbf{F}}_{2,1}$$

Newton's Second Law can be applied individually to the two bodies:

$$\vec{\mathbf{F}}_{2,1} = m_1 \frac{d^2 \vec{\mathbf{r}}_1}{dt^2}$$
$$\vec{\mathbf{F}}_{1,2} = m_2 \frac{d^2 \vec{\mathbf{r}}_2}{dt^2}$$

Dividing through by the mass in each of Equations (25.2.4) and (25.2.5) yields

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} = \frac{d^2 \vec{\mathbf{r}}_1}{dt^2}$$

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2 \vec{\mathbf{r}}_2}{dt^2}$$

Subtracting the expression in Equation (25.2.7) from that in Equation (25.2.6) yields

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} - \frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2 \vec{\mathbf{r}}_1}{dt^2} - \frac{d^2 \vec{\mathbf{r}}_2}{dt^2} = \frac{d^2 \vec{\mathbf{r}}}{dt^2}$$

Using Newton's Third Law, Equation (25.2.3), Equation (25.2.8) becomes

$$\vec{\mathbf{F}}_{2,1}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2 \vec{\mathbf{r}}}{dt^2}$$

Using the reduced mass μ , as defined in Equation (25.2.1), Equation (25.2.9) becomes

$$\frac{\vec{\mathbf{F}}_{2,1}}{\mu} = \frac{d^2 \vec{\mathbf{r}}}{dt^2}$$
$$\vec{\mathbf{F}}_{2,1} = \mu \frac{d^2 \vec{\mathbf{r}}}{dt^2}$$

where $\vec{\mathbf{F}}_{2,1}$ is given by Equation (25.2.2).

Our result has a special interpretation using Newton's Second Law. Let μ be the mass of a single body with position vector $\vec{\mathbf{r}} = r\hat{\mathbf{r}}$ with respect to an origin O , where $\hat{\mathbf{r}}$ is the unit vector pointing from the origin O to the single body. Then the equation of motion, Equation (25.2.10), implies that the single body of mass μ is under the influence of an attractive gravitational force pointing toward the origin. So, the original two-body gravitational problem has now been reduced to an equivalent one-body problem, involving a single body with mass μ under the influence of a central force $\vec{\mathbf{F}}^G = -\mathrm{F}_{2,1}\hat{\mathbf{r}}$ Note that in this reformulation, there is no body located at the central point (the origin O). The parameter r in the two-body problem is the relative distance between the original two bodies, while the same parameter r in the one-body problem is the distance between the single body and the central point. This reduction generalizes to all central forces.
