23.9: Appendix 23B - Complex Numbers

A complex number z can be written as a sum of a real number x and a purely imaginary number iy where $i=\sqrt{-1}$

$$z=x+iy$$

The complex number can be represented as a point in the x-y plane as show in Figure 23B.1.

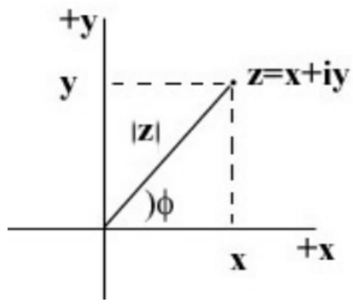

Figure 23B.1 Complex numbers

The complex conjugate $\bar{z}$ of a complex number z is defined to be

$$\bar{z}=x-iy$$

The modulus of a complex number is

$$|z|=(z\bar{z})^{1/2}=((x+iy)(x-iy))^{1/2}=\left(x^{2}+y^{2}\right)^{1/2}$$

where we used the fact that $i^{2}=-1$. The modulus $|z|$ represents the length of the ray from the origin to the complex number z in Figure 23B.1. Let ϕ denote the angle that the ray with the positive x -axis in Figure 23B.1. Then

$$x=|z|\cos\phi$$

$$y=|z|\sin\phi$$

Hence the angle ϕ is given by

$$\phi=\tan^{-1}(y/x)$$

The inverse of a complex number is then

$$\frac{1}{z}=\frac{\bar{z}}{z\bar{z}}=\frac{x-iy}{(x^{2}+y^{2})}$$

The modulus of the inverse is the inverse of the modulus;

$$\left|\frac{1}{z}\right|=\frac{1}{(x^{2}+y^{2})^{1/2}}=\frac{1}{|z|}$$

The sum of two complex numbers, $z_{1}=x_{1}+iy_{1}$ and $z_{2}=x_{2}+iy_{2}$ is the complex number

$$z_{3}=z_{1}+z_{2}=(x_{1}+x_{2})+i\,(y_{1}+y_{2})=x_{3}+iy_{3}$$

where $x_{3}=x_{1}+x_{2}$, $y_{3}=y_{1}+y_{2}$. We can represent this by the vector sum in Figure 23B.2,

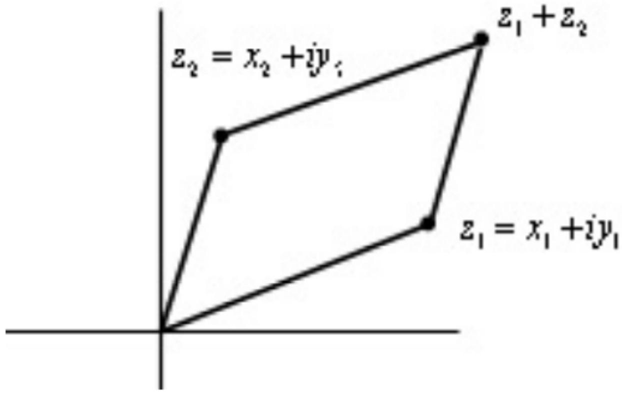

Figure 23B.2 Sum of two complex numbers

The product of two complex numbers is given by

$$z_3 = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = x_3 + iy_3$$

where $x_3 = x_1 x_2 - y_1 y_2$ and $y_3 = x_1 y_2 + x_2 y_1$

One of the most important identities in mathematics is the Euler formula,

$$e^{i\phi} = \cos\phi + i\sin\phi$$

This identity follows from the power series representations for the exponential, sine, and cosine functions,

$$e^{i\phi} = \sum_{n=0}^{n=\infty} \frac{1}{n!}(i\phi)^n = 1 + i\phi - \frac{\phi^2}{2} - i\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + i\frac{\phi^5}{5!}\ldots$$

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{4!} - \ldots$$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots$$

We define two projection operators. The first one takes the complex number $e^{i\phi}$ and gives its real part,

$$\mathrm{Re}\, e^{i\phi} = \cos\phi$$

The second operator takes the complex number $e^{i\phi}$ and gives its imaginary part, which is the real number

$$\mathrm{Im}\, e^{i\phi} = \sin\phi$$

A complex number $z = x + iy$ can also be represented as the product of a modulus $|z|$ and a phase factor $e^{i\phi}$

$$z = |z| e^{i\phi}$$

The inverse of a complex number is then

$$\frac{1}{z} = \frac{1}{|z|e^{i\phi}} = \frac{1}{|z|}e^{-i\phi}$$

where we used the fact that

$$\frac{1}{e^{i\phi}} = e^{-i\phi}$$

In terms of modulus and phase, the sum of two complex numbers, $z_1 = |z_1|\, e^{i\phi_1}$ and $z_2 = |z_2|\, e^{i\phi_2}$, is

$$z_1 + z_2 = |z_1|\, e^{i\phi_1} + |z_2|\, e^{i\phi_2}$$

A special case of this result is when the phase angles are equal, $\phi_1 = \phi_2$ then the sum $z_1 + z_2$ has the same phase factor $e^{i\phi_1}$ as z_1 and z_2

$$z_1+z_2=|z_1|\,e^{i\phi_1}+|z_2|\,e^{i\phi_1}=(|z_1|+|z_2|)\,e^{i\phi_1}$$

The product of two complex numbers, $z_1=|z_1|\,e^{i\phi_1}$, and $z_2=|z_2|\,e^{i\phi_2}$ is

$$z_1z_2=|z_1|\,e^{i\phi_1}\,|z_2|\,e^{i\phi_2}=|z_1||z_2|\,e^{i\phi_1+\phi_2}$$

When the phases are equal, the product does not have the same factor as z_1 and z_2

$$z_1z_2=|z_1|\,e^{i\phi_1}\,|z_2|\,e^{i\phi_1}=|z_1|\,|z_2|\,e^{i2\phi_1}$$