# 6.5: Angular Velocity and Angular Acceleration

## . Angular Velocity

We shall always choose a right-handed cylindrical coordinate system. If the positive z - axis points up, then we choose θ to be increasing in the counterclockwise direction as shown in Figures 6.6.

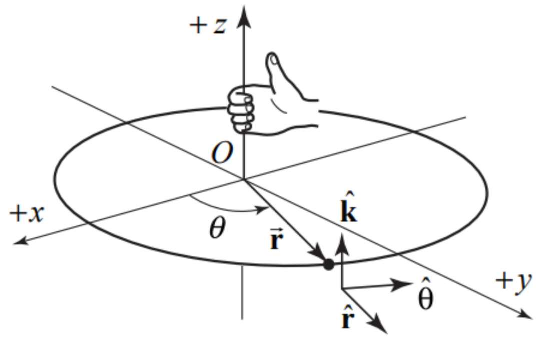


Figure 6.6 Right handed coordinate system

For a point object undergoing circular motion about the z -axis, the angular velocity vector $\vec{\omega}$ is directed along the z -axis with z - component equal to the time derivative of the angle θ,

$$\vec{\omega}=\frac{d \theta}{d t} \hat{\mathbf{k}}=\omega_{z} \hat{\mathbf{k}}$$

The SI units of angular velocity are $\left[\mathrm{rad} \cdot \mathrm{s}^{-1}\right]$ Note that the angular speed is just the magnitude of the z -component of the angular velocity,

$$\omega \equiv\left|\omega_{z}\right|=\left|\frac{d \theta}{d t}\right|$$

If the velocity of the object is in the $+\hat{\boldsymbol{\theta}}$-direction, (rotating in the counterclockwise direction in Figure 6.7(a)), then the z - component of the angular velocity is positive, $\omega_{z}=d \theta / d t>0$ The angular velocity vector then points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 6.7(a). If the velocity of the object is in the $-\hat{\boldsymbol{\theta}}$-direction, (rotating in the clockwise direction in Figure 6.7(b)), then the z -component of the angular velocity angular velocity is negative, $\omega_{z}=d \theta / d t<0$ . The angular velocity vector then points in the $-\hat{\mathbf{k}}$-direction as shown in Figure 6.7(b).

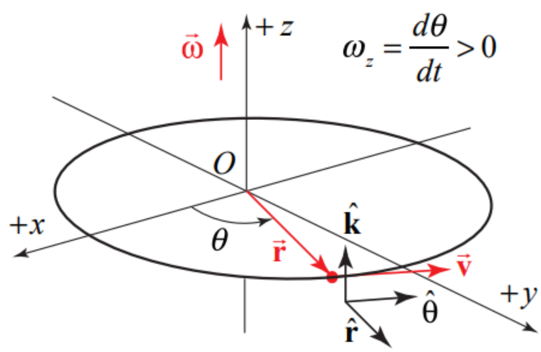


Figure 6.7(b) Angular velocity vector for motion with dθ / dt > 0 .

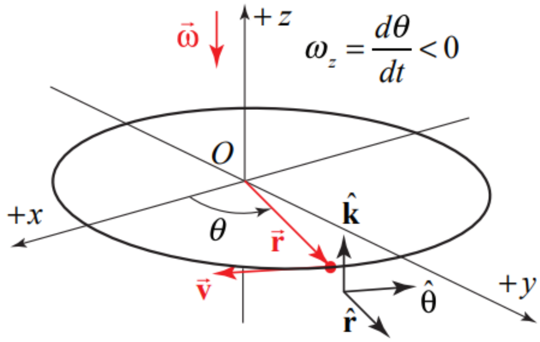


Figure 6.7(b) Angular velocity for motion with dθ / dt < 0 .

The velocity and angular velocity are related by

$$\overrightarrow{\mathbf{v}} = \overrightarrow{\boldsymbol{\omega}} \times \overrightarrow{\mathbf{r}} = \frac{d\theta}{dt}\hat{\mathbf{k}} \times r\hat{\mathbf{r}} = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

**Example 6.2 Angular Velocity**

A particle is moving in a circle of radius R. At t = 0 , it is located on the x -axis. The angle the particle makes with the positive x -axis is given by $\theta(t) = At - Bt^3$ where A and B are positive constants. Determine (a) the angular velocity vector, and (b) the velocity vector. Express your answer in polar coordinates. (c) At what time, $t = t_1$ is the angular velocity zero? (d) What is the direction of the angular velocity for 1. $t < t_1$ 2. $t > t_1$? ?

**Solution**:

The derivative of $\theta(t) = At - Bt^3$ is

$$\frac{d\theta(t)}{dt} = A - 3Bt^2$$

Therefore the angular velocity vector is given by

$$\overrightarrow{\boldsymbol{\omega}}(t) = \frac{d\theta(t)}{dt}\hat{\mathbf{k}} = \left(A - 3Bt^2\right)\hat{\mathbf{k}}$$

The velocity is given by

$$\overrightarrow{\mathbf{v}}(t) = R\frac{d\theta(t)}{dt}\hat{\theta}(t) = R\left(A - 3Bt^2\right)\hat{\boldsymbol{\theta}}(t)$$

The angular velocity is zero at time $t = t_1$ when

$$A - 3Bt_1^2 = 0 \Rightarrow t_1 = \sqrt{A/3B}$$

For $t < t_1$, $\frac{d\theta(t)}{dt} = A - 3Bt_1^2 > 0$ hence $\overrightarrow{\boldsymbol{\omega}}(t)$ points in the positive $\hat{\mathbf{k}}$-direction.

For $t > t_1$, $\frac{d\theta(t)}{dt} = A - 3Bt_1^2 < 0$ hence $\overrightarrow{\boldsymbol{\omega}}(t)$ points in the positive $\hat{\mathbf{k}}$-direction.

## Angular Acceleration

In a similar fashion, for a point object undergoing circular motion about the fixed z -axis, the angular acceleration is defined as

$$\vec{\alpha} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}} = \alpha_z\hat{\mathbf{k}}$$

The SI units of angular acceleration are $\left[\mathrm{rad} \cdot \mathrm{s}^{-2}\right]$ The magnitude of the angular acceleration is denoted by the Greek symbol alpha,

$$\alpha \equiv |\vec{\alpha}| = \left|\frac{d^2\theta}{dt^2}\right|$$

There are four special cases to consider for the direction of the angular velocity. Let's first consider the two types of motion with $\vec{\alpha}$ pointing in the $\hat{\mathbf{k}}$-direction: (i) if the object is rotating counterclockwise and speeding up then both $d\theta/dt > 0$ and $d^2\theta/dt^2 > 0$ (Figure 6.8(a)) (ii) if the object is rotating clockwise and slowing down then $d\theta/dt < 0$ but $d^2\theta/dt^2 > 0$ (Figure 6.8(b). There are two corresponding cases in which $\vec{\alpha}$ pointing in the $-\hat{\mathbf{k}}$-direction (iii) if the object is rotating counterclockwise and slowing down then $d\theta/dt > 0$ but $d^2\theta/dt^2 < 0$ (Figure 6.9(a), (iv) if the object is rotating clockwise and speeding up then both $d\theta/dt < 0$ and $d^2\theta/dt^2 < 0$ (Figure 6.9(b).

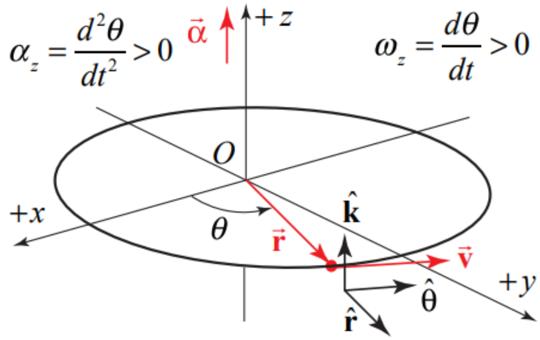


Figure 6.8(a) Angular acceleration vector vector for motion with $d\theta/dt > 0$, and $d^2\theta/dt^2 > 0$

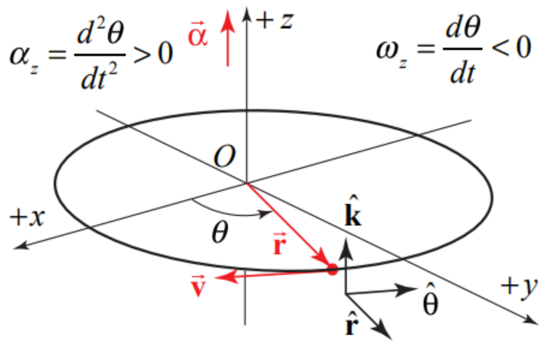


Figure 6.8(b) Angular velocity vector vector for motion with $d\theta/dt < 0$, and $d^2\theta/dt^2 > 0$

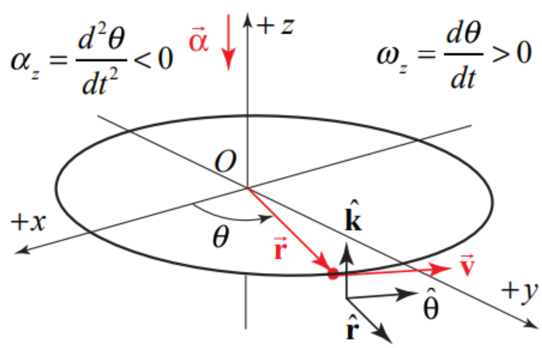


Figure 6.9(a) Angular acceleration vector vector for motion with $d\theta/dt > 0$, and $d^2\theta/dt^2 < 0$

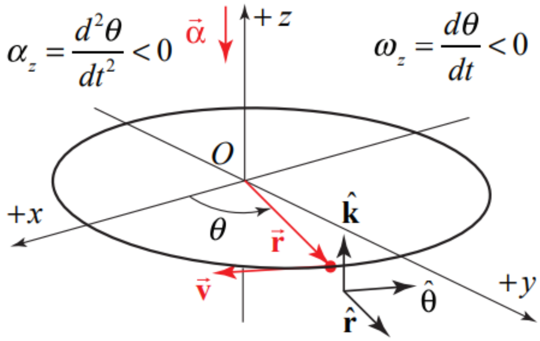


Figure 6.9(b) Angular velocity vector vector for motion with $d\theta/dt < 0$, and $d^2\theta/dt^2 < 0$

## Example 6.3 Integration and Circular Motion Kinematics

A point-like object is constrained to travel in a circle. The z -component of the angular acceleration of the object for the time interval $[0, t_1]$ is given by the function

$$\alpha_z(t) = \begin{cases} b\left(1 - \frac{t}{t_1}\right); 0 \leq t \leq t_1 \\ 0; t > t_1 \end{cases}$$

where b is a positive constant with units rad·S$^{-2}$.

a) Determine an expression for the angular velocity of the object at $t = t_1$ .

b) Through what angle has the object rotated at time $t = t_1$ ?

**Solution**:

a) The angular velocity at time $t = t_1$ is given by

$$\omega_z(t_1) - \omega_z(t=0) = \int_{t'=0}^{t'=t_1} \alpha_z(t')\, dt' = \int_{t'=0}^{t'=t_1} b\left(1 - \frac{t'}{t_1}\right) dt' = b\left(t_1 - \frac{t_1^2}{2t_1}\right) = \frac{bt_1}{2}$$

b) In order to find the angle $\theta(t_1) - \theta(t=0)$ that the object has rotated through at time $t = t_1$ , you first need to find $\omega_z(t)$ by integrating the z-component of the angular acceleration

$$\omega_z(t) - \omega_z(t=0) = \int_{t'=0}^{t'=t} \alpha_z(t')\, dt' = \int_{t=0}^{t'=t} b\left(1 - \frac{t'}{t_1}\right) dt' = b\left(t - \frac{t^2}{2t_1}\right)$$

Because it started from rest, $\omega_z(t=0) = 0$ , hence $\omega_z(t) = b\left(t - \frac{t^2}{2t_1}\right); 0 \leq t \leq t_1$

Then integrate $\omega_z(t)$ between t = 0 and $t = t_1$ to find that

$$\theta(t_1) - \theta(t=0) = \int_{t'=0}^{t'=t_1} \omega_z(t')\, dt' = \int_{t'=0}^{t'=t_1} b\left(t' - \frac{t'^2}{2t_1}\right) dt' = b\left(\frac{t_1^2}{2} - \frac{t_1^3}{6t_1}\right) = \frac{bt_1^2}{3}$$

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