6.3: Circular Motion- Tangential and Radial Acceleration

When the motion of an object is described in polar coordinates, the acceleration has two components, the tangential component a_θ, and the radial component, a_r. We can write the acceleration vector as

$$\overrightarrow{\mathbf{a}}=a_r\hat{\mathbf{r}}(t)+a_\theta\hat{\boldsymbol{\theta}}(t)$$

Keep in mind that as the object moves in a circle, the unit vectors $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ change direction and hence are not constant in time.

We will begin by calculating the tangential component of the acceleration for circular motion. Suppose that the tangential velocity $v_\theta=rd\theta/dt$ is changing in magnitude due to the presence of some tangential force; we shall now consider that $d\theta/dt$ is changing in time, (the magnitude of the velocity is changing in time). Recall that in polar coordinates the velocity vector Equation (6.2.8) can be written as

$$\overrightarrow{\mathbf{v}}(t)=r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t)$$

We now use the product rule to determine the acceleration.

$$\overrightarrow{\mathbf{a}}(t)=\frac{d\overrightarrow{\mathbf{v}}(t)}{dt}=r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t)+r\frac{d\theta(t)}{dt}\frac{d\hat{\boldsymbol{\theta}}(t)}{dt}$$

Recall from Equation (6.2.3) that $\hat{\boldsymbol{\theta}}(t)=-\sin\theta(t)\hat{\mathbf{i}}+\cos\theta(t)\hat{\mathbf{j}}$. So we can rewrite Equation (6.3.3) as

$$\overrightarrow{\mathbf{a}}(t)=r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t)+r\frac{d\theta(t)}{dt}\frac{d}{dt}(-\sin\theta(t)\hat{\mathbf{i}}+\cos\theta(t)\hat{\mathbf{j}})$$

We again use the chain rule (Equations (6.2.5) and (6.2.6)) and find that

$$\overrightarrow{\mathbf{a}}(t)=r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t)+r\frac{d\theta(t)}{dt}\left(-\cos\theta(t)\frac{d\theta(t)}{dt}\hat{\mathbf{i}}-\sin\theta(t)\frac{d\theta(t)}{dt}\hat{\mathbf{j}}\right)$$

Recall that $\omega\equiv d\theta/dt$, and from Equation (6.2.2), $\hat{\mathbf{r}}(t)=\cos\theta(t)\hat{\mathbf{i}}+\sin\theta(t)\hat{\mathbf{j}}$ therefore the acceleration becomes

$$\overrightarrow{\mathbf{a}}(t)=r\frac{d^2\theta(t)}{dt^2}\hat{\boldsymbol{\theta}}(t)-r\left(\frac{d\theta(t)}{dt}\right)^2\hat{\mathbf{r}}(t)$$

The tangential component of the acceleration is then

$$a_\theta=r\frac{d^2\theta(t)}{dt^2}$$

The radial component of the acceleration is given by

$$a_r=-r\left(\frac{d\theta(t)}{dt}\right)^2=-r\omega^2<0$$

Because $a_r<0$, that radial vector component $\overrightarrow{\mathbf{a}}_r(t)=-r\omega^2\hat{\mathbf{r}}(t)$ is always directed towards the center of the circular orbit.

Example 6.1 Circular Motion Kinematics

A particle is moving in a circle of radius R. At t = 0 , it is located on the x -axis. The angle the particle makes with the positive x -axis is given by $\theta(t)=At^3-Bt$ where A and B are positive constants. Determine (a) the velocity vector, and (b) the acceleration vector. Express your answer in polar coordinates. At what time is the centripetal acceleration zero?

Solution:

The derivatives of the angle function $\theta(t)=At^3-Bt$ are $d\theta/dt=3At^2-B$ and $d^2\theta/dt^2=6At$. Therefore the velocity vector is given by

$$\vec{\mathbf{v}}(t)=R\frac{d\theta(t)}{dt}\hat{\theta}(t)=R\left(3At^2-Bt\right)\hat{\theta}(t)$$

The acceleration is given by

$$\begin{aligned}\vec{\mathbf{a}}(t)&=R\frac{d^2\theta(t)}{dt^2}\hat{\theta}(t)-R\left(\frac{d\theta(t)}{dt}\right)^2\hat{\mathbf{r}}(t)\\&=R(6At)\hat{\boldsymbol{\theta}}(t)-R\left(3At^2-B\right)^2\hat{\mathbf{r}}(t)\end{aligned}$$

The centripetal acceleration is zero at time $t=t_1$ when

$$3At_1^2-B=0\Rightarrow t_1=\sqrt{B/3A}$$
