

21.3: Translational and Rotational Equations of Motion

For a system of particles, the torque about a point S can be written as

$$\vec{\tau}_S^{\text{ext}} = \sum_{i=1}^N \left(\vec{r}_i \times \vec{F}_i \right)$$

where we have assumed that all internal torques cancel in pairs. Let choose the point S to be the origin of the reference frame O , then $\vec{r}_{S,\text{cm}} = \vec{R}_{\text{cm}}$ (Figure 21.1). (You may want to recall the main properties of the center of mass reference frame by reviewing Chapter 15.2.1.)

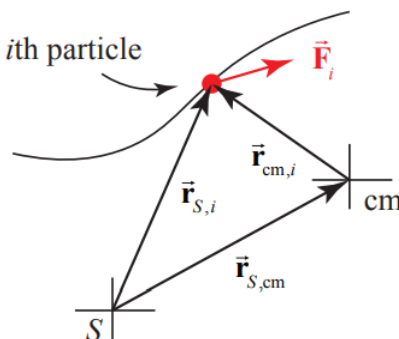


Figure 21.1 Torque diagram for center of mass reference frame

We can now apply $\vec{r}_{S,i} = \vec{r}_{S,\text{cm}} + \vec{r}_{\text{cm},i}$ to Equation (21.3.1) yielding

$$\vec{\tau}_S^{\text{ext}} = \sum_{i=1}^N \left(\vec{r}_{S,i} \times \vec{F}_i \right) = \sum_{i=1}^N \left(\left(\vec{r}_{S,\text{cm}} + \vec{r}_{\text{cm},i} \right) \times \vec{F}_i \right) = \sum_{i=1}^N \left(\vec{r}_{S,\text{cm}} \times \vec{F}_i \right) + \sum_{i=1}^N \left(\vec{r}_{\text{cm},i} \times \vec{F}_i \right)$$

The term

$$\vec{\tau}_{S,\text{cm}}^{\text{ext}} = \vec{r}_{S,\text{cm}} \times \vec{F}^{\text{ext}}$$

in Equation (21.3.2) corresponds to the external torque about the point S where all the external forces act at the center of mass (Figure 21.2).

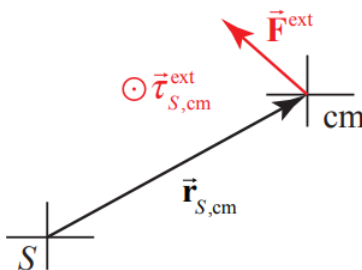


Figure 21.2 Torque diagram for "point-like" particle located at center of mass

The term,

$$\vec{\tau}_{\text{cm}}^{\text{ext}} = \sum_{i=1}^N \left(\vec{r}_{\text{cm},i} \times \vec{F}_i \right)$$

is the sum of the torques on the individual particles in the center of mass reference frame. If we assume that all internal torques cancel in pairs, then

$$\vec{\tau}_{\text{cm}}^{\text{ext}} = \sum_{i=1}^N \left(\vec{r}_{\text{cm},i} \times \vec{F}_i^{\text{ext}} \right)$$

We conclude that the external torque about the point S can be decomposed into two pieces,

$$\vec{\tau}_S^{\text{ext}} = \vec{\tau}_{S,\text{cm}}^{\text{ext}} + \vec{\tau}_{\text{cm}}^{\text{ext}}$$

We showed in Chapter 20.3 that

$$\vec{L}_S^{\text{sys}} = \vec{r}_{S,\text{cm}} \times \vec{p}^{\text{sys}} + \sum_{i=1}^N \left(\vec{r}_{\text{cm},i} \times m \vec{v}_{\text{cm},i} \right)$$

where the first term in Equation (21.3.7) is the orbital angular momentum of the center of mass about the point S

$$\vec{L}_S^{\text{orbital}} = \vec{r}_{S,\text{cm}} \times \vec{p}^{\text{sys}}$$

and the second term in Equation (21.3.7) is the spin angular momentum about the center of mass (independent of the point S)

$$\vec{L}_S^{\text{spin}} = \sum_{i=1}^N \left(\vec{r}_{\text{cm},i} \times m \vec{v}_{\text{cm},i} \right)$$

The angular momentum about the point S can therefore be decomposed into two terms

$$\vec{L}_S^{\text{sys}} = \vec{L}_S^{\text{orbital}} + \vec{L}_S^{\text{spin}}$$

Recall that that we have previously shown that it is always true that

$$\vec{\tau}_S^{\text{ext}} = \frac{d\vec{L}_S^{\text{sys}}}{dt}$$

Therefore we can therefore substitute Equation (21.3.6) on the LHS of Equation (21.3.11) and substitute Equation (21.3.10) on the RHS of Equation (21.3.11) yielding as

$$\vec{\tau}_{S,\text{cm}}^{\text{ext}} + \vec{\tau}_{\text{cm}}^{\text{ext}} = \frac{d\vec{L}_S^{\text{orbital}}}{dt} + \frac{d\vec{L}_S^{\text{spin}}}{dt}$$

We shall now show that Equation (21.3.12) can also be decomposed in two separate conditions. We begin by analyzing the first term on the RHS of Equation (21.3.12). We differentiate Equation (21.3.8) and find that

$$\frac{d\vec{L}_S^{\text{orbital}}}{dt} = \frac{d}{dt} \left(\vec{r}_{S,\text{cm}} \times \vec{p}^{\text{sys}} \right)$$

We apply the vector identity

$$\frac{d}{dt} (\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

to Equation (21.3.13) yielding

$$\frac{d\vec{L}_S^{\text{orbital}}}{dt} = \frac{d\vec{r}_{S,\text{cm}}}{dt} \times \vec{p}^{\text{sys}} + \vec{r}_{S,\text{cm}} \times \frac{d\vec{p}^{\text{sys}}}{dt}$$

The first term in Equation (21.3.21) is zero because

$$\frac{d\vec{r}_{S,\text{cm}}}{dt} \times \vec{p}^{\text{sys}} = \vec{V}_{\text{cm}} \times m^{\text{total}} \vec{V}_{\text{cm}} = \vec{0}$$

Therefore the time derivative of the orbital angular momentum about a point S , Equation (21.3.15), becomes

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \vec{\mathbf{r}}_{S,\text{cm}} \times \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt}$$

In Equation (21.3.17), the time derivative of the momentum of the system is the external force,

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt}$$

The expression in Equation (21.3.17) then becomes the first of our relations

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}^{\text{ext}} = \vec{\tau}_{S,\text{cm}}^{\text{ext}}$$

Thus the time derivative of the orbital angular momentum about the point S is equal to the external torque about the point S where all the external forces act at the center of mass, (we treat the system as a point-like particle located at the center of mass).

We now consider the second term on the RHS of Equation (21.3.12), the time derivative of the spin angular momentum about the center of mass. We differentiate Equation (21.3.9),

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt} \sum_{i=1}^N \left(\vec{\mathbf{r}}_{\text{cm},i} \times m_i \vec{\mathbf{v}}_{\text{cm},i} \right)$$

We again use the product rule for taking the time derivatives of a vector product (Equation (21.3.14)). Then Equation (21.3.20) the becomes

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \sum_{i=1}^N \left(\frac{d\vec{\mathbf{r}}_{\text{cm},i}}{dt} \times m_i \vec{\mathbf{v}}_{\text{cm},i} \right) + \sum_{i=1}^N \left(\vec{\mathbf{r}}_{\text{cm},i} \times \frac{d}{dt} (m_i \vec{\mathbf{v}}_{\text{cm},i}) \right)$$

The first term in Equation (21.3.21) is zero because

$$\sum_{i=1}^N \left(\frac{d\vec{\mathbf{r}}_{\text{cm},i}}{dt} \times m_i \vec{\mathbf{v}}_{\text{cm},i} \right) = \sum_{i=1}^N \left(\vec{\mathbf{v}}_{\text{cm},i} \times m_i \vec{\mathbf{v}}_{\text{cm},i} \right) = \vec{\mathbf{0}}$$

Therefore the time derivative of the spin angular momentum about the center of mass, Equation (21.3.21), becomes

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \sum_{i=1}^N \left(\vec{\mathbf{r}}_{\text{cm},i} \times \frac{d}{dt} (m_i \vec{\mathbf{v}}_{\text{cm},i}) \right)$$

The force, acting on an element of mass m_i , is

$$\vec{\mathbf{F}}_i = \frac{d}{dt} (m_i \vec{\mathbf{v}}_{\text{cm},i})$$

The expression in Equation (21.3.23) then becomes

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \sum_{i=1}^N \left(\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_i \right)$$

The term, $\sum_{i=1}^N \left(\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_i \right)$ is the sum of the torques on the individual particles in the center of mass reference frame. If we again assume that all internal torques cancel in pairs, Equation (21.3.25) may be expressed as

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \sum_{i=1}^N \left(\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_i^{\text{ext}} \right) = \sum_{i=1}^N \vec{\tau}_{\text{cm},i}^{\text{ext}} = \vec{\tau}_{\text{cm}}^{\text{ext}}$$

which is the second of our two relations.

Summary

For a system of particles, there are two conditions that always hold (Equations (21.3.19) and (21.3.26)) when we calculate the torque about a point S ; we treat the system as a point-like particle located at the center of mass of the system. All the external forces $\vec{\mathbf{F}}^{\text{ext}}$ act at the center of mass. We calculate the orbital angular momentum of the center of mass and determine its time derivative and then apply

$$\vec{\tau}_{S,\text{cm}}^{\text{ext}} = \vec{\mathbf{r}}_{S,\text{cm}} \times \vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{otbial}}}{dt}$$

In addition, we calculate the torque about the center of mass due to all the forces acting on the particles in the center of mass reference frame. We calculate the time derivative of the angular momentum of the system with respect to the center of mass in the center of mass reference frame and then apply

$$\vec{\tau}_{\text{cm}}^{\text{ext}} = \sum_{i=1}^N \left(\vec{\mathbf{r}}_{\text{cm},i} \times \vec{\mathbf{F}}_i^{\text{ext}} \right) = \frac{d\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}}}{dt}$$

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