21.5: Work-Energy Theorem

For a rigid body, we can also consider the work-energy theorem separately for the translational motion and the rotational motion. Once again treat the rigid body as a point-like particle moving with velocity $\overrightarrow{\mathbf{V}}_{cm}$ in reference frame O . We can use the same technique that we used when treating point particles to show that the work done by the external forces is equal to the change in kinetic energy

$$\begin{aligned} W_{\text{tans}}^{\text{ext}} &= \int_i^f \overrightarrow{\mathbf{F}}^{\text{ext}} \cdot d\overrightarrow{\mathbf{r}} = \int_i^f \frac{d\left(m\overrightarrow{\mathbf{V}}_{cm}\right)}{dt} \cdot d\overrightarrow{\mathbf{R}}_{cm} = m\int_i^f \frac{d\left(\overrightarrow{\mathbf{V}}_{cm}\right)}{dt} \cdot \overrightarrow{\mathbf{V}}_{cm} dt \\ &= \frac{1}{2} m \int_i^f d\left(\overrightarrow{\mathbf{V}}_{cm} \cdot \overrightarrow{\mathbf{V}}_{cm}\right) = \frac{1}{2} m V_{\mathrm{cm}, f}^2 - \frac{1}{2} m V_{\mathrm{cm}, i}^2 = \Delta K_{\text{trans}} \end{aligned}$$

For the rotational motion we go to the center of mass reference frame and we determine the rotational work done i.e. the integral of the z -component of the torque about the center of mass with respect to dθ as we did for fixed axis rotational work. Then

$$\begin{aligned} \int_i^f \left(\vec{\tau}_{\mathrm{cm}}^{\text{ext}}\right)_z d\theta &= \int_i^f I_{\mathrm{cm}} \frac{d\omega_{\mathrm{cm}, z}}{dt} d\theta = \int_i^f I_{\mathrm{cm}} d\omega_{\mathrm{cm}, z} \frac{d\theta}{dt} = \int_i^f I_{\mathrm{cm}} d\omega_{\mathrm{cm}, z} \omega_{\mathrm{cm}, z} \\ &= \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm}, f}^2 - \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm}, i}^2 = \Delta K_{\text{rot}} \end{aligned}$$

In Equation (21.5.2) we expressed our result in terms of the angular speed ω_{cm} because it appears as a square. Therefore we can combine these two separate results, Equations (21.5.1) and (21.5.2), and determine the work-energy theorem for a rotating and translating rigid body that undergoes fixed axis rotation about the center of mass.

$$\begin{aligned} W &= \left(\frac{1}{2} m V_{\mathrm{cm,f}}^2 + \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm}, f}^2\right) - \left(\frac{1}{2} m V_{\mathrm{cm,f}}^2 + \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm}, i}^2\right) \\ &= \Delta K_{\text{tans}} + \Delta K_{\text{rot}} = \Delta K \end{aligned}$$

Equations (21.4.1), (21.4.4), and (21.5.3) are principles that we shall employ to analyze the motion of a rigid bodies undergoing translation and fixed axis rotation about the center of mass.