8.3: Constraint Forces

Knowledge of all the external and internal forces acting on each of the objects in a system and applying Newton's Second Law to each of the objects determine a set of equations of motion. These equations of motion are not necessarily independent due to the fact that the motion of the objects may be limited by equations of constraint. In addition there are forces of constraint that are determined by their effect on the motion of the objects and are not known beforehand or describable by some force law. For example: an object sliding down an inclined plane is constrained to move along the surface of the inclined plane (Figure 8.6a) and the surface exerts a contact force on the object; an object that slides down the surface of a sphere until it falls off experiences a contact force until it loses contact with the surface (Figure 8.6b); gas particles in a sealed vessel are constrained to remain inside the vessel and therefore the wall must exert force on the gas molecules to keep them inside the vessel (8.6c); and a bead constrained to slide outward along a rotating rod is acting on by time dependent forces of the rod on the bead (Figure 8.6d). We shall develop methods to determine these constraint forces although there are many examples in which the constraint forces cannot be determined.

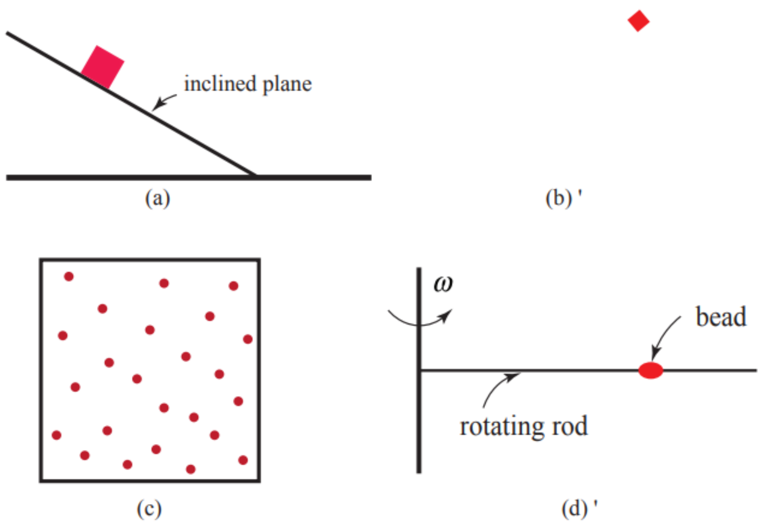


Figure 8.6 Constrained motions: (a) particle sliding down inclined plane, (b) particles sliding down surface of sphere, (c) gas molecules in a sealed vessel, and (d) bead sliding on a rotating rod

Contact Forces

Pushing, lifting and pulling are contact forces that we experience in the everyday world. Rest your hand on a table; the atoms that form the molecules that make up the table and your hand are in contact with each other. If you press harder, the atoms are also pressed closer together. The electrons in the atoms begin to repel each other and your hand is pushed in the opposite direction by the table.

According to Newton's Third Law, the force of your hand on the table is equal in magnitude and opposite in direction to the force of the table on your hand. Clearly, if you push harder the force increases. Try it! If you push your hand straight down on the table, the table pushes back in a direction perpendicular (normal) to the surface. Slide your hand gently forward along the surface of the table. You barely feel the table pushing upward, but you do feel the friction acting as a resistive force to the motion of your hand. This force acts tangential to the surface and opposite to the motion of your hand. Push downward and forward. Try to estimate the magnitude of the force acting on your hand.

The force of the table acting on your hand, $\overrightarrow{\mathbf{F}}^{C} \equiv \overrightarrow{\mathbf{C}}$ is called the contact force. This force has both a normal component to the surface, $\overrightarrow{\mathbf{C}}_{\perp} \equiv \overrightarrow{\mathbf{N}}$, called the normal force, and a tangential component to the surface, $\overrightarrow{\mathbf{C}}_{\|} \equiv \overrightarrow{\mathbf{f}}$ called the friction force (Figure 8.6).

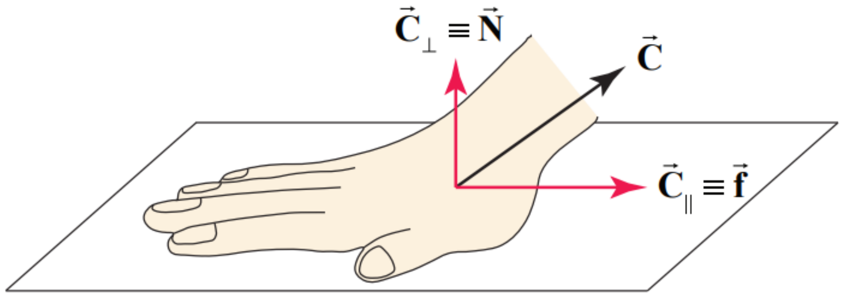


Figure 8.6 Normal and tangential components of the contact force

The contact force, written in terms of its component forces, is therefore

$$\overrightarrow{\mathbf{C}}=\overrightarrow{\mathbf{C}}_{\perp}+\overrightarrow{\mathbf{C}}_{\|} \equiv \overrightarrow{\mathbf{N}}+\overrightarrow{\mathbf{f}}$$

Any force can be decomposed into component vectors so the normal component, $\overrightarrow{\mathbf{N}}$, and the tangential component, $\overrightarrow{\mathbf{f}}$ are not independent forces but the vector components of the contact force, perpendicular and parallel to the surface of contact. The contact force is a distributed force acting over all the points of contact between your hand and the surface.

For most applications we shall treat the contact force as acting at single point but precaution must be taken when the distributed nature of the contact force plays a key role in constraining the motion of a rigid body.

In Figure 8.7, the forces acting on your hand are shown. These forces include the contact force, $\overrightarrow{\mathbf{C}}$ of the table acting on your hand, the force of your forearm, $\overrightarrow{\mathbf{F}}_{\text{forearm}}$ acting on your hand (which is drawn at an angle indicating that you are pushing down on your hand as well as forward), and the gravitational interaction, $\overrightarrow{\mathbf{F}}^{g}$ between the earth and your hand.

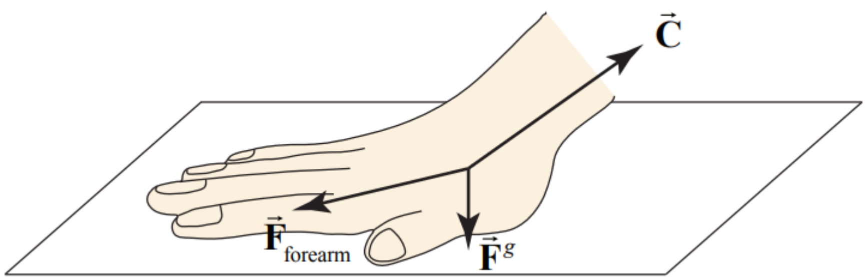


Figure 8.7 Forces on hand when moving towards the left

One point to keep in mind is that the magnitudes of the two components of the contact force depend on how hard you push or pull your hand and in what direction, a characteristic of constraint forces, in which the components are not specified by a force law but dependent on the particular motion of the hand.

Example 8.2 Normal Component of the Contact Force and Weight

Hold a block in your hand such that your hand is at rest (Figure 8.8). You can feel the “weight” of the block against your palm. But what exactly do we mean by “weight”?

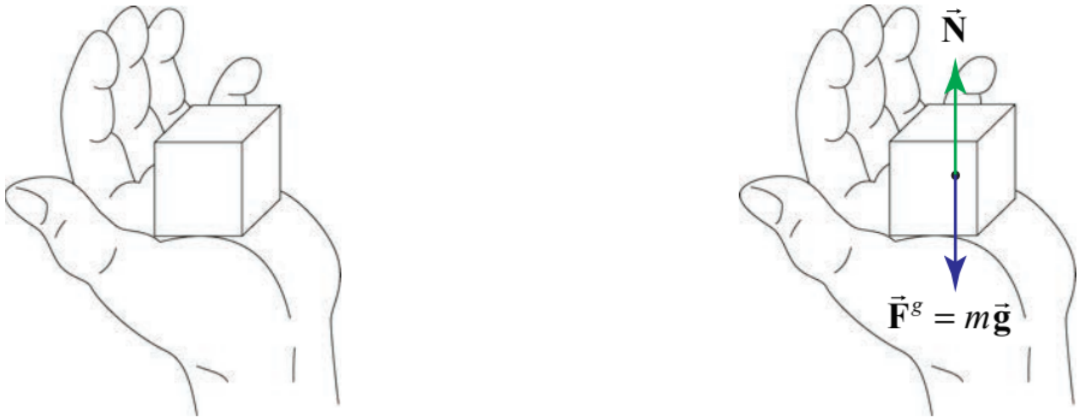


Figure 8.9 Forces on block

There are two forces acting on the block as shown in Figure 8.9. One force is the gravitational force between the earth and the block, and is denoted by $\overrightarrow{\mathbf{F}}^g = m\overrightarrow{\mathbf{g}}$. The other force acting on the block is the contact force between your hand and the block. Because our hand is at rest, this contact force on the block points perpendicular to the surface, and hence has only a normal component, $\overrightarrow{\mathbf{N}}$ Let N denote the magnitude of the normal force. Because the object is at rest in your hand, the vertical acceleration is zero. Therefore Newton's Second Law states that

$$\overrightarrow{\mathbf{N}} + \overrightarrow{\mathbf{F}}^g = \overrightarrow{\mathbf{0}}$$

Choose the positive direction to be upwards and then in terms of vertical components we have that

$$N - mg = 0$$

which can be solved for the magnitude of the normal force

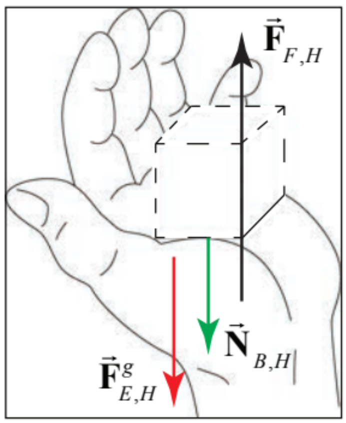
$$N = mg$$

When we talk about the "weight" of the block, we often are referring to the effect the block has on a scale or on the feeling we have when we hold the block. These effects are actually effects of the normal force. We say that a block "feels lighter" if there is an additional force holding the block up. For example, you can rest the block in your hand, but use your other hand to apply a force upwards on the block to make it feel lighter in your supporting hand.

The word "weight," is often used to describe the gravitational force that Earth exerts on an object. We shall always refer to this force as the gravitational force instead of "weight." When you jump in the air, you feel "weightless" because there is no normal force acting on you, even though Earth is still exerting a gravitational force on you; clearly, when you jump, you do not turn gravity off!

This example may also give rise to a misconception that the normal force is always equal to the mass of the object times the magnitude of the gravitational acceleration at the surface of the earth. The normal force and the gravitational force are two completely different forces. In this particular example, the normal force is equal in magnitude to the gravitational force and directed in the opposite direction because the object is at rest. The normal force and the gravitational force do not form a Third Law interaction pair of forces. In this example, our system is just the block and the normal force and gravitational force are external forces acting on the block.

Let's redefine our system as the block, your hand, and Earth. Then the normal force and gravitational force are now internal forces in the system and we can now identify the various interaction pairs of forces. We explicitly introduce our interaction pair notation to enable us to identify these interaction pairs: for example, let $\overrightarrow{\mathbf{F}}^g_{E,B}$ denote the gravitational force on the block due to the interaction with Earth. The gravitational force on Earth due to the interaction with the block is denoted by $\overrightarrow{\mathbf{F}}^g_{B,E}$ and these two forces form an interaction pair. By Newton's Third Law, $\overrightarrow{\mathbf{F}}^g_{E,B} = -\overrightarrow{\mathbf{F}}^g_{B,E}$. Note that these two forces are acting on different objects, the block and Earth. The contact force on the block due to the interaction between the hand and the block is then denoted by $\overrightarrow{\mathbf{N}}_{H,B}$. The force of the block on the hand, which we denote by $\overrightarrow{\mathbf{N}}_{B,H}$, satisfies $\overrightarrow{\mathbf{N}}_{B,H} = -\overrightarrow{\mathbf{N}}_{H,B}$. Because we are including your hand as part of the system, there are two additional forces acting on the hand. There is the gravitational force on your hand $\overrightarrow{\mathbf{F}}^g_{E,H}$, satisfying $\overrightarrow{\mathbf{F}}^g_{E,H} = -\overrightarrow{\mathbf{F}}^g_{H,E}$, where $\overrightarrow{\mathbf{F}}^g_{H,E}$ is the gravitational force on Earth due to your hand. Finally there is the force of your forearm holding your hand up, which we denote $\overrightarrow{\mathbf{F}}_{F,H}$. Because we are not including the forearm in our system, this force is an external force to the system. The forces acting on your hand are shown in diagram on your hand is shown in Figure 8.10, and the just the interaction pairing of forces acting on Earth is shown in Figure 8.11 (we are not representing all other external forces acting on the Earth).



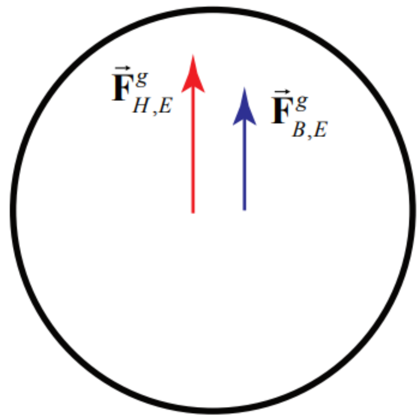


Figure 8.11 Gravitational forces on earth due to object and hand

Kinetic and Static Friction

When a block is pulled along a horizontal surface or sliding down an inclined plane there is a lateral force resisting the motion. If the block is at rest on the inclined plane, there is still a lateral force resisting the motion. This resistive force is known as dry friction, and there are two distinguishing types when surfaces are in contact with each other. The first type is when the two objects are moving relative to each other; the friction in that case is called **kinetic friction** or **sliding friction**. When the two surfaces are non-moving but there is still a lateral force as in the example of the block at rest on an inclined plane, the force is called, **static friction**.

Leonardo da Vinci was the first to record the results of measurements on kinetic friction over a twenty-year period between 1493–4 and about 1515. Based on his measurements, the force of kinetic friction, $\overrightarrow{\mathbf{f}}^k$ between two surfaces, he identified two key properties of kinetic friction. The magnitude of kinetic friction is proportional to the normal force between the two surfaces,

$$f_{\mathrm{k}} = \mu_{\mathrm{k}} N$$

where μ_{k} is called the coefficient of kinetic friction. The second result is rather surprising in that the magnitude of the force is independent of the contact surface. Consider two blocks of the same mass, but different surface areas. The force necessary to move the blocks at a constant speed is the same. The block in Figure 8.12a has twice the contact area as the block shown in Figure 8.12b, but when the same external force is applied to either block, the blocks move at constant speed. These results of da Vinci were rediscovered by Guillaume Amontons and published in 1699. The third property that kinetic friction is independent of the speed of moving objects (for ordinary sliding speeds) was discovered by Charles Augustin Coulomb.

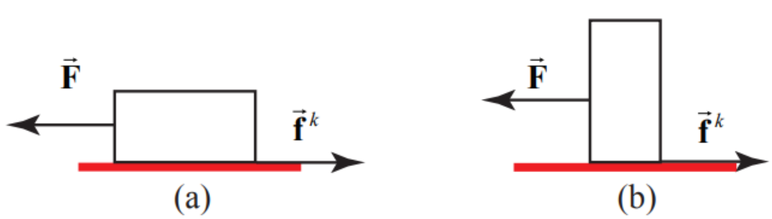


Figure 8.12 (a) and (b): kinetic friction is independent of the contact area

The kinetic friction on surface 2 moving relative to surface 1 is denoted by, $\overrightarrow{\mathbf{f}}^{k}_{1,2}$ The direction of the force is always opposed to the relative direction of motion of surface 2 relative to the surface 1. When one surface is at rest relative to our choice of reference frame we will denote the friction force on the moving object by $\overrightarrow{\mathbf{f}}^{k}$.

The second type of dry friction, static friction occurs when two surfaces are static relative to each other. Because the static friction force between two surfaces forms a third law interaction pair, will use the notation $\overrightarrow{\mathbf{f}}^{s}_{1,2}$ to denote the static friction force on surface 2 due to the interaction between surfaces 1 and 2. Push your hand forward along a surface; as you increase your pushing force, the frictional force feels stronger and stronger. Try this! Your hand will at first stick until you push hard enough, then your hand slides forward. The magnitude of the static frictional force, f_{s}, depends on how hard you push.

If you rest your hand on a table without pushing horizontally, the static friction is zero. As you increase your push, the static friction increases until you push hard enough that your hand slips and starts to slide along the surface. Thus the magnitude of static friction can vary from zero to some maximum value, $\left(f_{\mathrm{s}}\right)_{\max}$ when the pushed object begins to slip,

$$0 \leq f_{\mathrm{s}} \leq \left(f_{\mathrm{s}}\right)_{\max}$$

Is there a mathematical model for the magnitude of the maximum value of static friction between two surfaces? Through experimentation, we find that this magnitude is, like kinetic friction, proportional to the magnitude of the normal force

$$\left(f_{\mathrm{s}}\right)_{\max} = \mu_{\mathrm{s}} N$$

Here the constant of proportionality is μ_{s}, the coefficient of static friction. This constant is slightly greater than the constant μ_{k} associated with kinetic friction, $\mu_{\mathrm{s}} > \mu_{\mathrm{k}}$. This small difference accounts for the slipping and catching of chalk on a blackboard, fingernails on glass, or a violin bow on a string.

The direction of static friction on an object is always opposed to the direction of the applied force (as long as the two surfaces are not accelerating). In Figure 8.13a, an external force, $\overrightarrow{\mathbf{F}}$ is applied the left and the static friction, $\overrightarrow{\mathbf{f}}^{s}$ is shown pointing to the right opposing the external force. In Figure 8.13b, the external force, $\overrightarrow{\mathbf{F}}$ is directed to the right and the static friction, $\overrightarrow{\mathbf{f}}^{s}$, is now pointing to the left.

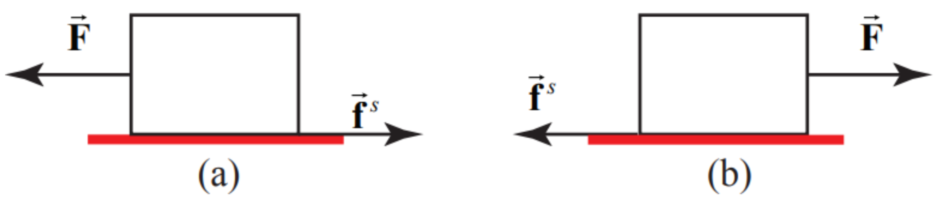


Figure 8.13 (a) and (b): External forces and the direction of static friction.

Although the force law for the maximum magnitude of static friction resembles the force law for sliding friction, there are important differences:

1. The direction and magnitude of static friction on an object always depends on the direction and magnitude of the applied forces acting on the object, where the magnitude of kinetic friction for a sliding object is fixed.

2. The magnitude of static friction has a maximum possible value. If the magnitude of the applied force along the direction of the contact surface exceeds the magnitude of the maximum value of static friction, then the object will start to slip (and be subject to kinetic friction.) We call this the just slipping condition.