

16.5: Appendix 16A- Proof of the Parallel Axis Theorem

Identify an infinitesimal volume element of mass dm . The vector from the point S to the mass element is $\vec{r}_{S, dm}$ the vector from the center of mass to the mass element is \vec{r}_{dm} , and the vector from the point S to the center of mass is $\vec{r}_{S, cm}$.

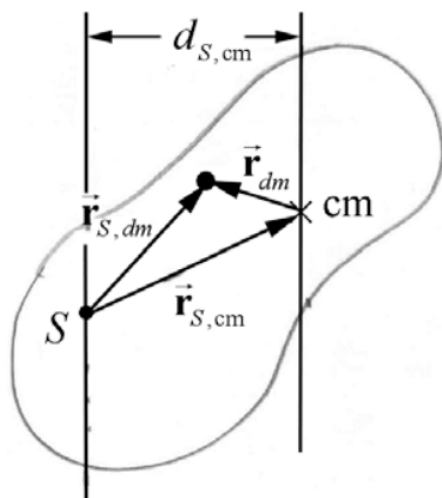


Figure 16A.1 Geometry of the parallel axis theorem.

From Figure 16A.1, we see that

$$\vec{r}_{S, dm} = \vec{r}_{S, cm} + \vec{r}_{dm}$$

The notation gets complicated at this point. The vector \vec{r}_{dm} has a component vector $\vec{r}_{\parallel, dm}$ parallel to the axis through the center of mass and a component vector $\vec{r}_{\perp, dm}$ perpendicular to the axis through the center of mass. The magnitude of the perpendicular component vector is

$$|\vec{r}_{cm, \perp, dm}| = r_{\perp, dm}$$

The vector $\vec{r}_{S, dm}$ has a component vector $\vec{r}_{S, \parallel, dm}$ parallel to the axis through the point S and a component vector $\vec{r}_{S, \perp, dm}$ perpendicular to the axis through the point S . The magnitude of the perpendicular component vector is

$$|\vec{r}_{S, \perp, dm}| = r_{S, \perp, dm}$$

The vector $\vec{r}_{S, cm}$ has a component vector $\vec{r}_{S, \parallel, cm}$ parallel to both axes and a perpendicular component vector $\vec{r}_{S, \perp, cm}$ that is perpendicular to both axes (the axes are parallel, of course). The magnitude of the perpendicular component vector is

$$|\vec{r}_{S, \perp, cm}| = d_{S, cm}$$

Equation (16.A.1) is now expressed as two equations,

$$\begin{aligned}\vec{r}_{S, \perp, dm} &= \vec{r}_{S, \perp, cm} + \vec{r}_{\perp, dm} \\ \vec{r}_{S, \parallel, dm} &= \vec{r}_{S, \parallel, cm} + \vec{r}_{\parallel, dm}\end{aligned}$$

At this point, note that if we had simply decided that the two parallel axes are parallel to the z -direction, we could have saved some steps and perhaps spared some of the notation with the triple subscripts. However, we want a more general result, one valid for cases where the axes are not fixed, or when different objects in the same problem have different axes. For example, consider the turning bicycle, for which the two wheel axes will not be parallel, or a spinning top that precesses (wobbles). Such cases will be considered in later on, and we will show the general case of the parallel axis theorem in anticipation of use for more general situations.

The moment of inertia about the point S is

$$I_S = \int_{\text{body}} dm (r_{S,\perp, dm})^2$$

From (16.A.5) we have

$$\begin{aligned} (r_{S,\perp, dm})^2 &= \vec{r}_{S,\perp, dm} \cdot \vec{r}_{S,\perp, dm} \\ &= \left(\vec{r}_{S,\perp, cm} + \vec{r}_{\perp, dm} \right) \cdot \left(\vec{r}_{S,\perp, cm} + \vec{r}_{\perp, dm} \right) \\ &= d_{S, cm}^2 + (r_{\perp, dm})^2 + 2 \vec{r}_{S,\perp, cm} \cdot \vec{r}_{\perp, dm} \end{aligned}$$

Thus we have for the moment of inertia about S ,

$$I_S = \int_{\text{body}} dm d_{S, cm}^2 + \int_{\text{body}} dm (r_{\perp, dm})^2 + 2 \int_{\text{body}} dm \left(\vec{r}_{S,\perp, cm} \cdot \vec{r}_{\perp, dm} \right)$$

In the first integral in Equation (16.A.8), $r_{S,\perp, cm} = d_{S, cm}$ is the distance between the parallel axes and is a constant. Therefore we can rewrite the integral as

$$d_{S, cm}^2 \int_{\text{body}} dm = m d_{S, cm}^2$$

The second term in Equation (16.A.8) is the moment of inertia about the axis through the center of mass,

$$I_{cm} = \int_{\text{body}} dm (r_{\perp, dm})^2$$

The third integral in Equation (16.A.8) is zero. To see this, note that the term $\vec{r}_{S,\perp, cm}$ is a constant and may be taken out of the integral,

$$2 \int_{\text{body}} dm \left(\vec{r}_{S,\perp, cm} \cdot \vec{r}_{\perp, dm} \right) = \vec{r}_{S,\perp, cm} \cdot 2 \int_{\text{body}} dm \vec{r}_{\perp, dm}$$

The integral $\int_{\text{body}} dm \vec{r}_{\perp, dm}$ is the perpendicular component of the position of the center of mass with respect to the center of mass, and hence $\vec{0}$, with the result that

$$2 \int_{\text{body}} dm \left(\vec{r}_{S,\perp, cm} \cdot \vec{r}_{\perp, dm} \right) = 0$$

Thus, the moment of inertia about S is just the sum of the first two integrals in Equation (16.A.8)

$$I_S = I_{cm} + m d_{S, cm}^2$$

proving the parallel axis theorem.

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