16.3: Rotational Kinetic Energy and Moment of Inertia

Rotational Kinetic Energy and Moment of Inertia

We have already defined translational kinetic energy for a point object as $K = (1/2)mv^2$; we now define the rotational kinetic energy for a rigid body about its center of mass.

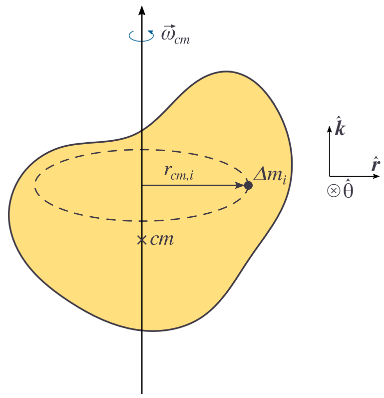

Figure 16.3.1: Volume element undergoing fixed-axis rotation about the z-axis that passes through the center of mass. (CC BY-NC; Ümit Kaya)

Choose the z-axis to lie along the axis of rotation passing through the center of mass. As discussed previously, divide the body into volume elements of mass Δm_i (Figure $16.3.1$). Each individual mass element Δm_i undergoes circular motion about the center of mass with z-component of angular velocity ω_{cm} in a circle of radius $r_{\text{cm},i}$. Therefore the velocity of each element is given by $\overrightarrow{\mathbf{v}}_{\text{cm},i} = r_{\text{cm},i}\omega_{\text{cm}}\hat{\boldsymbol{\theta}}$. The rotational kinetic energy is then

$$K_{\text{cm},i} = \frac{1}{2}\Delta m_i v_{\text{cm},i}^2 = \frac{1}{2}\Delta m r_i r_{\text{cm},i}^2 \omega_{\text{cm}}^2$$

We now add up the kinetic energy for all the mass elements,

$$\begin{aligned} K_{\text{cm}} &= \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{i=N} K_{\text{cm},i} = \lim_{i\to\infty} \sum_{\Delta m_i \to 0}^{i=N} \left(\sum_i \frac{1}{2}\Delta m_i r_{\text{cm},i}^2\right)\omega_{\text{cm}}^2 \\ &= \left(\frac{1}{2}\int_{\text{body}} dm r_{\text{dm}}^2\right)\omega_{\text{cm}}^2 \end{aligned}$$

where dm is an infinitesimal mass element undergoing a circular orbit of radius r_{dm} about the axis passing through the center of mass.

The quantity

$$I_{cm} = \int_{body} r_{dm}^2 \, dm$$

is called the **moment of inertia** of the rigid body about a fixed axis passing through the center of mass, and is a physical property of the body. The SI units for moments of inertia are $\left[\text{kg}\cdot\text{m}^2\right]$.

Thus

$$K_{\text{cm}} = \left(\frac{1}{2}\int_{\text{body}} r_{\text{dm}}^2 \, dm\right)\omega_{\text{cm}}^2 \equiv \frac{1}{2} I_{cm}\omega_{\text{cm}}^2$$

Moment of Inertia of a Rod of Uniform Mass Density

Consider a thin uniform rod of length L and mass m. In this problem, we will calculate the moment of inertia about an axis perpendicular to the rod that passes through the center of mass of the rod. A sketch of the rod, volume element, and axis is shown in Figure $16.3.2$. Choose Cartesian coordinates, with the origin at the center of mass of the rod, which is midway between the endpoints since the rod is uniform. Choose the x-axis to lie along the length of the rod, with the positive x-direction to the right, as in Figure $16.3.2$

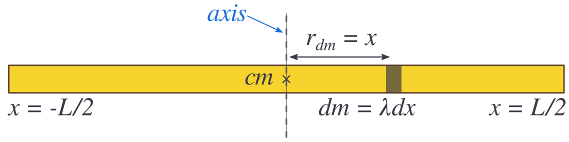

Figure 16.3.2: Moment of inertia of a uniform rod about center of mass. (CC BY-NC; Ümit Kaya)

Identify an infinitesimal mass element $dm = \lambda dx$, located at a displacement x from the center of the rod, where the mass per unit length $\lambda = m/L$ is a constant, as we have assumed the rod to be uniform. When the rod rotates about an axis perpendicular to the rod that passes through the center of mass of the rod, the element traces out a circle of radius $r_{dm} = x$. We add together the contributions from each infinitesimal element as we go from $x = -L/2$ to $x = L/2$. The integral is then

$$I_{\mathrm{cm}} = \int_{\mathrm{body}} r_{\mathrm{dm}}^2 dm \tag{16.3.1}$$

$$= \lambda \int_{-L/2}^{L/2} \left(x^2\right) dx = \lambda \frac{x^3}{3}\bigg|_{-L/2}^{L/2} \tag{16.3.2}$$

$$= \frac{m}{L}\frac{(L/2)^3}{3} - \frac{m}{L}\frac{(-L/2)^3}{3} = \frac{1}{12} m L^2 \tag{16.3.3}$$

By using a constant mass per unit length along the rod, we need not consider variations in the mass density in any direction other than the x-axis. We also assume that the width is the rod is negligible. (Technically we should treat the rod as a cylinder or a rectangle in the xy-plane if the axis is along the z-axis. The calculation of the moment of inertia in these cases would be more complicated.)

Example 16.3.1: Moment of Inertia of a Uniform Disk

A thin uniform disk of mass M and radius R is mounted on an axle passing through the center of the disk, perpendicular to the plane of the disk. Calculate the moment of inertia about an axis that passes perpendicular to the disk through the center of mass of the disk

Solution

As a starting point, consider the contribution to the moment of inertia from the mass element dm show in $16.3.3$ Let r denote the distance form the center of mass of the disk to the mass element.

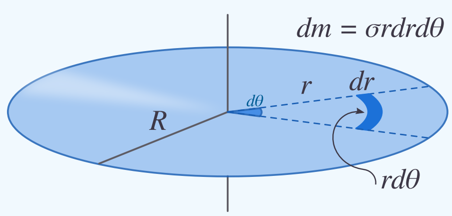

Figure 16.3.3: Infinitesimal mass element and coordinate system for disk. (CC BY-NC; Ümit Kaya)

Choose cylindrical coordinates with the coordinates (r, θ) in the plane and the z-axis perpendicular to the plane. The area element

$$da = r dr d\theta$$

may be thought of as the product of arc length $r d\theta$ and the radial width dr. Since the disk is uniform, the mass per unit area is a constant,

$$\sigma=\frac{dm}{da}=\frac{m_{\text{total}}}{\text{Area}}=\frac{M}{\pi R^2}$$

Therefore the mass in the infinitesimal area element as given in Equation 16.3.4, a distance r from the axis of rotation, is given by

$$dm=\sigma r dr d\theta=\frac{M}{\pi R^2} r\,dr\,d\theta \tag{16.3.4}$$

When the disk rotates, the mass element traces out a circle of radius $r_{dm}=r$; that is, the distance from the center is the perpendicular distance from the axis of rotation. The moment of inertia integral is now an integral in two dimensions; the angle θ varies from $\theta=0$ to $\theta=2\pi$, and the radial coordinate r varies from $r=0$ to $r=R$. Thus the limits of the integral are

$$I_{\text{cm}}=\int_{\text{body}} r_{dm}^2 dm=\frac{M}{\pi R^2}\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3 d\theta dr$$

The integral can now be explicitly calculated by first integrating the θ-coordinate

$$I_{\text{cm}}=\frac{M}{\pi R^2}\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) r^3 dr=\frac{M}{\pi R^2}\int_{r=0}^{r=R} 2\pi r^3 dr=\frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr$$

and then integrating the r-coordinate,

$$I_{\text{cm}}=\frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr=\frac{2M}{R^2}\frac{r^4}{4}\bigg|_{r=0}^{r=R}=\frac{2M}{R^2}\frac{R^4}{4}=\frac{1}{2}MR^2$$

Remark: Instead of taking the area element as a small patch $da=r dr d\theta$, choose a ring of radius r and width dr. Then the area of this ring is given by

$$da_{\text{ring}}=\pi(r+dr)^2-\pi r^2=\pi r^2+2\pi r dr+\pi(dr)^2-\pi r^2=2\pi r dr+\pi(dr)^2$$

In the limit that $dr\rightarrow 0$, the term proportional to $(dr)^2$ can be ignored and the area is $da=2\pi r dr$. This equivalent to first integrating the $d\theta$ variable

$$da_{\text{ring}}=r dr\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right)=2\pi r dr$$

Then the mass element is

$$dm_{\text{ring}}=\sigma da_{\text{ring}}=\frac{M}{\pi R^2}2\pi r dr$$

The moment of inertia integral is just an integral in the variable r,

$$I_{\text{cm}}=\int_{\text{body}}(r_{\perp})^2 dm=\frac{2\pi M}{\pi R^2}\int_{r=0}^{r=R} r^3 dr=\frac{1}{2}MR^2$$

Parallel Axis Theorem

Consider a rigid body of mass m undergoing fixed-axis rotation. Consider two parallel axes. The first axis passes through the center of mass of the body, and the moment of inertia about this first axis is I_{cm} The second axis passes through some other point S in the body. Let $d_{S,\text{cm}}$ denote the perpendicular distance between the two parallel axes (Figure 16.3.4).

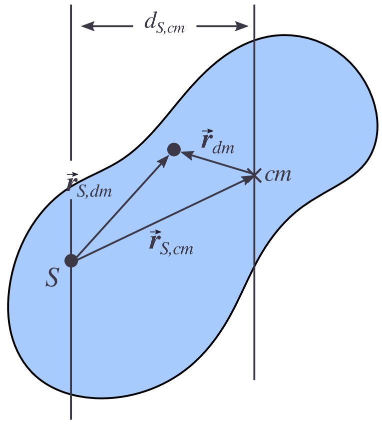

Figure 16.3.4: Geometry of the parallel axis theorem. (CC BY-NC; Ümit Kaya)

Then the moment of inertia I_S about an axis passing through a point S is related to I_{cm} by

$$I_S = I_{\mathrm{cm}} + m d_{S,\mathrm{cm}}^2 \tag{16.3.5}$$

Parallel Axis Theorem Applied to a Uniform Rod

Let point S be the left end of the rod of Figure $16.3.4$ Then the distance from the center of mass to the end of the rod is $d_{S,\mathrm{cm}} = L/2$ The moment of inertia $I_S = I_{\mathrm{end}}$ about an axis passing through the endpoint is related to the moment of inertia about an axis passing through the center of mass, $I_{\mathrm{cm}} = (1/12)mL^2$, according to Equation 16.3.5,

$$I_S = \frac{1}{12}mL^2 + \frac{1}{4}mL^2 = \frac{1}{3}mL^2$$

In this case it is easy and useful to check by direct calculation. Use Equation 16.3.1 but with the limits changed to $x' = 0$ and $x' = L$, where $x' = x + L/2$,

$$\begin{aligned} I_{\mathrm{end}} &= \int_{\mathrm{body}} r_\perp^2 \, dm = \lambda \int_0^L x'^2 dx' \\ &= \lambda \left.\frac{x'^3}{3}\right|_0^L = \frac{m}{L}\frac{(L)^3}{3} - \frac{m}{L}\frac{(0)^3}{3} = \frac{1}{3}mL^2 \end{aligned}$$

Example 16.3.2: Rotational Kinetic Energy of Disk

A disk with mass M and radius R is spinning with angular speed ω about an axis that passes through the rim of the disk perpendicular to its plane. The moment of inertia about cm is $I_{cm} = (1/2)mR^2$. What is the kinetic energy of the disk?

Solution

The parallel axis theorem states the moment of inertia about an axis passing perpendicular to the plane of the disk and passing through a point on the edge of the disk is equal to

$$I_{edge} = I_{cm} + mR^2$$

The moment of inertia about an axis passing perpendicular to the plane of the disk and passing through the center of mass of the disk is equal to $I_{cm} = (1/2)mR^2$. Therefore

$$I_{edge} = (3/2)mR^2$$

The kinetic energy is then

$$K = (1/2)I_{edge}\omega^2 = (3/4)mR^2\omega^2$$