8.5: Tension in a Rope

Definition of Tension in a Rope

Let's return to our example of the very light rope (object 2 with $m_2 \simeq 0$) that is attached to a block (object 1) at the point B , and pulled by an applied force at point A $\overrightarrow{\mathbf{F}}_{\mathrm{A},2}$ (Figure 8.18a).

Figure 8.18a Massless rope pulling a block

Choose a coordinate system with the $\hat{\mathbf{j}}$-unit vector pointing upward in the normal direction to the surface, and the $\hat{\mathbf{i}}$ -unit vector pointing in the positive x -direction, (Figure 8.18b). The force diagrams for the system consisting of the rope and block is shown in Figure 8.19, and for the rope and block separately in Figure 8.20, where $\overrightarrow{\mathbf{F}}_{2,1}$ the force on the block (object 1) due to the rope (object 2), and $\overrightarrow{\mathbf{F}}_{1,2}$ is the force on the rope due to the block.

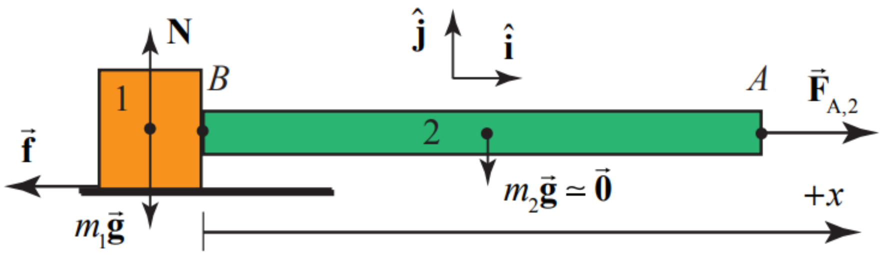


Figure 8.18b Forces acting on system consisting of block and rope

The forces on the rope and the block must each sum to zero. Because the rope is not accelerating, Newton's Second Law applied to the rope requires that $F_{\mathrm{A},2} - F_{1,2} = m_2 a$ (where we are using magnitudes for all the forces).

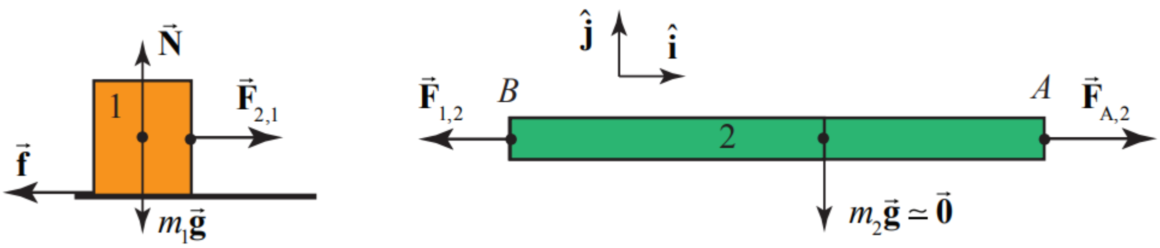


Figure 8.19 Separate force diagrams for rope and block

Because we are assuming the mass of the rope is negligible therefore

$$F_{\mathrm{A},2} - F_{1,2} = 0; \quad \text{(massless rope)}$$

If we consider the case that the rope is very light, then the forces acting at the ends of the rope are nearly horizontal. Then if the rope-block system is moving at constant speed or at rest, Newton's Second Law is now

$$F_{\mathrm{A},2} - F_{1,2} = 0$$

Newton's Second Law applied to the block in the $+\hat{\mathbf{i}}$-direction requires that $F_{2,1} - f = 0$ Newton's Third Law, applied to the block-rope interaction pair requires that $F_{1,2} = F_{2,1}$. Therefore

$$F_{\mathrm{A},2} = F_{1,2} = F_{2,1} = f$$

Thus the applied pulling force is transmitted through the rope to the block since it has the same magnitude as the force of the rope on the block. In addition, the applied pulling force is also equal to the friction force on the block.

How do we define “tension” at some point in a rope? Suppose make an imaginary slice of the rope at a point P , a distance x_P from point B , where the rope is attached to the block. The imaginary slice divides the rope into two sections, labeled L (left) and R (right), as shown in Figure 8.20.

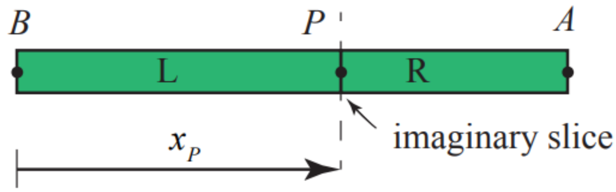


Figure 8.20 Imaginary slice through the rope

There is now a Third Law pair of forces acting between the left and right sections of the rope. Denote the force acting on the left section by $\overrightarrow{\mathbf{F}}_{\mathrm{R,L}}(x_P)$ and the force acting on the right section by $\overrightarrow{\mathbf{F}}_{\mathrm{L,R}}(x_P)$ Newton’s Third Law requires that the forces in this interaction pair are equal in magnitude and opposite in direction.

$$\overrightarrow{\mathbf{F}}_{\mathrm{R,L}}(x_p)=-\overrightarrow{\mathbf{F}}_{\mathrm{L,R}}(x_P)$$

The force diagram for the left and right sections are shown in Figure 8.21 where $\overrightarrow{\mathbf{F}}_{1,\mathrm{L}}$ is the force on the left section of the rope due to the block-rope interaction. (We had previously denoted that force by $\overrightarrow{\mathbf{F}}_{1,2}$) Now denote the force on the right section of the rope side due to the pulling force at the point A by $\overrightarrow{\mathbf{F}}_{\mathrm{A,R}}$ (which we had previously denoted by $\overrightarrow{\mathbf{F}}_{\mathrm{A},2}$).

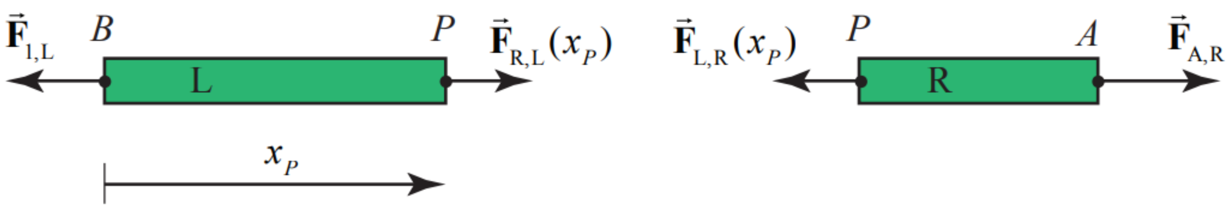


Figure 8.21 Force diagram for the left and right sections of rope

The tension $T(x_P)$ at a point P in rope lying a distance x from one the left end of the rope, is the magnitude of the action -reaction pair of forces acting at the point P ,

$$T(x_p)=\left|\overrightarrow{\mathbf{F}}_{\mathrm{R,L}}(x_P)\right|=\left|\overrightarrow{\mathbf{F}}_{\mathrm{L,R}}(x_P)\right|$$

For a rope of negligible mass, under tension, as in the above case, (even if the rope is accelerating) the sum of the horizontal forces applied to the left section and the right section of the rope are zero, and therefore the tension is uniform and is equal to the applied pulling force,

$$T=F_{\mathrm{A,R}}$$

Example 8.3 Tension in a Massive Rope

Figure 8.22a Massive rope pulling a block

Consider a block of mass m_1 that is lying on a horizontal surface. The coefficient of kinetic friction between the block and the surface is μ_k. A uniform rope of mass m_2 and length d is attached to the block. The rope is pulled from the side opposite the block with an applied force of magnitude $\left|\overrightarrow{\mathbf{F}}_{\mathrm{A},2}\right| = F_{\mathrm{A},2}$. Because the rope is now massive, the pulling force makes an angle ϕ with respect to the horizontal in order to balance the gravitational force on the rope, (Figure 8.22a). Determine the tension in the rope as a function of distance x from the block.

Solution: In the following analysis, we shall assume that the angle ϕ is very small and depict the pulling and tension forces as essentially acting in the horizontal direction even though there must be some small vertical component to balance the gravitational forces.

The key point to realize is that the rope is now massive and we must take in to account the inertia of the rope when applying Newton's Second Law. Consider an imaginary slice through the rope at a distance x from the block (Figure 8.22b), dividing the rope into two sections. The right section has length $d-x$ and mass $m_{\mathrm{R}} = (m_2/d)(d-x)$.

$$m_{\mathrm{R}} = (m_2/d)(d-x) \tag{8.5.1}$$

. The left section has length x and mass $m_{\mathrm{L}} = (m_2/d)(x)$.

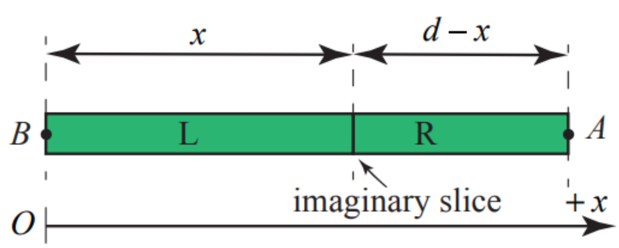


Figure 8.22b Imaginary slice through the rope

The free body force diagrams for the two sections of the rope are shown in Figure 8.22c, where T (x) is the tension in the rope at a distance x from the block, and $F_{1,\mathrm{L}} = \left|\overrightarrow{\mathbf{F}}_{1,\mathrm{L}}\right| \equiv \left|\overrightarrow{\mathbf{F}}_{1,2}\right|$ is the magnitude of the force on the left-section of the rope due to the rope-block interaction.

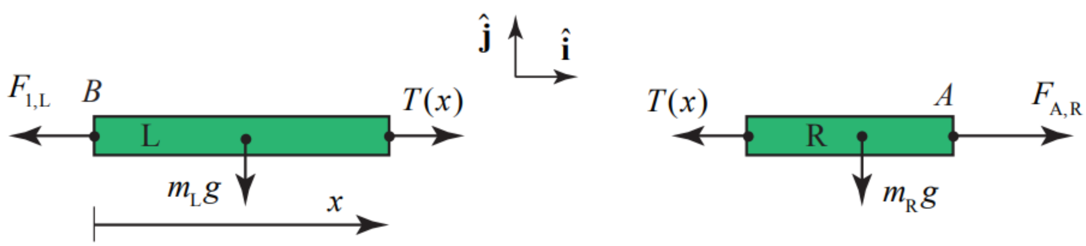


Figure 8.22c Force diagram for the left and right sections of rope

Apply Newton's Second Law to the right section of the rope yielding

$$F_{\mathrm{A,R}} - T(x) = m_{\mathrm{R}} a_{\mathrm{R}} = \frac{m_2}{d}(d-x) a_{\mathrm{R}}$$

where a_{R} is the x -component of the acceleration of the right section of the rope. Apply Newton's Second Law to the left slice of the rope yielding

$$T(x) - F_{1,\mathrm{L}} = m_{\mathrm{L}} a_{\mathrm{L}} = (m_2/d)\, x a_{\mathrm{L}}$$

where a_{L} is the x -component of the acceleration of the left piece of the rope.

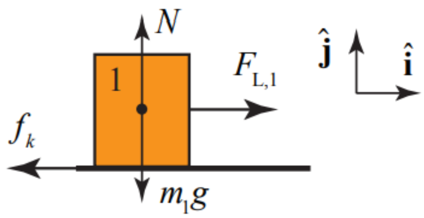


Figure 8.23 Force diagram on sliding block

The force diagram on the block is shown in Figure 8.23. Newton's Second Law on the block in the $+\hat{\mathbf{i}}$- direction is $F_{\mathrm{L},1} - f_k = m_1 a_1$ and in the $+\hat{\mathbf{j}}$- direction is $N - m_1 g = 0$ The kinetic friction force acting on the block is $f_k = \mu_k N = \mu_k m_1 g$ Newton's Second Law on the block in the $+\hat{\mathbf{i}}$-direction becomes

$$F_{\mathrm{L},1} - \mu_k m_1 g = m_1 a_1$$

Newton's Third Law for the block-rope interaction is given by $F_{\mathrm{L},1} = F_{1,\mathrm{L}}$. Equation (8.5.8) then becomes

$$T(x) - (\mu_k m_1 g + m_1 a_1) = (m_2/d)\, x a_{\mathrm{L}}$$

Because the rope and block move together, the accelerations are equal which we denote by the symbol $a \equiv a_1 = a_{\mathrm{L}}$. Then Equation (8.5.10) becomes

$$T(x) = \mu_k m_1 g + (m_1 + (m_2/d)\, x)\, a$$

This result is not unexpected because the tension is accelerating both the block and the left section and is opposed by the frictional force.

Alternatively, the force diagram on the system consisting of the rope and block is shown in Figure 8.24.

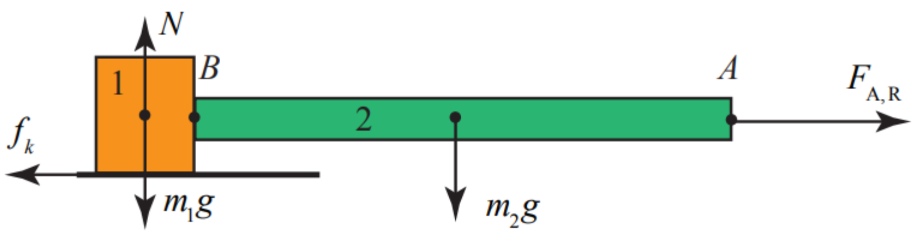


Figure 8.24 Force diagram on block-rope system

Newton's Second Law becomes

$$F_{\mathrm{A,R}} - \mu_k m_1 g = (m_2 + m_1)\, a$$

Solve Equation (8.5.12) for $F_{\mathrm{A,R}}$ and substitute into Equation (8.5.7), and solve for the tension yielding Equation (8.5.11).

Example 8.4 Tension in a Suspended Rope

A uniform rope of mass M and length L is suspended from a ceiling (Figure 8.25). The magnitude of the acceleration due to gravity is g . (a) Find the tension in the rope at the upper end where the rope is fixed to the ceiling. (b) Find the tension in the rope as a function of the distance from the ceiling. (c) Find an equation for the rate of change of the tension with respect to distance from the ceiling in terms of M , L , and g .

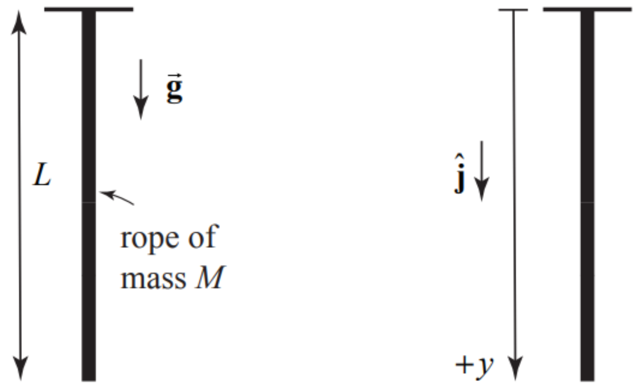


Figure 8.26 Coordinate system for suspended rope

Solution: (a) Begin by choosing a coordinate system with the origin at the ceiling and the positive y -direction pointing downward (Figure 8.26). In order to find the tension at the upper end of the rope, choose as a system the entire rope. The forces acting on the rope are the force at y = 0 holding the rope up, T (y = 0) , and the gravitational force on the entire rope. The free-body force diagram is shown in Figure 8.27.

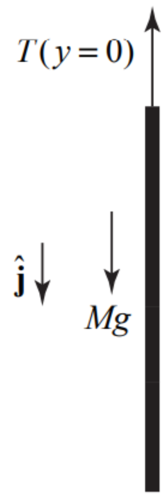


Figure 8.27 Force diagram on rope

Because the acceleration is zero, Newton's Second Law on the rope is $Mg - T(y=0) = 0$. Therefore the tension at the upper end is $T(y=0) = Mg$.

(b) Recall that the tension at a point is the magnitude of the action-reaction pair of forces acting at that point. Make an imaginary slice in the rope a distance y from the ceiling separating the rope into an upper segment 1, and lower segment 2 (Figure 8.28a). Choose the upper segment as a system with mass $m_1 = (M/L)y$ The forces acting on the upper segment are the gravitational force, the force T (y = 0) holding the rope up, and the tension T (y) at the point y , that is pulling the upper segment down. The free-body force diagram is shown in Figure 8.28b.

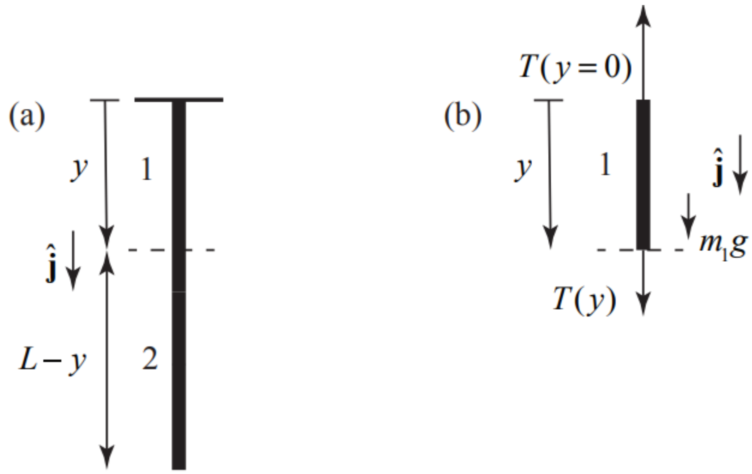


Figure 8.28 (a) Imaginary slice separates rope into two pieces. (b) Free-body force diagram on upper piece of rope

Apply Newton's Second Law to the upper segment: $m_1 g + T(y) - T(y=0) = 0$. Therefore the tension at a distance y from the ceiling is $T(y) = T(y=0) - m_1 g$. Because $m_1 = (M/L)y$ is the mass of the segment piece and Mg is the tension at the upper end, Newton's Second Law becomes

$$T(y) = Mg(1 - y/L)$$

As a check, we note that when $y = L$ the tension $T(y=L) = 0$ which is what we expect because there is no force acting at the lower end of the rope.

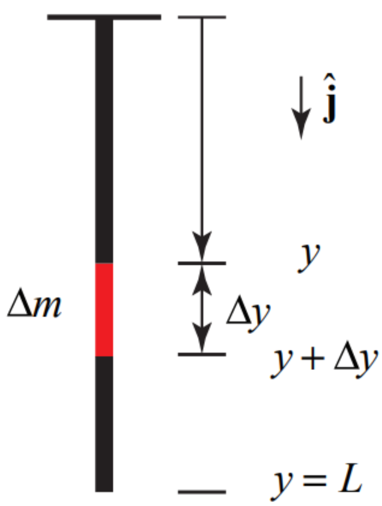
(c) Differentiate Equation (8.5.13) with respect to y yielding

$$\frac{dT}{dy} = -(M/L)g$$

The rate that the tension is changing at a constant rate with respect to distance from the top of the rope.

Continuous Systems and Newton's Second Law as a Differential Equations

We can determine the tension at a distance y from the ceiling in Example 8.4, by an alternative method, a technique that will generalize to many types of "continuous systems". Choose a coordinate system with the origin at the ceiling and the positive y - direction pointing downward as in Figure 8.25. Consider as the system a small element of the rope between the points y and $y + \Delta y$. This small element has length Δy The small element has mass $\Delta m = (M/L)\Delta y$ and is shown in Figure 8.29.



$$\Delta m = (M/L)\Delta y$$

The forces acting on the small element are the tension, T (y) at y directed upward, the tension $T(y+\Delta y)$ at $y+\Delta y$ directed downward, and the gravitational force Δmg directed downward. The tension $T(y+\Delta y)$ is equal to the tension T (y) plus a small difference ΔT

$$T(y+\Delta y)=T(y)+\Delta T$$

The small difference in general can be positive, zero, or negative. The free body force diagram is shown in Figure 8.30.

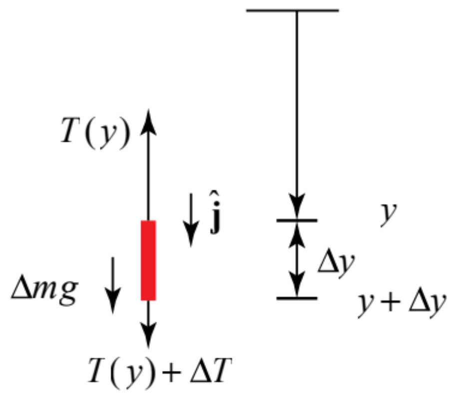


Figure 8.30 Free body force diagram on small mass element

Now apply Newton's Second Law to the small element

$$\Delta mg+T(y)-(T(y)+\Delta T)=0$$

The difference in the tension is then $\Delta T=-\Delta mg$. We now substitute our result for the mass of the element $\Delta m=(M/L)\Delta y$, and find that that

$$\Delta T=-(M/L)\Delta yg$$

Divide through by Δy yielding $\Delta T/\Delta y=-(M/L)g$. Now take the limit in which the length of the small element goes to zero, $\Delta y\rightarrow 0$

$$\lim_{\Delta y\rightarrow 0}\frac{\Delta T}{\Delta y}=-(M/L)g$$

Recall that the left hand side of Equation (8.5.18) is the definition of the derivative of the tension with respect to y , and so we arrive at Equation (8.5.14),

$$\frac{dT}{dy}=-(M/L)g$$

We can solve the differential equation, Equation (8.5.14), by a technique called separation of variables. We rewrite the equation as $dT=-(M/L)gdy$ and integrate both sides. Our integral will be a definite integral in which we integrate a 'dummy' integration variable y' and $y'=0$ to $y'=y$ and the corresponding T' from $T'=T(y=0)$ to $T'=T(y)$:

$$\int_{T'=T(y=0)}^{T'=T(y)}dT'=-(M/L)g\int_{y'=0}^{y'=y}dy'$$

After integration and substitution of the limits, we have that

$$T(y)-T(y=0)=-(M/L)gy$$

Us the fact that tension at the top of the rope is $T(y=0)=Mg$ and find that

$$T(y)=Mg(1-y/L)$$

in agreement with our earlier result, Equation (8.5.13).