23.8: Appendix 23A- Solution to Simple Harmonic Oscillator Equation

In our analysis of the solution of the simple harmonic oscillator equation of motion, Equation (23.2.1),

$$-kx = m\frac{d^2x}{dt^2}$$

we assumed that the solution was a linear combination of sinusoidal functions,

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/m}$. We shall now derive Equation (23.A.2).

Assume that the mechanical energy of the spring-object system is given by the constant E . Choose the reference point for potential energy to be the unstretched position of the spring. Let x denote the amount the spring has been compressed $(x < 0)$ or stretched $(x > 0)$ from equilibrium at time t and denote the amount the spring has been compressed or stretched from equilibrium at time $t = 0$ by $x(t = 0) \equiv x_0$. Let $v_x = dx/dt$ denote the x -component of the velocity at time t and denote the x -component of the velocity at time $t = 0$ by $v_x(t = 0) \equiv v_{x,0}$. The constancy of the mechanical energy is then expressed as

$$E = K + U = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

We can solve Equation (23.A.3) for the square of the x -component of the velocity,

$$v_x^2 = \frac{2E}{m} - \frac{k}{m}x^2 = \frac{2E}{m}\left(1 - \frac{k}{2E}x^2\right)$$

Taking square roots, we have

$$\frac{dx}{dt} = \sqrt{\frac{2E}{m}}\sqrt{1 - \frac{k}{2E}x^2}$$

(why we take the positive square root will be explained below).

Let $a_1 \equiv \sqrt{2E/m}$ and $a_2 \equiv k/2E$. It's worth noting that a_1 has dimensions of velocity and w has dimensions of [length] to the power −2 . Equation (23.A.5) is separable,

$$\begin{aligned} \frac{dx}{dt} &= a_1\sqrt{1 - a_2x^2} \\ \frac{dx}{\sqrt{1 - a_2x^2}} &= a_1 dt \end{aligned}$$

We now integrate Equation (23.A.6),

$$\int \frac{dx}{\sqrt{1 - a_1x^2}} = \int a_1 dt$$

The integral on the left in Equation (23.A.7) is well known, and a derivation is presented here. We make a change of variables $\cos\theta = \sqrt{a_2}x$ with the differentials $d\theta$ and dx related by $-\sin\theta d\theta = \sqrt{a_2}dx$. The integration variable is

$$\theta = \cos^{-1}(\sqrt{a_2}x)$$

Equation (23.A.7) then becomes

$$\int \frac{-\sin\theta d\theta}{\sqrt{1 - \cos^2\theta}} = \int \sqrt{a_2}a_1 dt$$

This is a good point at which to check the dimensions. The term on the left in Equation (23.A.9) is dimensionless, and the product $\sqrt{a_2}a_1$ on the right has dimensions of inverse time, $[\text{ length }]^{-1}\left[\text{length} \cdot \text{time}^{-1}\right] = \left[\text{time }^{-1}\right]$ so $\sqrt{a_2}a_1$ dt is dimensionless. Using the trigonometric identity $\sqrt{1 - \cos^2\theta} = \sin\theta$, Equation (23.A.9) reduces to

$$\int d\theta = -\int \sqrt{a_2}a_1 dt$$

Although at this point in the derivation we don't know that $\sqrt{a_2}a_1$, which has dimensions of frequency, is the angular frequency of oscillation, we'll use some foresight and make the identification

$$\omega_0 \equiv \sqrt{a_2}a_1 = \sqrt{\frac{k}{2E}}\sqrt{\frac{2E}{m}} = \sqrt{\frac{k}{m}}$$

and Equation (23.A.10) becomes

$$\int_{\theta=\theta_0}^{\theta} d\theta = -\int_{t=0}^{t} \omega_0 dt$$

After integration we have

$$\theta - \theta_0 = -\omega_0 t$$

where $\theta_0 \equiv -\phi$ is the constant of integration. Because $\theta = \cos^{-1}(\sqrt{a_2}x(t))$ Equation (23.A.13) becomes

$$\cos^{-1}(\sqrt{a_2}x(t)) = -(\omega_0 t + \phi)$$

Take the cosine of each side of Equation (23.A.14), yielding

$$x(t) = \frac{1}{\sqrt{a_2}}\cos(-(\omega_0 t + \phi)) = \sqrt{\frac{2E}{k}}\cos(\omega_0 t + \phi)$$

At t = 0

$$x_0 \equiv x(t=0) = \sqrt{\frac{2E}{k}}\cos\phi$$

The x -component of the velocity as a function of time is then

$$v_x(t) = \frac{dx(t)}{dt} = -\omega_0\sqrt{\frac{2E}{k}}\sin(\omega_0 t + \phi)$$

At t = 0,

$$v_{x,0} \equiv v_x(t=0) = -\omega_0\sqrt{\frac{2E}{k}}\sin\phi$$

We can determine the constant ϕ by dividing the expressions in Equations (23.A.18) and (23.A.16),

$$-\frac{v_{x,0}}{\omega_0 x_0} = \tan\phi$$

Thus the constant ϕ can be determined by the initial conditions and the angular frequency of oscillation,

$$\phi = \tan^{-1}\left(-\frac{v_{x,0}}{\omega_0 x_0}\right)$$

Use the identity

$$\cos(\omega_0 t + \phi) = \cos(\omega_0 t)\cos(\phi) - \sin(\omega_0 t)\sin(\phi)$$

to expand Equation (23.A.15) yielding

$$x(t) = \sqrt{\frac{2E}{k}}\cos(\omega_0 t)\cos(\phi) - \sqrt{\frac{2E}{k}}\sin(\omega_0 t)\sin(\phi)$$

and substituting Equations (23.A.16) and (23.A.18) into Equation (23.A.22) yields

$$x(t) = x_0\cos\omega_0 t + \frac{v_{x,0}}{\omega_0}\sin\omega_0 t$$

agreeing with Equation (23.2.21).

So, what about the missing $\pm$ that should have been in Equation (23.A.5)? Strictly speaking, we would need to redo the derivation for the block moving in different directions. Mathematically, this would mean replacing ϕ by $\pi-\phi$ (or $\phi-\pi$) when the block's velocity changes direction. Changing from the positive square root to the negative and changing ϕ to $\pi-\phi$ have the collective action of reproducing Equation (23.A.23).