10.4: System of Particles

Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \cdots, N$. The force on the i^{th} particle is

$$\overrightarrow{\mathbf{F}}_i = \overrightarrow{\mathbf{F}}_i^{\text{ext}} + \sum_{j=1, j\neq i}^{j=N} \overrightarrow{\mathbf{F}}_{i,j}$$

In this expression $\overrightarrow{\mathbf{F}}_{j,i}$ is the force on the i^{th} particle due to the interaction between the i^{th} and j^{th} particles. We sum over all j particles with $j \neq i$ since a particle cannot exert a force on itself (equivalently, we could define $\overrightarrow{\mathbf{F}}_{i,i} = \overrightarrow{\mathbf{0}}$), yielding the internal force acting on the i^{th} particle,

$$\overrightarrow{\mathbf{F}}_i^{\text{int}} = \sum_{j=1, j\neq i}^{j=N} \overrightarrow{\mathbf{F}}_{j,i}$$

The force acting on the system is the sum over all i particles of the force acting on each particle,

$$\overrightarrow{\mathbf{F}} = \sum_{i=1}^{i=N} \overrightarrow{\mathbf{F}}_i = \sum_{i=1}^{i=N} \overrightarrow{\mathbf{F}}_i^{\text{ext}} + \sum_{i=1}^{i=N} \sum_{j=1, j\neq i}^{j=N} \overrightarrow{\mathbf{F}}_{j,i} = \overrightarrow{\mathbf{F}}^{\text{ext}}$$

Note that the double sum vanishes,

$$\sum_{i=1}^{i=N} \sum_{j=1, j\neq i}^{j=N} \overrightarrow{\mathbf{F}}_{j,i} = \overrightarrow{\mathbf{0}}$$

because all internal forces cancel in pairs,

$$\overrightarrow{\mathbf{F}}_{j,i} + \overrightarrow{\mathbf{F}}_{i,j} = \overrightarrow{\mathbf{0}}$$

The force on the i^{th} particle is equal to the rate of change in momentum of the i^{th} particle,

$$\overrightarrow{\mathbf{F}}_i = \frac{d\overrightarrow{\mathbf{p}}_i}{dt}$$

When can now substitute Equation (10.4.6) into Equation (10.4.3) and determine that that the external force is equal to the sum over all particles of the momentum change of each particle,

$$\overrightarrow{\mathbf{F}}^{\text{ext}} = \sum_{i=1}^{i=N} \frac{d\overrightarrow{\mathbf{p}}_i}{dt}$$

The momentum of the system is given by the sum

$$\overrightarrow{\mathbf{p}}_{\text{sys}} = \sum_{i=1}^{i=N} \overrightarrow{\mathbf{p}}_i$$

momenta add as vectors. We conclude that the external force causes the momentum of the system to change, and we thus restate and generalize Newton's Second Law for a system of objects as

$$\overrightarrow{\mathbf{F}}^{\text{ext}} = \frac{d\overrightarrow{\mathbf{p}}_{\text{sys}}}{dt}$$

In terms of impulse, this becomes the statement

$$\Delta\overrightarrow{\mathbf{p}}_{\text{sys}} = \int_{t_0}^{t_f} \overrightarrow{\mathbf{F}}^{\text{ext}} dt \equiv \overrightarrow{\mathbf{I}}$$