13.5: Work done by Non-Constant Forces

Consider a body moving in the x -direction under the influence of a non-constant force in the x -direction, $\overrightarrow{\mathbf{F}}=F_x\hat{\mathbf{i}}$. The body moves from an initial position x_i to a final position x_f. In order to calculate the work done by a non-constant force, we will divide up the displacement of the point of application of the force into a large number N of small displacements Δx_j where the index j marks the j_{th} displacement and takes integer values from 1 to N . Let $\left(F_{x,j}\right)_{\text{ave}}$ denote the average value of the x -component of the force in the displacement interval $[x_{j-1}, x_j]$. For the j_{th} displacement interval we calculate the contribution to the work.

$$W_j=\left(F_{x,j}\right)_{\text{ave}}\Delta x_j$$

This contribution is a scalar so we add up these scalar quantities to get the total work

$$W_N=\sum_{j=1}^{j=N}W_j=\sum_{j=1}^{j=N}\left(F_{x,j}\right)_{\text{ave}}\Delta x_j$$

The sum in Equation (13.5.2) depends on the number of divisions N and the width of the intervals Δx_j. In order to define a quantity that is independent of the divisions, we take the limit as $N\rightarrow\infty$ and $|\Delta x_j|\rightarrow 0$ for all j . The work is then

$$W=\lim_{\substack{N\rightarrow\infty\\|\Delta x_j|\rightarrow 0}}\sum_{j=1}^{j=N}\left(F_{x,j}\right)_{\text{ave}}\Delta x_j=\int_{x=x_i}^{x=x_f}F_x(x)dx$$

This last expression is the definite integral of the x -component of the force with respect to the parameter x . In Figure 13.5 we graph the x -component of the force as a function of the parameter x . The work integral is the area under this curve between $x=x_i$ and $x=x_f$.

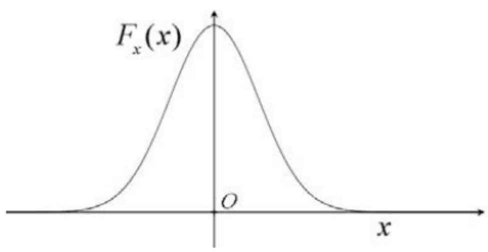

Figure 13.5 Plot of x -component of a sample force $F_x(x)$ as a function of x .

Example 13.5.1: Work done by the Spring Force

Connect one end of an unstretched spring of length l_0 with spring constant k to an object resting on a smooth frictionless table and fix the other end of the spring to a wall. Choose an origin as shown in the figure. Stretch the spring by an amount x_i and release the object. How much work does the spring do on the object when the spring is stretched by an amount x_f?

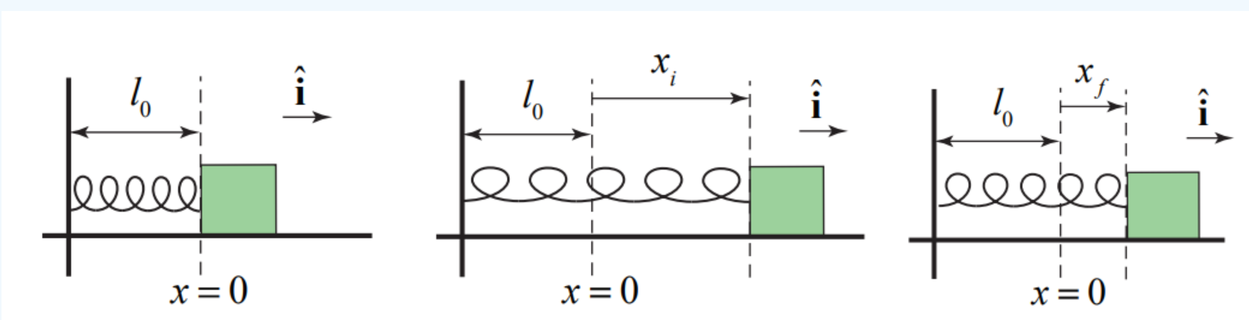

Figure 13.6 Equilibrium, initial and final states for a spring

Solution

We first begin by choosing a coordinate system with our origin located at the position of the object when the spring is unstretched (or uncompressed). We choose the $\hat{\mathbf{i}}$ unit vector to point in the direction the object moves when the spring is being stretched. We choose the coordinate function x to denote the position of the object with respect to the origin. We show the coordinate function and free-body force diagram in the figure below.

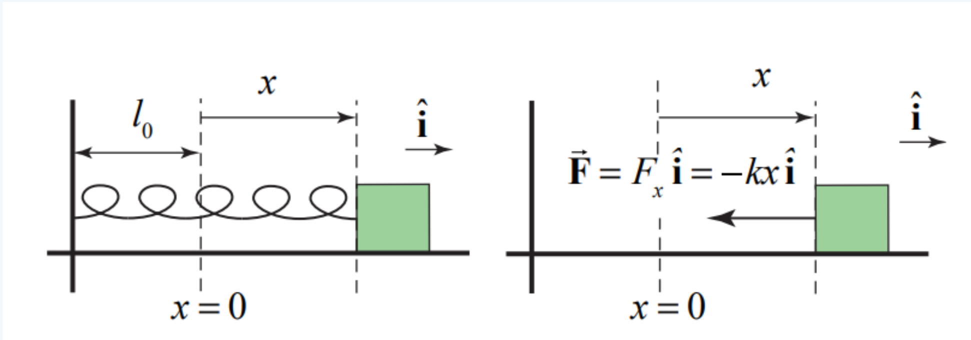

Figure 13.6a Spring force

The spring force on the object is given by (Figure 13.6a)

$$\overrightarrow{\mathbf{F}} = F_x \hat{\mathbf{i}} = -kx\hat{\mathbf{i}}$$

In Figure 13.7 we show the graph of the x -component of the spring force, $F_x(x)$, as a function of x

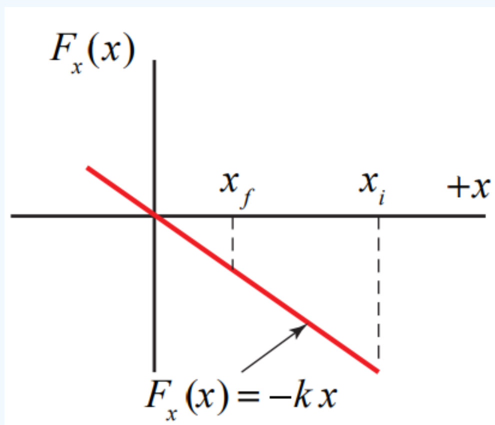

Figure 13.7 Plot of spring force $F_x(x)$ vs. displacement x

The work done is just the area under the curve for the interval x_i to x_f,

$$W = \int_{x'=x_i}^{x'=x_f} F_x\left(x'\right) dx' = \int_{x'=x_i}^{x'=x_f} -kx'\,dx' = -\frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

This result is independent of the sign of x_i and x_f because both quantities appear as squares. If the spring is less stretched or compressed in the final state than in the initial state, then the absolute value, $|x_f| < |x_i|$ and the work done by the spring force is positive. The spring force does positive work on the body when the spring goes from a state of “greater tension” to a state of “lesser tension.”