15.2: Reference Frames and Relative Velocities

We shall recall our definition of relative inertial reference frames. Let $\overrightarrow{\mathbf{R}}$ be the vector from the origin of frame S to the origin of reference frame S′. Denote the position vector of the j^{th} particle with respect to the origin of reference frame S by $\overrightarrow{\mathbf{r}}_j$ and similarly, denote the position vector of the particle with respect to the origin of reference frame S' by $\overrightarrow{\mathbf{r}}_j^{\prime}$(Figure 15.1).

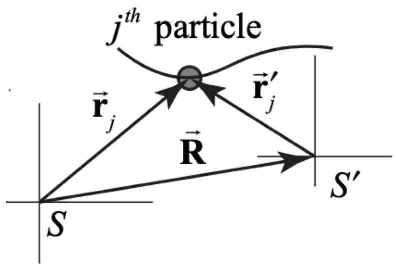

Figure 15.1 Position vector of j^{th} particle in two reference frames.

The position vectors are related by

$$\overrightarrow{\mathbf{r}}_j=\overrightarrow{\mathbf{r}}_j^{\prime}+\overrightarrow{\mathbf{R}}$$

The relative velocity (call this the boost velocity) between the two reference frames is given by

$$\overrightarrow{\mathbf{V}}=\frac{d\overrightarrow{\mathbf{R}}}{dt}$$

Assume the boost velocity between the two reference frames is constant. Then, the relative acceleration between the two reference frames is zero,

$$\overrightarrow{\mathbf{A}}=\frac{d\overrightarrow{\mathbf{V}}}{dt}=\overrightarrow{\mathbf{0}}$$

When Equation (15.2.3) is satisfied, the reference frames S and S′ are called relatively inertial reference frames. Suppose the j^{th} particle in Figure 15.1 is moving; then observers in different reference frames will measure different velocities. Denote the velocity of j^{th} particle in frame S by $\overrightarrow{\mathbf{v}}_j=d\overrightarrow{\mathbf{r}}_j/dt$, and the velocity of the same particle in frame S′ by $\overrightarrow{\mathbf{v}}_j^{\prime}=d\overrightarrow{\mathbf{r}}_j^{\prime}/dt$. Taking derivative, the velocities of the particles in two different reference frames are related according to

$$\overrightarrow{\mathbf{v}}_j=\overrightarrow{\mathbf{v}}_j^{\prime}+\overrightarrow{\mathbf{V}}$$

Relative Velocities

Consider two particles of masses m_1 and m_2 interacting via some force (Figure 15.2).

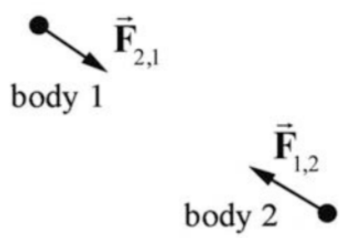

Figure 15.2 Two interacting particles

Choose a coordinate system (Figure 15.3) in which the position vector of body 1 is given by $\overrightarrow{\mathbf{r}}_1$ and the position vector of body 2 is given by $\overrightarrow{\mathbf{r}}_2$ The relative position of body 1 with respect to body 2 is given by $\overrightarrow{\mathbf{r}}_{1,2}=\overrightarrow{\mathbf{r}}_1-\overrightarrow{\mathbf{r}}_2$

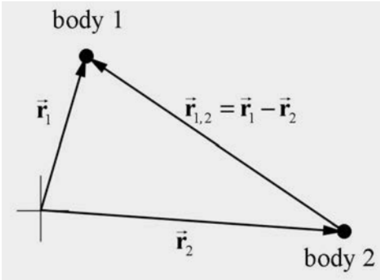

Figure 15.3 Coordinate system for two bodies.

During the course of the interaction, body 1 is displaced by $d\overrightarrow{\mathbf{r}}_1$ and body 2 is displaced by $d\overrightarrow{\mathbf{r}}_2$ so the **relative displacement** of the two bodies during the interaction is given by $d\overrightarrow{\mathbf{r}}_{1,2} = d\overrightarrow{\mathbf{r}}_1 - d\overrightarrow{\mathbf{r}}_2$. The **relative velocity** between the particles is

$$\overrightarrow{\mathbf{v}}_{1,2} = \frac{d\overrightarrow{\mathbf{r}}_{1,2}}{dt} = \frac{d\overrightarrow{\mathbf{r}}_1}{dt} - \frac{d\overrightarrow{\mathbf{r}}_2}{dt} = \overrightarrow{\mathbf{v}}_1 - \overrightarrow{\mathbf{v}}_2$$

We shall now show that the relative velocity between the two particles is independent of the choice of reference frame providing that the reference frames are relatively inertial. The relative velocity $\overrightarrow{\mathbf{v}}'_{12}$ in reference frame S′ can be determined from using Equation 12 (15.2.4) to express Equation (15.2.5) in terms of the velocities in the reference frame S′,

$$\overrightarrow{\mathbf{v}}_{1,2} = \overrightarrow{\mathbf{v}}_1 - \overrightarrow{\mathbf{v}}_2 = \left(\overrightarrow{\mathbf{v}}'_1 + \overrightarrow{\mathbf{V}}\right) - \left(\overrightarrow{\mathbf{v}}'_2 + \overrightarrow{\mathbf{V}}\right) = \overrightarrow{\mathbf{v}}'_1 - \overrightarrow{\mathbf{v}}'_2 = \overrightarrow{\mathbf{v}}'_{1,2}$$

and is equal to the relative velocity in frame S.

For a two-particle interaction, the relative velocity between the two vectors is independent of the choice of relatively inertial reference frames

Center-of-mass Reference Frame

Let $\overrightarrow{\mathbf{r}}_{cm}$ be the vector from the origin of frame S to the center-of-mass of the system of particles, a point that we will choose as the origin of reference frame S_{cm}, called the center-of-mass reference frame. Denote the position vector of the j^{th}particle with respect to origin of reference frame S by $\overrightarrow{\mathbf{r}}_j$ and similarly, denote the position of the j^{th} vector of the particle with respect to origin of reference frame S_{cm} by $\overrightarrow{\mathbf{r}}'_j$. (Figure 15.4)

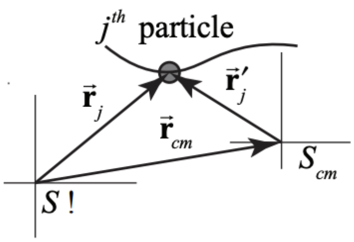

Figure 15.4 Position vector of j^{th} particle in the center-of-mass reference frame.

The position vector of the j^{th} particle in the center-of-mass frame is then given by

$$\overrightarrow{\mathbf{r}}'_j = \overrightarrow{\mathbf{r}}_j - \overrightarrow{\mathbf{r}}_{cm}$$

The velocity of the j^{th} particle in the center-of-mass reference frame is then given by

$$\overrightarrow{\mathbf{v}}'_j = \overrightarrow{\mathbf{v}}_j - \overrightarrow{\mathbf{v}}_{cm}$$

There are many collision problems in which the center-of-mass reference frame is the most convenient reference frame to analyze the collision.

Consider a system consisting of two particles, which we shall refer to as particle 1 and particle 2. We can use Equation (15.2.8) to determine the velocities of particles 1 and 2 in the center-of-mass,

$$\overrightarrow{\mathbf{v}}_{1}^{\prime}=\overrightarrow{\mathbf{v}}_{1}-\overrightarrow{\mathbf{v}}_{cm}=\overrightarrow{\mathbf{v}}_{1}-\frac{m_{1}\overrightarrow{\mathbf{v}}_{1}+m_{2}\overrightarrow{\mathbf{v}}_{2}}{m_{1}+m_{2}}=\frac{m_{2}}{m_{1}+m_{2}}\left(\overrightarrow{\mathbf{v}}_{1,}-\overrightarrow{\mathbf{v}}_{2}\right)=\frac{\mu}{m_{1}}\overrightarrow{\mathbf{v}}_{1,2}$$

where $\overrightarrow{\mathbf{v}}_{12}=\overrightarrow{\mathbf{v}}_{1}-\overrightarrow{\mathbf{v}}_{2}$ is the relative velocity of particle 1 with respect to particle 2 . A similar result holds for particle 2:

$$\overrightarrow{\mathbf{v}}_{2}^{\prime}=\overrightarrow{\mathbf{v}}_{2}-\overrightarrow{\mathbf{v}}_{cm}=\overrightarrow{\mathbf{v}}_{2}-\frac{m_{1}\overrightarrow{\mathbf{v}}_{1}+m_{2}\overrightarrow{\mathbf{v}}_{2}}{m_{1}+m_{2}}=-\frac{m_{1}}{m_{1}+m_{2}}\left(\overrightarrow{\mathbf{v}}_{1}-\overrightarrow{\mathbf{v}}_{2}\right)=-\frac{\mu}{m_{2}}\overrightarrow{\mathbf{v}}_{1,2}$$

The momentum of the system the center-of-mass reference frame is zero as we expect,

$$m_{1}\overrightarrow{\mathbf{v}}_{1}^{\prime}+m_{2}\overrightarrow{\mathbf{v}}_{2}^{\prime}=\mu\overrightarrow{\mathbf{v}}_{12}-\mu\overrightarrow{\mathbf{v}}_{12}=\overrightarrow{\mathbf{0}}$$

Kinetic Energy in the Center-of-Mass Reference Frame

The kinetic energy in the center of mass reference frame is given by

$$K_{cm}=\frac{1}{2}m_{1}\overrightarrow{\mathbf{v}}_{1}^{\prime}\cdot\overrightarrow{\mathbf{v}}_{1}^{\prime}+\frac{1}{2}m_{2}\overrightarrow{\mathbf{v}}_{2}^{\prime}\cdot\overrightarrow{\mathbf{v}}_{2}^{\prime}$$

We now use Equations (15.2.9) and (15.2.10) to rewrite the kinetic energy in terms of the relative velocity $\overrightarrow{\mathbf{v}}_{12}^{\prime}=\overrightarrow{\mathbf{v}}_{1}^{\prime}-\overrightarrow{\mathbf{v}}_{2}^{\prime}$,

$$\begin{aligned}K_{cm}=\frac{1}{2}m_{1}\left(\frac{\mu}{m_{1}}\overrightarrow{\mathbf{v}}_{1,2}\right)\cdot\left(\frac{\mu}{m_{1}}\overrightarrow{\mathbf{v}}_{1,2}\right)+\frac{1}{2}m_{2}\left(-\frac{\mu}{m_{2}}\overrightarrow{\mathbf{v}}_{1,2}\right)\cdot\left(-\frac{\mu}{m_{2}}\overrightarrow{\mathbf{v}}_{1,2}\right)=\frac{1}{2}\mu^{2}\overrightarrow{\mathbf{v}}_{1,2}\cdot\overrightarrow{\mathbf{v}}_{1,2}\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)\\=\frac{1}{2}\mu v_{1,2}^{2}\end{aligned}$$

where we used the fact that we defined the reduced mass by

$$\frac{1}{\mu}\equiv\frac{1}{m_{1}}+\frac{1}{m_{2}}$$

Change of Kinetic Energy and Relatively Inertial Reference Frames

The kinetic energy of the two particles in reference frame S is given by

$$K_{S}=\frac{1}{2}m_{1}v_{1}^{2}+\frac{1}{2}m_{2}v_{2}^{2}$$

We can take the scalar product of Equation (15.2.8) to rewrite Equation (15.2.15) as

$$\begin{aligned}K_{S}&=\tfrac{1}{2}m_{1}\left(\overrightarrow{\mathbf{v}}_{1}^{\prime}+\overrightarrow{\mathbf{v}}_{cm}\right)\cdot\left(\overrightarrow{\mathbf{v}}_{1}^{\prime}+\overrightarrow{\mathbf{v}}_{cm}\right)+\tfrac{1}{2}m_{2}\left(\overrightarrow{\mathbf{v}}_{2}^{\prime}+\overrightarrow{\mathbf{v}}_{cm}\right)\cdot\left(\overrightarrow{\mathbf{v}}_{2}^{\prime}+\overrightarrow{\mathbf{v}}_{cm}\right)\\&=\tfrac{1}{2}m_{1}v_{1}^{\prime 2}+\tfrac{1}{2}m_{2}v_{2}^{\prime 2}+\tfrac{1}{2}\left(m_{1}+m_{2}\right)v_{cm}^{2}+\left(m_{1}\overrightarrow{\mathbf{v}}_{1}^{\prime}+m_{2}\overrightarrow{\mathbf{v}}_{2}^{\prime}\right)\cdot\overrightarrow{\mathbf{v}}_{cm}\end{aligned}$$

The last term is zero due to the fact that the momentum of the system in the center of mass reference frame is zero (Equation (15.2.11)). Therefore Equation (15.2.16) becomes

$$K_{S}=\frac{1}{2}m_{1}v_{1}^{\prime 2}+\frac{1}{2}m_{2}v_{2}^{\prime 2}+\frac{1}{2}\left(m_{1}+m_{2}\right)v_{cm}^{2}$$

The first two terms correspond to the kinetic energy in the center of mass frame, thus the kinetic energies in the two reference frames are related by

$$K_{S}=K_{cm}+\frac{1}{2}\left(m_{1}+m_{2}\right)v_{cm}^{2}$$

We now use Equation (15.2.13) to rewrite Equation (15.2.18) as

$$K_{S}=\frac{1}{2}\mu\nu_{1,2}^{2}+\frac{1}{2}\left(m_{1}+m_{2}\right)v_{cm}^{2}$$

Even though kinetic energy is a reference frame dependent quantity, because the second term in Equation (15.2.19) is a constant, the change in kinetic energy in either reference frame is equal to

$$\Delta K=\frac{1}{2} \mu\left(\left(v_{1,2}^{2}\right)_{f}-\left(v_{1,2}^{2}\right)_{i}\right)$$

This generalizes to any two relatively inertial reference frames because the relative velocity is a reference frame independent quantity,

the change in kinetic energy is independent of the choice of relatively inertial reference frames.

We showed in Appendix 13A that when two particles of masses m_1 and m_2 interact, the work done by the interaction force is equal to

$$W=\frac{1}{2} \mu\left(\left(v_{1,2}^{2}\right)_{f}-\left(v_{1,2}^{2}\right)_{i}\right)$$

Hence we explicitly verified that for our two-particle system

$$W=\Delta K_{sys}$$
