25.9: Appendix 25B Properties of an Elliptical Orbit

25B.1 Coordinate System for the Elliptic Orbit

We now consider the special case of an elliptical orbit. Choose coordinates with the central point located at one focal point and coordinates (r, θ) for the position of the single body (Figure 25B.1a). In Figure 25B.1b, let a denote the semi-major axis, b denote the semi-minor axis and x_0 denote the distance from the center of the ellipse to the origin of our coordinate system.

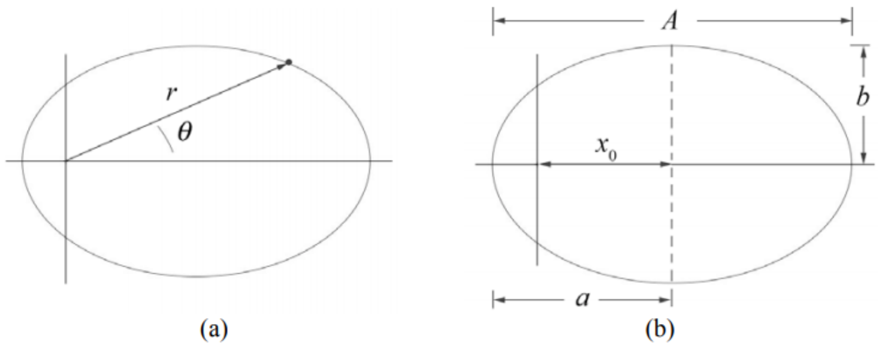

Figure 25B.1 (a) Coordinate system for elliptic orbit, (b) semi-major axis

25B.2 The Semi-major Axis

Recall the orbit equation, Eq, (25.A.9), describes r(θ),

$$r(\theta) = \frac{r_0}{1 - \varepsilon \cos \theta}$$

The major axis A = 2a is given by

$$A = 2a = r_a + r_p$$

where the distance of furthest approach r_a occurs when $\theta = 0$, hence

$$r_a = r(\theta = 0) = \frac{r_0}{1 - \varepsilon}$$

and the distance of nearest approach $\boldsymbol{r}_p$ occurs when $\theta = \pi$, hence

$$r_p = r(\theta = \pi) = \frac{r_0}{1 + \varepsilon}$$

Figure 25B.2 shows the distances of nearest and furthest approach.

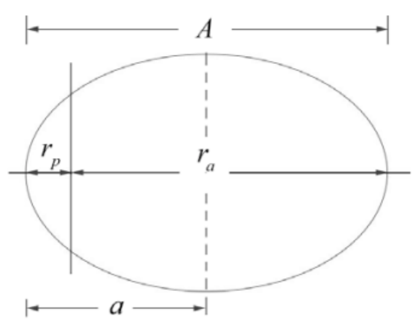

Figure 25B.2 Furthest and nearest approach

We can now determine the semi-major axis

$$a = \frac{1}{2}\left(\frac{r_0}{1 - \varepsilon} + \frac{r_0}{1 + \varepsilon}\right) = \frac{r_0}{1 - \varepsilon^2}$$

The semilatus rectum r_0 can be re-expressed in terms of the semi-major axis and the eccentricity,

$$r_0 = a\left(1-\varepsilon^2\right)$$

We can now express the distance of nearest approach, Equation (25.B.4), in terms of the semi-major axis and the eccentricity,

$$r_p = \frac{r_0}{1+\varepsilon} = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon} = a(1-\varepsilon)$$

In a similar fashion the distance of furthest approach is

$$r_a = \frac{r_0}{1-\varepsilon} = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon} = a(1+\varepsilon)$$

25B.2.3 The Location x_0 of the Center of the Ellipse

From Figure 25B.3a, the distance from a focus point to the center of the ellipse is

$$x_0 = a - r_p$$

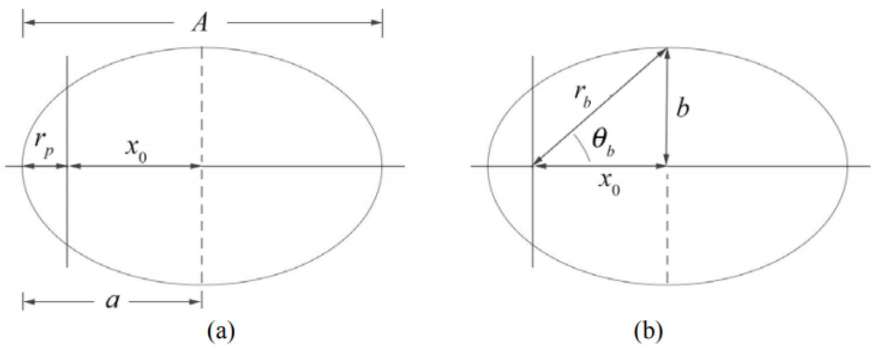

Figure 25B.3 Location of the center of the ellipse and semi-minor axis.

Using Equation (25.B.7) for r_p, we have that

$$x_0 = a - a(1-\varepsilon) = \varepsilon a$$

25B.2.4 The Semi-minor Axis

From Figure 25B.3b, the semi-minor axis can be expressed as

$$b = \sqrt{\left(r_b^2 - x_0^2\right)}$$

where

$$r_b = \frac{r_0}{1-\varepsilon\cos\theta_b}$$

We can rewrite Equation (25.B.12) as

$$r_b - r_b\varepsilon\cos\theta_b = r_0$$

The horizontal projection of r_b is given by (Figure 25B.2b),

$$x_0 = r_b\cos\theta_b$$

which upon substitution into Equation (25.B.13) yields

$$r_b = r_0 + \varepsilon x_0$$

Substituting Equation (25.B.10) for x_0 and Equation (25.B.6) for r_0 into Equation (25.B.15) yields

$$r_b = a\left(1-\varepsilon^2\right) + a\varepsilon^2 = a$$

The fact that $r_b = a$ is a well-known property of an ellipse reflected in the geometric construction, that the sum of the distances from the two foci to any point on the ellipse is a constant. We can now determine the semi-minor axis b by substituting Equation

(25.B.16) into Equation (25.B.11) yielding

$$b=\sqrt{\left(r_b^2-x_0^2\right)}=\sqrt{a^2-\varepsilon^2 a^2}=a\sqrt{1-\varepsilon^2}$$

25B.2.5 Constants of the Motion for Elliptic Motion

We shall now express the parameters a , b and x_0 in terms of the constants of the motion L , E , μ, m_1 and m_2. Using our results for r_0 and ε from Equations (25.3.13) and (25.3.14) we have for the semi-major axis

$$\begin{aligned} a &= \frac{L^2}{\mu G m_1 m_2} \frac{1}{\left(1-\left(1+2EL^2/\mu(Gm_1m_2)^2\right)\right)} \\ &= -\frac{Gm_1m_2}{2E} \end{aligned}$$

The energy is then determined by the semi-major axis,

$$E=-\frac{Gm_1m_2}{2a}$$

The angular momentum is related to the semilatus rectum r_0 by Equation (25.3.13). Using Equation (25.B.6) for r_0, we can express the angular momentum (25.B.4) in terms of the semi-major axis and the eccentricity,

$$L=\sqrt{\mu G m_1 m_2 r_0}=\sqrt{\mu G m_1 m_2 a\left(1-\varepsilon^2\right)}$$

Note that

$$\sqrt{\left(1-\varepsilon^2\right)}=\frac{L}{\sqrt{\mu G m_1 m_2 a}}$$

Thus, from Equations (25.3.14), (25.B.10), and (25.B.18), the distance from the center of the ellipse to the focal point is

$$x_0=\varepsilon a=-\frac{Gm_1m_2}{2E}\sqrt{\left(1+2EL^2/\mu(Gm_1m_2)^2\right)}$$

a result we will return to later. We can substitute Equation (25.B.21) for $\sqrt{1-\varepsilon^2}$ into Equation (25.B.17), and determine that the semi-minor axis is

$$b=\sqrt{aL^2/\mu G m_1 m_2}$$

We can now substitute Equation (25.B.18) for a into Equation (25.B.23), yielding

$$b=\sqrt{aL^2/\mu G m_1 m_2}=L\sqrt{-\frac{Gm_1m_2}{2E}/\mu G m_1 m_2}=L\sqrt{-\frac{1}{2\mu E}}$$

25B.2.6 Speeds at Nearest and Furthest Approaches

At nearest approach, the velocity vector is tangent to the orbit (Figure 25B.4), so the magnitude of the angular momentum is

$$L=\mu r_p v_p$$

and the speed at nearest approach is

$$v_p=L/\mu r_p$$

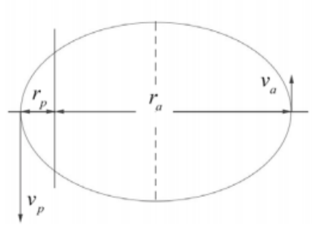

Figure 25B.4 Speeds at nearest and furthest approach

Using Equation (25.B.20) for the angular momentum and Equation (25.B.7) for r p , Equation (25.B.26) becomes

$$v_p = \frac{L}{\mu r_p} = \frac{\sqrt{\mu G m_1 m_2 \left(1-\varepsilon^2\right)}}{\mu a(1-\varepsilon)} = \sqrt{\frac{G m_1 m_2 \left(1-\varepsilon^2\right)}{\mu a(1-\varepsilon)^2}} = \sqrt{\frac{G m_1 m_2 (1+\varepsilon)}{\mu a(1-\varepsilon)}}$$

A similar calculation show that the speed v_a at furthest approach,

$$v_a = \frac{L}{\mu r_a} = \frac{\sqrt{\mu G m_1 m_2 \left(1-\varepsilon^2\right)}}{\mu a(1+\varepsilon)} = \sqrt{\frac{G m_1 m_2 1-\varepsilon^2}{\mu a(1+\varepsilon)^2}} = \sqrt{\frac{G m_1 m_2 (1-\varepsilon)}{\mu a(1+\varepsilon)}}$$