25.3: Energy and Angular Momentum, Constants of the Motion

The equivalent one-body problem has two constants of the motion, energy E and the angular momentum L about the origin O . Energy is a constant because in our original two-body problem, the gravitational interaction was an internal conservative force. Angular momentum is constant about the origin because the only force is directed towards the origin, and hence the torque about the origin due to that force is zero (the vector from the origin to the single body is anti-parallel to the force vector and $\sin \pi = 0$). Because angular momentum is constant, the orbit of the single body lies in a plane with the angular momentum vector pointing perpendicular to this plane.

In the plane of the orbit, choose polar coordinates (r, θ) for the single body (see Figure 25.3), where r is the distance of the single body from the central point that is now taken as the origin O , and θ is the angle that the single body makes with respect to a chosen direction, and which increases positively in the counterclockwise direction.

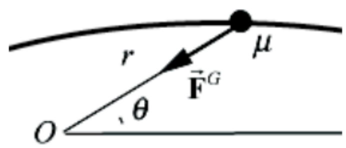

Figure 25.3 Coordinate system for the orbit of the single body

There are two approaches to describing the motion of the single body. We can try to find both the distance from the origin, $r(t)$ and the angle, $\theta(t)$, as functions of the parameter time, but in most cases explicit functions can't be found analytically. We can also find the distance from the origin, $r(\theta)$, as a function of the angle θ. This second approach offers a spatial description of the motion of the single body (see Appendix 25A).

The Orbit Equation for the One-Body Problem

Consider the single body with mass μ given by Equation (25.2.1), orbiting about a central point under the influence of a radially attractive force given by Equation (25.2.2). Since the force is conservative, the potential energy (from the two-body problem) with choice of zero reference point $U(\infty) = 0$ is given by

$$U(r) = -\frac{G m_1 m_2}{r}$$

The total energy E is constant, and the sum of the kinetic energy and the potential energy is

$$E = \frac{1}{2}\mu v^2 - \frac{G m_1 m_2}{r}$$

The kinetic energy term $\mu v^2/2$ is written in terms of the mass μ and the relative speed v of the two bodies. Choose polar coordinates such that

$$\overrightarrow{\mathbf{v}} = v_r \hat{\mathbf{r}} + v_\theta \hat{\theta}$$
$$v = |\overrightarrow{\mathbf{v}}| = \left|\frac{d\overrightarrow{\mathbf{r}}}{dt}\right|$$

where $v_r = dr/dt$ and $v_\theta = r(d\theta/dt)$. Equation (25.3.2) then becomes

$$E = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{G m_1 m_2}{r}$$

The angular momentum with respect to the origin O is given by

$$\overrightarrow{\mathbf{L}}_O = \overrightarrow{\mathbf{r}}_O \times \mu\overrightarrow{\mathbf{v}} = r\hat{\mathbf{r}} \times \mu\left(v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}\right) = \mu r v_\theta \hat{\mathbf{k}} = \mu r^2 \frac{d\theta}{dt}\hat{\mathbf{k}} \equiv L\hat{\mathbf{k}}$$

with magnitude

$$L = \mu r v_\theta = \mu r^2 \frac{d\theta}{dt}$$

We shall explicitly eliminate the θ dependence from Equation (25.3.4) by using our expression in Equation (25.3.6),

$$\frac{d\theta}{dt}=\frac{L}{\mu r^2}$$

The mechanical energy as expressed in Equation (25.3.4) then becomes

$$E=\frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2+\frac{1}{2}\frac{L^2}{\mu r^2}-\frac{Gm_1m_2}{r}$$

Equation (25.3.8) is a separable differential equation involving the variable r as a function of time t and can be solved for the first derivative dr/dt,

$$\frac{dr}{dt}=\sqrt{\frac{2}{\mu}}\left(E-\frac{1}{2}\frac{L^2}{\mu r^2}+\frac{Gm_1m_2}{r}\right)^{\frac{1}{2}}$$

Equation (25.3.9) can in principle be integrated directly for $r(t)$. In fact, doing the integrals is complicated and beyond the scope of this book. The function $r(t)$ can then, in principle, be substituted into Equation (25.3.7) and can then be integrated to find $\theta(t)$.

Instead of solving for the position of the single body as a function of time, we shall find a geometric description of the orbit by finding $r(\theta)$. We first divide Equation (25.3.7) by Equation (25.3.9) to obtain

$$\frac{d\theta}{dr}=\frac{\frac{d\theta}{dt}}{\frac{dr}{dt}}=\frac{L}{\sqrt{2\mu}}\frac{\left(1/r^2\right)}{\left(E-\frac{L^2}{2\mu r^2}+\frac{Gm_1m_2}{r}\right)^{1/2}}$$

The variables r and θ are separable;

$$d\theta=\frac{L}{\sqrt{2\mu}}\frac{\left(1/r^2\right)}{\left(E-\frac{L^2}{2\mu r^2}+\frac{Gm_1m_2}{r}\right)^{1/2}}dr$$

Equation (25.3.11) can be integrated to find the radius as a function of the angle θ; see Appendix 25A for the exact integral solution. The result is called the orbit equation for the reduced body and is given by

$$r=\frac{r_0}{1-\varepsilon\cos\theta}$$

where

$$r_0=\frac{L^2}{\mu Gm_1m_2}$$

is a constant (known as the semilatus rectum) and

$$\varepsilon=\left(1+\frac{2EL^2}{\mu(Gm_1m_2)^2}\right)^{\frac{1}{2}}$$

is the eccentricity of the orbit. The two constants of the motion, angular momentum L and mechanical energy E , in terms of r_0 and ε, are

$$L=\left(\mu Gm_1m_2r_0\right)^{1/2}$$
$$E=\frac{Gm_1m_2\left(\varepsilon^2-1\right)}{2r_0}$$

The orbit equation as given in Equation (25.3.12) is a general conic section and is perhaps somewhat more familiar in Cartesian coordinates. Let $x=r\cos\theta$ and $y=r\sin\theta$, with $r^2=x^2+y^2$. The orbit equation can be rewritten as

$$r=r_0+\varepsilon r\cos\theta$$

Using the Cartesian substitutions for x and y , rewrite Equation (25.3.17) as

$$\left(x^2+y^2\right)^{1/2}=r_0+\varepsilon x$$

Squaring both sides of Equation (25.3.18),

$$x^2+y^2=r_0^2+2\varepsilon x r_0+\varepsilon^2 x^2$$

After rearranging terms, Equation (25.3.19) is the general expression of a conic section with axis on the x -axis,

$$x^2\left(1-\varepsilon^2\right)-2\varepsilon x r_0+y^2=r_0^2$$

(We now see that the horizontal axis in Figure 25.3 can be taken to be the x -axis).

For a given $r_0>0$, corresponding to a given nonzero angular momentum as in Equation (25.3.12), there are four cases determined by the value of the eccentricity.

Case 1: when $\varepsilon=0, \quad E=E_{\min}<0$ and $r=r_0$ Equation (25.3.20) is the equation for a circle,

$$x^2+y^2=r_0^2$$

Case 2: when $0<\varepsilon<1, E_{\min}<E<0$ Equation (25.3.20) describes an ellipse,

$$y^2+A x^2-B x=k$$

where $A>0$ and k is a positive constant. (Appendix 25C shows how this expression may be expressed in the more traditional form involving the coordinates of the center of the ellipse and the semi-major and semi-minor axes.)

Case 3: when $\varepsilon=1, E=0$, Equation (25.3.20) describes a parabola,

$$x=\frac{y^2}{2 r_0}-\frac{r_0}{2}$$

Case 4: when $\varepsilon>1, E>0$, Equation (25.3.20) describes a hyperbola,

$$y^2-A x^2-B x=k$$

where $A>0$ and k is a positive constant.

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