# 25.8: Appendix 25A Derivation of the Orbit Equation

## 25A.1 Derivation of the Orbit Equation: Method 1

Start from Equation (25.3.11) in the form

$$d\theta=\frac{L}{\sqrt{2\mu}}\frac{\left(1/r^{2}\right)}{\left(E-\frac{L^{2}}{2\mu r^{2}}+\frac{Gm_{1}m_{2}}{r}\right)^{1/2}}dr$$

What follows involves a good deal of hindsight, allowing selection of convenient substitutions in the math in order to get a clean result. First, note the many factors of the reciprocal of r. So, we'll try the substitution $u=1/r, du=-\left(1/r^{2}\right)dr$, with the result

$$d\theta=-\frac{L}{\sqrt{2\mu}}\frac{du}{\left(E-\frac{L^{2}}{2\mu}u^{2}+Gm_{1}m_{2}u\right)^{1/2}}$$

Experience in evaluating integrals suggests that we make the absolute value of the factor multiplying $u^2$ inside the square root equal to unity. That is, multiplying numerator and denominator by $\sqrt{2\mu}/L$

$$d\theta=-\frac{du}{\left(2\mu E/L^{2}-u^{2}+2\left(\mu Gm_{1}m_{2}/L^{2}\right)u\right)^{1/2}}$$

As both a check and a motivation for the next steps, note that the left side $d\theta$ of Equation (25.A.3) is dimensionless, and so the right side must be. This means that the factor of $\mu Gm_1m_2/L^2$ in the square root must have the same dimensions as u , or $length^{-1}$ so, define $r_0\equiv L^2/\mu Gm_1m_2$ This is of course the semilatus rectum as defined in Equation (25.3.12), and it's no coincidence; this is part of the "hindsight" mentioned above. The differential equation then becomes

$$d\theta=-\frac{du}{\left(2\mu E/L^{2}-u^{2}+2u/r_{0}\right)^{1/2}}$$

We now rewrite the denominator in order to express it terms of the eccentricity.

$$\begin{aligned}d\theta&=-\frac{du}{\left(2\mu E/L^{2}+1/r_{0}^{2}-u^{2}+2u/r_{0}-1/r_{0}^{2}\right)^{1/2}}\\&=-\frac{du}{\left(2\mu E/L^{2}+1/r_{0}^{2}-\left(u-1/r_{0}\right)^{2}\right)^{1/2}}\\&=-\frac{r_{0}du}{\left(2\mu Er_{0}^{2}/L^{2}+1-\left(r_{0}u-1\right)^{2}\right)^{1/2}}\end{aligned}$$

We note that the combination of terms $2\mu Er_0^2/L^2+1$ is dimensionless, and is in fact equal to the square of the eccentricity $\varepsilon$ as defined in Equation (25.3.13); more hindsight. The last expression in (25.A.5) is then

$$d\theta=-\frac{r_{0}du}{\left(\varepsilon^{2}-\left(r_{0}u-1\right)^{2}\right)^{1/2}}$$

From here, we'll combine a few calculus steps, going immediately to the substitution $r_0u-1=\varepsilon\cos\alpha, r_0du=-\varepsilon\sin\alpha d\alpha$ with the final result that

$$d\theta=-\frac{-\varepsilon\sin\alpha d\alpha}{\left(\varepsilon^{2}-\varepsilon^{2}\cos^{2}\alpha\right)^{1/2}}=d\alpha$$

We now integrate Equation (25.A.7) with the very simple result that

$$\theta=\alpha+constant$$

We have a choice in selecting the constant, and if we pick $\theta=\alpha-\pi,\quad \alpha=\theta+\pi\quad \cos\alpha=-\cos\theta$, the result is

$$r=\frac{1}{u}=\frac{r_0}{1-\varepsilon\cos\theta}$$

which is our desired result, Equation (25.3.11). Note that if we chose the constant of integration to be zero, the result would be

$$r=\frac{1}{u}=\frac{r_0}{1+\varepsilon\cos\theta}$$

which is the same trajectory reflected about the “vertical” axis in Figure 25.3, indeed the same as rotating by $\pi$

## 25A.2 Derivation of the Orbit Equation: Method 2

The derivation of Equation (25.A.9) in the form

$$u=\frac{1}{r_0}(1-\varepsilon\cos\theta)$$

suggests that the equation of motion for the one-body problem might be manipulated to obtain a simple differential equation. That is, start from

$$\overrightarrow{\mathbf{F}}=\mu\overrightarrow{\mathbf{a}}$$

$$-G\frac{m_1m_2}{r^2}\hat{\mathbf{r}}=\mu\left(\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}}$$

Setting the components equal, using the constant of motion $L=\mu r^2(d\theta/dt)$ and rearranging, Equation (25.A.12) becomes

$$\mu\frac{d^2r}{dt^2}=\frac{L^2}{\mu r^3}-\frac{Gm_1m_2}{r^2}$$

We now use the same substitution u =1/r and change the independent variable from t to r , using the chain rule twice, since Equation (25.A.13) is a second-order equation. That is, the first time derivative is

$$\frac{dr}{dt}=\frac{dr}{du}\frac{du}{dt}=\frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}$$

From $r=1/u$ we have $dr/du=-1/u^2$ Combining with $d\theta/dt$ in terms of L and u,

$$d\theta/dt=Lu^2/\mu \tag{25.8.1}$$

, Equation (25.A.14) becomes

$$\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{Lu^2}{\mu}=-\frac{du}{d\theta}\frac{L}{\mu}$$

a very tidy result, with the variable u appearing linearly. Taking the second derivative with respect to t ,

$$\frac{d^2r}{dt^2}=\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{d}{d\theta}\left(\frac{dr}{dt}\right)\frac{d\theta}{dt}$$

Now substitute Equation (25.A.15) into Equation (25.A.16) with the result that

$$\frac{d^2r}{dt^2}=-\frac{d^2u}{d\theta^2}\left(u^2\frac{L^2}{\mu^2}\right)$$

Substituting into Equation (25.A.13), with $r=1/u$ yields

$$-\frac{d^2u}{d\theta^2}u^2\frac{L^2}{\mu}=\frac{L^2}{\mu}u^3-Gm_1m_2u^2$$

Canceling the common factor of $u^2$ and rearranging, we arrive at

$$-\frac{d^2u}{d\theta^2}=u-\frac{\mu Gm_1m_2}{L^2}$$

Equation (25.A.19) is mathematically equivalent to the simple harmonic oscillator equation with an additional constant term. The solution consists of two parts: the angleindependent solution

$$u_0 = \frac{\mu G m_1 m_2}{L^2}$$

and a sinusoidally varying term of the form

$$u_{\mathrm{H}} = A\cos(\theta - \theta_0)$$

where A and $\theta_0$ are constants determined by the form of the orbit. The expression in Equation (25.A.20) is the inhomogeneous solution and represents a circular orbit. The expression in Equation (25.A.21) is the homogeneous solution (as hinted by the subscript) and must have two independent constants. We can readily identify $1/u_0$ as the semilatus rectum $r_0$, with the result that

$$u = u_0 + u_{\mathrm{H}} = \frac{1}{r_0}(1 + r_0 A(\theta - \theta_0)) \Rightarrow$$
$$r = \frac{1}{u} = \frac{r_0}{1 + r_0 A(\theta - \theta_0)}$$

Choosing the product $r_0 A$ to be the eccentricity $\varepsilon$ and $\theta_0 = \pi$ (much as was done leading to Equation (25.A.9) above), Equation (25.A.9) is reproduced.

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