

23.11: Solution to the Forced Damped Oscillator Equation

We shall now use complex numbers to solve the differential equation

$$F_0 \cos(\omega t) = m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx$$

We begin by assuming a solution of the form

$$x(t) = x_0 \cos(\omega t + \phi)$$

where the amplitude x_0 and the phase constant ϕ need to be determined. We begin by defining the complex function

$$z(t) = x_0 e^{i(\omega t + \phi)}$$

Our desired solution can be found by taking the real projection

$$x(t) = \text{Re}(z(t)) = x_0 \cos(\omega t + \phi)$$

Our differential equation can now be written as

$$F_0 e^{i\omega t} = m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + kz$$

We take the first and second derivatives of Equation (23.D.3),

$$\begin{aligned} \frac{dz}{dt}(t) &= i\omega x_0 e^{i(\omega t + \phi)} = i\omega z \\ \frac{d^2 z}{dt^2}(t) &= -\omega^2 x_0 e^{i(\omega t + \phi)} = -\omega^2 z \end{aligned}$$

We substitute Equations (23.D.3), (23.D.6), and (23.D.7) into Equation (23.D.5) yielding

$$F_0 e^{i\omega t} = (-\omega^2 m + bi\omega + k) z = (-\omega^2 m + bi\omega + k) x_0 e^{i(\omega t + \phi)}$$

We divide Equation (23.D.8) through by $e^{i\omega t}$ and collect terms using yielding

$$x_0 e^{i\phi} = \frac{F_0/m}{((\omega_0^2 - \omega^2) + i(b/m)\omega)}$$

where we have used $\omega_0^2 = k/m$. Introduce the complex number

$$z_1 = (\omega_0^2 - \omega^2) + i(b/m)\omega$$

Then Equation (23.D.9) can be written as

$$x_0 e^{i\phi} = \frac{F_0}{m z_1}$$

Multiply the numerator and denominator of Equation (23.D.11) by the complex conjugate $\bar{z}_1 = (\omega_0^2 - \omega^2) - i(b/m)\omega$ yielding

$$x_0 e^{i\phi} = \frac{F_0 \bar{z}_1}{m z_1 \bar{z}_1} = \frac{F_0}{m} \frac{((\omega_0^2 - \omega^2) - i(b/m)\omega)}{((\omega_0^2 - \omega^2)^2 + (b/m)^2 \omega^2)} \equiv u + iv$$

where

$$\begin{aligned} u &= \frac{F_0}{m} \frac{(\omega_0^2 - \omega^2)}{((\omega_0^2 - \omega^2)^2 + (b/m)^2 \omega^2)} \\ v &= -\frac{F_0}{m} \frac{(b/m)\omega}{((\omega_0^2 - \omega^2)^2 + (b/m)^2 \omega^2)} \end{aligned}$$

Therefore the modulus x_0 is given by

$$x_0 = (u^2 + v^2)^{1/2} = \frac{F_0/m}{\left((\omega_0^2 - \omega^2)^2 + (b/m)^2 \omega^2 \right)^{1/2}}$$

and the phase is given by

$$\phi = \tan^{-1}(v/u) = \frac{-(b/m)\omega}{(\omega_0^2 - \omega^2)}$$

This page titled [23.11: Solution to the Forced Damped Oscillator Equation](#) is shared under a [not declared](#) license and was authored, remixed, and/or curated by [Peter Dourmashkin \(MIT OpenCourseWare\)](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.