17.5: Torque and Rotational Work

When a constant torque $\tau_{s,z}$ is applied to an object, and the object rotates through an angle $\Delta\theta$ about a fixed z -axis through the center of mass, then the torque does an amount of work $\Delta W = \tau_{S,z}\Delta\theta$ on the object. By extension of the linear work-energy theorem, the amount of work done is equal to the change in the rotational kinetic energy of the object,

$$W_{\text{rot}} = \frac{1}{2}I_{\text{cm}}\omega_f^2 - \frac{1}{2}I_{\text{cm}}\omega_i^2 = K_{\text{rot},f} - K_{\text{rot},i}$$

The rate of doing this work is the rotational power exerted by the torque,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \lim_{\Delta t \to 0}\frac{\Delta W_{\text{rot}}}{\Delta t} = \tau_{S,z}\frac{d\theta}{dt} = \tau_{S,z}\omega_z$$

Rotational Work

Consider a rigid body rotating about an axis. Each small element of mass Δm_i in the rigid body is moving in a circle of radius $(r_{S,i})_\perp$ about the axis of rotation passing through the point S. Each mass element undergoes a small angular displacement $\Delta\theta$ under the action of a tangential force, $\overrightarrow{\mathbf{F}}_{\theta,i} = F_{\theta,i}\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is the unit vector pointing in the tangential direction (Figure 17.20). The element will then have an associated displacement vector for this motion, $\Delta\overrightarrow{\mathbf{r}}_{S,i} = r_i\Delta\theta\hat{\boldsymbol{\theta}}$ and the work done by the tangential force is

$$\Delta W_{\text{rot},\,i} = \overrightarrow{\mathbf{F}}_{\theta,i}\cdot\Delta\overrightarrow{\mathbf{r}}_{S,i} = \left(F_{\theta,i}\hat{\theta}\right)\cdot\left(r_i\Delta\theta\hat{\theta}\right) = r_iF_{\theta,i}\Delta\theta$$

Recall the result of Equation (17.3.8) that the component of the torque (in the direction along the axis of rotation) about S due to the tangential force, $\overrightarrow{\mathbf{F}}_{\theta,i}$, acting on the mass element Δm_i is

$$(\tau_{S,i})_z = r_iF_{\theta,i}$$

and so Equation (17.4.10) becomes

$$\Delta W_{\text{rot}\,,i} = (\tau_{S,i})_z\Delta\theta$$

Summing over all the elements yields

$$W_{\text{rot}} = \sum_i \Delta W_{\text{rot},i} = \sum_i \left((\tau_{S,i})_z\right)\Delta\theta = \tau_{S,z}\Delta\theta$$

the rotational work is the product of the torque and the angular displacement. In the limit of small angles, $\Delta\theta \to d\theta, \Delta W_{\text{rot}} \to dW_{\text{rot}}$ and the differential rotational work is

$$dW_{\text{rot}} = \tau_{S,z}d\theta$$

We can integrate this amount of rotational work as the angle coordinate of the rigid body changes from some initial value $\theta = \theta_i$ to some final value $\theta = \theta_f$,

$$W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\theta_i}^{\theta_f}\tau_{S,z}d\theta$$

Rotational Work-Kinetic Energy Theorem

We will now show that the rotational work is equal to the change in rotational kinetic energy. We begin by substituting our result from Equation (17.3.14) into Equation (17.4.14) for the infinitesimal rotational work,

$$dW_{\text{rot}} = I_S\alpha_z d\theta$$

Recall that the rate of change of angular velocity is equal to the angular acceleration, $\alpha_z \equiv d\omega_z/dt$ and that the angular velocity is $\omega_z \equiv d\theta/dt$. Note that in the limit of small displacement,

$$\frac{d\omega_z}{dt}d\theta = d\omega_z\frac{d\theta}{dt} = d\omega_z\omega_z$$

Therefore the infinitesimal rotational work is

$$dW_{\mathrm{rot}} = I_S \alpha_z d\theta = I_S \frac{d\omega_z}{dt} d\theta = I_S d\omega_z \frac{d\theta}{dt} = I_S d\omega_z \omega_z$$

We can integrate this amount of rotational work as the angular velocity of the rigid body changes from some initial value $\omega_z = \omega_{z,i}$ to some final value $\omega_z = \omega_{z,f}$,

$$W_{\mathrm{rot}} = \int dW_{\mathrm{rot}} = \int_{\omega_{z,i}}^{\omega_{z,f}} I_S d\omega_z \omega_z = \frac{1}{2} I_S \omega_{z,f}^2 - \frac{1}{2} I_S \omega_{z,i}^2$$

When a rigid body is rotating about a fixed axis passing through a point S in the body, there is both rotation and translation about the center of mass unless S is the center of mass. If we choose the point S in the above equation for the rotational work to be the center of mass, then

$$W_{\mathrm{rot}} = \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm},f}^2 - \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm},i}^2 = K_{\mathrm{rot},f} - K_{\mathrm{rot},i} \equiv \Delta K_{\mathrm{rot}}$$

Note that because the z -component of the angular velocity of the center of mass appears as a square, we can just use its magnitude in Equation (17.4.20).

Rotational Power

The rotational power is defined as the rate of doing rotational work,

$$P_{\mathrm{rot}} \equiv \frac{dW_{\mathrm{rot}}}{dt}$$

We can use our result for the infinitesimal work to find that the rotational power is the product of the applied torque with the angular velocity of the rigid body,

$$P_{\mathrm{rot}} \equiv \frac{dW_{\mathrm{rot}}}{dt} = \tau_{S,z} \frac{d\theta}{dt} = \tau_{S,z} \omega_z$$

Example 17.12 Work Done by Frictional Torque

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular velocity ω_0, at t = 0, the power to the motor is shut off, and the washer slows down during a time interval $\Delta t_1 = t_a$ until it reaches an angular velocity of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{\mathrm{int}} = t_b - t_a$ after which the two washers and rotor rotate with angular speed ω_b. Assume the frictional torque on the axle (magnitude τ_f) is independent of speed, and remains the same when the second washer is dropped. (a) What angle does the rotor rotate through during the collision? (b) What is the work done by the friction torque from the bearings during the collision? (c) Write down an equation for conservation of energy. Can you solve this equation for ω_b (d) What is the average rate that work is being done by the friction torque during the collision?

Solution: We begin by solving for the frictional torque during the first stage of motion when the rotor is slowing down. We choose a coordinate system shown in Figure 17.29.

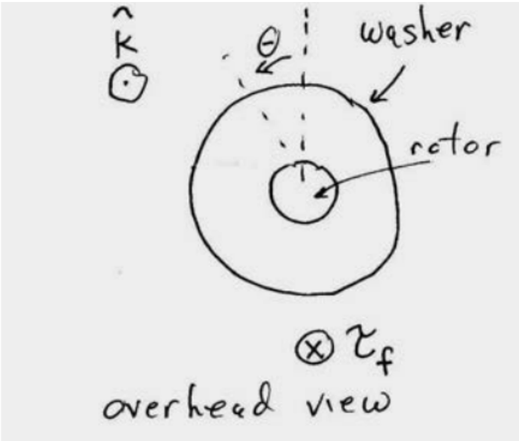

Figure 17.29 Coordinate system for Example 17.12

The component of average angular acceleration is given by

$$\alpha_1=\frac{\omega_a-\omega_0}{t_a}<0$$

We can use the rotational equation of motion, and find that the frictional torque satisfies

$$-\tau_f=I_0\left(\frac{\omega_a-\omega_0}{\Delta t_1}\right)$$

During the collision, the component of the average angular acceleration of the rotor is given by

$$\alpha_2=\frac{\omega_b-\omega_a}{(\Delta t_{\text{int}})}<0$$

The angle the rotor rotates through during the collision is (analogous to linear motion with constant acceleration)

$$\Delta\theta_2=\omega_a\Delta t_{\text{int}}+\frac{1}{2}\alpha_2\Delta t_{\text{int}}^2=\omega_a\Delta t_{\text{int}}+\frac{1}{2}\left(\frac{\omega_b-\omega_a}{\Delta t_{\text{int}}}\right)\Delta t_{\text{int}}^2=\frac{1}{2}(\omega_b+\omega_a)\Delta t_{\text{int}}>0$$

The non-conservative work done by the bearing friction during the collision is

$$W_{f,b}=-\tau_f\Delta\theta_{\text{rotor}}=-\tau_f\frac{1}{2}(\omega_a+\omega_b)\Delta t_{\text{int}}$$

Using our result for the frictional torque, the work done by the bearing friction during the collision is

$$W_{f,b}=\frac{1}{2}I_0\left(\frac{\omega_a-\omega_0}{\Delta t_1}\right)(\omega_a+\omega_b)\Delta t_{\text{int}}<0$$

The negative work is consistent with the fact that the kinetic energy of the rotor is decreasing as the rotor is slowing down. Using the work energy theorem during the collision the kinetic energy of the rotor has deceased by

$$W_{f,b}=\frac{1}{2}(I_0+I_w)\omega_b^2-\frac{1}{2}I_0\omega_a^2$$

Using our result for the work, we have that

$$\frac{1}{2}I_0\left(\frac{\omega_a-\omega_0}{\Delta t_1}\right)(\omega_a+\omega_b)\Delta t_{\text{int}}=\frac{1}{2}(I_0+I_w)\omega_b^2-\frac{1}{2}I_0\omega_a^2$$

This is a quadratic equation for the angular speed ω_b of the rotor and washer immediately after the collision that we can in principle solve. However remember that we assumed that the frictional torque is independent of the speed of the rotor. Hence the best practice would be to measure $\omega_0,\omega_a,\omega_b,\Delta t_1,\Delta t_{\text{int}},I_0$ and I_w and then determine how closely our model agrees with conservation of energy. The rate of work done by the frictional torque is given by

$$P_f = \frac{W_{f,b}}{\Delta t_{\text{int}}} = \frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1}\right) (\omega_a + \omega_b) < 0$$