24.4: Appendix 24A Higher-Order Corrections to the Period for Larger Amplitudes of a Simple Pendulum

In Section 24.1.1, we found the period for a simple pendulum that undergoes small oscillations is given by

$$T=\frac{2\pi}{\omega_0}\cong 2\pi\sqrt{\frac{l}{g}} \quad \text{(simple pendulum)}$$

How good is this approximation? If the pendulum is pulled out to an initial angle θ_0 that is not small (such that our first approximation $\sin\theta\cong\theta$ no longer holds) then our expression for the period is no longer valid. We shall calculate the first-order (or higherorder) correction to the period of the pendulum.

Let's first consider the mechanical energy, a conserved quantity in this system. Choose an initial state when the pendulum is released from rest at an angle θ_i ; this need not be at time t = 0 , and in fact later in this derivation we'll see that it's inconvenient to choose this position to be at t = 0 . Choose for the final state the position and velocity of the bob at an arbitrary time t . Choose the zero point for the potential energy to be at the bottom of the bob's swing (Figure 24A.1).

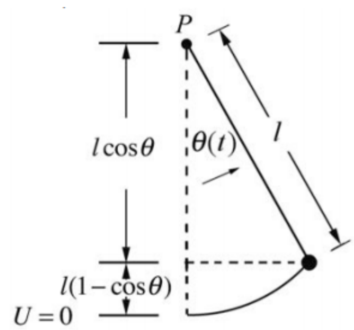


Figure 24A.1 Energy diagram for simple pendulum

The mechanical energy when the bob is released from rest at an angle θ_i is

$$E_i=K_i+U_i=mgl\,(1-\cos\theta_i)$$

The tangential component of the velocity of the bob at an arbitrary time t is given by

$$v_\theta=l\frac{d\theta}{dt}$$

and the kinetic energy at that time is

$$K_f=\frac{1}{2}mv_\theta^2=\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2$$

The mechanical energy at time t is then

$$E_f=K_f+U_f=\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2+mgl(1-\cos\theta)$$

Because the tension in the string is always perpendicular to the displacement of the bob, the tension does no work, we neglect any frictional forces, and hence mechanical energy is constant, $E_f=E_i$. Thus

$$\begin{aligned}&\tfrac{1}{2}m\left(l\tfrac{d\theta}{dt}\right)^2+mgl(1-\cos\theta)=mgl\,(1-\cos\theta_i)\\&\left(l\tfrac{d\theta}{dt}\right)^2=2\tfrac{g}{l}(\cos\theta-\cos\theta_i)\end{aligned}$$

We can solve Equation (24.C.41) for the angular velocity as a function of θ ,

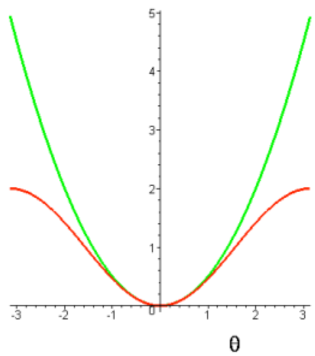
$$\frac{d\theta}{dt}=\sqrt{\frac{2g}{l}}\sqrt{\cos\theta-\cos\theta_i}$$

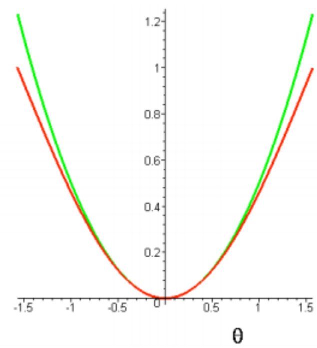
Note that we have taken the positive square root, implying that $d\theta/dt \geq 0$. This clearly cannot always be the case, and we should change the sign of the square root every time the pendulum's direction of motion changes. For our purposes, this is not an issue. If we wished to find an explicit form for either $\theta(t)$ or $t(\theta)$, we would have to consider the signs in Equation (24.C.42) more carefully.

Before proceeding, it's worth considering the difference between Equation (24.C.42) and the equation for the simple pendulum in the simple harmonic oscillator limit, where $\cos\theta \simeq 1-(1/2)\theta^2$. Then Equation (24.C.42) reduces to

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\frac{\theta_i^2}{2}-\frac{\theta^2}{2}}$$

In both Equations (24.C.42) and (24.C.43) the last term in the square root is proportional to the difference between the initial potential energy and the final potential energy. The final potential energy for the two cases is plotted in Figures 24A.2 for $-\pi<\theta<\pi$ on the left and $-\pi/2<\theta<\pi/2$ on the right (the vertical scale is in units of mgl).





Figures 24A.2 Potential energies as a function of displacement angle

It would seem to be to our advantage to express the potential energy for an arbitrary displacement of the pendulum as the difference between two squares. We do this by first recalling the trigonometric identity

$$1-\cos\theta = 2\sin^2(\theta/2)$$

with the result that Equation (24.C.42) may be re-expressed as

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{2\left(\sin^2(\theta_i/2)-\sin^2(\theta/2)\right)}$$

Equation (24.C.45) is separable,

$$\frac{d\theta}{\sqrt{\sin^2(\theta_i/2)-\sin^2(\theta/2)}} = 2\sqrt{\frac{g}{l}}dt$$

Rewrite Equation (24.C.46) as

$$\frac{d\theta}{\sin(\theta_i/2)\sqrt{1-\frac{\sin^2(\theta/2)}{\sin^2(\theta_i/2)}}} = 2\sqrt{\frac{g}{l}}dt$$

The ratio $\sin(\theta/2)/\sin(\theta_i/2)$ in the square root in the denominator will oscillate (but not with simple harmonic motion) between −1 and +1, and so we will make the identification

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_i/2)}$$

Let $b=\sin(\theta_i/2)$, so that

$$\begin{aligned}&\sin \tfrac{\theta}{2}=b \sin \phi \\ &\cos \tfrac{\theta}{2}=\left(1-\sin ^2 \tfrac{\theta}{2}\right)^{1 / 2}=\left(1-b^2 \sin ^2 \phi\right)^{1 / 2}\end{aligned}$$

Equation (24.C.47) can then be rewritten in integral form as

$$\int \frac{d \theta}{b \sqrt{1-\sin ^2 \phi}}=2 \int \sqrt{\frac{g}{l}} d t$$

From differentiating the first expression in Equation (24.C.49), we have that

$$\begin{aligned}&\tfrac{1}{2} \cos \tfrac{\theta}{2} d \theta=b \cos \phi d \phi \\ &d \theta=2 b \tfrac{\cos \phi}{\cos (\theta / 2)} d \phi=2 b \tfrac{\sqrt{1-\sin ^2 \phi}}{\sqrt{1-\sin ^2(\theta / 2)}} d \phi \\ &=2 b \tfrac{\sqrt{1-\sin ^2 \phi}}{\sqrt{1-b^2 \sin ^2 \phi}} d \phi\end{aligned}$$

Substituting the last equation in (24.C.51) into the left-hand side of the integral in (24.C.50) yields

$$\int \frac{2 b}{b \sqrt{1-\sin ^2 \phi}} \frac{\sqrt{1-\sin ^2 \phi}}{\sqrt{1-b^2 \sin ^2 \phi}} d \phi=2 \int \frac{d \phi}{\sqrt{1-b^2 \sin ^2 \phi}}$$

Thus the integral in Equation (24.C.50) becomes

$$\int \frac{d \phi}{\sqrt{1-b^2 \sin ^2 \phi}}=\int \sqrt{\frac{g}{l}} d t$$

This integral is one of a class of integrals known as elliptic integrals. We find a power series solution to this integral by expanding the function

$$\left(1-b^2 \sin ^2 \phi\right)^{-1 / 2}=1+\frac{1}{2} b^2 \sin ^2 \phi+\frac{3}{8} b^4 \sin ^4 \phi+\cdots$$

The integral in Equation (24.C.53) then becomes

$$\int\left(1+\frac{1}{2} b^2 \sin ^2 \phi+\frac{3}{8} b^4 \sin ^4 \phi+\cdots\right) d \phi=\int \sqrt{\frac{g}{l}} d t$$

Now let's integrate over one period. Set $t=0$ when $\theta=0$, the lowest point of the swing, so that $\sin \phi=0$ and $\phi=0$. One period T has elapsed the second time the bob returns to the lowest point, or when $\phi=2 \pi$. Putting in the limits of the φ -integral, we can integrate term by term, noting that

$$\begin{aligned}\int_0^{2 \pi} \frac{1}{2} b^2 \sin ^2 \phi d \phi &=\int_0^{2 \pi} \frac{1}{2} b^2 \frac{1}{2}(1-\cos (2 \phi)) d \phi \\ &=\frac{1}{2} b^2 \frac{1}{2}\left(\phi-\frac{\sin (2 \phi)}{2}\right)\Bigg|_0^{2 \pi} \\ &=\frac{1}{2} \pi b^2=\frac{1}{2} \pi \sin ^2 \frac{\theta_i}{2}\end{aligned}$$

Thus, from Equation (24.C.55) we have that

$$\begin{aligned}\int_0^{2 \pi}\left(1+\frac{1}{2} b^2 \sin ^2 \phi+\frac{3}{8} b^4 \sin ^4 \phi+\cdots\right) d \phi &=\int_0^T \sqrt{\frac{g}{l}} d t \\ 2 \pi+\frac{1}{2} \pi \sin ^2 \frac{\theta_i}{2}+\cdots &=\sqrt{\frac{g}{l}} T\end{aligned}$$

We can now solve for the period,

$$T=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_i}{2}+\cdots\right)$$

If the initial angle $\theta_i << 1$ (measured in radians), then $\sin^2(\theta_i/2)\simeq\theta_i^2/4$ and the period is approximately

$$T\cong 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\theta_i^2\right)=T_0\left(1+\frac{1}{16}\theta_i^2\right)$$

where

$$T_0=2\pi\sqrt{\frac{l}{g}}$$

is the period of the simple pendulum with the standard small angle approximation. The first order correction to the period of the pendulum is then

$$\Delta T_1=\frac{1}{16}\theta_i^2 T_0$$

Figure 24A.3 below shows the three functions given in Equation (24.C.60) (the horizontal, or red plot if seen in color), Equation (24.C.59) (the middle, parabolic or green plot) and the numerically-integrated function obtained by integrating the expression in Equation (24.C.53) (the upper, or blue plot) between $\phi=0$ and $\phi=2\pi$. The plots demonstrate that Equation (24.C.60) is a valid approximation for small values of θ_i, and that Equation (24.C.59) is a very good approximation for all but the largest amplitudes of oscillation. The vertical axis is in units of Note the displacement of the horizontal axis.

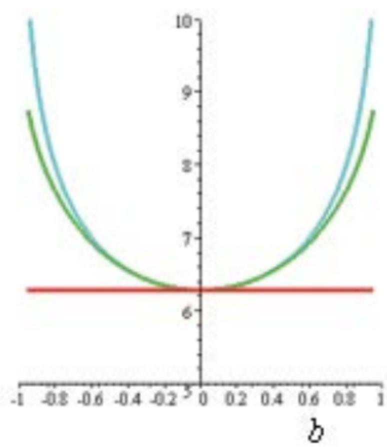


Figure 24A.3 Pendulum Period Approximations as Functions of Amplitude