23.11: Solution to the Forced Damped Oscillator Equation

We shall now use complex numbers to solve the differential equation

$$F_0 \cos(\omega t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx$$

We begin by assuming a solution of the form

$$x(t) = x_0 \cos(\omega t + \phi)$$

where the amplitude x_0 and the phase constant ϕ need to be determined. We begin by defining the complex function

$$z(t) = x_0 e^{i(\omega t+\phi)}$$

Our desired solution can be found by taking the real projection

$$x(t) = \mathrm{Re}(z(t)) = x_0 \cos(\omega t + \phi)$$

Our differential equation can now be written as

$$F_0 e^{i\omega t} = m\frac{d^2z}{dt^2} + b\frac{dz}{dt} + kz$$

We take the first and second derivatives of Equation (23.D.3),

$$\frac{dz}{dt}(t) = i\omega x_0 e^{i(\omega t+\phi)} = i\omega z$$

$$\frac{d^2z}{dt^2}(t) = -\omega^2 x_0 e^{i(\omega t+\phi)} = -\omega^2 z$$

We substitute Equations (23.D.3), (23.D.6), and (23.D.7) into Equation (23.D.5) yielding

$$F_0 e^{i\omega t} = \left(-\omega^2 m + bi\omega + k\right) z = \left(-\omega^2 m + bi\omega + k\right) x_0 e^{i(\omega t+\phi)}$$

We divide Equation (23.D.8) through by $e^{i\omega t}$ and collect terms using yielding

$$x_0 e^{i\phi} = \frac{F_0/m}{\left(\left(\omega_0^2 - \omega^2\right) + i(b/m)\omega\right)}$$

where we have used $\omega_0^2 = k/m$. Introduce the complex number

$$z_1 = \left(\omega_0^2 - \omega^2\right) + i(b/m)\omega$$

Then Equation (23.D.9) can be written as

$$x_0 e^{i\phi} = \frac{F_0}{my}$$

Multiply the numerator and denominator of Equation (23.D.11) by the complex conjugate $\bar{z}_1 = \left(\omega_0^2 - \omega^2\right) - i(b/m)\omega$ yeilding

$$x_0 e^{i\phi} = \frac{F_0 \bar{z}_1}{m z_1 \bar{z}_1} = \frac{F_0}{m}\frac{\left(\left(\omega_0^2 - \omega^2\right) - i(b/m)\omega\right)}{\left(\left(\omega_0^2 - \omega^2\right)^2 + (b/m)^2\omega^2\right)} \equiv u + iv$$

where

$$u = \frac{F_0}{m}\frac{\left(\omega_0^2 - \omega^2\right)}{\left(\left(\omega_0^2 - \omega^2\right)^2 + (b/m)^2\omega^2\right)}$$

$$v = -\frac{F_0}{m}\frac{(b/m)\omega}{\left(\left(\omega_0^2 - \omega^2\right)^2 + (b/m)^2\omega^2\right)}$$

Therefore the modulus x_0 is given by

$$x_0 = \left(u^2 + v^2\right)^{1/2} = \frac{F_0/m}{\left(\left(\omega_0^2 - \omega^2\right)^2 + (b/m)^2\omega^2\right)}$$

and the phase is given by

$$\phi = \tan^{-1}(v/u) = \frac{-(b/m)\omega}{\left(\omega_0^2 - \omega^2\right)}$$