

3.1: Vector Analysis

Introduction to Vectors

Certain physical quantities such as mass or the absolute temperature at some point in space only have magnitude. A single number can represent each of these quantities, with appropriate units, which are called **scalar** quantities. There are, however, other physical quantities that have both magnitude and direction. Force is an example of a quantity that has both direction and magnitude (strength). Three numbers are needed to represent the magnitude and direction of a vector quantity in a three dimensional space. These quantities are called **vector** quantities. Vector quantities also satisfy two distinct operations, vector addition and multiplication of a vector by a scalar. We can add two forces together and the sum of the forces must satisfy the rule for vector addition. We can multiply a force by a scalar thus increasing or decreasing its strength. Position, displacement, velocity, acceleration, force, and momentum are all physical quantities that can be represented mathematically by vectors. The set of vectors and the two operations form what is called a **vector space**. There are many types of vector spaces but we shall restrict our attention to the very familiar type of vector space in three dimensions that most students have encountered in their mathematical courses. We shall begin our discussion by defining what we mean by a vector in three dimensional space, and the rules for the operations of vector addition and multiplication of a vector by a scalar.

Properties of Vectors

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol \vec{A} . The magnitude of \vec{A} is $|\vec{A}| \equiv A$. We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector (Figure 3.1).

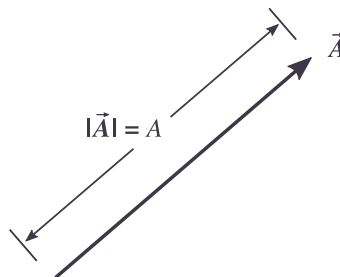


Figure 3.1: Vectors as arrows. (CC BY-NC; Ümit Kaya)

There are two defining operations for vectors:

(1) Vector Addition

Vectors can be added. Let \vec{A} and \vec{B} be two vectors. We define a new vector, $\vec{C} = \vec{A} + \vec{B}$, the “vector addition” of \vec{A} and \vec{B} , by a geometric construction. Draw the arrow that represents \vec{A} . Place the tail of the arrow that represents \vec{B} at the tip of the arrow for \vec{A} as shown in Figure 3.2a. The arrow that starts at the tail of \vec{A} and goes to the tip of \vec{B} is defined to be the “vector addition” $\vec{C} = \vec{A} + \vec{B}$. There is an equivalent construction for the law of vector addition. The vectors \vec{A} and \vec{B} can be drawn with their tails at the same point. The two vectors form the sides of a parallelogram. The diagonal of the parallelogram corresponds to the vector $\vec{C} = \vec{A} + \vec{B}$, as shown in Figure 3.2b.

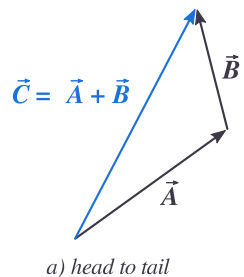
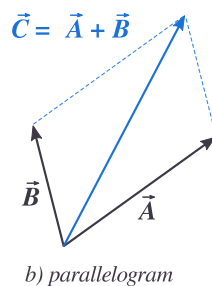


Figure 3.2a (CC BY-NC; Ümit Kaya)



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Figure 3.2b (CC BY-NC;

Vector addition satisfies the following four properties:

(i) Commutativity

The order of adding vectors does not matter;

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Our geometric definition for vector addition satisfies the commutative property (3.1.1). We can understand this geometrically because in the head to tail representation for the addition of vectors, it doesn't matter which vector you begin with, the sum is the same vector, as seen in Figure 3.3.

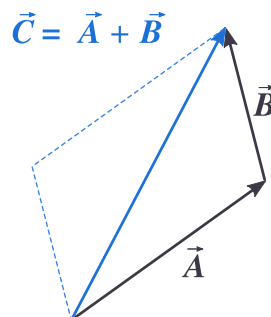
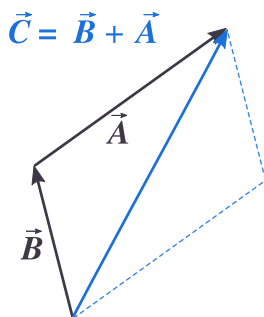


Figure 3.3: Commutative property of vector addition. (CC BY-NC; Ümit Kaya)

(ii) Associativity

When adding three vectors, it doesn't matter which two you start with

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

In Figure 3.4a, we add $(\vec{B} + \vec{C}) + \vec{A}$, and use commutativity to get $\vec{A} + (\vec{B} + \vec{C})$ to arrive at the same vector as in Figure 3.4a.

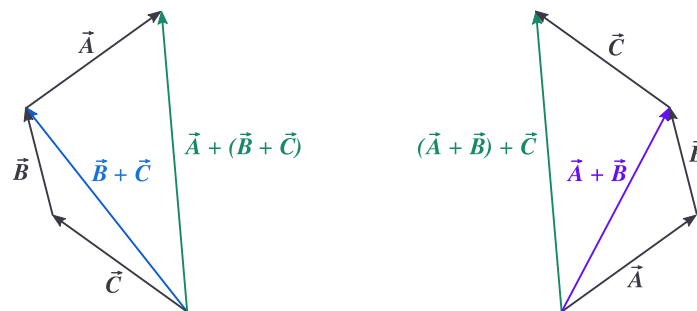


Figure 3.4a Associative law. (CC BY-NC; Ümit Kaya)

(iii) Identity Element for Vector Addition

There is a unique vector, $\vec{0}$, that acts as an identity element for vector addition. For all vectors \vec{A} ,

$$\vec{A} + \vec{0} = \vec{0} + \vec{A} = \vec{A}$$

(iv) Inverse Element for Vector Addition

For every vector \vec{A} there is a unique inverse vector $-\vec{A}$ such that

$$\vec{A} + (-\vec{A}) = \vec{0}$$

The vector $-\vec{A}$ has the same magnitude as \vec{A} , $|\vec{A}| = |-\vec{A}| = A$ but they point in opposite directions (Figure 3.5).

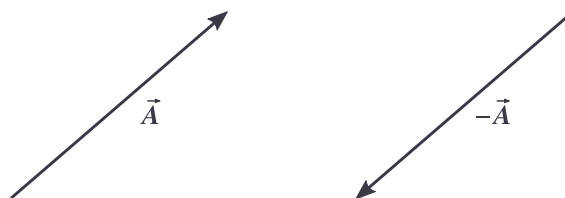


Figure 3.5 Additive inverse. (CC BY-NC; Ümit Kaya)

(2) Scalar Multiplication of Vectors

Vectors can be multiplied by real numbers. Let \vec{A} be a vector. Let c be a real positive number. Then the multiplication of \vec{A} by c is a new vector, which we denote by the symbol $c\vec{A}$. The magnitude of $c\vec{A}$ is c times the magnitude of \vec{A} (Figure 3.6a),

$$|c\vec{A}| = c|\vec{A}|$$

Let $c > 0$, then the direction of $c\vec{A}$ is the same as the direction of \vec{A} . However, the direction of $-c\vec{A}$ is opposite of \vec{A} (Figure 3.6).

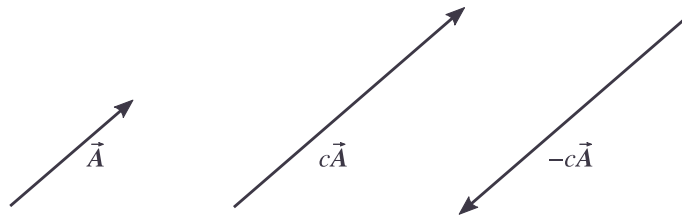


Figure 3.6 Multiplication of vector \vec{A} by $c > 0$, and $-c < 0$. (CC BY-NC; Ümit Kaya)

Scalar multiplication of vectors satisfies the following properties:

(i) Associative Law for Scalar Multiplication

The order of multiplying numbers is doesn't matter. Let b and c be real numbers. Then

$$b(c\vec{A}) = (bc)\vec{A} = (cb\vec{A}) = c(b\vec{A})$$

(ii) Distributive Law for Vector Addition

Vectors satisfy a distributive law for vector addition. Let c be a real number. Then

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$$

Figure 3.7 illustrates this property.

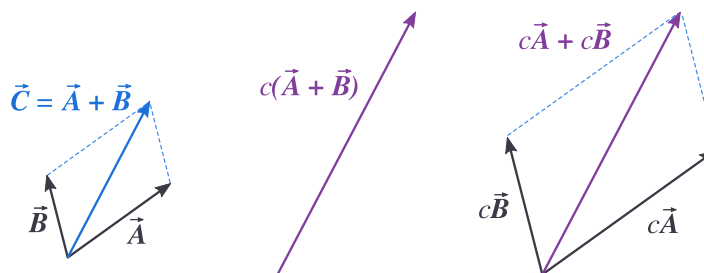


Figure 3.7 Distributive Law for vector addition. (CC BY-NC; Ümit Kaya)

(iii) Distributive Law for Scalar Addition

Vectors also satisfy a distributive law for scalar addition. Let b and c be real numbers. Then

$$(b + c)\vec{A} = b\vec{A} + c\vec{A}$$

Our geometric definition of vector addition and scalar multiplication satisfies this condition as seen in Figure 3.8.

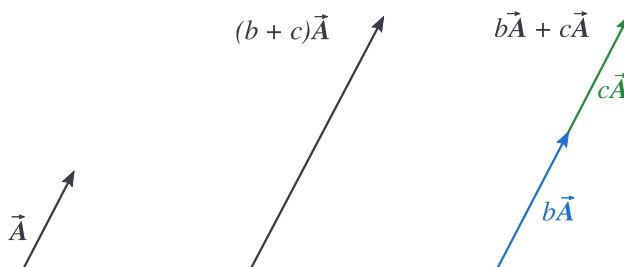


Figure 3.8 Distributive law for scalar multiplication. (CC BY-NC; Ümit Kaya)

(iv) Identity Element for Scalar Multiplication

The number 1 acts as an identity element for multiplication,

$$1 \vec{\mathbf{A}} = \vec{\mathbf{A}}$$

Definition: Unit Vector

Dividing a vector by its magnitude results in a vector of unit length which we denote with a caret symbol

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{|\vec{\mathbf{A}}|}$$

Note that $|\hat{\mathbf{A}}| = |\vec{\mathbf{A}}|/|\vec{\mathbf{A}}| = 1$

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