20.6: Appendix 20A Chasles's Theorem- Rotation and Translation of a Rigid Body

We now return to our description of the translating and rotating rod that we first considered when we began our discussion of rigid bodies. We shall now show that the motion of any rigid body consists of a translation of the center of mass and rotation about the center of mass.

We shall demonstrate this for a rigid body by dividing up the rigid body into point-like constituents. Consider two point-like constituents with masses m_1 and m_2. Choose a coordinate system with a choice of origin such that body 1 has position $\overrightarrow{\mathbf{r}}_1$ and body 2 has position $\overrightarrow{\mathbf{r}}_2$ (Figure 20A.1). The relative position vector is given by

$\overrightarrow{\mathbf{r}}_{1,2}=\overrightarrow{\mathbf{r}}_1-\overrightarrow{\mathbf{r}}_2$

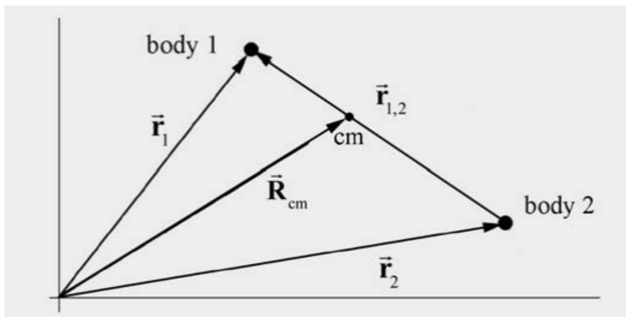

Figure 20A.1 Two-body coordinate system.

Recall we defined the center of mass vector, $\overrightarrow{\mathbf{R}}_{\mathrm{cm}}$ of the two-body system as

$$\overrightarrow{\mathbf{R}}_{cm}=\frac{m_1\overrightarrow{\mathbf{r}}_1+m_2\overrightarrow{\mathbf{r}}_2}{m_1+m_2}$$

In Figure 20A.2 we show the center of mass coordinate system.

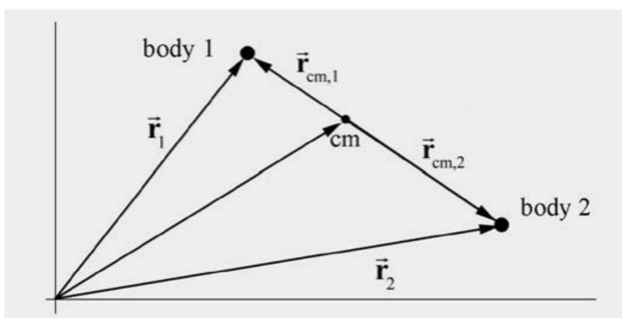

Figure 20A.2 Position coordinates with respect to center of mass

The position vector of the object 1 with respect to the center of mass is given by

$$\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}=\overrightarrow{\mathbf{r}}_1-\overrightarrow{\mathbf{R}}_{\mathrm{cm}}=\overrightarrow{\mathbf{r}}_1-\frac{m_1\overrightarrow{\mathbf{r}}_1+m_2\overrightarrow{\mathbf{r}}_2}{m_1+m_2}=\frac{m_2}{m_1+m_2}\left(\overrightarrow{\mathbf{r}}_1-\overrightarrow{\mathbf{r}}_2\right)=\frac{\mu}{m_1}\overrightarrow{\mathbf{r}}_{1,2}$$

where

$$\mu=\frac{m_1m_2}{m_1+m_2}$$

is the reduced mass. In addition, the relative position vector between the two objects is □ r independent of the choice of reference frame,

$$\overrightarrow{\mathbf{r}}_{12}=\overrightarrow{\mathbf{r}}_1-\overrightarrow{\mathbf{r}}_2=\left(\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}+\overrightarrow{\mathbf{R}}_{\mathrm{cm}}\right)-\left(\overrightarrow{\mathbf{r}}_{\mathrm{cm},2}+\overrightarrow{\mathbf{R}}_{\mathrm{cm}}\right)=\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}-\overrightarrow{\mathbf{r}}_{\mathrm{cm},2}=\overrightarrow{\mathbf{r}}_{\mathrm{cm},1,2}$$

Because the center of mass is at the origin in the center of mass reference frame,

$$\frac{m_1 \overrightarrow{\mathbf{r}}_{\mathrm{cm},1} + m_2 \overrightarrow{\mathbf{r}}_{\mathrm{cm},2}}{m_1 + m_2} = \overrightarrow{\mathbf{0}}$$

Therefore

$$m_1 \overrightarrow{\mathbf{r}}_{\mathrm{cm},1} = -m_2 \overrightarrow{\mathbf{r}}_{\mathrm{cm},2}$$

$$m_1 \left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}\right| = m_2 \left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},2}\right|$$

The displacement of object 1 about the center of mass is given by taking the derivative of Equation (20.A.3),

$$d\overrightarrow{\mathbf{r}}_{\mathrm{cm},1} = \frac{\mu}{m_1} d\overrightarrow{\mathbf{r}}_{1,2}$$

A similar calculation for the position of object 2 with respect to the center of mass yields for the position and displacement with respect to the center of mass

$$\overrightarrow{\mathbf{r}}_{\mathrm{cm},2} = \overrightarrow{\mathbf{r}}_2 - \overrightarrow{\mathbf{R}}_{\mathrm{cm}} = -\frac{\mu}{m_2} \overrightarrow{\mathbf{r}}_{1,2}$$

$$d\overrightarrow{\mathbf{r}}_{\mathrm{cm},2} = -\frac{\mu}{m_2} d\overrightarrow{\mathbf{r}}_{1,2}$$

Let $i = 1, 2$. An arbitrary displacement of the i^{th} object is given respectively by

$$d\overrightarrow{\mathbf{r}}_i = d\overrightarrow{\mathbf{r}}_{\mathrm{cm},i} + d\overrightarrow{\mathbf{R}}_{\mathrm{cm}}$$

which is the sum of a displacement about the center of mass $d\overrightarrow{\mathbf{r}}_{\mathrm{cm},i}$ and a displacement of the center of mass $d\overrightarrow{\mathbf{R}}_{\mathrm{cm}}$. The displacement of objects 1 and 2 are constrained by the condition that the distance between the objects must remain constant since the body is rigid. In particular, the distance between objects 1 and 2 is given by

$$\left|\overrightarrow{\mathbf{r}}_{1,2}\right|^2 = \left(\overrightarrow{\mathbf{r}}_1 - \overrightarrow{\mathbf{r}}_2\right) \cdot \left(\overrightarrow{\mathbf{r}}_1 - \overrightarrow{\mathbf{r}}_2\right)$$

Because this distance is constant we can differentiate Equation (20.A.13), yielding the rigid body condition that

$$0 = 2\left(\overrightarrow{\mathbf{r}}_1 - \overrightarrow{\mathbf{r}}_2\right) \cdot \left(d\overrightarrow{\mathbf{r}}_1 - d\overrightarrow{\mathbf{r}}_2\right) = 2\overrightarrow{\mathbf{r}}_{1,2} \cdot d\overrightarrow{\mathbf{r}}_{1,2}$$

20A.1. Translation of the Center of Mass

The condition (Equation (20.A.14)) can be satisfied if the relative displacement vector between the two objects is zero,

$$d\mathbf{r}_{1,2} = d\overrightarrow{\mathbf{r}}_1 - d\overrightarrow{\mathbf{r}}_2 = \overrightarrow{\mathbf{0}}$$

This implies, using, Equation (20.A.9) and Equation (20.A.11), that the displacement with respect to the center of mass is zero,

$$d\overrightarrow{\mathbf{r}}_{\mathrm{cm},1} = d\overrightarrow{\mathbf{r}}_{\mathrm{cm},2} = \overrightarrow{\mathbf{0}}$$

Thus by Equation (20.A.12), the displacement of each object is equal to the displacement of the center of mass,

$$d\overrightarrow{\mathbf{r}}_i = d\overrightarrow{\mathbf{R}}_{\mathrm{cm}}$$

which means that the body is undergoing pure translation.

20A.2 Rotation about the Center of Mass

Now suppose that $d\overrightarrow{\mathbf{r}}_{1,2} = d\overrightarrow{\mathbf{r}}_1 - d\overrightarrow{\mathbf{r}}_2 \neq \overrightarrow{\mathbf{0}}$ The rigid body condition can be expressed in terms of the center of mass coordinates. Using Equation (20.A.9), the rigid body condition (Equation (20.A.14)) becomes

$$0 = 2\frac{\mu}{m_1} \overrightarrow{\mathbf{r}}_{1,2} \cdot d\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}$$

Because the relative position vector between the two objects is independent of the choice of reference frame (Equation (20.A.5)), the rigid body condition Equation (20.A.14) in the center of mass reference frame is then given by

$$0=2\overrightarrow{\mathbf{r}}_{\mathrm{cm},1,2}\cdot d\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}$$

This condition is satisfied if the relative displacement is perpendicular to the line passing through the center of mass,

$$\overrightarrow{\mathbf{r}}_{\mathrm{cm},1,2}\perp d\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}$$

By a similar argument, $\overrightarrow{\mathbf{r}}_{\mathrm{cm},1,2}\perp d\overrightarrow{\mathbf{r}}_{\mathrm{cm},2}$. In order for these displacements to correspond to a rotation about the center of mass, the displacements must have the same angular displacement.

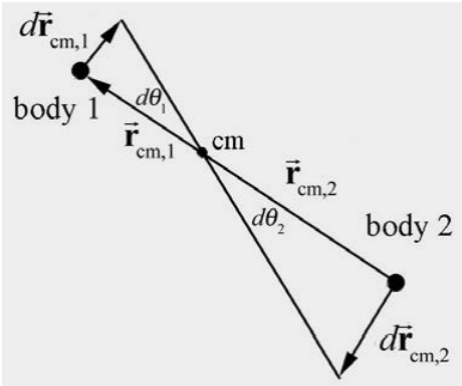

Figure 20A.3 Infinitesimal angular displacements in the center of mass reference frame

In Figure 20A.3, the infinitesimal angular displacement of each object is given by

$$d\theta_1=\frac{\left|d\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}\right|}{\left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}\right|}$$

$$d\theta_2=\frac{\left|d\overrightarrow{\mathbf{r}}_{\mathrm{cm},2}\right|}{\left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},2}\right|}$$

From Equation (20.A.9) and Equation (20.A.11), we can rewrite Equations (20.A.21) and (20.A.22) as

$$d\theta_1=\frac{\mu}{m_1}\frac{\left|d\overrightarrow{\mathbf{r}}_{1,2}\right|}{\left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}\right|}$$

$$d\theta_2=\frac{\mu}{m_2}\frac{\left|d\overrightarrow{\mathbf{r}}_{1,2}\right|}{\left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},2}\right|}$$

Recall that in the center of mass reference frame $m_1\left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},1}\right|=m_2\left|\overrightarrow{\mathbf{r}}_{\mathrm{cm},2}\right|$ (Equation (20.A.8)) and hence the angular displacements are equal,

$$d\theta_1=d\theta_2=d\theta$$

Therefore the displacement of the i^{th} object $d\overrightarrow{\mathbf{r}}_i$ differs from the displacement of the center of mass $d\overrightarrow{\mathbf{R}}_{\mathrm{cm}}$ by a vector that corresponds to an infinitesimal rotation in the center of mass reference frame

$$d\overrightarrow{\mathbf{r}}_i=d\overrightarrow{\mathbf{R}}_{\mathrm{cm}}+d\overrightarrow{\mathbf{r}}_{\mathrm{cm},i}$$

We have shown that the displacement of a rigid body is the vector sum of the displacement of the center of mass (translation of the center of mass) and an infinitesimal rotation about the center of mass.

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