# 20.3: Angular Momentum for a System of Particles Undergoing Translational and Rotational

We shall now show that the angular momentum of a body about a point $S$ can be decomposed into two vector parts, the angular momentum of the center of mass (treated as a point particle) about the point $S$, and the angular momentum of the rotational motion about the center of mass.

Consider a system of N particles located at the points labeled $i = 1, 2, \cdots, N$. The angular momentum about the point $S$ is the sum

$$\vec{\mathbf{L}}_S^{\text{total}} = \sum_{i=1}^{N} \vec{\mathbf{L}}_{S,i} = \left( \sum_{i=1}^{N} \vec{\mathbf{r}}_{S,i} \times m_i \vec{\mathbf{v}}_i \right)$$

where $\vec{\mathbf{r}}_{S,i}$ is the vector from the point $S$ to the $i^{th}$ particle (Figure 20.15) satisfying

$$\vec{\mathbf{r}}_{S,i} = \vec{\mathbf{r}}_{S,cm} + \vec{\mathbf{r}}_{\text{cm},i}$$

$$\vec{\mathbf{v}}_{S,i} = \vec{\mathbf{V}}_{cm} + \vec{\mathbf{v}}_{\text{cm},i}$$

where $\vec{\mathbf{v}}_{S,cm} = \vec{\mathbf{V}}_{cm}$ We can now substitute both Equations (20.3.2) and (20.3.3) into Equation (20.3.1) yielding

$$\vec{\mathbf{L}}_S^{\text{total}} = \sum_{i=1}^{N} \left( \vec{\mathbf{r}}_{S,cm} + \vec{\mathbf{r}}_{\text{cm},i} \right) \times m_i \left( \vec{\mathbf{V}}_{cm} + \vec{\mathbf{v}}_{\text{cm},i} \right)$$

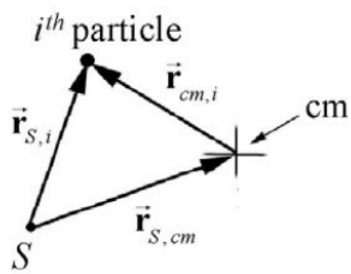


Figure 20.15 Vector Triangle

When we expand the expression in Equation (20.3.4), we have four terms,

$$\begin{aligned}\vec{\mathbf{L}}_S^{\text{total}} &= \sum_{i=1}^{N} \left( \vec{\mathbf{r}}_{S,cm} \times m \vec{\mathbf{v}}_{\text{cm},i} \right) + \sum_{i=1}^{N} \left( \vec{\mathbf{r}}_{S,cm} \times m_i \vec{\mathbf{V}}_{cm} \right) \\ &+ \sum_{i=1}^{N} \left( \vec{\mathbf{r}}_{\text{cm},i} \times m_i \vec{\mathbf{v}}_{\text{cm},i} \right) + \sum_{i=1}^{N} \left( \vec{\mathbf{r}}_{\text{cm},i} \times m_i \vec{\mathbf{V}}_{cm} \right)\end{aligned}$$

The vector $\vec{\mathbf{r}}_{S,cm}$ is a constant vector that depends only on the location of the center of mass and not on the location of the $i^{th}$ particle. Therefore in the first term in the above equation, $\vec{\mathbf{r}}_{S,cm}$ can be taken outside the summation. Similarly, in the second term the velocity of the center of mass $\vec{\mathbf{V}}_{cm}$ is the same for each term in the summation, and may be taken outside the summation,

$$\begin{aligned}\vec{\mathbf{L}}_S^{\text{total}} &= \vec{\mathbf{r}}_{S,cm} \times \left( \sum_{i=1}^{N} m_i \vec{\mathbf{v}}_{\text{cm},i} \right) + \vec{\mathbf{r}}_{S,cm} \times \left( \sum_{i=1}^{N} m_i \right) \vec{\mathbf{V}}_{cm} \\ &+ \sum_{i=1}^{N} \left( \vec{\mathbf{r}}_{\text{cm},i} \times m_i \vec{\mathbf{v}}_{\text{cm},i} \right) + \left( \sum_{i=1}^{N} m_i \vec{\mathbf{r}}_{\text{cm},i} \right) \times \vec{\mathbf{V}}_{cm}\end{aligned}$$

The first and third terms in Equation (20.3.6) are both zero due to the fact that

$$\sum_{i=1}^{N} m_i \vec{\mathbf{r}}_{\text{cm},i} = 0$$
$$\sum_{i=1}^{N} m_i \vec{\mathbf{v}}_{\text{cm},i} = 0$$

We first show that $\sum_{i=1}^{N} m_i \overrightarrow{\mathbf{r}}_{\mathrm{cm},i}$ is zero. We begin by using Equation (20.3.2),

$$\begin{aligned}\sum_{i=1}^{N}\left(m_i \overrightarrow{\mathbf{r}}_{\mathrm{cm},i}\right) &= \sum_{i=1}^{N}\left(m_i\left(\overrightarrow{\mathbf{r}}_i - \overrightarrow{\mathbf{r}}_{S,cm}\right)\right) \\ &= \sum_{i=1}^{N} m_i \overrightarrow{\mathbf{r}}_i - \left(\sum_{i=1}^{N}\left(m_i\right)\right)\overrightarrow{\mathbf{r}}_{S,cm} = \sum_{i=1}^{N} m_i \overrightarrow{\mathbf{r}}_i - m^{i\cot i|}\mathbf{r}_{S,cm}\end{aligned}$$

Substitute the definition of the center of mass (Equation 10.5.3) into Equation (20.3.8) yielding

$$\sum_{i=1}^{N}\left(m_i \overrightarrow{\mathbf{r}}_{\mathrm{cm},i}\right) = \sum_{i=1}^{N} m_i \overrightarrow{\mathbf{r}}_i - m^{total}\frac{1}{m^{total}}\sum_{i=1}^{N} m \overrightarrow{\mathbf{r}}_i = \overrightarrow{0}$$

The vanishing of $\sum_{i=1}^{N} m_i \overrightarrow{\mathbf{v}}_{\mathrm{cm},i} = 0$ follows directly from the definition of the center of mass

frame, that the momentum in the center of mass is zero. Equivalently the derivative of Equation (20.3.9) is zero. We could also simply calculate and find that

$$\begin{aligned}\sum_i m_i \overrightarrow{\mathbf{v}}_{\mathrm{cm},i} &= \sum_i m_i\left(\overrightarrow{\mathbf{v}}_i - \overrightarrow{\mathbf{V}}_{\mathrm{cm}}\right) \\ &= \sum_i m_i \overrightarrow{\mathbf{v}}_i - \overrightarrow{\mathbf{V}}_{\mathrm{cm}}\sum_i m_i \\ &= m^{\mathrm{total}}\overrightarrow{\mathbf{V}}_{\mathrm{cm}} - \overrightarrow{\mathbf{V}}_{\mathrm{cm}} m^{\mathrm{total}} \\ &= \overrightarrow{\mathbf{0}}\end{aligned}$$

We can now simplify Equation (20.3.6) for the angular momentum about the point $S$ using the fact that, $m_T = \sum_{i=1}^{N} m_i$ , and $\overrightarrow{\mathbf{p}}_{\mathrm{sys}} = m_T \overrightarrow{\mathbf{V}}_{cm}$ (in reference frame O ):

$$\overrightarrow{\mathbf{L}}_S^{\mathrm{total}} = \overrightarrow{\mathbf{r}}_{S,cm} \times \overrightarrow{\mathbf{p}}^{\mathrm{sys}} + \sum_{i=1}^{N}\left(\overrightarrow{\mathbf{r}}_{cm,i} \times m \overrightarrow{\mathbf{v}}_{cm,i}\right)$$

Consider the first term in Equation (20.3.11), $\overrightarrow{\mathbf{r}}_{S,\mathrm{cm}} \times \overrightarrow{\mathbf{p}}_{\mathrm{sys}}$ ; the vector $\overrightarrow{\mathbf{r}}_{S,\mathrm{cm}}$ is the vector from the point $S$ to the center of mass. If we treat the system as a point-like particle of mass $m_T$ located at the center of mass, then the momentum of this point-like particle is $\overrightarrow{\mathbf{p}}_{\mathrm{sss}} = m_T \overrightarrow{\mathbf{V}}_{\mathrm{cm}}$ Thus the first term is the angular momentum about the point $S$ of this “point-like particle”, which is called the orbital angular momentum about S,

$$\overrightarrow{\mathbf{L}}_S^{\mathrm{orbital}} = \overrightarrow{\mathbf{r}}_{S,\mathrm{cm}} \times \overrightarrow{\mathbf{p}}_{\mathrm{sys}}$$

for the system of particles.

Consider the second term in Equation (20.3.11), $\sum_{i=1}^{N}\left(\overrightarrow{\mathbf{r}}_{cm,i} \times m_i \overrightarrow{\mathbf{v}}_{cm,i}\right)$ ; the quantity inside the summation is the angular momentum of the $i^{th}$ particle with respect to the origin in the center of mass reference frame $O_{cm}$ (recall the origin in the center of mass reference frame is the center of mass of the system),

$$\overrightarrow{\mathbf{L}}_{cm,i} = \overrightarrow{\mathbf{r}}_{cm,i} \times m \overrightarrow{\mathbf{v}}_{cm,i}$$

Hence the total angular momentum of the system with respect to the center of mass in the center of mass reference frame is given by

$$\overrightarrow{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}} = \sum_{i=1}^{N}\overrightarrow{\mathbf{L}}_{cm,i} = \sum_{i=1}^{N}\left(\overrightarrow{\mathbf{r}}_{cm,i} \times m_i \overrightarrow{\mathbf{v}}_{cm,i}\right)$$

a vector quantity we call the spin angular momentum. Thus we see that the total angular momentum about the point $S$ is the sum of these two terms,

$$\overrightarrow{\mathbf{L}}_S^{+\mathrm{otal}} = \overrightarrow{\mathbf{L}}_S^{\mathrm{orbial}} + \overrightarrow{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}}$$

This decomposition of angular momentum into a piece associated with the translational motion of the center of mass and a second piece associated with the rotational motion about the center of mass in the center of mass reference frame is the key conceptual foundation for what follows.

## Example 20.5 Earth's Motion Around the Sun

The earth, of mass $m_{\mathrm{e}}=5.97 \times 10^{24}$ kg and (mean) radius $beginequation R_{\mathrm{e}}=6.38 \times 10^{6} \mathrm{~m}$ moves in a nearly circular orbit of radius $r_{\mathrm{s}, \mathrm{e}}=1.50 \times 10^{11} \mathrm{~m}$ around the sun with a period $\boxed{\text{T\_\{\textbackslash orbit\}\}=365.25 \textbackslash mathrm\{days\}}}$, and spins about its axis in a period $T_{\text{spin}}=23 \mathrm{hr} 56 \mathrm{min}$ the axis inclined to the normal to the plane of its orbit around the sun by $23.5^{\circ}$ (in Figure 20.16, the relative size of the earth and sun, and the radius and shape of the orbit are not representative of the actual quantities).

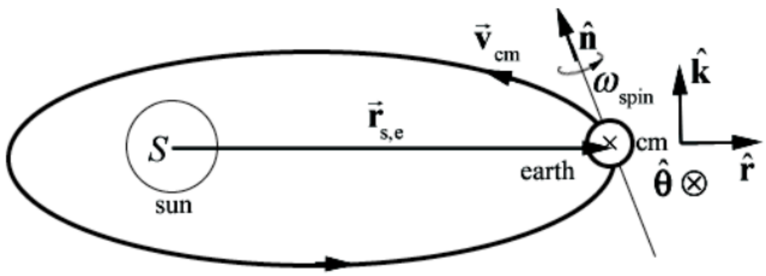


Figure 20.16 Example 20.5

If we approximate the earth as a uniform sphere, then the moment of inertia of the earth about its center of mass is

$$I_{\mathrm{cm}}=\frac{2}{5} m_{\mathrm{e}} R_{\mathrm{e}}^{2}$$

If we choose the point $S$ to be at the center of the sun, and assume the orbit is circular, then the orbital angular momentum is

$$\overrightarrow{\mathbf{L}}_{S}^{\text{orbital}}=\overrightarrow{\mathbf{r}}_{S, \mathrm{cm}} \times \overrightarrow{\mathbf{p}}_{\text{sys}}=r_{\mathrm{s}, \mathrm{e}} \hat{\mathbf{r}} \times m_{\mathrm{e}} v_{\mathrm{cm}} \hat{\boldsymbol{\theta}}=r_{\mathrm{s}, \mathrm{e}} m_{\mathrm{e}} v_{\mathrm{cm}} \hat{\mathbf{k}}$$

The velocity of the center of mass of the earth about the sun is related to the orbital angular velocity by

$$v_{\mathrm{cm}}=r_{\mathrm{s}, \mathrm{e}} \omega_{\text{orbit}}$$

where the orbital angular speed is

$$\begin{aligned}
\omega_{\text{otbit}} &=\frac{2 \pi}{T_{\text{otbit}}}=\frac{2 \pi}{(365.25 \mathrm{d})\left(8.640 \times 10^{4} \mathrm{s} \cdot \mathrm{d}^{-1}\right)} \\
&=1.991 \times 10^{-7} \mathrm{rad} \cdot \mathrm{s}^{-1}
\end{aligned}$$

The orbital angular momentum about $S$ is then

$$\begin{aligned}
\overrightarrow{\mathbf{L}}_{S}^{\text{objial}} &=m_{\mathrm{e}} r_{\mathrm{s}, \mathrm{e}}^{2} \omega_{\text{otit}} \hat{\mathbf{k}} \\
&=\left(5.97 \times 10^{24} \mathrm{kg}\right)\left(1.50 \times 10^{11} \mathrm{m}\right)^{2}\left(1.991 \times 10^{-7} \mathrm{rad} \cdot \mathrm{s}^{-1}\right) \hat{\mathbf{k}} \\
&=\left(2.68 \times 10^{40} \mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-1}\right) \hat{\mathbf{k}}
\end{aligned}$$

The spin angular momentum is given by

$$\overrightarrow{\mathbf{L}}_{\mathrm{cm}}^{\sin}=I_{\mathrm{cm}} \overrightarrow{\boldsymbol{\omega}}_{\text{spin}}=\frac{2}{5} m_{\mathrm{e}} R_{\mathrm{e}}^{2} \omega_{\text{spin}} \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit normal pointing along the axis of rotation of the earth and

$$\omega_{\text{spin}}=\frac{2 \pi}{T_{\text{spin}}}=\frac{2 \pi}{8.616 \times 10^{4} \mathrm{s}}=7.293 \times 10^{-5} \mathrm{rad} \cdot \mathrm{s}^{-1}$$

The spin angular momentum is then

$$\begin{aligned}\overrightarrow{\mathbf{L}}_{\mathrm{cm}}^{\mathrm{spin}} &= \frac{2}{5}\left(5.97 \times 10^{24}\mathrm{kg}\right)\left(6.38 \times 10^{6}\mathrm{m}\right)^{2}\left(7.293 \times 10^{-5}\mathrm{rad} \cdot \mathrm{s}^{-1}\right)\hat{\mathbf{n}} \\ &= \left(7.10 \times 10^{33}\mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-1}\right)\hat{\mathbf{n}}\end{aligned}$$

The ratio of the magnitudes of the orbital angular momentum about $S$ to the spin angular momentum is greater than a million,

$$\frac{L_{S}^{\text{orbial}}}{L_{\text{cm}}^{\text{spin}}} = \frac{m_{\mathrm{e}} r_{\mathrm{s,e}}^{2} \omega_{\text{orbit}}}{(2/5) m_{\mathrm{e}} R_{\mathrm{e}}^{2} \omega_{\text{spin}}} = \frac{5}{2} \frac{r_{\mathrm{s,e}}^{2}}{R_{\mathrm{e}}^{2}} \frac{T_{\text{spin}}}{T_{\text{orbit}}} = 3.77 \times 10^{6}$$

as this ratio is proportional to the square of the ratio of the distance to the sun to the radius of the earth. The angular momentum about $S$ is then

$$\overrightarrow{\mathbf{L}}_{S}^{\text{total}} = m_{\mathrm{e}} r_{\mathrm{s,e}}^{2} \omega_{\text{orbit}} \hat{\mathbf{k}} + \frac{2}{5} m_{\mathrm{e}} R_{\mathrm{e}}^{2} \omega_{\text{spin}} \hat{\mathbf{n}}$$

The orbit and spin periods are known to far more precision than the average values used for the earth's orbit radius and mean radius. Two different values have been used for one "day;" in converting the orbit period from days to seconds, the value for the solar day, $T_{\text{solar}} = 86,400\mathrm{s}$ was used. In converting the earth's spin angular frequency, the sidereal day, $T_{\text{sidereal}} = T_{\text{spin}} = 86,160\mathrm{s}$ was used. The two periods, the solar day from noon to noon and the sidereal day from the difference between the times that a fixed star is at the same place in the sky, do differ in the third significant figure.

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