25.10: Appendix 25C Analytic Geometric Properties of Ellipses

Consider Equation (25.3.20), and for now take $\varepsilon < 1$, so that the equation is that of an ellipse. We shall now show that we can write it as

$$\frac{\left(x-x_0\right)^2}{a^2}+\frac{y^2}{b^2}=1$$

where the ellipse has axes parallel to the x - and y -coordinate axes, center at $(x_0, 0)$, semi-major axis a and semi-minor axis b . We begin by rewriting Equation (25.3.20) as

$$x^2-\frac{2\varepsilon r_0}{1-\varepsilon^2}x+\frac{y^2}{1-\varepsilon^2}=\frac{r_0^2}{1-\varepsilon^2}$$

We next complete the square,

$$\begin{aligned}
&x^2-\frac{2\varepsilon r_0}{1-\varepsilon^2}x+\frac{\varepsilon^2 r_0^2}{\left(1-\varepsilon^2\right)^2}+\frac{y^2}{1-\varepsilon^2}=\frac{r_0^2}{1-\varepsilon^2}+\frac{\varepsilon^2 r_0^2}{\left(1-\varepsilon^2\right)^2} \Rightarrow \\
&\left(x-\frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2+\frac{y^2}{1-\varepsilon^2}=\frac{r_0^2}{\left(1-\varepsilon^2\right)^2} \Rightarrow \\
&\frac{\left(x-\frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2}{\left(r_0/\left(1-\varepsilon^2\right)\right)^2}+\frac{y^2}{\left(r_0/\sqrt{1-\varepsilon^2}\right)^2}=1
\end{aligned}$$

The last expression in (25.C.3) is the equation of an ellipse with semi-major axis

$$a=\frac{r_0}{1-\varepsilon^2}$$

semi-minor axis

$$b=\frac{r_0}{\sqrt{1-\varepsilon^2}}=a\sqrt{1-\varepsilon^2}$$

and center at

$$x_0=\frac{\varepsilon r_0}{\left(1-\varepsilon^2\right)}=\varepsilon a$$

as found in Equation (25.B.10).