6.6: Non-circular Central Motion

Let's now consider central motion in a plane that is non-circular. In Figure 6.10, we show the spiral motion of a moving particle. In polar coordinates, the key point is that the time derivative dr / dt of the position function r is no longer zero. The second derivative d^2r/dt^2 also may or may not be zero. In the following calculation we will drop all explicit references to the time dependence of the various quantities. The position vector is still given by Equation (6.2.1), which we shall repeat below

$$\overrightarrow{\mathbf{r}} = r\hat{\mathbf{r}}$$

Because $dr/dt \neq 0$ when we differentiate Equation (6.5.9), we need to use the product rule

$$\overrightarrow{\mathbf{v}} = \frac{d\overrightarrow{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}$$

Substituting Equation (6.2.4) into Equation (6.5.10)

$$\overrightarrow{\mathbf{v}} = \frac{d\overrightarrow{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}$$

The velocity is no longer tangential but now has a radial component as well

$$v_r = \frac{dr}{dt}$$

In order to determine the acceleration, we now differentiate Equation (6.5.11), again using the product rule, which is now a little more involved:

$$\overrightarrow{\mathbf{a}} = \frac{d\overrightarrow{\mathbf{v}}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt}$$

Now substitute Equations (6.2.4) and (6.2.7) for the time derivatives of the unit vectors in Equation (6.5.13), and after collecting terms yields

$$\begin{aligned} \overrightarrow{\mathbf{a}} &= \left(\tfrac{d^2r}{dt^2} - r\left(\tfrac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}} + \left(2\tfrac{dr}{dt}\tfrac{d\theta}{dt} + r\tfrac{d^2\theta}{dt^2}\right)\hat{\boldsymbol{\theta}} \\ &= a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} \end{aligned}$$

The radial and tangential components of the acceleration are now more complicated than then in the case of circular motion due to the non-zero derivatives of dr/dt and d^2r/dt^2. The radial component is

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

and the tangential component is

$$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

The firs term in the tangential component of the acceleration, $2(dr/dt)(d\theta/dt)$ has a special name, the coriolis acceleration,

$$a_{cor} = 2\frac{dr}{dt}\frac{d\theta}{dt}$$

Example 6.4 Spiral Motion

A particle moves outward along a spiral starting from the origin at t = 0 . Its trajectory is given by $r = b\theta$ where b is a positive constant with units $[\mathrm{m} \cdot \mathrm{rad}^{-1}] \cdot \theta$ increases in time according to $\theta = ct^2$, where $c > 0$ is a positive constant (with units $[\mathrm{rad} \cdot \mathrm{s}^{-2}]$)

a) Determine the acceleration as a function of time.

b) Determine the time at which the radial acceleration is zero.

c) What is the angle when the radial acceleration is zero?

d) Determine the time at which the radial and tangential accelerations have equal magnitude.

Solution:

a) The position coordinate as a function of time is given by $r = b\theta = bct^2$. The acceleration is given by Equation (6.5.14). In order to calculate the acceleration, we need to calculate the four derivatives $dr/dt = 2bct, d^2r/dt^2 = 2bc, d\theta/dt = 2ct$, and $d^2\theta/dt^2 = 2c$. The acceleration is then

$$\overrightarrow{\mathbf{a}} = \left(2bc - 4bc^3t^4\right)\hat{\mathbf{r}} + \left(8bc^2t^2 + 2bc^2t^2\right)\hat{\boldsymbol{\theta}} = \left(2bc - 4bc^3t^4\right)\hat{\mathbf{r}} + 10bc^2t^2\hat{\boldsymbol{\theta}}$$

b) The radial acceleration is zero when

$$t_1 = \left(\frac{1}{2c^2}\right)^{1/4}$$

c) The angle when the radial acceleration is zero is

$$\theta_1 = ct_1^2 = \sqrt{2}/2$$

d) The radial and tangential accelerations have equal magnitude when after some algebra

$$\left(2bc - 4bc^3t^4\right) = 10bc^2t^2 \Rightarrow 0 = t^4 + (5/2c)t^2 - \left(1/2c^2\right)$$

This equation has as only positive solution for t^2

$$t_2^2 = \frac{-(5/2c) \pm \left((5/2c)^2 + 2c^2\right)^{1/2}}{2} = \frac{\sqrt{33} - 5}{4c}$$

Therefore the magnitudes of the two components are equal when

$$t_2 = \sqrt{\frac{\sqrt{33} - 5}{4c}}$$
