9.4: Cavity resonators

9.4.1: Rectangular cavity resonators

Rectangular cavity resonators are hollow rectangular conducting boxes of width a, height b, and length d, where d ≥ a ≥ b by convention. Since they are simply rectangular waveguides terminated at both ends by conducting walls, and the electric fields must still obey the wave equation, $\left(\nabla^2+\omega^2 \mu \varepsilon\right) \underline{\overrightarrow{\mathrm{E}}}=0$, therefore $\underline{\overrightarrow{\mathrm{E}}}$ for TE modes must have the form of the TE waveguide fields (9.3.27), but with a sinusoidal z dependence that matches the boundary conditions at z = 0 and z = d; for example, equal forward- and backward-propagating waves would form the standing wave:

$$\underline{\overrightarrow{\mathrm{E}}}=\left(\underline{\mathrm{E}}_{\mathrm{o}} / \mathrm{k}_{\mathrm{o}}\right)\left(\hat{\mathrm{x}} \mathrm{k}_{\mathrm{y}} \sin \mathrm{k}_{\mathrm{y}} \mathrm{y} \cos \mathrm{k}_{\mathrm{x}} \mathrm{x}-\hat{\mathrm{y}} \mathrm{k}_{\mathrm{x}} \sin \mathrm{k}_{\mathrm{x}} \mathrm{x} \cos \mathrm{k}_{\mathrm{y}} \mathrm{y}\right)\left(\underline{\mathrm{A}} \sin \mathrm{k}_{\mathrm{z}} \mathrm{z}+\underline{\mathrm{B}} \cos \mathrm{k}_{\mathrm{z}} \mathrm{z}\right) \tag{9.4.1}$$

where B = 0 ensures $\overrightarrow{\mathrm{E}}_{//}=0$ at z = 0, and $k_z = p\pi/d$ ensures it for z = d, where p = 1, 2, ...

Unlike rectangular waveguides that propagate any frequency above cut-off for the spatial field distribution (mode) of interest, cavity resonators operate only at specific *resonant frequencies* or combinations of them in order to match all boundary conditions. The *resonant frequencies* ω_{mnp} for a rectangular cavity resonator follow from the dispersion relation:

$$\omega_{\mathrm{mnp}}^2 \mu \varepsilon=\mathrm{k}_{\mathrm{y}}^2+\mathrm{k}_{\mathrm{x}}^2+\mathrm{k}_{\mathrm{z}}^2=(\mathrm{m} \pi / \mathrm{a})^2+(\mathrm{n} \pi / \mathrm{b})^2+(\mathrm{p} \pi / \mathrm{d})^2 \tag{9.4.2}$$

$$\omega_{\mathrm{mnp}}=\left[(\mathrm{m} \pi \mathrm{c} / \mathrm{a})^2+(\mathrm{n} \pi \mathrm{c} / \mathrm{b})^2+(\mathrm{p} \pi \mathrm{c} / \mathrm{d})^2\right]^{0.5}\left[\mathrm{rs}^{-1}\right] \qquad \text{(cavity resonances)} \tag{9.4.3}$$

The *fundamental mode* for a cavity resonator is the lowest frequency mode. Since boundary conditions can not be met unless at least two of the quantum numbers m, n, and p are non-zero, the lowest resonant frequency is associated with the two longest dimensions, d and a. Therefore the lowest resonant frequency is:

$$\omega_{101}=\left[(\pi \mathrm{c} / \mathrm{a})^2+(\pi \mathrm{c} / \mathrm{d})^2\right]^{0.5}[\text { radians } / \mathrm{sec}] \qquad \text{(lowest resonance)} \tag{9.4.4}$$

Cavity resonators are therefore sometimes filled with dielectrics or magnetic materials to reduce their resonant frequencies by reducing c.

The fields for the fundamental mode of a rectangular cavity resonator, TE_{101}, follow from (9.4.1) and Faraday's law:

$$\underline{\overrightarrow{\mathrm{E}}}=\hat{x} \underline{\mathrm{E}}_0 \sin (\pi \mathrm{y} / \mathrm{a}) \sin (\pi z / \mathrm{d}) \qquad \text{(fundamental waveguide mode)} \tag{9.4.5}$$

$$\underline{\overrightarrow{\mathrm{H}}}=\mathrm{j} \underline{\mathrm{E}}_{\mathrm{o}}\left(\pi \omega \mathrm{c}^2 / \mathrm{n}\right)[\hat{y} \sin (\pi \mathrm{y} / \mathrm{a}) \cos (\pi \mathrm{z} / \mathrm{d}) / \mathrm{d}-\hat{z} \cos (\pi \mathrm{y} / \mathrm{a}) \sin (\pi \mathrm{z} / \mathrm{d}) / \mathrm{a}] \tag{9.4.6}$$

The total energy w [J] = $w_e(t) + w_m(t)$ in each mode m,n,p of a cavity resonator can be calculated using (2.7.28) and (2.7.29), and will decay exponentially at a rate that depends on total power dissipation P_d [W] due to losses in the walls and in any insulator filling the cavity interior:

$$\mathrm{w}(\mathrm{t}) \cong \mathrm{w}_{\mathrm{o}} \mathrm{e}^{-\mathrm{P}_{\mathrm{d}} \mathrm{t} / \mathrm{w}}=\mathrm{w}_{\mathrm{o}} \mathrm{e}^{-\omega \mathrm{t} / \mathrm{Q}} \tag{9.4.7}$$

Wall losses and any dissipation in insulators can be estimated by integrating (9.2.60) and (2.7.30), respectively, over the volume of the cavity resonator. The energy stored, power dissipation, and Q can be quite different for different modes, and are characterized by w_{mnp}, $P_{d,mnp}$, and Q_{mnp}, respectively, as defined by either (3.5.23) or (7.4.43):

$$\mathrm{Q}_{\mathrm{mnp}}=\omega \mathrm{w}_{\mathrm{mnp}} / \mathrm{P}_{\mathrm{d}_{\mathrm{mnp}}} \tag{9.4.8}$$

Example $9.4.A$

What are the lowest resonant frequency and its Q for a perfectly conducting metallic cavity of dimensions a, b, d if it is filled with a medium characterized by ε, μ, and σ. Assume Q >> 1.

Solution

The lowest resonant frequency ω_{101} is given by (9.4.4), where $c=(\mu \varepsilon)^{-0.5}$: $\omega_{101}=\pi(\mu \varepsilon)^{-0.5}\left(\mathrm{a}^{-2}+\mathrm{d}^{-2}\right)^{0.5}$. $\mathrm{Q}_{101}=\omega_{101} \mathrm{W}_{\mathrm{T} 101} / \mathrm{P}_{\mathrm{d} 101}$ where the total energy stored w_{T101} is twice the average electric energy stored since the total electric and magnetic energy storages are equal. At each point in the resonator the time-average electric energy density stored is

$\langle W_e \rangle = \varepsilon|\overrightarrow{\underline{E}}|^2/4\ \left[Jm^{-3}\right]$ and the time-average power dissipated is $\sigma|\overrightarrow{\underline{E}}|^2/2$, [W m^{-3}] so the electric-energy/dissipation density ratio everywhere is ε/2σ, and thus $w_{T101}/P_{d101} = \varepsilon/\sigma$, so $Q_{101} = \pi\varepsilon(\mu\varepsilon)^{-0.5}\left(a^{-2}+d^{-2}\right)^{0.5}/\sigma$.

9.4.2: Perturbation of resonator frequencies

Often we would like to tune a resonance to some nearby frequency. This can generally be accomplished by changing the shape of the resonator slightly. Although the relationship between shape and resonant frequency can be evaluated using Maxwell's equations, a simpler and more physical approach is taken here.

The energy stored in a resonator can be regarded as a population of N trapped photons at frequency f bouncing about inside. Since the energy E per photon is hf (1.1.10), the total energy in the resonator is:

$$w_T = Nhf\ [J] \tag{9.4.9}$$

If we force the walls of a resonator to move slowly toward its new shape, they will move either opposite to the forces imposed by the electromagnetic fields inside, or in the same direction, and thereby do positive or negative work, respectively, on those fields. If we do positive work, then the total electromagnetic energy w_T must increase. Since the number of photons remains constant if the shape change is slow compared to the frequency, positive work on the fields results in increased electromagnetic energy and frequency f. If the resonator walls move in the direction of the applied electromagnetic forces, the externally applied work on the fields is negative and the energy and resonant frequency decrease.

The paradigm above leads to a simple expression for the change in resonant frequency of any resonator due to small physical changes. Consider the case of an air-filled metallic cavity of any shape that is perturbed by pushing in or out the walls slightly in one or more places. The electromagnetic force on a conductor has components associated with both the attractive electric and repulsive magnetic pressures on conductors given by (4.1.15) and (4.1.23), respectively. For sinusoidal waves these pressures are:

$$P_e = -\varepsilon_o|\underline{E}_o|^2/4\ \left[Nm^{-2}\right] \qquad \text{(electric pressure)} \tag{9.4.10}$$

$$P_m = \mu_o|\underline{H}_o|^2/4\ \left[Nm^{-2}\right] \qquad \text{(magnetic pressure)} \tag{9.4.11}$$

But these pressures, except for the negative sign of P_e (corresponding to attraction), are the electric and magnetic energy densities [J m^{-3}].

The work Δw done in moving the cavity boundary slightly is the pressure $P_{e/m}$ applied, times the area over which it is applied, times the distance moved perpendicular to the boundary. For example, Δw equals the inward electromagnetic pressure (± energy density) times the increase in volume added by the moving boundary. But this increase in total stored electromagnetic energy is simply:

$$\Delta w_T = Nh\Delta f = -\left(P_e + P_m\right)\Delta v_{volume} = \Delta w_e - \Delta w_m \tag{9.4.12}$$

The signs for the increases in electric and magnetic energy storage Δw_e and Δw_m and pressures P_e and P_m are different because the pressures P_e and P_m are in opposite directions, where $\Delta w_e = W_e\Delta v_{ol}$, and $\Delta w_m = -P_m\Delta v_{ol} = -W_m\Delta v_{ol}$. Δw_e is defined as the electric energy stored in the increased volume of the cavity, Δv_{ol}, assuming the electric field strength remains constant as the wall moves slightly; Δw_m is defined similarly. The main restriction here is that the walls cannot be moved so far that the force density on the walls changes, nor can their shape change abruptly for the same reason. For example, a sharp point concentrates electric fields and would violate this constraint.

Dividing (9.4.12) by $w_T = Nhf$ yields the frequency perturbation equation:

$$\Delta w_T/w_T = \Delta f/f = \left(\Delta w_e - \Delta w_m\right)/w_T = \Delta v_{ol}\left(W_e - W_m\right)/w_T \qquad \text{(frequency perturbation)} \tag{9.4.13}$$

A simple example illustrates its use. Consider a rectangular cavity resonator operating in the TE_{101} mode with the fields given by (5.4.37) and (5.4.38). If we push in the center of the top or bottom of the cavity where $\overrightarrow{\underline{H}} \cong 0$ and $\overrightarrow{\underline{E}} \neq 0$ we are reducing the volume allocated to electric energy storage, so Δw_e is negative and the resonant frequency will drop in accord with (9.4.13). If we push in the sides, however, the resonant frequency will increase because we are reducing the volume where magnetic energy is stored and Δw_m is negative; the electric energy density at the sidewalls is zero. In physical terms, pushing in the top center where the electric fields pull inward on the wall means that those fields are doing work on the moving wall and therefore lose energy and frequency. Pushing in where the magnetic fields are pushing outward does work on the fields, increasing their energy and frequency. This technique can be used to determine experimentally the unknown resonant mode of a cavity as well as tuning it.