2.3: Maxwell's Equations, Waves, and Polarization in the Frequency Domain

2.3.1: Sinusoidal waves

Linear systems are easily characterized by the magnitude and phase of each output as a function of the frequency at which the input is sinusoidally stimulated. This simple characterization is sufficient because sinusoids of different frequencies can be superimposed to construct any arbitrary input waveform[5], and the output of a linear system is the superposition of its responses to each superimposed input. Systems with multiple inputs and outputs can be characterized in the same way. Nonlinear systems are more difficult to characterize because their output frequencies generally include harmonics of their inputs.

[5] The Fourier transform pair (10.4.17) and (10.4.18) relate arbitrary pulse waveforms h(t) to their corresponding spectra H(f), where each frequency f has its own magnitude and phase represented by H(f).

Fortunately free space is a linear system, and therefore it is fully characterized by its response to sinusoidal plane waves. For example, the arbitrary z-propagating x-polarized uniform plane wave of (2.2.9) and Figure 2.2.1 could be sinusoidal and represented by:

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{x} \mathrm{E}_{\mathrm{o}} \cos [\mathrm{k}(\mathrm{z}-\mathrm{ct})] \tag{2.3.1}$$

$$\overrightarrow{\mathrm{H}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{y} \sqrt{\varepsilon_{\mathrm{o}} / \mu_{\mathrm{o}}} \mathrm{E}_{\mathrm{o}} \cos [\mathrm{k}(\mathrm{z}-\mathrm{ct})] \tag{2.3.2}$$

where the *wave amplitude* E_o is a constant and the factor k is related to frequency, as shown below.

It is more common to represent sinusoidal waves using the argument (ωt - kz) so that their frequency and spatial dependences are more evident. The *angular frequency* ω is simply related to frequency f [Hz]:

$$\omega=2 \pi \mathrm{f}\left[\text { radians } \mathrm{s}^{-1}\right] \qquad \text{(angular frequency)} \tag{2.3.3}$$

and the *spatial frequency* k, often called the *wavenumber*, is simply related to ω and wavelength λ [m], which is the length of one period in space:

$$\mathrm{k}=2 \pi / \lambda=\omega / \mathrm{c}\left[\text { radians } \mathrm{m}^{-1}\right] \qquad \text{(wave number)} \tag{2.3.4}$$

The significance and dimensions of ω and k are directly analogous; they are radians s^{-1} and radians m^{-1}, respectively.

Therefore we can alternatively represent the wave of (2.3.1) and (2.3.2) as:

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{x} \mathrm{E}_{\mathrm{o}} \cos (\omega \mathrm{t}-\mathrm{kz})\left[\mathrm{vm}^{-1}\right] \tag{2.3.5}$$

$$\overrightarrow{\mathrm{H}}(\mathrm{z}, \mathrm{t})=\hat{y} \sqrt{\varepsilon_{\mathrm{o}} / \mu_{\mathrm{o}}} \mathrm{E}_{\mathrm{o}} \cos (\omega \mathrm{t}-\mathrm{kz})\left[\mathrm{Am}^{-1}\right] \tag{2.3.6}$$

Figure 2.3.1 suggests the form of this wave. Its *wavelength* is λ, the length of one cycle, where:

$$\lambda=\mathrm{c} / \mathrm{f}[\mathrm{m}] \qquad \text{(wavelength)} \tag{2.3.7}$$

The figure illustrates how these electric and magnetic fields are in phase but orthogonal to each other and to the direction of propagation. When the argument (ωt – kz) equals zero, the fields are maximum, consistent with cos(ωt - kz).

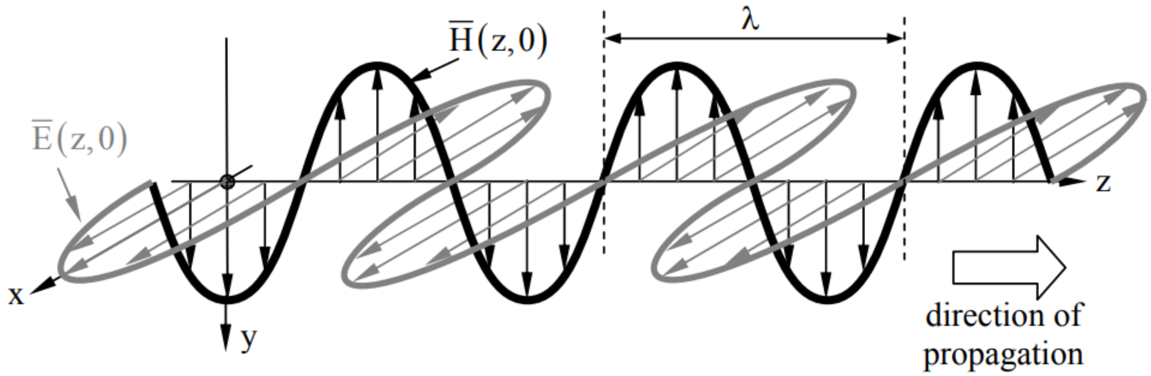

Figure 2.3.1: +z propagating y-polarized uniform plane wave of wavelength λ.

This notation makes it easy to characterize uniform plane waves propagating in other directions as well. For example:

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{x} \mathrm{E}_{\mathrm{o}} \cos (\omega \mathrm{t}+\mathrm{kz}) \qquad \text{(x -polarized wave in -z direction)} \tag{2.3.8}$$

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{y} \mathrm{E}_{\mathrm{o}} \cos (\omega \mathrm{t}-\mathrm{kz}) \qquad \text{(y -polarized wave in +z direction)} \tag{2.3.9}$$

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{y} \mathrm{E}_{\mathrm{o}} \cos (\omega \mathrm{t}-\mathrm{kx}) \qquad \text{(y-polarized wave in +x direction)} \tag{2.3.10}$$

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{z} \mathrm{E}_{\mathrm{o}} \cos (\omega \mathrm{t}+\mathrm{kx}) \qquad \text{(z-polarized wave in -x direction)} \tag{2.3.11}$$

2.3.2: Maxwell's equations in the complex-frequency domain

Electromagnetic fields are commonly characterized in the frequency domain in terms of their magnitudes and phases as a function of position $\vec{r}$ for frequency f. For example, the $\hat{x}$ component of a general sinusoidally varying $\vec{E}$ might be:

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{x} \mathrm{E}_{\mathrm{x}}(\overrightarrow{\mathrm{r}}) \cos [\omega \mathrm{t}+\phi(\overrightarrow{\mathrm{r}})] \tag{2.3.12}$$

This might become $\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\hat{x} \mathrm{E}_{\mathrm{x}} \cos (\omega \mathrm{t}-\mathrm{kz})$ for a uniform plane wave propagating in the +z direction.

It is generally more convenient to express phase using *complex notation* (see Appendix B). The x-component of the wave of (2.3.12) can also be represented as:

$$E_{x}(\overrightarrow{r}, t)=R_{e}\left\{E_{x}(\overrightarrow{r}) e^{j\left(\omega t+\phi_{x}(\overrightarrow{r})\right)}\right\}=\hat{x} R_{e}\left\{\underline{E}_{x}(\overrightarrow{r}) e^{j \omega t}\right\} \tag{2.3.13}$$

where the spatial and frequency parts of $E_{x}(\overrightarrow{r}, t)$ have been separated and $\underline{E}_{\mathbf{x}}(\overrightarrow{r})=\left|\underline{E}_{\mathbf{x}}(\overrightarrow{r})\right| e^{j \phi_{\mathrm{x}}(\overrightarrow{r})}$ is called a *phasor*. The simplicity will arise later when we omit $\mathrm{R}_{\mathrm{e}}\left\{\left[\quad\right] \mathrm{e}^{\mathrm{j} \omega \mathrm{t}}\right\}$ from our expressions as "understood", so only the phasors remain. The underoverline under $\underline{E}_{x}$ indicates $\underline{E}_{x}$ is not a function of time, but rather is a complex quantity with a real part and an imaginary part, where:

$$\underline{E}_{\mathbf{x}}(\overrightarrow{r})=R_{e}\left\{\underline{E}_{\mathbf{x}}(\overrightarrow{r})\right\}+j \mathrm{I}_{m}\left\{\underline{E}_{\mathrm{x}}(\overrightarrow{r})\right\}=\left|\underline{E}_{\mathbf{x}}(\overrightarrow{r})\right| \mathrm{e}^{j \phi_{\mathrm{x}}(\overrightarrow{r})} \tag{2.3.14}$$

and $\phi_{\mathrm{x}}(\vec{\mathrm{r}})=\tan ^{-1}\left(\operatorname{Im}\left\{\underline{\mathrm{E}}_{\mathrm{x}}(\vec{\mathrm{r}})\right\} / \operatorname{Re}\left\{\underline{\mathrm{E}}_{\mathrm{x}}(\vec{\mathrm{r}})\right\}\right)$. A general overlinetor can also be a phasor, e.g., $\underline{\overrightarrow{\mathrm{E}}}(\overrightarrow{\mathrm{r}})=\hat{x} \underline{\mathrm{E}}_{\mathbf{x}}(\overrightarrow{\mathrm{r}})+\hat{y} \underline{\mathrm{E}}_{\mathrm{y}}(\vec{\mathrm{r}})+\hat{z} \underline{\mathrm{E}}_{\mathrm{z}}(\vec{\mathrm{r}})$, where $\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\mathrm{R}_{\mathrm{e}}\left\{\underline{\overrightarrow{\mathrm{E}}}(\overrightarrow{\mathrm{r}}) \mathrm{e}^{\mathrm{j} \omega \mathrm{t}}\right\}$.

We can use such phasors to simplify Maxwell's equations. For example, we can express Faraday's law (2.2.1) as:

$$\nabla \times \mathrm{R}_{\mathrm{e}}\left\{\underline{\overrightarrow{\mathrm{E}}}(\overrightarrow{r}) \mathrm{e}^{\mathrm{j} \omega \mathrm{t}}\right\}=-\partial \mathrm{R}_{\mathrm{e}}\left\{\underline{\overrightarrow{\mathrm{B}}}(\mathrm{r}) \mathrm{e}^{\mathrm{j} \omega \dagger}\right\} / \partial \mathrm{t}=\mathrm{R}_{\mathrm{e}}\left\{\nabla \times \underline{\overrightarrow{\mathrm{E}}}(\overrightarrow{\mathrm{r}}) \mathrm{e}^{\mathrm{j} \omega \mathrm{t}}\right\}=\mathrm{R}_{\mathrm{e}}\left\{-\mathrm{j} \omega \underline{\overrightarrow{\mathrm{B}}}(\overrightarrow{\mathrm{r}}) \mathrm{e}^{\mathrm{j} \omega \mathrm{t}}\right\} \tag{2.3.15}$$

The other Maxwell equations can be similarly transformed, which suggests that the notation $R_e\{[\ \ ]e^{j\omega t}\}$ can be omitted and treated as understood. For example, removing this redundant notation from (2.3.15) results in: $\nabla\times\underline{\vec{E}}=-j\omega\underline{\vec{B}}$. Any problem solution expressed as a phasor, e.g. $\underline{\vec{E}}(\vec{r})$, can be converted back into a time-domain expression by the operator $R_e\{[\ \ ]e^{j\omega t}\}$. These omissions of the understood notation result in the complex or *time-harmonic Maxwell equations*:

$$\nabla\times\underline{\vec{E}}=-j\omega\underline{\vec{B}} \qquad \text{(Faraday's law)} \tag{2.3.16}$$

$$\nabla\times\underline{\vec{H}}=\underline{\vec{J}}+j\omega\underline{\vec{D}} \qquad \text{(Ampere's law)} \tag{2.3.17}$$

$$\nabla\bullet\underline{\vec{D}}=\underline{\rho} \qquad \text{(Gauss's law)} \tag{2.3.18}$$

$$\nabla\bullet\underline{\vec{B}}=0 \qquad \text{(Gauss's law)} \tag{2.3.19}$$

Note that these equations are the same as before [i.e., (2.2.1–4)], except that we have simply replaced the operator $\partial/\partial t$ with $j\omega$ and placed an underoverline under all variables, signifying that they are now phasors.

We can immediately derive the time-harmonic equation for conservation of charge (2.1.19) by computing the divergence of (2.3.17), noting that $\nabla\bullet(\nabla\times\vec{A})=0$ for any $\vec{A}$, and substituting $\nabla\bullet\underline{\vec{D}}=\rho$ (2.3.18):

$$\nabla\bullet\underline{\vec{J}}+j\omega\underline{\rho}=0 \tag{2.3.20}$$

Example $2.3.A$

Convert the following expressions into their time-domain equivalents: $j\omega\nabla\times\vec{Q}=\vec{R}j$, $\vec{R}e^{-jkz}$, and $\underline{\vec{E}}=\hat{x}3+\hat{y}\mathbf{j}4$.

Solution

$-\omega(\nabla\times\vec{Q})\sin(\omega t)=-\vec{R}\sin\omega t$, $\vec{R}\cos(\omega t-kz)$, and $3\hat{x}\cos\omega t-4\hat{y}\sin\omega t$.

Example $2.3.B$

Convert the following expressions into their complex frequency-domain equivalents: Acos (ω +t kz), and Bsin (ω +t φ).

Solution

Ae^{+jkz} , and $-jBe^{j\varphi}=-jB\cos\varphi+B\sin\varphi$.

2.3.3: Sinusoidal uniform plane waves

We can readily derive from Maxwell's equations the time-harmonic Helmholtz wave equation for vacuum (2.2.7) by substituting $j\omega$ for $\partial/\partial t$ or, as we did earlier, by taking the curl of Faraday's law, using the well known overlinetor identity (2.2.6) and Gauss's law, replacing $\underline{\vec{B}}$ by $\mu_o\underline{\vec{H}}$, and using Ampere's law to replace $\nabla\times\underline{\vec{H}}$. In both cases the Helmholtz wave equation becomes:

$$\left(\nabla^2+\omega^2\mu_o\varepsilon_o\right)\underline{\vec{E}}=0 \qquad \text{(wave equation)} \tag{2.3.21}$$

As before, the solution $\underline{\vec{E}}(\vec{r})$ to the wave equation can be any arbitrary function of space $(\vec{r})$ such that its second spatial derivative $\left(\nabla^2\underline{\vec{E}}\right)$ equals a constant $\left(-\omega^2\varepsilon_o\mu_o\right)$ times that same function $\underline{E}(\vec{r})$. One solution with these properties is the time-harmonic version of the timedomain expression $\vec{E}(\vec{r},t)=\hat{y}E_o\cos(\omega t-kz)$:

$$\underline{\vec{E}}(\vec{r})=\hat{y}E_oe^{-jkz}\left[vm^{-1}\right] \tag{2.3.22}$$

Substituting (2.3.22) into the wave equation (2.3.21) yields:

$$\left(\left[\partial^2/\partial z^2\right]+\omega^2\mu_o\varepsilon_o\right)\underline{\vec{E}}=\left([-jk]^2+\omega^2\mu_o\varepsilon_o\right)\underline{\vec{E}}=0 \tag{2.3.23}$$

which is satisfied if the wavenumber k is:

$$\mathrm{k}=\omega\sqrt{\mu_{\mathrm{o}}\varepsilon_{\mathrm{o}}}=\frac{\omega}{\mathrm{c}}=\frac{2\pi \mathrm{f}}{\mathrm{c}}=\frac{2\pi}{\lambda}\left[\text{radians m}^{-1}\right] \tag{2.3.24}$$

It is now an easy matter to find the magnetic field that corresponds to (2.3.22) by using Faraday's law (2.3.16), $\underline{\overrightarrow{\mathrm{B}}}=\mu_{\mathrm{o}}\underline{\overrightarrow{\mathrm{H}}}$, and the definition of the "∇×" operator (2.1.1):

$$\begin{aligned}\underline{\overrightarrow{\mathrm{H}}}(\overrightarrow{\mathrm{r}})&=-\frac{(\nabla\times\underline{\overrightarrow{\mathrm{E}}})}{\mathrm{j}\omega\mu_{\mathrm{o}}}=\frac{1}{\mathrm{j}\omega\mu_{\mathrm{o}}}\frac{\hat{x}\partial\mathrm{E}_{\mathrm{y}}}{\partial\mathrm{z}}=-\frac{\hat{x}\mathrm{k}\mathrm{E}_{\mathrm{o}}\mathrm{e}^{-\mathrm{jkz}}}{\omega\mu_{\mathrm{o}}}\\&=-\hat{x}\frac{1}{\eta_{\mathrm{o}}}\mathrm{E}_{\mathrm{o}}\mathrm{e}^{-\mathrm{jkz}}\left[\mathrm{Am}^{-1}\right]\end{aligned} \tag{2.3.25}$$

As before, $\underline{\overrightarrow{\mathrm{E}}}$ and $\underline{\overrightarrow{\mathrm{H}}}$ are orthogonal to each other and to the direction of propagation, and $|\underline{\overrightarrow{\mathrm{E}}}|=\eta_{\mathrm{o}}|\underline{\overrightarrow{\mathrm{H}}}|$.

As another example, consider a z-polarized wave propagating in the -x direction; then:

$$\underline{\overrightarrow{\mathrm{E}}}(\overrightarrow{\mathrm{r}})=\hat{z}\mathrm{E}_{\mathrm{o}}\mathrm{e}^{+\mathrm{jkx}},\ \underline{\overrightarrow{\mathrm{H}}}(\overrightarrow{\mathrm{r}})=\hat{y}\mathrm{E}_{\mathrm{o}}\mathrm{e}^{\mathrm{jkx}}/\eta_{\mathrm{o}} \tag{2.3.26}$$

It is easy to convert phasor expressions such as (2.3.26) into time-domain expressions. We simply divide the phasor expressions into their real and imaginary parts, and note that the real part varies as cos(ωt - kz) and the imaginary part varies as sin(ωt - kz). Thus the fields in (2.3.22) could be written instead as a real time-domain expression:

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}},\mathrm{t})=\hat{y}\mathrm{E}_{\mathrm{o}}\cos(\omega\mathrm{t}-\mathrm{kz}) \tag{2.3.27}$$

Had the electric field solution been instead the phasor $\hat{y}_j\mathrm{E}_{\mathrm{o}}\mathrm{e}^{-\mathrm{jkz}}$, the time domain expression $\mathrm{R}_{\mathrm{e}}\left\{\underline{\overrightarrow{\mathrm{E}}}(\overrightarrow{\mathrm{r}})\mathrm{e}^{\mathrm{j}\omega\mathrm{t}}\right\}$ would then be:

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}},\mathrm{t})=-\hat{y}\mathrm{E}_{\mathrm{o}}\sin(\omega\mathrm{t}-\mathrm{kz}) \tag{2.3.28}$$

The conversion of complex phasors to time-domain expressions, and vice-versa, is discussed further in Appendix B.

2.3.4: Wave polarization

Complex notation simplifies the representation of wave polarization, which characterizes the behavior of the sinusoidally varying electric field overlinetor as a function of time. It is quite distinct from the polarization of media discussed in Section 2.5.3. Previously we have seen waves for which the time-varying electric overlinetor points only in the ±x, ±y, or ±z directions, corresponding to x, y, or z polarization, respectively. By superimposing such waves at the same frequency and propagating in the same direction we can obtain any other desired time-harmonic polarization. Linear polarization results when the oscillating electric overlinetor points only along a single direction in the plane perpendicular to the direction of propagation, while elliptical polarization results when the x and y components of the electric overlinetor are out of phase so that the tip of the electric overlinetor traces an ellipse in the same plane. Circular polarization results only when the phase difference between x and y is 90 degrees and the two amplitudes are equal. These various polarizations for $+\hat{z}$ propagation are represented below at z = 0 in the time domain and as phasors, and in Figure 2.3.2.

$$\overrightarrow{\mathrm{E}}(\mathrm{t})=\hat{y}\mathrm{E}_{\mathrm{o}}\cos\omega\mathrm{t}\qquad\underline{\overrightarrow{\mathrm{E}}}=\hat{y}\mathrm{E}_{\mathrm{o}}\qquad\text{(y-polarized)} \tag{2.3.29}$$

$$\overrightarrow{\mathrm{E}}(\mathrm{t})=\hat{x}\mathrm{E}_{\mathrm{o}}\cos\omega\mathrm{t}\qquad\underline{\overrightarrow{\mathrm{E}}}=\hat{x}\mathrm{E}_{\mathrm{o}}\qquad\text{(x-polarized)} \tag{2.3.30}$$

$$\overrightarrow{\mathrm{E}}(\mathrm{t})=(\hat{x}+\hat{y})\mathrm{E}_{\mathrm{o}}\cos\omega\mathrm{t}\qquad\underline{\overrightarrow{\mathrm{E}}}=(\hat{x}+\hat{y})\mathrm{E}_{\mathrm{o}}\qquad\text{(45°-polarized)} \tag{2.3.31}$$

$$\overrightarrow{\mathrm{E}}(\mathrm{t})=\mathrm{E}_{\mathrm{o}}(\hat{x}\cos\omega\mathrm{t}+\hat{y}\sin\omega\mathrm{t})\qquad\underline{\overrightarrow{\mathrm{E}}}=(\hat{x}-\mathrm{j}\hat{y})\mathrm{E}_{\mathrm{o}}\qquad\text{(right-circular)} \tag{2.3.32}$$

$$\overrightarrow{\mathrm{E}}(\mathrm{t})=\mathrm{E}_{\mathrm{o}}(\hat{x}\cos\omega\mathrm{t}+1.5\hat{y}\sin\omega\mathrm{t})\qquad\underline{\overrightarrow{\mathrm{E}}}=(\hat{x}-1.5\mathrm{j}\hat{y})\mathrm{E}_{\mathrm{o}}\qquad\text{(right-elliptical)} \tag{2.3.33}$$

$$\overrightarrow{\mathrm{E}}(\mathrm{t})=\mathrm{E}_{\mathrm{o}}\left[\hat{x}\cos\omega\mathrm{t}+\hat{y}\cos(\omega\mathrm{t}+20^{\circ})\right]\qquad\underline{\overrightarrow{\mathrm{E}}}=\left(\hat{x}+\mathrm{e}^{0.35\mathrm{j}}\hat{y}\right)\mathrm{E}_{\mathrm{o}}\qquad\text{(left-elliptical)} \tag{2.3.34}$$

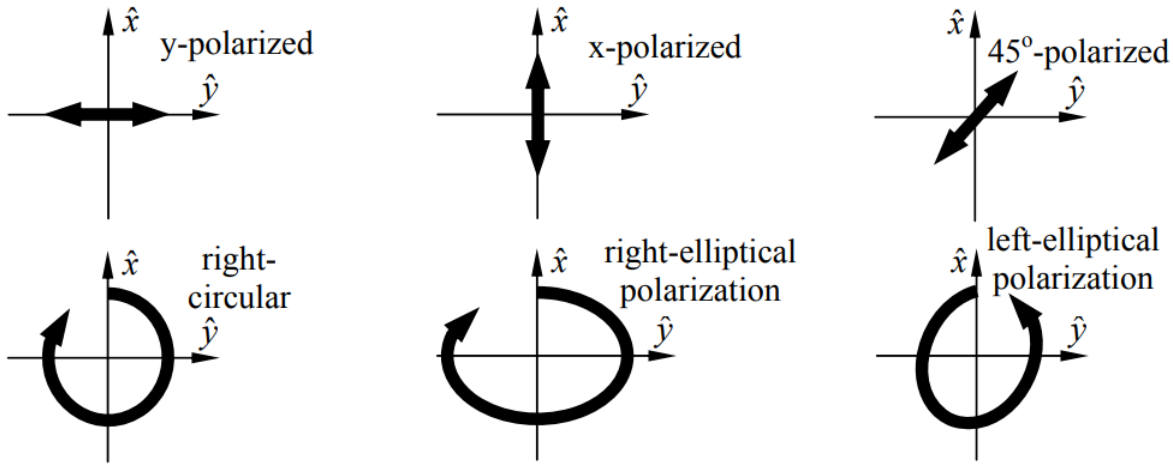

Figure 2.3.2: Polarization ellipses for +z-propagating plane waves (into the page).

The Institute of Electrical and Electronics Engineers (IEEE) has defined polarization as right-handed if the electric overlinetor traces a right-handed ellipse in the x-y plane for a wave propagating in the +z direction, as suggested in Figure 2.3.3. That is, for *right-handed polarization* the fingers of the right hand circle in the direction taken by the electric overlinetor while the thumb points in the direction of propagation. This legal definition is opposite that commonly used in physics, where that alternative definition is consistent with the handedness of the "screw" formed by the instantaneous three-dimensional loci of the tips of the electric overlinetors comprising a wave.

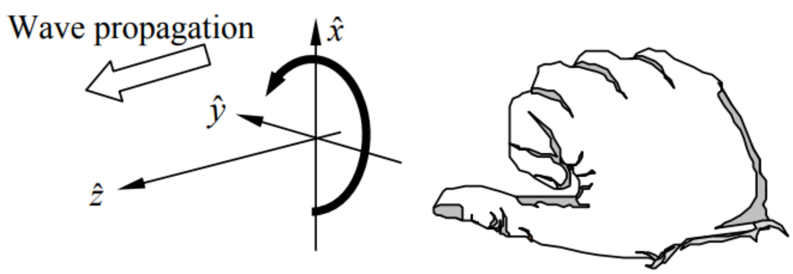

Figure 2.3.3: IEEE definition of right-handed polarization.

Example 2.3.C

If $\overrightarrow{\underline{E}} = \overrightarrow{\underline{E}}_o e^{-jkz}$, what polarizations correspond to: $\overrightarrow{\underline{E}}_o = \hat{y}$, $\overrightarrow{\underline{E}}_o = \hat{x} + 2\hat{y}$, and $\overrightarrow{\underline{E}}_o = \hat{x} - j\hat{y}$? ?

Solution

y polarization, linear polarization at angle $\tan^{-1}2$ relative to the x-z plane, and rightcircular polarization.