13.1: Acoustic Waves

13.1.1: Introduction

Wave phenomena are ubiquitous, so the wave concepts presented in this text are widely relevant. Acoustic waves offer an excellent example because of their similarity to electromagnetic waves and because of their important applications. Beside the obvious role of acoustics in microphones and loudspeakers, surface-acoustic-wave (SAW) devices are used as radio-frequency (RF) filters, acoustic-wave modulators diffract optical beams for real-time spectral analysis of RF signals, and mechanical crystal oscillators currently control the timing of most computers and clocks. Because of the great similarity between acoustic and electromagnetic phenomena, this chapter also reviews much of electromagnetics from a different perspective.

Section 13.1.2 begins with a simplified derivation of the two main differential equations that characterize linear acoustics. This pair of equations can be combined to yield the acoustic wave equation. Only longitudinal acoustic waves are considered here, not transverse or “shear” waves. These equations quickly yield the group and phase velocities of sound waves, the acoustic impedance of media, and an acoustic Poynting theorem. Section 13.2.1 then develops the acoustic boundary conditions and the behavior of acoustic waves at planar interfaces, including an acoustic Snell’s law, Brewster’s angle, the critical angle, and evanescent waves. Section 13.2.2 shows how acoustic plane waves can travel within pipes and be guided and manipulated much as plane waves can be manipulated within TEM transmission lines.

Acoustic waves can be totally reflected at firm boundaries, and Section 13.2.3 explains how they can be trapped and guided in a variety of propagation modes closely resembling those in electromagnetic waveguides, where they exhibit cutoff frequencies of propagation and evanescence below cutoff. Section 13.2.4 then explains how these guides can be terminated at their ends with open or closed orifices, thus forming resonators with Q’s that can be controlled as in electromagnetic resonators so as to yield band-stop or band-pass filters. The frequencies of acoustic resonances can be perturbed by distorting the shape of the cavity, as governed by nearly the same equation used for electromagnetic resonators except that the electromagnetic energy densities are replaced by acoustic energy density expressions. Section 13.3 discusses acoustic radiation and antennas, including antenna arrays, and Section 13.4 concludes the chapter with a brief introduction to representative electroacoustic devices.

13.1.2: Acoustic waves and power

Most waves other than electromagnetic waves involve perturbations. For example, acoustic waves involve perturbations in the pressure and velocity fields in gases, liquids, or solids. In gases we may express the total pressure $\mathrm{p_T}$, density ρ_T, and velocity $\vec{\mathrm{u}}_\mathrm{T}$ fields as the sum of a static component and a dynamic perturbation:

$$\mathrm{p_T}(\overrightarrow{\mathrm{r}}, \mathrm{t}) = \mathrm{P_o} + \mathrm{p}(\overrightarrow{\mathrm{r}}, \mathrm{t}) \ \left[\mathrm{N/m^2}\right] \tag{13.1.1}$$

$$\rho_\mathrm{T}(\overrightarrow{\mathrm{r}}, \mathrm{t}) = \rho_\mathrm{o} + \rho(\overrightarrow{\mathrm{r}}, \mathrm{t}) \ \left[\mathrm{kg/m^3}\right] \tag{13.1.2}$$

$$\overrightarrow{\mathrm{u}}(\overrightarrow{\mathrm{r}}, \mathrm{t}) = \overrightarrow{\mathrm{U}}_\mathrm{o} + \overrightarrow{\mathrm{u}}(\overrightarrow{\mathrm{r}}, \mathrm{t}) \ [\mathrm{m/s}] \tag{13.1.3}$$

Another complexity is that, unlike electromagnetic variables referenced to a particular location, gases move and compress, requiring further linearization.[73] Most important is the approximation that the mean velocity $\overrightarrow{\mathrm{U}}_\mathrm{o} = 0$. After these simplifying steps we are left with two linearized acoustic equations, *Newton’s law* (f = ma) and c*onservation of mass*:

$$\nabla \mathrm{p} \cong -\rho_\mathrm{o} \partial \overrightarrow{\mathrm{u}} / \partial \mathrm{t} \ \left[\mathrm{N/m^3}\right] \qquad \text{(Newton's law)} \tag{13.1.4}$$

$$\rho_\mathrm{o} \nabla \bullet \overrightarrow{\mathrm{u}} + \partial \rho / \partial \mathrm{t} \cong 0 \ \left[\mathrm{kg/m^3 s}\right] \qquad \text{(conservation of mass)} \tag{13.1.5}$$

[73] The Liebnitz identity facilitates taking time derivatives of integrals over volumes deforming in time.

Newton’s law states that the pressure gradient will induce mass acceleration, while conservation of mass states that velocity divergence $\nabla \bullet \overrightarrow{\mathrm{u}}$ is proportional to the negative time derivative of mass density.

These two basic equations involve three key variables: p, $\overrightarrow{\mathrm{u}}$, and ρ; we need the acoustic constitutive relation to reduce this set to two variables. Most acoustic waves involve frequencies sufficiently high that the heating produced by wave compression has no time to escape by conduction or radiation, and thus this heat energy returns to the wave during the subsequent expansion without significant loss. Such *adiabatic processes* involve no heat transfer across populations of particles. The resulting adiabatic *acoustic* *constitutive relation* states that the fractional change in density equals the fractional change in pressure, divided by a constant γ, called the adiabatic exponent:

$$\partial \rho / \partial p=\rho_{\mathrm{o}} / \gamma \mathrm{P}_{\mathrm{o}} \tag{13.1.6}$$

The reason γ is not unity is that gas heats when compressed, which further increases the pressure, so the gas thereby appears to be slightly "stiffer" or more resistant to compression than otherwise. This effect is diminished for gas particles that have internal rotational or vibrational degrees of freedom so the temperature rises less upon compression. Ideal monatomic molecules without such degrees of freedom exhibit $\gamma = 5/3$, and $1 < \gamma < 2$, in general.

Substituting this constitutive relation into the mass equation (13.1.5) replaces the variable ρ with p, yielding the acoustic differential equations:

$$\nabla \mathrm{p} \cong-\rho_{\mathrm{o}} \partial \overrightarrow{\mathrm{u}} / \partial \mathrm{t}\ \left[\mathrm{N} / \mathrm{m}^{3}\right] \qquad \text{(Newton's law)} \tag{13.1.7}$$

$$\nabla \bullet \overrightarrow{\mathrm{u}}=-\left(1 / \gamma \mathrm{P}_{\mathrm{o}}\right) \partial \mathrm{p} / \partial \mathrm{t} \tag{13.1.8}$$

These two differential equations are roughly analogous to Maxwell's equations (2.1.5) and (2.1.6), and can be combined. To eliminate u from Newton's law we operate on it with (∇ •), and then substitute (13.1.8) for $\nabla \bullet \overrightarrow{\mathrm{u}}$ to form the *acoustic wave equation*, analogous to the Helmholtz wave equation (2.2.7):

$$\nabla^{2} \mathrm{p}-\left(\rho_{\mathrm{o}} / \gamma \mathrm{P}_{\mathrm{o}}\right) \partial^{2} \mathrm{p} / \partial \mathrm{t}^{2}=0 \qquad \text{(acoustic wave equation)} \tag{13.1.9}$$

Wave equations state that the second spatial derivative equals the second time derivative times a constant. If the constant is not frequency dependent, then any arbitrary function of an argument that is the sum or difference of terms linearly proportional to time and space will satisfy this equation; for example:

$$\mathrm{p}(\overrightarrow{\mathrm{r}}, \mathrm{t})=\mathrm{p}(\omega \mathrm{t}-\overrightarrow{\mathrm{k}} \bullet \overrightarrow{\mathrm{r}})\ \left[\mathrm{N} / \mathrm{m}^{2}\right] \tag{13.1.10}$$

where p(•) is an arbitrary function of its argument (•), and $\overrightarrow{\mathrm{k}}=\mathrm{k}_{\mathrm{x}} \hat{x}+\mathrm{k}_{\mathrm{y}} \hat{y}+\mathrm{k}_{\mathrm{z}} \hat{z}$; this is analogous to the wave solution (9.2.4) using the notation (9.2.5). Substituting the solution (13.1.10) into the wave equation yields:

$$\left(\partial^{2} / \partial \mathrm{x}^{2}+\partial^{2} / \partial \mathrm{y}^{2}+\partial^{2} / \partial \mathrm{z}^{2}\right) \mathrm{p}(\omega \mathrm{t}-\overrightarrow{\mathrm{k}} \bullet \vec{\mathrm{r}})-\left(\rho_{\mathrm{o}} / \gamma \mathrm{P}_{\mathrm{o}}\right) \partial^{2} \mathrm{p}(\omega \mathrm{t}-\overrightarrow{\mathrm{k}} \bullet \vec{\mathrm{r}}) / \partial \mathrm{t}^{2}=0 \tag{13.1.11}$$

$$-\left(\mathrm{k}_{\mathrm{x}}^{2}+\mathrm{k}_{\mathrm{y}}^{2}+\mathrm{k}_{\mathrm{z}}^{2}\right) \mathrm{p}^{\prime \prime}(\omega \mathrm{t}-\overrightarrow{\mathrm{k}} \bullet \overrightarrow{\mathrm{r}})-\left(\rho_{\mathrm{o}} / \gamma \mathrm{P}_{\mathrm{o}}\right) \omega^{2} \mathrm{p}^{\prime \prime}(\omega \mathrm{t}-\overrightarrow{\mathrm{k}} \bullet \overrightarrow{\mathrm{r}})=0 \tag{13.1.12}$$

$$\mathrm{k}_{\mathrm{x}}^{2}+\mathrm{k}_{\mathrm{y}}^{2}+\mathrm{k}_{\mathrm{z}}^{2}=\mathrm{k}^{2}=\omega^{2} \rho_{\mathrm{o}} / \gamma \mathrm{P}_{\mathrm{o}}=\omega^{2} / \mathrm{v}_{\mathrm{p}}^{2} \tag{13.1.13}$$

This is analogous to the electromagnetic dispersion relation (9.2.8).

As in the case of electromagnetic waves [see (9.5.19) and (9.5.20)], the *acoustic phase velocity* v_p and *acoustic group velocity* v_g are simply related to k:

$$\mathrm{v}_{\mathrm{p}}=\omega / \mathrm{k}=\left(\gamma \mathrm{P}_{\mathrm{o}} / \rho_{\mathrm{o}}\right)^{0.5}=\mathrm{c}_{\mathrm{s}} \qquad \text{(acoustic phase velocity)} \tag{13.1.14}$$

$$\mathrm{v}_{\mathrm{g}}=(\partial \mathrm{k} / \partial \omega)^{-1}=\left(\gamma \mathrm{P}_{\mathrm{o}} / \rho_{\mathrm{o}}\right)^{0.5}=\mathrm{c}_{\mathrm{s}} \qquad \text{(acoustic group velocity)} \tag{13.1.15}$$

Adiabatic acoustic waves propagating in 0°C air near sea level experience $\gamma = 1.4$, $\rho_o = 1.29$ [kg/m^3], and $P_o = 1.01\times10^5$ [N/m^2], yielding $c_s \cong 330$ [m/s].

In solids or liquids the constitutive relation is:

$$\partial \rho / \partial \mathrm{p}=\rho / \mathrm{K} \qquad \text{(constitutive relation for solids and liquids)} \tag{13.1.16}$$

K [N m^{-2}] is the *bulk modulus* of the medium. The coefficient 1/K then replaces $1/\gamma P_o$ in (13.1.8–10), yielding the *acoustic velocity in solids* and liquids:

$$\mathrm{c}_{\mathrm{s}}=\left(\mathrm{K} / \rho_{\mathrm{o}}\right)^{0.5}\ \left[\mathrm{m\ s}^{-1}\right] \qquad \text{(acoustic velocity in solids and liquids)} \tag{13.1.17}$$

Typical acoustic velocities are 900 - 2000 m s^{-1} in liquids (~1500 m s^{-1} in water), and 1500– 13,000 m s^{-1} in solids (~5900 m s^{-1} in steel).

Analogous to (7.1.25) and (7.1.26), the acoustic differential equations (13.1.8) and (13.1.7) can be simplified for sinusoidal plane waves propagating along the z axis:

$$\nabla \underline{\mathrm{p}} \bullet \hat{\mathrm{z}}=\frac{\mathrm{d} \underline{\mathrm{p}}(\mathrm{z})}{\mathrm{dz}}=-\mathrm{j} \omega \rho_{\mathrm{o}} \underline{\mathrm{u}}_{\mathrm{z}}(\mathrm{z}) \tag{13.1.18}$$

$$\nabla \bullet \underline{\vec{u}}=\frac{\mathrm{d} \underline{\mathrm{u}}_{\mathrm{Z}}(\mathrm{z})}{\mathrm{dz}}=\frac{-\mathrm{j} \omega}{\gamma \mathrm{P}_{\mathrm{o}}} \underline{\mathrm{p}}(\mathrm{z}) \tag{13.1.19}$$

These can be combined to yield the wave equation for z-axis waves analogous to (7.1.27):

$$\frac{\mathrm{d}^{2} \underline{\mathrm{p}}(\mathrm{z})}{\mathrm{dz}^{2}}=-\omega^{2} \frac{\rho_{\mathrm{o}}}{\gamma \mathrm{P}_{\mathrm{o}}} \underline{\mathrm{p}}(\mathrm{z}) \tag{13.1.20}$$

Analogous to (7.1.28) and (7.1.29), the solution is a sum of exponentials of the form:

$$\underline{p}(z)=\underline{p}_{+} \mathrm{e}^{-\mathrm{jkz}}+\underline{\mathrm{p}}_{-} \mathrm{e}^{+\mathrm{jkz}}\ \left[\mathrm{N\ m}^{-2}\right] \tag{13.1.21}$$

$$\underline{\mathrm{u}}_{\mathrm{z}}(\mathrm{z})=-\frac{1}{\mathrm{j} \omega \rho_{\mathrm{o}}} \frac{\mathrm{d} \underline{\mathrm{p}}(\mathrm{z})}{\mathrm{dz}}=\frac{\mathrm{k}}{\omega \rho_{\mathrm{o}}}\left[\underline{\mathrm{p}}_{+} \mathrm{e}^{-\mathrm{jkz}}-\underline{\mathrm{p}}_{-} \mathrm{e}^{+\mathrm{jkz}}\right]\ [\mathrm{m} / \mathrm{s}] \tag{13.1.22}$$

Note that, unlike electromagnetic waves, where the key fields are vectors transverse to the direction of propagation, the velocity vector for acoustic waves is in the direction of propagation and pressure is a scalar.

Analogous to (7.1.31), the characteristic *acoustic impedance* of a gas is:

$$\eta_{\mathrm{s}}=\frac{\underline{\mathrm{p}}(\mathrm{z})}{\underline{\mathrm{u}}_{\mathrm{L}}(\mathrm{z})}=\frac{\omega \rho_{\mathrm{o}}}{\mathrm{k}}=\rho_{\mathrm{o}} \mathrm{c}_{\mathrm{s}}=\sqrt{\gamma \rho_{\mathrm{o}} \mathrm{P}_{\mathrm{o}}}\ \left[\mathrm{N\ s} / \mathrm{m}^{3}\right] \tag{13.1.23}$$

The acoustic impedance of air at room temperature is ~425 [N s m^{-3}]. The acoustic impedance for solids and liquids is $\eta_{\mathrm{s}}=\rho_{\mathrm{o}} \mathrm{c}_{\mathrm{s}}=\left(\rho_{\mathrm{o}} \mathrm{K}\right)^{0.5}$ [N s m^{-3}]. Note that the units are not ohms.

The instantaneous acoustic intensity [W m^{-2}] of this plane wave is $p(t)u_z(t)$, the complex power is $\underline{\mathrm{p}} \underline{\vec{u}}^{*} / 2$, and the time average acoustic power is $\mathrm{R}_{\mathrm{e}}\left\{\underline{\mathrm{p}} \underline{\vec{u}}^{*} / 2\right\}\ \left[\mathrm{W\ m}^{-2}\right]$, analogous to (2.7.41).

We can derive an *acoustic power conservation* law similar to the Poynting theorem (2.7.22) by computing the divergence of $\underline{\mathrm{p}}\ \underline{\vec{u}}^{*}$ [W m^{-2}] and substituting in (13.1.18) and (13.1.19):[74]

$$\nabla \bullet\left(\underline{\mathrm{p}} \underline{\mathrm{u}} *\right. =\underline{\vec{u}} * \bullet \nabla \underline{\mathrm{p}}+\underline{\mathrm{p}} \nabla \bullet \underline{\vec{u}}^{*}=\underline{\vec{u}} * \bullet\left(-\mathrm{j} \omega \rho_{\mathrm{o}} \underline{\vec{u}}\right)+\mathrm{j} \omega \underline{\mathrm{p}} \underline{\mathrm{p}}^{*} / \gamma \mathrm{P}_{\mathrm{o}} \tag{13.1.24}$$

$$=-4 \mathrm{j} \omega\left(\left[\rho_{\mathrm{o}}|\underline{\mathrm{u}}|^{2} / 4\right]-\left[|\underline{\mathrm{p}}|^{2} / 4 \gamma \mathrm{P}_{\mathrm{o}}\right]\right)=-4 \mathrm{j} \omega\left(\left\langle\mathrm{W}_{\mathrm{k}}\right\rangle-\left\langle\mathrm{W}_{\mathrm{p}}\right\rangle\right) \tag{13.1.25}$$

[74] Although these two equations apply to waves propagating in the z direction, their right-hand sides also apply to any direction if the subscript z is omitted.

The time average *acoustic kinetic energy density* of the wave is $\mathrm{W}_{\mathrm{k}}\ \left[\mathrm{J\ m}^{-3}\right]=\rho_{\mathrm{o}}|\underline{\vec{u}}|^{2} / 4$, and the time average *acoustic potential energy density* is $\mathrm{W}_{\mathrm{p}}=|\underline{\mathrm{p}}|^{2} / 4 \gamma \mathrm{P}_{\mathrm{o}}$. For liquids or solids $\gamma \mathrm{P}_{\mathrm{o}} \rightarrow$ K, so $\mathrm{W}_{\mathrm{p}}=|\underline{\mathrm{p}}|^{2} / 4 \mathrm{K}$. If there is no divergence of acoustic radiated power $\underline{\mathrm{p}} \underline{\vec{u}}^{*}$, then it follows from (13.1.25) that:

$$\left\langle\mathrm{W}_{\mathrm{k}}\right\rangle=\left\langle\mathrm{W}_{\mathrm{p}}\right\rangle \qquad \text{(energy balance in a lossless resonator)} \tag{13.1.26}$$

The *acoustic intensity* I [W m^{-2}] of an acoustic plane wave, analogous to (2.7.41), is:

$$\mathrm{I}=\mathrm{R}_{\mathrm{e}}\left\{\underline{\mathrm{p}} \underline{\mathrm{u}}^{*} / 2\right\}=|\underline{\mathrm{p}}|^{2} / 2 \eta_{\mathrm{s}}=\eta_{\mathrm{s}}|\underline{\mathrm{u}}|^{2} / 2\ \left[\mathrm{Wm}^{-2}\right] \qquad \text{(acoustic intensity)} \tag{13.1.27}$$

where the acoustic impedance $\eta_{\mathrm{s}}=\rho_{\mathrm{o}} \mathrm{c}_{\mathrm{s}}$. The instantaneous acoustic intensity is $p(t)u_z(t)$, as noted above.

Example 13.1.A

A loud radio radiates 100 acoustic watts at 1 kHz from a speaker 10-cm square near sea level where $\rho_{0}=1.29$ [kg m^{-3}] and cs ≅ 330 m s^{-1}. What are the: 1) wavelength, 2) peak pressure, particle velocity, and displacement, and 3) average energy density of this uniform acoustic plane wave in the speaker aperture?

Solution

$\lambda = c_s/f = 330/1000 = 33$ cm. (13.1.22) yields $|\underline{u}| = (2I/\eta_s)^{0.5}$, and (13.1.18) says $\eta_s = \rho_o c_s$, so $|\underline{u}| = [200/(1.29 \times 330)]^{0.5} = 0.69 \ \left[\mathrm{m\,s^{-1}}\right]$. $\underline{p} = \eta_s \underline{u} = 425.7 \times 0.69 = 292$ [N m^{-2}]. Note that this acoustic pressure is much less than the ambient pressure $P_o \cong 10^5$ N m^{-2}, as required for linearization of the acoustic equations. Displacement $\underline{d}$ is the integral of velocity $\underline{u}$, so $\underline{d} = \underline{u}/j\omega$ and the peak-to-peak displacement is $2|\underline{u}|/\omega = 2 \times 0.69/2\pi 1000 = 0.22\ \mathrm{mm}$ The average acoustic energy density stored equals $2\left\langle W_k \right\rangle = 2\rho_o|\underline{\vec{u}}|^2/4 = 1.29(0.69)^2/2 = 0.31 \ \left[\mathrm{J\ m^{-3}}\right]$.