

## 7.4: TEM Resonances

### 7.4.1: Introduction

Resonators are widely used for manipulating signals and power, although unwanted resonances can sometimes limit system performance. For example, resonators can be used either as bandpass filters that remove all frequencies from a signal except those near the desired resonant frequency  $\omega_n$ , or as band-stop filters that remove unwanted frequencies near  $\omega_n$  and let all frequencies pass. They can also be used effectively as step-up transformers to increase voltages or currents to levels sufficient to couple all available energy into desired loads without reflections. That is, the matching circuits discussed in Section 7.3.2 can become sufficiently reactive for badly mismatched loads that they act like band-pass resonators that match the load only for a narrow band of frequencies. Although simple RLC resonators have but one natural resonance and complex RLC circuits have many, distributed electromagnetic systems can have an infinite number.

A *resonator* is any structure that can trap oscillatory electromagnetic energy so that it escapes slowly or not at all. Section 7.4.2 discusses energy trapped in TEM lines terminated so that outbound waves are reflected back into the resonator, and Section 9.4 treats cavity resonators formed by terminating rectangular waveguides with short circuits that similarly reflect and trap otherwise escaping waves. In each of these cases boundary conditions restricted the allowed wave structure inside to patterns having integral numbers of half- or quarter-wavelengths along any axis of propagation, and thus only certain discrete resonant frequencies  $\omega_n$  can be present.

All resonators dissipate energy due to resistive losses, leakages, and radiation, as discussed in Section 7.4.3. The rate at which this occurs depends on where the peak currents or voltages in the resonator are located with respect to the resistive or radiating elements. For example, if the resistive element is in series at a current null or in parallel at a voltage null, there is no dissipation. Since dissipation is proportional to resonator energy content and to the squares of current or voltage, the decay of field strength and stored energy is generally exponential in time. Each resonant frequency  $f_n$  has its own rate of energy decay, characterized by the dimensionless *quality factor*  $Q_n$ , which is generally the number of radians  $\omega_n t$  required for the total energy  $w_{Tn}$  stored in mode  $n$  to decay by a factor of  $1/e$ . More importantly,  $Q \cong f_o/\delta f$ , where  $f_n$  is the resonant frequency and  $\delta f_n$  is the half-power full-width of resonance  $n$ .

Section 7.4.4 then discusses how resonators can be coupled to circuits for use as filters or transformers, and Section 7.4.5 discusses how arbitrary waveforms in resonators are simply a superposition of orthogonal modes, each decaying at its own rate.

### 7.4.2: TEM resonator frequencies

A *resonator* is any structure that traps electromagnetic radiation so it escapes slowly or not at all. Typical *TEM resonators* are terminated at their ends with lossless elements such as short- or open-circuits, inductors, or capacitors. Complex notation is used because resonators are strongly frequency-dependent. We begin with the expressions (7.1.55) and (7.1.58) for voltage and current on TEM lines:

$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz} \quad [V] \quad (7.4.1)$$

$$\underline{I}(z) = Y_o [\underline{V}_+ e^{-jkz} - \underline{V}_- e^{+jkz}] \quad [A] \quad (7.4.2)$$

For example, if both ends of a TEM line of length  $D$  are open-circuited, then  $\underline{I}(z) = 0$  at  $z = 0$  and  $z = D$ . Evaluating (7.4.1) at  $z = 0$  yields  $\underline{V}_- = \underline{V}_+$ . At the other boundary:<sup>36</sup>

$$\underline{I}(D) = 0 = Y_o \underline{V}_+ (e^{-jkD} - e^{+jkD}) = -2jY_o \underline{V}_+ \sin(kD) = -2jY_o \underline{V}_+ \sin(2\pi D/\lambda) \quad (7.4.3)$$

<sup>36</sup> We use the identity  $\sin \phi = (e^{j\phi} - e^{-j\phi}) / 2j$ .

To satisfy (7.4.3),  $\sin(2\pi D/\lambda) = 0$ , and so  $\lambda$  is restricted to specific resonances:

$$\lambda_n = 2D/n = c/f_n \text{ for } n = 0, 1, 2, 3, \dots \quad (7.4.4)$$

That is, at resonance the length of this open-circuited line is  $D = n\lambda_n/2$ , as suggested in Figure 7.4.1(a) for  $n = 1$ . The corresponding resonant frequencies are:

$$f_n = c/\lambda_n = nc/2D \quad [\text{Hz}] \quad (7.4.5)$$

By our definition, static storage of electric or magnetic energy corresponds to a resonance at zero frequency. For example, in this case the line can hold a static charge and store electric energy at zero frequency ( $n = 0$ ) because it is open-circuited at both ends. Because the different modes of a resonator are spatially orthogonal, the total energy stored in a resonator is the sum of the energies stored in each of the resonances separately, as shown later in (7.4.20).

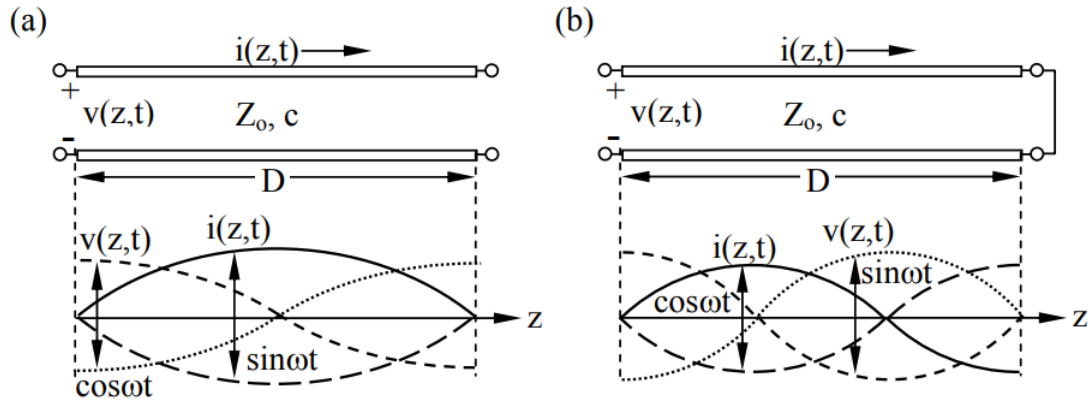


Figure 7.4.1: Voltage and current on TEM resonators.

The time behavior corresponding to (7.4.2) when  $2Y_0 \underline{V}_+ = I_0$  is:

$$i(t, z) = \text{Re} \{ \underline{I} e^{j\omega t} \} = I_0 \sin \omega t \sin(2\pi z / \lambda) \quad (7.4.6)$$

where  $\omega = 2\pi c / \lambda$ . The corresponding voltage  $v(t, z)$  follows from (7.4.1),  $\underline{V}_- = \underline{V}_+$ , and our choice that  $2Y_0 \underline{V}_+ = I_0$ :

$$\underline{V}(z) = \underline{V}_+ [e^{-jkz} + e^{+jkz}] = 2\underline{V}_+ \cos(2\pi z / \lambda) \quad (7.4.7)$$

$$v(t, z) = \text{Re} \{ \underline{V} e^{j\omega t} \} = 2V_+ \cos \omega t \cos(2\pi z / \lambda) \quad (7.4.8)$$

Both  $v(z, t)$  and  $i(z, t)$  are sketched in Figure 7.4.1(a) for  $n = 1$ . The behavior of  $i(t, z)$  resembles the motion of a piano string at resonance and is  $90^\circ$  out of phase with  $v(z, t)$  in both space and time.

Figure 7.4.1(b) illustrates one possible distribution of voltage and current on a TEM resonator short-circuited at one end and open-circuited at the other. Since  $i(t) = 0$  at the open circuit and  $v(t) = 0$  at the short circuit, boundary conditions are satisfied by the illustrated  $i(t, z)$  and  $v(t, z)$ . In this case:

$$D = (\lambda_n / 4) (2n + 1) \text{ for } n = 0, 1, 2, \dots \quad (7.4.9)$$

$$f_n = c / \lambda_n = c(2n + 1) / 4D \text{ [Hz]} \quad (7.4.10)$$

For  $n = 0$  the zero-frequency solution for Figure 7.4.1(a) corresponds to the line being charged to a DC voltage  $V_0$  with zero current. The electric energy stored on the line is then  $DCV_0^2 / 2$  [J], where the electric energy density on a TEM line (7.1.32) is:

$$W_e = C \langle v^2(t, z) \rangle / 2 = C |\underline{V}|^2 / 4 \text{ [Jm}^{-1}] \quad (7.4.11)$$

The extra factor of 1/2 in the right-hand term of (7.4.11) results because  $\langle \cos^2 \omega t \rangle = 0.5$  for nonzero frequencies. A transmission line short-circuited at both ends also has a zero-frequency resonance corresponding to a steady current flowing around the line through the two short circuits at the ends, and the voltage across the line is zero everywhere.<sup>37</sup> The circuit of Figure 7.4.1(b) cannot store energy at zero frequency, however, and therefore has no zero-frequency resonance.

<sup>37</sup> Some workers prefer not to consider the zero-frequency case as a resonance; by our definition it is.

There is also a simple relation between the electric and magnetic energy storage in resonators because  $Z_0 = (L/C)^{0.5}$  (7.1.31). Using (7.4.11), (7.4.6), and (7.4.8) for  $n > 0$ :

$$\langle W_e \rangle = C \langle v^2(t, z) \rangle / 2 = C \langle [Z_0 I_0 \cos \omega_n t \cos(2\pi z / \lambda_n)]^2 \rangle / 2 \quad (7.4.12)$$

$$= (Z_0^2 I_0^2 C / 4) \cos^2(2\pi z / \lambda_n) = (LI_0^2 / 4) [\cos^2(2\pi z / \lambda_n)] \text{ [Jm}^{-1}] \quad (7.4.13)$$

$$\langle W_m \rangle = L \langle i^2(t, z) \rangle / 2 = L \langle [I_0 \sin \omega_n t \sin(2\pi z / \lambda_n)]^2 \rangle / 2 \quad (7.4.14)$$

$$= (LI_0^2 / 4) \sin^2(2\pi z / \lambda_n) [\text{Jm}^{-1}] \quad (7.4.15)$$

Integrating these two time-average energy densities  $\langle W_e \rangle$  and  $\langle W_m \rangle$  over the length of a TEM resonator yields the important result that at any resonance the total time-average stored electric and magnetic energies  $w_e$  and  $w_m$  are equal; the fact that the lengths of all open- and/or shortcircuited TEM resonators are integral multiples of a quarter wavelength  $\lambda_n$  is essential to this result. Energy conservation also requires this because periodically the current or voltage is everywhere zero together with the corresponding energy; the energy thus oscillates between magnetic and electric forms at twice  $f_n$ .

All resonators, not just TEM, exhibit equality between their time-average stored electric and magnetic energies. This can be proven by integrating Poynting's theorem (2.7.24) over the volume of any resonator for the case where the surface integral of  $\vec{S} \cdot \hat{n}$  and the power dissipated  $P_d$  are zero:<sup>38</sup>

$$0.5 \oint_A \vec{S} \cdot \hat{n} da + \iiint_V [(P_d(t)) + 2j\omega(W_m - W_e)] dv = 0 \quad (7.4.16)$$

$$\therefore w_m \equiv \iiint_V W_m dv = \iiint_V W_e dv = w_e \quad (\text{energy balance at resonance}) \quad (7.4.17)$$

<sup>38</sup> We assume here that  $\mu$  and  $\epsilon$  are real quantities so  $W$  is real too.

This proof also applies, for example, to TEM resonators terminated by capacitors or inductors, in which case the reactive energy in the termination must be balanced by the line, which then is not an integral number of quarter wavelengths long.

Any system with spatially distributed energy storage exhibits multiple resonances. These resonance modes are generally orthogonal so the total stored energy is the sum of the separate energies for each mode, as shown below for TEM lines.

Consider first the open-ended TEM resonator of Figure 7.4.1(a), for which the voltage of the  $n$ th mode, following (7.4.7), might be:

$$\underline{V}_n(z) = \underline{V}_{no} \cos(n\pi z / D) \quad (7.4.18)$$

The total voltage is the sum of the voltages associated with each mode:

$$\underline{V}(z) = \sum_{n=0}^{\infty} \underline{V}(n) \quad (7.4.19)$$

The total electric energy on the TEM line is:

$$\begin{aligned} w_{eT} &= \int_0^D (C|\underline{V}(z)|^2 / 4) dz = (C/4) \int_0^D \sum_m \sum_n (\underline{V}_m(z) \underline{V}_n^*(z)) dz \\ &= (C/4) \int_0^D \sum_m \sum_n [\underline{V}_{mo} \cos(m\pi z / D) \underline{V}_{no}^* \cos(n\pi z / D)] dz \\ &= (C/4) \sum_n \int_0^D |\underline{V}_{no}|^2 \cos^2(n\pi z / D) dz = (CD/8) \sum_n |\underline{V}_{no}|^2 \\ &= \sum_n w_{eTn} \end{aligned} \quad (7.4.20)$$

where the total electric energy stored in the  $n$ th mode is:

$$w_{eTn} = CD|\underline{V}_{no}|^2 / 8 [\text{J}] \quad (7.4.21)$$

Since the time average electric and magnetic energies in any resonant mode are equal, the total energy is twice the value given in (7.4.21). Thus the total energy  $w_T$  stored on this TEM line is the sum of the energies stored in each resonant mode separately because all  $m \neq n$  cross terms in (7.4.20) integrate to zero. Superposition of energy applies here because all TEM<sub>m</sub> resonant modes are spatially orthogonal. The same is true for any TEM resonator terminated with short or open circuits. Although spatial orthogonality may not apply to the resonator of Figure 7.4.2(a), which is terminated with a lumped reactance, the modes are still orthogonal because they have different frequencies, and integrating  $v_m(t)v_n(t)$  over time also yields zero if  $m \neq n$ .

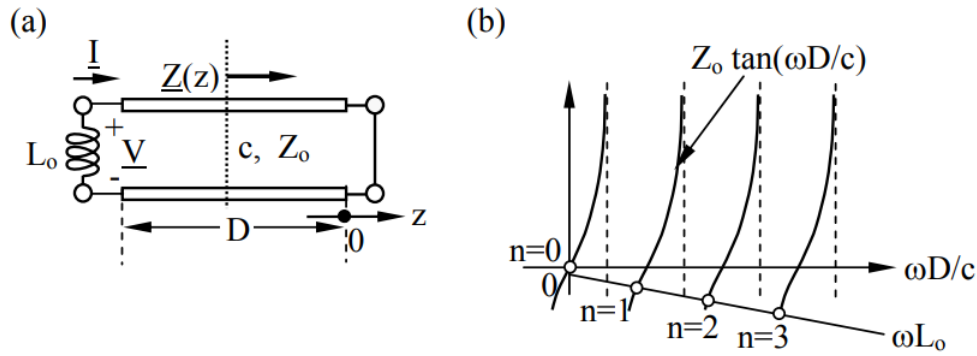


Figure 7.4.2: Inductively loaded TEM transmission line resonator.

Other types of resonator also generally have orthogonal resonant modes, so that in general:

$$w_T = \sum_n W_{Tn} \quad (7.4.22)$$

If a TEM resonator is terminated with a reactive impedance such as  $j\omega L$  or  $1/j\omega C$ , then energy is still trapped but the resonant frequencies are non-uniformly distributed. Figure 7.4.2(a) illustrates a lossless short-circuited TEM line of length  $D$  that is terminated with an inductor  $L_o$ . Boundary conditions immediately yield an expression for the resonant frequencies  $\omega_n$ . The impedance of the inductor is  $j\omega L_o$  and that of the TEM line follows from (7.3.6) for  $Z_L = 0$ :

$$\underline{Z}(z) = Z_o (\underline{Z}_L - jZ_o \tan kz) / (Z_o - jZ_L \tan kz) = -jZ_o \tan kz \quad (7.4.23)$$

Since the current  $\underline{I}$  and voltage  $\underline{V}$  at the inductor junction are the same for both the transmission line and the inductor, their ratios must also be the same except that we define  $\underline{I}$  to be flowing out of the inductor into the TEM line, which changes the sign of  $+j\omega L_o$ ; so:

$$\underline{V}/\underline{I} = -j\omega L_o = jZ_o \tan kD \quad (7.4.24)$$

$$\omega_n = -\frac{Z_o}{L_o} \tan kD = -\frac{Z_o}{L_o} \tan(\omega_n D/c) > 0 \quad (7.4.25)$$

The values of  $\omega_n$  that satisfy (7.4.25) are represented graphically in Figure 7.4.2(b), and are spaced non-uniformly in frequency. The resonant frequency  $\omega_o = 0$  corresponds to direct current and pure magnetic energy storage. Figure 7.4.2(b) yields  $\omega_n$  for a line shorted at both ends when  $L_o = 0$ , and shows that for small values of  $L_o$  (perturbations) that the shift in resonances  $\delta\omega_n$  are linear in  $L_o$ .

We generally can tune resonances to nearby frequencies by changing the resonator slightly. Section 9.4.2 derives the following expression (7.4.26) for the fractional change  $\delta f/f$  in any resonance  $f$  as a function of the incremental increases in average electric ( $\delta\omega_e$ ) and magnetic

( $\delta\omega_m$ ) energy storage and, equivalently, in terms of the incremental volume that was added to or subtracted from the structure, where  $W_e$  and  $W_m$  are the electric and magnetic energy densities in that added ( $+\delta v_{vol}$ ) or removed ( $-\delta v_{vol}$ ) volume, and  $w_T$  is the total energy associated with  $f$ . The energy densities can be computed using the unperturbed values of field strength to obtain approximate answers.

$$\Delta f/f = (\Delta w_e - \Delta w_m) / w_T = \Delta v_{vol} (W_e - W_m) / w_T \quad (\text{frequency perturbation}) \quad (7.4.26)$$

A simple example illustrates its use. Consider the TEM resonator of Figure 7.4.2(a), which is approximately short-circuited at the left end except for a small tuning inductance  $L_o$  having an impedance  $|j\omega L| \ll Z_o$ . How does  $L_o$  affect the resonant frequency  $f_1$ ? One approach is to use (7.4.25) or Figure 7.4.2(b) to find  $w_n$ . Alternatively, we may use (7.4.26) to find  $\Delta f = -f_1 \times \Delta w_m / w_T$ , where  $f_1 \cong c/\lambda \cong c/2D$  and  $\Delta w_m = L_o |\underline{I}'|^2 / 4 = |\underline{V}'|^2 / 4\omega^2 L$ , where  $\underline{I}'$  and  $\underline{V}'$  are exact. But the unperturbed voltage at the short-circuited end of the resonator is zero, so we must use  $\underline{I}'$  because perturbation techniques require that only small fractional changes exist in parameters to be computed, and a transition from zero to any other value is not a perturbation. Therefore  $\Delta w_m = L_o |\underline{I}_o|^2 / 4$ . To cancel  $|\underline{I}_o|^2$  in the expression for  $\delta f$ , we compute  $w_T$  in terms of voltage:  $w_T = 2w_m = 2 \int_0^D (L |\underline{I}(z)|^2 / 4) dz = DL \underline{I}_o^2 / 4$ . Thus:

$$\Delta f = \Delta f_n = -f_n \left( \frac{\Delta w_m}{w_T} \right) = -f_n \frac{|I_o|^2/4}{DL|I_o|^2/4} = -f_n \frac{L_o}{LD} \quad (7.4.27)$$

### 7.4.3: Resonator losses and Q

All resonators dissipate energy due to resistive losses, leakage, and radiation. Since dissipation is proportional to resonator energy content and to the squares of current or voltage, the decay of field strength and stored energy is generally exponential in time. Each resonant frequency  $f_n$  has its own rate of energy decay, characterized by the dimensionless *quality factor*  $Q_n$ , which is generally the number of radians  $\omega_n t$  required for the total energy  $w_{Tn}$  stored in mode  $n$  to decay by a factor of  $1/e$ :

$$w_{Tn}(t) = w_{Tn0} e^{-\omega_n t / Q_n} \text{ [J]} \quad (7.4.28)$$

$Q_n$  is easily related to  $P_n$ , the power dissipated by mode  $n$ :

$$P_n \cong -dw_{Tn}/dt = \omega_n w_{Tn} / Q_n \text{ [W]} \quad (7.4.29)$$

$$Q_n \cong \omega_n w_{Tn} / P_n \quad (\text{quality factor } Q) \quad (7.4.30)$$

The rate of decay for each mode depends on the location of the resistive or radiating elements relative to the peak currents or voltages for that mode. For example, if a resistive element experiences a voltage or current null, there is no dissipation. These relations apply to all resonators, for example, RLC resonators: (3.5.20–23).

Whether a resonator is used as a band-pass or band-stop filter, it has a bandwidth  $\delta\omega$  within which more than half the peak power is passed or stopped, respectively. This half-power bandwidth  $\delta\omega$  is simply related to  $Q$  by (3.5.36):

$$Q_n \cong \omega_n / \Delta\omega_n \quad (7.4.31)$$

The concept and utility of  $Q$  and the use of resonators in circuits are developed further in Section 7.4.4.

Loss in TEM lines arises because the wires are resistive or because the medium between the wires conducts slightly. In addition, lumped resistances may be present, as suggested in Figure 7.4.3(b) and (d). If these resistances do not significantly perturb the lossless voltage and current distributions, then the power dissipated and  $Q$  of each resonance  $\omega_n$  can be easily estimated using *perturbation techniques*. The perturbation method simply involves computing power dissipation using the voltages or currents appropriate for the lossless case under the assumption that the fractional change induced by the perturbing element is small (perturbations of zero-valued parameters are not allowed). The examples below illustrate that perturbing resistances can be either very large or very small.

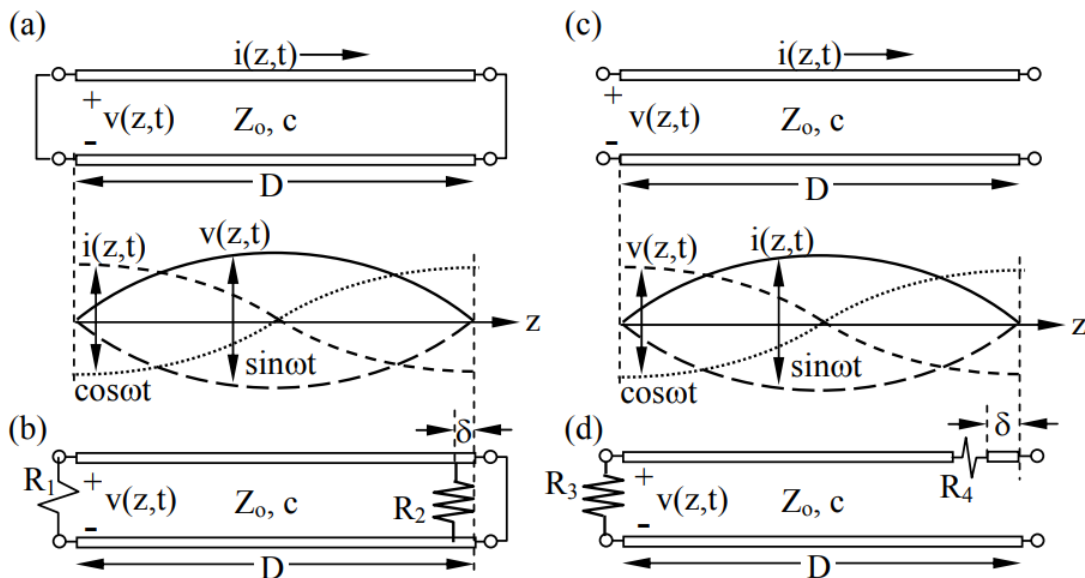


Figure 7.4.3: TEM resonators perturbed by loss.

Consider first the illustrated  $\omega_1$  resonance of Figure 7.4.3(a) as perturbed by the small resistor  $R_1 \ll Z_0$ ; assume  $R_2$  is absent. The nominal current on the TEM line is:

$$i(t, z) = \text{Re} \{ \underline{I} e^{j\omega t} \} = I_0 \sin \omega t \cos(\pi z/D) \quad (7.4.32)$$

The power  $P_1$  dissipated in  $R_1$  at  $z = 0$  using the unperturbed current is:

$$P_1 = \langle i^2(t, z=0) \rangle R_1 = I_0^2 R_1 / 2 \quad (7.4.33)$$

The corresponding total energy  $w_{T1}$  stored in this unperturbed resonance is twice the magnetic energy:

$$w_{T1} = 2 \int_0^D (L \langle i^2(t) \rangle / 2) dz = D L I_0^2 / 4 \text{ [J]} \quad (7.4.34)$$

Using (7.4.30) for  $Q$  and (7.4.5) for  $\omega$  we find:

$$Q_1 \cong \omega_1 w_{T1} / P_1 = (\pi c/D) (D L I_0^2 / 4) / (I_0^2 R_1 / 2) = \pi c L / 2 R_1 = (Z_0 / R_1) \pi / 2 \quad (7.4.35)$$

Thus  $Q_1 \cong Z_0 / R_1$  and is high when  $R_1 \ll Z_0$ ; in this case  $R_1$  is truly a perturbation, so our solution is valid.

A more interesting case involves the loss introduced by  $R_2$  in Figure 7.4.3(b) when  $R_1$  is zero. Since the unperturbed shunting current at that position on the line is zero, we must use instead the unperturbed voltage  $v(z, t)$  to estimate  $P_1$  for mode 1, where that nominal line voltage is:

$$v(z, t) = V_0 \sin \omega t \sin(\pi z/D) \quad (7.4.36)$$

The associated power  $P_1$  dissipated at position  $\delta$ , and total energy  $w_{T1}$  stored are:

$$P_1 \cong \langle v(\delta, t)^2 \rangle / R_2 = V_0^2 \sin^2(\pi \delta/D) / 2 R_2 \cong (V_0 \pi \delta/D)^2 / 2 R_2 \text{ [W]} \quad (7.4.37)$$

$$w_{T1} \cong 2 \int_0^D (C \langle v^2(z, t) \rangle / 2) dz = D C V_0^2 / 4 \text{ [J]} \quad (7.4.38)$$

Note that averaging  $v^2(z, t)$  over space and time introduces two factors of 0.5. Using (7.4.35) for  $Q$  and (7.4.5) for  $\omega$  we find:

$$Q_1 \cong \omega_1 w_{T1} / P_1 = (\pi c/D) (D C V_0^2 / 4) / [(V_0 \pi \delta/D)^2 / 2 R_2] = (D/\delta)^2 (R_2 / 2 \pi Z_0) \quad (7.4.39)$$

Thus  $Q_1$  is high and  $R_2$  is a small perturbation if  $D \gg \delta$ , even if  $R_2 < Z_0$ . This is because a leakage path in parallel with a nearby short circuit can be a perturbation even if its conductance is fairly high.

In the same fashion  $Q$  can be found for the loss perturbations of Figure 7.4.3(d). For example, if  $R_4 = 0$ , then [following (7.4.39)] the effect of  $R_3$  is:

$$Q_1 \cong \omega_1 w_{T1} / P_1 = (\pi c/D) (D C V_0^2 / 4) / (v_0^2 / 2 R_3) = (\pi/2) (R_3 / Z_0) \quad (7.4.40)$$

In this case  $R_3$  is a perturbation if  $R_3 \gg Z_0$ . Most  $R_4$  values are also perturbations provided  $\delta \ll D$ , similar to the situation for  $R_2$ , because any resistance in series with a nearby open circuit will dissipate little power because the currents there are so small.

#### Example 7.4. A

What is the  $Q$  of a TEM resonator of length  $D$  characterized by  $\omega_0$ ,  $C$ , and  $G$ ?

##### Solution

Equation (7.4.40) says  $Q = \omega_0 w_T / P_d$ , where the power dissipated is given by (7.1.61):  $P_d = \int_0^D (G |\underline{V}(z)|^2 / 2) dz$ . The total energy stored  $w_T$  is twice the average stored electric energy  $w_T = 2w_e = 2 \int_0^D (C |\underline{V}(z)|^2 / 4) dz$  [see 7.1.32]. The voltage distribution  $|\underline{V}(z)|$  in the two integrals cancels in the expression for  $Q$ , leaving  $Q = 2\omega_0 C / G$ .

### 7.4.4: Coupling to resonators

Depending on how resonators are coupled to circuits, they can either pass or stop a band of frequencies of width  $\sim \delta \omega_n$  centered on a resonant frequency  $\omega_n$ . This effect can be total or partial; that is, there might be total rejection of signals either near resonance or far away, or only a partial enhancement or attenuation. This behavior resembles that of the series and parallel RLC resonators discussed in Section 3.5.2.

Figure 7.4.4 shows how both series and parallel RLC resonators can block all the available power to the load resistor  $R_L$  near resonance, and similar behavior can be achieved with TEM resonators as suggested below; these are called *band-stop filters*. Alternatively, both series and parallel RLC resonators can pass to the load resistor the band near resonance, as suggested in Figure 3.5.3; these are called *band-pass filters*. In Figure 7.4.4(a) the series LC resonator shorts out the load  $R$  near resonance, while in (b) the parallel LC resonator open-circuits the load conductance  $G$ ; the resonant band is stopped in both cases.

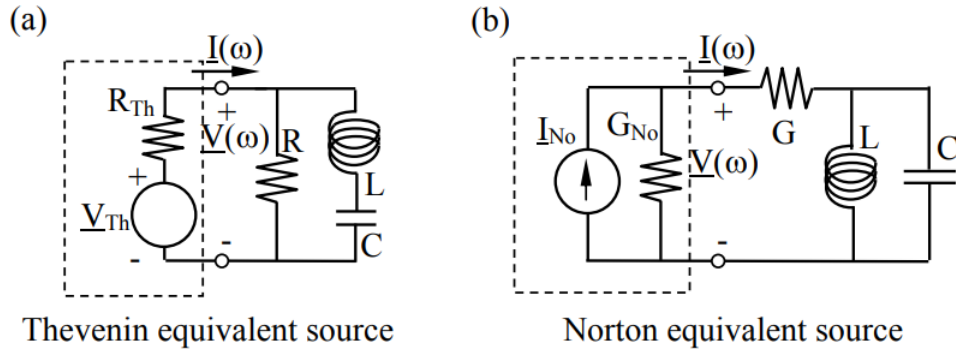


Figure 7.4.4: C-Band-stop RLC resonators.

The half-power width  $\delta\omega$  of each resonance is inversely proportional to the *loaded*  $Q$ , where  $Q_L$  was defined in (3.5.40),  $P_{DE}$  is the power dissipated externally (in the source resistance  $R_{Th}$ ), and  $P_{DI}$  is the power dissipated internally (in the load  $R$ ):

$$Q_L \equiv \omega\omega_T / (P_{DI} + P_{DE}) \quad (\text{loaded } Q) \quad (7.4.41)$$

$$\Delta\omega_n = \omega_n / Q_n \quad [\text{radians s}^{-1}] \quad (\text{half-power bandwidth}) \quad (7.4.42)$$

When  $\omega = (LC)^{-0.5}$  the LC resonators are either open- or short-circuit, leaving only the source and load resistors,  $R_{Th}$  and  $R_L$ . At the frequency  $f$  of maximum power transfer the fraction of the available power that can be passed to the load is determined by the ratio  $Z_n' = R_L / R_{Th}$ . For example, if the power source were a TEM transmission line of impedance  $Z_o \equiv R_{Th}$ , then the minimum fraction of incident power reflected from the load (7.2.22) would be:

$$|\Gamma|^2 = |(Z_n' - 1) / (Z_n' + 1)|^2 \quad (7.4.43)$$

The fraction reflected is zero only when the normalized load resistance  $Z_n' = 1$ , i.e., when  $R_L = R_{Th}$ . Whether the maximum transfer of power to the load occurs at resonance  $\omega_n$  (band-pass filter) or only at frequencies removed more than  $\sim\delta\omega$  from  $\omega_n$  (band-stop filter) depends on whether the current is blocked or passed at  $\omega_n$  by the LC portion of the resonator. For example, Figures 3.5.3 and 7.4.4 illustrate two forms of band-pass and band-stop filter circuits, respectively.

Resonators can be constructed using TEM lines simply by terminating them at both ends with impedances that reflect most or all incident power so that energy remains largely trapped inside, as illustrated in Figure 7.4.5(a). Because the load resistance  $R_L$  is positioned close to a short circuit ( $\delta \ll \lambda/4$ ), the voltage across  $R_L$  is very small and little power escapes, even if  $R_L \equiv Z_o$ . The  $Q$  for the  $\omega_1$  resonance is easily calculated by using (7.4.40) and the expression for line voltage (7.4.36):

$$v(z, t) = V_o \sin \omega t \sin(\pi z / D) \quad (7.4.44)$$

$$\begin{aligned} Q &\equiv \omega_1 W_T / P_D = (\pi c / D) (DCV_o^2 / 4) / [v_o^2 \sin^2(\pi \delta / D) / 2R_L] \\ &= (\pi R_L / 2Z_o) / \sin^2(\pi \delta / D) \cong (D/s)^2 R_L / 2\pi Z_o \quad (\text{for } \delta \ll D) \end{aligned} \quad (7.4.45)$$

Adjustment of  $\delta$  enables achievement of any desired  $Q$  for any given  $R_L$  in an otherwise lossless system. If we regard  $R_L$  as internal to the resonator then the  $Q$  calculated above is the *internal*  $Q$ ,  $Q_i$ .

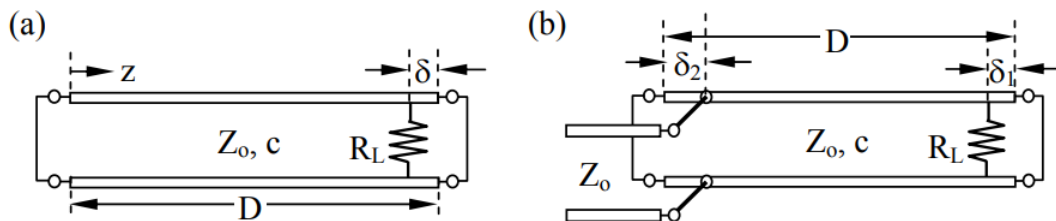


Figure 7.4.5: Coupled TEM resonator.



We may connect this resonator externally by adding a feed line at a short distance  $\delta_2$  and from its left end, as illustrated in Figure 7.4.5(b). If the feed line is matched at its left end then the *external*  $Q$ ,  $Q_E$ , associated with power dissipated there is given by (7.4.45) for  $\delta = \delta_2$  and  $R_L = Z_0$ . By adjusting  $\delta_2$  any  $Q_E$  can be obtained. Figures 3.5.3 and 7.4.4 suggest how the equivalent circuits for either band-pass or band-stop filters can match all the available power to the load if  $R_{Th} = R_L$  and therefore  $Q_E = Q_I$ . Thus all the available power can be delivered to  $R_L$  in Figure 7.4.5(b) for any small  $\delta_1$  by selecting  $\delta_2$  properly; if  $\delta_2$  yields a perfect match at resonance, we have a *critically coupled resonator*. If  $\delta_2$  is larger than the critically coupled value, then the input transmission line is too strongly coupled,  $Q_E < Q_I$ , and we have an *over-coupled resonator*; conversely, smaller values of  $\delta_2$  yield  $Q_E > Q_I$  and undercoupling. The bandwidth of this bandpass filter  $\delta\omega$  is related to the *loaded*  $Q$ ,  $Q_L$ , as defined in (7.4.41) where:

$$Q_L^{-1} = Q_I^{-1} + Q_E^{-1} = \Delta\omega/\omega \quad (7.4.46)$$

If the band-pass filter of Figure 7.4.5(b) is matched at resonance so  $Q_E = Q_I$ , it therefore has a bandwidth  $\delta\omega = 2\omega/Q_L$ , where  $Q_I$  is given by (7.4.39) and is determined by our choice of  $\delta_1$ . Smaller values of  $\delta_1$  yield higher values for  $Q_L$  and narrower bandwidths  $\delta\omega$ . In the special case where  $R_L$  corresponds to another matched transmission line with impedance  $Z_0$ , then a perfect match at resonance results here when  $\delta_1 = \delta_2$ .

Many variations of the coupling scheme in Figure 7.4.5 exist. For example, the feed line and resonator can be isolated by a shunt consisting of a large capacitor or a small inductor, both approximating short circuits relative to  $Z_0$ , or by a high-impedance block consisting of a small capacitor or large inductor in series. Alternatively, an external feed line can be connected in place of  $R_4$  in Figure 7.4.3(d). In each weakly coupled case perturbation methods quickly yield  $Q_I$  and  $Q_E$ , and therefore  $Q_L$ ,  $\delta\omega$ , and the impedance at resonance.

The impedance at resonance can be found once  $Q_E$ ,  $Q_I$ , and  $Z_0$  for the feedline are known, and once it is known whether the resonance is a series or parallel resonance. Referring to Figures 3.5.3 and 7.4.4 for equivalent circuits for band-pass and band-stop filters, respectively, it is clear that if  $Q_E = Q_I$ , then band-pass resonators are matched at resonance while band-stop series-resonance resonators are short circuits and parallel-resonance resonators are open circuits. Away from resonance band-pass resonators become open circuits for series resonances and short circuits for parallel resonances, while both types of band-stop resonator become matched loads if  $Q_E = Q_I$ . At resonance all four types of resonator have purely real impedances and reflection coefficients  $\Gamma$  that can readily be found by examining the four equivalent circuits cited above.

Sometimes unintended resonances can disrupt systems. For example, consider a waveguide that can propagate two modes, only one of which is desired. If a little bit of the unwanted mode is excited at one end of the waveguide, but cannot escape through the lines connected at each end, then the second mode is largely trapped and behaves as a weakly coupled resonator with its own losses. At each of its resonances it will dissipate energy extracted from the main waveguide. If the internal losses happen to cause  $Q_E = Q_I$  for these parasitic resonances, no matter how weakly coupled they are, they can appear as a matched load positioned across the main line; dissipation by parasitic resonances declines as their internal and external  $Q$ 's increasingly differ.

The ability of a weakly coupled resonance to have a powerful external effect arises because the field strengths inside a low-loss resonator can rise to values far exceeding those in the external circuit. For example, the critically coupled resonator of Figure 7.4.5(b) for  $R_L = Z_0$  and  $\delta_1 = \delta_2$ , has internal voltages  $v(z,t) = V_0 \sin\omega t \sin(\pi z/D)$  given by (7.4.44), where the maximum terminal voltage is only  $V_0 \sin(\pi\delta/D) \cong V_0 \pi\delta/D \ll V_0$ . Thus a parasitic resonance can slowly absorb energy from its surroundings at its resonant frequency until its internal fields build to the point that even with weak coupling it has a powerful effect on the external fields and thus reaches an equilibrium value. It is these potentially extremely strong resonant fields that enables critically coupled resonators to couple energy into poorly matched loads--the fields in the resonator build until the power dissipated in the load equals the available power provided. In some cases the fields can build to the point where the resonator arcs internally, as can happen with an empty microwave oven without an extra internal load to prevent it.

This analysis of the resonant behavior of TEM lines is approximate because the resonator length measured in wavelengths is a function of frequency within  $\delta\omega$ , so exact answers require use the TEM analysis methods of Sections 7.2–3, particularly when  $\delta\omega$  becomes a non-trivial fraction of the frequency difference between adjacent resonances.

#### Example 7.4.B

Consider a variation of the coupled resonator of Figure 7.4.5(b) where the resonator is open-circuited at both ends and the weakly coupled external connections at  $\delta_1$  and  $\delta_2$  from the ends are in series with the 100-ohm TEM resonator line rather than in parallel. Find  $\delta_1$  and  $\delta_2$  for:  $Q_L = 100$ ,  $Z_0 = 100$  ohms for both the feed line and resonator,  $R_L = 50$  ohms, and the resonator length is  $D \cong \lambda/2$ , where  $\lambda$  is the wavelength within the resonator.



### Solution

For critical coupling,  $Q_E = Q_L$ , so the resonator power lost to the input line,  $|I_2|^2 Z_o/2$ , must equal that lost to the load,  $|I_2|^2 R_L/2$ , and therefore  $|I_1|/|I_2| = (Z_o/R_L)^{0.5} = 2^{0.5}$ . Since the  $\lambda/2$  resonance of an open-circuited TEM line has  $I(z) \cong I_o \sin(\pi z/D)$  for  $\delta \ll D/\pi$  (high  $Q$ ), therefore  $|I_1|/|I_2| = [\sin(\pi\delta_1/D)]/[\sin(\pi\delta_2/D)] \cong \delta_1/\delta_2 \cong 2^{0.5}$ . Also,  $Q_L = 100 = 0.5 \times Q_I = 0.5 \omega_o W_T / P_{DI}$ , where:  $\omega_o = 2\pi f_o = 2\pi c/\lambda = \pi c/D$ ;  $W_T = 2w_m = 2 \int_0^D (L|I|^2/4) dz \cong LI_o^2 D/4$ ; and  $P_{DI} = |I(\delta_1)|^2 R_L/2 = I_o^2 \sin^2(\pi\delta_1/D) R_L/2 \cong (I_o \pi \delta_1/D)^2 R_L/2$ . Therefore

$$Q_I = \omega_o W_T / P_{DI} = 200 = (\pi c/D) (LI_o^2 D/4) / [(I_o \pi \delta_1/D)^2 R_L/2] = (D/\delta_1)^2 (Z_o/R_L) / \pi,$$

where  $cL = Z_o = 100$ . Thus  $\delta_1 = \pi^{-0.5} D/10$  and  $\delta_2 = \delta_1 2^{-0.5} = (2\pi)^{-0.5} D/10$ .

### 7.4.5: Transients in TEM resonators

TEM and cavity resonators have many resonant modes, all of which can be energized simultaneously, depending on initial conditions. Because Maxwell's equations are linear, the total fields can be characterized as the linear superposition of fields associated with each excited mode. This section illustrates how the relative excitation of each TEM resonator mode can be determined from any given set of initial conditions, e.g. from  $v(z, t = 0)$  and  $i(z, t = 0)$ , and how the voltage and current subsequently evolve. The same general method applies to modal excitation of cavity resonators. By using a similar orthogonality method to match boundary conditions in space rather than in time, the modal excitation of waveguides and optical fibers can be found, as discussed in Section 9.3.3.

The central concept developed below is that any initial condition in a TEM resonator at time zero can be replicated by superimposing some weighted set of voltage and current modes. Once the phase and magnitudes of those modes are known, the voltage and current are then known for all time. The key solution step uses the fact that the mathematical functions characterizing any two different modes  $a$  and  $b$ , e.g. the voltage distributions  $\underline{V}_a(z)$  and  $\underline{V}_b(z)$ , are spatially orthogonal:  $\int \underline{V}_a(z) \underline{V}_b^*(z) dz = 0$ .

Consider the open-circuited TEM resonator of Figure 7.4.3(c), for which  $\underline{V}_{n-} = \underline{V}_{n+}$  for any mode  $n$  because the reflection coefficient at the open circuit at  $z = 0$  is  $+1$ . The resulting voltage and current on the resonator for mode  $n$  are:<sup>39</sup>

<sup>39</sup> Where we recall  $\cos \phi = (e^{j\phi} + e^{-j\phi})/2$  and  $\sin \phi = (e^{j\phi} - e^{-j\phi})/2j$ .

$$\underline{V}_n(z) = \underline{V}_{n+} e^{-jk_n z} + \underline{V}_{n-} e^{+jk_n z} = 2\underline{V}_{n+} \cos k_n z \quad (7.4.47)$$

$$\underline{I}_n(z) = Y_0 (\underline{V}_{n+} e^{-jk_n z} - \underline{V}_{n-} e^{+jk_n z}) = -2jY_0 \underline{V}_{n+} \sin k_n z \quad (7.4.48)$$

where  $k_n = \omega_n/c$  and (7.4.5) yields  $\omega_n = n\pi c/D$ . We can restrict the general expressions for voltage and current to the moment  $t = 0$  when the given voltage and current distributions are  $v_o(z)$  and  $i_o(z)$ :

$$v(z, t = 0) = v_o(z) = \sum_{n=0}^{\infty} \text{Re} \{ \underline{V}_n(z) e^{j\omega_n t} \}_{t=0} = \sum_{n=0}^{\infty} \text{Re} \{ 2\underline{V}_{n+} \cos k_n z \} \quad (7.4.49)$$

$$i(z, t = 0) = i_o(z) = \sum_{n=0}^{\infty} \text{Re} \{ \underline{I}_n(z) e^{j\omega_n t} \}_{t=0} = Y_0 \sum_{n=0}^{\infty} \text{Im} \{ 2\underline{V}_{n+} \sin k_n z \} \quad (7.4.50)$$

We note that these two equations permit us to solve for both the real and imaginary parts of  $\underline{V}_{n+}$ , and therefore for  $v(z, t)$  and  $i(z, t)$ . Using spatial orthogonality of modes, we multiply both sides of (7.4.49) by  $\cos(m\pi z/D)$  and integrate over the TEM line length  $D$ , where  $k_n = n\pi z/D$ :

$$\begin{aligned} \int_0^D v_o(z) \cos(m\pi z/D) dz &= \int_0^D \sum_{n=0}^{\infty} \text{Re} \{ 2\underline{V}_{n+} \cos k_n z \} \cos(m\pi z/D) dz \\ &= \sum_{n=0}^{\infty} \text{Re} \{ 2\underline{V}_{n+} \} \int_0^D \cos(n\pi z/D) \cos(m\pi z/D) dz = 2\text{Re} \{ \underline{V}_{n+} \} (D/2) \delta_{mn} \end{aligned} \quad (7.4.51)$$

where  $\delta_{mn} \equiv 0$  if  $m \neq n$ , and  $\delta_{mn} \equiv 1$  if  $m = n$ . Orthogonality of modes thus enables this integral to single out the amplitude of each mode separately, yielding:

$$\text{Re} \{ \underline{V}_{n+} \} = D^{-1} \int_0^D v_o(z) \cos(n\pi z/D) dz \quad (7.4.52)$$

Similarly, we can multiply (7.4.50) by  $\sin(m\pi z/D)$  and integrate over the length  $D$  to yield:

$$\text{Im} \{ \underline{V}_{n+} \} = Z_o D^{-1} \int_0^D i_o(z) \sin(n\pi z/D) dz \quad (7.4.53)$$

Once  $\underline{V}_n$  is known for all  $n$ , the full expressions for voltage and current on the TEM line follow, where  $\omega_n = \pi n c/D$ :

$$v(z, t) = \sum_{n=0}^{\infty} \text{Re} \{ \underline{V}_n(z) e^{j\omega_n t} \} \quad (7.4.54)$$

$$i(z, t) = \sum_{n=0}^{\infty} \text{Re} \{ \underline{I}_n(z) e^{j\omega_n t} \} = Y_o \sum_{n=0}^{\infty} I_m \{ 2 \underline{V}_{n+} \sin k_n z \} \quad (7.4.55)$$

In general, each resonator mode decays exponentially at its own natural rate, until only the longest-lived mode remains.

As discussed in Section 9.3.3, the relative excitation of waveguide modes by currents can be determined in a similar fashion by expressing the fields in a waveguide as the sum of modes, and then matching the boundary conditions imposed by the given excitation currents at the spatial origin (not time origin). The real and imaginary parts of the amplitudes characterizing each waveguide propagation mode can then be determined by multiplying both sides of this boundary equation by spatial sines or cosines corresponding to the various modes, and integrating over the surface defining the boundary at  $z = 0$ . Arbitrary spatial excitation currents generally excite both propagating and evanescent modes in some combination. Far from the excitation point only the propagating modes are evident, while the evanescent modes are evident principally as a reactance seen by the current source.

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