4.5: Laplace's equation and separation of variables

4.5.1: Laplace's equation

Electric and magnetic fields obey Faraday's and Ampere's laws, respectively, and when the fields are static and the charge and current are zero we have:

$$\nabla \times \overrightarrow{\mathrm{E}} = 0 \tag{4.5.1}$$

$$\nabla \times \overrightarrow{\mathrm{H}} = 0 \tag{4.5.2}$$

These equations are satisfied by any $\overrightarrow{\mathrm{E}}$ and $\overrightarrow{\mathrm{H}}$ that can be expressed as the gradient of a potential:

$$\overrightarrow{\mathrm{E}} = -\nabla\Phi \tag{4.5.3}$$

$$\overrightarrow{\mathrm{H}} = -\nabla\Psi \tag{4.5.4}$$

Therefore Maxwell's equations for static charge-free regions of space are satisfied for any arbitrary differentiable potential function $\Phi(\overrightarrow{\mathrm{r}})$ and $\Psi(\overrightarrow{\mathrm{r}})$, which can be determined as discussed below.

Any potential function must be consistent with the given boundary conditions, and with Gauss's laws in static charge- and current-free spaces:

$$\nabla \bullet \overrightarrow{\mathrm{D}} = 0 \tag{4.5.5}$$

$$\nabla \bullet \overrightarrow{\mathrm{B}} = 0 \tag{4.5.6}$$

where $\overrightarrow{\mathrm{D}} = \varepsilon\overrightarrow{\mathrm{E}}$ and $\overrightarrow{\mathrm{B}} = \mu\overrightarrow{\mathrm{H}}$. Substituting (4.5.3) into (4.5.5), and (4.5.4) into (4.5.6) yields *Laplace's equation*:

$$\nabla^2\Phi = \nabla^2\Psi = 0 \qquad\qquad \text{(Laplace's equation)} \tag{4.5.7}$$

To find static electric or magnetic fields produced by any given set of boundary conditions we need only to solve Laplace's equation (4.5.7) for Φ or Ψ, and then use (4.5.3) or (4.5.4) to compute the gradient of the potential. One approach to solving Laplace's equation is developed in the following section.

Example $4.5.A$

Does the potential Φ = 1/r satisfy Laplace's equation $\nabla^2\Phi = 0$, where $r = (x^2 + y^2 + z^2)^{0.5}$?

Solution

$\nabla^2 = \partial^2/\partial \mathrm{x}^2 + \partial^2/\partial \mathrm{y}^2 + \partial^2/\partial \mathrm{z}^2$.

First: $(\partial/\partial x)\left(x^2+y^2+z^2\right)^{-0.5} = -0.5\left(x^2+y^2+z^2\right)^{-1.5}(2x)$,

so $\left(\partial^2/\partial x^2\right)\left(x^2+y^2+z^2\right)^{-0.5} = 0.75\left(x^2+y^2+z^2\right)^{-2.5}(2x)^2 - \left(x^2+y^2+z^2\right)^{-1.5}$.

Therefore

$$\begin{aligned}\left(\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2\right)\left(x^2+y^2+z^2\right)^{-0.5} &= 3\left(x^2+y^2+z^2\right)^{-2.5}\left(x^2+y^2+z^2\right) - 3\left(x^2+y^2+z^2\right)^{-1.5} \\ &= 0.\end{aligned}$$

So this potential satisfies Laplace's equation. The algebra could have been simplified if instead we wrote ∇^2 in spherical coordinates (see Appendix C), because only the radial term is potentially non-zero for Φ = 1/r: $\nabla^2 = r^{-2}(\partial/\partial r)(r^2\, \partial/\partial r)$. In this case the right-most factor is $r^2\partial r^{-1}/\partial r = r^2(-r^{-2}) = -1$, and $\partial(-1)/\partial r = 0$, so again $\nabla^2\Phi = 0$.

4.5.2: Separation of variables

We can find simple analytic solutions to Laplace's equation only in a few special cases for which the solutions can be factored into products, each of which is dependent only upon a single dimension in some coordinate system compatible with the geometry of the given boundaries. This process of separating Laplace's equation and solutions into uni-dimensional factors is called *separation of variables*. It is most easily illustrated in terms of two dimensions. Let's assume the solution can be factored:

$$\Phi(x, y) = X(x)Y(y) \tag{4.5.8}$$

Then Laplace's equation becomes:

$$\nabla^2\Phi = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = Y(y)d^2X/dx^2 + X(x)d^2Y/dy^2 = 0 \tag{4.5.9}$$

Dividing by X(x)Y(y) yields:

$$\left[d^2X(x)/dx^2\right]/X(x) = -\left[d^2Y(y)/dy^2\right]/Y(y) \tag{4.5.10}$$

Since (4.5.10) must be true for all values of x, y, it follows that each term must equal a constant k^2 , called the *separation constant*, so that:

$$d^2X/dx^2 = -k^2X \qquad d^2Y/dy^2 = k^2Y \tag{4.5.11}$$

Generic solutions to (4.5.11) are, for k ≠ 0:

$$X(x) = A\cos kx + B\sin kx \tag{4.5.12}$$

$$Y(y) = C\cosh ky + D\sinh ky \tag{4.5.13}$$

An equivalent alternative is $Y(y) = C'e^{ky} + D'e^{-ky}$. Generic solutions when k = 0 are:

$$X(x) = Ax + B \tag{4.5.14}$$

$$Y(y) = Cy + D \tag{4.5.15}$$

Note that by letting k → jk, the sinusoidal x-dependence becomes hyperbolic, and the hyperbolic y dependence becomes sinusoidal--the roles of x and y are reversed. Whether k is zero, real, imaginary, or complex depends upon boundary conditions. Linear combinations of solutions to differential equations are also solutions to those same equations, and such combinations are often required to match boundary conditions.

These univariable solutions can be combined to yield the three solution forms for x-y coordinates:

$$\Phi(x, y) = (A + Bx)(C + Dy) \qquad \text{for } k = 0 \tag{4.5.16}$$

$$\Phi(x, y) = (A\cos kx + B\sin kx)(C\cosh ky + D\sinh ky) \qquad \text{for } k^2 > 0 \tag{4.5.17}$$

$$\Phi(x, y) = (A\cosh qx + B\sinh qx)(C\cos qy + D\sin qy) \qquad \text{for } k^2 < 0 (k = jq) \tag{4.5.18}$$

This approach can be extended to three cartesian dimensions by letting Φ(x,y,z) = X(x)Y(y)Z(z); this leads to the solution[11]:

$$\Phi(x, y, z) = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y)(E\cosh k_z z + F\sinh k_z z) \tag{4.5.19}$$

where $k_x^2 + k_y^2 + k_z^2 = 0$. Since k_x^2, k_y^2, and k_2^2 must sum to zero, k_i^2 must be negative for one or two coordinates so that the solution is sinusoidal along either one or two axes and hyperbolic along the others.

[11] If $\Phi(x, y, z) = X(x)Y(y)Z(z)$, then $\nabla^2\Phi = YZd^2X/dx^2 + XZd^2Y/dy^2 + XYd^2Z/dz^2$. Dividing by XYZ yields $X^{-1}d^2X/dx^2 + Y^{-1}d^2Y/dy^2 + Z^{-1}d^2Z/dz^2 = 0$, which implies all three terms must be constants if the equation holds for all x,y,z; let these constants be k_x^2, k_y^2, and k_z^2, respectively. Then $d^2X(y)/dx^2 = k_x^2X(x)$, and the solution (4.5.19) follows when only $k_z^2 > 0$.

Once the form of the solution is established, the correct form, (4.5.16) to (4.5.19), is selected and the unknown constants are determined so that the solution matches the given boundary conditions, as illustrated in the following example.

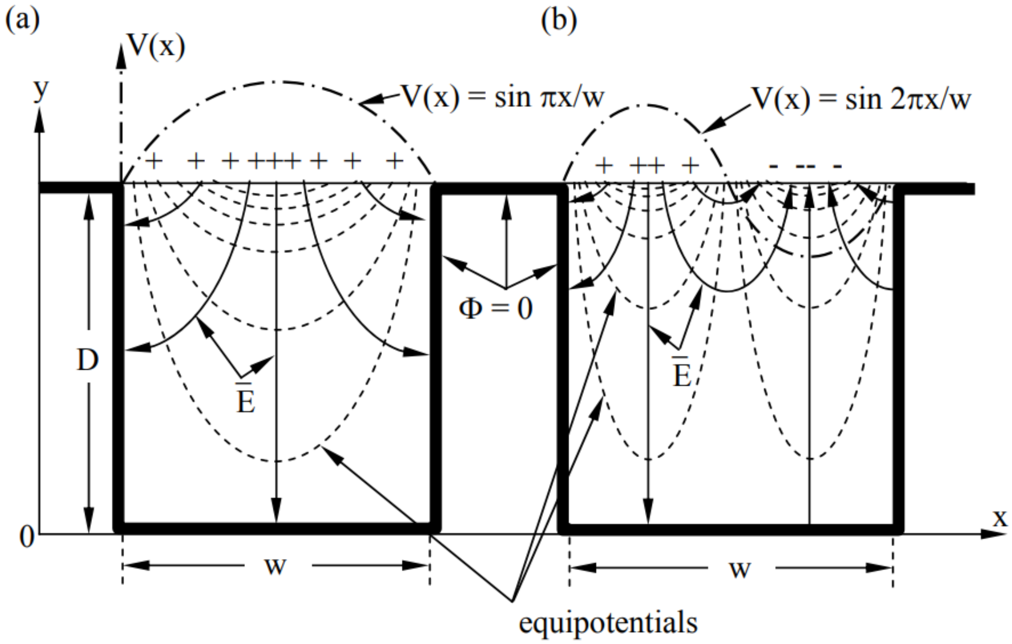

Figure 4.5.1: Static potentials and fields in a sinusoidally-driven conducting rectangular slot.

Consider an infinitely long slot of width w and depth d cut into a perfectly conducting slab, and suppose the cover to the slot has the voltage distribution V(x) = 5 sin(πx/w) volts, as illustrated in Figure 4.5.1(a). This is a two-dimensional cartesian-coordinate problem, so the solution (4.5.17) is appropriate, where we must ensure this expression yields potentials that have the given voltage across the top of the slot and zero potential over the side and bottom boundaries of the slot. Thus:

$$\Phi(\mathrm{x}, \mathrm{y})=\mathrm{A} \sin (\pi \mathrm{x} / \mathrm{w}) \sinh (\pi \mathrm{y} / \mathrm{w}) \quad[\text { volts }] \tag{4.5.20}$$

where the sine and sinh options[12] from (4.5.17) were chosen to match the given potentials on all four boundaries, and where A 5 sinh = (πDw) in order to match the given potential across the top of the slot.

Figure 4.5.1(b) illustrates the solution for the case where the potential across the open top of the slot is given as V(x) = sin 2πx/w. If an arbitrary voltage V(x) is applied across the opening at the top of the slot, then a sum of sine waves can be used to match the boundary conditions.

Although all of these examples were in terms of static electric fields $\overrightarrow{\mathrm{E}}$ and potentials Φ, they equally well could have been posed in terms of static $\overrightarrow{\mathrm{H}}$ and magnetic potential Ψ; the forms of solutions for Ψ are identical.

[12] $\sinh x=\left(e^{x}-e^{-x}\right) / 2$ and $\cosh x=\left(e^{x}+e^{-x}\right) / 2$.

Example 4.5.B

A certain square region obeys $\nabla^2\Phi = 0$ and has Φ = 0 along its two walls at x = 0 and at y = 0. Φ = V volts at the isolated corner x = y = L. Φ increases linearly from 0 to V along the other two walls. What are Φ(x,y) and $\overrightarrow{\mathrm{E}}(\mathrm{x}, \mathrm{y})$ within the square?

Solution

Separation of variables permits linear gradients in potentials in rectangular coordinates via (4.5.14) and (4.5.15), so the potential can have the form Φ = (Ax + B)(Cy + D) where B = D = 0 for this example. Boundary conditions are matched for Φ(x,y) = (V/L^2)xy [V]. It follows that: $\overrightarrow{\mathrm{E}}=-\nabla \Phi=\left(\mathrm{V} / \mathrm{L}^{2}\right)(\hat{x} \mathrm{y}+\hat{y} \mathrm{x})$.

4.5.3: Separation of variables in cylindrical and spherical coordinates

Laplace’s equation can be separated only in four known coordinate systems: cartesian, cylindrical, spherical, and elliptical. Section 4.5.2 explored separation in cartesian coordinates, together with an example of how boundary conditions could then be applied to determine a total solution for the potential and therefore for the fields. The same procedure can be used in a few other coordinate systems, as illustrated below for cylindrical and spherical coordinates.

When there is no dependence on the z coordinate, Laplace's equation in cylindrical coordinates reduces to circular coordinates and is:

$$\nabla^2\Phi = \mathrm{r}^{-1}(\partial/\partial \mathrm{r})(\mathrm{r}\partial\Phi/\partial \mathrm{r}) + \mathrm{r}^{-2}\left(\partial^2\Phi/\partial\phi^2\right) = 0 \tag{4.5.21}$$

Appendix C reviews the del operator in several coordinate systems. We again assume the solution can be separated:

$$\Phi = \mathrm{R(r)}\Phi(\phi) \tag{4.5.22}$$

Substitution of (4.5.22) into (4.5.21) and dividing by $\mathrm{R(r)}\Phi(\phi)$ yields:

$$\mathrm{R}^{-1}(\mathrm{d/dr})(\mathrm{rdR/dr}) = -\Phi^{-1}\left(\mathrm{d}^2\Phi/\mathrm{d}\phi^2\right) = \mathrm{m}^2 \tag{4.5.23}$$

where m^2 is the separation constant.

The solution to (4.5.23) depends on whether m^2 is zero, positive, or negative:

$$\Phi(\mathrm{r},\phi) = [\mathrm{A}+\mathrm{B}\phi][\mathrm{C}+\mathrm{D}(\ln \mathrm{r})] \qquad (\text{for } \mathrm{m}_2 = 0) \tag{4.5.24}$$

$$\Phi(\mathrm{r},\phi) = (\mathrm{A}\sin \mathrm{m}\phi + \mathrm{B}\cos \mathrm{m}\phi)\left(\mathrm{Cr}^{\mathrm{m}} + \mathrm{Dr}^{-\mathrm{m}}\right) \qquad \left(\text{for } \mathrm{m}^2 > 0\right) \tag{4.5.25}$$

$$\Phi(\mathrm{r},\phi) = [\mathrm{A}\sinh \mathrm{p}\phi + \mathrm{B}\cosh \mathrm{p}\phi][\mathrm{C}\cos(\mathrm{p}\ln \mathrm{r}) + \mathrm{D}\sin(\mathrm{p}\ln \mathrm{r})] \qquad \left(\text{for } \mathrm{m}^2 < 0\right) \tag{4.5.26}$$

where A, B, C, and D are constants to be determined and $m \equiv jp$ for $m^2 < 0$.

A few examples of boundary conditions and the resulting solutions follow. The simplest case is a uniform field in the $+\hat{x}$ direction; the solution that matches these boundary conditions is (4.5.25) for m = 1:

$$\Phi(\mathrm{r},\phi) = \mathrm{Br}\cos\phi \tag{4.5.27}$$

Another simple example is that of a conducting cylinder of radius R and potential V. Then the potential inside the cylinder is V and that outside decays as ln r, as given by (1.3.12), when m = C = 0:

$$\Phi(\mathrm{r},\phi) = (\mathrm{V}/\ln \mathrm{R})\ln \mathrm{r} \tag{4.5.28}$$

The electric field associated with this electric potential is:

$$\overrightarrow{\mathrm{E}} = -\nabla\Phi = -\hat{\mathrm{r}}\partial\Phi/\partial \mathrm{r} = \hat{\mathrm{r}}(\mathrm{V}/\ln \mathrm{R})\mathrm{r}^{-1} \tag{4.5.29}$$

Thus $\overrightarrow{\mathrm{E}}$ is radially directed away from the conducting cylinder if V is positive, and decays as r^{-1}.

A final interesting example is that of a dielectric cylinder perpendicular to an applied electric field $\overrightarrow{\mathrm{E}} = \hat{\mathrm{x}}\mathrm{E_o}$. Outside the cylinder the potential follows from (4.5.25) for m = 1 and is:

$$\Phi(\mathrm{r},\phi) = -\mathrm{E_o}\mathrm{r}\cos\phi + (\mathrm{AR/r})\cos\phi \tag{4.5.30}$$

The potential inside can have no singularity at the origin and is:

$$\Phi(\mathrm{r},\phi) = -\mathrm{E_o}(\mathrm{Br/R})\cos\phi \tag{4.5.31}$$

which corresponds to a uniform electric field. The unknown constants A and B can be found by matching the boundary conditions at the surface of the dielectric cylinder, where both Φ and $\overrightarrow{\mathrm{D}}$ must be continuous across the boundary between regions 1 and 2. The two linear equations for must be continuous across the boundary between regions 1 and 2. The two linear equations for continuity ($\Phi_1 = \Phi_2$, and $\overrightarrow{\mathrm{D}}_1 = \overrightarrow{\mathrm{D}}_2$) can be solved for the two unknowns A and B. The electric fields for this case are sketched in Figure 4.5.2.

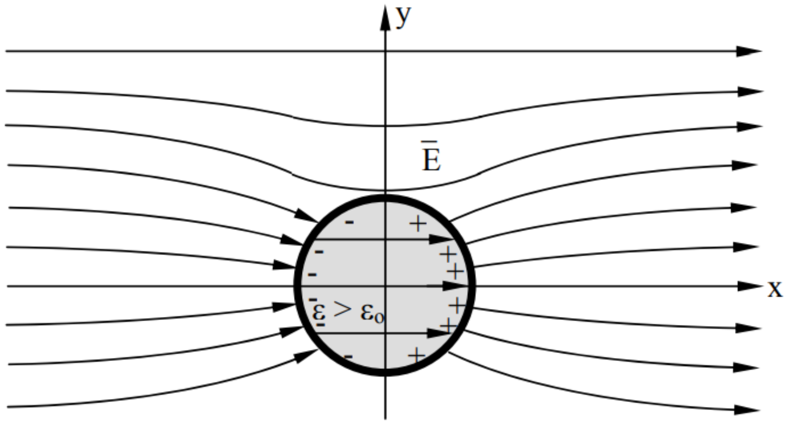

Figure 4.5.2: Electric fields perpendicular to a dielectric cylinder.

If these cylindrical boundary conditions also vary with z, the solution to Laplace's equation becomes:

$$\Phi(\mathrm{r}, \phi, \mathrm{z})=\Phi_{\mathrm{o}}\left[\mathrm{C}_1 \mathrm{e}^{\mathrm{kz}}+\mathrm{C}_2 \mathrm{e}^{-\mathrm{kz} 2}\right]\left[\mathrm{C}_3 \cos \mathrm{n} \phi+\mathrm{C}_4 \sin \mathrm{n} \phi\right]\left[\mathrm{C}_5 \mathrm{J}_{\mathrm{n}}(\mathrm{kr})+\mathrm{C}_6 \mathrm{N}_{\mathrm{n}}(\mathrm{kr})\right] \tag{4.5.32}$$

where J_n and N_n are Bessel functions of order n of the first and second kind, respectively, and C_i are dimensionless constants that match the boundary conditions. The rapidly growing complexity of these solutions as the dimensionality of the problem increases generally mandates numerical solutions of such boundary value problems in practical cases.

Our final example involves spherical coordinates, for which the solutions are:

$$\Phi(\mathrm{r}, \theta, \phi)=\Phi_{\mathrm{o}}\left[\mathrm{C}_1 \mathrm{r}^{\mathrm{n}}+\mathrm{C}_2 \mathrm{r}^{-\mathrm{n}-1}\right]\left[\mathrm{C}_3 \cos \mathrm{m} \phi+\mathrm{C}_4 \sin \mathrm{m} \phi\right]\left[\mathrm{C}_5 \mathrm{P}_{\mathrm{n}}^{\mathrm{m}}(\cos \theta)+\mathrm{C}_6 \mathrm{Q}_{\mathrm{n}}^{\mathrm{m}}(\cos \theta)\right] \tag{4.5.33}$$

where P_n^m and Q_n^m are associated Legendre functions of the first and second kind, respectively, and C_i are again dimensionless constants chosen to match boundary conditions. Certain spherical problems do not invoke Legendre functions, however, as illustrated below.

A dielectric sphere inserted in a uniform electric field $\hat{x}\mathrm{E}_\mathrm{o}$ exhibits the same general form of solution as does the dielectric rod perpendicular to a uniform applied electric field; the solution is the sum of the applied field and the dipole field produced by the induced polarization charges on the surface of the rod or sphere. Inside the sphere the field is uniform, as suggested in Figure 4.5.2. Polarization charges are discussed more fully in Section 2.5.3. The potential follows from (4.5.33) with n = 1 and m = 0, and is simply:

$$\Phi(\mathrm{r}, \theta, \phi)=-\mathrm{E}_{\mathrm{o}} \cos \theta\left(\mathrm{C}_1 \mathrm{r}-\mathrm{C}_2 \mathrm{R}^3 \mathrm{r}^{-2}\right) \tag{4.5.34}$$

where C_2 = 0 inside, and for the region outside the cylinder C_2 is proportional to the induced electric dipole. C_1 outside is unity and inside diminishes below unity as ε increases.

If the sphere in the uniform electric field is conducting, then in (4.5.34) $C_1 = C_2 = 0$ inside the sphere, and the field there is zero; the surface charge is:

$$\rho_{\mathrm{s}}=-\varepsilon_{\mathrm{o}} \hat{n} \bullet \nabla \Phi|_{\mathrm{r}=\mathrm{R}}=\varepsilon_{\mathrm{o}} \mathrm{E}_{\mathrm{r}}=3 \varepsilon_{\mathrm{o}} \mathrm{E}_{\mathrm{o}} \cos \theta\ \left[\mathrm{Cm}^{-2}\right] \tag{4.5.35}$$

Outside the conducting sphere C_1 = 1, and to ensure Φ(r = R) = 0, C_2 must also be unity.

The same considerations also apply to magnetic potentials. For example, a sphere of permeability μ and radius R placed in a uniform magnetic field would also have an induced magnetic dipole that produces a uniform magnetic field inside, and produces outside the superposition of the original uniform field with a magnetic dipole field produced by the sphere. A closely related example involves a sphere of radius R having surface current:

$$\overrightarrow{\mathrm{J}}_{\mathrm{S}}=\hat{\phi} \sin \theta\ \left[\mathrm{Am}^{-1}\right] \tag{4.5.36}$$

This can be produced approximately by a coil wound on the surface of the sphere with a constant number of turns per unit length along the z axis.

For a permeable sphere in a uniform magnetic field $\overrightarrow{\mathrm{H}}=-\hat{z} \mathrm{H}_{\mathrm{o}}$, the solution to Laplace's equation for magnetic potential $\nabla^2\Psi = 0$ has a form similar to (4.5.34):

$$\Psi(\mathrm{r}, \theta)=\mathrm{Cr} \cos \theta \qquad \text{(inside the sphere; } \mathrm{r}<\mathrm{R}) \tag{4.5.37}$$

$$\Psi(\mathrm{r}, \theta)=\mathrm{Cr}^{-2} \cos \theta+\mathrm{H}_{\mathrm{o}} \mathrm{r} \cos \theta \qquad \text{(outside the sphere; } \mathrm{r}>\mathrm{R}) \tag{4.5.38}$$

Using $\overrightarrow{\mathrm{H}}=-\nabla\Psi$, we obtain:

$$\overrightarrow{\mathrm{H}}(\mathrm{r},\theta)=-2\mathrm{C} \qquad \text{(inside the sphere; } \mathrm{r}<\mathrm{R}) \tag{4.5.39}$$

$$\overrightarrow{\mathrm{H}}(\mathrm{r},\theta)=-\mathrm{C}(\mathrm{R}/\mathrm{r})^{2}(\hat{r}\cos\theta+0.5\hat{\theta}\sin\theta)-\hat{z}\mathrm{H}_{\mathrm{o}} \qquad \text{(outside the sphere; } \mathrm{r}>\mathrm{R}) \tag{4.5.40}$$

Matching boundary conditions at the surface of the sphere yields C; e.g. equate $\overrightarrow{\mathrm{B}}=\mu\overrightarrow{\mathrm{H}}$ inside to $\overrightarrow{\mathrm{B}}=\mu_{\mathrm{o}}\mathrm{H}$ outside by equating (4.5.39) to (4.5.40) for θ = 0.