2.8: Uniqueness Theorem

Throughout this text we often implicitly assume uniqueness when we first guess the solution to Maxwell's equations for a given set of boundary conditions and then test that solution against those equations. This process does not guarantee that the resulting solution is unique, and often there are an infinite number of possible solutions, of which we might guess only one. The uniqueness theorem is quite useful for it sets forth constraints on the boundary conditions that guarantee there is only one solution to Maxwell's equations, which we can find as usual.

To prove the uniqueness theorem we begin by considering a volume V enclosed by surface S and governed by Maxwell's equations:

$$\nabla \bullet \overrightarrow{\mathrm{D}}_{\mathrm{i}}=\rho \tag{2.8.1}$$

$$\nabla \bullet \overrightarrow{\mathrm{B}}_{\mathrm{i}}=0 \quad \nabla \times \overrightarrow{\mathrm{E}}_{\mathrm{i}}=-\frac{\partial \overrightarrow{\mathrm{B}}_{\mathrm{i}}}{\partial \mathrm{t}} \quad \nabla \times \overrightarrow{\mathrm{H}}_{\mathrm{i}}=\overrightarrow{\mathrm{J}}+\frac{\partial \overrightarrow{\mathrm{D}}_{\mathrm{i}}}{\partial \mathrm{t}} \tag{2.8.2}$$

where i = 1, 2 correspond to two possible solutions consistent with the given source distributions ρ and $\vec{J}$. We can now show that the difference $\overrightarrow{\mathrm{A}}_{\mathrm{d}}=\overrightarrow{\mathrm{A}}_{1}-\overrightarrow{\mathrm{A}}_{2}$ between these two solutions must be zero under certain conditions, and therefore there can then be no more than one solution: $\vec{A}$ represents $\overrightarrow{\mathrm{D}}$, $\overrightarrow{\mathrm{B}}$, $\overrightarrow{\mathrm{E}}$, $\overrightarrow{\mathrm{H}}$, or $\overrightarrow{\mathrm{J}}$.

If we subtract (2.8.1) for i = 2 from (2.8.1) for i = 1 we obtain:

$$\nabla \bullet\left(\overrightarrow{D}_{1}-\overrightarrow{D}_{2}\right)=\nabla \bullet \overrightarrow{D}_{d}=0 \tag{2.8.3}$$

Similar subtraction of corresponding equations for (2.8.2) yield three more Maxwell's equations that the difference fields $\overrightarrow{\mathrm{B}}_{\mathrm{d}}$ and $\overrightarrow{\mathrm{D}}_{\mathrm{d}}$ must satisfy:

$$\nabla \bullet \overrightarrow{B}_{d}=0 \qquad \nabla \times \overrightarrow{E}_{d}=-\frac{\partial \overrightarrow{B}_{d}}{\partial t} \qquad \nabla \times \overrightarrow{H}_{d}=\frac{\partial \overrightarrow{D}_{d}}{\partial t} \tag{2.8.4}$$

where we note that the source terms ρ and $\vec{J}$ have vanished from (2.8.3) and (2.8.4) because they are given and fixed.

The boundary constraints that ensure uniqueness are:

1. At some time t = 0 the fields are known everywhere so that at that instant $\overrightarrow{\mathrm{E}}_{\mathrm{d}}=\overrightarrow{\mathrm{D}}_{\mathrm{d}}=\overrightarrow{\mathrm{H}}_{\mathrm{d}}=\overrightarrow{\mathrm{B}}_{\mathrm{d}}=0$.
2. At all times and at each point on the surface S either the tangential $\vec{E}$ or the tangential $\vec{H}$ is known.

Applying Poynting's theorem (2.7.10) to the difference fields at time t proves uniqueness subject to these constraints:

$$\iiint_{V}\left[\overrightarrow{H}_{d} \bullet \frac{\partial \overrightarrow{B}_{d}}{\partial t}+\overrightarrow{E}_{d} \bullet \frac{\partial \overrightarrow{D}_{d}}{\partial t}\right] d v+\oiint_{S}\left(\overrightarrow{E}_{d} \times \overrightarrow{H}_{d}\right) \bullet d \overrightarrow{a}=0 \tag{2.8.5}$$

Boundary constraint (2) ensures that the tangential component of either $\overrightarrow{\mathrm{E}}_{\mathrm{d}}$ or $\overrightarrow{\mathrm{H}}_{\mathrm{d}}$ is always zero, thus forcing the cross product in the second integral of (2.8.5) to zero everywhere on the enclosing surface S. The first integral can be simplified if $\overrightarrow{\mathrm{D}}=\varepsilon \overrightarrow{\mathrm{E}}$ and $\overrightarrow{\mathrm{B}}=\mu \overrightarrow{\mathrm{H}}$, where both ε and μ can be functions of position. Because this volume integral then involves only the time derivative of the squares of the difference fields $\left(\left|\overrightarrow{\mathrm{H}}_{\mathrm{d}}\right|^{2} \text{ and } \left|\overrightarrow{\mathrm{E}}_{\mathrm{d}}\right|^{2}\right)$, and because these fields are zero at t = 0 by virtue of constraint (1), the difference fields can never depart from zero while satisfying (2.8.5). Since (2.8.5) holds for all time, the difference fields must therefore always be zero everywhere, meaning there can be no more than one solution to Maxwell's equations subject to the two constraints listed above.