7.1: TEM Waves on Structures

7.1.1: Introduction

Transmission lines typically convey electrical signals and power from point to point along arbitrary paths with high efficiency, and can also serve as circuit elements. In most transmission lines, the electric and magnetic fields point purely transverse to the direction of propagation; such waves are called transverse electromagnetic or *TEM waves*, and such transmission lines are called *TEM lines*. The basic character of TEM waves is discussed in Section 7.1, the effects of junctions are introduced in Section 7.2, and the uses and analysis of TEM lines with junctions are treated in Section 7.3. Section 7.4 concludes by discussing TEM lines that are terminated at both ends so as to form resonators.

Transmission lines in communications systems usually exhibit frequency-dependent behavior, so complex notation is commonly used. Such lines are the subject of this chapter. For broadband signals such as those propagating in computers, complex notation can be awkward and the physics obscure. In this case the signals are often analyzed in the time domain, as introduced in Section 7.1.2 and discussed further in Section 8.1. Non-TEM transmission lines are commonly called waveguides; usually the waves propagate inside some conducting envelope, as discussed in Section 9.3, although sometimes they propagate partly outside their guiding structure in an "open" waveguide such as an optical fiber, as discussed in Section 12.2.

7.1.2: TEM waves between parallel conducting plates

The sinusoidal uniform plane wave of equations (7.1.1) and (7.1.2) is consistent with the presence of thin parallel conducting plates orthogonal to the electric field $\overrightarrow{\mathrm{E}}(\mathrm{z}, \mathrm{t})$, as illustrated in Figure 7.1.1(a)[31].

$$\overrightarrow{\mathrm{E}}(\mathrm{z}, \mathrm{t})=\hat{\mathrm{x}} \mathrm{E}_{\mathrm{o}} \cos (\omega \mathrm{t}-\mathrm{kz})\ [\mathrm{V} / \mathrm{m}] \tag{7.1.1}$$

$$\overrightarrow{\mathrm{H}}(\mathrm{z}, \mathrm{t})=\hat{\mathrm{y}} \frac{\mathrm{E}_{\mathrm{o}}}{\eta_{\mathrm{o}}} \cos (\omega \mathrm{t}-\mathrm{kz})\ [\mathrm{A} / \mathrm{m}] \tag{7.1.2}$$

Although perfect consistency requires that the plates be infinite, there is approximate consistency so long as the plate separation d is small compared to the plate width W and the fringing fields outside the structure are negligible. The more general wave $\overrightarrow{\mathrm{E}}(\mathrm{z}, \mathrm{t})=\hat{x} \mathrm{E}_{\mathrm{x}}(\mathrm{z}-\mathrm{ct})$, $\overrightarrow{\mathrm{H}}(\mathrm{z}, \mathrm{t})=\hat{z} \times \overrightarrow{\mathrm{E}}(\mathrm{z}, \mathrm{t}) / \eta_{\mathrm{o}}$ is also consistent [see (2.2.13), (2.2.18)], since any arbitrary waveform E(z - ct) can be expressed as the superposition of sinusoidal waves at all frequencies. In both cases all boundary conditions of Section 2.6 are satisfied because $\overrightarrow{\mathrm{E}}_{//}=\overrightarrow{\mathrm{H}}_{\perp}=0$ at the conductors. The voltage between two plates v(z,t) for this sinusoidal wave can be found by integrating $\overrightarrow{\mathrm{E}}(\mathrm{z}, \mathrm{t})$ over the distance d from the lower plate, which we associate here with the voltage +v, to the upper plate:

[31] See Section 2.3.1 for an introduction to uniform sinusoidal electromagnetic plane waves.

$$\mathrm{v}(\mathrm{t}, \mathrm{z})=\hat{x} \bullet \overrightarrow{\mathrm{E}}(\mathrm{z}, \mathrm{t}) \mathrm{d}=\mathrm{E}_{0} \mathrm{d} \cos (\omega \mathrm{t}-\mathrm{kz})\ [\mathrm{V}] \tag{7.1.3}$$

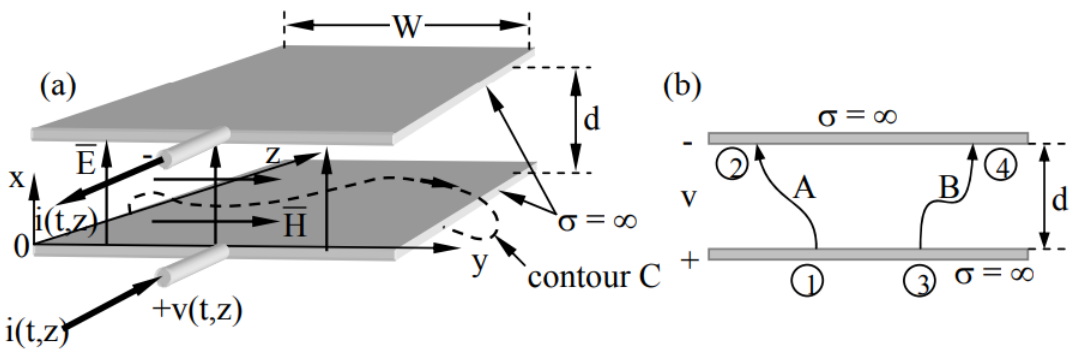

Figure 7.1.1: Parallel-plate TEM transmission line.

Although this computed voltage v(t,z) does not depend on the path of integration connecting the two plates, provided it is at constant z, it does depend on z itself. Thus there can be two different voltages between the same pair of plates at different positions

z. Kirchoff's voltage law says that the sum of voltage drops around a loop is zero; this law is violated here because such a loop in the x-z plane encircles time varying magnetic fields, $\overrightarrow{\mathrm{H}}(\mathrm{z}, \mathrm{t})$, as illustrated. In contrast, the sum of voltage drops around a loop confined to constant z is zero because it circles no; therefore the voltage v(z,t), computed by integrating $\overrightarrow{\mathrm{E}}(\mathrm{z})$ between the two plates, does not depend on the path of integration at constant z. For example, the integrals of $\vec{E} \bullet d \vec{s}$ along contours A and B in Figure 7.1.1(b) must be equal because the integral around the loop 1, 2, 4, 3, 1 is zero and the path integrals within the perfect conductors both yield zero.

If the electric and magnetic fields are zero outside the two plates and uniform between them, then equal and opposite currents i(t,z) flow in the two plates in the ±z direction. The surface current is determined by the boundary condition (2.6.17): $\overrightarrow{\mathrm{J}}_{\mathrm{s}}=\hat{n} \times \overrightarrow{\mathrm{H}}$ [A m^{-1}]. If the two conducting plates are spaced close together compared to their widths W so that d << W, then the fringing fields at the plate edges can be neglected and the total current flowing in the plates can be found from the given magnetic field $\overrightarrow{\mathrm{H}}(\mathrm{z}, \mathrm{t})=\hat{y}\left(\mathrm{E}_{\mathrm{o}} / \eta_{\mathrm{o}}\right) \cos (\omega \mathrm{t}-\mathrm{kz})$, and the integral form of Ampere's law:

$$\int_{C} \vec{H} \bullet d \vec{s}=\int \int_{A}[\vec{J}+(\partial \vec{D} / \partial t)] \bullet \hat{n} d a \tag{7.1.4}$$

If the integration contour C encircles the lower plate and surface A at constant z in a clockwise (right-hand) sense with respect to the +z axis as illustrated in Figure 7.1.1, then $\overrightarrow{\mathrm{D}} \bullet \hat{n}=0$ and the current flowing in the +z direction in the lower plate is simply:

$$\mathrm{i}(\mathrm{z}, \mathrm{t})=\mathrm{WJ}_{\mathrm{sz}}(\mathrm{z}, \mathrm{t})=\mathrm{WH}_{\mathrm{y}}(\mathrm{z}, \mathrm{t})=\left(\mathrm{WE}_{\mathrm{o}} / \eta_{\mathrm{o}}\right) \cos (\omega \mathrm{t}-\mathrm{kz})\ [\mathrm{A}] \tag{7.1.5}$$

An equal and opposite current flows in the upper plate.

Note that the computed current does not depend on the integration contour C chosen so long as C circles the plate at constant z. Also, the current flowing into a section of conducting plate at z_1 does not generally equal the current flowing out at z_2, seemingly violating Kirchoff's current law (the sum of currents flowing into a node is zero). This inequality exists because any section of parallel plates exhibits capacitance that conveys a displacement current $\partial \overrightarrow{\mathrm{D}} / \partial \mathrm{t}$ between the two plates; the right-hand side of Equation (2.1.6) suggests the equivalent nature of the conduction current density $\overrightarrow{\mathrm{J}}$ and the displacement current density $\partial \overrightarrow{\mathrm{D}} / \partial \mathrm{t}$.

Such a two-conductor structure conveying waves that are purely transverse to the direction of propagation, i.e., $E_z = H_z = 0$, is called a *TEM transmission line* because it is propagating transverse electromagnetic waves (*TEM waves*). Such lines generally have a physical crosssection that is independent of z. This particular TEM transmission line is called a *parallel-plate TEM line*.

Because there are no restrictions on the time structure of a plane wave, any v(t) can propagate between parallel conducting plates. The ratio between v(z,t) and i(z,t) for this or any other sinusoidal or non-sinusoidal forward traveling wave is the *characteristic impedance* Z_o of the TEM structure:

$$\mathrm{v}(\mathrm{z}, \mathrm{t}) / \mathrm{i}(\mathrm{z}, \mathrm{t})=\eta_{\mathrm{o}} \mathrm{d} / \mathrm{W}=\mathrm{Z}_{\mathrm{o}}[\text { ohms }] \qquad (\text { characteristic impedance }) \tag{7.1.6}$$

In the special case d = W, Z_o equals the characteristic impedance η_0 of free space, 377 ohms. Usually W >> d in order to minimize fringing fields, yielding Z_o << 377.

Since the two parallel plates can be perfectly conducting and lossless, the physical significance of Z_o ohms may be unclear. Z_o is defined as the ratio of line voltage to line current for a forward wave only, and is non-zero because the plates have inductance L per meter associated with the magnetic fields within the line. The value of Z_o also depends on the capacitance C per meter of this structure. Section 7.1.3 shows (7.1.59) that $Z_o = (L/C)^{0.5}$ for any lossless TEM line and (7.1.19) shows it for a parallel-plate line. The product of voltage and current v(z,t)i(z,t) represents power P(z,t) flowing past any point z toward infinity; this power is not being converted to heat by resistive losses, it is simply propagating away without reflections.

It is easy to demonstrate that the power P(z,t) carried by this forward traveling wave is the same whether it is computed by multiplying v and i, or by integrating the Poynting vector $\overrightarrow{\mathrm{S}}=\overrightarrow{\mathrm{E}} \times \overrightarrow{\mathrm{H}}$ [W m^{-2}] over the cross-sectional area Wd of the TEM line:

$$\mathrm{P}(\mathrm{z}, \mathrm{t})=\mathrm{v}(\mathrm{z}, \mathrm{t}) \mathrm{i}(\mathrm{z}, \mathrm{t})=[\mathrm{E}(\mathrm{z}, \mathrm{t}) \mathrm{d}][\mathrm{H}(\mathrm{z}, \mathrm{t}) \mathrm{W}]=[\mathrm{E}(\mathrm{z}, \mathrm{t}) \mathrm{H}(\mathrm{z}, \mathrm{t})] \mathrm{Wd}=\mathrm{S}\ \mathrm{Wd} \tag{7.1.7}$$

The differential equations governing v and i on TEM lines are easily derived from Faraday's and Ampere's laws for the fields between the plates of this line:

$$\nabla \times \overrightarrow{\mathrm{E}}=-(\partial / \partial \mathrm{t}) \mu \overrightarrow{\mathrm{H}}=\hat{y}(\partial / \partial \mathrm{z}) \mathrm{E}_{\mathrm{x}}(\mathrm{z}, \mathrm{t}) \tag{7.1.8}$$

$$\nabla \times \overrightarrow{\mathrm{H}}=(\partial / \partial \mathrm{t}) \varepsilon \overrightarrow{\mathrm{E}}=-\hat{x}(\partial / \partial \mathrm{z}) \mathrm{H}_{\mathrm{y}}(\mathrm{z}, \mathrm{t}) \tag{7.1.9}$$

Because all but one term in the curl expressions are zero, these two equations are quite simple. By substituting $v = E_x d$ (7.1.3) and $i = H_y W$ (7.1.5), (7.1.8) and (7.1.9) become:

$$\mathrm{dv} / \mathrm{dz}=-(\mu \mathrm{d} / \mathrm{W})(\mathrm{di} / \mathrm{dt})=-\mathrm{Ldi} / \mathrm{dt} \tag{7.1.10}$$

$$\mathrm{di} / \mathrm{dz}=-(\varepsilon \mathrm{W} / \mathrm{d})(\mathrm{dv} / \mathrm{dt})=-\mathrm{Cdv} / \mathrm{dt} \tag{7.1.11}$$

where we have used the expressions for *inductance per meter* L [Hy m^{-1}] and *capacitance per meter* C [F m^{-1}] of a parallel-plate TEM line [see (3.2.11)[32] and (3.1.10)]. This form of the differential equations in terms of L and C applies to any lossless TEM line, as shown in Section 7.1.3.

[32] Note: (3.2.11) gives the total inductance L for a length D of line, where area A = Dd. The inductance per unit length L = μd/W in both cases.

These two differential equations can be solved for v by eliminating i. The current i can be eliminated by differentiating (7.1.10) with respect to z, and (7.1.11) with respect to t, thus introducing $d^2i/(dt\,dz)$ into both expressions permitting its substitution. That is:

$$\mathrm{d}^{2} \mathrm{v} / \mathrm{dz}^{2}=-\mathrm{Ld}^{2} \mathrm{i} /(\mathrm{dtdz}) \tag{7.1.12}$$

$$\mathrm{d}^{2} \mathrm{i} /(\mathrm{dzdt})=-\mathrm{Cd}^{2} \mathrm{v} / \mathrm{dt}^{2} \tag{7.1.13}$$

Combining these two equations by eliminating $d^2i/(dt\,dz)$ yields the wave equation:

$$\mathrm{d}^{2} \mathrm{v} / \mathrm{dz}^{2}=\mathrm{LCd}^{2} \mathrm{v} / \mathrm{dt}^{2}=\mu \varepsilon \mathrm{d}^{2} \mathrm{v} / \mathrm{dt}^{2} \qquad \text{(wave equation)} \tag{7.1.14}$$

Wave equations relate the second spatial derivative to the second time derivative of the same variable, and the solution therefore can be any arbitrary function of an argument that has the same dependence on space as on time, except for a constant multiplier. That is, one solution to (7.1.14) is:

$$\mathrm{v}(\mathrm{z}, \mathrm{t})=\mathrm{v}_{+}(\mathrm{z}-\mathrm{ct}) \tag{7.1.15}$$

where v_+ is an arbitrary function of the argument (z − ct) and is associated with waves propagating in the +z direction at velocity c. This is directly analogous to the propagating waves characterized in Figure 2.2.1 and in Equation (2.2.9). Demonstration that (7.1.15) satisfies (7.1.14) for $c = (\mu\varepsilon)^{-0.5}$ follows the same proof provided for (2.2.9) in (2.2.10–12).

The general solution to (7.1.14) is any arbitrary waveform of the form (7.1.15) plus an independent arbitrary waveform propagating in the -z direction:

$$\mathrm{v}(\mathrm{z}, \mathrm{t})=\mathrm{v}_{+}(\mathrm{z}-\mathrm{ct})+\mathrm{v}_{-}(\mathrm{z}+\mathrm{ct}) \tag{7.1.16}$$

The general expression for current i(z,t) on a TEM line can be found, for example, by substituting (7.1.16) into the differential equation (7.1.11) and integrating over z. Thus, using the notation that $v'(q) \equiv dv(q)/dq$:

$$\mathrm{di} / \mathrm{dz}=-\mathrm{Cdv} / \mathrm{dt}=\mathrm{cC}\left[\mathrm{v}_{+}^{\prime}(\mathrm{z}-\mathrm{ct})-\mathrm{v}_{-}^{\prime}(\mathrm{z}+\mathrm{ct})\right] \tag{7.1.17}$$

$$\mathrm{i}(\mathrm{z}, \mathrm{t})=\mathrm{cC}\left[\mathrm{v}_{+}(\mathrm{z}-\mathrm{ct})-\mathrm{v}_{-}(\mathrm{z}+\mathrm{ct})\right]=\mathrm{Z}_{\mathrm{o}}^{-1}\left[\mathrm{v}_{+}(\mathrm{z}-\mathrm{ct})-\mathrm{v}_{-}(\mathrm{z}+\mathrm{ct})\right] \tag{7.1.18}$$

Equation (7.1.18) defines the characteristic impedance $\mathrm{Z}_{\mathrm{o}}=(\mathrm{cC})^{-1}=\sqrt{\mathrm{L} / \mathrm{C}}$ for the TEM line. Both the forward and backward waves alone have the ratio Z_o between v and i, although the sign of i is reversed for the negative-propagating wave because a positive voltage then corresponds to a negative current. These same TEM results are derived differently in Sections 7.1.3 and 8.1.1.

The characteristic impedance Z_o of a parallel-plate line can be usefully related using (7.1.18) to the capacitance C and inductance L per meter, where C = εW/d and L = μd/W for parallel plate structures (7.1.10–11):

$$\mathrm{Z}_{\mathrm{o}}=\sqrt{\frac{\mathrm{L}}{\mathrm{C}}}[\mathrm{ohms}]=\frac{\mathrm{d}}{\mathrm{c} \varepsilon \mathrm{W}}=\sqrt{\frac{\mu}{\varepsilon}} \frac{\mathrm{d}}{\mathrm{W}} \qquad \text{(characteristic impedance)} \tag{7.1.19}$$

All lossless TEM lines have this simple relationship, as seen in (8.3.9) for R = G = 0. It is also consistent with (7.1.6), where $\eta_{\mathrm{o}}=1 / \mathrm{c} \varepsilon=\left(\mu_{\mathrm{o}} / \varepsilon_{\mathrm{o}}\right)^{0.5}$.

The electric and magnetic energies per meter on a parallel-plate TEM line of plate separation d and plate width W are:[33]

$$W_e(\mathrm{t},\mathrm{z})=\frac{1}{2}\varepsilon|\overrightarrow{\mathrm{E}}(\mathrm{t},\mathrm{z})|^2=\frac{1}{2}\varepsilon\left(\frac{\mathrm{v}(\mathrm{t},\mathrm{z})}{\mathrm{d}}\right)^2\mathrm{Wd}\ \left[\mathrm{Jm}^{-1}\right] \tag{7.1.20}$$

$$W_m(\mathrm{t},\mathrm{z})=\frac{1}{2}\mu|\overrightarrow{\mathrm{H}}(\mathrm{t},\mathrm{z})|^2=\frac{1}{2}\mu\left(\frac{\mathrm{i}(\mathrm{t},\mathrm{z})}{\mathrm{d}}\right)^2\mathrm{Wd}\ \left[\mathrm{Jm}^{-1}\right] \tag{7.1.21}$$

[33] Italicized symbols for W_e and W_m [J m^{-1}] distinguish them from We and Wm [J m^{-3}].

Substituting C = cW/d and L = μd/W into (7.1.20) and (7.1.21) yields:

$$W_e(\mathrm{t},\mathrm{z})=\frac{1}{2}\mathrm{Cv}^2\ \left[\mathrm{Jm}^{-1}\right] \qquad (\text{ TEM electric energy density }) \tag{7.1.22}$$

$$W_m(\mathrm{t},\mathrm{z})=\frac{1}{2}\mathrm{Li}^2\ \left[\mathrm{Jm}^{-1}\right] \qquad (\text{ TEM magnetic energy density }) \tag{7.1.23}$$

If there is only a forward-moving wave, then v(t,z) = Z_oi(t,z) and so:

$$W_e(\mathrm{t},\mathrm{z})=\frac{1}{2}\mathrm{Cv}^2=\frac{1}{2}\mathrm{CZ}_o^2\mathrm{i}^2=\frac{1}{2}\mathrm{Li}^2=W_m(\mathrm{t},\mathrm{z}) \tag{7.1.24}$$

These relations (7.1.22) to (7.1.24) are true for any TEM line.

The same derivations can be performed using complex notation. Thus (7.1.10) and (7.1.11) can be written:

$$\frac{\mathrm{d}\underline{\mathrm{V}}(\mathrm{z})}{\mathrm{dz}}=-\frac{\mu\mathrm{d}}{\mathrm{W}}\mathrm{j}\omega\underline{\mathrm{I}}(\mathrm{z})=-\mathrm{j}\omega\mathrm{L}\underline{\mathrm{I}}(\mathrm{z}) \tag{7.1.25}$$

$$\frac{\mathrm{dI}(\mathrm{z})}{\mathrm{dz}}=-\frac{\varepsilon\mathrm{W}}{\mathrm{d}}\mathrm{j}\omega\underline{\mathrm{V}}(\mathrm{z})=-\mathrm{j}\omega\mathrm{C}\underline{\mathrm{V}}(\mathrm{z}) \tag{7.1.26}$$

Eliminating I(z) from this pair of equations yields the wave equation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{dz}^2}+\omega^2\mathrm{LC}\right)\underline{\mathrm{V}}(\mathrm{z})=0 \qquad (\text{wave equation}) \tag{7.1.27}$$

The solution to the wave equation (7.1.27) is the sum of forward and backward propagating waves with complex magnitudes that indicate phase:

$$\underline{\mathrm{V}}(\mathrm{z})=\underline{\mathrm{V}}_+\mathrm{e}^{-\mathrm{jkz}}+\underline{\mathrm{V}}_-\mathrm{e}^{+\mathrm{jkz}} \tag{7.1.28}$$

$$\underline{I}(z)=Y_o\left(\underline{V}_+\mathrm{e}^{-jkz}-\underline{V}_-e^{+jkz}\right) \tag{7.1.29}$$

where the *wavenumber* k follows from $k^2 = \omega^2 LC$, which is obtained by substituting (7.1.28) into (7.1.27):

$$\mathrm{k}=\omega\sqrt{\mathrm{LC}}=\frac{\omega}{\mathrm{c}}=\frac{2\pi}{\lambda} \tag{7.1.30}$$

The characteristic impedance of the line, as seen in (7.1.19) is:

$$\mathrm{Z}_o=\sqrt{\frac{\mathrm{L}}{\mathrm{C}}}=\frac{1}{\mathrm{Y}_o}\ [\text{ohms}] \tag{7.1.31}$$

and the time average stored electric and magnetic energy densities are:

$$W_e=\frac{1}{4}\mathrm{C}|\underline{\mathrm{V}}|^2\ [\mathrm{J}/\mathrm{m}], \qquad W_m=\frac{1}{4}\mathrm{L}|\underline{I}|^2\ [\mathrm{J}/\mathrm{m}] \tag{7.1.32}$$

The behavior of these arbitrary waveforms at TEM junctions is discussed in the next section and the practical application of these general solutions for arbitrary waveforms is discussed further in Section 8.1. Their practical application to sinusoidal waveforms is discussed in Sections 7.2–4.

Example 7.1.A

A certain TEM line consists of two parallel metal plates that are 10 cm wide, separated in air by d = 1 cm, and extremely long. A voltage v(t) = 10 cos ωt volts is applied to the plates at one end (z = 0). What currents i(t,z) flow? What power P(t) is being fed to the line? If the plate resistance is zero, where is the power going? What is the inductance L per unit length for this line?

Solution

In a TEM line the ratio v/i = Z_o for a single wave, where $Z_o = \eta_o d/W$ [see (7.1.6)], and $\eta_o = (\mu/\varepsilon)^{0.5} \cong 377$ ohms in air. Therefore

$$i(t,z) = Z_o^{-1} v(t,z) = (W/d\eta_o)\,10\cos(\omega t - kz) \cong [0.1/(0.01\times 377)]10\cos(\omega t - kz) \cong 0.27\cos[\omega(t - z/c)]\ [A].$$

$$P = vi = v^2/Z_0 \cong 2.65\cos^2[\omega(t - z/c)]\ [W].$$

The power is simply propagating losslessly along the line toward infinity. Since c = $(LC)^{-0.5} = 3\times 10^8$, and $Z_o = (L/C)^{0.5} \cong 37.7$, therefore $L = Z_o/c = 1.3\times 10^{-7}$ [Henries m^{-1}].

7.1.3: TEM waves in non-planar transmission lines

TEM waves can propagate in any perfectly conducting structure having at least two noncontacting conductors with an arbitrary cross-section independent of z, as illustrated in Figure 7.1.2, if they are separated by a uniform medium characterized by ε, μ, and σ. The parallel plate TEM transmission line analyzed in Section 7.1.2 is a special case of this configuration, and we shall see that the behavior of non-planar TEM lines is characterized by the same differential equations for v(z,t) and i(z,t), (7.1.10) and (7.1.11), when expressed in terms of L and C. This result follows from the derivation below.

We first divide the del operator into its transverse and longitudinal (z-axis) components:

$$\nabla = \nabla_T + \hat{z}\partial/\partial z \tag{7.1.33}$$

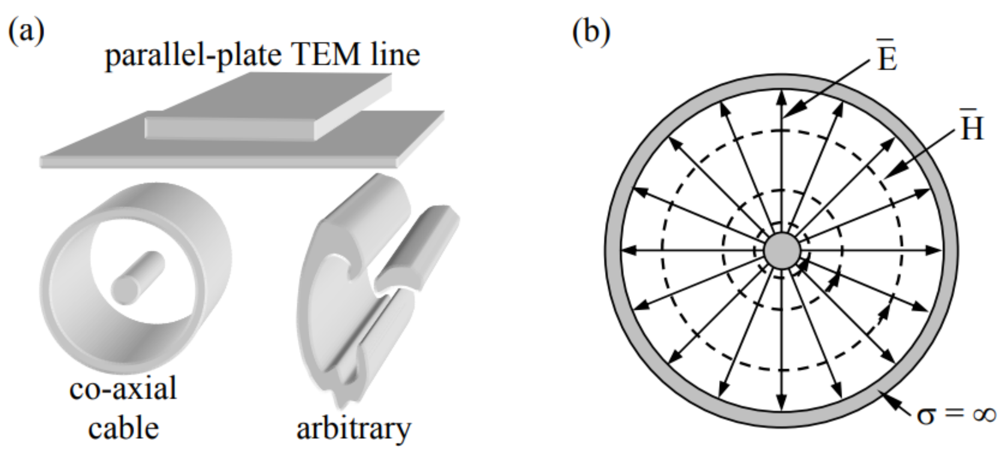

Figure 7.1.2: TEM lines with arbitrary cross-sections.

where $\nabla_T \equiv \hat{x}\partial/\partial x + \hat{y}\partial/\partial y$. Faraday's and Ampere's laws then become:

$$\nabla\times\vec{E} = \nabla_T\times\vec{E}_T + (\partial/\partial z)\left(\hat{z}\times\vec{E}_T\right) = -\mu\partial\vec{H}_T/\partial t \tag{7.1.34}$$

$$\nabla\times\vec{H} = \nabla_T\times\vec{H}_T + (\partial/\partial z)\left(\hat{z}\times\vec{H}_T\right) = \sigma\vec{E}_T + \varepsilon\partial\vec{E}_T/\partial t \tag{7.1.35}$$

The right-hand sides of these two equations have no $\hat{z}$ components, and therefore the transverse curl components on the left-hand side are zero because they lie only along the z axis:

$$\nabla_T\times\vec{E}_T = \nabla_T\times\vec{H}_T = 0 \tag{7.1.36}$$

Moreover, the divergences of $\vec{E}_T$ and $\vec{H}_T$ are also zero since $\hat{z}\bullet\vec{H}_T = \hat{z}\bullet\vec{E}_T = 0$, and:

$$\nabla \bullet \vec{H} = 0 = \nabla_T \bullet \vec{H}_T + (\partial / \partial z)\left(\hat{z} \bullet \vec{H}_T\right) \tag{7.1.37}$$

$$\nabla \bullet \vec{E} = \rho / \varepsilon = 0 = \nabla_T \bullet \vec{E}_T + (\partial / \partial z)\left(\hat{z} \bullet \vec{E}_T\right) \tag{7.1.38}$$

Since the curl and divergence of $\vec{E}_T$ and $\vec{H}_T$ are zero, both these fields must independently satisfy Laplace's equation (4.5.7), which governs electrostatics and magnetostatics; these field solutions will differ because their boundary conditions differ. Thus we can find the transverse electric and magnetic fields for TEM lines with arbitrary cross-sections using the equationsolving and field mapping methods described in Sections 4.5 and 4.6.

The behavior of $\vec{E}$ and $\vec{H}$ for an arbitrary TEM line can be expressed more simply if we first define the line's *capacitance per meter* C and the *inductance per meter* L. C is the charge Q' per unit length divided by the voltage v between the two conductors of interest, and L is the flux linkage Λ' per unit length divided by the current i. Capacitance, inductance, and flux linkage are discussed more fully in Sections 3.1 and 3.2.

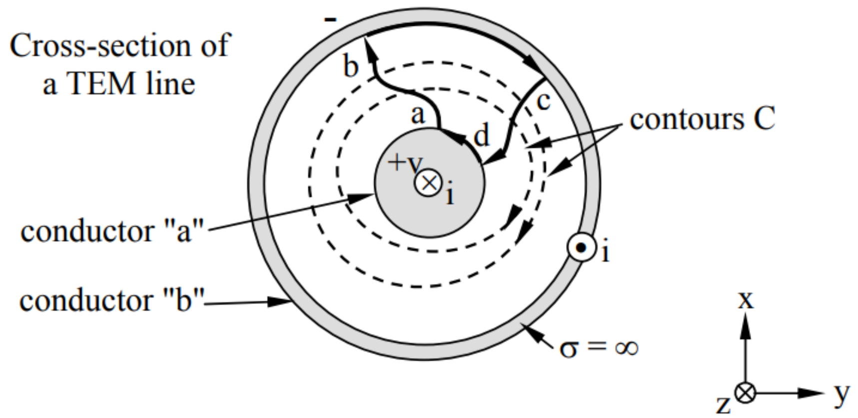

Figure 7.1.3: Integration paths for computing TEM line voltages and currents.

To compute Q' and Λ' we consider a differential element of length δ along the z axis of the TEM line illustrated in Figure 7.1.3, and then compute for Q' and Λ', respectively, surface and line integrals encircling the central positively charged conducting element "a" in a right-hand sense relative to $\hat{z}$. To compute the voltage v we integrate $\vec{E}_T$ from element a to element b, and to compute the current i we integrate $\vec{H}_T$ in a right-hand sense along the contour C circling conductor a:

$$\begin{aligned} C &= Q'/v = \left(\delta^{-1} \int\int_A \varepsilon \vec{E}_T \bullet \hat{n} da\right) \Big/ \left(\int_a^b \vec{E}_T \bullet d\vec{s}\right) \qquad \text{(capacitance/m)} \\ &= \left[\oint_C \varepsilon \hat{z} \bullet \left(\vec{E}_T \times d\vec{s}\right)\right] \Big/ \left(\int_a^b \vec{E}_T \bullet d\bar{s}\right) \left[\mathrm{Fm}^{-1}\right] \end{aligned} \tag{7.1.39}$$

$$\begin{aligned} L = \Lambda'/i &= \left[-\int_a^b \mu \hat{z} \bullet (\vec{H}T \times d\vec{s})\right] \Big/ \left(\oint_C \vec{H}T \bullet d\vec{s}\right) \qquad \text{(inductance/m)} \\ &= \left[\int_a^b \mu \vec{H}_T \bullet (\hat{z} \times d\vec{s})\right] \Big/ \left(\oint_C \vec{H}_T \bullet d\vec{s}\right) \left[\mathrm{Hm}^{-1}\right] \end{aligned} \tag{7.1.40}$$

It is also useful to define G, the line *conductance per meter*, in terms of the leakage current density Jσ' [A m^{-1}] conveyed between the two conductors by the conductivity σ of the medium, where we can use (7.1.39) to show:

$$G = J'_\sigma / v = \left(\delta^{-1} \int\int_A \sigma \vec{E}_T \bullet \hat{n}\, da\right) \Big/ \left(\int_a^b \vec{E}T \bullet d\vec{s}\right) = C\sigma/\varepsilon \tag{7.1.41}$$

We can readily prove that the voltage and current computed using line integrals in (7.1.39– 41) do not depend on the integration path. Figure 7.1.3 illustrates two possible paths of integration for computing v within a plane corresponding to a single value of z, the paths ab and dc. Since the curl of $\overrightarrow{\mathrm{E}}_{\mathrm{T}}$ is zero in the transverse plane we have:

$$\oint_{\mathrm{C}} \overrightarrow{\mathrm{E}} \mathrm{T} \bullet \mathrm{d} \overrightarrow{\mathrm{s}}=\int_{\mathrm{a}}^{\mathrm{b}} \overrightarrow{\mathrm{E}} \mathrm{T} \bullet \mathrm{d} \overrightarrow{\mathrm{s}}+\int_{\mathrm{b}}^{\mathrm{c}} \overrightarrow{\mathrm{E}}_{\mathrm{T}} \bullet \mathrm{d} \overrightarrow{\mathrm{s}}+\int_{\mathrm{c}}^{\mathrm{d}} \overrightarrow{\mathrm{E}}_{\mathrm{T}} \bullet \mathrm{d} \overrightarrow{\mathrm{s}}+\int_{\mathrm{d}}^{\mathrm{a}} \overrightarrow{\mathrm{E}}_{\mathrm{T}} \bullet \mathrm{d} \overrightarrow{\mathrm{s}}=0 \tag{7.1.42}$$

The line integrals along the conductors are zero (paths bc and da), and the cd path is the reverse of the dc path. Therefore voltage is uniquely defined because for any path dc we have:

$$\int_{\mathrm{a}}^{\mathrm{b}} \overrightarrow{\mathrm{E}}_{\mathrm{T}} \bullet \mathrm{d} \overline{\mathrm{s}}=\int_{\mathrm{d}}^{\mathrm{c}} \overrightarrow{\mathrm{E}}_{\mathrm{T}} \bullet \mathrm{d} \overline{\mathrm{s}}=\mathrm{v}(\mathrm{z}, \mathrm{t}) \tag{7.1.43}$$

The current i(z,t) is also uniquely defined because all possible contours C in Figure 7.1.3 circle the same current flowing in conductor a:

$$\mathrm{i}(\mathrm{z}, \mathrm{t})=\oint_{\mathrm{C}} \overrightarrow{\mathrm{H}}_{\mathrm{T}} \bullet \mathrm{d} \overrightarrow{\mathrm{s}} \tag{7.1.44}$$

To derive the differential equations governing v(z,t) and i(z,t) we begin with (7.1.34) and (7.1.35), noting that $\nabla_{\mathrm{T}} \times \overrightarrow{\mathrm{E}}_{\mathrm{T}}=\nabla_{\mathrm{T}} \times \overrightarrow{\mathrm{H}}_{\mathrm{T}}=0$:

$$(\partial / \partial \mathrm{z})\left(\hat{z} \times \overrightarrow{\mathrm{E}}_{\mathrm{T}}\right)=-\mu \partial \overrightarrow{\mathrm{H}}_{\mathrm{T}} / \partial \mathrm{t} \tag{7.1.45}$$

$$(\partial / \partial \mathbf{z})\left(\hat{z} \times \overrightarrow{\mathrm{H}}_{\mathrm{T}}\right)=(\sigma+\varepsilon \partial / \partial \mathrm{t}) \overrightarrow{\mathrm{E}}_{\mathrm{T}} \tag{7.1.46}$$

To convert (7.1.45) into an equation in terms of v we can compute the line integral of $\overrightarrow{\mathrm{E}}_{\mathrm{T}}$ from a to b: the first step is to use the identity $\overrightarrow{\mathrm{A}} \times(\overrightarrow{\mathrm{B}} \times \overrightarrow{\mathrm{C}})=\overrightarrow{\mathrm{B}}(\overrightarrow{\mathrm{A}} \bullet \overrightarrow{\mathrm{C}})-\overrightarrow{\mathrm{C}}(\overrightarrow{\mathrm{A}} \bullet \overrightarrow{\mathrm{B}})$ to show $\left(\hat{z} \times \overrightarrow{\mathrm{E}}_{\mathrm{T}}\right) \times \hat{z}=\overrightarrow{\mathrm{E}}_{\mathrm{T}}$. Using this we operate on (7.1.45) to yield:

$$\begin{aligned}
(\partial / \partial \mathrm{z}) \int_{\mathbf{a}}^{\mathrm{b}}\left[\left(\hat{z} \times \overrightarrow{\mathrm{E}}_{\mathrm{T}}\right) \times \hat{z}\right] \bullet \mathrm{d} \overrightarrow{\mathrm{s}} &=(\partial / \partial \mathrm{z}) \int_{\mathrm{a}}^{\mathrm{b}} \overrightarrow{\mathrm{E}} \mathrm{T} \bullet \mathrm{d} \overrightarrow{\mathrm{s}} \\
&=\partial \mathrm{v}(\mathrm{z}, \mathrm{t}) / \partial \mathrm{z} \\
&=-\mu(\partial / \partial \mathrm{t}) \int_{\mathrm{a}}^{\mathrm{b}}(\overrightarrow{\mathrm{H}} \mathrm{T} \times \hat{z}) \bullet \mathrm{d} \overrightarrow{\mathrm{s}}
\end{aligned} \tag{7.1.47}$$

Then the right-hand integral in (7.1.47), in combination with (7.1.40) and (7.1.44), becomes:

$$\int_{\mathrm{a}}^{\mathrm{b}}\left(\mathrm{H}_{\mathrm{T}} \times \hat{z}\right) \bullet \mathrm{d} \overrightarrow{\mathrm{s}}=\int_{\mathrm{a}}^{\mathrm{b}} \overrightarrow{\mathrm{H}}_{\mathrm{T}} \bullet(\hat{z} \times \mathrm{d} \overrightarrow{\mathrm{s}})=\mu^{-1} \mathrm{L} \oint_{\mathrm{C}} \overrightarrow{\mathrm{H}}_{\mathrm{T}} \bullet \mathrm{d} \overrightarrow{\mathrm{s}}=\mu^{-1} \mathrm{Li}(\mathrm{z}, \mathrm{t}) \tag{7.1.48}$$

Combining (7.1.47) and (7.1.48) yields:

$$\partial \mathrm{v}(\mathrm{z}, \mathrm{t}) / \partial \mathrm{z}=-\mathrm{L} \partial \mathrm{i}(\mathrm{z}, \mathrm{t}) / \partial \mathrm{t} \tag{7.1.49}$$

A similar contour integration of $\overrightarrow{\mathrm{H}}_{\mathrm{T}}$ to yield i(z,t) simplifies (7.1.46):

$$(\partial / \partial \mathrm{z}) \int_{\mathrm{C}}\left[\left(\hat{z} \times \overrightarrow{\mathrm{H}}_{\mathrm{T}}\right) \times \hat{z}\right] \bullet \mathrm{d} \overrightarrow{\mathrm{s}}=(\partial / \partial \mathrm{z}) \oint_{\mathrm{C}} \overrightarrow{\mathrm{H}}_{\mathrm{T}} \bullet \mathrm{d} \overrightarrow{\mathrm{s}}=\partial_{i} / \partial \mathrm{z}=(\sigma+\varepsilon \partial / \partial \mathrm{z}) \oint_{\mathrm{C}}\left(\overrightarrow{\mathrm{E}}_{\mathrm{T}} \times \hat{z}\right) \bullet \mathrm{d} \overrightarrow{\mathrm{s}} \tag{7.1.50}$$

The definitions of C (7.1.39) and G (7.1.41), combined with $(\vec{E} \times \hat{z}) \bullet d \vec{s}=\left(\vec{E}_{T} \times d \vec{s}\right) \bullet \hat{z}$ and the definition (7.1.43) of v, yields:

$$\partial \mathrm{i}(\mathrm{z}, \mathrm{t}) / \partial \mathrm{z}=-(\mathrm{G}+\mathrm{C} \partial / \partial \mathrm{t}) \mathrm{v}(\mathrm{z}, \mathrm{t}) \tag{7.1.51}$$

This pair of equations, (7.1.49) and (7.1.51), can then be combined to yield a more complete description of wave propagation on general TEM lines.

Because the characteristic impedance and phase velocity for general TEM lines are frequency dependent, the simple solutions (7.1.49) and (7.1.51) are not convenient. Instead it is useful to express them as complex functions of ω:

$$\partial \underline{V}(z)/\partial z = -j\omega L \underline{I}(z) \tag{7.1.52}$$

$$\partial \underline{I}(z)/\partial z_Z = -(G + j\omega C)\underline{V}(z) \tag{7.1.53}$$

Combining this pair of equations yields the wave equation:

$$\partial^2 \underline{V}(z)/\partial z^2 = j\omega L(G + j\omega C)\underline{V}(z) \qquad (\text{ TEM wave equation }) \tag{7.1.54}$$

The solution to this *TEM wave equation* must be a function that equals a constant times its own second derivative, such as:

$$\underline{V}(z) = \underline{V}_+ e^{-j\underline{k}z} + \underline{V}_- e^{+j\underline{k}z} \qquad (\text{wave equation solution}) \tag{7.1.55}$$

Substituting this assumed solution into the wave equation yields the *dispersion relation* for general TEM lines made with perfect conductors:

$$\underline{k}^2 = -j\omega L(G + j\omega C) \qquad (\text{ TEM dispersion relation }) \tag{7.1.56}$$

This equation yields a complex value for the *TEM propagation constant* $\underline{k} = k' - jk''$, the significance of which is that the forward (V_+) and backward (V_-) propagating waves are exponentially attenuated with distance:

$$\underline{V}(z) = \underline{V}_+ e^{-jk'z - k''z} + \underline{V}_- e^{+j\underline{k}'z + k''z} \tag{7.1.57}$$

The current can be found by substituting (7.1.57) into (7.1.53) to yield:

$$\underline{I}(z) = (\underline{k}/j\omega L)\left(\underline{V}_+ e^{-j\underline{k}z} - \underline{V}_- e^{+j\underline{k}z}\right) = \left(\underline{V}_+ e^{-j\underline{k}z} - \underline{V}_- e^{+j\underline{k}z}\right)/\underline{Z}_0 \tag{7.1.58}$$

$$\underline{Z}_0 = [j\omega L/(G + j\omega C)]^{0.5} \tag{7.1.59}$$

These expressions reduce to those for lossless TEM lines as G → 0.

Another consequence of this dispersion relation (7.1.56) is that the *TEM phase velocity* v_p is frequency dependent and thus most lossy lines are dispersive:

$$v_p = \omega/k' = (LC)^{-0.5}(1 - jG/\omega C)^{-0.5} \tag{7.1.60}$$

Although most TEM lines also have resistance R per unit length, this introduces $E_z \neq 0$, so analysis becomes much more complex. In this case the approximate Telegrapher's equations (8.3.3–4) are often used.

Example 7.1.B

What is the characteristic impedance Z_o for the air-filled *co-axial cable* illustrated in Figure 7.1.3 if the relevant diameters for the inner and outer conductors are a and b, respectively, where b/a = e? "Co-axial" means cylinders a and b share the same axis of symmetry.

Solution

$Z_o = (L/C)^{0.5}$ from (7.1.59). Since $c = (LC)^{-0.5}$ it follows that $L = (c^{-2}C^{-1})$ and $Z_o = 1/cC$ ohms. C follows from (7.1.39), which requires knowledge of the transverse electric field $\overrightarrow{E}_T$ (for TEM waves, there are no non-transverse fields). Symmetry in this cylindrical geometry requires $\overrightarrow{E}T = \hat{r}E_o/r$. Thus

$C = Q'/v = \left[\text{\oiint}_A \varepsilon_o \overrightarrow{E}_T \bullet \hat{r}da\right]\left[\int_a^b \overrightarrow{E}_T \bullet d\overrightarrow{s}\right] = a^{-1}[\varepsilon_o E_o 2\pi a] \Big/ \left[\int_a^b E_o r^{-1} dr\right]$

$= \varepsilon_0 2\pi \ln(b/a) = 2\pi\varepsilon_o = 56 \times 10^{-12}[F]$. Therefore $Z_o = \left(56 \times 10^{-12} \times 3 \times 10^8\right)^{-1}$

$\cong 60$ ohms, and $L \cong 2 \times 10^{-7}[H]$

7.1.4: Loss in transmission lines

Transmission line losses can be computed in terms of the resistance R, Ohms per meter, of TEM line length, or conductance G, Siemens/m, of the medium separating the two conductors. As discussed in Section 8.3.1, the time average power P_d dissipated per meter of length is simply the sum of the two contributions from the series and parallel conductances:

$$\mathrm{P_d(z)\ [W/m]} = |\underline{\mathrm{I}}(\mathrm{z})|^2\mathrm{R}/2 + |\underline{\mathrm{V}}(\mathrm{z})|^2\mathrm{G}/2 \tag{7.1.61}$$

When R and G are unknown, resistive losses in transmission lines can be estimated by integrating $|\overrightarrow{\underline{J}}|^2/2\sigma$ [W m^{-3}] over the volume of interest, where σ is the material conductivity [S m^{-1}] and J is the current density [A m^{-2}]. This surface loss density P_d [W m^{-2}] is derived for good conductors in Section 9.2 and is shown in (9.2.61) to be equal to the power dissipated by the same surface current $\underline{J}_s$ flowing uniformly through a slab of thickness δ, where $\delta = (2/\omega\mu\sigma)^{0.5}$ is the skin depth. The surface current $\overrightarrow{\mathrm{J}}_{\mathrm{S}}$ equals $\left|\overrightarrow{\underline{H}}_s\right|$, which is the magnetic field parallel to the conductor surface. Therefore:

$$\mathrm{P_d} \cong \left|\overrightarrow{\underline{\mathrm{H}}}_{\mathrm{s}}\right|^2 \sqrt{\frac{\omega\mu}{8\sigma}}\ \left[\mathrm{W/m^2}\right] \qquad \text{(power dissipation in conductors)} \tag{7.1.62}$$

For example, it is easy to compute with (7.1.62) the power dissipated in a 50-ohm copper TEM coaxial cable carrying P_o = 10 watts of entertainment over a 500-MHz band with an inner conductor diameter of one millimeter. First we note that $\left|\overrightarrow{\underline{\mathrm{H}}}_{\mathrm{s}}\right| = \underline{\mathrm{I}}/2\pi\mathrm{r}$ [A/m] where $|\underline{\mathrm{I}}^2|\,\mathrm{Z_o}/2 = \mathrm{P_o} = 10$, and 2r = 10^{-3} [m]. Therefore $\left|\overrightarrow{\underline{\mathrm{H}}}_{\mathrm{s}}\right| = (\mathrm{P_o/Z_o})^{0.5}/2\pi\mathrm{r}\ [\mathrm{A/m}] \cong 142$ Also, since the diameter of the outer sheath is typically ~5 times that of the inner conductor, the surface current density there, J_s, is one fifth that for the inner conductor, and the power dissipation per meter length is also one fifth. Therefore the total power dissipated per meter, P_L, in both conductors is ~1.2 times that dissipated in the inner conductor alone. If we consider only the highest and most lossy frequency, and assume σ = 5×10^7, then substituting $\left|\overrightarrow{\underline{\mathrm{H}}}_{\mathrm{s}}\right|$ into (7.1.62) and integrating over both conductors yields the power loss:

$$\begin{aligned}\mathrm{P_L} &\cong 1.2\times 2\pi\mathrm{r}\left|\overrightarrow{\underline{\mathrm{H}}}_{\mathrm{S}}\right|^2(\omega\mu_{\mathrm{o}}/4\sigma)^{0.5} = 1.2\times 2\pi\mathrm{r}\left[(2\mathrm{P_o/Z_o})^{0.5}/2\pi\mathrm{r}\right]^2(\omega\mu_{\mathrm{o}}/8\sigma)^{0.5} \\ &= 1.2\times\mathrm{P_o}(\mathrm{Z_o}\pi\mathrm{r})^{-1}(\omega\mu_{\mathrm{o}}/2\sigma)^{0.5} = 12\left(50\pi 10^{-3}\right)^{-1}\ \left(2\pi\times 5\times 10^8\times 4\pi\times 10^{-7}/10^8\right)^{0.5} \\ &= 0.48\ \text{watts / meter}\end{aligned} \tag{7.1.63}$$

The loss L [dB m^{-1}] is proportional to the ratio of P_L [W m^{-1}] to P_o [W]:

$$\mathrm{L}\ \left[\mathrm{dBm^{-1}}\right] = 4.34\mathrm{P_L/P_o} \tag{7.1.64}$$

Thus P_L is 0.48 watts/meter, a large fraction of the ten watts propagating on the line. This loss of 4.8 percent of the power per meter, including the outer conductor, corresponds to $10\log_{10}(1 - 0.048) \cong -0.21$dB per meter. If we would like amplifiers along a cable to provide no more than ~50 dB gain, we need amplifiers every ~234 meters. Dropping the top frequency to 100 MHz, or increasing the diameter of the central wire could reduce these losses by perhaps a factor of ~4. These loss issues and desires for broad bandwidth are motivating substitution of low-loss optical fiber over long cable lines, and use of co-axial cables only for short hops from a local fiber to the home or business.

Example 7.1.C

A perfectly conducting 50-ohm coaxial cable is filled with slightly conducting dielectric that gives the line a shunt conductivity G = 10^{-6} Siemens m^{-1} between the two conductors. What is the attenuation of this cable (dB m^{-1})?

Solution

The attenuation L[dB m^{-1}] = 4.34 P_d/P_o (7.1.64), where the power on the line $P_o\ [W] = |\underline{V}|^2/2Z_0$, and the dissipation here is $\mathrm{P_d}\ \left[\mathrm{Wm^{-1}}\right] = |\underline{\mathrm{V}}|^2\mathrm{G}/2$ (7.1.61); see Figure 8.3.1 for the incremental model of a lossy TEM transmission line. Therefore L = 4.34 GZ_o = 2.2×10^{-4} dB m^{-1}. This is generally independent of frequency and therefore might dominate at lower frequencies if the frequency-dependent dissipative losses in the wires become sufficiently small.