14.3: Mathematical Identities

$$\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

$$\vec{A} \bullet \vec{B} = A_xB_x + A_yB_y + A_zB_z = \hat{a} \times \hat{b}|\vec{A}||\vec{B}|\cos\theta$$

$$\begin{aligned}\vec{A} \times \vec{B} &= \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \hat{x}\left(A_yB_z - A_zB_y\right) + \hat{y}\left(A_zB_x - A_xB_z\right) + \hat{2}\left(A_xB_y - A_yB_x\right) \\ &= \hat{a} \times \hat{b}|\vec{A}||\vec{B}|\sin\theta\end{aligned}$$

$$\vec{A} \bullet (\vec{B} \times \vec{C}) = \vec{B} \bullet (\vec{C} \times \vec{A}) = \vec{C} \bullet (\vec{A} \times \vec{B})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \bullet \vec{C})\vec{B} - (\vec{A} \bullet \vec{B})\vec{C}$$

$$(\vec{A} \times \vec{B}) \bullet (\vec{C} \times \vec{D}) = (\vec{A} \bullet \vec{C})(\vec{B} \bullet \vec{D}) - (\vec{A} \bullet \vec{D})(\vec{B} \bullet \vec{C})$$

$$\nabla \times \nabla\Psi = 0$$

$$\nabla \bullet (\nabla \times \vec{A}) = 0$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \bullet \vec{A}) - \nabla^2\vec{A}$$

$$-\vec{A} \times (\nabla \times \vec{A}) = (\vec{A} \bullet \nabla)\vec{A} - \frac{1}{2}\nabla(\vec{A} \bullet \vec{A})$$

$$\nabla(\Psi\Phi) = \Psi\nabla\Phi + \Phi\nabla\Psi$$

$$\nabla \bullet (\Psi\vec{A}) = \vec{A} \bullet \nabla\Psi + \Psi\nabla \bullet \vec{A}$$

$$\nabla \times (\Psi\vec{A}) = \nabla\Psi \times \vec{A} + \Psi\nabla \times \vec{A}$$

$$\nabla^2\Psi = \nabla \bullet \nabla\Psi$$

$$\nabla(\vec{A} \bullet \vec{B}) = (\vec{A} \bullet \nabla)\vec{B} + (\vec{B} \bullet \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$$

$$\nabla \bullet (\vec{A} \times \vec{B}) = \vec{B} \bullet (\nabla \times \vec{A}) - \vec{A} \bullet (\nabla \times \vec{B})$$

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \bullet \vec{B}) - \vec{B}(\nabla \bullet \vec{A}) + (\vec{B} \bullet \nabla)\vec{A} - (\vec{A} \bullet \nabla)\vec{B}$$

14.3.1: Cartesian Coordinates (x,y,z):

$$\nabla\Psi = \hat{x}\frac{\partial\Psi}{\partial x} + \hat{y}\frac{\partial\Psi}{\partial y} + \hat{z}\frac{\partial\Psi}{\partial z}$$

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}$$

14.3.2: Cylindrical coordinates (r,φ,z):

$$
\begin{aligned}
\nabla \Psi &= \hat{\rho} \frac{\partial \Psi}{\partial \mathrm{r}} + \hat{\phi} \frac{1}{\mathrm{r}} \frac{\partial \Psi}{\partial \mathrm{y}} + \hat{z} \frac{\partial \Psi}{\partial \mathrm{z}} \\
\nabla \bullet \overrightarrow{\mathrm{A}} &= \frac{1}{\mathrm{r}} \frac{\partial\left(\mathrm{rA}_{\mathrm{r}}\right)}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}} \frac{\partial \mathrm{A}_\phi}{\partial \phi} + \frac{\partial \mathrm{A}_{\mathrm{z}}}{\partial \mathrm{z}} \\
\nabla \times \overrightarrow{\mathrm{A}} &= \hat{r}\left(\frac{1}{\mathrm{r}} \frac{\partial \mathrm{A}_{\mathrm{z}}}{\partial \phi} - \frac{\partial \mathrm{A}_\phi}{\partial \mathrm{z}}\right) + \hat{\phi}\left(\frac{\partial \mathrm{A}_{\mathrm{r}}}{\partial \mathrm{z}} - \frac{\partial \mathrm{A}_{\mathrm{z}}}{\partial \mathrm{r}}\right) + \hat{z} \frac{1}{\mathrm{r}}\left(\frac{\partial\left(\mathrm{rA}_\phi\right)}{\partial \mathrm{r}} - \frac{\partial \mathrm{A}_{\mathrm{r}}}{\partial \phi}\right) = \frac{1}{\mathrm{r}} \operatorname{det}\left|\begin{array}{ccc} \hat{r} / \partial \mathrm{r} & \partial / \partial \phi & \partial(\partial \mathrm{z} \mid \\ \mathrm{A}_{\mathrm{r}} & \mathrm{rA}_\phi & \mathrm{A}_{\mathrm{z}} \end{array}\right| \\
\nabla^2 \Psi &= \frac{1}{\mathrm{r}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r} \frac{\partial \Psi}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{\partial^2 \Psi}{\partial \mathrm{z}^2}
\end{aligned}
$$

14.3.3: Spherical coordinates (r,θ,φ):

$$
\begin{aligned}
\nabla \Psi &= \hat{r} \frac{\partial \Psi}{\partial \mathrm{r}} + \hat{\theta} \frac{1}{\mathrm{r}} \frac{\partial \Psi}{\partial \theta} + \hat{\phi} \frac{1}{\mathrm{r} \sin \theta} \frac{\partial \Psi}{\partial \phi} \\
\nabla \bullet \overrightarrow{\mathrm{A}} &= \frac{1}{\mathrm{r}^2} \frac{\partial\left(\mathrm{r}^2 \mathrm{A}_{\mathrm{r}}\right)}{\partial \mathrm{r}} + \frac{1}{\mathrm{r} \sin \theta} \frac{\partial\left(\sin \theta \mathrm{A}_\theta\right)}{\partial \theta} + \frac{1}{\mathrm{r} \sin \theta} \frac{\partial \mathrm{A}_\phi}{\partial \phi} \\
\nabla \times \overrightarrow{\mathrm{A}} &= \hat{r} \frac{1}{\mathrm{r} \sin \theta}\left(\frac{\partial\left(\mathrm{r} \sin \theta \mathrm{A}_\phi\right)}{\partial \theta} - \frac{\partial \mathrm{A}_\theta}{\partial \phi}\right) + \hat{\theta}\left(\frac{1}{\mathrm{r} \sin \theta} \frac{\partial \mathrm{A}_{\mathrm{r}}}{\partial \phi} - \frac{1}{\mathrm{r}} \frac{\partial\left(\mathrm{rA}_\phi\right)}{\partial \mathrm{r}}\right) + \hat{\phi} \frac{1}{\mathrm{r}}\left(\frac{\partial\left(\mathrm{rA}_\theta\right)}{\partial \mathrm{r}} - \frac{\partial \mathrm{A}_{\mathrm{r}}}{\partial \theta}\right) \\
&= \frac{1}{\mathrm{r}^2 \sin \theta} \operatorname{det}\left|\begin{array}{ccc} \hat{r} & r \hat{\theta} & \mathrm{r} \sin \theta \hat{\phi} \\ \partial / \partial \mathrm{r} & \partial / \partial \theta & \partial / \partial \phi \\ \mathrm{A}_{\mathrm{r}} & \mathrm{rA}_\theta & \mathrm{r} \sin \theta \mathrm{A}_\phi \end{array}\right| \\
\nabla^2 \Psi &= \frac{1}{\mathrm{r}^2} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2 \frac{\partial \Psi}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2 \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{\mathrm{r}^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2}
\end{aligned}
$$

14.3.4: Gauss' Divergence Theorem:

$$\int_V \nabla \bullet \vec{G}\, d v = \oint_A \vec{G} \bullet \hat{n} d a$$

14.3.5: Stokes' Theorem:

$$\int_{\mathrm{A}}(\nabla \times \overrightarrow{\mathrm{G}}) \bullet \hat{\mathrm{n}}\, \mathrm{da} = \oint_{\mathrm{C}} \overrightarrow{\mathrm{G}} \bullet \mathrm{d} \vec{\ell}$$

14.3.6: Fourier Transforms for pulse signals h(t):

$$
\begin{aligned}
\underline{\mathrm{H}}(\mathrm{f}) &= \int_{-\infty}^{\infty} \mathrm{h}(\mathrm{t}) \mathrm{e}^{-\mathrm{j} 2 \pi \mathrm{t}}\, \mathrm{dt} \\
\mathrm{h}(\mathrm{t}) &= \int_{-\infty}^{\infty} \underline{\mathrm{H}}(\mathrm{f}) \mathrm{e}^{+\mathrm{j} 2 \pi \mathrm{ft}}\, \mathrm{df}
\end{aligned}
$$