14.4: Basic Equations for Electromagnetics and Applications

14.4.1: Fundamentals

$$\vec{f} = q\left(\vec{E} + \vec{v} \times \mu_o \vec{H}\right) [N]$$

$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

$$\oint_c \vec{E} \bullet d\vec{s} = -\frac{d}{dt} \int_A \vec{B} \bullet d\vec{a}$$

$$\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t$$

$$\oint_c \vec{H} \bullet d\vec{s} = \int_A \vec{J} \bullet d\vec{a} + \frac{d}{dt} \int_A \vec{D} \bullet d\vec{a}$$

$$\nabla \bullet \vec{D} = \rho \rightarrow \oint_A \vec{D} \bullet d\vec{a} = \int_V \rho dv$$

$$\nabla \bullet \vec{B} = 0 \rightarrow \oint_A \vec{B} \bullet d\vec{a} = 0$$

$$\nabla \bullet \vec{J} = -\partial \rho / \partial t$$

$$\vec{E} = \text{ electric field } \left(Vm^{-1}\right)$$

$$\vec{H} = \text{ magnetic field } \left(Am^{-1}\right)$$

$$\vec{D} = \text{ electric displacement } \left(Cm^{-2}\right)$$

$$\vec{B} = \text{ magnetic flux density (T) Tesla (T)} = \text{ Weber } m^{-2} = 10,000 \text{ gauss}$$

$$\rho = \text{ charge density } \left(Cm^{-3}\right)$$

$$\vec{J} = \text{ current density } \left(Am^{-2}\right)$$

$$\sigma = \text{ conductivity (Siemens } m^{-1})$$

$$\vec{J}_s = \text{ surface current density } \left(Am^{-1}\right)$$

$$\rho_s = \text{ surface charge density } \left(Cm^{-2}\right) \varepsilon_0 = 8.85 \times 10^{-12} \text{ Fm}^{-1}$$

$$\mu_o = 4\pi \times 10^{-7} \text{ Hm}^{-1}$$

$$c = (\varepsilon_o \mu_o)^{-0.5} \cong 3 \times 10^8 \text{ ms}^{-1}$$

$$e = -1.60 \times 10^{-19} \text{ C}$$

$$E_y(z,t) = E_+(z-ct) + E_-(z+ct) = R_e\left\{\underline{E}_y(z) e^{j\omega t}\right\}$$

$$H_x(z,t) = \eta_o^{-1}\left[E_+(z-ct) - E_-(z+ct)\right] \text{ [or } (\omega t - kz) \text{ or } (t - z/c)]$$

$$\int_A (\vec{E} \times \vec{H}) \bullet d\vec{a} + (d/dt) \int_V \left(\varepsilon |\vec{E}|^2/2 + \mu |\vec{H}|^2/2\right) dv = -\int_V \vec{E} \bullet \vec{J} \text{ dv (Poynting Theorem)}$$

14.4.2: Media and Boundaries

$$\overrightarrow{\mathrm{D}} = \varepsilon_{\mathrm{o}} \overrightarrow{\mathrm{E}} + \overrightarrow{\mathrm{P}}$$

$$\nabla \bullet \overrightarrow{\mathrm{D}} = \rho_{\mathrm{f}}, \ \tau = \varepsilon / \sigma$$

$$\nabla \bullet \varepsilon_{\mathrm{o}} \overrightarrow{\mathrm{E}} = \rho_{\mathrm{f}} + \rho_{\mathrm{p}}$$

$$\nabla \bullet \overrightarrow{\mathrm{P}} = -\rho_{\mathrm{p}}, \ \ \overrightarrow{\mathrm{J}} = \sigma \overrightarrow{\mathrm{E}}$$

$$\overrightarrow{\mathrm{B}} = \mu \overrightarrow{\mathrm{H}} = \mu_{\mathrm{o}} (\overrightarrow{\mathrm{H}} + \overrightarrow{\mathrm{M}})$$

$$\varepsilon = \varepsilon_{\mathrm{o}} \left(1 - \omega_{\mathrm{p}}^2 / \omega^2\right)$$

$$\omega_{\mathrm{p}} = \left(\mathrm{Ne}^2 / \mathrm{m} \varepsilon_{\mathrm{o}}\right)^{0.5}$$

$$\varepsilon_{\mathrm{eff}} = \varepsilon (1 - \mathrm{j} \sigma / \omega \varepsilon)$$

skin depth $\delta = (2/\omega\mu\sigma)^{0.5}$ [m]

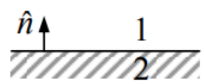
$$\overline{\mathrm{E}}_{1//} - \overline{\mathrm{E}}_{2//} = 0$$

$$\overline{\mathrm{H}}_{1//} - \overline{\mathrm{H}}_{2//} = \overline{\mathrm{J}}_{\mathrm{s}} \times \hat{\mathrm{n}}$$

$$\mathrm{B}_{1\perp} - \mathrm{B}_{2\perp} = 0$$

$$\left(\mathrm{D}_{1\perp} - \mathrm{D}_{2\perp}\right) = \rho_{\mathrm{s}}$$

$$\hookrightarrow 0 = \ \text{if } \sigma = \infty$$



14.4.3: Electromagnetic Quasistatics

$$\nabla^2 \Phi = 0$$

$$\mathrm{KCL}: \sum_{\mathrm{i}} \mathrm{I}_{\mathrm{i}}(\mathrm{t}) = 0 \text{ at node}$$

$$\mathrm{KVL}: \sum_{\mathrm{i}} \mathrm{V}_{\mathrm{i}}(\mathrm{t}) = 0 \text{ around loop}$$

$$\mathrm{C} = \mathrm{Q}/\mathrm{V} = \mathrm{A}\varepsilon/\mathrm{d}[\mathrm{F}]$$

$$\mathrm{L} = \Lambda / \mathrm{I}$$

$$\mathrm{i}(\mathrm{t}) = \mathrm{C}\mathrm{d}\mathrm{v}(\mathrm{t})/\mathrm{dt}$$

$$\mathrm{v}(\mathrm{t}) = \mathrm{L}\,\mathrm{di}(\mathrm{t})/\mathrm{dt} = \mathrm{d}\Lambda/\mathrm{dt}$$

$$\mathrm{C}_{\mathrm{parallel}} = \mathrm{C}_1 + \mathrm{C}_2$$

$$\mathrm{C}_{\mathrm{series}} = \left(\mathrm{C}_1^{-1} + \mathrm{C}_2^{-1}\right)^{-1}$$

$$\mathrm{w}_{\mathrm{e}} = \mathrm{C}\mathrm{v}^2(\mathrm{t})/2; \mathrm{w}_{\mathrm{m}} = \mathrm{L}\mathrm{i}^2(\mathrm{t})/2$$

$$\mathrm{L}_{\mathrm{solenoid}} = \mathrm{N}^2 \mu \mathrm{A}/\mathrm{W}$$

$$\tau = \mathrm{RC}, \tau = \mathrm{L}/\mathrm{R}$$

$$\Lambda = \int_{\mathrm{A}} \overrightarrow{\mathrm{B}} \bullet \mathrm{d}\bar{\mathrm{a}} \text{ (per turn)}$$

$$\overrightarrow{\mathrm{f}} = \mathrm{q}\left(\overrightarrow{\mathrm{E}} + \overrightarrow{\mathrm{v}} \times \mu_{\mathrm{o}} \overrightarrow{\mathrm{H}}\right) [\mathrm{N}]$$

$$\mathrm{f_z = -dw_T/dz}$$

$$\overrightarrow{\mathrm{F}} = \overrightarrow{\mathrm{I}} \times \mu_o \overrightarrow{\mathrm{H}} \left[\mathrm{Nm^{-1}}\right]$$

$$\overrightarrow{\mathrm{E}}_{\mathrm{e}} = -\overrightarrow{\mathrm{v}} \times \mu_o \overrightarrow{\mathrm{H}} \text{ inside wire}$$

$$\mathrm{P = \omega T = W_T dV_{olume}/dt[W]}$$

$$\mathrm{Max\ f/A = B^2/2\mu_o, D^2/2\varepsilon_o}\left[\mathrm{Nm^{-2}}\right]$$

$$\mathrm{vi} = \frac{\mathrm{dw_T}}{\mathrm{dt}} + \mathrm{f}\frac{\mathrm{dz}}{\mathrm{dt}}$$

14.4.4: Electromagnetic Waves

$$\left(\nabla^2 - \mu\varepsilon\partial^2/\partial t^2\right)\overrightarrow{\mathrm{E}} = 0 \text{ [Wave Eqn.]}$$

$$\left(\nabla^2 + \mathrm{k}^2\right)\underline{\overrightarrow{\mathrm{E}}} = 0,\ \underline{\overrightarrow{\mathrm{E}}} = \underline{\overrightarrow{\mathrm{E}}}_0 \mathrm{e}^{-\mathrm{j}\overrightarrow{\mathrm{k}}\bullet \mathrm{r}}$$

$$\mathrm{k} = \omega(\mu\varepsilon)^{0.5} = \omega/\mathrm{c} = 2\pi/\lambda$$

$$\mathrm{k_x^2 + k_y^2 + k_z^2 = k_o^2} = \omega^2\mu\varepsilon$$

$$\mathrm{v_p} = \omega/\mathrm{k},\ \mathrm{v_g} = (\partial\mathrm{k}/\partial\omega)^{-1}$$

$$\theta_\mathrm{r} = \theta_\mathrm{i}$$

$$\sin\theta_\mathrm{t}/\sin\theta_\mathrm{i} = \mathrm{k_i/k_t = n_i/n_t}$$

$$\theta_\mathrm{c} = \sin^{-1}(\mathrm{n_t/n_i})$$

$$\theta > \theta_\mathrm{c} \Rightarrow \underline{\overrightarrow{\mathrm{E}}}_\mathrm{t} = \underline{\overrightarrow{\mathrm{E}}}_\mathrm{i}\underline{\mathrm{T}}\mathrm{e}^{+\alpha x - \mathrm{jk_z z}}$$

$$\underline{\overrightarrow{\mathrm{k}}} = \overrightarrow{\mathrm{k}}' - \mathrm{j}\overrightarrow{\mathrm{k}}''$$

$$\underline{\Gamma} = \underline{\mathrm{T}} - 1$$

$$\underline{\mathrm{T}}_{\mathrm{TE}} = 2/\left(1 + [\eta_o\cos\theta_\mathrm{t}/\eta_\mathrm{t}\cos\theta_\mathrm{i}]\right)$$

$$\underline{\mathrm{T}}_{\mathrm{TM}} = 2/\left(1 + [\eta_\mathrm{t}\cos\theta_\mathrm{t}/\eta_\mathrm{i}\cos\theta_\mathrm{i}]\right)$$

$$\theta_\mathrm{B} = \tan^{-1}\left(\varepsilon_\mathrm{t}/\varepsilon_\mathrm{i}\right)^{0.5} \text{ for TM}$$

$$\mathrm{P_d} \cong \left|\underline{\overrightarrow{\mathrm{J}}}_\mathrm{S}\right|^2/2\sigma\delta\ \left[\mathrm{Wm^{-2}}\right]$$

$$\overrightarrow{\mathrm{E}} = -\nabla\phi - \partial\overrightarrow{\mathrm{A}}/\partial\mathrm{t},\quad \overrightarrow{\mathrm{B}} = \nabla\times\overrightarrow{\mathrm{A}}$$

$$\underline{\Phi}(\mathrm{r}) = \int_{\mathrm{V'}}\left(\underline{\rho}(\overrightarrow{\mathrm{r}})\mathrm{e}^{-\mathrm{jk}\left|\overrightarrow{\mathrm{r}}' - \overrightarrow{\mathrm{r}}\right|}\Big/4\pi\varepsilon_o\left|\overrightarrow{\mathrm{r}}' - \overrightarrow{\mathrm{r}}\right|\right)\mathrm{dv'}$$

$$\underline{\overrightarrow{\mathrm{A}}}(\mathrm{r}) = \int_{\mathrm{V'}}\mu_o\left(\underline{\vec{\mathrm{J}}}(\overrightarrow{\mathrm{r}})\mathrm{e}^{-\mathrm{jk}\left|\overrightarrow{\mathrm{r}}' - \overrightarrow{\mathrm{r}}\right|}\Big/4\pi\left|\overrightarrow{\mathrm{r}}' - \overrightarrow{\mathrm{r}}\right|\right)\mathrm{dv'}$$

$$\underline{\overrightarrow{\mathrm{E}}}_{\mathrm{ff}} = \hat{\vartheta}\left(\mathrm{j}\eta_o\mathrm{k}\underline{\mathrm{I}}\mathrm{d}/4\pi\mathrm{r}\right)\mathrm{e}^{-\mathrm{jkr}}\sin\theta$$

$$\nabla^2\underline{\Phi} + \omega^2\mu_0\varepsilon_0\underline{\Phi} = -\rho/\varepsilon_0$$

$$\nabla^2\underline{\overrightarrow{\mathrm{A}}} + \omega^2\mu_o\varepsilon_o\underline{\overrightarrow{\mathrm{A}}} = -\mu_o\underline{\overrightarrow{\mathrm{J}}}$$

14.4.5: Forces, Motors, and Generators

$$\vec{f} = q\left(\vec{E} + \vec{v} \times \mu_o \vec{H}\right)\,[N]$$

$$f_z = -dw_T/dz$$

$$\vec{F} = \vec{I} \times \mu_o \vec{H}\,\left[Nm^{-1}\right]$$

$$\vec{E}_e = -\vec{v} \times \mu_o \vec{H} \text{ inside wire}$$

$$P = \omega T = W_T dV_{olume}/dt[W]$$

$$\text{Max } f/A = B^2/2\mu_o,\ D^2/2\varepsilon_o\,\left[Nm^{-2}\right]$$

$$vi = \frac{dw_T}{dt} + f\frac{dz}{dt}$$

$$f = ma = d(mv)/dt$$

$$x = x_o + v_o t + at^2/2$$

$$P = fv\,[W] = T\omega$$

$$w_k = mv^2/2$$

$$T = I\,d\omega/dt$$

$$I = \sum_i m_i r_i^2$$

14.4.6: Circuits

$$KCL: \sum_i I_i(t) = 0 \text{ at node}$$

$$KVL: \sum_i V_i(t) = 0 \text{ around loop}$$

$$C = Q/V = A\varepsilon/d[F]$$

$$L = \Lambda/I$$

$$i(t) = C\,dv(t)/dt$$

$$v(t) = L\,di(t)/dt = d\Lambda/dt$$

$$C_{parallel} = C_1 + C_2$$

$$C_{series} = \left(C_1^{-1} + C_2^{-1}\right)^{-1}$$

$$w_e = Cv^2(t)/2;\ w_m = Li^2(t)/2$$

$$L_{solenoid} = N^2\mu A/W$$

$$\tau = RC, \tau = L/R$$

$$\Lambda = \int_A \vec{B} \bullet d\vec{a} \text{ (per turn)}$$

$$\underline{Z}_{series} = R + j\omega L + 1/j\omega C$$

$$\underline{Y}_{par} = G + j\omega C + 1/j\omega L$$

$$Q = \omega_o w_T/P_{diss} = \omega_o/\Delta\omega$$

$$\omega_o = (LC)^{-0.5}$$

$$\langle v^2(t) \rangle / R = kT$$

14.4.7: Limits to Computation Speed

$$dv(z)/dz = -L di(z)/dt$$

$$di(z)/dz = -C dv(z)/dt$$

$$d^2v/dz^2 = LC\ d^2v/dt^2$$

$$\begin{aligned} v(z,t) &= f_+(t - z/c) + f_-(t + z/c) \\ &= g_+(z - ct) + g_-(z - ct) \end{aligned}$$

$$i(t,z) = Y_o [f_+(t - z/c) - f_-(t + z/c)]$$

$$c = (LC)^{-0.5} = 1/\sqrt{\mu\varepsilon}$$

$$Z_o = Y_o^{-1} = (L/C)^{0.5}$$

$$\Gamma_L = f/f_+ = (R_L - Z_o) / (R_L + Z_o)$$

$$v(z,t) = g_+(z - ct) + g_-(z + ct)$$

$$V_{Th} = 2f_+(t),\ R_{Th} = Z_o$$

14.4.8: Power Transmission

$$\left(d^2/dz^2 + \omega^2 LC\right) \underline{V}(z) = 0$$

$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz}$$

$$\underline{I}(z) = Y_o \left[\underline{V}_+ e^{-jkz} - \underline{V}_- e^{+jkz}\right]$$

$$k = 2\pi/\lambda = \omega/c = \omega(\mu\varepsilon)^{0.5}$$

$$\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = Z_o \underline{Z}_n(z)$$

$$\underline{Z}_n(z) = [1 + \underline{\Gamma}(z)]/[1 - \underline{\Gamma}(z)] = R_n + jX_n$$

$$\underline{\Gamma}(z) = \left(\underline{V} - \underline{V}_+\right) e^{2\,jkz} = [\underline{Z}_n(z) - 1] / [\underline{Z}_n(z) + 1]$$

$$\underline{Z}(z) = Z_0 (\underline{Z}_L - jZ_0 \tan kz) / (\underline{Z}_0 - jZ_L \tan kz)$$

$$VSWR = |\underline{V}_{max}| / |\underline{V}_{min}| = R_{max}$$

14.4.9: Wireless Communications and Radar

$$G(\theta,\phi) = P_r / \left(P_R/4\pi r^2\right)$$

$$P_R = \int_{4\pi} P_r(\theta,\phi,r) r^2 \sin\theta\, d\theta d\phi$$

$$P_{rec} = P_r(\theta,\phi) A_e(\theta,\phi)$$

$$A_e(\theta,\phi) = G(\theta,\phi)\lambda^2/4\pi$$

$$R_r = P_R / \langle i^2(t) \rangle$$

$$E_{ff}(\theta \cong 0) = \left(je^{jkr}/\lambda r\right) \int_A E_t(x,y) e^{jk_x x + jk_y y} dxdy$$

$$P_{rec} = P_R \left(G\lambda/4\pi r^2\right)^2 \sigma_s/4\pi$$

$$\underline{\vec{E}} = \sum_i a_i \underline{\overrightarrow{E}}_i e^{-jkr_1} = (\text{ element factor })(\text{ array f})$$

$$E_{bit} \geq \sim 4 \times 10^{-20} [J]$$

$$\underline{Z}_{12} = \underline{Z}_{21} \text{ if reciprocity}$$

$$\left(d^2/dz^2 + \omega^2 LC\right) \underline{V}(z) = 0$$

$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz}$$

$$\underline{I}(z) = Y_o \left[\underline{V}_+ e^{-jkz} - \underline{V}_- e^{+jkz}\right]$$

$$k = 2\pi/\lambda = \omega/c = \omega(\mu\varepsilon)^{0.5}$$

$$\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = Z_o \underline{Z}_n(z)$$

$$\underline{Z}_n(z) = [1 + \underline{\Gamma}(z)]/[1 - \underline{\Gamma}(z)] = R_n + jX_n$$

$$\underline{\Gamma}(z) = \left(\underline{V}_- / \underline{V}_+\right) e^{2jkz} = [\underline{Z}_n(z) - 1] / [\underline{Z}_n(z) + 1]$$

$$\underline{Z}(z) = Z_0 \left(\underline{Z}_L - jZ_0 \tan kz\right) / \left(\underline{Z}_0 - jZ_L \tan kz\right)$$

$$VSWR = |\underline{V}_{max}| / |\underline{V}_{min}| = R_{max}$$

$$\theta_r = \theta_i$$

$$\sin\theta_t / \sin\theta_i = k_i/k_t = n_i/n_t$$

$$\theta_c = \sin^{-1}(n_t/n_i)$$

$$\theta > \theta_c \Rightarrow \underline{\overrightarrow{E}}_t = \underline{\overrightarrow{E}}_i \underline{T} e^{+\alpha x - jk_z z}$$

$$\underline{\vec{k}} = \overrightarrow{k}' - j\overrightarrow{k}''$$

$$\underline{\Gamma} = \underline{T} - 1$$

$$\text{At } \omega_0, \ \langle w_e \rangle = \langle w_m \rangle$$

$$\langle w_e \rangle = \int_V \left(\varepsilon |\underline{\overrightarrow{E}}|^2 / 4\right) dv$$

$$\langle w_m \rangle = \int_V \left(\mu |\underline{\overrightarrow{H}}|^2 / 4\right) dv$$

$$Q_n = \omega_n W_{Tn}/P_n = \omega_n / 2\alpha_n$$

$$f_{mnp} = (c/2)\left([m/a]^2 + [n/b]^2 + [p/d]^2\right)^{0.5}$$

$$S_n = j\omega_n - \alpha_n$$

14.4.10: Optical Communications

$$E = hf, \text{ photons or phonons}$$

$$hf/c = \text{ momentum } \left[\text{kg ms}^{-1}\right]$$

$$dn_2/dt = -\left[An_2 + B(n_2 - n_1)\right]$$

14.4.11: Acoustics

$$P = P_o + p, \ \overrightarrow{U} = \overrightarrow{U}_o + u \ \left(\overrightarrow{U}_o = 0 \text{ here}\right)$$

$$\nabla p=-\rho_o \partial \vec{u} / \partial t$$

$$\nabla \bullet \vec{u}=-\left(1 / \gamma P_o\right) \partial p / \partial t$$

$$\left(\nabla^2-k^2 \partial^2 / \partial t^2\right) p=0$$

$$k^2=\omega^2 / c_s^2=\omega^2 \rho_o / \gamma P_o$$

$$c_s=v_p=v_g=\left(\gamma P_o / \rho_o\right)^{0.5} \text { or }\left(K / \rho_o\right)^{0.5}$$

$$\eta_s=p / u=\rho_o c_s=\left(\rho_o \gamma P_o\right)^{0.5} \text { gases}$$

$$\eta_s=\left(\rho_o K\right)^{0.5} \text { solids, liquids}$$

$$p, \vec{u}_{\perp} \text { continuous at boundaries}$$

$$\underline{p}=\underline{p}_{+} e^{-jkz}+\underline{p}_{-} e^{+jkz}$$

$$\underline{u}_z=\eta_s^{-1}\left(\underline{p}_{+} e^{-jkz}-\underline{p}_{-} e^{+jkz}\right)$$

$$\int_A \vec{u} p \bullet d \vec{a}+(d / dt) \int_V\left(\rho_o|\vec{u}|^2 / 2+p^2 / 2 \gamma P_o\right) dV$$

14.4.12: Mathematical Identities

$$\sin^2 \theta+\cos^2 \theta=1$$

$$\cos \alpha+\cos \beta=2 \cos[(\alpha+\beta) / 2] \cos[(\alpha-\beta) / 2]$$

$$\underline{H}(f)=\int_{-\infty}^{+\infty} h(t) e^{-j \omega t} dt$$

$$e^x=1+x+x^2 / 2!+x^3 / 3!+\ldots$$

$$\sin \alpha=\left(e^{j \alpha}-e^{-j \alpha}\right) / 2j$$

$$\cos \alpha=\left(e^{j \alpha}+e^{-j \alpha}\right) / 2$$

14.4.13: Vector Algebra

$$\nabla=\hat{x} \partial / \partial x+\hat{y} \partial / \partial y+\hat{z} \partial / \partial z$$

$$\bar{A} \bullet \bar{B}=A_x B_x+A_y B_y+A_z B_z$$

$$\nabla^2 \phi=\left(\partial^2 / \partial x^2+\partial^2 / \partial y^2+\partial^2 / \partial z^2\right) \phi$$

$$\nabla \bullet(\nabla \times \bar{A})=0$$

$$\nabla \times(\nabla \times \bar{A})=\nabla(\nabla \bullet \bar{A})-\nabla^2 \bar{A}$$

14.4.14: Gauss and Stokes' Theorems

$$\oiint_V(\nabla \bullet \vec{G}) dv=\oiint_A \vec{G} \bullet d \vec{a}$$

$$\oiint_A(\nabla \times \vec{G}) \bullet d \vec{a}=\oint_c \vec{G} \bullet d \vec{s}$$

14.4.15: Complex Numbers and Phasors

$$v(t)=R_e\left\{\underline{V} e^{j \omega t}\right\} \text { where } \underline{V}=|V| e^{j \phi}$$

$$e^{j \omega t}=\cos \omega t+j \sin \omega t$$

14.4.16: Spherical Trigonometry

$$\int_{4\pi} \mathrm{r}^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi = 4\pi$$