5.5: Electric and magnetic pressure

5.5.1: Electromagnetic pressures acting on conductors

Forces on materials can be computed in several different ways, all of which can be derived using Maxwell's equations and the Lorentz force law. The pressure method for computing forces arising from static fields is useful because it expresses prior results in ways that are easy to evaluate and remember, and that have physical significance. The method simply notes that the electromagnetic force density (pressure) acting on the interface between two materials equals the difference in the electromagnetic energy densities on the two sides of the interface. Both energy density [J m^{-3}] and pressure [N m^{-2}] have identical units because [J] = [N m].

For example, both the Lorentz force law and the energy method yield the same expression, (5.2.4) and (5.4.3) respectively, for the *electric pressure* P_e due to a static electric field E pushing on a conductor:

$$P_e=-\varepsilon_0 E^2/2\ \left[Nm^{-2}\right] \qquad \text{(electric pressure on conductors)} \tag{5.5.1}$$

The Lorentz force law yields a similar expression (5.2.13) for the *magnetic pressure* pushing on a conductor:

$$\mathrm{P_m}=\mu_o \mathrm{H}^2/2\ \left[\mathrm{Nm}^{-2}\right] \qquad \text{(magnetic pressure on conductors)} \tag{5.5.2}$$

Thus motor and actuator forces are limited principally by the ability of material systems to sustain large static fields without breaking down in some way. Because large magnetic systems can sustain larger energy densities than comparable systems based on electric fields, essentially all large motors, generators, and actuators are magnetic. Only for devices with gaps on the order of a micron or less is the electrical breakdown field strength sufficiently high that electrostatic and magnetic motors compete more evenly with respect to power density, as discussed in Section 6.2.5.

5.5.2: Electromagnetic pressures acting on permeable and dielectric media

The Kelvin polarization and magnetization force densities, (5.3.7) and (5.3.13) respectively, can also be expressed in terms of energy densities and pressures. First we recall that $\overrightarrow{\mathrm{D}}=\varepsilon \overrightarrow{\mathrm{E}}=\varepsilon_o \overrightarrow{\mathrm{E}}+\overrightarrow{\mathrm{P}}$, so $\overrightarrow{\mathrm{P}}=(\varepsilon-\varepsilon_o) \overrightarrow{\mathrm{E}}$. Then it follows from (5.3.7) that the Kelvin polarization force density is:

$$\overrightarrow{\mathrm{F}}_{\mathrm{p}}=\overrightarrow{\mathrm{P}} \bullet \nabla \overrightarrow{\mathrm{E}}=(\varepsilon-\varepsilon_o) \overrightarrow{\mathrm{E}} \bullet \nabla \overrightarrow{\mathrm{E}}\ \left[\mathrm{Nm}^{-3}\right] \tag{5.5.3}$$

The special operator [•∇] is defined in (5.3.6) and explained in (5.5.4). The x component of force density for a curl-free electric field $\overrightarrow{\mathrm{E}}$ is:

$$\begin{aligned}
\mathrm{F_{px}} &=\overrightarrow{\mathrm{P}} \bullet (\nabla \mathrm{E_x})=(\varepsilon-\varepsilon_o) \overrightarrow{\mathrm{E}} \bullet \nabla \mathrm{E_x}=(\varepsilon-\varepsilon_o)\left(\mathrm{E_x} \partial / \partial \mathrm{x}+\mathrm{E_y} \partial / \partial \mathrm{y}+\mathrm{E_z} \partial / \partial \mathrm{z}\right) \mathrm{E_x} & (5.5.4)\\
&=(\varepsilon-\varepsilon_O)\left(\mathrm{E_x} \partial \mathrm{E_x} / \partial \mathrm{x}+\mathrm{E_y} \partial \mathrm{E_y} / \partial \mathrm{x}+\mathrm{E_z} \partial \mathrm{E_z} / \partial \mathrm{x}\right) & (5.5.5)\\
&=(\varepsilon-\varepsilon_O)\left(\mathrm{E_x} \partial \mathrm{E_x^2} / \partial \mathrm{x}+\mathrm{E_y} \partial \mathrm{E_y^2} / \partial \mathrm{x}+\mathrm{E_z} \partial \mathrm{E_z^2} / \partial \mathrm{x}\right) / 2=(\varepsilon-\varepsilon_o)\left(\partial|\overrightarrow{\mathrm{E}}|^2 / \partial \mathrm{x}\right) / 2 & (5.5.6)
\end{aligned}$$

In obtaining (5.5.5) we have used (5.3.8) for a curl-free electric field, for which $\partial E_x/\partial y = \partial E_y/\partial x$ and $\partial E_x/\partial z = \partial E_z/\partial x$.

Equations similar to (5.5.6) can be derived for the y and z components of the force density, which then add:

$$\overrightarrow{\mathrm{F}}_{\mathrm{p}}=(\varepsilon-\varepsilon_o) \nabla|\overrightarrow{\mathrm{E}}|^2 / 2\ \left[\mathrm{Nm}^{-3}\right] \qquad \text{(Kelvin polarization force density)} \tag{5.5.7}$$

A similar derivation applies to the Kelvin magnetization force density $\overrightarrow{\mathrm{F}}_{\mathrm{m}}$. We begin by recalling $\overrightarrow{\mathrm{B}}=\mu_o \overrightarrow{\mathrm{M}} \bullet \nabla \overrightarrow{\mathrm{H}}$, so $\overrightarrow{\mathrm{M}}=\left[(\mu / \mu_o)-1\right] \overrightarrow{\mathrm{H}}$. Then it follows from (5.3.13) that the Kelvin magnetization force density is:

$$\overrightarrow{\mathrm{F}}_{\mathrm{m}}=\mu_o \overrightarrow{\mathrm{M}} \bullet \nabla \overrightarrow{\mathrm{H}}=(\mu-\mu_o) \overrightarrow{\mathrm{H}} \bullet \nabla \overrightarrow{\mathrm{H}}\ \left[\mathrm{Nm}^{-3}\right] \tag{5.5.8}$$

Repeating the steps of (5.5.4–7) yields for curl-free magnetic fields the parallel result:

$$\overrightarrow{\mathrm{F}}_{\mathrm{m}}=(\mu-\mu_o) \nabla|\overrightarrow{\mathrm{H}}|^2 / 2\ \left[\mathrm{Nm}^{-3}\right] \qquad \text{(Kelvin magnetization force density)} \tag{5.5.9}$$

Note that these force density expressions depend only on the field magnitudes $|\overrightarrow{\mathrm{E}}|$ and $|\overrightarrow{\mathrm{H}}|$, not on field directions.

Two examples treated in Chapter 6 using energy methods suggest the utility of simple pressure equations. Figure 6.2.4 shows a parallel-plate capacitor with a dielectric slab that fits snugly between the plates but that is only partially inserted in the z direction a distance D that is much less than the length L of both the slab and the capacitor plates. The electric field between the plates is $\overrightarrow{\mathrm{E}}$, both inside and outside the dielectric slab. The total force on the dielectric slab is the integral of the Kelvin polarization force density (5.5.7) over the volume V of the slab, where V = LA and A is the area of the endface of the slab. We find from (5.5.7) that $\overrightarrow{\mathrm{F}}_{\mathrm{p}}$ is in the $\hat{z}$ direction and is non-zero only near the end of the capacitor plates where z = 0:

$$\mathrm{f}_{\mathrm{z}}=\mathrm{A} \int_{0}^{\mathrm{D}} \mathrm{F}_{\mathrm{pz}} \mathrm{dz}=\mathrm{A}\left[\left(\varepsilon-\varepsilon_{\mathrm{o}}\right) / 2\right] \int_{0}^{\mathrm{D}}\left(\mathrm{d}|\overrightarrow{\mathrm{E}}|^{2} / \mathrm{dz}\right) \mathrm{dz}=\mathrm{A}\left(\varepsilon-\varepsilon_{\mathrm{o}}\right)|\overrightarrow{\mathrm{E}}|^{2} / 2\ [\mathrm{N}] \tag{5.5.10}$$

The integral is evaluated between the limit z = 0 where E ≅ 0 outside the capacitor plates, and the maximum value z = D where the electric field between the plates is $\overrightarrow{\mathrm{E}}$. Thus the pressure method yields the total force f_z on the dielectric slab; it is the area A of the end of the slab, times the electric pressure $\left(\varepsilon-\varepsilon_{\mathrm{o}}\right)|\overrightarrow{\mathrm{E}}|^{2} / 2\ \left[\mathrm{Nm}^{-2}\right]$ at the end of the slab that is pulling the slab further between the plates. This pressure is zero at the other end of the slab because $\overrightarrow{\mathrm{E}} \cong 0$ there. This pressure is the same as will be found in (6.2.21) using energy methods.

The second example is illustrated in Figure 6.4.1, where a snugly fitting cylindrical iron slug of area A has been pulled a distance D into a solenoidal coil that produces an axial magnetic field H. As in the case of the dielectric slab, one end of the slug protrudes sufficiently far from the coil that H at that end is approximately zero. The force pulling on the slug is easily found from (5.5.9):

$$\mathrm{f}_{\mathrm{z}}=\mathrm{A} \int_{0}^{\mathrm{D}} \mathrm{F}_{\mathrm{mz}} \mathrm{dz}=\mathrm{A}\left[\left(\mu-\mu_{\mathrm{o}}\right) / 2\right] \int_{0}^{\mathrm{D}}\left(\mathrm{d}|\overrightarrow{\mathrm{H}}|^{2} / \mathrm{dz}\right) \mathrm{dz}=\mathrm{A}\left(\mu-\mu_{\mathrm{o}}\right)|\overrightarrow{\mathrm{H}}|^{2} / 2\ [\mathrm{N}] \tag{5.5.11}$$

This is more exact than the answer found in (6.4.10), where the μ_o term was omitted in (6.4.10) when the energy stored in the air was neglected.

To summarize, the static electromagnetic pressure [N m^{-2}] acting on a material interface with either free space or mobile liquids or gases is the difference between the two electromagnetic energy densities [J m^{-3}] on either side of that interface, provided that the relevant $\overrightarrow{\mathrm{E}}$ and $\overrightarrow{\mathrm{H}}$ are curl-free. In the case of dielectric or magnetic media, the pressure on the material is directed away from the greater energy density. In the case of conductors, external magnetic fields press on them while electric fields pull; the energy density inside the conductor is zero in both cases because $\overrightarrow{\mathrm{E}}$ and $\overrightarrow{\mathrm{H}}$ are presumed to be zero there.

Note that the pressure method for calculating forces on interfaces is numerically correct even when the true physical locus of the force may lie elsewhere. For example, the Kelvin polarization forces for a dielectric slab being pulled into a capacitor are concentrated at the edge of the capacitor plates at z = 0 in Figure 6.2.4, which is physically correct, whereas the pressure method implies incorrectly that the force on the slab is concentrated at its end between the plate where z = D. The energy method does not address this issue.

Example $5.5.A$

At what radius r from a 1-MV high voltage line does the electric force acting on a dust particle having $\varepsilon = 10\varepsilon_o$ exceed the gravitational force if its density ρ is 1 gram/cm^3? Assume the electric field around the line is the same as between concentric cylinders having radii a = 1 cm and b = 10 m.

Solution

The Kelvin polarization force density (5.5.7) can be integrated over the volume v of the particle and equated to the gravitational force $f_g = \rho v g = \sim 10^{-3} v 10$ [N]. (5.5.7) yields the total Kelvin force:

$$\overrightarrow{\mathrm{f}}_{\mathrm{K}}=\mathrm{v}\left(\varepsilon-\varepsilon_{\mathrm{o}}\right) \nabla|\overrightarrow{\mathrm{E}}|^{2} / 2$$

where

$$\overrightarrow{\mathrm{E}}(\mathrm{r})=\hat{\mathrm{r}} \mathrm{V} /[\mathrm{r} \ln (\mathrm{b} / \mathrm{a})]\left[\mathrm{V}\, \mathrm{m}^{-1}\right].$$

$\nabla|\overrightarrow{\mathrm{E}}|^2=[\mathrm{V}/\ln(\mathrm{b}/\mathrm{a})]^2\nabla\mathrm{r}^{-2}=-2\hat{\mathrm{r}}[\mathrm{V}/\ln(\mathrm{b}/\mathrm{a})]^2\mathrm{r}^{-3}$,

where the gradient here, $\nabla=\hat{\mathrm{r}}\partial/\partial\mathrm{r}$, was computed using cylindrical coordinates (see Appendix C). Thus $\mathrm{f_g}=\left|\overrightarrow{\mathrm{f}}_{\mathrm{K}}\right|$ becomes $10^{-2}\mathrm{v}=\mathrm{v}9\varepsilon_0[\mathrm{V}/\ln(\mathrm{b}/\mathrm{a})]^2\mathrm{r}^{-3}$, so $\mathrm{r}=\left\{900\varepsilon_{\mathrm{o}}[\mathrm{V}/\ln(\mathrm{b}/\mathrm{a})]^2\right\}^{1/3}=\left\{900\times 8.85\times 10^{-12}\left[10^6/\ln(1000)\right]^2\right\}^{1/3}=5.5\text{ meters}$, independent of the size of the particle. Thus high voltage lines make excellent dust catchers for dielectric particles.