7.2: TEM Lines with Junctions

7.2.1: Boundary value problems

A junction between two transmission lines forces the fields in the first line to conform to the fields at the second line at the boundary between the two. This is a simple example of a broad class of problems called boundary value problems. The general electromagnetic *boundary value problem* involves determining exactly which, if any, combination of waves matches any given set of *boundary conditions*, which generally includes both active and passive boundaries, the active boundaries usually being sources. Boundary conditions generally constrain $\vec{E}$ and/or $\vec{H}$ for all time on the boundary of the one-, two- or three-dimensional region of interest.

The uniqueness theorem presented in Section 2.8 states that only one solution satisfies all Maxwell's equations if the boundary conditions are sufficient. Therefore we may solve boundary value problems simply by hypothesizing the correct combination of waves and testing it against Maxwell's equations. That is, we leave undetermined the numerical constants that characterize the chosen combination of waves, and then determine which values of those constraints satisfy Maxwell's equations. This strategy eases the challenge of hypothesizing the final answer directly. Moreover, symmetry and other considerations often suggest the nature of the wave combination required by the problem, thus reducing the numbers of unknown constants that must be determined.

The four basic steps for solving boundary value problems are:

1. Determine the natural behavior of each homogeneous section of the system without the boundaries.
2. Express this general behavior as the superposition of waves or static fields characterized by unknown constants; symmetry and other considerations can minimize the number of waves required. Here our basic building blocks are TEM waves.
3. Write equations for the boundary conditions that must be satisfied by these sets of superimposed waves, and then solve for the unknown constants.
4. Test the resulting solution against any of Maxwell's equations that have not already been imposed.

Variations of this four-step procedure can be used to solve almost any problem by replacing Maxwell's equations with their approximate equivalent for the given problem domain[34]. For example, profitability, available capital, technological constraints, employee capabilities, and customer needs are often "boundary conditions" when deriving strategies for start-up enterprises, while "natural behavior" could include the probable family of behaviors of the entrepreneurial team and its customers, financiers, and suppliers.

[34] A key benefit of a technical education involves learning precise ways of thinking and solving problems; this procedure, when generalized, is an excellent example applicable to almost any career.

7.2.2: Waves at TEM junctions in the time domain

The boundary value problem approach described in Section 7.2.1 can be used for waves at TEM junctions. We assume that an arbitrary incident wave will produce both reflected and transmitted waves. For this introductory problem we also assume that no waves are incident from the other direction, for their solution could be superimposed later. Section 7.2.3 treats the same problem in the complex domain. We represent TEM lines graphically by parallel lines and their characteristic impedance Z_o, as illustrated in Figure 7.2.1 for lines a and b.

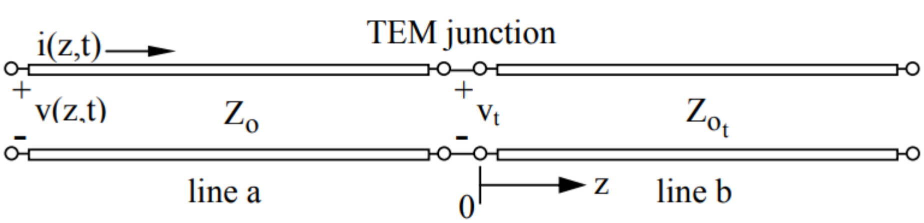

Figure 7.2.1: Junction of two TEM transmission lines.

Step one of the boundary value method involves characterizing the natural behavior of waves in the two media of interest, lines a and b. This follows from (7.1.16) for v(z,t) and (7.1.18) for i(z,t). Step two involves hypothesizing the form of the reflected and transmitted waves, $v_-(z,t)$ and vt(z,t). For simplicity we assume the source $v_+(z,t)$ is on the left, the TEM junction is at z = 0, and the line impedances Z_o are constants independent of time and frequency. Step three is to write the boundary conditions for the waves with unknown constants; v and i must both be constant across the junction at z = 0:

$$\mathrm{v}(\mathrm{z}, \mathrm{t})=\mathrm{v}_{+}(\mathrm{z}, \mathrm{t})+\mathrm{v}_{-}(\mathrm{z}, \mathrm{t})=\mathrm{v}_{\mathrm{t}}(\mathrm{z}, \mathrm{t}) \qquad (\text{ at } \mathrm{z}=0) \tag{7.2.1}$$

$$\mathrm{i}(\mathrm{z}, \mathrm{t})=\mathrm{Z}_{\mathrm{o}}^{-1}\left[\mathrm{v}_{+}(\mathrm{z}, \mathrm{t})-\mathrm{v}_{-}(\mathrm{z}, \mathrm{t})\right]=\mathrm{Z}_{\mathrm{t}}^{-1} \mathrm{v}_{\mathrm{t}}(\mathrm{t}) \qquad (\text{ at } z=0) \tag{7.2.2}$$

Step four involves solving (7.2.1) and (7.2.2) for the unknown waves $v_-(z,t)$ and $v_t(z,t)$. We can simplify the problem by taking the ratios of reflection and transmission relative to the incident wave and provide its amplitude later. If we regard the arguments (z=0, t) as understood, then (7.2.1) and (7.2.2) become:

$$1+\left(\mathrm{v}_{-} / \mathrm{v}_{+}\right)=\mathrm{v}_{\mathrm{t}} / \mathrm{v}_{+} \tag{7.2.3}$$

$$1-\left(\mathrm{v}_{-} / \mathrm{v}_{+}\right)=\left(\mathrm{Z}_{\mathrm{o}} / \mathrm{Z}_{\mathrm{t}}\right) \mathrm{v}_{\mathrm{t}} / \mathrm{v}_{+} \tag{7.2.4}$$

To make the algebra for these two equations still more transparent it is customary to define v_-/v_+ as the *reflection coefficient* Γ, v_t/v_+ as the *transmission coefficient* T, and $Z_t/Z_o = Z_n$ as the *normalized impedance* for line b. Note that v_-, v_+, Z_o, and Z_t are real, and the fraction of incident power that is reflected from a junction is $|\Gamma|^2$. Equations (7.2.3) and (7.2.4) then become:

$$1+\Gamma=T \tag{7.2.5}$$

$$1-\Gamma=\mathrm{T} / \mathrm{Z}_{\mathrm{n}} \tag{7.2.6}$$

Multiplying (7.2.6) by Z_n and subtracting the result from (7.2.5) eliminates T and yields:

$$\Gamma=\frac{\mathrm{v}_{-}}{\mathrm{v}_{+}}=\frac{\mathrm{Z}_{\mathrm{n}}-1}{\mathrm{Z}_{\mathrm{n}}+1} \tag{7.2.7}$$

$$\mathrm{v}_{-}(0, \mathrm{t})=\left[\left(\mathrm{Z}_{\mathrm{n}}-1\right) /\left(\mathrm{Z}_{\mathrm{n}}+1\right)\right] \mathrm{v}_{+}(0, \mathrm{t}) \tag{7.2.8}$$

$$\mathrm{v}_{-}(0+\mathrm{ct})=\left[\left(\mathrm{Z}_{\mathrm{n}}-1\right) /\left(\mathrm{Z}_{\mathrm{n}}+1\right)\right] \mathrm{v}_{+}(0+\mathrm{ct}) \tag{7.2.9}$$

$$\mathrm{v}_{-}(\mathrm{z}+\mathrm{ct})=\left[\left(\mathrm{Z}_{\mathrm{n}}-1\right) /\left(\mathrm{Z}_{\mathrm{n}}+1\right)\right] \mathrm{v}_{+}(\mathrm{z}+\mathrm{ct}) \tag{7.2.10}$$

The transitions to (7.2.9) and (7.2.10) utilized the fact that if two functions of two arguments are equal for all values of their arguments, then the functions remain equal as their arguments undergo the same numerical shifts. For example, if X(a) = Y(b) where a and b have the same units, then X(a + c) = Y(b + c). Combining (7.2.3) and (7.2.7) yields the transmitted voltage v_t in terms of the source voltage v_+:

$$\mathrm{v}_{\mathrm{t}}(\mathrm{z}-\mathrm{ct})=\left[2 \mathrm{Z}_{\mathrm{n}} /\left(\mathrm{Z}_{\mathrm{n}}+1\right)\right] \mathrm{v}_{+}(\mathrm{z}-\mathrm{ct}) \tag{7.2.11}$$

Example 7.2.A

Two parallel plates of width W and separation d_1 = 1 cm are connected at z = D to a similar pair of plates spaced only d_2 = 2 mm apart. If the forward wave on the first line is V_o cos(ωt - kz), what voltage $v_t(t,z)$ is transmitted beyond the junction at z = D?

Solution

$$\mathrm{v}_{\mathrm{t}}(\mathrm{t}, \mathrm{z})=\mathrm{T}_{\mathrm{V}+}(\mathrm{t}, \mathrm{z})=(1+\Gamma) \mathrm{v}_{+}(\mathrm{t}, \mathrm{z})=2 \mathrm{Z}_{\mathrm{n}} \mathrm{V}_{+}(\mathrm{t}, \mathrm{z}) /\left(\mathrm{Z}_{\mathrm{n}}+1\right)$$

where $\mathrm{Z}_{\mathrm{n}}=\mathrm{Z}_{\mathrm{t}} / \mathrm{Z}_{0}=\eta_{0} \mathrm{d}_{2} \mathrm{W} / \eta_{\mathrm{o}} \mathrm{d}_{1} \mathrm{W}=\mathrm{d}_{2} / \mathrm{d}_{1}=0.2$. Therefore for z > D,

$$\mathrm{v}_{\mathrm{t}}(\mathrm{t}, \mathrm{z})=\mathrm{v}_{+}(\mathrm{t}, \mathrm{z}) 2 \times 0.2 /(0.2+1)=\left(\mathrm{V}_{\mathrm{o}} / 3\right) \cos (\omega \mathrm{t}-\mathrm{kz})\ [\mathrm{V}]$$

7.2.3: Sinusoidal waves on TEM transmission lines and at junctions

The basic equations characterizing lossless TEM lines in the sinusoidal steady state correspond to the pair of differential equations (7.1.25) and (7.1.26):

$$\mathrm{d} \underline{\mathrm{V}}(\mathrm{z}) / \mathrm{dz}=-\mathrm{j} \omega \mathrm{L} \underline{\mathrm{I}}(\mathrm{z}) \tag{7.2.12}$$

$$\mathrm{d} \underline{\mathrm{I}}(\mathrm{z}) / \mathrm{dz}=-\mathrm{j} \omega \mathrm{C} \underline{\mathrm{V}}(\mathrm{z}) \tag{7.2.13}$$

L and C are the inductance and capacitance of the line per meter, respectively.

This pair of equations leads easily to the *transmission line wave equation*:

$$\mathrm{d}^2\underline{\mathrm{V}}(\mathrm{z})/\mathrm{dz}^2 = -\omega^2\mathrm{LC}\underline{\mathrm{V}}(\mathrm{z}) \qquad \text{(wave equation)} \tag{7.2.14}$$

The solution $\underline{\mathrm{V}}(\mathrm{z})$ to this wave equation involves exponentials in z because the second derivative of $\underline{\mathrm{V}}(\mathrm{z})$ equals a constant times $\underline{\mathrm{V}}(\mathrm{z})$. The exponents can be + or -, so in general a sum of these two alternatives is possible, where $\underline{\mathrm{V}}_+$ and $\underline{\mathrm{V}}_-$ are complex constants determined later by boundary conditions and k is given by (7.1.30):

$$\underline{V}(z) = \underline{V}_+\mathrm{e}^{-\mathrm{jkz}} + \underline{\mathrm{V}}_-\mathrm{e}^{+\mathrm{jkz}}\ [\mathrm{V}] \qquad \text{(TEM voltage)} \tag{7.2.15}$$

The corresponding current is readily found using (7.2.12):

$$\underline{\mathrm{I}}(\mathrm{z}) = (\mathrm{j}/\omega\mathrm{L})\mathrm{d}\underline{\mathrm{V}}(\mathrm{z})/\mathrm{dz} = (\mathrm{j}/\omega\mathrm{L})\left(-\mathrm{jk}\underline{\mathrm{V}}_+\mathrm{e}^{-\mathrm{jkz}} + \mathrm{jk}\underline{\mathrm{V}}_-\mathrm{e}^{+\mathrm{jkz}}\right) \tag{7.2.16}$$

$$\underline{\mathrm{I}}(z) = (1/Z_0)\left(\underline{\mathrm{V}}_+^{-jkz} - \underline{\mathrm{V}}_-e^{+jkz}\right) \qquad \text{(TEM current)} \tag{7.2.17}$$

where the *characteristic impedance* Z_o of the line is:

$$\mathrm{Z_o} = \mathrm{Y_o^{-1}} = \omega\mathrm{L}/\mathrm{k} = \mathrm{cL} = (\mathrm{L}/\mathrm{C})^{0.5}\ [\text{ohms}] \qquad \text{(characteristic impedance)} \tag{7.2.18}$$

The *characteristic admittance* Y_o of the line is the reciprocal of Z_o, and has units of Siemens or ohms^{-1}. It is important to appreciate the physical significance of Z_o; it is simply the ratio of voltage to current for a wave propagating in one direction only on the line, e.g., for the + wave only. This ratio does not correspond to dissipative losses in the line, although it is related to the power traveling down the line for any given voltage across the line.

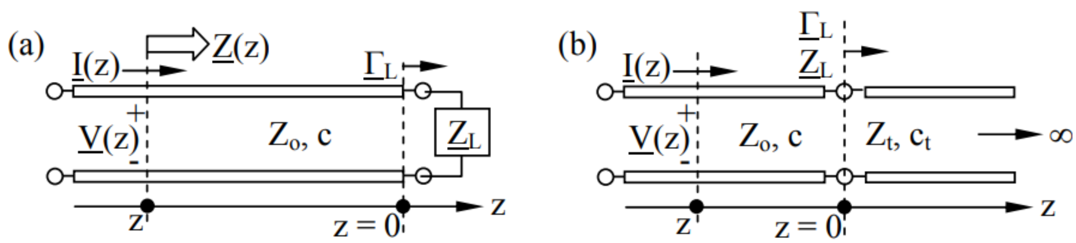

Figure 7.2.2: TEM transmission line impedances and coupling.

When there are both forward and backward waves on a line, the voltage/current ratio is called the complex impedance and varies with position, as suggested in Figure 7.2.2(a). The *impedance* at any point along the line is defined as:

$$\begin{aligned} \underline{Z}(z) &\equiv \underline{V}(z)/\underline{I}(z) = Z_0[1+\underline{\Gamma}(z)]/[1-\underline{\Gamma}(z)] \qquad \text{(line impedance)} \\ &= Z_0[1+\underline{I}(z)][1-\underline{\Gamma}(z)]\ \text{ohms} \end{aligned} \tag{7.2.19}$$

The complex *reflection coefficient* $\underline{\Gamma}(\mathrm{z})$ is defined as:

$$\underline{\Gamma}(z) \equiv \underline{V}_-\mathrm{e}^{+\mathrm{jkz}}/\underline{\mathrm{V}}_+\mathrm{e}^{-\mathrm{jkz}} = \left(\underline{\mathrm{V}}_-/\underline{\mathrm{V}}_+\right)\mathrm{e}^{2\mathrm{jkz}} = \underline{\Gamma}_{\mathrm{L}}\mathrm{e}^{2\mathrm{jkz}} \qquad \text{(reflection coefficient)} \tag{7.2.20}$$

When z = 0 at the load, then $\underline{\mathrm{V}}_-/\underline{\mathrm{V}}_+$ is defined at the load and $\underline{\Gamma}_{\mathrm{L}}$ is the load reflection coefficient, denoted by the subscript L.

Equation (7.2.20) leads to a simple algorithm for relating impedances at different points along the line. We first define normalized impedance $\underline{\mathrm{Z}}_{\mathrm{n}}$ and relate it to the reflection coefficient $\underline{\Gamma}(\mathrm{z})$ using (7.2.19); (7.2.22) follows from (7.2.21):

$$\underline{Z}_n(z) \equiv \frac{\underline{Z}(z)}{Z_o} = \frac{1+\underline{\Gamma}(z)}{1-\underline{\Gamma}(z)} \qquad \text{(normalized impedance)} \tag{7.2.21}$$

$$\underline{\Gamma}(\mathrm{z}) = \frac{\underline{\mathrm{Z}}_{\mathrm{n}}(\mathrm{z})-1}{\underline{\mathrm{Z}}_{\mathrm{n}}(\mathrm{z})+1} \tag{7.2.22}$$

For example, we can see the effect of the load impedance $\underline{Z}_{\mathrm{L}}$ (z = 0) at some other point z on the line by using (7.2.20–22) in an appropriate sequence:

$$\underline{Z}_{\mathrm{L}} \rightarrow \underline{Z}_{\mathrm{Ln}} \rightarrow \underline{\Gamma}_{\mathrm{L}} \rightarrow \underline{\Gamma}(\mathrm{z}) \rightarrow \underline{Z}_{\mathrm{n}}(\mathrm{z}) \rightarrow \underline{Z}(\mathrm{z}) \qquad \text{(impedance transformation)} \tag{7.2.23}$$

A simple example of the use of (7.2.23) is the transformation of a 50-ohm resistor by a 100-ohm line λ/4 long. Using (7.2.23) in sequence, we see $\underline{Z}_L = 50$, $\underline{Z}_{Ln} = 50/100 = 0.5$, $\underline{\Gamma}_L = -1/3$ from (7.2.22), $\underline{\Gamma}(z = -\lambda/4) = +1/3$ from (7.2.20) where

$e^{+2jkz} = e^{2j(2\pi/\lambda)(-\lambda/4)} = e^{-j\pi} = -1$, $\underline{Z}_n(-\lambda/4) = 2$ from (7.2.21), and therefore $\underline{Z}(-\lambda/4) = 200$ ohms .

Two other impedance transformation techniques are often used instead: a direct equation and the Smith chart (Section 7.3). The direct equation (7.2.24) can be derived by first substituting $\underline{\Gamma}_L = (\underline{Z}_L - Z_0)/(\underline{Z}_L + Z_0)$, i.e. (7.2.22), into $\underline{Z}(z) = \underline{V}(z)/\underline{I}(z)$, where $\underline{V}(z)$ and $\underline{I}(z)$ are given by (7.2.15) and (7.2.17), respectively, and $\underline{V}_-/\underline{V}_+ = \underline{\Gamma}_L$. The next step involves grouping the exponentials to yield sin kz and cos kz, and then dividing sin by cos to yield tan and the solution:

$$\underline{Z}(z) = Z_o \frac{\underline{Z}_L - jZ_o \tan kz}{Z_o - j\underline{Z}_L \tan kz} \qquad \text{(transformation equation)} \tag{7.2.24}$$

A closely related problem is illustrated in Figure 7.2.2(b) where two transmission lines are connected together and the right-hand line presents the impedance $\underline{Z}_t$ at z = 0. To illustrate the general method for solving boundary value problems outlined in Section 7.2.1, we shall use it to compute the reflection and transmission coefficients at this junction. The expressions (7.2.15) and (7.2.17) nearly satisfy the first two steps of that method, which involve writing trial solutions composed of superimposed waves with unknown coefficients that satisfy the wave equation within each region of interest. The third step is to write equations for these waves that satisfy the boundary conditions, and then to solve for the unknown coefficients. Here the boundary conditions are that both $\underline{V}$ and $\underline{I}$ are continuous across the junction at z = 0; the subscript t corresponds to the transmitted wave. The two waves on the left-hand side have amplitudes $\underline{V}_+$ and $\underline{V}_-$, whereas the wave on the right-hand side has amplitude $\underline{V}_t$. We assume no energy enters from the right. Therefore:

$$\underline{V}(0) = \underline{V}_+ + \underline{V}_- = \underline{V}_t \tag{7.2.25}$$

$$\underline{I}(0) = (\underline{V}_+ - \underline{V}_-)/Z_0 = \underline{V}_t/\underline{Z}_t \tag{7.2.26}$$

We define the complex reflection and transmission coefficients at the junction (z = 0) to be $\underline{\Gamma}$ and $\underline{T}$, respectively, where:

$$\underline{\Gamma} = \underline{V}_-/\underline{V}_+ \qquad \text{(complex reflection coefficient)} \tag{7.2.27}$$

$$\underline{T} = \underline{V}_t/\underline{V}_+ \qquad \text{(complex transmission coefficient)} \tag{7.2.28}$$

We may solve for $\underline{\Gamma}$ and $\underline{T}$ by first dividing (7.2.25) and (7.2.26) by $\underline{V}_+$:

$$1 + \underline{\Gamma} = \underline{T} \tag{7.2.29}$$

$$1 - \underline{\Gamma} = (Z_o/\underline{Z}_t)\,\underline{T} \tag{7.2.30}$$

This pair of equations is readily solved for $\underline{\Gamma}$ and $\underline{T}$:

$$\underline{\Gamma} = \frac{\underline{Z}_t - Z_o}{\underline{Z}_t + Z_o} = \frac{\underline{Z}_n - 1}{\underline{Z}_n + 1} \tag{7.2.31}$$

$$\underline{T} = \underline{\Gamma} + 1 = \frac{2\underline{Z}n}{\underline{Z}_n + 1} \tag{7.2.32}$$

where normalized impedance was defined in (7.2.21) as $\underline{Z}_n \equiv \underline{Z}_t/Z_o$. For example, (7.2.31) says that the reflection coefficient $\underline{\Gamma}$ is zero when the normalized impedance is unity and the line impedance is matched, so $\underline{Z}_t = Z_0$; (7.2.32) then yields $\underline{T} = 1$.

The complex coefficients $\underline{\Gamma}$ and $\underline{T}$ refer to wave amplitudes, but often it is power that is of interest. In general the time-average power incident upon the junction is:

$$P_+ = \underline{V}_+\underline{I}_+^*/2 = |\underline{V}_+|^2/2Z_o \text{ [W]} \qquad \text{(incident power)} \tag{7.2.33}$$

Similarly the reflected and transmitted powers are P- and P_t, where $P_- = |\underline{V}_-|^2/2Z_0$ and $P_t = |\underline{V}_t|^2/2Z_t$ $[W]$.

Another consequence of having both forward and backward moving waves on a TEM line is that the magnitudes of the voltage and current vary along the length of the line. The expression for voltage given in (7.2.15) can be rearranged as:

$$|\underline{V}(z)| = |\underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz}| = |\underline{V}_+ e^{-jkz}||1 + \underline{\Gamma}(z)| \tag{7.2.34}$$

The magnitude of $|\underline{V}_+ e^{-jk}|$ is independent of z, so the factor $|1 + \underline{\Gamma}(z)|$ controls the magnitude of voltage on the line, where $\underline{\Gamma}(z) = \underline{\Gamma}_L e^{2jkz}$ (7.2.20). Figure 7.2.3(a) illustrates the behavior of |V(z)|; it is quasi-sinusoidal with period λ/2 because of the 2jkz in the exponent. The maximum value $|V(z)|_{max} = |V_+| + |V_-|$ occurs when $\underline{\Gamma}(z) = |\underline{\Gamma}|$.

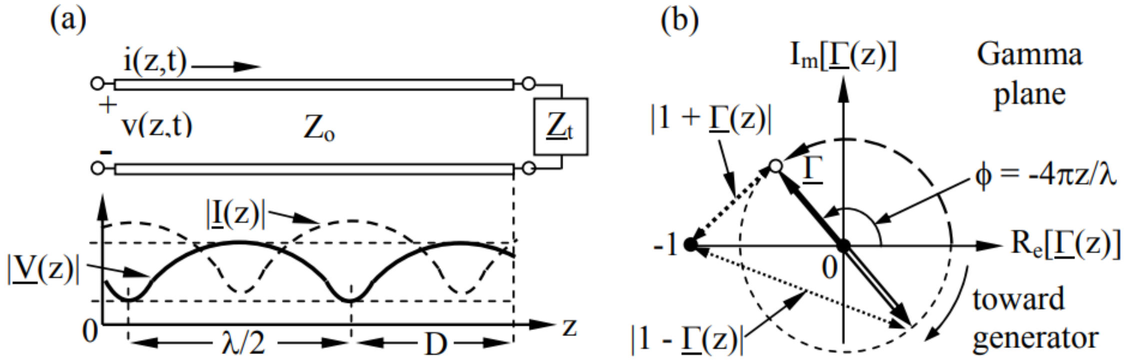

Figure 7.2.3: Standing waves on a TEM line and the Gamma plane.

The origins of this behavior of $|\underline{\mathrm{V}}(\mathrm{z})|$ is suggested in Figure 7.2.3(b), which illustrates the z dependence of $\underline{\Gamma}(\mathrm{z})$ in the complex *gamma plane*, where the horizontal and vertical axes are the real and imaginary parts of $\underline{\Gamma}(\mathrm{z})$, respectively. Increases in z simply rotate the vector $\underline{\Gamma}(\mathrm{z})$ clockwise, preserving its magnitude [see (7.2.20) and Figure 7.2.3(b)].

The quasi-sinusoidal form of $|\underline{\mathrm{V}}(\mathrm{z})|$ arises because $|\underline{V}(z)| \propto |1+\underline{\Gamma}(z)|$, which is the length of the vector linking $\underline{\Gamma}(\mathrm{z})$ with the point -1 on the gamma plane, as illustrated in Figure 7.2.3(b). As the phase ϕ of $\underline{\Gamma}$ varies with z and circles the diagram, the vector $1+\underline{\Gamma}(\mathrm{z})$ varies as might an arm turning a crank, and so it is sometimes called the "crank diagram". When $|\underline{\Gamma}| << 1$ then $|\underline{\mathrm{V}}(\mathrm{z})|$ resembles a weak sinusoid oscillating about a mean value of $|\underline{V}_+|$, whereas when $|\underline{\Gamma}| \cong 1$ then $|\underline{\mathrm{V}}(\mathrm{z})|$ resembles a fully rectified sinusoid. The voltage envelope $|\underline{\mathrm{V}}(\mathrm{z})|$ is called the standingwave pattern, and fields have a standing-wave component when $|\underline{\Gamma}| > 0$. The figure also illustrates how $|\underline{I}(z)| \propto |1-\underline{\Gamma}(z)|$ exhibits the same quasi-sinusoidal variation as $|\underline{V}(z)|$, but 180 degrees out of phase.

Because $|V(z)|$ and $|I(z)|$ are generally easy to measure along any transmission line, it is useful to note that such measurements can be used to determine not only the fraction of power that has been reflected from any load, and thus the efficiency of any connection, but also the impedance of the load itself. First we define the *voltage standing wave ratio* or *VSWR* as:

$$\mathrm{VSWR} \equiv |\underline{\mathrm{V}}(\mathrm{z})|_{\max}/\underline{\mathrm{V}}(\mathrm{z})|_{\min} = \left(|\underline{\mathrm{V}}_+|+|\underline{\mathrm{V}}_-|\right) / \left(|\underline{\mathrm{V}}_+|-|\underline{\mathrm{V}}_-|\right) = (1+|\underline{\Gamma})/(1-|\underline{\Gamma}) \tag{7.2.35}$$

Therefore:

$$|\underline{\Gamma}| = (\mathrm{VSWR}-1)/(\mathrm{VSWR}+1) \tag{7.2.36}$$

$$\mathrm{P}_-/\mathrm{P}_+ = |\underline{\Gamma}|^2 = \left[(\mathrm{VSWR}-1)/(\mathrm{VSWR}+1)\right]^2 \tag{7.2.37}$$

This simple relation between VSWR and fractional power reflected (P_-/P_+) helped make VSWR a common specification for electronic equipment.

To find the load impedance $\underline{\mathrm{Z}}_\mathrm{L}$ from observations of $|\underline{V}(\mathrm{z})|$ such as those plotted in Figure 7.2.3(a) we first associate any voltage minimum with that point on the gamma plane that corresponds to $-|\underline{\Gamma}|$. Then we can rotate on the gamma plane counter-clockwise (toward the load) an angle ϕ = 2kD = 4πD/λ radians that corresponds to the distance D between that voltage minimum and the load, where a full revolution in the gamma plane corresponds to D = λ/2. Once $\underline{\Gamma}$ for the load is determined, it follows from (7.2.21) that:

$$\underline{\mathrm{Z}}_\mathrm{L} = \mathrm{Z}_\mathrm{o}[1+\underline{\Gamma}]/[1-\underline{\Gamma}] \tag{7.2.38}$$

If more than two TEM lines join a single junction then their separate impedances combine in series or parallel, as suggested in Figure 7.2.4. The impedances add in parallel for Figure 7.2.4(a) so the impedance at the junction as seen from the left would be:

$$\underline{\mathrm{Z}}_{\text{parallel}} = \underline{\mathrm{Z}}_\mathrm{a}\mathrm{Z}_\mathrm{b}/\left(\mathrm{Z}_\mathrm{a}+\underline{\mathrm{Z}}_\mathrm{b}\right) \tag{7.2.39}$$

For Figure 7.2.4(b) the lines are connected in series so the impedance seen from the left would be $\underline{\mathrm{Z}}_\mathrm{a}+\underline{\mathrm{Z}}_\mathrm{b}$.

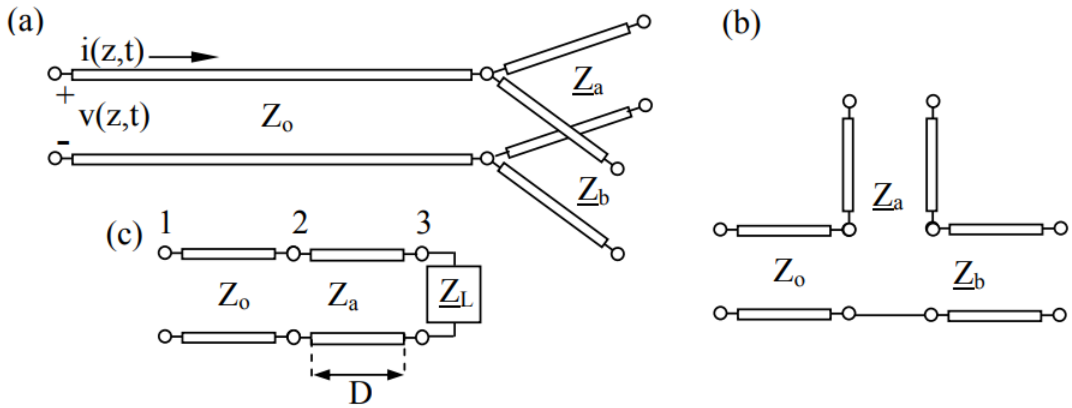

Figure 7.2.4: Multiple connected TEM lines.

Figure 7.2.4(c) illustrates how TEM lines can be concatenated. In this case the impedance $\underline{Z}_1$ seen at the left-hand terminals could be determined by transforming the impedance $\underline{Z}_L$ at terminals (3) to the impedance $\underline{Z}_2$ that would be seen at terminals (2). The impedance seen at (2) could then be transformed a second time to yield the impedance seen at the left-hand end. The algorithm for this might be:

$$\underline{Z}_L \rightarrow \underline{Z}_{Ln} \rightarrow \underline{\Gamma}_3 \rightarrow \underline{\Gamma}_2 \rightarrow \underline{Z}_{n2} \rightarrow \underline{Z}_2 \rightarrow \underline{Z}_{n2'} \rightarrow \underline{\Gamma}_{2'} \rightarrow \underline{\Gamma}_1 \rightarrow \underline{Z}_{nl} \rightarrow \underline{Z}_1 \tag{7.2.40}$$

Note that Z_{n2} is normalized with respect to Z_a and Z_{n2}' is normalized with respect to Z_o; both are defined at junction (2). Also, $\underline{\Gamma}_2$ is the reflection coefficient at junction (2) within the line Z_a, and $\underline{\Gamma}_2'$ is the reflection coefficient at junction (2) within the line Z_o.

Example 7.2.B

A 100-ohm air-filled TEM line is terminated at z = 0 with a capacitor C = 10-11 farads. What is $\underline{\Gamma}(z)$? At what positions z < 0 are voltage minima located on the line when f = 1/2π GHz? What is the VSWR? At z = - λ/4, what is the equivalent impedance?

Solution

The normalized load impedance $Z_L/Z_o \equiv Z_{Ln} = 1/j\omega C Z_o = -j/\left(10^9 \times 10^{-11} \times 100\right) = -j$, and (7.2.22) gives $\underline{\Gamma}_L = (\underline{Z}_{Ln} - 1)/(\underline{Z}_{Ln} + 1) = -(1+j)/(1-j) = -j$. $\underline{\Gamma}(z) = \underline{\Gamma}_L e^{2jkz} = -je^{2jkz}$. (7.2.34) gives $|\underline{V}(z)| \propto |1 + \underline{\Gamma}(z)| = \left|1 - je^{2jkz}\right| = 0$ when $e^{2jkz} = -j = e^{-j(\pi/2 + n2\pi)}$, where n =0,1,2… Therefore 2jkz = -j(π/2 + n2π), so z(nulls) = -(π/2 + n2π)λ/4π = -(λ/8)(1 + 4n). But f = $10^9/2\pi$, and so λ = c/f = $2\pi c\times10^{-9}$ = 0.6π [m]. (7.2.34) gives $\mathrm{VSWR} = (1+|\underline{\Gamma}|)/(1-|\underline{\Gamma}|) = \infty$. At z = -λ/4, $\underline{\Gamma} \rightarrow -\underline{\Gamma}_L = +j$ via (7.2.20), so by (7.2.38)

$$\underline{Z} = Z_o[1+\underline{\Gamma}]/[1-\underline{\Gamma}] = 100[1+j]/[1-j] = j100 = j\omega L_o \Rightarrow L_o = 100/\omega = 100/10^9 = 10^{-7}\ [\mathrm{H}]$$

Example 7.2.C

The VSWR observed on a 100-ohm air-filled TEM transmission line is 2. The voltage minimum is 15 cm from the load and the distance between minima is 30 cm. What is the frequency of the radiation? What is the impedance $\underline{Z}_L$ of the load?

Solution

The distance between minima is λ/2, so λ = 60 cm and f = c/λ = 3×10^8 /0.6 = 500 MHz. The load impedance is $\underline{Z}_L = Z_o\left[1+\underline{\Gamma}_L\right]/\left[1-\underline{\Gamma}_L\right]$ (7.2.38) where $|\underline{\Gamma}_L| = (\mathrm{VSWR}-1)/(\mathrm{VSWR}+1) = 1/3$ from (5.2.83). $\underline{\Gamma}_L$ is rotated on the Smith chart 180 degrees counter-clockwise (toward the load) from the voltage minimum, corresponding to a quarter wavelength. The voltage minimum must lie on the negative real $\underline{\Gamma}$ axis, and therefore $\underline{\Gamma}_L$ lies on the positive real $\underline{\Gamma}$ axis. Therefore $\underline{\Gamma}_L = 1/3$ and $\underline{Z}_L = 100(1+1/3)/(1-1/3) = 200$ ohms.