10.1: Radiation from charges and currents

10.1.1: Introduction to antennas and radiation

An antenna is a device that couples currents to electromagnetic waves for purposes of radiation or reception. The process by which antennas radiate can be easily understood in terms of the way in which accelerating charged particles or time-varying currents radiate, which is discussed in Section 10.1. The expressions for radiated electromagnetic fields derived in Section 10.1.4 are simple extensions of those derived in Sections 10.1.2 and 10.1.3 for the fields produced by static charges and currents, respectively.

Using the basic expressions for radiation derived in Section 10.1, simple short dipole antennas are shown in Section 10.2 to have stable directional properties far from the antenna (the antenna far field), and different directional properties closer than ~$\lambda/2\pi$ (the antenna near field). In Section 10.3 these properties are related to basic metrics that characterize each antenna, such as gain, effective area, and impedance. These metrics are then related to the performance of various communications systems. Antenna arrays are discussed in Section 10.4, followed by aperture and more complicated wire antennas in Sections 11.1 and 11.2, respectively.

10.1.2: Electric fields around static charges

One simple way to generate electromagnetic waves is to vibrate electric charges, creating time-varying current. The equation characterizing this radiation is very similar to that characterizing the electric fields produced by a single static charge, which is developed below. Section 10.1.3 extends this result to magnetic fields produced by moving charges.

Faraday's and Gauss's laws for static charges in vacuum are:

$$\nabla \times \overrightarrow{\mathrm{E}} = 0 \tag{10.1.1}$$

$$\nabla \bullet \varepsilon_0 \overrightarrow{\mathrm{E}} = \rho \tag{10.1.2}$$

Since the curl of $\overrightarrow{\mathrm{E}}$ is zero, $\overrightarrow{\mathrm{E}}$ can be the gradient of any arbitrary scalar function $\Phi(\overrightarrow{\mathrm{r}})$ and still satisfy (10.1.1). That is:

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}) = -\nabla\Phi(\overrightarrow{\mathrm{r}}) \tag{10.1.3}$$

where Φ is the *scalar electric potential* and is in units of Volts. The negative sign is consistent with $\overrightarrow{\mathrm{E}}$ pointing away from regions of high potential and toward lower potentials. Note that (10.1.3) satisfies (10.1.1) because $\nabla \times (-\nabla\Phi) \equiv 0$ is an identity, and that a simple three dimensional scalar field Φ fully characterizes the three-dimensional vector electric field $\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}})$. It is therefore often easiest to find the electric potential $\Phi(\overrightarrow{\mathrm{r}})$ before computing the electric field produced by static source charges.

If the charge q [Coulombs] is spherically symmetric, both Φ and $\overrightarrow{\mathrm{E}}$ must also be spherically symmetric. The only way a vector field can be spherically symmetric is for it to be directed radially, so:

$$\overrightarrow{\mathrm{E}} = \hat{r}\mathrm{E_r}(\mathrm{r}) \tag{10.1.4}$$

where r is the radius from the origin where the charge is centered and $\mathrm{E_r}(\mathrm{r})$ is the radial field. We can now relate $\overrightarrow{\mathrm{E}}$ to q by applying Gauss's divergence theorem (2.4.6) to the volume integral of Gauss's law (10.1.2):

$$\begin{aligned} \iiint f_{\mathrm{V}} \left(\nabla \bullet \varepsilon_0 \overrightarrow{\mathrm{E}}\right) \mathrm{dv} &= \iiint_{\mathrm{V}} \rho \mathrm{dv} = \mathrm{q} = \oiint_{\mathrm{A}} \varepsilon_0 \overrightarrow{\mathrm{E}} \bullet \hat{n} \mathrm{da}\, n \\ &= \oiint_{\mathrm{A}} \varepsilon_0 \hat{n} \bullet \hat{r} \mathrm{E_r}(\mathrm{r}) \mathrm{da} = 4\pi \mathrm{r}^2 \varepsilon_o \mathrm{E_r}(\mathrm{r}) \end{aligned} \tag{10.1.5}$$

Therefore the electric field produced by a charge q at the origin is:

$$\overrightarrow{\mathrm{E}}(\overrightarrow{\mathrm{r}}) = \hat{r}\mathrm{E_r}(\mathrm{r}) = \hat{r}\mathrm{q}/4\pi\varepsilon_0\mathrm{r}^2 = -\nabla\Phi \ \left[\mathrm{Vm}^{-1}\right] \tag{10.1.6}$$

To find the associated scalar potential Φ we integrate (10.1.6) using the definition of the *gradient operator*:

$$\nabla\Phi \equiv\left(\hat{x}\frac{\partial}{\partial \mathrm{x}}+\hat{y}\frac{\partial}{\partial \mathrm{y}}+\hat{z}\frac{\partial}{\partial \mathrm{z}}\right)\Phi \qquad \text{(gradient in Cartesian coordinates)} \tag{10.1.7}$$

$$=\left[\hat{r}\frac{\partial}{\partial \mathrm{r}}+\hat{\theta}\frac{1}{\mathrm{r}}\frac{\partial}{\partial\theta}+\hat{\phi}\frac{1}{\mathrm{r}\sin\theta}\frac{\partial}{\partial\phi}\right]\Phi \qquad \text{(gradient in spherical coordinates)} \tag{10.1.8}$$

Since the spherically symmetric potential Φ (10.1.6) is independent of θ and ϕ, it follows that $\partial/\partial\theta=\partial/\partial\phi=0$ and Equation (10.1.8) becomes:

$$\nabla\Phi=\hat{r}\partial\Phi/\partial r \tag{10.1.9}$$

This mathematical simplification occurs only in spherical coordinates, not Cartesian. Substitution of (10.1.9) into (10.1.6), followed by integration of (10.1.6) with respect to radius r, yields:

$$\Phi(\overrightarrow{\mathrm{r}})=\int\left(\mathrm{q}/4\pi\varepsilon_{\mathrm{o}}\mathrm{r}^{2}\right)\mathrm{dr}=\Phi_{\mathrm{o}}+\mathrm{q}/4\pi\varepsilon_{\mathrm{o}}\mathrm{r}=\mathrm{q}/\left(4\pi\varepsilon_{\mathrm{o}}\mid\overrightarrow{\mathrm{r}}\right) \tag{10.1.10}$$

where we define as zero the electric potential Φ_o contributed by any charge infinitely far away.

The solution for the electric potential Φ due to charge q at some position $\overrightarrow{\mathrm{r}}_{\mathrm{q}}$ other than the origin follows from (10.1.10):

$$\Phi(\overrightarrow{\mathrm{r}})=\mathrm{q}/\left(4\pi\varepsilon_{\mathrm{o}}\left|\overrightarrow{\mathrm{r}}-\overrightarrow{\mathrm{r}}_{\mathrm{q}}\right|\right)=\mathrm{q}/\left(4\pi\varepsilon_{\mathrm{o}}\mathrm{r}_{\mathrm{pq}}\right)\ [\mathrm{V}] \tag{10.1.11}$$

which can alternatively be written using subscripts p and q to refer to the locations $\overrightarrow{\mathrm{r}}_{\mathrm{p}}$ and $\overrightarrow{\mathrm{r}}_{\mathrm{q}}$ of the person (or observer) and the charge, respectively, and r_{pq} to refer to the distance $\left|\overrightarrow{\mathrm{r}}_{\mathrm{p}}-\overrightarrow{\mathrm{r}}_{\mathrm{q}}\right|$ between them.

If we replace the charge q with a charge density ρ_q in the infinitesimal volume dv, then we can integrate (10.1.11) over the source region to obtain the total static *electric potential* produced by an arbitrary charge distribution ρ_q:

$$\Phi_{\mathrm{p}}=\iiint_{\mathrm{V}_{\mathrm{q}}}\left[\rho_{\mathrm{q}}/\left(4\pi\varepsilon_{\mathrm{o}}\mathrm{r}_{\mathrm{pq}}\right)\right]\mathrm{dv}\ [\mathrm{V}] \qquad \text{(scalar Poisson integral)} \tag{10.1.12}$$

This integration to find Φ_p can be performed because Maxwell’s equations are linear so that superposition applies. Thus we have a simple way to compute Φ_p and $\overrightarrow{\mathrm{E}}$ for any arbitrary static charge density distribution ρ_q. This *scalar Poisson integral* for the potential function Φ is similar to that found for dynamic charge distributions in the next section. The integral (10.1.12) is also a solution to the *Poisson equation*:

$$\nabla^{2}\Phi=-\rho/\varepsilon_{0} \qquad \text{(Poisson equation)} \tag{10.1.13}$$

which follows from computing the divergence of Gauss’s law:

$$\nabla\bullet\{\nabla\Phi=-\overrightarrow{\mathrm{E}}\}\Rightarrow\nabla^{2}\Phi=-\nabla\bullet\overrightarrow{\mathrm{E}}=-\rho/\varepsilon_{0} \tag{10.1.14}$$

Poisson’s equation reduces to *Laplace’s equation*, $\nabla^2\Phi=0$, when ρ = 0.

10.1.3: Magnetic fields around static currents

Maxwell’s equations governing static magnetic fields in vacuum are:

$$\nabla\times\overrightarrow{\mathrm{H}}=\overrightarrow{\mathrm{J}} \qquad \text{(static Ampere's law)} \tag{10.1.15}$$

$$\nabla\bullet\mu_{0}\overrightarrow{\mathrm{H}}=0 \qquad \text{(Gauss's law)} \tag{10.1.16}$$

Because the divergence of $\overrightarrow{\mathrm{H}}$ is always zero, we can define the magnetic flux density in vacuum as being:

$$\overrightarrow{\mathrm{B}}=\mu_{\mathrm{o}}\overrightarrow{\mathrm{H}}=\nabla\times\overrightarrow{\mathrm{A}} \tag{10.1.17}$$

where $\overrightarrow{\mathrm{A}}$ is defined as the *magnetic vector potential*, which is a vector analog to Φ. This very general expression for Φ always satisfies Gauss’s law: $\nabla\bullet(\nabla\times\overrightarrow{\mathrm{A}})\equiv 0$.

Substituting (10.1.17) into Ampere’s law (10.1.15) results in:

$$\nabla \times (\nabla \times \vec{A}) = \mu_o \vec{J} \tag{10.1.18}$$

This can be simplified using the vector identity:

$$\nabla \times (\nabla \times \vec{A}) \equiv \nabla(\nabla \bullet \vec{A}) - \nabla^2 \vec{A} \tag{10.1.19}$$

where we note that $\nabla \bullet \vec{A}$ is arbitrary and does not impact any of our prior equations; therefore we set it equal to zero. Then (10.1.18) becomes the *vector Poisson equation*:

$$\nabla^2 \vec{A} = -\mu_o \vec{J} \qquad (\text{ vector Poisson equation }) \tag{10.1.20}$$

The three vector components of (10.1.20) are each scalar Poisson equations identical to (10.1.13) except for the constant, so the solution is nearly identical to (10.1.12) once the constants have been reconciled; this solution is:

$$\vec{A}_p = \iiint_{V_q} \left[\mu_o \vec{J}_q / (4\pi r_{pq})\right] dv \ \left[\mathrm{Vsm}^{-1}\right] \tag{10.1.21}$$

Thus we have a simple way to compute $\vec{A}$ and therefore $\vec{B}$ for any arbitrary static current distribution $\vec{J}_q$.

10.1.4: Electromagnetic fields produced by dynamic charges

In the static case of Section 10.1.2 it was very helpful to define the potential functions $\vec{A}$ and Φ, and the time-dependent Maxwell's equations for vacuum permit us to do so again:

$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t \qquad (\text{Faraday's law}) \tag{10.1.22}$$

$$\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t \qquad (\text{Ampere's law}) \tag{10.1.23}$$

$$\nabla \bullet \vec{E} = \rho / \varepsilon_o \qquad (\text{Gauss's law}) \tag{10.1.24}$$

$$\nabla \bullet \vec{B} = 0 \qquad (\text{Gauss's law}) \tag{10.1.25}$$

Although the curl of $\vec{E}$ is no longer zero so that $\vec{E}$ no longer equals the gradient of some potential Φ, we can satisfy $\nabla \bullet \vec{B} = 0$ if we define a vector potential $\vec{A}$ such that:

$$\vec{B} = \nabla \times \vec{A} = \mu_o \vec{H} \tag{10.1.26}$$

This definition of $\vec{A}$ always satsifies Gauss's law: $\nabla \bullet (\nabla \times \vec{A}) \equiv 0$. Substituting $\nabla \times \vec{A}$ for $\vec{B}$ in Faraday's law yields:

$$\nabla \times \vec{E} = -\partial(\nabla \times \vec{A}) / \partial t \tag{10.1.27}$$

Rearranging terms yields:

$$\nabla \times (\vec{E} + \partial \vec{A} / \partial t) = 0 \tag{10.1.28}$$

which implies that the quantity $(\vec{E} + \partial \vec{A} / \partial t)$ can be the gradient of any potential function Φ:

$$\vec{E} + \partial \vec{A} / \partial t = -\nabla \Phi \tag{10.1.29}$$

$$\vec{E} = -(\partial \vec{A} / \partial t + \nabla \Phi) \tag{10.1.30}$$

Thus dynamic electric fields have two components—one due to the instantaneous value of $\Phi(t)$, and one proportional to the time derivative of $\vec{A}$.

We can now use the vector identity (10.1.19) to simplify Ampere's law after $(\nabla \times \vec{A}) / \mu_o$ replaces $\vec{H}$:

$$\nabla \times (\nabla \times \vec{A}) = \mu_O(\vec{J} + \partial\vec{D}/\partial t) = \nabla(\nabla \bullet \vec{A}) - \nabla^2\vec{A} \tag{10.1.31}$$

In the earlier static case we let $\nabla \bullet \vec{A} = 0$ because specifying the curl of a vector field ($\vec{B} = \nabla \times \vec{A}$) does not constrain its divergence, which can be independently chosen[51]. Here we can let:

$$\nabla \bullet \vec{A} = -\mu_o\varepsilon_0\partial\Phi/\partial t \tag{10.1.32}$$

[51] Let $\vec{A} = \nabla\Phi + \nabla \times \vec{N}$; then $\nabla \times \vec{A} = \nabla \times (\nabla \times \vec{N})$ and $\nabla \bullet \vec{A} = \nabla^2\Phi$, so $\nabla \times \vec{A}$ and $\nabla \bullet \vec{A}$ can be chosen independently simply by choosing $\vec{N}$ and Φ independently.

This reduces (10.1.32) to a simple equation by eliminating its second term, yielding:

$$\nabla^2\vec{A} - \mu_0\varepsilon_0\partial^2\vec{A}/\partial t^2 = -\mu_o\vec{J} \tag{10.1.33}$$

which is called the inhomogeneous vector *Helmholtz equation* (the homogeneous version has no source term on the right hand side; $\vec{J} = 0$). It is a wave equation for $\vec{A}$ driven by the current source $\vec{J}$.

A similar inhomogeneous wave equation relating the electric potential Φ to the charge distribution ρ can also be derived. Substituting (10.1.30) into Gauss's law (10.1.24) yields:

$$\nabla \bullet \vec{E} = -\nabla \bullet (\partial\vec{A}/\partial t + \nabla\Phi) = -\partial(\nabla \bullet \vec{A})/\partial\tau - \nabla^2\Phi \tag{10.1.34}$$

Replacing $\nabla \bullet \vec{A}$ using (10.1.33) then produces:

$$\nabla \bullet \vec{E} = \mu_o\varepsilon_o\partial^2\Phi/\partial t^2 - \nabla^2\Phi = \rho/\varepsilon_o \tag{10.1.35}$$

which is more commonly written as the inhomogeneous scalar Helmholtz equation:

$$\nabla^2\Phi - \mu_0\varepsilon_0\partial^2\Phi/\partial t^2 = -\rho/\varepsilon_0 \tag{10.1.36}$$

analogous to the vector version (10.1.34) for $\vec{A}$. These inhomogeneous scalar and vector Helmholtz equations, (10.1.34) and (10.1.37), permit us to calculate the electric and magnetic potentials and fields produced anywhere in vacuum as a result of arbitrary source charges and currents, as explained below.

The solutions to the Helmholtz equations must reduce to: a) the traveling-wave solutions [e.g., (2.2.9)] for the wave equation [e.g., (2.2.7)] when the source terms are zero, and b) the static solutions (10.1.10) and (10.1.21) when $\partial/\partial t = 0$. The essential feature of solutions to wave equations is that their separate dependences on space and time must have the same form because their second derivatives with respect to space and time are identical within a constant multiplier. These solutions can therefore be expressed as an arbitrary function of a single argument that sums time and space, e.g. (z - ct) or ($t - r_{pq}/c$). The solutions must also have the form of the static solutions because they reduce to them when the source is static. Thus the solutions to the Helmholtz inhomogeneous equations are the static solutions expressed in terms of the argument ($t - r_{pq}/c$):

$$\Phi_p = \oiiint_{V_q} [\rho_q(t - r_{pq}/c)/(4\pi\varepsilon_o r_{pq})]\, dv\ [V] \tag{10.1.37}$$

$$\vec{A}_p = \oiiint_{V_q} \left[\mu_o\vec{J}_q(t - r_{pq}/c)/(4\pi r_{pq})\right] dv\ \left[Vsm^{-1}\right] \tag{10.1.38}$$

These solutions are the dynamic scalar *Poisson integral* and the dynamic vector Poisson integral, respectively. Note that Φ_p and $\vec{A}_p$ depend on the state of the sources at some time in the past, not on their instantaneous values. The delay r_{pq}/c is the ratio of the distance r_{pq} between the source and observer, and the velocity of light c. That is, r_{pq}/c is simply the propagation time between source and observer.