5.4: Forces Computed Using Energy Methods

5.4.1: Relationship between force and energy

Mechanics teaches that a force f in the z direction pushing an object a distance dz expends energy dw = f dz [J], so:

$$\mathrm{f}=\mathrm{dw} / \mathrm{dz} \qquad \text{(force/energy equation)} \tag{5.4.1}$$

Therefore the net force f_{be} applied by the environment to any object in the z direction can be found simply by differentiating the total system energy w with respect to motion of that object in the direction z. The total force vector $\vec{f}_{be}$ is the sum of its x, y, and z components.

Care must be taken, however, to ensure that the total system energy is differentiated, which can include the energy in any connected power supplies, mechanical elements, etc. Care must also be taken to carefully distinguish between forces f_{be} exerted by the environment, and forces f_{oe} exerted by objects on their environment; otherwise sign errors are readily introduced. This simple powerful approach to finding forces is illustrated in Section 5.4.2 for electrostatic forces and in Section 5.6 for photonic forces. The energy approach to calculating magnetic forces uses (5.4.1) in a straightforward way, but examples are postponed to Chapter 6 when magnetic fields in structures will be better understood.

Example $5.4.A$

A certain perfectly conducting electromagnet carrying one ampere exerts an attractive 100-N force f on a piece of iron while it moves away from the magnet at velocity v = 1 [m s^{-1}]. What voltage V is induced across the terminals of the electromagnet as a result of this velocity v? Is this voltage V positive or negative on that terminal where the current enters the magnet? Use conservation of power.

Solution

Conservation of power requires fv = VI, so V = fv/I = 100×1/1 = 100 volts. The voltage is negative because the magnet is acting as a generator since the motion of the iron is opposite to the magnetic force acting on it.

5.4.2: Electrostatic forces on conductors and dielectrics

The energy method easily yields the force f_{be} needed to separate in the z direction the two isolated capacitor plates oppositely charged with Q in vacuum and illustrated in Figure 5.2.1(a). Since the plates are attracted to one another, separating them does work and increases the stored energy w. The force needed to hold the plates apart is easily found using the force/energy equation (5.4.1):

$$\mathrm{f}_{\mathrm{be}}=\mathrm{dw} / \mathrm{dz}=\mathrm{d}\left(\mathrm{Q}^{2} \mathrm{s} / 2 \varepsilon_{\mathrm{o}} \mathrm{A}\right) / \mathrm{ds}=\mathrm{Q}^{2} / 2 \varepsilon_{\mathrm{o}} \mathrm{A}\ [\mathrm{N}] \tag{5.4.2}$$

where the plate separation is s and the plate area is A. The electric energy w_e stored in a capacitor C is $CV^2/2 = Q^2/2C = Q^2s/2\varepsilon A$, where Q = CV and C = εA/s, as shown in Section 3.1.3. Here we assumed $\varepsilon = \varepsilon_o$.

The derivative in (5.4.2) was easy to evaluate because Q remains constant as the disconnected plates are forced apart. It would be incorrect to use $w = CV^2/2$ when differentiating (5.4.2) unless we recognize that V increases as the plates separate because V = Q/C when C decreases. It is easier to express energy in terms of parameters that remain constant as z changes.

We can put (5.4.2) in the more familiar form (5.2.4) for the electric pressure P_e pushing on a conductor by noting that the force f_{be} needed to separate the plates is the same as the electric force attracting the oppositely charged plates. The force f_{be} thus balances the electric pressure on the same plates and $P_e = -f_{be}/A$. Since $Q = \varepsilon_o EA$ here we find:

$$\mathrm{P}_{\mathrm{e}}=-\mathrm{Q}^{2} / 2 \varepsilon_{\mathrm{o}} \mathrm{A}^{2}=-\varepsilon_{\mathrm{o}} \mathrm{E}^{2} / 2 \quad\left[\mathrm{Nm}^{-2}\right] \tag{5.4.3}$$

This static attractive pressure of electric fields remains the same if the plates are connected to a battery of voltage V instead of being isolated; the Lorentz forces are the same in both cases. A more awkward way to calculate the same force (5.4.2) is to assume (unnecessarily) that a battery is connected and that V remains constant as s changes. In this case Q must vary with dz, and dQ flows into the battery, increasing its energy by VdQ. Since dw in the force/energy expression (5.4.2) is the change in total system energy, the changes in both battery and electric field energy must be calculated to yield the correct energy; an example with a battery begins later with (5.4.5). As illustrated above, this complexity can be avoided by carefully restating the problem without the source, and by expressing w in terms of electrical variables (Q here) that do not vary with position (s here).

The power of the energy method (5.4.1) is much more evident when calculating the force $\vec{f}$ needed to pull two capacitor plates apart laterally, as illustrated in Figure 5.4.1(a). To use the Lorentz force law directly would require knowledge of the lateral components of $\vec{E}$ responsible for the lateral forces, but they are not readily determined. Since energy derivatives can often be computed accurately and easily (provided the fringing fields are relatively small), that is often the preferred method for computing electric and magnetic forces.

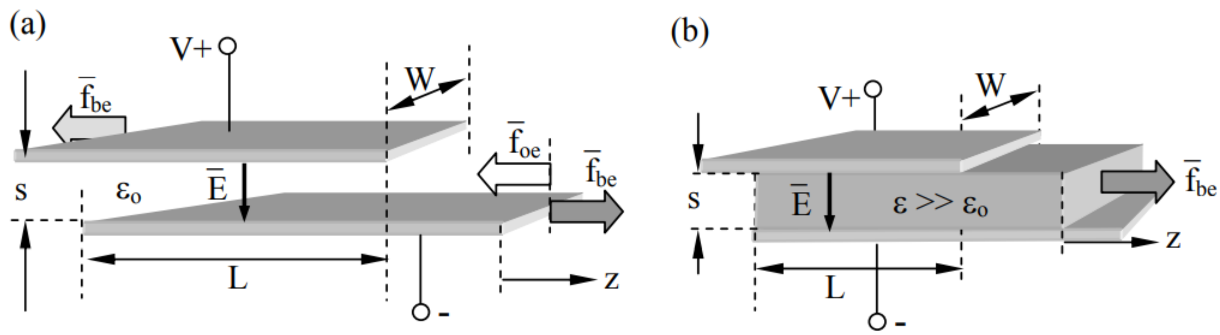

Figure 5.4.1: Capacitor plates and dielectrics being separated laterally.

The force/energy equation (5.4.1) can be expressed in terms of the area A = WL of the capacitor. Because L decreases as z increases, the sign of the derivative with respect to the plate overlap L is negative, and the force exerted on the plates by the environment is:

$$\mathrm{f_{be}} = \mathrm{dw/dz} = -\mathrm{d}\left(\mathrm{Q^2 s}/2\varepsilon_o \mathrm{WL}\right)/\mathrm{dL} = \mathrm{Q^2 s}/2\varepsilon_o \mathrm{WL^2}\ [\mathrm{N}] \tag{5.4.4}$$

where dz = dL and $w_e = Q^2s/2WL$. We again assumed that the plates were isolated in space so Q was constant, but the same force results when the plates are attached to a battery; in both cases the Lorentz forces arise from the very same charges so the two forces must be identical.

For purposes of illustration, let's solve the force/energy equation (5.4.1) for the same problem of Figure 5.4.1 the more difficult way by including the increase in battery energy as z increases. The incremental work $f_{be}dz$ involved in pulling the plates apart a distance dz is:

$$\mathrm{f_{be}} = \mathrm{dw_T/dz} = -\mathrm{d}\left(\varepsilon_o \mathrm{WLV^2}/2\mathrm{s}\right)/\mathrm{dL} - \mathrm{VdQ/dz} \tag{5.4.5}$$

where w_T is the total energy and the two terms on the right-hand side of (5.4.5) reflect the energy changes in the capacitor and battery respectively. The first negative sign in (5.4.5) arises because the overlap distance L decreases as z increases, and the second negative sign arises because the battery energy increases as Q decreases.

Since only L and Q vary with L, where $Q = CV = \varepsilon_o WLV/s$, (5.4.5) becomes:

$$\mathrm{f_{be}} = -\varepsilon_o \mathrm{WV^2}/2\mathrm{s} + \varepsilon_o \mathrm{WV^2/s} = \varepsilon_o \mathrm{WV^2}/2\mathrm{s}\ [\mathrm{N}] \tag{5.4.6}$$

where the sign of the second term ($\varepsilon_o WV^2/s$) reverses because Q decreases as z increases. This result when including the battery is the same as (5.4.4) without the battery, which can be seen by using V = Q/C and $C = \varepsilon_o WL/s$:

$$\mathrm{f_{be}} = \varepsilon_o \mathrm{WV^2}/2\mathrm{s} = \mathrm{Q^2 s}/2\varepsilon_o \mathrm{WL^2}\ [\mathrm{N}] \tag{5.4.7}$$

If the space between and surrounding the conducting plates were filled with a fluid having $\varepsilon > \varepsilon_o$, then for fixed V both the stored electric energy w_e and dw_e/dz, together with the force f_{be}, would obviously be increased by a factor of $\varepsilon/\varepsilon_o$ so that in this case the lateral force f_{be} would equal $\varepsilon WV^2/2s$.

Note that approximately the same force f_{be} is required to separate laterally two capacitor plates, one of which is coated with a dielectric having permittivity ε, as illustrated in Figure 5.4.1(b), because the force/energy equation (5.4.4) is largely unchanged except that $\varepsilon_o \rightarrow \varepsilon$:

$$\mathrm{f_{be}} = \mathrm{dw/dz} = -\mathrm{d}\left(\mathrm{Q^2 s}/2\varepsilon \mathrm{WL}\right)/\mathrm{dL} = \mathrm{Q^2 s}/2\varepsilon \mathrm{WL^2}\ [\mathrm{N}] \tag{5.4.8}$$

This page titled 5.4: Forces Computed Using Energy Methods is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by David H. Staelin (MIT OpenCourseWare) via source content that was edited to the style and standards of the LibreTexts platform.