

1.2: Binding energy and Semi-empirical mass formula

Binding Energy

Two important nuclear properties that we want to study are the nuclear binding energy and the mass of nuclides. You could think that since we know the masses of the proton and the neutron, we could simply find the masses of all nuclides with the simple formula:

$$m_N \stackrel{?}{=} Zm_p + Nm_n. \quad (1.2.1)$$

However, it is seen experimentally that this is not the case. From special relativity theory, we know that to each mass corresponds some energy, $E = mc^2$. Then if we just sum up the masses of all the constituents of a nucleus we would have how much energy they represent. The mass of a nucleus is also related to its intrinsic energy. It thus makes sense that this is not only the sum of its constituent energies, since we expect that some other energy is spent to keep the nucleus together. If the energy were equal, then it wouldn't be favorable to have bound nuclei, and all the nuclei would be unstable, constantly changing from their bound state to a sum of protons and neutrons.

The binding energy of a nucleus is then given by the difference in mass energy between the nucleus and its constituents. For a nucleus A_ZX_N the binding energy B is given by

$$B = [Zm_p + Nm_n - m_N({}^AX)] c^2$$

However, we want to express this quantity in terms of experimentally accessible quantities. Thus we write the nuclear mass in terms of the atomic mass, that we can measure,

$$m_N({}^AX) c^2 = [m_A({}^AX) - Zm_e] c^2 + B_e \quad (1.2.2)$$

where $m_A({}^AX)$ is the *atomic* mass of the nucleus. We further neglect the electronic binding energy B_e by setting

$$m_N({}^AX) c^2 = [m_A({}^AX) - Zm_e] c^2. \quad (1.2.3)$$

We finally obtain the expression for the **nuclear binding energy**:

$$B = \{Zm_p + Nm_n - [m_A({}^AX) - Zm_e]\} c^2$$

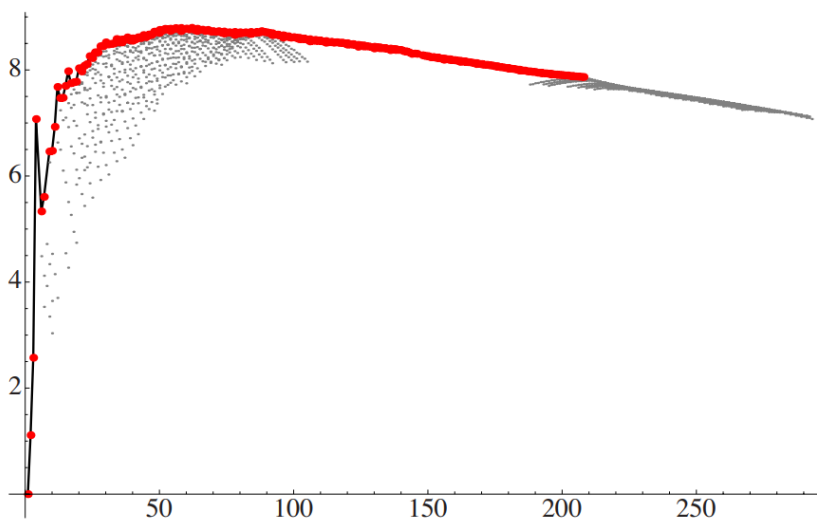


Figure 1.2.1: Binding energy per nucleon n (B/A in MeV vs. A) of stable nuclides (Red) and unstable nuclides (Gray). (CC BY-NC-ND; Paola Cappellaro)

Quantities of interest are also the neutron and proton separation energies:

$$S_n = B({}^A_ZX_N) - B({}^{A-1}_ZX_{N-1})$$

$$S_p = B({}^A_ZX_N) - B({}^{A-1}_{Z-1}X_N)$$

which are the analogous of the ionization energies in atomic physics, reflecting the energies of the *valence* nucleons. We will see that these energies show signatures of the shell structure of nuclei.

Semi-empirical mass formula

The binding energy is usually plotted as B/A or binding energy per nucleon. This illustrates that the binding energy is overall simply proportional to A , since B/A is mostly constant.

There are however corrections to this trend. The dependence of B/A on A (and Z) is captured by the **semi-empirical mass formula**. This formula is based on first principle considerations (a model for the nuclear force) and on experimental evidence to find the exact parameters defining it. In this model, the so-called **liquid-drop** model, all nucleons are uniformly distributed inside a nucleus and are bound together by the nuclear force while the Coulomb interaction causes repulsion among protons. Characteristics of the nuclear force (its short range) and of the Coulomb interaction explain part of the semi-empirical mass formula. However, other (smaller) corrections have been introduced to take into account variations in the binding energy that emerge because of its quantum-mechanical nature (and that give rise to the **nuclear shell model**).

The semi-empirical mass formula (SEMF) is

$$M(Z, A) = Zm({}^1H) + Nm_n - B(Z, A)/c^2$$

where the binding energy $B(Z, A)$ is given by the following formula:

$$B(A, Z) = \underset{\substack{\nearrow \\ \text{volume}}}{a_v A} - \underset{\substack{\uparrow \\ \text{surface}}}{a_s A^{2/3}} - \underset{\substack{\uparrow \\ \text{Coulomb}}}{a_c Z(Z-1)A^{-1/3}} - \underset{\substack{\uparrow \\ \text{symmetry}}}{a_{sym} \frac{(A-2Z)^2}{A}} + \underset{\substack{\nwarrow \\ \text{pairing}}}{\delta a_p A^{-3/4}}$$

We will now study each term in the SEMF.

Volume term

The first term is the volume term $a_v A$ that describes how the binding energy is mostly proportional to A . Why is that so?

Remember that the binding energy is a measure of the interaction among nucleons. Since nucleons are closely packed in the nucleus and the nuclear force has a very short range, each nucleon ends up interacting only with a few neighbors. This means that independently of the total number of nucleons, each one of them contribute in the same way. Thus the force is not proportional to $A(A-1)/2 \sim A^2$ (the total # of nucleons one nucleon can interact with) but it's simply proportional to A . The constant of proportionality is a fitting parameter that is found experimentally to be $a_v = 15.5 \text{ MeV}$.

This value is smaller than the binding energy of the nucleons to their neighbors as determined by the strength of the nuclear (strong) interaction. It is found (and we will study more later) that the energy binding one nucleon to the other nucleons is on the order of 50 MeV. The total binding energy is instead the difference between the interaction of a nucleon to its neighbor and the kinetic energy of the nucleon itself. As for electrons in an atom, the nucleons are fermions, thus they cannot all be in the same state with zero kinetic energy, but they will fill up all the kinetic energy levels according to Pauli's exclusion principle. This model, which takes into account the nuclear binding energy and the kinetic energy due to the filling of shells, indeed gives an accurate estimate for a_v .

Surface term

The surface term, $-a_s A^{2/3}$, also based on the strong force, is a correction to the volume term. We explained the volume term as arising from the fact that each nucleon interacts with a constant number of nucleons, independent of A . While this is valid for nucleons deep within the nucleus, those nucleons on the surface of the nucleus have fewer nearest neighbors. This term is similar to surface forces that arise for example in droplets of liquids, a mechanism that creates surface tension in liquids.

Since the volume force is proportional to $B_V \propto A$, we expect a surface force to be $\sim (B_V)^{2/3}$ (since the surface $S \sim V^{2/3}$). Also the term must be subtracted from the volume term and we expect the coefficient a_s to have a similar order of magnitude as a_v . In fact $a_s = 13 - 18 \text{ MeV}$.

Coulomb term

The third term $-a_c Z(Z-1)A^{-1/3}$ derives from the Coulomb interaction among protons, and of course is proportional to Z . This term is subtracted from the volume term since the Coulomb repulsion makes a nucleus containing many protons less favorable (more energetic).

To motivate the form of the term and estimate the coefficient a_c , the nucleus is modeled as a uniformly charged sphere. The potential energy of such a charge distribution is

$$E = \frac{1}{4\pi\epsilon_0} \frac{3}{5} \frac{Q^2}{R}$$

since from the uniform distribution inside the sphere we have the charge $q(r) = \frac{4}{3}\pi r^3 \rho = Q \left(\frac{r}{R}\right)^3$ and the potential energy is then:

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_0} \int dq(\vec{r}) \frac{q(\vec{r})}{|\vec{r}|} = \frac{1}{4\pi\epsilon_0} \int d^3\vec{r} \rho \frac{q(\vec{r})}{|\vec{r}|} = \frac{1}{4\pi\epsilon_0} \int_0^R dr 4\pi r^2 \rho \frac{q(r)}{r} \\ &= \frac{1}{4\pi\epsilon_0} \left(4\pi \int_0^R dr \frac{3Q}{4\pi R^3} r^2 Q \left(\frac{r}{R}\right)^3 \frac{1}{r} \right) = \frac{1}{4\pi\epsilon_0} \int_0^R dr \frac{3Q^2 r^4}{R^6} = \frac{1}{4\pi\epsilon_0} \frac{3}{5} \frac{Q^2}{R} \end{aligned}$$

Using the empirical radius formula $R = R_0 A^{1/3}$ and the total charge $Q^2 = e^2 Z(Z-1)$ (reflecting the fact that this term will appear only if $Z > 1$, i.e. if there are at least two protons) we have :

$$\frac{Q^2}{R} = \frac{e^2 Z(Z-1)}{R_0 A^{1/3}}$$

which gives the shape of the Coulomb term. Then the constant a_c can be estimated from $a_c \approx \frac{3}{5} \frac{e^2}{4\pi\epsilon_0 R_0}$, with $R_0 = 1.25$ fm, to be $a_c \approx 0.691$ MeV, not far from the experimental value.

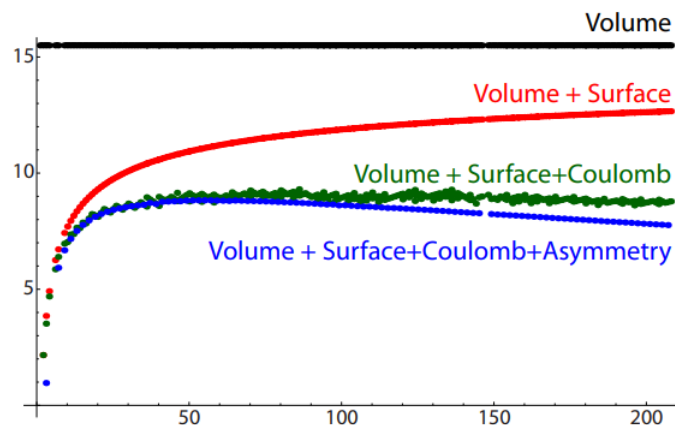


Figure 1.2.2: SEMF for stable nuclides. We plot $B(Z, A)/A$ vs. A . The various term contributions are added one by one to arrive at the final formula. (CC BY-NC-ND; Paola Cappellaro)

Symmetry term

The Coulomb term seems to indicate that it would be favorable to have less protons in a nucleus and more neutrons. However, this is not the case and we have to invoke something beyond the liquid-drop model in order to explain the fact that we have roughly the same number of neutrons and protons in stable nuclei. There is thus a correction term in the SEMF which tries to take into account the symmetry in protons and neutrons. This correction (and the following one) can only be explained by a more complex model of the nucleus, the **shell model**, together with the quantum-mechanical *exclusion principle*, that we will study later in the class. If we were to add more neutrons, they will have to be more energetic, thus increasing the total energy of the nucleus. This increase more than off-set the Coulomb repulsion, so that it is more favorable to have an approximately equal number of protons and neutrons.

The shape of the symmetry term is $\frac{(A-2Z)^2}{A}$. It can be more easily understood by considering the fact that this term goes to zero for $A = 2Z$ and its effect is smaller for larger A (while for smaller nuclei the symmetry effect is more important). The coefficient is $a_{\text{sym}} = 23$ MeV.

Pairing term

The final term is linked to the physical evidence that like-nucleons tend to pair off. Then it means that the binding energy is greater ($\delta > 0$) if we have an even-even nucleus, where all the neutrons and all the protons are paired-off. If we have a nucleus with both an odd number of neutrons and of protons, it is thus favorable to convert one of the protons into a neutrons or vice-versa (of course, taking into account the other constraints above). Thus, with all other factor constant, we have to subtract ($\delta < 0$) a term from the binding energy for odd-odd configurations. Finally, for even-odd configurations we do not expect any influence from this pairing energy ($\delta = 0$). The pairing term is then

$$+\delta a_p A^{-3/4} = \begin{cases} +a_p A^{-3/4} & \text{even-even} \\ 0 & \text{even-odd} \\ -a_p A^{-3/4} & \text{odd-odd} \end{cases}$$

with $a_p \approx 34\text{MeV}$. [Sometimes the form $\propto A^{-1/2}$ is also found].

Line of Stability in the Chart of nuclides

By taking the first derivative wrt Z we can calculate the optimal Z such that the mass is minimum. We obtain:

$$\begin{aligned} Z_{\min} &= \frac{A}{2} \left(\frac{1 + \frac{1}{4} A^{-1/3} \frac{a_c}{a_{\text{sym}}}}{1 + \frac{1}{4} A^{2/3} \frac{a_c}{a_{\text{sym}}}} \right) \\ &\approx \frac{A}{2} \left(1 + \frac{1}{4} A^{2/3} \frac{a_c}{a_{\text{sym}}} \right)^{-1} \\ &\approx \frac{A}{2} \left(1 - \frac{1}{4} A^{2/3} \frac{a_c}{a_{\text{sym}}} \right) \end{aligned}$$

which gives $Z \approx \frac{A}{2}$ at small A , but has a correction for larger A such that $Z \approx 0.41A$ for heavy nuclei. [Note the approximation and series expansion is taken because $a_c \ll a_{\text{sym}}$]

If we plot Z/A vs. A the nuclides lie between 1/2 and 0.41. There is a line of stability, following the stable isotopes (red in Figure 1.2.4 and black in Figure 1.2.3). The isotopes are then variously labeled, for example here by their lifetime. Interactive information is available at www.nndc.bnl.gov/chart/.

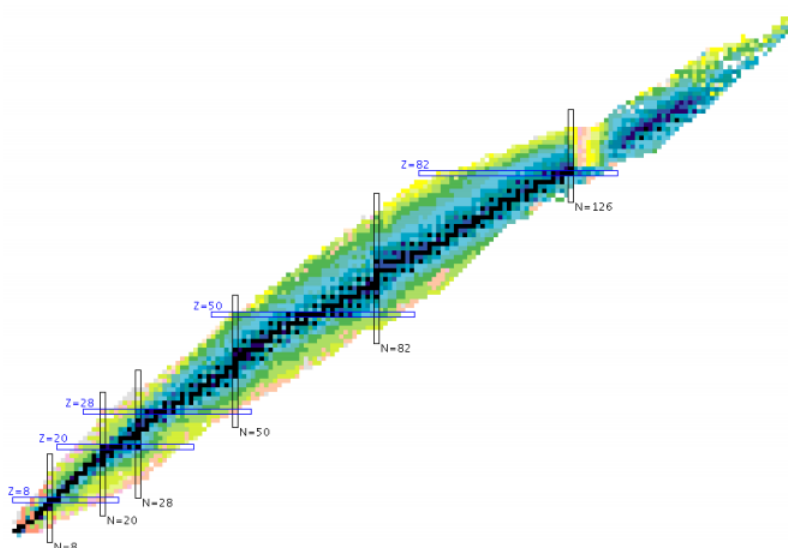


Figure 1.2.3: Chart of nuclides from www.nndc.bnl.gov/chart/ . Each nuclide is color-labeled by its half-life (black for stable nuclides) © Brookhaven National Laboratory. All rights reserved. This content is excluded from our Creative Commons license. For more information, see <http://ocw.mit.edu/fairuse>.

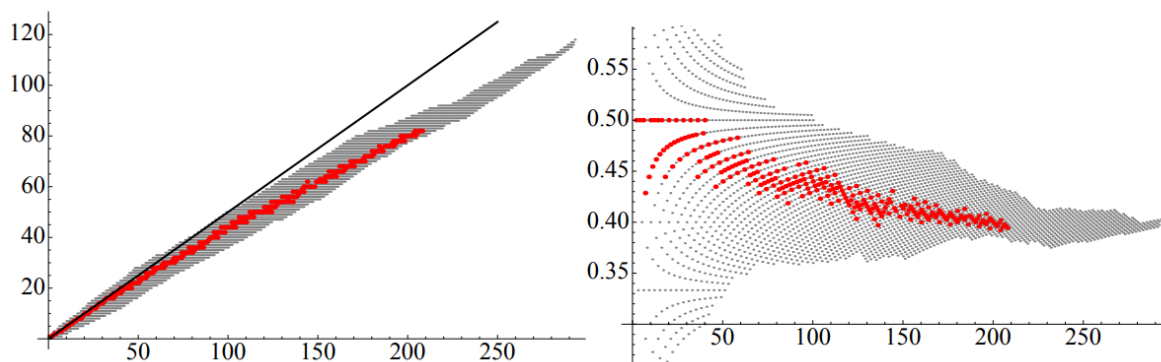


Figure 1.2.4: Nuclide chart (obtained with the software Mathematica). Left: Z vs. A , Right: Z/A vs. A . In red, stable nuclides. The black line represents $Z = A/2$. (CC BY-NC-ND; Paola Cappellaro)

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