3.1: Review - Energy Eigenvalue Problem

The time-independent wavefunction obeys the time-independent Schrödinger equation:

$$\boxed{\mathcal{H}\varphi(\vec{x}) = E\varphi(\vec{x})}$$

where E is identified as the energy of the system. If the wavefunction is given by just its time-independent part, $\psi(\vec{x}, t) = \varphi(\vec{x})$, the state is *stationary*. Thus, the time-independent Schrödinger equation allows us to find stationary states of the system, given a certain Hamiltonian.

Notice that the time-independent Schrödinger equation is nothing else than the eigenvalue equation for the Hamiltonian operator.

The energy of a particle has contributions from the kinetic energy as well as the potential energy:

$$\mathcal{H} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right) + V(\hat{x}, \hat{y}, \hat{z})$$

or more explicitly:

$$\mathcal{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x, y, z)$$

which can be written in a compact form as

$$\boxed{\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x, y, z)}$$

(Notice that V (x, y, z) is just a multiplicative operator, in the same way as the position is).

In 1D, for a free particle there is no potential energy, but only kinetic energy that we can rewrite as:

$$\mathcal{H} = \frac{1}{2m}p^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

The eigenvalue problem $\mathcal{H}w_n(x) = E_n w_n(x)$ is then the differential equation

$$\mathcal{H}w_n(x) = E_n w_n(x) \rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2 w_n(x)}{\partial x^2} = E_n w_n(x)$$

For a free particle there is no restriction on the possible energies, E_n can be any positive number. The solution to the eigenvalue problem is then the eigenfunction:

$$w_n(x) = A\sin(k_n x) + B\cos(k_n x) = A' e^{ik_n x} + B' e^{-ik_n x}$$

which represents two waves traveling in opposite directions.

We see that there are two independent functions for each eigenvalue E_n. Also there are two distinct momentum eigenvalues $\pm k_n$ for each energy eigenvalue, which correspond to two different directions of propagation of the wave function $e^{\pm ik_n x}$.