2.1: Laws of Quantum Mechanics

Every physical theory is formulated in terms of mathematical objects. It is thus necessary to establish a set of rules to map physical concepts and objects into mathematical objects that we use to represent them. Sometimes this mapping is evident, as in classical mechanics, while for other theories, such as quantum mechanics, the mathematical objects are not intuitive.

In the same way as classical mechanics is founded on Newton's laws or electrodynamics on the Maxwell-Boltzmann equations, quantum mechanics is also based on some fundamental laws, which are called the postulates or axioms of quantum mechanics.

We want in particular to develop a **mathematical model for the dynamics of closed quantum systems**[1]: therefore we are interested in defining

states – observables – measurements – evolution

Some subtleties will arise since we are trying to define measurement in a closed system, when the measuring person is instead outside the system itself. A more complete picture, that can explain some of the confusion arising from the measurement process, is possible, but we will not study it in this course.

We are interested in giving a description of physical phenomena and in particular in how they emerge during an experiment.

> **Note**
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> 1 We define a *closed* system any system that is isolated, thus not exchanging any input or output and not interacting with any other system. An *open* system instead interacts e.g., with an external environment.

Experiments – A physical experiment can be divided into two steps: preparation and measurement. In classical mechanics (CM):

- the first step determines the possible outcomes of the experiment,
- while the measurement retrieves the value of the outcome.

In quantum mechanics (QM) the situation is slightly different:

- the first step (preparation) determines the *probabilities* of the various possible outcomes,
- the second step (measurement) retrieve the *value* of a particular outcome, in a statistic manner.

This separation of the experiment in two steps is reflected into the two types of operators that we find in QM.

- The first step corresponds to the concept of a **state** of the system,
- while the second step corresponds to **observables**.

In CM the state of a system is described by a set of properties. For example, if we consider a ball, we can define its state by giving its position, momentum, energy, angular momentum (if for example the ball is spinning), its temperature etc. We can then perform a measurement on this ball, for example measuring its position. This will give us one value for one possible observable (the position).

We can express this process in mathematical terms. The state of the system is defined by a set of values: $\{\vec{r}, \vec{p}, E, \vec{L}, T, \ldots\}$. All of these values (and there might be of course more that I haven't written down) are needed to fully describe the state of the ball. Performing a measurement of the position, will retrieve the values $\{r_x, r_y, r_z\} = \vec{r}$ (the same values that describe the state).

If we now consider a nucleus, we can as well give a description of its state. In quantum mechanics, a complete description of the state of a quantum object (or system) is given mathematically by the state vector $|\psi\rangle$ (or wavefunction $\psi(\vec{r})$) . The situation is however different than in classical mechanics.

The state vector is no longer a collection of values for different properties of the system. The state gives instead a complete description of the set of *probabilities* for all the physical properties (or observables). All the information is contained in the state, irrespectively on how I got the state, of its previous history.

On the other hand, the observables are all the physical properties that in principle can be measured, in the same way as it was in classical mechanics. Since however the state only gives probabilities for all observables, the result of measurement will be a statistical variable.

All of these considerations are made more formal in the axioms of quantum mechanics that also indicate the mathematical formalism to be used.

1. The properties of a quantum system are completely defined by specification of its state vector $|\psi\rangle$. The state vector is an element of a complex Hilbert space H called the space of states.
2. With every physical property $\mathcal{A}$ (energy, position, momentum, angular momentum, ...) there exists an associated linear, Hermitian operator A (usually called observable), which acts in the space of states H. The eigenvalues of the operator are the possible values of the physical properties.
3. (a) If $|\psi\rangle$ is the vector representing the state of a system and if $|\varphi\rangle$ represents another physical state, there exists a probability $p(|\psi\rangle, |\varphi\rangle)$ of finding $|\psi\rangle$ in state $|\varphi\rangle$, which is given by the squared modulus of the inner product on $\mathcal{H}: p(|\psi\rangle, |\varphi\rangle) = |\langle\psi \mid \varphi\rangle|^2$ (Born Rule).

   (b) If A is an observable with eigenvalues a_n and eigenvectors $|n\rangle$ [such that the eigenvalue equation is $A|n\rangle = a_n|n\rangle$], given a system in the state $|\psi\rangle$, the probability of obtaining a_n as the outcome of the measurement of A is $p\left(a_n\right) = |\langle n \mid \psi\rangle|^2$. After the measurement the system is left in the state projected on the subspace of the eigenvalue a_n (Wave function collapse).
4. The evolution of a closed system is unitary (reversible). The evolution is given by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = \mathcal{H}|\psi\rangle$$

where $\mathcal{H}$ is the Hamiltonian of the system (the energy operator) and $\hbar$ is the reduced Planck constant $h/2\pi$ (with h the Planck constant, allowing conversion from energy to frequency units).