4.4: Identical Particles

We start first with the simplest case of a two-particle system. The wavefunction is then: $\psi\left(\vec{r}_1, \vec{r}_2\right)$ and if we assume that there is no interaction between the two particles, we will be able to describe the states using separation of variables:

$$\psi\left(\vec{r}_1, \vec{r}_2\right)=\psi_a\left(\vec{r}_1\right) \psi_b\left(\vec{r}_2\right)$$

where a and b label two different single-particle states. Implicit in this expression is the assumption that I can distinguish the two particles by some mean, and link particle one to the position 1 and the state a. However, if we consider two identical particles (2 electrons, two photons, two neutrons) there is no physical mean to distinguish them. Even if we try to measure them in order to keep track of which one is which, we know that in the process we destroy the state (by the wavefunction collapse) so not even this is a possibility.

Bosons, fermions

In quantum mechanics identical particle are fundamentally indistinguishable. Then the expression above does not correctly describe the state anymore. In order to faithfully describe a state in which we cannot know if particle a or b is at r_1 or r_2, we can take a linear combination of these two possibilities: $\psi\left(\vec{r}_1, \vec{r}_2\right)=A_1 \psi_a\left(\vec{r}_1\right) \psi_b\left(\vec{r}_2\right)+A_2 \psi_b\left(\vec{r}_1\right) \psi_a\left(\vec{r}_2\right)$. Now, since the two possibilities have the same probability, we have $\left|A_1\right|=\left|A_2\right|=\frac{1}{\sqrt{2}}$. Then there are two possible combinations:

$$\psi\left(\vec{r}_1, \vec{r}_2\right)=\frac{1}{\sqrt{2}}\left[\psi_a\left(\vec{r}_1\right) \psi_b\left(\vec{r}_2\right) \pm \psi_b\left(\vec{r}_1\right) \psi_a\left(\vec{r}_2\right)\right]$$

These two combinations describe two types of particle. The combination with the plus sign describes **bosons**, particles that are invariant under exchange of a particle pair. The combination with the minus sign describes **fermions**:

- all particles with integer spin are **bosons**
- all particles with half-integer spin are **fermions**

(This can be proved in relativistic QM).

Exchange operator

We can define an operator $\hat{P}$ that interchanges the two particles:

$$\hat{P}\left[\psi\left(\vec{r}_1, \vec{r}_2\right)\right]=\psi\left(\vec{r}_2, \vec{r}_1\right)$$

Since of course $\hat{P}\left[\hat{P}\left[\psi\left(\vec{r}_1, \vec{r}_2\right)\right]\right]=\psi\left(\vec{r}_1, \vec{r}_2\right)$, we have that $\hat{P}^2=1$. Then the eigenvalues of $\hat{P}$ must be ± 1. [If φ_n is an eigenfunction of $\hat{P}$ with eigenvalue p_n, we have $\hat{P}^2 \varphi_n=p_n^2 \varphi_n=\varphi_n$, from which $p_n^2=1$.] If two particles are identical, then the Hamiltonian is invariant with respect to their exchange and $[\mathcal{H}, \hat{P}]=0$. Then we can find energy eigenfunctions that are common eigenfunctions of the exchange operator, or $\psi\left(\vec{r}_1, \vec{r}_2\right)=\pm \psi\left(\vec{r}_2, \vec{r}_1\right)$. Then if the system is initially in such a state, it will be always be in a state with the same exchange symmetry. For the considerations above, however, we have seen that the wavefunction is not only allowed, but it must be in a state with a definite symmetry:

$$\psi\left(\vec{r}_1, \vec{r}_2\right)=\begin{cases}\psi\left(\vec{r}_2, \vec{r}_1\right) & \text{bosons} \\ -\psi\left(\vec{r}_2, \vec{r}_1\right) & \text{fermions}\end{cases}$$

Pauli exclusion principle

From the form of the allowed wavefunction for fermions, it follows that two fermions cannot occupy the same state. Assume that $\psi_a(\vec{r})=\psi_b(\vec{r})$, then we always have that

$$\psi_f\left(\vec{r}_1, \vec{r}_2\right)=\frac{1}{\sqrt{2}}\left[\psi_a\left(\vec{r}_1\right) \psi_b\left(\vec{r}_2\right)-\psi_b\left(\vec{r}_1\right) \psi_a\left(\vec{r}_2\right)\right]=0.$$

This is the well-known Pauli exclusion principle. Notice that of course it applies to any fermions. For example, it applies to electrons, and this is the reason why electrons do not pile up in the lowest energy level of the atomic structure, but form a shell model. We will see that the same applies as well to protons and neutrons, giving rise to the shell model for nuclei.