7.7: Drag Force and Terminal Speed

Learning Objectives

- Express the drag force mathematically
- Describe applications of the drag force
- Define terminal velocity
- Determine an object's terminal velocity given its mass

Another interesting force in everyday life is the force of drag on an object when it is moving in a fluid (either a gas or a liquid). You feel the drag force when you move your hand through water. You might also feel it if you move your hand during a strong wind. The faster you move your hand, the harder it is to move. You feel a smaller drag force when you tilt your hand so only the side goes through the air—you have decreased the area of your hand that faces the direction of motion.

Drag Forces

Like friction, the **drag force** always opposes the motion of an object. Unlike simple friction, the drag force is proportional to some function of the velocity of the object in that fluid. This functionality is complicated and depends upon the shape of the object, its size, its velocity, and the fluid it is in. For most large objects such as cyclists, cars, and baseballs not moving too slowly, the magnitude of the drag force F_D is proportional to the square of the speed of the object. We can write this relationship mathematically as $F_D \propto v^2$. When taking into account other factors, this relationship becomes

$$F_D = \frac{1}{2} C \rho A v^2, \tag{7.7.1}$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid. (Recall that density is mass per unit volume.) This equation can also be written in a more generalized fashion as $F_D = bv^2$, where b is a constant equivalent to $0.5C\rho A$. We have set the exponent n for these equations as 2 because when an object is moving at high velocity through air, the magnitude of the drag force is proportional to the square of the speed. As we shall see in Fluid Mechanics, for small particles moving at low speeds in a fluid, the exponent n is equal to 1.

Definition: Drag Force

Drag force F_D is proportional to the square of the speed of the object. Mathematically,

$$F_D = \frac{1}{2} C \rho A v^2, \tag{7.7.2}$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid.

Athletes as well as car designers seek to reduce the drag force to lower their race times (Figure $7.7.1A$). Aerodynamic shaping of an automobile can reduce the drag force and thus increase a car's gas mileage. The value of the drag coefficient C is determined empirically, usually with the use of a wind tunnel (Figure $7.7.1B$).

Figure 7.7.1: (A) From racing cars to bobsled racers, aerodynamic shaping is crucial to achieving top speeds. Bobsleds are designed for speed and are shaped like a bullet with tapered fins. (credit: “U.S. Army”/Wikimedia Commons) (B): NASA researchers test a model plane in a wind tunnel. (credit: NASA/Ames).

The drag coefficient can depend upon velocity, but we assume that it is a constant here. Table 7.7.1 lists some typical drag coefficients for a variety of objects. Notice that the drag coefficient is a dimensionless quantity. At highway speeds, over 50% of the power of a car is used to overcome air drag. The most fuel-efficient cruising speed is about 70–80 km/h (about 45–50 mi/h). For this reason, during the 1970s oil crisis in the United States, maximum speeds on highways were set at about 90 km/h (55 mi/h).

Table 7.7.1: Typical Values of Drag Coefficient C

Object	C
Airfoil	0.05
Toyota Camry	0.28
Ford Focus	0.32
Honda Civic	0.36
Ferrari Testarossa	0.37
Dodge Ram Pickup	0.43
Sphere	0.45
Hummer H2 SUV	0.64
Skydiver (feet first)	0.70
Bicycle	0.90
Skydiver (horizontal)	1.0
Circular flat plate	1.12

Substantial research is under way in the sporting world to minimize drag. The dimples on golf balls are being redesigned, as are the clothes that athletes wear. Bicycle racers and some swimmers and runners wear full bodysuits. Australian Cathy Freeman wore a full body suit in the 2000 Sydney Olympics and won a gold medal in the 400-m race. Many swimmers in the 2008 Beijing Olympics wore (Speedo) body suits; it might have made a difference in breaking many world records (Figure 7.7.2). Most elite swimmers (and cyclists) shave their body hair. Such innovations can have the effect of slicing away milliseconds in a race, sometimes making the difference between a gold and a silver medal. One consequence is that careful and precise guidelines must be continuously developed to maintain the integrity of the sport.

Figure 7.7.2: Body suits, such as this LZR Racer Suit, have been credited with aiding in many world records after their release in 2008. Smoother "skin" and more compression forces on a swimmer's body provide at least 10% less drag. (credit: NASA/Kathy Barnstorff)

Terminal Velocity

Some interesting situations connected to Newton's second law occur when considering the effects of drag forces upon a moving object. For instance, consider a skydiver falling through air under the influence of gravity. The two forces acting on him are the force of gravity and the drag force (ignoring the small buoyant force). The downward force of gravity remains constant regardless of the velocity at which the person is moving. However, as the person's velocity increases, the magnitude of the drag force increases until the magnitude of the drag force is equal to the gravitational force, thus producing a net force of zero. A zero net force means that there is no acceleration, as shown by Newton's second law. At this point, the person's velocity remains constant and we say that the person has reached his **terminal velocity** (v_T). Since F_D is proportional to the speed squared, a heavier skydiver must go faster for F_D to equal his weight. Let's see how this works out more quantitatively.

At the terminal velocity,

$$F_{net} = mg - F_D = ma = 0. \tag{7.7.3}$$

Thus,

$$mg = F_D. \tag{7.7.4}$$

Using the equation for drag force, we have

$$mg = \frac{1}{2} C\rho A v_T^2. \tag{7.7.5}$$

Solving for the velocity, we obtain

$$v_T = \sqrt{\frac{2mg}{\rho C A}}. \tag{7.7.6}$$

Assume the density of air is ρ = 1.21 kg/m^3. A 75-kg skydiver descending head first has a cross-sectional area of approximately A = 0.18 m^2 and a drag coefficient of approximately C = 0.70. We find that

$$v_T = \sqrt{\frac{2(75\ kg)(9.80\ m/s^2)}{(1.21\ kg/m^3)(0.70)(0.18\ m^2)}} = 98\ m/s = 350\ km/h. \tag{7.7.7}$$

This means a skydiver with a mass of 75 kg achieves a terminal velocity of about 350 km/h while traveling in a pike (head first) position, minimizing the area and his drag. In a spread-eagle position, that terminal velocity may decrease to about 200 km/h as the area increases. This terminal velocity becomes much smaller after the parachute opens.

✓ Example 7.7.1: Terminal Velocity of a Skydiver

Find the terminal velocity of an 85-kg skydiver falling in a spread-eagle position.

Strategy

At terminal velocity, $F_{net} = 0$. Thus, the drag force on the skydiver must equal the force of gravity (the person's weight). Using the equation of drag force, we find $mg = \frac{1}{2}\rho C A v^2$.

Solution

The terminal velocity v_T can be written as

$$v_T = \sqrt{\frac{2mg}{\rho CA}} = \sqrt{\frac{2(85\ kg)(9.80\ m/s^2)}{(1.21\ kg/m^3)(1.0)(0.70\ m^2)}} = 44\ m/s. \tag{7.7.8}$$

Significance

This result is consistent with the value for v_T mentioned earlier. The 75-kg skydiver going feet first had a terminal velocity of v_T = 98 m/s. He weighed less but had a smaller frontal area and so a smaller drag due to the air.

? Exercise 7.7.1

Find the terminal velocity of a 50-kg skydiver falling in spread-eagle fashion.

The size of the object that is falling through air presents another interesting application of air drag. If you fall from a 5-m-high branch of a tree, you will likely get hurt—possibly fracturing a bone. However, a small squirrel does this all the time, without getting hurt. You do not reach a terminal velocity in such a short distance, but the squirrel does.

The following interesting quote on animal size and terminal velocity is from a 1928 essay by a British biologist, J. B. S. Haldane, titled “On Being the Right Size.”

> *“To the mouse and any smaller animal, [gravity] presents practically no dangers. You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft. A rat is killed, a man is broken, and a horse splashes. For the resistance presented to movement by the air is proportional to the surface of the moving object. Divide an animal’s length, breadth, and height each by ten; its weight is reduced to a thousandth, but its surface only to a hundredth. So the resistance to falling in the case of the small animal is relatively ten times greater than the driving force.”*

The above quadratic dependence of air drag upon velocity does not hold if the object is very small, is going very slow, or is in a denser medium than air. Then we find that the drag force is proportional just to the velocity. This relationship is given by Stokes’ law.

Stokes’ law

For a spherical object falling in a medium, the drag force is

$$F_s = 6\pi r\eta v, \tag{7.7.9}$$

where r is the radius of the object, η is the viscosity of the fluid, and v is the object’s velocity.

Good examples of Stokes’ law are provided by microorganisms, pollen, and dust particles. Because each of these objects is so small, we find that many of these objects travel unaided only at a constant (terminal) velocity. Terminal velocities for bacteria (size about \(1\, \mu m) can be about \(2\, \mu m/s. To move at a greater speed, many bacteria swim using flagella (organelles shaped like little tails) that are powered by little motors embedded in the cell.

Sediment in a lake can move at a greater terminal velocity (about 5 μm/s), so it can take days for it to reach the bottom of the lake after being deposited on the surface.

If we compare animals living on land with those in water, you can see how drag has influenced evolution. Fish, dolphins, and even massive whales are streamlined in shape to reduce drag forces. Birds are streamlined and migratory species that fly large distances often have particular features such as long necks. Flocks of birds fly in the shape of a spearhead as the flock forms a streamlined pattern (Figure 7.7.3). In humans, one important example of streamlining is the shape of sperm, which need to be efficient in their use of energy.

Figure $7.7.3$: Geese fly in a V formation during their long migratory travels. This shape reduces drag and energy consumption for individual birds, and also allows them a better way to communicate. (credit: “Julo”/Wikimedia Commons)

In lecture demonstrations, we do measurements of the drag force on different objects. The objects are placed in a uniform airstream created by a fan. Calculate the Reynolds number and the drag coefficient.

Video $7.7.1$: Fluid Mechanics - Drag force - Flow simulation

The Calculus of Velocity-Dependent Frictional Forces

When a body slides across a surface, the frictional force on it is approximately constant and given by $\mu_k N$. Unfortunately, the frictional force on a body moving through a liquid or a gas does not behave so simply. This drag force is generally a complicated function of the body’s velocity. However, for a body moving in a straight line at moderate speeds through a liquid such as water, the frictional force can often be approximated by

$$f_R = -bv, \tag{7.7.10}$$

where b is a constant whose value depends on the dimensions and shape of the body and the properties of the liquid, and v is the velocity of the body. Two situations for which the frictional force can be represented this equation are a motorboat moving through water and a small object falling slowly through a liquid.

Let’s consider the object falling through a liquid. The free-body diagram of this object with the positive direction downward is shown in Figure $7.7.4$. Newton’s second law in the vertical direction gives the differential equation

$$mg - bv = m\frac{dv}{dt}, \tag{7.7.11}$$

where we have written the acceleration as $\frac{dv}{dt}$. As v increases, the frictional force $-bv$ increases until it matches mg. At this point, there is no acceleration and the velocity remains constant at the terminal velocity v_T. From the previous equation,

$$mg - bv_T = 0, \tag{7.7.12}$$

so

$$v_T = \frac{mg}{b}. \tag{7.7.13}$$

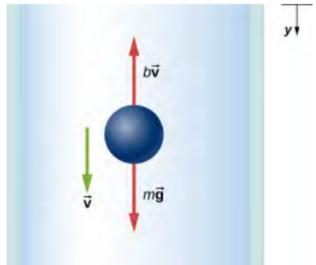

Figure 7.7.4: Free-body diagram of an object falling through a resistive medium.

We can find the object's velocity by integrating the differential equation for v. First, we rearrange terms in this equation to obtain

$$\frac{dv}{g-\left(\frac{b}{m}\right)v}=dt. \tag{7.7.14}$$

Assuming that $v=0$ at \9t = 0\), integration of Equation 7.7.14 yields

$$\int_0^v \frac{dv'}{g-\left(\frac{b}{m}\right)v'}=\int_0^t dt', \tag{7.7.15}$$

or

$$-\frac{m}{b}\ln\left(g-\frac{b}{m}v'\right)\Big|_0^v=t'\big|_0^t, \tag{7.7.16}$$

where v' and t' are dummy variables of integration. With the limits given, we find

$$-\frac{m}{b}[\ln\left(g-\frac{b}{m}v\right)-\ln g]=t. \tag{7.7.17}$$

Since $\ln A-\ln B=\ln(\left(\frac{A}{B}\right)$, and $\ln(\left(\frac{A}{B}\right)=x$ implies $e^x=\frac{A}{B}$, we obtain

$$\frac{g-\left(\frac{bv}{m}\right)}{g}=e^{-\frac{bt}{m}}, \tag{7.7.18}$$

and

$$v=\frac{mg}{b}\left(1-e^{-\frac{bt}{m}}\right). \tag{7.7.19}$$

Notice that as t → ∞, v → $\frac{mg}{b}$ = v_T, which is the terminal velocity.

The position at any time may be found by integrating the equation for v. With v = $\frac{dy}{dt}$,

$$dy=\frac{mg}{b}\left(1-e^{-\frac{bt}{m}}\right)dt. \tag{7.7.20}$$

Assuming y = 0 when t = 0,

$$\int_0^y dy'=\frac{mg}{b}\int_0^t\left(1-e^{-\frac{bt}{m}}\right)dt', \tag{7.7.21}$$

which integrates to

$$y=\frac{mg}{b}t+\frac{m^2g}{b^2}\left(e^{-\frac{bt}{m}}-1\right). \tag{7.7.22}$$

Example 7.7.2: Effect of the Resistive Force on a Motorboat

A motorboat is moving across a lake at a speed v_0 when its motor suddenly freezes up and stops. The boat then slows down under the frictional force $f_R = -bv$.

a. What are the velocity and position of the boat as functions of time?
b. If the boat slows down from 4.0 to 1.0 m/s in 10 s, how far does it travel before stopping?

Solution

a. With the motor stopped, the only horizontal force on the boat is $f_R = -bv$, so from Newton's second law,

$$m\frac{dv}{dt} = -bv, \tag{7.7.23}$$

which we can write as

$$\frac{dv}{v} = -\frac{b}{m}dt. \tag{7.7.24}$$

Integrating this equation between the time zero when the velocity is v_0 and the time t when the velocity is v, we have

$$\int_0^v \frac{dv'}{v'} = -\frac{b}{m}\int_0^t dt'. \tag{7.7.25}$$

Thus,

$$\ln\frac{v}{v_0} = -\frac{b}{m}t, \tag{7.7.26}$$

which, since lnA = x implies $e^x = A$, we can write this as

$$v = v_0 e^{-\frac{bt}{m}}. \tag{7.7.27}$$

Now from the definition of velocity,

$$\frac{dx}{dt} = v_0 e^{-\frac{bt}{m}}, \tag{7.7.28}$$

so we have

$$dx = v_0 e^{-\frac{bt}{m}} dt. \tag{7.7.29}$$

With the initial position zero, we have

$$\int_0^x dx' = v_0 \int_0^t e^{-\frac{bt'}{m}} dt', \tag{7.7.30}$$

and

$$x = -\frac{mv_0}{b}e^{-\frac{bt}{m}}\Big|_0^t = \frac{mv_0}{b}\left(1 - e^{-\frac{bt}{m}}\right). \tag{7.7.31}$$

As time increases, $e^{-\frac{bt}{m}} \rightarrow 0$, and the position of the boat approaches a limiting value

$$x_{max} = \frac{mv_0}{b}. \tag{7.7.32}$$

Although this tells us that the boat takes an infinite amount of time to reach x_{max}, the boat effectively stops after a reasonable time. For example, at t = $\frac{10m}{b}$, we have

$$v = v_0 e^{-10} \simeq 4.5 \times 10^{-5} v_0, \tag{7.7.33}$$

whereas we also have

$$x = x_{max}\left(1 - e^{-10}\right) \simeq 0.99995 x_{max}. \tag{7.7.34}$$

Therefore, the boat’s velocity and position have essentially reached their final values.

b. With v_0 = 4.0 m/s and v = 1.0 m/s, we have 1.0 m/s = (4.0 m/s) $e^{\left(-\frac{bt}{m}\right)(10\ s)}$, so

$$\ln 0.25 = -\ln 4.0 = -\frac{b}{m}(10\ s), \tag{7.7.35}$$

and

$$\frac{b}{m} = \frac{1}{10}\ln 4.0\ s^{-1} = 0.14\ s^{-1}. \tag{7.7.36}$$

Now the boat's limiting position is

$$x_{max} = \frac{mv_0}{b} = \frac{4.0\ m/s}{0.14\ s^{-1}} = 29\ m. \tag{7.7.37}$$

Significance

In the both of the previous examples, we found “limiting” values. The terminal velocity is the same as the limiting velocity, which is the velocity of the falling object after a (relatively) long time has passed. Similarly, the limiting distance of the boat is the distance the boat will travel after a long amount of time has passed. Due to the properties of exponential decay, the time involved to reach either of these values is actually not too long (certainly not an infinite amount of time!) but they are quickly found by taking the limit to infinity.

? Exercise 7.7.2

Suppose the resistive force of the air on a skydiver can be approximated by $f = -bv^2$. If the terminal velocity of a 100-kg skydiver is 60 m/s, what is the value of b?