

3.9: Products of Vectors (Part 2)

The Vector Products of Two Vectors (the Cross Product)

Vector multiplication of two vectors yields a vector product.

Vector Product (Cross Product)

The vector product of two vectors \vec{A} and \vec{B} is denoted by $\vec{A} \times \vec{B}$ and is often referred to as a cross product. The vector product is a vector that has its direction perpendicular to both vectors \vec{A} and \vec{B} . In other words, vector $\vec{A} \times \vec{B}$ is perpendicular to the plane that contains vectors \vec{A} and \vec{B} , as shown in Figure 3.9.1. The magnitude of the vector product is defined as

$$|\vec{A} \times \vec{B}| = AB \sin \varphi, \quad (3.9.1)$$

where angle φ , between the two vectors, is measured from vector \vec{A} (first vector in the product) to vector \vec{B} (second vector in the product), as indicated in Figure 3.9.1, and is between 0° and 180° .

According to Equation 3.9.1, the vector product vanishes for pairs of vectors that are either parallel ($\varphi = 0^\circ$) or antiparallel ($\varphi = 180^\circ$) because $\sin 0^\circ = \sin 180^\circ = 0$.

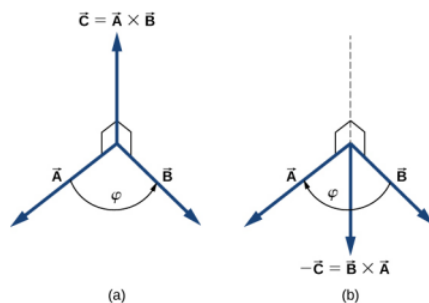


Figure 3.9.1: The vector product of two vectors is drawn in three-dimensional space. (a) The vector product $\vec{A} \times \vec{B}$ is a vector perpendicular to the plane that contains vectors \vec{A} and \vec{B} . Small squares drawn in perspective mark right angles between \vec{A} and \vec{C} , and between \vec{B} and \vec{C} so that if \vec{A} and \vec{B} lie on the floor, vector \vec{C} points vertically upward to the ceiling. (b) The vector product $\vec{B} \times \vec{A}$ is a vector antiparallel to vector $\vec{A} \times \vec{B}$.

On the line perpendicular to the plane that contains vectors \vec{A} and \vec{B} there are two alternative directions—either up or down, as shown in Figure 3.9.1—and the direction of the vector product may be either one of them. In the standard right-handed orientation, where the angle between vectors is measured counterclockwise from the first vector, vector $\vec{A} \times \vec{B}$ points **upward**, as seen in Figure 3.9.1(a). If we reverse the order of multiplication, so that now \vec{B} comes first in the product, then vector $\vec{B} \times \vec{A}$ must point **downward**, as seen in Figure 3.9.1(b). This means that vectors $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ are **antiparallel** to each other and that vector multiplication is **not** commutative but **anticommutative**. The **anticommutative property** means the vector product reverses the sign when the order of multiplication is reversed:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}. \quad (3.9.2)$$

The **corkscrew right-hand rule** is a common mnemonic used to determine the direction of the vector product. As shown in Figure 3.9.2, a corkscrew is placed in a direction perpendicular to the plane that contains vectors \vec{A} and \vec{B} , and its handle is turned in the direction from the first to the second vector in the product. The direction of the cross product is given by the progression of the corkscrew.

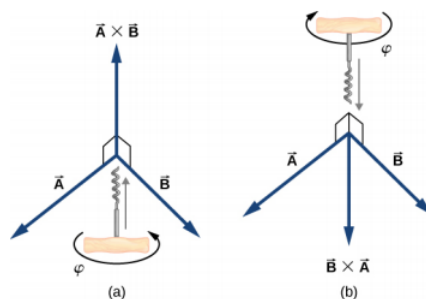


Figure 3.9.2: The corkscrew right-hand rule can be used to determine the direction of the cross product $\vec{A} \times \vec{B}$. Place a corkscrew in the direction perpendicular to the plane that contains vectors \vec{A} and \vec{B} , and turn it in the direction from the first to the second vector in the product. The direction of the cross product is given by the progression of the corkscrew. (a) Upward movement means the cross-product vector points up. (b) Downward movement means the cross-product vector points downward.

✓ Example 3.9.1: The Torque of a Force

The mechanical advantage that a familiar tool called a **wrench** provides (Figure 3.9.3) depends on magnitude F of the applied force, on its direction with respect to the wrench handle, and on how far from the nut this force is applied. The distance R from the nut to the point where force vector \vec{F} is attached is called the **lever arm** and is represented by the radial vector \vec{R} . The physical vector quantity that makes the nut turn is called **torque** (denoted by $\vec{\tau}$), and it is the vector product of the lever arm with the force: $\vec{\tau} = \vec{R} \times \vec{F}$.

To loosen a rusty nut, a 20.00-N force is applied to the wrench handle at angle $\varphi = 40^\circ$ and at a distance of 0.25 m from the nut, as shown in Figure 3.9.3(a). Find the magnitude and direction of the torque applied to the nut. What would the magnitude and direction of the torque be if the force were applied at angle $\varphi = 45^\circ$, as shown in Figure 3.9.3(b)? For what value of angle φ does the torque have the largest magnitude?

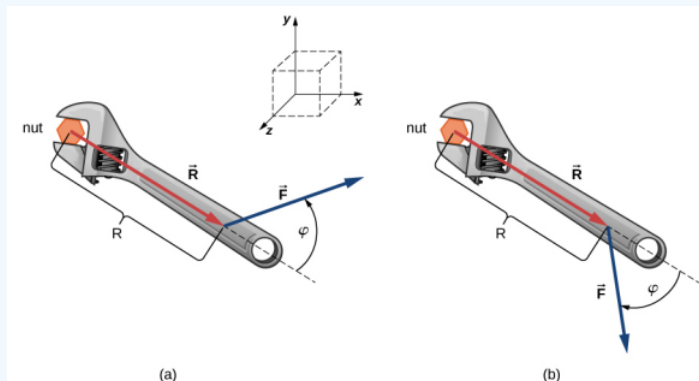


Figure 3.9.3: A wrench provides grip and mechanical advantage in applying torque to turn a nut. (a) Turn counterclockwise to loosen the nut. (b) Turn clockwise to tighten the nut.

Strategy

We adopt the frame of reference shown in Figure 3.9.3, where vectors \vec{R} and \vec{F} lie in the xy -plane and the origin is at the position of the nut. The radial direction along vector \vec{R} (pointing away from the origin) is the reference direction for measuring the angle φ because \vec{R} is the first vector in the vector product $\vec{\tau} = \vec{R} \times \vec{F}$. Vector $\vec{\tau}$ must lie along the z -axis because this is the axis that is perpendicular to the xy -plane, where both \vec{R} and \vec{F} lie. To compute the magnitude τ , we use Equation 3.9.1. To find the direction of $\vec{\tau}$, we use the corkscrew right-hand rule (Figure 3.9.2).

Solution

For the situation in (a), the corkscrew rule gives the direction of $\vec{R} \times \vec{F}$ in the positive direction of the z -axis. Physically, it means the torque vector $\vec{\tau}$ points out of the page, perpendicular to the wrench handle. We identify $F = 20.00 \text{ N}$ and $R = 0.25 \text{ m}$, and compute the magnitude using Equation 3.9.1:

$$\tau = |\vec{R} \times \vec{F}| = RF \sin \varphi = (0.25 \text{ m})(20.00 \text{ N}) \sin 40^\circ = 3.21 \text{ N} \cdot \text{m}. \quad (3.9.3)$$

For the situation in (b), the corkscrew rule gives the direction of $\vec{R} \times \vec{F}$ in the negative direction of the z-axis. Physically, it means the vector $\vec{\tau}$ points into the page, perpendicular to the wrench handle. The magnitude of this torque is

$$\tau = |\vec{R} \times \vec{F}| = RF \sin \varphi = (0.25 \text{ m})(20.00 \text{ N}) \sin 45^\circ = 3.53 \text{ N} \cdot \text{m}. \quad (3.9.4)$$

The torque has the largest value when $\sin \varphi = 1$, which happens when $\varphi = 90^\circ$. Physically, it means the wrench is most effective—giving us the best mechanical advantage—when we apply the force perpendicular to the wrench handle. For the situation in this example, this best-torque value is $\tau_{\text{best}} = RF = (0.25 \text{ m})(20.00 \text{ N}) = 5.00 \text{ N} \cdot \text{m}$.

Significance

When solving mechanics problems, we often do not need to use the corkscrew rule at all, as we'll see now in the following equivalent solution. Notice that once we have identified that vector $\vec{R} \times \vec{F}$ lies along the z-axis, we can write this vector in terms of the unit vector \hat{k} of the z-axis:

$$\vec{R} \times \vec{F} = RF \sin \varphi \hat{k}. \quad (3.9.5)$$

In this equation, the number that multiplies \hat{k} is the scalar z-component of the vector $\vec{R} \times \vec{F}$. In the computation of this component, care must be taken that the angle φ is measured counterclockwise from \vec{R} (first vector) to \vec{F} (second vector). Following this principle for the angles, we obtain $RF \sin (+40^\circ) = +3.2 \text{ N} \cdot \text{m}$ for the situation in (a), and we obtain $RF \sin (-45^\circ) = -3.5 \text{ N} \cdot \text{m}$ for the situation in (b). In the latter case, the angle is negative because the graph in Figure 3.9.3 indicates the angle is measured clockwise; but, the same result is obtained when this angle is measured counterclockwise because $+(360^\circ - 45^\circ) = +315^\circ$ and $\sin (+315^\circ) = \sin (-45^\circ)$. In this way, we obtain the solution without reference to the corkscrew rule. For the situation in (a), the solution is $\vec{R} \times \vec{F} = +3.2 \text{ N} \cdot \text{m} \hat{k}$; for the situation in (b), the solution is $\vec{R} \times \vec{F} = -3.5 \text{ N} \cdot \text{m} \hat{k}$.

? Exercise 2.15

For the vectors given in Figure 2.3.6, find the vector products $\vec{A} \times \vec{B}$ and $\vec{C} \times \vec{F}$.

Similar to the dot product (Equation 2.8.10), the cross product has the following distributive property:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}. \quad (3.9.6)$$

The distributive property is applied frequently when vectors are expressed in their component forms, in terms of unit vectors of Cartesian axes. When we apply the definition of the cross product, Equation 3.9.1, to unit vectors \hat{i} , \hat{j} , and \hat{k} that define the positive x-, y-, and z-directions in space, we find that

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0. \quad (3.9.7)$$

All other cross products of these three unit vectors must be vectors of unit magnitudes because \hat{i} , \hat{j} , and \hat{k} are orthogonal. For example, for the pair \hat{i} and \hat{j} , the magnitude is $|\hat{i} \times \hat{j}| = ij \sin 90^\circ = (1)(1)(1) = 1$. The direction of the vector product $\hat{i} \times \hat{j}$ must be orthogonal to the xy-plane, which means it must be along the z-axis. The only unit vectors along the z-axis are $-\hat{k}$ or $+\hat{k}$. By the corkscrew rule, the direction of vector $\hat{i} \times \hat{j}$ must be parallel to the positive z-axis. Therefore, the result of the multiplication $\hat{i} \times \hat{j}$ is identical to $+\hat{k}$. We can repeat similar reasoning for the remaining pairs of unit vectors. The results of these multiplications are

$$\begin{cases} \hat{i} \times \hat{j} = +\hat{k}, \\ \hat{j} \times \hat{k} = +\hat{i}, \\ \hat{k} \times \hat{i} = +\hat{j}. \end{cases} \quad (3.9.8)$$

Notice that in Equation 3.9.8, the three unit vectors \hat{i} , \hat{j} , and \hat{k} appear in the cyclic order shown in a diagram in Figure 3.9.4(a). The cyclic order means that in the product formula, \hat{i} follows \hat{k} and comes before \hat{j} , or \hat{k} follows \hat{j} and comes before \hat{i} , or \hat{j} follows \hat{i} and comes before \hat{k} . The cross product of two different unit vectors is always a third unit vector. When two unit vectors in the cross product appear in the cyclic order, the result of such a multiplication is the remaining unit vector, as illustrated in Figure 3.9.4(b). When unit vectors in the cross product appear in a different order, the result is a unit vector that is antiparallel to the remaining unit vector (i.e., the result is with the minus sign, as shown by the examples in Figure 3.9.4(c) and Figure 3.9.4(d).

In practice, when the task is to find cross products of vectors that are given in vector component form, this rule for the cross-multiplication of unit vectors is very useful.

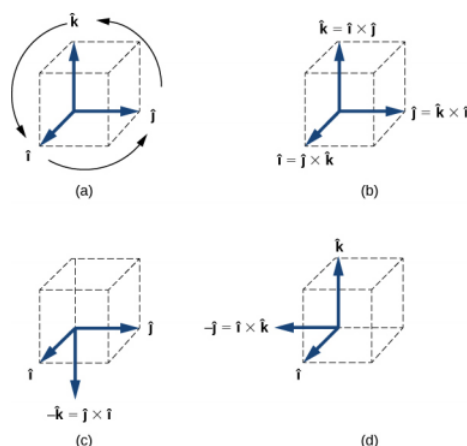


Figure 3.9.4: (a) The diagram of the cyclic order of the unit vectors of the axes. (b) The only cross products where the unit vectors appear in the cyclic order. These products have the positive sign. (c, d) Two examples of cross products where the unit vectors do not appear in the cyclic order. These products have the negative sign.

Suppose we want to find the cross product $\vec{A} \times \vec{B}$ for vectors $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$. We can use the distributive property (Equation 3.9.6), the anticommutative property (Equation 3.9.2), and the results in Equation 3.9.7 and Equation 3.9.8 for unit vectors to perform the following algebra:

$$\begin{aligned}
 \vec{A} \times \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\
 &= A_x \hat{i} \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) + A_y \hat{j} \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) + A_z \hat{k} \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\
 &= A_x B_x \hat{i} \times \hat{i} + A_x B_y \hat{i} \times \hat{j} + A_x B_z \hat{i} \times \hat{k} \\
 &\quad + A_y B_x \hat{j} \times \hat{i} + A_y B_y \hat{j} \times \hat{j} + A_y B_z \hat{j} \times \hat{k} \\
 &\quad + A_z B_x \hat{k} \times \hat{i} + A_z B_y \hat{k} \times \hat{j} + A_z B_z \hat{k} \times \hat{k} \\
 &= A_x B_x (0) + A_x B_y (+\hat{k}) + A_x B_z (-\hat{j}) \\
 &\quad + A_y B_x (-\hat{k}) + A_y B_y (0) + A_y B_z (+\hat{i}) \\
 &\quad + A_z B_x (+\hat{j}) + A_z B_y (-\hat{i}) + A_z B_z (0).
 \end{aligned}$$

When performing algebraic operations involving the cross product, be very careful about keeping the correct order of multiplication because the cross product is anticommutative. The last two steps that we still have to do to complete our task are, first, grouping the terms that contain a common unit vector and, second, factoring. In this way we obtain the following very useful expression for the computation of the cross product:

$$\vec{C} = \vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}. \quad (3.9.9)$$

In this expression, the scalar components of the cross-product vector are

$$\begin{cases} C_x = A_y B_z - A_z B_y, \\ C_y = A_z B_x - A_x B_z, \\ C_z = A_x B_y - A_y B_x. \end{cases} \quad (3.9.10)$$

When finding the cross product, in practice, we can use either Equation 3.9.1 or Equation 3.9.9, depending on which one of them seems to be less complex computationally. They both lead to the same final result. One way to make sure if the final result is correct is to use them both.

✓ Example 3.9.2: A Particle in a Magnetic Field

When moving in a magnetic field, some particles may experience a magnetic force. Without going into details—a detailed study of magnetic phenomena comes in later chapters—let's acknowledge that the magnetic field \vec{B} is a vector, the magnetic force \vec{F} is a vector, and the velocity \vec{u} of the particle is a vector. The magnetic force vector is proportional to the vector product

of the velocity vector with the magnetic field vector, which we express as $\vec{F} = \zeta \vec{u} \times \vec{B}$. In this equation, a constant ζ takes care of the consistency in physical units, so we can omit physical units on vectors \vec{u} and \vec{B} . In this example, let's assume the constant ζ is positive. A particle moving in space with velocity vector $\vec{u} = -5.0 \hat{i} - 2.0 \hat{j} + 3.5 \hat{k}$ enters a region with a magnetic field and experiences a magnetic force. Find the magnetic force \vec{F} on this particle at the entry point to the region where the magnetic field vector is (a) $\vec{B} = 7.2 \hat{i} - \hat{j} - 2.4 \hat{k}$ and (b) $\vec{B} = 4.5 \hat{k}$. In each case, find magnitude F of the magnetic force and angle θ the force vector \vec{F} makes with the given magnetic field vector \vec{B} .

Strategy

First, we want to find the vector product $\vec{u} \times \vec{B}$, because then we can determine the magnetic force using $\vec{F} = \zeta \vec{u} \times \vec{B}$. Magnitude F can be found either by using components, $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$, or by computing the magnitude $|\vec{u} \times \vec{B}|$ directly using Equation 3.9.1. In the latter approach, we would have to find the angle between vectors \vec{u} and \vec{B} . When we have \vec{F} , the general method for finding the direction angle θ involves the computation of the scalar product $\vec{F} \cdot \vec{B}$ and substitution into Equation 2.8.13. To compute the vector product we can either use Equation 3.9.9 or compute the product directly, whichever way is simpler.

Solution

The components of the velocity vector are $u_x = -5.0$, $u_y = -2.0$, and $u_z = 3.5$. (a) The components of the magnetic field vector are $B_x = 7.2$, $B_y = -1.0$, and $B_z = -2.4$. Substituting them into Equation 3.9.10 gives the scalar components of vector $\vec{F} = \zeta \vec{u} \times \vec{B}$:

$$\begin{cases} F_x = \zeta(u_y B_z - u_z B_y) = \zeta[(-2.0)(-2.4) - (3.5)(-1.0)] = 8.3\zeta \\ F_y = \zeta(u_z B_x - u_x B_z) = \zeta[(3.5)(7.2) - (-5.0)(-2.4)] = 13.2\zeta \\ F_z = \zeta(u_x B_y - u_y B_x) = \zeta[(-5.0)(-1.0) - (-2.0)(7.2)] = 19.4\zeta \end{cases} \quad (3.9.11)$$

Thus, the magnetic force is $\vec{F} = \zeta(8.3 \hat{i} + 13.2 \hat{j} + 19.4 \hat{k})$ and its magnitude is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \zeta \sqrt{(8.3)^2 + (13.2)^2 + (19.4)^2} = 24.9\zeta. \quad (3.9.12)$$

To compute angle θ , we may need to find the magnitude of the magnetic field vector

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(7.2)^2 + (-1.0)^2 + (-2.4)^2} = 7.6, \quad (3.9.13)$$

and the scalar product $\vec{F} \cdot \vec{B}$:

$$\vec{F} \cdot \vec{B} = F_x B_x + F_y B_y + F_z B_z = (8.3\zeta)(7.2) + (13.2\zeta)(-1.0) + (19.4\zeta)(-2.4) = 0. \quad (3.9.14)$$

Now, substituting into Equation 2.8.13 gives angle θ :

$$\cos \theta = \frac{\vec{F} \cdot \vec{B}}{FB} = \frac{0}{(18.2\zeta)(7.6)} = 0 \Rightarrow \theta = 90^\circ. \quad (3.9.15)$$

Hence, the magnetic force vector is perpendicular to the magnetic field vector. (We could have saved some time if we had computed the scalar product earlier.)

(b) Because vector $\vec{B} = 4.5 \hat{k}$ has only one component, we can perform the algebra quickly and find the vector product directly:

$$\begin{aligned} \vec{F} &= \zeta \vec{u} \times \vec{B} = \zeta(-5.0 \hat{i} - 2.0 \hat{j} + 3.5 \hat{k}) \times (4.5 \hat{k}) \\ &= \zeta[(-5.0)(4.5) \hat{i} \times \hat{k} + (-2.0)(4.5) \hat{j} \times \hat{k} + (3.5)(4.5) \hat{k} \times \hat{k}] \\ &= \zeta[-22.5(-\hat{j}) - 9.0(\hat{i}) + 0] = \zeta(-9.0 \hat{i} + 22.5 \hat{j}). \end{aligned}$$

The magnitude of the magnetic force is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \zeta \sqrt{(-9.0)^2 + (22.5)^2 + (0.0)^2} = 24.2\zeta. \quad (3.9.16)$$

Because the scalar product is

$$\vec{F} \cdot \vec{B} = F_x B_x + F_y B_y + F_z B_z = (-9.0\zeta)(90) + (22.5\zeta)(0) + (0)(4.5) = 0, \quad (3.9.17)$$

the magnetic force vector \vec{F} is perpendicular to the magnetic field vector \vec{B} .

Significance

Even without actually computing the scalar product, we can predict that the magnetic force vector must always be perpendicular to the magnetic field vector because of the way this vector is constructed. Namely, the magnetic force vector is the vector product $\vec{F} = \zeta \vec{u} \times \vec{B}$ and, by the definition of the vector product (see Figure 3.9.1), vector \vec{F} must be perpendicular to both vectors \vec{u} and \vec{B} .

? Exercise 2.16

Given two vectors $\vec{A} = -\hat{i} + \hat{j}$ and $\vec{B} = 3\hat{i} - \hat{j}$, find (a) $\vec{A} \times \vec{B}$, (b) $|\vec{A} \times \vec{B}|$, (c) the angle between \vec{A} and \vec{B} , and (d) the angle between $\vec{A} \times \vec{B}$ and vector $\vec{C} = \hat{i} + \hat{k}$.

In conclusion to this section, we want to stress that “dot product” and “cross product” are entirely different mathematical objects that have different meanings. The dot product is a scalar; the cross product is a vector. Later chapters use the terms **dot product** and **scalar product** interchangeably. Similarly, the terms **cross product** and **vector product** are used interchangeably.

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