

3.8: Products of Vectors (Part 1)

Learning Objectives

- Explain the difference between the scalar product and the vector product of two vectors.
- Determine the scalar product of two vectors.
- Determine the vector product of two vectors.
- Describe how the products of vectors are used in physics.

A vector can be multiplied by another vector but may not be divided by another vector. There are two kinds of products of vectors used broadly in physics and engineering. One kind of multiplication is a **scalar multiplication of two vectors**. Taking a scalar product of two vectors results in a number (a scalar), as its name indicates. Scalar products are used to define work and energy relations. For example, the work that a force (a vector) performs on an object while causing its displacement (a vector) is defined as a scalar product of the force vector with the displacement vector. A quite different kind of multiplication is a **vector multiplication of vectors**. Taking a vector product of two vectors returns as a result a vector, as its name suggests. Vector products are used to define other derived vector quantities. For example, in describing rotations, a vector quantity called **torque** is defined as a vector product of an applied force (a vector) and its lever arm (a vector). It is important to distinguish between these two kinds of vector multiplications because the scalar product is a scalar quantity and a vector product is a vector quantity.

The Scalar Product of Two Vectors (the Dot Product)

Scalar multiplication of two vectors yields a scalar product.

Definition: Scalar Product (Dot Product)

The scalar product $\vec{A} \cdot \vec{B}$ of two vectors \vec{A} and \vec{B} is a number defined by the equation

$$\vec{A} \cdot \vec{B} = AB \cos \varphi, \quad (3.8.1)$$

where φ is the angle between the vectors (shown in Figure 3.8.1). The scalar product is also called the **dot product** because of the dot notation that indicates it.

In the definition of the dot product, the direction of angle φ does not matter, and φ can be measured from either of the two vectors to the other because $\cos \varphi = \cos(-\varphi) = \cos(2\pi - \varphi)$. The dot product is a negative number when $90^\circ < \varphi \leq 180^\circ$ and is a positive number when $0^\circ \leq \varphi < 90^\circ$. Moreover, the dot product of two parallel vectors is $\vec{A} \cdot \vec{B} = AB \cos 0^\circ = AB$, and the dot product of two antiparallel vectors is $\vec{A} \cdot \vec{B} = AB \cos 180^\circ = -AB$. The scalar product of two orthogonal vectors vanishes: $\vec{A} \cdot \vec{B} = AB \cos 90^\circ = 0$. The scalar product of a vector with itself is the square of its magnitude:

$$\vec{A}^2 \equiv \vec{A} \cdot \vec{A} = AA \cos 0^\circ = A^2 \quad (3.8.2)$$

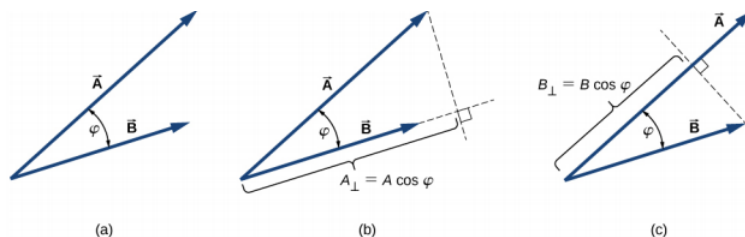


Figure 3.8.1: The scalar product of two vectors. (a) The angle between the two vectors. (b) The orthogonal projection A_\perp of vector \vec{A} onto the direction of vector \vec{B} . (c) The orthogonal projection B_\perp of vector \vec{B} onto the direction of vector \vec{A} .

✓ Example 3.8.1: The Scalar Product

For the vectors shown in Figure 2.3.6, find the scalar product $\vec{A} \cdot \vec{F}$.

Strategy

From Figure 2.3.6, the magnitudes of vectors \vec{A} and \vec{B} are $A = 10.0$ and $F = 20.0$. Angle θ , between them, is the difference: $\theta = \varphi - \alpha = 110^\circ - 35^\circ = 75^\circ$. Substituting these values into Equation 3.8.1 gives the scalar product.

Solution

A straightforward calculation gives us

$$\vec{A} \cdot \vec{F} = AF \cos \theta = (10.0)(20.0) \cos 75^\circ = 51.76. \quad (3.8.3)$$

? Exercise 2.11

For the vectors given in Figure 2.3.6, find the scalar products $\vec{A} \cdot \vec{B}$ and $\vec{B} \cdot \vec{C}$.

In the Cartesian coordinate system, scalar products of the unit vector of an axis with other unit vectors of axes always vanish because these unit vectors are orthogonal:

$$\hat{i} \cdot \hat{j} = |\hat{i}| |\hat{j}| \cos 90^\circ = (1)(1)(0) = 0, \quad (3.8.4)$$

$$\hat{i} \cdot \hat{k} = |\hat{i}| |\hat{k}| \cos 90^\circ = (1)(1)(0) = 0, \quad (3.8.5)$$

$$\hat{k} \cdot \hat{j} = |\hat{k}| |\hat{j}| \cos 90^\circ = (1)(1)(0) = 0. \quad (3.8.6)$$

In these equations, we use the fact that the magnitudes of all unit vectors are one: $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$. For unit vectors of the axes, Equation 3.8.2 gives the following identities:

$$\hat{i} \cdot \hat{i} = i^2 = \hat{j} \cdot \hat{j} = j^2 = \hat{k} \cdot \hat{k} = 1. \quad (3.8.7)$$

The scalar product $\vec{A} \cdot \vec{B}$ can also be interpreted as either the product of B with the projection A_{\parallel} of vector \vec{A} onto the direction of vector \vec{B} (Figure 3.8.1(b)) or the product of A with the projection B_{\parallel} of vector \vec{B} onto the direction of vector \vec{A} (Figure 3.8.1(c)):

$$\begin{aligned} \vec{A} \cdot \vec{B} &= AB \cos \varphi \\ &= B(A \cos \varphi) = BA_{\parallel} \\ &= A(B \cos \varphi) = AB_{\parallel}. \end{aligned}$$

For example, in the rectangular coordinate system in a plane, the scalar x-component of a vector is its dot product with the unit vector \hat{i} , and the scalar y-component of a vector is its dot product with the unit vector \hat{j} :

$$\begin{cases} \vec{A} \cdot \hat{i} = |\vec{A}| |\hat{i}| \cos \theta_A = A \cos \theta_A = A \cos \theta_A = A_x \\ \vec{A} \cdot \hat{j} = |\vec{A}| |\hat{j}| \cos(90^\circ - \theta_A) = A \sin \theta_A = A_y \end{cases} \quad (3.8.8)$$

Scalar multiplication of vectors is commutative,

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}, \quad (3.8.9)$$

and obeys the distributive law:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}. \quad (3.8.10)$$

We can use the commutative and distributive laws to derive various relations for vectors, such as expressing the dot product of two vectors in terms of their scalar components.

? Exercise 2.12

For vector $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ in a rectangular coordinate system, use Equation 3.8.4 through Equation 3.8.10 to show that $\vec{A} \cdot \hat{i} = A_x$, $\vec{A} \cdot \hat{j} = A_y$ and $\vec{A} \cdot \hat{k} = A_z$.

When the vectors in Equation 3.8.1 are given in their vector component forms,

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \text{ and } \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}, \quad (3.8.11)$$

we can compute their scalar product as follows:

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) \\ &= A_x B_x \hat{i} \cdot \hat{i} + A_x B_y \hat{i} \cdot \hat{j} + A_x B_z \hat{i} \cdot \hat{k} \\ &\quad + A_y B_x \hat{j} \cdot \hat{i} + A_y B_y \hat{j} \cdot \hat{j} + A_y B_z \hat{j} \cdot \hat{k} \\ &\quad + A_z B_x \hat{k} \cdot \hat{i} + A_z B_y \hat{k} \cdot \hat{j} + A_z B_z \hat{k} \cdot \hat{k}.\end{aligned}$$

Since scalar products of two different unit vectors of axes give zero, and scalar products of unit vectors with themselves give one (see Equation 3.8.4 and Equation 3.8.7), there are only three nonzero terms in this expression. Thus, the scalar product simplifies to

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z. \quad (3.8.12)$$

We can use Equation 3.8.12 for the scalar product in terms of scalar components of vectors to find the angle between two vectors. When we divide Equation 3.8.1 by AB , we obtain the equation for $\cos \varphi$, into which we substitute Equation 3.8.12

$$\cos \varphi = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{A_x B_x + A_y B_y + A_z B_z}{AB}. \quad (3.8.13)$$

Angle φ between vectors \vec{A} and \vec{B} is obtained by taking the inverse cosine of the expression in Equation 3.8.13

✓ Example 3.8.2

Three dogs are pulling on a stick in different directions, as shown in Figure 3.8.2. The first dog pulls with force $\vec{F}_1 = (10.0 \hat{i} - 20.4 \hat{j} + 2.0 \hat{k})\text{N}$, the second dog pulls with force $\vec{F}_2 = (-15.0 \hat{i} - 6.2 \hat{k})\text{N}$, and the third dog pulls with force $\vec{F}_3 = (5.0 \hat{i} + 12.5 \hat{j})\text{N}$. What is the angle between forces \vec{F}_1 and \vec{F}_2 ?

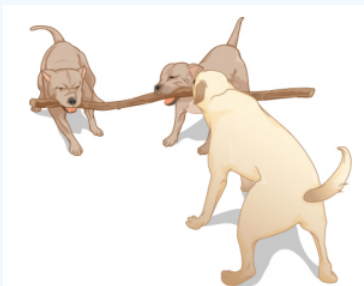


Figure 3.8.2: Three dogs are playing with a stick.

Strategy

The components of force vector \vec{F}_1 are $F_{1x} = 10.0 \text{ N}$, $F_{1y} = -20.4 \text{ N}$, and $F_{1z} = 2.0 \text{ N}$, whereas those of force vector \vec{F}_2 are $F_{2x} = -15.0 \text{ N}$, $F_{2y} = 0.0 \text{ N}$, and $F_{2z} = -6.2 \text{ N}$. Computing the scalar product of these vectors and their magnitudes, and substituting into Equation 3.8.13 gives the angle of interest.

Solution

The magnitudes of forces \vec{F}_1 and \vec{F}_2 are

$$F_1 = \sqrt{F_{1x}^2 + F_{1y}^2 + F_{1z}^2} = \sqrt{10.0^2 + 20.4^2 + 2.0^2} \text{ N} = 22.8 \text{ N} \quad (3.8.14)$$

and

$$F_2 = \sqrt{F_{2x}^2 + F_{2y}^2 + F_{2z}^2} = \sqrt{15.0^2 + 6.2^2} \text{ N} = 16.2 \text{ N}. \quad (3.8.15)$$

Substituting the scalar components into Equation 3.8.12 yields the scalar product

$$\begin{aligned}\vec{F}_1 \cdot \vec{F}_2 &= F_{1x} F_{2x} + F_{1y} F_{2y} + F_{1z} F_{2z} \\ &= (10.0 \text{ N})(-15.0 \text{ N}) + (-20.4 \text{ N})(0.0 \text{ N}) + (2.0 \text{ N})(-6.2 \text{ N}) \\ &= -162.4 \text{ N}^2.\end{aligned}$$

Finally, substituting everything into Equation 3.8.13 gives the angle

$$\cos \varphi = \frac{\vec{F}_1 \cdot \vec{F}_2}{F_1 F_2} = \frac{-162.4 \text{ N}^2}{(22.8 \text{ N})(16.2 \text{ N})} = -0.439 \Rightarrow \varphi = \cos^{-1}(-0.439) = 116.0^\circ. \quad (3.8.16)$$

Significance

Notice that when vectors are given in terms of the unit vectors of axes, we can find the angle between them without knowing the specifics about the geographic directions the unit vectors represent. Here, for example, the +x-direction might be to the east and the +y-direction might be to the north. But, the angle between the forces in the problem is the same if the +x-direction is to the west and the +y-direction is to the south.

? Exercise 2.13

Find the angle between forces \vec{F}_1 and \vec{F}_3 in Example 3.8.2.

✓ Example 3.8.3: The Work of a Force

When force \vec{F} pulls on an object and when it causes its displacement \vec{D} , we say the force performs work. The amount of work the force does is the scalar product $\vec{F} \cdot \vec{D}$. If the stick in Example 3.8.2 moves momentarily and gets displaced by vector $\vec{D} = (-7.9 \hat{j} - 4.2 \hat{k})\text{cm}$, how much work is done by the third dog in Example 3.8.2?

Strategy

We compute the scalar product of displacement vector \vec{D} with force vector $\vec{F}_3 = (5.0 \hat{i} + 12.5 \hat{j})\text{N}$, which is the pull from the third dog. Let's use W_3 to denote the work done by force \vec{F}_3 on displacement \vec{D} .

Solution

Calculating the work is a straightforward application of the dot product:

$$\begin{aligned} W_3 &= \vec{F}_3 \cdot \vec{D} = F_{3x}D_x + F_{3y}D_y + F_{3z}D_z \\ &= (5.0 \text{ N})(0.0 \text{ cm}) + (12.5 \text{ N})(-7.9 \text{ cm}) + (0.0 \text{ N})(-4.2 \text{ cm}) \\ &= -98.7 \text{ N} \cdot \text{cm}. \end{aligned}$$

Significance

The SI unit of work is called the joule (J), where $1 \text{ J} = 1 \text{ N} \cdot \text{m}$. The unit $\text{cm} \cdot \text{N}$ can be written as $10^{-2} \text{ m} \cdot \text{N} = 10^{-2} \text{ J}$, so the answer can be expressed as $W_3 = -0.9875 \text{ J} \approx -1.0 \text{ J}$.

? Exercise 2.14

How much work is done by the first dog and by the second dog in Example 3.8.2 on the displacement in Example 3.8.3?

Contributors and Attributions

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