3.6: Algebra of Vectors

Learning Objectives

- Apply analytical methods of vector algebra to find resultant vectors and to solve vector equations for unknown vectors.
- Interpret physical situations in terms of vector expressions.

Vectors can be added together and multiplied by scalars. Vector addition is associative (Equation 2.2.8) and commutative (Equation 2.2.7), and vector multiplication by a sum of scalars is distributive (Equation 2.2.9). Also, scalar multiplication by a sum of vectors is distributive:

$$\alpha(\vec{A}+\vec{B})=\alpha\vec{A}+\alpha B. \tag{3.6.1}$$

In this equation, α is any number (a scalar). For example, a vector antiparallel to vector $\vec{A}$ = $A_x\,\hat{i}$ + $A_y\,\hat{j}$ + $A_z\,\hat{k}$ can be expressed simply by multiplying $\vec{A}$ by the scalar α = −1:

$$-\vec{A}=A_x\hat{i}-A_y\hat{j}-A_z\hat{k}. \tag{3.6.2}$$

Example 3.6.1: Direction of Motion

In a Cartesian coordinate system where $\hat{i}$ denotes geographic east, $\hat{j}$ denotes geographic north, and $\hat{k}$ denotes altitude above sea level, a military convoy advances its position through unknown territory with velocity $\vec{v}$ = (4.0 $\hat{i}$ + 3.0 $\hat{j}$ + 0.1 $\hat{k}$) km/h. If the convoy had to retreat, in what geographic direction would it be moving?

Solution

The velocity vector has the third component $\vec{v}_z$ = (+ 0.1 km/h) $\hat{k}$, which says the convoy is climbing at a rate of 100 m/h through mountainous terrain. At the same time, its velocity is 4.0 km/h to the east and 3.0 km/h to the north, so it moves on the ground in direction $\tan^{-1}(3/4) \approx 37°$ north of east. If the convoy had to retreat, its new velocity vector $\vec{u}$ would have to be antiparallel to $\vec{v}$ and be in the form $\vec{u}=-\alpha\vec{v}$, where α is a positive number. Thus, the velocity of the retreat would be $\vec{u}$ = α(−4.0 $\hat{i}$ − 3.0 $\hat{j}$ − 0.1 $\hat{k}$) km/h. The negative sign of the third component indicates the convoy would be descending. The direction angle of the retreat velocity is $\tan^{-1}(-3\,\alpha - 4\,\alpha) \approx 37°$ south of west. Therefore, the convoy would be moving on the ground in direction 37° south of west while descending on its way back.

The generalization of the number zero to vector algebra is called the **null vector**, denoted by $\vec{0}$. All components of the null vector are zero, $\vec{0}$ = 0 $\hat{i}$+ 0 $\hat{j}$ + 0 $\hat{k}$, so the null vector has no length and no direction.

Two vectors $\vec{A}$ and $\vec{B}$ are **equal vectors** if and only if their difference is the null vector: $\vec{0}$ = $\vec{A}$ − $\vec{B}$ = ($A_x\,\hat{i}$ + $A_y\,\hat{j}$ + $A_z\,\hat{k}$) − ($B_x\,\hat{i}$ + $B_y\,\hat{j}$ + $B_z\,\hat{k}$) = ($A_x - B_x$) $\hat{i}$ + ($A_y - B_y$) $\hat{j}$ + ($A_z - B_z$) $\hat{k}$. This vector equation means we must have simultaneously $A_x - B_x = 0$, $A_y - B_y = 0$, and $A_z - B_z = 0$. Hence, we can write $\vec{A}=\vec{B}$ if and only if the corresponding components of vectors $\vec{A}$ and $\vec{B}$ are equal:

$$\vec{A}=\vec{B}\Leftrightarrow\begin{cases}A_x=B_x\\A_y=B_y\\A_z=B_z\end{cases}. \tag{3.6.3}$$

Two vectors are equal when their corresponding scalar components are equal. Resolving vectors into their scalar components (i.e., finding their scalar components) and expressing them analytically in vector component form (given by Equation 2.5.4) allows us to use vector algebra to find sums or differences of many vectors **analytically** (i.e., without using graphical methods). For example, to find the resultant of two vectors $\vec{A}$ and $\vec{B}$, we simply add them component by component, as follows:

$$\vec{R}=\vec{A}+\vec{B}=(A_x\,\hat{i}+A_y\,\hat{j}+A_z\,\hat{k})+(B_x\,\hat{i}+B_y\,\hat{j}+B_z\,\hat{k})=(A_x+B_x)\,\hat{i}+(A_y+B_y)\,\hat{j}+(A_z+B_z)\,\hat{k}. \tag{3.6.4}$$

In this way, using Equation 3.6.3, scalar components of the resultant vector $\vec{R}$ = $R_x\,\hat{i}$ + $R_y\,\hat{j}$ + $R_z\,\hat{k}$ are the sums of corresponding scalar components of vectors $\vec{A}$ and $\vec{B}$:

$$\begin{cases} R_x = A_x + B_x, \\ R_y = A_y + B_y, \\ R_z = A_z + B_z \end{cases}. \tag{3.6.5}$$

Analytical methods can be used to find components of a resultant of many vectors. For example, if we are to sum up N vectors $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, … , $\vec{F}_N$, where each vector is $\vec{F}_k$ = $F_{kx}\ \hat{i}$ + $F_{ky}\ \hat{j}$ + $F_{kz}\ \hat{k}$, the resultant vector $\vec{F}_R$ is

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_N = \sum_{k=1}^{N} \vec{F}_k = \sum_{k=1}^{N} \left(F_{kx}\hat{i} + F_{ky}\hat{j} + F_{kz}\hat{k}\right) = \left(\sum_{k=1}^{N} F_{kx}\right)\hat{i} + \left(\sum_{k=1}^{N} F_{ky}\right)\hat{j} + \left(\sum_{k=1}^{N} F_{kz}\right)\hat{k}. \tag{3.6.6}$$

Therefore, scalar components of the resultant vector are

$$\begin{cases} F_{Rx} = \sum_{k=1}^{N} F_{kx} = F_{1x} + F_{2x} + \ldots + F_{Nx} \\ F_{Ry} = \sum_{k=1}^{N} F_{ky} = F_{1y} + F_{2y} + \ldots + F_{Ny} \\ F_{Rz} = \sum_{k=1}^{N} F_{kz} = F_{1z} + F_{2z} + \ldots + F_{Nz}. \end{cases} \tag{3.6.7}$$

Having found the scalar components, we can write the resultant in vector component form:

$$\vec{F}_R = F_{Rx}\ \hat{i} + F_{Ry}\ \hat{j} + F_{Rz}\ \hat{k}. \tag{3.6.8}$$

Analytical methods for finding the resultant and, in general, for solving vector equations are very important in physics because many physical quantities are vectors. For example, we use this method in kinematics to find resultant displacement vectors and resultant velocity vectors, in mechanics to find resultant force vectors and the resultants of many derived vector quantities, and in electricity and magnetism to find resultant electric or magnetic vector fields.

In many physical situations, we often need to know the direction of a vector. For example, we may want to know the direction of a magnetic field vector at some point or the direction of motion of an object. We have already said direction is given by a unit vector, which is a dimensionless entity—that is, it has no physical units associated with it. When the vector in question lies along one of the axes in a Cartesian system of coordinates, the answer is simple, because then its unit vector of direction is either parallel or antiparallel to the direction of the unit vector of an axis. For example, the direction of vector $\vec{d}$ = −5 m $\hat{i}$ is unit vector $\hat{d}$ = − $\hat{i}$. The general rule of finding the unit vector $\hat{V}$ of direction for any vector $\vec{V}$ is to divide it by its magnitude V:

$$\hat{V} = \frac{\vec{V}}{V}. \tag{3.6.9}$$

We see from this expression that the unit vector of direction is indeed dimensionless because the numerator and the denominator in Equation 3.6.9 have the same physical unit. In this way, Equation 3.6.9 allows us to express the unit vector of direction in terms of unit vectors of the axes. Example 2.7.6 illustrates this principle.

Contributors

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