10.10: Center of Mass (Part 2)

Center of Mass of Continuous Objects

If the object in question has its mass distributed uniformly in space, rather than as a collection of discrete particles, then $m_j \rightarrow dm$, and the summation becomes an integral:

$$\vec{r}_{CM} = \frac{1}{M} \int \vec{r} dm. \tag{10.10.1}$$

In this context, r is a characteristic dimension of the object (the radius of a sphere, the length of a long rod). To generate an integrand that can actually be calculated, you need to express the differential mass element dm as a function of the mass density of the continuous object, and the dimension r. An example will clarify this.

Example 10.10.1: CM of a Uniform Thin Hoop

Find the center of mass of a uniform thin hoop (or ring) of mass M and radius r.

Strategy

First, the hoop's symmetry suggests the center of mass should be at its geometric center. If we define our coordinate system such that the origin is located at the center of the hoop, the integral should evaluate to zero.

We replace dm with an expression involving the density of the hoop and the radius of the hoop. We then have an expression we can actually integrate. Since the hoop is described as "thin," we treat it as a one-dimensional object, neglecting the thickness of the hoop. Therefore, its density is expressed as the number of kilograms of material per meter. Such a density is called a **linear mass density**, and is given the symbol λ; this is the Greek letter "lambda," which is the equivalent of the English letter "l" (for "linear").

Since the hoop is described as uniform, this means that the linear mass density λ is constant. Thus, to get our expression for the differential mass element dm, we multiply λ by a differential length of the hoop, substitute, and integrate (with appropriate limits for the definite integral).

Solution

First, define our coordinate system and the relevant variables (Figure 10.10.1).

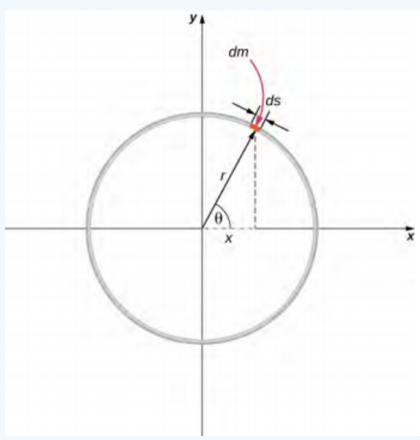

Figure 10.10.1: Finding the center of mass of a uniform hoop. We express the coordinates of a differential piece of the hoop, and then integrate around the hoop.

The center of mass is calculated with Equation 10.10.1:

$$\vec{r}_{CM} = \frac{1}{M} \int_a^b \vec{r} dm. \tag{10.10.2}$$

We have to determine the limits of integration a and b. Expressing $\vec{r}$ in component form gives us

$$\vec{r}_{CM}=\frac{1}{M}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]dm. \tag{10.10.3}$$

In the diagram, we highlighted a piece of the hoop that is of differential length ds; it therefore has a differential mass dm = λds. Substituting:

$$\vec{r}_{CM}=\frac{1}{M}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]\lambda ds. \tag{10.10.4}$$

However, the arc length ds subtends a differential angle d$theta$, so we have

$$ds=rd\theta \tag{10.10.5}$$

and thus

$$\vec{r}_{CM}=\frac{1}{M}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]\lambda rd\theta. \tag{10.10.6}$$

One more step: Since λ is the linear mass density, it is computed by dividing the total mass by the length of the hoop:

$$\lambda=\frac{M}{2\pi r} \tag{10.10.7}$$

giving us

$$\begin{aligned}\vec{r}_{CM}&=\frac{1}{M}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]\left(\frac{M}{2\pi r}\right)rd\theta\\&=\frac{1}{2\pi}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]d\theta.\end{aligned}$$

Notice that the variable of integration is now the angle θ. This tells us that the limits of integration (around the circular hoop) are θ = 0 to θ = 2π, so a = 0 and b = 2π. Also, for convenience, we separate the integral into the x- and y-components of $\vec{r}_{CM}$. The final integral expression is

$$\begin{aligned}\vec{r}_{CM}&=r_{CM,x}\hat{i}+r_{CM,y}\hat{j}\\&=\left[\frac{1}{2\pi}\int_0^{2\pi}(2\cos\theta d\theta\right]\hat{i}+\left[\frac{1}{2\pi}\int_0^{2\pi}(2\sin\theta d\theta\right]\hat{j}\\&=0\hat{i}+0\hat{j}=\vec{0}\end{aligned}$$

as expected.

Center of Mass and Conservation of Momentum

How does all this connect to conservation of momentum?

Suppose you have N objects with masses m_1, m_2, m_3, ...m_N and initial velocities $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, ..., $\vec{v}_N$. The center of mass of the objects is

$$\vec{r}_{CM}=\frac{1}{M}\sum_{j=1}^{N}m_j\vec{r}_j. \tag{10.10.8}$$

Its velocity is

$$\vec{v}_{CM}=\frac{d\vec{r}_{CM}}{dt}=\frac{1}{M}\sum_{j=1}^{N}m_j\frac{d\vec{r}_j}{dt} \tag{10.10.9}$$

and thus the initial momentum of the center of mass is

$$\left[M\frac{d\vec{r}_{CM}}{dt}\right]_i = \sum_{j=1}^{N} m_j \frac{d\vec{r}_{j,i}}{dt}$$
$$M\vec{v}_{CM,i} = \sum_{j=1}^{N} m_j \vec{v}_{j,i}.$$

After these masses move and interact with each other, the momentum of the center of mass is

$$M\vec{v}_{CM,f} = \sum_{j=1}^{N} m_j \vec{v}_{j,f}. \tag{10.10.10}$$

But conservation of momentum tells us that the right-hand side of both equations must be equal, which says

$$M\vec{v}_{CM,f} = M\vec{v}_{CM,i}. \tag{10.10.11}$$

This result implies that conservation of momentum is expressed in terms of the center of mass of the system. Notice that as an object moves through space with no net external force acting on it, an individual particle of the object may accelerate in various directions, with various magnitudes, depending on the net internal force acting on that object at any time. (Remember, it is only the vector sum of all the internal forces that vanishes, not the internal force on a single particle.) Thus, such a particle's momentum will not be constant—but the momentum of the entire extended object will be, in accord with Equation 10.10.11.

Equation 10.10.11 implies another important result: Since M represents the mass of the entire system of particles, it is necessarily constant. (If it isn't, we don't have a closed system, so we can't expect the system's momentum to be conserved.) As a result, Equation 10.10.11 implies that, for a closed system,

$$\vec{v}_{CM,f} = \vec{v}_{CM,i}. \tag{10.10.12}$$

That is to say, **in the absence of an external force, the velocity of the center of mass never changes**.

You might be tempted to shrug and say, "Well yes, that's just Newton's first law," but remember that Newton's first law discusses the constant velocity of a particle, whereas Equation 10.10.12 applies to the center of mass of a (possibly vast) collection of interacting particles, and that there may not be any particle at the center of mass at all! So, this really is a remarkable result.

Example 10.10.2: Fireworks Display

When a fireworks rocket explodes, thousands of glowing fragments fly outward in all directions, and fall to Earth in an elegant and beautiful display (Figure 10.10.2). Describe what happens, in terms of conservation of momentum and center of mass.

Figure 10.10.2: These exploding fireworks are a vivid example of conservation of momentum and the motion of the center of mass.

The picture shows radial symmetry about the central points of the explosions; this suggests the idea of center of mass. We can also see the parabolic motion of the glowing particles; this brings to mind projectile motion ideas.

Solution

Initially, the fireworks rocket is launched and flies more or less straight upward; this is the cause of the more-or-less-straight, white trail going high into the sky below the explosion in the upper-right of the picture (the yellow explosion). This trail is not parabolic because the explosive shell, during its launch phase, is actually a rocket; the impulse applied to it by the ejection of the burning fuel applies a force on the shell during the rise-time interval. (This is a phenomenon we will study in the next section.) The shell has multiple forces on it; thus, it is not in free-fall prior to the explosion.

At the instant of the explosion, the thousands of glowing fragments fly outward in a radially symmetrical pattern. The symmetry of the explosion is the result of all the internal forces summing to zero $\left(\sum_j \vec{f}_j^{int} = 0\right)$; for every internal force, there is another that is equal in magnitude and opposite in direction.

However, as we learned above, these internal forces cannot change the momentum of the center of mass of the (now exploded) shell. Since the rocket force has now vanished, the center of mass of the shell is now a projectile (the only force on it is gravity), so its trajectory does become parabolic. The two red explosions on the left show the path of their centers of mass at a slightly longer time after explosion compared to the yellow explosion on the upper right.

In fact, if you look carefully at all three explosions, you can see that the glowing trails are not truly radially symmetric; rather, they are somewhat denser on one side than the other. Specifically, the yellow explosion and the lower middle explosion are slightly denser on their right sides, and the upper-left explosion is denser on its left side. This is because of the momentum of their centers of mass; the differing trail densities are due to the momentum each piece of the shell had at the moment of its explosion. The fragment for the explosion on the upper left of the picture had a momentum that pointed upward and to the left; the middle fragment's momentum pointed upward and slightly to the right; and the right-side explosion clearly upward and to the right (as evidenced by the white rocket exhaust trail visible below the yellow explosion).

Finally, each fragment is a projectile on its own, thus tracing out thousands of glowing parabolas.

Significance

In the discussion above, we said, "…the center of mass of the shell is now a projectile (the only force on it is gravity)…." This is not quite accurate, for there may not be any mass at all at the center of mass; in which case, there could not be a force acting on it. This is actually just verbal shorthand for describing the fact that the gravitational forces on all the particles act so that the center of mass changes position exactly as if all the mass of the shell were always located at the position of the center of mass.

? Exercise 10.10.2

How would the firework display change in deep space, far away from any source of gravity?

You may sometimes hear someone describe an explosion by saying something like, "the fragments of the exploded object always move in a way that makes sure that the center of mass continues to move on its original trajectory." This makes it sound as if the process is somewhat magical: how can it be that, in every explosion, it always works out that the fragments move in just the right way so that the center of mass' motion is unchanged? Phrased this way, it would be hard to believe no explosion ever does anything differently.

The explanation of this apparently astonishing coincidence is: We defined the center of mass precisely so this is exactly what we would get. Recall that first we defined the momentum of the system:

$$\vec{p}_{CM} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}. \tag{10.10.13}$$

We then concluded that the net external force on the system (if any) changed this momentum:

$$\vec{F} = \frac{d\vec{p}_{CM}}{dt} \tag{10.10.14}$$

and then—and here's the point—we defined an acceleration that would obey Newton's second law. That is, we demanded that we should be able to write

$$\vec{a} = \frac{\vec{F}}{M} \tag{10.10.15}$$

which requires that

$$\vec{a} = \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j\right). \tag{10.10.16}$$

where the quantity inside the parentheses is the center of mass of our system. So, it’s not astonishing that the center of mass obeys Newton’s second law; we defined it so that it would.
