2.5: Motion with Constant Acceleration (Part 1)

Learning Objectives

- Identify which equations of motion are to be used to solve for unknowns.
- Use appropriate equations of motion to solve a two-body pursuit problem.

You might guess that the greater the acceleration of, say, a car moving away from a stop sign, the greater the car's displacement in a given time. But, we have not developed a specific equation that relates acceleration and displacement. In this section, we look at some convenient equations for kinematic relationships, starting from the definitions of displacement, velocity, and acceleration. We first investigate a single object in motion, called single-body motion. Then we investigate the motion of two objects, called **two-body pursuit problems**.

Notation

First, let us make some simplifications in notation. Taking the initial time to be zero, as if time is measured with a stopwatch, is a great simplification. Since elapsed time is $\Delta t = t_f - t_0$, taking $t_0 = 0$ means that $\Delta t = t_f$, the final time on the stopwatch. When initial time is taken to be zero, we use the subscript 0 to denote initial values of position and velocity. That is, x_0 **is the initial position** and v_0 **is the initial velocity**. We put no subscripts on the final values. That is, t **is the final time**, x **is the final position**, and v **is the final velocity**. This gives a simpler expression for elapsed time, $\Delta t = t$. It also simplifies the expression for x displacement, which is now $\Delta x = x - x_0$. Also, it simplifies the expression for change in velocity, which is now $\Delta v = v - v_0$. To summarize, using the simplified notation, with the initial time taken to be zero,

$$\Delta t = t \tag{2.5.1}$$

$$\Delta x = x - x_0 \tag{2.5.2}$$

$$\Delta v = v - v_0, \tag{2.5.3}$$

where the subscript 0 denotes an initial value and the absence of a subscript denotes a final value in whatever motion is under consideration.

We now make the important assumption that acceleration is constant. This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal—that is,

$$\bar{a} = a = constant. \tag{2.5.4}$$

Thus, we can use the symbol a for acceleration at all times. Assuming acceleration to be constant does not seriously limit the situations we can study nor does it degrade the accuracy of our treatment. For one thing, acceleration is constant in a great number of situations. Furthermore, in many other situations we can describe motion accurately by assuming a constant acceleration equal to the average acceleration for that motion. Lastly, for motion during which acceleration changes drastically, such as a car accelerating to top speed and then braking to a stop, motion can be considered in separate parts, each of which has its own constant acceleration.

Displacement and Position from Velocity

To get our first two equations, we start with the definition of average velocity:

$$\bar{v} = \frac{\Delta x}{\Delta t}. \tag{2.5.5}$$

Substituting the simplified notation for Δx and Δt yields

$$\bar{v} = \frac{x - x_0}{t}. \tag{2.5.6}$$

Solving for x gives us

$$x = x_0 + \bar{v}t, \tag{2.5.7}$$

where the average velocity is

$$\bar{v}=\frac{v_0+v}{2}. \tag{2.5.8}$$

The equation $\bar{v}=\frac{v_0+v}{2}$ reflects the fact that when acceleration is constant, v is just the simple average of the initial and final velocities. Figure 2.5.1 illustrates this concept graphically. In part (a) of the figure, acceleration is constant, with velocity increasing at a constant rate. The average velocity during the 1-h interval from 40 km/h to 80 km/h is 60 km/h:

$$\bar{v}=\frac{v_0+v}{2}=\frac{40\ km/h+80\ km/h}{2}=60\ km/h. \tag{2.5.9}$$

In part (b), acceleration is not constant. During the 1-h interval, velocity is closer to 80 km/h than 40 km/h. Thus, the average velocity is greater than in part (a).

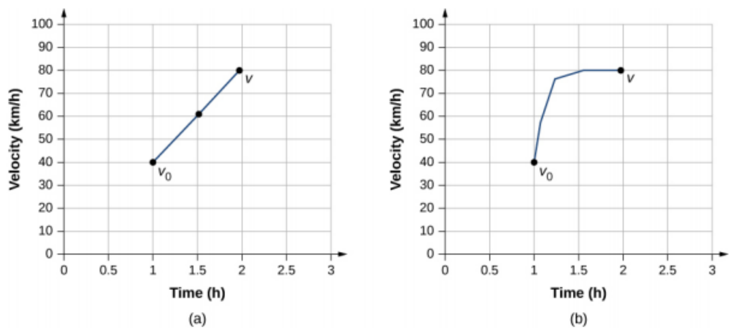

Figure 2.5.1: (a) Velocity-versus-time graph with constant acceleration showing the initial and final velocities v_0 and v. The average velocity is $\frac{1}{2}(v_0+v)=60$ km/h. (b) Velocity-versus-time graph with an acceleration that changes with time. The average velocity is not given by $\frac{1}{2}(v_0+v)$, but is greater than 60 km/h.

Solving for Final Velocity from Acceleration and Time

We can derive another useful equation by manipulating the definition of acceleration:

$$a=\frac{\Delta v}{\Delta t}. \tag{2.5.10}$$

Substituting the simplified notation for Δv and Δt gives us

$$a=\frac{v-v_0}{t}\ (constant\ a). \tag{2.5.11}$$

Solving for v yields

$$v=v_0+at\ (constant\ a). \tag{2.5.12}$$

Example 3.7: Calculating Final Velocity

An airplane lands with an initial velocity of 70.0 m/s and then decelerates at 1.50 m/s^2 for 40.0 s. What is its final velocity?

Strategy

First, we identify the knowns: v_0 = 70 m/s, a = −1.50 m/s^2, t = 40 s.

Second, we identify the unknown; in this case, it is final velocity v_f.

Last, we determine which equation to use. To do this we figure out which kinematic equation gives the unknown in terms of the knowns. We calculate the final velocity using Equation 2.5.12, v = v_0 + at.

Solution

Substitute the known values and solve:

$$v=v_0+at=70.0\ m/s+(-1.50\ m/s^2)(40.0\ s)=10.0\ m/s. \tag{2.5.13}$$

Figure 2.5.2 is a sketch that shows the acceleration and velocity vectors.

Figure 2.5.2: The airplane lands with an initial velocity of 70.0 m/s and slows to a final velocity of 10.0 m/s before heading for the terminal. Note the acceleration is negative because its direction is opposite to its velocity, which is positive.

Significance

The final velocity is much less than the initial velocity, as desired when slowing down, but is still positive (see figure). With jet engines, reverse thrust can be maintained long enough to stop the plane and start moving it backward, which is indicated by a negative final velocity, but is not the case here.

In addition to being useful in problem solving, the equation $v = v_0 + at$ gives us insight into the relationships among velocity, acceleration, and time. We can see, for example, that

- Final velocity depends on how large the acceleration is and how long it lasts
- If the acceleration is zero, then the final velocity equals the initial velocity ($v = v_0$), as expected (in other words, velocity is constant)
- If a is negative, then the final velocity is less than the initial velocity

All these observations fit our intuition. Note that it is always useful to examine basic equations in light of our intuition and experience to check that they do indeed describe nature accurately.

Solving for Final Position with Constant Acceleration

We can combine the previous equations to find a third equation that allows us to calculate the final position of an object experiencing constant acceleration. We start with

$$v = v_0 + at. \tag{2.5.14}$$

Adding v_0 to each side of this equation and dividing by 2 gives

$$\frac{v_0 + v}{2} = v_0 + \frac{1}{2}at. \tag{2.5.15}$$

Since $\frac{v_0+v}{2} = \bar{v}$ for constant acceleration, we have

$$\bar{v} = v_0 + \frac{1}{2}at. \tag{2.5.16}$$

Now we substitute this expression for $\bar{v}$ into the equation for displacement, $x = x_0 + \bar{v}t$, yielding

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \ (constant\ a). \tag{2.5.17}$$

✓ Example 3.8: Calculating Displacement of an Accelerating Object

Dragsters can achieve an average acceleration of 26.0 m/s^2. Suppose a dragster accelerates from rest at this rate for 5.56 s Figure 2.5.3. How far does it travel in this time?

Figure 2.5.3: U.S. Army Top Fuel pilot Tony “The Sarge” Schumacher begins a race with a controlled burnout. (credit: Lt. Col. William Thurmond. Photo Courtesy of U.S. Army.)

Strategy

First, let's draw a sketch Figure 2.5.4. We are asked to find displacement, which is x if we take x_0 to be zero. (Think about x_0 as the starting line of a race. It can be anywhere, but we call it zero and measure all other positions relative to it.) We can use the equation $x = x_0 + v_0 t + \frac{1}{2}at^2$ when we identify v_0, a, and t from the statement of the problem.

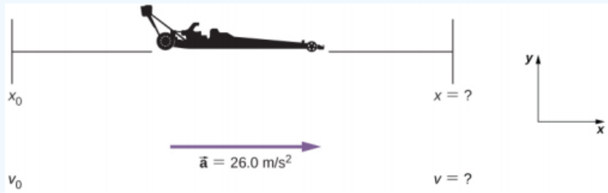

Figure 2.5.4: Sketch of an accelerating dragster.

Solution

First, we need to identify the knowns. Starting from rest means that $v_0 = 0$, a is given as 26.0 m/s^2 and t is given as 5.56 s.

Second, we substitute the known values into the equation to solve for the unknown:

$$x = x_0 + v_0 t + \frac{1}{2}at^2. \tag{2.5.18}$$

Since the initial position and velocity are both zero, this equation simplifies to

$$x = \frac{1}{2}at^2. \tag{2.5.19}$$

Substituting the identified values of a and t gives

$$x = \frac{1}{2}(26.0\ m/s^2)(5.56\ s)^2 = 402\ m. \tag{2.5.20}$$

Significance

If we convert 402 m to miles, we find that the distance covered is very close to one-quarter of a mile, the standard distance for drag racing. So, our answer is reasonable. This is an impressive displacement to cover in only 5.56 s, but top-notch dragsters can do a quarter mile in even less time than this. If the dragster were given an initial velocity, this would add another term to the distance equation. If the same acceleration and time are used in the equation, the distance covered would be much greater.

What else can we learn by examining the equation $x = x_0 + v_0 t + \frac{1}{2}at^2$? We can see the following relationships:

- Displacement depends on the square of the elapsed time when acceleration is not zero. In Example 3.8, the dragster covers only one-fourth of the total distance in the first half of the elapsed time.
- If acceleration is zero, then initial velocity equals average velocity ($v_0 = \bar{v}$) , and $x = x_0 + v_0 t + \frac{1}{2}at^2$ becomes $x = x_0 + v_0 t$.

Solving for Final Velocity from Distance and Acceleration

A fourth useful equation can be obtained from another algebraic manipulation of previous equations. If we solve $v = v_0 + at$ for t, we get

$$t = \frac{v - v_0}{a}. \tag{2.5.21}$$

Substituting this and $\bar{v} = \frac{v_0 + v}{2}$ into $x = x_0 + \bar{v}t$, we get

$$v^2 = v_0^2 + 2a(x - x_0)\ (constant\ a). \tag{2.5.22}$$

✔ Example 3.9: Calculating Final Velocity

Calculate the final velocity of the dragster in Example 3.8 without using information about time.

Strategy

The equation $v^2 = v_0^2 + 2a(x - x_0)$ is ideally suited to this task because it relates velocities, acceleration, and displacement, and no time information is required.

Solution

First, we identify the known values. We know that $v_0 = 0$, since the dragster starts from rest. We also know that $x - x_0 = 402$ m (this was the answer in Example 3.8). The average acceleration was given by a = 26.0 m/s^2. Second, we substitute the knowns into the equation $v^2 = v_0^2 + 2a(x - x_0)$ and solve for v:

$$v^2 = 0 + 2(26.0\ m/s^2)(402\ m). \tag{2.5.23}$$

Thus,

$$v^2 = 2.09 \times 10^4\ m/s^2 \tag{2.5.24}$$

$$v = \sqrt{2.09 \times 10^4\ m^2/s^2} = 145\ m/s. \tag{2.5.25}$$

Significance

A velocity of 145 m/s is about 522 km/h, or about 324 mi/h, but even this breakneck speed is short of the record for the quarter mile. Also, note that a square root has two values; we took the positive value to indicate a velocity in the same direction as the acceleration.

An examination of the equation $v^2 = v_0^2 + 2a(x - x_0)$ can produce additional insights into the general relationships among physical quantities:

- The final velocity depends on how large the acceleration is and the distance over which it acts.
- For a fixed acceleration, a car that is going twice as fast doesn't simply stop in twice the distance. It takes much farther to stop. (This is why we have reduced speed zones near schools.)