10.3: Impulse and Collisions (Part 1)

Learning Objectives

- Explain what an impulse is, physically
- Describe what an impulse does
- Relate impulses to collisions
- Apply the impulse-momentum theorem to solve problems

We have defined momentum to be the product of mass and velocity. Therefore, if an object's velocity should change (due to the application of a force on the object), then necessarily, its momentum changes as well. This indicates a connection between momentum and force. The purpose of this section is to explore and describe that connection.

Suppose you apply a force on a free object for some amount of time. Clearly, the larger the force, the larger the object's change of momentum will be. Alternatively, the more time you spend applying this force, again the larger the change of momentum will be, as depicted in Figure $10.3.1$. The amount by which the object's motion changes is therefore proportional to the magnitude of the force, and also to the time interval over which the force is applied.

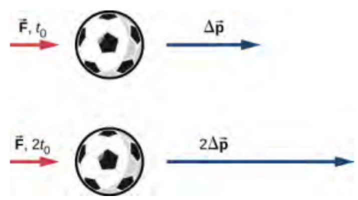


Figure $10.3.1$: The change in momentum of an object is proportional to the length of time during which the force is applied. If a force is exerted on the lower ball for twice as long as on the upper ball, then the change in the momentum of the lower ball is twice that of the upper ball.

Mathematically, if a quantity is proportional to two (or more) things, then it is proportional to the product of those things. The product of a force and a time interval (over which that force acts) is called impulse, and is given the symbol $\vec{J}$.

Definition: Impulse

Let $\vec{F}$(t) be the force applied to an object over some differential time interval dt (Figure $10.3.2$). The resulting impulse on the object is defined as

$$d\vec{J} \equiv \vec{F}(t)dt. \tag{10.3.1}$$

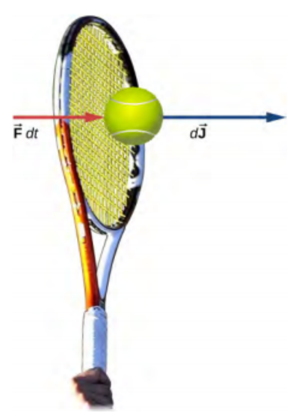


Figure $10.3.2$: A force applied by a tennis racquet to a tennis ball over a time interval generates an impulse acting on the ball.

The total impulse over the interval $t_f - t_i$ is

$$\vec{J} = \int_{t_i}^{t_f} d\vec{J} \tag{10.3.2}$$

or

$$\vec{J} \equiv \int_{t_i}^{t_f} \vec{F}(t)dt. \tag{10.3.3}$$

Equations 10.3.1 and 10.3.3 together say that when a force is applied for an infinitesimal time interval dt, it causes an infinitesimal impulse $d\vec{J}$, and the total impulse given to the object is defined to be the sum (integral) of all these infinitesimal impulses.

To calculate the impulse using Equation 10.3.3, we need to know the force function F(t), which we often don't. However, a result from calculus is useful here: Recall that the average value of a function over some interval is calculated by

$$f(x)_{ave} = \frac{1}{\Delta x} \int_{x_i}^{x_f} f(x)dx \tag{10.3.4}$$

where $\Delta x = x_f - x_i$. Applying this to the time-dependent force function, we obtain

$$\vec{F}_{ave} = \frac{1}{\Delta t} \int_{t_i}^{t_f} \vec{F}(t)dt. \tag{10.3.5}$$

Therefore, from Equation 10.3.3,

$$\vec{J} = \vec{F}_{ave}\Delta t. \tag{10.3.6}$$

The idea here is that you can calculate the impulse on the object even if you don't know the details of the force as a function of time; you only need the average force. In fact, though, the process is usually reversed: You determine the impulse (by measurement or calculation) and then calculate the average force that caused that impulse.

To calculate the impulse, a useful result follows from writing the force in Equation 10.3.3 as $\vec{F}(t) = m\,\vec{a}(t)$:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt = m \int_{t_i}^{t_f} \vec{a}(t)dt = m\left[\vec{v}(t_f) - \vec{v}(t_i)\right]. \tag{10.3.7}$$

For a constant force $\vec{F}_{ave} = \vec{F} = m\vec{a}$, this simplifies to

$$\vec{J} = m\vec{a}\Delta t = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i). \tag{10.3.8}$$

That is,

$$\vec{J} = m\Delta\vec{v}. \tag{10.3.9}$$

Note that the integral form, Equation 10.3.3, applies to constant forces as well; in that case, since the force is independent of time, it comes out of the integral, which can then be trivially evaluated.

✔ Example 10.3.1: The Arizona Meteor Crater

Approximately 50,000 years ago, a large (radius of 25 m) iron-nickel meteorite collided with Earth at an estimated speed of 1.28×10^4 m/s in what is now the northern Arizona desert, in the United States. The impact produced a crater that is still visible today (Figure 10.3.3); it is approximately 1200 m (three-quarters of a mile) in diameter, 170 m deep, and has a rim that rises 45 m above the surrounding desert plain. Iron-nickel meteorites typically have a density of ρ = 7970 kg/m^3. Use impulse considerations to estimate the average force and the maximum force that the meteor applied to Earth during the impact.

Figure 10.3.3: The Arizona Meteor Crater in Flagstaff, Arizona (often referred to as the Barringer Crater after the person who first suggested its origin and whose family owns the land). (credit: “Shane.torgerson”/Wikimedia Commons)

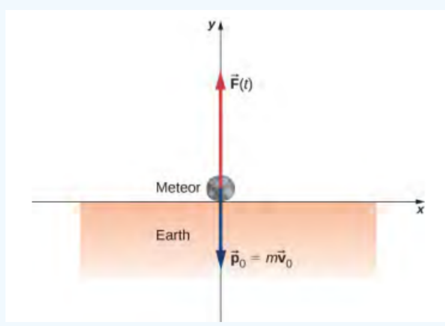
Strategy

It is conceptually easier to reverse the question and calculate the force that Earth applied on the meteor in order to stop it. Therefore, we’ll calculate the force on the meteor and then use Newton’s third law to argue that the force from the meteor on Earth was equal in magnitude and opposite in direction.

Using the given data about the meteor, and making reasonable guesses about the shape of the meteor and impact time, we first calculate the impulse using Equation 10.3.9. We then use the relationship between force and impulse Equation 10.3.6 to estimate the average force during impact. Next, we choose a reasonable force function for the impact event, calculate the average value of that function Equation 10.3.5, and set the resulting expression equal to the calculated average force. This enables us to solve for the maximum force.

Solution

Define upward to be the +y-direction. For simplicity, assume the meteor is traveling vertically downward prior to impact. In that case, its initial velocity is $\vec{v}_i = -v_i\,\hat{j}$, and the force Earth exerts on the meteor points upward, $\vec{F}(t) = +F(t)\,\hat{j}$. The situation at t = 0 is depicted below.



The average force during the impact is related to the impulse by

$$\vec{F}_{ave} = \frac{\vec{J}}{\Delta t}. \tag{10.3.10}$$

From Equation 10.3.9, $\vec{J} = m\Delta\vec{v}$, so we have

$$\vec{F}_{ave} = \frac{m\Delta\vec{v}}{\Delta t}. \tag{10.3.11}$$

The mass is equal to the product of the meteor’s density and its volume:

$$m = \rho V. \tag{10.3.12}$$

If we assume (guess) that the meteor was roughly spherical, we have

$$V = \frac{4}{3}\pi R^3. \tag{10.3.13}$$

Thus we obtain

$$\vec{F}_{ave} = \frac{\rho V \Delta \vec{v}}{\Delta t} = \frac{\rho \left(\frac{4}{3}\pi R^3\right)(\vec{v}_f - \vec{v}_i)}{\Delta t}. \tag{10.3.14}$$

The problem says the velocity at impact was -1.28×10^4 m/s $\hat{j}$ (the final velocity is zero); also, we guess that the primary impact lasted about $t_{max} = 2$ s. Substituting these values gives

$$\begin{aligned} \vec{F}_{ave} &= \frac{(7970\ kg/m^3)\left[\frac{4}{3}\pi(25\ m)^3\right]\left[0\ m/s - (-1.28 \times 10^4\ m/s\ \hat{j})\right]}{2\ s} \\ &= +(3.33 \times 10^{12}\ N)\hat{j} \end{aligned}$$

This is the average force applied during the collision. Notice that this force vector points in the same direction as the change of velocity vector $\Delta\vec{v}$.

Next, we calculate the maximum force. The impulse is related to the force function by

$$\vec{J} = \int_{t_i}^{t_{max}} \vec{F}(t)dt. \tag{10.3.15}$$

We need to make a reasonable choice for the force as a function of time. We define t = 0 to be the moment the meteor first touches the ground. Then we assume the force is a maximum at impact, and rapidly drops to zero. A function that does this is

$$F(t) = F_{max} e^{\frac{-t^2}{2\tau^2}}. \tag{10.3.16}$$

The parameter τ represents how rapidly the force decreases to zero.) The average force is

$$F_{ave} = \frac{1}{\Delta t}\int_0^{t_{max}} F_{max} e^{\frac{-t^2}{2\tau^2}}\, dt \tag{10.3.17}$$

where $\Delta t = t_{max} - 0$ s. Since we already have a numeric value for F_{ave}, we can use the result of the integral to obtain F_{max}. Choosing $\tau = \frac{1}{e} t_{max}$ (this is a common choice, as you will see in later chapters), and guessing that $t_{max} = 2$ s, this integral evaluates to

$$F_{avg} = 0.458\ F_{max}. \tag{10.3.18}$$

Thus, the maximum force has a magnitude of

$$\begin{aligned} 0.458\ F_{max} &= 3.33 \times 10^{12}\ N \\ F_{max} &= 7.27 \times 10^{12}\ N. \end{aligned}$$

The complete force function, including the direction, is

$$\vec{F}(t) = (7.27 \times 10^{12}\ N)e^{\frac{-t^2}{8\ s^2}}\hat{y}. \tag{10.3.19}$$

This is the force Earth applied to the meteor; by Newton's third law, the force the meteor applied to Earth is

$$\vec{F}(t) = -(7.27 \times 10^{12}\ N)e^{\frac{-t^2}{8\ s^2}}\hat{y} \tag{10.3.20}$$

which is the answer to the original question.

Significance

The graph of this function contains important information. Let's graph (the magnitude of) both this function and the average force together (Figure 10.3.4).

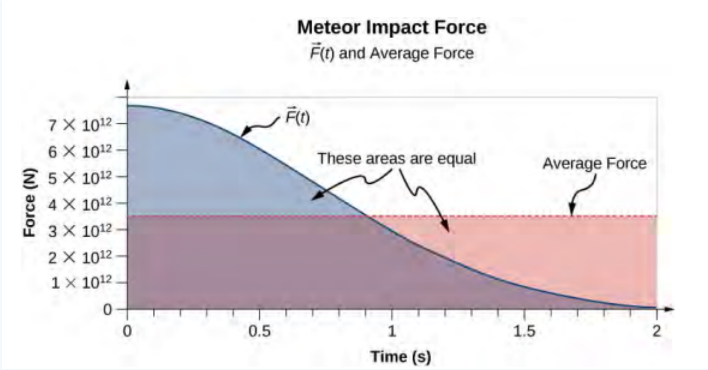


Figure 10.3.4: A graph of the average force (in red) and the force as a function of time (blue) of the meteor impact. The areas under the curves are equal to each other, and are numerically equal to the applied impulse.

Notice that the area under each plot has been filled in. For the plot of the (constant) force F_{ave}, the area is a rectangle, corresponding to F_{ave} Δt = J. As for the plot of F(t), recall from calculus that the area under the plot of a function is numerically equal to the integral of that function, over the specified interval; so here, that is $\int_0^{t_{max}}$ F(t)dt = J. Thus, the areas are equal, and both represent the impulse that the meteor applied to Earth during the two-second impact. The average force on Earth sounds like a huge force, and it is. Nevertheless, Earth barely noticed it. The acceleration Earth obtained was just

$$\vec{a} = \frac{-\vec{F}_{ave}}{M_{Earth}} = \frac{-(3.33 \times 10^{12}\ N)\hat{j}}{5.97 \times 10^{24}\ kg} = -(5.6 \times 10^{-13} m/s^2)\hat{j} \tag{10.3.21}$$

which is completely immeasurable. That said, the impact created seismic waves that nowadays could be detected by modern monitoring equipment.

✓ Example 10.3.2: The Benefits of Impulse

A car traveling at 27 m/s collides with a building. The collision with the building causes the car to come to a stop in approximately 1 second. The driver, who weighs 860 N, is protected by a combination of a variable-tension seatbelt and an airbag (Figure 10.3.5). (In effect, the driver collides with the seatbelt and airbag and not with the building.) The airbag and seatbelt slow his velocity, such that he comes to a stop in approximately 2.5 s.

a. What average force does the driver experience during the collision?
b. Without the seatbelt and airbag, his collision time (with the steering wheel) would have been approximately 0.20 s. What force would he experience in this case?

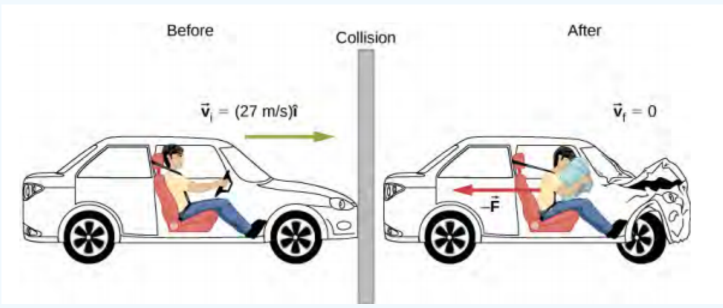


Figure 10.3.5: The motion of a car and its driver at the instant before and the instant after colliding with the wall. The restrained driver experiences a large backward force from the seatbelt and airbag, which causes his velocity to decrease to zero. (The forward force from the seatback is much smaller than the backward force, so we neglect it in the solution.)

Strategy

We are given the driver's weight, his initial and final velocities, and the time of collision; we are asked to calculate a force. Impulse seems the right way to tackle this; we can combine Equation 10.3.6 and Equation 10.3.9.

Solution

a. Define the +x-direction to be the direction the car is initially moving. We know

$$\vec{J} = \vec{F}\Delta t \tag{10.3.22}$$

and

$$\vec{J} = m\Delta\vec{v}. \tag{10.3.23}$$

Since J is equal to both those things, they must be equal to each other:

$$\vec{F}\Delta t = m\Delta\vec{v}. \tag{10.3.24}$$

We need to convert this weight to the equivalent mass, expressed in SI units:

$$\frac{860\ N}{9.8\ m/s^2} = 87.8\ kg. \tag{10.3.25}$$

Remembering that $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$, and noting that the final velocity is zero, we solve for the force:

$$\vec{F} = m\frac{0 - v_i\ \hat{i}}{\Delta t} = (87.8\ kg)\left(\frac{-(27\ m/s)\hat{i}}{2.5\ s}\right) = -(948\ N)\hat{i}. \tag{10.3.26}$$

The negative sign implies that the force slows him down. For perspective, this is about 1.1 times his own weight.

b. Same calculation, just the different time interval:

$$\vec{F} = (87.8\ kg)\left(\frac{-(27\ m/s)\hat{i}}{0.20\ s}\right) = -(11,853\ N)\hat{i}. \tag{10.3.27}$$

which is about 14 times his own weight. Big difference!

Significance

You see that the value of an airbag is how greatly it reduces the force on the vehicle occupants. For this reason, they have been required on all passenger vehicles in the United States since 1991, and have been commonplace throughout Europe and Asia since the mid-1990s. The change of momentum in a crash is the same, with or without an airbag; the force, however, is vastly different.