2.6: Motion with Constant Acceleration (Part 2)

Putting Equations Together

In the following examples, we continue to explore one-dimensional motion, but in situations requiring slightly more algebraic manipulation. The examples also give insight into problem-solving techniques. The note that follows is provided for easy reference to the equations needed. Be aware that these equations are not independent. In many situations we have two unknowns and need two equations from the set to solve for the unknowns. We need as many equations as there are unknowns to solve a given situation.

Summary of Kinematic Equations (constant a)

$$x = x_0 + \bar{v}t \tag{2.6.1}$$

$$\bar{v} = \frac{v_0 + v}{2} \tag{2.6.2}$$

$$v = v_0 + at \tag{2.6.3}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{2.6.4}$$

$$v^2 = v_0^2 + 2a(x - x_0) \tag{2.6.5}$$

Before we get into the examples, let’s look at some of the equations more closely to see the behavior of acceleration at extreme values. Rearranging $v = v_0 + at$, we have

$$a = \frac{v - v_0}{t}. \tag{2.6.6}$$

From this we see that, for a finite time, if the difference between the initial and final velocities is small, the acceleration is small, approaching zero in the limit that the initial and final velocities are equal. On the contrary, in the limit t → 0 for a finite difference between the initial and final velocities, acceleration becomes infinite.

Similarly, rearranging $v^2 = v_0^2 + 2a(x - x_0)$, we can express acceleration in terms of velocities and displacement:

$$a = \frac{v^2 - v_0^2}{2(x - x_0)}. \tag{2.6.7}$$

Thus, for a finite difference between the initial and final velocities acceleration becomes infinite in the limit the displacement approaches zero. Acceleration approaches zero in the limit the difference in initial and final velocities approaches zero for a finite displacement.

Example 3.10: How Far Does a Car Go?

On dry concrete, a car can decelerate at a rate of 7.00 m/s^2, whereas on wet concrete it can decelerate at only 5.00 m/s^2. Find the distances necessary to stop a car moving at 30.0 m/s (about 110 km/h) on (a) dry concrete and (b) wet concrete. (c) Repeat both calculations and find the displacement from the point where the driver sees a traffic light turn red, taking into account his reaction time of 0.500 s to get his foot on the brake.

Strategy

First, we need to draw a sketch Figure $2.6.1$. To determine which equations are best to use, we need to list all the known values and identify exactly what we need to solve for.

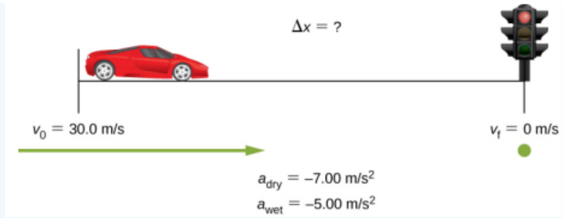

Figure 2.6.1: Sample sketch to visualize deceleration and stopping distance of a car.

Solution

a. First, we need to identify the knowns and what we want to solve for. We know that v_0 = 30.0 m/s, v = 0, and a = −7.00 m/s^2 (a is negative because it is in a direction opposite to velocity). We take x0 to be zero. We are looking for displacement Δx, or x − x_0. Second, we identify the equation that will help us solve the problem. The best equation to use is

$$v^2 = v_0^2 + 2a(x - x_0). \tag{2.6.8}$$

This equation is best because it includes only one unknown, x. We know the values of all the other variables in this equation. (Other equations would allow us to solve for x, but they require us to know the stopping time, t, which we do not know. We could use them, but it would entail additional calculations.) Third, we rearrange the equation to solve for x:

$$x - x_0 = \frac{v^2 - v_0^2}{2a} \tag{2.6.9}$$

and substitute the known values:

$$x - 0 = \frac{0^2 - (30.0\ m/s)^2}{2(-7.00\ m/s^2)}. \tag{2.6.10}$$

Thus,

$$x = 64.3\ m\ on\ dry\ concrete. \tag{2.6.11}$$

b. This part can be solved in exactly the same manner as (a). The only difference is that the acceleration is −5.00 m/s^2. The result is

$$x_{wet} = 90.0\ m\ on\ wet\ concrete. \tag{2.6.12}$$

c. When the driver reacts, the stopping distance is the same as it is in (a) and (b) for dry and wet concrete. So, to answer this question, we need to calculate how far the car travels during the reaction time, and then add that to the stopping time. It is reasonable to assume the velocity remains constant during the driver's reaction time. To do this, we, again, identify the knowns and what we want to solve for. We know that $\bar{v}$ = 30.0 m/s, $t_{reaction}$ = 0.500 s, and $a_{reaction}$ = 0. We take $x_{0\text{-}reaction}$ to be zero. We are looking for $x_{reaction}$. Second, as before, we identify the best equation to use. In this case, x = x_0 + $\bar{v}$t works well because the only unknown value is x, which is what we want to solve for. Third, we substitute the knowns to solve the equation:

$$x = 0 + (30.0\ m/s)(0.500\ s) = 15.0\ m. \tag{2.6.13}$$

This means the car travels 15.0 m while the driver reacts, making the total displacements in the two cases of dry and wet concrete 15.0 m greater than if he reacted instantly. Last, we then add the displacement during the reaction time to the displacement when braking (Figure 2.6.2),

$$x_{braking} + x_{reaction} = x_{total}, \tag{2.6.14}$$

and find (a) to be 64.3 m + 15.0 m = 79.3 m when dry and (b) to be 90.0 m + 15.0 m = 105 m when wet.

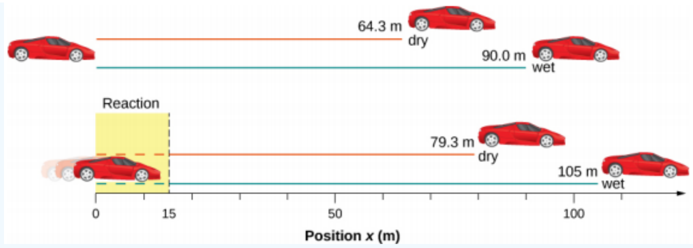

Figure 2.6.2: The distance necessary to stop a car varies greatly, depending on road conditions and driver reaction time. Shown here are the braking distances for dry and wet pavement, as calculated in this example, for a car traveling initially at 30.0 m/s. Also shown are the total distances traveled from the point when the driver first sees a light turn red, assuming a 0.500-s reaction time.

Significance

The displacements found in this example seem reasonable for stopping a fast-moving car. It should take longer to stop a car on wet pavement than dry. It is interesting that reaction time adds significantly to the displacements, but more important is the general approach to solving problems. We identify the knowns and the quantities to be determined, then find an appropriate equation. If there is more than one unknown, we need as many independent equations as there are unknowns to solve. There is often more than one way to solve a problem. The various parts of this example can, in fact, be solved by other methods, but the solutions presented here are the shortest.

Example 3.11: Calculating Time

Suppose a car merges into freeway traffic on a 200-m-long ramp. If its initial velocity is 10.0 m/s and it accelerates at 2.00 m/s^2, how long does it take the car to travel the 200 m up the ramp? (Such information might be useful to a traffic engineer.)

Strategy

First, we draw a sketch Figure 2.6.3. We are asked to solve for time t. As before, we identify the known quantities to choose a convenient physical relationship (that is, an equation with one unknown, t.)

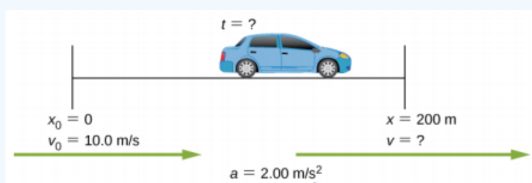

Figure 2.6.3: Sketch of a car accelerating on a freeway ramp.

Solution

Again, we identify the knowns and what we want to solve for. We know that $x_0 = 0$, $v_0 = 10$ m/s, $a = 2.00$ m/s^2, and $x = 200$ m.

We need to solve for t. The equation $x = x_0 + v_0t + \frac{1}{2}at^2$ works best because the only unknown in the equation is the variable t, for which we need to solve. From this insight we see that when we input the knowns into the equation, we end up with a quadratic equation.

We need to rearrange the equation to solve for t, then substituting the knowns into the equation:

$$200\ m = 0\ m + (10.0\ m/s)t + \frac{1}{2}(2.00\ m/s^2)t^2. \tag{2.6.15}$$

We then simplify the equation. The units of meters cancel because they are in each term. We can get the units of seconds to cancel by taking t = t s, where t is the magnitude of time and s is the unit. Doing so leaves

$$200 = 10t + t^2. \tag{2.6.16}$$

We then use the quadratic formula to solve for t,

$$t^2 + 10t - 200 = 0 \tag{2.6.17}$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{2.6.18}$$

which yields two solutions: t = 10.0 and t = −20.0. A negative value for time is unreasonable, since it would mean the event happened 20 s before the motion began. We can discard that solution. Thus,

$$t = 10.0\ s. \tag{2.6.19}$$

Significance

Whenever an equation contains an unknown squared, there are two solutions. In some problems both solutions are meaningful; in others, only one solution is reasonable. The 10.0-s answer seems reasonable for a typical freeway on-ramp.

Exercise 3.5

A manned rocket accelerates at a rate of 20 m/s^2 during launch. How long does it take the rocket to reach a velocity of 400 m/s?

Example 3.12: Acceleration of a Spaceship

A spaceship has left Earth's orbit and is on its way to the Moon. It accelerates at 20 m/s^2 for 2 min and covers a distance of 1000 km. What are the initial and final velocities of the spaceship?

Strategy

We are asked to find the initial and final velocities of the spaceship. Looking at the kinematic equations, we see that one equation will not give the answer. We must use one kinematic equation to solve for one of the velocities and substitute it into another kinematic equation to get the second velocity. Thus, we solve two of the kinematic equations simultaneously.

Solution

First we solve for v_0 using $x = x_0 + v_0 t + \frac{1}{2}at^2 = \frac{1}{2}t^2$:

$$x - x_0 = v_0 t + \frac{1}{2}at^2 = \frac{1}{2}t^2 \tag{2.6.20}$$

$$1.0 \times 10^6\ m = v_0(120.0\ s) + \frac{1}{2}(20.0\ m/s^2)(120.0\ s)^2 \tag{2.6.21}$$

$$v_0 = 7133.3\ m/s. \tag{2.6.22}$$

Then we substitute v_0 into v = v_0 + at to solve for the final velocity:

$$v = v_0 + at = 7133.3\ m/s + (20.0\ m/s^2)(120.0\ s) = 9533.3\ m/s. \tag{2.6.23}$$

Significance

There are six variables in displacement, time, velocity, and acceleration that describe motion in one dimension. The initial conditions of a given problem can be many combinations of these variables. Because of this diversity, solutions may not be easy as simple substitutions into one of the equations. This example illustrates that solutions to kinematics may require solving two simultaneous kinematic equations.

With the basics of kinematics established, we can go on to many other interesting examples and applications. In the process of developing kinematics, we have also glimpsed a general approach to problem solving that produces both correct answers and insights into physical relationships. The next level of complexity in our kinematics problems involves the motion of two interrelated bodies, called **two-body pursuit problems**.

Two-Body Pursuit Problems

Up until this point we have looked at examples of motion involving a single body. Even for the problem with two cars and the stopping distances on wet and dry roads, we divided this problem into two separate problems to find the answers. In a **two-body pursuit problem**, the motions of the objects are coupled—meaning, the unknown we seek depends on the motion of both objects. To solve these problems we write the equations of motion for each object and then solve them simultaneously to find the unknown. This is illustrated in Figure 2.6.4.

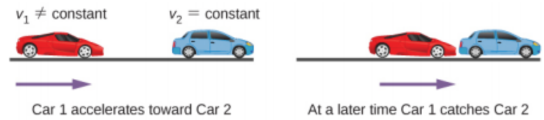

Figure 2.6.4: A two-body pursuit scenario where car 2 has a constant velocity and car 1 is behind with a constant acceleration. Car 1 catches up with car 2 at a later time.

The time and distance required for car 1 to catch car 2 depends on the initial distance car 1 is from car 2 as well as the velocities of both cars and the acceleration of car 1. The kinematic equations describing the motion of both cars must be solved to find these unknowns.

Consider the following example.

✓ Example 3.13: Cheetah Catching a Gazelle

A cheetah waits in hiding behind a bush. The cheetah spots a gazelle running past at 10 m/s. At the instant the gazelle passes the cheetah, the cheetah accelerates from rest at 4 m/s^2 to catch the gazelle. (a) How long does it take the cheetah to catch the gazelle? (b) What is the displacement of the gazelle and cheetah?

Strategy

We use the set of equations for constant acceleration to solve this problem. Since there are two objects in motion, we have separate equations of motion describing each animal. But what links the equations is a common parameter that has the same value for each animal. If we look at the problem closely, it is clear the common parameter to each animal is their position x at a later time t. Since they both start at $x_0 = 0$, their displacements are the same at a later time t, when the cheetah catches up with the gazelle. If we pick the equation of motion that solves for the displacement for each animal, we can then set the equations equal to each other and solve for the unknown, which is time.

Solution

a. Equation for the gazelle: The gazelle has a constant velocity, which is its average velocity, since it is not accelerating. Therefore, we use Equation 3.5.7 with $x_0 = 0$:

$$x = x_0 + \bar{v}t = \bar{v}t. \tag{2.6.24}$$

Equation for the cheetah: The cheetah is accelerating from rest, so we use Equation 3.5.17 with $x_0 = 0$ and $v_0 = 0$:

$$x = x_0 + v_0 t + \frac{1}{2}at^2 = \frac{1}{2}at^2. \tag{2.6.25}$$

Now we have an equation of motion for each animal with a common parameter, which can be eliminated to find the solution. In this case, we solve for t:

$$x = \bar{v}t = \frac{1}{2}at^2 \tag{2.6.26}$$

$$t = \frac{2\bar{v}}{a}. \tag{2.6.27}$$

The gazelle has a constant velocity of 10 m/s, which is its average velocity. The acceleration of the cheetah is 4 m/s^2. Evaluating t, the time for the cheetah to reach the gazelle, we have

$$t = \frac{2\bar{v}}{a} = \frac{2(10)}{4} = 5\ s. \tag{2.6.28}$$

b. To get the displacement, we use either the equation of motion for the cheetah or the gazelle, since they should both give the same answer. Displacement of the cheetah:

$$x = \frac{1}{2}at^2 = \frac{1}{2}(4)(5)^2 - 50\ m. \tag{2.6.29}$$

Displacement of the gazelle:

$$x = \bar{v}t = 10(5) = 50\ m. \tag{2.6.30}$$

We see that both displacements are equal, as expected.

Significance

It is important to analyze the motion of each object and to use the appropriate kinematic equations to describe the individual motion. It is also important to have a good visual perspective of the two-body pursuit problem to see the common parameter that links the motion of both objects.

Exercise 3.6

A bicycle has a constant velocity of 10 m/s. A person starts from rest and begins to run to catch up to the bicycle in 30 s when the bicycle is at the same position as the person. What is the acceleration of the person?
