3.S: Vectors (Summary)

Key Terms

anticommutative property	change in the order of operation introduces the minus sign
antiparallel vectors	two vectors with directions that differ by 180°
associative	terms can be grouped in any fashion
commutative	operations can be performed in any order
component form of a vector	a vector written as the vector sum of its components in terms of unit vectors
corkscrew right-hand rule	a rule used to determine the direction of the vector product
cross product	the result of the vector multiplication of vectors is a vector called a cross product; also called a vector product
difference of two vectors	vector sum of the first vector with the vector antiparallel to the second
direction angle	in a plane, an angle between the positive direction of the x-axis and the vector, measured counterclockwise from the axis to the vector
displacement	change in position
distributive	multiplication can be distributed over terms in summation
dot product	the result of the scalar multiplication of two vectors is a scalar called a dot product; also called a scalar product
equal vectors	two vectors are equal if and only if all their corresponding components are equal; alternately, two parallel vectors of equal magnitudes
magnitude	length of a vector
null vector	a vector with all its components equal to zero
orthogonal vectors	two vectors with directions that differ by exactly 90°, synonymous with perpendicular vectors
parallel vectors	two vectors with exactly the same direction angles
parallelogram rule	geometric construction of the vector sum in a plane
polar coordinate system	an orthogonal coordinate system where location in a plane is given by polar coordinates
polar coordinates	a radial coordinate and an angle
radical coordinate	distance to the origin in a polar coordinate system
resultant vector	vector sum of two (or more) vectors
scalar	a number, synonymous with a scalar quantity in physics
scalar component	a number that multiplies a unit vector in a vector component of a vector
scalar equation	equation in which the left-hand and right-hand sides are numbers
scalar product	the result of the scalar multiplication of two vectors is a scalar called a scalar product; also called a dot product
scalar quantity	quantity that can be specified completely by a single number with an appropriate physical unit
tail-to-head geometric construction	geometric construction for drawing the resultant vector of many vectors
unit vector	vector of a unit magnitude that specifies direction; has no physical unit
unit vectors of the axes	unit vectors that define orthogonal directions in a plane or in space
vector	mathematical object with magnitude and direction
vector components	orthogonal components of a vector; a vector is the vector sum of its vector components
vector equation	equation in which the left-hand and right-hand sides are vectors

vector product	the result of the vector multiplication of vectors is a vector called a vector product; also called a cross product
vector quantity	physical quantity described by a mathematical vector—that is, by specifying both its magnitude and its direction; synonymous with a vector in physics
vector sum	resultant of the combination of two (or more) vectors

Key Equations

Multiplication by a scalar (vector equation)	$$\vec{B} = \alpha\vec{A} \tag{3.S.1}$$
Multiplication by a scalar (scalar equation for magnitudes)	$$B = \lvert\alpha\rvert A \tag{3.S.2}$$
Resultant of two vectors	$$\vec{D}_{AD} = \vec{D}_{AC} + \vec{D}_{CD} \tag{3.S.3}$$
Commutative law	$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \tag{3.S.4}$$
Associative law	$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) \tag{3.S.5}$$
Distributive law	$$\alpha_1\vec{A} + \alpha_2\vec{A} = (\alpha_1 + \alpha_2)\vec{A} \tag{3.S.6}$$
The component form of a vector in two dimensions	$$\vec{A} = A_x\hat{i} + A_y\hat{j} \tag{3.S.7}$$
Scalar components of a vector in two dimensions	$$\begin{cases} A_x = x_e - x_b \\ A_y = y_e - y_b \end{cases} \tag{3.S.8}$$
Magnitude of a vector in a plane	$$A = \sqrt{A_x^2 + A_y^2} \tag{3.S.9}$$
The direction angle of a vector in a plane	$$\theta_A = \tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{3.S.10}$$
Scalar components of a vector in a plane	$$\begin{cases} A_x = A\cos\theta_A \\ A_y = A\sin\theta_A \end{cases} \tag{3.S.11}$$
Polar coordinates in a plane	$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \tag{3.S.12}$$
The component form of a vector in three dimensions	$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \tag{3.S.13}$$
The scalar z-component of a vector in three dimensions	$$A_z = z_e - z_b \tag{3.S.14}$$
Magnitude of a vector in three dimensions	$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{3.S.15}$$
Distributive property	$$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B} \tag{3.S.16}$$
Antiparallel vector to $\vec{A}$	$$-\vec{A} = A_x\hat{i} - A_y\hat{j} - A_z\hat{k} \tag{3.S.17}$$
Equal vectors	$$\vec{A} = \vec{B} \Leftrightarrow \begin{cases} A_x = B_x \\ A_y = B_y \\ A_z = B_z \end{cases} \tag{3.S.18}$$

Components of the resultant of N vectors	$\begin{cases} F_{Rx} = \sum_{k=1}^{N} F_{kx} = F_{1x} + F_{2x} + \ldots + F_{Nx} \\ F_{Ry} = \sum_{k=1}^{N} F_{ky} = F_{1y} + F_{2y} + \ldots + F_{Ny} \\ F_{Rz} = \sum_{k=1}^{N} F_{kz} = F_{1z} + F_{2z} + \ldots + F_{Nz} \end{cases}$	(3.S.19)
General unit vector	$\hat{V} = \frac{\vec{V}}{V}$	(3.S.20)
Definition of the scalar product	$\vec{A} \cdot \vec{B} = AB \cos\varphi$	(3.S.21)
Commutative property of the scalar product	$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$	(3.S.22)
Distributive property of the scalar product	$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$	(3.S.23)
Scalar product in terms of scalar components of vectors	$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$	(3.S.24)
Cosine of the angle between two vectors	$\cos\varphi = \frac{\vec{A} \cdot \vec{B}}{AB}$	(3.S.25)
Dot products of unit vectors	$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$	(3.S.26)
Magnitude of the vector product (definition)	$\lvert \vec{A} \times \vec{B} \rvert = AB \sin\varphi$	(3.S.27)
Anticommutative property of the vector product	$\lvert \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$	(3.S.28)
Distributive property of the vector product	$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$	(3.S.29)
Cross products of unit vectors	$\begin{cases} \hat{i} \times \hat{j} = +\hat{k}, \\ \hat{j} \times \hat{l} = +\hat{i}, \\ \hat{l} \times \hat{i} = +\hat{j}. \end{cases}$	(3.S.30)
The cross product in terms of scalar components of vectors	$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)$	

Summary

2.1 Scalars and Vectors

- A vector quantity is any quantity that has magnitude and direction, such as displacement or velocity.
- Geometrically, vectors are represented by arrows, with the end marked by an arrowhead. The length of the vector is its magnitude, which is a positive scalar. On a plane, the direction of a vector is given by the angle the vector makes with a reference direction, often an angle with the horizontal. The direction angle of a vector is a scalar.
- Two vectors are equal if and only if they have the same magnitudes and directions. Parallel vectors have the same direction angles but may have different magnitudes. Antiparallel vectors have direction angles that differ by 180°. Orthogonal vectors have direction angles that differ by 90°.
- When a vector is multiplied by a scalar, the result is another vector of a different length than the length of the original vector. Multiplication by a positive scalar does not change the original direction; only the magnitude is affected. Multiplication by a negative scalar reverses the original direction. The resulting vector is antiparallel to the original vector. Multiplication by a scalar is distributive. Vectors can be divided by nonzero scalars but cannot be divided by vectors.
- Two or more vectors can be added to form another vector. The vector sum is called the resultant vector. We can add vectors to vectors or scalars to scalars, but we cannot add scalars to vectors. Vector addition is commutative and associative.
- To construct a resultant vector of two vectors in a plane geometrically, we use the parallelogram rule. To construct a resultant vector of many vectors in a plane geometrically, we use the tail-to-head method.

2.2 Coordinate Systems and Components of a Vector

- Vectors are described in terms of their components in a coordinate system. In two dimensions (in a plane), vectors have two components. In three dimensions (in space), vectors have three components.
- A vector component of a vector is its part in an axis direction. The vector component is the product of the unit vector of an axis with its scalar component along this axis. A vector is the resultant of its vector components.

- Scalar components of a vector are differences of coordinates, where coordinates of the origin are subtracted from end point coordinates of a vector. In a rectangular system, the magnitude of a vector is the square root of the sum of the squares of its components.
- In a plane, the direction of a vector is given by an angle the vector has with the positive x-axis. This direction angle is measured counterclockwise. The scalar x-component of a vector can be expressed as the product of its magnitude with the cosine of its direction angle, and the scalar y-component can be expressed as the product of its magnitude with the sine of its direction angle.
- In a plane, there are two equivalent coordinate systems. The Cartesian coordinate system is defined by unit vectors $\hat{i}$ and $\hat{j}$ along the x-axis and the y-axis, respectively. The polar coordinate system is defined by the radial unit vector $\hat{r}$, which gives the direction from the origin, and a unit vector $\hat{t}$, which is perpendicular (orthogonal) to the radial direction.

2.3 Algebra of Vectors

- Analytical methods of vector algebra allow us to find resultants of sums or differences of vectors without having to draw them. Analytical methods of vector addition are exact, contrary to graphical methods, which are approximate.
- Analytical methods of vector algebra are used routinely in mechanics, electricity, and magnetism. They are important mathematical tools of physics.

2.4 Products of Vectors

- There are two kinds of multiplication for vectors. One kind of multiplication is the scalar product, also known as the dot product. The other kind of multiplication is the vector product, also known as the cross product. The scalar product of vectors is a number (scalar). The vector product of vectors is a vector.
- Both kinds of multiplication have the distributive property, but only the scalar product has the commutative property. The vector product has the anticommutative property, which means that when we change the order in which two vectors are multiplied, the result acquires a minus sign.
- The scalar product of two vectors is obtained by multiplying their magnitudes with the cosine of the angle between them. The scalar product of orthogonal vectors vanishes; the scalar product of antiparallel vectors is negative.
- The vector product of two vectors is a vector perpendicular to both of them. Its magnitude is obtained by multiplying their magnitudes by the sine of the angle between them. The direction of the vector product can be determined by the corkscrew right-hand rule. The vector product of two either parallel or antiparallel vectors vanishes. The magnitude of the vector product is largest for orthogonal vectors.
- The scalar product of vectors is used to find angles between vectors and in the definitions of derived scalar physical quantities such as work or energy.
- The cross product of vectors is used in definitions of derived vector physical quantities such as torque or magnetic force, and in describing rotations.

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