# 9.1: Background Material

## Text References

- balancing on the center of gravity

## A Classic Physics Problem

You are given $N$ identical uniform bricks of length $L$, and are asked to arrange them on top of one another such that the front end of the top brick extends as far over the edge of a table as possible.

**Figure 9.1.1 – Stacking Bricks to Extend off a Table's Edge**

The typical questions asked for this physical system are:

1. What is the farthest the front end of the top of these $N$ bricks can extend beyond the edge of the table? That is, what is the maximum value of $x$ in terms of $L$ for $N$ bricks?
2. Is there a limit to this extension? That is, what is the maximum value of $x$ in terms of $L$ as $N \rightarrow \infty$)?

The trick to analyzing this problem is to start with the top brick and work your way down. Suppose $N = 1$ – there is only one brick. The farthest this brick can extend beyond the edge of the table is of course $\frac{1}{2}L$. This is because the normal force from the table needs to align with the gravity force on the brick (which acts through its center), for there to be no net torque and the brick not to rotate off the table.

Now suppose there are two bricks. Well, the top brick still can't be allowed to rotate off the bottom brick, and the maximum extension it can have from the edge of that brick is again $\frac{1}{2}L$. We also need to arrange the bricks so that the combination of both bricks doesn't rotate off the table edge, so how far out can the second brick extend from the table edge? We can treat the two bricks as a single system, and as we know from the first brick, the we maximize the extension for which the system will not rotate if the system's center of mass is right at the edge. So we need to calculate where the *two-brick center of mass* is, and place *that* at the edge of the table. It isn't hard to determine that the center of mass of the two bricks is $\frac{3}{4}L$ from the open end of the top brick.

We can continue this process as we add more bricks, basically treating each new brick as if it is the table's edge for the previous bricks, and computing the new center of gravity of the full collection of bricks to determine the correct placement, and with it the maximum extension. Note that this means that the distance from the center of mass of all the bricks to the front end of the top brick the maximum $x$ value we are looking for.

## The Surprising Result

The general solution to this problem for $N$ bricks requires a bit of fancy footwork, so it will be presented here. We'll start by numbering the bricks from the top-down, with the $N^{th}$ brick resting on the table. The distance that the $n^{th}$ brick extends beyond the surface below it we will call:

$$\Delta x_n = \alpha_n L \;, \tag{9.1.1}$$

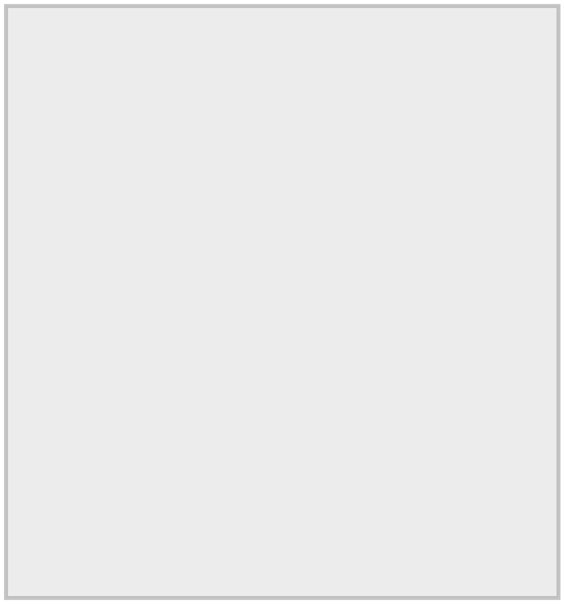
where the value $\alpha_n$ is the fractional overhang, and as we said above, $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{4}$.

The maximum overhang we are looking for is therefore:

$$x = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_N = (\alpha_1 + \alpha_2 + \cdots + \alpha_N)\, L \tag{9.1.2}$$

Perhaps a diagram is called for here:

**Figure 9.1.2 – N Bricks**



For maximum extension, the center of gravity of the top $N-1$ bricks is directly above the front edge of the bottom brick, as shown above. Let's choose the front end of the top brick as the origin along the $x$-direction. We'll call the distance separating the origin and the center of gravity of the top $N-1$ bricks $x_{cm}(N-1)$. We can use this to determine the distance from the origin of the center of gravity of all $N$ bricks, in the usual way. Calling the mass of a single brick $m$, then the mass of the $N-1$ bricks is $(N-1)m$, and the center of mass of the bottom brick is $\frac{1}{2}L$ farther from the origin than the center of gravity of the $N-1$ bricks. [*For a reminder of how to find the center of mass of two extended objects with their own centers of mass, go here.*]

$$x_{cm}(N) = \frac{[m]\left[x_{cm}(N-1) + \frac{1}{2}L\right] + [(N-1)\,m]\left[x_{cm}(N-1)\right]}{Nm} = x_{cm}(N-1) + \frac{1}{2N}L \tag{9.1.3}$$

So this shows that we can obtain the distance from the origin of the center of gravity of $N$ bricks if we know the distance of the center of gravity of the first $N-1$ bricks. This means we can express the distance as a sum by adding more and more terms (bricks). Clearly the formula works for $N=1$ (where $x_{cm}(0)=0$). Then we use this answer to solve for two bricks, and that answer to solve for three bricks, and so on. For $N$ bricks, we have the answer to question #1:

$$x(N) = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right) L \tag{9.1.4}$$

[*Note: Using our earlier notation,* $\alpha_n = \frac{1}{2n}$.]

The answer to question #2 is surprising. When the series in parentheses is taken to the limit as $N \rightarrow \infty$, it is called the *harmonic series*, and it diverges. This means that there is no limit to how far bricks can be extended, if we have enough of them! What makes this counterintuitive is how slowly this sum diverges. One can get the stack of bricks extended to more than one brick length ($x > L$) with just 4 bricks, but to get it to two brick lengths requires 31 bricks, and three brick lengths requires 227 bricks!