GRADUATE CLASSICAL MECHANICS

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Graduate Classical Mechanics

Michael Fowler

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Glossary

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Licensing

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The course begins with an introduction to the Calculus of Variations, and Hamilton's Principle. Next we go to material covered in the *last* chapter of Landau, the Hamilton-Jacobi formalism that makes clear the intimate connection between classical mechanics and quantum mechanics. (The students are taking quantum simultaneously, so this works well in helping appreciation of classical mechanics, for example how least action is a limit of the sum over paths, and how classical adiabatic invariants are immediately understandable from a quantum perspective.)

The rest of the course follows the sequence of the book, beginning with Keplerian orbits, which we cover in more detail than Landau. (Perhaps his students were already familiar with this material?)

Then on to small oscillations, but including some interesting nonlinear systems, for example parametric resonance, and the ponderomotive force. Landau treats these analytically, using perturbation theory-type approximations. The last part of the course covers rotational motion: free body, tops, nutation, Coriolis, etc.

We have added some material using the direct Newtonian vectorial approach to Newtonian mechanics (as opposed to the Lagrangian formulation),following Milne. In discussing orbits, we derive Hamilton's equation, a very quick route to the Runge-Lenz vector. At the end of the course, we give Milne's elegant analysis of a ball rolling on a tilted rotating plane. The surprising *cycloidal* path can be derived in a few lines from Newton's equations. (It's tough to do this nonholonomic problem using Lagrangian methods.)

About the Book

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CHAPTER OVERVIEW

1: Introductory Statics - the Catenary and the Arch

1.1: The Catenary
1.2: A Tight String
1.3: Not So Tight
1.4: Ideal Arches

1.1: The Catenary

What is the shape of a chain of small links hanging under gravity from two fixed points (one not directly below the other)? The word **catenary** (Latin for chain) was coined as a description for this curve by none other than Thomas Jefferson! Despite the image the word brings to mind of a chain of links, the word catenary is actually defined as the curve the chain approaches in the limit of taking smaller and smaller links, keeping the length of the chain constant. In other words, it describes a hanging rope. A *real* chain of identical rigid links is then a sort of discretization of the catenary.

We're going to analyze this problem as an introduction to the calculus of variations. First, we're going to solve it by a method you already know and love—just (mentally) adding up the forces on one segment of the rope. The tension is pulling at both ends, the segment's weight acts downwards. Since it's at rest, these three forces must add to zero. We'll show that writing down the balance of forces equation gives sufficient information to find the curve of the chain $y(x)$ meaning height y above ground as a function of horizontal position x. So we understand the mechanics of the problem.

But next we take a completely different approach: we assume the shape of the chain is described by an arbitrary function $y(x)$, required to go between the two fixed endpoints and to have total length equal to that of the (assumed inextensible) rope, and we work out its gravitational energy. We know, of course, that the true curve the rope settles into will be the one of minimum potential energy. The rope is at the bottom of a multidimensional potential well. This means that any slight variation from this minimum shape will only affect the potential energy to second order, in precise analogy with a slight change in position of a *particle* at the minimum point of a potential energy well.

So the technique, called the **Calculus of Variations**, is to find where the derivative of the potential energy with respect to variations of this curve becomes zero—that will be the minimum energy configuration we're looking for. Conceptually, this is a lot more complicated than ordinary differentiation with respect to one, or a few, variables. *We're varying a whole function*. This is why it's was worth solving the problem using traditional statics to begin with—it reassures us that the variational approach works.

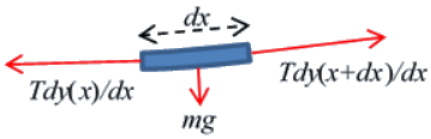
A Cambridge mathematician, William Whewell, famously stated in unconscious rhyme,

> *And so no force, however great,*
>
> *can pull a string, however fine,*
>
> *into a horizontal line,*
>
> *that shall be absolutely straight.*

(*Trivial fact*: Whewell had a way with words—he even invented some of the words you use every day, for example the words scientist, physicist, ion, cathode, anode, dielectric, and lots more.)

1.2: A Tight String

Let's take the hint and look first at a piece of uniform string at rest under high tension between two points at the same height, so that it's almost horizontal.



Each little bit of the string is in static equilibrium, so the forces on it balance. First, its weight acting downwards is $mg = \lambda g dx$, λ being the uniform mass per unit length. Second, the tension forces at the two ends don't quite balance because of the small change in slope.

Representing the string configuration as a curve $y(x)$, the balance of forces gives

$$T\frac{dy(x+dx)}{dx} - T\frac{dy(x)}{dx} = T\frac{d^2y}{dx^2}dx = \lambda g dx \tag{1.2.1}$$

so $d^2y/dx^2 = \lambda g/T,$ and $y = (\lambda g/2T)x^2$, taking the lowest point of the string as the origin.

So the curve is a parabola (but keep reading!).

1.3: Not So Tight

In fact, there are hidden approximations in the above analysis—for one thing, we've assumed that the length of string between x and $x+dx$ is dx , but it's really ds, where the distance parameter s is measured along the string. Second, we took the tension to be nonvarying. That's a pretty good approximation for a string that's almost horizontal, but think about a string a meter long hanging between two points 5 cm apart, and it becomes obvious that both these approximations are only good for a near-horizontal string.

Obviously, with the string nearly vertical, the tension is balancing the weight of string below it, and must be close to zero at the bottom, increasing approximately linearly with height. Not to mention, it's clear that this is no parabola, the two sides are very close to parallel near the top. The constant T approximation is evidently no good—but Whewell solved this problem exactly, back in the 1830's.

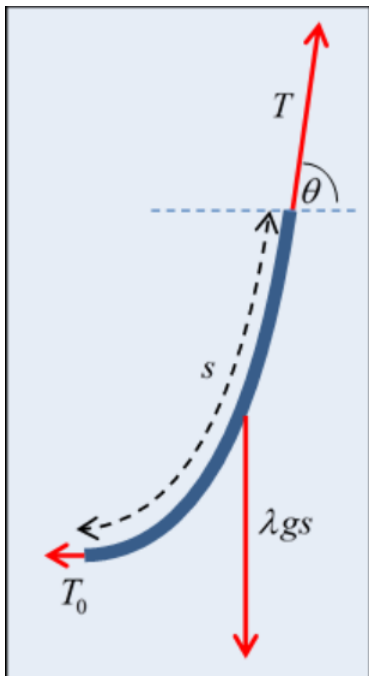


Figure 1.3.1

What he did was to work with the static equilibrium equation for a finite length of string, one end at the bottom.

If the tension at the bottom is T_0 and at a distance *s* away, *measured along the string*, the tension is T, and the string's angle to the horizontal there is θ (see diagram), then the equilibrium balance of force components is

$$T\cos\theta = T_0, \quad T\sin\theta = \lambda g s \tag{1.3.1}$$

from which the string slope

$$\tan\theta = \frac{\lambda g}{T_0}s = \frac{s}{a} \tag{1.3.2}$$

where we have introduced the constant $a = T_0/\lambda g$, which sets the length scale of the problem.

So we now have an equation for the catenary, θ in terms of *s*, distance along the string. What we want, though is an equation for vertical position y in terms of horizontal position x, the function $y(x)$ for the chain.

Now we've shown the slope is

$$dy/dx = \tan\theta = s/a \tag{1.3.3}$$

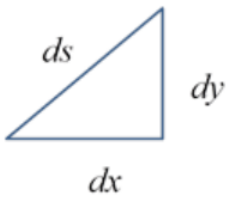


Figure 1.3.2

and the infinitesimals are related by $ds^2 = dx^2 + dy^2$, so putting these equations together

$$\left(\frac{s}{a}\right)^2 = \left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2 - 1 \tag{1.3.4}$$

that is,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{s}{a}\right)^2 = \frac{a^2 + s^2}{a^2} \tag{1.3.5}$$

Taking the square root and rearranging

$$\frac{ds}{\sqrt{a^2 + s^2}} = \frac{dx}{a} \tag{1.3.6}$$

which can be integrated immediately with the substitution

$$s = a\sinh\xi, \quad ds = a\cosh\xi d\xi \tag{1.3.7}$$

to give just

$$d\xi = dx/a \tag{1.3.8}$$

which integrates trivially to $\xi = (x/a) + b,$ with b a constant of integration, or

$$s = a\sinh\xi = a\sinh(x/a) \tag{1.3.9}$$

choosing the origin $x = 0$ at $s = 0,$ which makes $b = 0$

But of course what we want is the curve shape $y(x),$ not $s(x)$. We need to eliminate s in the favor of y. That is, we need to write y as a function of s, then substitute $s = a\sinh(x/a)$

Recall one of our first equations was for the slope $dy/dx = s/a$, and putting that together with $s = a\sinh(x/a)$ gives

$$ady = sdx = a\sinh(x/a)dx \tag{1.3.10}$$

integrating to

$$y = a\cosh(x/a) \tag{1.3.11}$$

This is the desired equation for the catenary curve $y(x)$

We've dropped the possible constant of integration, which is just the vertical positioning of the origin.

Question: is this the same as the curve of the chain in a suspension bridge?

Figure 1.3.3: Golden Gate Bridge

(Notice the vertical ropes are uniformly spaced horizontally.)

1.4: Ideal Arches

Now let's consider some upside-down curves, arches. Start with a Roman arch, an upside-down U.

Figure 1.4.1

Typically, Roman arches are found in sets like this, but let's consider one free standing arch. Assume it's made of blocks having the same cross-section throughout. What is the force between neighboring blocks? We'll do an upside-down version of the chain force analysis we just did.

Equating the pressure forces on the arch segment colored dark in the figure, we see pressure on the lowest block in the segment *must have a horizontal component, to balance the forced at the top point*, so the cement between blocks is under shear stress, or, for no cement, there's a strong sideways static frictional force. (So a series of arches, as shown in the photograph above, support each other with horizontal pressure.)

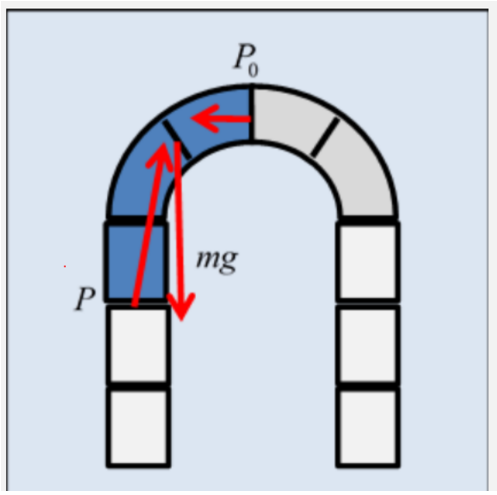


Figure 1.4.2: Pressure and weight forces on a segment of a Roman arch

A single Roman arch like this is therefore not an ideal design—it could fall apart sideways.

Let's define **an ideal arch** as one that doesn't have a tendency to fall apart sideways, outward or inward. This means no shear (sideways) stress between blocks, and that means the pressure force between blocks in contact is a normal force—it acts along the line of the arch. That should sound familiar! For a hanging string, obviously the tension acts along the line of the string.

Adding to our ideal arch definition that the blocks have the same mass per unit length along the entire arch, you can perhaps see that the static force balance for the arch is identical to that for the uniform hanging string, except that everything's reversed—the tension is now pressure, the whole thing is upside down.

Nevertheless, apart from the signs, the equations are mathematically identical, and the ideal arch shape is a catenary. Of course, some actual constructed arches, like the famous one in St. Louis, do not have uniform mass per unit length (It's thicker at the bottom) so the curve deviates somewhat from the ideal arch catenary.

Figure 1.4.3

Here’s a picture of catenary arches—these arches are in Barcelona, in a house designed by the architect Gaudi.

Figure 1.4.4: Gaudi's string model for the Sagreda Famila Church

In fact, Gaudi designed a church in Barcelona using a web of strings and weights to find correct shapes for the arches—he placed a large horizontal mirror above the strings, so looking in the mirror he could see what the actual right-way-up building would look like!—The lower image shows how he did it. (This model is in the church.)

CHAPTER OVERVIEW

2: The Calculus of Variations

We've seen how Whewell solved the problem of the equilibrium shape of chain hanging between two places, by finding how the forces on a length of chain, the tension at the two ends and its weight, balanced. We're now going to look at a completely different approach: the equilibrium configuration is an energy minimum, so small deviations from it can only make second-order changes in the gravitational potential energy. Here we'll find how analyzing that leads to a differential equation for the curve, and how the technique developed can be successfully applied of a vast array of problems.

2.1: The Catenary and the Soap Film
2.2: A Soap Film Between Two Horizontal Rings- the Euler-Lagrange Equation
2.3: General Method for the Minimization Problem
2.4: An Important First Integral of the Euler-Lagrange Equation
2.5: Fastest Curve for Given Horizontal Distance
2.6: The Perfect Pendulum
2.7: Calculus of Variations with Many Variables
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2.10: Lagrange Multipliers
2.11: Lagrange Multiplier for the Chain
2.12: The Brachistochrone

2.1: The Catenary and the Soap Film

The catenary is the curved configuration $y = y(x)$ of a uniform inextensible rope with two fixed endpoints at rest in a constant gravitational field. That is to say, it is the curve that minimizes the gravitational potential energy

$$J[y(x)] = 2\pi \int_{x_1}^{x_2} y ds = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx, \quad (y' = dy/dx) \tag{2.1.1}$$

where we have taken the rope density and gravity both equal to unity for mathematical convenience. Usually in calculus we minimize a function with respect to a single variable, or several variables. Here the potential energy is a function of a function, equivalent to an infinite number of variables, and our problem is to minimize it with respect to arbitrary small variations of that function. In other words, if we nudge the chain somewhere, and its motion is damped by air or internal friction, it will settle down again in the catenary configuration.

Formally speaking, there will be no change in that potential energy to leading order if we make an infinitesimal change in the curve, $y(x) \to y(x) + \delta y(x)$ (subject of course to keeping the length the same, that is $\delta \int ds = 0.$) .

This method of solving the problem is called the *calculus of variations*: in ordinary calculus, we make an infinitesimal change in a variable, and compute the corresponding change in a function, and if it's zero to leading order in the small change, we're at an extreme value.

(*Nitpicking footnote*: Actually this assumes the second order term is nonzero—what about x^3 near the origin? But such situations are infrequent in the problems we're likely to encounter.)

The difference here is that the potential energy of the hanging change isn't just a function of a variable, or even of a number of variables—it's a *function of a function*, it depends on the position of every point on the chain (in the limit of infinitely small links, that is, or equivalently a continuous rope).

So, we're looking for the configuration where the potential energy doesn't change to first order for *any* infinitesimal change in the curve of its position, subject to fixed endpoints, and a fixed chain length.

As a warm up, we'll consider a simpler—but closely related—problem.

2.2: A Soap Film Between Two Horizontal Rings- the Euler-Lagrange Equation

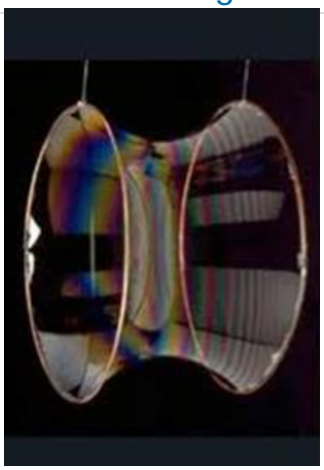

Figure 2.2.1: Photograph of a soap film between two horizontal rings

This problem is very similar to the catenary: surface tension will pull the soap film to the minimum possible total area compatible with the fixed boundaries (and neglecting gravity, which is a small effect).

(Interestingly, this problem is also closely related to *string theory*: as a closed string propagates, its path traces out as "world sheet" and the string dynamics is determined by that sheet having minimal area.)

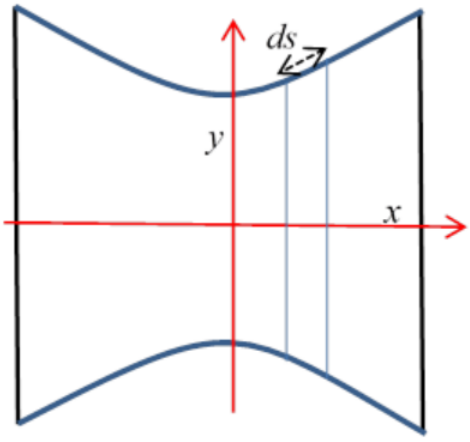

Figure 2.2.2

Taking the axis of rotational symmetry to be the x -axis, and the radius $y(x)$, we need to find the function $y(x)$ that minimizes the total area (ds is measured along the curve of the surface). Think of the soap film as a sequence of rings or collars, of radius y, and therefore area $2\pi y ds$.The total area is given by integrating, adding all these incremental collars,

$$J[y(x)] = 2\pi \int_{x_1}^{x_2} y ds = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx \tag{2.2.1}$$

subject to given values of y at the two ends. (You might be thinking at this point: isn't this *identical* to the catenary equation? The answer is yes, *but* the chain has an additional requirement: *it has a fixed length*. The soap film is not constrained in that way, it can stretch or contract to minimize the total area, so this is a different problem!)

That is, we want $\delta J = 0$ to first order, if we make a change $y(x) \to y(x) + \delta y(x)$. Of course, this also means $y'(x) \to y'(x) + \delta y'(x)$ where $\delta y' = \delta(dy/dx) = (d/dx)\delta y$

2.3: General Method for the Minimization Problem

To emphasize the generality of the method, we'll just write

$$J[y] = \int_{x_1}^{x_2} f\,(y, y')\,dx \quad (y' = dy/dx) \tag{2.3.1}$$

Then under any infinitesimal variation $\delta y(x)$ (equal to zero at the fixed endpoints)

$$\delta J[y] = \int_{x_1}^{x_2} \left[\frac{\partial f\,(y, y')}{\partial y} \delta y(x) + \frac{\partial f\,(y, y')}{\partial y'} \delta y'(x) \right] dx = 0 \tag{2.3.2}$$

To make further progress, we write $\delta y' = \delta(dy/dx) = (d/dx)\delta y$, then integrate the second term by parts, remembering $\delta y = 0$ at the endpoints, to get

$$\delta J[y] = \int_{x_1}^{x_2} \left[\frac{\partial f\,(y, y')}{\partial y} - \frac{d}{dx} \left(\frac{\partial f\,(y, y')}{\partial y'} \right) \right] \delta y(x) dx = 0 \tag{2.3.3}$$

Since this is true for any infinitesimal variation, we can choose a variation which is only nonzero near one point in the interval, and deduce that

$$\frac{\partial f\,(y, y')}{\partial y} - \frac{d}{dx} \left(\frac{\partial f\,(y, y')}{\partial y'} \right) = 0 \tag{2.3.4}$$

This general result is called the **Euler-Lagrange equation**. It's very important—you'll be seeing it again.

2.4: An Important First Integral of the Euler-Lagrange Equation

It turns out that, *since the function* \(f \) *does not contain x explicitly*, there is a simple first integral of this equation. Multiplying throughout by $y' = dy/dx$

$$\frac{\partial f\left(y, y'\right)}{\partial y}\frac{dy}{dx} - \frac{d}{dx}\left(\frac{\partial f\left(y, y'\right)}{\partial y'}\right)y' = 0 \tag{2.4.1}$$

Since f doesn't depend explicitly on x, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{2.4.2}$$

and using this to replace $\dfrac{\partial f\left(y, y'\right)}{\partial y}\dfrac{dy}{dx}$ in the preceding equation gives

$$\frac{df}{dx} - \frac{\partial f}{\partial y'}\frac{dy'}{dx} - \frac{d}{dx}\left(\frac{\partial f\left(y, y'\right)}{\partial y'}\right)y' = 0 \tag{2.4.3}$$

then multiplying by − (to match the equation as usually written) we have

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} - f\right) = 0 \tag{2.4.4}$$

giving a **first integral**

$$y'\frac{\partial f}{\partial y'} - f = \text{constant}. \tag{2.4.5}$$

For the soap film between two rings problem,

$$f\left(y, y'\right) = y\sqrt{1 + y'^2} \tag{2.4.6}$$

so the Euler-Lagrange equation is

$$\sqrt{1 + y'^2} - \frac{d}{dx}\frac{yy'}{\sqrt{1 + y'^2}} = 0 \tag{2.4.7}$$

and has first integral

$$y'\frac{\partial f}{\partial y'} - f = \frac{yy'^2}{\sqrt{1 + y'^2}} - y\sqrt{1 + y'^2} = -\frac{y}{\sqrt{1 + y'^2}} = \text{constant}. \tag{2.4.8}$$

We'll write

$$\frac{y}{\sqrt{1 + y'^2}} = a \tag{2.4.9}$$

with *a* the constant of integration, which will depend on the endpoints.

This is a first-order differential equation, and can be solved.

Rearranging,

$$\frac{dy}{dx} = \sqrt{\left(\frac{y}{a}\right)^2 - 1} \tag{2.4.10}$$

or

$$dx = \frac{ady}{\sqrt{y^2 - a^2}} \tag{2.4.11}$$

The standard substitution here is $y = a\cosh\xi$ from which

$$y = a\cosh\left(\frac{x-b}{a}\right) \tag{2.4.12}$$

Here b is the second constant of integration, the fixed endpoints determine a, b.

2.5: Fastest Curve for Given Horizontal Distance

Suppose we want to find the curve a bead slides down to minimize the time from the origin to some specified horizontal displacement X, but we don't care what vertical drop that entails.

Recall how we derived the equation for the curve:

At the minimum, under any infinitesimal variation $\delta y(x)$.

$$\delta J[y] = \int_{x_1}^{x_2} \left[\frac{\partial f\left(y, y'\right)}{\partial y} \delta y(x) + \frac{\partial f\left(y, y'\right)}{\partial y'} \delta y'(x) \right] dx = 0 \tag{2.5.1}$$

Writing $\delta y' = \delta(dy/dx) = (d/dx)\delta y$, and integrating the second term by parts,

$$\delta J[y] = \int_{x_1}^{x_2} \left[\frac{\partial f\left(y, y'\right)}{\partial y} - \frac{d}{dx}\left(\frac{\partial f\left(y, y'\right)}{\partial y'} \right) \right] \delta y(x) dx + \left[\frac{\partial f\left(y, y'\right)}{\partial y'} \delta y(x) \right]_0^X = 0 \tag{2.5.2}$$

In the earlier treatment, both endpoints were fixed, $\delta y(0) = \delta y(X) = 0$ so we dropped that final term.

However, we are now trying to find the fastest time for a given horizontal distance, so the final vertical distance is an adjustable parameter: $\delta y(X) \neq 0$

As before, since $\delta J[y] = 0$ or arbitrary $\delta y,$ we can still choose a $\delta y(x)$ which is only nonzero near some point not at the end, so we must still have

$$\frac{\partial f\left(y, y'\right)}{\partial y} - \frac{d}{dx}\left(\frac{\partial f\left(y, y'\right)}{\partial y'} \right) = 0 \tag{2.5.3}$$

However, we must also have $\dfrac{\partial f\left(y(X), y'(X)\right)}{\partial y'} \delta y(X) = 0$, to first order for arbitrary infinitesimal $\delta y(X)$, (imagine a variation δy only nonzero near the endpoint), this can only be true if $\dfrac{\partial f\left(y, y'\right)}{\partial y'} = 0$ at $x = X$

For the brachistochrone,

$$f = \sqrt{\frac{1 + y'^2}{2gy}}, \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2gy\left(1 + y'^2\right)}} \tag{2.5.4}$$

so $\dfrac{\partial f\left(y, y'\right)}{\partial y'} = 0$ at $x = X$ means that $f' = 0$, *the curve is horizontal at the end* $x = X$

So the curve that delivers the bead a given horizontal distance the fastest is the half-cycloid (inverted) flat at the end. It's easy to see this fixes the curve uniquely: think of the curve as generated by a rolling wheel, one half-turn of the wheel takes the top point to the bottom in distance X

Exercise: how low does it go?

2.6: The Perfect Pendulum

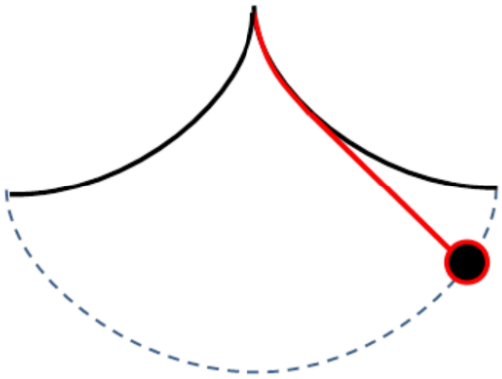

Figure 2.6.1

Around the time of Newton, the best timekeepers were pendulum clocks—but the time of oscillation of a simple pendulum depends on its amplitude, although of course the correction is small for small amplitude. The pendulum takes longer for larger amplitude. This can be corrected for by having the string constrained between enclosing surfaces to steepen the pendulum's path for larger amplitudes, and thereby speed it up.

It turns out (and was proved geometrically by Newton) that the ideal pendulum path is a cycloid. Thinking in terms of the equivalent bead on a wire problem, with a symmetric cycloid replacing the circular arc of an ordinary pendulum, if the bead is let go from rest at any point on the wire, it will reach the center in the same time as from any other point. So a clock with a pendulum constrained to such a path will keep very good time, and not be sensitive to the amplitude of swing.

The proof involves similar integrals and tricks to those used above:

$$T\left(y_0\right)=\int \frac{ds}{\sqrt{2g\left(y-y_0\right)}}=\int \frac{\sqrt{1+y'^2}}{\sqrt{2g\left(y-y_0\right)}}dx \tag{2.6.1}$$

and with the parameterization above, $y'=\sin\theta/(1-\cos\theta)$, the integral becomes

$$\sqrt{\frac{a}{g}}\int_{\theta_0}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_0-\cos\theta}}d\theta \tag{2.6.2}$$

As before, we can now write $1-\cos\theta=2\sin^2(\theta/2)$, etc., to find that $T\left(y_0\right)$ *is in fact independent of* y_0

This is left as an *exercise* for the reader. (*Hint*: you may find $\int_a^b dx/\sqrt{(x-a)(b-x)}=\pi$ to be useful. Can you prove this integral is correct? Why doesn't it depend on a,b?)

Exercise: As you well know, a simple harmonic oscillator, a mass on a linear spring with restoring force $-kx$, has a period independent of amplitude. Does this mean that a particle sliding on a cycloid is equivalent to a simple harmonic oscillator? Find out by expressing the motion as an equation $F=ma$ where the distance variable from the origin is *s measured along the curve*.

2.7: Calculus of Variations with Many Variables

We've found the equations defining the curve $y(x)$ along which the integral

$$J[y] = \int_{x_1}^{x_2} f\left(y, y'\right) dx \tag{2.7.1}$$

has a stationary value, and we've seen how it works in some two-dimensional curve examples.

But most dynamical systems are parameterized by more than one variable, so we need to know how to go from a curve in (x, y) to one in a space $(x, y_1, y_2, \ldots y_n)$, and we need to minimize (say)

$$J\left[y_1, y_2, \ldots y_n\right] = \int_{x_1}^{x_2} f\left(y_1, y_2, \ldots y_n, y_1', y_2', \ldots y_n'\right) dx \tag{2.7.2}$$

In fact, the generalization is straightforward: the path deviation simply becomes a vector,

$$\delta\vec{y}(x) = (\delta y_1(x), \delta y_2(x), \ldots, \delta y_n(x)) \tag{2.7.3}$$

Then under any infinitesimal variation

$$\delta\mathbf{y}(x) \text{ (writing also } \mathbf{y} = (y_1, \ldots y_n)) \tag{2.7.4}$$

$$\delta J[\vec{y}] = \int_{x_1}^{x_2} \sum_{i=1}^{n} \left[\frac{\partial f\left(\vec{y}, \vec{y}'\right)}{\partial y_i} \delta y_i(x) + \frac{\partial f\left(\vec{y}, \vec{y}'\right)}{\partial y_i'} \delta y_i'(x) \right] dx = 0 \tag{2.7.5}$$

Just as before, we take the variation zero at the endpoints, and integrate by parts to get now *n* separate equations for the stationary path:

$$\frac{\partial f\left(\vec{y}, \vec{y}'\right)}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f\left(\vec{y}, \vec{y}'\right)}{\partial y_i'} \right) = 0, \quad i = 1, \ldots, n \tag{2.7.6}$$

2.8: Multivariable First Integral

Following and generalizing the one-variable derivation, multiplying the above equations one by one by the corresponding $y_i' = dy_i/dx$ we have the *n* equations

$$\frac{\partial f\left(\vec{y},\vec{y}'\right)}{\partial y_i}\frac{dy_i}{dx} - \frac{d}{dx}\left(\frac{\partial f\left(\vec{y},\vec{y}'\right)}{\partial y_i'}\right)y_i' = 0 \tag{2.8.1}$$

Since f doesn't depend explicitly on x, we have

$$\frac{df}{dx} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\frac{dy_i}{dx} + \frac{\partial f}{\partial y_i'}\frac{dy_i'}{dx}\right) \tag{2.8.2}$$

and just as for the one-variable case, these equations give

$$\frac{d}{dx}\left(\sum_{i=1}^{n} y_i'\frac{\partial f}{\partial y_i'} - f\right) = 0 \tag{2.8.3}$$

and the (important!) **first integral** $\sum_{i=1}^{n} y_i'\dfrac{\partial f}{\partial y_i'} - f = \text{constant}$.

2.9: The Soap Film and the Chain

We see that the soap film profile function and the hanging chain have identical analytic form. This is not too surprising, because the potential energy of the hanging chain in simplified units is just

$$\int yds = \int y\left(1+y'^2\right)^{\frac{1}{2}}dx \tag{2.9.1}$$

the same as the area function for the soap film. But there's an important *physical* difference: the chain has a fixed length. The soap film is free to adjust its "length" to minimize the total area. The chain isn't—it's *constrained*. How do we deal with that?

2.10: Lagrange Multipliers

The problem of finding minima (or maxima) of a function subject to constraints was first solved by Lagrange. A simple example will suffice to show the method.

Imagine we have some smooth curve in the (x, y) plane that does not pass through the origin, and we want to find the point on the curve that is its closest approach to the origin. A standard illustration is to picture a winding road through a bowl shaped valley, and ask for the low point on the road. (We'll also assume that $\backslash x$ determines $\backslash y$ uniquely, the road doesn't double back, etc. If it does, the method below would give a series of *locally* closest points to the origin, we would need to go through them one by one to find the *globally* closest point.)

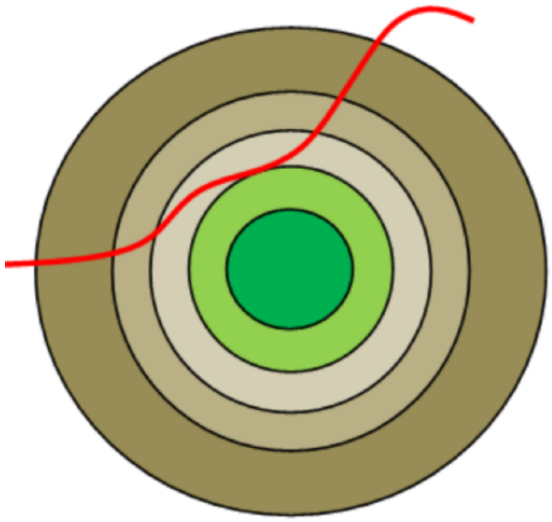

Figure 2.10.1: Road through valley: deep green is valley bottom, hills darken with height

Let's write the curve, the road, $g(x, y) = 0$ (the wiggly red line in the figure below).

To find the closest approach point algebraically, we need to minimize $f(x, y) = x^2 + y^2$ (square of distance to origin) subject to the constraint $g(x, y) = 0$.

In the figure, we've drawn curves

$$f(x, y) = x^2 + y^2 = a^2 \tag{2.10.1}$$

for a range of values of a (the circles centered at the origin). We need to find the point of intersection of $g(x, y) = 0$ with the smallest circle it intersects—and it's clear from the figure that it must **touch** that circle (if it crosses, it will necessarily get closer to the origin).

Therefore, at that point, the curves $g(x, y) = 0$ and $f(x, y) = a^2_{\text{min}}$ are parallel.

Therefore the normals to the curves are *also* parallel:

$$(\partial f/\partial x, \partial f/\partial y) = \lambda(\partial g/\partial x, \partial g/\partial y) \tag{2.10.2}$$

(***Note***: yes, those *are* the directions of the normals— for an infinitesimal displacement along the curve $f(x, y) = \text{ constant }, 0 = df = (\partial f/\partial x)dx + (\partial f/\partial y)dy$, so the vector $(\partial f/\partial x, \partial f/\partial y)$ is perpendicular to (dx, dy). This is also analogous to the electric field $\vec{E} = -\vec{\nabla}\varphi$ being perpendicular to the equipotential $\varphi(x, y) = \text{ constant. }$).

The constant λ introduced here is called a **Lagrange multiplier**. It's just the ratio of the lengths of the two normal vectors (of course, "normal" here means the vectors are perpendicular to the curves, they are *not* normalized to unit length!) We can find λ in terms of x, y but at this point we don't know their values.

The equations determining the closest approach to the origin can now be written:

$$\begin{aligned}\frac{\partial}{\partial x}(f-\lambda g)&=0\\ \frac{\partial}{\partial y}(f-\lambda g)&=0\\ \frac{\partial}{\partial \lambda}(f-\lambda g)&=0\end{aligned} \tag{2.10.3}$$

(The third equation is just $g\left(x_{\min}, y_{\min}\right)=0$, meaning we're on the road.)

We have transformed a *constrained* minimization problem in two dimensions to an *unconstrained* minimization problem in three dimensions! The first two equations can be solved to find λ and the ratio $x/y$the third equation then gives x, y separately.

Exercise for the reader: Work through this for $g(x,y)=x^2-2xy-y^2-1$ (There are two solutions because the curve $g=0$ is a hyperbola with two branches.)

Lagrange multipliers are widely used in economics, and other useful subjects such as traffic optimization.

2.11: Lagrange Multiplier for the Chain

The catenary is generated by minimizing the potential energy of the hanging chain given above,

$$J[y(x)] = \int y ds = \int y\left(1+y'^2\right)^{\frac{1}{2}} dx \tag{2.11.1}$$

but now subject to the constraint of fixed chain length, $L[y(x)] = \int ds = \ell$

The Lagrange multiplier method generalizes in a straightforward way from variables to variable functions. In the curve example above, we minimized $f(x,y) = x^2 + y^2$ subject to the constraint $g(x,y) = 0$. What we need to do now is minimize $J[y(x)]$ subject to the constraint $L[y(x)] - \ell = 0$

For the minimum curve $y(x)$ and the correct (so far unknown) value of λ an arbitrary infinitesimal variation of the curve will give zero first-order change in $J - \lambda L$, we write this as

$$\delta\{J[y(x)] - \lambda L[y(x)]\} = \delta\left\{\int_{x_1}^{x_2} (y-\lambda) ds\right\} = \delta\left\{\int_{x_1}^{x_2} (y-\lambda)\sqrt{1+y'^2}\, dx\right\} = 0 \tag{2.11.2}$$

Remarkably, the effect of the constraint is to give a simple adjustable parameter, the origin in the y direction, so that we can satisfy the endpoint and length requirements.

The solution to the equation follows exactly the route followed for the soap film, leading to the first integral

$$\frac{y-\lambda}{\left(1+y'^2\right)^{\frac{1}{2}}} = a \tag{2.11.3}$$

with a a constant of integration, which will depend on the endpoints.

Rearranging,

$$\frac{dy}{dx} = \sqrt{\left(\frac{y-\lambda}{a}\right)^2 - 1} \tag{2.11.4}$$

or

$$dx = \frac{a dy}{\sqrt{(y-\lambda)^2 - a^2}} \tag{2.11.5}$$

The standard substitution here is $y - \lambda = c\cosh\xi$, we find

$$y = \lambda + a\cosh\left(\frac{x-b}{a}\right) \tag{2.11.6}$$

Here b is the second constant of integration, the fixed endpoints and length give λ, a, b. In general, the equations must be solved numerically. To get some feel for why this will always work, note that changing a varies how rapidly the cosh curve climbs from its low point of $(x,y) = (b, \lambda + a)$, increasing a "fattens" the curve, then by varying b, λ we can move that lowest point to the lowest point of the chain (or rather of the catenary, since it might be outside the range covered by the physical chain).

Algebraically, we know the curve can be written as $y = a\cosh(x/a)$, although at this stage we don't know the constant a or where the origin is. What we do know is the length of the chain, and the horizontal and vertical distances $(x_2 - x_1)$ and $(y_2 - y_1)$ between the fixed endpoints. It's straightforward to calculate that the length of the chain is $\ell = a\sinh(x_2/a) - a\sinh(x_1/a)$, and the vertical distance v between the endpoints is $v = a\cosh(x_2/a) - a\cosh(x_1/a)$ from which $\ell^2 - v^2 = 4a^2\sinh^2[(x_2 - x_1)/2a]$. All terms in this equation are known except a, which can therefore be found numerically. (This is in Wikipedia, among other places.)

Exercise: try applying this reasoning to finding a for the soap film minimization problem. In that case, we know (x_1, y_1) and (x_2, y_2) , there is no length conservation requirement, to find a we must eliminate the unknown b from the equations $y_1 = a\cosh((x_1 - b)/a), y_2 = a\cosh((x_2 - b)/a)$. This is not difficult, but, in contrast to the chain, does *not* give a in terms of

$y_1 - y_2$, instead y_1, y_2 appear separately. Explain, in terms of the physics of the two systems, why this is so different from the chain.

2.12: The Brachistochrone

Suppose you have two points, A and B, B is below A, but not directly below. You have some smooth, let's say frictionless, wire, and a bead that slides on the wire. The problem is to curve the wire from A down to B in such a way that the bead makes the trip as quickly as possible.

This optimal curve is called the "brachistochrone", which is just the Greek for "shortest time".

But what, exactly, is this curve, that is, what is $y(x)$ in the obvious notation?

This was the challenge problem posed by Johann Bernoulli to the mathematicians of Europe in a Journal run by Leibniz in June 1696. Isaac Newton was working fulltime running the Royal Mint, recoining England, and hanging counterfeiters. Nevertheless, ending a full day's work at 4 pm, and finding the problem delivered to him, he solved it by 4am the next morning, and sent the solution anonymously to Bernoulli. Bernoulli remarked of the anonymous solution "I recognize the lion by his clawmark".

This was the beginning of the Calculus of Variations.

Here's how to solve the problem: we'll take the starting point A to be the origin, and for convenience measure the y -axis positive *downwards*. This means the velocity at any point on the path is given by

$$\frac{1}{2}mv^2 = mgy, \quad v = \sqrt{2gy} \tag{2.12.1}$$

So measuring length along the path as ds as usual, the time is given by

$$T = \int_A^B \frac{ds}{v} = \int_A^B \frac{ds}{\sqrt{2gy}} = \int_0^X \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}} \tag{2.12.2}$$

Notice that this has the same form as the catenary equation, the only difference being that y is replaced by $1/\sqrt{2gy}$ the integrand does not depend on x, so we have the first integral:

$$y'\frac{\partial f}{\partial y'} - f = \text{ constant }, \quad f = \sqrt{\frac{1+y'^2}{2gy}} \tag{2.12.3}$$

That is,

$$\frac{y'^2}{\sqrt{(1+y'^2)\,2gy}} - \sqrt{\frac{1+y'^2}{2gy}} = -\frac{1}{\sqrt{(1+y'^2)\,2gy}} = \text{ constant} \tag{2.12.4}$$

so

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{2a}{y} \tag{2.12.5}$$

$2a$ being a constant of integration (the 2 proves convenient).

Recalling that the curve starts at the origin A, it must begin by going vertically downward, since $y = 0$. For small enough y, we can approximate by ignoring the 1, so $\sqrt{2a}dx \cong \sqrt{y}dy$, $\sqrt{2a}x \cong 2/3 y^{3/2}$. The curve must however become horizontal if it gets as far down as $y = 2a$, and it cannot go below that level.

Rearranging in order to integrate,

$$dx = \frac{dy}{\sqrt{\frac{2a}{y} - 1}} = \sqrt{\frac{y}{2a-y}}dy \tag{2.12.6}$$

This is not a very appealing integrand. It looks a little nicer on writing $y = a - az$

$$dx = -a\sqrt{\frac{1-z}{1+z}}dz \tag{2.12.7}$$

Now what? We'd prefer for the expression inside the square root to be a perfect square, of course. You may remember from high school trig that $1+\cos\theta = 2\cos^2(\theta/2), \quad 1-\cos\theta = 2\sin^2(\theta/2)$. This gives immediately that

$$\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2} \tag{2.12.8}$$

so the substitution $z=\cos\theta$ is what we need.

Then $dz = -\sin\theta d\theta = -2\sin(\theta/2)\cos(\theta/2)d\theta$

$$dx = -a\tan\frac{\theta}{2}dz = 2a\tan\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta = 2a\sin^2\frac{\theta}{2}d\theta = a(1-\cos\theta)d\theta \tag{2.12.9}$$

This integrates to give

$$\begin{aligned} x &= a(\theta-\sin\theta) \\ y &= a(1-\cos\theta) \end{aligned} \tag{2.12.10}$$

where we've fixed the constant of integration so that the curve goes through the origin $(\text{ at } \theta = 0)$

To see what this curve looks like, first ignore the θ terms in x, leaving $x=-a\sin\theta, y=-a\cos\theta$. Evidently as θ increases from zero, the point (x,y) goes anticlockwise around a circle of radius a centered at $(0,-a)$ that is, touching the x -axis at the origin.

Now adding the θ back in, this circular motion move steadily to the right, in such a way that the initial direction of the path is vertically down. $(\text{For very small } \theta, y\sim\theta^2 \gg x\sim\theta^3)$

Visualizing the total motion as θ steadily increases, the center moves from its original position at $(0,-a)$ to the right at a speed $a\theta$. Meanwhile, the point is moving round the circle anticlockwise at this same speed. Putting together the center's linear velocity with the corresponding angular velocity, we see the motion $(x(\theta), y(\theta))$ is the path of a point on the rim of a wheel rolling without sliding along a road (upside down in our case, of course). This is a *cycloid*.

CHAPTER OVERVIEW

3: Fermat's Principle of Least Time

3.1: Another Minimization Problem…
3.2: Huygens' Picture of Wave Propagation
3.3: Fermat’s Principle
3.4: Reflection, Too
3.5: The Bottom Line- Geometric Optics and Wave Optics

3.1: Another Minimization Problem…

Here's another minimization problem from the 1600's, even earlier than the brachistochrone. Fermat famously stated in the 1630's that a ray of light going from point A to point B always takes the route of least time -- OK, it's trivially trivially true in a single medium, light rays go in a straight line, but it's a lot less obvious if, say, A is in air and B in glass. Notice that this is closely related to our previous topic, the calculus of variations -- if this is a minimal time path, varying the path by a small amount will not change the time taken to first order. (*Historical note*: actually what amounted to Fermat's principle was first stated by Alhazen, in Baghdad, around 1000 AD.)

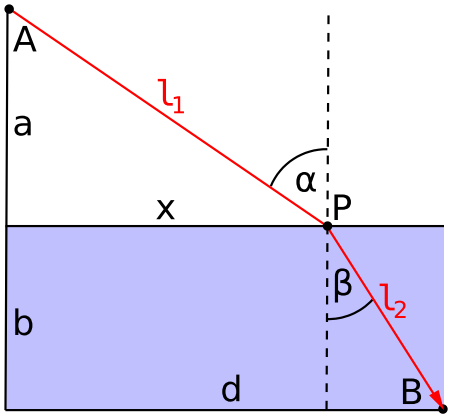

Figure 3.1.1: Fermat's principle in the case of refraction of light at a flat surface between (say) air and water. Given an object-point A in the air, and an observation point B in the water, the refraction point P is that which minimizes the time taken by the light to travel the path APB. If we seek the required value of x, we find that the angles α and β satisfy Snell's law. (Public Domain; Klaus-Dieter Keller via Wikipedia)

This seemed very mysterious when first extensively discussed, in the 1600's. In the last part of that century, and through the 1700's, Newton was the dominant figure, and he believed that light was a stream of particles. But how could the particle figure out the shortest time path from A to B?

In fact, there was one prominent physicist, Huygens', who thought light might be a wave, and, much later, this turned out to be the crucial insight. The main objection was that waves go around corners, at least to some extent, it seemed that light didn't. (Also, they exhibit diffraction effects, which no one thought they'd seen for light, although in fact Newton himself had observed diffraction -- Newton's rings -- but had an ingenious explanation, as always, of why his particle picture could explain what he saw.) Anyway, in 1678, Huygens' suggested the following picture: it's a simple beginning to understanding wave propagation, most notably it omits phases (later added by Fresnel) but it was a beginning.

3.2: Huygens' Picture of Wave Propagation

If a point source of light is switched on, the wavefront is an expanding sphere centered at the source. Huygens suggested that this could be understood if at any instant in time each point on the wavefront was regarded as a source of secondary wavelets, and the new wavefront a moment later was to be regarded as built up from the sum of these wavelets. For a light shining continuously, the process just keeps repeating.

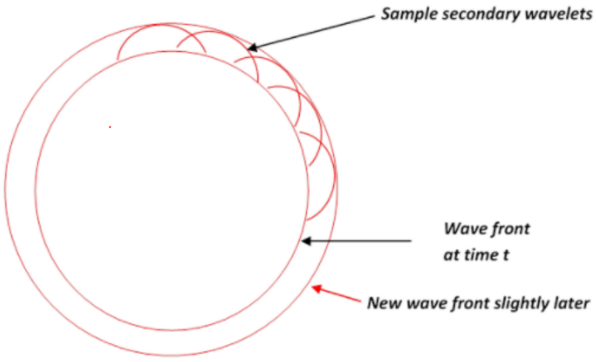

Figure 3.2.1: Huygens' picture of how a spherical wave propagates: each point of the wave front is a source of secondary wavelets that generate the new wave front

You might think that if a point on the wavefront is a new source, wouldn't the disturbance it generates be as likely to go backwards as forwards? Huygens did not address this point. In fact, it's not easy to give a short satisfactory answer. We'll discuss propagation of light (and of course other electromagnetic waves) fully in the second semester of E&M.

Huygens' principle *does* explain why the wavefront stays spherical, and more important, it explains *refraction* -- the change in direction of a wavefront on entering a different medium, such as a ray of light going from air into glass. Here's how: If the light moves more slowly in the glass, velocity v instead of c, with $v < c$, then Huygens' picture predicts Snell's Law, that the ratio of the sines of the angles to the normal of incident and transmitted beams is constant, and in fact is the ratio c/v. This is evident from the diagram below: in the time the wavelet centered at ***A*** has propagated to ***C***, that from ***B*** has reached ***D***, the ratio of lengths ***AC***/***BD*** being c/v.

But the angles in Snell's Law are in fact the angles ***ABC***, ***BCD***, and those right-angled triangles have a common hypotenuse ***BC***, from which the Law follows.

Notice, though, the crucial fact: we get Snell's law on the assumption that the speed of light is slower in glass than in air. If light was a stream of particles, the picture would have to be that they encountered a potential change on going into the glass, like a ball rolling on a horizontal floor encountering a step, smoothed out a bit, to a different level. This would give a force perpendicular to the interface on going from one level to the other, and if the path is bent towards the normal, as is observed, the ball must speed up -- so this predicts light moves faster in glass. It wasn't until the nineteenth century, though, that measuring the speed of light in glass (actually I think water) was technologically possible.

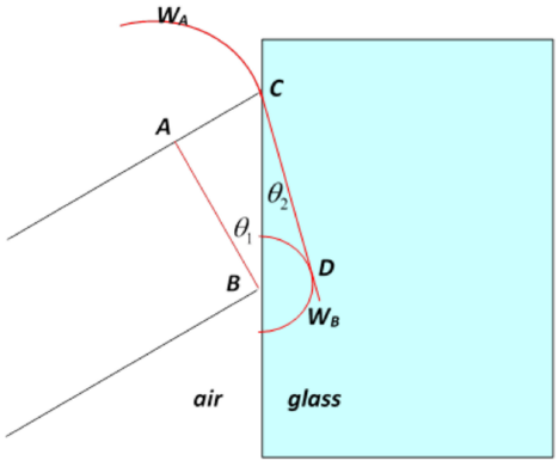

Figure 3.2.1: Huygen' explanation of refraction: showing two wavelets from the wavefront AB. W_B is slowed down compared with W_A, since it is propagating in glass. This turns the wavefront through an angle.

In fact, even in the early nineteenth century, the wave nature of light was widely doubted. Fresnel greatly improved Huygens' crude picture, fully taking into account the interference between secondary wavelets having different phases. One of the principal skeptics of the wave theory, the mathematician Poisson, pointed out that it was obvious nonsense because, using Fresnel's own arguments, it predicted that in the very center of the dark shadow of a sphere illuminated by a point source of light, there should be a bright spot: all the "light waves" grazing the edge of the sphere would generate secondary wavelets which would land at that spot in phase. A bright spot at the center of the dark disk seemed obvious nonsense, but an experimentalist colleague in Paris, Arago, decided to try the experiment anyway -- and the spot was there. It's now called the **Poisson spot**, and it gave a big boost to the wave theory in France (it was already fully accepted in England, where Thomas Young did the double slit interference pattern, and compared it to the wave pattern in a similarly configured ripple tank, presenting the results to the Royal Society in 1803).

3.3: Fermat's Principle

We will now temporarily forget about the wave nature of light, and consider a narrow ray or beam of light shining from point *A* to point *B*, where we suppose *A* to be in air, *B* in glass. Fermat showed that the path of such a beam is given by the Principle of Least Time: a ray of light going from *A* to *B* by any other path would take longer. How can we see that? It's obvious that any deviation from a straight line path in air or in the glass is going to add to the time taken, but what about moving slightly the point at which the beam enters the glass?

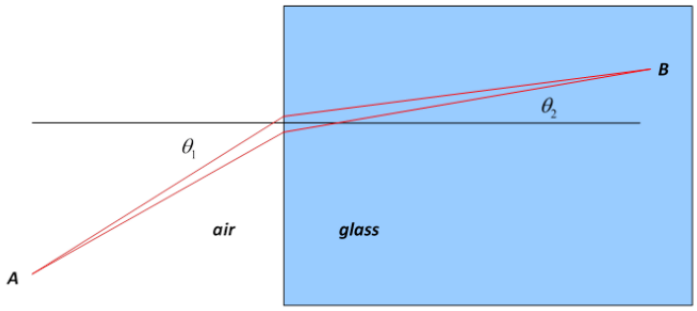

Figure 3.3.1

Where the air meets the glass, the two rays, separated by a small distance ***CD*** = *d* along that interface, will look parallel:

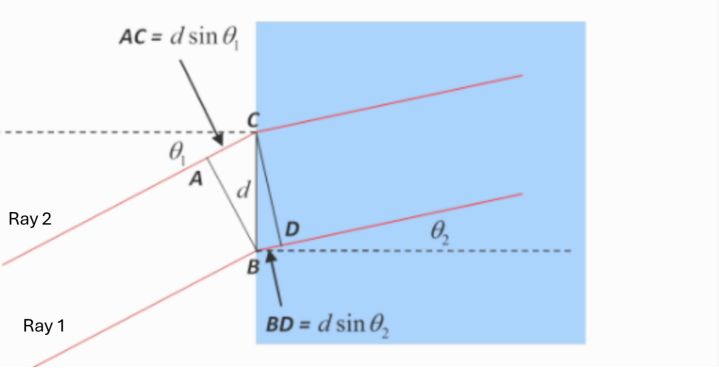

Figure 3.3.2: Magnified vies of two rays passing through interface: ray 1 is the minimum time path. Rays encounter the interface a distance CB=d apart

(Feynman gives a nice illustration: a lifeguard on a beach spots a swimmer in trouble some distance away, in a diagonal direction. He can run three times faster than he can swim. What is the quickest path to the swimmer?)

Moving the point of entry up a small distance d, the light has to travel an extra $d\sin\theta_1$ in air, but a distance less by $d\sin\theta_2$ in the glass, giving an extra travel time $\Delta t = d\sin\theta_1/c - d\sin\theta_2/v$. For the *classical* path, Snell's Law gives $\sin\theta_1/\sin\theta_2 = n = c/v$, so $\Delta t = 0$ to first order. But if we look at a series of possible paths, each a small distance d away from the next at the point of crossing from air into glass, Δt becomes of order d/c away from the classical path.

But now let's take a closer look at the Huygens picture of light propagation: it would suggest that the light reaching a point actually comes from many wavelets generated at different points on the previous wavefront. A handwaving generalization might be that the light reaching a point from another point actually includes multiple paths. To keep things manageable, let's suppose the light from A to B actually goes along all the paths that are straight in each medium, but different crossing point. Also, we'll make the approximation that they all reach B with equal amplitude. What will be the total contribution of all the paths at *B*? Since the times along the paths are different, the signals along the different paths will arrive at *B* with different phases, and to get the total wave amplitude we must add a series of unit 2*D* vectors, one from each path. (Representing the amplitude and phase of the wave by a complex number for convenience -- for a real wave, we can take the real part at the end.)

When we map out these unit 2*D* vectors, we find that in the neighborhood of the classical path, the phase varies little, but as we go away from it the phase spirals more and more rapidly, so those paths interfere amongst themselves destructively. To formulate this a little more precisely, let us assume that some close by path has a phase difference φ from the least time path, and goes from air to glass a distance x away from the least time path: then for these close by paths, $\varphi = ax^2$, where a depends on the geometric arrangement and the wavelength. From this, the sum over the close by paths is an integral of the form $\int e^{iax^2}dx$ (We are assuming

the wavelength of light is far less than the size of the equipment.) This is a standard integral, its value is $\sqrt{\pi/ia}$ all its weight is concentrated in a central area of width $1/\sqrt{a},$ exactly as for the real function e^{-ax^2} .

This is the explanation of Fermat's Principle -- only near the path of least time do paths stay approximately in phase with each other and add constructively. So this classical path rule has an underlying wave-phase explanation. In fact, the central role of phase in this analysis is sometimes emphasized by saying the light beam follows *the path of stationary phase*.

Of course, we're not summing over *all* paths here -- we assume that the path in air from the source to the point of entry into the glass is a straight line, clearly the subpath of stationary phase.

3.4: Reflection, Too

Suppose you look at a point of light reflected in a mirror. Imagine the point sending out rays in all directions, as it does. The ray that enters your eye from the mirror goes along the shortest bouncing-off-the-mirror path. You can prove that this is equivalent to angle of incidence equals angle of reflection by considering the path difference for a nearby path.

Of course, for a curved mirror there may be more than one shortest path. To take an extreme case, consider the two-dimensional scenario of a perfectly reflecting ellipse with a point light source inside. If the source is at one focus of the ellipse, all the light will be reflected to the other focus. And, all the paths will have the same length! (Recall an ellipse can be constructed with a piece of string, the ends nailed down at the foci, the string stretched taut.) A parabolic mirror is the limiting case of an ellipse with the other focus sent to infinity, so parallel rays coming in along the axis from a distant star will all go to the focus in phase with each other.

3.5: The Bottom Line- Geometric Optics and Wave Optics

In geometric optics, mirrors, lenses, telescopes and so on are analyzed by tracking narrow rays of light through the system, applying the standard rules of reflection and refraction. Despite Huygens' picture, most people using this well-established technique before 1800 thought the rays were streams of particles. Fermat's Principle of Least Time was an elegant formulation of the laws of motion of this stream -- it reduced all observed deflections, etc., to a single statement. It even included phenomena caused by a variable refractive index, and consequent curved paths for light rays, such as mirages, reflections of distant mountains in the middle-distance ground on hot days caused by a layer of hotter air close to the ground.

But despite its elegance, no theoretical explanation of Fermat's Principle was forthcoming until it was established that light was a wave -- then it became clear. The waves went out over all possible paths, but phase differences caused almost perfect cancellation except for paths in the vicinity of the shortest possible.

We shall find a similar connection between classical mechanics and quantum mechanics.

CHAPTER OVERVIEW

4: Hamilton's Principle and Noether's Theorem

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4.1: Introduction- Galileo and Newton

In the discussion of calculus of variations, we anticipated some basic dynamics, using the potential energy mgh for an element of the catenary, and conservation of energy $\frac{1}{2}mv^2 + mgh = E$ for motion along the brachistochrone. Of course, we haven't actually covered those things yet, but you're already very familiar with them from your undergraduate courses, and my aim was to give easily understood physical realizations of minimization problems, and to show how to find the minimal shapes using the calculus of variations.

At this point, we'll begin a full study of dynamics, starting with the laws of motion. The text, Landau, begins (page 2!) by stating that the laws come from the principle of least action, Hamilton's principle. This is certainly one possible approach, but confronted with it for the first time, one might well wonder where *it* came from. I prefer a gentler introduction, more or less following the historical order: Galileo, then Newton, then Lagrange and his colleagues, then Hamilton. The two approaches are of course equivalent. Naturally, you've seen most of this earlier stuff before, so here is a very brief summary.

To begin, then, with Galileo. His two major contributions to dynamics were:

1. The realization, and experimental verification, that falling bodies have constant acceleration (provided air resistance can be ignored) and *all falling bodies accelerate at the same rate*.

2. *Galilean relativity*. As he put it himself, if you are in a closed room below decks in a ship moving with steady velocity, no experiment on dropping or throwing something will look any different because of the ship's motion: you can't detect the motion. As we would put it now, the laws of physics are the same in all inertial frames.

Newton's major contributions were his *laws of motion*, and his *law of universal gravitational attraction.*

His laws of motion:

1. *The law of inertia*: a body moving at constant velocity will continue at that velocity unless acted on by a force. (Actually, Galileo essentially stated this law, but just for a ball rolling on a horizontal plane, with zero frictional drag.)

2. $\vec{F} = m\vec{a}$

3. *Action = reaction.*

In terms of Newton's laws, Galilean relativity is clear: if the ship is moving at steady velocity $\vec{v}$ relative to the shore, than an object moving at $\vec{u}$ relative to the ship is moving at $\vec{u} + \vec{v}$ relative to the shore. If there is no force acting on the object, it is moving at steady velocity in both frames: both are inertial frames, defined as frames in which Newton's first law holds. And, since $\vec{v}$ is constant, the acceleration is the same in both frames, so if a force is introduced the second law is the same in the two frames.

(Needless to say, all this is classical, meaning nonrelativistic, mechanics.)

Newton's Laws Explain Everything (in Principle...)

...in a classical mechanical system. Any dynamical system can be analyzed as a (possibly infinite) collection of parts, or particles, having mutual interactions, so in principle Newton's laws can provide a description of the motion developing from an initial configuration of positions and velocities.

The problem is, though, that the equations may be intractable—we can't do the mathematics. It is evident that in fact the Cartesian coordinate positions and velocities might not be the best choice of parameters to specify the system's configuration. For example, a simple pendulum is obviously more naturally described by the angle the string makes with the vertical, as opposed to the Cartesian coordinates of the bob. After Newton, a series of French mathematicians reformulated his laws in terms of more useful coordinates—culminating in Lagrange's equations.

The Irish mathematician Hamiltonian then established that these improved dynamical equations could be derived using the calculus of variations to minimize an integral of a function, the *Lagrangian*, along a path in the system's configuration space.

This integral is called the *action*, so the rule is that the system follows the path of least action from the initial to the final configuration.

Note

Nitpicking footnote: strictly, we need the path to be a stationary point in the space of possible paths, usually it's the least-action path.

4.2: Derivation of Hamilton's Principle from Newton's Laws in Cartesian Coordinates- Calculus of Variations Done Backwards!

We've shown how, given an integrand, we can find differential equations for the path in space time between two fixed points that minimizes the corresponding path integral between those points.

Now we'll do *the reverse*: we already know the differential equations in Cartesian coordinates describing the path taken by a Newtonian particle in some potential. We'll show how to use that knowledge to construct the integrand such that the action integral is a minimum along that path. (This follows Jeffreys and Jeffreys, *Mathematical Physics*.)

We begin with the simplest nontrivial system, a particle of mass m moving in one dimension from one point to another in a specified time, we'll assume it's in a time-independent potential $U(x)$, so

$$m\ddot{x} = -dU(x)/dx \tag{4.2.1}$$

Its path can be represented as a graph $x(t)$ against time—for example, for a ball thrown directly upwards in a constant gravitational field this would be a parabola.

Initial and final positions are given: $x\left(t_1\right) = x_1, x\left(t_2\right) = x_2,$ and the elapsed time is $t_2 - t_1$

Notice we have *not* specified the initial velocity—we don't have that option. The differential equation is only second order, so its solution is completely determined by the two (beginning and end) boundary conditions.

We're now ready to embark on the calculus of variations in reverse.

Trivially, multiplying both sides of the equation of motion by an arbitrary infinitesimal function the equality still holds:

$$m\ddot{x}\delta x(t) = -(dU(x)/dx)\delta x(t) \tag{4.2.2}$$

and in fact if *this* equation is true for *arbitrary* $\delta x(t)$, the original equation of motion holds throughout, because we can always choose a $\delta x(t)$ nonzero only in the neighborhood of a particular time t, from which the original equation must be true at that t.

By analogy with Fermat's principle in the preceding section, we can picture this $\delta x(t)$ as a slight variation in the path from the Newtonian trajectory, $x(t) \to x(t) + \delta x(t)$ and take the variation zero at the fixed ends, $\delta x\left(t_1\right) = \delta x\left(t_2\right) = 0$

In Fermat's case, the integrated time elapsed along the path was minimized—there was zero change to first order on going to a neighboring path. Developing the analogy, we're looking for some dynamical quantity that has zero change to first order on going to a neighboring path having the same endpoints in space and time. We've fixed the time, what's left to integrate along the path?

For such a simple system, we don't have many options! As we've discussed above, the equation of motion is equivalent to (putting in an overall minus sign that will prove convenient)

$$\int_{t_1}^{t_2} (-m\ddot{x}(t) - dU(x(t))/dx)\delta x(t)dt = 0 \quad \text{to leading order, for all variations } \delta x(t) \tag{4.2.3}$$

Integrating the first term by parts (recalling $\delta(x) = 0$ at the endpoints):

$$-\int_{t_1}^{t_2} m\ddot{x}(t)\delta x(t)dt = \int_{t_1}^{t_2} m\dot{x}(t)\delta\dot{x}(t)dt = \int_{t_1}^{t_2} \delta\left(\frac{1}{2}m\dot{x}^2(t)\right)dt = \int_{t_1}^{t_2} \delta T(x(t))dt \tag{4.2.4}$$

using the standard notation T for kinetic energy.

The second term integrates trivially:

$$-\int_{t_1}^{t_2} (dU(x)/dx)\delta x(t)dt = -\int_{t_1}^{t_2} \delta U(x)dt \tag{4.2.5}$$

establishing that on making an infinitesimal variation from the physical path (the one that satisfies Newton's laws) there is zero first order change in the integral of kinetic energy minus potential energy.

The standard notation is

$$\delta S = \delta\int_{t_1}^{t_2} (T-U)dt = \delta\int_{t_1}^{t_2} Ldt = 0 \tag{4.2.6}$$

The integral S is called the *action integral*, (also known as Hamilton's Principal Function) and the integrand T−U=L is called the *Lagrangian*.

This equation is Hamilton's Principle.

The derivation can be extended straightforwardly to a particle in three dimensions, in fact to n interacting particles in three dimensions. We shall assume that the forces on particles can be derived from potentials, including possibly time-dependent potentials, but we exclude frictional energy dissipation in this course. (It can be handled—see for example Vujanovic and Jones, *Variational Methods in Nonconservative Phenomena*, Academic press, 1989.)

4.3: But Why?

Fermat's principle was easy to believe once it was clear that light was a wave. Imagining that the wave really propagates along all paths, and for light the phase change along a particular path is simply the time taken to travel that path measured in units of the light wave oscillation time. That means that if neighboring paths have the same length to first order the light waves along them will add coherently, otherwise they will interfere and essentially cancel. So the path of least time is heavily favored, and when we look on a scale much greater than the wavelength of the light, we don't even see the diffraction effects caused by imperfect cancellation, the light rays might as well be streams of particles, mysteriously choosing the path of least time.

So what has this to do with Hamilton's principle? Everything. A standard method in quantum mechanics these days is the so-called sum over paths, for example to find the probability amplitude for an electron to go from one point to another in a given time under a given potential, you can sum over all possible paths it might take, multiplying each path by a phase factor: and that phase factor is none other than Hamilton's action integral divided by Planck's constant, $S/\hbar$. So the true wave nature of all systems in quantum mechanics ensures that in the classical limit $S \gg \hbar$ the well-defined path of a dynamical system will be that of least action.

Historical footnote: Lagrange developed these methods in a classic book that Hamilton called a "scientific poem". Lagrange thought mechanics properly belonged to pure mathematics, it was a kind of geometry in four dimensions (space and time). Hamilton was the first to use the principle of least action to derive Lagrange's equations in the present form. He built up the least action formalism directly from Fermat's principle, considered in a medium where the velocity of light varies with position and with direction of the ray. He saw mechanics as represented by geometrical optics in an appropriate space of higher dimensions. But it didn't apparently occur to him that this might be because it was really a wave theory! (See Arnold, *Mathematical Methods of Classical Mechanics*, for details.)

4.4: Lagrange's Equations from Hamilton's Principle Using Calculus of Variations

We started with Newton's equations of motion, expressed in Cartesian coordinates of particle positions. For many systems, these equations are mathematically intractable. Running the calculus of variations argument in reverse, we established Hamilton's principle: the system moves along the path through configuration space for which the action integral, with integrand the Lagrangian $L = T - U$, is a minimum.

We're now free to *begin* from Hamilton's principle, expressing the Lagrangian in variables that more naturally describe the system, taking advantage of any symmetries (such as using angle variables for rotationally invariant systems). Also, some forces do not need to be included in the description of the system: a simple pendulum is fully specified by its position and velocity, we do not need to know the tension in the string, although that *would* appear in a Newtonian analysis. The greater efficiency (and elegance) of the Lagrangian method, for most problems, will become evident on working through actual examples.

We'll define a set of *generalized coordinates* $q = (q_1, \ldots q_n)$ by requiring that they give a complete description of the configuration of the system (where everything is in space). The state of the system is specified by this set plus the corresponding velocities $\dot{q} = (\dot{q}_1, \ldots \dot{q}_n)$

For example, the x -coordinate of a particular particle a is given by some function of the q_i 's,

$x_a = f_{x_a}(q_1, \ldots q_n)$, and the corresponding velocity component $\dot{x}_a = \sum_k \dfrac{\partial f_{x_a}}{\partial q_k} \dot{q}_k$

The Lagrangian will depend on all these variables in general, and also possibly on time explicitly, for example if there is a time-dependent external potential. (But usually that isn't the case.)

Hamilton's principle gives

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = 0 \tag{4.4.1}$$

that is,

$$\int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt = 0 \tag{4.4.2}$$

Integrating by parts,

$$\delta S = \left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i dt = 0 \tag{4.4.3}$$

Requiring the path deviation to be zero at the endpoints gives Lagrange's equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \tag{4.4.4}$$

4.5: Generalized Momenta and Forces

For the above orbital Lagrangian, $dL/d\dot{r} = m\dot{r} = p_r$ the momentum in the r -direction, and $dL/d\dot{\theta} = mr^2\dot{\theta} = p_\theta$, the angular momentum associated with the variable θ.

The *generalized momenta* for a mechanical system are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{4.5.1}$$

Less frequently used are the generalized *forces*, $F_i = \partial L/\partial q_i$, defined to make the Lagrange equations look Newtonian, $F_i = \dot{p}_i$.

4.6: Non-uniqueness of the Lagrangian

The Lagrangian is not uniquely defined: *two Lagrangians differing by the total derivative with respect to time of some function will give the same identical equations on minimizing the action*,

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t) dt = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt + \int_{t_1}^{t_2} \frac{df(q,t)}{dt} dt = S + f\left(q\left(t_2\right), t_2\right) - f\left(q\left(t_1\right), t_1\right) \tag{4.6.1}$$

and since $q\left(t_1\right), t_1, q\left(t_2\right), t_2$ are all fixed, the integral over df/dt is trivially independent of path variations, and varying the path to minimize S' gives the same result as minimizing S. This turns out to be important later—it gives us a useful new tool to change the variables in the Lagrangian.

4.7: First Integral- Energy Conservation and the Hamiltonian

Since Lagrange's equations are precisely a calculus of variations result, it follows from our earlier discussion that *if the Lagrangian has no explicit time dependence* then:

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{constant} \tag{4.7.1}$$

(This is just the first integral $y' \partial f / \partial y' - f = \text{ constant }$ discussed earlier, now with n variables.)

This constant of motion is called the *energy* of the system, and denoted by E. We say the energy is *conserved*, even in the presence of external potentials—provided those potentials are time-independent.

(We'll just mention that the function on the left-hand side, $\sum_i \dot{q}_i \partial L / \partial \dot{q}_i - L$ is the Hamiltonian. We don't discuss it further at this point because, as we'll find out, it is more naturally treated in other variables.)

We'll now look at a couple of simple examples of the Lagrangian approach.

4.8: Example 1- One Degree of Freedom- Atwood's Machine

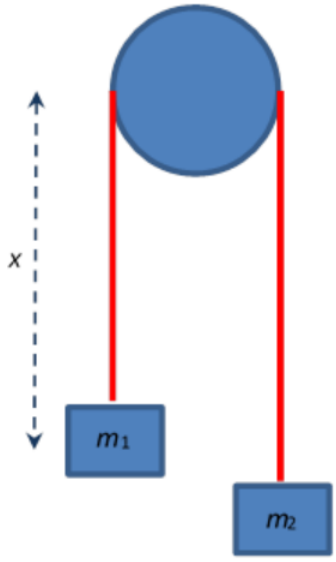

Figure 4.8.1

In 1784, the Rev. George Atwood, tutor at Trinity College, Cambridge, came up with a great demo for finding g. It's still with us. The traditional Newtonian solution of this problem is to write $F = ma$ for the two masses, then eliminate the tension T. (To keep things simple, we'll neglect the rotational inertia of the top pulley.)

The Lagrangian approach is, of course, to write down the Lagrangian, and derive the equation of motion.

Measuring gravitational potential energy from the top wheel axle, the potential energy is

$$U(x) = -m_1 g x - m_2 g(\ell - x) \tag{4.8.1}$$

and the Lagrangian

$$L = T - U = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2 + m_1 g x + m_2 g(\ell - x) \tag{4.8.2}$$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = (m_1 + m_2)\,\ddot{x} - (m_1 - m_2)\,g = 0 \tag{4.8.3}$$

gives the equation of motion in just one step.

It's usually pretty easy to figure out the kinetic energy and potential energy of a system, and thereby write down the Lagrangian. This is definitely less work than the Newtonian approach, which involves constraint forces, such as the tension in the string. This force doesn't even appear in the Lagrangian approach! Other constraint forces, such as the normal force for a bead on a wire, or the normal force for a particle moving on a surface, or the tension in the string of a pendulum—none of these forces appear in the Lagrangian. Notice, though, that these forces never do any work.

On the other hand, if you actually are interested in the tension in the string (will it break?) you use the Newtonian method, or maybe work backwards from the Lagrangian solution.

4.9: Example 2- Lagrangian Formulation of the Central Force Problem

A simple example of Lagrangian mechanics is provided by the central force problem, a mass m acted on by a force

$F_r = -dU(r)/dr$

To contrast the Newtonian and Lagrangian approaches, we'll first look at the problem using just $\vec{F} = m\vec{a}$. To take advantage of the rotational symmetry we'll use (r, θ) coordinates, and find the expression for acceleration by the standard trick of differentiating the complex number $z = re^{i\theta}$ twice, to get

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -dU(r)/dr \tag{4.9.1}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{4.9.2}$$

The second equation integrates immediately to give

$$mr^2\dot{\theta} = \ell \tag{4.9.3}$$

a constant, the angular momentum. This can then be used to eliminate $\dot{\theta}$ in the first equation, giving a differential equation for $r(t)$.

The Lagrangian approach, on the other hand, is first to write

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \tag{4.9.4}$$

and put it into the equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0 \end{aligned} \tag{4.9.5}$$

Note now that since L doesn't depend on θ, the second equation gives immediately:

$$\frac{\partial L}{\partial \dot{\theta}} = \text{constant} \tag{4.9.6}$$

and in fact $\partial L/\partial\dot{\theta} = mr^2\dot{\theta}$ the angular momentum, we'll call it ℓ

The first integral (see above) gives another constant:

$$\dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \text{constant} \tag{4.9.7}$$

This is just

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = E \tag{4.9.8}$$

the energy.

Angular momentum conservation, $mr^2\dot{\theta} = \ell$, then gives

$$\frac{1}{2}m\left(\dot{r}^2 + \frac{\ell^2}{m^2r^2}\right) + U(r) = E \tag{4.9.9}$$

giving a first-order differential equation for the radial motion as a function of time. We'll deal with this in more detail later. Note that it is equivalent to a particle moving in one dimension in the original potential plus an effective potential from the angular momentum term:

$$E = \frac{1}{2}mv^2 + U(r) + \frac{\ell^2}{m^2r^2} \tag{4.9.10}$$

This can be understood by realizing that for a fixed angular momentum, the closer the particle approaches the center the greater its speed in the tangential direction must be, so, to conserve total energy, its speed in the radial direction has to go down, unless it is in a *very* strongly attractive potential (the usual gravitational or electrostatic potential isn't strong enough) so the radial motion is equivalent to that with the existing potential plus the ℓ^2/m^2r^2

term, often termed the "centrifugal barrier".

Exercise: how strong must the potential be to overcome the centrifugal barrier? (This can happen in a black hole!)

4.10: Conservation Laws and Noether's Theorem

The two integrals of motion for the orbital example above can be stated as follows:

First: if the Lagrangian does not depend on the variable $\theta, \partial L/\partial\theta = 0$, that is, it's *invariant under rotation*, meaning it has circular symmetry, then

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \text{constant} \tag{4.10.1}$$

angular momentum is conserved.

Second: As stated earlier, if the Lagrangian is independent of time, that is, it's *invariant under time translation*, then *energy is conserved*. (This is nothing but the first integral of the calculus of variations, recall that for an integrand function $f(y, y')$ not explicitly dependent on $x, y'\partial f/\partial y' - f$ is constant.)

$$\sum_i \dot{q}_i \partial L/\partial \dot{q}_i - L = E, \quad \text{a constant} \tag{4.10.2}$$

Note

Both these results link symmetries of the Lagrangian——invariance under rotation and time translation respectively——with conserved quantities.

This connection was first spelled out explicitly, and proved generally, by Emmy Noether, published in 1915. The essence of the theorem is that if the Lagrangian (which specifies the system completely) does not change when some continuous parameter is altered, then some function of the $q_i, \dot{q}_i$ stays the same—it is called a constant of the motion, or an integral of the motion.

To look further at this expression for energy, we take a closed system of particles interacting with each other, but "closed" means no interaction with the outside world (except possibly a time-independent potential).

The Lagrangian for the particles is, in Cartesian coordinates,

$$L = \sum_2^1 m_i v_i^2 - U(\vec{r}_1, \vec{r}_2, \ldots) \tag{4.10.3}$$

A set of general coordinates $(q_1, \ldots, q_n)$, by definition, uniquely specifies the system configuration, so the coordinate and velocity of a particular particle a are given by

$$x_a = f_{x_a}(q_1, \ldots q_n), \quad \dot{x}_a = \sum_k \frac{\partial f_{x_a}}{\partial q_k}\dot{q}_k \tag{4.10.4}$$

From this it is clear that the kinetic energy term $T = \sum_2^1 m_i v_i^2$ is a homogeneous quadratic function of the $\dot{q}$ (meaning every term is of degree two), so

$$L = \frac{1}{2}\sum_{i,k} a_{ik}(q)\dot{q}_i\dot{q}_k - U(q) \tag{4.10.5}$$

This being of degree two in the time derivatives means

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T \tag{4.10.6}$$

(If this isn't obvious to you, check it out with a couple of terms: $\dot{q}_1^2, \dot{q}_1\dot{q}_2$)

Therefore for this system of interacting particles

$$E = \sum_{i=1}^n \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = 2T - (T - U) = T + U \tag{4.10.7}$$

This expression for the energy is called the *Hamiltonian*:

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \tag{4.10.8}$$

4.11: Momentum Conservation

Another conservation law follows if the Lagrangian is unchanged by displacing the whole system through a distance $\delta\vec{r} = \vec{\varepsilon}$. This means, of course, that the system cannot be in some spatially varying external field—it must be mechanically isolated.

It is natural to work in Cartesian coordinates to analyze this, each particle is moved the same distance $\vec{r}_i \to \vec{r}_i + \delta\vec{r}_i = \vec{r}_i + \vec{\varepsilon}$, so , so

$$\delta L = \sum_i \frac{\partial L}{\partial \vec{r}_i} \cdot \delta\vec{r}_i = \vec{\varepsilon} \cdot \sum_i \frac{\partial L}{\partial \vec{r}_i} \tag{4.11.1}$$

where the "differentiation by a vector" notation means differentiating with respect to each component, then adding the three terms. (I'm not crazy about this notation, but it's Landau's, so get used to it.)

For an isolated system, we must have $\delta L = 0$ on displacement, moving the whole thing through empty space in any direction $\vec{\varepsilon}$ changes nothing, so it must be that the vector sum $\sum_i \partial L/\partial \vec{r}_i = 0$, so from the Cartesian Euler-Lagrange equations, writing $\vec{r} = \vec{v}$

$$0 = \sum_i \partial L/\partial \vec{r}_i = \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_i}\right) = \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_i}\right) = \frac{d}{dt}\sum \frac{\partial L}{\partial \vec{v}_i} \tag{4.11.2}$$

so, taking the system to be composed of particles of mass m_i and velocity $\vec{v}_i$

$$\sum_i \frac{\partial L}{\partial \vec{v}_i} = \sum_i m_i \vec{v}_i = \vec{P} = \text{constant} \tag{4.11.3}$$

the *momentum* of the system.

This vector conservation law is of course three separate directional conservation laws, so even if there *is* an external field, if it doesn't vary in a particular direction, the component of total momentum in that direction will be conserved.

In the Newtonian picture, conservation of momentum in a closed system follows from Newton's third law. In fact, the above Lagrangian analysis is really Newton's third law in disguise. Since we're working in Cartesian coordinates,

$\partial L/\partial \vec{r}_i = -\partial V/\partial \vec{r}_i = \vec{F}_i$ the force on the ith particle, and if there are no external fields, $\sum_i \partial L/\partial \vec{r}_i = 0$ just means that if you add all the forces on all the particles, the sum is zero. For the Lagrangian of a two particle system to be invariant under translation through space, the potential must have the form $V(\vec{r}_1 - \vec{r}_2)$ from which automatically $\vec{F}_{12} = -\vec{F}_{21}$.

4.12: Center of Mass

If an inertial frame of reference K' is moving at constant velocity $\vec{V}$ relative to inertial frame K, the velocities of individual particles in the frames are related by $\vec{v}_i = \vec{v}_i' + \vec{V}$, so the total momenta are related by

$$\vec{P} = \sum_i m_i \vec{v}_i = \sum_i m_i \vec{v}_i' + \vec{V} \sum_i m_i = \vec{P}' + M\vec{V}, \quad M = \sum_i m_i \tag{4.12.1}$$

If we choose $\vec{V} = \vec{P}/M$, then $\vec{P}' = \sum_i m_i \vec{v}_i' = 0$, the system is "at rest" in the frame K'. Of course, the individual particles might be moving, what is at rest in K' is the *center of mass* defined by

$M\vec{R}_{\text{cm}} = \sum_i m_i \vec{r}_i$

(Check this by differentiating both sides with respect to time.)

The energy of a mechanical system in its rest frame is often called its *internal energy*, we'll denote it by E_{int} (This includes kinetic and potential energies.) The total energy of a moving system is then

$$E = \frac{1}{2} M \vec{V}^2 + E_{\text{int}} \tag{4.12.2}$$

(*Exercise*: verify this.)

4.13: Angular Momentum Conservation

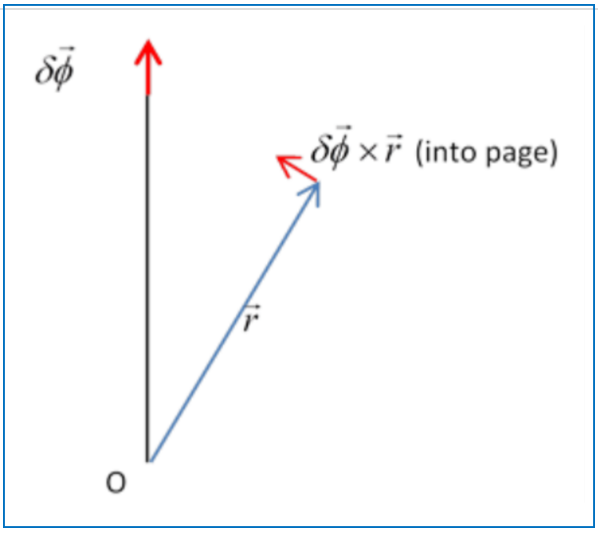

Figure 4.13.1: An infinitesimal rotation is represented by a vector of length $\delta\phi$ along the axis

Conservation of momentum followed from the invariance of the Lagrangian on being displaced in arbitrary directions in space, the homogeneity of space, angular momentum conservation is the consequence of the *isotropy* of space—there is no preferred direction.

So angular momentum of an isolated body in space is invariant even if the body is not symmetric itself.

The strategy is just as before, except now instead of an infinitesimal displacement we make an infinitesimal *rotation*,

$$\delta\vec{r} = \delta\vec{\phi}\times\vec{r} \tag{4.13.1}$$

and of course the velocities will also be rotated:

$$\delta\vec{v} = \delta\vec{\phi}\times\vec{v} \tag{4.13.2}$$

We must have

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \vec{r}_i}\cdot\delta\vec{r}_i + \frac{\partial L}{\partial \vec{v}_i}\cdot\delta\vec{v}_i \right) = 0 \tag{4.13.3}$$

Now $\partial L/\partial\vec{v}_i = \partial L/\partial\dot{\vec{r}}_i = \vec{p}_i$ by definition, and from Lagrange's equations

$$\partial L/\partial\vec{r}_i = (d/dt)\left(\partial L/\partial\dot{\vec{r}}_i\right) = \dot{\vec{p}}_i \tag{4.13.4}$$

so the isotropy of space implies that

$$\sum_i \left(\vec{p}_i\cdot\delta\vec{\phi}\times\vec{r}_i + \vec{p}_i\cdot\delta\vec{\phi}\times\vec{v}_i \right) = 0 \tag{4.13.5}$$

Notice the second term is identically zero anyway, since two of the three vectors in the triple product are parallel:

$$(d\vec{r}_i/dt)\times\vec{p}_i = \vec{v}_i\times m\vec{v}_i = 0 \tag{4.13.6}$$

That leaves the first term. The equation can be written:

$$\delta\vec{\phi}\cdot\frac{d}{dt}\sum_i \vec{r}_i\times\vec{p}_i = 0 \tag{4.13.7}$$

Integrating, we find that

$$\sum_i \vec{r}_i\times\vec{p}_i = \vec{L} \tag{4.13.8}$$

is a constant of motion, the *angular momentum*.

The angular momentum of a system is different about different origins. (Think of a single moving particle.) The angular momentum in the rest frame is often called the intrinsic angular momentum, the angular momentum in a frame in which the center of mass is at position $\vec{R}$ and moving with velocity $\vec{V}$ is .

$$\vec{L} = \vec{L}_{\text{cmframe}} + \vec{R} \times \vec{P} \tag{4.13.9}$$

(*Exercise*: check this.)

For a system of particles in a fixed external central field $V(r)$, the system is invariant with respect to rotations *about that point*, so angular momentum about that point is conserved. For a field "cylindrically" invariant for rotations about an axis, angular momentum about that axis is conserved.

CHAPTER OVERVIEW

5: Mechanical Similarity and the Virial Theorem

5.1: Some Examples

5.2: Lagrangian Treatment

5.3: The Virial Theorem

5.1: Some Examples

Similar triangles are just scaled up (or down) versions of each other, meaning they have the same angles. Scaling means the same thing in a mechanical system: if a planet can go around the sun in a given elliptical orbit, another planet can go in a scaled up version of that ellipse (the sun remaining at the focus). But it will take longer: so we can't just scale the spatial dimensions, to get the same equation of motion we must scale time as well, and not in general by the same factor.

In fact, we can establish the relative scaling of space and time in this instance with very simple dimensional analysis. We know the planet's radial acceleration goes as the inverse square of the distance, so

$$(\text{radial acceleration}) \times (\text{distance})^{\,2} = \text{constant} \tag{5.1.1}$$

the dimensionality of this expression is

$$LT^{-2}L^2 = L^3T^{-2}, \tag{5.1.2}$$

so

$$T^2 \propto L^3. \tag{5.1.3}$$

the square of the time of one orbit is proportional to the cube of the size of the orbit. A little more explicitly, the acceleration

$$\propto GM_{\text{Sun}}/r^2, \tag{5.1.4}$$

so for the same GM_{Sun}, if we double the orbit size, the equation will be the same but with orbital time up by $2\sqrt{2}$.

Galileo established that real mechanical systems, such as a person, are *not* scale invariant. A giant ten times the linear dimensions of a human would break his hip on the first step. The point is that the weight would be up by a factor of 1,000, the bone strength, going as cross sectional area, only by 100.

Mechanical similarity is important is constructing small models of large systems. A particularly important application is to fluid flow, for example in assessing fluid drag forces on a moving ship, plane or car. There are two different types of fluid drag: viscous frictional drag, and inertial drag, the latter caused by the body having to deflect the medium as it moves through. The patterns of flow depend on the relative importance of these two drag forces, this dimensionless ratio, inertial/viscous, is called the Reynolds number. To give meaningful results, airflow speeds around models must be adjusted to give the model the same Reynolds number as the real system.

5.2: Lagrangian Treatment

(Here we follow Landau.) Since the equations of motion are generated by minimizing the action, which is an integral of the Lagrangian along a trajectory, the motion won't be affected if the Lagrangian is multiplied by a constant. If the potential energy is a homogeneous function of the coordinates, rescaling would multiply it by a constant factor. If our system consists of particles interacting via such a potential energy, it will be possible to rescale time so that, rescaling both space and time, the Lagrangian is multiplied by an overall constant, so the equations of motion will look the same.

Specifically, if the potential energy U is homogeneous of degree k and the spatial coordinates are scaled by a factor α

$$(\alpha/\beta)^2 = \alpha^k, \quad \beta = \alpha^{1-\frac{1}{2}k} \tag{5.2.1}$$

For planetary orbits, $k = -1,$ so $\beta^2 = \alpha^3$, confirming our hand waving derivation above.

For the simple harmonic oscillator, $U(\vec{r}) \propto \vec{r}^2$ so $k = 2$ and $\beta = 0$ What does that mean? Scaling up the orbit does not affect the time—the oscillation time is always the same.

Falling under gravity: $k = 1, \beta = \sqrt{\alpha}, x \propto t^2$ So doubling the time scale requires quadrupling the length scale to get the scaled motion identical to the original.

5.3: The Virial Theorem

For a potential energy homogeneous in the coordinates, of degree k, say, and spatially bounded motion, there is a simple relation between the time averages of the kinetic energy, $\bar{T}$, and potential energy, $\bar{U}$. It's called the **virial theorem.**

Theorem 5.3.1: Virial Theorem

$$2\bar{T} = k\bar{U} \tag{5.3.1}$$

Proof

Since

$$T = \sum_i \frac{1}{2} m_i \vec{v}_i^2, \quad \vec{p}_i = m_i \vec{v}_i = \partial T / \partial \vec{v}_i \tag{5.3.2}$$

we have

$$2T = \sum_i \vec{p}_i \cdot \vec{v}_i = \frac{d}{dt}\left(\sum_i \vec{p}_i \cdot \vec{r}_i\right) - \sum_i \vec{r}_i \cdot \dot{\vec{p}}_i \tag{5.3.3}$$

We now average the terms in this equation over a very long time, that is, take

$$\bar{f} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(t) dt \tag{5.3.4}$$

Since we've said the orbits are bounded in space, and we assume also in momentum, the exact differential term contributes

$$\frac{1}{\tau}\left[\left(\sum_i \vec{p}_i \cdot \vec{r}_i\right)_{\text{at final}} - \left(\sum_i \vec{p}_i \cdot \vec{r}_i\right)_{\text{at initial}}\right] \to 0 \tag{5.3.5}$$

in the limit of infinite time.

So we have the time averaged

$$2\bar{T} = \sum_i \vec{r}_i \cdot \partial U / \partial \vec{r}_i \tag{5.3.6}$$

and for a potential energy a homogeneous function of degree k in the coordinates, from Euler's theorem:

$$2\bar{T} = k\bar{U} \tag{5.3.7}$$

So, for example, in a simple harmonic oscillator the average kinetic energy equals the average potential energy, and for an inverse-square system, the average kinetic energy is half the average potential energy in magnitude, and of opposite sign (being of course positive).

CHAPTER OVERVIEW

6: Hamilton’s Equations

6.1: A Dynamical System’s Path in Configuration Space and in State Space
6.2: Phase Space
6.3: Going From State Space to Phase Space
6.4: How It's Done in Thermodynamics
6.5: Math Note - the Legendre Transform
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6.7: Checking that We Can Eliminate the q˙i's
6.8: Hamilton’s Equations
6.9: A Simple Example

6.1: A Dynamical System's Path in Configuration Space and in State Space

The story so far: For a mechanical system with n degrees of freedom, the spatial configuration at some instant of time is completely specified by a set of n variables we'll call the q_i 's. . The n -dimensional q_i space is (naturally) called *configuration space*. It's like a freeze frame, a snapshot of the system at a given instant. Subsequent time evolution from that state is uniquely determined if we're also given the initial velocities $\dot{q}_i$.

The set of q_i 's and $\dot{q}_i$'s together define the *state* of the system, meaning both its configuration and how fast it's changing, therefore fully determining its future (and past) as well as its present. The 2n -dimensional space spanned by $(q_i, \dot{q}_i)$ is the *state space*.

The system's time evolution is along a path in *configuration* space parameterized by the time t. That, of course, fixes the corresponding path in *state* space, since differentiating the functions $q_i(t)$ along that path determines the $\dot{q}_i(t)$.

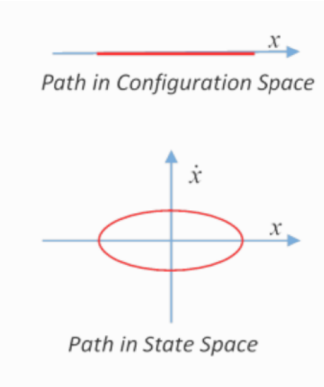

Figure 6.1.1: 1-D Simple Harmonic Oscillator

Trivial one-dimensional examples of these spaces are provided by the one-dimensional simple harmonic oscillator, where configuration space is just the x axis, say, the state space is the $(x, \dot{x})$ plane, the system's time path in the state space is an ellipse.

For a stone falling vertically down, the configuration space is again a line, the path in the $(x, \dot{x})$ state space is parabolic, $\dot{x} \propto \sqrt{x}$.

Exercise: sketch the paths in state space for motions of a pendulum, meaning a mass at the end of a light rod, the other end fixed, but free to rotate in one vertical plane. Sketch the paths in $(\theta, \dot{\theta})$ coordinates.

In principle, the system's path through configuration space can always be computed using Newton's laws of motion, but in practice the math may be intractable. As we've shown above, the elegant alternative created by Lagrange and Hamilton is to integrate the Lagrangian

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i) - V(q_i, t) \tag{6.1.1}$$

along different paths in configuration space from a given initial state to a given final state in a given time: as Hamilton proved, the actual path followed by the physical system between the two states in the given time is the one for which this integral, called the action, is minimized. This minimization, using the standard calculus of variations method, generates the Lagrange equations of motion in $q_i, \dot{q}_i$ and so determines the path.

Notice that specifying both the initial q_i 's and the final q_i 's fixes $2n$ variables. . That's all the degrees of freedom there are, so the motion is completely determined, just as it would be if we'd specified instead the initial q_i 's and $\dot{q}_i$'s. .

6.2: Phase Space

Newton wrote his equation of motion not as force equals mass times acceleration, but as force equals *rate of change of momentum*. Momentum, mass times velocity, is the natural "quantity of motion" associated with a time-varying dynamical parameter. It is some measure of how important that coordinate's motion is to the future dynamical development of the system.

Hamilton recast Lagrange's equations of motion in these more natural variables (q_i, p_i), positions and momenta, instead of $(q_i, \dot{q}_i)$. The q 's and p 's are called *phase space* coordinates.

So phase space is the same identical underlying space as state space, just with a different set of coordinates. Any particular state of the system can be completely specified either by giving all the variables $(q_i, \dot{q}_i)$ or by giving the values of all the (q_i, p_i).

6.3: Going From State Space to Phase Space

Now, the momenta are the *derivatives of the Lagrangian* with respect to the velocities, $p_i = \partial L\left(q_i, \dot{q}_i\right) / \partial \dot{q}_i$. So, how do we get from a function $L\left(q_i, \dot{q}_i\right)$ of positions and velocities to a function of positions and the derivatives of that function L with respect to the velocities?

6.4: How It's Done in Thermodynamics

To see how, we'll briefly review a very similar situation in thermodynamics: recall the expression that naturally arises for incremental energy, say for the gas in a heat engine, is

$$dE(S,V) = TdS - PdV \tag{6.4.1}$$

where S is the entropy and $T = \partial E/\partial S$ is the temperature. But S is not a handy variable in real life -- temperature T is a lot easier to measure! We need an energy-like function whose incremental change is some function of dT, dV rather than dS, dV The early thermodynamicists solved this problem by introducing the concept of the *free energy*,

$$F = E - TS \tag{6.4.2}$$

so that $dF = -SdT - PdV$. This change of function (and variable) was important: the free energy turns out to be more practically relevant than the total energy, it's what's available to do work.

So we've transformed from a function $E(S)$ to a function $F(T) = F(\partial E/\partial S)$ (ignoring P, V which are passive observers here).

6.5: Math Note - the Legendre Transform

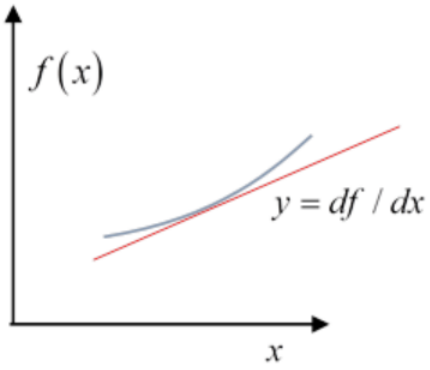

Figure 6.5.1

The change of variables described above is a standard mathematical routine known as the **Legendre transform**. Here's the essence of it, for a function of one variable. Suppose we have a function $f(x)$ that is convex, which is math talk for it always curves upwards, meaning $d^2f(x)/dx^2$ is positive. Therefore its slope, we'll call it

$$y = df(x)/dx \tag{6.5.1}$$

is a monotonically increasing function of x. For some physics (and math) problems, this slope y, rather than the variable x, is the interesting parameter. To shift the focus to y, Legendre introduced a new function, $g(y)$ defined by

$$g(y) = xy - f(x) \tag{6.5.2}$$

The function $g(y)$ is called the *Legendre transform* of the function $f(x)$.

To see how they relate, we take increments:

$$\begin{aligned} dg(y) &= ydx + xdy - df(x) \\ &= ydx + xdy - ydx \\ &= xdy \end{aligned}$$

(Looking at the diagram, an increment dx gives a related increment dy as the slope increases on moving up the curve.)

From this equation,

$$x = dg(y)/dy \tag{6.5.3}$$

Comparing this with $y = df(x)/dx$, it's clear that a *second* application of the Legendre transformation would get you back to the original $f(x)$. So no information is lost in the Legendre transformation $g(y)$ in a sense contains $f(x)$, and vice versa.

6.6: Hamilton's Use of the Legendre Transform

We have the Lagrangian $L\left(q_i, \dot{q}_i\right)$, and Hamilton's insight that these are not the best variables, we need to replace the Lagrangian with a closely related function (like going from the energy to the free energy), that is a function of the qi (that's not going to change) and, instead of the $\dot{q}_i$'s, the p_i 's, with $p_i = \partial L\left(q_i, \dot{q}_i\right) / \partial \dot{q}_i$. This is exactly a Legendre transform like the one from $f \rightarrow g$ discussed above.

The new function is

$$H\left(q_i, p_i\right) = \sum_{i=1}^{n} p_i \dot{q}_i - L\left(q_i, \dot{q}_i\right) \tag{6.6.1}$$

from which

$$dH\left(p_i, q_i\right) = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i \tag{6.6.2}$$

analogous to $dF = -SdT - PdV$ This new function is of course the *Hamiltonian*.

6.7: Checking that We Can Eliminate the q˙i's

We should check that we *can* in fact write

$$H\left(p_i, q_i\right) = \sum_{i=1}^{n} p_i \dot{q}_i - L\left(q_i, \dot{q}_i\right) \tag{6.7.1}$$

as a function of just the variables (q_i, p_i), with all trace of the $\dot{q}_i$'s eliminated. Is this always possible? The answer is yes.

Recall the $\dot{q}_i$'s only appear in the Lagrangian in the kinetic energy term, which has the general form

$$T = \sum_{i,j} a_{ij}\left(q_k\right) \dot{q}_i \dot{q}_j \tag{6.7.2}$$

where the coefficients a_{ij} depend in general on some of the q_k's but are independent of the velocities, the $\dot{q}_k$'s. Therefore, from the definition of the generalized momenta,

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{n} a_{ij}\left(q_k\right) \dot{q}_j \tag{6.7.3}$$

and we can write this as a vector-matrix equation,

$$\mathbf{p} = \mathbf{A}\dot{\mathbf{q}} \tag{6.7.4}$$

That is, p_i is a linear function of the $\dot{q}_j$'s. . Hence, the inverse matrix $\mathbf{A}^{-1}$ will give us $\dot{q}_i$ as a linear function of the pj's, and then putting this expression for the $\dot{q}_i$ into the Lagrangian gives the Hamiltonian as a function only of the qi's and the pi's, that is, the phase space variables.

The matrix A is always invertible because the kinetic energy is positive definite (as is obvious from its Cartesian representation) and a symmetric positive definite matrix has only positive eigenvalues, and therefore is invertible.

6.8: Hamilton's Equations

Having finally established that we can write, for an incremental change along the dynamical path of the system in phase space,

$$dH\left(q_i, p_i\right) = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i \tag{6.8.1}$$

we have immediately the so-called **canonical form** of Hamilton's equations of motion:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i$$

Evidently going from state space to phase space has replaced the second order Euler-Lagrange equations with this equivalent set of pairs of first order equations.

6.9: A Simple Example

For a particle moving in a potential in one dimension, $L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q)$.

Hence

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}, \quad \dot{q} = \frac{p}{m} \tag{6.9.1}$$

Therefore

$$\begin{aligned} H = p\dot{q} - L = p\dot{q} - \frac{1}{2}m\dot{q}^2 + V(q) \\ = \frac{p^2}{2m} + V(q) \end{aligned} \tag{6.9.2}$$

(Of course, this is just the total energy, as we expect.)

The Hamiltonian equations of motion are

$$\begin{aligned} \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial q} = -V'(q) \end{aligned} \tag{6.9.3}$$

So, as we've said, the second order Lagrangian equation of motion is replaced by two first order Hamiltonian equations. Of course, they amount to the same thing (as they must!): differentiating the first equation and substituting in the second gives immediately $-V'(q) = m\ddot{q}$, that is, $F = ma$, the original Newtonian equation (which we derived earlier from the Lagrange equations).

CHAPTER OVERVIEW

7: Time Evolution in Phase Space- Poisson Brackets and Constants of the Motion

7.1: The Poisson Bracket
7.2: Interlude - a Bit of History of Quantum Mechanics
7.3: The Jacobi Identity
7.4: Poisson's Theorem
7.5: Example- Angular Momentum Components

7.1: The Poisson Bracket

A function *[Math Processing Error]* of the phase space coordinates of the system and time has total time derivative

[Math Processing Error]

This is often written as

[Math Processing Error]

where

[Math Processing Error]

is called the *Poisson bracket*.

Caution

Equation *[Math Processing Error]* is Landau's definition for the Poisson bracket. It *differs in sign* from Goldstein, Wikipedia and others.

If, for a phase space function *[Math Processing Error]* (that is, no explicit time dependence) *[Math Processing Error]* is a constant of the motion, also called an *integral of the motion*.

In fact, the Poisson bracket can be defined for *any* two functions defined in phase space:

[Math Processing Error]

It's straightforward to check the following properties of the Poisson bracket:

[Math Processing Error]

The Poisson brackets of the basic variables are easily found to be:

[Math Processing Error]

Now, using

[Math Processing Error]

and the basic variable P.B.'s, we find

[Math Processing Error]

and, in fact, the bracket of p with any reasonably smooth function of q is:

[Math Processing Error]

7.2: Interlude - a Bit of History of Quantum Mechanics

It should be clear at this point that the Poisson bracket is very closely related to the commutator in quantum mechanics. In the usual quantum mechanical notation, the momentum operator

[Math Processing Error]

so the commutator (which acts on a wavefunction, remember)

[Math Processing Error]

identical to the Poisson bracket result multiplied by the constant *[Math Processing Error]*.

The first successful mathematical formulation of quantum mechanics, in 1925 (before Schrödinger's equation!) was by Heisenberg. As you know, he was the guy with the Uncertainty Principle: he realized that you couldn't measure momentum and position of anything simultaneously. He represented the states of a quantum system as vectors in some Hilbert space, and the dynamical variables as matrices acting on these vectors. He interpreted the result of a measurement as finding an eigenvalue of a matrix. If two variables couldn't be measured at the same time, the matrices had a nonzero commutator. In particular, for a particle's position and momentum the matrix representations satisfied *[Math Processing Error]*.

Dirac made the connection with Poisson brackets on a long Sunday walk, mulling over Heisenberg's uv−vu (as it was written). He suddenly but dimly remembered what he called "these strange quantities"—the Poisson brackets—which he felt might have properties corresponding to the quantum mathematical formalism Heisenberg was building. But he didn't have access to advanced dynamics books until the college library opened the next morning, so he spent a sleepless night. First thing Monday, he read the relevant bit of Whittaker's *Analytical Dynamics*, and saw he was correct. (From the biography by Helge Kragh.)

Dirac went on to adapt the equation

[Math Processing Error]

to quantum mechanics: for time-independent functions,

[Math Processing Error]

becomes

[Math Processing Error]

for time development of an operator in the Heisenberg picture, where state vectors of closed systems do not vary in time (as opposed to the Schrödinger picture, where the vectors vary and the operators remain constant).

7.3: The Jacobi Identity

Another important identity satisfied by the Poisson brackets is the **Jacobi identity**

[Math Processing Error]

This can be proved by the incredibly tedious method of just working it out. A more thoughtful proof is presented by Landau, but we're not going through it here. Ironically, the Jacobi identity is a lot easier to prove in its quantum mechanical incarnation (where the bracket just signifies the commutator of two matrix operators, *[Math Processing Error]* Jacobi's identity plays an important role in general relativity.

7.4: Poisson's Theorem

If f and g are two constants of the motion (i.e., they both have zero Poisson brackets with the Hamiltonian), then the Poisson bracket $[f, g]$ is *also* a constant of the motion. Of course, it could be trivial, like $[p, q] = 1$ or it could be a function of the original variables. But sometimes it's a new constant of motion. If f,g are time-independent, the proof follows immediately from Jacobi's identity. A proof for time *dependent* functions is given in Landau—it's not difficult.

7.5: Example- Angular Momentum Components

A moving particle has angular momentum about the origin $\vec{L} = \vec{r} \times \vec{p},$ so

$$L_1 = r_2 p_3 - r_3 p_2, \quad L_2 = r_3 p_1 - r_1 p_3 \tag{7.5.1}$$

Using the Poisson brackets found above,

$$[r_i, r_j] = [p_i, p_j] = 0, \quad [p_i, r_j] = \delta_{ij} \tag{7.5.2}$$

we have

$$\begin{aligned} &= [r_2 p_3, r_3 p_1] + [r_3 p_2, r_1 p_3] \\ &= r_2 p_1 - p_2 r_1 \\ &= -L_3 \end{aligned} \tag{7.5.3}$$

(*Note*: we remind the reader that we are following Landau's convention, in which the Poisson brackets have the *opposite sign* to the more common use, for example in Goldstein and Wikipedia.)

We conclude that if two components of angular momentum are conserved, so is the third.

CHAPTER OVERVIEW

8: A New Way to Write the Action Integral

Following Landau, we'll first find how the action integral responds to incremental changes in the *endpoint* coordinates and times, then use the result to write the action integral itself in a new, more intuitive way. This new formulation shows very directly the link to quantum mechanics, and variation of the action in this form gives Hamilton's equations immediately.

8.1: Function of Endpoint Position
8.2: Function of Endpoint Time
8.3: Varying Both Ends
8.4: Another Way of Writing the Action Integral
8.5: How this Classical Action Relates to Phase in Quantum Mechanics
8.6: Hamilton’s Equations from Action Minimization
8.7: How Can p, q Really Be Independent Variables?

8.1: Function of Endpoint Position

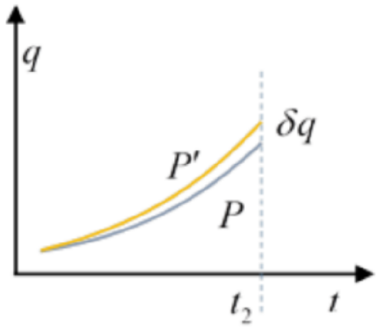

Figure 8.1.1: Both P and P' are physically realizable paths

We'll now think of varying the action in a slightly different way. (*Note*: We're using Landau's notation.) Previously, we considered the integral of the Lagrangian over all possible different paths from the initial place and time $q^{(1)}, t_1$ to the final place and time $q^{(2)}, t_2$

and found the path of minimum action. Now, though, we'll *start* with that path, the actual physical path, and investigate the corresponding action *as a function of the final endpoint variables*, given a fixed beginning place and time.

Taking one degree of freedom (the generalization is straightforward), for a small path variation the incremental change in action

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q dt \tag{8.1.1}$$

(Recall that first term comes from the calculus of variations when we allow the end point to vary -- it's exactly the same point we previously discussed in the brachistochrone problem of fastest time for a given horizontal distance, allowing the vertical position of the endpoint to be a free parameter.)

With the incremental variation, we've gone from the physical path P (followed by the system in configuration space from $q^{(1)}, t_1$ to $q^{(2)}, t_2\big)$ to a second path P' beginning at the same place and time, and ending at the same time t_2 as P, but at a slightly different place $q^{(2)} + \delta q\,(t_2)$.

Both paths P, P' are fully determined by their initial and final positions and times, so P, P' must correspond to *slightly different initial velocities*. The important point is that since both paths describe the natural dynamical development of the system from the initial conditions, the system obeys the equations of motion at all times along both paths, and therefore *the integral term in the above equation is identically zero*.

Writing $\delta q\,(t_2) = \delta q, p^{(2)} = p$ the action, *regarded as a function of the final position variable*, with the final time fixed at t_2, has the differential

$$\delta S\left(q^{(2)}, t_2\right) = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} = p^{(2)}\delta q^{(2)} = p\delta q \tag{8.1.2}$$

For the multidimensional case, the incremental change in the action on varying the *final position variable* is given by (dropping the superscript)

$$\partial S/\partial q_i = p_i \tag{8.1.3}$$

8.2: Function of Endpoint Time

What about the action as a function of the final point *arrival time*?

Since *[Math Processing Error]*, the value of the Lagrangian at the endpoint. *Remember we are defining the action at a point as that from integrating along the true path up to that point.*

Landau denotes *[Math Processing Error]*

and we'll be doing this, but it's crucial to keep in mind that the *endpoint* position and time are the variables here!

If we now allow an incremental time increase, [Math Processing Error], with the final coordinate position as a free parameter, the dynamical path will now continue on, to an incrementally different finishing point.

This will give (*with t understood from now on to mean [Math Processing Error]*

[Math Processing Error]

Putting this together with *[Math Processing Error]* gives immediately the *partial* time derivative

[Math Processing Error]

and therefore, combining this with the result *[Math Processing Error]* from the previous section,

[Math Processing Error]

This, then, is the total differential of the action as a function of the spatial and time coordinates of the end of the path.

8.3: Varying Both Ends

The argument given above for the incremental change in action from varying the endpoint is clearly equally valid for varying the beginning point of the integral (there will be a sign change, of course), so

[Math Processing Error]

The initial and final coordinates and times specify the action and the time development of the system uniquely.

(*Note*: We'll find this equation again in the section on canonical transformations -- the action will be seen there to be the generating function of the time-development canonical transformation, this will become clear when we get to it.)

8.4: Another Way of Writing the Action Integral

Up to this point, we've always written the action as an integral of the Lagrangian with respect to time along the path,

[Math Processing Error]

However, the expression derived in the last section for the increment of action generated by an incremental change in the path endpoint is clearly equally valid for the contribution to the action from some *interior* increment of the path, say from *[Math Processing Error]* so we can write the total action integral as the sum of these increments:

[Math Processing Error]

In this integral, of course, the *[Math Processing Error]* add up to cover the whole path.

(In writing *[Math Processing Error]* we're following Landau's default practice of taking the action as a function of the final endpoint coordinates and time, assuming the beginning point to be fixed. This is almost always fine—we'll make clear when it isn't.)

8.5: How this Classical Action Relates to Phase in Quantum Mechanics

The link between classical and quantum mechanics is particularly evident in the expression for the action integral given above. In the so-called semi-classical regime of quantum mechanics, the de Broglie waves oscillate with wavelengths much smaller than typical sizes in the system. This means that locally it's an adequate approximation to treat the Schrödinger wave function as a plane wave,

[Math Processing Error]

where the amplitude function *[Math Processing Error]* only varies over distances much greater than the wavelength, and times far longer than the oscillation period. This expression is valid in almost all the classically accessible regions, invalid in the neighborhood of turning points, but the size of those neighborhoods goes to zero in the classical limit.

As we've discussed earlier, in the Dirac-Feynman formulation of quantum mechanics, to find the probability amplitude of a particle propagating from one point to another, we add contributions from all possible paths between the two points, each path contributing a term with phase equal to *[Math Processing Error]* times the action integral along the path.

From the semi-classical Schrödinger wave function above, it's clear that the change in phase from a small change in the endpoint is *[Math Processing Error]* coinciding exactly with the incremental contribution to the action in

[Math Processing Error]

So again we see, here very directly, how the action along a classical path is a multiple of the quantum mechanical phase change along the path.

8.6: Hamilton's Equations from Action Minimization

For arbitrary small path variations *[Math Processing Error]* in phase space, the minimum action condition using the form of action given above generates Hamilton's equations.

(*Note for nitpickers*: This may seem a bit surprising, since we generated this form of the action using the equations along the actual dynamical path, how can we vary it and still use them? Bear with me, you'll see.)

We'll prove this for a one dimensional system, it's trivial to go to many variables, but it clutters up the equations.

For a small path deviation *[Math Processing Error]* the change in the action *[Math Processing Error]* is

[Math Processing Error]

and integrating *[Math Processing Error]* by parts, with *[Math Processing Error]* at the endpoints,

[Math Processing Error]

The path variations *[Math Processing Error]* are independent and arbitrary, so must have identically zero coefficients—Hamilton's equations follow immediately, *[Math Processing Error]* Again, it's worth emphasizing the close parallel with quantum mechanics: Hamilton's equations written using Poisson brackets are:

[Math Processing Error]

In quantum mechanics, the corresponding Heisenberg equations of motion for position and momentum operators in terms of *commutators* are

[Math Processing Error]

8.7: How Can p, q Really Be Independent Variables?

It may seem a little odd at first that varying p,q as independent variables leads to the same equations as the Lagrangian minimization, where we only varied q, and that variation "locked in" the variation of *[Math Processing Error]*. And, isn't p *defined* in terms of *[Math Processing Error]* which is some function of *[Math Processing Error]*? So wouldn't varying q automatically determine the variation of p?

The answer is, no, p is *not* defined as *[Math Processing Error]* from the start in Hamilton's formulation. In this Hamiltonian approach, p,q really are taken as independent variables, then varying them to find the minimum path gives the equations of motion, *including the relation between p and [Math Processing Error]*

This comes about as follows: Along the minimum action path, we just established that

[Math Processing Error]

We also have that *[Math Processing Error]* so (Legendre transformation!)

[Math Processing Error]

from which, along the physical path, *[Math Processing Error]*. So this identity, previously written as the *definition* of p, now arises as a consequence of the action minimization in phase space.

CHAPTER OVERVIEW

9: Maupertuis' Principle - Minimum Action Path at Fixed Energy

9.1: Divine Guidance

9.2: It's Not About Time

9.3: The Abbreviated Action

9.4: Maupertuis' Principle and the Time-Independent Schrödinger Equation

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9.1: Divine Guidance

Incredibly, Maupertuis came up with a kind of principle of least action in 1747, long before the work of Lagrange and Hamilton. Maupertuis thought a body moved along a path such that the sum of products of mass, speed and displacement taken over time was minimized, and he saw that as the hand of God at work. This didn't go over well with his skeptical fellow countrymen, such as Voltaire, and in fact his formulation wasn't quite right, but history has given him partial credit, his name on a least action principle.

Suppose we are considering the motion of a particle moving around in a plane from some initial point and time *[Math Processing Error]*. Suppose its potential energy is a function of position, *[Math Processing Error]*. For example, imagine aiming for the hole on a rather bumpy putting green, but also *requiring that the ball take a definite time*, say two seconds, from being hit to falling in the hole. The action principle we've talked about so far will give the path, parameterized by time, *[Math Processing Error]*.

9.2: It's Not About Time

But that's *not* what you're interested in doing! Of course you want the ball to get in the hole, but you're not obsessed with how long it takes to get there. Yet without that time requirement, there are obviously many possible paths. If you hit it really fast, so its kinetic energy is far greater than the gravitational potential energy variations in the bumpy green, it will go close to a straight line (we're assuming that when it gets over the hole, it will drop in). As you slow down, the winning path will deviate from a straight line because of the uneven terrain. So the physical path to the hole will vary continuously with initial kinetic energy.

Maupertuis' principle is about what is the path $y(x)$ *to the hole, say from* (x_1, y_1, t_1) to $(x_2, y_2,$ any $t)$ *for a given initial energy* E .

So now we're fixing the beginning and end points in *space*, but allowing possible variation in the final *time*. Also, we're *fixing the energy*: $H(x, y, p_x, p_y) = E$ This means that in varying the path to minimize the action, we must restrict ourselves to the class of paths having energy E. In the bumpy putting green, you're giving the ball a fixed initial speed v_0, and trying different initial directions to get it in the hole.

From the expression for the differential of action in terms of varying the *endpoint* (as well as the rest of the path—remember, that gives the integral term that disappears along the dynamical path), we have all $\delta q_i = 0$ (the endpoint is fixed at the hole), leaving

$$\delta S = \sum_i p_i \delta q_i - H\delta t = -E\delta t \tag{9.2.1}$$

(Since we're restricting to paths with energy E, H can be replaced by E.)

9.3: The Abbreviated Action

Writing the action in the integral form along this constant energy path, we can trivially do the time integral:

[Math Processing Error]

Therefore, from the result *[Math Processing Error]* it necessarily follows that

[Math Processing Error]

[Math Processing Error] is called the *abbreviated action*: this is *Maupertuis' principle*.

The abbreviated action for the physical path is the minimum among all paths satisfying energy conservation with total energy E and passing through the designated final point—we don't care when. Note that not all values of E will work—for example, if we start putting the ball from a low point in the green, we'll need to give it enough energy to get out of the hollow to begin with. But there will be valid physical paths for a wide range of energy values, since the final arrival time is flexible.

Naturally, since this is a path through configuration space, to evaluate the abbreviated action

[Math Processing Error]

it must be expressed in terms of the q's. For the usual Lagrangian *[Math Processing Error]*, with *[Math Processing Error]*, and momenta

[Math Processing Error]

we find the abbreviated action

[Math Processing Error]

This is indeed an integral along a path in configuration space, but we need to get rid of the dt. Physically, we can see how to do this—since we know the total energy E, the kinetic energy at a point is *[Math Processing Error]* so that determines the speed, hence the time *[Math Processing Error]*.

That is, (following Landau)

[Math Processing Error]

from which

[Math Processing Error]

(This doesn't look like a very healthy mathematical object, but as you'll see, it's fine.)

Hence

[Math Processing Error]

To take a very simple case; if there is no potential, and just a free particle, *[Math Processing Error]* this is nothing but the length of the path multiplied by *[Math Processing Error]*, minimized by a straight line between the two points.

If we have a particle of mass m in a spatially varying potential *[Math Processing Error]*, the abbreviated action reduces to

[Math Processing Error]

where *[Math Processing Error]* is an element of path length. (This is obvious, really—the square root is the absolute value of the momentum, and the momentum vector, of course, points along the path.)

The matrix *[Math Processing Error]* sometimes called the **mass matrix**, is evidently a metric, a measure in the configuration space, by which the "length" of the paths, and particularly the minimum action path, are measured.

Exercise *[Math Processing Error]*

Use Maupertuis' principle to find the path of a cannonball, energy E, fired at a target which is x meters distant horizontally, both cannon and target being at sea level (think ships!).

9.4: Maupertuis' Principle and the Time-Independent Schrödinger Equation

Recall that the action, multiplied by *[Math Processing Error]* is equivalent to the phase in quantum mechanics. The case we're discussing here is evidently related to the time-*independent* Schrödinger equation, the one for an energy eigenstate, with the time-dependent phase factored out. In other words, imagine solving the time-independent Schrödinger equation for a particle in a potential by summing over paths. In the classical limit, the abbreviated action gives the total phase change along a path. Minimizing this to find the classical path exactly parallels our earlier discussion of Fermat's principle of least time, paths close to Maupertuis' minimum have total phase change along them all the same to leading order, and so add coherently.

CHAPTER OVERVIEW

10: Canonical Transformations

10.1: Point Transformations
10.2: General and Canonical Transformations
10.3: Generating Functions for Canonical Transformations
10.4: Generating Functions in Different Variables
10.5: Poisson Brackets under Canonical Transformations
10.6: Time Development is a Canonical Transformation Generated by the Action

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10.1: Point Transformations

It’s clear that Lagrange’s equations are correct for any reasonable choice of parameters labeling the system configuration. Let’s call our first choice *[Math Processing Error]*. Now transform to a new set, maybe even time dependent, *[Math Processing Error]*. The derivation of Lagrange’s equations by minimizing the action still works, so Hamilton’s equations must still also be OK too. This is called a *point transformation*: we’ve just moved to a different coordinate system, we’re relabeling the *points* in configuration space (but possibly in a time-dependent way).

10.2: General and Canonical Transformations

In the Hamiltonian approach, we're in *phase space* with a coordinate system having positions and momenta on an equal footing. It is therefore possible to think of more general transformations than the point transformation (which was restricted to the position coordinates).

We can have transformations that mix up position and momentum variables:

[Math Processing Error]

where *[Math Processing Error]* means the whole set of the original variables.

In those original variables, the equations of motion had the nice *canonical* Hamilton form,

[Math Processing Error]

Things won't usually be that simple in the new variables, but it does turn out that many of the "natural" transformations that arise in dynamics, such as that corresponding to going forward in time, *do* preserve the form of Hamilton's canonical equations, that is to say

[Math Processing Error]

A transformation that retains the canonical form of Hamilton's equations is said to be ***canonical***.

(*Jargon note*: these transformations are occasionally referred to as *contact* transformations.)

10.3: Generating Functions for Canonical Transformations

In this section, we go back to considering the full action (not the abbreviated--fixed energy--action used earlier).

Now, we've established that Hamilton's equations in the original parameterization follow from minimizing the action in the form

$$\delta \int \left(\sum_i p_i dq_i - H dt \right) = 0 \tag{10.3.1}$$

For a canonical transformation, by definition the new variables must also satisfy Hamilton's equations, so, working backwards, action minimization must be expressible in the new variables exactly as in the old ones:

$$\delta \int \left(\sum_i P_i dQ_i - H' dt \right) = 0 \tag{10.3.2}$$

Now, we've previously stated that two actions lead to the same equations of motion if the integrands differ by the total differential of some function F of coordinates, momenta and time. (That's because in adding such a function to the integrand, the function's contribution to the integral is just the difference between its values at the two (fixed) ends, so in varying the path between the ends to minimize the total integral and so generate the equations of motion, this exact differential dF makes no contribution.)

That is to say, the two action integrals will be *minimized on the same path* through phase space provided the integrands differ by an exact differential:

$$\sum_i p_i dq_i - H dt = \sum_i P_i dQ_i - H' dt + dF \tag{10.3.3}$$

F is called the *generating function* of the transformation. Rearranging the equation above,

$$dF = \sum_i p_i dq_i - \sum_i P_i dQ_i + (H' - H)\, dt \tag{10.3.4}$$

Notice that the differentials here are dq_i, dQ_i, dt *so these are the natural variables for expressing the generating function.*

We will therefore write it as $F(q, Q, t)$,

and from the expression for dF above,

$$p_i = \frac{\partial F(q,Q,t)}{\partial q_i}, \quad P_i = -\frac{\partial F(q,Q,t)}{\partial Q_i}, \quad H' = H + \frac{\partial F(q,Q,t)}{\partial t} \tag{10.3.5}$$

Let's reemphasize here that a canonical transformation will in general mix up coordinates and momenta—they are the same kind of variable, from this Hamiltonian perspective. They can even be *exchanged*: for a system with one degree of freedom, for example, the transformation

$$Q = p, \quad P = -q \tag{10.3.6}$$

is a perfectly good canonical transformation (check out Hamilton's equations in the new variables), even though it turns a position into a momentum and vice versa!

If this particular transformation is applied to a simple harmonic oscillator, the Hamiltonian remains the same (we're taking $H = \frac{1}{2}\left(p^2 + q^2\right)$ so the differential dF of the generating function (given above) has no $H' - H$ term, it is just

$$dF(q, Q) = pdq - PdQ \tag{10.3.7}$$

The generating function for this transformation is easily found to be

$$F(q, Q) = Qq \tag{10.3.8}$$

from which

$$dF = Qdq + qdQ = pdq - PdQ \tag{10.3.9}$$

as required.

Another canonical transformation for a simple harmonic oscillator is $q = \sqrt{2P}\sin Q, \quad p = \sqrt{2P}\cos Q$. You will investigate this in homework.

10.4: Generating Functions in Different Variables

This *[Math Processing Error]* is only one example of a generating function—in discussing Liouville's theorem later, we'll find it convenient to have a generating function expressed in the q 's and P 's. We get that generating function, often labeled *[Math Processing Error]* by a *Legendre transformation*:

[Math Processing Error]

Then, for this new generating function

[Math Processing Error]

Evidently, we can similarly use the Legendre transform to find generating functions depending on the other possible mixes of old and new variables: p,Q, and p,P.

What's the Point of These Canonical Transformations?

It will become evident with a few examples: it is often possible to transform to a set of variables where the equations of motion are a lot simpler, and, for some variables, trivial. The canonical approach also gives a neat proof of Liouville's theorem, which we'll look at shortly.

10.5: Poisson Brackets under Canonical Transformations

First, note that if Hamilton's equations have the standard canonical form

[Math Processing Error]

with respect to a pair of variables *[Math Processing Error]*

then those variables are said to be *canonically conjugate*.

The Poisson bracket is invariant under a canonical transformation, meaning

[Math Processing Error]

Let's begin by establishing that

[Math Processing Error]

We'll show the method by taking just one pair of variables p,q, and a generating function *[Math Processing Error]*

Then

[Math Processing Error]

With the generating function *[Math Processing Error]*

[Math Processing Error]

and

[Math Processing Error]

Putting these results into the Poisson bracket,

[Math Processing Error]

These basic results can then be used to prove the general Poisson bracket is independent of the parametrization of phase space, details in Goldstein and elsewhere.

Landau, on the other hand, offers a one-line proof of the invariance of the Poisson bracket of two dynamical functions *[Math Processing Error]* under a canonical transformation: imagine a fictitious system having g as its Hamiltonian. Then *[Math Processing Error]* and cannot depend on the coordinate system used, so must equal *[Math Processing Error]*.

10.6: Time Development is a Canonical Transformation Generated by the Action

The transformation from the variables *[Math Processing Error]* has to be canonical, since the system obeys Hamilton's (canonical!) equations at all times.

In fact, the variation of the action along the true path from *[Math Processing Error]* with respect to final *and* initial coordinates and times was found earlier to be

[Math Processing Error]

and, comparing that expression with the differential form of a canonical transformation corresponding to *[Math Processing Error]* in the discussion above, which was

[Math Processing Error]

we see that *the action itself is the generating function* for the canonical transformation from the variables *[Math Processing Error]* time *[Math Processing Error]* actually −S generates the forward motion in time, the equivalent variables in the two equations above being

[Math Processing Error]

CHAPTER OVERVIEW

11: Introduction to Liouville's Theorem

11.1: Paths in Simple Phase Spaces - the SHO and Falling Bodies
11.2: Following Many Systems- a “Gas” in Phase Space
11.3: Liouville’s Theorem- Local Gas Density is Constant along a Phase Space Path
11.4: Landau’s Proof Using the Jacobian
11.5: Jacobian for Time Evolution
11.6: Jacobians 101
11.7: Jacobian proof of Liouville’s Theorem
11.8: Simpler Proof of Liouville’s Theorem
11.9: Energy Gradient and Phase Space Velocity

11.1: Paths in Simple Phase Spaces - the SHO and Falling Bodies

Let's first think further about paths in phase space. For example, the simple harmonic oscillator, with Hamiltonian $H = p^2/2m + m\omega^2 q^2/2$ describes circles in phase space parameterized with the variables $(p, m\omega q)$. (A more usual notation is to write the potential term as $\frac{1}{2}kq^2$)

Question: are these circles the only possible paths for the oscillator to follow?

Answer: *yes*: any other path would intersect a circle, and at that point, with both position and velocity defined, there is only one path forward (and back) in time possible, so the intersection can't happen.

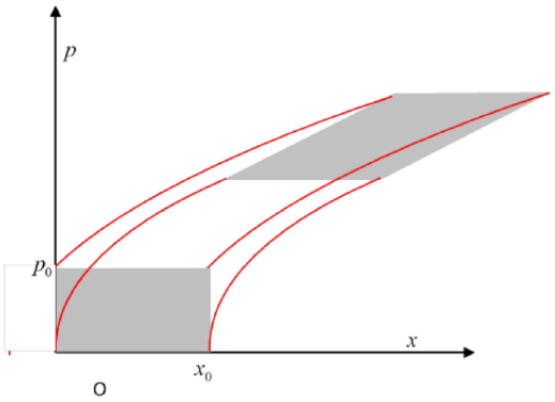


Figure 11.1.1: Four phase space trajectories for falling objects

Here's an example from Taylor of paths in phase space: four identical falling bodies are released simultaneously, see figure, x measures distance vertically down. Two are released with zero momentum from the origin O and from a point x_0 meters down, the other two are released with initial momenta p_0 again from the points O, x_0

(Note the difference in initial slope.)

Check: convince yourself that all these paths are parts of parabolas centered on the x -axis. (Just as the simple harmonic oscillator phase space is filled with circular paths, this one is filled with parabolas.)

Bodies released with the same initial velocity at the same time will keep the same vertical distance apart, those released with different initial velocities will keep the same velocity difference, since all accelerate at g. Therefore, *the area of the parallelogram formed by the four phase space points at a later time will have the same area as the initial square*.

Exercise: convince yourself that all the points of an initial vertical side of the square all stay in line as time goes on, even though the line does not stay vertical.

The four sides of the square deform with time to the four sides of the parallelogram, point by point. This means that if we have a falling stone corresponding initially to a point inside the square, it will go to a point inside the parallelogram, because if somehow its path reached the boundary, we would have two paths in phase space intersecting, and a particle at one point in phase space has a uniquely defined future path (and past).

11.2: Following Many Systems- a “Gas” in Phase Space

We’ve looked at four paths in phase space, corresponding to four falling bodies, all beginning at $t = 0$, but with different initial co-ordinates in (p, x) Suppose now we have many falling bodies, so that at t=0 a region of phase space can be imagined as filled with a “gas” of points, each representing one falling body, initially at (p_i, x_i), $\quad i = 1, \ldots, N$

The argument above about the phase space path of a point within the square at t=0 staying inside the square as time goes on and the square distorts to a parallelogram must also be true for any dynamical system, and any closed volume in phase space, since it depends on phase space paths never intersecting: that is,

if at t = 0 some closed surface in phase space contains a number of points of the gas, those same points remain inside the surface as it develops in time -- none exit or enter.

For the number of points N sufficiently large, the phase space time development looks like the flow of a fluid.

11.3: Liouville's Theorem- Local Gas Density is Constant along a Phase Space Path

The falling bodies phase space square has one more lesson for us: visualize now a uniformly dense gas of points inside the initial square. Not only does the gas stay within the distorting square, the area it covers in phase space remains constant, as discussed above, so *the local gas density stays constant as the gas flows through phase space*.

Liouville's theorem is that this constancy of local density is true for general dynamical systems.

11.4: Landau's Proof Using the Jacobian

Landau gives a very elegant proof of elemental volume invariance under a general canonical transformation, proving the Jacobian multiplicative factor is always unity, by clever use of the generating function of the canonical transformation.

Jacobians have wide applicability in different areas of physics, so this is a good time to review their basic properties, which we do below, as a preliminary to giving the proof.

It must be admitted that there are simpler ways of deriving Liouville's theorem, directly from Hamilton's equations, *the reader may prefer to skip the Jacobian proof at first reading*.

11.5: Jacobian for Time Evolution

As we've established, time development is equivalent to a canonical coordinate transformation,

$$(p_t, q_t) \rightarrow (p_{t+\tau}, q_{t+\tau}) \equiv (P, Q) \tag{11.5.1}$$

Since we already know that the number of points inside a closed volume is constant in time, Liouville's theorem is proved if we can show that the volume enclosed by the closed surface is constant, that is, with V'

denoting the volume V evolves to become, we must prove

$$\int_{V'} dQ_1 \dots dQ_s dP_1 \dots dP_s = \int_V dq_1 \dots dq_s dp_1 \dots dp_s ? \tag{11.5.2}$$

If you're familiar with Jacobians, you know that (by definition)

$$\int dQ_1 \dots dQ_s dP_1 \dots dP_s = \int D dq_1 \dots dq_s dp_1 \dots dp_s \tag{11.5.3}$$

where the Jacobian

$$D = \frac{\partial (Q_1, \dots, Q_s, P_1, \dots, P_s)}{\partial (q_1, \dots, q_s, p_1, \dots, p_s)} \tag{11.5.4}$$

Liouville's theorem is therefore proved if we can establish that D=1. If you're not familiar with Jacobians, or need reminding, read the next section!

11.6: Jacobians 101

Suppose we are integrating a function over some region of ordinary three-dimensional space,

[Math Processing Error]

but we want to change variables of integration to a different set of coordinates *[Math Processing Error]* such as, for example, *[Math Processing Error]*. The new coordinates are of course functions of the original ones *[Math Processing Error]* etc., and we assume that in the region of integration they are smooth, well-behaved functions. We can't simply re-express f in terms of the new variables, and replace the volume differential *[Math Processing Error]* that gives the wrong answer—in a plane, you can't replace *[Math Processing Error]* with *[Math Processing Error]*, you have to use *[Math Processing Error]*. That extra factor *[Math Processing Error]* is called the **Jacobian**, it's clear that in the plane a small element with sides of fixed lengths *[Math Processing Error]* is bigger the further it is from the origin, not all *[Math Processing Error]* elements are equal, so to speak. Our task is to construct the Jacobian for a general change of coordinates.

We need to think carefully about the volumes in the three-dimensional space represented by *[Math Processing Error]* and by *[Math Processing Error]*. Of course, the *[Math Processing Error]*'s are just ordinary perpendicular Cartesian axes so the volume is just the product of the three sides of the little box, *[Math Processing Error]*.Imagine this little box, its corner closest to the origin at *[Math Processing Error]* and its furthest point at the other end of the body diagonal at *[Math Processing Error]* Let's take these two points in the qi coordinates to be at *[Math Processing Error]*. In visualizing this, bear in mind that the q axes need not be perpendicular to each other (but they cannot all lie in a plane, that would not be well-behaved).

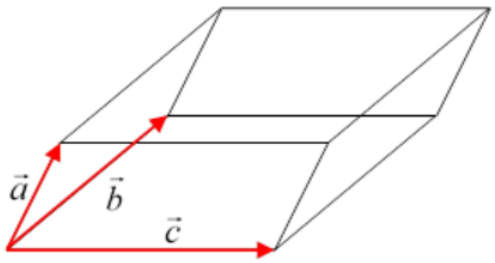
For the *[Math Processing Error]* coordinate integration, we imagine filling the space with little cubical boxes. For the *[Math Processing Error]* integration, we have a system of space filling infinitesimal parallelepipeds, in general pointing different ways in different regions (think *[Math Processing Error]*. What we need to find is the volume of the incremental parallelepiped with sides we'll write as vectors in x -coordinates, *[Math Processing Error]*. These three incremental vectors are along the corresponding *[Math Processing Error]* coordinate axes, and the three added together are the displacement from *[Math Processing Error]*

[Math Processing Error]

Hence, in components,

[Math Processing Error]

Now the volume of the parallelepiped with sides the three vectors from the origin *[Math Processing Error]* area of the parallelogram, then the dot product singles out the component of *[Math Processing Error]* perpendicular to the plane of *[Math Processing Error]*).



So, the volume corresponding to the increments *[Math Processing Error]* in *[Math Processing Error]* space is

[Math Processing Error]

writing *[Math Processing Error]* (Landau's notation) for the *determinant*, which is in fact the *Jacobian*, often denoted by *[Math Processing Error]*.

The standard notation for this determinantal Jacobian is

[Math Processing Error]

So the appropriate replacement for the three dimensional incremental volume element represented in the integral by *[Math Processing Error]* is

[Math Processing Error]

The *inverse*

[Math Processing Error]

is easily established using the chain rule for differentiation.

Exercise *[Math Processing Error]*

check this!

Thus the change of variables in an integral is accomplished by rewriting the integrand in the new variables, and replacing

[Math Processing Error]

The argument in higher dimensions is just the same: on going to dimension *[Math Processing Error]*, the hypervolume element is equal to that of the *[Math Processing Error]* dimensional element multiplied by the component of the new vector perpendicular to the *[Math Processing Error]* dimensional element. The determinantal form does this automatically, since a determinant with two identical rows is zero, so in adding a new vector only the component perpendicular to all the earlier vectors contributes.

We've seen that the chain rule for differentiation gives the inverse as just the Jacobian with numerator and denominator reversed, it also readily yields

[Math Processing Error]

and this extends trivially to n dimensions.

It's also evident form the determinantal form of the Jacobian that

[Math Processing Error]

identical variables in numerator and denominator can be canceled. Again, this extends easily to *[Math Processing Error]* dimensions.

11.7: Jacobian proof of Liouville's Theorem

After this rather long detour into Jacobian theory, recall we are trying to establish that the volume of a region in phase space is unaffected by a canonical transformation, we need to prove that

[Math Processing Error]

and that means we need to show that the Jacobian

[Math Processing Error]

Using the theorems above about the inverse of a Jacobian and the chain rule product,

[Math Processing Error]

Now invoking the rule that if the same variables appear in both numerator and denominator, they can be cancelled,

[Math Processing Error]

Up to this point, the equations are valid for any nonsingular transformation—but to prove the numerator and denominator are equal in this expression requires that the equation be canonical, that is, be given by a generating function, as explained earlier.

Recall now the properties of the generating function *[Math Processing Error]*

[Math Processing Error]

from which

[Math Processing Error]

In the expression for the Jacobian *[Math Processing Error]*, the *[Math Processing Error]*,*[Math Processing Error]* element of the numerator is *[Math Processing Error]*.

In terms of the generating function *[Math Processing Error]*.

Exactly the same procedure for the denominator gives the *[Math Processing Error]*,*[Math Processing Error]* element to be *[Math Processing Error]*

In other words, the two determinants are the same (rows and columns are switched, but that doesn't affect the value of a determinant). This means D=1, and Liouville's theorem is proved.

11.8: Simpler Proof of Liouville's Theorem

Landau's proof given above is extremely elegant: since phase space paths cannot intersect, point inside a volume stay inside, no matter how the volume contorts, and since time development is a canonical transformation, the total volume, given by integrating over volume elements *[Math Processing Error]*, stays the same, since it's an integral over the corresponding volume elements *[Math Processing Error]* and we've just shown that *[Math Processing Error]*.

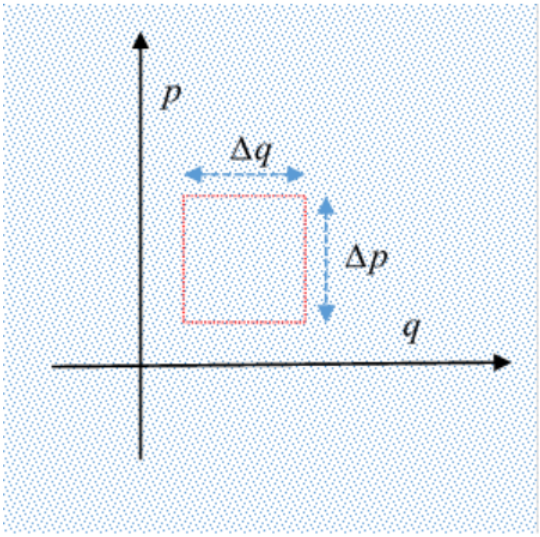


Figure *[Math Processing Error]*

Here we'll take a slightly different point of view: we'll look at a small square in phase space and track how its edges are moving, to prove its volume isn't changing. (We'll stick to one dimension, but the generalization is straightforward.)

The points here represent a "gas" of many systems in the two dimensional *[Math Processing Error]* phase space, and with a small square area *[Math Processing Error]* tagged by having all the systems on its boundary represented by dots of a different color. What is the incremental change in area of this initially square piece of phase space in time *[Math Processing Error]*?

Begin with the top edge: the particles are all moving with velocities *[Math Processing Error]* but of course the only change in *area* comes from the *[Math Processing Error]* term, that's the outward movement of the boundary, so the area change in *[Math Processing Error]* from the movement of this boundary will be *[Math Processing Error]*. Meanwhile, there will be a similar term from the bottom edge, and the *net* contribution, top plus bottom edges, will depend on the change in *[Math Processing Error]* from bottom to top, that is, a net area change from movement of these edges *[Math Processing Error]*.

Adding in the other two edges (the sides), with an exactly similar argument, the total area change is

[Math Processing Error]

But from Hamilton's equations *[Math Processing Error]*

[Math Processing Error]

and therefore

[Math Processing Error]

establishing that the total incremental area change as the square distorts is zero.

The conclusion is that the flow of the gas of systems in phase space is like an incompressible fluid, *but* with one important qualification: the density may vary with position! It just doesn't vary along a dynamical path.

11.9: Energy Gradient and Phase Space Velocity

For a time-independent Hamiltonian, the path in phase space (q, p) is a constant energy line, and we can think of the whole phase space as delineated by many such lines, exactly analogous to contour lines joining points at the same level on a map of uneven terrain, energy corresponding to height above sea level. The gradient at any point, the vector pointing exactly uphill and therefore perpendicular to the constant energy path, is

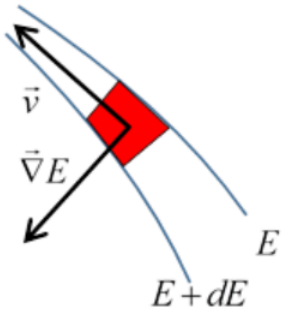


Figure 11.9.1

$$\vec{\nabla} H = (\partial H/\partial q, \partial H/\partial p) \tag{11.9.1}$$

here $H = E$. The velocity of a system's point moving through phase space is

$$\vec{v} = (\dot{q}, \dot{p}) = (\partial H/\partial p, -\partial H/\partial q) \tag{11.9.2}$$

This vector is perpendicular to the gradient vector, as it must be, of course, since the system moves along a constant energy path. But, interestingly, it has the same magnitude as the gradient vector! What is the significance of that? Imagine a small square sandwiched between two phase space paths close together in energy, and suppose the distance between the two paths is decreasing, so the square is getting squeezed, at a rate equal to the rate of change of the energy gradient. But at the same time it must be getting stretched along the direction of the path, an amount equal to the rate of change of phase space velocity along the path—and they are equal. So, this is just Liouville again, its area doesn't change.

CHAPTER OVERVIEW

12: The Hamilton-Jacobi Equation

12.1: Back to Configuration Space…

12.2: The Central Role of These Constants of Integration

12.3: Separation of Variables for a Central Potential; Cyclic Variables

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12.1: Back to Configuration Space…

We've established that the action, regarded as a function of its coordinate endpoints and time, satisfies

[Math Processing Error]

and at the same time *[Math Processing Error]* obeys the first-order differential equation

[Math Processing Error]

This is the *Hamilton-Jacobi equation*.

Notice that we're now back in *configuration space*!

For example, the Hamilton-Jacobi equation for the simple harmonic oscillator in one dimension is

[Math Processing Error]

(Notice that this has some resemblance to the Schrödinger equation for the same system.)

If the Hamiltonian has no explicit time dependence *[Math Processing Error]* so the action has the form *[Math Processing Error]*, and the Hamilton-Jacobi equation is

[Math Processing Error]

(This is analogous to the *time independent* Schrödinger equation for energy eigenstates.)

The Hamilton-Jacobi equation is therefore a *third* complete description of the dynamics, equivalent to Lagrange's equations and to Hamilton's equations.

Since *[Math Processing Error]* only appears differentiated, if we have a solution to the equation, we can always add an arbitrary constant term, to give an equally valid solution. For the general case, there will be a further s constants of integration, so a complete solution has the form

[Math Processing Error]

the α's and *[Math Processing Error]* being the constants of integration. We're not saying it's easy to solve this differential in general, just that we know how many constants of integration there must be in a final solution. Since the action determines the motion of the system completely, the constants of integration will be determined by the given initial and final coordinates, or, they could equally be regarded as functions of the initial coordinates and momenta (the initial momenta themselves being determined by the given initial and final coordinates).

12.2: The Central Role of These Constants of Integration

To describe the time development of a dynamical system in the simplest way possible, it is desirable to find parameters that are constant or change in a simple way. For example, motion in a spherically symmetric potential is described in terms of (constant) angular momentum components.

Now, these constant *[Math Processing Error]*'s are functions of the initial coordinates and momenta. Since they remain constant during the motion, they are evidently among the "variables" that describe the dynamical development in the simplest possible way. So, we need to construct a canonical transformation from our current set of variables (final coordinates and momenta) to a new set of variables that includes these constant of integration "momenta". (The corresponding canonical "positions" will then be given by differentiating the generating function with respect to the "momenta".)

How do we find the generating function for this transformation? A clue comes from one we've already discussed: that corresponding to development in time, going from the initial set of variables to the final set, or back. That transformation was generated *by the action itself*, expressed in terms of the two sets of positions. That is, we allowed *both* ends of the action integral path to vary, and wrote the action as a function of the final (2) and initial (1) endpoint variables and times:

[Math Processing Error]

In the present section, the final endpoint positions are denoted simply by *[Math Processing Error]* these are the same as the earlier *[Math Processing Error]*. Explicitly, we're writing

[Math Processing Error]

Compare this expression for the action with the formal expression we just derived from the Hamilton Jacobi equation,

[Math Processing Error]

These two expressions for *[Math Processing Error]* have just the same form: the action is expressed as a function of the endpoint position variables, plus another *[Math Processing Error]* variables needed to determine the motion uniquely. This time, instead of the original position variables, though, the second set of variables is these constants of integration, the *[Math Processing Error]*'s.

Now, just as we showed the action generated the transformation (either way) between the initial set of coordinates and momenta and the final set, it will also generate a canonical transformation from the final set of coordinates and momenta to another canonical set, having the *[Math Processing Error]*'s as the new "momenta". We'll label the new "coordinates" (the canonical conjugates of the *[Math Processing Error]*'s *[Math Processing Error]*

Taking then the action (neglecting the constant *[Math Processing Error]* which does nothing) *[Math Processing Error]* as the generating function, it depends on the old coordinates *[Math Processing Error]* and the new momenta *[Math Processing Error]*. This is the same set of variables—old coordinates and new momenta—as those of the (previously discussed) generating function *[Math Processing Error]*.

Recall

[Math Processing Error]

so here

[Math Processing Error]

and

[Math Processing Error]

This defines the new "coordinates" *[Math Processing Error]* *and* ensures that the transformation is canonical.

To find the new Hamiltonian *[Math Processing Error]*.

But

[Math Processing Error]

where *[Math Processing Error]* is just a constant, so

[Math Processing Error]

The first equation in this section was

[Math Processing Error]

so the new Hamiltonian

[Math Processing Error]

We have made a canonical transformation that has led to a zero Hamiltonian!

What does that mean? It means that the *neither the new momenta nor the new coordinates vary in time*:

[Math Processing Error]

(The fact that all momenta and coordinates are fixed in this representation does *not* mean that the system doesn't move—as will become evident in the following simple example, the original coordinates are functions of these new (nonvarying!) variables *and time*.)

The *[Math Processing Error]* equations *[Math Processing Error]* can then be used to find the *[Math Processing Error]* as functions of *[Math Processing Error]* To see how all this works, it is necessary to work through an example.

A Simple Example of the Hamilton-Jacobi Equation: Motion Under Gravity

The Hamiltonian for motion under gravity in a vertical plane is

[Math Processing Error]

so the Hamilton-Jacobi equation is

[Math Processing Error]

First, this Hamiltonian has no explicit time dependence (gravity isn't changing!), so from *[Math Processing Error]*, we can replace the last term in the equation by *[Math Processing Error]*.

A Simple Separation of Variables

Since the potential energy term depends only on *[Math Processing Error]*, the equation is solvable using separation of variables. To see this works, try

[Math Processing Error]

Putting this form into the equation, the resulting first term depends only on the variable *[Math Processing Error]*, the second plus third depend only on *[Math Processing Error]*, the last term is just the constant *[Math Processing Error]*. A function depending only on *[Math Processing Error]* can only equal a function independent of *[Math Processing Error]* if both are constants, similarly for *[Math Processing Error]*.

Labeling the constants *[Math Processing Error]*

[Math Processing Error]

So these *[Math Processing Error]*'s are constants of the motion, they are our new "momenta" (although they have dimensions of energy).

Solving,

[Math Processing Error]

(We could add in constants of integration, but adding constants to the action changes nothing.)

So now we have

[Math Processing Error]

This is our generating function (equivalent to *[Math Processing Error]*), in terms of old coordinates and these new "momenta", *[Math Processing Error]* Following the Hamilton-Jacobi analysis, this action will generate a canonical transformation which reduces the Hamiltonian to zero, meaning that not only these new momenta stay constant, but so do their conjugate "coordinate" variables,

[Math Processing Error]

These equations solve the problem. Rearranging, the trajectory is

[Math Processing Error]

The four “constants of motion” *[Math Processing Error]* are uniquely fixed by the initial coordinates and velocities, and they parameterize the subsequent time evolution of the system.

12.3: Separation of Variables for a Central Potential; Cyclic Variables

Landau presents in some details the separation of variables method for a *[Math Processing Error]* potential, interesting here because it results in equations you've met before—those arising in the standard quantum treatment of the hydrogen atom.

How do we make any progress with these formidable differential equations? One possibility is that some coordinates are *cyclic*, meaning that *[Math Processing Error]* say, does not appear explicitly in the Hamiltonian—for example, an angle variable in a spherically symmetric field. Then we have immediately that the corresponding momentum, *[Math Processing Error]*, a constant.

The Hamiltonian for a central potential is:

[Math Processing Error]

The Hamilton-Jacobi equation is therefore

[Math Processing Error]

The first thing to note is that *[Math Processing Error]* is cyclic (it doesn't appear in the Hamiltonian), so we can immediately replace *[Math Processing Error]*.

Then we have:

[Math Processing Error]

Now we seek a solution of the form

[Math Processing Error]

Substituting in the equation, notice that the expression in square brackets will become

[Math Processing Error]

independent of *[Math Processing Error]*, but on multiplying the full equation by *[Math Processing Error]*, and staring at the result, we see that in fact it is purely a function of *[Math Processing Error]*. This means that it's a constant, say

[Math Processing Error]

and then

[Math Processing Error]

These first-order equations can then be solved, at least numerically (and of course exactly for some cases). Physically, *[Math Processing Error]* being the total angular momentum, and *[Math Processing Error]* is the total energy.

Note: recall that in quantum mechanics, for example in solving the Schrödinger equation for the hydrogen atom, the separation of variables was achieved by writing the wave function as a *product* of functions belonging to the different variables. Here we use a *sum*—remember that the action corresponds closely to the *phase* of a quantum mechanical system, so a *sum* of actions is analogous to a *product* of wave functions.

CHAPTER OVERVIEW

13: Adiabatic Invariants and Action-Angle Variables

13.1: Adiabatic Invariants
13.2: Adiabatic Invariance and Quantum Mechanics
13.3: Action-Angle Variables
13.4: *Kepler Orbit Action-Angle Variables

13.1: Adiabatic Invariants

Imagine a particle in one dimension oscillating back and forth in some potential. The potential doesn't have to be harmonic, but it must be such as to trap the particle, which is executing periodic motion with period T. Now suppose we *gradually* change the potential, but keeping the particle trapped. That is, the potential depends on some parameter *[Math Processing Error]*, which we change gradually, meaning over a time much greater than the time of oscillation: *[Math Processing Error]*

A crude demonstration is a simple pendulum with a string of variable length, for example one hanging from a fixed support, but the string passing through a small loop that can be moved vertically to change the effective length (Figure *[Math Processing Error]*).

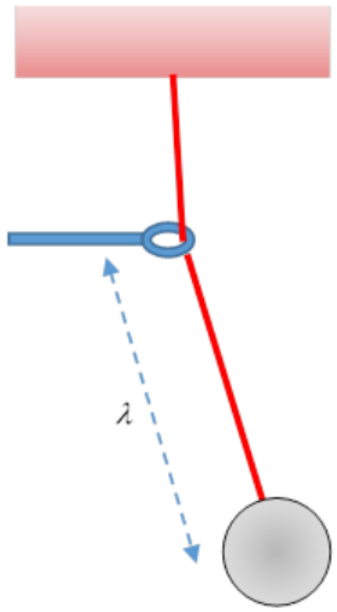

Figure *[Math Processing Error]*

If *[Math Processing Error]* were fixed, the system would have constant energy *[Math Processing Error]* and period *[Math Processing Error]*. As λ is gradually changed from outside, there will be energy exchange in general, we'll write the Hamiltonian *[Math Processing Error]*, the energy of the system will be *[Math Processing Error]* (Of course, *[Math Processing Error]* also depends on the initial energy before the variation began.) Remember now that from Hamilton's equations *[Math Processing Error]* so during the variation

[Math Processing Error]

It's clear from the diagram that the energy fed into the system as the ring moves slowly down varies throughout the cycle—for example, when the pendulum is close to vertically down, its energy will be almost unaffected by moving the ring.

Moving slowly down means *[Math Processing Error]* varies very little in one cycle of the system, we can average over a cycle:

[Math Processing Error]

where

[Math Processing Error]

Now Hamilton's equation *[Math Processing Error]* means that we can replace *[Math Processing Error]* with *[Math Processing Error]*, so the time for going round one complete cycle is

[Math Processing Error]

(This won't integrate to zero, because on the return leg both *[Math Processing Error]* will be negative.)

Therefore, replacing *[Math Processing Error]* as well,

[Math Processing Error]

Now, we assume *[Math Processing Error]* are varying slowly enough that they are close to constant over one cycle, meaning that at a given point q on the circuit, the momentum can be written *[Math Processing Error]* as constant and independent parameters. (We can always adjust *[Math Processing Error]*

If we now partially differentiate *[Math Processing Error]* constant (appropriate infinitesimal pushes required!), we get, at point *[Math Processing Error]* on the circuit,

[Math Processing Error]

which is the integrand in the numerator of our expression for *[Math Processing Error]*

[Math Processing Error]

In the denominator, we've replaced *[Math Processing Error]*

Rearranging,

[Math Processing Error]

This can be written

[Math Processing Error]

[Math Processing Error] is an *adiabatic invariant*: That means it stays constant when the parameters of the system change gradually, even though the system's energy changes.

Important! The partial derivative with respect to energy *[Math Processing Error]* determines the period of the motion:

[Math Processing Error]

(*Note*: here is another connection with quantum mechanics. If the system is connected to the outside world, for example if the orbiting particle is charged, as it usually is, and can therefore emit radiation, since in quantum mechanics successive action numbers *[Math Processing Error]* differ by integers, and the quantum of action is *[Math Processing Error]*, the energy radiated per quantum drop in action is *[Math Processing Error]*. This is of course in the classical limit of high quantum numbers.)

Notice that *[Math Processing Error]* *is the area of phase space enclosed by the integral*,

[Math Processing Error]

For the SHO, it's easy to check from the area of the ellipse that *[Math Processing Error]*

Take

[Math Processing Error]

The phase space elliptical orbit has semi-axes with lengths *[Math Processing Error]*, so the area enclosed is *[Math Processing Error]*.

The bottom line is that as we *gradually* change the spring strength (or, for that matter, the mass) of an oscillator (not necessarily harmonic), the *energy changes proportionally with the frequency*.

13.2: Adiabatic Invariance and Quantum Mechanics

This finding, the invariance of *[Math Processing Error]* for slow variation of the potential strength in a simple harmonic oscillator, connects directly with quantum mechanics, as was first pointed out be Einstein in 1911. Suppose the (quantum mechanical) oscillator is in the energy eigenstate with *[Math Processing Error]*. Then the spatial wave function has *[Math Processing Error]* zeros. If the potential is changed slowly enough (meaning little change over one cycle of oscillation) the oscillator will not jump to another eigenstate (or, more precisely, the probability will go to zero with the speed of change). The wave function will gradually stretch (or compress) but the number of zeroes will not change. Therefore the energy will stay at *[Math Processing Error]*, and track with *[Math Processing Error]*. Of course, the classical system is a little different: the quantum system is "locked in" to a particular state if the perturbation has vanishingly small frequency components corresponding to the energy differences *[Math Processing Error]* to available states. The classical system, on the other hand, can move to states arbitrarily close in energy. Landau gives a nontrivial analysis of the classical system, concluding that the change in the adiabatic "invariant" is of order *[Math Processing Error]* for an external change acting over a time *[Math Processing Error]*.

13.3: Action-Angle Variables

For a closed one-dimensional system undergoing finite motion (essentially a bound state), the equations of motion can be reformulated using the action variable *[Math Processing Error]* in place of the energy *[Math Processing Error]*. *[Math Processing Error]* is a function of energy alone in a closed one-dimensional system, and vice versa.

We're visualizing here a particle moving back and forth in a one-dimensional well with potential zero at the origin, and the potential never decreasing on going out from the origin to infinity. Obviously, if a potential has *two* low points, local bound states can arise in different places, and the *[Math Processing Error]* relationship is complicated, with different branches, possibly coming together at high energies.

Caution

Notice the integral sign in the expression for the action variable *[Math Processing Error]* is *[Math Processing Error]* signifying an integral around a *closed path*, a circuit. *Don't* confuse this integral with the abbreviated action integral, which has the same integrand, but is an integral *[Math Processing Error]* along a contour from a fixed starting point, say the origin, to the endpoint *[Math Processing Error]*, not going around a closed path. (Apologies for using the same letter for the differential and the endpoint, just following Landau.)

In the spirit of the discussion of constants of motion above, we make a canonical transformation to *[Math Processing Error]* as the new "momentum", using as generating function the abbreviated action *[Math Processing Error]*

The original momentum

[Math Processing Error]

The new "coordinate" conjugate to the momentum *[Math Processing Error]* will be

[Math Processing Error]

This is called an *angle variable*, *[Math Processing Error]* is the *action variable*, they are canonical.

To find Hamilton's equations in the transformed variables, since there is no time-dependence in the transformation, and the system is closed, the energy remains constant. Also, the energy is a function of *[Math Processing Error]* (meaning not of *[Math Processing Error]*.)

Hence

[Math Processing Error]

so the angle is a linear function of time: *[Math Processing Error]*

One further point about the action variable and the action: since we define the action as

[Math Processing Error]

it follows that if we track the change in this integral as time goes on and the system moves round and round the circuit in phase space, an additional term *[Math Processing Error]* will be added to the action for each time round, so the action is multi-valued.

13.4: *Kepler Orbit Action-Angle Variables

We have not yet covered Kepler orbits, so skip this section for now: it's here to refer back to later. It's from Landau, p 167. For motion confined to a plane, we can take the central potential analysis with *[Math Processing Error]*, the angular momentum, so the Hamiltonian is

[Math Processing Error]

The Hamilton-Jacobi equation is therefore

[Math Processing Error]

So, following the previous analysis of separation of variables for motion in a central potential, here

[Math Processing Error]

The action variable for the angular motion is just the angular momentum itself,

[Math Processing Error]

And the radial action variable, with potential *[Math Processing Error]*

[Math Processing Error]

(Details on doing the integral are given in the Appendix, *Mathematica* can do it too.)

So the energy is

[Math Processing Error]

The motion is *degenerate*: the two fundamental frequencies coincide, *[Math Processing Error]*

This has major consequences in quantum mechanics: the actions are all quantized in units of Planck's constant, for the hydrogen atom, from the formula above, the energy depends only on the *sum* of the quantum numbers: above the ground state, energy levels are degenerate, which is why the energy spectrum has the deceptively simple form so successfully explained by the Bohr model.

The orbital parameters, semi-latus rectum and eccentricity, from *[Math Processing Error]*, are

[Math Processing Error]

Recall the semi-major axis is given by *[Math Processing Error]* and from the above expression

[Math Processing Error]

in the hydrogen atom quantum number notation.

Appendix: Doing the Integral for The Radial Action Ir

The integral can be put in the form

[Math Processing Error]

which can be integrated by taking a contour encircling the cut from *[Math Processing Error]* to *[Math Processing Error]*. The integral will have a contribution from the pole at the origin equal to *[Math Processing Error]* and another from the circle at infinity, which is

[Math Processing Error]

Equating coefficients (multiplying the term inside the square root by *[Math Processing Error]*)

[Math Processing Error]

So the contribution from the origin gives the *[Math Processing Error]*.

CHAPTER OVERVIEW

14: Mathematics for Orbits

14.1: Preliminaries- Conic Sections
14.2: The Ellipse
14.3: The Parabola
14.4: The Hyperbola

14.1: Preliminaries- Conic Sections

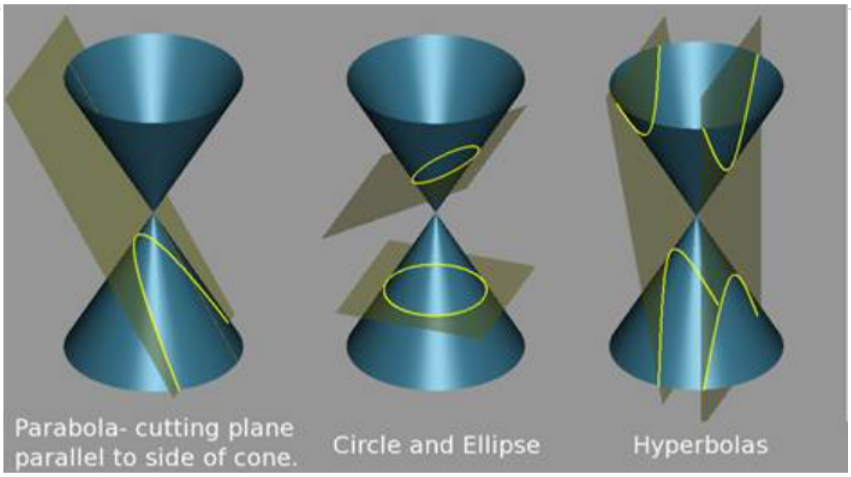

Figure *[Math Processing Error]*

Ellipses, parabolas and hyperbolas can all be generated by cutting a cone with a plane (see diagrams, from Wikimedia Commons). Taking the cone to be *[Math Processing Error]*, and substituting the z in that equation from the planar equation *[Math Processing Error]* is the vector perpendicular to the plane from the origin to the plane, gives a quadratic equation in *[Math Processing Error]*. This translates into a quadratic equation in the plane—take the line of intersection of the cutting plane with the *[Math Processing Error]* plane as the *[Math Processing Error]* axis in both, then one is related to the other by a scaling *[Math Processing Error]*. To identify the conic, diagonalized the form, and look at the coefficients of *[Math Processing Error]*. If they are the same sign, it is an ellipse, opposite, a hyperbola. The parabola is the exceptional case where one is zero, the other equates to a linear term. It is instructive to see how an important property of the ellipse follows immediately from this construction.

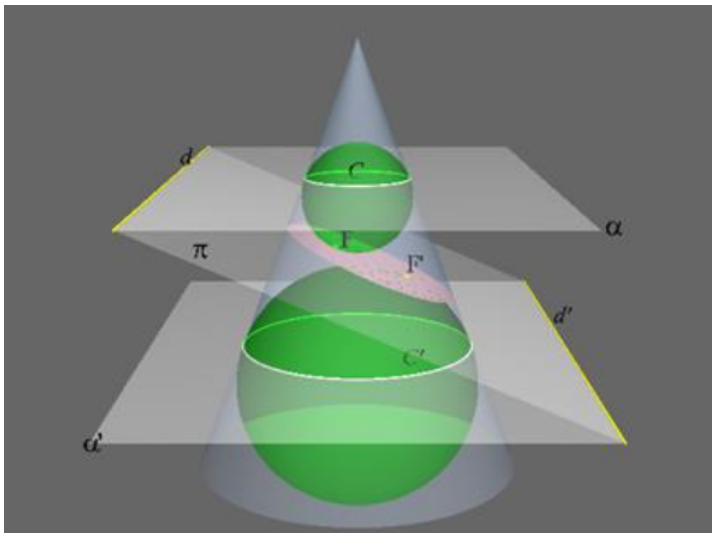

Figure *[Math Processing Error]*

The slanting plane in the figure cuts the cone in an ellipse. Two spheres inside the cone, having circles of contact with the cone *[Math Processing Error]*, are adjusted in size so that they both just touch the plane, at points *[Math Processing Error]* respectively.

It is easy to see that such spheres exist, for example start with a tiny sphere inside the cone near the point, and gradually inflate it, keeping it spherical and touching the cone, until it touches the plane. Now consider a point *[Math Processing Error]* on the ellipse. Draw two lines: one from *[Math Processing Error]* to the point *[Math Processing Error]* where the small sphere touches, the other up the cone, aiming for the vertex, but stopping at the point of intersection with the circle *[Math Processing Error]*. *Both these lines are tangents to the small sphere, and so have the same length*. (The tangents to a sphere from a point outside it form a cone, they are all of equal length.) Now repeat with *[Math Processing Error]*. We find that *[Math Processing Error]*, the distances to the circles measured along the line through the vertex. So *[Math Processing Error]* are therefore evidently the foci of the ellipse.

14.2: The Ellipse

Squashed Circles and Gardeners

The simplest nontrivial planetary orbit is a circle: *[Math Processing Error]* is centered at the origin and has radius *[Math Processing Error]*. An ellipse is a circle scaled (squashed) in one direction, so an ellipse centered at the origin with semimajor axis *[Math Processing Error]* and semiminor axis *[Math Processing Error]* has equation

[Math Processing Error]

in the standard notation, a circle of radius *[Math Processing Error]* scaled by a factor *[Math Processing Error]* in the *[Math Processing Error]* direction. (It's usual to orient the larger axis along *[Math Processing Error]*.)

A *circle* can also be defined as the set of points which are the same distance a from a given point, and an *ellipse* can be defined as the set of points such that the *sum of the distances from two fixed points is a constant length* (which must obviously be greater than the distance between the two points!). This is sometimes called the *gardener's definition*: to set the outline of an elliptic flower bed in a lawn, a gardener would drive in two stakes, tie a loose string between them, then pull the string tight in all different directions to form the outline.

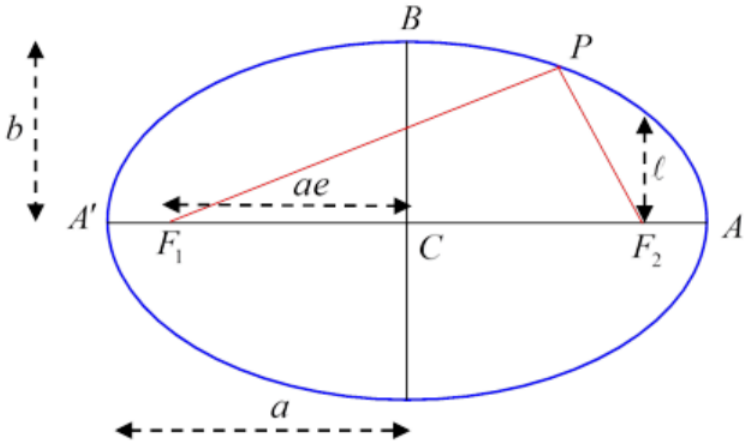

Figure *[Math Processing Error]*

In the diagram, the stakes are at *[Math Processing Error]* the **red lines** are the **string**, *[Math Processing Error]* is an arbitrary point on the ellipse.

[Math Processing Error] is called the semimajor axis length a, *[Math Processing Error]* the semiminor axis, length *[Math Processing Error]*.

[Math Processing Error] are called the foci (plural of focus).

Notice first that *the string has to be of length [Math Processing Error]*, because it must stretch along the major axis from *[Math Processing Error]* then back to *[Math Processing Error]* and for that configuration there's a double length of string along *[Math Processing Error]*. But the length *[Math Processing Error]*, so the total length of string is the same as the total length *[Math Processing Error]*.

Suppose now we put *[Math Processing Error]* at *[Math Processing Error]*. Since *[Math Processing Error]*, and the string has length *[Math Processing Error]*

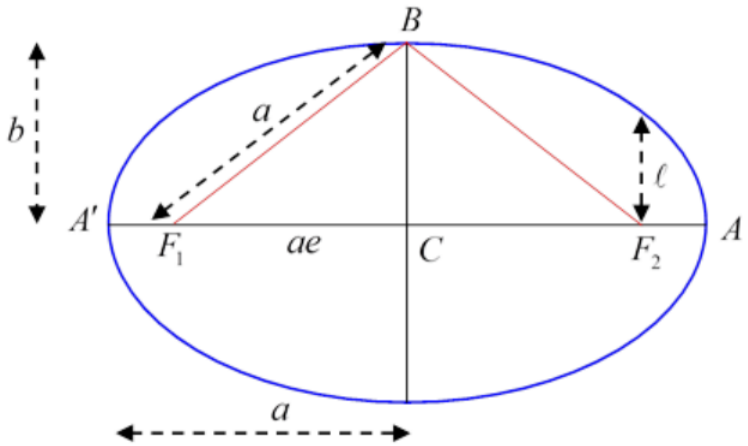

Figure *[Math Processing Error]*

We get a useful result by applying Pythagoras' theorem to the triangle *[Math Processing Error]*

[Math Processing Error]

(We shall use this shortly.)

Evidently, for a circle, *[Math Processing Error]*

Eccentricity

The **eccentricity** *[Math Processing Error]* of the ellipse is defined by

[Math Processing Error]

Eccentric just means off center, this is *how far the focus is off the center of the ellipse*, as a fraction of the semimajor axis. The eccentricity of a circle is zero. The eccentricity of a long thin ellipse is just below one.

[Math Processing Error] on the diagram are called the foci of the ellipse (plural of focus) because if a point source of light is placed at [Math Processing Error], and the ellipse is a mirror, it will reflect—and therefore focus—all the light to [Math Processing Error].

Equivalence of the Two Definitions

We need to verify, of course, that this gardener's definition of the ellipse is equivalent to the squashed circle definition. From the diagram, the total string length

[Math Processing Error]

and squaring both sides of

[Math Processing Error]

then rearranging to have the residual square root by itself on the left-hand side, then squaring again,

[Math Processing Error]

[Math Processing Error]

Ellipse in Polar Coordinates

In fact, in analyzing planetary motion, it is more natural to *take the origin of coordinates at the center of the Sun* rather than the center of the elliptical orbit.

It is also more convenient to take *[Math Processing Error]* coordinates instead of *[Math Processing Error]* coordinates, because the strength of the gravitational force depends only on *[Math Processing Error]*. Therefore, the relevant equation describing a planetary orbit is the *[Math Processing Error]* equation with the origin at one focus, here we follow the standard usage and choose the origin at *[Math Processing Error]*.

For an ellipse of semi major axis *[Math Processing Error]* and eccentricity *[Math Processing Error]* the equation is:

[Math Processing Error]

This is also often written

[Math Processing Error]

where *[Math Processing Error]* is the **semi-latus rectum**, the perpendicular distance from a focus to the curve *[Math Processing Error]*, see the diagram below: but notice again that *this equation has [Math Processing Error]*

(It's easy to prove *[Math Processing Error]* using Pythagoras' theorem, *[Math Processing Error]*

The directrix: writing *[Math Processing Error]*, the equation for the ellipse can also be written as

[Math Processing Error]

where *[Math Processing Error]* (the origin x=0 being the focus).

The line *[Math Processing Error]* is called the **directrix**.

For any point on the ellipse, its distance from the focus is e times its distance from the directrix.

Deriving the Polar Equation from the Cartesian Equation

Note first that (following standard practice) coordinates *[Math Processing Error]* and *[Math Processing Error]* have different origins!

Writing *[Math Processing Error]* in the Cartesian equation,

[Math Processing Error]

that is, with slight rearrangement,

[Math Processing Error]

This is a quadratic equation for *[Math Processing Error]* and can be solved in the usual fashion, but looking at the coefficients, it's evidently a little easier to solve the corresponding quadratic for *[Math Processing Error]*

The solution is:

[Math Processing Error]

from which

[Math Processing Error]

where we drop the other root because it gives negative *[Math Processing Error]*, for example for *[Math Processing Error]*. This establishes the equivalence of the two equations.

14.3: The Parabola

The parabola can be defines as the limiting curve of an ellipse as one focus (in the case we're examining, that would be F_1) going to infinity. The eccentricity evidently goes to one, $e \to 1$ since the center of the ellipse has gone to infinity as well. The semi-latus rectum ℓ is still defined as the perpendicular distance from the focus to the curve, the equation is

$$\ell = r(1+\cos\theta) \tag{14.3.1}$$

Note that this describes a parabola opening to the *left*. Taking $OF = 1$, the equation of this parabola is $y^2 = -4x$.

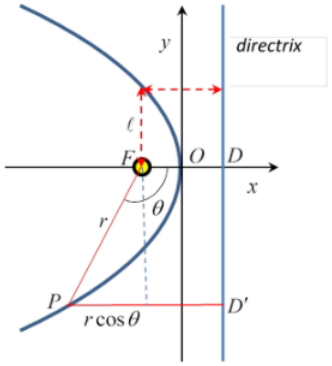

Figure 14.3.1

All parabolas look the same, apart from scaling (maybe just in one direction). The line perpendicular to the axis and the same distance from the curve along the axis as the focus is, but outside the curve, is the parabola's **directrix**. That is, $FO = OD$.

Each point on the curve is the same distance from the focus as it is from the directrix. This can be deduced from the limit of the ellipse property that the sum of distances to the two foci is constant. Let's call the other focus ∞. Then $FP + P\infty = FO + O\infty = D\infty = D'\infty$. So from the diagram, $FP = PD'$

? Exercise 14.3.1

Prove by finding the slope, etc., that any ray of light emitted by a point lamp at the focus will be reflected by a parabolic mirror to go out parallel to the axis.

? Exercise 14.3.1

From the diagram above, show that the equality $FP = PD'$ easily gives the equation for the parabola, both in (r,θ) and in (x,y) coordinates.

14.4: The Hyperbola

Cartesian Coordinates

The hyperbola has eccentricity *[Math Processing Error]*. In Cartesian coordinates, it has equation

[Math Processing Error]

and has two branches, both going to infinity approaching asymptotes *[Math Processing Error]*. The curve intersects the x axis at *[Math Processing Error]* for any point on the curve,

[Math Processing Error]

the sign being opposite for the two branches.

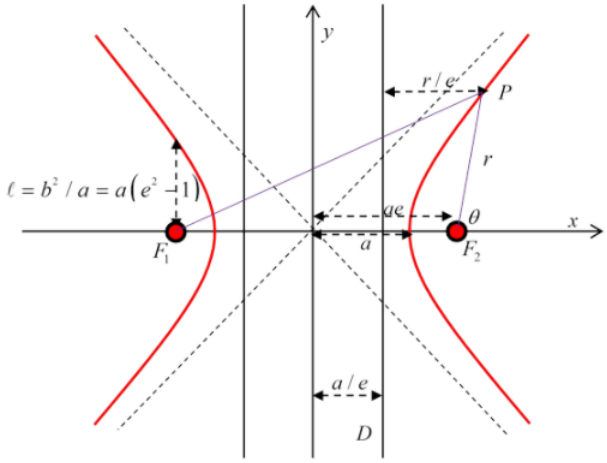

Figure *[Math Processing Error]*

The semi-latus rectum, as for the earlier conics, is the perpendicular distance from a focus to the curve, and is *[Math Processing Error]*. Each focus has an associated directrix, the distance of a point on the curve from the directrix multiplied by the eccentricity gives its distance from the focus.

Polar Coordinates

The *[Math Processing Error]* equation with respect to a focus can be found by substituting *[Math Processing Error]* in the Cartesian equation and solving the quadratic for *[Math Processing Error]*

Notice that *[Math Processing Error]* *has a limited range*: the equation for the right-hand curve with respect to its own focus *[Math Processing Error]* has

[Math Processing Error]

The equation for this curve is

[Math Processing Error]

in the range

[Math Processing Error]

This equation comes up with various signs! The left hand curve, with respect to the left hand focus, would have a positive sign *[Math Processing Error]*. With origin at *[Math Processing Error]* (on the *left*) the equation of the *right*-hand curve is *[Math Processing Error]* finally with the origin at *[Math Processing Error]* the left-hand curve is *[Math Processing Error]*. These last two describe *repulsive* inverse square scattering (Rutherford).

Note: A Useful Result for Rutherford Scattering

If we define the hyperbola by

[Math Processing Error]

then *the perpendicular distance from a focus to an asymptote is just* b.

This equation is the same (including scale) as

[Math Processing Error]

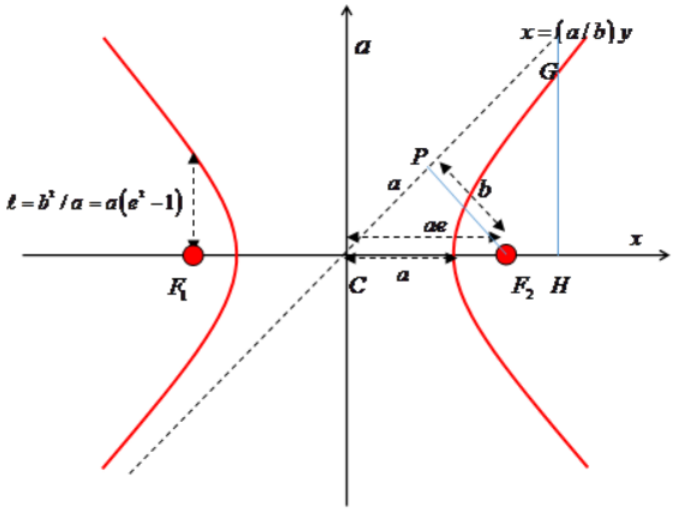

Figure *[Math Processing Error]*

Proof: The triangle *[Math Processing Error]* is similar to triangle *[Math Processing Error]* and since the square of the hypotenuse *[Math Processing Error]*

I find this a surprising result because in analyzing Rutherford scattering (and other scattering) the impact parameter, the distance of the ingoing particle path from a parallel line through the scattering center, is denoted by *[Math Processing Error]*. Surely this can't be a coincidence? But I can't find anywhere that this was the original motivation for the notation.

CHAPTER OVERVIEW

15: Keplerian Orbits

15.1: Preliminary- Polar Equations for Conic Section Curves
15.2: Summary
15.3: Kepler’s Statement of his Three Laws
15.4: Dynamics of Motion in a Central Potential- Deriving Kepler’s Laws
15.5: A Vectorial Approach- Hamilton’s Equation and the Runge Lenz Vector

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15.1: Preliminary- Polar Equations for Conic Section Curves

As we shall find, Newton's equations for particle motion in an inverse-square central force give orbits that are conic section curves. Properties of these curves are fully discussed in the accompanying "Math for Orbits" lecture, here for convenience we give the relevant polar equations for the various possibilities.

For an ellipse, with eccentricity e and semilatus rectum (perpendicular distance from focus to curve) ℓ:

$$\frac{\ell}{r} = 1 + e\cos\theta \tag{15.1.1}$$

Recall the eccentricity e is defined by the distance from the center of the ellipse to the focus being ae, where a is the semi-major axis, and $\ell = a\left(1-e^2\right) = b^2/a$.

For a parabola,

$$\ell = r(1+\cos\theta) \tag{15.1.2}$$

For a hyperbolic orbit with an attractive inverse square force, the polar equation with origin at the center of attraction is

$$\frac{\ell}{r} = 1 - e\cos\theta \tag{15.1.3}$$

where $\theta_{\text{asymptote}} < \theta < 2\pi - \theta_{\text{asymptote}}$ (Of course, the physical path of the planet (say) is only one branch of the hyperbola.)

The (r,θ) origin is at the center of attraction (the Sun), geometrically this is one focus of the hyperbola, and for this attractive case it's the focus "inside" the curve.

For a hyperbolic orbit with a *repulsive* inverse square force (such as Rutherford scattering), the origin is the focus "outside" the curve, and to the right (in the usual representation):

$$\frac{\ell}{r} = -e\cos\theta - 1 \tag{15.1.4}$$

with angular range $-\theta_{\text{asymptote}} < \theta < \theta_{\text{asymptote}}$.

15.2: Summary

We'll begin by stating Kepler's laws, then apply Newton's Second Law to motion in a central force field. Writing the equations vectorially leads easily to the conservation laws for angular momentum and energy.

Next, we use Bernoulli's change of variable $u = 1/r$ to prove that the inverse-square law gives conic section orbits.

A further vectorial investigation of the equations, following Hamilton, leads naturally to an unsuspected *third* conserved quantity, after energy and angular momentum, the Runge Lenz vector.

Finally, we discuss the rather surprising behavior of the momentum vector as a function of time.

15.3: Kepler's Statement of his Three Laws

1. The planets all move in elliptical orbits with the Sun at one focus.

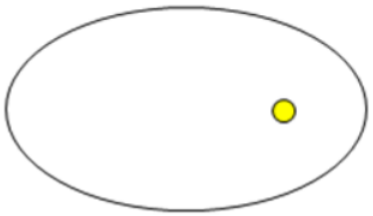

Figure 15.3.1

2. As a planet moves in its orbit, the line from the center of the Sun to the center of the planet sweeps out equal areas in equal times, so if the area SAB(with curved side AB) equals the area SCD, the planet takes the same time to move from A to B as it does from $CtoD$.

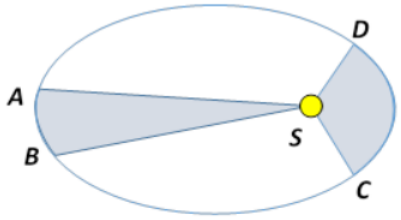

Figure 15.3.1

3. The time it takes a planet to make one complete orbit around the sun T (one planet year) is related to the length of the semimajor axis of the ellipse a:

$$T^2 \propto a^3 \tag{15.3.1}$$

In other words, if a table is made of the length of year T for each planet in the Solar System, and the length of the semimajor axis of the ellipse a, and T^2/a^3 is computed for each planet, the numbers are all the same.

These laws of Kepler's are precise (apart from tiny relativistic corrections, undetectable until centuries later) but they are only *descriptive*—Kepler did not understand why the planets should behave in this way. Newton's great achievement was to prove that all this complicated behavior followed from one simple law of attraction.

15.4: Dynamics of Motion in a Central Potential- Deriving Kepler's Laws

Conserved Quantities

The equation of motion is:

$$m\ddot{\vec{r}} = -f(r)\hat{\vec{r}} \tag{15.4.1}$$

Here we use the hat ^ to denote a unit vector, so $f(r)$ gives the magnitude (and sign) of the force. For Kepler's problem, $f(r) = GMm/r^2$

(Strictly speaking, we should be using the *reduced* mass for planetary motion, for our Solar System, that is a small correction. It can be put in at the end if needed.)

Let's see how using vector methods we can easily find constants of motion: first, angular momentum - just act on the equation of motion with $\vec{r}\times$:

$$m\vec{r}\times\ddot{\vec{r}} = -f(r)\vec{r}\times\hat{\vec{r}} \tag{15.4.2}$$

Since $\vec{r}\times\hat{\vec{r}} = 0$, we have $\vec{r}\times m\ddot{\vec{r}} = 0$, which immediately integrates to

$$\vec{r}\times m\dot{\vec{r}} = \vec{L} \tag{15.4.3}$$

a constant, the angular momentum, and note that $\vec{L}\cdot\vec{r} = 0 = \vec{L}\cdot\dot{\vec{r}}$ so the motion will always stay in a plane, with $\vec{L}$ perpendicular to the plane.

This establishes that motion in a purely *central* force obeys *a conservation law: that of angular momentum.*

(As we've discussed earlier in the course, *conserved quantities in dynamical systems are always related to some underlying symmetry* of the Hamiltonian. The conservation of angular momentum comes from the spherical symmetry of the system: the attraction depends only on distance, not angle. In quantum mechanics, the angular momentum operator is a rotation operator: the three components of the angular momentum vector are conserved, are constants of the motion, because the Hamiltonian is invariant under rotation. That is, the angular momentum operators commute with the Hamiltonian. The classical analogy is that they have zero Poisson brackets with the Hamiltonian.)

To get back to Kepler's statement of his Laws, notice that when the planet moves through an incremental distance $d\vec{r}$ it "sweeps out" an area $\frac{1}{2}\vec{r}\times d\vec{r}$, so the rate of sweeping out area is $dA/dt = \frac{1}{2}|\vec{r}\times\dot{\vec{r}}| = L/2m$. Kepler's Second Law is just conservation of angular momentum!

Second, conservation of energy: this time, we act on the equation of motion with $\dot{\vec{r}}\cdot$:

$$m\dot{\vec{r}}\cdot\ddot{\vec{r}} = -f(r)\hat{\vec{r}}\cdot\dot{\vec{r}} \tag{15.4.4}$$

This immediately integrates to

$$\frac{1}{2}m\dot{\vec{r}}^2 - \int_r^\infty f(r)dr = E \tag{15.4.5}$$

Another conservation law coming from a simple integral: *conservation of energy*. What symmetry does *that* correspond to? The answer is the invariance of the Hamiltonian under *time*: the central force is time invariant, and we're assuming there are time-dependent potential terms, (such as from another star passing close by).

Standard Calculus Derivation of Kepler's First Law

The first mathematical proof that an elliptic orbit about a focus meant an inverse-square attraction was given by Newton, using Euclidean geometry (even though he invented calculus!). The proof is notoriously difficult to follow. Bernoulli found a fairly straightforward calculus proof in polar coordinates by changing the variable to $u = 1/r$.

The first task is to express $\vec{F} = m\vec{a}$ in polar, meaning (r,θ) coordinates

The simplest way to find the expression for acceleration is to parameterize the planar motion as a complex number: position $re^{i\theta}$, velocity $\dot{r}e^{i\theta}+ir\dot{\theta}e^{i\theta}$, notice this means $(\dot{r},r\dot{\theta})$ since the i ensures the $r\dot{\theta}$ term is in the positive θ direction, and differentiating again gives

$$\vec{a}=\ddot{\vec{r}}=\left(\ddot{r}-r\dot{\theta}^2,\vec{r}\ddot{\theta}+2\dot{r}\dot{\theta}\right). \tag{15.4.6}$$

For a central force, the only acceleration is in the r direction, so $r\ddot{\theta}+2\dot{r}\dot{\theta}=0$, which integrates to give

$$mr^2\dot{\theta}=L, \tag{15.4.7}$$

the constancy of angular momentum.

Equating the radial components,

$$\frac{d^2r}{dt^2}-r\dot{\theta}^2=-\frac{GM}{r^2} \tag{15.4.8}$$

This isn't ready to integrate yet, because $\dot{\theta}$ varies too. But since the angular momentum $L=mr^2\dot{\theta}$ is constant, we can eliminate $\dot{\theta}$ from the equation, giving:

$$\begin{aligned}\frac{d^2r}{dt^2}&=-\frac{GM}{r^2}+r\left(\frac{L}{mr^2}\right)^2\\&=-\frac{GM}{r^2}+\frac{L^2}{m^2r^3}\end{aligned} \tag{15.4.9}$$

This doesn't look too promising, but Bernoulli came up with two clever tricks. The first was to change from the variable r to its inverse, $u=1/r$. The other was to use the constancy of angular momentum to change the variable t to θ.

Putting these together:

$$L=mr^2\dot{\theta}=\frac{m}{u^2}\frac{d\theta}{dt} \tag{15.4.10}$$

so

$$\frac{d}{dt}=\frac{Lu^2}{m}\frac{d}{d\theta} \tag{15.4.11}$$

.

Therefore

$$\frac{dr}{dt}=\frac{d}{dt}\left(\frac{1}{u}\right)=-\frac{1}{u^2}\frac{du}{dt}=-\frac{L}{m}\frac{du}{d\theta} \tag{15.4.12}$$

and similarly

$$\frac{d^2r}{dt^2}=-\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}. \tag{15.4.13}$$

Going from r to u in the equation of motion

$$\frac{d^2r}{dt^2}=-\frac{GM}{r^2}+\frac{L^2}{m^2r^3} \tag{15.4.14}$$

we get

$$-\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}=-GMu^2+\frac{L^2u^3}{m^2}, \tag{15.4.15}$$

or

$$\frac{d^2u}{d\theta^2}+u=\frac{GMm^2}{L^2}. \tag{15.4.16}$$

This equation is easy to solve! The solution is

$$u = \frac{1}{r} = \frac{GMm^2}{L^2} + C\cos\theta. \tag{15.4.17}$$

where C is a constant of integration, determined by the initial conditions.

This proves that *Kepler's First Law follows from the inverse-square nature of the force*, because (see beginning of lecture) the equation above is *exactly* the standard (r, θ) equation of an ellipse of semi major axis a and eccentricity e, with the origin at one focus:

$$\frac{a\left(1-e^2\right)}{r} = 1 + e\cos\theta. \tag{15.4.18}$$

Comparing the two equations, we can find the geometry of the ellipse in terms of the angular momentum, the gravitational attraction, and the initial conditions. The angular momentum is

$$L^2 = GMm^2 a\left(1-e^2\right) = GMm^2 b^2/a. \tag{15.4.19}$$

15.5: A Vectorial Approach- Hamilton's Equation and the Runge Lenz Vector

(Mainly following Milne, Vectorial Mechanics, p 235 on.)

Laplace and Hamilton developed a rather different approach to this inverse-square orbit problem, best expressed vectorially, and made a surprising discovery: even though conservation of angular momentum and of energy were enough to determine the motion completely, *for the special case of an inverse-square central force, something else was conserved.* So the system has another symmetry!

Hamilton's approach (actually vectorized by Gibbs) was to apply the operator $\vec{L}\times$ to the equation of motion $m\ddot{\vec{r}} = -f(r)\hat{\vec{r}}$

$$\vec{L}\times m\ddot{\vec{r}} = -f(r)[(\vec{r}\times m\dot{\vec{r}})\times\hat{\vec{r}}] \tag{15.5.1}$$

Now

$$(\vec{r}\times\dot{\vec{r}})\times\hat{\vec{r}} = -\vec{r}(\hat{\vec{r}}\cdot\dot{\vec{r}}) + r\dot{\vec{r}}$$
$$= r^2\left(\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^2}\frac{dr}{dt}\right) = r^2\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = r^2\frac{d\hat{\vec{r}}}{dt}$$

so

$$\vec{L}\times m\ddot{\vec{r}} = -mr^2 f(r)\frac{\hat{\vec{r}}}{dt} \tag{15.5.2}$$

This is known as *Hamilton's equation.*

In fact, it's pretty easy to understand on looking it over: $d\hat{\vec{r}}/dt$ has magnitude $\dot{\theta}$ and direction perpendicular to $\vec{r}$, $m\ddot{\vec{r}} = -f(r)\hat{\vec{r}}, mr^2\dot{\theta} = L$, etc.

It isn't very useful, though—*except* in one case, the inverse-square: $f(r) = k/r^2 \quad$ (so $k = GM.$)

Then it becomes tractable: $\vec{L}\times m\ddot{\vec{r}} = -mr^2 f(r)\frac{d\hat{\vec{r}}}{dt} = -km\frac{d\hat{\vec{r}}}{dt}$ and—surprise—this *integrates immediately* to

$$\vec{L}\times m\dot{\vec{r}} = -km\hat{\vec{r}} - \vec{A} \tag{15.5.3}$$

where $\vec{A}$ is a vector constant of integration, that is to say we find

$$\vec{A} = \vec{p}\times\vec{L} - km\vec{r} \tag{15.5.4}$$

is *constant throughout the motion!*

This is unexpected: we found the usual conserved quantities, energy and angular momentum, and indeed they were sufficient for us to find the orbit. But for the special case of the inverse-square law, something else is conserved. It's called the *Runge Lenz* vector (sometimes Laplace Runge Lenz, and in fact Runge and Lenz don't really deserve the fame—they just rehashed Gibbs' work in a textbook).

From our earlier discussion, this conserved vector must correspond to a symmetry. Finding the orbit gives some insight into what's special about the inverse-square law.

Deriving the Orbital Equation from the Runge-Lenz Vector

The Runge Lenz vector gives a very quick derivation of the elliptic orbit, without Bernoulli's unobvious tricks in the standard derivation presented above.

First, taking the dot product of $\vec{A} = \vec{p}\times\vec{L} - km\hat{\vec{r}}$ with the angular momentum $\vec{L}$, we find $\vec{A}\cdot\vec{L} = km\hat{\vec{r}}\cdot\vec{L} = 0$, meaning that the constant vector $\vec{A}$ lies in the plane of the orbit.

Next take the dot product of $\vec{A}$ with $\vec{r},$ and since $\vec{p}\times\vec{L}\cdot\vec{r} = \vec{L}\cdot\vec{r}\times\vec{p} = L^2$, we find $L^2 = kmr + \vec{A}\cdot\vec{r},$ or

$$\frac{\ell}{r} = 1 + e\cos\theta \tag{15.5.5}$$

where $\ell = L^2/km, e = A/km$ and θ is the angle between the planet's orbital position and the Runge Lenz vector $\vec{A}$.

This is the standard (r, θ) equation for an ellipse, with ℓ the semi-latus rectum (the perpendicular distance from a focus to the ellipse), *e* the eccentricity.

Evidently $\vec{A}$ points along the major axis.

The point is that the direction of the major axis remains the same: the elliptical orbit repeats indefinitely.

If the force law is changed slightly from inverse-square, the orbit *precesses*: the whole elliptical orbit rotates around the central focus, the Runge Lenz vector is no longer a conserved quantity. Strictly speaking, of course, the orbit isn't quite elliptical even for once around in this case. The most famous example, historically, was an extended analysis of the precession of Mercury's orbit, most of which precession arises from gravitational pulls from other planets, but when all this was taken into account, there was left over precession that led to a lengthy search for a planet closer to the Sun (it didn't exist), but the discrepancy was finally, and precisely, accounted for by Einstein's theory of general relativity.

Variation of the Momentum Vector in the Orbit (Hodograph)

It's interesting and instructive to track how the *momentum* vector changes as time progresses, this is easy from the Runge Lenz equation. (Hamilton did this.)

From $\vec{p} \times \vec{L} = km\hat{\vec{r}} + \vec{A}$, we have

$$\vec{L} \times (\vec{p} \times \vec{L}) = km\vec{L} \times \hat{\vec{r}} + \vec{L} \times \vec{A} \tag{15.5.6}$$

That is,

$$\vec{p} = \frac{\vec{L} \times \vec{A}}{L^2} + \frac{km\vec{L}}{L^2} \times \hat{\vec{r}} \tag{15.5.7}$$

Staring at this expression, we see that $\vec{p}$ goes in a circle of radius km/L about a point distance A/L from the momentum plane origin.

Of course, $\vec{p}$ is not moving in this circle at a uniform rate (except for a planet in a circular orbit), its angular progression around its circle matches the angular progression of the planet in its elliptical orbit (because its location on the circle is always perpendicular to the $\hat{\vec{r}}$ direction from the circle center).

An orbit plotted in momentum space is called a *hodograph*.

Orbital Energy as a Function of Orbital Parameters Using Runge-Lenz

We'll prove that the total energy, and the time for a complete orbit, *only depend on the length of the major axis* of the ellipse. So a circular orbit and a very thin one going out to twice the circular radius take the same time, and have the same total energy per unit mass.

Take $\vec{p} \times \vec{L} = \vec{A} + km\hat{\vec{r}}$ and square both sides, giving

$$\begin{aligned} p^2L^2 &= A^2 + k^2m^2 + 2km\vec{A} \cdot \hat{\vec{r}} \\ &= A^2 + k^2m^2 + 2km\left(\frac{L^2 - kmr}{r}\right) \\ &= A^2 - k^2m^2 + 2kmL^2/r \end{aligned} \tag{15.5.8}$$

Dividing both sides by $2mL^2$

$$\frac{p^2}{2m} - \frac{k}{r} = \frac{A^2 - k^2m^2}{2mL^2} \tag{15.5.9}$$

Putting in the values found above, $A = kme, L^2 = km\ell, \ell = a\left(1 - e^2\right)$ we find

$$\frac{p^2}{2m} - \frac{k}{r} = -\frac{k}{2a} \tag{15.5.10}$$

So the total energy, kinetic plus potential, depends only on the length of the major axis of the ellipse.

Now for the time in orbit: we've shown area is swept out at a rate $L/2m$ so one orbit takes time $T = \pi ab/(L/2m)$ and $b = a\sqrt{1-e^2}$, $L = \sqrt{\mathrm{km}\, a\,(1-e^2)}$, so

$$T = 2\pi a^{3/2}\sqrt{m/k} = 2\pi a^{3/2}/\sqrt{GM} \tag{15.5.11}$$

This is Kepler's famous Third Law: $T^2 \propto a^3$, easily proved for circular orbits, not so easy for ellipses.

Important Hint!

Always remember that for Kepler problems with a given massive Sun, both *the time in orbit and the total orbital energy/unit mass* only *depend on the length of the major axis*, they are independent of the length of the minor axis. This can be very useful in solving problems.

The Runge-Lenz Vector in Quantum Mechanics

This is fully discussed in advanced quantum mechanics texts, we just want to mention that, just as spherical symmetry ensures that the total angular momentum and its components commute with the Hamiltonian, and as a consequence there are degenerate energy levels connected by the raising operator, an analogous operator can be constructed for the Runge-Lenz vector, connecting states having the same energy. Furthermore, this raising operator, although it commutes with the Hamiltonian, does *not* commute with the total angular momentum, meaning that states with different total angular momentum can have the same energy. This is the degeneracy in the hydrogen atom energy levels that led to the simple Bohr atom correctly predicting all the energy levels (apart from fine structure, etc.). It's also worth mentioning that these two vectors, angular momentum and Runge-Lenz, both sets of rotation operators in three dimensional spaces, combine to give a complete set of operators in a four dimensional space, and the inverse-square problem can be formulated as the mechanics of a free particle on the surface of a sphere in four-dimensional space.

CHAPTER OVERVIEW

16: Elastic Scattering

16.1: Billiard Balls
16.2: Discovery of the Nucleus
16.3: The Differential Cross Section
16.4: Analyzing Inverse-Square Repulsive Scattering- Kepler Again
16.5: Vectorial Derivation of the Scattering Angle

16.1: Billiard Balls

“Elastic” means no internal energy modes of the scatterer or of the scatteree are excited—so total kinetic energy is conserved. As a simple first exercise, think of two billiard balls colliding. The best way to see it is in the center of mass frame of reference. If they’re equal mass, they come in from opposite directions, scatter, then move off in opposite directions. In the early days of particle accelerators (before colliders) a beam of particles was directed at a stationary target. So, the frame in which one particle is initially at rest is called the lab frame. What happens if we shoot one billiard ball at another which is initially at rest? (We’ll ignore possible internal energies, including spinning.) The answer is that they come off at right angles. This follows trivially from conservation of energy (in an obvious notation)

$$\frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\vec{v}_1^2 + \frac{1}{2}m\vec{v}_2^2 \tag{16.1.1}$$

and momentum

$$m\vec{v} = m\vec{v}_1 + m\vec{v}_2 \tag{16.1.2}$$

and Pythagoras’ theorem.

16.2: Discovery of the Nucleus

The first significant use of scattering to learn about the internal structure of matter was Rutherford's use of α particles directed at gold atoms. This experiment revealed the atomic nucleus for the first time. Our plan here is to analyze this kind of scattering, to understand why it indicated the presence of a nucleus. Similar much later analyses have established that the proton itself has point like constituents, so this is not just of distant historical interest.

For α particles on gold atoms, it's an excellent approximation to take the scatterer as being fixed. This is not an essential requirement, but it simplifies the calculation, and can be corrected for later.

To visualize what's going on, think of the scatterer as a bowling ball with tiny marbles directed towards it, they're moving fast horizontally, along parallel but random paths. (Let's take zero gravity here—the α particles we're modeling are moving at about one-twentieth the speed of light!). We observe the rate at which marbles are being scattered in various directions. Call the **scattering angle** χ.

So, let's assume the width of the "beam" of marbles is much greater than the size of the bowling ball. We'll also take the intensity of the beam to be uniform, with n marbles crossing unit area perpendicular to the beam per second. Now, if the bowling ball has radius R, and we ignore the radius of the tiny marbles, the number of marbles that strike the bowling ball and are scattered is clearly $\pi R^2 n$ per second. Not surprisingly, πR^2 is called the *total cross-section* and usually denoted by σ.

16.3: The Differential Cross Section

In a real scattering experiment, information about the scatterer can be figured out from the different rates of scattering to different angles. Detectors are placed at various angles (θ, ϕ). Of course, a physical detector collects scattered particles over some nonzero solid angle. The usual notation for infinitesimal solid angle is $d\Omega = \sin\theta d\theta d\phi$. The full solid angle (all possible scatterings) is $\int d\Omega = 4\pi$ the area of a sphere of unit radius. (*Note*: Landau uses do for solid angle increment, but $d\Omega$ has become standard.)

The differential cross section, written $d\sigma/d\Omega$ is the fraction of the total number of scattered particles that come out in the solid angle $d\Omega$, so the rate of particle scattering to this detector is $nd\sigma/d\Omega,$ with n the beam intensity as defined above.

Now, we'll assume the potential is spherically symmetric. Imagine a line parallel to the incoming particles going through the center of the atom. For a given ingoing particle, its **impact parameter** is defined as the distance its ingoing line of flight is from this central line. Landau calls this ρ, we'll follow modern usage and call it b.

A particle coming in with impact parameter between b and $b + db$ will be scattered through an angle between χ and $\chi + d\chi$ where we're going to calculate, $\chi(b)$ by solving the equation of motion of a single particle in a repulsive inverse-square force.

Note: we've switched for this occasion from θ to χ for the angle scattered through because we want to save θ for the (r, θ) coordinates describing the complete trajectory, or orbit, of the scattered particle.

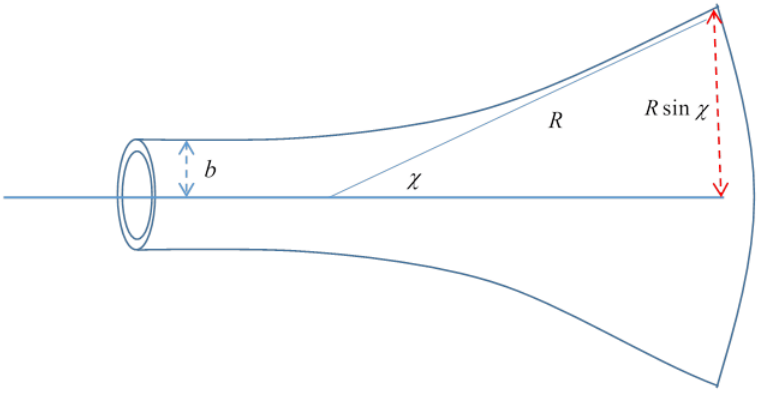

Figure 16.3.1

So, an ingoing cross section $d\sigma = 2\pi b db$ scatters particles into an outgoing spherical area (centered on the scatterer) $2\pi R \sin\chi R d\chi$, that is, a solid angle $d\Omega = 2\pi \sin\chi d\chi$

Therefore the scattering *differential cross section*

$$\frac{d\sigma}{d\Omega} = \frac{b(\chi)}{\sin\chi}\left|\frac{db}{d\chi}\right| \tag{16.3.1}$$

(Note that $d\chi/db$ is clearly negative—increasing b means increasing distance from the scatterer, so a smaller χ)

16.4: Analyzing Inverse-Square Repulsive Scattering- Kepler Again

To make further progress, we must calculate $b(\chi)$, or equivalently $\chi(b)$: what is the angle of scattering, the angle between the outgoing velocity and the ingoing velocity, for a given impact parameter? χ will of course also depend on the strength of the repulsion, and the ingoing particle energy.

Recall our equation for Kepler orbits:

$$\frac{d^2u}{d\theta^2}+u=\frac{GMm^2}{L^2} \tag{16.4.1}$$

Let's now switch from gravitational scattering with an attractive force GMm/r^2 to an electrical repulsive force between two charges Z_1e, Z_2e, force strength $\dfrac{1}{4\pi\varepsilon_0}\dfrac{Z_1Z_2e^2}{r^2}=\dfrac{k}{r^2}$, say. Since this is repulsive, *the sign will change* in the radial acceleration equation,

$$\frac{d^2u}{d\theta^2}+u=-\frac{km}{L^2} \tag{16.4.2}$$

Also, we want the scattering parameterized in terms of the impact parameter b and the incoming speed v_∞, so putting $L=mv_\infty b$ this is

$$\frac{d^2u}{d\theta^2}+u=-\frac{k}{mb^2v_\infty^2} \tag{16.4.3}$$

So just as with the Kepler problem, the orbit is given by

$$\frac{1}{r}=u=-\frac{k}{mb^2v_\infty^2}+C\cos(\theta-\theta_0)=-\kappa+C\cos(\theta-\theta_0),\ \text{say.} \tag{16.4.4}$$

From the lecture on Orbital Mathematics, the polar equation for the left hyperbola branch relative to the external (right) focus is

$$\mathrm{l}/r=-e\cos\theta-1 \tag{16.4.5}$$

this is a branch symmetric about the x-axis:

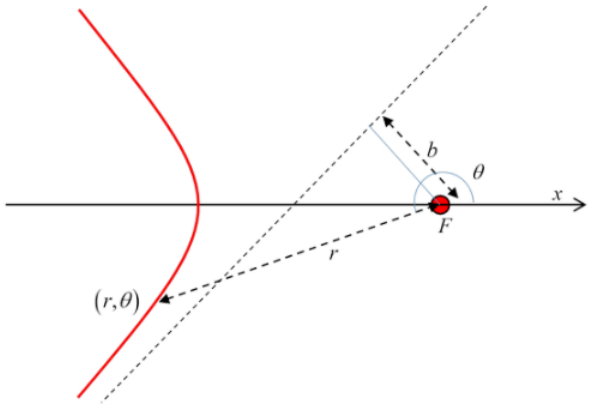

Figure 16.4.1

But we want the incoming branch to be *parallel* to the axis, which we do by suitable choice of θ_0. In other words, we rotate the hyperbola clockwise through half the angle between its asymptotes, keeping the scattering center (right-hand focus) fixed.

From the lecture on orbital mathematics (last page), the perpendicular distance from the focus to the asymptote is the hyperbola parameter b!. Presumably, this is why we use b for the impact parameter. Hence the particle goes in a hyperbolic path with parameters $e/\mathrm{l}=-C,\quad 1/\mathrm{l}=\kappa$. This is not enough information to fix the path uniquely: we've only fed in the angular momentum mbv_∞ not the energy, so this is a family of paths having different impact parameters but the same angular momentum .

We can, however, fix the path uniquely by equating the leading order correction to the incoming zeroth order straight path: the particle is coming in parallel to the x-axis far away to the left, perpendicular distance b from the axis, that is, from the line $\theta=\pi$.

So, going back to that pre-scattering time,

$$u \to 0, \quad \pi - \theta \to b/r = bu \tag{16.4.6}$$

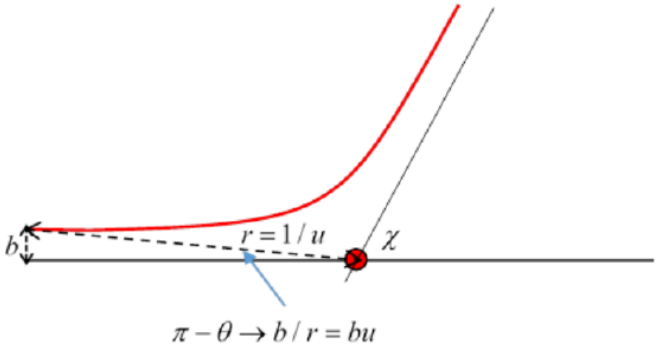

Figure 16.4.1

and in this small u limit,

$$u = C\cos(\pi - bu - \theta_0) - \kappa \cong C\cos(\pi - \theta_0) - \kappa + bCu\sin(\pi - \theta_0) \tag{16.4.7}$$

Matching the zeroth order *and* the first order terms

$$C\cos(\pi - \theta_0) = \kappa, \quad u = bCu\sin(\pi - \theta_0) \tag{16.4.8}$$

eliminates C and fixes the angle θ_0, which is the angle the hyperbola had to be rotated through to align the asymptote with the negative x-axis, and therefore half the angle between the asymptotes, which would be π minus the angle of scattering χ (see the earlier diagram),

$$\tan(\pi - \theta_0) = -\tan\theta_0 = \frac{1}{b\kappa} = \frac{mbv_\infty^2}{k} \tag{16.4.9}$$

$$\chi = \pi - 2\cot^{-1} b\kappa = 2\tan^{-1} b\kappa \tag{16.4.10}$$

So this is the scattering angle in terms of the impact parameter b, that is, in the diagram above

$$\chi(b) = 2\tan^{-1}\left(\frac{k}{mbv_\infty^2}\right) \tag{16.4.11}$$

Equivalently,

$$b = \frac{k}{mv_\infty^2}\cot\frac{\chi}{2}, \text{ so } db = \frac{k}{2mv_\infty^2}\operatorname{cosec}^2\frac{\chi}{2}d\chi \tag{16.4.12}$$

and the incremental cross sectional area

$$d\sigma = 2\pi b db = \pi\left(\frac{k}{mv_\infty^2}\right)^2 \operatorname{cosec}^2\frac{1}{2}\chi\cot\frac{1}{2}\chi d\chi = \pi\left(\frac{k}{mv_\infty^2}\right)^2 \frac{\cos\frac{1}{2}\chi}{\sin^3\frac{1}{2}\chi}d\chi = \left(\frac{k}{2mv_\infty^2}\right)^2 \frac{1}{\sin^4\frac{1}{2}\chi}d\Omega \tag{16.4.13}$$

This is Rutherford's formula: the incremental cross section for scattering into an incremental solid angle, the differential cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{2mv_\infty^2}\right)^2 \frac{1}{\sin^4\frac{1}{2}\chi} \tag{16.4.14}$$

(Recall $k = \dfrac{1}{4\pi\varepsilon_0}Z_1Z_2e^2$ in MKS units.)

16.5: Vectorial Derivation of the Scattering Angle

The essential result of the above analysis was the scattering angle as a function of impact parameter, for a given incoming energy. It's worth noting that this can be found more directly by vectorial methods from Hamilton's equation.

Recall from the last lecture Hamilton's equation

$$\vec{L}\times m\ddot{\vec{r}} = -mr^2 f(r)\frac{d\hat{\vec{r}}}{dt} \tag{16.5.1}$$

and the integral for an inverse square force $f(r) = k/r^2$ (changing the sign of $\vec{A}$ for later convenience)

$$\vec{L}\times m\dot{\vec{r}} = km\hat{\vec{r}} + \vec{A} \tag{16.5.2}$$

As previously discussed, multiplying by $\vec{L}$. establishes that $\vec{A}$ is in the plane of the orbit, and multiplying by $\vec{r}$ gives

$$-L^2 = kmr + Ar\cos\theta \tag{16.5.3}$$

This corresponds to the equation

$$1/r = -e\cos\theta - 1 \tag{16.5.4}$$

(the left-hand branch with the right-hand focus as origin, note from diagram above that $\cos\theta$ is negative throughout) and

$$\frac{L^2}{kmr} = -1 - \frac{A}{km}\cos\theta \tag{16.5.5}$$

To find the scattering angle, suppose the unit vector pointing parallel to the asymptote is $\hat{\vec{r}}_\infty$, so the asymptotic velocity is $v_\infty\hat{\vec{r}}_\infty$.

Note that as before, $\vec{A}$ is along the major axis (to give the correct form for the (r,θ) equation), and $r=\infty$ gives the asymptotic angles from

$$\cos\theta_{r=\infty} = -km/A \tag{16.5.6}$$

We're not rotating the hyperbola as we did in the alternative treatment above: here we keep it symmetric about the x-axis, and find its asymptotic angle to that axis, which is one-half the scattering angle.

Now take Hamilton's equation in the asymptotic limit, where the velocity is parallel to the displacement:

the vector product of Hamilton's equation $\times\hat{\vec{r}}_\infty$ yields

$$\vec{A}\times\hat{\vec{r}}_\infty = \left(\vec{L}\times mv_\infty\hat{\vec{r}}_\infty\right)\times\hat{\vec{r}}_\infty = -\vec{L}(L/b) \tag{16.5.7}$$

It follows that

$$\sin\theta_{r=\infty} = -L^2/Ab \tag{16.5.8}$$

And together with $\cos\theta_{r=\infty} = -km/A$, we find

$$\tan\theta_{r=\infty} = \frac{L^2}{kmb} = \frac{mbv_\infty^2}{k} \tag{16.5.9}$$

This is the angle between the asymptote and the major axis: the scattering angle

$$\chi = \pi - 2\theta_{r=\infty} = 2\left(\frac{\pi}{2} - \theta_{r=\infty}\right) = 2\tan^{-1}\left(\frac{k}{mbv_\infty^2}\right) \tag{16.5.10}$$

agreeing with the previous result.

CHAPTER OVERVIEW

17: Small Oscillations

17.1: Particle in a Well
17.2: Two Coupled Pendulums
17.3: Normal Modes
17.4: Principle of Superposition
17.5: Three Coupled Pendulums
17.6: Normal Coordinates
17.7: Three Equal Pendulums Equally Coupled

17.1: Particle in a Well

We begin with the one-dimensional case of a particle oscillating about a local minimum of the potential energy $V(x)$. We'll assume that near the minimum, call it x_0 the potential is well described by the leading second-order term, $V(x) = \frac{1}{2}V''(x_0)(x-x_0)^2$ so we're taking the zero of potential at x_0, assuming that the second derivative $V''(x_0) \neq 0$, and (for now) neglecting higher order terms.

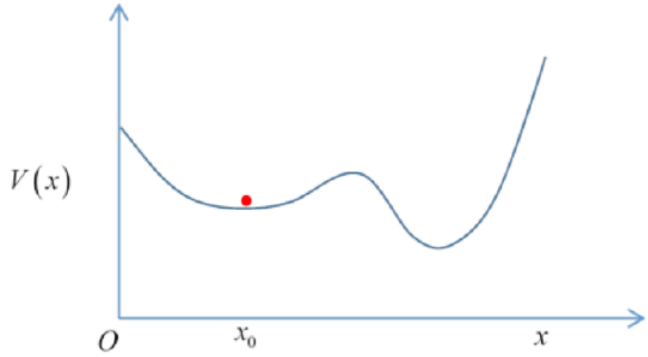

Figure 17.1.1

To simplify the equations, we'll also move the x origin to x_0, so

$$m\ddot{x} = -V''(0)x = -kx \tag{17.1.1}$$

replacing the second derivative with the standard "spring constant" expression.

This equation has solution

$$x = A\cos(\omega t + \delta), \text{ or } x = \mathrm{Re}\left(Be^{i\omega t}\right), \quad B = Ae^{i\delta}, \quad \omega = \sqrt{k/m} \tag{17.1.2}$$

(This can, of course, also be derived from the Lagrangian, easily shown to be $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$.

17.2: Two Coupled Pendulums

We'll take two equal pendulums, coupled by a light spring. We take the spring restoring force to be directly proportional to the angular difference between the pendulums. (This turns out to be a good approximation.)

For small angles of oscillation, we take the Lagrangian to be

$$L = \frac{1}{2}m\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell^2\dot{\theta}_2^2 - \frac{1}{2}mg\ell\theta_1^2 - \frac{1}{2}mg\ell\theta_2^2 - \frac{1}{2}C\left(\theta_1 - \theta_2\right)^2 \tag{17.2.1}$$

Denoting the single pendulum frequency by ω_0, the equations of motion are (writing $\omega_0^2 = g/\ell, k = C/m\ell^2$, so $[k] = T^{-2}$)

$$\begin{aligned} \ddot{\theta}_1 &= -\omega_0^2\theta_1 - k\left(\theta_1 - \theta_2\right) \\ \ddot{\theta}_2 &= -\omega_0^2\theta_2 - k\left(\theta_2 - \theta_1\right) \end{aligned} \tag{17.2.2}$$

We look for a periodic solution, writing

$$\theta_1(t) = A_1 e^{i\omega t}, \quad \theta_2(t) = A_2 e^{i\omega t} \tag{17.2.3}$$

(The final physical angle solutions will be the real part.)

The equations become (in matrix notation):

$$\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \omega_0^2 + k & -k \\ -k & \omega_0^2 + k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{17.2.4}$$

Denoting the 2×2 matrix by $\mathbf{M}$

$$\mathbf{M}\vec{A} = \omega^2\vec{A}, \quad \vec{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{17.2.5}$$

This is an eigenvector equation, with ω^2 the eigenvalue, found by the standard procedure:

$$\det\left(\mathbf{M} - \omega^2\mathbf{I}\right) = \begin{vmatrix} \omega_0^2 + k - \omega^2 & -k \\ -k & \omega_0^2 + k - \omega^2 \end{vmatrix} = 0 \tag{17.2.6}$$

Solving, $\omega^2 = \omega_0^2 + k \pm k$, that is

$$\omega^2 = \omega_0^2, \quad \omega^2 = \omega_0^2 + 2k \tag{17.2.7}$$

The corresponding eigenvectors are (1,1) and (1,−1).

17.3: Normal Modes

The physical motion corresponding to the amplitudes eigenvector (1,1) has two constants of integration (amplitude and phase), often written in terms of a single complex number, that is,

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix} \mathrm{Re}\, Be^{i\omega_0 t} = \begin{pmatrix}A\cos(\omega_0 t+\delta)\\ A\cos(\omega_0 t+\delta)\end{pmatrix}, \quad B = Ae^{i\delta} \tag{17.3.1}$$

with A, δ real.

Clearly, this is the mode in which the two pendulums are in sync, oscillating at their natural frequency, with the spring playing no role.

In physics, this mathematical eigenstate of the matrix is called a *normal mode* of oscillation. In a normal mode, all parts of the system oscillate at a single frequency, given by the eigenvalue.

The other normal mode,

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix} \mathrm{Re}\, Be^{i\omega' t} = \begin{pmatrix}A\cos(\omega' t+\delta)\\ -A\cos(\omega' t+\delta)\end{pmatrix}, \quad B = Ae^{i\delta} \tag{17.3.2}$$

where we have written $\omega' = \sqrt{\omega_0^2 + 2k}$. Here the system is oscillating with the single frequency ω', the pendulums are now exactly *out* of phase, so when they part the spring pulls them back to the center, thereby increasing the system oscillation frequency.

The matrix structure can be clarified by separating out the spring contribution:

$$\mathbf{M} = \begin{pmatrix}\omega_0^2+k & -k\\ -k & \omega_0^2+k\end{pmatrix} = \omega_0^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + k\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \tag{17.3.3}$$

All vectors are eigenvectors of the identity, of course, so the first matrix just contributes ω_0^2 to the eigenvalue. The second matrix is easily found to have eigenvalues are 0,2, and eigenstates (1,1) and (1,−1).

17.4: Principle of Superposition

The equations of motion are *linear* equations, meaning that if you multiply a solution by a constant, that's still a solution, and if you have two different solutions to the equation, the sum of the two is also a solution. This is called the *principle of superposition*.

The general motion of the system is therefore

$$\begin{aligned}\theta_1(t) &= Ae^{i\omega_0 t} + Be^{i\sqrt{\omega_0^2+2k}t} \\ \theta_2(t) &= Ae^{i\omega_0 t} - Be^{i\sqrt{\omega_0^2+2k}t}\end{aligned} \tag{17.4.1}$$

where it is understood that A,B are complex numbers and the physical motion is the real part.

This is a four-parameter solution: the initial positions and velocities can be set arbitrarily, completely determining the motion.

Exercise: begin with one pendulum straight down, the other displaced, both momentarily at rest. Find values for A,B and describe the subsequent motion.

Solution: At $t=0$, the pendulums are at $A \pm B$. Take $A=B=1$. This ensures zero initial velocities too, since the physical parameters are the real parts of the complex solution, and at the initial instant the derivatives are pure imaginary.

The solution for the motion of the first pendulum is

$$\theta_1(t) = \cos\omega_0 t + \cos\left(\omega_0^2+2k\right)^{\frac{1}{2}} t = 2\cos\frac{1}{2}\left(\left(\omega_0^2+2k\right)^{\frac{1}{2}}+\omega_0\right)t\cos\frac{1}{2}\left(\left(\omega_0^2+2k\right)^{\frac{1}{2}}-\omega_0\right)t \tag{17.4.2}$$

and for small k,

$$\theta_1(t) \cong 2\cos\omega_0 t\cos(k/\omega_0)t \tag{17.4.3}$$

Here the pendulum is oscillating at approximately ω_0, but the second term sets the overall oscillation amplitude: it's slowly varying, going to zero periodically (at which point the *other* pendulum has maximum kinetic energy).

To think about: What happens if the two have different masses? Do we still get these beats -- can the larger pendulum transfer all its kinetic energy to the smaller?

Exercise: try pendulums of different lengths, hung so the bobs are at the same level, small oscillation amplitude, same spring as above.

17.5: Three Coupled Pendulums

Let's now move on to the case of three equal mass coupled pendulums, the middle one connected to the other two, but they're not connected to each other.

The Lagrangian is

$$L = \frac{1}{2}m\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell^2\dot{\theta}_2^2 + \frac{1}{2}m\ell^2\dot{\theta}_3^2 - \frac{1}{2}mg\ell\theta_1^2 - \frac{1}{2}mg\ell\theta_2^2 - \frac{1}{2}mg\ell\theta_3^2 - \frac{1}{2}C\left(\theta_1-\theta_2\right)^2 - \frac{1}{2}C\left(\theta_3-\theta_2\right)^2 \quad (17.5.1)$$

Putting $\omega_0^2 = g/\ell, \quad k = C/m\ell^2$

$$L = \frac{1}{2}\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \frac{1}{2}\dot{\theta}_3^2 - \frac{1}{2}\omega_0^2\theta_1^2 - \frac{1}{2}\omega_0^2\theta_2^2 - \frac{1}{2}\omega_0^2\theta_3^2 - \frac{1}{2}k\left(\theta_1-\theta_2\right)^2 - \frac{1}{2}k\left(\theta_3-\theta_2\right)^2$$

The equations of motion are

$$\begin{aligned}
\ddot{\theta}_1 &= -\omega_0^2\theta_1 - k\left(\theta_1-\theta_2\right) \\
\ddot{\theta}_2 &= -\omega_0^2\theta_2 - k\left(\theta_2-\theta_1\right) - k\left(\theta_2-\theta_3\right) \\
\ddot{\theta}_3 &= -\omega_0^2\theta_3 - k\left(\theta_3-\theta_2\right)
\end{aligned} \quad (17.5.2)$$

Putting $\theta_i(t) = A_i e^{i\omega t}$, the equations can be written in matrix form

$$\begin{pmatrix} \omega_0^2+k & -k & 0 \\ -k & \omega_0^2+2k & -k \\ 0 & -k & \omega_0^2+k \end{pmatrix} = \omega_0^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad (17.5.3)$$

The normal modes of oscillation are given by the eigenstates of that second matrix.

The one obvious normal mode is all the pendulums swinging together, at the original frequency ω_0, so the springs stay at the rest length and play no role. For this mode, evidently the second matrix has a zero eigenvalue, and eigenvector (1,1,1).

The full eigenvalue equation is

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0 \quad (17.5.4)$$

that is,

$$(1-\lambda)^2(2-\lambda) - 2(1-\lambda) = 0 = (1-\lambda)[(1-\lambda)(2-\lambda)-2] = (1-\lambda)\left(\lambda^2-3\lambda\right) \quad (17.5.5)$$

so the eigenvalues are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$, with frequencies

$$\omega_1^2 = \omega_0^2, \omega_2^2 = \omega_0^2 + k, \omega_3^2 = \omega_0^2 + 3k \quad (17.5.6)$$

The normal mode eigenvectors satisfy

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0 \quad (17.5.7)$$

They are $(1,1,1)/\sqrt{3}, \quad (1,0,-1)/\sqrt{2}, \quad (1,-2,1)/\sqrt{6}$normalizing them to unity.

The equations of motion are linear, so the general solution is a superposition of the normal modes:

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathrm{Re}\left(C_1 e^{i\omega_1 t}\right) + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathrm{Re}\left(C_2 e^{i\omega_2 t}\right) + \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \mathrm{Re}\left(C_3 e^{i\omega_3 t}\right) \quad (17.5.8)$$

17.6: Normal Coordinates

Landau writes $Q_\alpha = \mathrm{Re}\, C_\alpha e^{i\omega_\alpha t}$. (Actually he brings in an intermediate variable Θ_α, but we'll skip that.) These “normal coordinates” can have any amplitude and phase, but oscillate at a single frequency $\ddot{Q}_\alpha = -\omega_\alpha^2 Q_\alpha$.

The components of the above vector equation read:

$$\begin{aligned} \theta_1 &= Q_1/\sqrt{3} + Q_2/\sqrt{2} + Q_3/\sqrt{6} \\ \theta_2 &= Q_1/\sqrt{3} - 2Q_3/\sqrt{6} \\ \theta_3 &= Q_1/\sqrt{3} - Q_2/\sqrt{2} + Q_3/\sqrt{6} \end{aligned} \tag{17.6.1}$$

It’s worth going through the exercise of writing the Lagrangian in terms of the normal coordinates:

recall the Lagrangian:

$$L = \frac{1}{2}\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \frac{1}{2}\dot{\theta}_3^2 - \frac{1}{2}\omega_0^2\theta_1^2 - \frac{1}{2}\omega_0^2\theta_2^2 - \frac{1}{2}\omega_0^2\theta_3^2 - \frac{1}{2}k(\theta_1-\theta_2)^2 - \frac{1}{2}k(\theta_3-\theta_2)^2 \tag{17.6.2}$$

Putting in the above expressions for the θ_α, after some algebra

$$L = \frac{1}{2}\left[\dot{Q}_1^2 - \omega_0^2 Q_1^2\right] + \frac{1}{2}\left[\dot{Q}_2^2 - \left(\omega_0^2 + k\right) Q_2^2\right] + \frac{1}{2}\left[\dot{Q}_3^2 - \left(\omega_0^2 + 3k\right) Q_3^2\right] \tag{17.6.3}$$

We’ve achieved a separation of variables. The Lagrangian is manifestly a sum of three simple harmonic oscillators, which can have independent amplitudes and phases. Incidentally, this directly leads to the action angle variables -- recall that for a simple harmonic oscillator the action $I = E/\omega$, and the angle is that of rotation in phase space.

17.7: Three Equal Pendulums Equally Coupled

What if we had a third identical spring connecting the two end pendulums (we could have small rods extending down so the spring went below the middle pendulum?

What would the modes of oscillation look like in this case?

Obviously, all three swinging together is still an option, the eigenvector (1,1,1), corresponding to eigenvalue zero of the "interaction matrix" above. But actually we have to extend that matrix to include the new spring -- it's easy to check this gives the equation:

$$\begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0 \tag{17.7.1}$$

The equation for the eigenvalues is easily found to be $\lambda(\lambda-3)^2=0$. Putting $\lambda=3$ into the matrix yields the equation $A_1+A_2+A_3=0$ This is telling us that any vector perpendicular to the all-swinging-together vector (1,1,1) is an eigenvector. This is because the other two eigenvectors have the same eigenvalue, meaning that any linear combination of them also has that eigenvalue—this is a *degeneracy*.

Think about the physical situation. If we set the first two pendulums swinging exactly out of phase, the third pendulum will feel no net force, so will stay at rest. But we could equally choose another pair. And, the eigenvector (1,−2,1) we found before is still an eigenvector: it's in this degenerate subspace, equal to (1,−1,0)−(0,1,−1).

CHAPTER OVERVIEW

18: Driven Oscillator

Michael Fowler (*closely following Landau para 22*)

Consider a one-dimensional simple harmonic oscillator with a variable external force acting, so the equation of motion is

$$\ddot{x}+\omega^2 x = F(t)/m \tag{18.1}$$

which would come from the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + xF(t) \tag{18.2}$$

(Landau "derives" this as the leading order non-constant term in a time-dependent external potential.)

The general solution of the differential equation is $x = x_0 + x_1$, where $x_0 = a\cos(\omega t+\alpha)$, the solution of the homogeneous equation, and x_1 is some particular integral of the inhomogeneous equation.

An important case is that of a periodic driving force $F(t) = f\cos(\gamma t+\beta)$. A trial solution \(x_{1}(t)=b \cos (\gamma t+\beta) \text { yields } b=f / m\left(\omega^{2}-\gamma^{2}\right) \text { so }\]

$$x(t) = a\cos(\omega t+\alpha) + \frac{f}{m\left(\omega^2-\gamma^2\right)}\cos(\gamma t+\beta) \tag{18.3}$$

But what happens when $\gamma = \omega$? To find out, take part of the first solution into the second, that is,

\[x(t)=a^{\prime} \cos \left(\omega t+\alpha^{\prime}\right)+\dfrac{f}{m\left(\omega^{2}-\gamma^{2}\right)}[\cos (\gamma t+\beta)-\cos (\omega t+\beta)]\)

The second term now goes to $0/0$ as $\gamma \to \omega$, so becomes the ratio of its first derivatives with respect to ω (or, equivalently, γ).

$$x(t) = a'\cos(\omega t+\alpha') + \frac{f}{2m\omega}t\sin(\omega t+\beta) \tag{18.4}$$

The amplitude of the oscillations grows linearly with time. Obviously, this small oscillations theory will crash eventually.

But what if the external force frequency is slightly *off* resonance?

Then (real part understood)

$$x = Ae^{i\omega t} + Be^{i(\omega+\varepsilon)t} = \left(A+Be^{i\varepsilon t}\right)e^{i\omega t}, \quad A = ae^{i\alpha}, \quad B = be^{i\beta} \tag{18.5}$$

with a, b, α, β real.

The wave amplitude squared

$$C^2 = \left|A+Be^{i\varepsilon t}\right|^2 = a^2+b^2+2ab\cos(\varepsilon t+\beta-\alpha) \tag{18.6}$$

We're seeing beats, with beat frequency ε. Note that if the oscillator begins at the origin, $x(t=0)=0$, then $A+B=0$ and the amplitude periodically goes to zero, this evidently only occurs when $|A| = |B|$.

Energy is exchanged back and forth with the driving external force.

18.1: More General Energy Exchange

18.2: Damped Driven Oscillator

18.1: More General Energy Exchange

We'll derive a formula for the energy fed into an oscillator by an *arbitrary* time-dependent external force.

The equation of motion can be written

$$\frac{d}{dt}(\dot{x}+i\omega x)-i\omega(\dot{x}+i\omega x)=\frac{1}{m}F(t) \tag{18.1.1}$$

and defining $\xi=\dot{x}+i\omega x$, this is

$$(d\xi/dt-i\omega\xi=F(t)/m \tag{18.1.2}$$

This first-order equation integrates to

$$\xi(t)=e^{i\omega t}\left(\int_0^t\frac{1}{m}F\left(t'\right)e^{-i\omega t'}dt'+\xi_0\right) \tag{18.1.3}$$

The energy of the oscillator is

$$E=\frac{1}{2}m\left(\dot{x}^2+\omega^2x^2\right)=\frac{1}{2}m|\xi|^2 \tag{18.1.4}$$

So if we drive the oscillator over all time, with beginning energy zero,

$$E=\frac{1}{2m}\left|\int_{-\infty}^{\infty}F(t)e^{-i\omega t}dt\right|^2 \tag{18.1.5}$$

This is equivalent to the quantum mechanical time-dependent perturbation theory result: ξ,ξ^* are equivalent to the annihilation and creation operators.

18.2: Damped Driven Oscillator

The linear damped driven oscillator:

$$\ddot{x}+2\lambda\dot{x}+\omega_0^2 x=(f/m)e^{i\Omega t} \tag{18.2.1}$$

(Following Landau's notation here—note it means the actual frictional drag force is $2\lambda m\dot{x}$)

Looking near resonance for steady state solutions at the driving frequency, with amplitude b, phase lag δ, that is,

$x(t)=be^{i(\Omega t+\delta)}$, we find

$$be^{i\delta}\left(-\Omega^2+2i\lambda\Omega+\omega_0^2\right)=(f/m) \tag{18.2.2}$$

For a near-resonant driving frequency $\Omega=\omega_0+\varepsilon$, and assuming the damping to be sufficiently small that we can drop the $\varepsilon\lambda$ term along with ε^2, the leading order terms give

$$be^{i\delta}=-f/2m(\varepsilon-i\lambda)\omega_0 \tag{18.2.3}$$

so the response, the dependence of amplitude of oscillation on frequency, is to this accuracy

$$b=\frac{f}{2m\omega_0\sqrt{(\Omega-\omega_0)^2+\lambda^2}}=\frac{f}{2m\omega_0\sqrt{\varepsilon^2+\lambda^2}} \tag{18.2.4}$$

(We might also note that the resonant frequency is itself lowered by the damping, but this is another second-order effect we ignore here.)

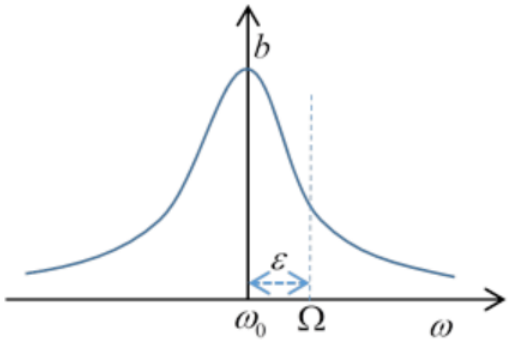

Figure 18.2.1

The rate of absorption of energy equals the frictional loss. The friction force $2\lambda m\dot{x}$ on the mass moving at $\dot{x}$ is doing work at a rate:

$$2\lambda m\dot{x}^2=\lambda mb^2\Omega^2 \tag{18.2.5}$$

The half width of the resonance curve as a function of driving frequency Ω is given by the damping. The total area under the curve is independent of damping.

For future use, we'll write the above equation for the amplitude as

$$b^2\left(\varepsilon^2+\lambda^2\right)=\frac{f^2}{4m^2\omega_0^2} \tag{18.2.6}$$

CHAPTER OVERVIEW

19: One-Dimensional Crystal Dynamics

19.1: The Model
19.2: The Circulant Matrix- Nature of its Eigenstates
19.3: Comparison with Raising Operators
19.4: Finding the Eigenvectors
19.5: Eigenvectors of the Linear Chain
19.6: Allowed Wavenumbers from Boundary Conditions
19.7: Finding the Eigenvalues
19.8: The Discrete Fourier Transform
19.9: A Note on the Physics of These Waves

19.1: The Model

Notation! In this lecture, I use κ for the spring constant (k is a wave number) and Ω for frequency (ω is a root of unity).

A good classical model for a crystal is to represent the atoms by balls held in place by light springs, representing valence bonds, between nearest neighbors. The simplest such crystal that has some realistic features is a single chain of connected identical atoms. To make the math easy, we'll connect the ends of the chain to make it a circle. This is called "imposing periodic boundary conditions". It is common practice in condensed matter theory, and makes little difference to the physics for a large system.

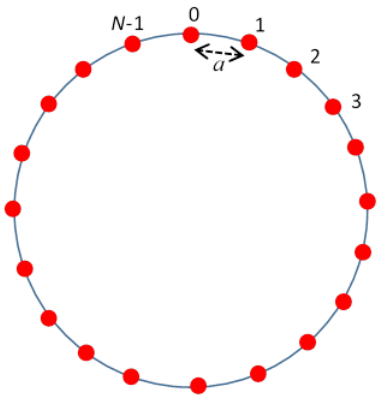

Figure 19.1.1

We'll take the rest positions of the atoms to be uniformly spaced, a apart, with the first atom at a, the n^{th} atom at na, the final N^{th} atom at the origin.

Away from the lowest energy state, we denote the position of the n^{th} atom $na + x_n$, so, as in our earlier discussion of oscillating systems, x_n is the displacement from equilibrium (which we take to be along the line—we are not considering transverse modes of vibration at this time).

The Lagrangian of this circular chain system is:

$$L = \sum_{n=1}^{N} \frac{1}{2} m \dot{x}_n^2 - \sum_{n=1}^{N} \frac{1}{2} \kappa (x_{n+1} - x_n)^2 \quad N+1 \equiv 1 \tag{19.1.1}$$

We're going to call the spring constant κ, we'll need k for something else. We'll also call the frequency Ω

Looking for eigenstates with frequency Ω, we find the set of equations

$$m\ddot{x}_n = -\kappa \left(2x_n - x_{n-1} - x_{n+1}\right) \tag{19.1.2}$$

Taking a solution $x_n = A_n e^{i\Omega t}$, with the understanding that A_n may be complex, and at the end x_n is just the real part of the formal solution, we find the eigenvalue equation for a chain of four atoms (the biggest Mathtype can handle!)

$$-m\Omega^2 \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} -2\kappa & \kappa & 0 & \kappa \\ \kappa & -2\kappa & \kappa & 0 \\ 0 & \kappa & -2\kappa & \kappa \\ \kappa & 0 & \kappa & -2\kappa \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{19.1.3}$$

Actually we'd have a much bigger matrix, with lots of zeroes, but hopefully the pattern is already clear: -2κ or each diagonal element and κ's in two diagonal-slanting lines flanking the main diagonal (corresponding to the links between nearest neighbors) and finally κ's in the two far corners, these coming from the spring joining $Nto1$ to complete the circle.

Notice first that if $\Omega = 0$ there is an eigenvector $(1,1,1,1)^T$, since the sum of the elements in one row is zero (the T means transpose, that is, it's really a column vector, but row vectors are a lot easier to fit into the text here).

This eigenvector is just uniform displacement of the whole system, which costs zero energy since the system isn't anchored to a particular place on the ring. We'll assume, though, that the system as a whole is at rest, meaning the center of mass is stationary, and the atoms have well-defined rest positions as in the picture at $a, 2a, 3a, \ldots, Na, Na \equiv 0$

19.2: The Circulant Matrix- Nature of its Eigenstates

The matrix we've constructed above has a very special property: each row is identical to the preceding row with the elements moved over one place, that is, it has the form

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{pmatrix} \tag{19.2.1}$$

Such matrices are called **circulants**, and their properties are well known. In particular, we'll show that the *eigenvectors* have the form $\left(1, \omega_j, \omega_j^2, \omega_j^3, \ldots, \omega_j^{N-1}\right)^T$ where $\omega_j^N = 1$.

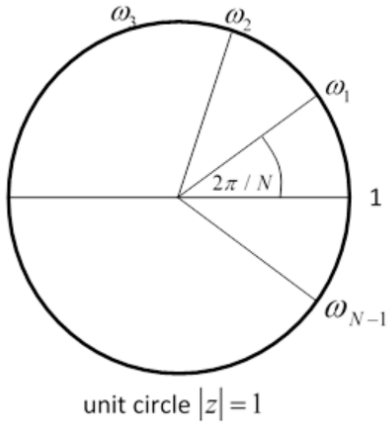

Figure 19.2.1

Recall the roots of the equation $z^N = 1$ are N points equally spaced around the unit circle,

$$e^{2\pi i n/N}, n = 0, 1, 2, \ldots N-1 \tag{19.2.2}$$

The standard mathematical notation is to label these points $1, \omega_1, \omega_2, \omega_3, \ldots, \omega_{N-1}$ as shown in the figure, but notice that $\omega_j = \omega_1^j$

19.3: Comparison with Raising Operators

Actually these matrices are related to the raising and lowering operator for angular momentum (and simple harmonic oscillators) in quantum mechanics. For example, the 4×4 matrices would be for a spin 3/2, with four S_z eigenstates.

The quantum mechanical raising and lowering matrices look like

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{19.3.1}$$

They move the spin S_z component up (and down) by one notch, except that on applying the raising operator, the top state $S_z = 3/2$ is annihilated, similarly the lowering operator on the bottom state.

Our circular generalizations have one extra element:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{19.3.2}$$

This makes the matrices circulants, and gives them a “recycling” property: the top element isn’t thrown away, it just goes to the bottom of the pile.

(And bear in mind that the standard notation for a vector has the lowest index (0 or 1) for the top element, so when we bend the ladder into a circle, the “raising” operator actually moves to the next *lower* number, in other words, it’s a shift to the *left*.)

We’ll take this shift operator P as our basic matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ from which } P^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{19.3.3}$$

It should be evident from this that the circulant matrix having top row c_0, c_1, c_2, c_3 is just the matrix $c_0 I + c_1 P + c_2 P^2 + c_3 P^3$.

This generalizes trivially to $N \times N$ matrices.

19.4: Finding the Eigenvectors

Now let's look at the eigenvectors, we'll start with those of P.Let's call the eigenvalue λ

Then for an eigenstate of the shift operator, the shifted vector must be just a multiple of the original vector:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} A_2 \\ A_3 \\ A_4 \\ A_1 \end{pmatrix} = \lambda \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{19.4.1}$$

Reading off the element by element equivalence of the two vectors,

$$A_2 = \lambda A_1, \quad A_3 = \lambda A_2, \quad A_4 = \lambda A_3, \quad A_1 = \lambda A_4 \tag{19.4.2}$$

The first three equalities tell us the eigenvector has the form $\left(1, \lambda, \lambda^2, \lambda^3\right)^T$, the last tells us that $\lambda^4 = 1$.

From our earlier discussion of circulant matrices, writing the smallest phase nontrivial N^{th} root of unity as $\omega = e^{2\pi i/N}$, *the roots of the equation* $\lambda^N = 1$ *are just this basic root raised to N different powers*: the roots are $1, \omega, \omega^2, \omega^3, \ldots, \omega^{N-1}$

This establishes that the eigenvectors of P have the form

$$\left(1, \omega^j, \left(\omega^j\right)^2, \left(\omega^j\right)^3, \ldots, \left(\omega^j\right)^{N-1}\right)^T \tag{19.4.3}$$

where $j = 0, 1, 2, 3, \ldots, N-1$ with corresponding eigenvalue the basic root raised to the j^{th} power, $\omega^j = e^{2\pi i j/N}$. Try it out for 3×3: the eigenvalues are given by

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0, \quad \lambda^3 = 1, \quad \lambda = 1, \omega, \omega^2; \quad \omega^3 = 1 \tag{19.4.4}$$

The corresponding eigenvectors are found to be

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \tag{19.4.5}$$

For the $N \times N$ case, there are N different, linearly independent, vectors of this form, so this is a complete set of eigenvectors of P.

They are also, of course, eigenvectors of P^2, P^3 all $N-1$ powers of P and therefore of all the circulant matrices! This means that all $N \times N$ circulant matrices commute.

19.5: Eigenvectors of the Linear Chain

Let's get back to our chain, with eigenfunction equation of motion the N dimensional equivalent of

$$-m\Omega^2 \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} -2\kappa & \kappa & 0 & \kappa \\ \kappa & -2\kappa & \kappa & 0 \\ 0 & \kappa & -2\kappa & \kappa \\ \kappa & 0 & \kappa & -2\kappa \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{19.5.1}$$

. We see the matrix is a circulant, so we know the eigenvectors are of the form

$$\left(1, \omega^j, \left(\omega^j\right)^2, \left(\omega^j\right)^3, \ldots, \left(\omega^j\right)^{N-1}\right)^T \tag{19.5.2}$$

, which we'll now write

$$\left(1, e^{2\pi i j/N}, e^{2\pi i 2j/N}, e^{2\pi i 3j/N}, \ldots, e^{2\pi i j(N-1)/N}\right) \tag{19.5.3}$$

What does this mean for our chain system? Remember that the n^{th} element of the eigenvector represents the displacement of the n^{th} atom of the chain from its equilibrium position, that would be proportional to $e^{2\pi i j n/N}$

The steady phase progression on going around the chain $n = 0, 1, 2, 3, \ldots$ makes clear that this is essentially a (longitudinal) wave. (The actual n^{th} particle displacement is the real part of the n^{th} element, but there could be an overall complex factor fixing the phase.)

19.6: Allowed Wavenumbers from Boundary Conditions

The usual way of representing a wave on a line in physics is to have displacement proportional to e^{ikx}, and k is called the *wavenumber*. For our discretized system, the displacement parameter for the n^{th} atom, at position na, would therefore be proportional to e^{ikna}.

But we know this is an eigenvector of a circulant, so we must have $e^{iNka} = 1$, and the allowed values of k are

$$k_n = \frac{2\pi}{Na} n = \frac{2\pi}{L} n \tag{19.6.1}$$

with n an integer.

The circulant structure of the matrix has determined the eigenvectors, but not the eigenvalues ω_n

19.7: Finding the Eigenvalues

The eigenvalues are found by operating on the eigenvector we just found with the matrix, meaning the N dimensional generalization of

$$-m\Omega^2 \begin{pmatrix} 1 \\ e^{ik_n a} \\ e^{ik_n 2a} \\ e^{ik_n 3a} \end{pmatrix} = \begin{pmatrix} -2\kappa & \kappa & 0 & \kappa \\ \kappa & -2\kappa & \kappa & 0 \\ 0 & \kappa & -2\kappa & \kappa \\ \kappa & 0 & \kappa & -2\kappa \end{pmatrix} \begin{pmatrix} 1 \\ e^{ik_n a} \\ e^{ik_n 2a} \\ e^{ik_n 3a} \end{pmatrix} \tag{19.7.1}$$

Applying the matrix to the column vector

$$\left(1, e^{ik_n a}, e^{2ik_n a}, e^{3ik_n a}, \ldots, e^{i(N-1)k_n a}\right)^T \tag{19.7.2}$$

, and cancelling out the common $e^{ik_n na}$ factor, we have

$$-m\Omega_n^2 = \kappa \left(e^{ik_n a} + e^{-ik_n a} - 2\right) \tag{19.7.3}$$

(Of course, this same result comes from every row.)

The complete set of eigenvalues is given by inserting in the above expression

$$k_n = 2\pi n/Na, \quad n = 0, 1, 2, \ldots, N-1 \text{ so } e^{ik_n a} = e^{2\pi i n/N} \tag{19.7.4}$$

so $n = 0$ is displacement of the system as a whole, as is $n = N$.

Wavenumber values k_n beyond $n = N$ repeats the eigenstates we already have, since

$$e^{ik_{N+n}a} = e^{i\dfrac{2\pi(N+n)a}{Na}} = e^{2\pi i} e^{2\pi i n/N} = e^{2\pi i n/N} = e^{ik_n a} \tag{19.7.5}$$

k are restricted to

$$\Omega_n = 2\sqrt{\frac{\kappa}{m}} \sin\left(\frac{k_n a}{2}\right) = 2\sqrt{\frac{\kappa}{m}} \sin\left(\frac{n\pi}{N}\right) \tag{19.7.6}$$

$$0 \le k < 2\pi/a \tag{19.7.7}$$

or equivalently

$$-\pi/a < k \le \pi/a \tag{19.7.8}$$

The eigenvalue equation is

$$\Omega_n^2 = 2(\kappa/m)\left(1 - \cos k_n a\right) \tag{19.7.9}$$

or

$$\Omega_n = 2\sqrt{\frac{\kappa}{m}} \sin\left(\frac{k_n a}{2}\right) = 2\sqrt{\frac{\kappa}{m}} \sin\left(\frac{n\pi}{N}\right) \tag{19.7.10}$$

To see the dynamics of this eigenstate

$$\left(1, e^{ik_n a}, e^{2ik_n a}, e^{3ik_n a}, \ldots, e^{ik_n (N-1)a}\right) \tag{19.7.11}$$

, we need to multiply by the time dependence $e^{i\Omega_n t}$, then finally take the real part of the solution:

$$(\cos\Omega_n t, \quad \cos(k_n a + \Omega_n t), \quad \cos(2k_n a + \Omega_n t), \quad \cos(3k_n a + \Omega_n t), \ldots, \cos((N-1)k_n a + \Omega_n t)) \tag{19.7.12}$$

Notice that in the continuum limit, meaning large N and small a, the atom displacement as a function of position has the form $\cos(kx + \Omega t)$ in other words we're looking at a sinusoidal wave disturbance with wavenumber k_n here.

Now, $-k_n$ is also a solution, but that is the same as $n' = N - n$ so one must be careful not to overcount. The two frequencies $\pm\Omega_n$ correspond to waves going in opposite directions.

19.8: The Discrete Fourier Transform

It's worth looking over this one more time from a slightly different perspective. In finding the energy of an oscillating continuous string, a standard approach is to analyze the motion of the string in terms of an infinite Fourier series of shorter and shorter wavelength oscillations, find the energy in each of these modes, and add to find the total energy. We'll apply the same approach here—but with a difference. Since the waves only have meaning in our chain at a discrete set of uniformly spaced points, the set of waves needed to fully account for all possible motions is finite. In fact, it's the same as the number of points. As we've discussed above, a wave with a higher wavenumber gives an identical set of displacements of the atoms as some lower one. So a complete Fourier analysis of the displacements at these N equally spaces points only needs linear combinations of N waves. This is the *Discrete Fourier Transform (DFT)*.

Writing the complex (amplitude and phase) coefficient of the n^{th} frequency eigenstate X_n, the position of the j^{th} atom in a superposition of such waves (with the standard normalization convention)

$$x_j = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{ik_n ja} = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{i2\pi nj/N} \tag{19.8.1}$$

Given the positions x_j of the atoms, the amplitude coefficients can be found using the inverse mapping:

$$X_n = \sum_{j=0}^{N-1} x_j e^{-i2\pi jn/N} = \sum_{j=0}^{N-1}\frac{1}{N}\sum_{n'=0}^{N-1} X_{n'} e^{i2\pi n'j/N} e^{-i2\pi jn/N} \tag{19.8.2}$$

then using

$$\sum_{j=1}^{N} e^{-i2\pi nj/N} e^{i2\pi n'j/N} = \sum_{j=1}^{N} e^{i2\pi (n'-n)j/N} = N\delta_{n'n} \tag{19.8.3}$$

gives $X_n = X_n$, establishing that we have the correct form for the inverse transformation.

The instantaneous configuration of the system is completely defined by the set $(x_1, x_2, \ldots, x_N)$ and equally by the set $(X_1, X_2, \ldots, X_N)$. All possible particle displacements at the N equally spaced sites can be mapped into N amplitudes of the N distinct waves (eigenvectors).

(This DFT mapping is widely used in the time domain in signal processing: the signal amplitude is sampled, say every millisecond, then the data can be DFT'd to give the wave components down to a minimum frequency around one millisecond. A good quality voice signal would need a shorter time interval, maybe 0.2 milliseconds.)

Now, from

$$x_j = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{i2\pi nj/N} \tag{19.8.4}$$

$$\sum_{j=1}^{N} x_j^* x_j = \frac{1}{N^2}\sum_{j=1}^{N}\sum_{m=0}^{N-1} X_m^* e^{-i2\pi mj/N}\sum_{n=0}^{N-1} X_n e^{i2\pi nj/N} \tag{19.8.5}$$

and again using

$$\sum_{j=1}^{N} e^{-i2\pi mj/N} e^{i2\pi nj/N} = \sum_{j=1}^{N} e^{i2\pi (n-m)j/N} = N\delta_{mn} \tag{19.8.6}$$

we find

$$\sum_{j=1}^{N} x_j^* x_j = \frac{1}{N}\sum_{n=1}^{N} X_n^* X_n \tag{19.8.7}$$

Back to our chain: for a physical configuration of the atoms, all the x_j must be real, so from

$$X_n = \sum_{j=0}^{N-1} x_j e^{-i2\pi jn/N} \tag{19.8.8}$$

we

see that $X_n^* = X_{-n} = X_{N-n}$. (This reduces the number of apparent degrees of freedom in the X representation to the correct N. X_0 is real, $X_1^* = X_{N-1}$, etc., and if there is a middle X_n, it must be real.)

The kinetic energy of the chain particles,

$$\sum_{j=1}^{N} |\dot{x}_j|^2 = \frac{1}{N} \sum_{n=1}^{N} \left|\dot{X}_n\right|^2 \tag{19.8.9}$$

We can find potential energy similarly:

$$\sum_{j=1}^{N} (x_{j+1} - x_j) = \sum_{j=1}^{N} \sum_{n=0}^{N-1} X_n e^{i2\pi nj/N} \left(e^{2\pi in/N} - 1\right) \tag{19.8.10}$$

and using the same routine as before,

$$\sum_{j=1}^{N} (x_{j+1} - x_j)^2 = \sum_{n=0}^{N-1} |X_n|^2 \left|e^{2\pi in/N} - 1\right|^2 = \sum_{n=0}^{N-1} |X_n|^2 \left|e^{ik_n a} - 1\right|^2 \tag{19.8.11}$$

Finally,

$$\left|e^{ik_n a} - 1\right|^2 = 4\sin^2\left(\frac{k_n a}{2}\right) \tag{19.8.12}$$

Putting all this together, the Lagrangian can be written in terms of the transformed variables:

$$L = \frac{1}{N} \sum_{n=1}^{N} \left[\frac{1}{2} m \left|\dot{X}_n\right|^2 - 2\kappa \sin^2\left(\frac{k_n a}{2}\right) |X_n|^2\right] \tag{19.8.13}$$

The equation of motion is then

$$m\ddot{X}_n = -4\kappa \sin^2\left(\frac{k_n a}{2}\right) X_n \tag{19.8.14}$$

with eigenvalues

$$\Omega_n = \sqrt{\frac{\kappa}{m}} \cdot 2\sin\left(\frac{k_n a}{2}\right) = \sqrt{\frac{\kappa}{m}} \cdot 2\sin\left(\frac{n\pi}{N}\right) \tag{19.8.15}$$

This is of course the same result we found earlier, but it is perhaps worth seeing how it comes from the (mathematically equivalent) DFT analysis.

19.9: A Note on the Physics of These Waves

For wavelength long compared to the interparticle spacing, $\Omega \cong vk$ these are like sound waves (and indeed they are what are called **acoustic phonons** in a crystal). As the wavelength shortens, the wave eigenstates are moving more slowly, remember the group velocity of a wavepacket goes as $d\Omega/dk$. This is because there is some Bragg reflection of the waves by the lattice. At $ka = \pi$, we have a standing wave. This is the highest energy mode, with even numbered sites all in sync with each other, and the odd numbered sites all half a cycle behind, so the restoring force experienced by an atom as a function of displacement is the maximum possible.

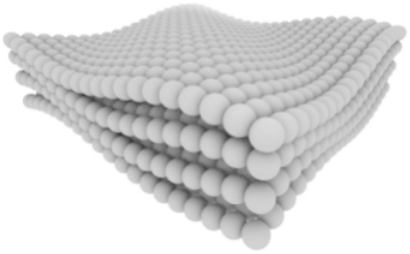

Figure 19.9.1: Collective motions called phonons (Public Domain; Sean Kelley via NIST)

CHAPTER OVERVIEW

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20.1: Introduction to Parametric Resonance

(*Following Landau para 27*)

A one-dimensional simple harmonic oscillator, a mass on a spring,

$$\frac{d}{dt}(m\dot{x}) + kx = 0 \tag{20.1.1}$$

has two parameters, m and k. For some systems, the parameters can be changed externally (an example being the length of a pendulum if at the top end the string goes over a pulley).

We are interested here in the system's response to some externally imposed periodic variation of its parameters, and in particular we'll be looking at *resonant* response, meaning large response to a small imposed variation.

Note first that imposed variation in the mass term is easily dealt with, by simply redefining the time variable to $d\tau = dt/m(t)$ meaning, $\tau = \int \frac{dt}{m(t)}$. Then

$$\frac{d}{dt}\left(m\frac{dx}{dt}\right) = \frac{1}{m}\frac{d}{d\tau}\left(m\frac{1}{m}\frac{dx}{d\tau}\right) = \frac{1}{m}\frac{d^2x}{d\tau^2} \tag{20.1.2}$$

and the equation of motion becomes $\frac{d^2x}{d\tau^2} + m(\tau)kx = 0$

This means we can always transform the equation so all the parametric variation is in the spring constant, so we'll just analyze the equation

$$\frac{d^2x}{dt^2} + \omega^2(t)x = 0 \tag{20.1.3}$$

Furthermore, since we're looking for *resonance* phenomena, we will only consider a small parametric variation at a single frequency, that is, we'll take

$$\omega^2(t) = \omega_0^2(1 + h\cos\Omega t) \tag{20.1.4}$$

where $h \ll 1,$ and h is positive (a trivial requirement—just setting the time origin).

(*Note*: We prefer Ω where Landau uses γ which is often used for a resonance *width* these days.)

We have now a driven oscillator:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = -\omega_0^2 x h\cos\Omega t \tag{20.1.5}$$

How does this differ from our previous analysis of a driven oscillator? In a very important way!

The amplitude x is a factor in the driving force.

For one thing, this means that if the oscillator is initially at rest, it stays that way, in contrast to an ordinary externally driven oscillator. But if the amplitude increases, so does the driving force. This can lead to an *exponential* increase in amplitude, unlike the linear increase we found earlier with an external driver. (Of course, in a real system, friction and nonlinear potential terms will limit the growth.)

What frequencies will prove important in driving the oscillator to large amplitude? It responds best, of course, to its natural frequency ω_0. But if it is in fact already oscillating at that frequency, then the driving force, *including the factor of* x, is proportional to

$$\cos\omega_0 t\cos\Omega t = \frac{1}{2}\cos(\Omega - \omega_0)t + \frac{1}{2}\cos(\Omega + \omega_0)t \tag{20.1.6}$$

with no component at the natural frequency ω_0 for a general Ω

The simplest way to get resonance is to take $\Omega = 2\omega_0$. Can we understand this physically? Yes. Imagine a mass oscillating backwards and forwards on a spring, and the spring force increases just after those points where the mass is furthest away from

equilibrium, so it gets an extra tug inwards twice a cycle. This will feed in energy. (You can drive a swing this way.) In contrast, if you drive at the natural frequency, giving little push inwards just after it begins to swing inwards from one side, then you'll be giving it a little push *outwards* just after it begins to swing back from the other side. Of course, if you push only from one side, like swinging a swing, this works—but it isn't a single frequency force, the next harmonic is doing most of the work.

20.2: Resonance near Double the Natural Frequency

From the above argument, the place to look for resonance is close to $\Omega = 2\omega_0$. Landau takes

$\ddot{x} + \omega_0^2 \left[1 + h\cos(2\omega_0 + \varepsilon)t\right] x = 0$

and, bearing in mind that we're looking for oscillations close to the natural frequency, puts in

$$x = a(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \tag{20.2.1}$$

with $a(t), b(t)$ slowly varying.

It's important to realize that this is an *approximate* approach. It neglects nonresonant frequencies which must be present in small amounts, for example

$$\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\cos(2\omega_0 + \varepsilon)t = \frac{1}{2}\cos 3\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + \frac{1}{2}\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \tag{20.2.2}$$

and the $3\left(\omega_0 + \frac{1}{2}\varepsilon\right)$ term is thrown away.

And, since the assumption is that $a(t),\ \ b(t)$ are slowly varying, their second derivatives are dropped too, leaving just

$$\begin{aligned}\ddot{x} = &-2\dot{a}(t)\omega_0 \sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - a(t)\left(\omega_0^2 + \omega_0\varepsilon\right)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \\ &+2\dot{b}(t)\omega_0 \cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - b(t)\left(\omega_0^2 + \omega_0\varepsilon\right)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\end{aligned} \tag{20.2.3}$$

This must equal

$$-\omega_0^2\left[1 + h\cos(2\omega_0 + \varepsilon)t\right]\left[a(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right] \tag{20.2.4}$$

Keeping only the resonant terms, we take $\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \cdot \cos(2\omega_0 + \varepsilon)t = \frac{1}{2}\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t$ and $\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \cdot \cos(2\omega_0 + \varepsilon)t = -\frac{1}{2}\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t$

so this expression becomes

$$\begin{aligned}&-\omega_0^2\left[1 + h\cos(2\omega_0 + \varepsilon)t\right]\left[a(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right] \\ &= -\omega_0^2\left[a(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + \frac{1}{2}ha(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - \frac{1}{2}hb(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right]\end{aligned} \tag{20.2.5}$$

The equation becomes:

$$\begin{aligned}\ddot{x} = &-2\dot{a}(t)\omega_0 \sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - a(t)\left(\omega_0^2 + \omega_0\varepsilon\right)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \\ &+2\dot{b}(t)\omega_0 \cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - b(t)\left(\omega_0^2 + \omega_0\varepsilon\right)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \\ = &\ \ = \omega_0^2\left[a(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + b(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + \frac{1}{2}ha(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - \frac{1}{2}hb(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right]\end{aligned} \tag{20.2.6}$$

The zeroth-order terms cancel between the two sides, leaving

$$\begin{aligned}&-2\dot{a}(t)\omega_0 \sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - a(t)\omega_0\varepsilon\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t + 2\dot{b}(t)\omega_0\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - b(t)\omega_0\varepsilon\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t \\ &= -\omega_0^2\left[\frac{1}{2}ha(t)\cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t - \frac{1}{2}hb(t)\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t\right]\end{aligned} \tag{20.2.7}$$

Collecting the terms in $\sin\left(\omega_0 + \frac{1}{2}\varepsilon\right)t, \quad \cos\left(\omega_0 + \frac{1}{2}\varepsilon\right)t$

$$-\left(2\dot{a}+b\varepsilon+\frac{1}{2}h\omega_0 b\right)\omega_0\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t+\left(2\dot{b}(t)-a\varepsilon+\frac{1}{2}h\omega_0 a\right)\omega_0\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t=0 \tag{20.2.8}$$

The sine and cosine can't cancel each other, so the two coefficients must both be identically zero. This gives two first order differential equations for the functions $a(t),\ b(t)$, and we look for exponentially increasing functions, proportional to $a(t)=ae^{st},\quad b(t)=be^{st}$, which will be solutions provided

$$\begin{aligned} sa+\frac{1}{2}\left(\varepsilon+\frac{1}{2}h\omega_0\right)b&=0\\ \frac{1}{2}\left(\varepsilon-\frac{1}{2}h\omega_0\right)a-sb&=0\end{aligned} \tag{20.2.9}$$

The amplitude growth rate is therefore

$$s^2=\frac{1}{4}\left[\left(\frac{1}{2}h\omega_0\right)^2-\varepsilon^2\right] \tag{20.2.10}$$

Parametric resonance will take place if s is real, that is, if

$$-\frac{1}{2}h\omega_0<\varepsilon<\frac{1}{2}h\omega_0 \tag{20.2.11}$$

a band of width $h\omega_0$ about $2\omega_0$

20.3: Example- Pendulum Driven at near Double the Natural Frequency

A simple pendulum of length ℓ, mass m is attached to a point which oscillates vertically $y = a\cos\Omega t$. Measuring y downwards, the pendulum position is

$$x = \ell\sin\phi, y = a\cos\Omega t + \ell\cos\phi \tag{20.3.1}$$

The Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mg\ell\cos\phi \\ &= \frac{1}{2}m\left(\ell^2\cos^2\phi\right)\dot{\phi}^2 + \frac{1}{2}m(a\Omega\sin\Omega t + \ell\dot{\phi}\sin\phi)^2 + mg\ell\cos\phi \\ &= \frac{1}{2}m\ell^2\dot{\phi}^2 - ma\ell\Omega\sin\Omega t\frac{d}{dt}\cos\phi + a^2\Omega^2\sin^2\Omega t + mg\ell\cos\phi \end{aligned} \tag{20.3.2}$$

The purely time-dependent term will not affect the equations of motion, so we drop it, and since the equations are not affected by adding a total derivative to the Lagrangian, we can integrate the second term by parts (meaning we're dropping a term $\frac{d}{dt}\left(\text{mal}\,\Omega\sin\Omega t\cos\phi\right)\Big)$ to get

$$L = \frac{1}{2}m\ell^2\dot{\phi}^2 + ma\ell\Omega^2\cos\Omega t\cos\phi + mg\ell\cos\phi \tag{20.3.3}$$

(We've also dropped the term $mga\cos\Omega t$ from the potential energy term—it has no ϕ or $\dot{\phi}$ dependence, so will not affect the equations of motion.)

The equation for small oscillations is

$$\ddot{\phi} + \omega_0^2\left[1 + (4a/\ell)\cos(2\omega_0 + \varepsilon)t\right]\phi = 0, \quad \omega_0^2 = g/\ell \tag{20.3.4}$$

Comparing this with

$$\ddot{x} + \omega_0^2\left[1 + h\cos(2\omega_0 + \varepsilon)t\right]x = 0 \tag{20.3.5}$$

we see that $4a/\ell \equiv h$, so the parametric resonance range around $2\omega_0 = 2\sqrt{g/\ell}$ is of width $\frac{1}{2}h\omega_0 = 2a\sqrt{g/\ell^3}$.

CHAPTER OVERVIEW

21: The Ponderomotive Force

21.1: Introduction to the Ponderomotive Force
21.2: Finding the Effective Potential Generated by the Oscillating Force
21.3: Stability of a Pendulum with a Rapidly Oscillating Vertical Driving Force
21.4: Hand-Waving Explanation of the Ponderomotive Force
21.5: Pendulum with Top Point Oscillating Rapidly in a Horizontal Direction

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21.1: Introduction to the Ponderomotive Force

Imagine first a particle of mass m moving along a line in a smoothly varying potential $V(x)$, so $m\ddot{x} = -\nabla V(x)$. Now add in a rapidly oscillating force, not necessarily small, acting on the particle:

$$f = f_1 \cos\omega t + f_2 \sin\omega t \tag{21.1.1}$$

where f_1, f_2 are in general functions of position. This force is oscillating much more rapidly than any oscillation of the particle in the original potential, and we'll assume that the position of the particle as a function of time can be written as a sum of a "slow motion" $X(t)$ and a rapid oscillation $\xi(t)$.

$$x(t) = X(t) + \xi(t) \tag{21.1.2}$$

We'll also assume that the amplitude of the oscillations, determined by the strength of the force and the frequency, is small compared with distances over which the original fixed potential and the coefficients f_1, f_2 vary substantially.

You might be thinking at this point, well, isn't $X(t)$ just the path the particle would describe in $V(x)$ alone, and the force f just jiggles it about that path? Surprisingly, the answer is no. For example, a rigid pendulum confined to rotation in a vertical plane, but with its point of support driven in fairly small amplitude rapid up-and-down oscillations from the outside, can be stable pointing *upwards*. For motion on the slow timescale associated with the original potential, the rapidly oscillating imposed force is equivalent to an effective potential.

This turns out to have important practical consequences. For a charged particle in a rapidly oscillating electric field, the effective potential from the oscillation is proportional to $e^2\overline{E^2}$, generating a force driving the particle towards regions of weaker field. It is termed the *ponderomotive force*.

For plasma physicists, the ponderomotive force has one very important property—*it drives the positive and negative particles in the same direction*, and so gives a different tool from the usual electric and magnetic fields for containing a plasma.

In the analysis below, following Landau, we have a fixed potential and a fast oscillating field superimposed. However, we could just have a non-uniform fast oscillating field, with an equation of motion $\ddot{x} = g(x)\cos\omega t$, and still write the particle path as a sum of slow moving and jiggling components, $x(t) = X(t) + \xi(t)$. Fast oscillating electric fields (crossed laser beams) are used to trap ultracold ions and atoms, using the ponderomotive force. It has been suggested that atoms trapped in this way could be part of a quantum computer (Turker, arXiv: 1308.0573v1).

21.2: Finding the Effective Potential Generated by the Oscillating Force

As stated above, our system is a particle of mass m moving in one dimension in a time-independent potential $V(x)$ and subject to a rapidly oscillating force $f = f_1 \cos\omega t + f_2 \sin\omega t$.

The oscillation's strength and frequency are such that the particle only moves a small distance in $V(x)$ during one cycle, and the oscillation is much faster than any oscillation possible in the potential alone.

The equation of motion is

$$m\ddot{x} = -dV/dx + f \tag{21.2.1}$$

The particle will follow a path

$$x(t) = X(t) + \xi(t) \tag{21.2.2}$$

where $\xi(t)$ describes rapid oscillations about a smooth path $X(t)$, and the average value $\overline{\xi(t)}$ of $\xi(t)$ over a period $2\pi/\omega$ is zero.

Expanding to first order in ξ,

$$m\ddot{X} + m\ddot{\xi} = -\frac{dV}{dx} - \xi\frac{d^2V}{dx^2} + f(X,t) + \xi\frac{\partial f}{\partial X} \tag{21.2.3}$$

This equation has smooth terms and rapidly oscillating terms on both sides, and we can equate them separately. The leading oscillating terms are

$$m\ddot{\xi} = f(X,t) \tag{21.2.4}$$

We've dropped the terms on the right of order ξ, but kept $\ddot{\xi}$, because $\ddot{\xi} \sim \omega^2\xi \gg \xi$.

So to leading order in the rapid oscillation,

$$\xi = -f/m\omega^2 \tag{21.2.5}$$

Now, averaging the full equation of motion with respect to time (smoothing out the jiggle, matching the slow-moving terms), the $m\ddot{\xi}$ on the left and the $f(X,t)$ on the right both disappear (but cancel each other anyway), the $\xi d^2V/dx^2$ term averages to zero on the assumption that the variation of d^2V/dx^2 over a cycle of the fast oscillation is negligible, but we cannot drop the average

$$\overline{\xi\frac{\partial f}{\partial X}} = -\frac{1}{m\omega^2}\overline{f\frac{\partial f}{\partial X}} = -\frac{1}{m\omega^2}\nabla_X\overline{f^2} \tag{21.2.6}$$

Incorporating this nonzero term, we have an equation of "slow motion"

$$m\ddot{X} = -dV_{\text{eff}}/dX \tag{21.2.7}$$

where, using $|\dot{\xi}| = |f|/m\omega$,

$$V_{\text{eff}} = V + \overline{f^2}/2m\omega^2 = V + \frac{1}{2}m\overline{\dot{\xi}^2} \tag{21.2.8}$$

The effective potential is the original plus a term proportional to the kinetic energy of the oscillation.

21.3: Stability of a Pendulum with a Rapidly Oscillating Vertical Driving Force

Recall now the Lagrangian for the simple (rigid)pendulum of length ℓ, mass m, angle from vertically down ϕ, constrained to move in a vertical plane, point of support driven to oscillate vertically with amplitude α and frequency Ω (from the section on parametric resonance),

$$L = \frac{1}{2} m\ell^2 \dot{\phi}^2 + ma\ell\Omega^2 \cos\Omega t \cos\phi + mg\ell\cos\phi \tag{21.3.1}$$

Our previous analysis of this system was for driving frequencies near double the natural frequency. Now we'll investigate the behavior for driving frequencies *far more rapid* than the natural frequency.

The equation of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \tag{21.3.2}$$

is

$$m\ell^2\ddot{\phi} = -ma\ell\Omega^2\cos\Omega t\sin\phi - mg\ell\sin\phi \tag{21.3.3}$$

so evidently the external driving force is $f = -ma\Omega^2\cos\Omega t\sin\phi$

(Landau has a misprint—an extra ℓ in this, p 95) and, from the previous section, (except that for the pendulum we are using Ω, not ω, for the external driving frequency)

$$V_{\text{eff}} = V + \overline{f^2}/2m\Omega^2 = mg\ell\left[-\cos\phi + \left(a^2\Omega^2/4g\ell\right)\sin^2\phi\right] \tag{21.3.4}$$

For $\phi = \pi + \varepsilon$, $\quad \varepsilon$ small,

$$V_{\text{eff}}(\varepsilon) \cong mg\ell\left[1 - \frac{1}{2}\varepsilon^2 + \left(a^2\Omega^2/4g\ell\right)\varepsilon^2\right] \tag{21.3.5}$$

and for $a^2\Omega^2 > 2g\ell$

the *upward* position is stable!

At first glance, this may seem surprising: the extra term in the potential from the oscillations is like a kinetic energy term for the oscillating movement. Surely the pendulum is oscillating more in the vertically up position than when it's to one side? So why isn't *that* a maximum of the added effective potential? The point is that the relevant variable is not the pendulum's height above some fixed point, the variable is ϕ —and the rapid oscillations of ϕ are minimum (zero) in the vertically up position.

21.4: Hand-Waving Explanation of the Ponderomotive Force

Let's look again at the vertically stable pendulum—the quiver force has sufficient frequency that although the quivering motion is of small amplitude, it drives the pendulum to the vertical position. To see what's going on, we'll replace the oscillating force with a series of discrete impulses of alternating sign. Remember, the impulse on the pendulum will be in a vertical direction, but the pendulum is constrained to move along the circular arc. Therefore, the impulse it feels is the component along this path. If it is away from the vertical, the greater its deviation the greater the effective impulse, so as it quivers back and forth it feels greater drive pushing it back up towards the vertical, since it feels that impulse when it's further down. If it does feel a downward impulse at its low point, that will set it up for a greater upward impulse as it goes down.

This can also be understood for a charged particle in an oscillating electromagnetic field in terms of radiation pressure. Where the oscillating field is more intense, there is more radiation pressure, so the particle will be driven by the pressure imbalance towards the regions where the field is weakest.

21.5: Pendulum with Top Point Oscillating Rapidly in a Horizontal Direction

Take the coordinates of m to be

$$x = a\cos\Omega t + \ell\sin\phi, y = \ell\cos\phi \tag{21.5.1}$$

The Lagrangian, omitting the term depending only on time, and performing an integration by parts and dropping the total derivative term, (following the details of the analysis above for the vertically driven pendulum) is

$$L = \frac{1}{2}m\ell^2\dot{\phi}^2 + ma\ell\Omega^2\cos\Omega t\sin\phi + mg\ell\cos\phi \tag{21.5.2}$$

It follows that $f = m\ell a\Omega^2\cos\Omega t\cos\phi$ (the only difference in f from the *vertically* driven point of support is the final $\cos\phi$ instead of $\sin\phi$)) and

$$V_{\text{eff}} = mg\ell\left[-\cos\phi + \overline{f^2}/2m\omega^2\right] = mg\ell\left[-\cos\phi + \left(a^2\Omega^2/4g\ell\right)\cos^2\phi\right] \tag{21.5.3}$$

If $a^2\Omega^2 < 2g\ell, \quad \phi = 0$ is stable. If $a^2\Omega^2 > 2g\ell$ the stable position is $\cos\phi = 2g\ell/a^2\Omega^2$

That is, at high frequency, the rest position is *at an angle* to the vertical!

In this case, the ponderomotive force towards the direction of least angular quiver (in this case the *horizontal* direction) is balanced by the gravitational force.

CHAPTER OVERVIEW

22: Resonant Nonlinear Oscillations

22.1: Introduction to Resonant Nonlinear Oscillations
22.2: Frequency of Oscillation of a Particle is a Slightly Anharmonic Potential
22.3: Resonance in a Damped Driven Linear Oscillator- A Brief Review
22.4: Damped Driven Nonlinear Oscillator- Qualitative Discussion
22.5: Nonlinear Case - Landau's Analysis
22.6: Frequency Multiples

22.1: Introduction to Resonant Nonlinear Oscillations

Landau' next sections (Chapter 6, sections 28,29) address nonlinear one-dimensional systems. In particular, he focusses on driven damped oscillators with nonlinear, but small, added potential terms. Using ingenious semiquantitative techniques, he predicts some unexpected results: for example, a *discontinuity* in the oscillation amplitude on slowly varying the driving frequency at constant driving force (and constant damping). He also finds resonances when the driving frequency is a fraction, for example a third, of the oscillator's natural frequency.

Fortunately, this system is easy to analyze numerically, and we have an applet to do just that. The parameters are set by sliders, and one can immediately find the large discontinuity in amplitude (factor of two or so) as the frequency is slightly changed. At the end of this lecture, we show simple plots of amplitude response to a constant driving force as the frequency is varied. These were found using the applet, the reader can easily check them, and venture into parts of the parameter space. The applet provides a measure of Landau's (semiquantitative) accuracy, of course surprisingly good (of order 20% error or less) given the nature of the problem.

It should be added that this is one area where, thanks to computers, major advances have been made since Landau wrote the book, in particular the discovery for some systems of period doubling and chaos as the driving force is increased. We've added a lecture (22a) on a particular system, the driven damped pendulum, a natural extension of Landau's oscillator. This illustrates some of the novel features. We will follow part of chapter 12 of Taylor's excellent text, *Classical Mechanics*. Taylor provides many computer-generated graphs of the pendulum's response as parameters are varied. We provide applets that can generate these graphs. The reader can easily use these applets to explore other parameter inputs.

In this lecture, to gain a bit of intuition about these nonlinear potentials, we'll begin (following Landau) with no driving and no damping: just a particle oscillating in a potential that's simple harmonic plus small and (positive) terms. The basic questions are, how do these terms change the frequency of oscillation, and how does that frequency depend on the amplitude of oscillation? The answers will guide us in understanding how a particle in such a potential will respond to a harmonic driving term, plus damping.

Next, we briefly review the driven damped *linear* oscillator (covered in detail in lecture 18, this is really just a reminder of the notation). Then we add small cubic and quartic terms. We present Landau's argument that--above a certain driving force--gradually increasing the driving frequency leads at a critical value to a *discontinuous drop* in the amplitude of the response, then use an applet to confirm and quantify his result.

22.2: Frequency of Oscillation of a Particle is a Slightly Anharmonic Potential

See the applet illustrating this section.

Landau (para 28) considers a simple harmonic oscillator with added small potential energy terms $\frac{1}{3}m\alpha x^3+\frac{1}{4}m\beta x^4$. In leading orders, these terms contribute separately, and differently, so it's easier to treat them one at a time. We'll first consider the quartic term, an equation of motion

$$\ddot{x}+\omega_0^2 x=-\beta x^3 \tag{22.2.1}$$

(We'll always take β positive)

Writing a perturbation theory expansion (following Landau):

$$x=x^{(1)}+x^{(2)}+\cdots \tag{22.2.2}$$

(Standard practice in most books would be to write $x=x^{(0)}+x^{(1)}+\ldots$ with the superscript indicating the order of the perturbation--we're following Landau's notation, hopefully reducing confusion…) We take as the leading term

$$x^{(1)}=a\cos\omega t \tag{22.2.3}$$

with the exact value of $\omega, \omega=\omega_0+\Delta\omega$. Of course, we don't know the value of ω yet—this is what we're trying to find!

And, as Landau points out, you can't just write $\cos(\omega_0+\Delta\omega)t=\cos\omega_0 t-(\Delta\omega)t\omega_0\sin\omega_0 t$ because that implies motion increasing in time, and our system is a particle oscillating in a fixed potential, with no energy supply. Furthermore, even if we did somehow have the value of ω exactly right, this expression would not be a full solution to the equation: the motion is certainly periodic with period $2\pi/\omega$, but the complete description of the motion is a Fourier series including frequencies nω,n an integer, since the potential is no longer simple harmonic.

Anyway, putting this correct frequency into the equation of motion $\ddot{x}+\omega_0^2 x=-\beta x^3$ gives a nonzero left-hand side, so we rearrange. We subtract $\left(1-\left(\omega_0^2/\omega^2\right)\right)\ddot{x}$ from both sides to get:

$$\frac{\omega_0^2}{\omega^2}\ddot{x}+\omega_0^2 x=-\beta x^3-\left(1-\frac{\omega_0^2}{\omega^2}\right)\ddot{x} \tag{22.2.4}$$

Now putting the leading term $x^{(1)}=a\cos\omega t$ into the left-hand side does give zero: if the equation had zero on the right hand side, this would just be a free (undamped) oscillator with natural frequency ω not ω_o. This doesn't look very promising, but keep reading.

The equation for the first-order correction $x^{(2)}$ is:

$$\frac{\omega_0^2}{\omega^2}\ddot{x}^{(2)}+\omega_0^2 x^{(2)}=-\beta\left(x^{(1)}\right)^3-\left(1-\frac{\omega_0^2}{\omega^2}\right)\ddot{x}^{(1)} \tag{22.2.5}$$

Notice that the second term on the right-hand side includes $\ddot{x}^{(1)}=-\omega^2 a\cos\omega t$ .This equation now represents a driving force on an undamped oscillator exactly at its resonant frequency, so would cause the amplitude to increase linearly, obviously an unphysical result, since we're just modeling a particle sliding back and forth in a potential, with no energy being supplied from outside!

The key is that there is also a resonant driver in that first term $-\beta\left(x^{(1)}\right)^3$.

Clearly these two driving terms have to cancel, and this requirement nails $\Delta\omega$: here's how:

$$-\beta\left(x^{(1)}\right)^3=-\beta a^3\cos^3\omega t=-\beta a^3\left(\frac{3}{4}\cos\omega t+\frac{1}{4}\cos 3\omega t\right) \tag{22.2.6}$$

so the resonant driving terms cancel provided

$$-\beta a^3\frac{3}{4}\cos\omega t-\left(1-\frac{\omega_0^2}{\omega^2}\right)\left(-\omega^2 a\cos\omega t\right)=0 \tag{22.2.7}$$

Remembering $\omega=\omega_0+\Delta\omega$, this gives (to this order)

$$\Delta\omega = \frac{3\beta a^2}{8\omega_0} \tag{22.2.8}$$

(Strictly, $\omega_0 + \frac{1}{2}\Delta\omega$ in the denominator, but that's a higher order correction.)

Note that the frequency increases with amplitude: the x^4 potential gives an increasingly stronger restoring force with amplitude than the harmonic well. You can check this with the applet. Now let's consider the equation for a small cubic perturbation,

$$\ddot{x} + \omega_0^2 x = -\alpha x^2 \tag{22.2.9}$$

This represents an added potential $-\frac{1}{3}\alpha x^3$, which is an odd function, so to leading order it won't change the period, speeding up one half of the oscillation and slowing the other half the same amount in leading order. The first correction to the position as a function of time is the solution of

$$\ddot{x}^{(2)} + \omega_0^2 x^{(2)} = -\alpha a^2 \cos^2 \omega t = -\frac{1}{2}\alpha a^2 - \frac{1}{2}\alpha a^2 \cos 2\omega t \tag{22.2.10}$$

The solution is

$$x^{(2)} = -\frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos 2\omega t \tag{22.2.11}$$

Physically, adding this to the leading term, the particle is spending more of its time in the softer half of the potential, giving an amplitude-dependent correction to its average position.

To get the correction to the frequency, we need to go to the next order,

$$\omega = \omega_0 + \omega^{(2)} \tag{22.2.12}$$

. Dropping terms of higher order, the equation of motion for the next correction is

$$\ddot{x}^{(3)} + \omega_0^2 x^{(3)} = -2\alpha x^{(1)} x^{(2)} + 2\omega_0 \omega^{(2)} x^{(1)} \tag{22.2.13}$$

and with

$$x = x^{(1)} + x^{(2)} + x^{(3)}, \omega = \omega_0 + \omega^{(2)} \tag{22.2.14}$$

, following Landau,

$$\ddot{x}^{(3)} + \omega_0^2 x^{(3)} = -\frac{\alpha^2 a^3}{6\omega_0^2}\cos 3\omega t + a\left[2\omega_0\omega^{(2)} + \frac{5a^2\alpha^2}{6\omega_0^2}\right]\cos\omega t \tag{22.2.15}$$

Again, there cannot be a nonzero term driving the system at resonance, so the quantity in the square brackets must be zero, this gives us $\Delta\omega = \omega^{(2)} = -5a^2\alpha^2/12\omega_0^3$

The total correction to frequency to leading order for the additional small potentials

$$\frac{1}{3}m\alpha x^3 + \frac{1}{4}m\beta x^4 \tag{22.2.16}$$

is therefore (they add independently to this order)

$$\Delta\omega = \left(\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3}\right)a^2 = \kappa a^2 \tag{22.2.17}$$

(the κ here being a convenient notation Landau employs later).

How Good Are These Approximations?

We have an applet that solves this equation numerically, so it is straightforward to check.

Beginning with the quartic perturbation potential $\frac{1}{4}m\beta x^4$, Landau finds a frequency correction $\Delta\omega = 3\beta a^2/8\omega_0$. Taking a rather large perturbation $a = \beta = \omega_0 = 1$ we find from the applet that $\Delta\omega = 0.33$ whereas Landau's perturbation theory predicts $\Delta\omega = \frac{3}{8} = 0.375$. However, if we correct Landau's denominator (as mentioned above, he pointed out it should be ω, but said that was secondorder) the error is very small.

Taking $\alpha = 0.3$, $\beta = 0.1$, $\omega_0 = 0$, $a = 1$ the formula gives $\Delta\omega = 0.018$ so less than two percent error,
and for amplitude 0.2, the effort is less than 0.1%.

Explore with the applet here.

22.3: Resonance in a Damped Driven Linear Oscillator- A Brief Review

This is just to remind you of what we covered in lecture 18, before we add anharmonic terms in the next section.

The *linear* damped driven oscillator has equation of motion:

$$\ddot{x}+2\lambda\dot{x}+\omega_0^2 x=(f/m)e^{i\gamma t} \tag{22.3.1}$$

(Following Landau's notation here note it means the actual frictional drag force is $2\lambda m\dot{x}$)

Looking near resonance for steady state solutions at the driving frequency, with amplitude b, phase lag δ, that is, $x(t)=be^{i(\gamma t+\delta)}$, we find

$$be^{i\delta}\left(-\gamma^2+2i\lambda\gamma+\omega_0^2\right)=(f/m) \tag{22.3.2}$$

For a near-resonant driving frequency

$$\gamma=\omega_0+\varepsilon \tag{22.3.3}$$

and assuming the damping to be sufficiently small that we can drop the term along with , the leading order terms give

$$be^{i\delta}=-f/2m(\varepsilon-i\lambda)\omega_0 \tag{22.3.4}$$

so the response, the dependence of amplitude b on driving frequency $\Omega=\omega_0+\varepsilon$ is to this accuracy

$$b=\frac{f}{2m\omega_0\sqrt{(\gamma-\omega_0)^2+\lambda^2}}=\frac{f}{2m\omega_0\sqrt{\varepsilon^2+\lambda^2}} \tag{22.3.5}$$

(Note also that the resonant frequency is itself lowered by the damping, another second-order effect we'll ignore.)

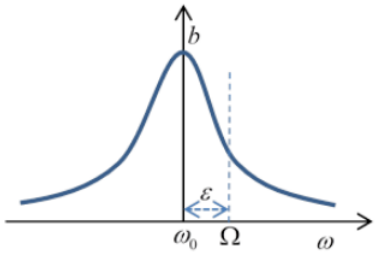

Figure 22.3.1

The rate of absorption of energy equals the frictional loss. The friction force $2\lambda m\dot{x}$ on the mass moving at $\dot{x}$ is doing work at an average rate:

$$2\lambda m\overline{\dot{x}^2}=\lambda m b^2\gamma^2 \tag{22.3.6}$$

The half width of the resonance curve as a function of γ is given by the damping. The total area under the curve is independent of damping.

For future use, we'll write the above equation for the amplitude b in terms of deviation ε from the resonant frequency ω_0.

$$b^2\left(\varepsilon^2+\lambda^2\right)=\frac{f^2}{4m^2\omega_0^2},\quad \varepsilon=\gamma-\omega_0 \tag{22.3.7}$$

22.4: Damped Driven Nonlinear Oscillator- Qualitative Discussion

We now add to the damped driven linear oscillator a positive quartic potential term, giving equation of motion

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = (f/m)\cos\gamma t - \beta x^3 \tag{22.4.1}$$

As mentioned above, for a particle oscillating in this potential $\frac{1}{2}\omega_0^2 x^2 + \frac{1}{4}\beta x^4$ the frequency *increases with amplitude*: the bigger swings encounter a potential becoming stronger and stronger than the simple harmonic oscillator.

So if we drive the oscillator from rest at the frequency that resonates with its small amplitude oscillations (where the $\frac{1}{4}\beta x^4$ potential term has negligible effect), as the amplitude builds up, the oscillator frequency increases, and the driving force falls out of sync.

The way to keep the amplitude increasing is evidently to gradually increase the frequency of the driving force to match the natural frequency at the increased amplitude. (*Side note*: this is the principle of the synchrocyclotron except, in that case the frequency is lowered as the energy increases, because the particles go to bigger and bigger orbits as their mases increase relativistically.) This way a small external driving force (enough to overcome frictional damping) can maintain a large amplitude oscillation at a frequency well above the frequency ω_0 of small oscillations.

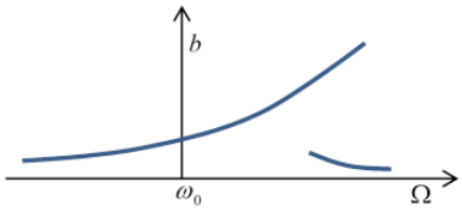

Figure 22.4.1

But what if we apply this high frequency to a system initially at rest, rather than gradually ramping up in sync with the oscillations? Then for a small driving force, we can treat the system as a damped simple harmonic oscillator, and this off-resonance force will drive relatively small amplitude oscillations.

The bottom line is that for the same external driving force, with frequency in some range above ω_0, there can be two possible steady state oscillation amplitudes, depending on the *history* of the system.

22.5: Nonlinear Case - Landau's Analysis

The equation of motion is:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = (f/m)\cos\gamma t - \beta x^3 \tag{22.5.1}$$

We established earlier that the nonlinear quartic term brings in a correction to the oscillator's frequency that depends on the amplitude b:

$$\omega = \omega_0 + \frac{3\beta b^2}{8\omega_0} = \omega_0 + \kappa b^2 \tag{22.5.2}$$

in Landau's notation $\kappa = 3\beta/8\omega_0$,

The equation for the amplitude in the *linear* case (from the previous section) was, with $\varepsilon = \gamma - \omega_0$,

$$b^2\left(\varepsilon^2 + \lambda^2\right) = \frac{f^2}{4m^2\omega_0^2} \tag{22.5.3}$$

For the *nonlinear* case, the maximum amplitude will clearly be at the true (amplitude dependent!) resonance frequency $\omega(b) = \omega_0 + \kappa b^2$ so with $\varepsilon = \gamma - \omega_0$ before, we now have a *cubic* equation for b^2:

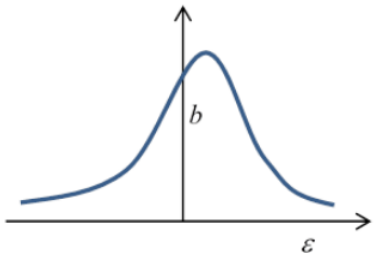

Figure 22.5.1

$$b^2\left(\left(\varepsilon - \kappa b^2\right)^2 + \lambda^2\right) = \frac{f^2}{4m^2\omega_0^2} \tag{22.5.4}$$

Note that for small driving force $f \ll 2m\omega_0\lambda, b$ is small ($b_{\max}^2 \approx f^2/4m^2\omega_0^2\lambda^2$) but the center of the peak has shifted slightly upwards, to $\varepsilon = \kappa b^2,$ that is, at a driving frequency $\gamma = \omega_0 + \kappa b^2$. The cubic equation for b^2 has only this one real solution.

However, as the driving force is increased, the coefficients of the cubic equation change and at a critical force f_k two more real roots appear.

The b, ε curve for driving force above f_k looks like:

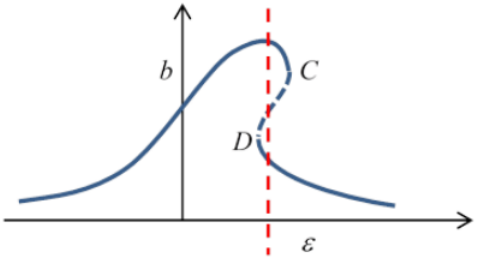

Figure 22.5.2

So what's going on here? For a range of frequencies, including the vertical dashed red line in the figure, there appear to be three possible amplitudes of steady oscillation at one frequency. However, it turns out that the middle one is unstable, so will exponentially deviate, going to one of the other two, both of which are stable.

If the oscillator is being driven at ω_0, and the driving frequency is gradually increased, the amplitude will follow the upper curve to the point C , then drop discontinuously to the lower curve. Further frequency increase (with the same strength driving force, of

course) will give diminishing amplitude of oscillation just as happens for the ordinary simple harmonic oscillator on going away from the resonant frequency.

If the frequency is now gradually lowered, the amplitude gradually will increase to point D, where it will jump discontinuously to the upper curve. The overall response to driving frequency is sometimes called hysteresis, by analogy with the response of a magnetic material to a varying imposed external field.

To put in some numbers, the maximum amplitude for any of these curves is when $db/d\varepsilon = 0,$ that is, at $\varepsilon = \kappa b^2,$ or

$$b_{\max} = f/2m\omega_0\lambda \tag{22.5.5}$$

the same result as for small oscillations.

To find the critical value of the driving force for which the multiple solutions appear, in the graph above that's when C, D coincide. That is, $db/d\varepsilon = \infty$ has coincident roots.

Differentiating the equation $b(\varepsilon)$ for amplitude as a function of frequency (and of course this is at constant driving force f)

$$\frac{db}{d\varepsilon} = -\frac{-\varepsilon b + \kappa b^2}{\varepsilon^2 + \lambda^2 - 4\kappa\varepsilon b^2 + 3\kappa^2 b^4} \tag{22.5.6}$$

C,D coincide when the discriminant in the denominator quadratic is zero, that is, at $\kappa^2 b^4 = \lambda^2$ where $\varepsilon = 2\kappa b^2$

Putting these values into the equation for $b(\varepsilon)$ as a function of driving force f, the critical driving force is

$$f_\kappa^2 = \frac{8m^2\omega_0^2\lambda^3}{|\kappa|} \tag{22.5.7}$$

Numerical Applet Results

The results above are all from Landau's book, and are semiquantitative. They can easily be checked using our online applet, which is accurate to one percent or better. The curves below are plotted from applet results, and certainly exhibit the behavior predicted by Landau.

These plots are for $\omega_0^2 = m = \beta = 1, \alpha = 0, 2\lambda = 0.34$

For these values, Landau's f_k is approximately $0.3.$ Ours looks a bit more. Note for $f = 0.3,$ we show $\kappa b^2 = 0.38 \cdot (0.75)^2 = 0.21,$ close to graph peak.

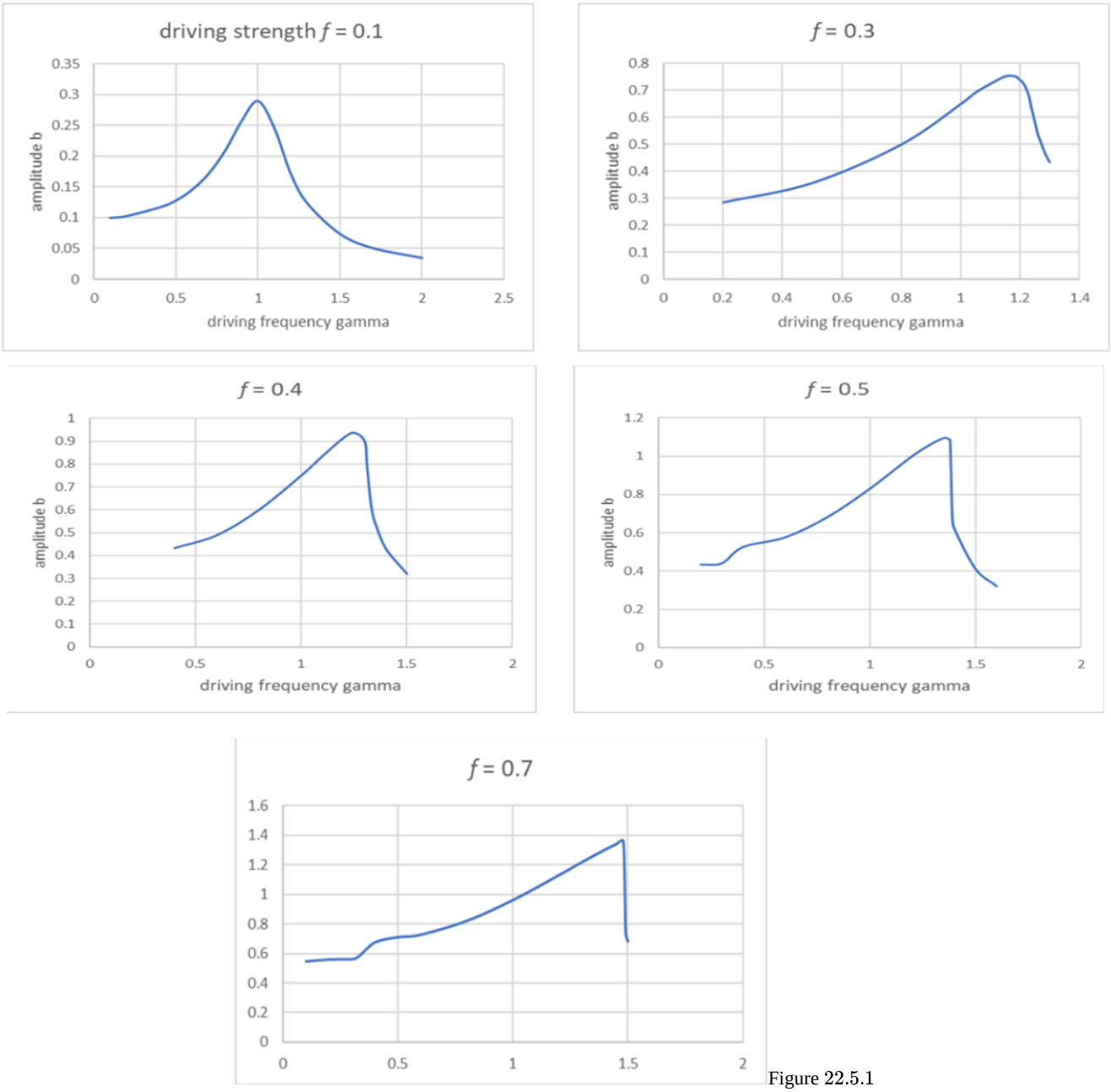

Figure 22.5.1

22.6: Frequency Multiples

The above analysis is for frequencies not very far from ω_0. But nonlinear terms can cause resonance to occur at frequencies which are rational multiples of ω_0. Landau shows that a small $\frac{1}{3}\alpha x^3$ in the potential (so an additional force αx^2 in the equation of motion) can generate a resonance near $\gamma = \frac{1}{2}\omega_0$. We've only considered a quartic addition to the potential, $\frac{1}{4}\beta x^4$, a force βx^3, we can show that gives a resonance near $\gamma = \frac{1}{3}\omega_0$, and presumably this is the small bump near the beginning of the curves above for large driving strength.

We have $\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = (f/m)\cos\gamma t - \beta x^3$

We'll write $x = x^{(0)} + x^{(1)} + \ldots$

Let's define $x^{(0)}$ by

$$\ddot{x}^{(0)} + 2\lambda\dot{x}^{(0)} + \omega_0^2 x^{(0)} = (f/m)\cos\gamma t \tag{22.6.1}$$

So $x^{(0)} = b\cos(\gamma t + \delta)$. Then

$$\begin{aligned}
\ddot{x}^{(1)} + 2\lambda\dot{x}^{(1)} + \omega_0^2 x^{(1)} &= -\beta\left(x^{(0)}\right)^3 \\
&= -\beta b^3\cos^3(\gamma t + \delta) \\
&= -\beta b^3\left[\frac{3}{4}\cos(\gamma t + \delta) + \frac{1}{4}\cos(3\gamma t + \delta)\right]
\end{aligned}$$

Then, for $\gamma = \frac{1}{3}\omega_0$, the second term, $-\beta b^3\frac{1}{4}\cos(3\gamma t + \delta) = -\beta b^3\frac{1}{4}\cos(\omega_0 t + \delta)$, will have a resonant response, although it is proportional to the (small) amplitude cubed. Similar arguments work for other fractional frequencies.

CHAPTER OVERVIEW

23: Damped Driven Pendulum- Period Doubling and Chaos

23.1: Introduction

23.2: The Road to Chaos

23.3: Lyapunov Exponents and Dimensions of Strange Attractors

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23.1: Introduction

We've previously discussed the driven damped simple harmonic oscillator, and in the last lecture we extended that work (following Landau) to an anharmonic oscillator, adding a quartic term to the potential. Here we make a different extension of the simple oscillator: we go to a driven damped *pendulum*. (A weight at one end of a light rigid rod, the other end of the rod being at a fixed position but the rod free to rotate about that fixed point in a vertical plane). That is, we replace the potential term $-\omega_0^2 x$ in the linear oscillator with $-\omega_0^2 \sin x$, or rather $-\omega_0^2 \sin\phi$, to make clear we have an *angular* system. Going to a driven damped pendulum leads to many surprises!

For a sufficiently weak driving force, the behavior of the driven damped pendulum is of course close to that of the driven damped linear oscillator, but on gradually increasing the driving force, at a certain strength the period of the response *doubles*, then, with further increase, it doubles again and again, at geometrically decreasing intervals, going to a chaotic (nonperiodic) response at a definite driving strength. But that is not the end of the story—the chaotic response regime has a lot of structure: many points within a generally chaotic region are in fact nonchaotic, well-defined cyclical patterns. And, as we'll see later, the response pattern itself can be fractal in nature, see for example the strange attractor discussed at the end of this lecture. You can use the accompanying applet to generate this attractor and its cousins easily.

Obviously, this is a very rich subject, we provide only a brief overview. We closely follow the treatment in Taylor's *Classical Mechanics*, but with the addition of our applets for illustrative purposes, and also to encourage further exploration: the applets accurately describe the motion and exhibit the strange attractors in the chaotic regime. With the applets, it is easy to check how these strange attractors change (or collapse) on varying the driving force or the damping. We end with some discussion of the fractal dimension of the attractors, and how it relates to the dynamics, in particular to the rate of divergence of initially close trajectories, here following Baker and Gollub, *Chaotic Dynamics*.

23.2: The Road to Chaos

Equation of Motion

For the driven pendulum, *the natural measure of the driving force is its ratio to the weight mg,* Taylor calls this the *drive strength*, so for driving force

$$F(t) = F_0 \cos \omega t, \tag{23.2.1}$$

the drive strength γ is defined by

$$\gamma = F_0/mg. \tag{23.2.2}$$

The equation of motion (with resistive damping force $-bv$ and hence resistive torque $-bL^2\dot{\phi}$) is:

$$mL^2\ddot{\phi} = -bL^2\dot{\phi} - mgL\sin\phi + LF(t) \tag{23.2.3}$$

Dividing by mL^2 and writing the damping term $b/m = 2\beta$ (to coincide with Taylor's notation, his equation 12.12) we get (with $\omega_0^2 = g/L$)

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t. \tag{23.2.4}$$

Behavior on Gradually Increasing the Driving Force: Period Doubling

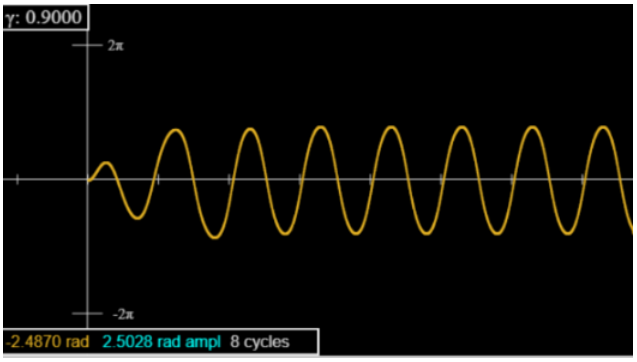
The driving force $\gamma = F_0/mg$ is the dimensionless ratio of drive strength to weight, so if this is small the pendulum will not be driven to large amplitudes, and indeed we find that after initial transients it settles to motion at the driving frequency, close to the linearized case. We would expect things to get more complicated when the oscillations have amplitude of order a radian, meaning a driving force comparable to the weight. And indeed they do.

Here we'll show that our applet reproduces the sequence found by Taylor as the driving strength is increased.

In the equation of motion

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t, \tag{23.2.5}$$

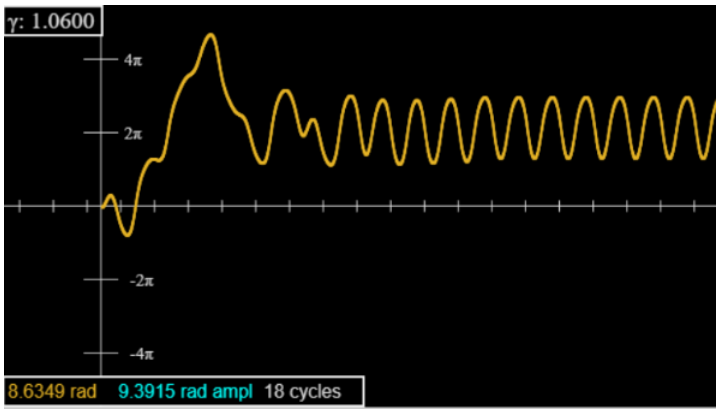
We therefore choose his values $\omega_0 = 1.5, 2\beta = 0.75, \omega = 1$ and gradually increase γ from 0.9 to 1.0829, where chaos begins.



For $\gamma = 0.9$, (see picture) the oscillation (after brief initial transients) looks like a sine wave, although it's slightly flatter, and notice the amplitude (turquoise in box) is greater than $\pi/2$, the positive and negative swings are equal in magnitude to five figure accuracy after five or so oscillations.

You can see this for yourself by opening the applet! Click here. The applet plots the graph, and simultaneously shows the swinging pendulum (click bar at top right).

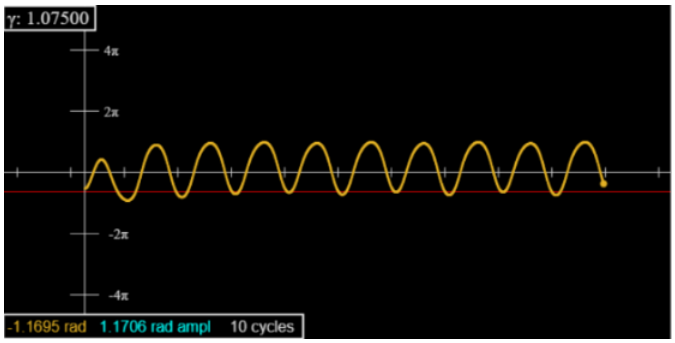
For $\gamma = 1.06$, there is a larger initial transient, vaguely resembling the three-cycle we'll meet shortly. In fact, this can be suppressed by starting at $\varphi = -\pi/2$ (bottom slider on the right), but there are still transients: the peaks are not uniform to five-figure accuracy until about forty cycles.



For $\gamma = 1.0662$, there are *very long* transients: not evident on looking at the graph (on the applet), but revealed by monitoring the amplitude readout (turquoise figures in the box on the graph), the value at the most recent extremum.

To get past these transients, set the applet speed to 50 (this does *not* affect accuracy, only prints a lot less dots). run 350 cycles then pause and go to speed = 2. You will find the peaks now alternate in height to the five-figure accuracy displayed, looks like period doubling—but it isn't, run a few thousand cycles, if you have the patience, and you find all peaks have the same height. That was a long lived transient precursor of the period doubling transition.

Going to 1.0664, you'll see both peaks and dips now alternating in amplitude, for a well-defined period 2, which lasts until 1.0792. (Look at 1.075 after 70 or so cycles—and set the initial angle at -90.)

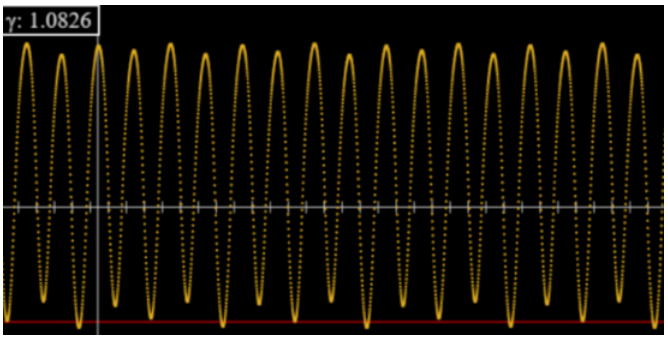


Use the " Red Line" slider just below the graph to check the heights of successive peaks or dips.

For $\gamma = 1.0793$, the period doubles again: successive peaks are now of order 0.4 radians apart, but successive "high" peaks are suddenly about 0.01 radians apart. (To see these better on the applet, magnify the vertical scale and move the red line.)

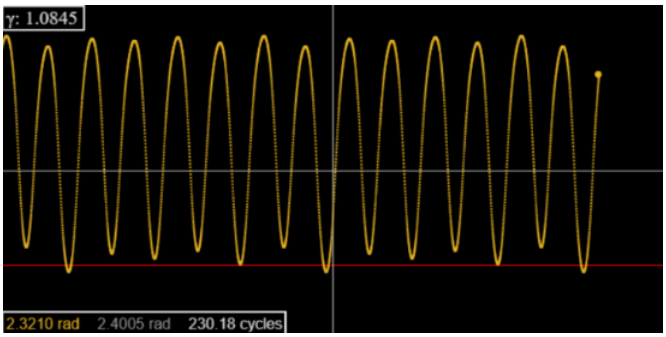
For $\gamma = 1.0821$ there is a further doubling to an 8 cycle, then at 1.0827 to a 16 cycle.

Look at the graph for 1.0826: in particular, the faint red horizontal line near the bottom. Every fourth dip goes below the line, but the next lowest dips alternate between touching the line and not quite reaching it. This is the 8 cycle.



It is known that the intervals between successive doublings decrease by a factor $\delta = 4.6692$, found *universally* in period doubling cascades, and called the *Feigenbaum number*, after its discoverer. Our five-figure accuracy is too crude to follow this sequence further, but we can establish (or at least make very plausible!) that beyond the geometric series limit at $\gamma_c = 1.0829$ the periodicity

disappears (temporarily, as we'll see), the system is chaotic. (Of course, the values of γ depend on the chosen damping parameter, etc., only *the ratio of doubling intervals* is universal.)



In fact, the full picture is complex: there are further intervals of periodicity, for example a 6 cycle at $\gamma = 1.0845$, pictured here.

Different Attractors

The periodic solutions described above are called *"attractors"*: configurations where the system settles down after initially wandering around.

Clearly the attractors change with the driving strength, what is less obvious is that they may be different for different initial conditions. Taylor shows that for $\gamma = 1.077$ taking $\varphi(0) = 0$ gives a 3-cycle after transients, but $\varphi(0) = -\pi/2$ gives a 2-cycle. (Easily checked with the applet!)

Looking at the initial wanderings, which can be quite different for very small changes in the driving strength (compare 1.0730 to 1.0729 and 1.0731, use speed 5, it doesn't affect the accuracy). But you can see these initial wanderings include elements from both attractors.

Exercise: use the next applet to plot at the same time 1.0729 and 1.0730, all other parameters the same, speed 5.

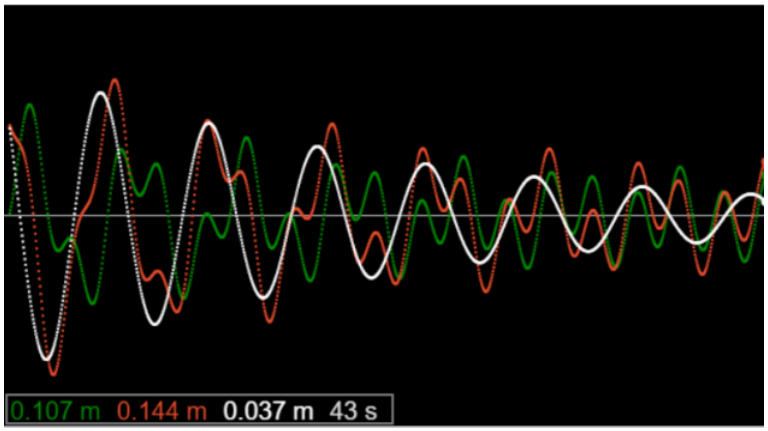
Exercise: Use the applet to find at what angle the transition from one attractor to the other takes place. And, explore what happens over a range of driving strengths.

These are the simplest attractors: there are far more complex entities, called strange attractors, we'll discuss later.

Exercises: Try different values of the damping constant and see how this affects the bifurcation sequence.

Sensitivity to Initial Conditions

Recall that for the *linear* damped driven oscillator, which can be solved exactly, we found that changing the initial conditions changed the path, of course, but the *difference* between paths decayed exponentially: the curves converged.



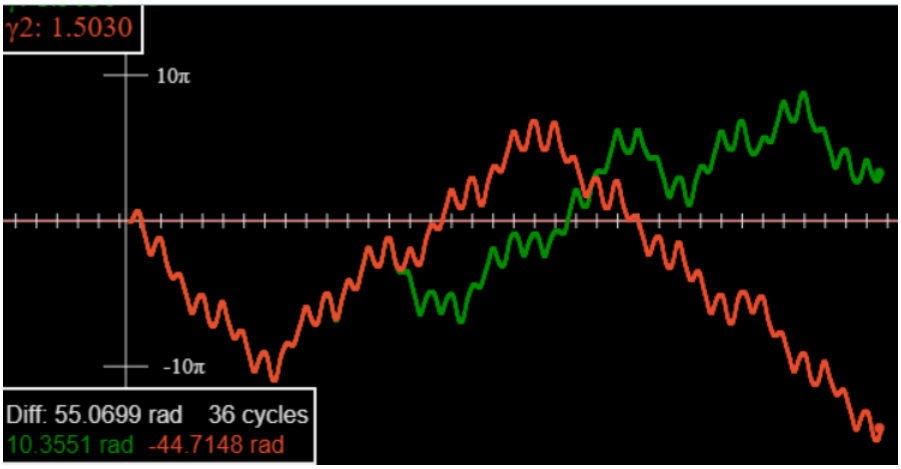
This illustration is from an earlier applet: the red and green curves correspond to different initial conditions, the white curve is the difference between the two, obviously exponentially decreasing, as can be verified analytically.

For the damped driven pendulum, the picture is more complicated. For $\gamma < \gamma_c = 1.0829$ curves corresponding to slightly different initial conditions will converge (except, for example, $\gamma = 1.077$ where, as mentioned above, varying the initial angle at a certain

point switches form a final three-cycle to a two-cycle).

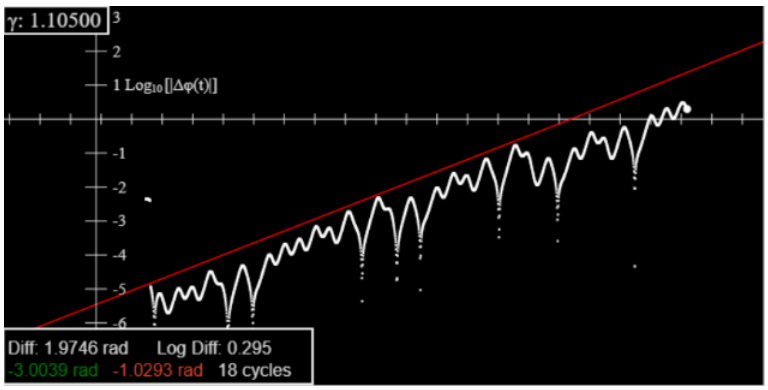
The Liapunov Exponent

For $\gamma > \gamma_c$, curves even with very small initial differences (say, 10^{-4} radians) separate *exponentially*, as $e^{\lambda t}$, λ is called the *Liapunov* exponent.



Bear in mind, though, that this is *chaotic* motion, the divergence is not as smooth as the convergence pictured above for the linear system. This graph is from our two-track applet.

Nevertheless, the (average) exponential nature can be discerned by plotting the logarithm of the difference against time:



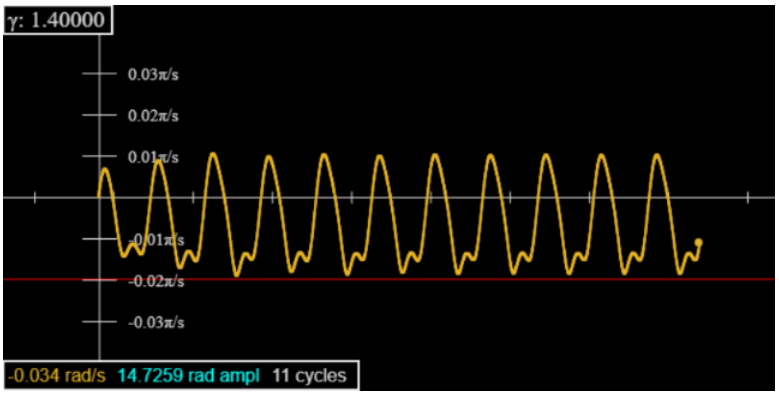
This is from our log difference applet. It is very close to Taylor's fig 12.13. The red line slope gives the Lyapunov exponent.

(The red line has adjustable position and slope.)

Plotting Velocity against Time

As discussed in Taylor, further increase in the driving force beyond the chaotic threshold can lead to brief nonchaotic intervals, such as that containing the six cycle at 1.0845 illustrated above, but there are two long stretches of nonchaotic behavior in Taylor's parameter range, from 1.1098 to 1.1482 and from 1.3 to 1.48.

In the stronger driving force range, the pendulum is driven *completely around* in each cycle, so plotting the position against time gives a "rounded staircase" kind of graph. Check this with the applet.



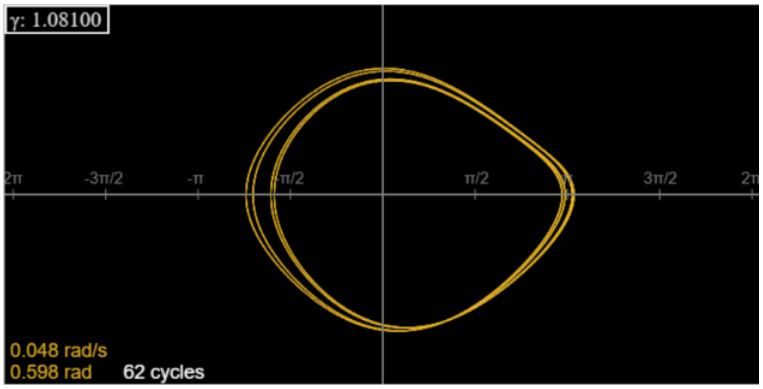
The solution is to plot *velocity* against time, and thereby discover that there is a repetition of the period doubling route to chaos at the upper end of this interval. Click to plot $d\phi/dt$ instead of ϕ.

State-Space Trajectories

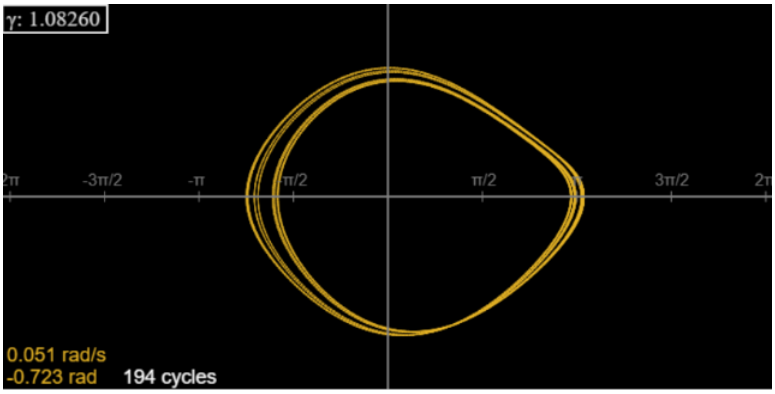
It can be illuminating to watch the motion develop in time in the *two-dimensional state space* $(\phi, \dot{\phi})$. (Equally called *phase* space.) See the State Space applet!

Now for a particle in a *time-independent potential*, specifying position and velocity at a given instant determines the future path—but that is *not* the case here, the acceleration is determined by the *phase of the driving force*, which is time varying, so the system really needs *three* parameters to specify its subsequent motion.

That means *the phase space is really three-dimensional*, the third direction being the driving phase, or, equivalently, time, but periodic with the period of the driving force. In this three-dimensional space, paths cannot cross, at any point the future path is uniquely defined. Our two-dimensional plots are projections of these three-dimensional paths on to the $(\phi, \dot{\phi})$ plane.



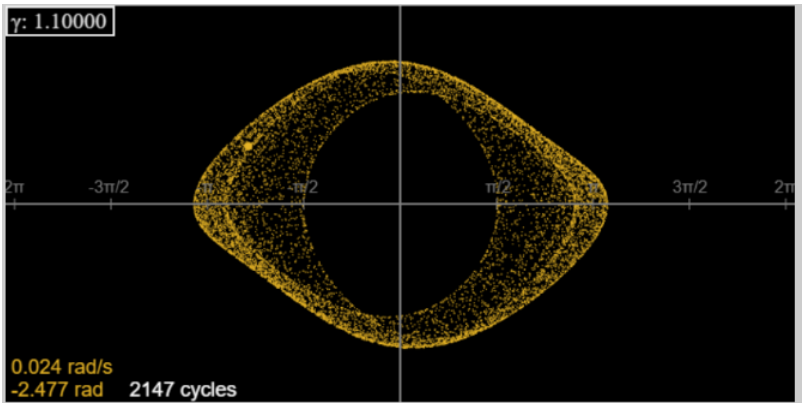
Here's the 4-cycle at $\gamma = 1.081$, minus the first 20 cycles, to eliminate transients.



For $\gamma = 1.0826$, there is an 8-cycle. Run the applet for 40 cycles or so to get rid of transients, then look at the far-left end of the curve generated by running, say, to 200 cycles. It doesn't look like there are 8 lines, but the outermost line and the two innermost lines are doubles (notice the thickness). You can check this with the applet, by pausing at the axis and noticing the position readout:

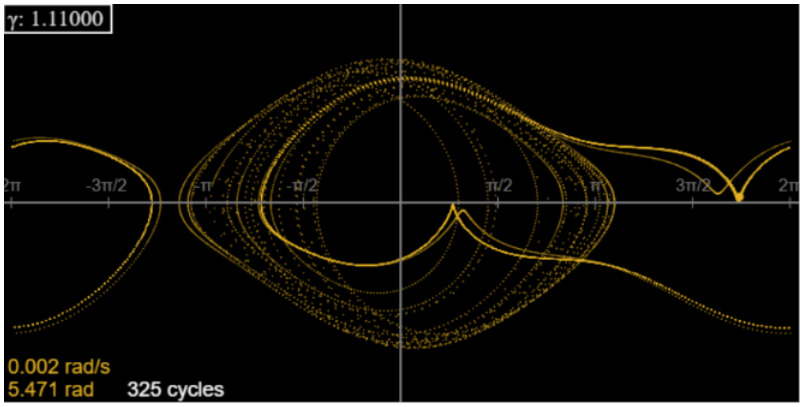
there are 8 different values that repeat. (You don't have to stop right on the axis—you can stop close to it, then use the step buttons to get there.)

For $\gamma = 1.0830$, the path is *chaotic*—but doesn't look too different on this scale! Check it out with the applet. The chaos becomes more evident on further increasing γ. For $\gamma = 1.087$ the pattern is "fattened" as repeated cycles differ slightly.



For further increase in γ, the orbital motion covers more territory: at $\gamma = 1.1$, here are the first three hundred or so cycles.

Plotting many orbits at high speed we find:



However, here the story gets complicated: it turns out that this chaotic series of orbits is in fact a transient, and after about 300 cycles the system skips to a 3-cycle, where it stays. In fact, we have reached a range of γ (approximately 1.1098 to 1.1482) where after initial transients (which can be quite long) the motion is *not* chaotic, but a 3-cycle.

Obviously, there is a lot to explore for this system! To get a better grasp of these complexities, we try a different approach, developed by Poincaré.

Poincaré Sections

Looking at the above pictures, as we go from a single orbit to successive period doublings and then chaos, the general shape of the orbit doesn't change dramatically (until that final three-cycle). The interesting things are the doubling sequence, chaos, and attractors—perhaps we're plotting too much information.

To focus on what's essential, Poincaré plotted a *single point* from each cycle, this is now called a *Poincaré section*. To construct this, we begin with the $t = 0$ position, label it $P_0 = (\varphi_0, \dot{\varphi}_0)$. Then add the point $P_1 = (\varphi_1, \dot{\varphi}_1)$ precisely one cycle later, and so on—points one cycle apart, $P_2 = (\varphi_2, \dot{\varphi}_2)$, etc. Now, knowing the position $P = (\varphi, \dot{\varphi})$ in state space is not enough information to plot the future orbit—we also need to know the phase of the driving force. But by plotting points *one cycle in time apart*, they will all see the same phase starting force, so the transformation that takes us from P_0 to P_1 just repeats in going from P_1 to P_2, etc.

To see these single points in the State Space applet, click **"Show Red Dot"**: on running, the applet will show one dot per cycle red.

Thinking momentarily of the full three-dimensional phase space, the Poincaré section is a cross-section at the initial phase of the driving cycle. We've added to the applet a ***red dot phase option***, to find the Poincaré section at a different phase. Doing this a few times, and looking at the movement of the red dots, gives a fuller picture of the motion.

So the Poincaré section, on increasing γ through the doubling sequence (and always omitting initial transients) goes from a single point to two points, to four, etc.

To see all this a bit more clearly, the next applet, called Poincaré Section, shows *only* one dot per cycle, but has a phase option so you can look at any stage in the cycle.

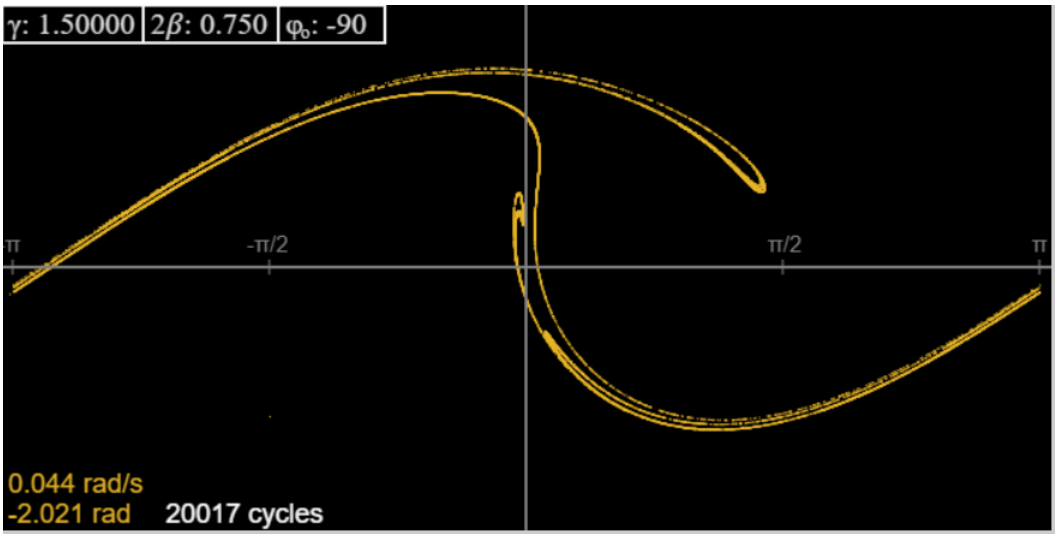
Exercise: check this all out with the Poincaré applet! To see it well, click Toggle Origin/Scale, use the sliders to center the pattern, more or less, then scale it up. Run for a few hundred cycles, then click Hide Trace followed by Show Trace to get rid of transients.

Start with $\gamma = 1.0826$. The Poincaré section is eight points on a curve, at 1.0827 you can discern 16. By 1.0829, we have chaos, and the section has finite stretches of the curve, not just points. It looks very similar at 1.0831, but—surprise—at 1.0830 it's a set of points (32?) some doubled. This tells us we're in a minefield. There's nothing smooth about chaos.

Apart from interruptions, as γ increases, the Poincaré section fills a curve, which then evolves towards the strange attractor shown by Taylor and others. Going from 1.090 in steps to 1.095, the curve evolves a second, closely parallel, branch. At 1.09650 the two branches are equal in length, then at 1.097 the lower branch suddenly extends a long curve, looks much the same at 1.100, then by 1.15 we recognize the emergence of the strange attractor.

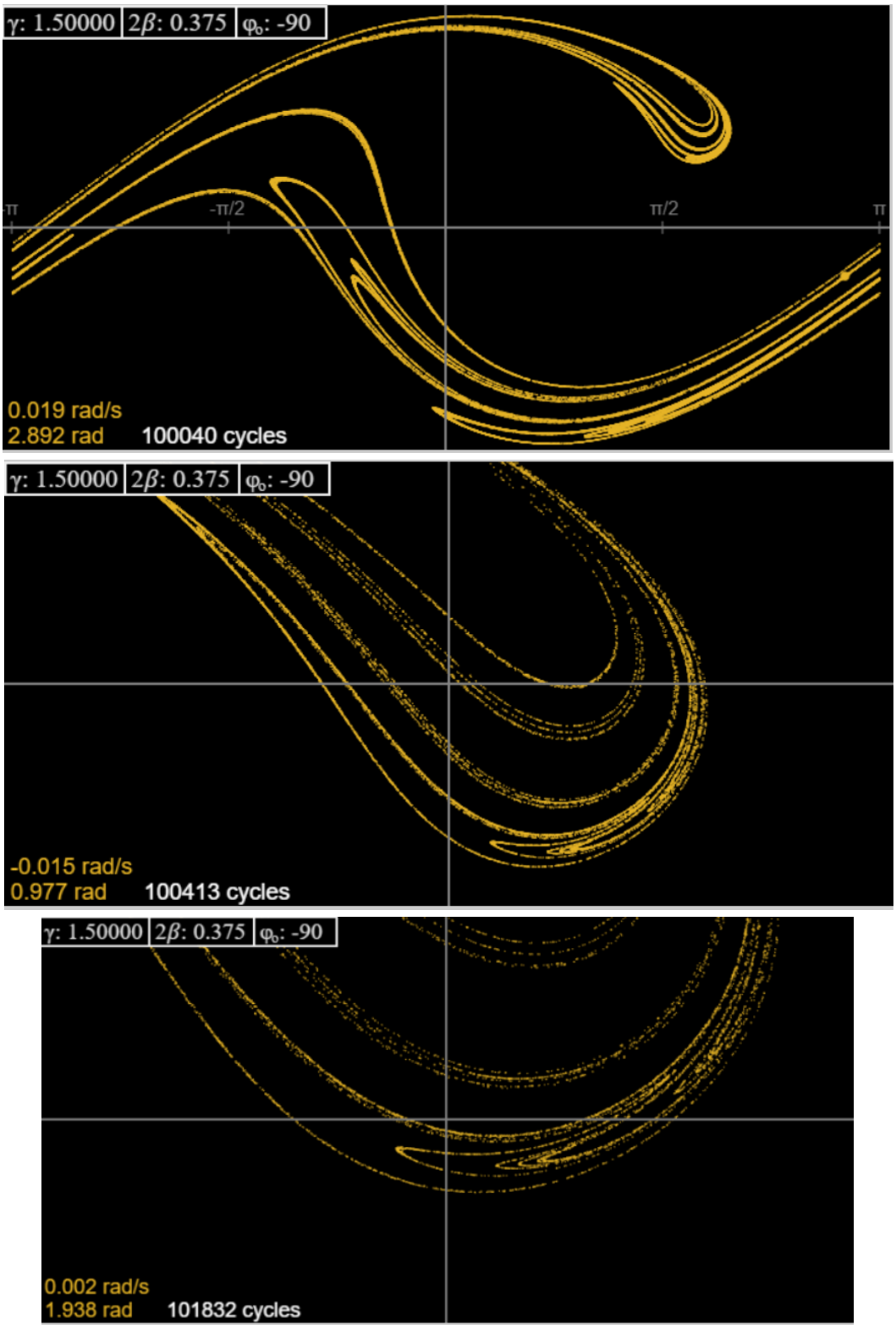
But in fact this is a simplified narrative: there are interruptions. For example, at $\gamma = 1.0960$ there's a 5-cycle, no chaos (it disappears on changing by 0.0001). And at 1.12 we're in the 3-cycle interval mentioned earlier.

Anyway, the strange attractor keeps reappearing, and at 1.5 it looks like this:



This looks rather thin compared to Taylor's picture for the same γ: the reason is we have kept the strong damping.

Changing to $2\beta = 0.375$, we recover Taylor's picture. Like him, we show here (from the applet) the attractor and successive magnifications of one part, to give some hint of the fractal structure:



We have to admit that at the highest magnification Taylor's picture is certainly superior to that generated by the applet, but does not contradict it—actually, it reassures us that the applet is reliable up to the level of definition that is self-evident on looking at it.

Exercise: Use the last two applets to explore other regions of parameter space. What happens on varying the damping? Going from $2\beta = 0.375$ to $2\beta = 0.75$ the attractor has the same general form but is much narrower. Then go in 0.001 steps to 0.76. For some values you see the attractor, but for others a cycle, different lengths. There is a two-cycle from 0.76 to 0.766, then a four cycle, then

at 0.78 back to a two-cycle, at 0.814 a one-cycle. If you up the driving strength, and the damping, you can find extremely narrow attractors.

Exercise: Baker and Gollub, *Chaotic Dynamics*, page 52, give a sequence of Poincaré sections at 0.2π intervals for $\gamma = 1.5$ and $2\beta = 0.75$. You can check them (and intermediate ones) with the applet, and try to imagine putting them in a stack to visualize the full three-dimensional attractor!

23.3: Lyapunov Exponents and Dimensions of Strange Attractors

Scaling the Attractor

Looking at the strange attractor pictured in the previous section, we found that on magnifying a small part of it we saw the same kind of structure the attractor has as a whole: if we look at a $\gamma = 1.5$ attractor, with damping 0.75, there are long thin stretches, they end by looping over. We notice that halving the damping to 0.375 fattens the previously quasi-one-dimensional stretches and reveals complicated looping *at several levels*. (Remember that what we are looking at here is a Poincaré section of the attractor, the other dimension is periodic time (or driver phase, the same thing) so a curve here is a section of some sheet). If more computing power is used, going to smaller and smaller scales, it turns out that the magnified tiny part of the attractor looks much the same as the attractor. This kind of scale invariance is a characteristic of a *fractal*. A mysterious aspect of fractals is their dimensionality. Look at the strange attractor. There are no places where it solidly fills a stretch of two-dimensional space, this is clearer on going to greater and greater magnification: we see more and more one-dimensional structures, with no end, so it surely has dimension less than two, but greater than one—how do we make sense of that?

Fractals: the Cantor Set

To try to find a generalized concept of dimension of a set (i.e. not just an integer), we begin with perhaps the simplest example of a fractal, the *Cantor set*: take the numbers between 0 and 1, and cut out the middle third. You now have two strips of numbers, from 0 to 1/3, and from 2/3 to 1. For each of *those* strips, cut out the middle third. You now have four strips—cut out the middle third of each of them (it might help to draw this). Do this forever. What's left is the Cantor set. You can see this is scale invariant: after doing this many times, take one of the remaining tiny strips, what happens to it on continuing the process is identical (scaled down suitably) to what happened to the initial strip.

How big is this Cantor set? At each step, we cut the total length of line included by 2/3. Since $(2/3)^n$ goes to zero as n goes to infinity, it clearly has size zero, right? But clearly there's more to the Cantor set than there is to a single point, or for that to matter a finite number of points. What about a countably infinite number of points—for example, the *rational* numbers between 0 and 1? Well, you can write them out in a list, ordered by increasing denominators, and for one denominator by increasing numerators. Then you can put them one by one into tiny intervals, 1/2 goes into an interval of length ε, 1/3 in an interval $\varepsilon/2$, 2/3 in one of length $\varepsilon/2^2$, 1/4 in $\varepsilon/2^3$, and so on, the total length of the infinite number of intervals being 2ε, so all the rationals can be covered by an arbitrarily small set of intervals. Can we count in order the numbers in the Cantor set in the same way? The answer is no, and to see why think first about *all* the numbers between 0 and 1, rationals and irrationals. If you make an infinite list of them, I can show you missed some out: I just take your list and write down a decimal that differs from your n^{th} number in the n^{th} place. So we can't put all the numbers in the interval in little boxes that add to zero, which is obvious anyway!

But now to the Cantor set: suppose we write all numbers between 0 and 1 using base 3, instead of the traditional base 10. That is, each number is a string of 0's, 1's and 2's. Then the Cantor set is all those numbers that don't have any 1's, such as 0.2, 0.02202, etc. (Check this yourself.) But the number of *these* numbers is exactly the same as *all* the numbers between 0 and 1 in *binary* notation! So surely the Cantor set has dimension 1? (These infinities are tricky.)

The bottom line from the above argument is that we can plausibly argue both that the Cantor set has dimension 0, and that it has dimension 1. To understand and categorize fractals better, we need a working definition of *dimension* for fractals. One approach is the *capacity* dimension.

Dimensions: Capacity and Correlation

Suppose we cover the interval 0,1 with a set of small boxes, length ε, there are clearly $N(\varepsilon) = 1/\varepsilon$ such boxes (assume it's an integer). Now consider a *subset* of the numbers between 0 and 1, choose ε, and find how many boxes $N(\varepsilon)$ are necessary to cover this subset. The *capacity dimension* is defined as

$$d_C = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}. \tag{23.3.1}$$

For simplicity, we choose $1/\varepsilon = 3, 9, 27, \ldots$ so the necessary numbers of boxes to cover the Cantor set described in the previous section are $2, 4, 8 \ldots$ out of total numbers of boxes $3, 9, 27, \ldots$. Therefore

$$d_C(\text{Cantor}) = \frac{n \log 2}{n \log 3} = \frac{\log 2}{\log 3}, \tag{23.3.2}$$

of course between 0 and 1. (There are many ways to define dimensionality of sets of numbers—this definition gives zero for a finite set of points, and one for all the numbers between 0 and 1, *but* also 1 for the set of rationals, which we've shown above can be covered by an infinite set of intervals of arbitrarily small total length.)

Another measure used is the *correlation dimension*, in which for a large number of points (such as our representation of the attractor) a correlation integral $C(r)$ is defined as the total number of pairs of two points less than r apart. For small r, this goes as a power r^ν, and it turns out that in many cases ν is close to the capacity definition of the fractal dimension. (Grassberger and Procaccia.)

Time Development of Systems in Phase Space

Recall first that the state space or phase space we have been plotting is really a projection of the full orbit space into two dimensions, the third dimension necessary to predict future motion being the phase of the sine-wave driving force, so this is just time (although of course cyclic).

Suppose now we populate this three-dimensional space with many points, like a gas, each representing a driven damped pendulum. As time goes on the gas will flow, each gas atom's path completely determined, and no two will ever be at the same point in this full space (except perhaps asymptotically at infinite time).

Take now a small volume, say a cube with sides parallel to the axes, containing many points. Consider first the *undamped* system: then Liouville's Theorem (link to my lecture) tells us that as time goes on the cube will generally distort, but it will *not* change in volume. In other words, the gas of systems flows like an incompressible fluid. (Details of the derivation are given in the linked lecture—briefly, the motions of the sides in time Δt come from the equations of motion, etc.)

However, if the system has *damping*—as ours does—the same analysis leads to the conclusion that the volume the systems occupy in phase space (remember, this is now three-dimensional) shrinks at a rate determined by the damping. As a trivial example, think of undriven damped pendula—they will all tend to the low point rest position. Lightly *driven* pendula will go to a one-dimensional cycle.

We can prove this shrinkage from the equation of motion:

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t. \tag{23.3.3}$$

In the three-dimensional phase space, a pendulum's position can be written in coordinates $(\phi, \dot{\phi}, \psi)$ where $\psi = \omega t$, the driving phase, between 0 and 2π.

The local phase space velocity $\vec{F}$ can be written in terms of the coordinates (this is just the above equation rewritten!):

$$\frac{\partial\phi}{\partial t} = \dot{\phi}, \tag{23.3.4}$$

$$\frac{\partial\dot{\phi}}{\partial t} = -2\beta\dot{\phi} - \omega_0^2 \sin\phi + \gamma\omega_0^2 \cos\psi, \tag{23.3.5}$$

$$\frac{\partial\psi}{\partial t} = \omega, \tag{23.3.6}$$

This is therefore the local velocity of the atoms of the gas (meaning the systems), and it is trivial to check that

$$\vec{\nabla} \cdot \vec{F} = -2\beta. \tag{23.3.7}$$

This means that if we have a small sphere containing many points corresponding to systems (a "gas" of systems) then the volume of the (now distorting) sphere enclosing those points is *decreasing* in volume at an exponential rate $V(t) = V_0 e^{-2\beta t}$.

Relating this Picture to Lyapunov Exponents

Continuing to think about the development of a small sphere (containing many points corresponding to systems) in phase space, it will be moving along an orbit, but at the same time distorting, let's say to an ellipsoid as an initial first approximation, and tumbling around. In the chaotic regime, we know it must be growing in *some* direction, at least on average (the rates will vary along the orbit) because we know that points initially close together separate on average at an exponential rate given by the first

Lyapunov exponent, $\propto e^{\lambda_1 t}$. We'll make the simplifying assumption that the ellipsoid has its axes initially varying in time as $(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t})$, with $\lambda_1 > \lambda_2 > \lambda_3$. From the result above $\vec{\nabla} \cdot \vec{F} = -2\beta$, we conclude that

$$\lambda_1 + \lambda_2 + \lambda_3 = -2\beta. \tag{23.3.8}$$

We need to say something more about λ_1. We're taking it as defined by the growth rate of distance between trajectories *after* any initial transients but *before* the distance is comparable to the size of the system (finding this interval plausibly has been termed a "dark art"). We envision our initially small sphere of gas elongating and tumbling around as it moves along. Hopefully its rate of elongation correlates well with what we actually measure, that is, the rate of growth of net ϕ displacement, the coordinate separation of two initially close orbits, which we plot and approximately fit with an exponential, $\propto e^{\lambda_1 t}$.

For the pendulum, the ψ direction is just time, not scaled, so $\lambda_3 = \lambda_\psi = 0$. Then necessarily $\lambda_2 < 0$ to satisfy the damping equation.

So taking a local (in phase space) collection of systems, those inside a given closed surface, like a little sphere, and following their evolution in time in the chaotic regime, the sphere will *expand* in one direction; a direction, however, that varies with time, but *contract* or stay constant in the other directions. As the surface grows this gets more complicated because it's confined to a finite total phase space. And it continues to expand at the same rate as time goes on, so the continual increase in surface must imply tighter and tighter foldings to stay in the phase space. And this is what the strange attractor looks like.

A Fractal Conjecture

In 1979, Kaplan and Yorke conjectured that the dimensionality of the strange attractor followed from the Lyapunov exponents taking part in its creation. In our case—the driven damped pendulum—there are only two relevant exponents, $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_1 + \lambda_2 = -2\beta$.

A plausibility argument is given in Baker and Gollub's book, *Chaotic Dynamics*. They define a Lyapunov dimension dL of the attractor by

$$dL = \lim_{\varepsilon \to 0} \left[\frac{d(\log N(\varepsilon))}{d(\log(1/\varepsilon))} \right], \tag{23.3.9}$$

exactly analogous to the definition of capacity dimension in the previous section.

Now, as time passes a small square element will have its area multiplied by a factor $e^{(\lambda_1+\lambda_2)t}$. (No scaling takes place in the third (time) direction.) At the same time, they argue that the length unit ε changes as $e^{\lambda_2 t}$. Then $N(\varepsilon)$ is the area $\varepsilon_0^2 e^{(\lambda_1+\lambda_2)t}$ divided by the shrinking basic area $\varepsilon_0^2 e^{2\lambda_2 t}$. The differential of $\log N(\varepsilon)$ is $\lambda_1 - \lambda_2$, that of $\log(1/\varepsilon)$ is $-\lambda_2$, so their argument gives

$$dL = 1 - \frac{\lambda_1}{\lambda_2}. \tag{23.3.10}$$

The Lyapunov applet is designed to measure λ_1 by tracking separation of initially close trajectories. Try it a few times: it becomes clear that there is considerable uncertainty in this approach. Given λ_1, λ_2 follows from $\lambda_1 + \lambda_2 = -2\beta$. There are various ways to assign a dimension to the attractor, such as the capacity and correlation dimensions mentioned above. Various attempts to verify this relationship have been made, but the uncertainties are considerable, and although results seem to be in the right ballpark, the results are off by ten or twenty percent typically. It seems there is work still to be done on this fascinating problem.

Recommended reading: chapter 5 of Baker and Gollub, *Chaotic Dynamics*. The brief discussion above is based on their presentation.

CHAPTER OVERVIEW

24: Motion of a Rigid Body - the Inertia Tensor

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24.1: Definition of Rigid

We're thinking here of an idealized solid, in which the distance between any two internal points stays the same as the body moves around. That is, we ignore vibrations, or strains in the material resulting from inside or outside stresses. In fact, this is almost always an excellent approximation for ordinary solids subject to typical stresses—obvious exceptions being rubber, flesh, etc. Following Landau, we'll usually begin by representing the body as a collection of particles of different masses m_i held in their places $\vec{r}_i$ by massless bonds. This approach has the merit that the dynamics can be expressed cleanly in terms of sums over the particles, but for an ordinary solid we'll finally take a continuum limit, replacing the finite sums over the constituent particles by integrals over a continuous mass distribution.

24.2: Rotation of a Body about a Fixed Axis

As a preliminary, let's look at a body firmly attached to a rod fixed in space, and rotating with angular velocity Ω radians/sec. about that axis. You'll recall from freshman physics that the angular momentum and rotational energy are $L_z = I\Omega, \quad E_{\text{rot}} = \frac{1}{2} I\Omega^2$ where

$$I = \sum_i m_i r_{\perp i}^2 = \int dxdydz\rho(x,y,z) r_\perp^2 \tag{24.2.1}$$

(here $r_\perp = \sqrt{x^2+y^2}$ is the distance from the axis).

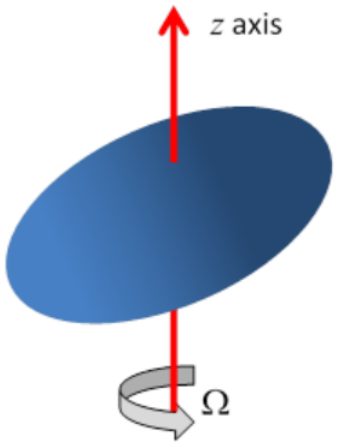


Figure 24.2.1

But you also know that both angular velocity and angular momentum are vectors. Obviously, for this example, the angular velocity is a vector pointing along the axis of rotation, $\vec{\Omega} = (0, 0, \Omega_z)$. One might be tempted to conclude that the angular momentum also points along the axis, but *this is not always the case*. An instructive example is provided by two masses m at the ends of a rod of length 2α held at a fixed angle θ to the z axis, which is the axis of rotation.

Evidently,

$$L_z = 2ma^2 \sin^2\theta \cdot \Omega \tag{24.2.2}$$

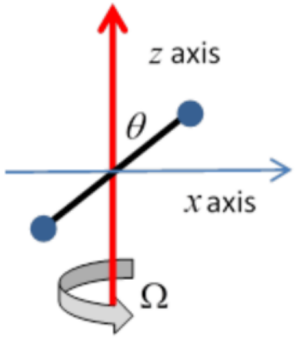


Figure 24.2.1: Masses are momentarily in (x,z) plane

But notice that, assuming the rod is momentarily in the xz plane, as shown, then

$$L_x = -2ma^2 \cos^2\theta \cdot \Omega \tag{24.2.3}$$

The total angular *momentum* is not parallel to the total angular *velocity*!

In fact, as should be evident, the total angular momentum is rotating around the constant angular velocity vector, so the axis must be providing a torque. This is why unbalanced car wheels stress the axle.

24.3: General Motion of a Rotating Rigid Body

We'll follow the Landau notation (which itself tends to be bilingual between coordinates (x, y, z) and (x_1, x_2, x_3) .). Notice that we'll label the components by (x_1, x_2, x_3) , not (r_1, r_2, r_3) *even though we call the vector* $\vec{r}$. Again, we're following Landau.

We take a fixed, inertial (or lab) coordinate system labeled (X, Y, Z) and in this system the rigid body's center of mass, labeled O, is at $\vec{R}$. We have a Cartesian set of axes fixed in the body, origin at the center of mass, and coordinates in this system, vectors from O to a point in the body denoted by $\vec{r}$ are labeled (x, y, z) or (x_1, x_2, x_3).

A vector from the *external* inertial fixed origin to a point in the body is then

$$\vec{R} + \vec{r} = \vec{\rho} \tag{24.3.1}$$

say, as shown in the figure.

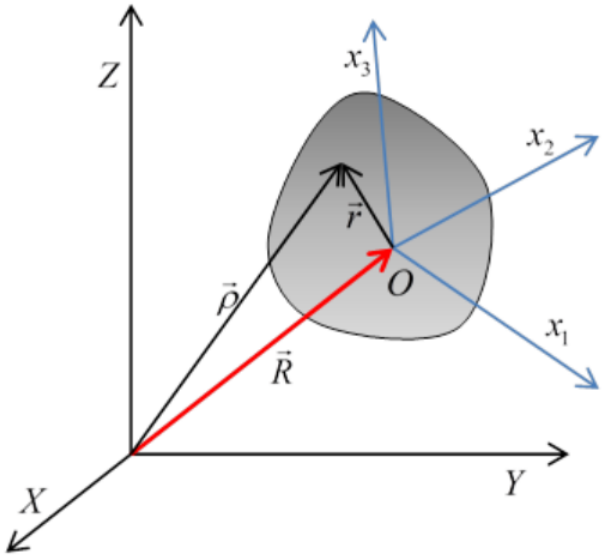


Figure 24.3.1

Suppose now that in infinitesimal time dt, the center of mass of the body moves $d\vec{R}$ and the body rotates through $d\vec{\phi}$. Then a particle at $\vec{r}$ as measured from the center of mass will move through

$$d\vec{\rho} = d\vec{R} + d\vec{\phi} \times \vec{r} \tag{24.3.2}$$

Therefore, the velocity of that particle in the fixed frame, writing the center of mass velocity and the angular velocity as

$$d\vec{R}/dt = \vec{V}, \quad d\vec{\phi}/dt = \vec{\Omega} \tag{24.3.3}$$

is

$$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r} \tag{24.3.4}$$

Now, in deriving the above equation, we have not used the fact that the origin O fixed in the body is at the center of mass. (That turns out to be useful shortly.) What if instead we had taken some other origin O' fixed in the body? Would we find the angular velocity $\vec{\Omega'}$ about O' to be the same as $\vec{\Omega}$? The answer turns out to be *yes*, but we need to prove it! Here's the proof:

If the position of O' relative to O is $\vec{a}$ (a vector fixed in the body and so moving with it) then the velocity $\vec{V'}$ of O' is

$$\vec{V'} = \vec{V} + \vec{\Omega} \times \vec{a} \tag{24.3.5}$$

A particle at $\vec{r}$ relative to O is at $\vec{r'} = \vec{r} - \vec{a}$ relative to O'

Its velocity relative to the fixed external axes is

$$\vec{v} = \overrightarrow{V'} + \overrightarrow{\Omega'} \times \overrightarrow{r'} \tag{24.3.6}$$

this must of course equal

$$\vec{V} + \vec{\Omega} \times \vec{r} = \vec{V} + \vec{\Omega} \times \overrightarrow{r'} + \vec{\Omega} \times \vec{a} = \vec{V'} + \vec{\Omega} \times \overrightarrow{r'} \tag{24.3.7}$$

It follows that $\overrightarrow{\Omega'} = \vec{\Omega}$

This means that if we describe the motion of any particle in the body in terms of some origin fixed in the body, plus rotation about that origin, *the angular velocity vector describing the body's motion is the same irrespective of the origin we choose*. So we can, without ambiguity, talk about the angular velocity of the body.

From now on, we'll assume that the origin fixed in the body is at the center of mass.

24.4: The Inertia Tensor

Regarding a rigid body as a system of individual particles, we find the kinetic energy

$$T = \sum_n \frac{1}{2} m_n v_n^2 = \sum_n \frac{1}{2} m_n \left(\vec{V} + \vec{\Omega} \times \vec{r}_n \right) \\ = \sum_n \frac{1}{2} m_n V^2 + \sum_n m_n \vec{V} \cdot \vec{\Omega} \times \vec{r}_n + \sum_n \frac{1}{2} m_n \left(\vec{\Omega} \times \vec{r}_n \right)^2 \tag{24.4.1}$$

The first term in the last line is

$$\sum_n \frac{1}{2} m_n V^2 = \frac{1}{2} M V^2 \tag{24.4.2}$$

where M is the total mass of the body.

The second term is

$$\sum_n m_n \vec{V} \cdot \vec{\Omega} \times \vec{r}_n = \vec{V} \cdot \vec{\Omega} \times \sum_n m_n \vec{r}_n = 0 \tag{24.4.3}$$

from the definition of the center of mass (our origin here) $\sum_n m_n \vec{r}_n = 0$

The third term can be rewritten:

$$\sum_n \frac{1}{2} m_n \left(\vec{\Omega} \times \vec{r}_n \right)^2 = \sum_n \frac{1}{2} m_n \left[\Omega^2 r_n^2 - \left(\vec{\Omega} \cdot \vec{r}_n \right)^2 \right] \tag{24.4.4}$$

Here we have used

$$|\vec{\Omega} \times \vec{r}| = \Omega r \sin\theta, \quad |\vec{\Omega} \cdot \vec{r}| = \Omega r \cos\theta \tag{24.4.5}$$

Alternatively, you could use the vector product identity

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{a}(\vec{b} \cdot \vec{c}) + \vec{b}(\vec{a} \cdot \vec{c}) \tag{24.4.6}$$

together with

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{c} \cdot \vec{d} \tag{24.4.7}$$

to find

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \tag{24.4.8}$$

The bottom line is that the kinetic energy

$$T = \frac{1}{2} M V^2 + \sum_n \frac{1}{2} m_n \left[\Omega^2 r_n^2 - \left(\vec{\Omega} \cdot \vec{r}_n \right)^2 \right] = T_{\text{tr}} + T_{\text{rot}} \tag{24.4.9}$$

a translational kinetic energy plus a rotational kinetic energy.

Warning about notation: at this point, things get a bit messy. The reason is that to make further progress in dealing with the rotational kinetic energy, we need to write it in terms of the individual components of the n particle position vectors $\vec{r}_n$. Following Landau and others, we'll write these components in two different ways:

$$\vec{r}_n = (x_n, y_n, z_n) \equiv (x_{n1}, x_{n2}, x_{n3}) \tag{24.4.10}$$

The x,y,z notation is helpful in giving a clearer picture of rotational energy, but the x_{ni} notation is essential in handling the math, as will become evident.

Landau's solution to the too many suffixes for clarity problem is to omit the suffix n labeling the individual particles, I prefer to keep it in.

Double Suffix Summation Notation: to cut down on the number of Σ's in expressions, we'll follow Landau and others in using Einstein's rule that if a suffix like i, j, k appears twice in a product, it is to be summed over the values 1,2,3. It's called a "*dummy*

suffix" because it doesn't matter what you label it, as long as it appears twice. For example,

the inner product of two vectors $\vec{A}\cdot\vec{B}=\sum_{i=1}^{3}A_iB_i$ can be written as A_iB_i or equally as A_kB_k. Furthermore, Ω_i^2 means $\Omega_1^2+\Omega_2^2+\Omega_3^2=\Omega^2$.

But do not use Greek letters for dummy suffixes in this context: the standard is that they are used in relativistic equations to signify sums over the four dimensions of space time, Latin letters for sums over the three spatial dimensions, as we are doing here.

The rotational kinetic energy is then

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\sum_n m_n\left(\Omega_i^2x_{ni}^2-\Omega_ix_{ni}\Omega_kx_{nk}\right)\\ &= \frac{1}{2}\sum_n m_n\left(\Omega_i\Omega_k\delta_{ik}x_{nl}^2-\Omega_i\Omega_kx_{ni}x_{nk}\right)\\ &= \frac{1}{2}\Omega_i\Omega_k\sum_n m_n\left(\delta_{ik}x_{nl}^2-x_{ni}x_{nk}\right) \end{aligned} \tag{24.4.11}$$

Warning: That first line is a bit confusing: copying Landau, I've written $\Omega_i^2x_{ni}^2$, you might think that's $\Omega_1^2x_{n1}^2+\Omega_2^2x_{n2}^2+\Omega_3^2x_{n3}^2$, but a glance at the previous equation (and the second line of this equation) makes clear it's actually Ω^2r^2. Landau should have written $\Omega_i^2x_{nl}^2$. Actually I'm not even keen on Ω_i^2 implying a double summation. Standard use in relativity, for example, is that both of the two suffixes be explicit for summation to be implied. In GR one would write $\Omega_i\Omega_i$. (Well, actually $\Omega_i\Omega^i$, but that's another story.)

Anyway, moving on, we introduce the *inertia tensor*

$$I_{ik}=\sum_n m_n\left(x_{nl}^2\delta_{ik}-x_{ni}x_{nk}\right) \tag{24.4.12}$$

In terms of which the kinetic energy of the moving, rotating rigid body is

$$T=\frac{1}{2}MV^2+\frac{1}{2}I_{ik}\Omega_i\Omega_k \tag{24.4.13}$$

As usual, the Lagrangian $L=T-V$ where the potential energy V is a function of six variables in general, the center of mass location and the orientation of the body relative to the center of mass.

Landau writes the inertia tensor explicitly as:

$$I_{ik}=\begin{bmatrix} \sum m\left(y^2+z^2\right) & -\sum mxy & -\sum mxz\\ -\sum mxy & \sum m\left(z^2+x^2\right) & -\sum myz\\ -\sum mxz & -\sum myz & \sum m\left(x^2+y^2\right) \end{bmatrix} \tag{24.4.14}$$

but you should bear in mind that $-\sum mxz$ means $-\sum_n m_nx_nz_n$.

24.5: Tensors 101

We see that the “inertia tensor” defined above as

$$I_{ik} = \sum_n m_n \left(x_{nl}^2 \delta_{ik} - x_{ni} x_{nk} \right) \tag{24.5.1}$$

is a 3×3 two-dimensional array of terms, called *components*, each of which is made up (for this particular tensor) of products of vector components.

Obviously, if we had chosen a different set of Cartesian axes from the same origin O the vector components would be different: we know how a vector transforms under such a change of axes, $(x, y, z) \rightarrow (x', y', z')$ where

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{24.5.2}$$

This can be written more succinctly as

$$x_i' = R_{ij} x_j, \text{ or } \mathbf{x}' = \mathbf{R}\mathbf{x} \tag{24.5.3}$$

the bold font indicating a vector or matrix.

In fact, a transformation from any set of Cartesian axes to any other set having the same origin is a rotation about some axis. This can easily be seen by first rotating so that the x' axis coincides with the x axis, then rotating about *that* axis. (Of course, both sets of axes must have the same handedness.) We’ll discuss these rotation transformations in more detail later, for now we’ll just mention that the inverse of a rotation is given by the transpose matrix (check for the example above),

$$\mathbf{R^T} = \mathbf{R}^{-1}, \quad \text{or} \quad R_{ji} = R_{ij}^{-1} \tag{24.5.4}$$

so if the column vector

$$x_i' = R_{ij} x_j, \text{ or } \mathbf{x}' = \mathbf{R}\mathbf{x} \tag{24.5.5}$$

the row vector

$$\mathbf{x'^T} = \mathbf{x}^{\mathrm{T}} \mathbf{R^T} = \mathbf{x^T R^{-1}} \tag{24.5.6}$$

a.k.a. $x_i' = R_{ij} x_j = x_j R_{ji}^T = x_j R_{ji}^{-1}$, and the *length* of the vector doesn’t change:

$x_i' x_i' = \mathbf{x'}^{\mathrm{T}} \mathbf{x}' = \mathbf{x}^{\mathrm{T}} \mathbf{R}^{\mathrm{T}} \mathbf{R}\mathbf{x} = \mathbf{x}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{R}\mathbf{x} = \mathbf{x}^{\mathrm{T}} \mathbf{x} = x_i x_i$

It might be worth spelling out explicitly here that the transpose of a square matrix (and almost all our matrices are square) is found by just swapping the rows and columns, or equivalently swapping elements which are the reflections of each other in the main diagonal, but the transpose of a vector, written as a column, has the same elements as a row, and the *product of vectors follows the standard rules for matrix multiplication*:

$$(AB)_{ij} = A_{ik} B_{kj} \tag{24.5.7}$$

with the dummy suffix k summed over.

Thus,

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \tag{24.5.8}$$

and

$$\mathbf{a^T a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1^2 + a_2^2 + a_3^2 \tag{24.5.9}$$

but

$$\mathbf{a}\mathbf{a}^{\mathbf{T}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{pmatrix} \tag{24.5.10}$$

This will perhaps remind you of the Hilbert space vectors in quantum mechanics: the transposed vector above is analogous to the bra, the initial column vector being the ket. One difference from quantum mechanics is that all our vectors here are real, if that were not the case it would be natural to add complex conjugation to the transposition, to give $\mathbf{a}^*\mathbf{a} = |a_1|^2 + |a_2|^2 + |a_3|^2$, the length squared of the vector.

The difference shown above between $\mathbf{a}^{\mathbf{T}}\mathbf{a}$ and $\mathbf{a}\mathbf{a}^{\mathbf{T}}$ is exactly parallel to the difference between $\langle a \mid a \rangle$ and $|a\rangle\langle a|$ in quantum mechanics—the first is a number, the norm of the vector, the second is an operator, a projection into the state $|a\rangle$

24.6: Definition of a Tensor

We have a definite rule for how vector components transform under a change of basis: $x_i' = R_{ij}x_j$. What about the components of the inertia tensor $I_{ik} = \sum_n m_n \left(x_{nl}^2 \delta_{ik} - x_{ni}x_{nk}\right)$?

We'll do it in two parts, and one particle at a time. First, take that second term for one particle, it has the form $-mx_ix_k$. But we already know how vector components transform, so this must go to

$$-mx_i'x_k' = R_{il}R_{jm}\left(-mx_lx_m\right) \tag{24.6.1}$$

The same rotation matrix R_{ij} is applied to all the particles, so we can add over n.

In fact, the inertia tensor is made up of elements exactly of this form in all nine places, plus diagonal terms mr_i^2, obviously invariant under rotation. To make this clear, we write the inertia tensor:

$$\begin{bmatrix} \sum m\left(y^2+z^2\right) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m\left(z^2+x^2\right) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m\left(x^2+y^2\right) \end{bmatrix} = \sum m\left(x^2+y^2+z^2\right)\mathbf{1} - \begin{bmatrix} \sum mx^2 & \sum mxy & \sum mxz \\ \sum mxy & \sum my^2 & \sum myz \\ \sum mxz & \sum myz & \sum mz^2 \end{bmatrix} \tag{24.6.2}$$

where 1 is the 3×3 identity matrix. (*Not* to be confused with I!)

Exercise: convince yourself that this is the same as $\mathbf{I} = \sum m\left[\left(\mathbf{x^T x}\right)\mathbf{1} - \mathbf{x x^T}\right]$

This transformation property is the definition of a two-suffix Cartesian three-dimensional tensor: just as a vector in this space can be defined as an array of three components that are transformed under a change of basis by applying the rotation matrix, $x_i' = R_{ij}x_j$, a tensor with two suffixes in the same space is a two-dimensional array of nine numbers that transform as

$T_{ij}' = R_{il}R_{jm}T_{lm}$

Writing this in matrix notation, and keeping an eye on the indices, we see that with the standard definition of a matrix product, $(\mathbf{AB})_{ij} = \mathbf{A}_{ik}\mathbf{B}_{kj}$

$$\mathbf{T'} = \mathbf{RTR^T} = \mathbf{RTR^{-1}} \tag{24.6.3}$$

(The transformation property for our tensor followed immediately from that for a vector, since our tensor is constructed from vectors, but by definition the same rule applies to *all* Cartesian tensors, which are not always expressible in terms of vector components.)

24.7: Diagonalizing the Inertia Tensor

The inertial tensor has the form of a real symmetric matrix. By an appropriate choice of axes (x_1, x_2, x_3) any such tensor can be put in diagonal form, so that

$$T_{\text{rot}} = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) \tag{24.7.1}$$

These axes, with respect to which the inertia tensor is diagonal, are called the *principal axes* of inertia, the moments about them I_1, I_2, I_3 the principal moments of inertia.

If you're already familiar with the routine for diagonalizing a real symmetric matrix, you can skip this review.

The diagonalization of the tensor/matrix proceeds as follows.

First, find the eigenvalues λ_i and corresponding eigenvectors $\mathbf{e}_i$ of the inertial tensor I :

$$\mathbf{I}\mathbf{e_i} = \lambda_i \mathbf{e_i} (i = 1, 2, 3, \text{ not summed }) \tag{24.7.2}$$

(The λ_i turn out to be the principal moments I_i, but we'll leave them as λ_i for now, we need first to establish that they're real.)

Now since I is real and symmetric, $\mathbf{I}^{\mathrm{T}} = \mathbf{I}$ the eigenvalues are real. To prove this, take the equation for $\mathbf{e}_1$ above and premultiply by the *row* vector $\mathbf{e}_1^{*\mathrm{T}}$, the complex conjugate transpose:

$$\mathbf{e_1^{*\mathrm{T}} I e_1} = \lambda_1 \mathbf{e_1^{*\mathrm{T}} e_1} \tag{24.7.3}$$

The left hand side is a real number: this can be established by taking its complex conjugate. The fact that the tensor is real and symmetric is crucial!

$$\left(e_{1i}^* I_{ij} e_{1j}\right)^* = e_{1i} I_{ij}^* e_{1j}^* = e_{1i} I_{ji} e_{1j}^* = e_{1j}^* I_{ji} e_{1i} \tag{24.7.4}$$

And since these are dummy suffixes, we can swap the i 's and j 's to establish that this number is identical to its complex conjugate, hence it's real. Clearly, $\mathrm{e}_1^{*\mathrm{T}}\mathrm{e}_1$ is real and positive, so the eigenvalues are real.

(*Note*: a real symmetric matrix does not necessarily have *positive* roots: for example $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Taking the eigenvalues to be distinct (the degenerate case is easy to deal with) the eigenvectors are orthogonal, by the standard proof, for this matrix left eigenvectors (rows) have the same eigenvalues as their transpose, so

$$\mathbf{e_2^{\mathrm{T}} I e_1} = \lambda_2 \mathbf{e_2^{\mathrm{T}} e_1} = \lambda_1 \mathbf{e_2^{\mathrm{T}} e_1} \tag{24.7.5}$$

and $\mathbf{e}_2^{\mathrm{T}}\mathbf{e}_1 = 0$.

The diagonalizing matrix is made up of these eigenvectors (assumed normalized):

$$\mathbf{R} = \begin{pmatrix} \mathbf{e_1^{\mathrm{T}}} \\ \mathbf{e_2^{\mathrm{T}}} \\ \mathbf{e_3^{\mathrm{T}}} \end{pmatrix} \tag{24.7.6}$$

a column of row vectors.

To check that this is indeed a rotation vector, from one orthogonal set of axes to another, notice first that its transpose $\mathbf{R}^{\mathrm{T}} = (\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3\,)$ is its inverse (as required for a rotation), since the eigenvectors form an orthonormal set.

Now apply this R to an arbitrary vector:

$$\mathbf{x}' = \mathbf{R}\mathbf{x} = \begin{pmatrix} \mathbf{e_1^{\mathrm{T}}} \\ \mathbf{e_2^{\mathrm{T}}} \\ \mathbf{e_3^{\mathrm{T}}} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{e_1^{\mathrm{T}} x} \\ \mathbf{e_2^{\mathrm{T}} x} \\ \mathbf{e_3^{\mathrm{T}} x} \end{pmatrix} \tag{24.7.7}$$

In vector language, these elements are just $\vec{e}_1 \cdot \vec{x}$, etc., so $x_1' = \vec{e}_1 \cdot \vec{x}$, the primed components are just the components of $\vec{x}$ along the eigenvector axes, so the operator R gives the vector components relative to these axes, meaning it has rotated the coordinate

system to one with the principal axes of the body are now the x_1, x_2, x_3 axes.

We can confirm this by applying the rotation to the inertia tensor itself:

$$\mathbf{I}' = \mathbf{RTR}^{\mathbf{T}} = \begin{pmatrix} \mathbf{e}_1^{\mathrm{T}} \\ \mathbf{e}_2^{\mathrm{T}} \\ \mathbf{e}_3^{\mathrm{T}} \end{pmatrix} \mathbf{I} \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1^{\mathrm{T}} \\ \mathbf{e}_2^{\mathrm{T}} \\ \mathbf{e}_3^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \lambda_1 \mathbf{e}_1 & \lambda_2 \mathbf{e}_2 & \lambda_3 \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{24.7.8}$$

Let's examine the contribution of one particle to the inertia tensor:

$$\mathbf{I_1} = m \left[\left(\mathbf{x}^{\mathbf{T}} \mathbf{x} \right) \mathbf{1} - \mathbf{x} \mathbf{x}^{\mathbf{T}} \right] \tag{24.7.9}$$

Note that x here represents the column vector of the particle coordinates, in other words, it's just $\vec{r}$! And, watch out for the inertia tensor I and the unit tensor 1.

They transform as $\mathbf{x}' = \mathbf{Rx}$, note that this agrees with $\mathbf{I}' = \mathbf{RIR}^{\mathbf{T}}$. Since under rotation the length of a vector is invariant $\mathbf{x}'^{\mathbf{T}} \mathbf{x}' = \mathbf{x}^{\mathbf{T}} \mathbf{x}$, and $\mathbf{R} \mathbf{x} \mathbf{x}^{\mathbf{T}} \mathbf{R}^{\mathbf{T}} = \mathbf{x}' \mathbf{x}'^{\mathbf{T}}$ it is evident that in the rotated frame (the eigenvector frame) the single particle contributes to the *diagonal* elements

$$m \left[\left(x_2^2 + x_3^2 \right), \left(x_3^2 + x_1^2 \right), \left(x_1^2 + x_2^2 \right) \right] \tag{24.7.10}$$

. We've dropped the primes, since we'll be working in this natural frame from now on.

24.8: Principal Axes Form of Moment of Inertia Tensor

We already know that the transformed matrix is diagonal, so its form has to be

$$\sum_n m_n \begin{pmatrix} x_{n2}^2 + x_{n3}^2 & 0 & 0 \\ 0 & x_{n3}^2 + x_{n1}^2 & 0 \\ 0 & 0 & x_{n1}^2 + x_{n2}^2 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{24.8.1}$$

The moments of inertia, the diagonal elements, are of course all positive. Note that no one of them can exceed the sum of the other two, although it can be equal in the (idealized) case of a two-dimensional object. For that case, taking it to lie in the (x, y) plane,

$$I_z = \sum_n \left(x_n^2 + y_n^2 \right) = I_x + I_y \tag{24.8.2}$$

24.9: Relating Angular Momentum to Angular Velocity

It's easy to check that the angular momentum vector is

$L_i = I_{ij}\Omega_j$

since

$$\mathbf{L} = \sum \vec{r}_n \times m_n \vec{v}_n = \sum m_n \vec{r}_n \times \left(\vec{\Omega} \times \vec{r}_n\right) = \vec{\Omega} \sum m_n r_n^2 - \sum m_n \vec{r}_n \left(\vec{\Omega} \cdot \vec{r}_n\right) = \mathbf{I}\vec{\Omega} \tag{24.9.1}$$

Exercise: verify this by putting in all the suffixes.

24.10: Symmetries, Other Axes, the Parallel Axis Theorem

If a body has an axis of symmetry, the center of mass must be on that axis, and it is a principal axis of inertia. To prove the center of mass statement, note that the body is made up of pairs of equal mass particles on opposite sides of the axis, each pair having its center of mass on the axis, and the body's center of mass is that of all these pairs centers of mass, all of which are on the axis.

Taking this axis to be the x axis, symmetry means that for each particle at (x, y, z) there is one of equal mass at $(x, -y, -z)$, so the off-diagonal terms in the x row and column, $-\sum mxy, \quad -\sum mxz$ all add up to zero, meaning this is indeed a principal axis.

The moment of inertia about an arbitrary axis through the center of mass, in the direction of the unit vector $\hat{\vec{b}}$ is

$$\sum m\left(\vec{r}^2-(\vec{r}\cdot\hat{b})^2\right)=\hat{\vec{b}}^{\mathbf{T}}\mathbf{I}\hat{\vec{b}}=b_x^2I_x+b_y^2I_y+b_z^2I_z \tag{24.10.1}$$

The inertia tensor about some origin O' located at position a relative to the center of mass is easily found to be

$$I'_{ik}=I_{ik}+M\left(\mathbf{a}^2\delta_{ik}-\mathbf{a}_i\mathbf{a}_k\right) \tag{24.10.2}$$

In particular, we have the *parallel axis theorem*: the moment of inertia about any axis through some point O' equals that about the parallel axis through the center of mass O plus $Ma_\perp^2$, where $a_\perp$ is the perpendicular distance between the axes.

Exercise: check this!

CHAPTER OVERVIEW

25: Moments of Inertia and Rolling Motion

25.1: Examples of Moments of Inertia
25.2: Analyzing Rolling Motion
25.3: Rolling Without Slipping - Two Views
25.4: Cylinder Rolling Inside another Cylinder

25.1: Examples of Moments of Inertia

Molecules

The moment of inertia of the hydrogen molecule was historically important. It's trivial to find: the nuclei (protons) have 99.95% of the mass, so a classical picture of two point masses m a fixed distance a apart gives $I = \frac{1}{2}ma^2$. In the nineteenth century, the mystery was that equipartition of energy, which gave an excellent account of the specific heats of almost all gases, didn't work for hydrogen—at low temperatures, apparently these diatomic molecules didn't spin around, even though they constantly collided with each other. The resolution was that the moment of inertia was so low that a lot of energy was needed to excite the first quantized angular momentum state, $L = \hbar$ This was not the case for heavier diatomic gases, since the energy of the lowest angular momentum state $E = L^2/2I = \hbar^2/2I$, is lower for molecules with bigger moments of inertia .

Here's a simple *planar* molecule:

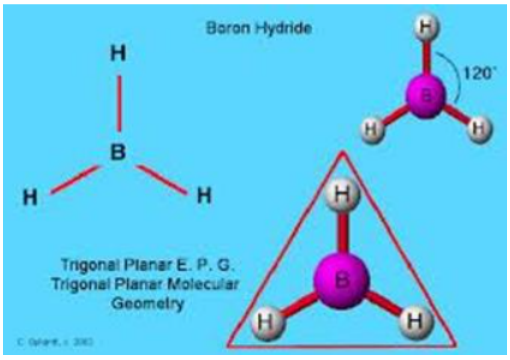


Figure 25.1.1

Obviously, one principal axis is through the centroid, perpendicular to the plane. We've also established that any axis of symmetry is a principal axis, so there are evidently three principal axes in the plane, one along each bond! The only interpretation is that there is a degeneracy: there are two equal-value principal axes in the plane, and any two perpendicular axes will be fine. The moment of inertial about either of these axes will be one-half that about the perpendicular-to-the-plane axis.

What about a symmetrical three dimensional molecule?

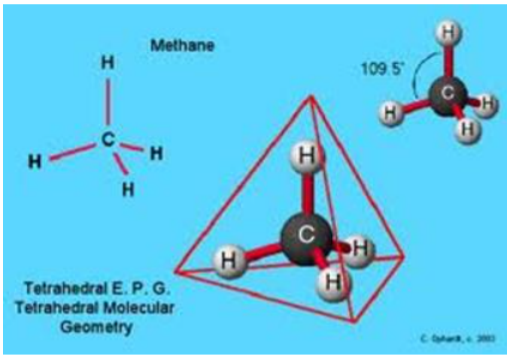


Figure 25.1.1

Here we have four obvious principal axes: only possible if we have spherical degeneracy, meaning all three principal axes have the same moment of inertia.

Various Shapes

A thin rod, linear mass density λ, length ℓ:

$$I = 2\int_0^{\ell/2} \lambda x^2 dx = 2\lambda\ell^3/24 = \frac{1}{12}m\ell^2 \tag{25.1.1}$$

A square of mass m, side ℓ, about an axis in its plane, through the center, perpendicular to a side:

$$I = \frac{1}{12} m\ell^2 \tag{25.1.2}$$

(It's just a row of rods.) in fact, the moment is the same about any line in the plane through the center, from the symmetry, and the moment about a line perpendicular to the plane through the center is twice this—that formula will then give the moment of inertia of a cube, about any axis through its center.

A disc of mass M, radius a and surface density σ has

$$I = \int_0^a r^2 \cdot \sigma \cdot 2\pi r dr = \frac{1}{2}\pi a^4 \sigma = \frac{1}{2} M a^2 \tag{25.1.3}$$

This is also correct for a cylinder (think of it as a stack of discs) about its axis.

A disc about a line through its center *in its plane* must be $\frac{1}{4}Ma^2$ from the perpendicular axis theorem. A solid cylinder about a line through its center *perpendicular* to its main axis can be regarded as a stack of discs, of radius a, height h, taking the mass of a disc as ρdz and using the parallel axes theorem,

$$I = 2\int_0^{h/2} \rho dz \left(\frac{1}{4}a^2 + z^2\right) = \frac{1}{4}Ma^2 + \frac{1}{12}Mh^2 \tag{25.1.4}$$

For a sphere, a stack of discs of varying radii,

$$I = \int_{-a}^{a} dz \frac{1}{2}\rho\pi \left(a^2 - z^2\right)^2 = \frac{8}{15}\rho\pi a^5 = \frac{2}{5}Ma^2 \tag{25.1.5}$$

An ellipsoid of revolution and a sphere of the *same mass and radius* clearly have the same motion of inertial about their common axis (shown).

Figure 25.1.1

Moments of Inertia of a Cone

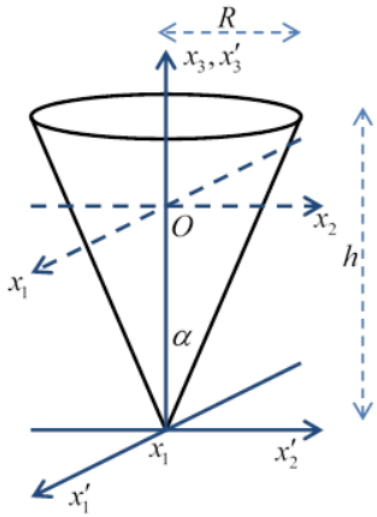


Figure 25.1.1

Following Landau, we take height h and base radius R and semivertical angle α so that $R = h\tan\alpha$.

As a preliminary, the volume of the cone is

$V = \int_0^h \pi r^2 dz = \int_0^h \pi \left(\frac{Rz}{h}\right)^2 dz = \frac{1}{3}\pi R^2 h$

The center of mass is distance a from the vertex, where

$$aV = a \cdot \frac{1}{3}\pi R^2 h = \int_0^h z dV = \int_0^h \pi z \left(\frac{Rz}{h}\right)^2 dz = \frac{1}{4}\pi R^2 h^2, \quad a = \frac{3}{4}h \tag{25.1.6}$$

The moment of inertia about the central axis of the cone is (taking density ρ) that of a stack of discs each having mass

$$m(dz) = \pi r^2 \rho dz = \pi \left(\frac{Rz}{h}\right)^2 \rho dz \tag{25.1.7}$$

and moment of inertia $I(dz) = \frac{1}{2} m(dz) r^2$

$$\int_0^h \frac{1}{2}\pi\rho \left(\frac{Rz}{h}\right)^4 dz = \frac{1}{10}\pi\rho R^4 h = \frac{3}{10} M R^2 \tag{25.1.8}$$

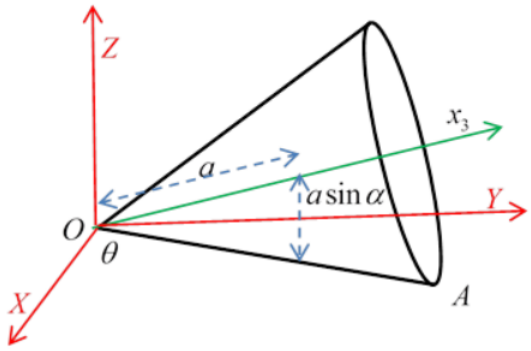
The moment of inertia about the axis x_1' through the vertex, perpendicular to the central axis, can be calculated using the stack-of-discs parallel axis approach, the discs having mass $\pi\rho\left(\frac{Rz}{h}\right)^2 dz$, it is

$$\int_0^h \pi\rho \left(\frac{Rz}{h}\right)^2 \left[\frac{1}{4}\left(\frac{Rz}{h}\right)^2 + z^2\right] dz = \frac{1}{20}\pi\rho R^4 h + \frac{1}{5}\pi\rho R^2 h^3 = \frac{3}{20} M R^2 + \frac{3}{5} M h^2 \tag{25.1.9}$$

25.2: Analyzing Rolling Motion

Kinetic Energy of a Cone Rolling on a Plane

The cone rolls without slipping on the horizontal XY plane. The momentary line of contact with the plane is OA, at an angle θ in the horizontal plane from the X axis.



The important point is that this line of contact, *regarded as part of the rolling cone*, is momentarily at rest when it's in contact with the plane. This means that, *at that moment*, the cone is rotating about the stationary line OA. Therefore, the angular velocity vector $\vec{\Omega}$ points along OA.

Taking the cone to have semi-vertical angle α (meaning this is the angle between OA and the central axis of the cone) the center of mass, which is a distance a from the vertex, and on the central line, moves along a circle at height $a\sin\alpha$ above the plane, this circle being centered on the Z axis, and having radius $a\cos\alpha$. The center of mass moves at velocity $V = \dot{\theta} a\cos\alpha$, so contributes *translational* kinetic energy

$$\frac{1}{2}MV^2 = \frac{1}{2}M\dot{\theta}^2 a^2\cos^2\alpha \tag{25.2.1}$$

Now visualize the rolling cone turning around the momentarily fixed line OA: the center of mass, at height $a\sin\alpha$, moves at V, so the angular velocity

$$\Omega = \frac{V}{a\sin\alpha} = \dot{\theta}\cot\alpha. \tag{25.2.2}$$

Next, we first define a new set of axes with origin O: one, $x_3$is the cone's own center line, another, x_2 is perpendicular to that *and* to OA, this determines x_1 (For these last two, since they're through the vertex, the moment of inertia is the one worked out at the end of the previous section, see above.)

Since $\vec{\Omega}$ is along $OA,$ its components with respect to these axes (x_1, x_2, x_3) are $(\Omega\sin\alpha, 0, \quad \Omega\cos\alpha)$

However, to compute the total kinetic energy, for the rotational contribution we need to use a parallel set of axes *through the center of mass*. This just means subtracting from the vertex perpendicular moments of inertia found above a factor Ma^2.

The total kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}M\dot{\theta}^2 a^2\cos^2\alpha + \frac{1}{2}I_1\dot{\theta}^2\cos^2\alpha + \frac{1}{2}I_3\dot{\theta}^2\frac{\cos^4\alpha}{\sin^2\alpha} \\ &= 3Mh^2\dot{\theta}^2\left(1+5\cos^2\alpha\right)/40 \end{aligned}$$

using

$$I_1 = \frac{3}{20}MR^2 + \frac{3}{80}Mh^2, \quad I_3 = \frac{3}{10}MR^2, \quad a = \frac{3}{4}h, \quad R = h\tan\alpha \tag{25.2.3}$$

25.3: Rolling Without Slipping - Two Views

Think of a hoop, mass M radius R, rolling along a flat plane at speed V. It has translational kinetic energy $\frac{1}{2}MV^2$, angular velocity $\Omega = V/R$, and moment of inertia $I = MR^2$ so its angular kinetic energy $\frac{1}{2}I\Omega^2 = \frac{1}{2}MV^2$ and its total kinetic energy is MV^2.

But we could also have thought of it as *rotating about the point of contact*—remember, that point of the hoop is momentarily at rest. The angular velocity would again be Ω, but now with moment of inertia, from the parallel axes theorem, $I = MR^2 + MR^2 = 2MR^2$, giving same total kinetic energy, but now all rotational.

25.4: Cylinder Rolling Inside another Cylinder

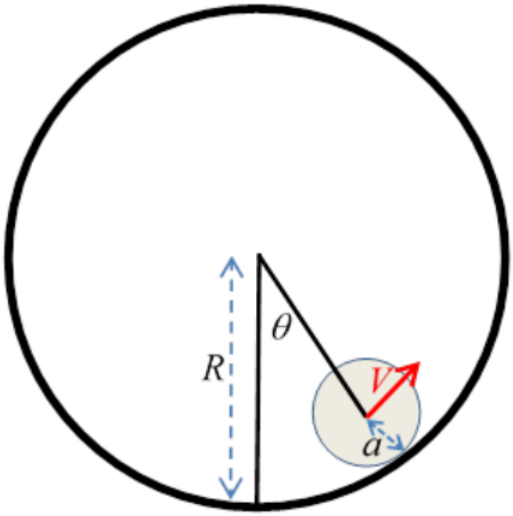


Figure 25.4.1

Now consider a solid cylinder radius a rolling inside a hollow cylinder radius R, angular distance from the lowest point θ, the solid cylinder axis moving at $V=(R-a)\dot{\theta}$ and therefore having angular velocity (compute about the point of contact) $\Omega=V/a$.

The kinetic energy is

$$\frac{1}{2}MV^2+\frac{1}{2}I(V/a)^2=\frac{1}{2}\left(M+\frac{I}{a^2}\right)(R-a)^2\dot{\theta}^2 \tag{25.4.1}$$

The potential energy is $-Mg(R-a)\cos\theta$.

The Lagrangian $L=T-V$, the equation of motion is

$$\left(M+\frac{I}{a^2}\right)(R-a)^2\ddot{\theta}=-Mg(R-a)\sin\theta\cong-Mg(R-a)\theta \tag{25.4.2}$$

so small oscillations are at frequency $\omega=\sqrt{\dfrac{g}{\left(1+\dfrac{I}{Ma^2}\right)(R-a)}}$.

CHAPTER OVERVIEW

26: Rigid Body Moving Freely

26.1: Angular Momentum and Angular Velocity

26.2: Precession of a Symmetrical Top

26.3: Throwing a Football…

26.4: Equations of Motion for Rigid Body with External Forces

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26.1: Angular Momentum and Angular Velocity

In contrast to angular velocity, the angular momentum of a body depends on the point with respect to which it is defined. For now, we take it (following Landau, of course) as relative to the center of mass, but we denote it by $\vec{L}$, following modern usage. This "intrinsic" angular momentum is like the Earth's angular momentum from its diurnal rotation, as distinct from its orbital angular momentum in going around the Sun.

That is

$$\vec{L} = \sum_n \vec{r}_n \times m_n \vec{v}_n = \sum_n \vec{r}_n \times m_n \left(\vec{\Omega} \times \vec{r}_n \right) = \sum_n m_n \left[r_n^2 \vec{\Omega} - \vec{r}_n \left(\vec{r}_n \cdot \vec{\Omega} \right) \right] = \mathbf{I} \vec{\Omega} \tag{26.1.1}$$

where I is the inertia tensor: this just means $L_i = I_{ik} \Omega_k$

Explicitly, taking the principal axes as the (x_1, x_2, x_3) axes,

$$L_1 = I_1 \Omega_1, \quad L_2 = I_2 \Omega_2, \quad L_3 = I_3 \Omega_3 \tag{26.1.2}$$

For anything with spherical inertial symmetry (such as a cube or a tetrahedron!) $\vec{L} = I \vec{\Omega}$

Landau defines a *rotator* as a collection of massive particles all on a line. (I guess that includes diatomic molecules, and, for example, CO_2, neglecting electrons and nuclear size). We know there are only two physical rotational degrees of freedom for these molecular rotators (thanks to quantum mechanics) and obviously the two principal axes are perpendicular to the line of masses, and degenerate. Again, then, $\vec{L} = I \vec{\Omega}$.

26.2: Precession of a Symmetrical Top

A more interesting case is the free rotation (zero external torque) of a symmetrical top, meaning $I_1 = I_2 \neq I_3$.

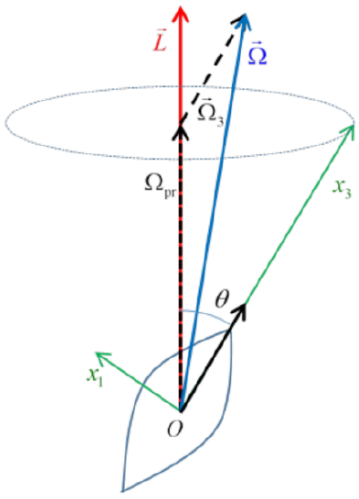


Figure 26.2.1

We can take any pair of orthogonal axes, perpendicular to the body's symmetry axis, as the x_1, x_2 axes. We'll choose x_2 following Landau, as perpendicular to the plane containing $\vec{L}$ and the momentary position of the x_3 axis, so in the diagram here x_2 is perpendicularly out from the paper/screen, towards the viewer.

This means the angular momentum component $L_2 = 0$ and therefore $\Omega_2 = 0$. Hence $\vec{\Omega}$ is in the same plane as $\vec{L}, x_3,$ and so the velocity $\vec{v} = \vec{\Omega} \times \vec{r}$ of every point on the axis of the top is perpendicular to this plane (into the paper/screen). The axis of the top Ox_3 must be rotating uniformly about the direction of $\vec{L}$.

The spin rate of the top around its own axis is

$$\Omega_3 = L_3/I_3 = (L/I_3)\cos\theta \tag{26.2.1}$$

The angular velocity vector $\vec{\Omega}$ can be written as a sum of two components, one along the body's axis Ox_3 and one parallel to the angular momentum $\vec{L}$ (these components are shown dashed in the figure)

$$\vec{\Omega} = \vec{\Omega}_{\text{precession}} + \vec{\Omega}_3 \tag{26.2.2}$$

The component along the body's axis Ox_3 does not contribute to the precession, which all comes from the component along the (fixed in space) angular momentum vector.

The speed of precession follows from

$$\Omega_{\text{precession}} \sin\theta = \Omega_1 \tag{26.2.3}$$

and

$$\Omega_1 = L_1/I_1 = (L/I_1)\sin\theta \tag{26.2.4}$$

so

$$\Omega_{\text{precession}} = L/I_1 \tag{26.2.5}$$

Note also the ratio of precession rate to spin around axis is

$$\Omega_{\text{precession}}/\Omega_3 = (I_3/I_1)\sec\theta \tag{26.2.6}$$

This means the precession rate and the spin are very comparable, except when θ is near $\pi/2$, when the precession becomes much faster. Remember this is the body's precession with *no external torque*, and is clearly completely different—much faster precession—than the familiar case of a fast spinning top under gravity.

26.3: Throwing a Football…

If you throw a football and manage to give it only spin about the long axis, it will stay pointing that way (apart from drag effects, which will tend to line up spin direction with velocity). If when you throw it you also add some angular velocity along a shorter axis, it will precess (wobble). Given the angle, the ratio of precession to spin is fixed.

If you take a ball that is already spinning fast about its long axis, have the two ends of the long axis (its points) in your palms, then, as you throw it, give it a quick twist by moving one hand downwards and the other up as you throw, to give it significant angular velocity about a short axis, at the same time keeping the fast spin about the long axis, once the ball leaves your hands, the angle between the spin and the total angular momentum, the angle of wobble, is completely determined by the ratio of the two angular velocities.

26.4: Equations of Motion for Rigid Body with External Forces

Translation

A free rigid body has six degrees of freedom (for instance, the coordinates of the center of mass and the orientation of the body). Therefore, there are six equations of motion, three for the rate of change of spatial position of the center of mass, in other words for the components of the velocity $\vec{V}$, and three for the rate of change of orientation, the angular velocity $\vec{\Omega}$.

These equations are of course nothing but Newton's laws, easily derived by summing over the set of equations $\vec{f}_i = d\left(m_i \vec{v}_i\right)/dt$ for each particle.

Denoting the total momentum of the body by $\vec{P}$.

$$\sum_n \frac{d}{dt}(m_n \vec{v}_n) = \frac{d\vec{P}}{dt} = \sum_n \vec{f}_n = \vec{F} \tag{26.4.1}$$

and $\vec{P} = M\vec{V}$, where $\vec{V} = d\vec{R}/dt$ is the velocity of the center of mass. (This can be established by differentiating with respect to time the definition of the center of mass, $M\vec{R} = \sum_n m_n \vec{r}_n$.)

The total force on all the particles is a sum of the total external force on the body and the sum of internal forces between particles—but these internal forces come in equal and opposite pairs, from Newton's Third Law, and therefore add to zero.

The bottom line, then, is that the rate of change of momentum of a rigid body equals the total external force on the body. If this force is from a time-independent potential, then

$$\vec{F} = -\partial V/\partial \vec{R} \tag{26.4.2}$$

because if the body is moved through $\delta\vec{R}$ (without rotation, hence the *partial* derivative), each individual particle moves through the same $\delta\vec{R}$, the work done by the external potential on the n^{th} particle is $\vec{f}_n^{\text{ext}} \cdot \delta\vec{R} = -\delta V_n$, and summing over all the particles gives $\vec{F} \cdot \delta\vec{R} = -\delta V_{\text{tot}}$, giving the above equation as $\delta\vec{R} \to 0$.

Rotation

To derive the equation of motion for *rotation* of a rigid body, *we choose the inertial frame in which the center of mass is momentarily at rest*, and take the center of mass as the origin.

The rate of change of angular momentum about the center of mass (origin),

$$\vec{L} = (d/dt)\sum_n \vec{r}_n \times \vec{p}_n = \sum_n \left[\left(\dot{\vec{r}}_n \times \vec{p}_n\right) + \left(\vec{r}_n \times \overrightarrow{\vec{p}}_n\right)\right] = \sum_n \vec{r}_n \times \vec{f}_n = \vec{K} \tag{26.4.3}$$

where we dropped the $\dot{\vec{r}}_n \times \vec{p}_n$ term because $\dot{\vec{r}}_n = \vec{v}_n$ is parallel to $\vec{p}_n = m\vec{v}_n$, then we used $\vec{f}_n = \vec{p}_n$ to get the total moment of the external forces about the center of mass, the *torque*.

The angular momentum *about the center of mass* is the same in any inertial frame, since the extra term on adding a velocity $\vec{v}_0$ to each mass is

$$\sum \vec{r}_n \times m_n \vec{v}_0 = -\vec{v}_0 \times \sum m_n \vec{r}_n = 0 \tag{26.4.4}$$

from the definition of the center of mass.

If the center of mass is *not* at the origin, denote the particle coordinates by $\vec{\rho}_n = \vec{R} + \vec{r}_n$ in the usual notation, so

$$\vec{L}_{\text{new origin}} = \sum_n \vec{\rho}_n \times m_n \vec{v}_n = \sum_n \vec{r}_n \times m_n \vec{v}_n + \sum_n \vec{R} \times m_n \vec{v}_n = \vec{L}_{\text{cm}} + \vec{R} \times \vec{P} \tag{26.4.5}$$

a sum of an intrinsic ("spin") angular momentum and an extrinsic ("orbital")angular momentum.

Similarly, if the torque of external forces relative to the center of mass is $\vec{K} = \sum_n \vec{r}_n \times \vec{f}_n$ as defined above, then relative to the new origin the torque is

$$\vec{K}_{\text{new origin}} = \sum_n \vec{\rho}_n \times \vec{f}_n = \sum_n \vec{R} \times \vec{f}_n + \sum_n \vec{r}_n \times \vec{f}_n = \vec{R} \times \vec{F} + \vec{K}_{\text{cm}} \tag{26.4.6}$$

that is, the torque about the new origin is the torque about the center of mass *plus* the torque about the new origin of the total external force acting at the center of mass.

An important special case is that of a *couple*: a pair of equal but oppositely directed forces, acting along parallel but separated lines (like two hands oppositely placed turning a steering wheel). The forces add to zero, so from the above equation a couple exerts the same torque about any origin.

More generally, the term couple is often used (including by Landau) to refer to any set of forces that add to zero, but give a nonzero torque because of their lines of action, and such a set give the same torque about any origin.

Exercise: prove that for a rigid body freely falling in a uniform gravitational field, the angular momentum about the center of mass remains constant, but about another point it will in general be changing. What about a charged rigid body moving in space (no gravity) through a uniform *electric* field?

CHAPTER OVERVIEW

27: Euler Angles

So far, our analysis of rotational motion has been of essentially one dimensional, or more precisely one angular parameter, motion: rotating about an axis, rolling, precessing and so on. But this leaves out many interesting phenomena, for example the wobbling of a slowing down top, nutation, and so on. We need a well-defined set of parameters for the orientation of a rigid body in space to make further progress in analyzing the dynamics.

The standard set is Euler's Angles. What you see as you watch a child's top beginning to wobble as it slows down is the direction of the axis—this is given by the first two of Euler's angles: θ, ϕ the usual spherical coordinates, the angle θ from the vertical direction and the azimuthal angle ϕ about that vertical axis. Euler's third angle, ψ, specifies the orientation of the top about its own axis, completing the description of the precise positioning of the top. To describe the motion of the wobbling top as we see it, we evidently need to cast the equations of motion in terms of these angles.

27.1: Definition of Euler Angles

27.2: Angular Velocity and Energy in Terms of Euler's Angles

27.3: Free Motion of a Symmetrical Top

27.4: Motion of Symmetrical Top around a Fixed Base with Gravity - Nutation

27.5: Steady Precession

27.6: Stability of Top Spinning about Vertical Axis

27.1: Definition of Euler Angles

The rotational motion of a rigid body is completely defined by tracking the set of principal axes (x_1, x_2, x_3), with origin at the center of mass, as they turn relative to a set of fixed axes (X,Y,Z). The principal axes can be completely defined relative to the fixed set by three angles: the two angles (θ, ϕ) fix the direction of x_3, but that leaves the pair x_1, x_2 free to turn in the plane perpendicular to x_3, the angle ψ fixes their orientation.

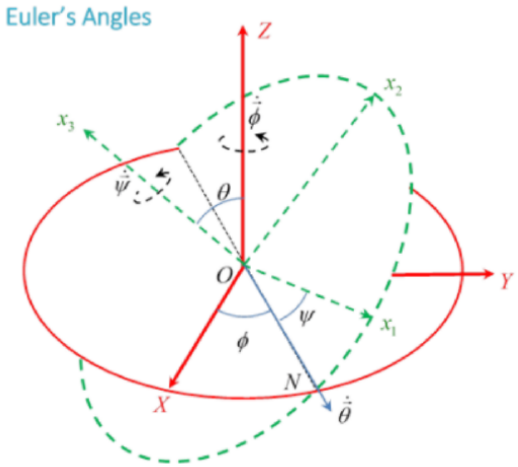


Figure 27.1.1: θ, ϕ follow standard physics practice for labeling the direction of body axis x_3 relative to lab axes X, Y, Z, ψ is the body rotation angle from ON to the x_1 axis in the x_1, x_2 plane, about its x_3 axis.

To see these angles, start with the fixed axes, draw a circle centered at the origin in the horizontal X,Y plane. Now draw a circle of the same size, also centered at the same origin, but in the principal axes x_1, x_2 plane. Landau calls the line of intersection of these circles (or discs) the *line of nodes*. It goes through the common origin, and is a diameter of both circles.

The angle between these two planes, which is also the angle between Z, x_3 (since they're the perpendiculars to the planes) is labeled θ.

The angle between this line of nodes and the X axis is ϕ. It should be clear that θ, ϕ together fix the direction of x_3, then the other axes are fixed by giving ψ, the angle between x_1 and the line of nodes ON. The direction of measurement of ϕ, ψ around Z, x_3 are given by the right-hand or corkscrew rule.

27.2: Angular Velocity and Energy in Terms of Euler's Angles

Since the position is uniquely defined by Euler's angles, angular velocity is expressible in terms of these angles and their derivatives.

The strategy here is to find the angular velocity components along the body axes x_1, x_2, x_3 of $\dot{\theta}, \dot{\phi}, \dot{\psi}$ in turn. Once we have the angular velocity components along the principal axes, the kinetic energy is easy.

Note

You might be thinking: wait a minute, aren't the axes embedded in the body? Don't they turn with it? How can you talk about rotation about these axes? Good point: what we're doing here is finding the components of angular velocity about a set of axes *fixed in space*, not the body, but *momentarily coinciding* with the principal axes of the body.

From the diagram, $\dot{\theta}$ is along the line ON, and therefore in the x_1, x_2 plane: notice it is at an angle $-\psi$ with respect to x_1. Its components are therefore $\vec{\dot{\theta}} = (\dot{\theta}\cos\psi, -\dot{\theta}\sin\psi, 0)$.

Now $\dot{\phi}$ is about the Z axis. The principal axis x_3 is at angle θ to the Z axis, so $\vec{\phi}$ has component $\dot{\phi}\cos\theta$ about x_3, and $\dot{\phi}\sin\theta$ in the x_1, x_2 plane, that latter component along a line perpendicular to ON, and therefore at angle $-\psi$ from the x_2 axis. Hence $\vec{\dot{\phi}} = (\dot{\phi}\sin\theta\sin\psi, \dot{\phi}\sin\theta\cos\psi, \dot{\phi}\cos\theta)$

The angular velocity $\dot{\psi}$ is already along a principal axis, x_3.

To summarize, the Euler angle angular velocities (components along the body's principal axes) are:

$$\begin{aligned}\vec{\dot{\theta}} &= (\dot{\theta}\cos\psi, -\dot{\theta}\sin\psi, 0)\\ \vec{\dot{\phi}} &= (\dot{\phi}\sin\theta\sin\psi, \dot{\phi}\sin\theta\cos\psi, \dot{\phi}\cos\theta)\\ \dot{\psi} &= (0, 0, \dot{\psi})\end{aligned} \tag{27.2.1}$$

from which, the angular velocity components along those in-body axes x_1, x_2, x_3 are:

$$\begin{aligned}\Omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\\ \Omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\\ \Omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}\end{aligned}$$

For a symmetric top, meaning $I_1 = I_2 \neq I_3$, the rotational kinetic energy is therefore

$$T_{\text{rot}} = \frac{1}{2}I_1\left(\Omega_1^2 + \Omega_2^2\right) + \frac{1}{2}I_3\Omega_3^2 = \frac{1}{2}I_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$

For this symmetrical case, as Landau points out, we could have taken the x_1 axis momentarily along the line of nodes ON, giving

$$\vec{\Omega} = (\dot{\theta}, \dot{\phi}\sin\theta, \dot{\phi}\cos\theta + \dot{\psi})$$

27.3: Free Motion of a Symmetrical Top

As a warm up in using Euler's angles, we'll redo the free symmetric top covered in the last lecture. With *no external torques acting* the top will have constant angular momentum $\vec{L}$.

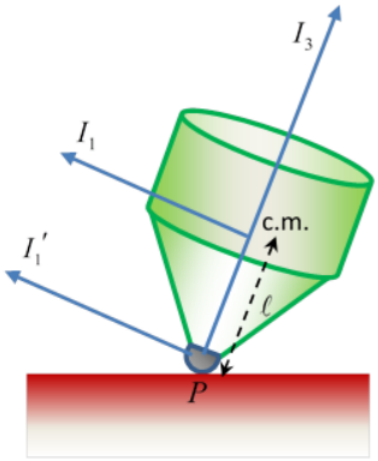


Figure 27.3.1:

We'll take $\vec{L}$ in the fixed Z direction. The axis of the top is along x_3.

Taking the x_1 axis along the line of nodes ON (Figure 27.3.1) at the instant considered, the constant angular

$$\begin{aligned}\vec{L} &= (I_1\Omega_1, I_1\Omega_2, I_3\Omega_3) \\ &= \left(I_1\dot{\theta}, \quad I_1\dot{\phi}\sin\theta, \quad I_3(\dot{\phi}\cos\theta + \dot{\psi})\right)\end{aligned}$$

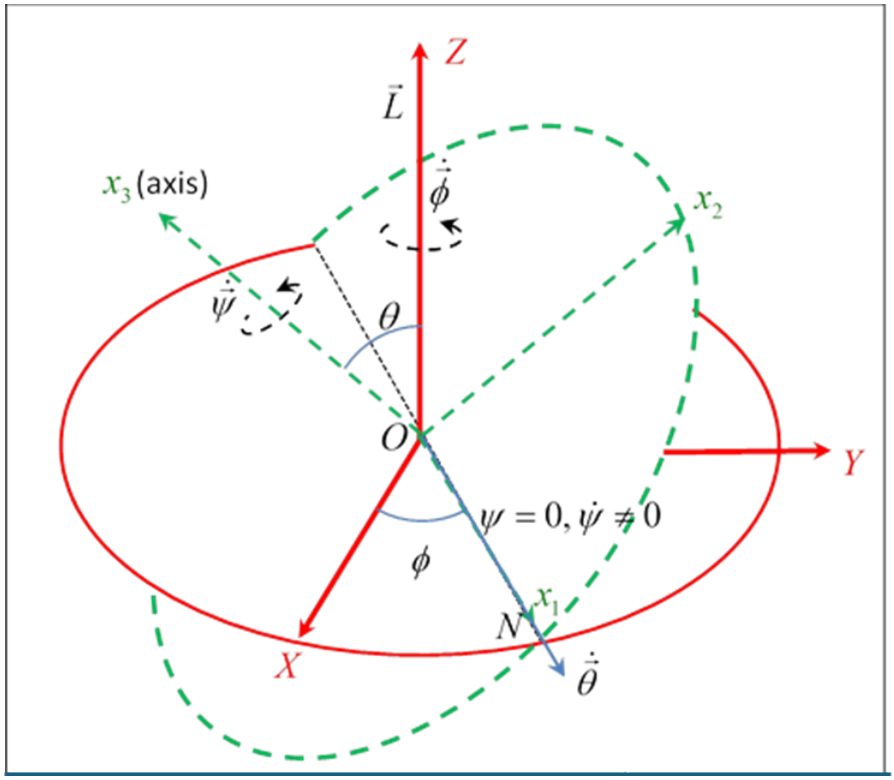


Figure 27.3.1: Free motion of symmetric top: Constant $vecL$ along fixed Z

Remember, this new x_1 axis (Figure 27.3.1) is perpendicular to the Z axis we've taken $\vec{L}$ along, so $L_1 = I_1\dot{\theta} = 0$, and θ is constant, meaning that the principal axis x_3 describes a cone around the constant angular momentum vector $\vec{L}$. The rate of precession follows from the constancy of $L_2 = I_1\dot{\phi}\sin\theta$. Writing the absolute magnitude of the angular momentum as L, $L_2 = L\sin\theta$ (remember L is in the Z direction, and x_1 is momentarily along ON) so the rate of precession $\dot{\phi} = L/I_1$. Finally, the component of $\vec{L}$ along the x_3 axis of symmetry of the top is $L\cos\theta = I_3\Omega_3$, so the top's spin along its own axis is $\Omega_3 = (L/I_3)\cos\theta$.

27.4: Motion of Symmetrical Top around a Fixed Base with Gravity - Nutation

Denoting the distance of the center of mass from the fixed bottom point P as ℓ (along the axis) the moment of inertia about a line perpendicular to the axis at the base point is

$$I_1' = I_1 + M\ell^2 \tag{27.4.1}$$

(I_1 being usual center of mass moment.)

The Lagrangian is (P being the origin, I_3 in direction θ, ϕ)

$$L = \frac{1}{2} I_1' \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2 - Mg\ell\cos\theta \tag{27.4.2}$$

Notice that the coordinates ψ, ϕ do not appear explicitly, so there are two constants of motion:

$$\begin{aligned} p_\psi &= \partial L/\partial\dot{\psi} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = L_3 \\ p_\phi &= \partial L/\partial\dot{\phi} = \left(I_1'\sin^2\theta + I_3\cos^2\theta\right)\dot{\phi} + I_3\dot{\psi}\cos\theta = L_Z \end{aligned} \tag{27.4.3}$$

That is, the angular momentum about x_3 is conserved, because the two forces acting on the top, the gravitational pull at the center of mass and the floor reaction at the bottom point, both act along lines intersecting the axis, so never have torque about x_3. The angular momentum about Z is conserved because the gravitational torque acts perpendicular to this line.

We have two linear equations in $\dot{\psi}, \dot{\phi}$ with coefficients depending on θ and the two constants of motion L_3, L_Z. The solution is straightforward, giving

$$\dot{\phi} = \frac{L_Z - L_3\cos\theta}{I_1'\sin^2\theta} \tag{27.4.4}$$

and

$$\dot{\psi} = \frac{L_3}{I_3} - \cos\theta\left(\frac{L_Z - L_3\cos\theta}{I_1'\sin^2\theta}\right) \tag{27.4.5}$$

The (conserved) energy

$$E = \frac{1}{2} I_1' \left(\Omega_1^2 + \Omega_2^2\right) + \frac{1}{2} I_3 \Omega_3^2 + Mg\ell\cos\theta \tag{27.4.6}$$

$$= \frac{1}{2} I_1' \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2 + Mg\ell\cos\theta \tag{27.4.7}$$

Using the constants of motion to express $\dot{\psi}, \dot{\phi}$ in terms of θ and the constants L_Z, L_3, then subtracting a θ independent term to reduce clutter,

$$E' = E - Mg\ell - \left(L_3^2/2I_3\right) \tag{27.4.8}$$

we have

$$\begin{aligned} E' &= \frac{1}{2} I_1' \dot{\theta}^2 + V_{\text{eff}}(\theta), \quad V_{\text{eff}}(\theta) \\ &= \frac{(L_Z - L_3\cos\theta)^2}{2I_1'\sin^2\theta} - Mg\ell(1-\cos\theta) \end{aligned}$$

The range of motion in θ is given by $E' > V_{\text{eff}}(\theta)$. For $L_3 \neq L_Z, V_{\text{eff}}(\theta)$ goes to infinity at $\theta = 0, \pi$ It has a single minimum between these points. (This isn't completely obvious—one way to see it is to change variable to $u = \cos\theta$, following Goldstein. Multiplying throughout by $\sin^2\theta$, and writing $\dot{\theta}^2\sin^2\theta = \dot{u}^2$ gives a one dimensional particle in a potential problem, and the potential is a cubic in u. Of course some roots of $E' = V_{\text{eff}}(\theta)$ could be in the unphysical region $|u| > 1$. In any case, there are at most three roots, so since the potential is positive and infinite at $\theta = 0, \pi$ it has at most two roots in the physical range.)

From the one-dimensional particle in a potential analogy, it's clear that θ oscillates between these two points θ_1 and θ_2. This oscillation is called **nutation**. Now

$$\dot{\phi} = \left(L_Z - L_3 \cos\theta\right) / I_1' \sin^2\theta \tag{27.4.9}$$

could change sign during this oscillation, depending on whether or not the angle $\cos^{-1}\left(L_Z / L_3\right)$ is in the range. Visualizing the path of the top center point on a spherical surface centered at the fixed point, as it goes around it oscillates up and down, but if there is this sign change, it will "loop the loop", going backwards on the top part of the loop.

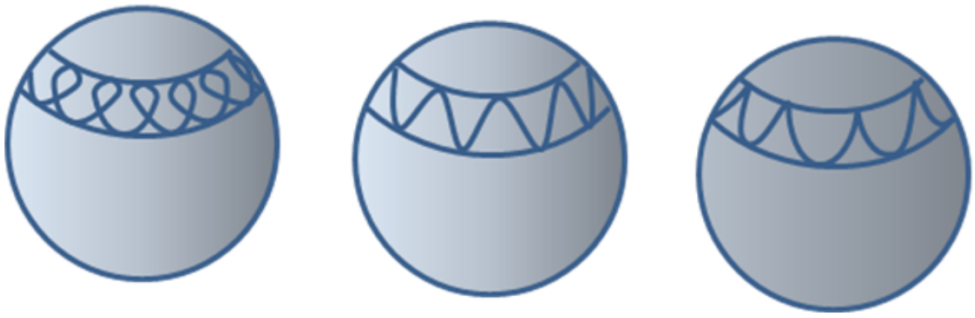


Figure 27.4.1

27.5: Steady Precession

Under what conditions will a top, spinning under gravity, precess at a steady rate? The constancy of L_3, L_Z mean that $\Omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$, and $\Omega_{\text{pr}} = \dot{\phi}$ are constants.

The θ Lagrange equation is

$$I_1'\ddot{\theta} = I_1'\dot{\phi}^2 \sin\theta\cos\theta - I_3(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi}\sin\theta + Mg\ell\sin\theta \tag{27.5.1}$$

For constant θ, $\ddot{\theta} = 0$, so, with $\Omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$, and $\Omega_{\text{pr}} = \dot{\phi}$.

$$I_1'\Omega_{pr}^2\cos\theta - I_3\Omega_3\Omega_{pr} + Mg\ell = 0 \tag{27.5.2}$$

Since Equation 27.5.2 is a *quadratic* equation for the precession rate, there are *two* solutions in general: on staring at a precessing top, this is a bit surprising! We know that for the top, when it's precessing nicely, the spin rate Ω_3 far exceeds the precession rate Ω_{pr}. Assuming I_1', I_3 to be of similar size, this means the first term in the quadratic is much smaller than the second. If we just drop the first term, we get the precession rate

$$\Omega_{\text{precess (slow)}} = \frac{Mg\ell}{I_3\Omega_3}, \quad (\Omega_3 \gg \Omega_{\text{precess}}) \tag{27.5.3}$$

Note that this is independent of angle—the torque varies as $\sin\theta$, but so does the horizontal component of the angular momentum, which is what's changing.

This is the familiar solution for a child's fast-spinning top precessing slowly. But this is a quadratic equation, there's another possibility: in this large Ω_3 limit, this other possibility is that Ω_{pr} is itself of order Ω_3 , so now in the equation the last term, the gravitational one, is negligible, and

$$\Omega_{\text{precess (fast)}} \cong I_3\Omega_3/I_1'\cos\theta \tag{27.5.4}$$

This is just the nutation of a *free* top! In fact, of course, both of these are approximate solutions, only exact in the limit of infinite spin (where one goes to zero, the other to infinity), and a more precise treatment will give corrections to each arising from the other. Landau indicates the leading order gravitational correction to the free body nutation mode.

27.6: Stability of Top Spinning about Vertical Axis

(from Landau) For $\theta = \dot{\theta} = 0, \quad L_3 = L_Z, E' = 0.$ Near $\theta = 0$,

$$\begin{aligned} V_{\text{effective}}\left(\theta\right) &= \frac{\left(L_Z - L_3\cos\theta\right)^2}{2I_1'\sin^2\theta} - Mg\ell(1-\cos\theta) \\ &\cong \frac{L_3^2\left(\frac{1}{2}\theta^2\right)^2}{2I_1'\theta^2} - \frac{1}{2}Mg\ell\theta^2 \\ &= \left(L_3^2/8I_1' - \frac{1}{2}Mg\ell\right)\theta^2 \end{aligned} \tag{27.6.1}$$

The vertical position is stable against small oscillations provided $L_3^2 > 4I_1'Mg\ell,$ or , or $\Omega_3^2 > 4I_1'Mg\ell/I_3^2$

Exercise 27.6.1

Suppose you set the top vertical, but spinning at less than $\Omega_{3\,\text{crit}}$, the value at which it is just stable. It will fall away, but bounce back, and so on. Show the maximum angle it reaches is given by $\cos(\theta/2) = \Omega_3/\Omega_{3\text{crit}}$.

CHAPTER OVERVIEW

28: Euler’s Equations

28.1: Introduction to Euler’s Equations
28.2: Free Rotation of a Symmetric Top Using Euler’s Equations
28.3: Using Energy and Angular Momentum Conservation
28.4: The Asymmetrical Top

28.1: Introduction to Euler's Equations

We've just seen that by specifying the rotational direction and the angular phase of a rotating body using Euler's angles, we can write the Lagrangian in terms of those angles and their derivatives, and then derive equations of motion. These can be solved to describe precession, nutation, etc.

One might hope for a more direct Newtonian approach—we know, for example, that the steadily precessing child's top is easy to understand in terms of the gravitational torque rotating the angular momentum vector.

What about applying $(d\vec{L}/dt)_{\text{lab}} = \vec{K}$ (the external torque on the system) more generally? It's certainly valid. The problem is that in the *lab frame* $\vec{L} = \mathbf{I}\vec{\Omega}$ is

$$L_i = I_{ij}\Omega_j \tag{28.1.1}$$

and the elements of the inertia tensor relative to the lab axes are constantly changing as the body rotates.

The Newtonian approach is only practicable if the connection between $\vec{L}, \vec{\Omega}$ can be made in the *body frame* defined by the principal axes of inertia (x_1, x_2, x_3), in which $\vec{L} = \mathbf{I}\vec{\Omega}$ is

$$L_1 = I_1\Omega_1, L_2 = I_2\Omega_2, L_3 = I_3\Omega_3 \tag{28.1.2}$$

With the body frame rotating at $\vec{\Omega}$ relative to the fixed-in-space (X,Y,Z) frame, the rates of change of a vector in the two frames satisfy

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{lab}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{body}} + \vec{\Omega} \times \vec{A} \tag{28.1.3}$$

To understand this equation, think first of a moving particle, say a bug crawling about on the rotating body. The bug's movement relative to the center of rotation is equal to its movement relative to axes fixed in the rotating body, plus the rotational movement of that body relative to the fixed-in-space axes.

You might be thinking at this point: yes, I can see this is true if the vector represents the position of a particle that's moving around in space, but we're looking at the changing angular momentum, why isn't the angular momentum just zero in a frame in which the body is at rest? And the angular velocity, too?

But what is meant here by the vector "in the body frame" is the components of the vector in an inertial frame that is momentarily coincident with the principal axes.

The $\vec{\Omega} \times \vec{A}$ term represents the change on going through this *succession of inertial frames*.

Think of a long forward pass of an (American) football. The ball is spinning about its long axis (usually), but that axis itself is precessing about the line of flight. Of course, air resistance presumably helps it line up this way, but is not the main effect, which is that the angular momentum vector points in a constant direction in space, the axis of symmetry is precessing around it, as we saw on throwing the ball in class. Imagine now running alongside the ball, holding a pencil pointing in the direction of the constant angular momentum vector. As seen by an observer in the ball, relative to the ball's principal axes frame of reference, the pencil will be describing a cone—this is what we mean by the path of the angular momentum vector relative to the body axes.

The equations of motion in the body frame are then

$$\left(\frac{d\vec{P}}{dt}\right)_{\text{body}} + \vec{\Omega} \times \vec{P} = \vec{F}, \quad \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\Omega} \times \vec{L} = \vec{K} \tag{28.1.4}$$

where $\vec{F}, \vec{K}$ are the external force and couple respectively.

Writing the angular momentum equation in components along the principal axes:

$$\begin{aligned} I_1 d\Omega_1/dt + (I_3 - I_2)\,\Omega_2\Omega_3 &= K_1 \\ I_2 d\Omega_2/dt + (I_1 - I_3)\,\Omega_3\Omega_1 &= K_2 \\ I_3 d\Omega_3/dt + (I_2 - I_1)\,\Omega_1\Omega_2 &= K_3 \end{aligned} \tag{28.1.5}$$

These are *Euler's Equations*.

In the important special case of zero torque:

$$\begin{aligned} I_1 d\Omega_1/dt + (I_3 - I_2)\,\Omega_2\Omega_3 &= 0 \\ I_2 d\Omega_2/dt + (I_1 - I_3)\,\Omega_3\Omega_1 &= 0 \\ I_3 d\Omega_3/dt + (I_2 - I_1)\,\Omega_1\Omega_2 &= 0 \end{aligned} \tag{28.1.6}$$

28.2: Free Rotation of a Symmetric Top Using Euler's Equations

This is a problem we've already solved, using Lagrangian methods and Euler *angles*, but it's worth seeing just how easy it is using Euler's equations.

For $I_1 = I_2$, the third equation gives immediately $\Omega_3 = \text{ constant.}$.

Then, writing for convenience

$$\Omega_3 \left(I_3 - I_1\right)/I_1 = \omega \tag{28.2.1}$$

the first two equations are

$$\dot{\Omega}_1 = -\omega\Omega_2, \quad \dot{\Omega}_2 = \omega\Omega_1 \tag{28.2.2}$$

These equations can be combined to give

$$d\left(\Omega_1 + i\Omega_2\right)/dt = i\omega\left(\Omega_1 + i\Omega_2\right), \text{ so } \left(\Omega_1 + i\Omega_2\right) = Ae^{i\omega t} \tag{28.2.3}$$

That is, (Ω_1, Ω_2) moves around a circle centered at the origin with constant angular velocity. So $\Omega_1^2 + \Omega_2^2 = |A|^2$ stays constant, and Ω_3 is constant, the angular velocity vector has constant length and rotates steadily about the axis x_3.

From

$$L_1 = I_1\Omega_1, L_2 = I_2\Omega_2, L_3 = I_3\Omega_3 \tag{28.2.4}$$

it follows that the angular *momentum* vector also precesses at a steady rate about x_3. This is, remember, in the *body* frame—we know that in the fixed space frame, the angular momentum vector is constant! It follows that, as viewed from the outside, the x_3 axis precesses around the fixed angular momentum vector at a steady rate.

Of course, the rate is the same as that found using Euler's angles, recall from the previous lecture that

$$\vec{L} = (I_1\Omega_1, I_1\Omega_2, I_3\Omega_3) = \left(I_1\dot{\theta}, \quad I_1\dot{\phi}\sin\theta, \quad I_3(\dot{\phi}\cos\theta + \dot{\psi})\right) \tag{28.2.5}$$

so in precession

$$L_3 = L\cos\theta = I_3(\dot{\phi}\cos\theta + \dot{\psi}) \text{ and } \dot{\phi} = L/I_1 \tag{28.2.6}$$

so

$$\dot{\psi} = L\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right) = -\Omega_3\left(I_3 - I_1\right)/I_1 \tag{28.2.7}$$

28.3: Using Energy and Angular Momentum Conservation

We can also gain some insight into the motion of the free spinning top just from conservation of energy and angular momentum.

The equations are:

$$\begin{aligned} \frac{L_1^2}{I_1} + \frac{L_2^2}{I_1} + \frac{L_3^2}{I_3} &= 2E \\ L_1^2 + L_2^2 + L_3^2 &= L^2 \end{aligned} \tag{28.3.1}$$

Visualize these equations as surfaces in (L_1, L_2, L_3) space.

The second is a sphere, radius L, centered at the origin.

The first is an ellipsoid, also centered at the origin, with semimajor axes

$$\left(\sqrt{2EI_1}, \sqrt{2EI_1}, \sqrt{2EI_3}\right) \tag{28.3.2}$$

Do these two surfaces intersect?

The answer is *yes*, they always do.

To see that, assume first that $I_1 > I_3,$ then $2EI_1 \geq L,$ and further $L \geq 2EI_3$. The sphere intersects the ellipsoid in two circles. These degenerate to one circle, the equator, when $L_3 = 0$, and two points (the poles) when $L_1 = L_2 = 0$.

We conclude that the path of the angular momentum vector is a circle around the axis of symmetry in the body coordinate system, and since we know that relative to the fixed space axes the angular momentum is in fact constant, this means that actually the *body* is precessing about its axis of symmetry as seen by an observer.

28.4: The Asymmetrical Top

Here we'll generalize the above equations and argument to the general case of three different moments of inertia about the three principal axes. That is,

$$\begin{aligned} \frac{L_1^2}{I_1}+\frac{L_2^2}{I_2}+\frac{L_3^2}{I_3} &= 2E \\ L_1^2+L_2^2+L_3^2 &= L^2 \end{aligned} \tag{28.4.1}$$

We'll take

$$I_3 > I_2 > I_1 \tag{28.4.2}$$

From the limitations on energy for given angular momentum, the maximum sphere radius is the maximum semimajor axis of the ellipsoid, the ellipsoid touches the sphere at its two extreme poles. For a slightly smaller sphere, the lines of intersection arte two small ellipses centered at the poles, evidently the major axis will go around this elliptical cone in fixed space. At the other extreme, the minimum allowed angular momentum for a given energy, the sphere is entirely inside the ellipsoid except that it touches at the ends of the shortest axes. Again, for a top with slightly more angular momentum, it will precess (elliptically) around this minimum axis.

In both these cases, spin about the axis is stable against small perturbations. This is *not* the case for spin about the middle axis — for that energy, the intersection of the sphere and ellipsoid does not reduce to two points.

The equations for pure spin about the middle axis can be written

$$\begin{aligned} \frac{L_1^2}{I_1}+\frac{L_2^2}{I_2}+\frac{L_3^2}{I_3} &= 2E \\ \frac{L_1^2}{I_2}+\frac{L_2^2}{I_2}+\frac{L_3^2}{I_2} = \frac{L^2}{I_2} &= 2E \end{aligned} \tag{28.4.3}$$

These equations define the points common to the sphere and the ellipsoid at this energy.

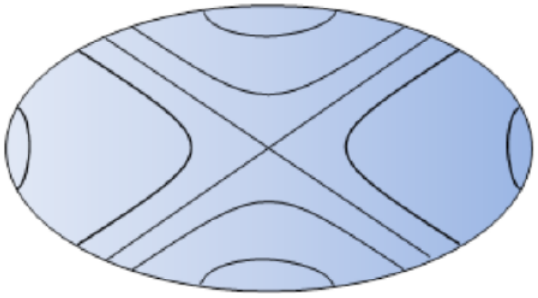


Figure 28.4.1: Contours of constant angular momentum on a constant-energy ellipsoidal surface, viewed along the intermediate axis.

Subtracting the second from the first, we find

$$L_1^2\left(\frac{1}{I_1}-\frac{1}{I_2}\right)+L_3^2\left(\frac{1}{I_3}-\frac{1}{I_2}\right)=0 \tag{28.4.4}$$

with solutions

$$L_1=\pm cL_3, \quad c=I_1\left(I_3-I_2\right)/I_2\left(I_2-I_1\right) \tag{28.4.5}$$

So for this energy and total angular momentum squared, *the intersection of the sphere and ellipsoid is in two planes*, both containing the intermediate axis. This means that any perturbation of this motion will send the system along one of these paths, or a trajectory close to it—in other words, it will deviate far from its original motion, in contrast to spin about either of the other two axes.

In all cases, the path followed by $\vec{L}$, the intersection of the sphere and the ellipsoid, is closed, so the angular momentum cycles back to its original value, in periodic motion. Landau calculates the period, a straightforward (but too lengthy to repeat here)

solution of Euler's equations, giving elliptic functions. Actually, the top does not return to its original configuration: the angular momentum returns, but the top has a different angular *position* about its axis. Landau states this (Berry like?) result, but refers to Whittaker for the derivation.

CHAPTER OVERVIEW

29: Non-Inertial Frame and Coriolis Effect

29.1: The Lagrangian in Accelerating and Rotating Frames
29.2: Uniformly Rotating Frame
29.3: Coriolis Effect - Particle Moving near Earth's Surface
29.4: Gyroscopes and Gyrocompasses in Navigation

29.1: The Lagrangian in Accelerating and Rotating Frames

This section concerns the motion of a single particle in some potential $U(\vec{r})$ in a non-inertial frame of reference. (We'll use $U(\vec{r})$ rather than $V(\vec{r})$ for potential in this section, since we'll be using $\vec{V}$ for relative frame velocity.) The most general noninertial frame has both linear acceleration and rotation, and the angular velocity of rotation may itself be changing.

Our strategy is to begin with an inertial frame K_0, then go to a frame K' having linear acceleration relative to K_0 then finally to a frame K rotating relative to K'. We will construct the Lagrangian in K, and from it the equations of motion in that noninertial frame.

First, suppose the noninertial frame K' to be moving relative to K_0 at time-varying velocity $\vec{V}(t)$. In the *inertial* frame K_0, the Lagrangian is as usual

$$L_0 = \frac{1}{2}m\vec{v}_0^2 - U \tag{29.1.1}$$

so Lagrange's equations give the standard result,

$$md\vec{v}_0/dt = -\partial U/\partial\vec{r} \tag{29.1.2}$$

the subscript 0 denoting quantities in this inertial frame.

The Principle of Least Action is a frame-independent concept, so the calculus of variations Lagrangian equations it leads to,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\vec{v}}\right) = \frac{\partial L}{\partial\vec{r}} \tag{29.1.3}$$

must also be correct in a non-inertial frame.

How can this be true? The reason is that in a non-inertial frame, the Lagrangian has a different form.

To find the Lagrangian in terms of the velocity $\vec{v}'$, meaning the velocity measured in the frame K', we just add the velocity of K' relative to K.

$$\vec{v}_0 = \vec{v}' + \vec{V} \tag{29.1.4}$$

and putting this into L_0, gives the Lagrangian L' in the accelerating frame K' :

$$L' = \frac{1}{2}m\vec{v}'^2 + m\vec{v}'\cdot\vec{V}(t) + \frac{1}{2}m\vec{V}^2(t) - U\left(\vec{r}'\right) \tag{29.1.5}$$

Following Landau, $\vec{V}^2(t)$ is purely a function of time, so can be expressed as the derivative of a function of time, recall terms of that form do not affect the minimization of the action giving the equations of motion, and so can be dropped from the Lagrangian.

The second term,

$$m\vec{V}(t)\cdot\vec{v}' = m\vec{V}\cdot d\vec{r}'/dt = d\left(m\vec{V}\cdot\vec{r}'\right)/dt - m\vec{r}'\cdot d\vec{V}/dt \tag{29.1.6}$$

Again, the total derivative term can be dropped, giving

$$L' = \frac{1}{2}m\vec{v}'^2 - m(d\vec{V}(t)/dt)\cdot\vec{r}' - U\left(\vec{r}'\right) \tag{29.1.7}$$

from which the equation of motion is

$$m\frac{d\vec{v}'}{dt} = -\frac{\partial U\left(\vec{r}'\right)}{\partial\vec{r}'} - m\left(\frac{d\vec{V}}{dt}\right) \tag{29.1.8}$$

Landau writes this as

$$m\frac{d\vec{v}'}{dt} = -\frac{\partial U}{\partial\vec{r}'} - m\vec{W} \tag{29.1.9}$$

So the motion in the accelerating frame is *the same as if an extra force is added*—this extra force is just the product of the particle's mass and the frame's acceleration, it's just the "force" that pushes you back in your seat when you step on the gas, the linear equivalent of the "centrifugal force" in a rotating frame.

Speaking of centrifugal force, we now bring in our final frame K, having the same origin as K', (so we can take $\vec{r}' = \vec{r}$ at a given instant) but rotating relative to it with angular velocity $\vec{\Omega}(t)$.

What is the Lagrangian translated into K variables? The velocities in K', K are related by

$$\vec{v}' = \vec{v} + \vec{\Omega} \times \vec{r} \tag{29.1.10}$$

and, putting this in the Lagrangian above,

$$L = \frac{1}{2} m \vec{v}^2 + m\vec{v} \cdot \vec{\Omega} \times \vec{r} + \frac{1}{2} m (\vec{\Omega} \times \vec{r})^2 - m\vec{W} \cdot \vec{r} - U(\vec{r}) \tag{29.1.11}$$

From this,

$$\partial L / \partial \vec{v} = m\vec{v} + m\vec{\Omega} \times \vec{r} \tag{29.1.12}$$

(Note that this is the canonical momentum, $p_i = \partial L / \partial \dot{x}_i = \partial L / \partial v_i$

Using

$$\vec{v} \cdot \vec{\Omega} \times \vec{r} = \vec{v} \times \vec{\Omega} \cdot \vec{r}, \quad (\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r}) = (\vec{\Omega} \times \vec{r}) \times \vec{\Omega} \cdot \vec{r} \tag{29.1.13}$$

we have

$$\partial L / \partial \vec{r} = m\vec{v} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} - m\vec{W} - \partial U / \partial \vec{r} \tag{29.1.14}$$

The equation of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \vec{v}} \right) = \frac{\partial L}{\partial \vec{r}} \tag{29.1.15}$$

is therefore:

$$m d\vec{v}/dt = -\partial U / \partial \vec{r} - m\vec{W} + m\vec{r} \times \vec{\vec{\Omega}} + 2m\vec{v} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} \tag{29.1.16}$$

29.2: Uniformly Rotating Frame

For the important case of a frame having uniform rotation and no translation motion,

$$m d\vec{v}/dt = -\partial U/\partial \vec{r} + 2m\vec{v}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega} \tag{29.2.1}$$

The last term is (as Landau states) the “centrifugal force”, but this term is now politically incorrect, since it isn’t a “real force”, just an effect of being in a rotating frame. (It’s still OK to say gravitational force, though, although that isn’t a real force either, I guess, since it disappears in the local inertial “freely falling” frame, as was first noticed by Galileo, and centuries later by Einstein, who called it "the happiest thought of my life".)

The second term, $2m\vec{v}\times\vec{\Omega}$ Landau calls the *Coriolis force*. (Again, the politically correct tend to talk about the Coriolis *effect*, meaning deviation of a projectile, say, from an inertial frame trajectory resulting from the operation of this “force”.) A very nice illustration of this “force” is in the Frames of Reference 2 movie, starting at time 3:50.

Notice the Coriolis force depends on the velocity of the particle, and is reminiscent of the magnetic force on a charged particle. For example, it does no work on the particle, but *does* curve the particle’s path.

The energy of the particle can be found from the standard Lagrangian equation

$$E = \sum \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L = \vec{v}\cdot\vec{p} - L \tag{29.2.2}$$

where

$$\vec{p} = \partial L/\partial \vec{v} = m\vec{v} + m\vec{\Omega}\times\vec{r} \tag{29.2.3}$$

This is interesting! Remembering $\vec{v}_0 = \vec{v} + \vec{\Omega}\times\vec{r}$, *the momentum*, defined in this way as a canonical variable, not as just $m\vec{v}$ in the frame we’re in, *is the same in the two frames* K_0, K

$$\vec{p}_0 = \vec{p} \tag{29.2.4}$$

The angular momenta $\vec{L}_0 = \vec{r}\times\vec{p}_0$ and $\vec{L} = \vec{r}\times\vec{p}$ are also equal in the two frames.

The Lagrangian is

$$L = \frac{1}{2}m\vec{v}^2 + m\vec{v}\cdot\vec{\Omega}\times\vec{r} + \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 - U(\vec{r}) \tag{29.2.5}$$

so

$$\begin{aligned} E &= \vec{p}\cdot\vec{v} - L \\ &= m\vec{v}^2 + m\vec{\Omega}\times\vec{r}\cdot\vec{v} - \left(\frac{1}{2}m\vec{v}^2 + m\vec{v}\cdot\vec{\Omega}\times\vec{r} + \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 - U(\vec{r})\right) \\ &= \frac{1}{2}m\vec{v}^2 - \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 + U(\vec{r}) \end{aligned} \tag{29.2.6}$$

The new term is the centrifugal potential energy. It’s negative because it takes work to bring something towards the axis of rotation. To see how this energy relates to the energy in the original fixed frame, substitute in this equation $\vec{v} = \vec{v}_0 - \vec{\Omega}\times\vec{r}$ to find

$$E = E_0 - \vec{L}\cdot\vec{\Omega} \tag{29.2.7}$$

true for one particle, and by addition for any system of particles.

Exercise 29.2.1

Confirme that that $\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times\vec{p}_0$. Notice the difference can be positive or negative—give a simple one-particle illustration of this.

29.3: Coriolis Effect - Particle Moving near Earth's Surface

The Earth's surface is a rotating frame of reference, but the angular velocity is sufficiently small that we can often drop second order terms. Recall the equation of motion in a rotating frame is

$$m d\vec{v}/dt = -\partial U/\partial \vec{r} + 2m\vec{v}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega} \tag{29.3.1}$$

which becomes, close to the Earth and dropping the second-order term,

$$d\vec{v}/dt = \vec{g} + 2\vec{v}\times\vec{\Omega} \tag{29.3.2}$$

The leading order solution, ignoring the small rotation term, is the familiar

$$\vec{v}_1 = \vec{v}_0 + \vec{g}t \tag{29.3.3}$$

The second term is much smaller than the first, so it's OK to replace the $\vec{v}$ there by $\vec{v}_1$, and to find the leading correction to the path put

$$\vec{v} = \vec{v}_1 + \vec{v}_2 \tag{29.3.4}$$

in the equation $d\vec{v}/dt = \vec{g} + 2\vec{v}\times\vec{\Omega}$, giving

$$d\vec{v}_2/dt = 2\vec{v}_1\times\vec{\Omega} \cong 2t\vec{g}\times\vec{\Omega} + 2\vec{v}_0\times\vec{\Omega} \tag{29.3.5}$$

The full equation can now be integrated to give

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{g}t^2 + \frac{1}{3}t^3\vec{g}\times\vec{\Omega} + t^2\vec{v}_0\times\vec{\Omega} \tag{29.3.6}$$

Let's try some numbers: in 1803, an experiment was conducted in Schlebusch, Germany that attracted the interest of the scientific community. Twenty-nine iron pebbles were dropped into a 90-meter deep mineshaft.

In 1831 the experiment was repeated in a 158.5 m deep mine in Freiburg, Saxony. From 106 drops an average deflection of 28.3 mm was estimated, close to the theoretical value of 27.5 mm. (This agrees exactly with our formula, from Landau's book.)

Exercise 29.3.1

Where would you expect the particle to fall, compared with a straight down plumb line? To make visualizing a little easier, imagine the mine to be on the equator. Then the ground is moving east faster than the bottom of the mine— so the pebble will fall to the east.

Exercise 29.3.2: Naval Gunnery - HMS Dreadnought, 1906

The BL 12 inch Mk X gun on the HMS Dreadnought. Shells at 800 m/sec, range about 23 km. For vertical velocity of say 400 m/sec, time in air of order 80 secs. The two terms are about equal magnitude, around 100 meters.

I pick this ship because there is a rumor that in a 1915 naval battle near the Falkland islands, off Argentina, between the British and German navies, the British kept missing because they corrected their aim for Coriolis deflections appropriate to the northern hemisphere. I'm not sure if it's true.

29.4: Gyroscopes and Gyrocompasses in Navigation

A gyroscope is a fast-spinning disc supported in such a way that there is no external torque, except an occasional boost to keep up its speed (jet of air, or magnetic). Since there is no external torque, it always points the same way. A device based on this principle and used in aircraft is termed a heading indicator.

Suppose the gyroscope is set spinning about an initially vertical axis. If it's nighttime, this means its axis is pointing to a particular star, overhead at that moment. It will continue to point to that star, so, unless you're at the North or South Pole, the axis will move from the local vertical to return 24 hours later. Since the direction of the gyroscope axis is fixed in space, and the Earth's axis of rotation is fixed in space, the angle between the two is obviously constant, so as seen in the lab, say, the gyroscope axis describes a cone about a line parallel to the axis (the line a sundial pointer points along, vertical at the North Pole (so the gyro doesn't change) down to horizontal at the Equator.

This is useful, but it would be better to have a pointer that just points North (or South). This is achieved by damping the gyroscopes motion—put one end in viscous liquid. Then, when it moves relative to its container, there is a couple opposing the motion. What this does is move it inwards, relative to its cone of motion, that is, the cone shrinks, so it goes to a stable orientation parallel to the Earth's axis, that is, it points North (or South).

A gyrocompass is the same idea, but now constrained to lie in a horizontal plane. This plus some damping forces the compass to orient as close to the Earth's axis as possible in the horizontal plane, meaning it points North, in this hemisphere.

CHAPTER OVERVIEW

30: A Rolling Sphere on a Rotating Plane

30.1: Introduction
30.2: Holonomic Constraints and non-Holonomic Constraints
30.3: D'Alembert's Principle
30.4: Ball with External Forces Rolling on Horizontal Plane
30.5: Ball Rolling on Rotating Plane
30.6: Ball Rolling on Inclined Rotating Plane

30.1: Introduction

We'll now consider an interesting dynamics problem not covered in most introductory texts, a rolling ball on a rotating, possibly tilted, surface. As we'll see, this tough sounding problem is not that difficult to solve using Newtonian methods, and leads to some surprising results. For example, a ball rolling on a steadily rotating horizontal plane moves in a circle, and not a circle centered at the axis of rotation. We'll prove this—and demonstrate it in class. Even more remarkably, if the rotating plane is tilted, the ball follows a *cycloidal* path, keeping at the same average height—not rolling downhill. This is exactly analogous to an electron in crossed electric and magnetic fields. One reason the rolling ball problems are generally avoided is that they do not readily lend themselves to Lagrangian analysis, but can in fact be solved quite quickly with a vectorized application of Newton's laws. The appropriate techniques are described in Milne's book *Vectorial Mechanics*, which we follow.

30.2: Holonomic Constraints and non-Holonomic Constraints

A sphere rolling on a plane without slipping is constrained in its translational and rotational motion by the requirement that the point of the sphere momentarily in contact with the plane is at rest. How do we incorporate this condition in the dynamical analysis: the least action approach, for example, or the direct Newtonian equations of motion?

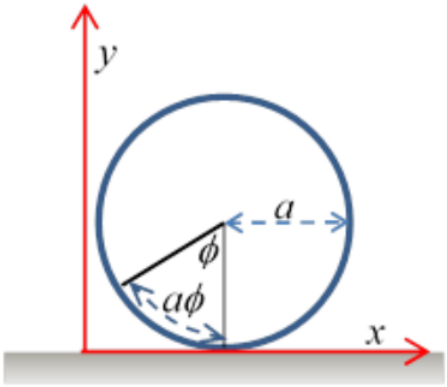


Figure 30.2.1

We'll begin with a simpler example, that of a cylinder rolling in the x direction, its orientation ϕ defined as zero as it passes the x origin, and its radius a. We see immediately that its orientation is uniquely given by its position (for no slipping) by $x = a\phi,$ or $v - a\dot{\phi} = 0$. The constraint enables us to eliminate one of the dynamical variables from the equation. If we measure its position at some later time, we know the angle it turned through. The same argument works for a cylinder rolling inside a larger cylinder.

A constraint on a dynamical system that can be integrated in this way to eliminate one of the variables is called a *holonomic* constraint. A constraint that cannot be integrated is called a *nonholonomic* constraint.

For a sphere rolling on a rough plane, the no-slip constraint turns out to be nonholonomic.

To see this, imagine a sphere placed at the origin in the (x,y) plane. Call the point at the top of the sphere the North Pole. Now roll the sphere along the x axis until it has turned through ninety degrees. Its NS axis is now parallel to the x axis, the N pole pointing in the positive x direction. Now roll it through ninety degrees in a direction parallel to the y axis. The N pole is still pointing in the positive x direction, the sphere, taken to have unit radius, is at $(\pi/2, \pi/2)$

Now start again at the origin, the N pole on top. This time, first roll the sphere through ninety degrees in the y direction. The N pole now points along the positive y axis. Next, roll the sphere through ninety degrees in the x direction: we're back to the point $(\pi/2, \pi/2)$but this time the N pole is pointing in the y direction.

The bottom line is that, in contrast to the cylindrical case, for a rolling sphere the no-slip constraint does not allow us to eliminate any dynamical variables—given that initially the sphere is at the origin with the N pole at the top, there is *no unique relationship between orientation* (θ, ϕ) *and position (x,y) at a later point*, we would have to know the rolling history, and in fact we can roll back to the origin by a different route and in general the N pole will *not* be at the top when we return.

So the constraint equation, which can be written

$$\vec{V} - a\vec{\Omega} \times \vec{n} = 0 \tag{30.2.1}$$

does not allow us to eliminate a variable, but it certainly plays a role in the dynamics! As we've seen, the identical equation for the cylinder, $dx/dt - ad\phi/dt = 0,$ trivially integrates to $x = a\phi + c$, uniquely linking change in orientation with change in position. We see that for a ball rolling in two dimensions, there can be no such integral.

A possible approach is to use Lagrange multipliers to take account of the constraint, just as in deriving the equation for the catenary the fixed length of the string entered as a constraint. Doing this for the rolling ball turns out to lead to a very messy problem—for once, the advanced approach to dynamics doesn't pay off. But there's a better way.

30.3: D’Alembert’s Principle

The “better way” is simply to write down Newton’s equations, $\vec{F} = m\vec{a}$ and the rotational equivalent $\vec{K} = I\dot{\vec{\Omega}}$ for each component of the system, now using, of course, *total* force and torque, including constraint reaction forces, etc. This approach Landau calls “d’Alembert’s principle”.

Footnote: We’re not going to pursue this here, but the “principle” stems from the concept of **virtual work**: if a system is in equilibrium, then making tiny displacements of all parameters, subject to the system constraints (but not necessarily an infinitesimal set of displacements that would arise in ordinary dynamical development in time), the total work done by all forces acting on parts of the system is zero. This is just saying that in equilibrium, it is at a local minimum (or stationary point if we allow unstable equilibrium) in the energy “landscape”. D’Alembert generalized this to the dynamical case by adding in effective forces corresponding to the coordinate accelerations, he wrote essentially $\sum \vec{F} - m\vec{a} = 0,$ representing $m\vec{a}$ as a “force”, equivalent to Newton’s laws of motion.

Having written down the equations, the reaction forces can be cancelled out to derive equations of motion.

30.4: Ball with External Forces Rolling on Horizontal Plane

Here's how it works for a simple example (done in Landau, and see diagram below): the equation of motion of a sphere rolling on a fixed horizontal plane under an external force $\vec{F}$ and torque $\vec{K}$\).

Taking the reaction at the plane to be $\vec{R}$ (and note that this can be in any upward direction, not in general vertical), we have

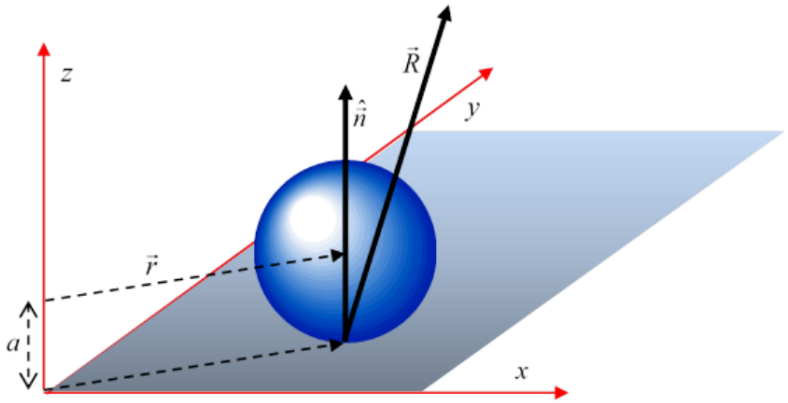
$$\begin{aligned} M d\vec{V}/dt &= \vec{F} + \vec{R} \\ I d\vec{\Omega}/dt &= \vec{K} - a\vec{n} \times \vec{R} \end{aligned} \tag{30.4.1}$$

The constraint equation, differentiated, gives $\dot{\vec{V}} = a\dot{\vec{\Omega}} \times \vec{n}$, so the first equation can be written

$$Ma\dot{\vec{\Omega}} \times \vec{n} = \vec{F} + \vec{R} \tag{30.4.2}$$

then substituting $\dot{\vec{\Omega}}$ from the second equation,

$$(I/aM)(\vec{F} + \vec{R}) = \vec{K} \times \vec{n} - a\vec{R} + a\vec{n}(\vec{n} \cdot \vec{R}) \tag{30.4.3}$$



Ball of radius a rolling on horizontal plane

Figure 30.4.1

This equation gives the components of the reaction force as functions of the external force and couple: the velocities have been eliminated. So we can now put $\vec{R}$ in the first equation of motion giving the translational acceleration in terms of the external force and torque. Note that any vertical component of the torque $\vec{K}$ will not affect the reaction at the plane $\vec{R}$ (it would just spin the ball about the point of contact) so we have, using $I = \frac{2}{5}Ma^2$.

$$R_x = \frac{5}{7}(K_y/a) - \frac{2}{7}F_x, \quad R_y = -\frac{5}{7}(K_x/a) - \frac{2}{7}F_y \tag{30.4.4}$$

and substitution in the original equations of motion gives

$$\begin{aligned} \frac{dV_x}{dt} &= \frac{5}{7M}\left(F_x + \frac{K_y}{a}\right) \\ \frac{dV_y}{dt} &= \frac{5}{7M}\left(F_y - \frac{K_x}{a}\right) \end{aligned} \tag{30.4.5}$$

Exercise: interpret this for the zero torque case, and for the zero force case.

Landau goes on to solve three statics problems which could be in an introductory physics course. We'll skip them.

30.5: Ball Rolling on Rotating Plane

(The following examples are from Milne, *Vectorial Mechanics*.)

A sphere is rolling without slipping on a horizontal plane. The plane *is itself rotating* at constant angular velocity ω.

We have three vector equations: Newton's equations for linear and angular acceleration, and the rolling condition. We want to find the path taken by the rolling ball on the rotating surface, that is, $\vec{r}(t)$. We'll use our three equations to eliminate two (vector) variables: the reaction force between the plane and the ball $\vec{R}$, and the angular velocity $\vec{\Omega}$.

The equations of motion of the sphere (radius a, mass m, center at $\vec{r}$ measured in the lab, horizontally from the axis of the plane's rotation) with $\vec{R}$ the contact force of the plane on the sphere, are

$$m\ddot{\vec{r}} = \vec{R} - mg\hat{\vec{n}}, \quad I\dot{\vec{\Omega}} = -a\hat{\vec{n}} \times \vec{R} \tag{30.5.1}$$

(Of course, the gravitational force here is just balancing the vertical component of the reaction force, but this is no longer the case for the tilted plane, treated in the next section.)

First, we'll eliminate the reaction force $\vec{R}$ to get an equation of motion:

$$I\dot{\vec{\Omega}} = -am\hat{\vec{n}} \times (\ddot{\vec{r}} + g\hat{\vec{n}}) = \mathrm{am}\,\ddot{\vec{r}} \times \hat{\vec{n}} \tag{30.5.2}$$

The rolling condition is:

$$\dot{\vec{r}} - a\vec{\Omega} \times \hat{\vec{n}} = \omega\hat{\vec{n}} \times \vec{r} \tag{30.5.3}$$

the right-hand side being the local velocity of the turntable, $\vec{r}$ measured from an origin at the center of rotation.

We'll use the rolling condition to eliminate $\dot{\vec{\Omega}}$ and give us an equation for the actual path of the sphere.

First, differentiate it (remember $\hat{\vec{n}}, \omega$ are both constant) to get

$$\ddot{\vec{r}} - a\dot{\vec{\Omega}} \times \hat{\vec{n}} = \omega\hat{\vec{n}} \times \dot{\vec{r}} \tag{30.5.4}$$

Next, take the equation of motion $I\dot{\vec{\Omega}} = am\ddot{\vec{r}} \times \hat{\vec{n}}$ and $\times\hat{\vec{n}}$ to get

$$I\dot{\vec{\Omega}} \times \hat{\vec{n}} = am(\ddot{\vec{r}} \times \hat{\vec{n}}) \times \hat{\vec{n}} = -am\ddot{\vec{r}} \tag{30.5.5}$$

and putting these together to get rid of the angular velocity,

$$\left(1 + a^2 m/I\right)\ddot{\vec{r}} = \omega\hat{\vec{n}} \times \dot{\vec{r}} \tag{30.5.6}$$

This integrates to

$$\dot{\vec{r}} = \left(\frac{\omega}{1 + a^2 m/I}\right)\hat{\vec{n}} \times (\vec{r} - \vec{r}_0) \tag{30.5.7}$$

which is just the equation for steady *circular* motion about the point $\vec{r}_0$.

For a uniform sphere, $I = \dfrac{2}{5}Ma^2$, so $\dot{\vec{r}} = \dfrac{2}{7}\omega\hat{\vec{n}} \times (\vec{r} - \vec{r}_0)$

So the ball rolling on the rotating plate goes around in a circle, which could be *any* circle. If it is put down gently at any point on the rotating plane, and held in place until it is up to speed (meaning no slipping) it will stay at that point for quite a while (until the less than perfect conditions, such as air resistance or vibration, cause noticeable drift). If it is nudged, it will move in a circle. In class, we saw it circle many times—eventually, it fell off, a result of air resistance plus the shortcomings of our apparatus, but the circular path was very clear.

30.6: Ball Rolling on Inclined Rotating Plane

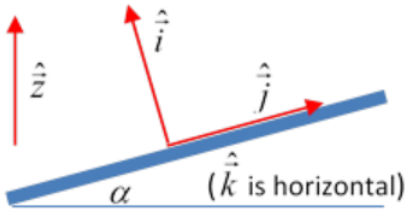


Figure 30.6.1

We'll take unit vectors $\hat{z}$ pointing vertically up, $\hat{i}$ perpendicularly up from the plane, the angle between these two unit vectors being α. (We will need a set of orthogonal unit vectors $\hat{i}, \hat{j}, \hat{k}$, not fixed in the plane, but appropriately oriented, with $\hat{k}$ horizontal.) The vector to the center of the sphere (radius a, mass m) from an origin on the axis of rotation, at a point a above the plane, is $\vec{r}$. The contact reaction force of the plane on the sphere is $\vec{R}$.

The equations of motion are:

$$m\ddot{\vec{r}} = \vec{R} - mg\hat{z}, \quad I\dot{\vec{\Omega}} = -a\hat{i} \times \vec{R} \tag{30.6.1}$$

and the equation of rolling contact is $\dot{\vec{r}} - a\vec{\Omega} \times \hat{i} = \widehat{\omega i} \times \vec{r}$.

First, we eliminate $\vec{R}$ from the equations of motion to give

$$\dot{\vec{\Omega}} = (am/I)(\ddot{\vec{r}} + g\hat{z}) \times \hat{i} \tag{30.6.2}$$

Note that $\dot{\vec{\Omega}} \cdot \hat{i} = 0$, so the spin in the direction normal to the plane is constant, $\vec{\Omega} \cdot \hat{i} = n$ say. (Both forces on the sphere have zero torque about this axis.)

Integrating,

$$\vec{\Omega} + \text{ const. } = (ma/I)(\dot{\vec{r}} + gt\hat{z}) \times \hat{i} \tag{30.6.3}$$

Now eliminate $\vec{\Omega}$ by multiplying both sides by $\times\vec{i}$ and using the equation of rolling contact

$$\dot{\vec{r}} - a\vec{\Omega} \times \hat{i} = \omega\hat{i} \times \vec{r} \tag{30.6.4}$$

to find:

$$\left(ma^2/I\right)\left[(\dot{\vec{r}} + gt\hat{z}) \times \hat{i}\right] \times \hat{i} = a\vec{\Omega} \times \hat{i} + \text{ const. } = \dot{\vec{r}} - \omega\hat{i} \times \vec{r} + \text{ const.} \tag{30.6.5}$$

then using $(\dot{\vec{r}} \times \hat{i}) \times \hat{i} = -\dot{\vec{r}}$, $(\hat{z} \times \hat{i}) \times \hat{i} = -\hat{j}$, we find

$$\dot{\vec{r}}\left(1 + ma^2/I\right) + \left(ma^2/I\right) gt\hat{j}\sin\alpha + \text{ const. } = \widehat{\omega veci} \times \vec{r} \tag{30.6.6}$$

The constant is fixed by the initial position $\vec{r}_0$, giving finally

$$\dot{\vec{r}} = \frac{\omega}{1 + ma^2/I}\hat{i} \times \left[(\vec{r} - \vec{r}_0) + \frac{ma^2/I}{\omega} gt\hat{k}\sin\alpha\right] \tag{30.6.7}$$

The first term in the square brackets would give the same circular motion we found for the horizontal rotating plane, the second term adds a steady motion of the center of this circle, in a *horizontal* direction (*not* down the plane!) at constant speed $\left(ma^2/I\omega\right) g\sin\alpha$.

(This is identical to the motion of a charged particle in crossed electric and magnetic fields.)

Bottom line: the intuitive notion that a ball rolling on a rotating inclined turntable would tend to roll downhill is wrong! Recall that for a particle circling in a magnetic field, if an electric field is added perpendicular to the magnetic field, the particle moves in a

cycloid at the same average electrical potential—it has *no net movement in the direction of the electric field* , only perpendicular to it. Our rolling ball follows an identical cycloidal path—keeping the same average gravitational potential.

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