21.3: Stability of a Pendulum with a Rapidly Oscillating Vertical Driving Force

Recall now the Lagrangian for the simple (rigid)pendulum of length ℓ, mass m, angle from vertically down ϕ, constrained to move in a vertical plane, point of support driven to oscillate vertically with amplitude α and frequency Ω (from the section on parametric resonance),

$$L=\frac{1}{2} m \ell^2 \dot{\phi}^2+m a \ell \Omega^2 \cos \Omega t \cos \phi+m g \ell \cos \phi \tag{21.3.1}$$

Our previous analysis of this system was for driving frequencies near double the natural frequency. Now we'll investigate the behavior for driving frequencies *far more rapid* than the natural frequency.

The equation of motion,

$$\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\phi}}\right)=\frac{\partial L}{\partial \phi} \tag{21.3.2}$$

is

$$m \ell^2 \ddot{\phi}=-m a \ell \Omega^2 \cos \Omega t \sin \phi-m g \ell \sin \phi \tag{21.3.3}$$

so evidently the external driving force is $f=-m a \Omega^2 \cos \Omega t \sin \phi$

(Landau has a misprint—an extra ℓ in this, p 95) and, from the previous section, (except that for the pendulum we are using Ω, not ω, for the external driving frequency)

$$V_{\text{eff}}=V+\overline{f^2} / 2 m \Omega^2=m g \ell\left[-\cos \phi+\left(a^2 \Omega^2 / 4 g \ell\right) \sin ^2 \phi\right] \tag{21.3.4}$$

For $\phi=\pi+\varepsilon, \quad \varepsilon$ small,

$$V_{\text{eff}}(\varepsilon) \cong m g \ell\left[1-\frac{1}{2} \varepsilon^2+\left(a^2 \Omega^2 / 4 g \ell\right) \varepsilon^2\right] \tag{21.3.5}$$

and for $a^2 \Omega^2>2 g \ell$

the *upward* position is stable!

At first glance, this may seem surprising: the extra term in the potential from the oscillations is like a kinetic energy term for the oscillating movement. Surely the pendulum is oscillating more in the vertically up position than when it's to one side? So why isn't *that* a maximum of the added effective potential? The point is that the relevant variable is not the pendulum's height above some fixed point, the variable is ϕ —and the rapid oscillations of ϕ are minimum (zero) in the vertically up position.