5.2: Lagrangian Treatment

(Here we follow Landau.) Since the equations of motion are generated by minimizing the action, which is an integral of the Lagrangian along a trajectory, the motion won't be affected if the Lagrangian is multiplied by a constant. If the potential energy is a homogeneous function of the coordinates, rescaling would multiply it by a constant factor. If our system consists of particles interacting via such a potential energy, it will be possible to rescale time so that, rescaling both space and time, the Lagrangian is multiplied by an overall constant, so the equations of motion will look the same.

Specifically, if the potential energy U is homogeneous of degree k and the spatial coordinates are scaled by a factor α

$$(\alpha/\beta)^2 = \alpha^k, \quad \beta = \alpha^{1-\frac{1}{2}k} \tag{5.2.1}$$

For planetary orbits, $k=-1,$ so $\beta^2 = \alpha^3$, confirming our hand waving derivation above.

For the simple harmonic oscillator, $U(\vec{r}) \propto \vec{r}^2$ so $k=2$ and $\beta = 0$ What does that mean? Scaling up the orbit does not affect the time—the oscillation time is always the same.

Falling under gravity: $k=1, \beta = \sqrt{\alpha}, x \propto t^2$ So doubling the time scale requires quadrupling the length scale to get the scaled motion identical to the original.