2.7: Calculus of Variations with Many Variables

We’ve found the equations defining the curve $y(x)$ along which the integral

$$J[y]=\int_{x_1}^{x_2} f\left(y, y^{\prime}\right) d x \tag{2.7.1}$$

has a stationary value, and we’ve seen how it works in some two-dimensional curve examples.

But most dynamical systems are parameterized by more than one variable, so we need to know how to go from a curve in (x, y) to one in a space $\left(x, y_1, y_2, \ldots y_n\right)$, and we need to minimize (say)

$$J\left[y_1, y_2, \ldots y_n\right]=\int_{x_1}^{x_2} f\left(y_1, y_2, \ldots y_n, y_1^{\prime}, y_2^{\prime}, \ldots y_n^{\prime}\right) d x \tag{2.7.2}$$

In fact, the generalization is straightforward: the path deviation simply becomes a vector,

$$\delta \vec{y}(x)=\left(\delta y_1(x), \delta y_2(x), \ldots, \delta y_n(x)\right) \tag{2.7.3}$$

Then under any infinitesimal variation

$$\delta \mathbf{y}(x) \text { (writing also } \mathbf{y}=\left(y_1, \ldots y_n\right)) \tag{2.7.4}$$

$$\delta J[\vec{y}]=\int_{x_1}^{x_2} \sum_{i=1}^n\left[\frac{\partial f\left(\vec{y}, \vec{y}^{\prime}\right)}{\partial y_i} \delta y_i(x)+\frac{\partial f\left(\vec{y}, \vec{y}^{\prime}\right)}{\partial y_i^{\prime}} \delta y_i^{\prime}(x)\right] d x=0 \tag{2.7.5}$$

Just as before, we take the variation zero at the endpoints, and integrate by parts to get now *n* separate equations for the stationary path:

$$\frac{\partial f\left(\vec{y}, \vec{y}^{\prime}\right)}{\partial y_i}-\frac{d}{d x}\left(\frac{\partial f\left(\vec{y}, \vec{y}^{\prime}\right)}{\partial y_i^{\prime}}\right)=0, \quad i=1, \ldots, n \tag{2.7.6}$$