

2.1: The Catenary and the Soap Film

The catenary is the curved configuration $y = y(x)$ of a uniform inextensible rope with two fixed endpoints at rest in a constant gravitational field. That is to say, it is the curve that minimizes the gravitational potential energy

$$J[y(x)] = 2\pi \int_{x_1}^{x_2} y ds = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + y'^2} dx, \quad (y' = dy/dx) \quad (2.1.1)$$

where we have taken the rope density and gravity both equal to unity for mathematical convenience. Usually in calculus we minimize a function with respect to a single variable, or several variables. Here the potential energy is a function of a function, equivalent to an infinite number of variables, and our problem is to minimize it with respect to arbitrary small variations of that function. In other words, if we nudge the chain somewhere, and its motion is damped by air or internal friction, it will settle down again in the catenary configuration.

Formally speaking, there will be no change in that potential energy to leading order if we make an infinitesimal change in the curve, $y(x) \rightarrow y(x) + \delta y(x)$ (subject of course to keeping the length the same, that is $\delta \int ds = 0$.)

This method of solving the problem is called the *calculus of variations*: in ordinary calculus, we make an infinitesimal change in a variable, and compute the corresponding change in a function, and if it's zero to leading order in the small change, we're at an extreme value.

(*Nitpicking footnote*: Actually this assumes the second order term is nonzero—what about x^3 near the origin? But such situations are infrequent in the problems we're likely to encounter.)

The difference here is that the potential energy of the hanging chain isn't just a function of a variable, or even of a number of variables—it's a *function of a function*, it depends on the position of every point on the chain (in the limit of infinitely small links, that is, or equivalently a continuous rope).

So, we're looking for the configuration where the potential energy doesn't change to first order for *any* infinitesimal change in the curve of its position, subject to fixed endpoints, and a fixed chain length.

As a warm up, we'll consider a simpler—but closely related—problem.

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