# 17.2: Two Coupled Pendulums

We'll take two equal pendulums, coupled by a light spring. We take the spring restoring force to be directly proportional to the angular difference between the pendulums. (This turns out to be a good approximation.)

For small angles of oscillation, we take the Lagrangian to be

$$L=\frac{1}{2}m\ell^2\dot{\theta}_1^2+\frac{1}{2}m\ell^2\dot{\theta}_2^2-\frac{1}{2}mg\ell\theta_1^2-\frac{1}{2}mg\ell\theta_2^2-\frac{1}{2}C(\theta_1-\theta_2)^2 \tag{17.2.1}$$

Denoting the single pendulum frequency by $\omega_0$, the equations of motion are (writing $\omega_0^2=g/\ell, k=C/m\ell^2$ , so $[k]=T^{-2}$)

$$\begin{aligned}\ddot{\theta}_1&=-\omega_0^2\theta_1-k(\theta_1-\theta_2)\\ \ddot{\theta}_2&=-\omega_0^2\theta_2-k(\theta_2-\theta_1)\end{aligned} \tag{17.2.2}$$

We look for a periodic solution, writing

$$\theta_1(t)=A_1e^{i\omega t},\quad \theta_2(t)=A_2e^{i\omega t} \tag{17.2.3}$$

(The final physical angle solutions will be the real part.)

The equations become (in matrix notation):

$$\omega^2\begin{pmatrix}A_1\\A_2\end{pmatrix}=\begin{pmatrix}\omega_0^2+k & -k\\ -k & \omega_0^2+k\end{pmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix} \tag{17.2.4}$$

Denoting the $2\times 2$ matrix by $\mathbf{M}$

$$\mathbf{M}\vec{A}=\omega^2\vec{A},\quad \vec{A}=\begin{pmatrix}A_1\\A_2\end{pmatrix} \tag{17.2.5}$$

This is an eigenvector equation, with $\omega^2$ the eigenvalue, found by the standard procedure:

$$\det(\mathbf{M}-\omega^2\mathbf{I})=\begin{vmatrix}\omega_0^2+k-\omega^2 & -k\\ -k & \omega_0^2+k-\omega^2\end{vmatrix}=0 \tag{17.2.6}$$

Solving, $\omega^2=\omega_0^2+k\pm k$ , that is

$$\omega^2=\omega_0^2,\quad \omega^2=\omega_0^2+2k \tag{17.2.7}$$

The corresponding eigenvectors are (1,1) and (1,−1).