

22.2: Frequency of Oscillation of a Particle is a Slightly Anharmonic Potential

See the [applet](#) illustrating this section.

Landau (para 28) considers a simple harmonic oscillator with added small potential energy terms $\frac{1}{3}m\alpha x^3 + \frac{1}{4}m\beta x^4$. In leading orders, these terms contribute separately, and differently, so it's easier to treat them one at a time. We'll first consider the quartic term, an equation of motion

$$\ddot{x} + \omega_0^2 x = -\beta x^3 \quad (22.2.1)$$

(We'll always take β positive)

Writing a perturbation theory expansion (following Landau):

$$x = x^{(1)} + x^{(2)} + \dots \quad (22.2.2)$$

(Standard practice in most books would be to write $x = x^{(0)} + x^{(1)} + \dots$ with the superscript indicating the order of the perturbation--we're following Landau's notation, hopefully reducing confusion...) We take as the leading term

$$x^{(1)} = a \cos \omega t \quad (22.2.3)$$

with the exact value of ω , $\omega = \omega_0 + \Delta\omega$. Of course, we don't know the value of ω yet--this is what we're trying to find!

And, as Landau points out, you can't just write $\cos(\omega_0 + \Delta\omega)t = \cos \omega_0 t - (\Delta\omega)t \sin \omega_0 t$ because that implies motion increasing in time, and our system is a particle oscillating in a fixed potential, with no energy supply. Furthermore, even if we did somehow have the value of ω exactly right, this expression would not be a full solution to the equation: the motion is certainly periodic with period $2\pi/\omega$, but the complete description of the motion is a Fourier series including frequencies $n\omega$, n an integer, since the potential is no longer simple harmonic.

Anyway, putting this correct frequency into the equation of motion $\ddot{x} + \omega_0^2 x = -\beta x^3$ gives a nonzero left-hand side, so we rearrange. We subtract $(1 - (\omega_0^2/\omega^2)) \ddot{x}$ from both sides to get:

$$\frac{\omega_0^2}{\omega^2} \ddot{x} + \omega_0^2 x = -\beta x^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right) \ddot{x} \quad (22.2.4)$$

Now putting the leading term $x^{(1)} = a \cos \omega t$ into the left-hand side does give zero: if the equation had zero on the right hand side, this would just be a free (undamped) oscillator with natural frequency ω not ω_0 . This doesn't look very promising, but keep reading.

The equation for the first-order correction $x^{(2)}$ is:

$$\frac{\omega_0^2}{\omega^2} \ddot{x}^{(2)} + \omega_0^2 x^{(2)} = -\beta (x^{(1)})^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right) \ddot{x}^{(1)} \quad (22.2.5)$$

Notice that the second term on the right-hand side includes $\ddot{x}^{(1)} = -\omega^2 a \cos \omega t$. This equation now represents a driving force on an undamped oscillator exactly at its resonant frequency, so would cause the amplitude to increase linearly, obviously an unphysical result, since we're just modeling a particle sliding back and forth in a potential, with no energy being supplied from outside!

The key is that there is also a resonant driver in that first term $-\beta (x^{(1)})^3$.

Clearly these two driving terms have to cancel, and this requirement nails $\Delta\omega$: here's how:

$$-\beta (x^{(1)})^3 = -\beta a^3 \cos^3 \omega t = -\beta a^3 \left(\frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t \right) \quad (22.2.6)$$

so the resonant driving terms cancel provided

$$-\beta a^3 \frac{3}{4} \cos \omega t - \left(1 - \frac{\omega_0^2}{\omega^2}\right) (-\omega^2 a \cos \omega t) = 0 \quad (22.2.7)$$

Remembering $\omega = \omega_0 + \Delta\omega$, this gives (to this order)

$$\Delta\omega = \frac{3\beta a^2}{8\omega_0} \quad (22.2.8)$$

(Strictly, $\omega_0 + \frac{1}{2}\Delta\omega$ in the denominator, but that's a higher order correction.)

Note that the frequency increases with amplitude: the x^4 potential gives an increasingly stronger restoring force with amplitude than the harmonic well. You can check this with the applet. Now let's consider the equation for a small cubic perturbation,

$$\ddot{x} + \omega_0^2 x = -\alpha x^2 \quad (22.2.9)$$

This represents an added potential $-\frac{1}{3}\alpha x^3$, which is an odd function, so to leading order it won't change the period, speeding up one half of the oscillation and slowing the other half the same amount in leading order. The first correction to the position as a function of time is the solution of

$$\ddot{x}^{(2)} + \omega_0^2 x^{(2)} = -\alpha a^2 \cos^2 \omega t = -\frac{1}{2}\alpha a^2 - \frac{1}{2}\alpha a^2 \cos 2\omega t \quad (22.2.10)$$

The solution is

$$x^{(2)} = -\frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2} \cos 2\omega t \quad (22.2.11)$$

Physically, adding this to the leading term, the particle is spending more of its time in the softer half of the potential, giving an amplitude-dependent correction to its average position.

To get the correction to the frequency, we need to go to the next order,

$$\omega = \omega_0 + \omega^{(2)} \quad (22.2.12)$$

. Dropping terms of higher order, the equation of motion for the next correction is

$$\ddot{x}^{(3)} + \omega_0^2 x^{(3)} = -2\alpha x^{(1)} x^{(2)} + 2\omega_0 \omega^{(2)} x^{(1)} \quad (22.2.13)$$

and with

$$x = x^{(1)} + x^{(2)} + x^{(3)}, \omega = \omega_0 + \omega^{(2)} \quad (22.2.14)$$

, following Landau,

$$\ddot{x}^{(3)} + \omega_0^2 x^{(3)} = -\frac{\alpha^2 a^3}{6\omega_0^2} \cos 3\omega t + a \left[2\omega_0 \omega^{(2)} + \frac{5a^2 \alpha^2}{6\omega_0^2} \right] \cos \omega t \quad (22.2.15)$$

Again, there cannot be a nonzero term driving the system at resonance, so the quantity in the square brackets must be zero, this gives us $\Delta\omega = \omega^{(2)} = -5a^2 \alpha^2 / 12\omega_0^3$

The total correction to frequency to leading order for the additional small potentials

$$\frac{1}{3} m \alpha x^3 + \frac{1}{4} m \beta x^4 \quad (22.2.16)$$

is therefore (they add independently to this order)

$$\Delta\omega = \left(\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3} \right) a^2 = \kappa a^2 \quad (22.2.17)$$

(the κ here being a convenient notation Landau employs later).

How Good Are These Approximations?

We have an applet that solves this equation numerically, so it is straightforward to check.

Beginning with the quartic perturbation potential $\frac{1}{4}m\beta x^4$, Landau finds a frequency correction $\Delta\omega = 3\beta a^2/8\omega_0$. Taking a rather large perturbation $a = \beta = \omega_0 = 1$ we find from the applet that $\Delta\omega = 0.33$ whereas Landau's perturbation theory predicts $\Delta\omega = \frac{3}{8} = 0.375$. However, if we correct Landau's denominator (as mentioned above, he pointed out it should be ω , but said that was secondorder) the error is very small.

Taking $\alpha = 0.3$, $\beta = 0.1$, $\omega_0 = 0$, $a = 1$ the formula gives $\Delta\omega = 0.018$ so less than two percent error, and for amplitude 0.2, the effort is less than 0.1%.

Explore with the [applet here](#).

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