30.5: Ball Rolling on Rotating Plane

(The following examples are from Milne, *Vectorial Mechanics*.)

A sphere is rolling without slipping on a horizontal plane. The plane *is itself rotating* at constant angular velocity ω.

We have three vector equations: Newton's equations for linear and angular acceleration, and the rolling condition. We want to find the path taken by the rolling ball on the rotating surface, that is, $\vec{r}(t)$. We'll use our three equations to eliminate two (vector) variables: the reaction force between the plane and the ball $\vec{R}$, and the angular velocity $\vec{\Omega}$.

The equations of motion of the sphere (radius a, mass m, center at $\vec{r}$ measured in the lab, horizontally from the axis of the plane's rotation) with $\vec{R}$ the contact force of the plane on the sphere, are

$$m\ddot{\vec{r}} = \vec{R} - mg\widehat{\vec{n}}, \quad I\dot{\vec{\Omega}} = -a\widehat{\vec{n}} \times \vec{R} \tag{30.5.1}$$

(Of course, the gravitational force here is just balancing the vertical component of the reaction force, but this is no longer the case for the tilted plane, treated in the next section.)

First, we'll eliminate the reaction force $\vec{R}$ to get an equation of motion:

$$I\dot{\vec{\Omega}} = -am\widehat{\vec{n}} \times (\ddot{\vec{r}} + g\widehat{\vec{n}}) = \mathrm{am}\,\ddot{\vec{r}} \times \widehat{\vec{n}} \tag{30.5.2}$$

The rolling condition is:

$$\dot{\vec{r}} - a\vec{\Omega} \times \widehat{\vec{n}} = \omega\widehat{\vec{n}} \times \vec{r} \tag{30.5.3}$$

the right-hand side being the local velocity of the turntable, $\vec{r}$ measured from an origin at the center of rotation.

We'll use the rolling condition to eliminate $\dot{\vec{\Omega}}$ and give us an equation for the actual path of the sphere.

First, differentiate it (remember $\widehat{\vec{n}}, \omega$ are both constant) to get

$$\ddot{\vec{r}} - a\dot{\vec{\Omega}} \times \widehat{\vec{n}} = \omega\widehat{\vec{n}} \times \dot{\vec{r}} \tag{30.5.4}$$

Next, take the equation of motion $I\dot{\vec{\Omega}} = am\ddot{\vec{r}} \times \widehat{\vec{n}}$ and $\times\widehat{\vec{n}}$ to get

$$I\dot{\vec{\Omega}} \times \widehat{\vec{n}} = am(\ddot{\vec{r}} \times \widehat{\vec{n}}) \times \widehat{\vec{n}} = -am\ddot{\vec{r}} \tag{30.5.5}$$

and putting these together to get rid of the angular velocity,

$$\left(1 + a^2m/I\right)\ddot{\vec{r}} = \omega\widehat{\vec{n}} \times \dot{\vec{r}} \tag{30.5.6}$$

This integrates to

$$\dot{\vec{r}} = \left(\frac{\omega}{1 + a^2m/I}\right)\widehat{\vec{n}} \times (\vec{r} - \vec{r}_0) \tag{30.5.7}$$

which is just the equation for steady *circular* motion about the point $\vec{r}_0$.

For a uniform sphere, $I = \frac{2}{5}Ma^2$, so $\dot{\vec{r}} = \frac{2}{7}\omega\widehat{\vec{n}} \times (\vec{r} - \vec{r}_0)$

So the ball rolling on the rotating plate goes around in a circle, which could be *any* circle. If it is put down gently at any point on the rotating plane, and held in place until it is up to speed (meaning no slipping) it will stay at that point for quite a while (until the less than perfect conditions, such as air resistance or vibration, cause noticeable drift). If it is nudged, it will move in a circle. In class, we saw it circle many times—eventually, it fell off, a result of air resistance plus the shortcomings of our apparatus, but the circular path was very clear.