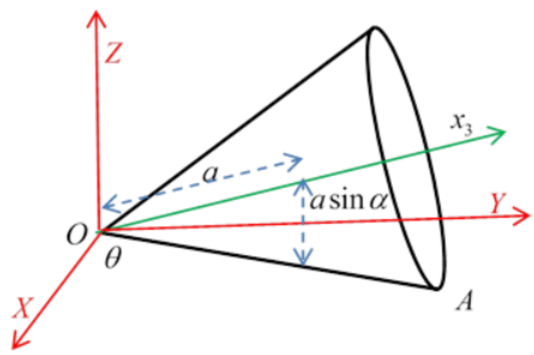
25.2: Analyzing Rolling Motion

Kinetic Energy of a Cone Rolling on a Plane

The cone rolls without slipping on the horizontal XY plane. The momentary line of contact with the plane is OA, at an angle θ in the horizontal plane from the X axis.



The important point is that this line of contact, *regarded as part of the rolling cone*, is momentarily at rest when it's in contact with the plane. This means that, *at that moment*, the cone is rotating about the stationary line OA. Therefore, the angular velocity vector $\vec{\Omega}$ points along OA.

Taking the cone to have semi-vertical angle α (meaning this is the angle between OA and the central axis of the cone) the center of mass, which is a distance a from the vertex, and on the central line, moves along a circle at height $a\sin\alpha$ above the plane, this circle being centered on the Z axis, and having radius $a\cos\alpha$. The center of mass moves at velocity $V=\dot{\theta} a\cos\alpha$, so contributes *translational* kinetic energy

$$\frac{1}{2}MV^2=\frac{1}{2}M\dot{\theta}^2a^2\cos^2\alpha \tag{25.2.1}$$

Now visualize the rolling cone turning around the momentarily fixed line OA: the center of mass, at height $a\sin\alpha$, moves at V, so the angular velocity

$$\Omega=\frac{V}{a\sin\alpha}=\dot{\theta}\cot\alpha. \tag{25.2.2}$$

Next, we first define a new set of axes with origin O: one, $x_3$is the cone's own center line, another, x_2 is perpendicular to that *and* to OA, this determines x_1 (For these last two, since they're through the vertex, the moment of inertia is the one worked out at the end of the previous section, see above.)

Since $\vec{\Omega}$ is along OA, its components with respect to these axes (x_1,x_2,x_3) are $(\Omega\sin\alpha, 0, \quad \Omega\cos\alpha)$

However, to compute the total kinetic energy, for the rotational contribution we need to use a parallel set of axes *through the center of mass*. This just means subtracting from the vertex perpendicular moments of inertia found above a factor Ma^2.

The total kinetic energy is

$$\begin{aligned} T &=\frac{1}{2}M\dot{\theta}^2a^2\cos^2\alpha+\frac{1}{2}I_1\dot{\theta}^2\cos^2\alpha+\frac{1}{2}I_3\dot{\theta}^2\frac{\cos^4\alpha}{\sin^2\alpha} \\ &=3Mh^2\dot{\theta}^2\left(1+5\cos^2\alpha\right)/40 \end{aligned}$$

using

$$I_1=\frac{3}{20}MR^2+\frac{3}{80}Mh^2,\quad I_3=\frac{3}{10}MR^2,\quad a=\frac{3}{4}h,\quad R=h\tan\alpha \tag{25.2.3}$$