28.1: Introduction to Euler's Equations

We've just seen that by specifying the rotational direction and the angular phase of a rotating body using Euler's angles, we can write the Lagrangian in terms of those angles and their derivatives, and then derive equations of motion. These can be solved to describe precession, nutation, etc.

One might hope for a more direct Newtonian approach—we know, for example, that the steadily precessing child's top is easy to understand in terms of the gravitational torque rotating the angular momentum vector.

What about applying $(d\vec{L}/dt)_{\text{lab}} = \vec{K}$ (the external torque on the system) more generally? It's certainly valid. The problem is that in the *lab frame* $\vec{L} = \mathbf{I}\vec{\Omega}$ is

$$L_i = I_{ij}\Omega_j \tag{28.1.1}$$

and the elements of the inertia tensor relative to the lab axes are constantly changing as the body rotates.

The Newtonian approach is only practicable if the connection between $\vec{L}, \vec{\Omega}$ can be made in the *body frame* defined by the principal axes of inertia (x_1, x_2, x_3), in which $\vec{L} = \mathbf{I}\vec{\Omega}$ is

$$L_1 = I_1\Omega_1, L_2 = I_2\Omega_2, L_3 = I_3\Omega_3 \tag{28.1.2}$$

With the body frame rotating at $\vec{\Omega}$ relative to the fixed-in-space (X,Y,Z) frame, the rates of change of a vector in the two frames satisfy

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{lab}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{body}} + \vec{\Omega} \times \vec{A} \tag{28.1.3}$$

To understand this equation, think first of a moving particle, say a bug crawling about on the rotating body. The bug's movement relative to the center of rotation is equal to its movement relative to axes fixed in the rotating body, plus the rotational movement of that body relative to the fixed-in-space axes.

You might be thinking at this point: yes, I can see this is true if the vector represents the position of a particle that's moving around in space, but we're looking at the changing angular momentum, why isn't the angular momentum just zero in a frame in which the body is at rest? And the angular velocity, too?

But what is meant here by the vector "in the body frame" is the components of the vector in an inertial frame that is momentarily coincident with the principal axes.

The $\vec{\Omega} \times \vec{A}$ term represents the change on going through this *succession of inertial frames*.

Think of a long forward pass of an (American) football. The ball is spinning about its long axis (usually), but that axis itself is precessing about the line of flight. Of course, air resistance presumably helps it line up this way, but is not the main effect, which is that the angular momentum vector points in a constant direction in space, the axis of symmetry is precessing around it, as we saw on throwing the ball in class. Imagine now running alongside the ball, holding a pencil pointing in the direction of the constant angular momentum vector. As seen by an observer in the ball, relative to the ball's principal axes frame of reference, the pencil will be describing a cone—this is what we mean by the path of the angular momentum vector relative to the body axes.

The equations of motion in the body frame are then

$$\left(\frac{d\vec{P}}{dt}\right)_{\text{body}} + \vec{\Omega} \times \vec{P} = \vec{F}, \quad \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\Omega} \times \vec{L} = \vec{K} \tag{28.1.4}$$

where $\vec{F}, \vec{K}$ are the external force and couple respectively.

Writing the angular momentum equation in components along the principal axes:

$$\begin{aligned} I_1 d\Omega_1/dt + (I_3 - I_2)\Omega_2\Omega_3 &= K_1 \\ I_2 d\Omega_2/dt + (I_1 - I_3)\Omega_3\Omega_1 &= K_2 \\ I_3 d\Omega_3/dt + (I_2 - I_1)\Omega_1\Omega_2 &= K_3 \end{aligned} \tag{28.1.5}$$

These are *Euler's Equations*.

In the important special case of zero torque:

$$\begin{aligned} I_1 d\Omega_1/dt + (I_3 - I_2)\,\Omega_2\Omega_3 = 0 \\ I_2 d\Omega_2/dt + (I_1 - I_3)\,\Omega_3\Omega_1 = 0 \\ I_3 d\Omega_3/dt + (I_2 - I_1)\,\Omega_1\Omega_2 = 0 \end{aligned} \tag{28.1.6}$$
