21.2: Finding the Effective Potential Generated by the Oscillating Force

As stated above, our system is a particle of mass m moving in one dimension in a time-independent potential $V(x)$ and subject to a rapidly oscillating force $f=f_1\cos\omega t+f_2\sin\omega t$.

The oscillation's strength and frequency are such that the particle only moves a small distance in $V(x)$ during one cycle, and the oscillation is much faster than any oscillation possible in the potential alone.

The equation of motion is

$$m\ddot{x}=-dV/dx+f \tag{21.2.1}$$

The particle will follow a path

$$x(t)=X(t)+\xi(t) \tag{21.2.2}$$

where $\xi(t)$ describes rapid oscillations about a smooth path $X(t)$, and the average value $\overline{\xi(t)}$ of $\xi(t)$ over a period $2\pi/\omega$ is zero.

Expanding to first order in ξ,

$$m\ddot{X}+m\ddot{\xi}=-\frac{dV}{dx}-\xi\frac{d^2V}{dx^2}+f(X,t)+\xi\frac{\partial f}{\partial X} \tag{21.2.3}$$

This equation has smooth terms and rapidly oscillating terms on both sides, and we can equate them separately. The leading oscillating terms are

$$m\ddot{\xi}=f(X,t) \tag{21.2.4}$$

We've dropped the terms on the right of order ξ, but kept $\ddot{\xi}$, because $\ddot{\xi}\sim\omega^2\xi\gg\xi$.

So to leading order in the rapid oscillation,

$$\xi=-f/m\omega^2 \tag{21.2.5}$$

Now, averaging the full equation of motion with respect to time (smoothing out the jiggle, matching the slow-moving terms), the $m\ddot{\xi}$ on the left and the $f(X,t)$ on the right both disappear (but cancel each other anyway), the $\xi d^2V/dx^2$ term averages to zero on the assumption that the variation of d^2V/dx^2 over a cycle of the fast oscillation is negligible, but we cannot drop the average

$$\overline{\xi\frac{\partial f}{\partial X}}=-\frac{1}{m\omega^2}\overline{f\frac{\partial f}{\partial X}}=-\frac{1}{m\omega^2}\nabla_X\overline{f^2} \tag{21.2.6}$$

Incorporating this nonzero term, we have an equation of "slow motion"

$$m\ddot{X}=-dV_{\text{eff}}/dX \tag{21.2.7}$$

where, using $|\dot{\xi}|=|f|/m\omega$,

$$V_{\text{eff}}=V+\overline{f^2}/2m\omega^2=V+\frac{1}{2}m\overline{\dot{\xi}^2} \tag{21.2.8}$$

The effective potential is the original plus a term proportional to the kinetic energy of the oscillation.