11.9: Energy Gradient and Phase Space Velocity

For a time-independent Hamiltonian, the path in phase space (q, p) is a constant energy line, and we can think of the whole phase space as delineated by many such lines, exactly analogous to contour lines joining points at the same level on a map of uneven terrain, energy corresponding to height above sea level. The gradient at any point, the vector pointing exactly uphill and therefore perpendicular to the constant energy path, is

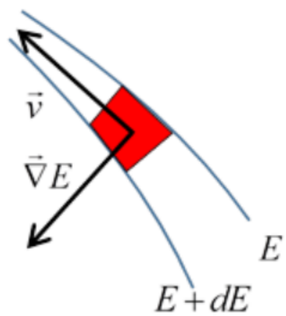

Figure 11.9.1

$$\vec{\nabla} H=(\partial H / \partial q, \partial H / \partial p) \tag{11.9.1}$$

here $H=E$. The velocity of a system's point moving through phase space is

$$\vec{v}=(\dot{q}, \dot{p})=(\partial H / \partial p,-\partial H / \partial q) \tag{11.9.2}$$

This vector is perpendicular to the gradient vector, as it must be, of course, since the system moves along a constant energy path. But, interestingly, it has the same magnitude as the gradient vector! What is the significance of that? Imagine a small square sandwiched between two phase space paths close together in energy, and suppose the distance between the two paths is decreasing, so the square is getting squeezed, at a rate equal to the rate of change of the energy gradient. But at the same time it must be getting stretched along the direction of the path, an amount equal to the rate of change of phase space velocity along the path—and they are equal. So, this is just Liouville again, its area doesn't change.