30.6: Ball Rolling on Inclined Rotating Plane

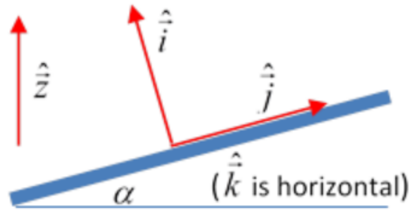

Figure 30.6.1

We'll take unit vectors $\widehat{\vec{z}}$ pointing vertically up, $\hat{\vec{i}}$ perpendicularly up from the plane, the angle between these two unit vectors being α. (We will need a set of orthogonal unit vectors $\hat{\vec{i}}, \hat{\vec{j}}, \hat{\vec{k}}$, not fixed in the plane, but appropriately oriented, with $\hat{\vec{k}}$ horizontal.) The vector to the center of the sphere (radius a, mass m) from an origin on the axis of rotation, at a point a above the plane, is $\vec{r}$. The contact reaction force of the plane on the sphere is $\vec{R}$.

The equations of motion are:

$$m\ddot{\vec{r}} = \vec{R} - mg\widehat{\vec{z}}, \quad I\dot{\vec{\Omega}} = -a\hat{\vec{i}} \times \vec{R} \tag{30.6.1}$$

and the equation of rolling contact is $\dot{\vec{r}} - a\vec{\Omega} \times \hat{\vec{i}} = \widehat{\omega i} \times \vec{r}$.

First, we eliminate $\vec{R}$ from the equations of motion to give

$$\dot{\vec{\Omega}} = (am/I)(\ddot{\vec{r}} + g\widehat{\vec{z}}) \times \hat{\vec{i}} \tag{30.6.2}$$

Note that $\dot{\vec{\Omega}} \cdot \hat{\vec{i}} = 0$, so the spin in the direction normal to the plane is constant, $\vec{\Omega} \cdot \hat{\vec{i}} = n$ say. (Both forces on the sphere have zero torque about this axis.)

Integrating,

$$\vec{\Omega} + \text{ const. } = (ma/I)(\dot{\vec{r}} + gt\widehat{\vec{z}}) \times \hat{\vec{i}} \tag{30.6.3}$$

Now eliminate $\vec{\Omega}$ by multiplying both sides by $\times\vec{i}$ and using the equation of rolling contact

$$\dot{\vec{r}} - a\vec{\Omega} \times \hat{\vec{i}} = \omega\hat{\vec{i}} \times \vec{r} \tag{30.6.4}$$

to find:

$$\left(ma^2/I\right)\left[(\dot{\vec{r}} + gt\widehat{\vec{z}}) \times \hat{\vec{i}}\right] \times \hat{\vec{i}} = a\vec{\Omega} \times \hat{\vec{i}} + \text{ const. } = \dot{\vec{r}} - \omega\hat{\vec{i}} \times \vec{r} + \text{ const.} \tag{30.6.5}$$

then using $(\dot{\vec{r}} \times \hat{\vec{i}}) \times \hat{\vec{i}} = -\dot{\vec{r}},\ (\widehat{\vec{z}} \times \hat{\vec{i}}) \times \hat{\vec{i}} = -\hat{\vec{j}}$, we find

$$\dot{\vec{r}}\left(1 + ma^2/I\right) + \left(ma^2/I\right) gt\hat{\vec{j}} \sin\alpha + \text{ const. } = \widehat{\omega veci} \times \vec{r} \tag{30.6.6}$$

The constant is fixed by the initial position $\vec{r}_0$, giving finally

$$\dot{\vec{r}} = \frac{\omega}{1 + ma^2/I}\hat{\vec{i}} \times \left[(\vec{r} - \vec{r}_0) + \frac{ma^2/I}{\omega} gt\hat{\vec{k}} \sin\alpha\right] \tag{30.6.7}$$

The first term in the square brackets would give the same circular motion we found for the horizontal rotating plane, the second term adds a steady motion of the center of this circle, in a *horizontal* direction (*not* down the plane!) at constant speed $\left(ma^2/I\omega\right) g\sin\alpha$.

(This is identical to the motion of a charged particle in crossed electric and magnetic fields.)

Bottom line: the intuitive notion that a ball rolling on a rotating inclined turntable would tend to roll downhill is wrong! Recall that for a particle circling in a magnetic field, if an electric field is added perpendicular to the magnetic field, the particle moves in a cycloid at the same average electrical potential—it has *no net movement in the direction of the electric field* , only perpendicular to it. Our rolling ball follows an identical cycloidal path—keeping the same average gravitational potential.
