27.5: Steady Precession

Under what conditions will a top, spinning under gravity, precess at a steady rate? The constancy of L_3, L_Z mean that $\Omega_3=\dot{\phi}\cos\theta+\dot{\psi}$, and $\Omega_{\mathrm{pr}}=\dot{\phi}$ are constants.

The θ Lagrange equation is

$$I_1'\ddot{\theta}=I_1'\dot{\phi}^2\sin\theta\cos\theta-I_3(\dot{\phi}\cos\theta+\dot{\psi})\dot{\phi}\sin\theta+Mg\ell\sin\theta \tag{27.5.1}$$

For constant θ, $\ddot{\theta}=0$, so, with $\Omega_3=\dot{\phi}\cos\theta+\dot{\psi}$, and $\Omega_{\mathrm{pr}}=\dot{\phi}$.

$$I_1'\Omega_{pr}^2\cos\theta-I_3\Omega_3\Omega_{pr}+Mg\ell=0 \tag{27.5.2}$$

Since Equation 27.5.2 is a *quadratic* equation for the precession rate, there are *two* solutions in general: on staring at a precessing top, this is a bit surprising! We know that for the top, when it's precessing nicely, the spin rate Ω_3 far exceeds the precession rate Ω_{pr}. Assuming I_1', I_3 to be of similar size, this means the first term in the quadratic is much smaller than the second. If we just drop the first term, we get the precession rate

$$\Omega_{\text{precess (slow)}}=\frac{Mg\ell}{I_3\Omega_3},\quad(\Omega_3\gg\Omega_{\text{precess}}) \tag{27.5.3}$$

Note that this is independent of angle—the torque varies as $\sin\theta$, but so does the horizontal component of the angular momentum, which is what's changing.

This is the familiar solution for a child's fast-spinning top precessing slowly. But this is a quadratic equation, there's another possibility: in this large Ω_3 limit, this other possibility is that Ω_{pr} is itself of order Ω_3 , so now in the equation the last term, the gravitational one, is negligible, and

$$\Omega_{\text{precess (fast)}}\cong I_3\Omega_3/I_1'\cos\theta \tag{27.5.4}$$

This is just the nutation of a *free* top! In fact, of course, both of these are approximate solutions, only exact in the limit of infinite spin (where one goes to zero, the other to infinity), and a more precise treatment will give corrections to each arising from the other. Landau indicates the leading order gravitational correction to the free body nutation mode.