12.2: The Central Role of These Constants of Integration

To describe the time development of a dynamical system in the simplest way possible, it is desirable to find parameters that are constant or change in a simple way. For example, motion in a spherically symmetric potential is described in terms of (constant) angular momentum components.

Now, these constant *[Math Processing Error]*'s are functions of the initial coordinates and momenta. Since they remain constant during the motion, they are evidently among the "variables" that describe the dynamical development in the simplest possible way. So, we need to construct a canonical transformation from our current set of variables (final coordinates and momenta) to a new set of variables that includes these constant of integration "momenta". (The corresponding canonical "positions" will then be given by differentiating the generating function with respect to the "momenta".)

How do we find the generating function for this transformation? A clue comes from one we've already discussed: that corresponding to development in time, going from the initial set of variables to the final set, or back. That transformation was generated *by the action itself*, expressed in terms of the two sets of positions. That is, we allowed *both* ends of the action integral path to vary, and wrote the action as a function of the final (2) and initial (1) endpoint variables and times:

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In the present section, the final endpoint positions are denoted simply by *[Math Processing Error]* these are the same as the earlier *[Math Processing Error]*. Explicitly, we're writing

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Compare this expression for the action with the formal expression we just derived from the Hamilton Jacobi equation,

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These two expressions for *[Math Processing Error]* have just the same form: the action is expressed as a function of the endpoint position variables, plus another *[Math Processing Error]* variables needed to determine the motion uniquely. This time, instead of the original position variables, though, the second set of variables is these constants of integration, the *[Math Processing Error]*'s.

Now, just as we showed the action generated the transformation (either way) between the initial set of coordinates and momenta and the final set, it will also generate a canonical transformation from the final set of coordinates and momenta to another canonical set, having the *[Math Processing Error]*'s as the new "momenta". We'll label the new "coordinates" (the canonical conjugates of the *[Math Processing Error]*'s *[Math Processing Error]*

Taking then the action (neglecting the constant *[Math Processing Error]* which does nothing) *[Math Processing Error]* as the generating function, it depends on the old coordinates *[Math Processing Error]* and the new momenta *[Math Processing Error]*. This is the same set of variables—old coordinates and new momenta—as those of the (previously discussed) generating function *[Math Processing Error]*.

Recall

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so here

[Math Processing Error]

and

[Math Processing Error]

This defines the new "coordinates" *[Math Processing Error]* *and* ensures that the transformation is canonical.

To find the new Hamiltonian *[Math Processing Error]*.

But

[Math Processing Error]

where *[Math Processing Error]* is just a constant, so

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The first equation in this section was

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so the new Hamiltonian

[Math Processing Error]

We have made a canonical transformation that has led to a zero Hamiltonian!

What does that mean? It means that the *neither the new momenta nor the new coordinates vary in time*:

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(The fact that all momenta and coordinates are fixed in this representation does *not* mean that the system doesn't move—as will become evident in the following simple example, the original coordinates are functions of these new (nonvarying!) variables *and time*.)

The *[Math Processing Error]* equations *[Math Processing Error]* can then be used to find the *[Math Processing Error]* as functions of *[Math Processing Error]* To see how all this works, it is necessary to work through an example.

A Simple Example of the Hamilton-Jacobi Equation: Motion Under Gravity

The Hamiltonian for motion under gravity in a vertical plane is

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so the Hamilton-Jacobi equation is

[Math Processing Error]

First, this Hamiltonian has no explicit time dependence (gravity isn't changing!), so from *[Math Processing Error]*, we can replace the last term in the equation by *[Math Processing Error]*.

A Simple Separation of Variables

Since the potential energy term depends only on *[Math Processing Error]*, the equation is solvable using separation of variables. To see this works, try

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Putting this form into the equation, the resulting first term depends only on the variable *[Math Processing Error]*, the second plus third depend only on *[Math Processing Error]*, the last term is just the constant *[Math Processing Error]*. A function depending only on *[Math Processing Error]* can only equal a function independent of *[Math Processing Error]* if both are constants, similarly for *[Math Processing Error]*.

Labeling the constants *[Math Processing Error]*

[Math Processing Error]

So these *[Math Processing Error]*'s are constants of the motion, they are our new "momenta" (although they have dimensions of energy).

Solving,

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(We could add in constants of integration, but adding constants to the action changes nothing.)

So now we have

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This is our generating function (equivalent to *[Math Processing Error]*), in terms of old coordinates and these new "momenta", *[Math Processing Error]* Following the Hamilton-Jacobi analysis, this action will generate a canonical transformation which reduces the Hamiltonian to zero, meaning that not only these new momenta stay constant, but so do their conjugate "coordinate" variables,

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These equations solve the problem. Rearranging, the trajectory is

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The four “constants of motion” *[Math Processing Error]* are uniquely fixed by the initial coordinates and velocities, and they parameterize the subsequent time evolution of the system.
