6.9: A Simple Example

For a particle moving in a potential in one dimension, $L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q)$.

Hence

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}, \quad \dot{q} = \frac{p}{m} \tag{6.9.1}$$

Therefore

$$\begin{aligned} H &= p\dot{q} - L = p\dot{q} - \frac{1}{2}m\dot{q}^2 + V(q) \\ &= \frac{p^2}{2m} + V(q) \end{aligned} \tag{6.9.2}$$

(Of course, this is just the total energy, as we expect.)

The Hamiltonian equations of motion are

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} &= -\frac{\partial H}{\partial q} = -V'(q) \end{aligned} \tag{6.9.3}$$

So, as we've said, the second order Lagrangian equation of motion is replaced by two first order Hamiltonian equations. Of course, they amount to the same thing (as they must!): differentiating the first equation and substituting in the second gives immediately $-V'(q) = m\ddot{q}, \text{ that is, } F = ma$, the original Newtonian equation (which we derived earlier from the Lagrange equations).