11.6: Jacobians 101

Suppose we are integrating a function over some region of ordinary three-dimensional space,

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but we want to change variables of integration to a different set of coordinates *[Math Processing Error]* such as, for example, *[Math Processing Error]*. The new coordinates are of course functions of the original ones *[Math Processing Error]* etc., and we assume that in the region of integration they are smooth, well-behaved functions. We can't simply re-express f in terms of the new variables, and replace the volume differential *[Math Processing Error]* that gives the wrong answer—in a plane, you can't replace *[Math Processing Error]* with *[Math Processing Error]*, you have to use *[Math Processing Error]*. That extra factor *[Math Processing Error]* is called the **Jacobian**, it's clear that in the plane a small element with sides of fixed lengths *[Math Processing Error]* is bigger the further it is from the origin, not all *[Math Processing Error]* elements are equal, so to speak. Our task is to construct the Jacobian for a general change of coordinates.

We need to think carefully about the volumes in the three-dimensional space represented by *[Math Processing Error]* and by *[Math Processing Error]*. Of course, the *[Math Processing Error]*'s are just ordinary perpendicular Cartesian axes so the volume is just the product of the three sides of the little box, *[Math Processing Error]*.Imagine this little box, its corner closest to the origin at *[Math Processing Error]* and its furthest point at the other end of the body diagonal at *[Math Processing Error]* Let's take these two points in the qi coordinates to be at *[Math Processing Error]*. In visualizing this, bear in mind that the q axes need not be perpendicular to each other (but they cannot all lie in a plane, that would not be well-behaved).

For the *[Math Processing Error]* coordinate integration, we imagine filling the space with little cubical boxes. For the *[Math Processing Error]* integration, we have a system of space filling infinitesimal parallelepipeds, in general pointing different ways in different regions (think *[Math Processing Error]*. What we need to find is the volume of the incremental parallelepiped with sides we'll write as vectors in x -coordinates, *[Math Processing Error]*. These three incremental vectors are along the corresponding *[Math Processing Error]* coordinate axes, and the three added together are the displacement from *[Math Processing Error]*

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Hence, in components,

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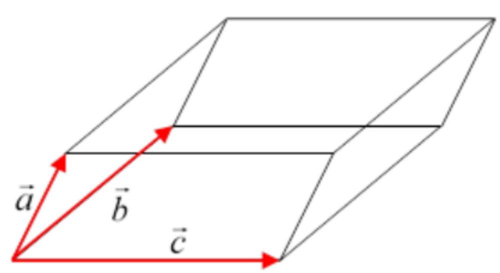
Now the volume of the parallelepiped with sides the three vectors from the origin *[Math Processing Error]* area of the parallelogram, then the dot product singles out the component of *[Math Processing Error]* perpendicular to the plane of *[Math Processing Error]*).



So, the volume corresponding to the increments *[Math Processing Error]* in *[Math Processing Error]* space is

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writing *[Math Processing Error]* (Landau's notation) for the *determinant*, which is in fact the *Jacobian*, often denoted by *[Math Processing Error]*.

The standard notation for this determinantal Jacobian is

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So the appropriate replacement for the three dimensional incremental volume element represented in the integral by *[Math Processing Error]* is

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The *inverse*

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is easily established using the chain rule for differentiation.

Exercise *[Math Processing Error]*

check this!

Thus the change of variables in an integral is accomplished by rewriting the integrand in the new variables, and replacing

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The argument in higher dimensions is just the same: on going to dimension *[Math Processing Error]*, the hypervolume element is equal to that of the *[Math Processing Error]* dimensional element multiplied by the component of the new vector perpendicular to the *[Math Processing Error]* dimensional element. The determinantal form does this automatically, since a determinant with two identical rows is zero, so in adding a new vector only the component perpendicular to all the earlier vectors contributes.

We've seen that the chain rule for differentiation gives the inverse as just the Jacobian with numerator and denominator reversed, it also readily yields

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and this extends trivially to n dimensions.

It's also evident form the determinantal form of the Jacobian that

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identical variables in numerator and denominator can be canceled. Again, this extends easily to *[Math Processing Error]* dimensions.