4.2: Derivation of Hamilton's Principle from Newton's Laws in Cartesian Co-ordinates- Calculus of Variations Done Backwards!

We've shown how, given an integrand, we can find differential equations for the path in space time between two fixed points that minimizes the corresponding path integral between those points.

Now we'll do *the reverse*: we already know the differential equations in Cartesian coordinates describing the path taken by a Newtonian particle in some potential. We'll show how to use that knowledge to construct the integrand such that the action integral is a minimum along that path. (This follows Jeffreys and Jeffreys, *Mathematical Physics*.)

We begin with the simplest nontrivial system, a particle of mass m moving in one dimension from one point to another in a specified time, we'll assume it's in a time-independent potential $U(x)$, so

$$m\ddot{x} = -dU(x)/dx \tag{4.2.1}$$

Its path can be represented as a graph $x(t)$ against time—for example, for a ball thrown directly upwards in a constant gravitational field this would be a parabola.

Initial and final positions are given: $x(t_1) = x_1, x(t_2) = x_2,$ and the elapsed time is $t_2 - t_1$

Notice we have *not* specified the initial velocity—we don't have that option. The differential equation is only second order, so its solution is completely determined by the two (beginning and end) boundary conditions.

We're now ready to embark on the calculus of variations in reverse.

Trivially, multiplying both sides of the equation of motion by an arbitrary infinitesimal function the equality still holds:

$$m\ddot{x}\delta x(t) = -(dU(x)/dx)\delta x(t) \tag{4.2.2}$$

and in fact if *this* equation is true for *arbitrary* $\delta x(t)$, the original equation of motion holds throughout, because we can always choose a $\delta x(t)$ nonzero only in the neighborhood of a particular time t, from which the original equation must be true at that t.

By analogy with Fermat's principle in the preceding section, we can picture this $\delta x(t)$ as a slight variation in the path from the Newtonian trajectory, $x(t) \rightarrow x(t) + \delta x(t)$ and take the variation zero at the fixed ends, $\delta x(t_1) = \delta x(t_2) = 0$

In Fermat's case, the integrated time elapsed along the path was minimized—there was zero change to first order on going to a neighboring path. Developing the analogy, we're looking for some dynamical quantity that has zero change to first order on going to a neighboring path having the same endpoints in space and time. We've fixed the time, what's left to integrate along the path?

For such a simple system, we don't have many options! As we've discussed above, the equation of motion is equivalent to (putting in an overall minus sign that will prove convenient)

$$\int_{t_1}^{t_2} (-m\ddot{x}(t) - dU(x(t))/dx)\delta x(t)dt = 0 \quad \text{to leading order, for all variations } \delta x(t) \tag{4.2.3}$$

Integrating the first term by parts (recalling $\delta(x) = 0$ at the endpoints):

$$-\int_{t_1}^{t_2} m\ddot{x}(t)\delta x(t)dt = \int_{t_1}^{t_2} m\dot{x}(t)\delta\dot{x}(t)dt = \int_{t_1}^{t_2} \delta\left(\frac{1}{2}m\dot{x}^2(t)\right)dt = \int_{t_1}^{t_2} \delta T(x(t))dt \tag{4.2.4}$$

using the standard notation T for kinetic energy.

The second term integrates trivially:

$$-\int_{t_1}^{t_2} (dU(x)/dx)\delta x(t)dt = -\int_{t_1}^{t_2} \delta U(x)dt \tag{4.2.5}$$

establishing that on making an infinitesimal variation from the physical path (the one that satisfies Newton's laws) there is zero first order change in the integral of kinetic energy minus potential energy.

The standard notation is

$$\delta S = \delta\int_{t_1}^{t_2} (T - U)dt = \delta\int_{t_1}^{t_2} Ldt = 0 \tag{4.2.6}$$

The integral S is called the *action integral*, (also known as Hamilton's Principal Function) and the integrand $T-U=L$ is called the *Lagrangian*.

This equation is Hamilton's Principle.

The derivation can be extended straightforwardly to a particle in three dimensions, in fact to n interacting particles in three dimensions. We shall assume that the forces on particles can be derived from potentials, including possibly time-dependent potentials, but we exclude frictional energy dissipation in this course. (It can be handled—see for example Vujanovic and Jones, *Variational Methods in Nonconservative Phenomena*, Academic press, 1989.)
