3.3: Fermat's Principle

We will now temporarily forget about the wave nature of light, and consider a narrow ray or beam of light shining from point A to point B, where we suppose A to be in air, B in glass. Fermat showed that the path of such a beam is given by the Principle of Least Time: a ray of light going from A to B by any other path would take longer. How can we see that? It's obvious that any deviation from a straight line path in air or in the glass is going to add to the time taken, but what about moving slightly the point at which the beam enters the glass?

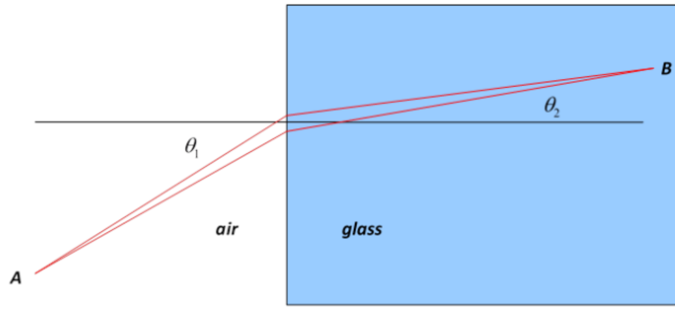

Figure 3.3.1

Where the air meets the glass, the two rays, separated by a small distance ***CD*** = d along that interface, will look parallel:

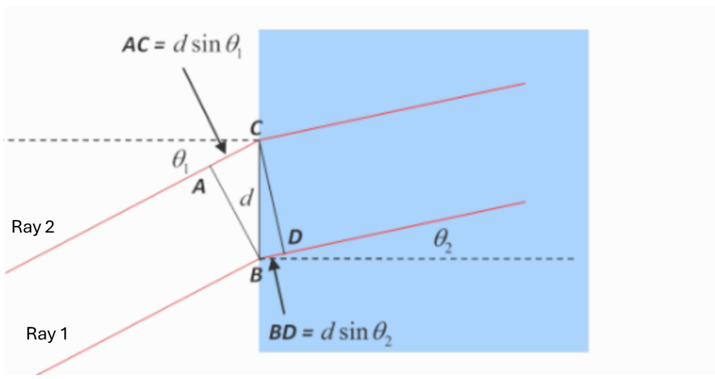

Figure 3.3.2: Magnified vies of two rays passing through interface: ray 1 is the minimum time path. Rays encounter the interface a distance CB=d apart

(Feynman gives a nice illustration: a lifeguard on a beach spots a swimmer in trouble some distance away, in a diagonal direction. He can run three times faster than he can swim. What is the quickest path to the swimmer?)

Moving the point of entry up a small distance d, the light has to travel an extra $d\sin\theta_1$ in air, but a distance less by $d\sin\theta_2$ in the glass, giving an extra travel time $\Delta t = d\sin\theta_1/c - d\sin\theta_2/v$. For the *classical* path, Snell's Law gives $\sin\theta_1/\sin\theta_2 = n = c/v$, so $\Delta t = 0$ to first order. But if we look at a series of possible paths, each a small distance d away from the next at the point of crossing from air into glass, Δt becomes of order d/c away from the classical path.

But now let's take a closer look at the Huygens picture of light propagation: it would suggest that the light reaching a point actually comes from many wavelets generated at different points on the previous wavefront. A handwaving generalization might be that the light reaching a point from another point actually includes multiple paths. To keep things manageable, let's suppose the light from A to B actually goes along all the paths that are straight in each medium, but different crossing point. Also, we'll make the approximation that they all reach B with equal amplitude. What will be the total contribution of all the paths at B? Since the times along the paths are different, the signals along the different paths will arrive at B with different phases, and to get the total wave amplitude we must add a series of unit 2D vectors, one from each path. (Representing the amplitude and phase of the wave by a complex number for convenience -- for a real wave, we can take the real part at the end.)

When we map out these unit 2D vectors, we find that in the neighborhood of the classical path, the phase varies little, but as we go away from it the phase spirals more and more rapidly, so those paths interfere amongst themselves destructively. To formulate this a little more precisely, let us assume that some close by path has a phase difference φ from the least time path, and goes from air to glass a distance x away from the least time path: then for these close by paths, $\varphi = ax^2$, where a depends on the geometric arrangement and the wavelength. From this, the sum over the close by paths is an integral of the form $\int e^{iax^2}dx$ (We are assuming the wavelength of light is far less than the size of the equipment.) This is a standard integral, its value is $\sqrt{\pi / i a}$ all its weight is concentrated in a central area of width $1 / \sqrt{a},$ exactly as for the real function $e^{-a x^{2}}$.

This is the explanation of Fermat's Principle -- only near the path of least time do paths stay approximately in phase with each other and add constructively. So this classical path rule has an underlying wave-phase explanation. In fact, the central role of phase in this analysis is sometimes emphasized by saying the light beam follows *the path of stationary phase*.

Of course, we're not summing over *all* paths here -- we assume that the path in air from the source to the point of entry into the glass is a straight line, clearly the subpath of stationary phase.
