24.6: Definition of a Tensor

We have a definite rule for how vector components transform under a change of basis: $x_i' = R_{ij} x_j$. What about the components of the inertia tensor $I_{ik} = \sum_n m_n \left(x_{nl}^2 \delta_{ik} - x_{ni} x_{nk} \right)$?

We'll do it in two parts, and one particle at a time. First, take that second term for one particle, it has the form $-m x_i x_k$. But we already know how vector components transform, so this must go to

$$-m x_i' x_k' = R_{il} R_{jm} \left(-m x_l x_m \right) \tag{24.6.1}$$

The same rotation matrix R_{ij} is applied to all the particles, so we can add over n.

In fact, the inertia tensor is made up of elements exactly of this form in all nine places, plus diagonal terms $m r_i^2$, obviously invariant under rotation. To make this clear, we write the inertia tensor:

$$\begin{bmatrix} \sum m \left(y^2 + z^2 \right) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m \left(z^2 + x^2 \right) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m \left(x^2 + y^2 \right) \end{bmatrix} = \sum m \left(x^2 + y^2 + z^2 \right) \mathbf{1} - \begin{bmatrix} \sum mx^2 & \sum mxy & \sum mxz \\ \sum mxy & \sum my^2 & \sum myz \\ \sum mxz & \sum myz & \sum mz^2 \end{bmatrix} \tag{24.6.2}$$

where 1 is the 3×3 identity matrix. (*Not* to be confused with I!)

Exercise: convince yourself that this is the same as $\mathbf{I} = \sum m \left[\left(\mathbf{x}^{\mathbf{T}} \mathbf{x} \right) \mathbf{1} - \mathbf{x} \mathbf{x}^{\mathbf{T}} \right]$

This transformation property is the definition of a two-suffix Cartesian three-dimensional tensor: just as a vector in this space can be defined as an array of three components that are transformed under a change of basis by applying the rotation matrix, $x_i' = R_{ij} x_j$, a tensor with two suffixes in the same space is a two-dimensional array of nine numbers that transform as

$T_{ij}' = R_{il} R_{jm} T_{lm}$

Writing this in matrix notation, and keeping an eye on the indices, we see that with the standard definition of a matrix product, $(\mathbf{AB})_{ij} = \mathbf{A}_{ik} \mathbf{B}_{kj}$

$$\mathbf{T}' = \mathbf{R} \mathbf{T} \mathbf{R}^{\mathbf{T}} = \mathbf{R} \mathbf{T} \mathbf{R}^{-1} \tag{24.6.3}$$

(The transformation property for our tensor followed immediately from that for a vector, since our tensor is constructed from vectors, but by definition the same rule applies to *all* Cartesian tensors, which are not always expressible in terms of vector components.)