24.2: Rotation of a Body about a Fixed Axis

As a preliminary, let's look at a body firmly attached to a rod fixed in space, and rotating with angular velocity Ω radians/sec. about that axis. You'll recall from freshman physics that the angular momentum and rotational energy are $L_z = I\Omega, \quad E_{\text{rot}} = \frac{1}{2} I\Omega^2$ where

$$I = \sum_i m_i r_{\perp i}^2 = \int dxdydz\rho(x,y,z) r_\perp^2 \tag{24.2.1}$$

(here $r_\perp = \sqrt{x^2+y^2}$ is the distance from the axis).

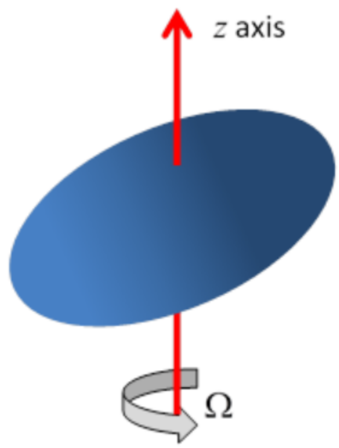

Figure 24.2.1

But you also know that both angular velocity and angular momentum are vectors. Obviously, for this example, the angular velocity is a vector pointing along the axis of rotation, $\vec{\Omega} = (0, 0, \Omega_z)$. One might be tempted to conclude that the angular momentum also points along the axis, but *this is not always the case*. An instructive example is provided by two masses m at the ends of a rod of length 2α held at a fixed angle θ to the z axis, which is the axis of rotation.

Evidently,

$$L_z = 2ma^2 \sin^2\theta \cdot \Omega \tag{24.2.2}$$

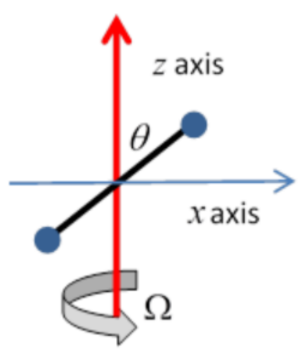

Figure 24.2.1: Masses are momentarily in (x,z) plane

But notice that, assuming the rod is momentarily in the xz plane, as shown, then

$$L_x = -2ma^2 \cos^2\theta \cdot \Omega \tag{24.2.3}$$

The total angular *momentum* is not parallel to the total angular *velocity*!

In fact, as should be evident, the total angular momentum is rotating around the constant angular velocity vector, so the axis must be providing a torque. This is why unbalanced car wheels stress the axle.