2.11: Lagrange Multiplier for the Chain

The catenary is generated by minimizing the potential energy of the hanging chain given above,

$$J[y(x)] = \int y ds = \int y\left(1 + y'^2\right)^{\frac{1}{2}} dx \tag{2.11.1}$$

but now subject to the constraint of fixed chain length, $L[y(x)] = \int ds = \ell$

The Lagrange multiplier method generalizes in a straightforward way from variables to variable functions. In the curve example above, we minimized $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = 0$. What we need to do now is minimize $J[y(x)]$ subject to the constraint $L[y(x)] - \ell = 0$

For the minimum curve $y(x)$ and the correct (so far unknown) value of λ an arbitrary infinitesimal variation of the curve will give zero first-order change in $J - \lambda L$, we write this as

$$\delta\{J[y(x)] - \lambda L[y(x)]\} = \delta\left\{\int_{x_1}^{x_2} (y - \lambda) ds\right\} = \delta\left\{\int_{x_1}^{x_2} (y - \lambda)\sqrt{1 + y'^2}\, dx\right\} = 0 \tag{2.11.2}$$

Remarkably, the effect of the constraint is to give a simple adjustable parameter, the origin in the y direction, so that we can satisfy the endpoint and length requirements.

The solution to the equation follows exactly the route followed for the soap film, leading to the first integral

$$\frac{y - \lambda}{\left(1 + y'^2\right)^{\frac{1}{2}}} = a \tag{2.11.3}$$

with a a constant of integration, which will depend on the endpoints.

Rearranging,

$$\frac{dy}{dx} = \sqrt{\left(\frac{y - \lambda}{a}\right)^2 - 1} \tag{2.11.4}$$

or

$$dx = \frac{a dy}{\sqrt{(y - \lambda)^2 - a^2}} \tag{2.11.5}$$

The standard substitution here is $y - \lambda = c\cosh\xi$, we find

$$y = \lambda + a\cosh\left(\frac{x - b}{a}\right) \tag{2.11.6}$$

Here b is the second constant of integration, the fixed endpoints and length give λ, a, b. In general, the equations must be solved numerically. To get some feel for why this will always work, note that changing a varies how rapidly the cosh curve climbs from its low point of $(x, y) = (b, \lambda + a)$, increasing a "fattens" the curve, then by varying b, λ we can move that lowest point to the lowest point of the chain (or rather of the catenary, since it might be outside the range covered by the physical chain).

Algebraically, we know the curve can be written as $y = a\cosh(x/a)$, although at this stage we don't know the constant a or where the origin is. What we do know is the length of the chain, and the horizontal and vertical distances $(x_2 - x_1)$ and $(y_2 - y_1)$ between the fixed endpoints. It's straightforward to calculate that the length of the chain is $\ell = a\sinh(x_2/a) - a\sinh(x_1/a)$, and the vertical distance v between the endpoints is $v = a\cosh(x_2/a) - a\cosh(x_1/a)$ from which $\ell^2 - v^2 = 4a^2\sinh^2[(x_2 - x_1)/2a]$. All terms in this equation are known except a, which can therefore be found numerically. (This is in Wikipedia, among other places.)

Exercise: try applying this reasoning to finding a for the soap film minimization problem. In that case, we know (x_1, y_1) and (x_2, y_2) , there is no length conservation requirement, to find a we must eliminate the unknown b from the equations $y_1 = a\cosh((x_1 - b)/a), y_2 = a\cosh((x_2 - b)/a)$. This is not difficult, but, in contrast to the chain, does *not* give a in terms of

$y_1 - y_2$, instead y_1, y_2 appear separately. Explain, in terms of the physics of the two systems, why this is so different from the chain.