17.5: Three Coupled Pendulums

Let's now move on to the case of three equal mass coupled pendulums, the middle one connected to the other two, but they're not connected to each other.

The Lagrangian is

$$L=\frac{1}{2} m \ell^2 \dot{\theta}_1^2+\frac{1}{2} m \ell^2 \dot{\theta}_2^2+\frac{1}{2} m \ell^2 \dot{\theta}_3^2-\frac{1}{2} m g \ell \theta_1^2-\frac{1}{2} m g \ell \theta_2^2-\frac{1}{2} m g \ell \theta_3^2-\frac{1}{2} C\left(\theta_1-\theta_2\right)^2-\frac{1}{2} C\left(\theta_3-\theta_2\right)^2 \tag{17.5.1}$$

Putting $\omega_0^2=g / \ell, \quad k=C / m \ell^2$

$L=\frac{1}{2} \dot{\theta}_1^2+\frac{1}{2} \dot{\theta}_2^2+\frac{1}{2} \dot{\theta}_3^2-\frac{1}{2} \omega_0^2 \theta_1^2-\frac{1}{2} \omega_0^2 \theta_2^2-\frac{1}{2} \omega_0^2 \theta_3^2-\frac{1}{2} k(\theta_1-\theta_2)^2-\frac{1}{2} k(\theta_3-\theta_2)^2$

The equations of motion are

$$\begin{aligned}
& \ddot{\theta}_1=-\omega_0^2 \theta_1-k\left(\theta_1-\theta_2\right) \\
& \ddot{\theta}_2=-\omega_0^2 \theta_2-k\left(\theta_2-\theta_1\right)-k\left(\theta_2-\theta_3\right) \\
& \ddot{\theta}_3=-\omega_0^2 \theta_3-k\left(\theta_3-\theta_2\right)
\end{aligned} \tag{17.5.2}$$

Putting $\theta_i(t)=A_i e^{i \omega t}$, the equations can be written in matrix form

$$\left(\begin{array}{ccc}
\omega_0^2+k & -k & 0 \\
-k & \omega_0^2+2 k & -k \\
0 & -k & \omega_0^2+k
\end{array}\right)=\omega_0^2\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)+k\left(\begin{array}{ccc}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{array}\right) \tag{17.5.3}$$

The normal modes of oscillation are given by the eigenstates of that second matrix.

The one obvious normal mode is all the pendulums swinging together, at the original frequency ω_0, so the springs stay at the rest length and play no role. For this mode, evidently the second matrix has a zero eigenvalue, and eigenvector (1,1,1).

The full eigenvalue equation is

$$\left|\begin{array}{ccc}
1-\lambda & -1 & 0 \\
-1 & 2-\lambda & -1 \\
0 & -1 & 1-\lambda
\end{array}\right|=0 \tag{17.5.4}$$

that is,

$$(1-\lambda)^2(2-\lambda)-2(1-\lambda)=0=(1-\lambda)[(1-\lambda)(2-\lambda)-2]=(1-\lambda)\left(\lambda^2-3 \lambda\right) \tag{17.5.5}$$

so the eigenvalues are $\lambda_1=0, \lambda_2=1, \lambda_3=3$, with frequencies

$$\omega_1^2=\omega_0^2, \omega_2^2=\omega_0^2+k, \omega_3^2=\omega_0^2+3 k \tag{17.5.6}$$

The normal mode eigenvectors satisfy

$$\left(\begin{array}{ccc}
1-\lambda & -1 & 0 \\
-1 & 2-\lambda & -1 \\
0 & -1 & 1-\lambda
\end{array}\right)\left(\begin{array}{l}
A_1 \\
A_2 \\
A_3
\end{array}\right)=0 \tag{17.5.7}$$

They are $(1,1,1) / \sqrt{3}, \quad(1,0,-1) / \sqrt{2}, \quad(1,-2,1) / \sqrt{6}$normalizing them to unity.

The equations of motion are linear, so the general solution is a superposition of the normal modes:

$$\left(\begin{array}{l}
\theta_1 \\
\theta_2 \\
\theta_3
\end{array}\right)=\frac{1}{\sqrt{3}}\left(\begin{array}{l}
1 \\
1 \\
1
\end{array}\right) \operatorname{Re}\left(C_1 e^{i \omega_1 t}\right)+\frac{1}{\sqrt{2}}\left(\begin{array}{c}
1 \\
0 \\
-1
\end{array}\right) \operatorname{Re}\left(C_2 e^{i \omega_2 t}\right)+\frac{1}{\sqrt{6}}\left(\begin{array}{c}
1 \\
-2 \\
1
\end{array}\right) \operatorname{Re}\left(C_3 e^{i \omega_3 t}\right) \tag{17.5.8}$$