2.5: Fastest Curve for Given Horizontal Distance

Suppose we want to find the curve a bead slides down to minimize the time from the origin to some specified horizontal displacement X, but we don't care what vertical drop that entails.

Recall how we derived the equation for the curve:

At the minimum, under any infinitesimal variation $\delta y(x)$.

$$\delta J[y]=\int_{x_1}^{x_2}\left[\frac{\partial f\left(y, y^{\prime}\right)}{\partial y} \delta y(x)+\frac{\partial f\left(y, y^{\prime}\right)}{\partial y^{\prime}} \delta y^{\prime}(x)\right] d x=0 \tag{2.5.1}$$

Writing $\delta y^{\prime}=\delta(d y / d x)=(d / d x) \delta y$, and integrating the second term by parts,

$$\delta J[y]=\int_{x_1}^{x_2}\left[\frac{\partial f\left(y, y^{\prime}\right)}{\partial y}-\frac{d}{d x}\left(\frac{\partial f\left(y, y^{\prime}\right)}{\partial y^{\prime}}\right)\right] \delta y(x) d x+\left[\frac{\partial f\left(y, y^{\prime}\right)}{\partial y^{\prime}} \delta y(x)\right]_0^X=0 \tag{2.5.2}$$

In the earlier treatment, both endpoints were fixed, $\delta y(0)=\delta y(X)=0$ so we dropped that final term.

However, we are now trying to find the fastest time for a given horizontal distance, so the final vertical distance is an adjustable parameter: $\delta y(X) \neq 0$

As before, since $\delta J[y]=0$ or arbitrary $\delta y,$ we can still choose a $\delta y(x)$ which is only nonzero near some point not at the end, so we must still have

$$\frac{\partial f\left(y, y^{\prime}\right)}{\partial y}-\frac{d}{d x}\left(\frac{\partial f\left(y, y^{\prime}\right)}{\partial y^{\prime}}\right)=0 \tag{2.5.3}$$

However, we must also have $\dfrac{\partial f\left(y(X), y^{\prime}(X)\right)}{\partial y^{\prime}} \delta y(X)=0$, to first order for arbitrary infinitesimal $\delta y(X)$, (imagine a variation δy only nonzero near the endpoint), this can only be true if $\dfrac{\partial f\left(y, y^{\prime}\right)}{\partial y^{\prime}}=0 \text{ at } x=X$

For the brachistochrone,

$$f=\sqrt{\frac{1+y^{\prime 2}}{2 g y}}, \quad \frac{\partial f}{\partial y^{\prime}}=\frac{y^{\prime}}{\sqrt{2 g y\left(1+y^{\prime 2}\right)}} \tag{2.5.4}$$

so $\dfrac{\partial f\left(y, y^{\prime}\right)}{\partial y^{\prime}}=0 \text{ at } x=X \text{ means that } f^{\prime}=0$, *the curve is horizontal at the end* $x=X$

So the curve that delivers the bead a given horizontal distance the fastest is the half-cycloid (inverted) flat at the end. It's easy to see this fixes the curve uniquely: think of the curve as generated by a rolling wheel, one half-turn of the wheel takes the top point to the bottom in distance X

Exercise: how low does it go?