20.2: Resonance near Double the Natural Frequency

From the above argument, the place to look for resonance is close to $\Omega=2\omega_0$. Landau takes

$\ddot{x}+\omega_0^2\left[1+h\cos(2\omega_0+\varepsilon)t\right]x=0$

and, bearing in mind that we're looking for oscillations close to the natural frequency, puts in

$$x=a(t)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t \tag{20.2.1}$$

with $a(t),b(t)$ slowly varying.

It's important to realize that this is an *approximate* approach. It neglects nonresonant frequencies which must be present in small amounts, for example

$$\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t\cos(2\omega_0+\varepsilon)t=\frac{1}{2}\cos 3\left(\omega_0+\frac{1}{2}\varepsilon\right)t+\frac{1}{2}\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t \tag{20.2.2}$$

and the $3\left(\omega_0+\frac{1}{2}\varepsilon\right)$ term is thrown away.

And, since the assumption is that $a(t),\ b(t)$ are slowly varying, their second derivatives are dropped too, leaving just

$$\begin{aligned}\ddot{x}=&-2\dot{a}(t)\omega_0\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t-a(t)\left(\omega_0^2+\omega_0\varepsilon\right)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t\\&+2\dot{b}(t)\omega_0\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t-b(t)\left(\omega_0^2+\omega_0\varepsilon\right)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\end{aligned} \tag{20.2.3}$$

This must equal

$$-\omega_0^2\left[1+h\cos(2\omega_0+\varepsilon)t\right]\left[a(t)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\right] \tag{20.2.4}$$

Keeping only the resonant terms, we take $\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t\cdot\cos(2\omega_0+\varepsilon)t=\frac{1}{2}\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t$ and $\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\cdot\cos(2\omega_0+\varepsilon)t=-\frac{1}{2}\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t$

so this expression becomes

$$\begin{aligned}&-\omega_0^2\left[1+h\cos(2\omega_0+\varepsilon)t\right]\left[a(t)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\right]\\&=-\omega_0^2\left[a(t)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t+\frac{1}{2}ha(t)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t-\frac{1}{2}hb(t)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\right]\end{aligned} \tag{20.2.5}$$

The equation becomes:

$$\begin{aligned}\ddot{x}=&\ -2\dot{a}(t)\omega_0\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t-a(t)\left(\omega_0^2+\omega_0\varepsilon\right)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t\\&+2\dot{b}(t)\omega_0\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t-b(t)\left(\omega_0^2+\omega_0\varepsilon\right)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\\=&\ =\omega_0^2\left[a(t)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t+b(t)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t+\frac{1}{2}ha(t)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t-\frac{1}{2}hb(t)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\right]\end{aligned} \tag{20.2.6}$$

The zeroth-order terms cancel between the two sides, leaving

$$\begin{aligned}&-2\dot{a}(t)\omega_0\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t-a(t)\omega_0\varepsilon\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t+2\dot{b}(t)\omega_0\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t-b(t)\omega_0\varepsilon\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\\&=-\omega_0^2\left[\frac{1}{2}ha(t)\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t-\frac{1}{2}hb(t)\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t\right]\end{aligned} \tag{20.2.7}$$

Collecting the terms in $\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t,\quad\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t$

$$-\left(2\dot{a}+b\varepsilon+\frac{1}{2}h\omega_0 b\right)\omega_0\sin\left(\omega_0+\frac{1}{2}\varepsilon\right)t+\left(2\dot{b}(t)-a\varepsilon+\frac{1}{2}h\omega_0 a\right)\omega_0\cos\left(\omega_0+\frac{1}{2}\varepsilon\right)t=0 \tag{20.2.8}$$

The sine and cosine can't cancel each other, so the two coefficients must both be identically zero. This gives two first order differential equations for the functions $a(t),\ b(t)$, and we look for exponentially increasing functions, proportional to $a(t)=ae^{st},\quad b(t)=be^{st}$, which will be solutions provided

$$\begin{aligned} sa+\frac{1}{2}\left(\varepsilon+\frac{1}{2}h\omega_0\right)b=0 \\ \frac{1}{2}\left(\varepsilon-\frac{1}{2}h\omega_0\right)a-sb=0 \end{aligned} \tag{20.2.9}$$

The amplitude growth rate is therefore

$$s^2=\frac{1}{4}\left[\left(\frac{1}{2}h\omega_0\right)^2-\varepsilon^2\right] \tag{20.2.10}$$

Parametric resonance will take place if s is real, that is, if

$$-\frac{1}{2}h\omega_0<\varepsilon<\frac{1}{2}h\omega_0 \tag{20.2.11}$$

a band of width $h\omega_0$ about $2\omega_0$