23.3: Lyapunov Exponents and Dimensions of Strange Attractors

Scaling the Attractor

Looking at the strange attractor pictured in the previous section, we found that on magnifying a small part of it we saw the same kind of structure the attractor has as a whole: if we look at a $\gamma = 1.5$ attractor, with damping 0.75, there are long thin stretches, they end by looping over. We notice that halving the damping to 0.375 fattens the previously quasi-one-dimensional stretches and reveals complicated looping *at several levels*. (Remember that what we are looking at here is a Poincaré section of the attractor, the other dimension is periodic time (or driver phase, the same thing) so a curve here is a section of some sheet). If more computing power is used, going to smaller and smaller scales, it turns out that the magnified tiny part of the attractor looks much the same as the attractor. This kind of scale invariance is a characteristic of a *fractal*. A mysterious aspect of fractals is their dimensionality. Look at the strange attractor. There are no places where it solidly fills a stretch of two-dimensional space, this is clearer on going to greater and greater magnification: we see more and more one-dimensional structures, with no end, so it surely has dimension less than two, but greater than one—how do we make sense of that?

Fractals: the Cantor Set

To try to find a generalized concept of dimension of a set (i.e. not just an integer), we begin with perhaps the simplest example of a fractal, the *Cantor set*: take the numbers between 0 and 1, and cut out the middle third. You now have two strips of numbers, from 0 to 1/3, and from 2/3 to 1. For each of *those* strips, cut out the middle third. You now have four strips—cut out the middle third of each of them (it might help to draw this). Do this forever. What's left is the Cantor set. You can see this is scale invariant: after doing this many times, take one of the remaining tiny strips, what happens to it on continuing the process is identical (scaled down suitably) to what happened to the initial strip.

How big is this Cantor set? At each step, we cut the total length of line included by 2/3. Since $(2/3)^n$ goes to zero as n goes to infinity, it clearly has size zero, right? But clearly there's more to the Cantor set than there is to a single point, or for that to matter a finite number of points. What about a countably infinite number of points—for example, the *rational* numbers between 0 and 1? Well, you can write them out in a list, ordered by increasing denominators, and for one denominator by increasing numerators. Then you can put them one by one into tiny intervals, 1/2 goes into an interval of length ε, 1/3 in an interval $\varepsilon/2$, 2/3 in one of length $\varepsilon/2^2$, 1/4 in $\varepsilon/2^3$, and so on, the total length of the infinite number of intervals being 2ε, so all the rationals can be covered by an arbitrarily small set of intervals. Can we count in order the numbers in the Cantor set in the same way? The answer is no, and to see why think first about *all* the numbers between 0 and 1, rationals and irrationals. If you make an infinite list of them, I can show you missed some out: I just take your list and write down a decimal that differs from your n^{th} number in the n^{th} place. So we can't put all the numbers in the interval in little boxes that add to zero, which is obvious anyway!

But now to the Cantor set: suppose we write all numbers between 0 and 1 using base 3, instead of the traditional base 10. That is, each number is a string of 0's, 1's and 2's. Then the Cantor set is all those numbers that don't have any 1's, such as 0.2, 0.02202, etc. (Check this yourself.) But the number of *these* numbers is exactly the same as *all* the numbers between 0 and 1 in *binary* notation! So surely the Cantor set has dimension 1? (These infinities are tricky.)

The bottom line from the above argument is that we can plausibly argue both that the Cantor set has dimension 0, and that it has dimension 1. To understand and categorize fractals better, we need a working definition of *dimension* for fractals. One approach is the *capacity* dimension.

Dimensions: Capacity and Correlation

Suppose we cover the interval 0,1 with a set of small boxes, length ε, there are clearly $N(\varepsilon) = 1/\varepsilon$ such boxes (assume it's an integer). Now consider a *subset* of the numbers between 0 and 1, choose ε, and find how many boxes $N(\varepsilon)$ are necessary to cover this subset. The *capacity dimension* is defined as

$$d_C = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}. \tag{23.3.1}$$

For simplicity, we choose $1/\varepsilon = 3, 9, 27, \ldots$ so the necessary numbers of boxes to cover the Cantor set described in the previous section are $2, 4, 8\ldots$ out of total numbers of boxes $3, 9, 27, \ldots$. Therefore

$$d_C(\text{Cantor}) = \frac{n \log 2}{n \log 3} = \frac{\log 2}{\log 3}, \tag{23.3.2}$$

of course between 0 and 1. (There are many ways to define dimensionality of sets of numbers—this definition gives zero for a finite set of points, and one for all the numbers between 0 and 1, *but* also 1 for the set of rationals, which we've shown above can be covered by an infinite set of intervals of arbitrarily small total length.)

Another measure used is the *correlation dimension*, in which for a large number of points (such as our representation of the attractor) a correlation integral $C(r)$ is defined as the total number of pairs of two points less than r apart. For small r, this goes as a power r^{ν}, and it turns out that in many cases ν is close to the capacity definition of the fractal dimension. (Grassberger and Procaccia.)

Time Development of Systems in Phase Space

Recall first that the state space or phase space we have been plotting is really a projection of the full orbit space into two dimensions, the third dimension necessary to predict future motion being the phase of the sine-wave driving force, so this is just time (although of course cyclic).

Suppose now we populate this three-dimensional space with many points, like a gas, each representing a driven damped pendulum. As time goes on the gas will flow, each gas atom's path completely determined, and no two will ever be at the same point in this full space (except perhaps asymptotically at infinite time).

Take now a small volume, say a cube with sides parallel to the axes, containing many points. Consider first the *undamped* system: then Liouville's Theorem (link to my lecture) tells us that as time goes on the cube will generally distort, but it will *not* change in volume. In other words, the gas of systems flows like an incompressible fluid. (Details of the derivation are given in the linked lecture—briefly, the motions of the sides in time Δt come from the equations of motion, etc.)

However, if the system has *damping*—as ours does—the same analysis leads to the conclusion that the volume the systems occupy in phase space (remember, this is now three-dimensional) shrinks at a rate determined by the damping. As a trivial example, think of undriven damped pendula—they will all tend to the low point rest position. Lightly *driven* pendula will go to a one-dimensional cycle.

We can prove this shrinkage from the equation of motion:

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 \sin\phi = \gamma\omega_0^2 \cos\omega t. \tag{23.3.3}$$

In the three-dimensional phase space, a pendulum's position can be written in coordinates $(\phi, \dot{\phi}, \psi)$ where $\psi = \omega t$, the driving phase, between 0 and 2π.

The local phase space velocity $\vec{F}$ can be written in terms of the coordinates (this is just the above equation rewritten!):

$$\frac{\partial\phi}{\partial t} = \dot{\phi}, \tag{23.3.4}$$

$$\frac{\partial\dot{\phi}}{\partial t} = -2\beta\dot{\phi} - \omega_0^2 \sin\phi + \gamma\omega_0^2 \cos\psi, \tag{23.3.5}$$

$$\frac{\partial\psi}{\partial t} = \omega, \tag{23.3.6}$$

This is therefore the local velocity of the atoms of the gas (meaning the systems), and it is trivial to check that

$$\vec{\nabla} \cdot \vec{F} = -2\beta. \tag{23.3.7}$$

This means that if we have a small sphere containing many points corresponding to systems (a "gas" of systems) then the volume of the (now distorting) sphere enclosing those points is *decreasing* in volume at an exponential rate $V(t) = V_0 e^{-2\beta t}$.

Relating this Picture to Lyapunov Exponents

Continuing to think about the development of a small sphere (containing many points corresponding to systems) in phase space, it will be moving along an orbit, but at the same time distorting, let's say to an ellipsoid as an initial first approximation, and tumbling around. In the chaotic regime, we know it must be growing in *some* direction, at least on average (the rates will vary along the orbit) because we know that points initially close together separate on average at an exponential rate given by the first

Lyapunov exponent, $\propto e^{\lambda_1 t}$. We'll make the simplifying assumption that the ellipsoid has its axes initially varying in time as $\left(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\right)$, with $\lambda_1 > \lambda_2 > \lambda_3$. From the result above $\vec{\nabla} \cdot \vec{F} = -2\beta$, we conclude that

$$\lambda_1 + \lambda_2 + \lambda_3 = -2\beta. \tag{23.3.8}$$

We need to say something more about λ_1. We're taking it as defined by the growth rate of distance between trajectories *after* any initial transients but *before* the distance is comparable to the size of the system (finding this interval plausibly has been termed a "dark art"). We envision our initially small sphere of gas elongating and tumbling around as it moves along. Hopefully its rate of elongation correlates well with what we actually measure, that is, the rate of growth of net ϕ displacement, the coordinate separation of two initially close orbits, which we plot and approximately fit with an exponential, $\propto e^{\lambda_1 t}$.

For the pendulum, the ψ direction is just time, not scaled, so $\lambda_3 = \lambda_\psi = 0$. Then necessarily $\lambda_2 < 0$ to satisfy the damping equation.

So taking a local (in phase space) collection of systems, those inside a given closed surface, like a little sphere, and following their evolution in time in the chaotic regime, the sphere will *expand* in one direction; a direction, however, that varies with time, but *contract* or stay constant in the other directions. As the surface grows this gets more complicated because it's confined to a finite total phase space. And it continues to expand at the same rate as time goes on, so the continual increase in surface must imply tighter and tighter foldings to stay in the phase space. And this is what the strange attractor looks like.

A Fractal Conjecture

In 1979, Kaplan and Yorke conjectured that the dimensionality of the strange attractor followed from the Lyapunov exponents taking part in its creation. In our case—the driven damped pendulum—there are only two relevant exponents, $\lambda_1 > 0, \quad \lambda_2 < 0$ and $\lambda_1 + \lambda_2 = -2\beta$.

A plausibility argument is given in Baker and Gollub's book, *Chaotic Dynamics.* They define a Lyapunov dimension dL of the attractor by

$$dL = \lim_{\varepsilon \to 0}\left[\frac{d(\log N(\varepsilon))}{d(\log(1/\varepsilon))}\right], \tag{23.3.9}$$

exactly analogous to the definition of capacity dimension in the previous section.

Now, as time passes a small square element will have its area multiplied by a factor $e^{(\lambda_1+\lambda_2)t}$. (No scaling takes place in the third (time) direction.) At the same time, they argue that the length unit ε changes as $e^{\lambda_2 t}$. Then $N(\varepsilon)$ is the area $\varepsilon_0^2 e^{(\lambda_1+\lambda_2)t}$ divided by the shrinking basic area $\varepsilon_0^2 e^{2\lambda_2 t}$. The differential of $\log N(\varepsilon)$ is $\lambda_1 - \lambda_2$, that of $\log(1/\varepsilon)$ is $-\lambda_2$, so their argument gives

$$dL = 1 - \frac{\lambda_1}{\lambda_2}. \tag{23.3.10}$$

The Lyapunov applet is designed to measure λ_1 by tracking separation of initially close trajectories. Try it a few times: it becomes clear that there is considerable uncertainty in this approach. Given λ_1, λ_2 follows from $\lambda_1 + \lambda_2 = -2\beta$. There are various ways to assign a dimension to the attractor, such as the capacity and correlation dimensions mentioned above. Various attempts to verify this relationship have been made, but the uncertainties are considerable, and although results seem to be in the right ballpark, the results are off by ten or twenty percent typically. It seems there is work still to be done on this fascinating problem.

Recommended reading: chapter 5 of Baker and Gollub, *Chaotic Dynamics*. The brief discussion above is based on their presentation.