19.4: Finding the Eigenvectors

Now let's look at the eigenvectors, we'll start with those of P.Let's call the eigenvalue λ

Then for an eigenstate of the shift operator, the shifted vector must be just a multiple of the original vector:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = \begin{pmatrix} A_2 \\ A_3 \\ A_4 \\ A_1 \end{pmatrix} = \lambda \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \tag{19.4.1}$$

Reading off the element by element equivalence of the two vectors,

$$A_2 = \lambda A_1, \quad A_3 = \lambda A_2, \quad A_4 = \lambda A_3, \quad A_1 = \lambda A_4 \tag{19.4.2}$$

The first three equalities tell us the eigenvector has the form $(1, \lambda, \lambda^2, \lambda^3)^T$, the last tells us that $\lambda^4 = 1$.

From our earlier discussion of circulant matrices, writing the smallest phase nontrivial N^{th} root of unity as $\omega = e^{2\pi i/N}$, *the roots of the equation $\lambda^N = 1$ are just this basic root raised to N different powers*: the roots are $1, \omega, \omega^2, \omega^3, \ldots, \omega^{N-1}$

This establishes that the eigenvectors of P have the form

$$\left(1, \omega^j, \left(\omega^j\right)^2, \left(\omega^j\right)^3, \ldots, \left(\omega^j\right)^{N-1}\right)^T \tag{19.4.3}$$

where $j = 0, 1, 2, 3, \ldots, N-1$ with corresponding eigenvalue the basic root raised to the j^{th} power, $\omega^j = e^{2\pi ij/N}$. Try it out for 3×3: the eigenvalues are given by

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0, \quad \lambda^3 = 1, \quad \lambda = 1, \omega, \omega^2; \quad \omega^3 = 1 \tag{19.4.4}$$

The corresponding eigenvectors are found to be

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \tag{19.4.5}$$

For the $N \times N$ case, there are N different, linearly independent, vectors of this form, so this is a complete set of eigenvectors of P.

They are also, of course, eigenvectors of P^2, P^3 all $N-1$ powers of P and therefore of all the circulant matrices! This means that all $N \times N$ circulant matrices commute.