17.3: Normal Modes

The physical motion corresponding to the amplitudes eigenvector (1,1) has two constants of integration (amplitude and phase), often written in terms of a single complex number, that is,

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \operatorname{Re} Be^{i\omega_0 t} = \begin{pmatrix} A\cos(\omega_0 t+\delta) \\ A\cos(\omega_0 t+\delta) \end{pmatrix}, \quad B = Ae^{i\delta} \tag{17.3.1}$$

with A, δ real.

Clearly, this is the mode in which the two pendulums are in sync, oscillating at their natural frequency, with the spring playing no role.

In physics, this mathematical eigenstate of the matrix is called a *normal mode* of oscillation. In a normal mode, all parts of the system oscillate at a single frequency, given by the eigenvalue.

The other normal mode,

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \operatorname{Re} Be^{i\omega' t} = \begin{pmatrix} A\cos(\omega' t+\delta) \\ -A\cos(\omega' t+\delta) \end{pmatrix}, \quad B = Ae^{i\delta} \tag{17.3.2}$$

where we have written $\omega' = \sqrt{\omega_0^2 + 2k}$. Here the system is oscillating with the single frequency ω', the pendulums are now exactly *out* of phase, so when they part the spring pulls them back to the center, thereby increasing the system oscillation frequency.

The matrix structure can be clarified by separating out the spring contribution:

$$\mathbf{M} = \begin{pmatrix} \omega_0^2+k & -k \\ -k & \omega_0^2+k \end{pmatrix} = \omega_0^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{17.3.3}$$

All vectors are eigenvectors of the identity, of course, so the first matrix just contributes ω_0^2 to the eigenvalue. The second matrix is easily found to have eigenvalues are 0,2, and eigenstates (1,1) and (1,−1).