24.10: Symmetries, Other Axes, the Parallel Axis Theorem

If a body has an axis of symmetry, the center of mass must be on that axis, and it is a principal axis of inertia. To prove the center of mass statement, note that the body is made up of pairs of equal mass particles on opposite sides of the axis, each pair having its center of mass on the axis, and the body's center of mass is that of all these pairs centers of mass, all of which are on the axis.

Taking this axis to be the x axis, symmetry means that for each particle at (x, y, z) there is one of equal mass at $(x, -y, -z)$, so the off-diagonal terms in the x row and column, $-\sum mxy, \quad -\sum mxz$ all add up to zero, meaning this is indeed a principal axis.

The moment of inertia about an arbitrary axis through the center of mass, in the direction of the unit vector $\vec{\hat{b}}$ is

$$\sum m\left(\vec{r}^{2}-(\vec{r}\cdot\hat{b})^{2}\right)=\vec{\hat{b}}^{\mathbf{T}}\mathbf{I}\vec{\hat{b}}=b_{x}^{2}I_{x}+b_{y}^{2}I_{y}+b_{z}^{2}I_{z} \tag{24.10.1}$$

The inertia tensor about some origin O' located at position a relative to the center of mass is easily found to be

$$I_{ik}'=I_{ik}+M\left(\mathbf{a}^{2}\delta_{ik}-\mathbf{a}_{i}\mathbf{a}_{k}\right) \tag{24.10.2}$$

In particular, we have the *parallel axis theorem*: the moment of inertia about any axis through some point O' equals that about the parallel axis through the center of mass O plus $Ma_{\perp}^{2}$, where $a_{\perp}$ is the perpendicular distance between the axes.

Exercise: check this!