6.7: Checking that We Can Eliminate the q˙i's

We should check that we *can* in fact write

$$H\left(p_i, q_i\right)=\sum_{i=1}^n p_i \dot{q}_i-L\left(q_i, \dot{q}_i\right) \tag{6.7.1}$$

as a function of just the variables $\left(q_i, p_i\right)$, with all trace of the $\dot{q}_i$'s eliminated. Is this always possible? The answer is yes.

Recall the $\dot{q}_i$'s only appear in the Lagrangian in the kinetic energy term, which has the general form

$$T=\sum_{i, j} a_{ij}\left(q_k\right) \dot{q}_i \dot{q}_j \tag{6.7.2}$$

where the coefficients a_{ij} depend in general on some of the q_k's but are independent of the velocities, the $\dot{q}_k$'s. Therefore, from the definition of the generalized momenta,

$$p_i=\frac{\partial L}{\partial \dot{q}_i}=\sum_{j=1}^n a_{ij}\left(q_k\right) \dot{q}_j \tag{6.7.3}$$

and we can write this as a vector-matrix equation,

$$\mathbf{p}=\mathbf{A} \dot{\mathbf{q}} \tag{6.7.4}$$

That is, p_i is a linear function of the $\dot{q}_j$'s. . Hence, the inverse matrix $\mathbf{A}^{-1}$ will give us $\dot{q}_i$ as a linear function of the pj's, and then putting this expression for the $\dot{q}_i$ into the Lagrangian gives the Hamiltonian as a function only of the qi's and the pi's, that is, the phase space variables.

The matrix A is always invertible because the kinetic energy is positive definite (as is obvious from its Cartesian representation) and a symmetric positive definite matrix has only positive eigenvalues, and therefore is invertible.