13.4: *Kepler Orbit Action-Angle Variables

We have not yet covered Kepler orbits, so skip this section for now: it's here to refer back to later. It's from Landau, p 167. For motion confined to a plane, we can take the central potential analysis with *[Math Processing Error]*, the angular momentum, so the Hamiltonian is

[Math Processing Error]

The Hamilton-Jacobi equation is therefore

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So, following the previous analysis of separation of variables for motion in a central potential, here

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The action variable for the angular motion is just the angular momentum itself,

[Math Processing Error]

And the radial action variable, with potential *[Math Processing Error]*

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(Details on doing the integral are given in the Appendix, *Mathematica* can do it too.)

So the energy is

[Math Processing Error]

The motion is *degenerate*: the two fundamental frequencies coincide, *[Math Processing Error]*

This has major consequences in quantum mechanics: the actions are all quantized in units of Planck's constant, for the hydrogen atom, from the formula above, the energy depends only on the *sum* of the quantum numbers: above the ground state, energy levels are degenerate, which is why the energy spectrum has the deceptively simple form so successfully explained by the Bohr model.

The orbital parameters, semi-latus rectum and eccentricity, from *[Math Processing Error]*, are

[Math Processing Error]

Recall the semi-major axis is given by *[Math Processing Error]* and from the above expression

[Math Processing Error]

in the hydrogen atom quantum number notation.

Appendix: Doing the Integral for The Radial Action Ir

The integral can be put in the form

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which can be integrated by taking a contour encircling the cut from *[Math Processing Error]* to *[Math Processing Error]*. The integral will have a contribution from the pole at the origin equal to *[Math Processing Error]* and another from the circle at infinity, which is

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Equating coefficients (multiplying the term inside the square root by *[Math Processing Error]*)

[Math Processing Error]

So the contribution from the origin gives the *[Math Processing Error]*.