4.4: Lagrange's Equations from Hamilton's Principle Using Calculus of Variations

We started with Newton's equations of motion, expressed in Cartesian coordinates of particle positions. For many systems, these equations are mathematically intractable. Running the calculus of variations argument in reverse, we established Hamilton's principle: the system moves along the path through configuration space for which the action integral, with integrand the Lagrangian $L = T - U$, is a minimum.

We're now free to *begin* from Hamilton's principle, expressing the Lagrangian in variables that more naturally describe the system, taking advantage of any symmetries (such as using angle variables for rotationally invariant systems). Also, some forces do not need to be included in the description of the system: a simple pendulum is fully specified by its position and velocity, we do not need to know the tension in the string, although that *would* appear in a Newtonian analysis. The greater efficiency (and elegance) of the Lagrangian method, for most problems, will become evident on working through actual examples.

We'll define a set of *generalized coordinates* $q = (q_1, \dots q_n)$ by requiring that they give a complete description of the configuration of the system (where everything is in space). The state of the system is specified by this set plus the corresponding velocities $\dot{q} = (\dot{q}_1, \dots \dot{q}_n)$

For example, the x -coordinate of a particular particle a is given by some function of the q_i 's,

$x_a = f_{x_a}(q_1, \dots q_n)$, and the corresponding velocity component $\dot{x}_a = \sum_k \frac{\partial f_{x_a}}{\partial q_k} \dot{q}_k$

The Lagrangian will depend on all these variables in general, and also possibly on time explicitly, for example if there is a time-dependent external potential. (But usually that isn't the case.)

Hamilton's principle gives

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = 0 \tag{4.4.1}$$

that is,

$$\int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt = 0 \tag{4.4.2}$$

Integrating by parts,

$$\delta S = \left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i dt = 0 \tag{4.4.3}$$

Requiring the path deviation to be zero at the endpoints gives Lagrange's equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \tag{4.4.4}$$