11.7: Jacobian proof of Liouville's Theorem

After this rather long detour into Jacobian theory, recall we are trying to establish that the volume of a region in phase space is unaffected by a canonical transformation, we need to prove that

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and that means we need to show that the Jacobian

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Using the theorems above about the inverse of a Jacobian and the chain rule product,

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Now invoking the rule that if the same variables appear in both numerator and denominator, they can be cancelled,

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Up to this point, the equations are valid for any nonsingular transformation—but to prove the numerator and denominator are equal in this expression requires that the equation be canonical, that is, be given by a generating function, as explained earlier.

Recall now the properties of the generating function *[Math Processing Error]*

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from which

[Math Processing Error]

In the expression for the Jacobian *[Math Processing Error]*, the *[Math Processing Error]*,*[Math Processing Error]* element of the numerator is *[Math Processing Error]*.

In terms of the generating function *[Math Processing Error]*.

Exactly the same procedure for the denominator gives the *[Math Processing Error]*,*[Math Processing Error]* element to be *[Math Processing Error]*

In other words, the two determinants are the same (rows and columns are switched, but that doesn't affect the value of a determinant). This means D=1, and Liouville's theorem is proved.
