# 4.11: Momentum Conservation

Another conservation law follows if the Lagrangian is unchanged by displacing the whole system through a distance $\delta\vec{r} = \vec{\varepsilon}$. This means, of course, that the system cannot be in some spatially varying external field—it must be mechanically isolated.

It is natural to work in Cartesian coordinates to analyze this, each particle is moved the same distance $\vec{r}_i \rightarrow \vec{r}_i + \delta\vec{r}_i = \vec{r}_i + \vec{\varepsilon}$, so , so

$$\delta L = \sum_i \frac{\partial L}{\partial \vec{r}_i} \cdot \delta\vec{r}_i = \vec{\varepsilon} \cdot \sum_i \frac{\partial L}{\partial \vec{r}_i} \tag{4.11.1}$$

where the "differentiation by a vector" notation means differentiating with respect to each component, then adding the three terms. (I'm not crazy about this notation, but it's Landau's, so get used to it.)

For an isolated system, we must have $\delta L = 0$ on displacement, moving the whole thing through empty space in any direction $\vec{\varepsilon}$ changes nothing, so it must be that the vector sum $\sum_i \partial L/\partial \vec{r}_i = 0$ , so from the Cartesian Euler-Lagrange equations, writing $\dot{\vec{r}} = \vec{v}$

$$0 = \sum_i \partial L/\partial \vec{r}_i = \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_i}\right) = \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_i}\right) = \frac{d}{dt}\sum \frac{\partial L}{\partial \vec{v}_i} \tag{4.11.2}$$

so, taking the system to be composed of particles of mass $m_i$ and velocity $\vec{v}_i$

$$\sum_i \frac{\partial L}{\partial \vec{v}_i} = \sum_i m_i \vec{v}_i = \vec{P} = \text{constant} \tag{4.11.3}$$

the *momentum* of the system.

This vector conservation law is of course three separate directional conservation laws, so even if there *is* an external field, if it doesn't vary in a particular direction, the component of total momentum in that direction will be conserved.

In the Newtonian picture, conservation of momentum in a closed system follows from Newton's third law. In fact, the above Lagrangian analysis is really Newton's third law in disguise. Since we're working in Cartesian coordinates,

$\partial L/\partial \vec{r}_i = -\partial V/\partial \vec{r}_i = \vec{F}_i$ the force on the $i$th particle, and if there are no external fields, $\sum_i \partial L/\partial \vec{r}_i = 0$ just means that if you add all the forces on all the particles, the sum is zero. For the Lagrangian of a two particle system to be invariant under translation through space, the potential must have the form $V(\vec{r}_1 - \vec{r}_2)$ from which automatically $\vec{F}_{12} = -\vec{F}_{21}$.