14.2: The Ellipse

Squashed Circles and Gardeners

The simplest nontrivial planetary orbit is a circle: *[Math Processing Error]* is centered at the origin and has radius *[Math Processing Error]*. An ellipse is a circle scaled (squashed) in one direction, so an ellipse centered at the origin with semimajor axis *[Math Processing Error]* and semiminor axis *[Math Processing Error]* has equation

[Math Processing Error]

in the standard notation, a circle of radius *[Math Processing Error]* scaled by a factor *[Math Processing Error]* in the *[Math Processing Error]* direction. (It's usual to orient the larger axis along *[Math Processing Error]*.)

A *circle* can also be defined as the set of points which are the same distance a from a given point, and an *ellipse* can be defined as the set of points such that the *sum of the distances from two fixed points is a constant length* (which must obviously be greater than the distance between the two points!). This is sometimes called the *gardener's definition*: to set the outline of an elliptic flower bed in a lawn, a gardener would drive in two stakes, tie a loose string between them, then pull the string tight in all different directions to form the outline.

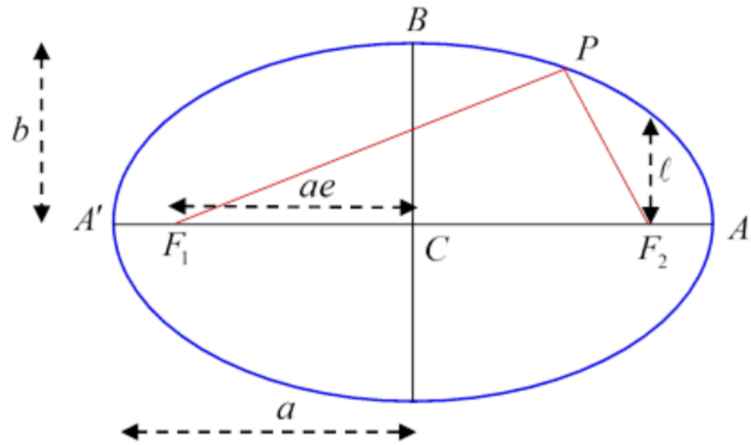

Figure *[Math Processing Error]*

In the diagram, the stakes are at *[Math Processing Error]* the **red lines** are the **string**, *[Math Processing Error]* is an arbitrary point on the ellipse.

[Math Processing Error] is called the semimajor axis length a, *[Math Processing Error]* the semiminor axis, length *[Math Processing Error]*.

[Math Processing Error] are called the foci (plural of focus).

Notice first that *the string has to be of length [Math Processing Error]*, because it must stretch along the major axis from *[Math Processing Error]* then back to *[Math Processing Error]* and for that configuration there's a double length of string along *[Math Processing Error]*. But the length *[Math Processing Error]*, so the total length of string is the same as the total length *[Math Processing Error]*.

Suppose now we put *[Math Processing Error]* at *[Math Processing Error]*. Since *[Math Processing Error]*, and the string has length *[Math Processing Error]*

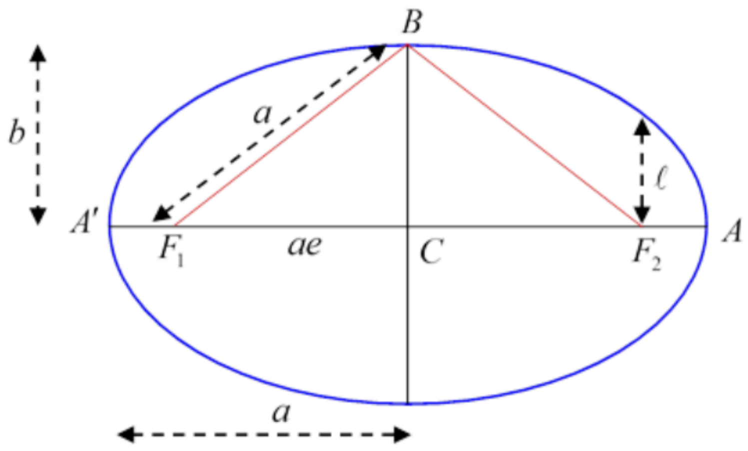

Figure *[Math Processing Error]*

We get a useful result by applying Pythagoras' theorem to the triangle *[Math Processing Error]*

[Math Processing Error]

(We shall use this shortly.)

Evidently, for a circle, *[Math Processing Error]*

Eccentricity

The **eccentricity** *[Math Processing Error]* of the ellipse is defined by

[Math Processing Error]

Eccentric just means off center, this is *how far the focus is off the center of the ellipse*, as a fraction of the semimajor axis. The eccentricity of a circle is zero. The eccentricity of a long thin ellipse is just below one.

[Math Processing Error] on the diagram are called the foci of the ellipse (plural of focus) because if a point source of light is placed at [Math Processing Error], and the ellipse is a mirror, it will reflect—and therefore focus—all the light to [Math Processing Error].

Equivalence of the Two Definitions

We need to verify, of course, that this gardener's definition of the ellipse is equivalent to the squashed circle definition. From the diagram, the total string length

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and squaring both sides of

[Math Processing Error]

then rearranging to have the residual square root by itself on the left-hand side, then squaring again,

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[Math Processing Error]

Ellipse in Polar Coordinates

In fact, in analyzing planetary motion, it is more natural to *take the origin of coordinates at the center of the Sun* rather than the center of the elliptical orbit.

It is also more convenient to take *[Math Processing Error]* coordinates instead of *[Math Processing Error]* coordinates, because the strength of the gravitational force depends only on *[Math Processing Error]*. Therefore, the relevant equation describing a planetary orbit is the *[Math Processing Error]* equation with the origin at one focus, here we follow the standard usage and choose the origin at *[Math Processing Error]*.

For an ellipse of semi major axis *[Math Processing Error]* and eccentricity *[Math Processing Error]* the equation is:

[Math Processing Error]

This is also often written

[Math Processing Error]

where *[Math Processing Error]* is the **semi-latus rectum**, the perpendicular distance from a focus to the curve *[Math Processing Error]*, see the diagram below: but notice again that *this equation has [Math Processing Error]*

(It's easy to prove *[Math Processing Error]* using Pythagoras' theorem, *[Math Processing Error]*

The directrix: writing *[Math Processing Error]*, the equation for the ellipse can also be written as

[Math Processing Error]

where *[Math Processing Error]* (the origin x=0 being the focus).

The line *[Math Processing Error]* is called the **directrix**.

For any point on the ellipse, its distance from the focus is e times its distance from the directrix.

Deriving the Polar Equation from the Cartesian Equation

Note first that (following standard practice) coordinates *[Math Processing Error]* and *[Math Processing Error]* have different origins!

Writing *[Math Processing Error]* in the Cartesian equation,

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that is, with slight rearrangement,

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This is a quadratic equation for *[Math Processing Error]* and can be solved in the usual fashion, but looking at the coefficients, it's evidently a little easier to solve the corresponding quadratic for *[Math Processing Error]*

The solution is:

[Math Processing Error]

from which

[Math Processing Error]

where we drop the other root because it gives negative *[Math Processing Error]*, for example for *[Math Processing Error]*. This establishes the equivalence of the two equations.