6.6: Hamilton's Use of the Legendre Transform

We have the Lagrangian $L\left(q_i, \dot{q}_i\right)$, and Hamilton's insight that these are not the best variables, we need to replace the Lagrangian with a closely related function (like going from the energy to the free energy), that is a function of the qi (that's not going to change) and, instead of the $\dot{q}_i$'s, the p_i 's, with $p_i = \partial L\left(q_i, \dot{q}_i\right) / \partial \dot{q}_i$. This is exactly a Legendre transform like the one from $f \rightarrow g$ discussed above.

The new function is

$$H\left(q_i, p_i\right)=\sum_{i=1}^n p_i \dot{q}_i-L\left(q_i, \dot{q}_i\right) \tag{6.6.1}$$

from which

$$d H\left(p_i, q_i\right)=-\sum_i \dot{p}_i d q_i+\sum_i \dot{q}_i d p_i \tag{6.6.2}$$

analogous to $dF = -SdT - PdV$ This new function is of course the *Hamiltonian*.