4.3: But Why?

Fermat's principle was easy to believe once it was clear that light was a wave. Imagining that the wave really propagates along all paths, and for light the phase change along a particular path is simply the time taken to travel that path measured in units of the light wave oscillation time. That means that if neighboring paths have the same length to first order the light waves along them will add coherently, otherwise they will interfere and essentially cancel. So the path of least time is heavily favored, and when we look on a scale much greater than the wavelength of the light, we don't even see the diffraction effects caused by imperfect cancellation, the light rays might as well be streams of particles, mysteriously choosing the path of least time.

So what has this to do with Hamilton's principle? Everything. A standard method in quantum mechanics these days is the so-called sum over paths, for example to find the probability amplitude for an electron to go from one point to another in a given time under a given potential, you can sum over all possible paths it might take, multiplying each path by a phase factor: and that phase factor is none other than Hamilton's action integral divided by Planck's constant, $S/\hbar$. So the true wave nature of all systems in quantum mechanics ensures that in the classical limit $S \gg \hbar$ the well-defined path of a dynamical system will be that of least action.

Historical footnote: Lagrange developed these methods in a classic book that Hamilton called a "scientific poem". Lagrange thought mechanics properly belonged to pure mathematics, it was a kind of geometry in four dimensions (space and time). Hamilton was the first to use the principle of least action to derive Lagrange's equations in the present form. He built up the least action formalism directly from Fermat's principle, considered in a medium where the velocity of light varies with position and with direction of the ray. He saw mechanics as represented by geometrical optics in an appropriate space of higher dimensions. But it didn't apparently occur to him that this might be because it was really a wave theory! (See Arnold, *Mathematical Methods of Classical Mechanics*, for details.)