17.6: Normal Coordinates

Landau writes $Q_\alpha = \operatorname{Re} C_\alpha e^{i\omega_\alpha t}$. (Actually he brings in an intermediate variable Θ_α, but we'll skip that.) These “normal coordinates” can have any amplitude and phase, but oscillate at a single frequency $\ddot{Q}_\alpha = -\omega_\alpha^2 Q_\alpha$.

The components of the above vector equation read:

$$\begin{aligned} \theta_1 &= Q_1/\sqrt{3} + Q_2/\sqrt{2} + Q_3/\sqrt{6} \\ \theta_2 &= Q_1/\sqrt{3} - 2Q_3/\sqrt{6} \\ \theta_3 &= Q_1/\sqrt{3} - Q_2/\sqrt{2} + Q_3/\sqrt{6} \end{aligned} \tag{17.6.1}$$

It’s worth going through the exercise of writing the Lagrangian in terms of the normal coordinates:

recall the Lagrangian:

$$L = \frac{1}{2}\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \frac{1}{2}\dot{\theta}_3^2 - \frac{1}{2}\omega_0^2\theta_1^2 - \frac{1}{2}\omega_0^2\theta_2^2 - \frac{1}{2}\omega_0^2\theta_3^2 - \frac{1}{2}k(\theta_1-\theta_2)^2 - \frac{1}{2}k(\theta_3-\theta_2)^2 \tag{17.6.2}$$

Putting in the above expressions for the θ_α, after some algebra

$$L = \frac{1}{2}\left[\dot{Q}_1^2 - \omega_0^2 Q_1^2\right] + \frac{1}{2}\left[\dot{Q}_2^2 - \left(\omega_0^2 + k\right)Q_2^2\right] + \frac{1}{2}\left[\dot{Q}_3^2 - \left(\omega_0^2 + 3k\right)Q_3^2\right] \tag{17.6.3}$$

We’ve achieved a separation of variables. The Lagrangian is manifestly a sum of three simple harmonic oscillators, which can have independent amplitudes and phases. Incidentally, this directly leads to the action angle variables -- recall that for a simple harmonic oscillator the action $I = E/\omega$, and the angle is that of rotation in phase space.