# CHAPTER OVERVIEW

## 18: Driven Oscillator

*Michael Fowler* (*closely following Landau para 22*)

Consider a one-dimensional simple harmonic oscillator with a variable external force acting, so the equation of motion is

$$\ddot{x}+\omega^{2} x=F(t) / m \tag{18.1}$$

which would come from the Lagrangian

$$L=\frac{1}{2} m \dot{x}^{2}-\frac{1}{2} k x^{2}+x F(t) \tag{18.2}$$

(Landau “derives” this as the leading order non-constant term in a time-dependent external potential.)

The general solution of the differential equation is $x=x_{0}+x_{1}$, where $x_{0}=a \cos (\omega t+\alpha)$ , the solution of the homogeneous equation, and $x_{1}$ is some particular integral of the inhomogeneous equation.

An important case is that of a periodic driving force $F(t)=f \cos (\gamma t+\beta)$ . A trial solution \(x_{1}(t)=b \cos (\gamma t+\beta) \text { yields } b=f / m\left(\omega^{2}-\gamma^{2}\right) \text { so }\]

$$x(t)=a \cos (\omega t+\alpha)+\frac{f}{m\left(\omega^{2}-\gamma^{2}\right)} \cos (\gamma t+\beta) \tag{18.3}$$

But what happens when $\gamma=\omega$? To find out, take part of the first solution into the second, that is,

\[x(t)=a^{\prime} \cos \left(\omega t+\alpha^{\prime}\right)+\dfrac{f}{m\left(\omega^{2}-\gamma^{2}\right)}[\cos (\gamma t+\beta)-\cos (\omega t+\beta)]\)

The second term now goes to $0/0$ as $\gamma \rightarrow \omega$, so becomes the ratio of its first derivatives with respect to $\omega$ (or, equivalently, $\gamma$).

$$x(t)=a^{\prime} \cos \left(\omega t+\alpha^{\prime}\right)+\frac{f}{2 m \omega} t \sin (\omega t+\beta) \tag{18.4}$$

The amplitude of the oscillations grows linearly with time. Obviously, this small oscillations theory will crash eventually.

But what if the external force frequency is slightly *off* resonance?

Then (real part understood)

$$x=A e^{i \omega t}+B e^{i(\omega+\varepsilon) t}=\left(A+B e^{i \varepsilon t}\right) e^{i \omega t}, \quad A=a e^{i \alpha}, \quad B=b e^{i \beta} \tag{18.5}$$

with $a, b, \alpha, \beta$ real.

The wave amplitude squared

$$C^{2}=\left|A+B e^{i \varepsilon t}\right|^{2}=a^{2}+b^{2}+2 a b \cos (\varepsilon t+\beta-\alpha) \tag{18.6}$$

We’re seeing beats, with beat frequency $\varepsilon$. Note that if the oscillator begins at the origin, $x(t=0)=0$, then $A+B=0$ and the amplitude periodically goes to zero, this evidently only occurs when $|A|=|B|$.

Energy is exchanged back and forth with the driving external force.