29.2: Uniformly Rotating Frame

For the important case of a frame having uniform rotation and no translation motion,

$$m d\vec{v}/dt = -\partial U/\partial\vec{r} + 2m\vec{v}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega} \tag{29.2.1}$$

The last term is (as Landau states) the "centrifugal force", but this term is now politically incorrect, since it isn't a "real force", just an effect of being in a rotating frame. (It's still OK to say gravitational force, though, although that isn't a real force either, I guess, since it disappears in the local inertial "freely falling" frame, as was first noticed by Galileo, and centuries later by Einstein, who called it "the happiest thought of my life".)

The second term, $2m\vec{v}\times\vec{\Omega}$ Landau calls the *Coriolis force*. (Again, the politically correct tend to talk about the Coriolis *effect*, meaning deviation of a projectile, say, from an inertial frame trajectory resulting from the operation of this "force".) A very nice illustration of this "force" is in the Frames of Reference 2 movie, starting at time 3:50.

Notice the Coriolis force depends on the velocity of the particle, and is reminiscent of the magnetic force on a charged particle. For example, it does no work on the particle, but *does* curve the particle's path.

The energy of the particle can be found from the standard Lagrangian equation

$$E = \sum \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L = \vec{v}\cdot\vec{p} - L \tag{29.2.2}$$

where

$$\vec{p} = \partial L/\partial\vec{v} = m\vec{v} + m\vec{\Omega}\times\vec{r} \tag{29.2.3}$$

This is interesting! Remembering $\vec{v}_0 = \vec{v} + \vec{\Omega}\times\vec{r}$, *the momentum*, defined in this way as a canonical variable, not as just $m\vec{v}$ in the frame we're in, *is the same in the two frames* K_0, K

$$\vec{p}_0 = \vec{p} \tag{29.2.4}$$

The angular momenta $\vec{L}_0 = \vec{r}\times\vec{p}_0$ and $\vec{L} = \vec{r}\times\vec{p}$ are also equal in the two frames.

The Lagrangian is

$$L = \frac{1}{2}m\vec{v}^2 + m\vec{v}\cdot\vec{\Omega}\times\vec{r} + \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 - U(\vec{r}) \tag{29.2.5}$$

so

$$\begin{gathered} E = \vec{p}\cdot\vec{v} - L \\ = m\vec{v}^2 + m\vec{\Omega}\times\vec{r}\cdot\vec{v} - \left(\frac{1}{2}m\vec{v}^2 + m\vec{v}\cdot\vec{\Omega}\times\vec{r} + \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 - U(\vec{r})\right) \\ = \frac{1}{2}m\vec{v}^2 - \frac{1}{2}m(\vec{\Omega}\times\vec{r})^2 + U(\vec{r}) \end{gathered} \tag{29.2.6}$$

The new term is the centrifugal potential energy. It's negative because it takes work to bring something towards the axis of rotation. To see how this energy relates to the energy in the original fixed frame, substitute in this equation $\vec{v} = \vec{v}_0 - \vec{\Omega}\times\vec{r}$ to find

$$E = E_0 - \vec{L}\cdot\vec{\Omega} \tag{29.2.7}$$

true for one particle, and by addition for any system of particles.

Exercise 29.2.1

Confirme that that $\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times\vec{p}_0$. Notice the difference can be positive or negative—give a simple one-particle illustration of this.