

## 29.1: The Lagrangian in Accelerating and Rotating Frames

This section concerns the motion of a single particle in some potential  $U(\vec{r})$  in a non-inertial frame of reference. (We'll use  $U(\vec{r})$  rather than  $V(\vec{r})$  for potential in this section, since we'll be using  $\vec{V}$  for relative frame velocity.) The most general noninertial frame has both linear acceleration and rotation, and the angular velocity of rotation may itself be changing.

Our strategy is to begin with an inertial frame  $K_0$ , then go to a frame  $K'$  having linear acceleration relative to  $K_0$  then finally to a frame  $K$  rotating relative to  $K'$ . We will construct the Lagrangian in  $K$ , and from it the equations of motion in that noninertial frame.

First, suppose the noninertial frame  $K'$  to be moving relative to  $K_0$  at time-varying velocity  $\vec{V}(t)$ . In the *inertial* frame  $K_0$ , the Lagrangian is as usual

$$L_0 = \frac{1}{2}m\vec{v}_0^2 - U \quad (29.1.1)$$

so Lagrange's equations give the standard result,

$$m d\vec{v}_0/dt = -\partial U/\partial \vec{r} \quad (29.1.2)$$

the subscript 0 denoting quantities in this inertial frame.

The Principle of Least Action is a frame-independent concept, so the calculus of variations Lagrangian equations it leads to,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \vec{v}} \right) = \frac{\partial L}{\partial \vec{r}} \quad (29.1.3)$$

must also be correct in a non-inertial frame.

How can this be true? The reason is that in a non-inertial frame, the Lagrangian has a different form.

To find the Lagrangian in terms of the velocity  $\vec{v}'$ , meaning the velocity measured in the frame  $K'$ , we just add the velocity of  $K'$  relative to  $K$ .

$$\vec{v}_0 = \vec{v}' + \vec{V} \quad (29.1.4)$$

and putting this into  $L_0$ , gives the Lagrangian  $L'$  in the accelerating frame  $K'$  :

$$L' = \frac{1}{2}m\vec{v}'^2 + m\vec{v}' \cdot \vec{V}(t) + \frac{1}{2}m\vec{V}^2(t) - U(\vec{r}') \quad (29.1.5)$$

Following Landau,  $\vec{V}^2(t)$  is purely a function of time, so can be expressed as the derivative of a function of time, recall terms of that form do not affect the minimization of the action giving the equations of motion, and so can be dropped from the Lagrangian.

The second term,

$$m\vec{V}(t) \cdot \vec{v}' = m\vec{V} \cdot d\vec{r}'/dt = d(m\vec{V} \cdot \vec{r}')/dt - m\vec{r}' \cdot d\vec{V}/dt \quad (29.1.6)$$

Again, the total derivative term can be dropped, giving

$$L' = \frac{1}{2}m\vec{v}'^2 - m(d\vec{V}(t)/dt) \cdot \vec{r}' - U(\vec{r}') \quad (29.1.7)$$

from which the equation of motion is

$$m \frac{d\vec{v}'}{dt} = -\frac{\partial U(\vec{r}')}{\partial \vec{r}'} - m \left( \frac{d\vec{V}}{dt} \right) \quad (29.1.8)$$

Landau writes this as

$$m \frac{d\vec{v}'}{dt} = -\frac{\partial U}{\partial \vec{r}'} - m\vec{W} \quad (29.1.9)$$

So the motion in the accelerating frame is *the same as if an extra force is added*—this extra force is just the product of the particle's mass and the frame's acceleration, it's just the "force" that pushes you back in your seat when you step on the gas, the linear equivalent of the "centrifugal force" in a rotating frame.

Speaking of centrifugal force, we now bring in our final frame  $K$ , having the same origin as  $K'$ , (so we can take  $\vec{r}' = \vec{r}$  at a given instant) but rotating relative to it with angular velocity  $\vec{\Omega}(t)$ .

What is the Lagrangian translated into  $K$  variables? The velocities in  $K'$ ,  $K$  are related by

$$\vec{v}' = \vec{v} + \vec{\Omega} \times \vec{r} \quad (29.1.10)$$

and, putting this in the Lagrangian above,

$$L = \frac{1}{2}m\vec{v}^2 + m\vec{v} \cdot \vec{\Omega} \times \vec{r} + \frac{1}{2}m(\vec{\Omega} \times \vec{r})^2 - m\vec{W} \cdot \vec{r} - U(\vec{r}) \quad (29.1.11)$$

From this,

$$\partial L / \partial \vec{v} = m\vec{v} + m\vec{\Omega} \times \vec{r} \quad (29.1.12)$$

(Note that this is the canonical momentum,  $p_i = \partial L / \partial \dot{x}_i = \partial L / \partial v_i$ )

Using

$$\vec{v} \cdot \vec{\Omega} \times \vec{r} = \vec{v} \times \vec{\Omega} \cdot \vec{r}, \quad (\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r}) = (\vec{\Omega} \times \vec{r}) \times \vec{\Omega} \cdot \vec{r} \quad (29.1.13)$$

we have

$$\partial L / \partial \vec{r} = m\vec{v} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} - m\vec{W} - \partial U / \partial \vec{r} \quad (29.1.14)$$

The equation of motion

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \vec{v}} \right) = \frac{\partial L}{\partial \vec{r}} \quad (29.1.15)$$

is therefore:

$$m d\vec{v} / dt = -\partial U / \partial \vec{r} - m\vec{W} + m\vec{r} \times \vec{\Omega} + 2m\vec{v} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega} \quad (29.1.16)$$

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