12.3: Separation of Variables for a Central Potential; Cyclic Variables

Landau presents in some details the separation of variables method for a *[Math Processing Error]* potential, interesting here because it results in equations you've met before—those arising in the standard quantum treatment of the hydrogen atom.

How do we make any progress with these formidable differential equations? One possibility is that some coordinates are *cyclic*, meaning that *[Math Processing Error]* say, does not appear explicitly in the Hamiltonian—for example, an angle variable in a spherically symmetric field. Then we have immediately that the corresponding momentum, *[Math Processing Error]*, a constant.

The Hamiltonian for a central potential is:

[Math Processing Error]

The Hamilton-Jacobi equation is therefore

[Math Processing Error]

The first thing to note is that *[Math Processing Error]* is cyclic (it doesn't appear in the Hamiltonian), so we can immediately replace *[Math Processing Error]*.

Then we have:

[Math Processing Error]

Now we seek a solution of the form

[Math Processing Error]

Substituting in the equation, notice that the expression in square brackets will become

[Math Processing Error]

independent of *[Math Processing Error]*, but on multiplying the full equation by *[Math Processing Error]*, and staring at the result, we see that in fact it is purely a function of *[Math Processing Error]*. This means that it's a constant, say

[Math Processing Error]

and then

[Math Processing Error]

These first-order equations can then be solved, at least numerically (and of course exactly for some cases). Physically, *[Math Processing Error]* being the total angular momentum, and *[Math Processing Error]* is the total energy.

Note: recall that in quantum mechanics, for example in solving the Schrödinger equation for the hydrogen atom, the separation of variables was achieved by writing the wave function as a *product* of functions belonging to the different variables. Here we use a *sum*—remember that the action corresponds closely to the *phase* of a quantum mechanical system, so a *sum* of actions is analogous to a *product* of wave functions.