28.2: Free Rotation of a Symmetric Top Using Euler's Equations

This is a problem we've already solved, using Lagrangian methods and Euler *angles*, but it's worth seeing just how easy it is using Euler's equations.

For $I_1 = I_2$, the third equation gives immediately $\Omega_3 = \text{ constant. }$.

Then, writing for convenience

$$\Omega_3 \left(I_3 - I_1\right) / I_1 = \omega \tag{28.2.1}$$

the first two equations are

$$\dot{\Omega}_1 = -\omega\Omega_2, \quad \dot{\Omega}_2 = \omega\Omega_1 \tag{28.2.2}$$

These equations can be combined to give

$$d\left(\Omega_1 + i\Omega_2\right) / dt = i\omega\left(\Omega_1 + i\Omega_2\right), \text{ so } \left(\Omega_1 + i\Omega_2\right) = Ae^{i\omega t} \tag{28.2.3}$$

That is, $\left(\Omega_1, \Omega_2\right)$ moves around a circle centered at the origin with constant angular velocity. So $\Omega_1^2 + \Omega_2^2 = |A|^2$ stays constant, and Ω_3 is constant, the angular velocity vector has constant length and rotates steadily about the axis x_3.

From

$$L_1 = I_1\Omega_1, L_2 = I_2\Omega_2, L_3 = I_3\Omega_3 \tag{28.2.4}$$

it follows that the angular *momentum* vector also precesses at a steady rate about x_3. This is, remember, in the *body* frame—we know that in the fixed space frame, the angular momentum vector is constant! It follows that, as viewed from the outside, the x_3 axis precesses around the fixed angular momentum vector at a steady rate.

Of course, the rate is the same as that found using Euler's angles, recall from the previous lecture that

$$\vec{L} = \left(I_1\Omega_1, I_1\Omega_2, I_3\Omega_3\right) = \left(I_1\dot{\theta}, \quad I_1\dot{\phi}\sin\theta, \quad I_3(\dot{\phi}\cos\theta + \dot{\psi})\right) \tag{28.2.5}$$

so in precession

$$L_3 = L\cos\theta = I_3(\dot{\phi}\cos\theta + \dot{\psi}) \text{ and } \dot{\phi} = L / I_1 \tag{28.2.6}$$

so

$$\dot{\psi} = L\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right) = -\Omega_3\left(I_3 - I_1\right) / I_1 \tag{28.2.7}$$