12.1: Back to Configuration Space…

We’ve established that the action, regarded as a function of its coordinate endpoints and time, satisfies

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and at the same time *[Math Processing Error]* obeys the first-order differential equation

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This is the *Hamilton-Jacobi equation*.

Notice that we’re now back in *configuration space*!

For example, the Hamilton-Jacobi equation for the simple harmonic oscillator in one dimension is

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(Notice that this has some resemblance to the Schrödinger equation for the same system.)

If the Hamiltonian has no explicit time dependence *[Math Processing Error]* so the action has the form *[Math Processing Error]*, and the Hamilton-Jacobi equation is

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(This is analogous to the *time independent* Schrödinger equation for energy eigenstates.)

The Hamilton-Jacobi equation is therefore a *third* complete description of the dynamics, equivalent to Lagrange’s equations and to Hamilton’s equations.

Since *[Math Processing Error]* only appears differentiated, if we have a solution to the equation, we can always add an arbitrary constant term, to give an equally valid solution. For the general case, there will be a further s constants of integration, so a complete solution has the form

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the α's and *[Math Processing Error]* being the constants of integration. We're not saying it's easy to solve this differential in general, just that we know how many constants of integration there must be in a final solution. Since the action determines the motion of the system completely, the constants of integration will be determined by the given initial and final coordinates, or, they could equally be regarded as functions of the initial coordinates and momenta (the initial momenta themselves being determined by the given initial and final coordinates).