5.3: The Virial Theorem

For a potential energy homogeneous in the coordinates, of degree k, say, and spatially bounded motion, there is a simple relation between the time averages of the kinetic energy, $\bar{T}$, and potential energy, $\bar{U}$. It's called the **virial theorem.**

Theorem 5.3.1: Virial Theorem

$$2\bar{T} = k\bar{U} \tag{5.3.1}$$

Proof

Since

$$T = \sum_i \frac{1}{2} m_i \vec{v}_i^2, \quad \vec{p}_i = m_i \vec{v}_i = \partial T / \partial \vec{v}_i \tag{5.3.2}$$

we have

$$2T = \sum_i \vec{p}_i \cdot \vec{v}_i = \frac{d}{dt}\left(\sum_i \vec{p}_i \cdot \vec{r}_i\right) - \sum_i \vec{r}_i \cdot \dot{\vec{p}}_i \tag{5.3.3}$$

We now average the terms in this equation over a very long time, that is, take

$$\bar{f} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau f(t)dt \tag{5.3.4}$$

Since we've said the orbits are bounded in space, and we assume also in momentum, the exact differential term contributes

$$\frac{1}{\tau}\left[\left(\sum_i \vec{p}_i \cdot \vec{r}_i\right)_{\text{at final}} - \left(\sum_i \vec{p}_i \cdot \vec{r}_i\right)_{\text{at initial}}\right] \to 0 \tag{5.3.5}$$

in the limit of infinite time.

So we have the time averaged

$$2\bar{T} = \sum_i \vec{r}_i \cdot \partial U / \partial \vec{r}_i \tag{5.3.6}$$

and for a potential energy a homogeneous function of degree k in the coordinates, from Euler's theorem:

$$2\bar{T} = k\bar{U} \tag{5.3.7}$$

So, for example, in a simple harmonic oscillator the average kinetic energy equals the average potential energy, and for an inverse-square system, the average kinetic energy is half the average potential energy in magnitude, and of opposite sign (being of course positive).