13.3: Action-Angle Variables

For a closed one-dimensional system undergoing finite motion (essentially a bound state), the equations of motion can be reformulated using the action variable *[Math Processing Error]* in place of the energy *[Math Processing Error]*. *[Math Processing Error]* is a function of energy alone in a closed one-dimensional system, and vice versa.

We're visualizing here a particle moving back and forth in a one-dimensional well with potential zero at the origin, and the potential never decreasing on going out from the origin to infinity. Obviously, if a potential has *two* low points, local bound states can arise in different places, and the *[Math Processing Error]* relationship is complicated, with different branches, possibly coming together at high energies.

Caution

Notice the integral sign in the expression for the action variable *[Math Processing Error]* is *[Math Processing Error]* signifying an integral around a *closed path*, a circuit. *Don't* confuse this integral with the abbreviated action integral, which has the same integrand, but is an integral *[Math Processing Error]* along a contour from a fixed starting point, say the origin, to the endpoint *[Math Processing Error]*, not going around a closed path. (Apologies for using the same letter for the differential and the endpoint, just following Landau.)

In the spirit of the discussion of constants of motion above, we make a canonical transformation to *[Math Processing Error]* as the new "momentum", using as generating function the abbreviated action *[Math Processing Error]*

The original momentum

[Math Processing Error]

The new "coordinate" conjugate to the momentum *[Math Processing Error]* will be

[Math Processing Error]

This is called an *angle variable*, *[Math Processing Error]* is the *action variable*, they are canonical.

To find Hamilton's equations in the transformed variables, since there is no time-dependence in the transformation, and the system is closed, the energy remains constant. Also, the energy is a function of *[Math Processing Error]* (meaning not of *[Math Processing Error]*.)

Hence

[Math Processing Error]

so the angle is a linear function of time: *[Math Processing Error]*

One further point about the action variable and the action: since we define the action as

[Math Processing Error]

it follows that if we track the change in this integral as time goes on and the system moves round and round the circuit in phase space, an additional term *[Math Processing Error]* will be added to the action for each time round, so the action is multi-valued.