2.3: General Method for the Minimization Problem

To emphasize the generality of the method, we'll just write

$$J[y]=\int_{x_1}^{x_2} f\left(y, y'\right) dx \quad \left(y'=dy/dx\right) \tag{2.3.1}$$

Then under any infinitesimal variation $\delta y(x)$ (equal to zero at the fixed endpoints)

$$\delta J[y]=\int_{x_1}^{x_2}\left[\frac{\partial f\left(y, y'\right)}{\partial y} \delta y(x)+\frac{\partial f\left(y, y'\right)}{\partial y'} \delta y'(x)\right] dx=0 \tag{2.3.2}$$

To make further progress, we write $\delta y'=\delta(dy/dx)=(d/dx)\delta y$, then integrate the second term by parts, remembering $\delta y=0$ at the endpoints, to get

$$\delta J[y]=\int_{x_1}^{x_2}\left[\frac{\partial f\left(y, y'\right)}{\partial y}-\frac{d}{dx}\left(\frac{\partial f\left(y, y'\right)}{\partial y'}\right)\right] \delta y(x) dx=0 \tag{2.3.3}$$

Since this is true for any infinitesimal variation, we can choose a variation which is only nonzero near one point in the interval, and deduce that

$$\frac{\partial f\left(y, y'\right)}{\partial y}-\frac{d}{dx}\left(\frac{\partial f\left(y, y'\right)}{\partial y'}\right)=0 \tag{2.3.4}$$

This general result is called the **Euler-Lagrange equation**. It's very important—you'll be seeing it again.