8.6: Hamilton’s Equations from Action Minimization

For arbitrary small path variations *[Math Processing Error]* in phase space, the minimum action condition using the form of action given above generates Hamilton’s equations.

(*Note for nitpickers*: This may seem a bit surprising, since we generated this form of the action using the equations along the actual dynamical path, how can we vary it and still use them? Bear with me, you’ll see.)

We’ll prove this for a one dimensional system, it’s trivial to go to many variables, but it clutters up the equations.

For a small path deviation *[Math Processing Error]* the change in the action *[Math Processing Error]* is

[Math Processing Error]

and integrating *[Math Processing Error]* by parts, with *[Math Processing Error]* at the endpoints,

[Math Processing Error]

The path variations *[Math Processing Error]* are independent and arbitrary, so must have identically zero coefficients—Hamilton’s equations follow immediately, *[Math Processing Error]* Again, it’s worth emphasizing the close parallel with quantum mechanics: Hamilton’s equations written using Poisson brackets are:

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In quantum mechanics, the corresponding Heisenberg equations of motion for position and momentum operators in terms of *commutators* are

[Math Processing Error]