4.9: Example 2- Lagrangian Formulation of the Central Force Problem

A simple example of Lagrangian mechanics is provided by the central force problem, a mass m acted on by a force

$F_r = -dU(r)/dr$

To contrast the Newtonian and Lagrangian approaches, we'll first look at the problem using just $\vec{F} = m\vec{a}$. To take advantage of the rotational symmetry we'll use (r, θ) coordinates, and find the expression for acceleration by the standard trick of differentiating the complex number $z = re^{i\theta}$ twice, to get

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -dU(r)/dr \tag{4.9.1}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{4.9.2}$$

The second equation integrates immediately to give

$$mr^2\dot{\theta} = \ell \tag{4.9.3}$$

a constant, the angular momentum. This can then be used to eliminate $\dot{\theta}$ in the first equation, giving a differential equation for $r(t)$.

The Lagrangian approach, on the other hand, is first to write

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \tag{4.9.4}$$

and put it into the equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0 \end{aligned} \tag{4.9.5}$$

Note now that since L doesn't depend on θ, the second equation gives immediately:

$$\frac{\partial L}{\partial \dot{\theta}} = \text{constant} \tag{4.9.6}$$

and in fact $\partial L/\partial\dot{\theta} = mr^2\dot{\theta}$ the angular momentum, we'll call it ℓ

The first integral (see above) gives another constant:

$$\dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \text{constant} \tag{4.9.7}$$

This is just

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = E \tag{4.9.8}$$

the energy.

Angular momentum conservation, $mr^2\dot{\theta} = \ell$, then gives

$$\frac{1}{2}m\left(\dot{r}^2 + \frac{\ell^2}{m^2r^2}\right) + U(r) = E \tag{4.9.9}$$

giving a first-order differential equation for the radial motion as a function of time. We'll deal with this in more detail later. Note that it is equivalent to a particle moving in one dimension in the original potential plus an effective potential from the angular momentum term:

$$E = \frac{1}{2}mv^2 + U(r) + \frac{\ell^2}{m^2r^2} \tag{4.9.10}$$

This can be understood by realizing that for a fixed angular momentum, the closer the particle approaches the center the greater its speed in the tangential direction must be, so, to conserve total energy, its speed in the radial direction has to go down, unless it is in a *very* strongly attractive potential (the usual gravitational or electrostatic potential isn't strong enough) so the radial motion is equivalent to that with the existing potential plus the ℓ^2/m^2r^2

term, often termed the "centrifugal barrier".

Exercise: how strong must the potential be to overcome the centrifugal barrier? (This can happen in a black hole!)