2.4: An Important First Integral of the Euler-Lagrange Equation

It turns out that, *since the function \(f \) does not contain x explicitly*, there is a simple first integral of this equation. Multiplying throughout by $y' = dy/dx$

$$\frac{\partial f\left(y, y'\right)}{\partial y} \frac{dy}{dx} - \frac{d}{dx}\left(\frac{\partial f\left(y, y'\right)}{\partial y'}\right) y' = 0 \tag{2.4.1}$$

Since f doesn't depend explicitly on x, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \tag{2.4.2}$$

and using this to replace $\frac{\partial f\left(y, y'\right)}{\partial y} \frac{dy}{dx}$ in the preceding equation gives

$$\frac{df}{dx} - \frac{\partial f}{\partial y'} \frac{dy'}{dx} - \frac{d}{dx}\left(\frac{\partial f\left(y, y'\right)}{\partial y'}\right) y' = 0 \tag{2.4.3}$$

then multiplying by − (to match the equation as usually written) we have

$$\frac{d}{dx}\left(y' \frac{\partial f}{\partial y'} - f\right) = 0 \tag{2.4.4}$$

giving a **first integral**

$$y' \frac{\partial f}{\partial y'} - f = \text{constant.} \tag{2.4.5}$$

For the soap film between two rings problem,

$$f\left(y, y'\right) = y\sqrt{1 + y'^2} \tag{2.4.6}$$

so the Euler-Lagrange equation is

$$\sqrt{1 + y'^2} - \frac{d}{dx} \frac{yy'}{\sqrt{1 + y'^2}} = 0 \tag{2.4.7}$$

and has first integral

$$y' \frac{\partial f}{\partial y'} - f = \frac{yy'^2}{\sqrt{1 + y'^2}} - y\sqrt{1 + y'^2} = -\frac{y}{\sqrt{1 + y'^2}} = \text{constant.} \tag{2.4.8}$$

We'll write

$$\frac{y}{\sqrt{1 + y'^2}} = a \tag{2.4.9}$$

with a the constant of integration, which will depend on the endpoints.

This is a first-order differential equation, and can be solved.

Rearranging,

$$\frac{dy}{dx} = \sqrt{\left(\frac{y}{a}\right)^2 - 1} \tag{2.4.10}$$

or

$$dx = \frac{a dy}{\sqrt{y^2 - a^2}} \tag{2.4.11}$$

The standard substitution here is $y = a\cosh\xi$ from which

$$y=a\cosh\left(\frac{x-b}{a}\right) \tag{2.4.12}$$

Here b is the second constant of integration, the fixed endpoints determine a, b.