4.10: Conservation Laws and Noether's Theorem

The two integrals of motion for the orbital example above can be stated as follows:

*First***:** if the Lagrangian does not depend on the variable $\theta, \partial L/\partial\theta = 0$, that is, it's *invariant under rotation*, meaning it has circular symmetry, then

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = \text{constant} \tag{4.10.1}$$

angular momentum is conserved.

Second: As stated earlier, if the Lagrangian is independent of time, that is, it's *invariant under time translation*, then *energy is conserved*. (This is nothing but the first integral of the calculus of variations, recall that for an integrand function $f(y, y')$ not explicitly dependent on $x, y'\partial f/\partial y' - f$ is constant.)

$$\sum_i \dot{q}_i \partial L/\partial\dot{q}_i - L = E, \quad \text{a constant} \tag{4.10.2}$$

Note

Both these results link symmetries of the Lagrangian——invariance under rotation and time translation respectively——with conserved quantities.

This connection was first spelled out explicitly, and proved generally, by Emmy Noether, published in 1915. The essence of the theorem is that if the Lagrangian (which specifies the system completely) does not change when some continuous parameter is altered, then some function of the $q_i, \dot{q}_i$ stays the same—it is called a constant of the motion, or an integral of the motion.

To look further at this expression for energy, we take a closed system of particles interacting with each other, but "closed" means no interaction with the outside world (except possibly a time-independent potential).

The Lagrangian for the particles is, in Cartesian coordinates,

$$L = \sum_2^1 m_i v_i^2 - U(\vec{r}_1, \vec{r}_2, \ldots) \tag{4.10.3}$$

A set of general coordinates $(q_1, \ldots, q_n)$, by definition, uniquely specifies the system configuration, so the coordinate and velocity of a particular particle a are given by

$$x_a = f_{x_a}(q_1, \ldots q_n), \quad \dot{x}_a = \sum_k \frac{\partial f_{x_a}}{\partial q_k} \dot{q}_k \tag{4.10.4}$$

From this it is clear that the kinetic energy term $T = \sum_2^1 m_i v_i^2$ is a homogeneous quadratic function of the $\dot{q}$ (meaning every term is of degree two), so

$$L = \frac{1}{2} \sum_{i,k} a_{ik}(q) \dot{q}_i \dot{q}_k - U(q) \tag{4.10.5}$$

This being of degree two in the time derivatives means

$$\sum_i \dot{q}_i \frac{\partial L}{\partial\dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial\dot{q}_i} = 2T \tag{4.10.6}$$

(If this isn't obvious to you, check it out with a couple of terms: $\dot{q}_1^2, \dot{q}_1\dot{q}_2$)

Therefore for this system of interacting particles

$$E = \sum_{i=1}^n \dot{q}_i \frac{\partial L}{\partial\dot{q}_i} - L = 2T - (T - U) = T + U \tag{4.10.7}$$

This expression for the energy is called the *Hamiltonian*:

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \tag{4.10.8}$$
