10.3: Generating Functions for Canonical Transformations

In this section, we go back to considering the full action (not the abbreviated--fixed energy--action used earlier).

Now, we've established that Hamilton's equations in the original parameterization follow from minimizing the action in the form

$$\delta \int \left(\sum_i p_i dq_i - H dt \right) = 0 \tag{10.3.1}$$

For a canonical transformation, by definition the new variables must also satisfy Hamilton's equations, so, working backwards, action minimization must be expressible in the new variables exactly as in the old ones:

$$\delta \int \left(\sum_i P_i dQ_i - H' dt \right) = 0 \tag{10.3.2}$$

Now, we've previously stated that two actions lead to the same equations of motion if the integrands differ by the total differential of some function F of coordinates, momenta and time. (That's because in adding such a function to the integrand, the function's contribution to the integral is just the difference between its values at the two (fixed) ends, so in varying the path between the ends to minimize the total integral and so generate the equations of motion, this exact differential dF makes no contribution.)

That is to say, the two action integrals will be *minimized on the same path* through phase space provided the integrands differ by an exact differential:

$$\sum_i p_i dq_i - H dt = \sum_i P_i dQ_i - H' dt + dF \tag{10.3.3}$$

F is called the *generating function* of the transformation. Rearranging the equation above,

$$dF = \sum_i p_i dq_i - \sum_i P_i dQ_i + (H' - H) dt \tag{10.3.4}$$

Notice that the differentials here are dq_i, dQ_i, dt *so these are the natural variables for expressing the generating function*.

We will therefore write it as $F(q, Q, t)$,

and from the expression for dF above,

$$p_i = \frac{\partial F(q, Q, t)}{\partial q_i}, \quad P_i = -\frac{\partial F(q, Q, t)}{\partial Q_i}, \quad H' = H + \frac{\partial F(q, Q, t)}{\partial t} \tag{10.3.5}$$

Let's reemphasize here that a canonical transformation will in general mix up coordinates and momenta—they are the same kind of variable, from this Hamiltonian perspective. They can even be *exchanged*: for a system with one degree of freedom, for example, the transformation

$$Q = p, \quad P = -q \tag{10.3.6}$$

is a perfectly good canonical transformation (check out Hamilton's equations in the new variables), even though it turns a position into a momentum and vice versa!

If this particular transformation is applied to a simple harmonic oscillator, the Hamiltonian remains the same (we're taking $H = \frac{1}{2}(p^2 + q^2)$ so the differential dF of the generating function (given above) has no $H' - H$ term, it is just

$$dF(q, Q) = p dq - P dQ \tag{10.3.7}$$

The generating function for this transformation is easily found to be

$$F(q, Q) = Qq \tag{10.3.8}$$

from which

$$dF = Q dq + q dQ = p dq - P dQ \tag{10.3.9}$$

as required.

Another canonical transformation for a simple harmonic oscillator is $q=\sqrt{2P}\sin Q, \quad p=\sqrt{2P}\cos Q$. You will investigate this in homework.