22.6: Frequency Multiples

The above analysis is for frequencies not very far from ω_0. But nonlinear terms can cause resonance to occur at frequencies which are rational multiples of ω_0. Landau shows that a small $\frac{1}{3}\alpha x^3$ in the potential (so an additional force αx^2 in the equation of motion) can generate a resonance near $\gamma=\frac{1}{2}\omega_0$. We've only considered a quartic addition to the potential, $\frac{1}{4}\beta x^4$, a force βx^3, we can show that gives a resonance near $\gamma=\frac{1}{3}\omega_0$, and presumably this is the small bump near the beginning of the curves above for large driving strength.

We have $\ddot{x}+2\lambda\dot{x}+\omega_0^2 x=(f/m)\cos\gamma t-\beta x^3$

We'll write $x=x^{(0)}+x^{(1)}+\ldots$

Let's define $x^{(0)}$ by

$$\ddot{x}^{(0)}+2\lambda\dot{x}^{(0)}+\omega_0^2 x^{(0)}=(f/m)\cos\gamma t \tag{22.6.1}$$

So $x^{(0)}=b\cos(\gamma t+\delta)$. Then

$$\begin{aligned}\ddot{x}^{(1)}+2\lambda\dot{x}^{(1)}+\omega_0^2 x^{(1)} &=-\beta\left(x^{(0)}\right)^3\\ &=-\beta b^3\cos^3(\gamma t+\delta)\\ &=-\beta b^3\left[\frac{3}{4}\cos(\gamma t+\delta)+\frac{1}{4}\cos(3\gamma t+\delta)\right]\end{aligned}$$

Then, for $\gamma=\frac{1}{3}\omega_0$, the second term, $-\beta b^3\frac{1}{4}\cos(3\gamma t+\delta)=-\beta b^3\frac{1}{4}\cos(\omega_0 t+\delta)$, will have a resonant response, although it is proportional to the (small) amplitude cubed. Similar arguments work for other fractional frequencies.