20.3: Example- Pendulum Driven at near Double the Natural Frequency

A simple pendulum of length ℓ, mass m is attached to a point which oscillates vertically $y = a\cos\Omega t$. Measuring y downwards, the pendulum position is

$$x = \ell\sin\phi, y = a\cos\Omega t + \ell\cos\phi \tag{20.3.1}$$

The Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mg\ell\cos\phi \\ &= \frac{1}{2}m\left(\ell^2\cos^2\phi\right)\dot{\phi}^2 + \frac{1}{2}m(a\Omega\sin\Omega t + \ell\dot{\phi}\sin\phi)^2 + mg\ell\cos\phi \\ &= \frac{1}{2}m\ell^2\dot{\phi}^2 - ma\ell\Omega\sin\Omega t\frac{d}{dt}\cos\phi + a^2\Omega^2\sin^2\Omega t + mg\ell\cos\phi \end{aligned} \tag{20.3.2}$$

The purely time-dependent term will not affect the equations of motion, so we drop it, and since the equations are not affected by adding a total derivative to the Lagrangian, we can integrate the second term by parts (meaning we're dropping a term $\frac{d}{dt}(\text{mal}\,\Omega\sin\Omega t\cos\phi)\Big)$ to get

$$L = \frac{1}{2}m\ell^2\dot{\phi}^2 + ma\ell\Omega^2\cos\Omega t\cos\phi + mg\ell\cos\phi \tag{20.3.3}$$

(We've also dropped the term $mga\cos\Omega t$ from the potential energy term—it has no ϕ or $\dot{\phi}$ dependence, so will not affect the equations of motion.)

The equation for small oscillations is

$$\ddot{\phi} + \omega_0^2\left[1 + (4a/\ell)\cos(2\omega_0 + \varepsilon)t\right]\phi = 0, \quad \omega_0^2 = g/\ell \tag{20.3.4}$$

Comparing this with

$$\ddot{x} + \omega_0^2\left[1 + h\cos(2\omega_0 + \varepsilon)t\right]x = 0 \tag{20.3.5}$$

we see that $4a/\ell \equiv h$, so the parametric resonance range around $2\omega_0 = 2\sqrt{g/\ell}$ is of width $\frac{1}{2}h\omega_0 = 2a\sqrt{g/\ell^3}$.