14.1: Preliminaries- Conic Sections

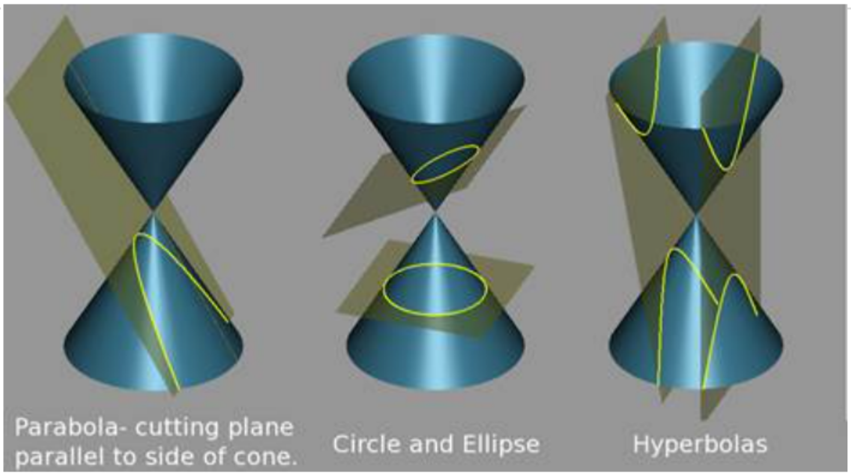

Figure [Math Processing Error]

Ellipses, parabolas and hyperbolas can all be generated by cutting a cone with a plane (see diagrams, from Wikimedia Commons). Taking the cone to be [Math Processing Error], and substituting the z in that equation from the planar equation [Math Processing Error] is the vector perpendicular to the plane from the origin to the plane, gives a quadratic equation in [Math Processing Error]. This translates into a quadratic equation in the plane—take the line of intersection of the cutting plane with the [Math Processing Error] plane as the [Math Processing Error] axis in both, then one is related to the other by a scaling [Math Processing Error]. To identify the conic, diagonalized the form, and look at the coefficients of [Math Processing Error]. If they are the same sign, it is an ellipse, opposite, a hyperbola. The parabola is the exceptional case where one is zero, the other equates to a linear term. It is instructive to see how an important property of the ellipse follows immediately from this construction.

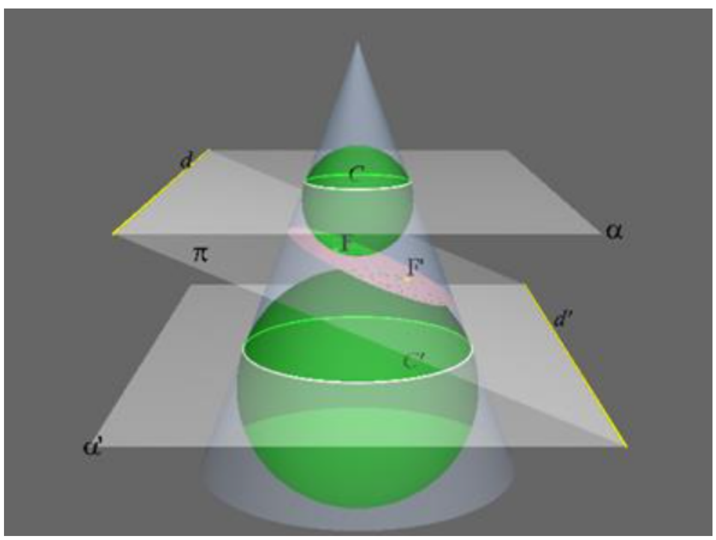

Figure [Math Processing Error]

The slanting plane in the figure cuts the cone in an ellipse. Two spheres inside the cone, having circles of contact with the cone [Math Processing Error], are adjusted in size so that they both just touch the plane, at points [Math Processing Error] respectively.

It is easy to see that such spheres exist, for example start with a tiny sphere inside the cone near the point, and gradually inflate it, keeping it spherical and touching the cone, until it touches the plane. Now consider a point [Math Processing Error] on the ellipse. Draw two lines: one from [Math Processing Error] to the point [Math Processing Error] where the small sphere touches, the other up the cone, aiming for the vertex, but stopping at the point of intersection with the circle [Math Processing Error]. *Both these lines are tangents to the small sphere, and so have the same length*. (The tangents to a sphere from a point outside it form a cone, they are all of equal length.) Now repeat with [Math Processing Error]. We find that [Math Processing Error], the distances to the circles measured along the line through the vertex. So [Math Processing Error] are therefore evidently the foci of the ellipse.