27.2: Angular Velocity and Energy in Terms of Euler's Angles

Since the position is uniquely defined by Euler's angles, angular velocity is expressible in terms of these angles and their derivatives.

The strategy here is to find the angular velocity components along the body axes x_1, x_2, x_3 of $\dot{\theta}, \dot{\phi}, \dot{\psi}$ in turn. Once we have the angular velocity components along the principal axes, the kinetic energy is easy.

Note

You might be thinking: wait a minute, aren't the axes embedded in the body? Don't they turn with it? How can you talk about rotation about these axes? Good point: what we're doing here is finding the components of angular velocity about a set of axes *fixed in space*, not the body, but *momentarily coinciding* with the principal axes of the body.

From the diagram, $\dot{\theta}$ is along the line ON, and therefore in the x_1, x_2 plane: notice it is at an angle $-\psi$ with respect to x_1. Its components are therefore $\vec{\dot{\theta}} = (\dot{\theta}\cos\psi, -\dot{\theta}\sin\psi, 0)$.

Now $\dot{\phi}$ is about the Z axis. The principal axis x_3 is at angle θ to the Z axis, so $\vec{\dot{\phi}}$ has component $\dot{\phi}\cos\theta$ about x_3, and $\dot{\phi}\sin\theta$ in the x_1, x_2 plane, that latter component along a line perpendicular to ON, and therefore at angle $-\psi$ from the x_2 axis. Hence $\vec{\dot{\phi}} = (\dot{\phi}\sin\theta\sin\psi, \dot{\phi}\sin\theta\cos\psi, \dot{\phi}\cos\theta)$

The angular velocity $\dot{\psi}$ is already along a principal axis, x_3.

To summarize, the Euler angle angular velocities (components along the body's principal axes) are:

$$\begin{aligned} \vec{\dot{\theta}} &= (\dot{\theta}\cos\psi, -\dot{\theta}\sin\psi, 0) \\ \vec{\dot{\phi}} &= (\dot{\phi}\sin\theta\sin\psi, \dot{\phi}\sin\theta\cos\psi, \dot{\phi}\cos\theta) \\ \vec{\dot{\psi}} &= (0, 0, \dot{\psi}) \end{aligned} \tag{27.2.1}$$

from which, the angular velocity components along those in-body axes x_1, x_2, x_3 are:

$$\begin{aligned} \Omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \Omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \Omega_3 &= \dot{\phi}\cos\theta + \dot{\psi} \end{aligned}$$

For a symmetric top, meaning $I_1 = I_2 \neq I_3$, the rotational kinetic energy is therefore

$$T_{\text{rot}} = \frac{1}{2} I_1 \left(\Omega_1^2 + \Omega_2^2\right) + \frac{1}{2} I_3 \Omega_3^2 = \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2$$

For this symmetrical case, as Landau points out, we could have taken the x_1 axis momentarily along the line of nodes ON, giving

$$\vec{\Omega} = (\dot{\theta}, \dot{\phi}\sin\theta, \dot{\phi}\cos\theta + \dot{\psi})$$