24.4: The Inertia Tensor

Regarding a rigid body as a system of individual particles, we find the kinetic energy

$$\begin{aligned} T=\sum_n \frac{1}{2} m_n v_n^2=\sum_n \frac{1}{2} m_n\left(\vec{V}+\vec{\Omega} \times \vec{r}_n\right) \\ =\sum_n \frac{1}{2} m_n V^2+\sum_n m_n \vec{V} \cdot \vec{\Omega} \times \vec{r}_n+\sum_n \frac{1}{2} m_n\left(\vec{\Omega} \times \vec{r}_n\right)^2 \end{aligned} \tag{24.4.1}$$

The first term in the last line is

$$\sum_n \frac{1}{2} m_n V^2=\frac{1}{2} M V^2 \tag{24.4.2}$$

where M is the total mass of the body.

The second term is

$$\sum_n m_n \vec{V} \cdot \vec{\Omega} \times \vec{r}_n=\vec{V} \cdot \vec{\Omega} \times \sum_n m_n \vec{r}_n=0 \tag{24.4.3}$$

from the definition of the center of mass (our origin here) $\sum_n m_n \vec{r}_n=0$

The third term can be rewritten:

$$\sum_n \frac{1}{2} m_n\left(\vec{\Omega} \times \vec{r}_n\right)^2=\sum_n \frac{1}{2} m_n\left[\Omega^2 r_n^2-\left(\vec{\Omega} \cdot \vec{r}_n\right)^2\right] \tag{24.4.4}$$

Here we have used

$$|\vec{\Omega} \times \vec{r}|=\Omega r \sin \theta, \quad|\vec{\Omega} \cdot \vec{r}|=\Omega r \cos \theta \tag{24.4.5}$$

Alternatively, you could use the vector product identity

$$(\vec{a} \times \vec{b}) \times \vec{c}=-\vec{a}(\vec{b} \cdot \vec{c})+\vec{b}(\vec{a} \cdot \vec{c}) \tag{24.4.6}$$

together with

$$(\vec{a} \times \vec{b}) \cdot(\vec{c} \times \vec{d})=(\vec{a} \times \vec{b}) \times \vec{c} \cdot \vec{d} \tag{24.4.7}$$

to find

$$(\vec{a} \times \vec{b}) \cdot(\vec{c} \times \vec{d})=(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d})-(\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \tag{24.4.8}$$

The bottom line is that the kinetic energy

$$T=\frac{1}{2} M V^2+\sum_n \frac{1}{2} m_n\left[\Omega^2 r_n^2-\left(\vec{\Omega} \cdot \vec{r}_n\right)^2\right]=T_{\mathrm{tr}}+T_{\mathrm{rot}} \tag{24.4.9}$$

a translational kinetic energy plus a rotational kinetic energy.

Warning about notation: at this point, things get a bit messy. The reason is that to make further progress in dealing with the rotational kinetic energy, we need to write it in terms of the individual components of the n particle position vectors $\vec{r}_n$. Following Landau and others, we'll write these components in two different ways:

$$\vec{r}_n=\left(x_n, y_n, z_n\right) \equiv\left(x_{n 1}, x_{n 2}, x_{n 3}\right) \tag{24.4.10}$$

The x,y,z notation is helpful in giving a clearer picture of rotational energy, but the x_{ni} notation is essential in handling the math, as will become evident.

Landau's solution to the too many suffixes for clarity problem is to omit the suffix n labeling the individual particles, I prefer to keep it in.

Double Suffix Summation Notation: to cut down on the number of Σ's in expressions, we'll follow Landau and others in using Einstein's rule that if a suffix like i, j, k appears twice in a product, it is to be summed over the values 1,2,3. It's called a "*dummy* *suffix*" because it doesn't matter what you label it, as long as it appears twice. For example,

the inner product of two vectors $\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i$ can be written as $A_i B_i$ or equally as $A_k B_k$. Furthermore, Ω_i^2 means $\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = \Omega^2$.

But do not use Greek letters for dummy suffixes in this context: the standard is that they are used in relativistic equations to signify sums over the four dimensions of space time, Latin letters for sums over the three spatial dimensions, as we are doing here.

The rotational kinetic energy is then

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2} \sum_n m_n \left(\Omega_i^2 x_{ni}^2 - \Omega_i x_{ni} \Omega_k x_{nk} \right) \\ &= \frac{1}{2} \sum_n m_n \left(\Omega_i \Omega_k \delta_{ik} x_{nl}^2 - \Omega_i \Omega_k x_{ni} x_{nk} \right) \\ &= \frac{1}{2} \Omega_i \Omega_k \sum_n m_n \left(\delta_{ik} x_{nl}^2 - x_{ni} x_{nk} \right) \end{aligned} \tag{24.4.11}$$

Warning: That first line is a bit confusing: copying Landau, I've written $\Omega_i^2 x_{ni}^2$, you might think that's $\Omega_1^2 x_{n1}^2 + \Omega_2^2 x_{n2}^2 + \Omega_3^2 x_{n3}^2$, but a glance at the previous equation (and the second line of this equation) makes clear it's actually $\Omega^2 r^2$. Landau should have written $\Omega_i^2 x_{nl}^2$. Actually I'm not even keen on Ω_i^2 implying a double summation. Standard use in relativity, for example, is that both of the two suffixes be explicit for summation to be implied. In GR one would write $\Omega_i \Omega_i$. (Well, actually $\Omega_i \Omega^i$, but that's another story.)

Anyway, moving on, we introduce the *inertia tensor*

$$I_{ik} = \sum_n m_n \left(x_{nl}^2 \delta_{ik} - x_{ni} x_{nk} \right) \tag{24.4.12}$$

In terms of which the kinetic energy of the moving, rotating rigid body is

$$T = \frac{1}{2} M V^2 + \frac{1}{2} I_{ik} \Omega_i \Omega_k \tag{24.4.13}$$

As usual, the Lagrangian $L = T - V$ where the potential energy V is a function of six variables in general, the center of mass location and the orientation of the body relative to the center of mass.

Landau writes the inertia tensor explicitly as:

$$I_{ik} = \begin{bmatrix} \sum m \left(y^2 + z^2 \right) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m \left(z^2 + x^2 \right) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m \left(x^2 + y^2 \right) \end{bmatrix} \tag{24.4.14}$$

but you should bear in mind that $-\sum mxz$ means $-\sum_n m_n x_n z_n$.