10.5: Poisson Brackets under Canonical Transformations

First, note that if Hamilton's equations have the standard canonical form

[Math Processing Error]

with respect to a pair of variables *[Math Processing Error]*

then those variables are said to be *canonically conjugate*.

The Poisson bracket is invariant under a canonical transformation, meaning

[Math Processing Error]

Let's begin by establishing that

[Math Processing Error]

We'll show the method by taking just one pair of variables p,q, and a generating function *[Math Processing Error]*

Then

[Math Processing Error]

With the generating function *[Math Processing Error]*

[Math Processing Error]

and

[Math Processing Error]

Putting these results into the Poisson bracket,

[Math Processing Error]

These basic results can then be used to prove the general Poisson bracket is independent of the parametrization of phase space, details in Goldstein and elsewhere.

Landau, on the other hand, offers a one-line proof of the invariance of the Poisson bracket of two dynamical functions *[Math Processing Error]* under a canonical transformation: imagine a fictitious system having g as its Hamiltonian. Then *[Math Processing Error]* and cannot depend on the coordinate system used, so must equal *[Math Processing Error]*.