24.5: Tensors 101

We see that the "inertia tensor" defined above as

$$I_{ik} = \sum_n m_n \left(x_{nl}^2 \delta_{ik} - x_{ni} x_{nk}\right) \tag{24.5.1}$$

is a 3×3 two-dimensional array of terms, called *components*, each of which is made up (for this particular tensor) of products of vector components.

Obviously, if we had chosen a different set of Cartesian axes from the same origin O the vector components would be different: we know how a vector transforms under such a change of axes, $(x, y, z) \rightarrow (x', y', z')$ where

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{24.5.2}$$

This can be written more succinctly as

$$x_i' = R_{ij} x_j, \text{ or } \mathbf{x}' = \mathbf{R}\mathbf{x} \tag{24.5.3}$$

the bold font indicating a vector or matrix.

In fact, a transformation from any set of Cartesian axes to any other set having the same origin is a rotation about some axis. This can easily be seen by first rotating so that the x' axis coincides with the x axis, then rotating about *that* axis. (Of course, both sets of axes must have the same handedness.) We'll discuss these rotation transformations in more detail later, for now we'll just mention that the inverse of a rotation is given by the transpose matrix (check for the example above),

$$\mathbf{R}^{\mathbf{T}} = \mathbf{R}^{-1}, \quad \text{ or } \quad R_{ji} = R_{ij}^{-1} \tag{24.5.4}$$

so if the column vector

$$x_i' = R_{ij} x_j, \text{ or } \mathbf{x}' = \mathbf{R}\mathbf{x} \tag{24.5.5}$$

the row vector

$$\mathbf{x}'^{\mathbf{T}} = \mathbf{x}^{\mathrm{T}} \mathbf{R}^{\mathbf{T}} = \mathbf{x}^{\mathbf{T}} \mathbf{R}^{-1} \tag{24.5.6}$$

a.k.a. $x_i' = R_{ij} x_j = x_j R_{ji}^T = x_j R_{ji}^{-1}$, and the *length* of the vector doesn't change:

$x_i' x_i' = \mathbf{x}'^{\mathrm{T}} \mathbf{x}' = \mathbf{x}^{\mathrm{T}} \mathbf{R}^{\mathrm{T}} \mathbf{R} \mathbf{x} = \mathbf{x}^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{R} \mathbf{x} = \mathbf{x}^{\mathrm{T}} \mathbf{x} = x_i x_i$

It might be worth spelling out explicitly here that the transpose of a square matrix (and almost all our matrices are square) is found by just swapping the rows and columns, or equivalently swapping elements which are the reflections of each other in the main diagonal, but the transpose of a vector, written as a column, has the same elements as a row, and the *product of vectors follows the standard rules for matrix multiplication*:

$$(AB)_{ij} = A_{ik} B_{kj} \tag{24.5.7}$$

with the dummy suffix k summed over.

Thus,

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \tag{24.5.8}$$

and

$$\mathbf{a}^{\mathbf{T}} \mathbf{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1^2 + a_2^2 + a_3^2 \tag{24.5.9}$$

but

$$\mathbf{a}\mathbf{a}^{\mathbf{T}}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}=\begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{pmatrix} \tag{24.5.10}$$

This will perhaps remind you of the Hilbert space vectors in quantum mechanics: the transposed vector above is analogous to the bra, the initial column vector being the ket. One difference from quantum mechanics is that all our vectors here are real, if that were not the case it would be natural to add complex conjugation to the transposition, to give $\mathbf{a}^{*}\mathbf{a}=|a_1|^2+|a_2|^2+|a_3|^2$, the length squared of the vector.

The difference shown above between $\mathbf{a}^{\mathbf{T}}\mathbf{a}$ and $\mathbf{a}\mathbf{a}^{\mathbf{T}}$ is exactly parallel to the difference between $\langle a \mid a\rangle$ and $|a\rangle\langle a|$ in quantum mechanics—the first is a number, the norm of the vector, the second is an operator, a projection into the state $|a\rangle$