26.4: Equations of Motion for Rigid Body with External Forces

Translation

A free rigid body has six degrees of freedom (for instance, the coordinates of the center of mass and the orientation of the body). Therefore, there are six equations of motion, three for the rate of change of spatial position of the center of mass, in other words for the components of the velocity $\vec{V}$, and three for the rate of change of orientation, the angular velocity $\vec{\Omega}$.

These equations are of course nothing but Newton's laws, easily derived by summing over the set of equations $\vec{f}_i = d\left(m_i\vec{v}_i\right)/dt$ for each particle.

Denoting the total momentum of the body by $\vec{P}$.

$$\sum_n \frac{d}{dt}\left(m_n\vec{v}_n\right) = \frac{d\vec{P}}{dt} = \sum_n \vec{f}_n = \vec{F} \tag{26.4.1}$$

and $\vec{P} = M\vec{V}$, where $\vec{V} = d\vec{R}/dt$ is the velocity of the center of mass. (This can be established by differentiating with respect to time the definition of the center of mass, $M\vec{R} = \sum_n m_n\vec{r}_n$.)

The total force on all the particles is a sum of the total external force on the body and the sum of internal forces between particles—but these internal forces come in equal and opposite pairs, from Newton's Third Law, and therefore add to zero.

The bottom line, then, is that the rate of change of momentum of a rigid body equals the total external force on the body. If this force is from a time-independent potential, then

$$\vec{F} = -\partial V/\partial\vec{R} \tag{26.4.2}$$

because if the body is moved through $\delta\vec{R}$ (without rotation, hence the *partial* derivative), each individual particle moves through the same $\delta\vec{R}$, the work done by the external potential on the n^{th} particle is $\vec{f}_n^{\text{ext}}\cdot\delta\vec{R} = -\delta V_n$, and summing over all the particles gives $\vec{F}\cdot\delta\vec{R} = -\delta V_{\text{tot}}$, giving the above equation as $\delta\vec{R}\to 0$.

Rotation

To derive the equation of motion for *rotation* of a rigid body, *we choose the inertial frame in which the center of mass is momentarily at rest*, and take the center of mass as the origin.

The rate of change of angular momentum about the center of mass (origin),

$$\vec{L} = (d/dt)\sum_n \vec{r}_n\times\vec{p}_n = \sum_n\left[\left(\dot{\vec{r}}_n\times\vec{p}_n\right) + \left(\vec{r}_n\times\vec{\dot{p}}_n\right)\right] = \sum_n \vec{r}_n\times\vec{f}_n = \vec{K} \tag{26.4.3}$$

where we dropped the $\dot{\vec{r}}_n\times\vec{p}_n$ term because $\dot{\vec{r}}_n = \vec{v}_n$ is parallel to $\vec{p}_n = m\vec{v}_n$, then we used $\vec{f}_n = \dot{\vec{p}}_n$ to get the total moment of the external forces about the center of mass, the *torque*.

The angular momentum *about the center of mass* is the same in any inertial frame, since the extra term on adding a velocity $\vec{v}_0$ to each mass is

$$\sum\vec{r}_n\times m_n\vec{v}_0 = -\vec{v}_0\times\sum m_n\vec{r}_n = 0 \tag{26.4.4}$$

from the definition of the center of mass.

If the center of mass is *not* at the origin, denote the particle coordinates by $\vec{\rho}_n = \vec{R}+\vec{r}_n$ in the usual notation, so

$$\vec{L}_{\text{new origin}} = \sum_n\vec{\rho}_n\times m_n\vec{v}_n = \sum_n\vec{r}_n\times m_n\vec{v}_n + \sum_n\vec{R}\times m_n\vec{v}_n = \vec{L}_{\text{cm}} + \vec{R}\times\vec{P} \tag{26.4.5}$$

a sum of an intrinsic ("spin") angular momentum and an extrinsic ("orbital")angular momentum.

Similarly, if the torque of external forces relative to the center of mass is $\vec{K} = \sum_n\vec{r}_n\times\vec{f}_n$ as defined above, then relative to the new origin the torque is

$$\vec{K}_{\text{new origin}} = \sum_n \vec{\rho}_n \times \vec{f}_n = \sum_n \vec{R} \times \vec{f}_n + \sum_n \vec{r}_n \times \vec{f}_n = \vec{R} \times \vec{F} + \vec{K}_{\text{cm}} \tag{26.4.6}$$

that is, the torque about the new origin is the torque about the center of mass *plus* the torque about the new origin of the total external force acting at the center of mass.

An important special case is that of a *couple*: a pair of equal but oppositely directed forces, acting along parallel but separated lines (like two hands oppositely placed turning a steering wheel). The forces add to zero, so from the above equation a couple exerts the same torque about any origin.

More generally, the term couple is often used (including by Landau) to refer to any set of forces that add to zero, but give a nonzero torque because of their lines of action, and such a set give the same torque about any origin.

Exercise: prove that for a rigid body freely falling in a uniform gravitational field, the angular momentum about the center of mass remains constant, but about another point it will in general be changing. What about a charged rigid body moving in space (no gravity) through a uniform *electric* field?