9.3: The Abbreviated Action

Writing the action in the integral form along this constant energy path, we can trivially do the time integral:

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Therefore, from the result *[Math Processing Error]* it necessarily follows that

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[Math Processing Error] is called the *abbreviated action*: this is *Maupertuis' principle*.

The abbreviated action for the physical path is the minimum among all paths satisfying energy conservation with total energy E and passing through the designated final point—we don't care when. Note that not all values of E will work—for example, if we start putting the ball from a low point in the green, we'll need to give it enough energy to get out of the hollow to begin with. But there will be valid physical paths for a wide range of energy values, since the final arrival time is flexible.

Naturally, since this is a path through configuration space, to evaluate the abbreviated action

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it must be expressed in terms of the q's. For the usual Lagrangian *[Math Processing Error]*, with *[Math Processing Error]*, and momenta

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we find the abbreviated action

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This is indeed an integral along a path in configuration space, but we need to get rid of the dt. Physically, we can see how to do this—since we know the total energy E, the kinetic energy at a point is *[Math Processing Error]* so that determines the speed, hence the time *[Math Processing Error]*.

That is, (following Landau)

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from which

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(This doesn't look like a very healthy mathematical object, but as you'll see, it's fine.)

Hence

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To take a very simple case; if there is no potential, and just a free particle, *[Math Processing Error]* this is nothing but the length of the path multiplied by *[Math Processing Error]*, minimized by a straight line between the two points.

If we have a particle of mass m in a spatially varying potential *[Math Processing Error]*, the abbreviated action reduces to

[Math Processing Error]

where *[Math Processing Error]* is an element of path length. (This is obvious, really—the square root is the absolute value of the momentum, and the momentum vector, of course, points along the path.)

The matrix *[Math Processing Error]* sometimes called the **mass matrix**, is evidently a metric, a measure in the configuration space, by which the "length" of the paths, and particularly the minimum action path, are measured.

Exercise *[Math Processing Error]*

Use Maupertuis' principle to find the path of a cannonball, energy E, fired at a target which is x meters distant horizontally, both cannon and target being at sea level (think ships!).