16.5: Vectorial Derivation of the Scattering Angle

The essential result of the above analysis was the scattering angle as a function of impact parameter, for a given incoming energy. It's worth noting that this can be found more directly by vectorial methods from Hamilton's equation.

Recall from the last lecture Hamilton's equation

$$\vec{L} \times m\ddot{\vec{r}} = -mr^2 f(r) \frac{d\hat{\vec{r}}}{dt} \tag{16.5.1}$$

and the integral for an inverse square force $f(r) = k/r^2$ (changing the sign of $\vec{A}$ for later convenience)

$$\vec{L} \times m\dot{\vec{r}} = km\hat{\vec{r}} + \vec{A} \tag{16.5.2}$$

As previously discussed, multiplying by $\vec{L}$. establishes that $\vec{A}$ is in the plane of the orbit, and multiplying by $\vec{r}$ gives

$$-L^2 = kmr + Ar\cos\theta \tag{16.5.3}$$

This corresponds to the equation

$$1/r = -e\cos\theta - 1 \tag{16.5.4}$$

(the left-hand branch with the right-hand focus as origin, note from diagram above that $\cos\theta$ is negative throughout) and

$$\frac{L^2}{kmr} = -1 - \frac{A}{km}\cos\theta \tag{16.5.5}$$

To find the scattering angle, suppose the unit vector pointing parallel to the asymptote is $\hat{\vec{r}}_\infty$, so the asymptotic velocity is $v_\infty \hat{\vec{r}}_\infty$.

Note that as before, $\vec{A}$ is along the major axis (to give the correct form for the (r,θ) equation), and $r = \infty$ gives the asymptotic angles from

$$\cos\theta_{r=\infty} = -km/A \tag{16.5.6}$$

We're not rotating the hyperbola as we did in the alternative treatment above: here we keep it symmetric about the x-axis, and find its asymptotic angle to that axis, which is one-half the scattering angle.

Now take Hamilton's equation in the asymptotic limit, where the velocity is parallel to the displacement:

the vector product of Hamilton's equation $\times\hat{\vec{r}}_\infty$ yields

$$\vec{A} \times \hat{\vec{r}}_\infty = \left(\vec{L} \times mv_\infty \hat{\vec{r}}_\infty\right) \times \hat{\vec{r}}_\infty = -\vec{L}(L/b) \tag{16.5.7}$$

It follows that

$$\sin\theta_{r=\infty} = -L^2/Ab \tag{16.5.8}$$

And together with $\cos\theta_{r=\infty} = -km/A$, we find

$$\tan\theta_{r=\infty} = \frac{L^2}{kmb} = \frac{mbv_\infty^2}{k} \tag{16.5.9}$$

This is the angle between the asymptote and the major axis: the scattering angle

$$\chi = \pi - 2\theta_{r=\infty} = 2\left(\frac{\pi}{2} - \theta_{r=\infty}\right) = 2\tan^{-1}\left(\frac{k}{mbv_\infty^2}\right) \tag{16.5.10}$$

agreeing with the previous result.