19.7: Finding the Eigenvalues

The eigenvalues are found by operating on the eigenvector we just found with the matrix, meaning the N dimensional generalization of

$$-m\Omega^2\begin{pmatrix}1\\ e^{ik_na}\\ e^{ik_n2a}\\ e^{ik_n3a}\end{pmatrix}=\begin{pmatrix}-2\kappa & \kappa & 0 & \kappa\\ \kappa & -2\kappa & \kappa & 0\\ 0 & \kappa & -2\kappa & \kappa\\ \kappa & 0 & \kappa & -2\kappa\end{pmatrix}\begin{pmatrix}1\\ e^{ik_na}\\ e^{ik_n2a}\\ e^{ik_n3a}\end{pmatrix} \tag{19.7.1}$$

Applying the matrix to the column vector

$$\left(1, e^{ik_na}, e^{2ik_na}, e^{3ik_na}, \ldots, e^{i(N-1)k_na}\right)^T \tag{19.7.2}$$

, and cancelling out the common e^{ik_nna} factor, we have

$$-m\Omega_n^2=\kappa\left(e^{ik_na}+e^{-ik_na}-2\right) \tag{19.7.3}$$

(Of course, this same result comes from every row.)

The complete set of eigenvalues is given by inserting in the above expression

$$k_n=2\pi n/Na,\quad n=0,1,2,\ldots,N-1 \text{ so } e^{ik_na}=e^{2\pi in/N} \tag{19.7.4}$$

so $n=0$ is displacement of the system as a whole, as is $n=N$.

Wavenumber values k_n beyond $n=N$ repeats the eigenstates we already have, since

$$e^{ik_{N+n}a}=e^{i\dfrac{2\pi(N+n)a}{Na}}=e^{2\pi i}e^{2\pi in/N}=e^{2\pi in/N}=e^{ik_na} \tag{19.7.5}$$

k are restricted to

$$\Omega_n=2\sqrt{\frac{\kappa}{m}}\sin\left(\frac{k_na}{2}\right)=2\sqrt{\frac{\kappa}{m}}\sin\left(\frac{n\pi}{N}\right) \tag{19.7.6}$$

$$0\le k<2\pi/a \tag{19.7.7}$$

or equivalently

$$-\pi/a<k\le\pi/a \tag{19.7.8}$$

The eigenvalue equation is

$$\Omega_n^2=2(\kappa/m)(1-\cos k_na) \tag{19.7.9}$$

or

$$\Omega_n=2\sqrt{\frac{\kappa}{m}}\sin\left(\frac{k_na}{2}\right)=2\sqrt{\frac{\kappa}{m}}\sin\left(\frac{n\pi}{N}\right) \tag{19.7.10}$$

To see the dynamics of this eigenstate

$$\left(1, e^{ik_na}, e^{2ik_na}, e^{3ik_na}, \ldots, e^{ik_n(N-1)a}\right) \tag{19.7.11}$$

, we need to multiply by the time dependence $e^{i\Omega_nt}$, then finally take the real part of the solution:

$$\left(\cos\Omega_nt,\quad \cos(k_na+\Omega_nt),\quad \cos(2k_na+\Omega_nt),\quad \cos(3k_na+\Omega_nt),\ldots,\cos((N-1)k_na+\Omega_nt)\right) \tag{19.7.12}$$

Notice that in the continuum limit, meaning large N and small a, the atom displacement as a function of position has the form $\cos(kx+\Omega t)$ in other words we're looking at a sinusoidal wave disturbance with wavenumber k_n here.

Now, $-k_n$ is also a solution, but that is the same as $n'=N-n$ so one must be careful not to overcount. The two frequencies $\pm\Omega_n$ correspond to waves going in opposite directions.