17.7: Three Equal Pendulums Equally Coupled

What if we had a third identical spring connecting the two end pendulums (we could have small rods extending down so the spring went below the middle pendulum?

What would the modes of oscillation look like in this case?

Obviously, all three swinging together is still an option, the eigenvector (1,1,1), corresponding to eigenvalue zero of the "interaction matrix" above. But actually we have to extend that matrix to include the new spring -- it's easy to check this gives the equation:

$$\begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0 \tag{17.7.1}$$

The equation for the eigenvalues is easily found to be $\lambda(\lambda-3)^2=0$. Putting $\lambda=3$ into the matrix yields the equation $A_1+A_2+A_3=0$ This is telling us that any vector perpendicular to the all-swinging-together vector (1,1,1) is an eigenvector. This is because the other two eigenvectors have the same eigenvalue, meaning that any linear combination of them also has that eigenvalue—this is a *degeneracy*.

Think about the physical situation. If we set the first two pendulums swinging exactly out of phase, the third pendulum will feel no net force, so will stay at rest. But we could equally choose another pair. And, the eigenvector (1,−2,1) we found before is still an eigenvector: it's in this degenerate subspace, equal to (1,−1,0)−(0,1,−1).