2.12: The Brachistochrone

Suppose you have two points, A and B, B is below A, but not directly below. You have some smooth, let's say frictionless, wire, and a bead that slides on the wire. The problem is to curve the wire from A down to B in such a way that the bead makes the trip as quickly as possible.

This optimal curve is called the "brachistochrone", which is just the Greek for "shortest time".

But what, exactly, is this curve, that is, what is $y(x)$ in the obvious notation?

This was the challenge problem posed by Johann Bernoulli to the mathematicians of Europe in a Journal run by Leibniz in June 1696. Isaac Newton was working fulltime running the Royal Mint, recoining England, and hanging counterfeiters. Nevertheless, ending a full day's work at 4 pm, and finding the problem delivered to him, he solved it by 4am the next morning, and sent the solution anonymously to Bernoulli. Bernoulli remarked of the anonymous solution "I recognize the lion by his clawmark".

This was the beginning of the Calculus of Variations.

Here's how to solve the problem: we'll take the starting point A to be the origin, and for convenience measure the y -axis positive *downwards*. This means the velocity at any point on the path is given by

$$\frac{1}{2} m v^2 = m g y, \quad v = \sqrt{2 g y} \tag{2.12.1}$$

So measuring length along the path as ds as usual, the time is given by

$$T = \int_A^B \frac{ds}{v} = \int_A^B \frac{ds}{\sqrt{2 g y}} = \int_0^X \frac{\sqrt{1 + y'^2}\, dx}{\sqrt{2 g y}} \tag{2.12.2}$$

Notice that this has the same form as the catenary equation, the only difference being that y is replaced by $1/\sqrt{2gy}$ the integrand does not depend on x, so we have the first integral:

$$y' \frac{\partial f}{\partial y'} - f = \text{ constant }, \quad f = \sqrt{\frac{1 + y'^2}{2 g y}} \tag{2.12.3}$$

That is,

$$\frac{y'^2}{\sqrt{\left(1 + y'^2\right) 2 g y}} - \sqrt{\frac{1 + y'^2}{2 g y}} = -\frac{1}{\sqrt{\left(1 + y'^2\right) 2 g y}} = \text{ constant} \tag{2.12.4}$$

so

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{2a}{y} \tag{2.12.5}$$

$2a$ being a constant of integration (the 2 proves convenient).

Recalling that the curve starts at the origin A, it must begin by going vertically downward, since $y = 0$. For small enough y, we can approximate by ignoring the 1, so $\sqrt{2a}dx \cong \sqrt{y}dy$, $\sqrt{2a}x \cong 2/_3 y^{3/2}$. The curve must however become horizontal if it gets as far down as $y = 2a$, and it cannot go below that level.

Rearranging in order to integrate,

$$dx = \frac{dy}{\sqrt{\frac{2a}{y} - 1}} = \sqrt{\frac{y}{2a - y}} dy \tag{2.12.6}$$

This is not a very appealing integrand. It looks a little nicer on writing $y = a - az$

$$dx = -a \sqrt{\frac{1 - z}{1 + z}} dz \tag{2.12.7}$$

Now what? We'd prefer for the expression inside the square root to be a perfect square, of course. You may remember from high school trig that $1+\cos\theta=2\cos^2(\theta/2), \quad 1-\cos\theta=2\sin^2(\theta/2)$. This gives immediately that

$$\frac{1-\cos\theta}{1+\cos\theta}=\tan^2\frac{\theta}{2} \tag{2.12.8}$$

so the substitution $z=\cos\theta$ is what we need.

Then $dz=-\sin\theta d\theta=-2\sin(\theta/2)\cos(\theta/2)d\theta$

$$dx=-a\tan\frac{\theta}{2}dz=2a\tan\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta=2a\sin^2\frac{\theta}{2}d\theta=a(1-\cos\theta)d\theta \tag{2.12.9}$$

This integrates to give

$$\begin{aligned} x&=a(\theta-\sin\theta)\\ y&=a(1-\cos\theta) \end{aligned} \tag{2.12.10}$$

where we've fixed the constant of integration so that the curve goes through the origin $(\text{ at }\theta=0)$

To see what this curve looks like, first ignore the θ terms in x, leaving $x=-a\sin\theta, y=-a\cos\theta$. Evidently as θ increases from zero, the point (x,y) goes anticlockwise around a circle of radius a centered at $(0,-a)$ that is, touching the x -axis at the origin.

Now adding the θ back in, this circular motion move steadily to the right, in such a way that the initial direction of the path is vertically down. $(\text{For very small }\theta, y\sim\theta^2\gg x\sim\theta^3)$

Visualizing the total motion as θ steadily increases, the center moves from its original position at $(0,-a)$ to the right at a speed $a\theta$. Meanwhile, the point is moving round the circle anticlockwise at this same speed. Putting together the center's linear velocity with the corresponding angular velocity, we see the motion $(x(\theta),y(\theta))$ is the path of a point on the rim of a wheel rolling without sliding along a road (upside down in our case, of course). This is a *cycloid*.