19.3: Comparison with Raising Operators

Actually these matrices are related to the raising and lowering operator for angular momentum (and simple harmonic oscillators) in quantum mechanics. For example, the 4×4 matrices would be for a spin 3/2, with four S_z eigenstates.

The quantum mechanical raising and lowering matrices look like

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{19.3.1}$$

They move the spin S_z component up (and down) by one notch, except that on applying the raising operator, the top state $S_z = 3/2$ is annihilated, similarly the lowering operator on the bottom state.

Our circular generalizations have one extra element:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{19.3.2}$$

This makes the matrices circulants, and gives them a “recycling” property: the top element isn’t thrown away, it just goes to the bottom of the pile.

(And bear in mind that the standard notation for a vector has the lowest index (0 or 1) for the top element, so when we bend the ladder into a circle, the “raising” operator actually moves to the next *lower* number, in other words, it’s a shift to the *left*.)

We’ll take this shift operator P as our basic matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ from which } P^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{19.3.3}$$

It should be evident from this that the circulant matrix having top row c_0, c_1, c_2, c_3 is just the matrix $c_0 I + c_1 P + c_2 P^2 + c_3 P^3$.

This generalizes trivially to $N \times N$ matrices.