

11.3: Quantum Effects Near Black Holes

Learning Objectives

- You will be able to calculate the temperatures of black holes.
- You will understand that black holes can evaporate, and that their rate of evaporation depends strongly on their mass.
- You will understand why particle accelerators are not going to devour the world by producing mini black holes.
- You will understand that general relativity does not completely describe what ultimately happens to material that falls into a black hole at the singularity because it is not a quantum theory.
- You will understand that a mathematical singularity is a term that goes to infinity, but that the singularity of a black hole would not be infinitely dense in real life.
- You will understand that a quantum theory of gravity has not yet been successfully developed, but that we have an idea of the length and time scales where it would be applicable.

An important aspect of black holes that lies outside the scope of general relativity is their quantum nature. General relativity is not a quantum theory of gravity, and so our understanding of how black holes will behave on extremely small scales is still far from complete.

11.3.1: The Temperature of Black Holes

One of the first attempts to understand the quantum aspects of black holes was undertaken by Stephen Hawking (b. January 8, 1942, d. March 14, 2018) in the early 1970s. Hawking was studying the thermodynamics of black holes. He found that black holes have a temperature given by:

$$T_{bh} = \frac{hc^3}{16\pi^2 GM k_B} \quad (11.3.1)$$

where h is Planck's constant ($h = 6.626 \times 10^{-34}$ J s), and k_B is known as Boltzmann's constant ($k_B = 1.38 \times 10^{-23}$ J/K). The other symbols have their usual meaning: c is the speed of light, G is the gravitational constant, and M is the mass of the black hole. If SI units are used for these quantities, then the temperature is in kelvin (K). If we combine all of the constants into a single number, Equation 11.3.1 becomes:

$$T_{bh} = \frac{1.23 \times 10^{23}}{M}$$

where the temperature of the black hole (T_{bh}) is in kelvin, its mass (M) is in kg, and the constant in the numerator has units kg K. From the expression, we see that the temperature of a black hole is inversely proportional to its mass: the greater the mass, the lower the temperature of the black hole. In the next activity, we will explore this relationship for black holes with different masses.

✓ Black Hole Mass and Temperature

Worked Example:

Find the temperature of a one solar mass black hole.

- Given: A black hole with mass, $M = 2 \times 10^{30}$ kg (one solar mass)
- Find: T , the temperature of this black hole in Kelvin (K)
- Concept(s):

$$T_{bh} = \frac{1.23 \times 10^{23} \text{ kg K}}{M}$$

- Solution:

$$\begin{aligned} T &= \frac{1.23 \times 10^{23} \text{ kg K}}{2 \times 10^{30} \text{ kg}} \\ &= 6.1 \times 10^{-8} \text{ K} \end{aligned}$$

This is a tiny temperature. The average temperature in the Universe is about 3 K, so a one solar mass black hole has a temperature even colder, by a factor of about 50 million, than the average temperature of the Universe.

Questions:

1.

2.

We discussed earlier that a black hole is an object where the spacetime curvature is so extreme that not even light can escape. Yet, Hawking found that black holes emit radiation in a spectrum identical to a Planck emitter of the same temperature. So a black hole like the one whose mass you found in the previous activity (with a temperature of 6000 K) would emit light with a spectrum

identical to that of the Sun; recall that a Planck emitter (sometimes referred to as a blackbody) has a spectrum that depends only on its temperature. How can this be if black holes by definition do not allow light to escape? The answer to that question requires us to consider the quantum aspects of the problem.

11.3.2: Hawking Radiation and the Evaporation of Black Holes

Special relativity allows for the conversion of mass to energy and vice versa. So if we imagine an electron, say, and its antiparticle, a positron, coming together, we know that they will annihilate and form two photons with equal energy and opposite direction of travel. Each photon will have an energy γmc^2 , where m is the mass of the electron or positron. The two particles are identical except for their electric charge, which is negative for an electron and positive for a positron. The γ factor is the relativistic boosting factor that depends on a particle's speed. It is 1 if the particles are not moving, but larger than 1 if they are moving. We will assume for simplicity that the particles approach each other with equal speeds: we can always consider the collision from a frame of reference where this is true. Our frame of reference cannot affect the outcome of the collision.

In this interaction there are several quantities that must be conserved, or in other words, there are several quantities that must be the same before the collision and after it. These include energy, momentum and electric charge. There are some others, too, but these three are the important ones to consider at the moment. Momentum is described in [Going Further 7.1: Newton's Third Law](#). If you have not read it, now would be a good time to do so.

Because each photon has the energy of one of the initial particles, we know that energy will be conserved. Similarly, because photons have no electric charge, we can see that total charge will also be conserved. Before the collision, the total charge (we call the charge of an electron “ e ”) was: $e - e = 0$. Afterward it is still zero. The fact that the two photons move in opposite directions with equal energy means that momentum is also conserved. Recall that momentum is a vector, so it has strength and direction. The strength of the momentum of a photon is E/c , its energy divided by the speed of light. The direction of each photon is opposite the other, both before and after the collision. Beforehand they approach one another. Afterward they move away. In both cases the particle momenta add to zero.

Of course, we could also imagine the opposite process, where two photons of sufficient energy collide and are converted into a positron and electron pair. That process would be just like the first if we imagined filming it and running the film backwards. The reverse process would also satisfy all known laws of physics. Furthermore, it is observed to happen.

Quantum mechanics adds an interesting twist to this scenario by relaxing the requirement that energy be strictly conserved. It does not completely do away with conservation of energy: clearly, it is not possible to create something from nothing, at least not as we observe in the laboratory and our everyday existence. But it is possible to cheat a bit, and create something out of nothing for a very short time. This absurd sounding statement is one possible phrasing of the Heisenberg uncertainty principle, first discovered by the German physicist Werner Heisenberg (1901–1976). It can be stated more succinctly with mathematics.

$$\Delta E \Delta t \geq \frac{h}{4\pi}$$

In this expression, h is Planck's constant. Planck's constant is always present when quantum considerations are important. The ΔE on the left side of the equation is the uncertainty in the energy for a particular process, and Δt is the uncertainty in the time required for the process to complete. There are always uncertainties associated with any measurement, but the uncertainty principle is more fundamental than that. It states that no matter how carefully you build your measuring device, no matter how great its precision, there will always be an inherent uncertainty imposed by nature itself. That uncertainty is quantified by the uncertainty principle.

Notice that the quantum uncertainty is very small. Planck's constant is only about 10^{-34} in SI units. While this is not zero, it is too small to be noticed in day-to-day life. Notice further that it is the *product* of time and energy uncertainty that is constrained. So the uncertainty in one can be made arbitrarily small, but only at the cost of making the uncertainty in the other arbitrarily large. That is the important consideration for the radiation emitted from black holes.

✓ The Uncertainty Principle

Imagine a volume of empty space. The volume contains no light, no particles, nothing. According to the uncertainty principle, if we watch that empty volume of space, we should from time to time see a positron–electron pair (for example) spontaneously pop into being, briefly exist, and then annihilate and disappear. This happens in empty vacuum and is allowed as long as the

pair does not exist for longer than the limit set by the uncertainty principle. So just how long is that? We can answer that using the uncertainty principle.

Worked Example:

1. How long can an electron–positron pair spontaneously pop into being and exist in a vacuum? The mass of each is 9×10^{-31} kg.

- Given: $m_e = 9 \times 10^{-31}$ kg .
- Find: The time limit set by the uncertainty principle in seconds, Δt
- Concept(s):

If we wish the uncertainty in the energy to be large enough to allow for the creation of an electron and positron, then the uncertainty in the time must be small enough such that the uncertainty principle is not violated. So we can use the uncertainty principle, with the equals sign as a limit on the range.

For the energy, we can use the familiar relationship between matter and energy, $E = mc^2$.

The uncertainty principle is: $(\Delta E)(\Delta t) = h/4\pi$, where $h = 6.626 \times 10^{-34}$ J s.

The energy of a positron–electron pair is: $\Delta E = 2m_e c^2$ where $c = 3 \times 10^8$ m/s

- Solution:

$$\begin{aligned}
 (\Delta t) &= \frac{h}{4\pi(\Delta E)} \\
 &= \frac{h}{4\pi(2m_e c^2)} \\
 &= \frac{h}{8\pi m_e c^2} \\
 &= \frac{6.626 \times 10^{-34} \text{ Js}}{(8)(\pi)(9.1 \times 10^{-31} \text{ kg})(3 \times 10^8 \text{ m/s})^2} \\
 &= 3.2 \times 10^{-22} \text{ s}
 \end{aligned}$$

Questions

1.

2.

In this activity we have found an *upper limit* on the time that particles that have been spontaneously created out of empty space can exist: if the energy is large enough to create an electron and positron, for example, then the uncertainty in the time must be large enough that the product $\Delta E \Delta t$ remains no smaller than $\hbar/4\pi$. That means if the particles exist for less time than this uncertainty, there is no violation of the uncertainty principle. An electron and positron of typical mass must annihilate within about 10^{-22} seconds to satisfy the uncertainty principle. You should have found that more massive particles, such as a proton—anti-proton pair would exist for less time. Such particles are not *real* in the sense that they cannot exist indefinitely, but only for a time so brief that

they cannot even be measured individually. To distinguish them from real (long lasting) particles, physicists call these sorts of particles **virtual particles**.

Of course, the uncertainty principle is a theoretical prediction, and this has all been a bit of mathematics. How do we know that the spontaneous creation/annihilation process really happens, and that virtual particles actually exist at all? See [Going Further 11.1: Testing the Uncertainty Principle](#), to learn why physicists think that the uncertainty principle gives a correct description of virtual particles, and that even they, in some sense, are “real.”

Going Further 11.1: Testing the Uncertainty Principle

The Heisenberg uncertainty principle is one of the more bizarre aspects of quantum mechanics, a fairly non-intuitive description of the world to begin with. Why should anyone believe any of the strange predictions of the theory? Well, nobody should believe them; scientists are forced to accept the strangeness of the theory because its predictions are borne out by experiment. The uncertainty principle is basic to quantum mechanics and affects all aspects of the theory, but it has two consequences that are particularly useful in demonstrating that the theory has merit.

One consequence of the uncertainty principle is the Lamb shift, a tiny shift in the energy level of the electron in a hydrogen atom. The shift is caused by the sort of quantum vacuum fluctuations associated with the production of virtual particle–antiparticle pairs as discussed for electrons and positrons. Even though any particle–antiparticle pair exists for a time that is too short to measure directly, new pairs are popping up (and then disappearing) all the time. These fluctuations of the vacuum perturb the electrons in an atom, gently shifting its energy—in effect, the hydrogen atom constantly absorbs and then re-emits virtual photons.

Because the effect is averaged over many virtual particle pairs constantly appearing and disappearing, the energy shift looks like a slight broadening of the energy of an atomic orbital. Using the quantum theory of the electromagnetic interaction, called **quantum electrodynamics (QED)**, it is possible to calculate the size of this broadening. In a hydrogen atom, the shift slightly separates the orbitals known as $^2S_{1/2}$ and $^2P_{1/2}$, which would otherwise have the same energy. The splitting was actually measured in 1947 before the theory of QED was fully developed. However, once the theory was available, its prediction for the splitting matched the observed amount to one part in a trillion. Or in other words, the theory matched the measurement to the eleventh decimal place, which is the limit of experimental precision. Such spectacular agreement between theory and experiment suggests that the notion of virtual particles is a valid description of nature, no matter the philosophical discomfort it might cause.

An additional verification of the quantum view of nature is the Casimir effect. In this experiment, two grounded conducting plates are placed very close together, in effect creating a region of space where only certain energy states of particles, including virtual particles, can exist. The reason is that quantum mechanics predicts that particles should have a wave-like nature in addition to their particle nature. The wavelength of a particle depends on its energy. Within an enclosed space, an even number of half-wavelengths must be able to fit. This is a bit like the vibrating string on a violin or guitar, which is fixed at both ends, as shown in Figure B.11.1. The restriction on wavelengths is a restriction on energy states of any particles within the space between the plates. Since any particle energies are possible outside the two plates, the exclusion of certain states between them causes a pressure difference between the inside and outside of the plates. As a result they are pushed together. This has been shown to happen in accord with the predictions of quantum electrodynamics, just as the Lamb shift does.

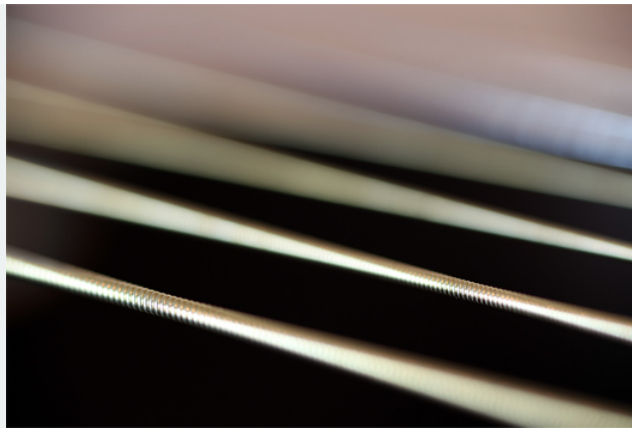


Figure B.11.1. The wave-like nature of particles is analogous to waves on a stringed instrument. Just as the strings on the guitar pictured here can only vibrate with certain frequencies because the strings are tied down at two ends, only certain particle energies are possible in the Casimir effect, where the particles are constrained between two plates. Credit: Shutterstock

11.3.2.1: Heisenberg Uncertainty and Hawking Radiation

Stephen Hawking used this strange quantum uncertainty effect to explore the radiation emitted by black holes. He imagined what would happen if a particle–antiparticle pair was created near the event horizon of a black hole, as in Figure 11.9. Under these circumstances it would be possible for one of the particles to cross the horizon, while the other did not. In this case, the two could not recombine to annihilate. The one particle would have been lost inside the black hole. Its partner would have to become a real particle. But that would require the energy to create it, which would have to come from somewhere. The only source of that energy is the mass–energy of the black hole itself. So these particle pairs provide a way for black holes to evaporate. The process is quite slow for a large black hole.

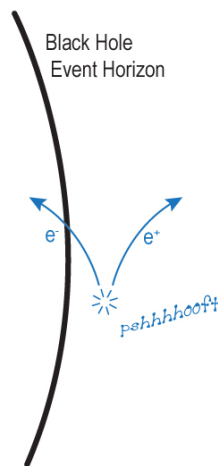


Figure 11.9: Hawking radiation. A virtual particle pair is formed near the event horizon of a black hole. One of the particles crosses the horizon and cannot get back out to annihilate with its partner. The partner must therefore become a real particle, stealing the energy required to do so from the only source available, the black hole itself. The figure shows an electron–positron pair, but any pair of antiparticles will work. Credit: NASA/SSU/Aurore Simonnet

To estimate the time required for a black hole to evaporate, we can use the temperature expression for a black hole and several pieces of information that you have learned previously:

- The relationship between the temperature (T) and flux (F) of an object emitting energy with a Planck spectrum

$$F = \sigma T^4$$

- The relationship between flux (F), luminosity (L), and distance (r)

$$L = (F)(4\pi r^2)$$

- The relationship between mass (m) and energy (E)

$$E = mc^2$$

- The relationship between energy (E), luminosity (L), and lifetime (t)

$$t = \frac{E}{L}$$

To combine these ideas, we use the mass of the black hole (M), the temperature of the black hole (T_{bh}), and the Schwarzschild radius (R_s) as the distance. We find that the thermal luminosity of a black hole is:

$$L_{th} = \frac{3.56 \times 10^{32}}{M^2}$$

where the thermal luminosity L has units of W, the mass M has units of kg, and the constant has units of W kg^2 .

The lifetime of a black hole is given by:

$$t \approx 2.5 \times 10^{-16} M^3$$

where the lifetime t has units of seconds, the mass M has units of kg, and the constant has units of s/kg^3 .

The thermal luminosity of a black hole is inversely proportional to mass squared. The time for a black hole to evaporate is proportional to mass cubed. This means that the *larger* the mass of a black hole, the *lower* its thermal luminosity and the *greater* its lifetime.

If you are interested in a mathematical derivation of these relationships where we have included each of the steps, see [Going Further 11.2: Deriving the Thermal Luminosity and Lifetime of a Black Hole](#).

You might object that we cannot simply divide the rest energy of the black hole by the thermal luminosity because both change as the black hole evaporates and loses mass. You would be right to make this objection, however, a more careful mathematical treatment that takes these changes into account shows that we have overestimated the lifetime by only a factor of 3. We will ignore this factor for our purposes of illustration, but you can keep it in mind if you like. The next activity will show why we are not going to concern ourselves with it.

Going Further 11.2: Deriving the Thermal Luminosity and Lifetime of a Black Hole

Recall that a black hole is a Planck emitter (or blackbody). The amount of energy lost by a Planck emitter per unit area per unit time (the energy flux, F) is given by the Stefan-Boltzmann equation:

$$F = \sigma T^4$$

where the constant $\sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$ and is needed to account for the system of units we use.

The relationship between flux and luminosity is: $F = L/(4\pi r^2)$. This means that the total amount of energy emitted by the black hole, its luminosity, L , is this energy flux multiplied by the total area of the event horizon, essentially the area of the black hole, A_{BH} . Since a black hole is spherical, we can use the area of a sphere: $A = 4\pi R^2$.

$$L = A_{BH} \sigma T^4 = 4\pi R_{Sch}^2 \sigma T^4$$

We can now substitute into this expression for the Schwarzschild radius, the appropriate radius for the event horizon of a black hole, and the temperature of the black hole to find the black hole thermal luminosity:

$$L_{th} = 4\pi \sigma \left(\frac{2GM}{c^2} \right)^2 \left(\frac{hc^3}{16\pi^2 GM k_B} \right)^4$$

Or simplifying:

$$L_{th} = \frac{\sigma h^4 c^8}{2^{12} \pi^7 G^2 k_B^4 M^2}$$

This expression looks complicated, but it is mostly just a bunch of physical constants, or in other words, a product of a bunch of numbers. That means it will simplify to a single number divided by the square of the black hole mass. In SI units, it becomes:

$$L_{th} = \frac{3.56 \times 10^{32}}{M^2}$$

where the thermal luminosity L is in W, the mass M is in kg, and the constant is in W kg^2 .

We have found the thermal luminosity of a black hole, the amount of energy lost to the black hole per unit time due to Hawking radiation. Now we would like to determine how long it can radiate its energy. That depends on how much energy it has to radiate. The total amount of energy in the black hole is contained in its rest mass: $E = Mc^2$. We can estimate its lifetime, t , if we divide the energy in the black hole by its thermal luminosity. This is analogous to the way we estimate the lifetime of a star.

$$t \approx \frac{E}{L} = \frac{Mc^2}{\left(\frac{\sigma h^4 c^8}{2^{12} \pi^7 G^2 k_B^4 M^2}\right)}$$

Simplifying, we have:

$$t \approx \frac{2^{12} \pi^7 G^2 k_B^4 M^3}{\sigma h^4 c^6}$$

Again this looks complicated, but we can simplify further by multiplying out all the constants as we did previously:

$$t \approx 2.5 \times 10^{-16} M^3$$

where the lifetime t is in seconds, the mass M is in kg, and the constant is in s/kg^3 .

✓ Evaporating Black Holes

Worked Example:

1. Determine the luminosity for an evaporating one solar mass black hole.

- Given: mass $M = M_{Sun} = 2 \times 10^{30} \text{ kg}$
- Find: luminosity, L
- Concept(s): $L \sim 3.56 \times 10^{32} / M^2$
- Solution:

$$\begin{aligned} L &\sim \frac{3.56 \times 10^{32} \text{ W } \cancel{\text{kg}^2}}{(2 \times 10^{30} \cancel{\text{ kg}})^2} \\ &= 8.9 \times 10^{-29} \text{ W} \end{aligned}$$

2. Determine the evaporation time in years for a one solar mass black hole.

- Given: mass $M = M_{Sun} = 2 \times 10^{30} \text{ kg}$
- Find: t , the estimated evaporation time in units of seconds
- Concept(s): $t \sim 2.5 \times 10^{-16} M^3$
- Solution:

$$t \sim (2.5 \times 10^{-16} \text{ s } \cancel{\text{kg}^{-3}})(2 \times 10^{30} \cancel{\text{ kg}})^3 = 2 \times 10^{75} \text{ s}$$

Converting to years

$$(2 \times 10^{75} \cancel{\text{ s}})(1\text{yr} / 3.15 \times 10^7 \cancel{\text{ s}}) = 6.3 \times 10^{67} \text{ years}$$

Astronomers have measured the current age of the Universe to be about 14 billion years. Clearly the time for a one solar mass black hole to evaporate is enormously long compared to the age of the Universe. The ignored factor of three plays no role in this conclusion.

Questions

1.

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3.

In the previous activity, we saw that it takes an extremely long time for solar mass black holes to evaporate. So we should not expect to see any of them evaporating anytime soon. However, the time required for a black hole to evaporate depends on its mass cubed, so as the mass of a black hole approaches zero, the time required for it to evaporate by Hawking radiation also approaches zero, and very quickly. Furthermore, if you look at the luminosity we derived, we expect that the black hole gets brighter and brighter as it gets smaller. In fact, the luminosity grows without bound as the mass of the black hole shrinks to zero. So just before a black hole disappears, it emits a very brief, but very bright, flash of light. Its temperature also grows without bound as its mass shrinks, so we would expect this light to be mostly X-rays and gamma-rays.

But do we expect to see any such objects? Given that the Universe is about 14 billion years old, it is possible that small enough black holes might just be evaporating now, assuming they were formed as part of whatever process formed the Universe itself. Astronomers refer to these hypothetical objects as primordial black holes because they would have been formed from processes that formed the Universe. In the next activity you will compute the size of such black holes.

Primordial Black Holes

Use the expression for the evaporation time of black holes to find the mass of primordial black holes, given the age of the Universe. Remember to convert to SI units (kg, m, s) where necessary.

Questions

1.

2.

What about even smaller black holes, such as proton-mass black holes? There were stories in the news when the LHC at CERN was turned on (and also several decades ago when the Tevatron at Fermilab came online) about the possibility of Earth being swallowed by mini-black holes that might be produced in the particle collisions. The black holes, it was argued by some, would start out with masses around the mass of a proton, but would then grow in size until they eventually consumed the entire Earth. What a scary thought—but is it reasonable?

Some of those making these arguments went so far as to file lawsuits asking that the accelerators be barred from starting. How reasonable were their claims? We can answer that question with a calculation like the ones we have just done.

MINI BLACK HOLES

Questions:

1.

You should have discovered that it takes a very short time for a proton-mass black hole to evaporate. It is as close to instantaneous as we are likely to encounter. So long before anything can fall into such a tiny black hole it will evaporate.

Given the short lifetime of tiny black holes, why were some people concerned about the possibility that mini-black holes in the Tevatron and LHC could grow large enough to consume the whole planet? If you are unable to understand why they were concerned, then you are in agreement with physicists. You are also in agreement with the judges for the lawsuits, which were all thrown out.

11.3.3: Singularities and Quantum Gravity

As we have already explored, close to the event horizon of a black hole we expect quantum effects to become important. If we imagine journeying beyond the horizon and down into a black hole, we can ask what happens as we approach the center. We have seen that before we even arrive there, we are stretched radially and compressed from the sides by spaghettification. And we have discussed how this process of destruction will continue on to smaller and smaller scales, growing more extreme as we approach $r = 0$. This happens because the spacetime curvature increases without bound as we approach $r = 0$, and there is energy in spacetime curvature. In fact, if we look at the Schwarzschild interval, we see that both the time and space curvature terms become infinite at $r = 0$.

This point is called a **singularity**, a mathematical term used to describe places where mathematical expressions become undefined. If an expression becomes infinite at some point, that certainly satisfies the conditions for being singular there. One often hears that all the mass that falls into a black hole gets squeezed into a point of infinite density at the singularity. Certainly infinite density, or infinite anything, would constitute a singularity. But is a singularity really a meaningful concept when applied to the actual Universe?

At some point we expect that the energy of the curved spacetime will begin to interact with the fluctuating virtual particles, or quantum fluctuations, in the spacetime, converting gravitational energy into particles. Of course, those particles cannot get across the horizon; they are imprisoned inside the black hole near its center. They fall back to the center and are converted back to curvature; the situation could be extremely chaotic.

In order to understand the conditions near a singularity, we require a **quantum theory of gravity** —such a theory would encompass both gravitational and quantum phenomena. Physicists have not yet created such a theory. Still, there are some rudimentary considerations that we can make to give us an idea of what a quantum theory of gravity would encompass and where it would be applicable.

The first such consideration is to think about the size scale on which we expect quantum fluctuations to begin to interact strongly with gravity. We can use the relevant physical constants to determine this scale. Any theory of quantum gravity must contain combinations of h , c , and G : the h because the theory encompasses quantum phenomena, the c because it must include relativity, and the G because it must include gravity. These quantities are strictly determined by various properties we measure for the Universe. They are independent of any specific model we might devise, and so we expect them to be in some sense unbiased and to provide fundamental information that any successful future theory must incorporate. But how can we combine these constants in such a way to give us a size scale, or in other words, a length? We can assume that the constants combine in the following way.

$$[l] = [h^\alpha c^\beta G^\gamma]$$

We use the square brackets to mean that we are comparing only the dimensions of the quantities (essentially their units), not their numerical values. That reduces our task to finding values of the exponents α , β , and γ (alpha, beta and gamma) that give the right-hand side of the equation the value “length to the first power.” Note that we could have used a , b and c instead, but we are already using c to mean the speed of light, and we want to avoid confusion. Speaking of which, don't confuse γ with the relativistic γ factor we learned about in Chapter 9. Here we are using it to mean a simple parameter for this exercise.

If we use SI units, then the quantities must combine to give meters. It turns out that the appropriate values of α , β and γ are $1/2$, $-3/2$ and $1/2$, respectively. See [Going Further 11.3: Planck Units and Dimensional Analysis](#) for the details of how we determine this.

We can check this result by inserting units for each of the constants, still ignoring their numerical values:

- $[h] = \text{J s}$
- $[c] = \text{m/s}$
- $[G] = \text{N m}^2 / \text{kg}^2$

Now we can use these values to check that our exponents are correct.

$$(\text{J s})^{1/2} (\text{m/s})^{-3/2} (\text{N m}^2 / \text{kg}^2)^{1/2} = ((\text{kg m}^2/\text{s}^2) (\text{s}))^{1/2} (\text{m/s})^{-3/2} ((\text{kg m/s}^2) (\text{m}^2/\text{kg}^2))^{1/2} = \text{m}$$

We expressed joules and newtons in terms of meters, kilograms, and seconds in the second expression. Combining terms we find that they cancel except for one factor of meters, just as we had desired.

Now that we know this set of exponents result in a length, we can determine what the length scale is by inserting the numerical values for the constants. This is the length scale on which gravity and quantum mechanics interact strongly. It is referred to as the

Planck length because it was Max Planck who first determined it:

$$l_{\text{planck}} = \left(\frac{hG}{c^3} \right)^{1/2}$$

Numerically it is equal to 4×10^{-35} m. On size scales smaller than this, we expect that quantum effects will become important for gravity. In a sense, our idea of spacetime as smooth and continuous breaks down on this scale; space becomes grainy. On these scales, general relativity should no longer be adequate to explain gravity, and so any singularities contained in the theory should not be important. Notice that the Planck length is tiny. A proton is about 10^{20} Planck lengths across, so we do not expect to see any quantum gravity effects in atoms or even in currently known subatomic particles. But l_{planck} is still much larger than zero, and therefore long before we get to any singularity, general relativistic predictions should cease to be a valid description of nature. We should not have to worry about things like points of infinite density.

From the Planck length one can construct a Planck time, the time required for light to travel one planck length:

$$t_{\text{planck}} = \frac{l_{\text{planck}}}{c} = \left(\frac{hG}{c^5} \right)^{1/2}$$

Numerically the Planck time is 10^{-43} s. Again, this is a tiny quantity. We expect that quantum gravitational effects will be important for processes with durations similar to this timescale. We also expect that time itself will be grainy on this scale, not the smooth continuous stream we generally think about.

We have not made any predictions about what happens at the center of a black hole. All we have done is make an argument that we do not expect our current understanding of gravity, as expressed in the theory of general relativity, to be an adequate description there. On these scales we might see the structure of spacetime change abruptly, forming tiny black holes or creating other abrupt changes in the local curvature, but these changes would only last for a time of order t_{planck} and affect length scales of order l_{planck} . We will likely have to wait for physicists to create a quantum theory of gravity before a full theoretical description of black holes emerges, and before we know if our assumptions about the quantum nature of spacetime are in any sense correct.

For an artist's depiction of what space and time might be like on scales where quantum gravity is important, see Figure 11.10. Both space and time are grainy, resembling foam, and are not continuous as in our everyday experience. The concept of quantum gravity not only has important implications for the nature of the interiors of black holes, but also for the earliest moments in the history of the Universe.

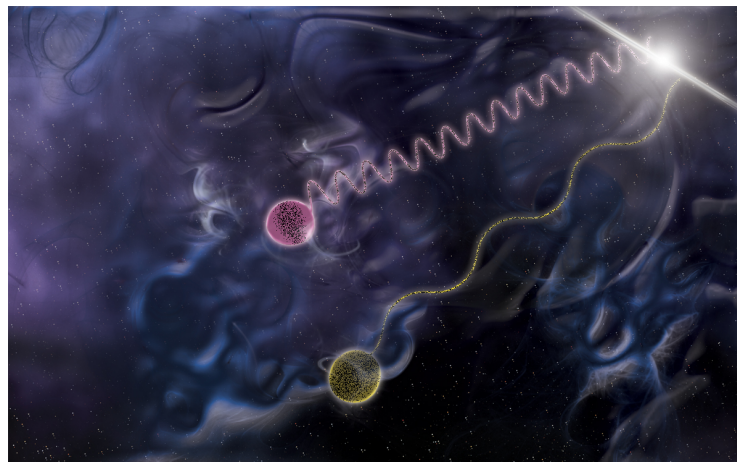


Figure 11.10: On tiny length scales and short timescales it is possible that spacetime has a quantum nature, giving it the structure of a foam. Random bubbles form and disappear. The bubbles have a size comparable to the Planck length and exist for times comparable to the Planck time. Thus spacetime might look something like the foam that forms in a rapidly boiling pot of sauce, but on minuscule scales in space and time. Credit: NASA/SSU/Aurore Simonnet

Going Further 11.3: Planck Units and Dimensional Analysis

To find the proper combination of physical constants to find the Planck length, we can use a procedure called dimensional analysis. In this procedure we use the dimensions (or units) of physical quantities to find how they must be combined in order

to give us another physical quantity of interest to us. This is a technique used by scientists when they do not have a theory to describe some phenomenon.

As an aside, the technique can (and should) be used by students of science to check the answers to computations they make, because if the units of their answer come out wrong, the answer itself must be wrong. Dimensional analysis is a quick and easy self-check, although it does not guarantee an answer is correct just because the units are.

Generally in dimensional analysis square brackets surrounding a quantity indicate that we should consider only its dimensions (or units), not its numerical value. The numerical value will be used at the end, but first we consider just the units. So, in determining the Planck length, we have a relation that says:

$$[l] = [h^\alpha c^\beta G^\gamma] = [h]^\alpha [c]^\beta [G]^\gamma$$

We will use the SI units for each dimension to simplify the analysis. Of course, this procedure will work with other units as well; it is only the dimension, whether length, time, mass, etc., that matters. We pick SI for convenience and specificity.

The units for the three quantities we have are:

- $[h] = \text{J s}$
- $[c] = \text{m/s}$
- $[G] = \text{N m}^2 / \text{kg}^2$

Substituting these units into the above expression, we get:

$$m = (\text{J s})^\alpha \cdot (\text{m/s})^\beta \cdot (\text{Nm}^2/\text{kg}^2)^\gamma$$

We have to break down joules and newtons into their equivalent combinations of meters, kilograms, and seconds to proceed. These are

$$\text{J} = \text{kg m}^2/\text{s}^2$$

$$\text{N} = \text{kg m/s}^2$$

And now we substitute these in.

$$m = ((\text{kg m}^2/\text{s}^2)(\text{s}))^\alpha \cdot (\text{m/s})^\beta \cdot ((\text{kg m/s}^2)(\text{m}^2/\text{kg}^2))^\gamma$$

We have to combine like terms. This gives us

$$m = (\text{kg m}^2/\text{s})^\alpha \cdot (\text{m/s})^\beta \cdot (\text{m}^3/\text{kg s}^2)^\gamma$$

and finally

$$m = \text{kg}^{(\alpha-\gamma)} \cdot \text{m}^{(2\alpha+\beta+3\gamma)} \cdot \text{s}^{(-\alpha-\beta-2\gamma)}$$

We have derived three expressions, and these can be used to relate the exponents to each other. By comparing the units on either side of the equation, we find that

$$\alpha - \gamma = 0$$

since there are no kilograms to the left of the equal sign.

$$-\alpha - \beta - 2\gamma = 0$$

since there are no seconds to the left of the equal sign.

$$2\alpha + \beta + 3\gamma = 1$$

since there is only one power of meters to the left of the equal sign.

This gives us three equations in three unknowns, and we can solve them in the normal way of substitution. We find the values $\alpha = \gamma = 1/2$, $\beta = -3/2$.

This leads us to the Planck length:

$$l_{\text{planck}} = h^{1/2} G^{1/2} c^{-3/2} = \left(\frac{hG}{c^3} \right)^{1/2}$$

If instead of a length scale we had wanted to find a mass scale, we could have used the same procedure, except that the left-hand side of our equation would have had one power of kilograms instead of one power of meters. Everything else would have been the same. In that case we would have found the Planck mass:

$$m_{\text{planck}} = h^{1/2} c^{1/2} G^{-3/2} = \left(\frac{hc}{G^3} \right)^{1/2}$$

If you are interested, see if you can figure out how this expression is determined. The procedure is identical to the previous one, but for the minor difference we have already pointed out. The Planck mass will be important in our consideration of the earliest moments of the Universe, when its age was of order the Planck time. If you like you can also try to work out the Planck time this way and compare it to the value mentioned in the main text—they should be the same.

The German physicist Max Planck first undertook dimensional analysis of this sort in 1899. He thought of these units, l_{planck} , t_{planck} , m_{planck} , etc., as natural units, in that they are independent of any particular theory or particle. The basic units of length, mass, and time can be combined in various ways to derive expressions for energy, force, pressure, etc., so we can also compute a Planck energy, a Planck force, and so on.

In addition to the three mechanical constants we have used here, Planck also used the Coulomb constant (like the gravitational constant G , but for the electrical force) and the Boltzmann constant, which is important in thermodynamics and statistical physics. These are used when we wish to venture beyond mechanics and include electromagnetic and thermal phenomena.

Planck units are just one set of many other natural units that are possible, but in some sense they are the most fundamental because, as mentioned already, they do not depend on the properties of any particular particle or theory. They depend only on physical constants related to space, time, and the interactions between particles.

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