9.2: Time Dilation

Learning Objectives

- You will know that moving clocks are slower.
- You will be able to correctly use the time dilation formula to compare times in different reference frames.

What Do You Think: Constant Speed of Light

9.2.1: A Little Thought-Experiment

The first result of relativity that we will explore is its implication for time. The primary tool we will use in our explorations will be what Einstein called a Gedankenexperiment, or "thought-experiment." Einstein used many of these experiments to guide his thinking about physics. Only later did he flesh out the details through mathematical calculations. His imagined trip alongside a light wave is a kind of Gedankenexperiment.

For our next thought-experiment, we will imagine a special kind of clock. This clock is very simple, consisting of only three parts: two mirrors that are 100% reflecting and perfectly flat, and a photon that bounces back and forth between the mirrors, as depicted in Figure 9.2.

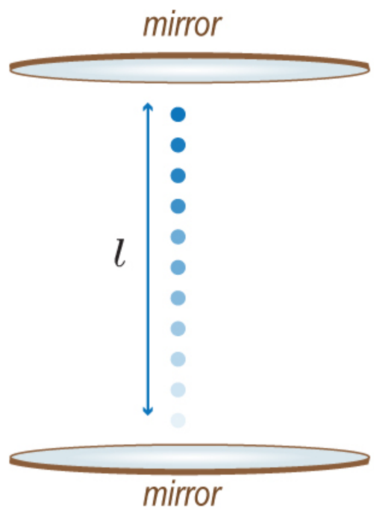

Figure 9.2: An ideal clock can be imagined from a photon bouncing back and forth between two parallel mirrors. This is called a light clock. Credit: NASA/SSU/Aurore Simonnet

This clock keeps time by registering a “tick” each time the photon bounces off the bottom mirror (we could use the top mirror just as well). If the distance between the mirrors remains constant and the speed of light is also constant, then this will be a perfect clock, keeping perfect time. It is not important whether we lack the technology to actually construct such a clock. This is a *thought*-experiment, so this experiment takes place in the ideal laboratory of our mind.

If we could observe such a stationary clock in our lab, we would note that it registers a tick after a time span (Δt) of:

$$\Delta t = \frac{2l}{c}$$

Here c is the speed of light, as usual. The numerator on the right side of the equation, $2l$, is the distance the photon travels between ticks. We use the Greek letter Δ (“delta”) in addition to the letter *t* to represent the time span, for reasons of convention: In physics and astronomy, *Δ* often implies an amount that a quantity has changed—in this case, an interval of time. This is distinct from an initial or final value (starting or ending time), which would be represented by just the letter *t* itself.

The equation above is the result we get for a clock that is not moving relative to us. We say it is in our rest frame because we measure the clock to be at rest. The term "frame" is shorthand for frame of reference, or simply, reference frame. So far, this result is completely consistent with what we expect from classical physics.

Stationary Light Clock

Worked Example:

How big would our light clock have to be for the time between ticks to equal 1 second?

- Given: $c = 300,000\,\text{km/s}$, $\Delta t = 1\,\text{sec}$
- Find: l
- Concept: $2l = c(\Delta t)$
- Solution:

$$\begin{aligned} 2l &= (300,000\,\text{km}/\cancel{\text{s}})(1\,\cancel{\text{s}}) \\ &= 300,000\,\text{km} \end{aligned}$$

Therefore $l = 150,000\,\text{km}$.

We can also answer this question by realizing that the total distance the light travels must be one light-second of travel time, so the size of the light clock, l, must be half a light-second since the clock is traversed by the photon once on the way out and once on the way back. So, we can think of the size as half a light-second, or as 150,000 km.

Question:

Now we will look at what happens if the clock moves past us with some constant velocity, v. Figure 9.3 shows the situation that we observe if the clock moves relative to us.

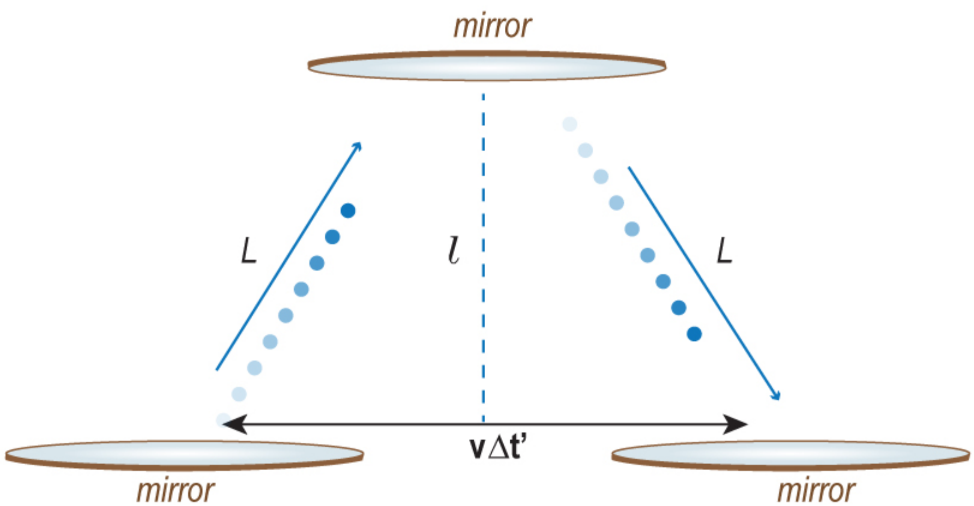

Figure 9.3: A moving ideal clock with a photon bouncing back and forth between two parallel mirrors. The clock is moving to the right with speed v. Credit: NASA/SSU/Aurore Simonnet

In the time that the light travels from the bottom mirror to the top and back again ($\Delta t'$), the clock has moved a distance (d) given by:

$$d = v\Delta t'$$

Here, we have put a prime on the time interval to distinguish it from the stationary case because, as we shall see, the two are not the same.

The light travels directly from one mirror to the other. This is the distance labeled L in the figure. Notice that the path traveled by the light (L) is related to the distance the clock moves (d) and the distance between the two mirrors (l) because they form a (right) triangle. We can write *d*, *L*, and *l* in terms of *v*, the time intervals and *c*, and eventually derive an expression for $\Delta t'$ in terms of Δt, *v*, and *c*. The derivation is shown in Going Further 9.1: Moving Clocks. The expression is:

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

According to this equation, and thus, to the special theory of relativity, the time interval measured when the clock is stationary and the time interval measured when the clock is moving are not the same; the two observers have measured *different* time intervals for the same two events—successive arrivals of a photon at the bottom mirror.

Some care is required here to understand exactly what this result is saying. The time interval Δt is that needed for a photon to travel from the bottom mirror to the top one and back again *in a frame in which the clock is at rest*. Let us call any frame in which an observer is at rest the *rest frame of the observer.* If a clock is in an observer's rest frame (at rest relative to that observer), then that observer will measure a time interval Δt between successive ticks of that clock. Now imagine the case when the clock is not in the observer's rest frame, but rather, is moving with some constant speed v in a constant direction (uniform motion) relative to the observer. In that case, the observer would measure a time interval $\Delta t'$ between successive ticks of the clock.

Clearly, from the expression above, Δt and $\Delta t'$ are not the same when the relative speed between the two frames is not zero. Since the velocity between the clock and any observer will always be less than c, we can reason that the term multiplying Δt is always greater than or equal to one. That means that a "moving" observer measures the round-trip time for the photon in the clock to always be larger than the round-trip time for the photon measured by an observer in the clock's rest frame: keep in mind that "moving" in this case means moving relative to the clock. The clock therefore ticks more slowly when it moves relative to the person observing it. This means, and this is the tricky part, a clock *ticking slower* will record *less time passing* than a clock ticking more rapidly. Since clocks that are moving relative to an observer always tick slower than clocks at rest relative to that observer, they record less time passing than clocks in the observer's rest frame.

However, it is important to consider more carefully what we mean by "at rest" and "moving." Consider two observers moving uniformly relative to each other. Each can have a clock in her own rest frame, and each will consider herself to be at rest and the other observer to be in motion. Since, according to special relativity, the laws of physics must be the same for both observers, there is no experiment that either can do to show that she is "really" at rest, while the other observer is "really" in motion. Special relativity implies that the claims of both observers to be "at rest" are equally valid. That means that the time distortion effects are completely symmetric. Each observer sees her own clock, in her own rest frame, ticking more rapidly than the clock of her "moving" counterpart.

So, which clock is really ticking slower? That is the wrong question to ask. The "relativity" part of special relativity is there because motion, and also the measurement of space and time, is relative, not absolute. If an observer in uniform motion notices another observer, also in uniform motion *relative to herself*, then she will measure time for that observer to pass more slowly *relative to her own time*. This statement is true for all observers in uniform motion, and there is no absolute measure of either space or time. That is a critical result of special relativity to keep in mind.

Going Further 9.1

For the case of the moving light-clock, the path traveled by the light (L) is related to the distance the clock moves (d) and the distance between the two mirrors (l) by the Pythagorean theorem:

$$L^2 = l^2 + \left(\frac{d}{2}\right)^2$$

We have written d/2 because the right triangle is only half of the light's path.

We can write these distances in terms of v, the time intervals, and c, to derive an expression for Δt' in terms of Δt, v, and c.

Writing the distance moved by the clock in the time $\Delta t'$ we have $d = v\Delta t'$, which we can substitute as below.

$$L^2 = l^2 + \left(\frac{v\Delta t'}{2}\right)^2$$

Traveling the distance L takes only half the time between ticks because the light has to traverse it once on the way up and then again on the way down to make one tick. That is what the factor of two represents in the expression for d: we have only traveled half of the full time interval needed for a clock tick by traveling up to the top mirror. Since we assume that the speed of light is constant (according to the second principle of relativity), we can write a relation between L, c, and Δt' like the one we wrote for l, c, and Δt:

$$\Delta t' = \frac{2L}{c}$$

This looks similar to our expression for l (as measured when the clock is at rest), but it is for the case of the moving clock. Solving these for L and l and substituting them into the equation for L2, we get:

$$\left(\frac{c\Delta t'}{2}\right)^2 = \left(\frac{c\Delta t}{2}\right)^2 + \left(\frac{v\Delta t'}{2}\right)^2$$

Now rearranging by collecting the common time terms we have the following.

$$c^2 \Delta t'^2 \left(1 - \frac{v^2}{c^2}\right) = c^2 \Delta t^2$$

Finally, we may cancel the common term of c^2, rearrange and take the square root.

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

9.2.2: The Relativistic Gamma Factor

Examining the relationship between *Δt'* and *Δt*, we see that the term inside the radical (square root symbol) is always between zero and one. It is zero when the velocity of the "moving" clock is *c*, the speed of light (which, according to special relativity, can never happen; see Section 9.7). The term equals 1 when the velocity of the "moving" clock is zero (i.e., when the moving clock is not moving!). At all other velocities, it is some number between zero and one. The square root of a number between zero and one is also between zero and one, and its reciprocal is always greater than or equal to one. This factor plays a central role in relativistic physics. Since it comes up over and over again, it has been given its own name: gamma (γ).

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The symbols used in this expression are what we expect: *c* is the speed of light and *v* is the relative speed between two reference frames of interest. The "equal sign" with three bars instead of two is actually a "defined as" sign. This expression is the definition of the gamma factor in relativity.

A graph of the gamma factor vs. *v/c* is shown in Figure 9.4. An interactive version of this graph will be available as you work through the activities below, in case you prefer to use the graph instead of the equation to determine *γ* or *v* as needed.

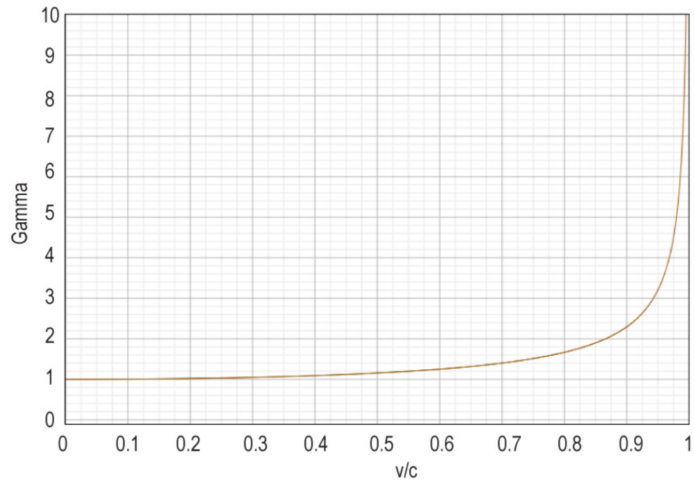

Figure 9.4: The gamma factor (γ) is plotted vs. *v/c*. For velocities much lower than that of light, gamma is very nearly one. In that case the time difference in the two frames is small. Only as *v* approaches *c* does gamma begin to grow, slowly at first, and then quite abruptly after about 0.9c. Credit: NASA/SSU/Aurore Simonnet/Kevin McLin

We can rewrite the equation for time intervals between successive clock ticks on two clocks in relative motion in terms of the gamma factor.

$$\Delta t' = \gamma \Delta t$$

As before, $\Delta t'$ is the length of time it takes for one tick on a moving clock, and Δt is the length of time it takes for one tick on a clock at rest.

Since gamma is always greater than or equal to one, the length of time it takes for one tick on a moving clock is always greater than the length of time it takes for one tick on a clock at rest. This effect is called **time dilation**. Clocks tick more slowly for the moving clock. As we have already discussed, this result is true for both observers: two observers in relative uniform motion each sees her own clock (the one that is not moving for her) ticking more rapidly (time passing faster) and the other clock (moving relative to her) ticking more slowly. If the relative velocity between the observers is zero, then they measure the same time. In this case, they are in the same rest frame. Remember that the motion referred to here is relative motion. Any observer always has her own clock, say a wristwatch, that is *not moving* relative to herself.

Time dilation allows us to understand the problem we first encountered at the opening of the chapter: how muons manage to make it all the way down to the ground before decaying. According to an observer on the ground, the muons can travel to the ground because time for them is passing much more slowly than it passes for that observer, who is at rest on Earth. This idea opens our minds to a new way of thinking about time, as you will see in the next set of activities.

Time Dilation

Worked Examples:

The fastest spacecraft that humans have launched so far is New Horizons, which visited Pluto in 2015. It traveled at a speed of about 16 km/s. If a spaceship in the future could go 1,000 times faster than New Horizons, what would the gamma factor be for that speed?

- Given: v = 16,000 km/s
- Find: γ
- Concept: $\gamma = 1/(\ 1 - (v)^2/c^2\ )^{0.5}$
- Solution: $\gamma = 1/(\ 1 - (16{,}000\ \cancel{km/s})^2/(300{,}000\ \cancel{km/s})^2\ )^{0.5} = 1.0014$
- Note: here we have used the exponent of 0.5 to denote the square root

Alternatively, you could find *v/c* and then use the clickable gamma plot tool:

USE GRAPH

v/c = (16,000)/(300,000) = 0.053

From reading the graph, $\gamma = 1.00125$ (at 0.05c)

2. If one tick on a clock in mission control takes 1 second, how much time would it take for a tick on a clock in the ultra-fast (16,000 km/s) spacecraft (from the perspective of someone at mission control)?

The observer at mission control thinks that she is at rest and that the spacecraft is moving.

- Given: $\gamma = 1.0014$, $\Delta t = 1$ s
- Find: $\Delta t'$
- Concept: $\Delta t' = (\Delta t)(\gamma)$
- Solution: $\Delta t' = (1\ s)(1.0014) = 1.0014$ s. This means that time on the moving spacecraft is slower (ticks take longer) compared to the time on the ground.

3. If 1 second passes on a clock in mission control, how much time would pass on a clock in the spacecraft (from the perspective of someone at mission control)?

The observer at mission control thinks that she is at rest and that the spacecraft is moving. Since the moving clocks tick more slowly, less time will pass on the spaceship than in mission control, at least, according to an observer sitting in mission control. The two are related to each other by gamma. If we call the time that passes at mission control Δt_m and the time on the ship Δt_s, then we have:

- Given: $\gamma = 1.0014$, $\Delta t_m = 1$ s
- Find: Δt_s
- Concept: The time that passes on the spaceship (Δt_s) will be *smaller* than Δt_m by a factor γ (the time dilation equation is for the time of ticks, not for the time that passes).
- Solution: $\Delta t_s = (1\ s)/(1.0014) = 0.9986$ s This means that time on the moving spacecraft is slower (less time has passed) compared to the time on the ground.

Questions:

Time Dilation and Reference Frames

In this activity, you will change the speed at which clocks move and see how the elapsed time intervals are affected. A small window with a clock will open when you load the activity.

Play Activity

A. Ticking time

A light clock ticks every time a bouncing photon returns to the first mirror. These ticks define the length of a second for the clock, so one second is the round trip travel time for a photon. The duration of this second is adjustable with time dilation, which in turn depends on the relative motion between the clock and observer.

After you have understood how time dilation of the clock works, click on the OK button to load the main activity window. Note that the OK button will not be enabled until you adjust the time dilation with the slider.

If you want to go back to the opening example clock at any point, you can reload the page.

B. Moving light clocks

When the main page loads, you will see windows for two clocks. Each can have its speed (relative to you, the observer) adjusted independently.

Adjust the slider bar on the upper clock so that it will be traveling at 10% of the speed light (i.e. 0.1c or 10% c). Then play the animation.

Now repeat with v = 0.2c, but this time, set the velocity for the top clock to zero, and adjust the slider bar for the bottom clock.

Predictions, observations, and explanations. You may find the clickable gamma graph useful.

USE GRAPH

You might protest that our light clock thought-experiment with mirrors is not an accurate depiction of the real world. For instance, if the mirror moved through space so quickly that it was no longer in the photon's path by the time the photon traveled from the bottom mirror to the top mirror, then the photon would not be reflected. You might conclude that this method would be a way to distinguish the moving mirror case from the stationary mirror case. However, this is not correct. In the case that the photon missed the mirror, it would do so for both observers, not just one. And, we could avoid this problem entirely by imagining two infinitely long mirrors. Everything in our argument above would still be valid, and we would arrive at the same conclusions as before.

The proper way to view the situation is as Albert Einstein did: It is not possible to say which observer is "at rest." Only the observers' relative motion can be determined. If one of them sees the photon bounce off one of the mirrors, then the other observer will see that too. What they will disagree about is the time at which each bounce occurs.

They will also disagree about where the photon is when it bounces, as we will see in Section 9.3. We now revisit the example from the opening of this chapter: muons produced in Earth's upper atmosphere.

✔ Muons From Cosmic Rays, as Seen From the Ground

Worked Example:

We now revisit the example from the opening of this chapter: muons produced in Earth's upper atmosphere. If 10 million muons are produced at an altitude of 10 km, how many of them will reach the ground?

1. First, we will examine this problem without using special relativity.

If we take the speed of the muons to be slightly slower than the speed of light (0.99c), then we can calculate the time required for the muons to reach the ground:

- Given: d = 10 km, v = (0.99)(300,000 km/s) = 2.97×10^5 km/s, N = 10^7 muons
- Find: t
- Concept: v = d/t

- Solve: $t = (10\,\mathrm{km})/(2.97 \times 10^5 \mathrm{km/s}) = 3.37 \times 10^{-5}\ \mathrm{s} = 33.7\,\mathrm{microseconds}$

But, this is a very long time for a muon. Muons have a half-life for decay of only 2.2 microseconds. This trip to Earth's surface takes more than 15 half-lives (33.7/2.2 = 15.3). The probability of a muon surviving so long is:

$P = (0.5)^{15.3} = 2.5 \times 10^{-5}$

Given this incredibly low probability, we should expect to see only about $(10^7$ muons) × (2.5×10^{-5}) = 250 muons reach the ground. However, many more muons do reach the ground.

2. We will do the calculation again, but this time, we will include the effects of special relativity.

We know that the muon lifetime will be increased by a factor of gamma, so we must calculate what this factor will be. You can either use the equation or the clickable graph to determine gamma. Here, we show you how to calculate gamma using the equation. You should verify that you get the same thing using the clickable graph.

- Given: v = 0.99c
- Find: γ
- Concept: $\gamma = 1/(1 - (v^2/c^2))^{0.5}$
- Solution: $\gamma = \dfrac{1}{\sqrt{1-(0.99c)^2/c^2}}$ = 7.1
- Again, the exponent of 0.5 means take the square root

USE GRAPH

So, now if we calculate the probability of the muons making it to the ground, but with a lifetime 7.1 times bigger, we have only about 2 half-lives (trip time / half-life = 33.7/(7.1 × 2.2) = 2.1):

$P = (0.5)^{2.1} = 0.22$

This result is a much larger probability, and many more muons reach the ground: (0.22) × (10^7 muons) = 2.2×10^6 or 2.2 million muons. This result is many more than would be seen if relativity did not affect the process, and it agrees with the fraction actually measured.

Questions: