# 8.3: Gravity and Models for Different Mass Distributions

**What Do You Think: Mass Distributions**


In the previous section we learned that the velocities of orbiting objects can be modeled and understood using mathematical formulae and graphs. In this section, we will see that the distribution of mass in astronomical objects can be modeled and understood in the same way. We will examine several example mass models to build our understanding. We will then link these example models to velocity using what we already know about gravity, just as we did in the case of the Solar System.

The mass models we describe here will seem abstract at first. However, they are very helpful for understanding the motions of stars and gas within galaxies, and they can also be used to understand the motions of galaxies within clusters of galaxies.

*An important note before we carry on*: In the everyday examples of rotation that we discussed in Section 8.1, gravity does not play a major role in causing the objects to rotate the way they do. For instance, in the case of a rigid disk like a wheel, the chemical bonds holding it together keep it rigid. In the case of water swirling around a drain, the physics describing the water's motion has to do with the way molecules interact in a fluid, though sometimes gravity plays a role.

Unlike the terrestrial examples discussed at the outset, gravity plays a major role on the motions of astronomical objects, including planets orbiting stars, stars and gas moving within galaxies, and galaxies moving inside clusters. The everyday examples we discussed at the outset are familiar and serve as a good introduction to thinking about rotation and revolution. They generally involve different physics, but they give us a good reference point for understanding the motion of astronomical objects, which is dominated by gravity.

## 8.3.1: Gravity and Velocities When Mass is Spread Evenly in Space

When we looked at the motion of the planets in the Solar System, we modeled the mass of the Sun as a point mass. This is because the Sun's mass is entirely enclosed within the orbits of each of the planets, and also because the Sun is small in comparison to the distances between the Sun and planets.

What about systems that cannot be modeled so simply? For example, what if the Sun was not a point mass? What if its mass was evenly spread out in a sphere the size of Earth's orbit? We would still use the following equation to describe the motion of the planets, as we did above.

$$v^2 = \frac{GM}{r}$$

However, we would have to think more carefully when applying this equation. The $M$ in the equation refers to the mass *enclosed* by the planet's orbit. In the case of Earth, Mars, Jupiter, or any of the planets farther out in the Solar System, we could plug in the total mass of the Sun for $M$ (note that this is the case even for Earth, which is orbiting just at the edge of our spread-out Sun). The motion of Earth and outer planets would not be any different than they are now, and they would still follow the Keplerian rotation relation.

Not so for the motions of Mercury and Venus. They *would* be affected because they are closer to the center of the solar system than Earth. Some of the mass of our imagined bloated Sun would lie outside the orbits of these two inner planets. If the mass beyond the orbit of Venus is evenly distributed (as we are imagining), its gravitational effect on Venus will cancel out and have no effect (see Figure 8.7). Only the mass enclosed by Venus' orbit will affect Venus' orbital speed. Similarly, only the mass enclosed by Mercury's orbit will affect Mercury's orbital speed.

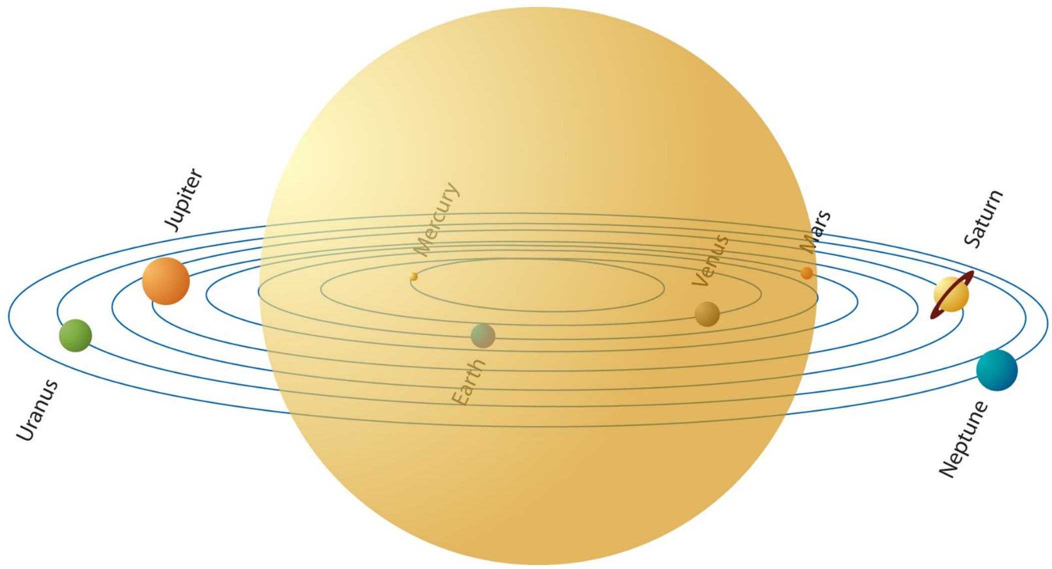


Figure 8.7: In this picture, planets orbit a Sun that is spread out in a sphere the size of Earth's orbit. Mercury's and Venus' orbital speeds are only affected by the mass enclosed within their orbits, respectively. Credit: NASA/SSU/Aurore Simonnet

### Worked Example: Planetary Orbits and a Spread-out Sun

We can determine Mercury's orbital speed in the case of the spread-out Sun. We already know that we can use the equation below to solve for orbital speed.

$$v^2 = \frac{GM}{r}$$

We can plug in the value of the gravitational constant for $G$ (6.67e-11 $m^3$ / kg $s^{2)}$, and the distance between Mercury and the Sun for $r$ (5.79e10 m). But what should we plug in for $M$? We will need to calculate the amount of the Sun's mass that is enclosed by Mercury's orbit.

To do this, it is helpful to note that when we say an object's mass is evenly spread out, that means it has a constant density. We must also remember that density, ρ, is equal to mass divided by volume, or

$$\rho = \frac{M}{V}$$

(Note: here we use upper-case $V$ for volume, to distinguish from lower-case $v$, which we have used for velocity.)

To determine the mass of the Sun that is enclosed by Mercury's orbit, we can use the relationship between mass and density written slightly differently:

$$M = \rho V$$

If the mass of the Sun were spread all the way out to Earth, its density would be 1.4e-4 kg/$m^3$. To learn why, see Math Exploration 8.2.

For $V$, we will want to plug in the expression for the volume of the Sun that is enclosed by Mercury's orbit:

$$V = \frac{4\pi}{3} r^3_{Mercury}$$

So the full expression for the mass of the Sun that is enclosed by Mercury's orbit is this:

$$\begin{aligned} M &= \left(1.4\mathrm{e}-4\,\mathrm{kg\,m^{-3}}\right)\left(\frac{4\pi}{3}\right)\left(r^3_{Mercury}\right) \\ &= \left(1.4\mathrm{e}-4\,\mathrm{kg\,m^{-3}}\right)\left(\frac{4\pi}{3}\right)(5.79e10\,\mathrm{m})^3 \\ &= 1.15\mathrm{e}29\,\mathrm{kg} \end{aligned}$$

Now, we are ready to plug everything into the equation for orbital speed:

$$\begin{aligned} v^2 &= \frac{GM}{r} \\ &= \left[\frac{(6.67\mathrm{e}-11\,\mathrm{N\,m^2\,kg^{-2}})(1.15\mathrm{e}29\,\mathrm{kg})}{5.79\mathrm{e}10\,\mathrm{m}}\right] \\ &= 1.32\mathrm{e}8\,\mathrm{m^2/s^2} \end{aligned}$$

or

$$v = 1.15 \times 10^4\,\mathrm{m\,s^{-1}}$$

This is smaller than Mercury's actual orbital speed ($4.79 \times 10^4$ m/s), which is what we expect. In our imaginary scenario, only a fraction of the Sun's mass is attracting Mercury inward — the fraction enclosed by Mercury's orbit. The rest, outside the orbit, has no net effect. As a result, Mercury does not have to move as fast to overcome the attraction of gravity.

Math Exploration 8.2

### Questions

Now we will look at the more general case of orbital motion when mass is evenly spread out in space. How do objects orbit in this case? Again, we can model this kind of system by recognizing the relationship between mass and density. Remember that if an object's mass is spread out evenly, that means it has a constant density. We can work again with our equation for velocity:

$$v^2 = \frac{GM}{r}$$

where we can substitute $M = \rho V$:

$$v^2 = \frac{G\rho V}{r}$$

and we can plug in the volume of a sphere for *V*:

$$v^2 = \left(\frac{4\pi r^3}{3}\right)\left(\frac{G\rho}{r}\right)$$

This leads to the following expression of proportionality.

$$v \propto r$$

So, in a case where the mass is evenly spread out in space, we would expect the velocity to increase with radius.

**Velocity Proportional to Radius**

What does the expression $v \propto r$ look like in graphical form? To make our example a bit easier to understand, we can take our expression of proportionality and turn it into an equation:

$$v \propto r$$

becomes

$$v = constant \times r$$

In this equation, the constant is a simple number. It does not depend on radius or any other variable.

Imagine we were studying a system that had a constant density, and the velocities of the objects within the system followed an equation like the one above, with the constant set equal to five.

$$v = 5 \times r$$

## 8.3.2: Gravity and Velocities When Mass is Distributed in Different Ways

In this section of the chapter, we are going to look at the rotation that results, due to gravity, when mass is distributed or spread out in different ways. In Section 8.3.1 above, we looked at what would happen to the velocities of the inner planets if the mass of the Sun was evenly spread out to the radius of Earth's orbit. In that case, we used the concept of density to mathematically describe the fact that the mass of our imaginary Sun was evenly spread. Mathematically, we described this by saying that the density of our spread-out Sun was constant within the radius of Earth's orbit.

Describing the spread-out Sun's *density* as constant (or unchanging at different radii) is not the same as describing the Sun's *mass* as constant. In the case of the spread-out Sun, the constant density leads to a mass that gets larger with radius. We can do the math to see why:

$$\rho = constant$$

and remember that mass ($M$) is equal to density ($\rho$) multiplied by volume ($V$):

$$M = \rho V$$

For a spherical spread-out Sun, we can plug in the volume of a sphere for $V$, just as we did in Section 8.3.1:

$$M = \frac{4\pi r^3}{3}\rho$$

In other words, as distance (r) from the center of the spread-out Sun increases, the enclosed mass increases quite a bit— the enclosed mass increases as a factor of $r^3$, or $r \times r \times r$! So, at a larger radius, we are enclosing a lot more mass in this system. As we saw above, this leads to orbital speeds that increase with radius, too.

$$v \propto r$$

Now we will look at some other ways mass could be distributed in a system such as a star or a galaxy.

### 8.3.2.1: **CASE 1: mass of the system increases directly with radius**

$$M \propto r$$

or

$$M = constant \times r$$

How would objects orbit in this system? Would their velocities be faster at larger radii, or smaller? Go back to looking at how orbital velocities and mass are related, through gravity:

$$v^2 = \frac{GM}{r}$$

We can plug in our expression for mass:

$$v^2 = \frac{G \times constant \times r}{r}$$

Since $G$ is a constant, we can consider $G \times constant = another\ constant$. We can also cancel out r in the top and bottom of the fraction above. This leads to:

$$v = constant$$

So for this mass profile, the orbital speeds of the objects in the system are all the same!

### 8.3.2.2: **CASE 2: Mass of the system does not change with radius**

In other words, imagine the following:

$$M = constant$$

This equation means that at any radius, the enclosed mass is the same. Our point-mass model of the Solar System is an example of a system in which the mass does not change with radius. In that model, at the radius of any planet, the mass enclosed by the planet's orbit is the mass of the Sun.

We can make a mathematical model of the orbital speeds in this type of system, and check whether they would match what we expect in the Solar System. Again, we go back to the equation that relates the gravitational force to the centripetal force:

$$v^2 = \frac{GM}{r}$$

and plug in M = constant:

$$v^2 = \frac{G \times constant}{r}$$

or

$$v \propto \sqrt{\frac{1}{r}}$$

This is the same relationship between velocity and radius (or distance from the Sun) that we find for the Solar System.

**Visualizing Mass and Velocity Vs. Distance (Radius)**

Below are several graphs that represent relationships, or proportionalities, between radius (on the x-axis) and another variable. In this section of the chapter, we have discussed how mass can be distributed with radius. We have also discussed the resulting relationships (due to gravity) between velocity and radius.

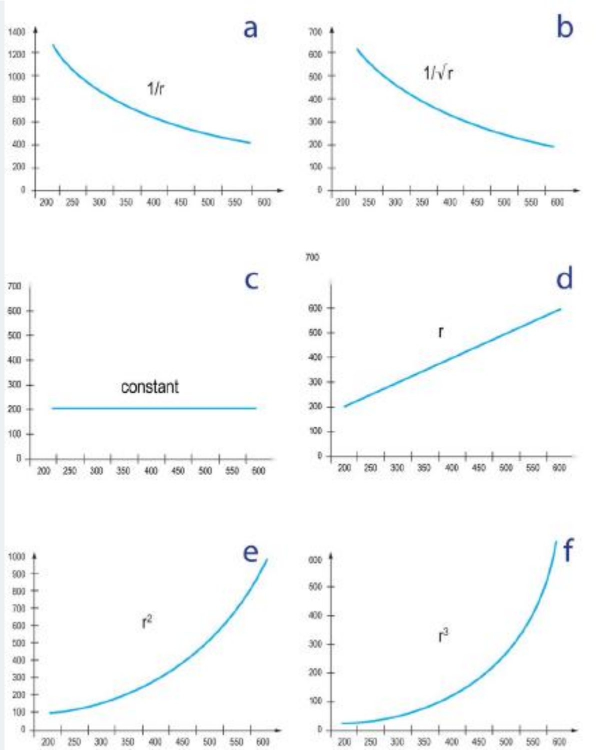


Figure A.8.3: Graphs of proportionalities with distance. Each of these graphs shows distance (radius) on the x-axis, and another variable on the y-axis. Compare these graphs to the relations we have studied in this chapter. Credit: NASA/SSU/Aurore Simonnet

***Worked Example:***

In Section 8.3.1 and the beginning of 8.3.2, we imagined what would happen if the density of the Sun were constant out to the radius of the Earth's orbit. Within that radius, we showed that a constant density leads to mass increasing with radius according to the following expression of proportionality:

$$M \propto r^3$$

If $M$ is the value plotted on the y-axis, which of the graphs in Figure A.8.3 shows this proportionality?

Answer: graph f

**Questions**