9.1: Mathematical Preliminaries

We will start with a theorem on differential forms which is needed to formulate Carathéodory's version of the second law.

Before proving Carathéodory's theorem, we will need the following result.

Theorem 9.1.1 — Integrating Factor Theorem

Let $A = A_i dx^i$ denote a differential one-form. If $A \wedge dA = 0$, then at least locally, one can find an integrating factor for A; i.e., there exist functions τ and φ such that $A = \tau\, d\varphi$.

The proof of this result is most easily done inductively in the dimension of the space. First, we consider the two-dimensional case, so that $i = 1, 2$. In this case, the condition $A \wedge dA = 0$ is vacuous. Write $A = A_1 dx^1 + A_2 dx^2$. We make a coordinate transformation to λ, φ where

$$\begin{aligned} \frac{dx^1}{d\lambda} &= -f(x^1, x^2) A_2 \\ \frac{dx^2}{d\lambda} &= f(x^1, x^2) A_1 \end{aligned} \tag{9.1.1}$$

where $f(x^1, x^2)$ is an arbitrary function which can be chosen in any convenient way. This equation shows that

$$A_1 \frac{\partial x^1}{\partial \lambda} + A_2 \frac{\partial x^2}{\partial \lambda} = 0 \tag{9.1.2}$$

Equations 9.1.1 define a set of nonintersecting trajectories, λ being the parameter along the trajectory. We choose ϕ as the coordinate on transverse sections of the flow generated by (9.1.1). Making the coordinate transformation from x^1, x^2 to λ, ϕ, we can now write the one-form A as

$$\begin{aligned} A &= \left(A_1 \frac{\partial x^1}{\partial \lambda} + A_2 \frac{\partial x^2}{\partial \lambda} \right) d\lambda + \left(A_1 \frac{\partial x^1}{\partial \phi} + A_2 \frac{\partial x^2}{\partial \phi} \right) d\phi \\ &= \tau\, d\phi \\ \tau &= A_i \frac{\partial x^i}{\partial \phi} \end{aligned} \tag{9.1.3}$$

This proves the theorem for two dimensions. In three dimensions, we have

$$A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \tag{9.1.4}$$

The strategy is to start by determining τ, ϕ for the A_1, A_2 subsystem. We choose the new coordinates as λ, ϕ, x^3 and impose Equation 9.1.1. Solving these, we will find x^1 and x^2 as functions of λ and x^3. The trajectories will also depend on the staring points which may be taken as points on the transverse section and hence labeled by ϕ. Thus we get

$$x^1 = x^1(\lambda, \phi, x^3), x^2 = x^2(\lambda, \phi, x^3) \tag{9.1.5}$$

The one-form A in Equation 9.1.4 now becomes

$$\begin{aligned} A &= \left(A_1 \frac{\partial x^1}{\partial \lambda} + A_2 \frac{\partial x^2}{\partial \lambda} \right) d\lambda + \left(A_1 \frac{\partial x^1}{\partial \phi} + A_2 \frac{\partial x^2}{\partial \phi} \right) d\phi + A_3 dx^3 + \left(A_1 \frac{\partial x^1}{\partial x^3} + A_2 \frac{\partial x^2}{\partial x^3} \right) dx^3 \\ &= \tau\, d\phi + \widetilde{A}_3 dx^3 \\ \widetilde{A}_3 &= A_3 \left(A_1 \frac{\partial x^1}{\partial x^3} + A_2 \frac{\partial x^2}{\partial x^3} \right) \end{aligned} \tag{9.1.6}$$

We now consider imposing the equations $A \wedge dA = 0$,

$$\begin{aligned} A \wedge dA &= \left[\widetilde{A}_3 (\partial_\lambda A_\phi - \partial_\phi A_\lambda) + A_\lambda (\partial_\phi \widetilde{A}_3 - \partial_3 A_\phi) + A_\phi (\partial_3 A_\lambda - \partial_\lambda \widetilde{A}_3) \right] dx^3 \wedge d\lambda \wedge d\phi \\ &= 0 \end{aligned} \tag{9.1.7}$$

Since $A_\lambda = 0$ and $A_\phi = \tau$ from Equation 9.1.6, this equation becomes

$$\widetilde{A}_3 \frac{\partial \tau}{\partial \lambda} - \tau \frac{\partial \widetilde{A}_3}{\partial \lambda} = 0 \tag{9.1.8}$$

Writing $\widetilde{A}_3 = \tau\, h$, this becomes

$$\tau^2 \frac{\partial h}{\partial \lambda} = 0 \tag{9.1.9}$$

Since τ is not identically zero for us, we get $\frac{\partial h}{\partial \lambda} = 0$ and, going back to Equation 9.1.6, we can write

$$A = \tau \left[d\phi + h(\phi, x^3) dx^3 \right] \tag{9.1.10}$$

The quantity in the square brackets is a one-form on the two-dimensional space defined by ϕ, x^3. For this we can use the two-dimensional result and write it as $\tilde{\tau} d\widetilde{\phi}$, so that

$$A = \tau\tau[d\phi + h(\phi, x^3)dx^3] = \tau\tilde{\tau} d\widetilde{\phi} \equiv T d\widetilde{\phi} \tag{9.1.11}$$

$T = \tau\tilde{\tau}$ This proves the theorem for the three-dimensional case.

The extension to four dimensions follows a similar pattern. The solutions to Equation 9.1.1 become

$$x^1 = x^1(\lambda, \phi, x^3, x^4), \qquad x^2 = x^2(\lambda, \phi, x^3, x^4) \tag{9.1.12}$$

so that we can bring A to the form

$$\begin{aligned} A &= \left(A_1 \frac{\partial x^1}{\partial \phi} + A_2 \frac{\partial x^2}{\partial \phi} \right) d\phi + \left(A_3 + A_1 \frac{\partial x^1}{\partial x^3} + A_2 \frac{\partial x^2}{\partial x^3} \right) dx^3 + \left(A_4 + A_1 \frac{\partial x^1}{\partial x^4} + A_2 \frac{\partial x^2}{\partial x^4} \right) dx^4 \\ &= \tau\, d\phi + \widetilde{A}_3 dx^3 + \widetilde{A}_4 dx^4 \end{aligned} \tag{9.1.13}$$

We now turn to imposing the condition $A \wedge dA = 0$. In local coordinates this becomes

$$A_\alpha (\partial_\mu A_\nu - \partial_\nu A_\mu) + A_\mu (\partial_\nu A_\alpha - \partial_\alpha A_\nu) + A_\nu (\partial_\alpha A_\mu - \partial_\mu A_\alpha) = 0 \tag{9.1.14}$$

There are four independent conditions here corresponding to $(\alpha, \mu, \nu) = (1,2,3), (4,1,2), (3,4,1), (3,2,4)$ Using $A_\lambda = 0$ and $A_\phi = \tau$, these four equations become

$$\widetilde{A}_3 \frac{\partial \tau}{\partial \lambda} - \tau \frac{\partial \widetilde{A}_3}{\partial \lambda} = 0 \tag{9.1.15}$$

$$\widetilde{A}_4 \frac{\partial \tau}{\partial \lambda} - \tau \frac{\partial \widetilde{A}_4}{\partial \lambda} = 0 \tag{9.1.16}$$

$$\widetilde{A}_4 \frac{\partial \widetilde{A}_3}{\partial \lambda} - \widetilde{A}_3 \frac{\partial \widetilde{A}_4}{\partial \lambda} = 0 \tag{9.1.17}$$

$$\widetilde{A}_3 \frac{\partial \widetilde{A}_4}{\partial \phi} - \widetilde{A}_4 \frac{\partial \widetilde{A}_3}{\partial \phi} + \tau \frac{\partial \widetilde{A}_3}{\partial x^4} - \widetilde{A}_3 \frac{\partial \tau}{\partial x^4} + \widetilde{A}_4 \frac{\partial \tau}{\partial x^3} - \tau \frac{\partial \widetilde{A}_4}{\partial x^3} = 0 \tag{9.1.18}$$

Again, we introduce h and g by \widetilde{A}_3 = τ h\), $\widetilde{A}_4 = \tau g$. Then, equations (9.1.15) and (9.1.16) become

$$\frac{\partial h}{\partial \lambda} = 0, \qquad \frac{\partial g}{\partial \lambda} = 0 \tag{9.1.19}$$

Equation 9.1.17 is then identically satisfied. The last equation, namely, 9.1.18, simplifies to

$$h \frac{\partial g}{\partial \phi} - g \frac{\partial h}{\partial \phi} + \frac{\partial h}{\partial x^4} - \frac{\partial g}{\partial x^3} = 0 \tag{9.1.20}$$

Using these results, Equation 9.1.13 becomes

$$A == [\tau d\phi + h dx^3 + g dx^4] \tag{9.1.21}$$

The quantity in the square brackets is a one-form on the three-dimensional space of ϕ, x^3, x^4 and we can use the previous result for an integrating factor for this. The condition for the existence of an integrating factor for $d\phi + h dx^3 + g dx^4$ is precisely 9.1.20. Thus if we have Equation 9.1.20, we can write $d\phi + h dx^3 + g dx^4$ as tds for some functions t and s, so that finally A takes the form $A = T dS$.

Thus the theorem is proved for four dimensions. The procedure can be extended to higher dimensions recursively, establishing the theorem for all dimensions.

Now we turn to the basic theorem needed for the Carathéodory formulation. Consider an n-dimensional manifold M with a one-form A on it. A solution curve to A is defined by $A=0$ along the curve. Explicitly, the curve may be taken as given by a set of function $x^i=\xi^i(t)$ where t is the parameter along the curve and

$$A_i\frac{dx^i}{dt}=A_i\dot{\xi}^i=0 \tag{9.1.22}$$

In other words, the tangent vector to the curve is orthogonal to A_i. The curve therefore lies on an $(n-1)$-dimensional surface. Two points, say, P and P' on M are said to be A accessible if there is a solution curve which contains P and P'. Carathéodory's theorem is the following:

Theorem 9.1.2 — Carathéodory's Theorem.

If in the neighborhood of a point P there are A-inaccessible points, then A admits an integrating factor; i.e., $A=TdS$ where T and S are well defined functions in the neighborhood.

The proof of the theorem involves a *reductio ad absurdum* argument which constructs paths connecting P to any other point in the neighborhood. (This proof is due to H.A. Buchdahl, Proc. Camb. Phil. Soc. **76**, 529 (1979).) For this, define

$$C_{ijk}=A_i(\partial_jA_k-\partial_kA_j)+A_k(\partial_iA_j-\partial_jA_i)+A_j(\partial_kA_i-\partial_iA_k) \tag{9.1.23}$$

Now consider a point P' near P We have a displacement vector $\epsilon\eta^i$ for the coordinates of P' (from P). η^i can in general have a component along A_i and some components orthogonal to A_i. The idea is to solve for these from the equation $A=0$. Let $\xi^i(t)$ be a path which begins and ends at P, i.e., $\xi^i(0)=\xi^i(1)=0$, $0\le t\le 1$, and which is orthogonal to A_i. Thus it is a solution curve. Any closed curve starting at P and lying in the $(n-1)$-dimensional space orthogonal to A_i can be chosen. Consider now a nearby path given by $x^i(t)=\xi^i(t)+\epsilon\eta^i(t)$. This will also be a solution curve if $A_I(\xi+\epsilon\eta)(\dot{\xi}+\epsilon\dot{\eta})^i=0$. Expanding to first order in ϵ, this is equivalent to

$$A_i\dot{\eta}^i+\dot{\xi}^i\left(\frac{\partial A_i}{\partial x^j}\right)\eta^j=0 \tag{9.1.24}$$

where we also used $A_i\dot{\xi}^i=0$. We may choose $\dot{\xi}^i$ to be of the form $\dot{\xi}^i=f^{ij}A_j$ where f^{ij} is antisymmetric, to be consistent with $A_i\dot{\xi}^i=0$. We can find quantities f^{ij} such that this is true; in any case, it is sufficient to show one path which makes P' accessible. So we may consider $\dot{\xi}^i$'s of this form. Thus Equation 9.1.24becomes

$$A_i\dot{\eta}^i+\eta^j(\partial_jA_i)f^{ik}A_k=0 \tag{9.1.25}$$

This is one equation for the n components of the displacement η^i. We can choose the $n-1$ components of η^i which are orthogonal to A_i as we like and view this equation as determining the remaining component, the one along A_i. So we rewrite this equation as an equation for $A_i\eta^i$ as follows.

$$\begin{aligned}\frac{d}{dt}(A_i\eta^i) &= \dot{A}_i\eta^i+A_i\dot{\eta}^i\\ &=(\partial_jA_i)\dot{\xi}^j\eta^i-\eta^j(\partial_jA_i)f^{ik}A_k)\\ &=-\eta^if^{jk}(\partial_iA_j-\partial_jA_i)A_k\\ &=\frac{1}{2}\eta^if^{jk}[A_k(\partial_iA_j-\partial_jA_i)+A_j(\partial_kA_i-\partial_iA_k)+A_i(\partial_jA_k-\partial_kA_j)]+\frac{1}{2}(A\cdot\eta)f^{jk}(\partial_jA_k-\partial_kA_j)\\ &=\frac{1}{2}\eta^if^{jk}C_{kij}+\frac{1}{2}(A\cdot\eta)f^{ij}(\partial_iA_j-\partial_jA_i)\end{aligned} \tag{9.1.26}$$

This can be rewritten as

$$\frac{d}{dt}(A\cdot\eta)-F(A\cdot\eta)=-\frac{1}{2}(C_{kij}\eta^if^{jk}) \tag{9.1.27}$$

where $F = \frac{1}{2} f^{ij} (\partial_i A_j - \partial_j A_i)$. The important point is that we can choose f^{ij}, along with a coordinate transformation if needed, such that $C_{kij}\eta^i f^{jk}$ has no component along A_i. For this, notice that

$$C_{kij}\eta^i f^{jk} A_i = A^2 F_{ij} - A_i A_k F_{kj} + A_j A_k F_{ki} f^{ij} \tag{9.1.28}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$. There are $\frac{1}{2} n(n-1)$ components for f^{ij}, for which we have one equation if we set $C_{kij}\eta^i f^{jk} A_i$ to zero. We can always find a solution; in fact, there are many solutions. Making this choice, $C_{kij}\eta^i f^{jk}$ has no component along A_i, so the components of η on the right hand side of Equation 9.1.27 are orthogonal to A_i. As mentioned earlier, there is a lot of freedom in how these components of η are chosen. Once they are chosen, we can integrate Equation 9.1.27 to get $(A \cdot \eta)$, the component along A_i. Integrating Equation 9.1.27, we get

$$A \cdot \eta(1) = \int_0^1 dt \, \exp\left(\int_t^1 dt' F(t')\right) \left(\frac{1}{2} C_{kij}\eta^i f^{jk}\right) \tag{9.1.29}$$

We have chosen $\eta(0) = 0$. It is important that the right-hand side of Equation 9.1.27 does not involve $(A \cdot \eta)$ for us to be able to integrate like this. We choose all components of η^i orthogonal to A_i to be such that

$$\epsilon\eta^i = \text{ coordinates of } P' \text{ orthogonal to } A \tag{9.1.30}$$

We then choose f^{jk}, if needed by scaling it, such that $A \cdot \eta(1)$ in Equation 9.1.30 gives $A_i (x_{P'} - x_P)^i$. We have thus shown that we can always access P' along a solution curve. The only case where the argument would fail is when $C_{ijk} = 0$. In this case, $A \cdot \eta(1)$ as calculated is zero and we have no guarantee of matching the component of the displacement of P' along the direction of A_i. Thus if there are inaccessible points in the neighborhood of P, then we must have $C_{ijk} = 0$. In this case, by the previous theorem, A admits an integrating factor and we can write $A = TdS$ for some functions T and S in the neighborhood of P. This completes the proof of the Carathéodory theorem.
