8.3: Fermi-Dirac Distribution

The counting of distinct arrangements for fermions is even simpler than for the Bose-Einstein case, since each state can have an occupation number of either zero or 1. Thus consider g states with n particles to be distributed among them. There are n states which are singly occupied and these can be chosen in $\frac{g!}{(n!(g-n)!)}$ ways. The total number of distinct arrangements is thus given by

$$W(\{n_\alpha\}) = \prod_\alpha \frac{g_\alpha}{n_\alpha!(g_\alpha - n_\alpha)} \tag{8.3.1}$$

The function to be maximized to identify the equilibrium distribution is therefore given by

$$\frac{S}{k} - \beta U + \beta \mu N = -n_\alpha \log n_\alpha - (g_\alpha - n_\alpha) \log(g_\alpha - n_\alpha) - \beta(\epsilon_\alpha - \mu) n_\alpha + \text{constant} \tag{8.3.2}$$

The extremization condition reads

$$\log\left[\frac{(g_\alpha - n_\alpha)}{n_\alpha}\right] = \beta(\epsilon_\alpha - \mu) \tag{8.3.3}$$

with the solution

$$n_\alpha = \frac{g_\alpha}{e^{\beta(\epsilon_\alpha - \mu)} + 1} \tag{8.3.4}$$

So, for fermions in equilibrium, we can take the occupation number to be given by

$$n = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} \tag{8.3.5}$$

with the degeneracy factor arising from summation over states of the same energy. This is the **Fermi-Dirac distribution**. The normalization conditions are again,

$$\begin{gathered} \sum \int \frac{d^3x d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\epsilon-\mu)} + 1} = N \\ \sum \int \frac{d^3x d^3p}{(2\pi\hbar)^3} \frac{\epsilon}{e^{\beta(\epsilon-\mu)} + 1} = U \end{gathered} \tag{8.3.6}$$

As in the case of the Bose-Einstein distribution, we can write down the partition function for free fermions as

$$\begin{gathered} \log Z = \sum \log\left(1 + e^{-\beta(\epsilon-\mu)}\right) \\ Z = \prod \frac{1}{1 + e^{-\beta(\epsilon-\mu)}} \end{gathered} \tag{8.3.7}$$

Notice that, here too, the partition function for each state is $\sum_n e^{-n\beta(\epsilon-\mu)}$; it is just that, in the present case, n can only be zero or 1.