

## 8.5: Applications of the Fermi-Dirac Distribution

We now consider some applications of the Fermi-Dirac distribution (8.2.5). It is useful to start by examining the behavior of this function as the temperature goes to zero. This is given by

$$n \rightarrow \begin{cases} 1 & \epsilon < \mu \\ 0 & \epsilon > \mu \end{cases} \quad (8.5.1)$$

Thus all states below a certain value, which is the zero-temperature value of the chemical potential, are filled with one fermion each. All states above this value are empty. This is a highly quantum state. The value of  $\epsilon$  for the highest filled state is called the Fermi level. Given the behavior in Equation 8.5.1, it is easy to calculate the Fermi level in terms of the number of particles. Let  $p_F$  correspond to the magnitude of the momentum of the highest filled level. Then

$$N = g_s \int_F^p \frac{d^3x d^3p}{(2\pi\hbar)^3} = g_s V \frac{p_F^3}{6\pi^2\hbar^3} \quad (8.5.2)$$

where  $g_s$  is the number of polarizations for spin,  $g_s = 2s + 1$ . Denoting  $\frac{N}{V} = \bar{n}$ , the Fermi level is thus given by

$$\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2}{2m} \left( \frac{6\pi^2\bar{n}}{g_s} \right)^{\frac{2}{3}} \quad (8.5.3)$$

The ground state energy is given by

$$U = g_s \int_F^p \frac{d^3x d^3p}{(2\pi\hbar)^3} \frac{p^2}{2m} \quad (8.5.4)$$

$$\begin{aligned} &= V g_s \frac{p_F^5}{20\pi^2 m \hbar^3} = \frac{3}{10} \frac{\hbar^2}{2m} \left( \frac{6\pi^2}{g_s} \right)^{\frac{2}{3}} \frac{N^{\frac{5}{3}}}{V^{\frac{2}{3}}} \\ &= V \frac{3}{5} \epsilon_F \bar{n} \end{aligned} \quad (8.5.5)$$

The pressure is then easily calculated as

$$p = \frac{\hbar^2}{5m} \left( \frac{6\pi^2}{g_s} \right)^{\frac{2}{3}} \bar{n}^{\frac{5}{3}} \quad (8.5.6)$$

(Since  $\epsilon_F$  depends on  $\bar{n}$ , it is easier to use Equation 8.5.4 for this.) The multiparticle state here is said to be highly degenerate as particles try to go to the single quantum state of the lowest energy possible subject to the constraints of the exclusion principle. The pressure from Equation 8.5.6 is referred to as the degeneracy pressure. Since fermions try to exclude each other, it is as if there is some repulsion between them and this is the reason for this pressure. It is entirely quantum mechanical in origin, due to the needed correlation between the electrons. As we will see, it plays an important role in astrophysics.

The Fermi energy  $\epsilon_F$  determines what temperatures can be considered as high or low. For electrons in a metal,  $\epsilon_F$  is of the order of  $eV$ , corresponding to temperatures around  $10^4 K$ . Thus, for most of the physics considerations, electrons in a metal are at low temperatures. For atomic gases, the Fermi level is much smaller due to the  $\frac{1}{m}$  factor in Equation 8.5.3, and room temperature is high compared to  $\epsilon_F$ . We will first consider the high temperature case, where we expect small deviations from the classical physics.

The expression for  $N$ , given by the normalization condition (8.2.6) is

$$\begin{aligned} \bar{n} = \frac{N}{V} &= g_s \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon - \mu)} + 1} = g_s \int \frac{d^3k}{(2\pi)^3} \frac{ze^{\beta\epsilon}}{1 + ze^{\beta\epsilon}} \\ &= g_s \frac{4}{\lambda^3 \sqrt{\pi}} \int_0^\infty du \, u^2 e^{-u} \left[ z - z^2 e^{-u^2} z^3 e^{-2u^2} + \dots \right] \\ &= \frac{g_s}{\lambda^3} \left( -\text{Li}_{\frac{3}{2}}(-z) \right) \end{aligned} \quad (8.5.7)$$

where  $\lambda$  is the thermal wavelength, defined as before, by  $\lambda = \sqrt{\frac{2\pi\hbar^2}{mkT}}$ . The partition function  $Z$ , from (8.2.7), is given by

$$\log Z = g_s \int \frac{d^3x d^3p}{(2\pi\hbar)^3} \log(1 + e^{-\beta(\epsilon - \mu)}) \quad (8.5.8)$$

This being  $\frac{pV}{kT}$ , the equation of state is given by

$$\begin{aligned} \frac{p}{kT} &= \frac{g_s}{V} \int \frac{d^3x d^3p}{(2\pi\hbar)^3} \log(1 + e^{-\beta(\epsilon - \mu)}) \\ &= \frac{g_s}{\lambda^3} \left( -\text{Li}_{\frac{5}{2}}(-z) \right) \end{aligned} \quad (8.5.9)$$

At low densities and high temperatures, we see from the power series expansion of the polylogarithms that it is consistent to take  $z$  to be small. Keeping terms up to the quadratic order in  $z$ , we get

$$z \approx \frac{\bar{n}\lambda^3}{g_s} + \frac{1}{2^{\frac{3}{2}}} \left( \frac{\bar{n}\lambda^3}{g_s} \right)^2 + \dots \quad (8.5.10)$$

$$\frac{p}{kT} = \bar{n} \left[ 1 + \bar{n} \frac{\lambda^3}{g_s 2^{\frac{5}{2}}} + \dots \right]$$

So, as in the bosonic case, we are not far from the ideal gas law. The correction may be identified in terms of the second virial coefficient as

$$B_2 = \frac{\lambda^3}{g_s 2^{\frac{5}{2}}} \quad (8.5.11)$$

This is positive; so, unlike the bosonic case, we would need a repulsive potential between classical particles to mimic this effect via the classical expression (8.3.27) for  $B_2$ .

### 8.4.1: Electrons in a Metal

Consider a two-state system in quantum mechanics and, to begin with, we take the states to be degenerate. Thus the Hamiltonian is just a diagonal  $2 \times 2$  matrix,

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} \quad (8.5.12)$$

If we consider a perturbation to this system such that the Hamiltonian becomes

$$H = H_0 + V = H_0 + \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} = \begin{pmatrix} E_0 & v \\ v & E_0 \end{pmatrix} \quad (8.5.13)$$

then the degeneracy between the two eigenstates of  $H_0$  is lifted and we have two eigenstates with eigenvalues

$$E_{\pm} = E_0 \pm v \quad (8.5.14)$$

Now consider a system with  $N$  states, with the Hamiltonian as an  $N \times N$  matrix. Starting with all states degenerate, a perturbation would split the levels by an amount depending on the perturbing term. We would still have  $N$  eigenstates, of different energies which will be close to each other if the perturbation is not large. As  $N$  becomes very large, the eigenvalues will be almost continuous; we get a band of states as the new eigenstates. This is basically what happens in a solid. Consider  $N$  atoms on a lattice. The electronic states, for each atom by itself, is identical to the electronic states of any other atom by itself. Thus we have a Hamiltonian with a very large degeneracy for any of the atomic levels. The interatomic forces act as a perturbation to these levels. The result is that, instead of each atomic level, the solid has a band of energy levels corresponding to each unperturbed single-atom state. Since typically  $N \sim$  the Avogadro number, it is a very good approximation to treat the band as having continuous energy eigenvalues between two fixed values. There are gaps between different bands, reflecting the energy gaps in the single-atom case. Thus the structure of electronic states in a solid is a series of well-separated bands with the energy levels within each band so close together as to be practically continuous. Many of these eigenstates will have wave functions localized around individual nuclei.

These correspond to the original single-atom energy states which are not perturbed very much by the neighboring atoms. Typically, inner shell electrons in a multi-electron atom would reside in such states. However, for the outer shell electrons, the perturbations can be significant enough that they can hop from one atomic nucleus to a neighbor, to another neighbor, and so on, giving essentially free electrons subject to a periodic potential due to the nuclei. In fact, for the calculation of these bands, it is a better approximation to start from free electrons in a periodic potential rather than perturbing individual atomic states. These nonlocalized bands are crucial for electrical conductivity. The actual calculation of the band structure of a solid is a formidable problem, but for understanding many physical phenomena, we only need the general structure.

Consider now a solid with the electronic states being a set of bands. We then consider filling in these bands with the available electrons. Assume that the number of electrons is such that a certain number of bands are completely filled, at zero temperature. Such a material is an insulator, because if an electric field is applied, then the electrons cannot respond to the field because of the exclusion principle, as there are no unoccupied states of nearby energy.

Table 8.4.1: Some Sample Fermi levels

Metal	$\epsilon_F$ in eV	Metal	$\epsilon_F$ in eV
Gold	5.53	Aluminum	11.7
Silver	5.49	Iron	11.1
Copper	7.00	Zinc	9.47

The only available unoccupied states are in the next higher band separated by an energy gap. As a result, the electrical conductivity is zero. If the field is strong enough to overcome the gap, then, of course, there can be conduction; this amounts to a dielectric breakdown.

However, when all the available electrons have been assigned to states, if there is a band of nonlocalized states which is not entirely filled, it would mean that there are unoccupied states very close to the occupied ones. Electrons can move into these when an electric field is applied, even if the amount of energy given by the potential is very small. This will lead to nonzero electrical conductivity. This is the case for conductors; they have bands which are not fully filled. Such bands are called conducting bands, while the filled ones are called valence bands.

The nonlocalized states of the conducting band can be labeled by the electron momentum  $\vec{k}$  with energy  $\epsilon(k)$ . The latter is, in general, not a simple function like  $\frac{\hbar^2 k^2}{2m}$ , because of interactions with the lattice of atoms and between electrons. In general, it is not isotropic either but will depend on the crystalline symmetry of the lattice. But for most metals, we can approximate it by the simple form

$$\epsilon = \frac{\hbar^2 k^2}{2m^*} \quad (8.5.15)$$

The effect of interactions can be absorbed into an effective electron mass  $m^*$ . (Showing that this can actually be done is a fairly complicated task; it goes by the name of fermi liquid theory, originally guessed, with supporting arguments, by Landau and proved to some extent by Migdal, Luttinger and others. We will not consider it here.) At zero temperature, when we have a partially filled band, the highest occupied energy level within the band is the Fermi level  $\epsilon_F$ . The value of  $\epsilon_F$  can be calculated from (8.5.3), knowing  $\bar{n}$ , the number of electrons (not bound to sites); this is shown in Table 8.4.1. Since 1 eV is of the order  $10^4$  K in terms of temperature, we see that, for phenomena at normal temperatures, we must consider the low temperature regime of the Fermi-Dirac distribution. The fugacity  $z = e^{\beta\mu}$  is very large and we need a large fugacity asymptotic expansion for various averages. This is done using a method due to Sommerfeld.

Consider the expression for  $\bar{n}$  from (8.5.7), which we can write as

$$\begin{aligned} \bar{n} &= g_s \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon - \mu)} + 1} \\ &= \frac{g_s}{\lambda^3} \frac{4}{\sqrt{\pi}} \int_0^\infty du \frac{u^2}{e^{u^2 - \beta\mu} + 1} = \frac{g_s}{\lambda^3} \frac{2}{\sqrt{\pi}} \int_0^\infty dw \frac{\sqrt{w}}{e^{w - \beta\mu} + 1} \end{aligned} \quad (8.5.16)$$

where  $u = k\sqrt{\frac{\hbar^2}{2mkT}}$  and  $w = u^2$ . The idea is to change the variable of integration to  $w - \beta\mu$ . The lower limit of integration will then be  $-\beta\mu$ , which may be replaced by  $-\infty$  as a first approximation. But in doing so, we need to ensure that the integrand vanishes at  $-\infty$ . For this one needs to do a partial integration first. Explicitly, we rewrite Equation 8.5.16 as

$$\begin{aligned}\frac{\bar{n}\lambda^3}{g_s} &= \frac{4}{3\sqrt{\pi}} \int_{-\beta\mu}^{\infty} dw \frac{(w + \beta\mu)^{\frac{3}{2}}}{(e^w + 1)(e^{-w} + 1)} \\ &= \frac{4}{3\sqrt{\pi}} \int_{-\infty}^{\infty} dw \frac{(w + \beta\mu)^{\frac{3}{2}}}{(e^w + 1)(e^{-w} + 1)} + \mathcal{O}(e^{-\beta\mu})\end{aligned}\quad (8.5.17)$$

In the first line, we have done a partial integration of the expression from Equation 8.5.16; in the second line, we replaced the lower limit by  $-\infty$ . The discrepancy in doing this is at least of order  $e^{-\beta\mu}$  due to the  $e^{-w}$  in the denominator of the integrand. This is why we needed a partial integration. We can now expand  $(w + \beta\mu)^{\frac{3}{2}}$  in powers of  $w$ ; the contribution from large values of  $|w|$  will be small because the denominator ensures the integrand is sharply peaked around  $w = 0$ . Odd powers of  $w$  give zero since integrand would be odd under  $w \rightarrow -w$ . Thus

$$\begin{aligned}\int_{-\infty}^{\infty} dw \frac{(w + \beta\mu)^{\frac{3}{2}}}{(e^w + 1)(e^{-w} + 1)} &= \int_{-\infty}^{\infty} \frac{dw}{(e^w + 1)(e^{-w} + 1)} \left[ (\beta\mu)^{\frac{3}{2}} + \frac{3}{8}(\beta\mu)^{-\frac{1}{2}} w^2 + \dots \right] \\ &= (\beta\mu)^{\frac{3}{2}} + (\beta\mu)^{-\frac{1}{2}} \frac{\pi^2}{8} + \dots\end{aligned}\quad (8.5.18)$$

This gives us the equation for  $\mu$  as

$$\frac{\bar{n}\lambda^3 3\sqrt{\pi}}{4 g_s} = (\beta\mu)^{\frac{3}{2}} + \frac{\pi^2}{8} (\beta\mu)^{-\frac{1}{2}} \quad (8.5.19)$$

By writing  $\mu = \mu_0 + \mu_1 + \dots$ , we can solve this to first order as

$$\beta\mu \approx \beta\epsilon_F \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{\epsilon_F} \right)^2 + \dots \right] \quad (8.5.20)$$

where we have used the expression for  $\epsilon_F$  in terms of  $\bar{n}$ . As expected, the value of  $\mu$  at zero temperature is  $\epsilon_F$ . Turning to the internal energy, by a similar procedure, we find

$$\begin{aligned}\frac{U}{V} &= g_s kT \frac{4}{5\sqrt{\pi}} \int_{-\infty}^{\infty} dw \frac{(w + \beta\mu)^{\frac{5}{2}}}{(e^w + 1)(e^{-w} + 1)} + \mathcal{O}(e^{-\beta\mu}) \\ &= g_s kT \frac{4}{5\sqrt{\pi}} \left[ (\beta\mu)^{\frac{5}{2}} + \frac{5\pi^2}{8} (\beta\mu)^{\frac{1}{2}} + \dots \right]\end{aligned}\quad (8.5.21)$$

Using the result (8.5.20) for  $\mu$ , this becomes

$$\frac{U}{V} = \frac{3}{5} \epsilon_F \bar{n} \left[ 1 + \frac{5\pi^2}{12} \left( \frac{kT}{\epsilon_F} \right)^2 + \dots \right] \quad (8.5.22)$$

The most interesting result of this calculation is that there is an electronic contribution to the specific heat which, at low temperatures, is given by

$$C_v = N k \frac{\pi^2}{2} \frac{kT}{\epsilon_F} + \mathcal{O}(T^3) \quad (8.5.23)$$

As expected from the third law, this too vanishes as  $T \rightarrow 0$ .

### 8.4.2: White Dwarf Stars

A gas of fermions which is degenerate is also important in many other physical phenomena, including astrophysics. Here we will briefly consider its role in white dwarf stars.

The absolute magnitude of a star which is proportional to its luminosity or total output of energy per unit time is related to its spectral characteristic, which is in turn related to the temperature of its photosphere. Thus a plot of luminosity versus spectral classification, known as a Hertzsprung-Russell diagram, is a useful guide to classifying stars. Generally, bluer stars or hotter stars have a higher luminosity compared to stars in the red part of the spectrum. They roughly fall into a fairly well-defined curve. Stars in this category are called main sequence stars. Our own star, the Sun, is a main sequence star. There are two main exceptions, white dwarfs which tend to have low luminosity even though they are white and red giants which have a higher luminosity than expected for the red part of the spectrum. White dwarfs have lower luminosity because they have basically run out of hydrogen for fusion and usually are not massive enough to pass the threshold for the fusion of higher nuclei. They are thus mostly made of helium. The radiation is primarily from gravitational contraction. (Red giants are rather low mass stars which have exhausted the hydrogen in their cores. But then the cores contract, hydrogen from outer layers get pulled in somewhat and compressed enough to sustain fusion outside the core. Because the star has a large radius, the total output is very high even though the photosphere is not very hot, only around 4000 K.)

Returning to white dwarfs, what keeps them from completely collapsing is the degeneracy pressure due to electrons. The stars are hot enough for most of the helium to be ionized and so there is a gas of electrons. The Fermi level is around 20 MeV or so, while the temperature in the core is of the order of  $10^7 \text{ K} \sim 10^3 \text{ eV}$ . Thus the electron gas is degenerate and the pressure due to this is important in maintaining equilibrium. Electron mass being  $\sim 0.5 \text{ MeV}$ , the gas is relativistic. If we use the extreme relativistic formula, the energy-momentum relation is

$$\epsilon \sim c|\vec{p}| \quad (8.5.24)$$

Calculating the Fermi level and energy density, we find

$$\begin{aligned} \frac{N}{V} \equiv \bar{n} &= g_s \int_0^{k_F} dw \frac{d^3k}{(2\pi)^3} = g_s \frac{k_F^3}{6\pi^2} \\ \frac{U}{V} &= g_s \hbar c \int_0^{k_F} dw \frac{d^3k}{(2\pi)^3} k = \frac{g_s \hbar c}{8\pi^2} \left( \frac{6\pi^2 \bar{n}}{g_s} \right)^{\frac{4}{3}} \equiv K \bar{n}^{\frac{4}{3}} \\ p &= \frac{K}{3} \bar{n}^{\frac{4}{3}} \end{aligned} \quad (8.5.25)$$

For the condition for hydrostatic equilibrium, consider a spherical shell of material in the star, of thickness  $dr$  at a radius  $r$  from the center. If the density (which is a function of the radius) is  $\rho(r)$ , then the mass of this shell is  $\rho 4\pi r^2 dr$ . The attractive force pulling this towards the center is  $-(\frac{Gm\rho}{r^2})4\pi r^2 dr$ , where  $m(r)$  is the mass enclosed inside the sphere of radius  $r$ . The pressure difference between the inside and outside of the shell under consideration is  $p(r) - p(r+dr)$ , with an outward force  $(\frac{dp}{dr})4\pi r^2 dr$ . Thus equilibrium requires

$$\frac{dp}{dr} = -\frac{G m \rho}{r^2} \quad (8.5.26)$$

Further, the mass enclosed can be written as

$$m(r) = \int_0^r dr 4\pi r^2 \rho \quad (8.5.27)$$

These two equations, along with the equation of state, give a second order equation for  $\rho(r)$ . The radius  $R$  of the star is defined by  $p(R) = 0$ .

What contributes to the pressure? This is the key issue in solving these equations. For a main sequence star which is still burning hydrogen, the kinetic pressure (due to the random movements of the material particles) and the radiation pressure contribute. For a white dwarf, it is basically the degeneracy pressure. Thus we must solve these equations, using the pressure from Equation 8.5.25. The result is then striking. If the mass of the star is beyond a certain value, then the electron degeneracy pressure is not enough to counterbalance it, and hence the star cannot continue as a white dwarf. This upper limit on the mass of a white dwarf is approximately 1.4 times the mass of the Sun. This limit is known as the Chandrasekhar limit.

What happens to white dwarfs with higher masses? They can collapse and ignite other fusion processes, usually resulting in a supernova. They could end up as a neutron star, where the electrons, despite the degeneracy pressure, have been squeezed to a point

where they combine with the protons and we get a star made of neutrons. This (very dense) star is held up by neutron degeneracy pressure. (The remnant from the Crab Nebula supernova explosion is such a neutron star.) There is an upper limit to the mass of neutron stars as well, by reasoning very similar to what led to the Chandrasekhar limit; this is known as the Tolman-Oppenheimer-Volkov limit. What happens for higher masses? They may become quark stars, and for even higher masses, beyond the stability limit of quark stars, they may completely collapse to form a black hole.

### 8.4.3: Diamagnetism and Paramagnetism

Diamagnetism and paramagnetism refer to the response of a material system to an external magnetic field  $B_i$ . To quantify this, we look at the internal energy  $U$  of the material, considered as a function of the magnetic field. The magnetization  $M_i$  of the material is then defined by

$$M_i = \frac{1}{V} \left( - \frac{\partial U}{\partial B_i} \right)_{S,V,N} \quad (8.5.28)$$

The magnetization is the average magnetic dipole moment (per unit volume) which the material develops in response to the field and, in general, is itself a function of the field  $B_i$ . For ferromagnetic materials, the magnetization can be nonzero even when we turn off the external field, but for other materials, for small values of the field, we can expect a series expansion in powers of  $B_i$ , so that

$$M_i = \chi_{ij} B_j + \mathcal{O}(B^2) \quad (8.5.29)$$

$\chi_{ij}$  is the magnetic susceptibility of the material. In cases where the linear approximation (8.5.29) is not adequate, we define

$$\chi_{ij} = \left( \frac{\partial M_i}{\partial B_j} \right) = \left( - \frac{1}{V} \right) \left( \frac{\partial^2 U}{\partial B_i \partial B_j} \right)_{S,V,N} \quad (8.5.30)$$

In general  $\chi_{ij}$  is a tensor, but for materials which are isotropic to a good approximation, we can take  $\chi_{ij} = \chi \delta_{ij}$ , defining a scalar susceptibility  $\chi$ . Materials for which  $\chi < 0$  are said to be diamagnetic while materials for which  $\chi > 0$  are said to be paramagnetic. The field  $\vec{H}$ , which appears in the Maxwell equation which has the free current  $\vec{J}$  as the source, is related to the field  $\vec{B}$  by  $\vec{H} = \vec{B}(1 - \chi) = \frac{\vec{B}}{\mu}$ ;  $\mu$  is the magnetic permeability.

Regarding magnetization and susceptibility, there is a theorem which is very simple but deep in its implications. It is originally due to Niels Bohr and later rediscovered by H.J. van Leeuwen. The theorem can be rephrased as follows.

#### Theorem 8.5.1 — Bohr-van Leeuwen Theorem

The equilibrium partition function of a system of charged particles obeying classical statistics in an external magnetic field is independent of the magnetic field.

It is very easy to prove this theorem. Consider the Hamiltonian of a system of  $N$  charged particles in an external magnetic field. It is given by

$$H = \sum_{\alpha=1}^N \frac{(p_{\alpha i} - q_{\alpha} A_i(x_{\alpha i}))^2}{2 m_{\alpha}} + V(x) \quad (8.5.31)$$

where  $\alpha$  refers to the particle,  $i = 1, 2, 3$ , as usual, and  $V$  is the potential energy. It could include the electrostatic potential energy for the particles as well as the contribution from any other source.  $A_i(x_{\alpha i})$  is the vector potential which is evaluated at the position of the  $\alpha$ -th particle. The classical canonical partition function is given by

$$Q_N = \frac{1}{N!} \int \prod_{\alpha} \frac{d^3 x_{\alpha} d^3 p_{\alpha}}{2\pi\hbar^3} e^{-\beta H} \quad (8.5.32)$$

The strategy is to change the variables of integration to

$$\Pi_{\alpha i} = p_{\alpha i} - q_{\alpha} A_i(x_{\alpha i}) \quad (8.5.33)$$

The Hamiltonian becomes

$$H = \sum_{\alpha=1}^N \frac{\Pi_{\alpha i} \Pi_{\alpha i}}{2 m_{\alpha}} + V(x) \quad (8.5.34)$$

Although this eliminates the external potential  $A_i$  from the Hamiltonian, we have to be careful about the Jacobian of the transformation. But in this case, we can see that the Jacobian is 1. For the phase space variables of one particle, we find

$$\begin{pmatrix} d\Pi_i \\ dx_i \end{pmatrix} = \begin{bmatrix} \delta_{ij} & -q \frac{\partial A_i}{\partial x_j} \\ 0 & \delta_{ij} \end{bmatrix} \begin{pmatrix} dp_j \\ dx_j \end{pmatrix} \quad (8.5.35)$$

The determinant of the matrix in this equation is easily verified to be the identity and the argument generalizes to  $N$  particles. Hence

$$Q_N = \frac{1}{N!} \int \prod_{\alpha} \frac{d^3 x_{\alpha} d^3 \Pi_{\alpha}}{2\pi\hbar^3} e^{-\beta H(\{\Pi, x\})} \quad (8.5.36)$$

We see that  $A_i$  has disappeared from the integral, proving the theorem. Notice that any mutual binding of the particles via electrostatic interactions, which is contained in  $V(x)$ , does not change this conclusion. The argument extends to the grand canonical partition since it is  $\sum_n z^n Q_N$ .

#### Note

For the cognoscenti, what we are saying is that one can describe the dynamics of charged particles in a magnetic field in two ways. We can use the Hamiltonian (8.5.31) with the symplectic form  $\omega = dp_i \wedge dx_i$  or one can use the Hamiltonian (8.5.34) with the symplectic form  $\Omega = d\Pi_i \wedge dx_i + q \frac{\partial A_j}{\partial x_i} dx_i \wedge dx_j$ . The equations of motion will be identical. But in the second form, the Hamiltonian does not involve the vector potential. The phase volume defined by  $\Omega$  is also independent of  $A_i$ . Thus the partition function is independent of  $A_i$ .

This theorem shows that the explanation for diamagnetism and paramagnetism must come from the quantum theory. We will consider these briefly, starting with diamagnetism. The full treatment for an actual material has to take account of the proper wave functions of the charged particles involved, for both the localized states and the extended states. We will consider a gas of charged particles, each of charge  $e$  and mass  $m$ , for simplicity. We take the magnetic field to be along the third axis. The energy eigenstates of a charged particle in an external uniform magnetic field are the so-called Landau levels, and these are labeled by  $p, k, \lambda$ . The energy eigenvalues are

$$E_{k,p} = \frac{p^2}{2m} + \hbar\omega(k + \frac{1}{2}), \quad k = 0, 1, \dots \quad (8.5.37)$$

where  $\omega = \frac{eB}{m}$ .  $p$  is the momentum along the third axis and  $k$  labels the Landau level. Each of these levels has a degeneracy equal to

$$\text{Degeneracy} = \frac{eB}{(2\pi\hbar)} \times \text{Area of Sample} \quad (8.5.38)$$

The states with the same energy eigenvalue are labeled by  $\lambda$ . The particles are fermions (electrons) and hence the occupation number of each state can be zero or one. Thus the partition function  $Z$  is given by

$$\begin{aligned} \log Z &= \sum_{\lambda, k, p} \log(1 + e^{-\beta E_{k,p} + \beta \mu}) \\ &= \int d^2 x \frac{eB}{(2\pi\hbar)} \frac{dp dx_3}{(2\pi\hbar)} \sum_k \log(1 + ze^{-\beta E_{k,p}}) \end{aligned} \quad (8.5.39)$$

where  $z$  is the fugacity as usual. For high temperatures, we can consider a small  $z$ -expansion. Retaining only the leading term,

$$\log Z = V \frac{eB}{(2\pi\hbar)^2} (2\pi m k T)^{\frac{1}{2}} z \frac{e^{-\frac{x}{2}}}{1 - e^{-x}} + \dots, \quad x = \frac{\hbar\omega}{kT} \quad (8.5.40)$$

For high temperatures, we can also use a small  $x$ -expansion,

$$\frac{e^{-\frac{x}{2}}}{1 - e^{-x}} \approx \frac{1}{x} \left( 1 - \frac{x^2}{24} + \dots \right) \quad (8.5.41)$$

This leads to

$$\log Z = V \left( \frac{2\pi m k T}{(2\pi \hbar)^2} \right)^{\frac{3}{2}} z \left( 1 - \frac{x^2}{24} + \dots \right) \quad (8.5.42)$$

The definition (8.5.28) is equivalent to  $dU = TdS - pdV + \mu dN - MVdB$ . From  $pV = kT \log Z$ , we have  $G - F = \mu N - F = kT \log Z$ , so that

$$d(kT \log Z) = dG - dF = Nd\mu + SdT + pdV + MVdB \quad (8.5.43)$$

which shows that

$$M = \frac{kT}{V} \left( \frac{\partial \log Z}{\partial B} \right)_{T,V,\mu} = \frac{kT}{V} \left( \frac{\partial \log Z}{\partial B} \right)_{T,V,z} \quad (8.5.44)$$

Using Equation (8.5.28) and (8.5.43), we see that

$$M = \left( \frac{2\pi m k T}{(2\pi \hbar)^2} \right)^{\frac{3}{2}} z \left[ -\frac{1}{12} \left( \frac{e\hbar}{m} \right)^2 \frac{B}{kT} + \dots \right] \quad (8.5.45)$$

Further, the average particle number is given by

$$N \equiv z \left( \frac{\partial \log Z}{\partial z} \right)_{T,V,B} = V \left( \frac{2\pi m k T}{(2\pi \hbar)^2} \right)^{\frac{3}{2}} z \left( 1 - \frac{x^2}{24} + \dots \right) \quad (8.5.46)$$

Using this to eliminate  $z$ , we find from Equation 8.5.45,

$$\begin{aligned} M &= \frac{N}{V} \left[ -\frac{1}{12} \left( \frac{e\hbar}{m} \right)^2 \frac{B}{kT} + \dots \right] \left( 1 - \frac{x^2}{24} \right)^{-1} \\ &\approx \frac{N}{V} \left[ -\frac{1}{12} \left( \frac{e\hbar}{m} \right)^2 \frac{B}{kT} \right] \end{aligned} \quad (8.5.47)$$

The diamagnetic susceptibility is thus

$$\chi \approx -\frac{N}{V} \left( \frac{e\hbar}{m} \right)^2 \frac{1}{12kT} \quad (8.5.48)$$

Although the quantum mechanical formula for the energy levels is important in this derivation, we have not really used the Fermi-Dirac distribution, since only the high temperature case was considered. At low temperatures, the Fermi-Dirac distribution will be important. The problem also becomes closely tied in with the quantum Hall effect, which is somewhat outside the scope of what we want to discuss here. So we will not consider the low temperature case for diamagnetism here. Instead we shall turn to a discussion of paramagnetism.

Paramagnetism can arise for the spin magnetic moment of the electron. Thus this is also very much a quantum effect. The Hamiltonian for a charged point particle including the spin-magnetic field coupling is

$$H = \frac{(p - eA)^2}{2m} - \frac{e}{2m} \gamma \vec{S} \cdot \vec{B} \quad (8.5.49)$$

Here  $\vec{S}$  is the spin vector and  $\gamma$  is the gyromagnetic ratio. For the electron  $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ ,  $\vec{\sigma}$  being the Pauli matrices, and  $\gamma$  is very close to 2; we will take  $\gamma = 2$ . Since we want to show how a positive  $\chi$  can arise from the spin magnetic moment, we will, for this argument, ignore the vector potential  $A$  in the first term of the Hamiltonian. The energy eigenvalues are thus

$$E_{p,\pm} = \frac{p^2}{2m} \mp \mu_0 B, \quad \mu_0 = \frac{e\hbar}{2m} \quad (8.5.50)$$



The partition function  $Z$  is thus given by

$$\log Z = \int \frac{d^3x d^3p}{(2\pi\hbar)^3} \left[ \log \left( 1 + z_+ e^{-\frac{\beta p^2}{2m}} \right) + \log \left( 1 + z_- e^{-\frac{\beta p^2}{2m}} \right) \right] \quad (8.5.51)$$

where  $Z_{\pm} = \exp(\beta\mu_{\pm})$  with

$$\mu_{\pm} = \mu \pm \mu_0 B \quad (8.5.52)$$

From  $\log Z$ , we get

$$M = \frac{1}{V} (kT) \left( \frac{\partial \log Z}{\partial B} \right)_z = \mu_0 (n_+ - n_-) \quad (8.5.53)$$

$$n_{\pm} = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\frac{p^2}{2m} - \mu_{\pm})} + 1}$$

By taking  $z(\frac{\partial \log Z}{\partial z})$ , we see that the number density of electrons for both spin states together is  $n = n_+ + n_-$ . For high temperatures, we can approximate the integral in Equation 8.5.53 by

$$n_{\pm} \approx \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\beta \frac{p^2}{2m}} e^{\beta \mu_{\pm}} = \left[ \frac{2\pi m kT}{(2\pi\hbar)^2} \right]^{\frac{3}{2}} e^{\beta \mu_{\pm}} \quad (8.5.54)$$

Using this, the magnetization becomes,

$$M = \mu_0 n \tanh \left( \frac{\mu_0 B}{kT} \right) \quad (8.5.55)$$

$$\approx \frac{\mu_0^2 n}{kT} B, \quad \text{for } \mu_0 B \ll kT$$

The susceptibility at high temperatures is thus given as

$$\chi = \frac{\mu_0^2 n}{kT} \quad (8.5.56)$$

Turning to low temperatures, notice that we have already obtained the required expansion for the integral in Equation 8.5.53; this is what we have done following Equation 8.5.16 so we can use the formula Equation 8.5.20 for  $\mu$ , along with Equation 8.5.3 for the Fermi level in terms of  $\bar{n}$ , applied in the present case to  $\mu_{\pm}$  separately. Thus

$$\mu_{\pm} = \epsilon_F(n_{\pm}) \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{\epsilon_F(n_{\pm})} \right)^2 + \dots \right] \quad (8.5.57)$$

$$\epsilon_F(n_{\pm}) = \frac{\hbar^2}{2m} (6\pi^2 n_{\pm})^{\frac{2}{3}}$$

Defining  $\Delta = n_+ - n_-$ , we can write

$$\epsilon_F(n_{\pm}) = \epsilon_F(n) \Big|_{n=(1 \pm \frac{\Delta}{n})}, \quad \epsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{\frac{2}{3}} \quad (8.5.58)$$

The fact that  $\mu_+ - \mu_- = 2\mu_0 B$  from Equation 8.5.52 now can be written as

$$2\mu_0 B = \epsilon_F \left[ \left( 1 + \frac{\Delta}{n} \right)^{\frac{2}{3}} - \left( 1 - \frac{\Delta}{n} \right)^{\frac{2}{3}} \right] - \frac{\pi^2}{12} \left( \frac{kT}{\epsilon_F} \right)^2 \left[ \frac{1}{\left( 1 + \frac{\Delta}{n} \right)^{\frac{2}{3}}} - \frac{1}{\left( 1 - \frac{\Delta}{n} \right)^{\frac{2}{3}}} \right] + \dots \quad (8.5.59)$$

After solving this for  $\frac{\Delta}{n}$ , we can get the magnetization as  $M = \mu_0 \Delta$  or  $\chi = \frac{\mu_0 \Delta}{B}$ . Since  $\Delta = 0$  for  $B = 0$ , to linear order in the magnetic field, we find

$$\frac{\Delta}{n} = \frac{3}{2} \frac{\mu_0 B}{\epsilon_F} \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{\epsilon_F} \right)^2 + \dots \right] \quad (8.5.60)$$

The susceptibility at low temperatures is then

$$\chi = \frac{3}{2} \frac{\mu_0 n}{\epsilon_F} \left[ 1 - \frac{\pi^2}{12} \left( \frac{kT}{\epsilon_F} \right)^2 + \dots \right] \quad (8.5.61)$$

The susceptibility from spin magnetic moment shows paramagnetic behavior both at high and low temperatures, as seen from Equation 8.5.56 and 8.5.61.

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