10.3: Entropy and Gravity

There is something deep about the concept of entropy which is related to gravity. This is far from being well understood, and is a topic of ongoing research, but there are good reasons to think that the Einstein field equations for gravity may actually emerge as some some sort of entropy maximization condition. A point of contact between gravity and entropy is for spacetimes with a horizon, an example being a black hole. In an ultimate theory of quantum gravity, spacetimes with a horizon may turn out to be nothing special, but for now, they may be the only window to the connection between entropy and gravity. To see something of the connection, we look at a spherical solution to the Einstein equations, corresponding to the metric around a point (or spherical distribution of) mass. This is the **Schwarzschild metric** given as

$$ds^2 = c^2 dt^2 \left(1 - \frac{2GM}{c^2 r}\right) - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{10.3.1}$$

We are writing this in the usual spherical coordinates (r, θ, φ) for the spatial dimensions. G is Newton's gravitational constant and c is the speed of light in vacuum. We can immediately see that there are two singularities in this expression. The first is obviously at $r = 0$, similar to what occurs in Newton's theory for the gravitational potential, and the second is at $r = \frac{2GM}{c^2}$. This second singularity is a two-sphere since it occurs at finite radius. Now, one can show that $r = 0$ is a genuine singularity of the theory, in the sense that it cannot be removed by a coordinate transformation. The singularity at $r = \frac{2GM}{c^2}$ is a coordinate singularity. It is like the singularity at $\theta = 0, \pi$ when we use spherical coordinates and can be eliminated by choosing a different set of coordinates. Nevertheless, the radius $\frac{2GM}{c^2}$ does have an important role. The propagation of light, in the ray optics approximation, is described by $ds = 0$. As a result, one can see that nothing can escape from $r < \frac{2GM}{c^2}$ to larger values of the radius, to be detected by observers far away. An observer far away who is watching an object falling to the center will see the light coming from it being redshifted due to the $(1 - \frac{2GM}{c^2 r})$ factor, eventually being redshifted to zero frequency as it crosses $r = \frac{2GM}{c^2}$; the object fades out. For this reason, and because it is not a real singularity, we say that the sphere at $r = \frac{2GM}{c^2}$ is a horizon. Because nothing can escape from inside the horizon, the region inside is a black hole. The value $\frac{2GM}{c^2}$ is called the **Schwarzschild radius**.

Are there examples of black holes in nature? The metric (10.3.1) can be used to describe the spacetime outside of a nearly spherical matter distribution such as a star or the Sun. For the Sun, with a mass of about $2 \times 10^{30} kg$, the Schwarzschild radius is about $1.4 km$. The form of the metric in (10.3.1) ceases to be valid once we pass inside the surface of the Sun, and so there is no horizon physically realized for the Sun (and for most stars). (Outside of the gravitating mass, one can use (10.3.1) which is how observable predictions of Einstein's theory such as the precession of the perihelion of Mercury are obtained.) But consider a star which is more massive than the **Chandrasekhar** and **Tolman-Oppenheimer-Volkov limits**. If it is massive enough to contract gravitationally overcoming even the quark degeneracy pressure, its radius can shrink below its Schwarzschild radius and we can get an black hole. The belief is that there is such a black hole at the center of our galaxy, and most other galaxies as well.

Returning to the physical properties of black holes, although classical theory tells us that nothing can escape a black hole, a most interesting effect is that black holes radiate. This is a quantum process. A full calculation of this process cannot be done without a quantum theory of gravity (which we do not yet have). So, while the fact that black holes must radiate can be argued in generality, the nature of the radiation can only be calculated in a semiclassical way. The result of such a semiclassical calculation is that irrespective of the nature of the matter which went into the formation of the black hole, the radiation which comes out is thermal, following the Planck spectrum, corresponding to a certain temperature

$$T_H = \frac{\hbar c^3}{8\pi k G M} \tag{10.3.2}$$

Although related processes were understood by many scientists, the general argument for radiation from black holes was due to Hawking and hence the radiation from any spacetime horizon and the corresponding temperature are referred to as the **Hawking radiation** and **Hawking temperature**, respectively.

Because there is a temperature associated with a black hole, we can think of it as a thermodynamic system obeying

$$dU = T\, dS \tag{10.3.3}$$

The internal energy can be taken as Mc^2 following the Einstein mass-energy equivalence. We can then use Equation 10.3.3 to calculate the entropy of a black hole as

$$S_{B-H} = \frac{c^3}{\hbar}\frac{A}{4\,G} \tag{10.3.4}$$

(This formula for the entropy is known as the Bekenstein-Hawking formula.) Here A is the area of the horizon, \(A = 4πR_S^2 , R_S = \frac{2GM}{c^2} being the Schwarzschild radius.

These results immediately bring up a number of puzzles.

1. *A priori*, there is nothing thermodynamic about the Schwarzschild metric or the radiation process. The radiation can obtained from the quantized version of the Maxwell equations in the background spacetime (10.3.1). So how do thermodynamic concepts arise in this case?
2. One could envisage forming a black hole from a very ordered state of very low entropy. Yet once the black hole forms, the entropy is given by (10.3.4). There is nothing wrong with generating more entropy, but how did we lose the information coded into the low entropy state? Further, the radiation coming out is thermal and hence carries no information. So is there any way to understand what happened to it? These questions can be sharpened further. First of all, we can see that the Schwarzschild black hole can evaporate away by Hawking radiation in a finite time. This is because the radiation follows the Planck spectrum and so we can use the Stefan-Boltzmann law (8.3.14) to calculate the rate of energy loss. Then from
$$\frac{d(M\,c^2)}{dt} = -\sigma T_H^4 A \tag{10.3.5}$$
we can obtain the evaporation time. Now, there is a problem with the radiation being thermal. Time-evolution in the quantum theory is by unitary transformations and these do not generate any entropy. So if we make a black hole from a very low entropy state and then it evaporates into thermal radiation which is a high entropy state, how is this compatible with unitary time-evolution? Do we need to modify quantum theory, or do we need to modify the theory of gravity?
3. Usually, when we have nonzero entropy, we can understand that in terms of microscopic counting of states. Are the number of states of a black hole proportional to S_{B-H}? Is there a quantitative way to show this?
4. The entropy is proportional to the area of the horizon. Usually, entropy is extensive and the number of states is proportional to the volume (via things like $\frac{d^3x d^3p}{(2\pi\hbar)^3}$). How can all the states needed for a system be realized in terms of a lower dimensional surface?

There are some tentative answers to some of these questions. Although seemingly there is a problem with unitary time-evolution, this may be because we cannot do a full calculation. The semiclassical approximation breaks down for very small black holes. So we cannot reliably calculate the late stages of black hole evaporation. Example calculations with black holes in lower dimensions can be done using string theory and this suggests that time-evolution is indeed unitary and that information is recovered in the correlations in the radiation which develop in later stages.

For most black hole solutions, there is no reliable counting of microstates which lead to the formula (10.3.4). But there are some supersymmetric black holes in string theory for which such a counting can be done using techniques special to string theory. For those cases, one does indeed get the formula (10.3.4). This suggests that string theory could provide a consistent quantum theory of black holes and, more generally, of spacetimes with horizons. It could also be that the formula (10.3.4) has such universality (as many things in thermodynamics do) that the microscopic theory may not matter and that if we learn to do the counting of states correctly, any theory which has quantum gravity will lead to (10.3.4), with perhaps, calculable additional corrections (which are subleading, i.e., less extensive than area).

The idea that a lower dimensional surface can encode enough information to reconstruct dynamics in a higher dimensional space is similar to what happens in a hologram. So perhaps to understand the entropy formula (10.3.4), one needs a holographic formulation of physical laws. Such a formulation is realized, at least for a restricted class of theories, in the so-called **AdS/CFT correspondence** (or **holographic correspondence**) and its later developments. The original conjecture for this is due to J. Maldacena and states that string theory on an anti-de Sitter(AdS) spacetime background in five dimensions (with an additional 5-sphere) is dual to the maximally supersymmetric Yang-Mills gauge theory (which is a conformal field theory (CFT)) on the boundary of the AdS space. One can, in principle, go back and forth, calculating quantities in one using the other. Although still a conjecture, this does seem to hold for all cases where calculations have been possible.

It is clear that this is far from a finished story. But from what has been said so far, there is good reason to believe that research over the next few years will discover some deep connection between gravity and entropy.