2.4: How to multiply vectors? Heuristic considerations

In evaluating the various methods of multiplying vectors with vectors, we start with a critical analysis of the procedure of elementary vector calculus based on the joint use of the inner or scalar product and the vector product.

The first of these is readily generalized to $V(n, R)$, and we refer to the literature for further detail. In contrast, the vector product is tied to three dimensions, and in order to generalize it, we have to recognize that it is commonly used in two contexts, to perform entirely different functions.

First to act as a rotation operator, to provide the increment $\delta\vec{a}$ of a vector $\vec{a}$ owing to a rotation by an angle $\delta\theta$ around an axis $\hat{n}$:

$$\delta\vec{a} = \delta\theta\hat{n} \times \vec{a} \tag{2.4.1}$$

Here $\delta\theta\hat{n}$ is a dimensionless operator that transforms a vector into another vector in the same space.

Second, to provide an "area", the dimension of which is the product of the dimension of the factors. In addition to the geometrical case, we have also such constructs as the angular momentum

$$\vec{L} = \vec{r} \times \vec{p} \tag{2.4.2}$$

The product is here "exterior" to tha original vector space.

There is an interesting story behind this double role of the vector product. Gibbs' vector algebra arose out of the attempt of reconciling and simplifying two ingenious, but complicated geometric algebras which were advanced almost simultaneously in the 1840's. Sir William Rowan Hamilton's theory of quaternions is adapted to problems of rotation in three- and four-dimensional spaces, whereas Hermann Grassman's Ausdehnungslehre (Theory of Extensions) deals with volumes in spaces of an arbitrary number of dimensions. The dichotomy corresponds to that of Equations 2.4.1 and 2.4.2.

The complementary character of the two calculi was not recognized at the time, and the adherents of the two methods were in fierce competition. Gibbs found his way out of the difficulty by removing all complicated and controversial elements from both calculi and by reducing them to their common core. The result is our well known elementary vector calculus with its dual-purpose vector product which seemed adequate for three-dimensional/space.

Ironically, the Gibbsian calculus became widely accepted at a time when the merit of Hamilton's four-dimensional rotations was being vindicated in the context of the Einstein-Minkowski fourdimensional world.

Although it is possible to adapt quaternions to deal with the Lorentz group, it is more practical to use instead the algebra of complex two-by-two matrices, the so-called Pauli algebra, and the complex vectors (spinors) on which these matrices operate. These methods are descendents of quaternion algebra, but they are more general, and more in line with quantum mechanical techniques. We shall turn to their development in the next Chapter.

In recent years, also some of Grassmann's ideas have been revived and the exterior calculus is now a standard technique of differential geometry (differential forms, calculus of manifolds). These matters are relevant to the geometry of phase space, and we shall discuss them later on.