5.3: Polarized Light

Polarization optics provides a most appropriate field of application for the Pauli algebra and the spinor formalism. Historically, of course, it went the other way around, and various aspects of the formalism had been advanced by many authors, often through independent discovery in response to a practical need.

In the present discussion we forego the historical approach and by using the mathematical formalism already developed, we arrive at the consolidation and streamlining of much disconnected material.

Another factor which simplifies our argument is that we do not attempt to describe polarization in all the complexity of a real situation, but concentrate first on a simple mathematical model, the two-dimensional isotropic, harmonic oscillator. This is, of course, the standard method of the elementary theory, however, by translating this description into the spinorial formalism, we set the stage for generalizations. A potential generalization would be to establish the connection with the statistical theory of coherence. However, at the present stage we shall be more concerned with applications to quantum mechanics.

Let us consider a monochromatic, polarized plane wave propagating in the z direction and write for the x and y components of the electric field

$$\begin{aligned} E_x &= p_1\cos(\omega t+\phi_1) = p_1\cos\tau \\ E_y &= p_2\cos(\omega t+\phi_2) = p_2\cos(\tau-\phi) \end{aligned} \tag{5.3.1}$$

where

$$\phi=\phi_1-\phi_2, \quad p_1,p_2\geq 0 \tag{5.3.2}$$

Let us define new parameters:

$$\begin{aligned} p_1 &= p\cos\frac{\theta}{2} \\ 0 &\leq \theta\leq\pi \\ p_2 &= p\sin\frac{\theta}{2} \end{aligned} \tag{5.3.3}$$

It is convenient to express the information contained in Equations 5.3.1–5.3.3 in terms of the spinor

$$|\hat{k}\rangle = p\begin{pmatrix} e^{-i\phi/2} & \cos(\theta/2) \\ e^{i\phi/2} & \sin(\theta/2)\end{pmatrix} e^{-i\psi/2} \tag{5.3.4}$$

Here $\psi=\omega t+\phi_1$ 1 represents the common phase of the two components which does not affect the state of polarization. However, the presence of this third angle is in line with our definition of spinor in Equations 5.3.10 and 5.3.11 in Section 5.1. It will prove to be of significance in the problem of beam splitting and composition. By normalizing the intensity and setting $p=1$, the spinor 5.3.4 conforms to our unitary normalization of Section 5.1.

By using Equations 5.3.30, 5.3.36 and 5.3.38 of Section 5.1 we obtain

$$|\hat{k}\rangle\langle\hat{k}| = \frac{1}{2}(1+\hat{k}\cdot\vec{\sigma}) \tag{5.3.5}$$

$$\begin{aligned} k_1 &= \left\langle \hat{k}\,|\sigma_1|\,\hat{k}\right\rangle = \sin\theta\cos\phi = 2p_1p_2\cos\phi \\ k_2 &= \left\langle \hat{k}\,|\sigma_2|\,\hat{k}\right\rangle = \sin\theta\sin\phi = 2p_1p_2\sin\phi \\ k_3 &= \left\langle \hat{k}\,|\sigma_3|\,\hat{k}\right\rangle = \cos\theta = p_2^2-p_1^2 \end{aligned} \tag{5.3.6}$$

In such a fashion the spinor 5.3.5, and hence each state of polarization is mapped on the surface of the unit sphere, the so-called Poincare sphere.

We see that the unit vector $(1,0,0)(\theta=\pi/2,\phi=0)$ corresponds to linear polarization along $\frac{1}{\sqrt{2}}(\hat{x}+\hat{y}),$ or $|45^\circ\rangle$ $(0,1,0)(\theta=\pi/2,\phi=\pi/2)$ corresponds to right circularly polarized light $|R\rangle$, and $(0,0,1)$, and $(0,0,1)$ or $\theta=0$ to linear polarization in the $\hat{x}$ direction: $|\hat{x}\rangle$. . (See Figures 5.3 and 5.3.)

There is an alternative, and even more favored method of parametrizing the Poincaré sphere, in which the preferred axis for the definition of spherical coordinates corresponds to light of positive helicity $|R\rangle$. This choice implies a new set of spherical angles, say α, β to replace ϕ, θ. Their relation is displayed geometrically in Figures 5.3 and 5.4. The corresponding algebraic treatment is summed up as follows.

We relabel the Cartesian axes in the "Poincare space" as

$$\begin{aligned} k_3 &= s_1 = \sin\beta\cos\alpha \\ k_1 &= s_2 = \sin\beta\sin\alpha \\ k_2 &= s_3 = \cos\beta \end{aligned} \tag{5.3.7}$$

The vector $\hat{s}$ is associated with the unitary spinor

$$|\hat{s}\rangle = \begin{pmatrix} \exp(-i\alpha/2) & \cos(\beta/2) \\ \exp(i\alpha/2) & \sin(\beta/2) \end{pmatrix} \tag{5.3.8}$$

and

$$|\hat{s}\rangle\langle\hat{s}| = \frac{1}{2}(1 + \hat{s}\cdot\sigma) \tag{5.3.9}$$

The advantage of this choice is that the angles α, β have a simple meaning. We claim that

$$\begin{aligned} a_1 &= a\cos\left(\tfrac{1}{2}\left(\tfrac{\pi}{2} - \beta\right)\right) \\ a_2 &= a\sin\left(\tfrac{1}{2}\left(\tfrac{\pi}{2} - \beta\right)\right) \end{aligned} \tag{5.3.10}$$

where a_1, a_2 are the half major and minor axes of the ellipse traced by the $\vec{E}$ vector; we associate a positive and negative a_2 with an ellipse circled in the positive and negative sense respectively. Moreover, the angle α is twice the angle of inclination of the major axis against the x axis (Figure 5.4-d). The angle γ refers to the overall phase in complete analogy to ψ.

The proof of these statements are found in Born and Wolf (see pp. 24-32 of [BW64], the later editions are almost unchanged). A somewhat simplified derivation follows.

First we prove that Equations 5.3.1 and 5.3.2 provide indeed a parametric representation of an ellipse. The elimination of τ from the two equations 5.3.1 yields

$$\left(\frac{E_1}{p_1\sin\phi}\right)^2 - \left(\frac{2E_1E_2\cos\phi}{p_1p_2\sin^2\phi}\right) + \left(\frac{E_2}{p_2\sin\phi}\right)^2 = 1 \tag{5.3.11}$$

This is an equation of the form

$$\sum_{i=1}^{2} a_{ik}x_ix_k = 1 \tag{5.3.12}$$

with the a_{ik} real, symmetric, and $a_{11} > 0$,

$$a_{11}a_{22} - a_{12}^2 > 0 \tag{5.3.13}$$

The axes of the ellipse are derived from the eigenvalue problem:

$$\begin{aligned} (a_{11} - \lambda)\,x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)\,x_2 &= 0 \end{aligned} \tag{5.3.14}$$

Hence

$$\lambda^2 - (a_{11} + a_{22})\,\lambda + a_{11}a_{22} - a_{12}^2 = 0 \tag{5.3.15}$$

with

$$\lambda_1 = \frac{1}{a_1^2}, \quad \lambda_2 = \frac{1}{a_2^2} \tag{5.3.16}$$

where a_1, a_2 are the half major and half minor axes respectively. We have, by inserting for the a_{ik}, from Equation 5.3.11

$$\lambda_1+\lambda_1=a_{11}+a_{22}=\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}\right)\frac{1}{\sin^2\phi} \tag{5.3.17}$$

$$\lambda_1+\lambda_1=a_{11}a_{22}-a_{12}^2=\frac{1}{p_1^2p_2^2\sin^2\phi} \tag{5.3.18}$$

From these equations we have

$$a_1^2a_2^2=p_1^2p_2^2\sin^2\phi \tag{5.3.19}$$

$$a_1^2+a_2^2=p_1^2+p_2^2 \tag{5.3.20}$$

From Equation 5.3.19 we have $a_1a_2=\pm p_1p_2\sin\phi$. We require

$$a_1a_2=p_1p_2\sin\phi \tag{5.3.21}$$

and let $a_2<0$ for $\sin\phi<0$

We introduce now the auxiliary angle 3 as defined in Equation 5.3.10. With such an assignment $\beta=0,\pi$ correspond indeed to right and left circularly polarized light $|R\rangle,|\bar{R}\rangle$ respectively. More over $a_1\geq|a_2|$. Hence a_1 is the half major axis.

From Equations 5.3.3, 5.3.19 and 5.3.10 we obtain

$$\cos\beta=\sin\theta\sin\phi \tag{5.3.22}$$

We complete the parametrization of ellipticity by introducing $\alpha/2$ for the angle between the major axis and the $\hat{x}$ direction (Figure 5.4-c).

From Equation 5.3.14 we have

$$\tan\frac{\alpha}{2}=\frac{x_2}{x_1}=\frac{\lambda-a_{11}}{a_{12}}=\frac{a_{12}}{\lambda-a_{22}} \tag{5.3.23}$$

and

$$\begin{aligned}\tan\alpha&=\frac{\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}=\frac{\frac{\lambda-a_{11}}{a_{12}}+\frac{a_{12}}{\lambda-a_{22}}}{\frac{\lambda-a_{11}}{\lambda-a_{22}}}\\&=\frac{(\lambda-a_{11})(\lambda-a_{22})\frac{1}{a_{12}}+a_{12}}{a_{11}-a_{22}}=\frac{2a_{12}}{a_{11}-a_{22}}\\&=\frac{2p_1p_2\cos\phi}{p_1^2-p_2^2}=\frac{\sin\theta\cos\phi}{\cos\theta}=\tan\theta\cos\phi\end{aligned} \tag{5.3.24}$$

It is apparent from Equation 5.3.22 that the axis s_3 can be indeed identified with k_2. Moreover Equation 5.3.24 yields

$$\frac{s_2}{s_1}=\frac{k_1}{k_3} \tag{5.3.25}$$

Since, $k_1^2+k_2^2+k_3^2=1=s_1^2=s_2^2=s_3^2$ we arrive at the rest of the identification suggested in Equation 5.3.7.

We shall refer to the formalism based on the parametrizations $\hat{k}(\phi,\theta,\psi)$ and $\hat{s}(\alpha,\beta,\gamma)$ as the $\hat{k}$ scheme and the $\hat{s}$-scheme respectively. Since either of the two pairs of angles ϕ,θ and α,β provide a satisfactory description of the polarization state, it is worthwhile to deal with both schemes.

The role of the “third angle” ψ or γ respectively, is more subtle. It is well known that a spinor can be visualized as a vector and an angle, a “flagpole” and a “flag” in Penrose’s terminology. However, the angle represents a phase, and as such has notoriously ambivalent properties. While a single phase is usually unimportant, phase relations are often most significant. Although one can solve particular problems in polarization optics in terms of the Poincaré sphere without an explicit use of the third angle, for us these problems are merely stepping stones for deeper problems and we prefer to present them as instances of the general formalism. No matter if this seems to be a somewhat heavy gun for the purpose.

In proceeding this way we have to ignore some fine distinctions; thus we assign $|\xi\rangle$ and $-|\xi\rangle 7$ to the same state of polarization. We consider it an advantage that the formalism has the reserve capacity to be used later to such problems as the electron spin.

We demonstrate the usefulness of the spinor formalism by translating one of its simple propositions into, what might be called the fundamental theorem of polarization optics.

Consider two pairs of conjugate spinors $|\xi\rangle, |\bar{\xi}\rangle$ and $|\xi'\rangle, \left|\bar{\xi}'\right\rangle$

Theorem 3. There is a uniquely determined unimodular unitary matrix U such that

$$\begin{aligned} |\xi'\rangle &= U|\xi\rangle \\ \left|\bar{\xi}'\right\rangle &= U|\bar{\xi}\rangle \end{aligned} \tag{5.3.26}$$

Proof. By using Equation 5.3.27of Section 5.1 we consider the unitary matrices associated with the spinor pairs:

$$V = (|\xi\rangle, |\bar{\xi}\rangle) \tag{5.3.27}$$

$$V' = \left(|\xi'\rangle, \left|\bar{\xi}'\right\rangle\right) \tag{5.3.28}$$

The matrix $U = V'V^{-1}$ has the desired properties, since $UV = V'$

Let the monoaxial parametrization of U be $U(\hat{u}, \chi/2)$. By using Equations 5.1.58 and 5.1.59 of Section 5.1 we see that U has two eigenspinors:

$$U|\hat{u}\rangle = \exp(-i\chi/2)|\hat{u}\rangle \tag{5.3.29}$$

$$U|\bar{\hat{u}}\rangle = \exp(i\chi/2)|\bar{\hat{u}}\rangle \tag{5.3.30}$$

Hence U produces a phase shift between the conjugate states $|\hat{u}\rangle$ and $|\bar{\hat{u}}\rangle$; moreover it rotates their linear combinations:

$$|\xi\rangle = a_0|\hat{u}\rangle + a_1|\bar{\hat{u}}\rangle \tag{5.3.31}$$

where

$$|a_0|^2 + |a_1|^2 = 1 \tag{5.3.32}$$

These results translate into polarization optics as follows. An arbitrary, fully polarized beam can be transformed into another beam of the same kind by a phase shifter, the axis $\hat{u}$ of which is uniquely determined in terms of the spinor representation of the given beams. Since the result is a rotation of the Poincaré sphere, the axis of the phase shifter can be determined also geometrically.

To counteract the complete generality of the Poincaré construction, let us consider the special cases

$$U = U\left(\hat{k}_3, \frac{\Delta\phi}{2}\right) \tag{5.3.33}$$

$$U = U\left(\hat{s}_3, \frac{\Delta\alpha}{2}\right) \tag{5.3.34}$$

The phase shifter, Equation 5.3.33, is called a linear retarder, it establishes a phase lag between one state of linear polarization and its antipodal state. For $\Delta\phi = \pi/2$ we have a quarter waveplate that transforms elliptic into linear polarization or vice versa.

The phase shifter, Equation 5.3.34, produces a phase lag between right and left circularly polarized beams. (A circularly biregringent crystal, say quartz is cut perpendicularly to the optic axis: spiral staircase effect.)

Since a linearly polarized beam is the linear composition $|R\rangle$ and $|L\rangle$ the phase lag manifests itself in a rotation of the plane of polarization, hence a rotation around $\hat{s}_3$. The device is called a rotator.

Thus rotations of the Poincare´ sphere may produce either a change of shape, or a change of orientation in ordinary space.

We may add that by combining two quarter waveplates with one general rotator we can realize an arbitrary phase shifter $U(\hat{u}, \chi/2)$.

Our main theorem on the representation of the transformation of fully polarized beams is evidently the counterpart of Euler's theorem on the displacements of the gyroscope mentioned on page 56.

Although we have a formal identity, in the sense that we have in both cases the rotation of a triad, there is a great deal of difference in the physical interpretation. The rotation takes place now in an abstract space, we may call it the Poincare´ space. Also it makes a great deal of difference that the angular velocities of the rotating object are now replaced by the time rates of change of the phase difference between pairs of conjugate polarizations. On going from rigid bodies to polarized waves (degenerate vibrations) we do not have to modify the formalism, but the new interpretation opens up new opportunities. The concept of phase difference evokes the idea of coherent superposition as contrasted with incoherent composition. These matters have no analog in the case of rigid rotation, and we now turn to the examination of the new features.

Let us consider a polarized beam represented in the $\hat{s}(\alpha, \beta)$ scheme by the spinor $|\hat{s}\rangle$ where

$$S=|\hat{s}\rangle\langle\hat{s}|=\frac{1}{2}(1+\hat{s}\cdot\vec{\sigma}) \tag{5.3.35}$$

or alternatively

$$S=\begin{pmatrix} s_0 s_0^* & s_0 s_1^* \\ s_1 s_0^* & s_0 s_1^* \end{pmatrix} \tag{5.3.36}$$

S is called the density matrix or coherency matrix associated with a polarized beam (a pure state in quantum mechanics). As we have seen already, it is idempotent and the determinant $|S|=0$

We analyze this beam with an instrument $U(u, \Delta\psi/2)$ where $\hat{u}\neq\hat{s}$ where $\hat{u}\neq\hat{s}$, and obtain

$$|\hat{s}\rangle=a_0|\hat{u}\rangle+a_1|\bar{\hat{u}}\rangle \tag{5.3.37}$$

with

$$a_0=\langle\hat{u}\mid\hat{s}\rangle \quad a_1=\langle\bar{\hat{u}}\mid\hat{s}\rangle \tag{5.3.38}$$

$$|a_0|^2+|a_1|^2=1 \tag{5.3.39}$$

From Equations 5.1.58 and 5.1.58 we have

$$\begin{aligned} |\xi'\rangle &= U\left(\hat{u}, \frac{\Delta\psi}{2}\right)|\hat{s}\rangle+ \\ &= a_0 \exp\left(-i\frac{\Delta\psi}{2}\right)|\hat{u}\rangle a_1 \exp\left(i\frac{\Delta\psi}{2}\right)|\bar{\hat{u}}\rangle \\ &= a_0\left|\hat{u}, \frac{\psi+\Delta\psi}{2}\right\rangle+a_1\left|\hat{u}, \frac{\psi+\Delta\psi}{2}\right\rangle \end{aligned} \tag{5.3.40}$$

Let us now assume that the instrument U is doubled up with a reverse instrument that reunites the two beams that have been separated in the first step. This reunification may happen after certain manipulations have been performed on the separated beams. Such a device, the so-called analyzing loop has been used more for the conceptual analysis of the quantum mechanical formalism than for the practical purposes of polarization optics.

Depending on the nature of the manipulations we have a number of distinct situations which we proceed to disentangle on hand of the following formulas.

We obtain from Equation 5.3.40

$$\frac{1}{2}S'=\left|\hat{s}'\right\rangle\left\langle\hat{s}'\right|=|a_0|^2\mid\hat{u}\big\rangle\big\langle\hat{u}|+|a_1|^2\mid\bar{\hat{u}}\big\rangle\big\langle\bar{\hat{u}}\,|+a_0 a_1^*\exp(-i\Delta\psi)|\,\hat{u}\big\rangle\langle\bar{\hat{u}}| \tag{5.3.41}$$

Here S' is idempotent and of determinant zero just as S is, since $\left|\hat{s}'\right\rangle$ arises out of $|\hat{s}\rangle$ by means of a unitary operation.

Let us consider now a different case in which the phase difference between the two partial beams has been randomized. In fact, take first the extreme case in which the interference terms vanish:

$$\left\langle a_0 a_1^* \exp(-i\Delta\psi)\right\rangle_{av}=\left\langle a_1 a_0^* \exp(i\Delta\psi)\right\rangle_{av}=0 \tag{5.3.42}$$

We obtain from Equations 5.3.41, 5.3.42and 5.3.39

$$S'=1+\left(|a_0|^2-|a_1|^2\right)\hat{u}\cdot\vec{\sigma} \tag{5.3.43}$$

We write S' as

$$S' = 1 + s'\hat{u} \cdot \vec{\sigma} \tag{5.3.44}$$

where $0 \leq s' < 1$ and

$$0 < |S| = 1 + s'^2 \leq 1 \tag{5.3.45}$$

We have now a generazized form of the density matrix associated with a partially polarized or even natural light (if $s' = 0$). In quantum mechanics we speak of a mixture of states.

It is usual in optics to change the normalization and set for a partially polarized beam

$$S = s_0 + s\hat{s} \cdot \vec{\sigma} \tag{5.3.46}$$

where s_0 is the total intensity and s the intensity of the polarized component. We have for the determinant

$$0 \leq |S| = s_0^2 - s^2 \leq s_0^2 (31) \tag{5.3.47}$$

which is zero for polarized light and positive otherwise.

In addition to conserving or destroying phase relations, one may operate directly on the intensity as well. If one of the components of the analyser, say $|\hat{u}\rangle$ or $|\bar{\hat{u}}\rangle$ is blocked off, the instrument acts as a perfect polarizer.

Formally, we can let the projection operator $\frac{1}{2}(1 \pm \hat{u} \cdot \vec{\sigma})$ act on the density matrix of the beam, which may be polarized fully, partially, or not at all. Nonpolarized or natural light can be considered as a statistical ensemble of polarized light beams uniformly distributed over the Poincare´ sphere. (See problem #15.)

An imperfect polarizer (such as a sheet of polaroid) exhibits an unequal absorption of two conjugate linear polarizations. It can be represented as a Hermitian operator acting on S.

We have seen above that the inchoherent composition of the two beams of an analyzer is accounted, for by the addition of the density matrix.

Conversely, every partially polarized beam can be constructed in such a fashion. (See problem #13.)

Yet we may wish to add incoherently an arbitrary set of partially polarized beams, and this is always accomplished by adding their density matrices. The question arises then: Could we not operate phenomenologically in terms of the density matrices alone?

The matter was considered already by Stokes (1852) who introduced a column vector with the four components I, M, C, S corresponding to our $s_0, \vec{s}$. A general instrument is represented by a real 4×4 matrix. Note that the "instrument" might be also a molecule producing a change of polarization on scattering.

The $4x4$ matrices are commonly called Mueller matrices. This formalism is usually mentioned along with the Jones calculus of 2×2 complex matrices. This was developed by R. Clark Jones of the Polaroid Co. and his collaborators in a long series of papers in Journal of the American Optical Society in the 1940's (quoted e.g., by Shurcliff, and C. Whitney). This is basically a two-component spinor theory to deal with instruments which modify the polarization without depolarization or loss of intensity. It was developed in close contact with experiment without reliance on an existing mathematical formalism.

Mueller liked to emphasize the purely phenomenological character of his formalism. The four Stokes parameters of a beam can be determined from measurements by four filters. However a difficulty of this phenomenological approach is that not every 4×4 matrix corresponds to a physically realizable instrument or scattering object. This means that a so-called passive instrument must neither increase total intensity nor create phase correlations. The situation is simpler in the 2×2 matrix formulation in which the redundant parameters have been eliminated.

However, we do not enter into such details, since polarization optics is not our primary concern. In fact, the two-valued ness of the full spinor formalism brings about a certain complication which is justified by the fact that our main interest is in the applications to quantum mechanics. We shall compare the different types of applications which are available at this juncture in Section 5.5.

Meanwhile in the next section we show that the concept of unitary spinor can be generalized to relativistic situations. This is indispensable if the formalism is to be applied also to the propagation rather than only the polarization of light.