2.1: Groups

When group theory was introduced into the formalism of quantum mechanics in the late 1920's to solve abstruse spectroscopic problems, it was considered to be the hardest and the most unwelcome branch of mathematical physics. Since that time group theory has been simplified and popularized and it is widely practiced in many branches of physics, although this practice is still limited mostly to difficult problems where other methods fail.

In contrast, I wish to emphasize that group theory has also simple aspects which prove to be eminently useful for the systematic presentation of the material of this course.

Postponing for a while the precise definition,- we state somewhat loosely that we call a set of elements a group if it is closed with respect to a single binary operation usually called multiplication. This multiplication is, in general not to be taken in the common sense of the word, and need not Mbe commutative. It is, however, associative and invertible.

The most common interpretation of such an operation is a transformation. Say, the translations and rotations of Euclidean space; the transformations that maintain the symmetry of an object such as a cube or a sphere. The transformations that connect the findings of different inertial observers with each other.

With some training we recognize groups anywhere we look. Thus we can consider the group of displacement of a rigid body, and also any particular subset of these displacements' that arise in the course of a particular motion.

We shall see indeed, that group theory provides a terminology that is invaluable for the precise and intuitive discussion of the most elementary and fundamental principles of physics. As to the discussion of specific problems we shall concentrate on those that can be adequately handled by stretching the elementary methods, and we shall not invoke advanced group theoretical results. Therefore we turn now to a brief outline of the principal definitions and theorems that we shall need in the sequel.

Let us consider a set of elements $A, B, C, \ldots$ and a binary operation that is traditionally called "multiplication". We refer to this set as a group $\mathcal{G}$ if the following requirements are satisfied.

1. For any ordered pair, A, B there is a product AB = C. The set is closed with respect to multiplication.
2. The associative law holds: (AB)C = A(BC).
3. There is a unit element $E \in \mathcal{G}$ such that $EA = AE = A$ for all $A \in \mathcal{G}$. .
4. For each element A there is an inverse $A^{-1}$with $A^{-1}A = AA^{-1} = E$.

The multiplication need not be commutative. If it is, the group is called **Abelian**.

The number of elements in $\mathcal{G}$ is called the **order of the group**. This may be finite or infinite, denumerable or continuous.

If a subset of $\mathcal{G}$ satisfies the group postulates, it is called a **subgroup**.

2.1.1 Criterion for Subgroups

If a subset of the elements of a group of finite order $\mathcal{G}$ is closed under multiplication, then it is a subgroup of $\mathcal{G}$.

Prove that the group postulates are satisfied. Discuss the case of groups of infinite order.

In order to explain the use of these concepts we list a few examples of sets chosen from various branches of mathematics of interest in physics, for which the group postulates are valid.

Examples

1. The set of integers (positive, negative and zero) is an Abelian group of infinite order where the common addition plays the role of multiplication. Zero serves as the unit and the inverse of a is $-a$.

2. The set of permutations of n objects, called also the **symmetric group** $S(n)$, is of order n!. It is non-Abelian for $n > 2$.

3. The infinite set of $n \times n$ matrices with non-vanishing determinants. The operation is matrix multiplication; it is in general non-commutative.

4. The set of covering operations of a symmetrical object such as a rectangular prism (fourgroup), a regular triangle, tetrahedron, a cube or a sphere, to mention only a few important cases. Expressing the symmetry of an object, they are called symmetry groups. Multiplication of two elements means that the corresponding operations are carried out in a definite sequence. Except for the first case, these groups are non-Abelian.

The concrete definitions given above specify the multiplication rule for each group. For finite groups the results are conveniently represented in multiplication tables, from which one extracts the entire group structure. One recognizes for instance that some of the groups of covering operations listed under (4) are subgroups of others.

It is easy to prove the rearrangement theorem: In the multiplication table each column or row contains each element once and only once. This theorem is very helpful in setting up multiplication tables. (Helps to spot errors!)

2.1.2 Cyclic Groups

For an arbitrary element A of a finite $\mathcal{G}$ form the sequence: $A, A^2, A^3 \dots$ let the numbers of distinct elements in the sequence be p. It is easy to show that $A^p = E$. The sequence

$$A, A^2, \dots, A^p = E \tag{2.1.1}$$

is called the period of $A; p$ is the order of A. The period is an Abelian group, a subgroup of $\mathcal{G}$. It may be identical to it, in which case $\mathcal{G}$ is called a cyclic group.

Corollary: Since periods are subgroups, the order of each element is a divisor of the order of the group.

2.1.3 Cosets

Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ with elements $E, H_2, \dots H_h$; the set of elements

$$EA, H_2A, \dots, H_hA \tag{2.1.2}$$

is called a right coset $\mathcal{H}_A$ provided A is not in $\mathcal{H}$. It is easily shown that $\mathcal{G}$ can be decomposed as

$$\mathcal{G} = \mathcal{H}_E + \mathcal{H}_{A2} + \mathcal{H}_{Ah} \tag{2.1.3}$$

into distinct cosets, each of which contains h elements. Hence the order g of the group is

$$g = hk \quad \text{and} \quad h = g/k \tag{2.1.4}$$

Thus we got the important result that the order of a subgroup is a divisor of the order of the group. Note that the cosets are not subgroups except for $\mathcal{H}_E = \mathcal{H}$ which alone contains the unit element. Similar results hold for left cosets.

2.1.4 Conjugate Elements and Classes

The element XAX^{-1} 1 is said to be an element conjugate to A. The relation of being conjugate is reflexive, symmetric and transitive. Therefore the elements conjugate to each other form a class.

A single element A determines the entire class:

$$EAE^{-1} = A, A_2AA_2^{-1}, \dots, A_nAA_n^{-1} \tag{2.1.5}$$

Here all elements occur at least once, possibly more than once. The elements of the group can be divided into classes, and every element appears in one and only one class.

In the case of groups of covering operations of symmetrical objects, elements of the same class correspond to rotations by the same angle around different axes that transform into each other by symmetry operations.

E.g. the three mirror planes of the regular triangle are in the same class and so are the four rotations by $2\pi/3$ in a tetrahedron, or the eight rotations by $\pm 2\pi/3$ in a cube.

It happens that the elements of two groups defined in different conceptual terms are in one-one relation to each other and obey the same multiplication rules. A case in point is the permutation group $\mathcal{S}(3)$ and the symmetry group of the regular triangle. Such groups are called isomorphic. Recognizing isomorphisms may lead to new insights and to practical economies in the study of individual groups.

It is confirmed in the above examples that the term "multiplication" is not to be taken in a literal sense. What is usually meant is the performance of operations in a specified sequence, a situation that arises in many practical and theoretical contexts.

The operations in question are often transformations in ordinary space, or in some abstract space (say, the configuration space of an object of interest). In order to describe these transformations in a quantitative fashion, it is important to develop an algebraic formalism dealing with vector spaces.

However, before turning to the algebraic developments in Section 2.3, we consider first a purely geometric discussion of the rotation group in ordinary three-dimensional space.