6.5: On Parameterization and Integration

The explicit performance of the bilateral multiplication provides the connection between the parameters of rotation and the elements of the 4×4 matrices. We consider here only the pure rotation generated by

$$U = \exp\left(-i\frac{\phi}{2}\hat{u}\cdot\vec{\sigma}\right) \tag{6.5.1}$$

Let

$$l_0 = \cos\phi/2, \quad l_1 = \sin\phi/2\hat{u}_1 \tag{6.5.2}$$

$$l_2 = \sin\phi/2\hat{u}_3, \quad l_3 = \sin\phi/2\hat{u}_3 \tag{6.5.3}$$

$$u_1 = \cos(\hat{u}\cdot\hat{x}_1), \ldots, \text{ etc.} \tag{6.5.4}$$

$$u_1^2 + u_2^2 + u_3^2 = 1 \tag{6.5.5}$$

$$\begin{pmatrix} k_1' \\ k_2' \\ k_3' \end{pmatrix} = \begin{pmatrix} l_0^2 + l_1^2 - l_2^2 - l_3^2 & 2(l_1 l_2 - l_0 l_3) & 2(l_1 l_3 + l_0 l_2) \\ 2(l_1 l_2 + l_0 l_3) & l_0^2 - l_1^2 + l_2^2 - l_3^2 & 2(l_2 l_3 - l_0 l_1) \\ 2(l_1 l_3 - l_0 l_3) & 2(l_2 l_3 + l_0 l_1) & l_0^2 - l_1^2 - l_2^2 + l_3^2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \tag{6.5.6}$$

Such expression are, of course not very practical. One usually considers infinitesimal relations with the parameters $d\phi\mu_k$. Integration of the infinitesimal operations into those of the finite group can be achieved within the general theory of Lie groups and Lie algebras.

In our approach the integration is achieved by explicit construction for the special case of the restricted Lorentz group. This is the first step in our program of using group theory to supplement or replace method of differential equations.