

4.1: Lorentz transformation and Lorentz force

The main importance of the Pauli algebra is to provide us with a stepping stone for the theory of spinor spaces to which we turn in Section 5. Yet it is useful to stop at this point to show that the formalism already developed provides us with an efficient framework for limited, yet important aspects of classical electrodynamics (CED).

We have seen on page 26 that the effect on electric field on a test charge, a “boost,” can be considered as an active Lorentz transformation, whereby the field is proportional to the “hyperbolic angular velocity $\dot{\mu}$.”

This is in close analogy with the well known relation between the magnetic field and the cyclotron frequency, i.e., a “circular angular velocity” $\dot{\phi}$. These results had been obtained under very special conditions. The Pauli algebra is well suited to state them in much greater generality.

The close connection between the algebra of the Lorentz group and that of the electromagnetic field, is well known. However, instead of developing the two algebras separately and noting the isomorphism of the results, we utilize the mathematical properties of the Lorentz group developed in Section 3 and translate them into the language of electrodynamics. The definition of the electromagnetic field implied by this procedure is, of course, hypothetical, and we turn to experience to ascertain its scope and limits. The proper understanding of the limitation of this conception is particularly important, as it serves to identify the direction for the deepening of the theory. The standard operational definition of the electromagnetic field involves the use of a test charge. Accordingly we assume the existence of particles that can act in such a capacity. The particle is to carry a charge e , a constant rest mass m , and the effect of the field acting during the time dt is to manifest itself in a change of the 4-momentum only, without involving any change in internal structure.

This means that the field has a sufficiently low frequency in the rest frame of the particle so as not to affect its internal structure. This is in harmony with the temporary exclusion of radiative interaction stated already.

Let the test charge be exposed to an electromagnetic field during a small time dt . We propose to describe the resulting change of the four-momentum $P \rightarrow P' = P + dP$ as an infinitesimal Lorentz transformation. In this preliminary form the statement would seem to be trivial, since it is valid for any force that does not affect the intrinsic structure, say a combination of gravitational and frictional forces. In order to characterize specifically the Lorentz force, we have to add that the characterization of the field is independent of the four-momentum of the test charge, moreover it is independent of the frame of reference of the observer. These conditions can be expressed formally in the following.

Postulate 4.1.1

The effect of the Lorentz force on a particle (test charge) is represented as the transformation of the four-momentum space of the particle unto itself, and the transformations are elements of the active Lorentz group. Moreover, matrix representations in different Lorentz frames are connected by similarity transformations. (See Section 3.4.4.)

We now proceed to show that this postulate implies the known properties of the Lorentz force.

First, we show that an infinitesimal Lorentz transformation indeed reduces to the Lorentz force provided we establish a “dictionary” between the parameters of the transformation and the electromagnetic field (see below Equation 4.1.12). Consider a pure Lorentz transformation along \hat{h} .

$$p' = p \cosh \mu + p_0 \sinh \mu \quad (4.1.1)$$

$$p'_0 = p \sinh \mu + p_0 \cosh \mu \quad (4.1.2)$$

where $\vec{p} = p\hat{h} + \vec{p}$ with $\vec{p} \cdot \vec{h} = 0$. For infinitesimal transformations $\mu \rightarrow d\mu$:

$$p' - p = p_0 d\mu \quad (4.1.3)$$

$$p'_0 - p_0 = p d\mu \quad (4.1.4)$$

or

$$\dot{\vec{p}} = p_0 \dot{\mu} \hat{h} \quad (4.1.5)$$

$$\dot{p}_0 = \vec{p} \cdot \hat{h} \dot{\mu} \quad (4.1.6)$$

By making use of

$$\vec{p} = mc \sinh \mu = \gamma m \vec{v} \quad (4.1.7)$$

$$p_0 = mc \cosh \mu = \gamma mc$$

we obtain

$$\dot{\vec{p}} = pmc \hat{h} \dot{\mu} \quad (4.1.8)$$

$$\dot{p}_0 = pm \frac{\vec{v}}{c} \cdot \hat{h} \dot{\mu}$$

Turning to rotation we have from Equation A.3.8 of Appendix A (See note on page 51).

$$\vec{p}_\perp = \vec{p}_\perp \cos \phi + \hat{u} \times \vec{p}_\perp \sin \phi \quad (4.1.9)$$

For an infinitesimal rotation $\phi \simeq d\phi$, and by using Equation 4.1.7 we obtain, since $\vec{p}_\parallel = \vec{p}_\parallel$ and $\vec{p}_\parallel \times \hat{u} = 0$

$$\vec{p}' - \vec{p} = -\vec{p} \times \hat{u} d\phi = -\gamma m \vec{v} \times \hat{u} d\phi \quad (4.1.10)$$

or

$$\dot{\vec{p}} = -\gamma m \vec{v} \times \hat{u} d\phi \quad (4.1.11)$$

With the definitions of 3.3.28 and 3.3.29 of page 26 written vectorially:

$$\vec{E} = \frac{\gamma mc}{e} \dot{\mu} \hat{h}$$

$$\vec{B} = \frac{-\gamma mc}{e} \dot{\phi} \hat{u} \quad (4.1.12)$$

Equations 4.1.8 and 4.1.9 reduce to the Lorentz force equations.

Let us consider now an infinitesimal Lorentz transformation generated by

$$V = 1 + \frac{\mu}{2} \hat{h} \cdot \vec{\sigma} - \frac{i\phi}{2} \hat{u} \cdot \vec{\sigma} \quad (4.1.13)$$

$$= 1 + \frac{edt}{2\gamma mc} (\vec{E} + i\vec{B}) \cdot \vec{\sigma} \quad (4.1.14)$$

$$= 1 + \frac{edt}{2\gamma mc} F \quad (4.1.15)$$

with

$$\vec{f} = \vec{E} + i\vec{B}, \quad F = \vec{f} \cdot \vec{\sigma} \quad (4.1.16)$$

It is apparent from Equations 4.1.15 and 4.1.16 that the transformation properties of V and F are identical. Since the transformation of V has been obtained already in Section 3.4.4, we can write down at once that of the field \vec{f}

Let us express the passive Lorentz transformation of the four-momentum P from the inertial frame Σ to Σ' as

$$P' = S P S^\dagger \quad (4.1.17)$$

where S is unimodular. The field matrix transforms by a similarity transformation:

$$F' = S F S^{-1} \quad (4.1.18)$$

with the complex reflections (contragradient entities) transforming as

$$\bar{P}' = \bar{S} \bar{P} \bar{S}^{-1} \quad (4.1.19)$$

$$\bar{F}' = \bar{S} \bar{F} \bar{S}^\dagger \quad (4.1.20)$$

For

$$S = H = \exp\left(-\frac{\mu}{2}\hat{h} \cdot \vec{\sigma}\right) \quad (4.1.21)$$

we obtain the passive Lorentz transformation for a frame Σ_2 moving with respect to Σ_1 , with-the velocity

$$\vec{v} = c\hat{h} \tanh \mu \quad (4.1.22)$$

$$F' = H F H^{-1} \quad (4.1.23)$$

We extract from here the standard expressions by using the familiar decomposition

$$\vec{f} = \vec{f}_{\parallel} + \vec{f}_{\perp} = (\vec{f} \cdot \hat{h})\hat{h} \quad (4.1.24)$$

we get

$$\vec{f}'_{\parallel} = \vec{f}_{\parallel} \quad (4.1.25)$$

$$\vec{f}_{\perp} \cdot \vec{\sigma} = H \left(\vec{f}_{\perp} \cdot \vec{\sigma} \right) H^{-1} = H^2 \left(\vec{f}_{\perp} \cdot \vec{\sigma} \right) \quad (4.1.26)$$

$$= (\cosh \mu - \sinh \mu \hat{h} \cdot \vec{\sigma}) \vec{f}_{\perp} \cdot \vec{\sigma} \quad (4.1.27)$$

Hence

$$\vec{f}'_{\perp} = \vec{f}_{\perp} \cosh \mu + i \sinh \mu \hat{h} \times \vec{f}_{\perp} \quad (4.1.28)$$

$$= \cosh \mu \left(\vec{f}_{\perp} + i \tanh \mu \hat{h} \times \vec{f}_{\perp} \right) \quad (4.1.29)$$

$$= \gamma \left(\vec{f}_{\perp} + i \frac{\vec{v}}{c} \times \vec{f}_{\perp} \right) \quad (4.1.30)$$

where we used Equation 4.1.22 Inserting from Equation 4.1.16 we get

$$\vec{E}'_{\perp} = \gamma \left(\vec{E}_{\perp} + \frac{\vec{v}}{c} \times \vec{B}_{\perp} \right)$$

$$\vec{B}'_{\perp} = \gamma \left(\vec{B}_{\perp} - \frac{\vec{v}}{c} \times \vec{E}_{\perp} \right) \quad (4.1.31)$$

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \quad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

It is interesting to compare the two compact forms 4.1.18 and 4.1.31. Whereas the latter may be the most convenient for solving specific problems, the former will be the best stepping stone for the deepening of the theory. The only Lorentz invariant of the field is the determinant, which we write for convenience with the negative sign:

$$-|F| = \frac{1}{2} \text{Tr} F^2 = \vec{f}^2 = \vec{E}^2 - \vec{B}^2 + 2i \vec{E} \cdot \vec{B} = g^2 e^2 \Psi \quad (4.1.32)$$

Hence we obtain the well know invariants

$$I_1 = \vec{E}^2 - \vec{B}^2 = g^2 \cos 2\psi \quad (4.1.33)$$

$$I_2 = 2\vec{E} \cdot \vec{B} = g^2 \sin 2\psi$$

We distinguish two cases

$$1. \vec{f}^2 \neq 0$$

$$2. \vec{f}^2 = 0$$

These cases can be associated with the similarity classes of Table 3.2. In the case (i) F is unimodular axial, for (ii) it is nonaxial singular. (Since F is traceless, the two other entries in the table do not apply.) We first dispose of case (ii). A field having this Lorentz invariant property is called a null-field. The F matrix generates an exceptional Lorentz transformation (Section 3.4.4). In

this field configuration \vec{E} and \vec{B} are perpendicular and are of equal size. This is a relativistically invariant property that is characteristic of plane waves to be discussed in Section 4.2.

In the “normal” case (i) it is possible to find a canonical Lorentz frame, in which the electric and the magnetic fields are along the same line, they are parallel, or antiparallel. The Lorentz screw corresponds to a Maxwell wrench. It is specified by a unit vector \hat{n} and the values of the fields in the canonical frame E_{can} and B_{can} . The wrench may degenerate can can with $E_{can} = 0$, or $B_{can} = 0$. The canonical frame is not unique, since a Lorentz transformation along \hat{n} leaves the canonical fields invariant.

We can evaluate the invariant eqn:iii-8-18ab in the canonical frame and obtain

$$E_{can}^2 - B_{can}^2 = I_1 = g^2 \cos 2\psi \quad (4.1.34)$$

$$2E_{can} B_{can} = I_2 = g^2 \sin 2\psi \quad (4.1.35)$$

One obtains from here

$$E_{can} = g \cos \psi \quad (4.1.36)$$

$$B_{can} = g \sin \psi \quad (4.1.37)$$

The invariant character of the field is determined by the ratio

$$\frac{B_{can}}{E_{can}} = \tan \psi \quad (4.1.38)$$

that has been called its pitch by Synge (op. cit. p. 333) who discussed the problem of canonical frames of the electromagnetic field with the standard tensorial method.

The definition of pitch in problem #8 is the reciprocal to the one here given and should be changed to agree with Eq. (4.1.20)

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