2.5: A Short Survey of Linear Groups

The linear vector space $V(n,F)$ provides us with the opportunity to define a number of linear groups which we shall use in the sequel.

We start with the group of nonsingular linear transformations defined by Equations 2.3.4 and 2.3.5 of Section 2.3 and designated as $\mathcal{GL}(n,R)$, for “general linear group over the field F.” If the matrices are required to have unit determinants, they are called unimodular, and the group is $\mathcal{SL}(n,F)$ for simple linear group.

Let us consider now the group $\mathcal{GL}(n,R)$ over the real field, and assume that an inner product is defined:

$$x_1y_1+x_2y_2+\ldots+x_ny_n=X^TY \tag{2.5.1}$$

Transformations which leave this form invariant are called orthogonal. By using Equations 2.3.10 and 2.3.12 of Sectionsec:vec-space, we see that they satisfy the condition

$$O^TO=\mathcal{I} \tag{2.5.2}$$

where $\mathcal{I}$ is the unit matrix. The corresponding group is called $\mathcal{O}(n)$.

It follows from 2.5.2 that the determinant of O is $\det O=|O|=\pm 1$. The matrices with positive determinant form a subgroup $\mathcal{SO}(n)$.

The orthogonal groups have an important geometrical meaning, they leave the so-called metric properties, lengths and angles invariant. The group $SO(n)$ corresponds to pure rotations, these operations can be continuously connected with the identity. In contrast, transformations with negative determinants involve the inversion, and hence mirrorings and improper rotations. The set of matrices with $|O|=-1$, does not form a group, since it does not contain the unit element.

The geometrical interpretation of $\mathcal{GL}(n,R)$ is not explained as easily. Instead of metric Euclidean geometry, we arrive at the less familiar affine geometry, the practical applications of which are not so direct. We shall return to these questions in Chapter VII. However, in the next section we shall show that the geometrical interpretation of the group of unimodular transformations $\mathcal{SL}(n,R)$ is to leave volume invariant.

We turn now to an extension of the concept of metric geometry. We note first that instead of requiring the invariance of the expression 2.5.1, we could have selected an arbitrary positive definite quadratic form in order to establish a metric. However, a proper choice of basis in $\mathcal{V}(n,R)$ leads us back to Equation 2.5.1.

If the invariant quadratic form is indefinite, it reduces to the canonical form

$$x_1^2+x_2^2+\ldots+x_k^2-x_{k+1}^2-\ldots-x_{k+l}^2 \tag{2.5.3}$$

The corresponding group of invariance is pseudo-orthogonal denoted as $O(k,l)$.

In this category the Lorentz group $SO(3,1)$ is of fundamental physical interest. At this point we accept this as a fact, and a sufficient incentive for us to examine the mathematical structure of $SO(3,1)$ in Section 3. However, subsequently, in Section 4, we shall review the physical principles which are responsible for the prominent role of this group. The nature of the mathematical study can be succinctly explained as follows.

The general $n\times n$ matrix over the real field contains n^2 independent parameters. The condition 2.5.2 cuts down this number to $n(n-l)/2$. For $n=3$ the number of parameters is cut down from nine to three, for $n=4$ from sixteen to six. The parameter count is the same for $SO(3,1)$ as for $SO(4)$. One of the practical problems involved in the applications of these groups is to avoid dealing with the redundant variables, and to choose such independent parameters that can be easily identified with geometrically and physically relevant quantities. This is the problem discussed in Section 3. We note that $SO(3)$ is a subgroup of the Lorentz group, and the two groups are best handled within the same framework.

It will turn out that the proper parametrization can be best attained in terms of auxiliary vector spaces defined over the complex field. Therefore we conclude our list of groups by adding the unitary groups.

Let us consider the group $\mathcal{GL}(n,C)$ and impose an invariant Hermitian form

$$\sum a_{ik}x_ix_k^* \tag{2.5.4}$$

that can be brought to the canonical form

$$x_1x_1^* + x_2x_2^* + \ldots + x_nx_n^* = X^\dagger X \tag{2.5.5}$$

where $X^\dagger = X^{*T}$ is the Hermitian adjoint of X and the star stands for the conjugate complex. Expression 2.5.5 is invariant under transformations by matrices that satisfy the condition

$$U^\dagger U = \mathcal{I} \tag{2.5.6}$$

These matrices are called unitary, they form the unitary group $\mathcal{U}(n)$. Their determinants have the absolute value one. If the determinant is equal to one, the unitary matrices are also, unimodular, we have the simple unitary group $\mathcal{SU}(n)$.