5.4: Relativistic triads and spinors. A preliminary discussion

We have arrived at the concept of unitary spinors by searching for the proper parametrization of a Euclidean triad. We shall arrive at relativistic spinors by parametrising the relativistic triad. This is not a standard term, but it seems appropriate to so designate the configuration $\vec{E}, \vec{B}, \vec{k}$ (electric and magnetic fields, and the wave vector) in a monochromatic electromagnetic plane wave in vacuum.

The propagation of light is a dynamic problem and we are not ready to discuss it within the geometric-kinematic context of this chapter.

The purpose of this section is only to show that the formalism of unitary spinors developed thus far can be extended to relativistic situations with only a few indispensable adjustments.

It is a remarkable fact that the mutual orthogonality of the above mentioned vectors is a Lorentz invariant property. However, we have to abandon the unitary normalization since the length of the vectors is affected by inertial transformations.

Accordingly, we set up the relativistic analog of the Equations 5.1.40. We consider first

$$|\xi\rangle\langle\xi| = \frac{1}{2}\left(k_0 + \vec{k}\cdot\vec{\sigma}\right) = \frac{1}{2}K \tag{5.4.1}$$

$$|\bar{\xi}\rangle\langle\bar{\xi}| = \frac{1}{2}\left(k_0 - \vec{k}\cdot\vec{\sigma}\right) = \frac{1}{2}\bar{K} \tag{5.4.2}$$

with the unitary normalization changed to

$$\langle\xi \mid \xi\rangle = \langle\bar{\xi} \mid \bar{\xi}\rangle = k_0 \tag{5.4.3}$$

The Lorentz transformation properties of the spinors follow from that of K:

$$|\xi'\rangle = V|\xi\rangle \tag{5.4.4}$$

$$\left|\bar{\xi}'\right\rangle = \bar{V}|\bar{\xi}\rangle \tag{5.4.5}$$

$$\langle\xi'| = \langle\xi|V^\dagger \tag{5.4.6}$$

$$\left\langle\bar{\xi}'\right| = \langle\bar{\xi}|V^{-1} \tag{5.4.7}$$

If $V = U$ is unitary, we have $\bar{U} = U, U^\dagger = U^{-1}$

Let us define a second spinor by

$$|\eta\rangle\langle\eta| = \frac{1}{2}(r_0 + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}R \tag{5.4.8}$$

The relativistic invariant, (2.2.3a) appears now as

$$\begin{aligned}\frac{1}{2}\operatorname{Tr}(R\bar{K}) &= \frac{1}{2}\operatorname{Tr}(|\eta\rangle\langle\eta \mid \bar{\xi}\rangle\langle\bar{\xi}|) \\ &= \langle\bar{\xi} \mid \eta\rangle\langle\eta \mid \bar{\xi}\rangle = |\langle\bar{\xi} \mid \eta\rangle|^2\end{aligned} \tag{5.4.9}$$

It follows from Equations 5.4.4 and 5.4.7 that even the amplitude is invariant

$$\langle\bar{\xi} \mid \eta\rangle = \text{ invariant} \tag{5.4.10}$$

Explicitly it is equal to

$$(-\xi_1, \xi_0)\begin{pmatrix}\eta_0 \\ \eta_1\end{pmatrix} = \xi_0\eta_1 - \xi_1\eta_0 = \begin{vmatrix}\xi_0 & \eta_0 \\ \xi_1 & \eta_1\end{vmatrix} \tag{5.4.11}$$

We turn now to the last two of the Equations 5.1.40 and write by analogy

$$|\xi\rangle\langle\bar{\xi}| \sim (\vec{E} + i\vec{B})\cdot\vec{\sigma} = F \tag{5.4.12}$$

$$|\bar{\xi}\rangle\langle\xi| \sim (\vec{E} - i\vec{B})\cdot\vec{\sigma} = -\bar{F} = F^\dagger \tag{5.4.13}$$

We see that the field quantities have, in view of Equations 5.4.4– 5.4.7 the correct transformation properties.

The occurrence of the same spinor in Equations ref{1}, 5.4.2, 5.4.12 and 5.4.13 ensures the expected orthogonality properties of the triad.

However, in Equations 5.4.12 and 5.4.13 we write proportionality instead of equality, because we have to admit a different normalization for the four-vector and the six-vector respectively. We are not ready to discuss the matter at this point.

If in Equation 5.4.10 we choose the two spinors to be identical, the invariant vanishes:

$$\langle \bar{\xi} \mid \xi \rangle = 0 \tag{5.4.14}$$

The same is true of the invariant of the electromagnetic field:

$$\begin{aligned} \frac{1}{2}\operatorname{Tr}(F\tilde{F}) &= -\frac{1}{2}\operatorname{Tr} F^2 \simeq -\frac{1}{2}\operatorname{Tr}(|\xi\rangle\langle\bar{\xi} \mid \xi\rangle\langle\bar{\xi}|) \\ &= -(\langle\bar{\xi} \mid \xi\rangle)^2 = 0 \end{aligned} \tag{5.4.15}$$

Thus from a single spinor we can build up only constructs corresponding toa plane wave. We do not enter here into the discussion of more complicated situations and note only that we cannot use the device of taking linear combination of conjugate spinors in the usual form $a_0|\xi\rangle + a_1|\bar{\xi}\rangle$, because the two terms have contragradient Lorentz transformation properties. We write them, displaying their Lorentz transformations as

$$\begin{pmatrix} |\xi'\rangle \\ |\bar{\xi}'\rangle \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} |\xi\rangle \\ |\bar{\xi}\rangle \end{pmatrix} \tag{5.4.16}$$

We have arrived at spinors of the Dirac type. We return to their discussion later.

Let us conclude this section by considering the relation of the formalism to the standard formalism of van der Waerden. (See e.g., [MTW73])

The point of departure is Equation 5.4.4 applied to two spinors yielding the determinantal invariant of 5.4.11. The first characteristic aspect of the theory is the rule for raising the indices:

$$\eta_1 = \eta^0, \quad \eta_0 = -\eta^1 \tag{5.4.17}$$

Hence the invariant appears as

$$\xi_0\eta^0 + \xi_1\eta^1 \tag{5.4.18}$$

The motivation for writing the invariant in this form is to harmonize the presentation with the standard tensor formalism. In contrast, our expression 5.4.10, is an extension of the bra-ket formalism of nonrelativistic quantum mechanics, that is also quite natural for the linear algebra of complex vector spaces.

A second distinctive feature is connected with the method of complexification. Van der Waerden takes the complex conjugate of the matrix V by taking the complex conjugate of its elements, whereas we deal with the Hermitian conjugate $V^\dagger$ and the complex reflection $\bar{V}$.

From the practical point of view we tend to develop unitary and relativistic spinors in as united a form as objectively possible.