4.2: The Free Maxwell Field

Our approach to CED thus far is unusual inasmuch as we have effectively defined, classified and transformed the electromagnetic field at a small region of space-time without having used the Maxwell equations. This is, of course, an indication of the effectiveness of our definition of the field in terms of active Lorentz transformations.

In order to arrive at the Maxwell equations we invoke the standard principle of relativistic invariance, involving the passive interpretation of the Lorentz group.

Postulate 4.2.1

The electromagnetic field satisfies a first order differential equation in the space-time coordinates that is covariant under Lorentz transformations.

We consider the four-dimensional del operator $\{\partial_0, \nabla\}$ with $\partial_0 = \partial/\partial(ct) = \partial/\partial r_0$ as a fourvector, with its matrix equivalent

$$D = \partial_0 - \nabla \cdot \vec{\sigma} \tag{4.2.1}$$

The rationale for the minus sign is as follows. Let D operate on a function representing a plane wave:

$$\psi = \exp -i(\omega t - \vec{k} \cdot \vec{r}) = \exp -i\left(k_0 r_0 - \vec{k} \cdot \vec{r}\right) \tag{4.2.2}$$

we have

$$iD\psi = \left(k_0 1 + \vec{k} \cdot \vec{\sigma}\right)\psi = K\psi \tag{4.2.3}$$

Thus D has the same transformation properties as K:

$$D' = SDS^{\dagger} \tag{4.2.4}$$

while the complex reflection

$$\bar{D} = \partial_0 1 + \nabla \cdot \vec{\sigma} \tag{4.2.5}$$

transforms as $\bar{K}$, i.e.

$$\bar{D}' = \bar{S}\bar{D}S^{-1} \tag{4.2.6}$$

By using the transformation rules 4.1.18, 4.1.20 of the last section we see that $\bar{D}F$ transforms as a four-vector J:

$$\left(\bar{S}\bar{D}S^{-1}\right)\left(SFS^{-1}\right) = \bar{S}\bar{J}S^{-1} \tag{4.2.7}$$

Thus

$$\bar{D}F = \bar{J} \tag{4.2.8}$$

is a differential equation satisfying the conditions of Postulate 2. Setting tentatively

$$J = \rho 1 + \frac{\vec{j}}{c} \cdot \vec{\sigma} \tag{4.2.9}$$

with $\rho, \vec{j}$ the densities of charge and current, 4.2.8 is indeed a compact form of the Maxwell equations.

This is easily verified by sorting out the terms with the factors $(1, \sigma_k)$ and by separating the real and imaginary parts.

By operating on the Equation 4.2.9 with D and taking the trace we obtain

$$D\bar{D}F = \left(\partial_0^2 - \nabla^2\right)F = D\bar{J} \tag{4.2.10}$$

and

$$\frac{1}{2} Tr D\bar{J} = \partial_0 \rho + \frac{1}{c}\vec{\nabla} \cdot \vec{j} = 0 \tag{4.2.11}$$

These are standard results which are easily provided by the formalism. However, we do not have an explicit expression for J that would be satisfactory for a theory of radiative interaction.

Therefore, in accordance with our program stated in Section 4.1 we set $J=0$ and examine only the free field that obeys the homogeneous equations $\bar{D}F=0$

$$\left(\partial_0^2-\nabla^2\right)F=0 \tag{4.2.12}$$

The simplest elementary solution of 4.2.12 are monochromatic plane waves from which more complicated solutions can be built up. Hence we consider

$$F(\vec{r},t)=F_+(\vec{k},\omega)\exp\{i(\omega t-\vec{k}\cdot\vec{r})\}+F_-(\vec{k},\omega)\exp\{i(\omega t-\vec{k}\cdot\vec{r})\} \tag{4.2.13}$$

where $F_\pm$ are matrices independent of space-time. Inserting into Equation 4.2.12 yields the condition

$$\omega^2-c^2k^2=0 \tag{4.2.14}$$

Introducing the notation

$$\theta=k_0r_0-\vec{k}\cdot\vec{r} \tag{4.2.15}$$

we write Equation 4.2.13 as

$$F(\vec{r},t)=F_+\exp(-i\theta)+F_-\exp(i\theta) \tag{4.2.16}$$

Inserting into 4.2.12 we have

$$KF_\pm=K\left(\vec{E}_\pm+i\vec{B}_\pm\right)\cdot\vec{\sigma}=0 \tag{4.2.17}$$

From here we get explicitly

$$\vec{k}\cdot\left(\vec{E}_\pm+i\vec{B}_\pm\right)=0 \tag{4.2.18}$$

$$\vec{E}_\pm+i\vec{B}_\pm=i\hat{k}x\left(\vec{E}_\pm+i\vec{B}_\pm\right) \tag{4.2.19}$$

and we infer the well known properties of plane waves: $\vec{E}$ and $\vec{B}$ are of equal magnitude, and $\vec{E},\vec{B},\vec{k}$ from a right-handed Cartesian triad. We note that this constellation corresponds to the field of the type (ii) with $\vec{f}^{\,2}=0$ mentioned on page 51.

Since the classical $\vec{E},\vec{B}$ are real, we have also the relations

$$\vec{E}_-=\vec{E}_+^*,\quad \vec{B}_-=\vec{B}_+^* \tag{4.2.20}$$

Consider now the case in which

$$\vec{f}_-=\vec{E}_-+i\vec{B}_-=0 \tag{4.2.21}$$

which, in view of 4.2.20 implies

$$\vec{E}_+=i\vec{B}_+ \tag{4.2.22}$$

Thus at a fixed point and direction in space the electric field lags the magnetic field by a phase $\pi/2$, and $\vec{f}_+=\left(\vec{E}_++i\vec{B}_+\right)$ is the amplitude of a circularly polarized wave of positive helicity, i.e., the rotation of the electric and magnetic vectors and the wave vector $\vec{k}$ form a right screw or the linear and angular momentum point in the same direction $+\vec{k}$. In the traditional optical terminology this is called a left circularly polarized wave. However, following current practice, we shall refer to positive helicity as right polarization R. The negative helicity state is represented by $\vec{f}_-=\left(\vec{E}_-+i\vec{B}_-\right)$.

Actually, we have the alternative of associating $\overrightarrow{f_-^*}=\left(\vec{E}_--i\vec{B}_-\right)$ with R and $\overrightarrow{f_+^*}=\left(\vec{E}_+-i\vec{B}_+\right)$ with L. However, we prefer the first choice and will use the added freedom in the formalism to describe the absorption and the emission process at a later stage.

Meanwhile, we turn to the discussion of polarization which has a number of interesting aspects, particularly if carried out in the context of spinor algebra.