02. Calculating the Rotational Inertia

Calculating the Rotational Inertia

To fully utilize Newton's second law in rotational form, we must be able to set up and evaluate the integral that determines the rotational inertia. (To be honest, this is a lie. For the vast majority of common shapes, and many quite uncommon shapes, these integrals have already been evaluated. A table of selected results is at the end of this section.) To test our understanding of the relationship for rotational inertia for a thin disk about an axis passing through its center of mass and perpendicular to its circular face.

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1. Choose the chunks of mass, dm, to be ring-shaped. This is because you must multiply each dm by the distance of the chunk from the rotation axis squared. To facilitate doing this, it's crucial that every point in the chunk be the same distance from the axis, i.e., have the same r. if the ring-shaped chunk is thin enough, for example dr (infinitesimally) thick, then this is true.
2. Realize that the mass of the little chunk is directly proportional to its volume, assuming the disk has a constant density. If it does have a constant density, the ratio of the chunk's mass to its volume must be the same as the ratio of the total mass of the disk to its volume.

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   where R is the radius of the disk and T is its thickness. The volume of the ring-shaped chunk, dV, is equal to the product of the circumference of the ring (2pr), the thickness of the disk (T), and the thickness of the ring (dr). Thus,

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3. Plug the expression for dm into

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To include all the chunks of mass, the integral must go from r = 0 m up r = R.

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Thus, the rotational inertia of a thin disk about an axis through its CM is the product of one-half the total mass of the disk and the square of its radius. Notice that the thickness of the disk does not effect its rotational inertia. A consequence of this fact is that a cyclinder has the same rotational inertia as a disk, when rotated about an axis through its CM and perpendicular to its circular face.

The Parallel-Axis Theorem

Now let's imagine we need to calculate the rotational inertia of a thin disk about an axis perpendicular to its circular face and along the edge of the disk. It would be convenient if we could determine the rotational inertia about an axis along the edge using the rotational inertia about an axis through the CM (which we've already calculated). In fact, there is a *very* convenient method to determine the rotational inertia about any axis *parallel* to an axis through the CM if we know the rotational inertia about an axis through the CM.

Imagine you want to determine the rotational inertia of an arbitrarily shaped object about an arbitrary axis. The solid circle denotes an axis of interest. The two axes are parallel.

Notice that

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with

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and

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Thus,

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The rotational inertia about the axis of interest is given by:

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Note that x_{CM}, y_{CM}, and r_{CM} are constants that depend only on the distance between the two axes. Thus, x_{CM}, y_{CM}, and r_{CM} can by brought outside of the integral.

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Now comes the key observation in the derivation. Examine the term $\int$ x dm. Remember that x is the horizontal distance from the CM. If this distance from the CM. If this distance is integrated over all the chunks of mass, dm, throughtout the entire object, this integral must equal zero because the CM is defined to be in exactly the spot where a mass-weighted average over distance is equal to zero. $\int$ x dm and $\int$ y dm are equal to zero by the definition of CM! (Pretty cool, huh?)

Thus,

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Noting that $\int$ dm is the total mass of the object, M, and $\int r^2$ dm is the rotational inertia about the CM, I_{CM}, then

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This result states that the rotational inertia about an axis parallel to an axis through the CM, I, is equal to the rotational inertia about an axis through the CM, I_{CM}, plus the product of the total mass of the object and the distance between the axes, r_{CM}, squared.

To answer the original question, let's determine the rotational inertia of a thin disk about an axis perpendicular to its circular face and along the egde of the disk using the parallel-axis theorem.

We know that the rotational inertia for a thin disk about an axis passing through its center of mass and perpendicular to its circular face is $^1/_2$ MR^2 and the axis of interest is R. Thus,

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