1.5: Potential Energy and Equilibrium

Another important role of the potential energy U, especially for dissipative systems whose total mechanical energy E is not conserved because it may be drained to the environment, is finding the positions of equilibrium (sometimes called the fixed points of the system under analysis) and analyzing their stability with respect to small perturbations. For a single particle, this is very simple: the force (22) vanishes at each extremum (either minimum or maximum) of the potential energy. [22] Of those fixed points, only the minimums of $U(\mathbf{r})$ are stable - see Sec. 3.2 below for a discussion of this point.

A slightly more subtle case is a particle with potential energy $U(\mathbf{r})$, subjected to an additional external force $\mathbf{F}^{(\mathrm{ext})}(\mathbf{r})$. In this case, the stable equilibrium is reached at the minimum of not the function $U(\mathbf{r})$, but of what is sometimes called the Gibbs potential energy

$$U_{\mathrm{G}}(\mathbf{r}) \equiv U(\mathbf{r})-\int^{\mathbf{r}} \mathbf{F}^{(\mathrm{ext})}\left(\mathbf{r}^{\prime}\right) \cdot d \mathbf{r}^{\prime}, \tag{1.5.1}$$

$$(1.39) \tag{1.5.2}$$

which is defined, just as $U(\mathbf{r})$ is, to an arbitrary additive constant. [23] The proof of Eq. (39) is very simple: in an extremum of this function, the total force acting on the particle,

$$\mathbf{F}^{(\mathrm{tot})}=\mathbf{F}+\mathbf{F}^{(\mathrm{ext})} \equiv-\nabla U+\nabla \int^{\mathbf{r}} \mathbf{F}^{(\mathrm{ext})}\left(\mathbf{r}^{\prime}\right) \cdot d \mathbf{r}^{\prime} \equiv-\nabla U_{\mathrm{G}}, \tag{1.5.3}$$

vanishes, as it should.

Physically, the difference $U_{\mathrm{G}}-U$ specified by Eq. (39) is the $\mathbf{r}$-dependent part of the potential energy $U^{(\mathrm{ext})}$ of the external system responsible for the force $\mathbf{F}^{(\mathrm{ext})}$, so that U_{G} is just the total potential energy $U+U^{(\mathrm{ext})}$, excluding its part that does not depend on $\mathbf{r}$ and hence is irrelevant for the analysis. According to the 3^{rd} Newton law, the force exerted by the particle on the external system equals $\left(-\mathbf{F}^{(\mathrm{ext})}\right)$, so that its work (and hence the change of $U^{(\mathrm{ext})}$ due to the change of $\mathbf{r}$) is given by the second term on the right-hand side of Eq. (39). Thus the condition of equilibrium, $-\nabla U_{\mathrm{G}}=0$, is just the condition of an extremum of the total potential energy, $U+U^{(\mathrm{ext})}+$ const, of the two interacting systems.

For the simplest (and very frequent) case when the applied force is independent of the particle's position, the Gibbs potential energy (39) is just [24]

$$U_{\mathrm{G}}(\mathbf{r}) \equiv U(\mathbf{r})-\mathbf{F}^{(\mathrm{ext})} \cdot \mathbf{r}+\text { const } \tag{1.5.4}$$

As the simplest example, consider a $1 \mathrm{D}$ deformation of the usual elastic spring providing the returning force $(-\kappa x)$, where x is the deviation from its equilibrium. As follows from Eq. (22), its potential energy is $U=\kappa x^{2} / 2+$ const, so that its minimum corresponds to $x=0$. Now let us apply an additional external force F, say independent of x. Then the equilibrium deformation of the spring, $x_{0}=F / \kappa$, corresponds to the minimum of not U, but rather of the Gibbs potential energy (41), in our particular case taking the form

$$U_{\mathrm{G}} \equiv U-F x=\frac{\kappa x^{2}}{2}-F x \tag{1.5.5}$$

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[22] Assuming that the additional, non-conservative forces (such as viscosity) responsible for the mechanical energy drain, vanish at equilibrium - as they typically do. (The static friction is one counter-example.)

[23] Unfortunately, in most textbooks, the association of the (unavoidably used) notion of U_{G} with the glorious name of Josiah Willard Gibbs is postponed until a course of statistical mechanics and/or thermodynamics, where U_{G} is a part of the Gibbs free energy, in contrast to U, which is a part of the Helmholtz free energy - see, e.g., SM Sec. 1.4. I use this notion throughout my series, because the difference between U_{G} and U, and hence that between the Gibbs and Helmholtz free energies, has nothing to do with statistics or thermal motion, and belongs to the whole physics, including not only mechanics but also electrodynamics and quantum mechanics.

[24] Note that Eq. (41) is a particular case of what mathematicians call the Legendre transformations.