6.6: Wave Decay and Attenuation

Now let us discuss the effects of dissipation on the 1D waves, on the example of the same uniform system shown in Figure 4. The effects are simplest for a linear drag that may be described, as it was done for a single oscillator in Sec.5.1, by adding the term $\eta dq_j/dt$, to Eq. (24) for each particle:

$$m\ddot{q}_j+\eta\dot{q}_j-\kappa_{\text{ef}}\left(q_{j+1}-q_j\right)+\kappa_{\text{ef}}\left(q_j-q_{j-1}\right)=0. \tag{6.6.1}$$

(In a uniform system, the drag coefficient η should be similar for all particles, though it may be different for the longitudinal and transverse oscillations.)

To analyze the dissipation effect on the standing waves, we may again use the variable separation method, i.e. look for the solution of Eq. (76) in the form similar to Eq. (67), naturally readjusting it for our current discrete case:

$$q\left(z_j,t\right)=\sum_n Z_n\left(z_j\right)T_n(t). \tag{6.6.2}$$

After dividing all terms by $mZ_n\left(z_j\right)T_n(t)$, and separating the time-dependent and space-dependent terms, we get

$$\frac{\ddot{T}_n}{T_n}+\frac{\eta}{m}\frac{\dot{T}_n}{T_n}=\frac{\kappa_{\text{ef}}}{m}\left[\frac{Z_n\left(z_{j+1}\right)}{Z_n\left(z_j\right)}+\frac{Z_n\left(z_{j+1}\right)}{Z_n\left(z_j\right)}-2\right]=\text{ const} \tag{6.6.3}$$

As we know from the previous section, the resulting equation for the function $Z_n\left(z_j\right)$ is satisfied if the variable separation constant is equal to $-{\omega_n}^2$, where ω_n obeys the dispersion relation (30) for the wave number k_n, properly calculated for the dissipation-free system, with the account of the given boundary conditions - see, e.g. Eqs. (62) and (72). Hence for the function $T_n(t)$ we are getting the ordinary differential equation

$$\ddot{T}_n+2\delta\dot{T}_n+\omega_n^2T_n=0,\qquad \text{with } \delta\equiv\frac{\eta}{2m}, \tag{6.6.4}$$

which is absolutely similar to Eq. (5.6b) for a single linear oscillator, which was studied in Sec. 5.1. As we already know, it has the solution (5.9), describing the free oscillation decay with the relaxation time given by $(5.10), \tau=1/\delta$, and hence similar for all modes. [23]

Hence, the above analysis of the dissipation effect on the free standing waves has not brought any surprises, but it gives us a hint of how their forced oscillations, induced by some external forces $F_j(t)$ exerted on the particles, may be analyzed. Indeed, representing the force as a sum of spatial harmonics proportional to the system's modes,

$$F_j(t)=m\sum_n f_n(t)Z_n\left(z_j\right) \tag{6.6.5}$$

and using the variable separation (77), we arrive at the equation

$$\ddot{T}_n+2\delta\dot{T}_n+\omega_n^2T_n=f_n(t), \tag{6.6.6}$$

similar to Eq. (5.13b) for a single oscillator. This fact enables using all the methods discussed in Sec. 5.1 for the forced oscillation analysis, besides that the temporal Green's function, defined by either of the equivalent equations (5.27) and (5.28), now acquires the index n, i.e. becomes mode-dependent: $G(\tau)\rightarrow G_n(\tau)$. Performing the weighed summation similar to Eq. (80),

$$G_j(\tau)=\sum_n G_n(\tau)Z_n\left(z_j\right), \tag{6.6.7}$$

we get the spatial-temporal Green's function of the system - in this case, for a discrete, 1D set of spatial points $z_j=jd$. As in the single-oscillator case, it has a simple physical sense of the oscillations induced by a delta-functional force (i.e. a very short pulse), exerted on the j^{th} particle. We will meet (and use) such spatial-temporal Green's functions in other parts of this series as well.

Now let us discuss the dissipation effects on the traveling waves, where they may take a completely different form of attenuation. Let us discuss it on a simple example when one end (located at $z=0$) of a very long chain $(l\rightarrow\infty)$ is externally-forced to perform sinusoidal oscillations of a certain frequency ω and a fixed amplitude A_0. In this case, it is natural to look for the particular solution of Eq. (76) in a form very different from Eq. (77):

$$q_j(z,t)=\operatorname{Re}\left[c_je^{-i\omega t}\right] \tag{6.6.8}$$

with time-independent but generally complex amplitudes c_j. As our discussion of a single oscillator in Sec. 5.1 implies, this is not the general, but rather a partial solution, which describes forced oscillations in the system, to which it settles after some initial transient process. (At non-zero damping, we may be sure that this process fades after a finite time, and thus may be ignored for most purposes.)

Plugging Eq. (83) into Eq. (76), we reduce it to an equation for the amplitudes c_j,

$$\left(-m\omega^2 - i\omega\eta + 2\kappa_{\text{ef}}\right) c_j - \kappa_{\text{ef}} c_{j+1} - \kappa_{\text{ef}} c_{j-1} = 0, \tag{6.6.9}$$

which is a natural generalization of Eq. (25). As a result, partial solutions of the set of these equations (for $j = 0, 1, 2, \ldots$) may be looked for in the form (26) again, but now, because of the new, imaginary term in Eq. (84), we should be ready to get a complex phase shift α, and hence a complex wave number $k \equiv \alpha/d$.[24] Indeed, the resulting characteristic equation for k,

$$\sin^2 \frac{kd}{2} = \frac{\omega^2}{\omega_{\max}^2} + i\frac{2\omega\delta}{\omega_{\max}^2} \tag{6.6.10}$$

(where $\omega_{\max}$ is defined by Eq. (30), and the damping coefficient is defined just as in a single oscillator, $\delta \equiv \eta/2m$), does not have a real solution even at $\omega < \omega_{\max}$. Using the well-known expressions for the sine function of a complex argument[25] Eq. (85) may be readily solved in the most important low-damping limit $\delta << \omega$. In the linear approximation in δ, it does not affect the real part of k, but makes its imaginary part different from zero:

$$k = \pm\frac{2}{d}\left(\sin^{-1}\frac{\omega}{\omega_{\max}} + i\frac{\delta}{\omega_{\max}}\right) \equiv \pm\left(\frac{2}{d}\sin^{-1}\frac{\omega}{\omega_{\max}} + i\frac{\delta}{v}\right), \quad \text{for } -\pi \leq \operatorname{Re} k \leq \pi, \tag{6.6.11}$$

with a periodic extension to other periods - see Figure 5. Just as was done in Eq. (28), due to two values of the wave number, generally we have to take c_j in the form of not a single wave (26), but of a linear superposition of two partial solutions:

$$c_j = \sum_{\pm} c_{\pm} \exp\left\{\pm i \operatorname{Re} k z_j \mp \frac{\delta}{v} z_j\right\} \tag{6.6.12}$$

where the constants $c_{\pm}$should be found from the boundary conditions. In our particular case, $|c_0| = A_0$ and $c_\infty = 0$, so that only one of these two waves, namely the wave exponentially decaying at its penetration into the system, is different from zero: $|c_+| = A_0, c_- = 0$. Hence our solution describes a single wave, with the real amplitude and the oscillation energy decreasing as

$$A_j \equiv |c_j| = A_0 \exp\left\{-\frac{\delta}{v} z_j\right\}, \quad E_j \propto A_j^2 \propto \exp\{-\alpha z_j\}, \quad \text{with } \alpha = \frac{2\delta}{v}, \tag{6.6.13}$$

i.e. with a frequency-independent attenuation constant $\alpha = 2\delta/v$,[26] so that the spatial scale of wave penetration into a dissipative system is given by $l_d \equiv 1/\alpha$. Certainly, our simple solution (88) is only valid for a system of length $l >> l_d$; otherwise, we would need the second term in the sum (87) to describe the wave reflected from its opposite end.

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[23] Even an elementary experience with acoustic guitars shows that for their strings this particular conclusion of our theory is not valid: higher modes ("overtones") decay substantially faster, leaving the fundamental mode oscillations for a slower decay. This is a result of another important energy loss (i.e. the wave decay) mechanism, not taken into account in Eq. (76) - the radiation of the sound into the guitar's body through the string supports, mostly through the bridge. Such radiation may be described by a proper modification of the boundary conditions (62), in terms of the ratio of the wave impedance (47) of the string and those of the supports.

[24] As a reminder, we have already met such a situation in the absence of damping, but at $\omega > \omega_{\max}-$ see Eq. (38).

[25] See, e.g., MA Eq. (3.5).

[26] I am sorry to use for the attenuation the same letter α as for the phase shift in Eq. (26) and a few of its corollaries, but both notations are traditional.
