2.4: Other Conservation Laws

Looking at the Lagrange equation (19), we immediately see that if $L \equiv T - U$ is independent of some generalized coordinate $q_j, \partial L / \partial q_j = 0,$ [15] then the corresponding generalized momentum is an integral of motion: 16

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \text{const.} \tag{2.4.1}$$

For example, for a 1D particle with the Lagrangian (21), the momentum p_x is conserved if the potential energy is constant (and hence the x-component of force is zero) - of course. As a less obvious example, let us consider a 2D motion of a particle in the field of central forces. If we use polar coordinates r and φ in the role of generalized coordinates, the Lagrangian function, [17]

$$L \equiv T - U = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r), \tag{2.4.2}$$

is independent of φ and hence the corresponding generalized momentum,

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}, \tag{2.4.3}$$

is conserved. This is just a particular (2D) case of the angular momentum conservation - see Eq. (1.24). Indeed, for the $2 \mathrm{D}$ motion within the $[x, y]$ plane, the angular momentum vector,

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix}, \tag{2.4.4}$$

has only one component different from zero, namely the component normal to the motion plane:

$$L_z = x(m\dot{y}) - y(m\dot{x}). \tag{2.4.5}$$

Differentiating the well-known relations between the polar and Cartesian coordinates,

$$x = r\cos\varphi, \quad y = r\sin\varphi, \tag{2.4.6}$$

over time, and plugging the result into Eq. (52), we see that

$$L_z = mr^2\dot{\varphi} \equiv p_\varphi. \tag{2.4.7}$$

Thus the Lagrangian formalism provides a powerful way of searching for non-evident integrals of motion. On the other hand, if such a conserved quantity is evident or known a priori, it is helpful for the selection of the most appropriate generalized coordinates, giving the simplest Lagrange equations. For example, in the last problem, if we knew in advance that p_φ had to be conserved, this could provide a motivation for using the angle φ as one of generalized coordinates.

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[16] This fact may be considered a particular case of a more general mathematical statement called the Noether theorem - named after its author, Emmy Nöther, sometimes called the "greatest woman mathematician ever lived". Unfortunately, because of time/space restrictions, for its discussion I have to refer the interested reader elsewhere - for example to Sec. 13.7 in H. Goldstein et al., Classical Mechanics, 3[rd] ed. Addison Wesley, 2002

[17] Note that here $\dot{r}^2$ is the square of the scalar derivative $\dot{r}$, rather than the square of the vector $\dot{\mathbf{r}} = \mathbf{v}$.