5.3: Reduced Equations

A much more important issue is the stability of the solutions described by Eq. (48). Indeed, Figure 4 shows that within a certain range of parameters, these equations give three different values for the oscillation amplitude (and phase), and it is important to understand which of these solutions are stable. Since these solutions are not the fixed points in the sense discussed in Sec. 3.2 (each point in Figure 4 represents a nearly-sinusoidal oscillation), their stability analysis needs a more general approach that would be valid for oscillations with amplitude and phase slowly evolving in time. This approach will also enable the analysis of non-stationary (especially the initial transient) processes, which are of importance for some dynamic systems.

First of all, let us formalize the way the harmonic balance equations, such as Eqs. (47), are obtained for the general case (38) - rather than for the particular Eq. (43) considered in the last section. After plugging in the 0^{th} approximation (41) into the right-hand side of equation (38) we have to require the amplitudes of both quadrature components of frequency ω to vanish. From the standard Fourier analysis, we know that these requirements may be represented as

$$\overline{f^{(0)}\sin\Psi}=0, \overline{f^{(0)}\cos\Psi}=0, \tag{5.3.1}$$

where the top bar means the time averaging - in our current case, over the period $2\pi/\omega$ of the right-hand side of Eq. (52), with the arguments calculated in the 0^{th} approximation:

$$f^{(0)}\equiv f\left(t, q^{(0)}, \dot{q}^{(0)}, \ldots\right)\equiv f(t, A\cos\Psi, -A\omega\sin\Psi, \ldots), \quad \text{with } \Psi=\omega t-\varphi. \tag{5.3.2}$$

Now, for a transient process the contribution of $q^{(0)}$ to the left-hand side of Eq. (38) is not zero any longer, because its amplitude and phase may be both slow functions of time - see Eq. (41). Let us calculate this contribution. The exact result would be

$$\begin{aligned}\ddot{q}^{(0)}+\omega^2 q^{(0)} &\equiv\left(\frac{d^2}{dt^2}+\omega^2\right)A\cos(\omega t-\varphi)\\ &=\left(\ddot{A}+2\dot{\varphi}\omega A-\dot{\varphi}^2 A\right)\cos(\omega t-\varphi)-2\dot{A}(\omega-\dot{\varphi})\sin(\omega t-\varphi).\end{aligned}$$

However, in the first approximation in ε, we may neglect the second derivative of A, and also the squares and products of the first derivatives of A and φ (which are all of the second order in ε), so that Eq. (54) is reduced to

$$\ddot{q}^{(0)}+\omega^2 q^{(0)}\approx 2A\dot{\varphi}\omega\cos(\omega t-\varphi)-2\dot{A}\omega\sin(\omega t-\varphi). \tag{5.3.3}$$

On the right-hand side of Eq. (53), we can neglect the time derivatives of the amplitude and phase at all, because this part is already proportional to the small parameter. Hence, in the first order in ε, Eq. (38) becomes

$$\ddot{q}^{(1)}+\omega^2 q^{(1)}=f_{\text{ef}}^{(0)}\equiv f^{(0)}-(2A\dot{\varphi}\omega\cos\Psi-2\dot{A}\omega\sin\Psi). \tag{5.3.4}$$

Now, applying Eqs. (52) to the function $f_{\text{ef}}^{(0)}$, and taking into account that the time averages of $\sin^2\Psi$ and $\cos^2\Psi$ are both equal to $1/2$, while the time average of the product $\sin\Psi\cos\Psi$ vanishes, we get a pair of so-called reduced equations (alternatively called "truncated", or "RWA", or "van der Pol" equations) for the time evolution of the amplitude and phase:

$$\dot{A}=-\frac{1}{\omega}\overline{f^{(0)}\sin\Psi}, \quad \dot{\varphi}=\frac{1}{\omega A}\overline{f^{(0)}\cos\Psi} \tag{5.3.5}$$

Extending the definition (4) of the complex amplitude of oscillations to their slow evolution in time, $a(t)\equiv A(t)\exp\{i\varphi(t)\}$, and differentiating this relation, the two equations (57a) may be also rewritten in the form of either one equation for a :

$$\dot{a}=\frac{i}{\omega}\overline{f^{(0)}e^{i(\Psi+\varphi)}}\equiv\frac{i}{\omega}\overline{f^{(0)}e^{i\omega t}} \tag{5.3.6}$$

or two equations for the real and imaginary parts of $a(t)=u(t)+iv(t)$:

$$\dot{u}=-\frac{1}{\omega}\overline{f^{(0)}\sin\omega t}, \quad \dot{v}=\frac{1}{\omega}\overline{f^{(0)}\cos\omega t}. \tag{5.3.7}$$

The first-order harmonic balance equations (52) are evidently just the particular case of the reduced equations (57) for stationary oscillations ($\dot{A}=\dot{\varphi}=0$). [21]

Superficially, the system (57a) of two coupled, first-order differential equations may look more complex than the initial, second-order differential equation (38), but actually, it is usually much simpler. For example, let us spell them out for the easy case of free oscillations a linear oscillator with damping. For that, we may reuse the ready Eq. (46) by taking $\alpha = f_0 = 0$, and thus turning Eqs. (57a) into

$$\begin{aligned} \dot{A} &= -\frac{1}{\omega}\overline{f^{(0)}\sin\Psi} \equiv -\frac{1}{\omega}\overline{(2\xi\omega A\cos\Psi + 2\delta\omega A\sin\Psi)\sin\Psi} \equiv -\delta A, \\ \dot{\varphi} &= \frac{1}{\omega A}\overline{f^{(0)}\cos\Psi} \equiv \frac{1}{\omega A}\overline{(2\xi\omega A\cos\Psi + 2\delta\omega A\sin\Psi)\cos\Psi} \equiv \xi. \end{aligned}$$

The solution of Eq. (58a) gives us the same "envelope" law $A(t) = A(0)e^{-\delta t}$ as the exact solution (10) of the initial differential equation, while the elementary integration of Eq. (58b) yields $\varphi(t) = \xi t + \varphi(0) \equiv \omega t - \omega_0 t + \varphi(0)$. This means that our approximate solution,

$$q^{(0)}(t) = A(t)\cos[\omega t - \varphi(t)] = A(0)e^{-\delta t}\cos[\omega_0 t - \varphi(0)], \tag{5.3.8}$$

agrees with the exact Eq. (9), and misses only the correction (8) of the oscillation frequency. (This correction is of the second order in δ, i.e. of the order of ε^2, and hence is beyond the accuracy of our first approximation.) It is remarkable how nicely do the reduced equations recover the proper frequency of free oscillations in this autonomous system - in which the very notion of ω is ambiguous.

The result is different at forced oscillations. For example, for the (generally, nonlinear) Duffing oscillator described by Eq. (43) with $f_0 \neq 0$, Eqs. (57a) yield the reduced equations,

$$\dot{A} = -\delta A + \frac{f_0}{2\omega}\sin\varphi, \quad A\dot{\varphi} = \xi(A)A + \frac{f_0}{2\omega}\cos\varphi, \tag{5.3.9}$$

which are valid for an arbitrary function $\xi(A)$, provided that this nonlinear detuning remains much smaller than the oscillation frequency. Here (after a transient), the amplitude and phase tend to the stationary states described by Eqs. (47). This means that φ becomes a constant, so that $q^{(0)} \rightarrow A\cos(\omega t -$ const), i.e. the reduced equations again automatically recover the correct frequency of the solution, in this case equal to the external force frequency.

Note that each stationary oscillation regime, with certain amplitude and phase, corresponds to a fixed point of the reduced equations, so that the stability of those fixed points determines that of the oscillations. In the next three sections, we will carry out such an analysis for several simple systems of key importance for physics and engineering.

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[21] One may ask why cannot we stick to just one, most compact, complex-amplitude form (57b) of the reduced equations. The main reason is that when the function $f(q, \dot{q}, t)$ is nonlinear, we cannot replace its real arguments, such as $q = A\cos(\omega t - \varphi)$, with their complex-function representations like $a\exp\{-i\omega t\}$ (as could be done in the linear problems considered in Sec. 5.1), and need to use real variables, such as either $\{A, \varphi\}$ or $\{u, v\}$, anyway.
