3.4: Planetary Problems

Leaving a more detailed study of oscillations for Chapter 5, let us now discuss the so-called planetary systems [10] whose description, somewhat surprisingly, may be also reduced to an effectively 1D problem. Consider two particles that interact via a conservative, central force $\mathbf{F}_{21}=-\mathbf{F}_{12}=\mathbf{n}_r F(r)$, where r and $\mathbf{n}_r$ are, respectively, the magnitude and direction of the distance vector $\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$ connecting the two particles (Figure 3).

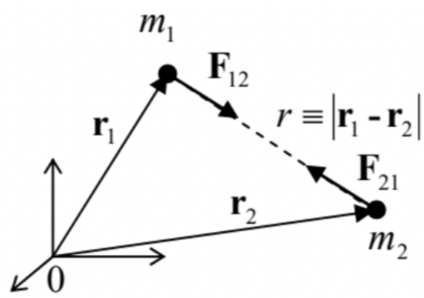

Figure 3.3. Vectors in the "planetary" problem.

Generally, two particles moving without constraints in 3D space, have $3+3=6$ degrees of freedom, which may be described, e.g., by their Cartesian coordinates $\{x_1, y_1, z_1, x_2, y_2, z_2\}$ However, for this particular form of interaction, the following series of tricks allows the number of essential degrees of freedom to be reduced to just one.

First, the central, conservative force of particle interaction may be described by a timeindependent potential energy $U(r)$, such that $F(r)=-\partial U(r)/\partial r$. [11] Hence the Lagrangian of the system is

$$L\equiv T-U(r)=\frac{m_1}{2}\dot{\mathbf{r}}_1^2+\frac{m_2}{2}\dot{\mathbf{r}}_2^2-U(r). \tag{3.4.1}$$

Let us perform the transfer from the initial six scalar coordinates of the particles to the following six generalized coordinates: three Cartesian components of the distance vector

$$\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2, \tag{3.4.2}$$

and three scalar components of the following vector:

$$\mathbf{R}\equiv\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{M}, \qquad \text{with } M\equiv m_1+m_2, \tag{3.4.3}$$

which defines the position of the center of mass of the system, with the total mass M. Solving the system of two linear equations (31) and (32) for $\mathbf{r}_1$ and $\mathbf{r}_2$, we get

$$\mathbf{r}_1=\mathbf{R}+\frac{m_2}{M}\mathbf{r}, \quad \mathbf{r}_2=\mathbf{R}-\frac{m_1}{M}\mathbf{r}. \tag{3.4.4}$$

Plugging these relations into Eq. (30), we see that it is reduced it to

$$L=\frac{M}{2}\dot{\mathbf{R}}^2+\frac{m}{2}\dot{\mathbf{r}}^2-U(r), \tag{3.4.5}$$

where m is the so-called reduced mass:

$$m\equiv\frac{m_1m_2}{M}, \text{ so that } \frac{1}{m}\equiv\frac{1}{m_1}+\frac{1}{m_2}. \tag{3.4.6}$$

Note that according to Eq. (35), the reduced mass is lower than that of the lightest component of the two-body system. If one of $m_{1,2}$ is much less than its counterpart (like it is in most star-planet or planetsatellite systems), then with a good precision $m\approx\min[m_1,m_2]$.

Since the Lagrangian function (34) depends only on $\mathbf{R}$ rather than $\mathbf{R}$ itself, according to our discussion in Sec. 2.4, all Cartesian components of R are cyclic coordinates, and the corresponding generalized momenta are conserved:

$$P_j\equiv\frac{\partial L}{\partial\dot{R}_j}\equiv M\dot{R}_j=\text{const}, \quad j=1,2,3. \tag{3.4.7}$$

Physically, this is just the conservation law for the full momentum $\mathbf{P} \equiv M\mathbf{R}$ of our system, due to the absence of external forces. Actually, in the axiomatics used in Sec. 1.3 this law is postulated - see Eq. (1.10) - but now we may attribute the momentum $\mathbf{P}$ to a certain geometric point, with the center-of-mass radius vector $\mathbf{R}$. In particular, since according to Eq. (36) the center moves with a constant velocity in the inertial reference frame used to write Eq. (30), we may consider a new inertial frame with the origin at point $\mathbf{R}$. In this new frame, $\mathbf{R} \equiv 0$, so that the vector $\mathbf{r}$ (and hence the scalar r) remain the same as in the old frame (because the frame transfer vector adds equally to $\mathbf{r}_1$ and $\mathbf{r}_2$, and cancels in $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$), and the Lagrangian (34) is now reduced to

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 - U(r) \tag{3.4.8}$$

Thus our initial problem has been reduced to just three degrees of freedom - three scalar components of the vector $\mathbf{r}$. Moreover, Eq. (37) shows that dynamics of the vector $\mathbf{r}$ of our initial, twoparticle system is identical to that of the radius vector of a single particle with the effective mass m, moving in the central potential field $U(r)$.

Two more degrees of freedom may be excluded from the planetary problem by noticing that according to Eq. (1.35), the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of our effective single particle of mass m is also conserved, both in magnitude and direction. Since the direction of $\mathbf{L}$ is, by its definition, perpendicular to both of $\mathbf{r}$ and $\mathbf{v} = \mathbf{p}/m$, this means that the particle's motion is confined to the plane whose orientation is determined by the initial directions of the vectors $\mathbf{r}$ and $\mathbf{v}$. Hence we can completely describe particle's position by just two coordinates in that plane, for example by the distance r to the origin, and the polar angle φ. In these coordinates, Eq. (37) takes the form identical to Eq. (2.49):

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r). \tag{3.4.9}$$

Moreover, the latter coordinate, polar angle φ, may be also eliminated by using the conservation of angular momentum's magnitude, in the form of Eq. (2.50) : [12]

$$L_z = mr^2\dot{\varphi} = \text{ const.} \tag{3.4.10}$$

A direct corollary of this conservation is the so-called 2^{nd} Kepler law: [13] the radius vector $\mathbf{r}$ sweeps equal areas A in equal times. Indeed, in the linear approximation in $dA << A$, the area differential dA is equal to the area of a narrow right triangle with the base being the arc differential $rd\varphi$, and the height equal to r-see Figure 4. As a result, according to Eq. (39), the time derivative of the area,

$$\frac{dA}{dt} = \frac{r(rd\varphi)/2}{dt} \equiv \frac{1}{2}r^2\dot{\varphi} = \frac{L_z}{2m}, \tag{3.4.11}$$

remains constant. Since the factor $L_z/2m$ is constant, integration of this equation over an arbitrary (not necessarily small!) time interval Δt proves the 2^{nd} Kepler law: $A \propto \Delta t$.

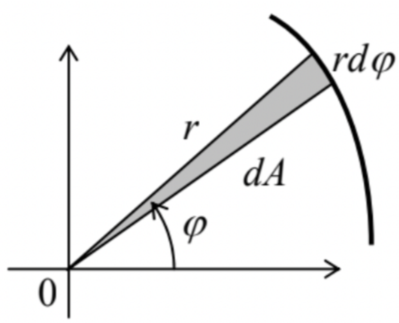

Figure 3.4. The area differential dA in the polar coordinates.

Now note that since $\partial L/\partial t = 0$, the Hamiltonian function H is also conserved, and since, according to Eq. (38), the kinetic energy of the system is a quadratic-homogeneous function of the generalized velocities $\dot{r}$ and $\dot{\varphi}$, we have $H = E$, so that the system's energy E,

$$E = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\varphi}^2 + U(r), \tag{3.4.12}$$

is also a first integral of motion. [14] But according to Eq. (39), the second term on the right-hand side of Eq. (41) may be represented as

$$\frac{m}{2} r^2 \dot{\varphi}^2 = \frac{L_z^2}{2mr^2}, \tag{3.4.13}$$

so that the energy (41) may be expressed as that of a 1D particle moving along axis r,

$$E = \frac{m}{2}\dot{r}^2 + U_{\text{ef}}(r), \tag{3.4.14}$$

in the following effective potential:

$$U_{\text{ef}}(r) \equiv U(r) + \frac{L_z^2}{2mr^2} \tag{3.4.15}$$

(The physical sense of the second term is similar to that of the first term in the U_{ef} spelled out in Eq. (6), and will be discussed again in Sec. 4.6 below.) So the planetary motion problem has been reduced to the dynamics of an effectively-1D system. [15]

Now we may proceed just like we did in Sec. 3, with due respect to the very specific effective potential (44) which, in particular, diverges at $r \rightarrow 0$ - besides the very special case of an exactly radial motion, $L_z = 0$. In particular, we may solve Eq. (43) for dr/dt to get

$$dt = \left(\frac{m}{2}\right)^{1/2} \frac{dr}{\left[E - U_{\text{ef}}(r)\right]^{1/2}} \tag{3.4.16}$$

This equation allows us not only to get a direct relationship between time t and distance r, similar to Eq. (26),

$$t = \pm\left(\frac{m}{2}\right)^{1/2} \int \frac{dr}{\left[E - U_{\text{ef}}(r)\right]^{1/2}} = \pm\left(\frac{m}{2}\right)^{1/2} \int \frac{dr}{\left[E - U(r) - L_z^2/2mr^2\right]^{1/2}}, \tag{3.4.17}$$

but also do a similar calculation of the angle φ. Indeed, integrating Eq. (39),

$$\varphi \equiv \int \dot{\varphi} dt = \frac{L_z}{m} \int \frac{dt}{r^2}, \tag{3.4.18}$$

and plugging dt from Eq. (45), we get an explicit expression for the particle trajectory $\varphi(r)$:

$$\varphi = \pm \frac{L_z}{(2m)^{1/2}} \int \frac{dr}{r^2\left[E - U_{\text{ef}}(r)\right]^{1/2}} = \pm \frac{L_z}{(2m)^{1/2}} \int \frac{dr}{r^2\left[E - U(r) - L_z^2/2mr^2\right]^{1/2}}. \tag{3.4.19}$$

Note that according to Eq. (39), the derivative $d\varphi/dt$ does not change sign at the reflection from any classical turning point $r \neq 0$, so that, in contrast to Eq. (46), the sign on the right-hand side of Eq. (48) is uniquely determined by the initial conditions and cannot change during the motion.

Let us use these results, valid for any interaction law $U(r)$, for the planetary motion's classification. (Following a good tradition, in what follows I will select the arbitrary constant in the potential energy in the way to provide $U \rightarrow 0$, and hence $U_{\text{ef}} \rightarrow 0$ at $r \rightarrow \infty$.) The following cases should be distinguished.

If $U(r) < 0$, i.e. the particle interaction is attractive (as it always is in the case of gravity), and the divergence of the attractive potential at $r \rightarrow 0$ is faster than $1/r^2$, then $U_{\text{ef}}(r) \rightarrow -\infty$ at $r \rightarrow 0$, so that at appropriate initial conditions the particle may drop on the center even if $L_z \neq 0$ - the event called the capture. On the other hand, with $U(r)$ either converging or diverging slower than $1/r^2$, at $r \rightarrow 0$, the effective energy profile $U_{\text{ef}}(r)$ has the shape shown schematically in Figure 5. This is true, in particular, for the very important case

$$U(r) = -\frac{\alpha}{r}, \quad \text{with } \alpha > 0 \tag{3.4.20}$$

which describes, in particular, the Coulomb (electrostatic) interaction of two particles with electric charges of opposite signs, and the Newton gravity law (1.15). This particular case will be analyzed below, but now let us return to the analysis of an arbitrary attractive potential $U(r) < 0$ leading to the effective potential shown in Figure 5 , when the angular-momentum term in Eq. (44) dominates at small distances r.

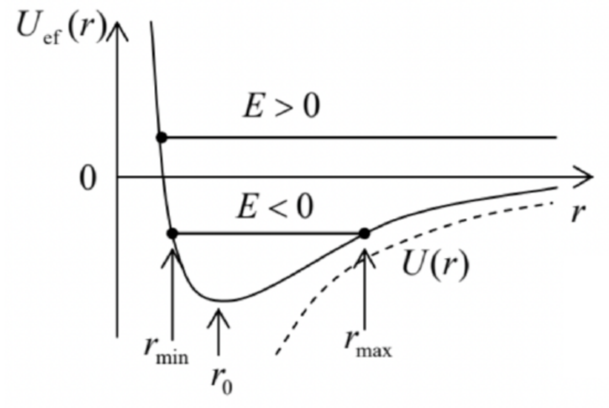

Figure 3.5. Effective potential profile of an attractive central field, and two types of motion in it.

According to the analysis in Sec. 3, such potential profile, with a minimum at some distance r_0, may sustain two types of motion, depending on the energy E (which is determined by initial conditions):

(i) If $E > 0$, there is only one classical turning point where $E = U_{ef}$, so that the distance r either grows with time from the very beginning or (if the initial value of $\dot{r}$ was negative) first decreases and then, after the reflection from the increasing potential U_{ef}, starts to grow indefinitely. The latter case, of course, describes the scattering of the effective particle by the attractive center. [16]

(ii) On the opposite, if the energy is within the range

$$U_{ef}(r_0) \leq E < 0, \tag{3.4.21}$$

the system moves periodically between two classical turning points r_{min} and r_{max} – see Figure 5. These oscillations of the distance r correspond to the bound orbital motion of our effective particle about the attracting center.

Let us start with the discussion of the bound motion, with the energy within the range (50). If the energy has its minimal possible value,

$$E = U_{ef}(r_0) \equiv \min\left[U_{ef}(r)\right], \tag{3.4.22}$$

the distance cannot change, $r = r_0 =$ const, so that the orbit is circular, with the radius r_0 satisfying the condition $dU_{ef}/dr = 0$. Using Eq. (44), we see that the condition for r_0 may be written as

$$\frac{L_z^2}{mr_0^3} = \left.\frac{dU}{dr}\right|_{r=r_0}. \tag{3.4.23}$$

Since at circular motion, the velocity $\mathbf{v}$ is perpendicular to the radius vector $\mathbf{r}$, L_z is just mr_0v, the lefthand side of Eq. (52) equals mv^2/r_0, while its right-hand side is just the magnitude of the attractive force, so that this equality expresses the well-known 2[nd] Newton law for the circular motion. Plugging this result into Eq. (47), we get a linear law of angle change, $\varphi = \omega t +$ const, with the angular velocity

$$\omega = \frac{L_z}{mr_0^2} = \frac{v}{r_0}, \tag{3.4.24}$$

and hence the rotation period $\tau_\varphi \equiv 2\pi/\omega$ obeys the elementary relation

$$\tau_\varphi = \frac{2\pi r_0}{v}. \tag{3.4.25}$$

Now let the energy be above its minimum value (but still negative). Using Eq. (46) just as in Sec. 3, we see that the distance r now oscillates with the period

$$\tau_r = (2m)^{1/2} \int_{r_{min}}^{r_{max}} \frac{dr}{\left[E - U(r) - L_z^2/2mr^2\right]^{1/2}}. \tag{3.4.26}$$

This period is not necessarily equal to another period, T_{φ}, that corresponds to the 2π-change of the angle. Indeed, according to Eq. (48), the change of the angle φ between two sequential points of the nearest approach,

$$|\Delta\varphi| = 2\frac{L_z}{(2m)^{1/2}} \int_{r_{\min}}^{r_{\max}} \frac{dr}{r^2\left[E - U(r) - L_z^2/2mr^2\right]^{1/2}} \tag{3.4.27}$$

is generally different from 2π. Hence, the general trajectory of the bound motion has a spiral shape see, e.g., an illustration in Figure 6.

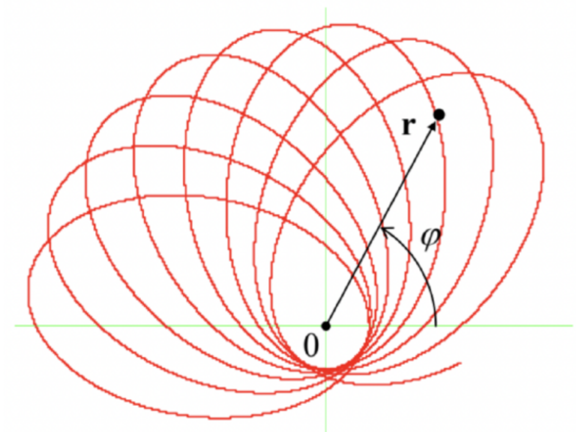

Figure 3.6. A typical open orbit of a particle moving in a non-Coulomb central field.

The situation is special, however, for a very important particular case, namely that of the Coulomb potential described by Eq. (49). Indeed, plugging this potential into Eq. (48), we get

$$\varphi = \pm\frac{L_z}{(2m)^{1/2}} \int \frac{dr}{r^2\left(E + \alpha/r - L_z^2/2mr^2\right)^{1/2}}. \tag{3.4.28}$$

This is a table integral, [17] giving

$$\varphi = \pm\cos^{-1}\frac{L_z^2/m\alpha r - 1}{\left(1 + 2EL_z^2/m\alpha^2\right)^{1/2}} + \text{const}. \tag{3.4.29}$$

The reciprocal function, $r(\varphi)$, is 2π-periodic:

$$r = \frac{p}{1 + e\cos(\varphi + \text{const})}, \tag{3.4.30}$$

so that at $E < 0$, the orbit is a closed line, [18] characterized by the following parameters: [19]

$$p \equiv \frac{L_z^2}{m\alpha}, \quad e \equiv \left(1 + \frac{2EL_z^2}{m\alpha^2}\right)^{1/2} \tag{3.4.31}$$

The physical meaning of these parameters is very simple. Indeed, the general Eq. (52), in the Coulomb potential for which $dU/dr = \alpha/r^2$, shows that p is just the circular orbit radius [20] for the given $L_z : r_0 = L_z^2/m\alpha \equiv p$, so that

$$\min\left[U_{\text{ef}}(r)\right] \equiv U_{\text{ef}}\left(r_0\right) = -\frac{\alpha^2 m}{2L_z^2}. \tag{3.4.32}$$

Using this equality together with the second of Eqs. (60), we see that the parameter e (called the eccentricity) may be represented just as

$$e = \left\{1 - \frac{E}{\min\left[U_{\text{ef}}(r)\right]}\right\}^{1/2}. \tag{3.4.33}$$

Analytical geometry tells us that Eq. (59), with $e < 1$, is one of the canonical representations of an ellipse, with one of its two focuses located at the origin. The fact that planets have such trajectories is known as the 1^{st} Kepler law. Figure 7 shows the relations between the dimensions of the ellipse and the parameters p and e.[21]

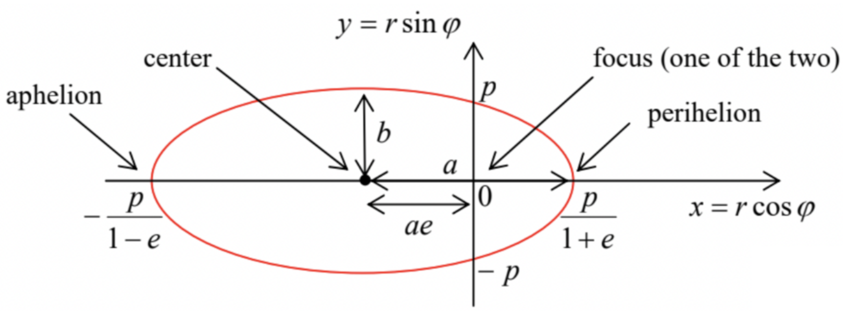

Figure 3.7. Ellipse, and its special points and dimensions.

In particular, the major semi-axis a and the minor semi-axis b are simply related to p and e and hence, via Eqs. (60), to the motion integrals E and L_z :

$$a=\frac{p}{1-e^2}=\frac{\alpha}{2|E|}, \quad b=\frac{p}{\left(1-e^2\right)^{1/2}}=\frac{L_z}{(2m|E|)^{1/2}}. \tag{3.4.34}$$

As was mentioned above, at $E \rightarrow \min\left[U_{\text{ef}}(r)\right]$ the orbit is almost circular, with $r(\varphi) \cong r_0 \approx p$. On the contrary, as E is increased to approach zero (its maximum value for the closed orbit), then $e \rightarrow 1$, so that the aphelion point $r_{\max}=p/(1-e)$ tends to infinity, i.e. the orbit becomes extremely extended $-$ see the red lines in Figure 8 .

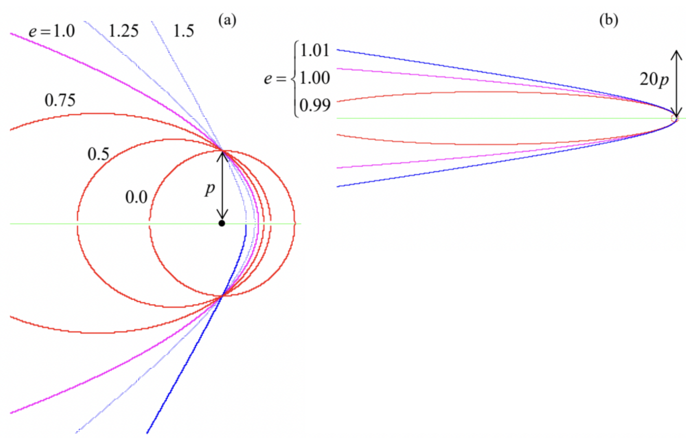

Figure 3.8. (a) Zoom-in and (b) zoom-out on the Coulombfield trajectories corresponding to the same parameter p (i.e., the same L_z), but different values of the eccentricity parameter e, i.e. of the energy E - see Eq. (60): ellipses ($e<1$, red lines), a parabola ($e=1$, magenta line), and hyperbolas ($e>1$, blue lines). Note that the transition from closed to open trajectories at $e=1$ is dramatic only at very large distances, $r >> p$.

The above relations enable, in particular, a ready calculation of the rotation period $T \equiv T_r=\tau_{\varphi}$. (In the case of a closed trajectory, τ_r and τ_{φ} coincide.) Indeed, it is well known that the ellipse area $A=\pi ab$. But according to the 2$^{\text{nd}}$ Kepler law (40), $dA/dt=L_z/2m=$ const. Hence

$$\tau=\frac{A}{dA/dt}=\frac{\pi ab}{L_z/2m}. \tag{3.4.35}$$

Using Eqs. (60) and (63), this important result may be represented in several other forms:

$$\tau=\frac{\pi p^2}{\left(1-e^2\right)^{3/2}\left(L_z/2m\right)}=\pi\alpha\left(\frac{m}{2|E|^3}\right)^{1/2}=2\pi a^{3/2}\left(\frac{m}{\alpha}\right)^{1/2}. \tag{3.4.36}$$

Since for the Newtonian gravity (1.15), $\alpha=Gm_1m_2=GmM$, at $m_1<<m_2$ (i.e. $m<<M$ this constant is proportional to m, and the last form of Eq. (64b) yields the 3$^{\text{rd}}$ Kepler law: periods of motion of different planets in the same central field, say that of our Sun, scale as $\tau \propto a^{3/2}$. Note that in contrast to the 2$^{\text{nd}}$ Kepler law (which is valid for any central field), the 1$^{\text{st}}$ and the 3$^{\text{rd}}$ Kepler laws are potentialspecific.

Now reviewing the above derivation of Eqs. (59)-(60), we see that they are also valid in the case of $E \geq 0$ - see the top horizontal line in Figure 5 and its discussion above, if we limit the results to the physically meaningful range $r \geq 0$. This means that if the energy is exactly zero, Eq. (59) (with $e=1$) is still valid for all values of φ (except for one special point $\varphi=\pi$ where r becomes infinite) and describes a parabolic (i.e. open) trajectory - see the magenta lines in Figure 8.

Moreover, if $E>0$, Eq. (59) is still valid within a certain sector of angles φ,

$$\Delta \varphi=2 \cos ^{-1} \frac{1}{e} \equiv 2 \cos ^{-1}\left(1+\frac{2 E L_z^2}{m \alpha^2}\right)^{-1 / 2}<\pi, \quad \text { for } E>0, \tag{3.4.37}$$

and describes an open, hyperbolic trajectory (see the blue lines in Figure 8). As was mentioned earlier, such trajectories are typical, in particular, for particle scattering.

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[10] This name is very conditional, because this group of problems includes, for example, charged particle scattering (see Sec. 3.7 below).

[11] See, e.g., MA Eq. (10.8) with $\partial / \partial \theta=\partial / \partial \varphi=0$.

[12] Here index z stands for the coordinate perpendicular to the motion plane. Since other components of the angular momentum equal zero, the index is not really necessary, but I will still use it - just to make a clear distinction between the angular momentum L_z and the Lagrangian function L.

[13] This is one of the three laws deduced, from the extremely detailed astronomical data collected by Tycho Brahe (1546-1601), by Johannes Kepler in the early 17^{th} century. In turn, the three Kepler laws have become the main basis for Newton's discovery, a few decades later, of the gravity law (1.15). That relentless march of physics...

[14] One may argue that this fact should have been evident from the very beginning because the effective particle of mass m moves in a potential field $U(r)$, which conserves energy.

[15] Note that this reduction has been done in a way different from that used for our testbed problem (shown in Figure 2.1) in Sec. 2 above. (The reader is encouraged to analyze this difference.) To emphasize this fact, I will keep writing E instead of H here, though for the planetary problem we are discussing now, these two notions coincide.

[16] In the opposite case when the interaction is repulsive, $U(r)>0$, the addition of the positive angular energy term only increases the trend, and the scattering scenario is the only one possible.

[17] See, e.g., MA Eq. (6.3a).

[18] It may be proved that for the power-law interaction, $U \propto r^v$, the orbits are closed curves only if $v=-1$ (our current case of the Coulomb potential) or if $v=+2$ (the 3D harmonic oscillator) - the so-called Bertrand theorem.

[19] Let me hope that the difference between the parameter p and the particle momentum's magnitude is absolutely clear from the context, so that using the same (traditional) notation for both notions cannot lead to confusion.

[20] Mathematicians prefer a more solemn terminology: the parameter $2p$ is called the latus rectum of the ellipse.

[21] In this figure, the constant participating in Eqs. (58)-(59) is assumed to be zero. A different choice of the constant corresponds just to a different origin of φ, i.e. a constant turn of the ellipse about the origin.
