7.4: Equilibrium

Now we are fully equipped to discuss the elastic deformation dynamics, but let us start with statics. The static (equilibrium) state may be described by requiring the right-hand side of Eq. (25) to vanish. To find the elastic deformation, we need to plug σ_{ij} ' from Hooke's law (49a), and then express the elements s_{ij} ' via the displacement distribution - see Eq. (9). For a uniform material, the result is [21]

$$\frac{E}{2(1+v)}\sum_{j'=1}^{3}\frac{\partial^2 q_j}{\partial r_{j'}^2}+\frac{E}{2(1+v)(1-2v)}\sum_{j'=1}^{3}\frac{\partial^2 q_{j'}}{\partial r_j\partial r_{j'}}+f_j=0. \tag{7.4.1}$$

Taking into account that the first sum in Eq. (51) is just the j^{th} component of $\nabla^2\mathbf{q}$, while the second sum is the j^{th} component of $\nabla(\nabla\cdot\mathbf{q})$, we see that all three equations (51) for three Cartesian components ($j=$ 1,2 and 3) of the deformation vector $\mathbf{q}$, may be conveniently merged into one vector equation

$$\frac{E}{2(1+v)}\nabla^2\mathbf{q}+\frac{E}{2(1+v)(1-2v)}\nabla(\nabla\cdot\mathbf{q})+\mathbf{f}=0 \tag{7.4.2}$$

For some applications, it is more convenient to recast this equation into a different form, using the wellknown vector identity [22] $\nabla^2\mathbf{q}=\nabla(\nabla\cdot\mathbf{q})-\nabla\times(\nabla\times\mathbf{q})$. The result is

$$\frac{E(1-v)}{(1+v)(1-2v)}\nabla(\nabla\cdot\mathbf{q})-\frac{E}{2(1+v)}\nabla\times(\nabla\times\mathbf{q})+\mathbf{f}=0. \tag{7.4.3}$$

It is interesting that in problems without volume-distributed forces $(\mathbf{f}=0)$, Young's modulus E cancels out. Even more fascinating, in this case the equation may be re-written in a form not involving the Poisson ratio v either. Indeed, calculating the divergence of the remaining terms of Eq. (53), taking into account MA Eqs. (9.2) and (11.2), we get a surprisingly simple equation

$$\nabla^2(\nabla\cdot\mathbf{q})=0 \tag{7.4.4}$$

A natural question here is how do the elastic moduli affect the deformation distribution if they do not participate in the differential equation describing it. The answer is different is two cases. If what is fixed at the body's boundary are deformations, then the moduli are irrelevant, because the deformation distribution through the body does not depend on them. On the other hand, if the boundary conditions describe fixed stress (or a combination of stress and strain), then the elastic constants creep into the solution via the recalculation of these conditions into the strain.

As a simple but representative example, let us calculate the deformation distribution in a (generally, thick) spherical shell under the different pressures inside and outside it $-$ see Figure 7 a.

image

(b)

Figure 7.7. The spherical shell problem: (a) the general case, and (b) the thin shell limit.

Due to the spherical symmetry of the problem, the deformation is obviously sphericallysymmetric and radial, $\mathbf{q}(\mathbf{r})=q(r)\mathbf{n}_r$, i.e. is completely described by one scalar function $q(r)$. Since the curl of such a radial vector field is zero, [23] Eq. (53) is reduced to

$$\nabla(\nabla\cdot\mathbf{q})=0, \tag{7.4.5}$$

This means that the divergence of the function $q(r)$ is constant within the shell. In the spherical coordinates: [24]

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2q\right)=\text{ const.} \tag{7.4.6}$$

Naming this constant $3a$ (with the numerical factor chosen just for the later notation's convenience), and integrating Eq. (56) over r, we get its solution,

$$q(r)=ar+\frac{b}{r^2}, \tag{7.4.7}$$

which also includes another integration constant, b. The constants a and b may be determined from the boundary conditions. Indeed, according to Eq. (19),

$$\sigma_{rr}=\begin{cases}-\mathcal{P}_1, & \text{at } r=R_1,\\ -\mathcal{P}_2, & \text{at } r=R_2.\end{cases} \tag{7.4.8}$$

In order to relate this stress to strain, let us use Hooke's law, but for that, we first need to calculate the strain tensor components for the deformation distribution (57). Using Eqs. (17), we get

$$s_{rr}=\frac{\partial q}{\partial r}=a-2\frac{b}{r^3},\quad s_{\theta\theta}=s_{\varphi\varphi}=\frac{q}{r}=a+\frac{b}{r^3}, \tag{7.4.9}$$

so that $\mathrm{Tr}(\mathrm{s})=3a$. Plugging these relations into Eq. (49a) for σ_{rr}, we get

$$\sigma_{rr}=\frac{E}{1+v}\left[\left(a-2\frac{b}{r^3}\right)+\frac{v}{1-2v}3a\right]. \tag{7.4.10}$$

Now plugging this relation into Eqs. (58), we get a system of two linear equations for the coefficients a and b. Solving this system, we get:

$$a=\frac{1-2v}{E}\frac{\mathcal{P}_1R_1^3-\mathcal{P}_2R_2^3}{R_2^3-R_1^3},\quad b=\frac{1+v}{2E}\frac{(\mathcal{P}_1-\mathcal{P}_2)R_1^3R_2^3}{R_2^3-R_1^3}. \tag{7.4.11}$$

Formulas (57) and (61) give a complete solution to our problem. (Note that the elastic moduli are back, as was promised.) The solution is rich in physical content and deserves at least some analysis. First of all, note that according to Eq. (48), the coefficient $(1-2v)/E$ in the expression for a is just $1/3K$, so that the first term in Eq. (57) for the net deformation describes the hydrostatic compression. Now note that the second of Eqs. (61) yields $b=0$ if $R_1=0$. Thus for a solid sphere, we have only the hydrostatic compression, which was discussed in the previous section. Perhaps less intuitively, making two pressures equal also gives $b=0$, i.e. the purely hydrostatic compression, for arbitrary $R_2>R_1$.

However, in the general case, $b\neq 0$, so that the second term in the deformation distribution (57), which describes the shear deformation,[25] is also substantial. In particular, let us consider the important thin-shell limit, when $R_2-R_1\equiv t<R_{1,2}\equiv R$ - see Figure 7 b. In this case, $q(R_1)\approx q(R_2)$ is just the change of the shell radius R, for which Eqs. (57) and (61) (with ${R_2}^3-{R_1}^3\approx 3R^2t$) give

$$\Delta R\equiv q(R)\approx aR+\frac{b}{R^2}\approx\frac{(\mathcal{P}_1-\mathcal{P}_2)R^2}{3t}\left(\frac{1-2v}{E}+\frac{1+v}{2E}\right)=(\mathcal{P}_1-\mathcal{P}_2)\frac{R^2}{t}\frac{1-v}{2E}. \tag{7.4.12}$$

Naively, one could think that at least in this limit the problem could be analyzed by elementary means. For example, the total force exerted by the pressure difference $(\mathcal{P}_1-\mathcal{P}_2)$ on the diametrical crosssection of the shell (see, e.g., the dashed line in Figure 7 b) is $F=\pi R^2(\mathcal{P}_1-\mathcal{P}_2)$, giving the stress,

$$\sigma=\frac{F}{A}=\frac{\pi R^2(\mathcal{P}_1-\mathcal{P}_2)}{2\pi Rt}=(\mathcal{P}_1-\mathcal{P}_2)\frac{R}{2t}, \tag{7.4.13}$$

directed along the shell's walls. One can check that this simple formula may be indeed obtained, in this limit, from the strict expressions for $\sigma_{\theta\theta}$ and $\sigma_{\varphi\varphi}$, following from the general treatment carried out above. However, if we now tried to continue this approach by using the simple relation (45) to find the small change Rs_{zz} of the sphere's radius, we would arrive at a result with the structure of Eq. (62), but without the factor $(1-v)<1$ in the numerator. The reason for this error (which may be as significant as 330% for typical construction materials - see Table 1) is that Eq. (45), while being valid for thin rods of arbitrary cross-section, is invalid for thin but broad sheets, and in particular the thin shell in our problem. Indeed, while at the tensile stress both lateral dimensions of a thin rod may contract freely, in our problem all dimensions of the shell are under stress - actually, under much more tangential stress than the radial one[26]

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[21] As follows from Eqs. (48), the coefficient before the first sum in Eq. (51) is just the shear modulus μ, while that before the second sum is equal to $(K+\mu/3)$.

[22] See, e.g., MA Eq. (11.3).

[23] If this is not immediately evident, please have a look at MA Eq. (10.11) with $\mathbf{f}=f_r(r)\mathbf{n}_r$.

[24] See, e.g., MA Eq. (10.10) with $\mathbf{f}=q(r)\mathbf{n}_r$

[25] Indeed, according to Eq. (48), the material-dependent factor in the second of Eqs. (61) is just $1/4\mu$.

[26] Strictly speaking, this is only true if the pressure difference is not too small, namely, if $|\mathcal{P}_1 - \mathcal{P}_2| >> \mathcal{P}_{1,2}t/R$.