1.3: Dynamics- Newton Laws

Generally, the classical dynamics is fully described (in addition to the kinematic relations discussed above) by three Newton laws. In contrast to the impression some textbooks on theoretical physics try to create, these laws are experimental in nature, and cannot be derived from purely theoretical arguments.

I am confident that the reader of these notes is already familiar with the Newton laws, [7] in one or another formulation. Let me note only that in some formulations, the 1^{st} Newton law looks just as a particular case of the 2^{nd} law - when the net force acting on a particle equals zero. To avoid this duplication, the 1^{st} law may be formulated as the following postulate:

There exists at least one reference frame, called inertial, in which any free particle (i.e. a particle fully isolated from the rest of the Universe) moves with $\mathbf{v}=$ const, i.e. with $\mathbf{a}=0$.

Note that according to Eq. (7), this postulate immediately means that there is also an infinite number of inertial frames because all frames 0 ' moving without rotation or acceleration relative to the postulated inertial frame 0 (i.e. having $\mathbf{a}_0 \mid$ in 0 , $=0$) are also inertial.

On the other hand, the 2^{nd} and 3^{rd} Newton laws may be postulated together in the following elegant way. Each particle, say number k, may be characterized by a scalar constant (called mass m_k), such that at any interaction of N particles (isolated from the rest of the Universe), in any inertial system,

$$\mathbf{P} \equiv \sum_{k=1}^N \mathbf{p}_k \equiv \sum_{k=1}^N m_k \mathbf{v}_k= \text{ const. } \tag{1.3.1}$$

(Each component of this sum,

$$\mathbf{p}_k \equiv m_k \mathbf{v}_k, \tag{1.3.2}$$

is called the mechanical momentum [8] of the corresponding particle, while the sum $\mathbf{P}$, the total momentum of the system.)

Let us apply this postulate to just two interacting particles. Differentiating Eq. (8), written for this case, over time, we get

$$\dot{\mathbf{p}}_1=-\dot{\mathbf{p}}_2. \tag{1.3.3}$$

Let us give the derivative $\dot{\mathbf{p}}_1$ (which is a vector) the name of force $\mathbf{F}$ exerted on particle 1 . In our current case, when the only possible source of the force is particle 2 , it may be denoted as $\mathbf{F}_{12}: \dot{\mathbf{p}}_1 \equiv \mathbf{F}_{12}$. Similarly, $\mathbf{F}_{21} \equiv \dot{\mathbf{p}}_2$, so that Eq. (10) becomes the 3^{rd} Newton law

$$\mathbf{F}_{12}=-\mathbf{F}_{21}. \tag{1.3.4}$$

Plugging Eq. (1.9) into these force definitions, and differentiating the products $m_k \mathbf{v}_k$, taking into account that particle masses are constants, [9] we get that for k and k ' taking any of values 1,2 ,

$$m_k \dot{\mathbf{v}}_k \equiv m_k \mathbf{a}_k=\mathbf{F}_{k k^{\prime}}, \quad \text{ where } k^{\prime} \neq k.. \tag{1.3.5}$$

Now, returning to the general case of several interacting particles, and making an additional (but very natural) assumption that all partial forces $\mathbf{F}_{k k^{\prime}}$ acting on particle k add up as vectors, we may generalize Eq. (12) into the 2^{nd} Newton law

$$m_k \mathbf{a}_k \equiv \dot{\mathbf{p}}_k=\sum_{k^{\prime} \neq k} \mathbf{F}_{k k^{\prime}} \equiv \mathbf{F}_k, \tag{1.3.6}$$

that allows a clear interpretation of the mass as a measure of particle's inertia.

As a matter of principle, if the dependence of all pair forces $\mathbf{F}_{k k^{\prime}}$ of particle positions (and generally of time as well) is known, Eq. (13), augmented with the kinematic relations (2) and (3), allows calculation of the laws of motion $\mathbf{r}_k(t)$ of all particles of the system. For example, for one particle the 2^{nd} law (13) gives an ordinary differential equation of the second order,

$$m \ddot{\mathbf{r}}=\mathbf{F}(\mathbf{r}, t), \tag{1.3.7}$$

which may be integrated - either analytically or numerically.In certain cases, this is very simple. As an elementary example, Newton's gravity force [10]

$$\mathbf{F}=-G\frac{mm'}{R^3}\mathbf{R} \tag{1.3.8}$$

(where $\mathbf{R}\equiv\mathbf{r}-\mathbf{r}'$ is the distance between particles of masses m and m')[11], is virtually uniform and may be approximated as be approximated as

$$\mathbf{F}=m\mathbf{g} \tag{1.3.9}$$

with the vector $\mathbf{g}\equiv\left(Gm'/r'^3\right)\mathbf{r}'$ being constant, for local motions with $r<<r'$.[12] As a result, m in Eq. (13) cancels, it is reduced to just $\ddot{\mathbf{r}}=\mathbf{g}=$ const, and may be easily integrated twice:

$$\dot{\mathbf{r}}(t)\equiv\mathbf{v}(t)=\int_0^t\mathbf{g}dt'+\mathbf{v}(0)=\mathbf{g}t+\mathbf{v}(0),\quad \mathbf{r}(t)=\int_0^t\mathbf{v}\left(t'\right)dt'+\mathbf{r}(0)=\mathbf{g}\frac{t^2}{2}+\mathbf{v}(0)t+\mathbf{r}(0), \tag{1.3.10}$$

thus giving the generic solution of all those undergraduate problems on the projectile motion, which should be so familiar to the reader.

All this looks (and indeed is) very simple, but in most other cases, Eq. (13) leads to more complex calculations. As an example, let us think about would we use it to solve another simple problem: a bead of mass m sliding, without friction, along a round ring of radius R in a gravity field obeying Eq. (16) - see Figure 3. (This system is equivalent to the usual point pendulum, i.e. a point mass suspended from point 0 on a light rod or string, and constrained to move in one vertical plane.)

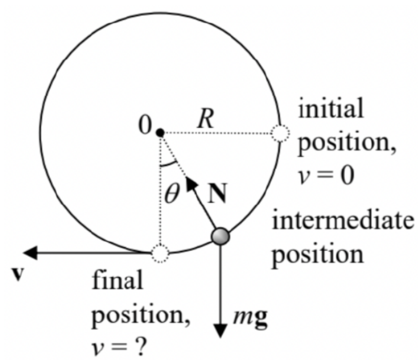

Figure 1.3. A bead sliding along a vertical ring.

Suppose we are only interested in the bead's velocity v at the lowest point, after it has been dropped from the rest at the rightmost position. If we want to solve this problem using only the Newton laws, we have to make the following steps:

(i) consider the bead in an arbitrary intermediate position on a ring, described, for example by the angle θ shown in Figure 3;

(ii) draw all the forces acting on the particle - in our current case, the gravity force mg and the reaction force $\mathbf{N}$ exerted by the ring - see Figure 3 above

(iii) write the Cartesian components of the 2nd Newton law (14) for the bead acceleration: $ma_x=N_x, ma_y=N_y-mg$,

(iv) recognize that in the absence of friction, the force $\mathbf{N}$ should be normal to the ring, so that we can use two additional equations, $N_x=-N\sin\theta$ and $N_y=N\cos\theta$;

(v) eliminate unknown variables N, N_x, and N_y from the resulting system of four equations, thus getting a single second-order differential equation for one variable, for example θ,

$$mR\ddot{\theta}=-mg\sin\theta; \tag{1.3.11}$$

(vi) use the mathematical identity $\ddot{\theta}=(d\dot{\theta}/d\theta)\dot{\theta}$ to integrate this equation over θ once to get an expression relating the velocity $\dot{\theta}$ and the angle θ; and, finally,

(vii) using our specific initial condition ($\dot{\theta}=0$ at $\theta=\pi/2$), find the final velocity as $v=R\dot{\theta}$ at $\theta=0$.

All this is very much doable, but please agree that the procedure it too cumbersome for such a simple problem. Moreover, in many other cases even writing equations of motion along relevant coordinates is very complex, and any help the general theory may provide is highly valuable. In many cases, such help is given by conservation laws; let us review the most general of them.

[7] Due to the genius of Sir Isaac, these laws were formulated in the same Principia (1687), well ahead of the physics of his time.

[8] The more extended term linear momentum is typically used only in cases when there is a chance of its confusion with the angular momentum of the same particle/system - see below. The present-day definition of the linear momentum and the term itself belong to John Wallis (1670), but the concept may be traced back to more vague notions of several previous scientists - all the way back to at least a 570 AD work by John Philoponus.

[9] Note that this may not be true for composite bodies of varying total mass M (e.g., rockets emitting jets, see Problem 11), in these cases the momentum's derivative may differ from $M\mathbf{a}$.

[10] Introduced in the same famous Principia!

[11] The fact that the masses participating in Eqs. (14) and (16) are equal, the so-called weak equivalence principle, is actually highly nontrivial, but has been repeatedly verified experimentally with gradually improved relative accuracy, currently reaching $\sim 10^{-14}$ - see P. Touboul et al., Phys. Rev. Lett. 119, 231101 (2017).

[12] Of course, the most important particular case of Eq. (16) is the motion of objects near the Earth's surface. In this case, using the fact that Eq. (15) remains valid for the gravity field created by a heavy sphere, we get $g = GM_{\mathrm{E}}/R_{\mathrm{E}}{}^2$, where M_{E} and R_{E} are the Earth's mass and radius. Plugging in their values, $M_{\mathrm{E}} \approx 5.97 \times 10^{24}\ \mathrm{kg}$ and $R_{\mathrm{E}} \approx 6.37 \times 10^{6}\ \mathrm{m}$, we get $g \approx 9.82\ \mathrm{m/s}^2$. The experimental value of g varies from 9.78 to $9.83\ \mathrm{m/s}^2$ at various locations on Earth's surface (due to the deviations of Earth's shape from a sphere, and the location-dependent effect of the centrifugal "inertial force" - see Sec. 4.5 below), with an average value of $g \approx 9.807\ \mathrm{m/s}^2$.