1.6: OK, Can We Go Home Now?

Sorry, not yet. In many cases, the conservation laws discussed above provide little help, even in systems without dissipation. Consider for example a generalization of the bead-on-the-ring problem shown in Figure 3, in which the ring is rotated by external forces, with a constant angular velocity ω, about its vertical diameter. [25] In this problem (to which I will repeatedly return below, using it as an analytical mechanics "testbed"), none of the three conservation laws listed in the last section, holds. In particular, bead's energy,

$$E=\frac{m}{2} v^2+m g h, \tag{1.6.1}$$

is not constant, because the external forces rotating the ring may change it. Of course, we still can solve the problem using the Newton laws, but this is even more complex than for the above case of the ring at rest, in particular because the force $\mathbf{N}$ exerted on the bead by the ring now may have three rather than two Cartesian components, which are not simply related. On the other hand, it is clear that the bead still has just one degree of freedom (angle θ), so that its dynamics should not be too complicated.

This fact gives the clue how situations like this one could be simplified: if we only could exclude the so-called reaction forces such as $\mathbf{N}$, that take into account the external constraints imposed on the particle motion, in advance, that should help a lot. Such a constraint exclusion may be provided by analytical mechanics, in particular its Lagrangian formulation, to which we will now proceed.

Of course, the value of the Lagrangian approach goes far beyond simple systems such as the bead on a rotating ring. Indeed, this system has just two externally imposed constrains: the fixed distance of the bead from the center of the ring, and the instant angle of rotation of the ring about its vertical diameter. Now let us consider the motion of a rigid body. It is essentially a system of a very large number, $N \gg 1$, of particles (10^{23} of them if we think about atoms in a 1-cm-scale body). If the only way to analyze its motion would be to write the Newton laws for each of the particles, the situation would be completely hopeless. Fortunately, the number of constraints imposed on its motion is almost similarly huge. (At negligible deformations of the body, the distances between each pair of its particles should be constant.) As a result, the number of actual degrees of freedom of such a body is small (at negligible deformations, just six - see Sec. 6.1), so that with the kind help from analytical mechanics, the motion of the body may be, in many important cases, analyzed even without numerical calculations.

One more important motivation for analytical mechanics is given by dynamics of "nonmechanical" systems, for example, of the electromagnetic field - possibly interacting with charged particles, conducting bodies, etc. In many such systems, the easiest (and sometimes the only practicable) way to find the equations of motion is to derive them from either the Lagrangian or Hamiltonian function of the system. Moreover, the Hamiltonian formulation of the analytical mechanics (to be discussed in Chapter 10) offers a direct pathway to deriving quantum-mechanical Hamiltonian operators of various systems, which are necessary for the analysis of their quantum properties.

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[25] This is essentially a simplified model of the mechanical control device called the centrifugal (or "flyball", or "centrifugal flyball") governor - see, e.g., http://en.wikipedia.org/wiki/Centrifugal governor. (Sometimes the device is called the "Watt's governor", after the famous James Watts who used it in 1788 in one of his first steam engines, but it had been used in European windmills at least since the early 1600s.) Just as a curiosity: the nowubiquitous term cybernetics was coined by Norbert Wiener in 1948 from the word "governor" (or rather from its ancient-Greek original κυß\varepsilon\rhoνή\taų) exactly in this meaning because it had been the first well-studied control device.
