

## 6.5: Standing Waves

Now let us consider the two limits in which Eqs. (55) predicts a total wave reflection ( $\tau = 0$ ) :  $Z'/Z \rightarrow \infty$  (when  $R = -1$ ) and  $Z'/Z \rightarrow 0$  (when  $R = +1$ ). According to Eq. (53), the former limit corresponds to  $f(t) + f_+(t) \equiv q(0, t) = 0$ , i.e. to vanishing oscillations at the interface. This means that this particular limit describes a perfectly rigid boundary, not allowing the system's end to oscillate at all. In this case, Eqs. (58)-(59) yield

$$q(z, t) = \text{Re}[a(e^{-ikz} - e^{+ikz})e^{-i\omega t}] \equiv -2\text{Re}[ae^{-i\omega t}] \sin kz,$$

$$F(z, t) = \text{Re}[i\omega Z a(e^{-ikz} + e^{+ikz})e^{-i\omega t}] \equiv 2\omega Z \text{Re}[ae^{-i(\omega t - \pi/2)}] \cos kz.$$

These equalities mean that we may interpret the process on the right of the interface using two mathematically equivalent, but physically different languages: either as the sum of two traveling waves (the incident one and the reflected one, traveling in opposite directions), or as a single standing wave. Note that in contrast with the traveling wave (Figure 9a, cf. Figure 7), in the standing sinusoidal wave (Figure 9b) all particles oscillate in time with one phase.

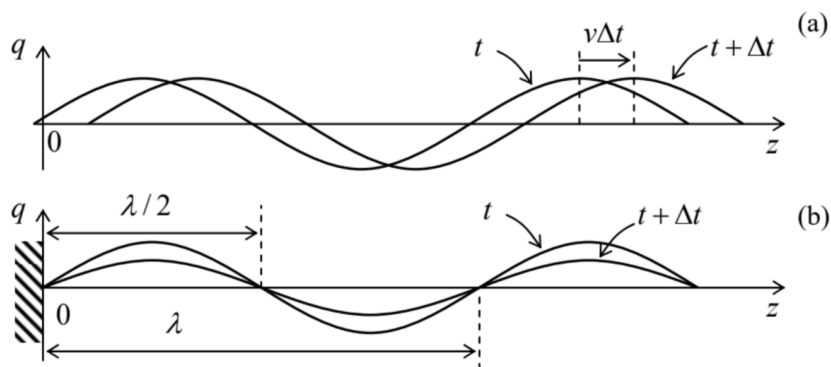


Figure 6.9. The time evolution of (a) a traveling sinusoidal wave, and (b) a standing sinusoidal wave at a rigid boundary.

Note also that the phase of the force oscillations (61) is shifted, both in space and in time, by  $\pi/2$  relatively to the particle displacement oscillations. (In particular, at the rigid boundary the force amplitude reaches its maximum.) As a result, the average power flow vanishes, so that the average energy of the standing wave does not change, though its instant energy still oscillates, at each spatial point, between its kinetic and potential components - just as at the usual harmonic oscillations of one particle. A similar standing wave, but with a maximum of the displacement  $q$ , and with a zero ("node") of the force  $F$ , is formed at the open boundary, with  $Z'/Z \rightarrow 0$ , and hence  $R = +1$ . Now I have to explain why I have used the sinusoidal waveform for the wave reflection analysis. Let us consider a 1D wave system, which obeys Eq. (40), of a finite length  $l$ , limited by two rigid walls (located, say, at  $z = 0$  and  $z = l$ ), which impose the corresponding boundary conditions,

$$q(0, t) = q(l, t) = 0, \quad (6.5.1)$$

on its motion. Naturally, a sinusoidal traveling wave, induced in the system, will be reflected from both ends, forming the standing wave patterns of the type (60) near each of them. These two patterns are compatible if  $l$  is exactly equal to an integer number (say,  $n$ ) of  $\lambda/2$ , where  $\lambda \equiv 2\pi/k$  is the wavelength:

$$l = n \frac{\lambda}{2} \equiv n \frac{\pi}{k}. \quad (6.5.2)$$

This requirement yields the following spectrum of possible wave numbers:

$$k_n = n \frac{\pi}{l}, \quad (6.5.3)$$

where the list of possible integers  $n$  may be limited to non-negative values:  $n = 1, 2, 3, \dots$  (Indeed, negative values give absolutely similar waves (60), while  $n = 0$  yields  $k_n = 0$ , and the corresponding wave vanishes at all points:  $\sin(0 \cdot z) \equiv 0$ .) In the acoustic wave limit we are discussing, Eq. (31),  $\omega = \pm vk$ , may be used to translate this wave-number spectrum into an equally simple spectrum of possible standing-wave frequencies: <sup>17</sup>

$$\omega_n = vk_n = n \frac{\pi v}{l}, \quad \text{with } n = 1, 2, 3, \dots \quad (6.5.4)$$

Now let us notice that this spectrum, and the corresponding standing-wave patterns,<sup>18</sup>

$$q^{(n)}(z, t) = 2 \operatorname{Re}[a_n \exp\{-i\omega_n t\}] \sin k_n z, \quad \text{for } 0 \leq z \leq l, \quad (6.5.5)$$

may be obtained in a different way, by a direct solution of the wave equation (41) with the boundary conditions (62). Indeed, let us look for the general solution of this partial differential equation in the so-called variable-separated form<sup>19</sup>

$$q(z, t) = \sum_n Z_n(z) T_n(t), \quad (6.5.6)$$

where each partial product  $Z_n(z) T_n(t)$  is supposed to satisfy the equation on its own. Plugging such partial solution into Eq. (40), and then dividing all its terms by the same product,  $Z_n T_n$ , we may rewrite the result as

$$\frac{1}{v^2} \frac{1}{T_n} \frac{d^2 T_n}{dt^2} = \frac{1}{Z_n} \frac{d^2 Z_n}{dz^2}. \quad (6.5.7)$$

Here comes the punch line of the variable separation method: since the left-hand side of the equation may depend only on  $t$ , while its right-hand side, only on  $z$ , Eq. (68) may be valid only if both its sides are constant. Denoting this constant as  $-k_n^2$ , we get two similar ordinary differential equations,

$$\frac{d^2 Z_n}{dz^2} + k_n^2 Z_n = 0, \quad \frac{d^2 T_n}{dt^2} + \omega_n^2 T_n = 0, \quad \text{where } \omega_n^2 \equiv v^2 k_n^2, \quad (6.5.8)$$

with well-known (and similar) sinusoidal solutions

$$Z_n = c_n \cos k_n z + s_n \sin k_n z, \quad T_n = u_n \cos \omega_n t + v_n \sin \omega_n t \equiv \operatorname{Re}[a_n \exp\{-i\omega_n t\}], \quad (6.5.9)$$

where  $c_n, v_n, u_n$ , and  $v_n$  (or, alternatively,  $a_n \equiv u_n + i v_n$ ) are constants. The first of these relations, with all  $k_n$  different, may satisfy the boundary conditions only if for all  $n$ ,  $c_n = 0$ , and  $\sin k_n l = 0$ , giving the same wave number spectrum (64) and hence the own frequency spectrum (65), so that the general solution (67) of the so-called boundary problem, given by Eqs. (40) and (62), takes the form

$$q(z, t) = \operatorname{Re} \sum_n a_n \exp\{-i\omega_n t\} \sin k_n z, \quad (6.5.10)$$

where the complex amplitudes  $a_n$  are determined by the initial conditions.

Hence such sinusoidal standing waves (Figure 10a) are not just an assumption, but a natural property of the 1D wave equation. It is also easy to verify that the result (71) is valid for the same system with different boundary conditions, though with a modified wave number spectrum. For example, if the rigid boundary condition ( $q = 0$ ) is implemented at  $z = 0$ , and the so-called free (or "open") boundary conditions ( $F = 0$ , i.e.  $\partial q / \partial z = 0$ ) is imposed at  $z = l$ , the spectrum becomes

$$k_n = \left(n - \frac{1}{2}\right) \frac{\pi}{l}, \quad \text{with } n = 1, 2, 3, \dots, \quad (6.5.11)$$

so that the lowest standing waves look like Figure 10b shows.<sup>20</sup>

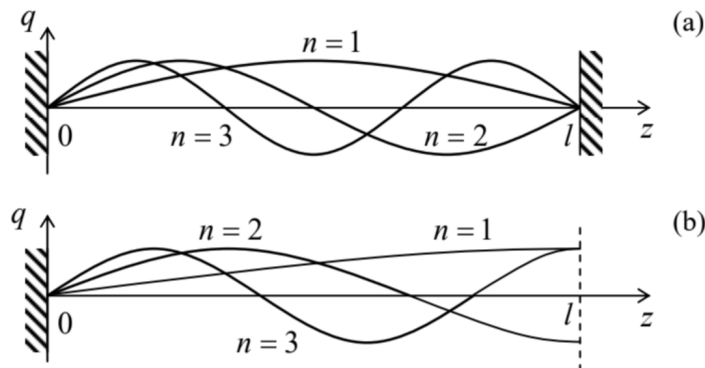


Figure 6.10. The lowest standing wave modes for the 1D systems with (a) two rigid boundaries, and (b) one rigid and one open boundary.

Note that the difference between the sequential values of  $k_n$  is still constant:

$$k_{n+1} - k_n = \frac{\pi}{l}, \quad (6.5.12)$$

i.e. the same as for the spectrum (64). This is natural because in both cases the transfer from the  $n^{\text{th}}$  mode to the  $(n+1)^{\text{th}}$  mode corresponds just to an addition of one more half-wave - see Figure 10. (This conclusion is valid for any combination of rigid and free boundary conditions.) As was discussed above, for the discrete-particle chain we have started with (Figure 4), the wave equation (40), and hence the above derivation of Eq. (71), are only valid in the acoustic wave limit, i.e. when the distance  $d$  between the particles is much less than the wavelengths  $\lambda_n \equiv 2\pi/k_n$  of the mode under analysis. For a chain of length  $l$ , this means that the number of particles,  $N \sim l/d$ , has to be much larger than 1. However, a remarkable property of Eq. (71) is that it remains valid, with the same wave number spectrum (64), not only in the acoustic limit but also for arbitrary  $N > 0$ . Indeed, since  $\sin k_n z \equiv (\exp\{+ik_n z\} - \exp\{-ik_n z\})/2$ , each  $n^{\text{th}}$  term of Eq. (71) may be represented as a sum of two traveling waves with equal but opposite wave vectors. As was discussed in Sec. 3, such a wave is a solution of equation (24) describing the discreteparticle system for any  $k_n$ , with the only condition that its frequency obeys the general dispersion relation (30), rather than its acoustic limit (65).

Moreover, the expressions for  $k_n$  (with appropriate boundary conditions), such as Eq. (64) or Eq. (72), also survive the transition to arbitrary  $N$ , because their derivation above was based only on the sinusoidal form of the standing wave. The only new factor arising in the case of arbitrary  $N$  is that due to the equidistant property (73) of the wave number spectrum, as soon as  $n$  exceeds  $N$ , the waveforms (71), at particle locations  $z_j = jd$ , start to repeat. For example,

$$\sin k_{n+N} z_j = \sin(k_n + N\Delta k)jd = \sin\left(k_n + N\frac{\pi}{d}\right)jd = \sin(k_n z_j + \pi jN) = \pm \sin k_n z_j. \quad (6.5.13)$$

Hence the system has only  $N$  different (linearly-independent) modes. But this result is in full compliance with the general conclusion made in Sec. 2, that any system of  $N$  coupled 1D oscillators has exactly  $N$  own frequencies and corresponding oscillation modes. So, our analysis of a particular system, shown in Figure 4, just exemplifies this general conclusion. Figure 11 below illustrates this result for a particular finite value of  $N$ ; the curve connecting the points shows exactly the same dispersion relation as was shown in Figure 5, but now it is just a guide for the eye, because for a system with a finite length  $l$ , the wave number spectrum is discrete, and the intermediate values of  $k$  and  $\omega$  do not have an immediate physical sense.<sup>21</sup> Note that the own frequencies of the system are not equidistant, while the wave numbers are.

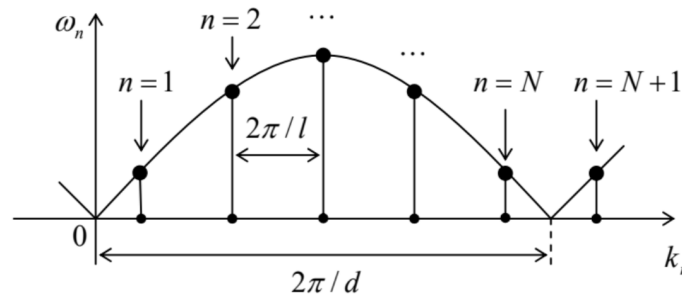


Figure 6.11. The wave numbers and own frequencies of a chain of a finite number  $N$  of particles in a chain with one rigid and one open boundary - schematically.

This insensitivity of the spacing (73) between the adjacent wave numbers to the particular physics of a macroscopically uniform system is a very general fact, common for waves of any nature, and is broadly used for analyses of systems with a very large number of particles (such as human-size crystals, with  $N \sim 10^{23}$ ). For  $N$  so large, the effect of the boundary conditions, e.g., the difference between the spectra (64) and (72) is negligible, and they may be summarized as the following rule for the number of different standing waves within some interval  $\Delta k \gg \pi/l$ :

$$\Delta N \equiv \frac{\Delta k|_{\text{standing}}}{k_{n+1} - k_n} = \frac{l}{\pi} \Delta k|_{\text{standing}}. \quad (6.5.14)$$

For such analyses, it is frequently more convenient to work with traveling waves rather than the standing ones. In this case, we have to take into account that (as was just discussed above) each standing wave (66) may be decomposed to two traveling waves with wave numbers  $\pm k_n$ , so that size of the interval  $\Delta k$  doubles, and Eq. (75a) becomes<sup>22</sup>

$$\Delta N = \frac{l}{2\pi} \Delta k \Big|_{\text{traveling}} \quad (6.5.15)$$

Note that this counting rule is valid for waves of just one type. As was discussed above, for the model system we have studied (Figure 4), there are 3 types of such waves - one longitudinal and two transverse, so that if we need to count them all,  $\Delta N$  should be multiplied by 3 .

---

<sup>17</sup> Again, negative values of  $\omega$  may be dropped, because they give similar real functions  $q(z, t)$ .

<sup>18</sup> They describe, in particular, the well-known transverse standing waves on a guitar string.

<sup>19</sup> This variable separation method is very general and is discussed in all parts of this series, especially in EM Chapter 2.

<sup>20</sup> The lowest standing wave of the system, with the smallest  $k_n$  and  $\omega_n$ , is usually called its fundamental mode.

<sup>21</sup> Note that Figure 11 shows the case of one rigid and one open boundary (see Figure 10b), where  $l = Nd$ ; for a conceptually simpler system with two rigid boundaries (Figure 10a) we would need to take  $l = (N + 1)d$  , because neither of the end points can oscillate.

<sup>22</sup> Note that this simple, but very important relation is frequently derived using the so-called Born-Carman boundary condition  $q_0(t) \equiv q_N(t)$ , which implies bending the system of interest into a closed loop. For a 1D system with  $N \gg 1$ , such mental exercise may be somehow justified, but for systems of higher dimension, it is hardly physically plausible – and is unnecessary.

---

This page titled [6.5: Standing Waves](#) is shared under a [CC BY-NC-SA 4.0](#) license and was authored, remixed, and/or curated by [Konstantin K. Likharev](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.