8.3: Kinematics

In contrast to the stress tensor, which is useful and simple - see Eq. (2), the strain tensor is not a very useful notion in fluid mechanics. Indeed, besides a very few situations, [12] typical problems of this field involve fluid flow, i.e. a state when the velocity of fluid particles has some nonzero time average. This means that the trajectory of each particle is a long line, and the notion of its displacement $\mathbf{q}$ becomes impracticable. However, particle's velocity $\mathbf{v} \equiv d\mathbf{q}/dt$ remains a very useful notion, especially if it is considered as a function of the observation point $\mathbf{r}$ and (generally) time t. In an important class of fluid dynamics problem, the so-called stationary (or "steady", or "static") flow, the velocity defined in this way does not depend on time, $\mathbf{v} = \mathbf{v}(\mathbf{r})$.

There is, however, a price to pay for the convenience of this notion: namely, due to the difference between the vectors $\mathbf{q}$ and $\mathbf{r}$, particle's acceleration $\mathbf{a} = d^2\mathbf{q}/dt^2$ (that participates, in particular, in the 2^{nd} Newton law) cannot be calculated just as the time derivative of the velocity $\mathbf{v}(\mathbf{r}, t)$. This fact is evident, for example, for the static flow case, in which the acceleration of individual fluid particles may be very significant even if $\mathbf{v}(\mathbf{r})$ does not depend on time - just think about the acceleration of a drop of water flowing over the Niagara Falls' rim, first accelerating fast and then virtually stopping below, while the water velocity $\mathbf{v}$ at every particular point, as measured from a bank-based reference frame, is nearly constant. Thus the main task of fluid kinematics is to express a via $\mathbf{v}(\mathbf{r}, t)$; let us do this.

Since each Cartesian component v_j of the velocity $\mathbf{v}$ has to be considered as a function of four independent scalar variables: three Cartesian components r_j of the vector $\mathbf{r}$ and time t, its full time derivative may be represented as

$$\frac{dv_j}{dt} = \frac{\partial v_j}{\partial t} + \sum_{j'=1}^{3} \frac{\partial v_j}{\partial r_{j'}} \frac{dr_{j'}}{dt}. \tag{8.3.1}$$

Let us apply this general relation to a specific set of infinitesimal changes $\{dr_1, dr_2, dr_3\}$ that follows a small displacement $d\mathbf{q}$ of a certain particular particle of the fluid, $d\mathbf{r} = d\mathbf{q} = \mathbf{v}dt$, i.e.

$$dr_j = v_j dt. \tag{8.3.2}$$

In this case, dv_j/dt is the j^{th} component a_j of the particle's acceleration a, so that Eq. (17) yields the following key relation of fluid kinematics:

$$a_j = \frac{\partial v_j}{\partial t} + \sum_{j'=1}^{3} v_{j'} \frac{\partial v_j}{\partial r_{j'}}. \tag{8.3.3}$$

Using the del operator ∇, this result may be rewritten in the following compact vector form: [13]

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}. \tag{8.3.4}$$

This relation already signals the main technical problem of the fluid dynamics: many equations involving particle's acceleration are nonlinear in velocity, excluding such a powerful tool as the linear superposition principle (which was used so frequently in the previous chapters of this course) from the applicable mathematical arsenal.

One more basic relation of the fluid kinematics is the so-called continuity equation, which is essentially just the differential version of the mass conservation law. Let us mark, inside a fluid flow, an arbitrary volume V limited by a stationary (time-independent) surface S. The total mass of the fluid inside the volume may change only due to its flow through the boundary:

$$\frac{dM}{dt} \equiv \frac{d}{dt} \int_V \rho d^3 r = -\int_S \rho v_n d^2 r \equiv -\int_S \rho \mathbf{v} \cdot d\mathbf{A}, \tag{8.3.5}$$

where the elementary area vector $d\mathbf{A}$ is defined just as in Sec. $7.2-$ see Figure 7. Now using the same divergence theorem that has been used several times in this course [14] the surface integral in Eq. (20a) may be transformed into the integral of $\nabla(\rho\mathbf{v})$ over the volume V, so that this relation may be rewritten as

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} \right) d^3 r = 0 \tag{8.3.6}$$

where the vector $\mathbf{j} \equiv \rho\mathbf{v}$ is called either the mass flux density (or the "mass current"). Since Eq. (20b) is valid for an arbitrary stationary volume V, the function under the integral has to vanish at any point:

$$\text{Continuity equation} \qquad \frac{\partial \rho}{\partial t}+\nabla \cdot \mathbf{j}=0. \tag{8.3.7}$$

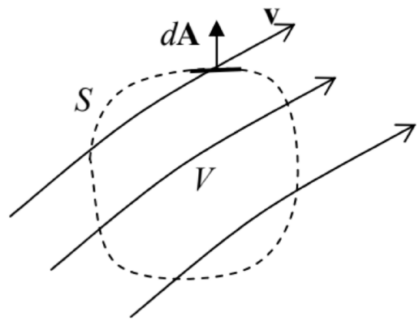

Fig. 8.7. Deriving the continuity equation.

Note that this continuity equation is valid not only for mass, but also for other conserved physics quantities (e.g., the electric charge, probability, etc.), with the proper re-definition of ρ and $\mathbf{j}$. [15]

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[12] One of them is sound propagation, where the particle displacements $\mathbf{q}$ are typically small, so that results of Sec. 7.7 are applicable. As a reminder, they show that in fluids, with $\mu=0$, the transverse sound cannot propagate, while the longitudinal sound can - see Eq. (7.114).

[13] Note that the operator relation $d / d t=\partial / \partial t+(\mathbf{v} \cdot \nabla)$ is applicable to an arbitrary (scalar or vector) function; it is frequently called the convective derivative. (Alternative adjectives, such as "Lagrangian", "substantial", or "Stokes", are sometimes used for this derivative as well.) The relation has numerous applications well beyond the fluid dynamics - see, e.g., EM Chapter 9 and QM Chapter 1 .

[14] If the reader still needs a reminder, see MA Eq. (12.1).

[15] See, e.g., EM Sec. 4.1, QM Sec. 1.4, and SM Sec. 5.6.
