1.4: Conservation Laws

(i) Energy conservation is arguably the most general law of physics, but in mechanics, it takes a more humble form of mechanical energy conservation, which has limited applicability. To derive it, we first have to define the kinetic energy of a particle as [13]

$$T \equiv \frac{m}{2} v^{2} \tag{1.4.1}$$

and then recast its differential as [14]

$$d T \equiv d\left(\frac{m}{2} v^{2}\right) \equiv d\left(\frac{m}{2} \mathbf{v} \cdot \mathbf{v}\right)=m \mathbf{v} \cdot d \mathbf{v}=m \frac{d \mathbf{r} \cdot d \mathbf{v}}{d t}=d \mathbf{r} \cdot \frac{d \mathbf{p}}{d t}. \tag{1.4.2}$$

Now plugging in the momentum's derivative from the 2$^{\text{nd}}$ Newton law, $d\mathbf{p}/dt = \mathbf{F}$, where $\mathbf{F}$ is the full force acting on the particle, we get $dT = \mathbf{F}\cdot d\mathbf{r}$. The integration of this equality along the particle's trajectory connecting some points A and B gives the formula that is sometimes called the work-energy principle:

$$\Delta T \equiv T\left(\mathbf{r}_{\mathrm{B}}\right)-T\left(\mathbf{r}_{\mathrm{A}}\right)=\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F} \cdot d \mathbf{r}, \tag{1.4.3}$$

where the integral on the right-hand side is called the work of the force $\mathbf{F}$ on the path from A to B.

The following step may be made only for a potential (also called "conservative") force that may be represented as the (minus) gradient of some scalar function $U(\mathbf{r})$, called the potential energy. [15] The vector operator ∇ (called either del or nabla) of spatial differentiation [16] allows a very compact expression of this fact:

$$\mathbf{F}=-\nabla U. \tag{1.4.4}$$

For example, for the uniform gravity field (16),

$$U=m g h+\text{ const }, \tag{1.4.5}$$

where h is the vertical coordinate directed "up" - opposite to the direction of the vector $\mathbf{g}$.Integrating the tangential component F_τ of the vector $\mathbf{F}$ given by Eq. (22), along an arbitrary path connecting the points A and $\boldsymbol{B}$, we get

$$\int_{\mathrm{A}}^{\mathrm{B}} F_{\tau} d r \equiv \int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F} \cdot d \mathbf{r}=U\left(\mathbf{r}_{\mathrm{A}}\right)-U\left(\mathbf{r}_{\mathrm{B}}\right), \tag{1.4.6}$$

i.e. work of potential forces may be represented as the difference of values of the function $U(\mathbf{r})$ in the initial and final points of the path. (Note that according to Eq. (24), the work of a potential force on any closed path, with $\mathbf{r}_{\mathrm{A}} = \mathbf{r}_{\mathrm{B}}$, is zero.)

Now returning to Eq. (21) and comparing it with Eq. (24), we see that

$$T\left(\mathbf{r}_{\mathrm{B}}\right)-T\left(\mathbf{r}_{\mathrm{A}}\right)=U\left(\mathbf{r}_{\mathrm{A}}\right)-U\left(\mathbf{r}_{\mathrm{B}}\right), \text{ i.e. } T\left(\mathbf{r}_{\mathrm{A}}\right)+U\left(\mathbf{r}_{\mathrm{A}}\right)=T\left(\mathbf{r}_{\mathrm{A}}\right)+U\left(\mathbf{r}_{\mathrm{A}}\right), \tag{1.4.7}$$

so that the total mechanical energy E, defined as

is indeed conserved:

$$\begin{gathered} E \equiv T+U, \\ E\left(\mathbf{r}_{\mathrm{A}}\right) \equiv E\left(\mathbf{r}_{\mathrm{B}}\right), \end{gathered}$$

but for conservative forces only. (Non-conservative forces may change E by either transferring energy from its mechanical form to another form, e.g., to heat in the case of friction, or by pumping the energy into the system under consideration from another, "external" system.)

The mechanical energy conservation allows us to return for a second to the problem shown in Figure 3 and solve it in one shot by writing Eq. (27) for the initial and final points: [17]

$$0+m g R=\frac{m}{2} v^{2}+0. \tag{1.4.8}$$

The (elementary) solution of Eq. (28) for v immediately gives us the desired answer. Let me hope that the reader agrees that this way of problem's solution is much simpler, and I have earned their attention to discuss other conservation laws - which may be equally effective.

(ii) Linear momentum. The conservation of the full linear momentum of any system of particles isolated from the rest of the world was already discussed in the previous section, and may serve as the basic postulate of classical dynamics - see Eq. (8). In the case of one free particle, the law is reduced to the trivial result $\mathbf{p}=$ const, i.e. $\mathbf{v}=$ const. If a system of N particles is affected by external forces $\mathbf{F}^{(\text{ext})}$, we may write

$$\mathbf{F}_k=\mathbf{F}_k^{(\text{ext})}+\sum_{k=1}^N \mathbf{F}_{kk'}. \tag{1.4.9}$$

If we sum up the resulting Eqs. (13) for all particles of the system then, due to the 3rd Newton law (11), valid for any indices $k\neq k'$, the contributions of all internal forces $\mathbf{F}_{kk}$ ' to the resulting double sum on the right-hand side cancel, and we get the following equation:

$$\dot{\mathbf{P}}=\mathbf{F}^{(\text{ext})}, \quad \text{ where } \mathbf{F}^{(\text{ext})}\equiv\sum_{k=1}^N \mathbf{F}_k^{(\text{ext})} \tag{1.4.10}$$

It tells us that the translational motion of the system as the whole is similar to that of a single particle, under the effect of the net external force $\mathbf{F}^{(\text{ext})}$. As a simple sanity check, if the external forces have a zero sum, we return to the postulate (8). Just one reminder: Eq. (30), as its precursor Eq. (13), is only valid in an inertial reference frame.

I hope that the reader knows numerous examples of application of the linear momentum's conservation law, including all these undergraduate problems on car collisions, where the large collision forces are typically not known so that the direct application of Eq. (13) to each car is impracticable.

(iii) The angular momentum of a particle [18] is defined as the following vector: [19]

$$\mathbf{L}\equiv\mathbf{r}\times\mathbf{p} \tag{1.4.11}$$

where $\mathbf{a}\times\mathbf{b}$ means the vector (or "cross-") product of the vector operands. [20] Differentiating Eq. (31) over time, we get

$$\dot{\mathbf{L}}=\dot{\mathbf{r}}\times\mathbf{p}+\mathbf{r}\times\dot{\mathbf{p}} \tag{1.4.12}$$

In the first product, $\dot{\mathbf{r}}$ is just the velocity vector $\mathbf{v}$, parallel to the particle momentum $\mathbf{p}=m\mathbf{v}$, so that this term vanishes since the vector product of any two parallel vectors equals zero. In the second product, $\dot{\mathbf{p}}$ is equal to the full force $\mathbf{F}$ acting on the particle, so that Eq. (32) is reduced to

where the vector

$$\dot{\mathbf{L}}=\boldsymbol{\tau},$$
$$\boldsymbol{\tau}\equiv\mathbf{r}\times\mathbf{F},$$

is called the torque exerted by force $\mathbf{F}$. [21] (Note that the torque is reference-frame specific - and again, the frame has to be inertial for Eq. (33) to be valid, because we have used Eq. (13) for its derivation.) For an important particular case of a central force $\mathbf{F}$ that is directed along the radius vector $\mathbf{r}$ of a particle, the torque vanishes, so that (in that particular reference frame only!) the angular momentum is conserved:

$$\mathbf{L}=\text{ const.} \tag{1.4.13}$$

For a system of N particles, the total angular momentum is naturally defined as

$$\mathbf{L}\equiv\sum_{k=1}^N \mathbf{L}_k \tag{1.4.14}$$

Differentiating this equation over time, using Eq. (33) for each $\dot{\mathbf{L}}_k$, and again partitioning each force per Eq. (29), we get

$$\dot{\mathbf{L}}=\sum_{\substack{k,k'=1\\k'\neq k}}^N \mathbf{r}_k\times\mathbf{F}_{kk'}+\boldsymbol{\tau}^{(\text{ext})}, \quad \text{ where } \boldsymbol{\tau}^{(\text{ext})}\equiv\sum_{k=1}^N \mathbf{r}_k\times\mathbf{F}_k^{(\text{ext})} \tag{1.4.15}$$

The first (double) sum may be always divided into pairs of the type $(\mathbf{r}_k\times\mathbf{F}_{kk'}+\mathbf{r}_{k'}\times\mathbf{F}_{k'k})$. With a natural assumption of the central forces, $\mathbf{F}_{kk'}\parallel(\mathbf{r}_k-\mathbf{r}_{k'})$, each of these pairs equals zero. Indeed, in this case, each component of the pair is a vector perpendicular to the plane containing the positions of both particles and the reference frame origin, i.e. to the plane of the drawing of Figure 4.

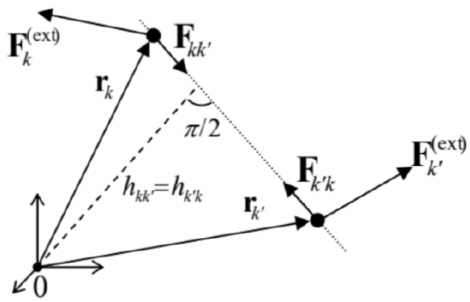

Figure 1.4. Internal and external forces, and the internal torque cancellation in a system of two particles.

Also, due to the 3^{rd} Newton law (11), these two forces are equal and opposite, and the magnitude of each term in the sum may be represented as $|F_{kk}|\,h_{kk}$, with equal to the "lever arms" $h_{kk}=h_{k'k}$. As a result, each sum $(\mathbf{r}_k\times\mathbf{F}_{kk'}+\mathbf{r}_k\times\mathbf{F}_{k'k})$, and hence the whole double sum in Eq. (37) vanish, and it is reduced to a very simple result,

$$\dot{\mathbf{L}}=\boldsymbol{\tau}^{(\mathrm{ext})}\tag{1.4.16}$$

which is similar to Eq. (33) for a single particle, and is the angular analog of Eq. (30).

In particular, Eq. (38) shows that if the full external torque $\tau^{(\mathrm{ext})}$ vanishes by some reason (e.g., if the system of particles is isolated from the rest of the Universe), the conservation law (35) is valid for the full angular momentum $\mathbf{L}$, even if its individual components $\mathbf{L}_k$ are not conserved due to interparticle interactions.

Note again that since the conservation laws may be derived from the Newton laws (as was done above), they do not introduce anything absolutely new to the dynamics of any system. Indeed, from the mathematical point of view, the conservation laws discussed above are just the first integrals of the second-order differential equations of motion following from the Newton laws, which may liberate us from the necessity to integrate the equations twice.

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[13] In such quantitative form, the kinetic energy was introduced (under the name "living force") by Gottfried Leibniz and Johann Bernoulli (circa 1700), though its main properties (21) and (27) had not been clearly revealed until an 1829 work by Gaspard-Gustave de Coriolis. The modern term "kinetic energy" was coined only in $1849-1851$ by Lord Kelvin (born William Thomson).

[14] In these notes, $\mathbf{a}\cdot\mathbf{b}$ denotes the scalar (or "dot-") product of vectors $\mathbf{a}$ and $\mathbf{b}$ - see, e.g., MA Eq. (7.1).

[15] Note that because of its definition via the gradient, the potential energy is only defined to an arbitrary additive constant. This notion had essentially been used already by G. Leibniz, though the term we are using for it now was introduced much later (in the mid- 19^{th} century) by William Rankine.

[16] Its basic properties are listed in MA Sec. 8.

[17] Here the arbitrary constant in Eq. (23) is chosen so that the potential energy is zero in the final point.

[18] Here we imply that the internal motions of the particle, including its rotation about its axis, are negligible. (Otherwise, it could not be represented by a point, as was postulated in Sec. 1.)

[19] Such explicit definition of the angular momentum (in different mathematical forms, and under the name of "moment of rotational motion") has appeared in scientific publications only in the 1740s, though the fact of its conservation (35) in the field of central forces, in the form of the 2^{nd} Kepler law (see Figure 3.4 below), was proved by I. Newton in his Principia.

[20] See, e.g., MA Eq. (7.3).

[21] Alternatively, especially in mechanical engineering, torque is called the force moment. This notion may be traced all the way back to Archimedes' theory of levers developed in the 3^{rd} century BC.
