3.1: One-dimensional and 1D-reducible Systems

If a particle is confined to motion along a straight line (say, axis x), its position is completely determined by this coordinate. In this case, as we already know, particle's Lagrangian is given by Eq. (2.21):

$$L=T(\dot{x})-U(x,t), \quad T(\dot{x})=\frac{m}{2}\dot{x}^2, \tag{3.1.1}$$

so that the Lagrange equation of motion, given by Eq. (2.22),

$$m\ddot{x}=-\frac{\partial U(x,t)}{\partial x} \tag{3.1.2}$$

is just the x-component of the 2^{nd} Newton law.

It is convenient to discuss the dynamics of such really-1D systems as a part of a more general class of effectively-1D systems, whose position, due to holonomic constraints and/or conservation laws, is also fully determined by one generalized coordinate q, and whose Lagrangian may be represented in a form similar to Eq. (1):

$$L=T_{\text{ef}}(\dot{q})-U_{\text{ef}}(q,t), \quad T_{\text{ef}}=\frac{m_{\text{ef}}}{2}\dot{q}^2, \tag{3.1.3}$$

where m_{ef} is some constant which may be considered as the effective mass of the system, and the function U_{ef} its effective potential energy. In this case, the Lagrange equation (2.19), describing the system's dynamics, has a form similar to Eq. (2):

$$m_{\text{ef}}\ddot{q}=-\frac{\partial U_{\text{ef}}(q,t)}{\partial q}. \tag{3.1.4}$$

As an example, let us return again to our testbed system shown in Figure 2.1. We have already seen that for this system, having one degree of freedom, the genuine kinetic energy T, expressed by the first of Eqs. (2.23), is not a quadratically-homogeneous function of the generalized velocity. However, the system's Lagrangian function (2.23) still may be represented in the form (3),

$$L=\frac{m}{2}R^2\dot{\theta}^2+\frac{m}{2}R^2\omega^2\sin^2\theta+mgR\cos\theta+\text{const}=T_{\text{ef}}-U_{\text{ef}}, \tag{3.1.5}$$

if we take

$$T_{\text{ef}}\equiv\frac{m}{2}R^2\dot{\theta}^2, \quad U_{\text{ef}}\equiv-\frac{m}{2}R^2\omega^2\sin^2\theta-mgR\cos\theta+\text{const} \tag{3.1.6}$$

In this new partitioning of the function L, which is legitimate because U_{ef} depends only on the generalized coordinate θ, but not on the corresponding generalized velocity, T_{ef} includes only a part of the full kinetic energy T of the bead, while U_{ef} includes not only its real potential energy U in the gravity field but also an additional term related to ring rotation. (As we will see in Sec. 4.6, this term may be interpreted as the effective potential energy due to the inertial centrifugal "force" arising at the problem's solution in the non-inertial reference frame rotating with the ring.)

Returning to the general case of effectively-1D systems with Lagrangians of the type (3), let us calculate their Hamiltonian function, using its definition (2.32):

$$H=\frac{\partial L}{\partial\dot{q}}\dot{q}-L=m_{\text{ef}}\dot{q}^2-(T_{\text{ef}}-U_{\text{ef}})=T_{\text{ef}}+U_{\text{ef}}. \tag{3.1.7}$$

So, H is expressed via T_{ef} and U_{ef} exactly as the energy E is expressed via genuine T and U.