

## 2.1: Green's Functions

I have spent so much time/space discussing the potential distributions created by a single point charge in various conductor geometries because for any of the geometries, the generalization of these results to the arbitrary distribution  $\rho(\mathbf{r})$  of free charges is straightforward. Namely, if a single charge  $q$ , located at some point  $\mathbf{r}'$ , creates the electrostatic potential

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} q G(\mathbf{r}, \mathbf{r}'), \quad (2.202)$$

then, due to the linear superposition principle, an arbitrary charge distribution (either discrete or continuous) creates the potential  
Spatial Green's function

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_j q_j G(\mathbf{r}, \mathbf{r}_j) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d^3r'. \quad (2.203)$$

The function  $G(\mathbf{r}, \mathbf{r}')$  is called the (spatial) Green's function – the notion very fruitful and hence popular in all fields of physics.<sup>65</sup> Evidently, as Eq. (1.35) shows, in the unlimited free space

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad (2.204)$$

i.e. the Green's function depends only on one scalar argument – the distance between the field-observation point  $\mathbf{r}$  and the field-source (charge) point  $\mathbf{r}'$ . However, as soon as there are conductors around, the situation changes. In this course, I will only discuss the Green's functions that vanish as soon as the radius-vector  $\mathbf{r}$  points to the surface (S) of any conductor:<sup>66</sup>

$$G(\mathbf{r}, \mathbf{r}')|_{\mathbf{r} \in S} = 0. \quad (2.205)$$

With this definition, it is straightforward to deduce the Green's functions for the solutions of the last section's problems in which conductors were grounded ( $\phi = 0$ ). For example, for a semi-space  $z \geq 0$  limited by a conducting plane  $z = 0$  (Fig. 26), Eq. (185) yields

$$G = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}''|}, \quad \text{with } \rho'' = \rho' \text{ and } z'' = -z'. \quad (2.206)$$

We see that in the presence of conductors (and, as we will see later, any other polarizable media), the Green's function may depend not only on the difference  $\mathbf{r} - \mathbf{r}'$ , but on each of these two arguments in a specific way.

So far, this looks just like re-naming our old results. The really non-trivial result of this formalism for electrostatics is that, somewhat counter-intuitively, the knowledge of the Green's function for a system with grounded conductors (Fig. 30a) enables the calculation of the field created by voltage-biased conductors (Fig. 30b), with the same geometry. To show this, let us use the so-called Green's theorem of the vector calculus.<sup>67</sup> The theorem states that for any two scalar, differentiable functions  $f(\mathbf{r})$  and  $g(\mathbf{r})$ , and any volume  $V$ ,

$$\int_V (f \nabla^2 g - g \nabla^2 f) d^3r = \oint_S (f \nabla g - g \nabla f)_n d^2r, \quad (2.207)$$

where  $S$  is the surface limiting the volume. Applying the theorem to the electrostatic potential  $\phi(\mathbf{r})$  and the Green's function  $G$  (also considered as a function of  $\mathbf{r}$ ), let us use the Poisson equation (1.41) to replace  $\nabla^2 \phi$  with  $(-\rho/\epsilon_0)$ , and notice that  $G$ , considered as a function of  $\mathbf{r}$ , obeys the Poisson equation with the  $\delta$ -functional source:

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}'). \quad (2.208)$$

(Indeed, according to its definition (202), this function may be formally considered as the field of a point charge  $q = 4\pi\epsilon_0$ .) Now swapping the notation of the radius-vectors,  $\mathbf{r} \leftrightarrow \mathbf{r}'$ , and using the Green's function symmetry,  $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$ ,<sup>68</sup> we get

$$-4\pi\phi(\mathbf{r}) - \int_V \left( -\frac{\rho(\mathbf{r}')}{\epsilon_0} \right) G(\mathbf{r}, \mathbf{r}') d^3r' = \oint_S \left[ \phi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - G(\mathbf{r}, \mathbf{r}') \frac{\partial \phi(\mathbf{r}')}{\partial n'} \right] d^2r'. \quad (2.209)$$

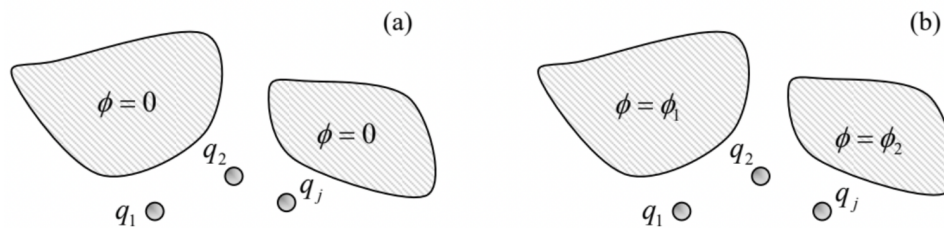


Fig. 2.30. Green's function method allows the solution of a simpler boundary problem (a) to be used to find the solution of a more complex problem (b), for the same conductor geometry.

Let us apply this relation to the volume  $V$  of free space between the conductors, and the boundary  $S$  drawn immediately outside of their surfaces. In this case, by its definition, Green's function  $G(\mathbf{r}, \mathbf{r}')$  vanishes at the conductor surface ( $\mathbf{r} \in S$ ) – see Eq. (205). Now changing the sign of  $\partial n'$  (so that it would be the outer normal for conductors, rather than free space volume  $V$ ), dividing all terms by  $4\pi$ , and partitioning the total surface  $S$  into the parts (numbered by index  $j$ ) corresponding to different conductors (possibly, kept at different potentials  $\phi_k$ ), we finally arrive at the famous result:<sup>69</sup>

Potential via Green's function

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d^3r' + \frac{1}{4\pi} \sum_k \phi_k \oint_{S_k} \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} d^2r'. \quad (2.210)$$

While the first term on the right-hand side of this relation is a direct and evident expression of the superposition principle, given by Eq. (203), the second term is highly non trivial: it describes the effect of conductors with non-zero potentials  $\phi_k$  (Fig. 30b), using the Green's function calculated for the

similar system with grounded conductors, i.e. with all  $\phi_k = 0$  (Fig. 30a). Let me emphasize that since our volume  $V$  excludes conductors, the first term on the right-hand side of Eq. (210) includes only the stand-alone charges in the system (in Fig. 30, marked  $q_1$ ,  $q_2$ , etc.), but not the surface charges of the conductors – which are taken into account, implicitly, by the second term.

In order to illustrate what a powerful tool Eq. (210) is, let us use to calculate the electrostatic field in two systems. In the first of them, a plane, circular, conducting disk, separated with a very thin cut from the remaining conducting plane, is biased with potential  $\phi = V$ , while the rest of the plane is grounded – see Fig. 31.

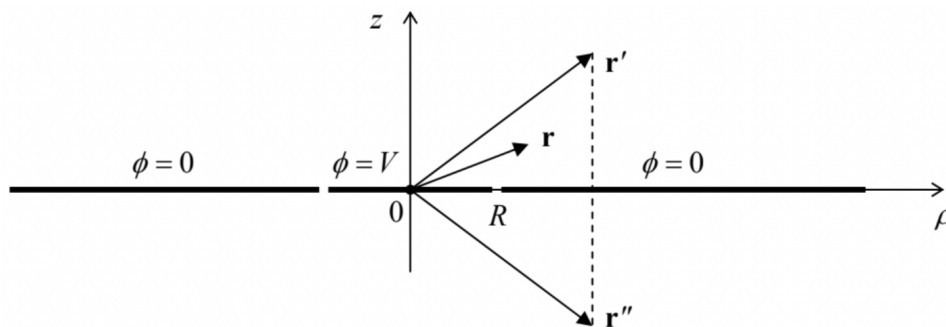


Fig. 2.31. A voltage-biased conducting disk separated from the rest of a conducting plane.

If the width of the gap between the disk and the rest of the plane is negligible, we may apply Eq. (210) without stand-alone charges,  $\rho(\mathbf{r}') = 0$ , and the Green's function for the uncut plane – see Eq. (206).<sup>70</sup> In the cylindrical coordinates, with the origin at the disk's center (Fig. 31), the function is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\left[ \rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + (z - z')^2 \right]^{1/2}} - \frac{1}{\left[ \rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + (z + z')^2 \right]^{1/2}}. \quad (2.211)$$

(The sum of the first three terms under each square root in Eq. (211) is just the squared distance between the horizontal projections  $\rho$  and  $\rho'$  of the vectors  $\mathbf{r}$  and  $\mathbf{r}'$  (or  $\mathbf{r}''$ ) correspondingly, while the last terms are the squares of their vertical displacements.)

Now we can readily calculate the derivative participating in Eq. (210), for  $z \geq 0$ :

$$\left. \frac{\partial G}{\partial n'} \right|_s = \left. \frac{\partial G}{\partial z'} \right|_{z'=+0} = \frac{2z}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + z^2)^{3/2}}. \quad (2.212)$$

Due to the axial symmetry of the system, we may take  $\varphi$  for zero. With this, Eqs. (210) and (212) yield

$$\phi = \frac{V}{4\pi} \oint_S \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} d^2 r' = \frac{Vz}{2\pi} \int_0^{2\pi} d\varphi' \int_0^R \frac{\rho' d\rho'}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos \varphi' + z^2)^{3/2}}. \quad (2.213)$$

This integral is not overly pleasing, but may be readily worked out at least for points on the symmetry axis ( $\rho = 0, z \geq 0$ ):<sup>71</sup>

$$\phi = Vz \int_0^R \frac{\rho' d\rho'}{(\rho'^2 + z^2)^{3/2}} = \frac{V}{2} \int_0^{R^2/z^2} \frac{d\xi}{(\xi + 1)^{3/2}} = V \left[ 1 - \frac{z}{(R^2 + z^2)^{1/2}} \right] \quad (2.214)$$

This result shows that if  $z \rightarrow 0$ , the potential tends to  $V$  (as it should), while at  $z \gg R$ ,

$$\phi \rightarrow V \frac{R^2}{2z^2}. \quad (2.215)$$

Now let us use the same Eq. (210) to solve the (in :-)-famous problem of the cut sphere (Fig. 32). Again, if the gap between the two conducting semi-spheres is very thin ( $t \ll R$ ), we may use Green's function for the grounded (and uncut) sphere. For a particular case  $\mathbf{r}' = d\mathbf{n}_z$ , this function is given by

Eqs. (197)-(198); generalizing the former relation for an arbitrary direction of vector  $\mathbf{r}'$ , we get

$$G = \frac{1}{[r^2 + r'^2 - 2rr' \cos \gamma]^{1/2}} - \frac{R/r'}{[r^2 + (R^2/r')^2 - 2r(R^2/r') \cos \gamma]^{1/2}}, \quad \text{for } r, r' \geq R, \quad (2.216)$$

where  $\gamma$  is the angle between the vectors  $\mathbf{r}$  and  $\mathbf{r}'$ , and hence  $\mathbf{r}''$  – see Fig. 32.

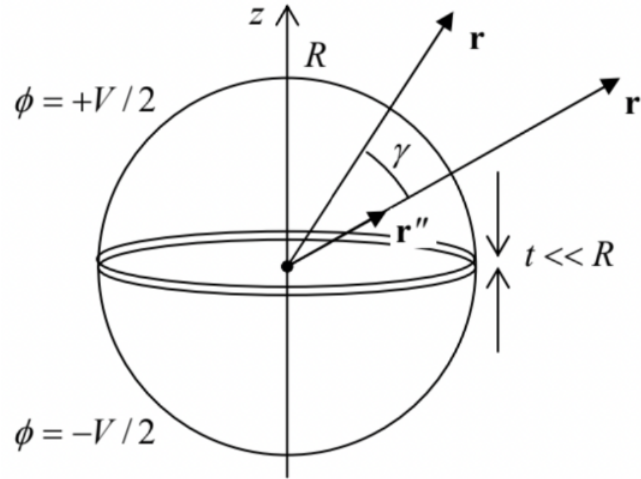


Fig. 2.32. A system of two separated, oppositely biased semi-spheres.

Now, calculating the Green's function's derivative,

$$\left. \frac{\partial G}{\partial r'} \right|_{r'=R+0} = - \frac{(r^2 - R^2)}{R[r^2 + R^2 - 2Rr \cos \gamma]^{3/2}}, \quad (2.217)$$

and plugging it into Eq. (210), we see that the integration is again easy only for the field on the symmetry axis ( $\mathbf{r} = z\mathbf{n}_z, \gamma = \theta$ ), giving:

$$\phi = \frac{V}{2} \left[ 1 - \frac{z^2 - R^2}{z(z^2 + R^2)^{1/2}} \right]. \quad (2.218)$$

For  $z \rightarrow R$ , this relation yields  $\phi \rightarrow V/2$  (just checking :-), while for  $z/R \rightarrow \infty$ ,

$$\phi \rightarrow \frac{3R^2}{4z^2}V. \quad (2.219)$$

As will be discussed in the next chapter, such a field is typical for an electric dipole.

## Reference

<sup>65</sup> See, e.g., CM Sec. 5.1, QM Secs. 2.2 and 7.4, and SM Sec. 5.5.

<sup>66</sup>  $G$  so defined is sometimes called the Dirichlet function.

<sup>67</sup> See, e.g., MA Eq. (12.3). Actually, this theorem is a ready corollary of the better-known divergence ("Gauss") theorem, MA Eq. (12.2).

<sup>68</sup> This symmetry, evident for the particular cases (204) and (206), may be readily proved for the general case by applying Eq. (207) to functions  $f(\mathbf{r}) \equiv G(\mathbf{r}, \mathbf{r}')$  and  $g(\mathbf{r}) \equiv G(\mathbf{r}, \mathbf{r}'')$ . With this substitution, the left-hand side of that equality becomes equal to  $-4\pi [G(\mathbf{r}'', \mathbf{r}') - G(\mathbf{r}', \mathbf{r}'')]$ , while the right-hand side is zero, due to Eq. (205).

<sup>69</sup> In some textbooks, the sign before the surface integral is negative, because their authors use the outer normal to the free-space region  $V$  rather than that occupied by conductors – as I do.

<sup>70</sup> Indeed, if all parts of the cut plane are grounded, a narrow cut does not change the field distribution, and hence the Green's function, significantly.

<sup>71</sup> There is no need to repeat the calculation for  $z \leq 0$ : from the symmetry of the problem,  $\phi(-z) = \phi(z)$ .

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