6.2: Magnetic Energy Revisited

Now we are sufficiently equipped to return to the issue of the magnetic energy, in particular, to finally prove Eqs. (5.57) and (5.140), and discuss the dichotomy of the signs in Eqs. (5.53) and (5.54). For that, let us consider a sufficiently slow, small magnetic field variation $\delta\mathbf{B}$. If we want to neglect the kinetic energy of the system of electric currents under consideration, as well as the wave radiation effects, we need to prevent its acceleration by the arising induction field $\mathbf{E}_{\text{ind}}$. Let us suppose that we do this by the virtual balancing of this field by an external electric field $\mathbf{E}_{\text{ext}} = -\mathbf{E}_{\text{ind}}$. According to Eq. (4.38), the work of that field[6] on the stand-alone currents of the system during time interval δt, and hence the change of the potential energy of the system, is

$$\delta U = \delta t \int_V \mathbf{j}\cdot\mathbf{E}_{\text{ext}}\, d^3r = -\delta t \int_V \mathbf{j}\cdot\mathbf{E}_{\text{ind}}\, d^3r, \tag{6.10}$$

where the integral is over the volume of the system. Now expressing the current density $\mathbf{j}$ from the macroscopic Maxwell equation (5.107), and then applying the vector algebra identity[7]

$$(\nabla\times\mathbf{H})\cdot\mathbf{E}_{\text{ind}} \equiv \mathbf{H}\cdot(\nabla\times\mathbf{E}_{\text{ind}}) - \nabla\cdot(\mathbf{E}_{\text{ind}}\times\mathbf{H}), \tag{6.11}$$

we get

$$\delta U = -\delta t \int_V \mathbf{H}\cdot(\nabla\times\mathbf{E})d^3r + \delta t \int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})d^3r. \tag{6.12}$$

According to the divergence theorem, the second integral in the right-hand of this equality is equal to the flux of the so-called Poynting vector $\mathbf{S}\equiv\mathbf{E}\times\mathbf{H}$ through the surface limiting the considered volume V. Later in the course we will see that this flux represents, in particular, the power of electromagnetic radiation through the surface. If such radiation is negligible (as it always is if the field variation is sufficiently slow), the surface may be selected sufficiently far, so that the flux of $\mathbf{S}$ vanishes. In this case, we may express $\nabla\times\mathbf{E}$ from the Faraday induction law (5) to get

$$\delta U = -\delta t \int_V \left(-\frac{\partial B}{\partial t}\right)\cdot\mathbf{H}d^3r = \int_V \mathbf{H}\cdot\delta\mathbf{B}d^3r. \tag{6.13}$$

Just as in the electrostatics (see Eqs. (1.65) and (3.73), and their discussion), this relation may be interpreted as the variation of the magnetic field energy U of the system, and represented in the form

$$\delta U = \int_V \delta u(\mathbf{r})d^3r, \quad \text{with } \delta u \equiv \mathbf{H}\cdot\delta\mathbf{B}. \qquad \text{Magnetic energy's variation} \tag{6.14}$$

This is a keystone result; let us discuss it in some detail.

First of all, for a system filled with a linear and isotropic magnetic material, we may use Eq. (14) together with Eq. (5.110): $\mathbf{B}=\mu\mathbf{H}$. Integrating the result over the variation of the field from 0 to a certain final value $\mathbf{B}$, we get Eq. (5.140) – so important that it is worthy of rewriting it again:

$$U = \int_V u(\mathbf{r})d^3r, \quad \text{with } u = \frac{B^2}{2\mu}. \tag{6.15}$$

In the simplest case of free space (no magnetics at all, so that j above is the complete current density), we may take $\mu=\mu_0$, and reduce Eq. (15) to Eq. (5.57). Now performing backward the transformations that took us, in Sec. 5.3, to derive that relation from Eq. (5.54), we finally have the latter formula proved – as was promised in the last chapter.

It is very important, however, to understand the limitations of Eq. (15). For example, let us try to apply it to a very simple problem, which was already analyzed in Sec. 5.6 (see Fig. 5.15): a very long cylindrical sample of a linear magnetic material placed into a fixed external field $\mathbf{H}_{\text{ext}}$, parallel to the sample's axis. It is evident that in this simple geometry, the field $\mathbf{H}$ and hence the field $\mathbf{B}=\mu\mathbf{H}$ have to be uniform inside the sample, besides negligible regions near its ends, so that Eq. (15) is reduced to

$$U = \frac{B^2}{2\mu}V, \tag{6.16}$$

where $V=Al$ is the cylinder's volume. Now if we try to calculate the static (equilibrium) value of the field from the minimum of this potential energy, we get evident nonsense: $\mathbf{B}=0$ (WRONG!).[8]

The situation may be readily rectified by using the notion of the Gibbs potential energy, just as it was done for the electric field in Sec. 3.5 (and implicitly in the end of Sec. 1.3). According to Eq. (14), in magnetostatics, the Cartesian components of the field $\mathbf{H}(\mathbf{r})$ play the role of the generalized forces, while those of the field $\mathbf{B}(\mathbf{r})$, of the generalized coordinates (per unit volume).[9] As the result, the Gibbs potential energy, whose minimum corresponds to the stable equilibrium of the system under the effect of a fixed generalized force (in our current case, of the fixed external field $\mathbf{H}_{\text{ext}}$, is

$$\text{Gibbs potential energy} \qquad U_{\mathrm{G}}=\int_V u_{\mathrm{G}}(\mathbf{r}) d^3 r, \quad \text{ with } u_{\mathrm{G}}(\mathbf{r}) \equiv u(\mathbf{r})-\mathbf{H}_{\text{ext}}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}), \tag{6.17}$$

- the expression parallel to Eq. (3.78). For a system with linear magnetics, we may use Eq. (15) for u, getting the following Gibbs energy's density:

$$u_{\mathrm{G}}(\mathbf{r})=\frac{1}{2 \mu} \mathbf{B} \cdot \mathbf{B}-\mathbf{H}_{\text{ext}} \cdot \mathbf{B} \equiv \frac{1}{2 \mu}\left(\mathbf{B}-\mu \mathbf{H}_{\text{ext}}\right)^2+\text{const}, \tag{6.18}$$

where "const" means a term independent of the field $\mathbf{B}$ inside the sample. For our simple cylindrical system, with its uniform fields, Eqs. (17)-(18) gives the following full Gibbs energy of the sample:

$$U_{\mathrm{G}}=\frac{\left(\mathbf{B}_{\text{int}}-\mu \mathbf{H}_{\text{ext}}\right)^2}{2 \mu} V+\text{const}, \tag{6.19}$$

whose minimum immediately gives the correct stationary value $\mathbf{B}_{\text{int}}=\mu \mathbf{H}_{\text{ext}}$, i.e. $\mathbf{H}_{\text{int}} \equiv \mathbf{B}_{\text{int}} / \mu=\mathbf{H}_{\text{ext}}$ - which was already obtained in Sec. 5.6 in a different way, from the boundary condition (5.117).

Now notice that with this result on hand, Eq. (18) may be rewritten in a different form:

$$u_{\mathrm{G}}(\mathbf{r})=\frac{1}{2 \mu} \mathbf{B} \cdot \mathbf{B}-\frac{\mathbf{B}}{\mu} \cdot \mathbf{B} \equiv-\frac{B^2}{2 \mu}, \tag{6.20}$$

similar to Eq. (15) for u(r), but with an opposite sign. This sign dichotomy explains that in Eqs. (5.53) and Eq. (5.54); indeed, as was already noted in Sec. 5.3, the former of these expressions gives the potential energy whose minimum corresponds to the equilibrium of a system with fixed currents. (In our current example, these are the external stand-alone currents inducing the field $\mathbf{H}_{\text{ext}}$.) So, the energy U_j given by Eq. (5.53) is essentially the Gibbs energy U_{G} defined by Eqs. (17) and (for the case of linear magnetics, or no magnetic media at all) by Eq. (20), while Eq. (5.54) is just another form of Eq. (15) – as was explicitly shown in Sec. 5.3.[10]

Let me complete this section by stating that the difference between the energies U and U_{G} is not properly emphasized (or even left obscure) in some textbooks, so that the reader is advised to seek additional clarity by solving additional simple problems – for example, by spelling out these energies for the simple case of a long straight solenoid (Fig. 5.6a), and then using these formulas to calculate the pressure exerted by the magnetic field on the solenoid's walls (windings) and the longitudinal forces exerted on its ends.

Reference

[6] As a reminder, the magnetic component of the Lorentz force (5.10), $\mathbf{v} \times \mathbf{B}$, is always perpendicular to particle's velocity $\mathbf{v}$, so that the magnetic field B itself cannot perform any work on moving charges, i.e. on currents.

[7] See, e.g., MA Eq. (11.7) with $\mathbf{f}=\mathbf{E}_{\text{ind}}$ and $\mathbf{g}=\mathbf{H}$.

[8] Note that this erroneous result cannot be corrected by just adding the energy (15) of the field outside the cylinder, because in the limit $A \rightarrow 0$, this field is not affected by the internal field $\mathbf{B}$.

[9] Note that in this respect, the analogy with electrostatics is not quite complete. Indeed, according to Eq. (3.76), in electrostatics the role of a generalized coordinate is played by "would-be" field $\mathbf{D}$, and that of the generalized force, by the actual electric field $\mathbf{E}$. This difference may be traced back to the fact that electric field $\mathbf{E}$ may perform work on a moving charged particle, while the magnetic field cannot. However, this difference does not affect the full analogy of expressions (3.73) and (15) for the field energy density in linear media.

[10] As was already noted in Sec. 5.4, one more example of the energy U_j (i.e. U_{G}) is given by Eq. (5.100).