6.4: Electrodynamics of Superconductivity, and the Gauge Invariance

The effect of superconductivity[20] takes place (in certain materials only, mostly metals) when temperature T is reduced below a certain critical temperature T_{c}, specific for each material. For most metallic superconductors, T_{c} is of the order of typically a few kelvins, though several compounds (the so-called high-temperature superconductors) with T_{c} above 100 K have been found since 1987. The most notable property of superconductors is the absence, at $T<T_{\mathrm{c}}$, of measurable resistance to (not very high) dc currents. However, the electromagnetic properties of superconductors cannot be described by just taking $\sigma=\infty$ in our previous results. Indeed, for this case, Eq. (33) would give $\delta_{\mathrm{s}}=0$, i.e., no ac magnetic field penetration at all, while for the dc field we would have the uncertainty $\sigma\omega\rightarrow ?$ Experiment shows something substantially different: weak magnetic fields do penetrate into superconductors by a material-specific distance $\delta_{\mathrm{L}}\sim 10^{-7}-10^{-6}\ \mathrm{m}$, the so-called London's penetration depth,[21] which is virtually frequency-independent until the skin depth δ_{s}, measured in the same material in its "normal" state, i.e. the absence of superconductivity, becomes less than δ_{L}. (This crossover happens typically at frequencies $\omega\sim 10^{13}-10^{14}\ \mathrm{s}^{-1}$.) The smallness of δ_{L} on the human scale means that the magnetic field is pushed out of macroscopic samples at their transition into the superconducting state.

This Meissner-Ochsenfeld effect, discovered experimentally in 1933,[22] may be partly understood using the following classical reasoning. The discussion of the Ohm law in Sec. 4.2 implied that the current's (and hence the electric field's) frequency ω is either zero or sufficiently low. In the classical Drude reasoning, this is acceptable while $\omega\tau<<1$, where τ is the effective carrier scattering time participating in Eqs. (4.12)-(4.13). If this condition is not satisfied, we should take into account the charge carrier inertia; moreover, in the opposite limit $\omega\tau>>1$, we may neglect the scattering at all. Classically, we can describe the charge carriers in such a "perfect conductor" as particles with a non-zero mass m, which are accelerated by the electric field, following the 2nd Newton law (4.11),

$$m\dot{\mathbf{v}}=\mathbf{F}=q\mathbf{E}, \tag{6.39}$$

so that the current density $\mathbf{j}=qn\mathbf{v}$ they create, changes in time as

$$\dot{\mathbf{j}}=qn\dot{\mathbf{v}}=\frac{q^2n}{m}\mathbf{E}. \tag{6.40}$$

In terms of the Fourier amplitudes of the functions $\mathbf{j}(t)$ and $\mathbf{E}(t)$, this means

$$-i\omega_{\mathbf{j}_\omega}=\frac{q^2n}{m}\mathbf{E}_\omega. \tag{6.41}$$

Comparing this formula with the relation $\mathbf{j}_\omega=\sigma\mathbf{E}_\omega$ implied in the last section, we see that we can use all its results with the following replacement:

$$\sigma\rightarrow i\frac{q^2n}{m\omega}. \tag{6.42}$$

This change replaces the characteristic equation (29) with

$$-i\omega=\frac{\kappa^2m\omega}{iq^2n\mu},\quad \text{i.e. }\kappa^2=\frac{\mu q^2n}{m}, \tag{6.43}$$

i.e. replaces the skin effect with the field penetration by the following frequency-independent depth:

$$\delta\equiv\frac{1}{\kappa}=\left(\frac{m}{\mu q^2n}\right)^{1/2}. \tag{6.44}$$

Superficially, this means that the field decay into the superconductor does not depend on frequency:

$$H(x,t)=H(0,t)e^{-x/\delta}, \tag{6.45}$$

thus explaining the Meissner-Ochsenfeld effect.

However, there are two problems with this result. First, for the parameters typical for good metals $\left(q=-e, n\sim 10^{29}\ \mathrm{m}^{-3}, m\sim m_{\mathrm{e}}, \mu\approx\mu_0\right)$, Eq. (44) gives $\delta\sim 10^{-8}\ \mathrm{m}$, one or two orders of magnitude lower than the experimental values of δ_{L}. Experiment also shows that the penetration depth diverges at $T \rightarrow T_{\mathrm{c}}$, which is not predicted by Eq. (44).

The second, much more fundamental problem with Eq. (44) is that it has been derived for $\omega\tau >> 1$. Even if we assume that somehow there is no scattering at all, i.e. $\tau = \infty$, at $\omega \rightarrow 0$ both parts of the characteristic equation (43) vanish, and we cannot make any conclusion about κ. This is not just a mathematical artifact we could ignore. For example, let us place a non-magnetic metal into a static external magnetic field at $T > T_{\mathrm{c}}$. The field would completely penetrate the sample. Now let us cool it. As soon as temperature is decreased below T_{c}, the above calculations would become valid, forbidding the penetration into the superconductor of any change of the field, so that the initial field would be "frozen" inside the sample. The experiment shows something completely different: as T is lowered below T_{c}, the initial field is being pushed out of the sample.

The resolution of these contradictions has been provided by quantum mechanics. As was explained in 1957 in a seminal work by J. Bardeen, L. Cooper, and J. Schrieffer (commonly referred to as the BSC theory), the superconductivity is due to the correlated motion of electron pairs, with opposite spins and nearly opposite momenta. Such Cooper pairs, each with the electric charge $q = -2e$ and zero spin, may form only in a narrow energy layer near the Fermi surface, of certain thickness $\Delta(T)$. This parameter $\Delta(T)$, which may be also considered as the binding energy of the pair, tends to zero at $T \rightarrow T_{\mathrm{c}}$, while at $T << T_{\mathrm{c}}$ it has a virtually constant value $\Delta(0) \approx 3.5 k_{\mathrm{B}} T_{\mathrm{c}}$, of the order of a few meV for most superconductors. This fact readily explains the relatively low spatial density of the Cooper pairs: $n_p \sim n\Delta(T)/\varepsilon_{\mathrm{F}} \sim 10^{26}\ \mathrm{m}^{-3}$. With the correction $n \rightarrow n_{\mathrm{p}}$, Eq. (44) for the penetration depth becomes

$$\delta \rightarrow \delta_{\mathrm{L}} = \left(\frac{m}{\mu q^2 n_{\mathrm{p}}}\right)^{1/2}. \qquad \text{London's penetration depth} \tag{6.46}$$

This result diverges at $T \rightarrow T_{\mathrm{c}}$, and generally fits the experimental data reasonably well, at least for the so-called "clean" superconductors (with the mean free path $l = \nu_{\mathrm{F}}\tau$, where $\nu_{\mathrm{F}} \sim (2m\varepsilon_{\mathrm{F}})^{1/2}$ is the r.m.s. velocity of electrons on the Fermi surface, much longer than the Cooper pair size ξ – see below).

The smallness of the coupling energy $\Delta(T)$ is also a key factor in the explanation of the Meissner-Ochsenfeld effect, as well as several macroscopic quantum phenomena in superconductors. Because of Heisenberg's quantum uncertainty relation $\delta r \delta p \sim \hbar$, the spatial extension of the Cooper-pair's wavefunction (the so-called coherence length of the superconductor) is relatively large: $\xi \sim \delta r \sim \hbar/\delta p \sim \hbar\nu_{\mathrm{F}}/\Delta(T) \sim 10^{-6}\ \mathrm{m}$. As a result, $n_{\mathrm{p}}\xi^3 >> 1$, meaning that the wavefunctions of the pairs are strongly overlapped in space. Due to their integer spin, Cooper pairs behave like bosons, which means in particular that at low temperatures they exhibit the so-called Bose-Einstein condensation onto the same ground energy level ε_{g}.[23] This means that the frequency $\omega = \varepsilon_{\mathrm{g}}/\hbar$ of the time evolution of each pair's wavefunction $\Psi = \psi \exp\{-i\omega t\}$ is exactly the same, and that the phases φ of the wavefunctions, defined by the relation

$$\psi = |\psi| e^{i\varphi}, \tag{6.47}$$

are equal, so that the electric current is carried not by individual Cooper pairs but rather their Bose-Einstein condensate described by a single wavefunction (47). Due to this coherence, the quantum effects (which are, in the usual Fermi-gases of single electrons, masked by the statistical spread of their energies, and hence of their phases), become very explicit – "macroscopic".

To illustrate this, let us write the well-known quantum-mechanical formula for the probability current density of a free, non-relativistic particle,[24]

$$\mathbf{j}_w = \frac{i\hbar}{2m}(\psi\nabla\psi^* - \text{ c.c. }) \equiv \frac{1}{2m}[\psi^*(-i\hbar\nabla)\psi - \text{ c.c. }], \tag{6.48}$$

where c.c. means the complex conjugate of the previous expression. Now let me borrow one result that will be proved later in this course (in Sec. 9.7) when we discuss the analytical mechanics of a charged particle moving in an electromagnetic field. Namely, to account for the magnetic field effects, the particle's kinetic momentum $\mathbf{p} \equiv m\mathbf{v}$ (where $\mathbf{v} \equiv d\mathbf{r}/dt$ is particle's velocity) has to be distinguished from its canonical momentum,[25]

$$\mathbf{P} \equiv \mathbf{p} + q\mathbf{A}. \tag{6.49}$$

where $\mathbf{A}$ is the field's vector potential defined by Eq. (5.27). In contrast with the Cartesian components $p_j = m\nu_j$ of the kinetic momentum $\mathbf{p}$, the canonical momentum's components are the generalized momenta corresponding to the Cartesian components r_j of the radius-vector $\mathbf{r}$, considered as generalized coordinates of the particle: $P_j = \partial\mathscr{L}/\partial\nu_j$, where $\mathscr{L}$ is the particle's Lagrangian function. According to the general rules of transfer from classical to quantum mechanics,[26] it is the vector $\mathbf{P}$ whose operator (in the coordinate representation) equals $-i\hbar\nabla$, so that the operator of the kinetic momentum $\mathbf{p}=\mathbf{P}-q\mathbf{A}$ is equal to $-i\hbar\nabla q\mathbf{A}$. Hence, to account for the magnetic field[27] effects, we should make the following replacement,

$$-i\hbar\nabla \rightarrow -i\hbar\nabla - q\mathbf{A}, \tag{6.50}$$

in all field-free quantum-mechanical relations. In particular, Eq. (48) has to be generalized as

$$\mathbf{j}_w = \frac{1}{2m}[\psi^*(-i\hbar\nabla - q\mathbf{A})\psi - \text{ c.c. }]. \tag{6.51}$$

This expression becomes more transparent if we take the wavefunction in form (47):

$$\mathbf{j}_w = \frac{\hbar}{m}|\psi|^2\left(\nabla\varphi - \frac{q}{\hbar}\mathbf{A}\right). \tag{6.52}$$

This relation means, in particular, that in order to keep $\mathbf{j}_w$ gauge-invariant, the transformation (8)-(9) has to be accompanied by a simultaneous transformation of the wavefunction's phase:

$$\varphi \rightarrow \varphi + \frac{q}{\hbar}\chi. \tag{6.53}$$

It is fascinating that the quantum-mechanical wavefunction (or more exactly, its phase) is not gauge-invariant, meaning that you may change it in your mind – at your free will! Again, this does not change any observable (such as $\mathbf{j}_w$ or the probability density $\psi\psi^*$, i.e. any experimental results.

Now for the electric current density of the whole superconducting condensate, Eq. (52) yields the following constitutive relation:

$$\mathbf{j} \equiv \mathbf{j}_w q n_\text{p} = \frac{\hbar q n_\text{p}}{m}|\psi|^2\left(\nabla\varphi - \frac{q}{\hbar}\mathbf{A}\right) \qquad \text{Supercurrent density} \tag{6.54}$$

The formula shows that this supercurrent may be induced by the dc magnetic field alone and does not require any electric field. Indeed, for the simplest, 1D geometry shown in Fig.2a, $\mathbf{j}(\mathbf{r})=j(x)\mathbf{n}_z$, $\mathbf{A}(\mathbf{r})=A(x)\mathbf{n}_z$, and $\partial/\partial z=0$, so that the Coulomb gauge condition (5.48) is satisfied for any choice of the gauge function $\chi(x)$, and for the sake of simplicity we can choose it to provide $\varphi(\mathbf{r})\equiv\text{const}$,[28] so that

$$\mathbf{j} = -\frac{q^2 n_\text{p}}{m}\mathbf{A} \equiv -\frac{1}{\mu\delta_\text{L}^2}\mathbf{A}. \tag{6.55}$$

where δ_L is given by Eq. (46), and the field is assumed to be small, and hence not affecting the probability $|\psi|^2$ (normalized to 1 in the absence of the field). This is the so-called London equation, proposed (in a different form) by F. and H. London in 1935 for the Meissner-Ochsenfeld effect's explanation. Combining it with Eq. (5.44), generalized for a linear magnetic medium by the replacement $\mu_0 \rightarrow \mu$, we get

$$\nabla^2\mathbf{A} = \frac{1}{\delta_\text{L}^2}\mathbf{A}, \tag{6.56}$$

This simple differential equation, similar to Eq. (23), for our 1D geometry has an exponential solution similar to Eq. (32):

$$A(x)=A(0)\exp\left\{-\frac{x}{\delta_\text{L}}\right\}, \quad B(x)=B(0)\exp\left\{-\frac{x}{\delta_\text{L}}\right\}, \quad j(x)=j(0)\exp\left\{-\frac{x}{\delta_\text{L}}\right\}, \tag{6.57}$$

which shows that the magnetic field and the supercurrent penetrate into a superconductor only by London's penetration depth δ_L, regardless of frequency.[29] By the way, integrating the last result through the penetration layer, and using the vector potential's definition, $\mathbf{B}=\nabla\times\mathbf{A}$ (for our geometry, giving $B(x)=dA(x)/dx=-\delta_\text{L}A(x)$) we may readily verify that the linear density $\mathbf{J}$ of the surface supercurrent still satisfies the universal coarse-grain relation (38).

This universality should bring to our attention the following common feature of the skin effect (in "normal" conductors) and the Meissner-Ochsenfeld effect (in superconductors): if the linear size of a bulk sample is much larger than, respectively, δ_s or δ_L, than $\mathbf{B}=0$ in the dominating part of its interior. According to Eq. (5.110), a formal description of such conductors (valid only on a coarse-grain scale much larger than either δ_s or δ_L), may be achieved by formally treating the sample as an ideal diamagnet, with $\mu=0$. In particular, we can use this description and Eq. (5.124) to immediately obtain the magnetic field's distribution outside of a bulk sphere:

$$\mathbf{B} = \mu_0 \mathbf{H} = -\mu_0 \nabla \phi_m, \quad \text{with } \phi_{\mathrm{m}} = H_0 \left(-r - \frac{R^3}{2r^2} \right) \cos\theta, \quad \text{for } r \geq R. \tag{6.58}$$

Figure 3 shows the corresponding surfaces of equal potential ϕ_m. It is evident that the magnetic field lines (which are normal to the equipotential surfaces) bend to become parallel to the surface near it.

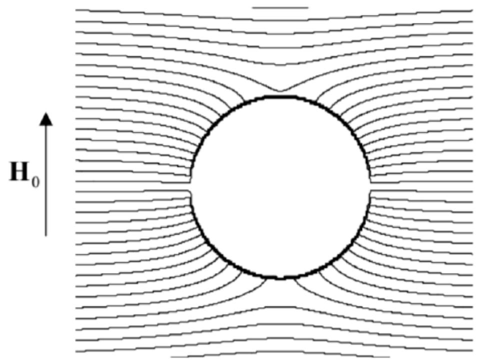

Fig. 6.3. Equipotential surfaces $\phi_{\mathrm{m}} = \text{const}$ around a (super)conducting sphere of radius $R >> \delta_{\mathrm{s}}$ (or δ_{L}), placed into a uniform magnetic field, within the coarse-grain model $\mu = 0$.

This pattern also helps to answer the question that might arise at making the assumption (24): what happens to bulk conductors placed into in a normal ac magnetic field – and to superconductors in a normal dc magnetic field as well? The answer is: the field is deformed outside of the conductor to sustain the following coarse-grain boundary condition:[30]

$$\text{Coarse-grain boundary condition} \qquad B_n|_{\text{surface}} = 0, \tag{6.59}$$

which follows from Eq. (5.118) and the coarse-grain requirement $\mathbf{B}|_{\text{inside}} = 0$.

This answer should be taken with reservations. For normal conductors it is only valid at sufficiently high frequencies where the skin depth (33) is sufficiently small: $\delta_{\mathrm{s}} << a$, where a is the scale of the conductor's linear size – for a sphere, $a \sim R$. In superconductors, this simple picture is valid not

only if $\delta_{\mathrm{s}} << a$, but also only in sufficiently low magnetic fields, because strong fields do penetrate into superconductors, destroying superconductivity (completely or partly), and as a result violating the Meissner-Ochsenfeld effect – see the next section.

Reference

[20] Discovered experimentally in 1911 by Heike Kamerlingh Onnes.

[21] Named so to acknowledge the pioneering theoretical work of brothers Fritz and Heinz London – see below.

[22] It is hardly fair to shorten the name to just the “Meissner effect”, as it is frequently done, because of the reportedly crucial contribution by Robert Ochsenfeld, then a Walther Meissner’s student, into the discovery.

[23] A quantitative discussion of the Bose-Einstein condensation of bosons may be found in SM Sec. 3.4, though the full theory of superconductivity is more complicated because it has to describe the condensation taking place simultaneously with the formation of effective bosons (Cooper pairs) from fermions (single electrons). For a detailed, but very readable coverage of the physics of superconductors, I can refer the reader to the monograph by M. Tinkham, Introduction to Superconductivity, 2nd ed., McGraw-Hill, 1996.

[24] See, e.g., QM Sec. 1.4, in particular Eq. (1.47).

[25] I am sorry to use traditional notations $\mathbf{p}$ and $\mathbf{P}$ for the momenta – the same symbols which were used for the electric dipole moment and polarization in Chapter 3. I hope there will be no confusion, because the latter notions are not used in this section.

[26] See, e.g., CM Sec. 10.1, in particular Eq. (10.26).

[27] The account of the electric field is easier, because the related energy $q\phi$ of the particle may be directly included in the potential energy operator.

[28] This is the so-called London gauge; for our simple geometry, it is also the Coulomb gauge (5.48).

[29] Since at $T > 0$ not all electrons in a superconductor form Cooper pairs, at any frequency $\omega \neq 0$ the unpaired electrons provide energy-dissipating Ohmic currents, which are not described by Eq. (54). These losses become very substantial when the frequency ω becomes so high that the skin-effect length δ_s of the material (as measured with superconductivity suppressed, say by high magnetic field) becomes less than δ_L. For typical metallic superconductors, this crossover takes place at frequencies of a few hundred GHz, so that even for microwaves, Eq. (57) still gives a fairly accurate description of the field penetration.

[30] Sometimes this boundary condition, as well as the (compatible) Eq. (38), are called "macroscopic". However, this term may lead to confusion with the genuine macroscopic boundary conditions (5.117)-(5.118), which also ignore the atomic-scale microstructure of the "effective currents" $\mathbf{j}_{\text{ef}} = \nabla \times \mathbf{M}$, but (as was shown earlier in this section) still allow explicit, detailed accounts of the skin-current (34) and supercurrent (55) distributions.
