2.9: Variable Separation – Polar Coordinates

If a system of conductors is cylindrical, the potential distribution is independent of the coordinate z along the cylinder axis: $\partial\phi/\partial z=0$, and the Laplace equation becomes two-dimensional. If the conductor's cross-section is rectangular, the variable separation method works best in Cartesian coordinates $\{x, y\}$, and is just a particular case of the 3D solution discussed above. However, if the cross-section is circular, much more compact results may be obtained by using the polar coordinates $\{\rho, \varphi\}$. As we already know from Sec. 3(ii), these 2D coordinates are orthogonal, so that the two-dimensional Laplace operator is a sum of two separable terms.[36] Requiring, just as we have done above, each component of the sum (84) to satisfy the Laplace equation, we get

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi_k}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\phi_k}{\partial\varphi^2}=0. \tag{2.104}$$

In a full analogy with Eq. (85), let us represent each particular solution ϕ_k as a product $\mathcal{R}(\rho)\mathcal{F}(\varphi)$. Plugging this expression into Eq. (104) and then dividing all its parts by $\mathcal{R}\mathcal{F}/\rho^2$, we get

$$\frac{\rho}{\mathcal{R}}\frac{d}{d\rho}\left(\rho\frac{d\mathcal{R}}{d\rho}\right)+\frac{1}{\mathcal{F}}\frac{d^2\mathcal{F}}{d\varphi^2}=0. \tag{2.105}$$

Following the same reasoning as for the Cartesian coordinates, we get two separated ordinary differential equations

$$\rho\frac{d}{d\rho}\left(\rho\frac{d\mathcal{R}}{d\rho}\right)=v^2\mathcal{R}, \tag{2.106}$$

$$\frac{d^2\mathcal{F}}{d\varphi^2}+v^2\mathcal{F}=0, \tag{2.107}$$

where v^2 is the variable separation constant.

Let us start their analysis from Eq. (106), plugging into it a probe solution $\mathcal{R}=c\rho^\alpha$, where c and α are some constants. The elementary differentiation shows that if $\alpha\neq 0$, the equation is indeed satisfied for any c, with just one requirement on the constant α, namely $\alpha^2=v^2$. This means that the following linear superposition

$$\mathcal{R}=a_v\rho^{+v}+b_v\rho^{-v}, \qquad \text{for } v\neq 0, \tag{2.108}$$

with any constant coefficients a_v and b_v, is also a solution to Eq. (106). Moreover, the general theory of linear ordinary differential equations tells us that the solution of a second-order equation like Eq. (106) may only depend on just two constant factors that scale two linearly-independent functions. Hence, for all values $v^2\neq 0$, Eq. (108) presents the general solution of that equation. The case when $v=0$, in which the functions ρ^{+v} and $\rho-^v$ are just constants and hence are not linearly-independent, is special, but in this case, the integration of Eq. (106) is straightforward,[37] giving

$$\mathcal{R}=a_0+b_0\ln\rho, \qquad \text{for } v=0. \tag{2.109}$$

In order to specify the separation constant, let us explore Eq. (107), whose general solution is

$$\mathcal{F}=\begin{cases} c_v\cos v\varphi+s_v\sin v\varphi, & \text{for } v\neq 0,\\ c_0+s_0\varphi, & \text{for } v=0.\end{cases} \tag{2.110}$$

There are two possible cases here. In many boundary problems solvable in cylindrical coordinates, the free space region, in which the Laplace equation is valid, extends continuously around the origin point $\rho=0$. In this region, the potential has to be continuous and uniquely defined, so that $\mathcal{F}$ has to be a 2π periodic function of φ. For that, one needs the product $v(\varphi+2\pi)$ to equal $v\varphi+2\pi n$, with n being an integer, immediately giving us a discrete spectrum of possible values of the variable separation constant:

$$v=n=0,\pm1,\pm2,\ldots \tag{2.111}$$

In this case, both functions $\mathcal{R}$ and $\mathcal{F}$ may be labeled with the integer index n. Taking into account that the terms with negative values of n may be summed up with those with positive n, and that s_0 has to equal zero (otherwise the 2π-periodicity of function $\mathcal{F}$ would be violated), we see that the general solution to the 2D Laplace equation for such geometries may be represented as

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$$\phi(\rho,\varphi)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}\left(a_n\rho^n+\frac{b_n}{\rho^n}\right)(c_n\cos n\varphi+s_n\sin n\varphi) \tag{2.112}$$

Let us see how all this machinery works on the famous problem of a round cylindrical conductor placed into an electric field that is uniform and perpendicular to the cylinder's axis at large distances (see Fig. 15a), as it is if created by a large plane capacitor. First of all, let us explore the effect of the system's symmetries on the coefficients in Eq. (112). Selecting the coordinate system as shown in Fig. 15a, and taking the cylinder's potential for zero, we immediately get $a_0=0$.

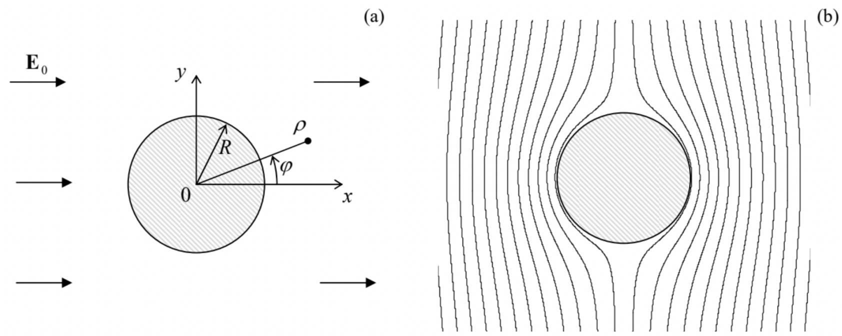

Fig. 2.15. A conducting cylinder inserted into an initially uniform electric field perpendicular to its axis: (a) the problem's geometry, and (b) the equipotential surfaces given by Eq. (117).

Moreover, due to the mirror symmetry about the plane [x, z], the solution has to be an even function of the angle φ, and hence all coefficients sn should also equal zero. Also, at large distances ($\rho>>R$) from the cylinder, its effect on the electric field should vanish, and the potential should approach that of the uniform external field $\mathbf{E}=E_0\mathbf{n}_x$:

$$\phi\rightarrow -E_0x\equiv -E_0\rho\cos\varphi,\qquad \text{for } \rho\rightarrow\infty. \tag{2.113}$$

This is only possible if in Eq. (112), $b_0=0$, and also all coefficients a_n with $n\neq 1$ vanish, while the product a_1c_1 should be equal to $(-E_0)$. Thus the solution is reduced to the following form

$$\phi(\rho,\varphi)=-E_0\rho\cos\varphi+\sum_{n=1}^{\infty}\frac{B_n}{\rho^n}\cos n\varphi, \tag{2.114}$$

in which the coefficients $B_n\equiv b_nc_n$ should be found from the boundary condition at $\rho=R$:

$$\phi(R,\varphi)=0. \tag{2.115}$$

This requirement yields the following equation,

$$\left(-E_0R+\frac{B_1}{R}\right)\cos\varphi+\sum_{n=2}^{\infty}\frac{B_n}{R^n}\cos n\varphi=0, \tag{2.116}$$

which should be satisfied for all φ. This equality, read backward, may be considered as an expansion of a function identically equal to zero into a series over mutually orthogonal functions cos $n\varphi$. It is evidently valid if all coefficients of the expansion, including $(-E_0R+B_1/R)$, and all B_n for $n\geq 2$ are equal to zero. Moreover, mathematics tells us that such expansions are unique, so this is the only possible solution of Eq. (116). So, $B_1=E_0R^2$, and our final answer (valid only outside of the cylinder, i.e. for $\rho\geq R$), is

$$\phi(\rho,\varphi)=-E_0\left(\rho-\frac{R^2}{\rho}\right)\cos\varphi\equiv -E_0\left(1-\frac{R^2}{x^2+y^2}\right)x. \tag{2.117}$$

This result, which may be represented with the equipotential surfaces shown in Fig. 15b, shows a smooth transition between the uniform field (113) far from the cylinder, to the equipotential surface of the cylinder (with $\phi=0$). Such smoothening is very typical for Laplace equation solutions. Indeed, as we know from Chapter 1, these solutions correspond to the lowest integral of the potential gradient's square, i.e. to the lowest potential energy (1.65), possible at the given boundary conditions.

To complete the problem, let us use Eq. (3) to calculate the distribution of the surface charge density over the cylinder's cross-section:

$$\sigma=\varepsilon_0 E_n|_{\text{surface}} \equiv -\varepsilon_0 \frac{\partial \phi}{\partial \rho}\bigg|_{\rho=R}=\varepsilon_0 E_0 \cos\varphi \frac{\partial}{\partial \rho}\left(\rho-\frac{R^2}{\rho}\right)_{\rho=R}=2\varepsilon_0 E_0 \cos\varphi. \tag{2.118}$$

This very simple formula shows that with the field direction shown in Fig. 15a $(E_0>0)$, the surface charge is positive on the right-hand side of the cylinder and negative on its left-hand side, thus creating a field directed from the right to the left, which exactly compensates the external field inside the conductor, where the net field is zero. (Please take one more look at the schematic Fig. 1a.) Note also that the net electric charge of the cylinder is zero, in correspondence with the problem symmetry. Another useful by-product of the calculation (118) is that the surface electric field equals $2E_0\cos\varphi$, and hence its largest magnitude is twice the field far from the cylinder. Such electric field concentration is very typical for all convex conducting surfaces.

The last observation gets additional confirmation from the second possible topology, when Eq. (110) is used to describe problems with no angular periodicity. A typical example of this situation is a cylindrical conductor with a cross-section that features a corner limited by straight lines (Fig. 16). Indeed, we may argue that at $\rho<R$ (where R is the radial extension of the straight sides of the corner, see Fig. 16), the Laplace equation may be satisfied by a sum of partial solutions $\mathcal{R}(\rho)\mathcal{F}(\varphi)$, if the angular components of the products satisfy the boundary conditions on the corner sides. Taking (just for the simplicity of notation) the conductor's potential to be zero, and one of the corner's sides as the x-axis ($\varphi=0$), these boundary conditions are

$$\mathcal{F}(0)=\mathcal{F}(\beta)=0, \tag{2.119}$$

where the angle β may be anywhere between 0 and 2π – see Fig. 16.

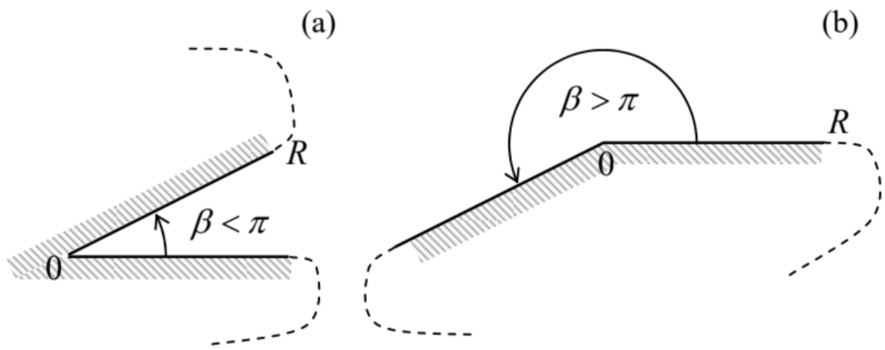

Fig. 2.16. The cross-sections of cylindrical conductors with (a) a corner and (b) a wedge.

Comparing this condition with Eq. (110), we see that it requires s_0 and all c_v to vanish, and v to take one of the values of the following discrete spectrum:

$$v_m\beta=\pi m, \qquad \text{with } m=1,2,\ldots \tag{2.120}$$

Hence the full solution of the Laplace equation for this geometry takes the form

$$\phi=\sum_{m=1}^{\infty} a_m \rho^{\pi m/\beta} \sin\frac{\pi m\varphi}{\beta}, \qquad \text{for } \rho<R, \quad 0\leq\varphi\leq\beta, \tag{2.121}$$

where the constants s_v have been incorporated into a_m. The set of coefficients a_m cannot be universally determined, because it depends on the exact shape of the conductor outside the corner, and the externally applied electric field. However, whatever the set is, in the limit $\rho\rightarrow 0$, the solution (121) is
almost[38] always dominated by the term with the lowest $m=1$:

$$\phi\rightarrow a_1\rho^{\pi/\beta}\sin\frac{\pi}{\beta}\varphi, \tag{2.122}$$

because the higher terms go to zero faster. This potential distribution corresponds to the surface charge density

$$\sigma=\varepsilon_0 E_n|_{\text{surface}}=-\varepsilon_0\frac{\partial\phi}{\partial(\rho\varphi)}\bigg|_{\rho=\text{const},\varphi\rightarrow+0}=-\varepsilon_0\frac{\pi a_1}{\beta}\rho^{(\pi/\beta-1)}. \tag{2.123}$$

(It is similar, with the opposite sign, on the opposite face of the angle.)

The result (123) shows that if we are dealing with a concave corner ($\beta<\pi$, see Fig. 16a), the charge density (and the surface electric field) tends to zero. On the other case, at a "convex corner" with $\beta>\pi$ (actually, a wedge – see Fig. 16b), both the charge and the field's strength concentrate, formally diverging at $\rho \rightarrow 0$. (So, do not sit on a roof's ridge during a thunderstorm; rather hide in a ditch!) We have already seen qualitatively similar effects for the thin round disk and the split plane.

Reference

[36] See, e.g., MA Eq. (10.3) with $\partial/\partial z = 0$.

[37] Actually, we have already done it in Sec. 3 – see Eq. (43).

[38] Exceptions are possible only for highly symmetric configurations when the external field is specially crafted to make $a_1 = 0$. In this case, the solution at $\rho \rightarrow 0$ is dominated by the first nonzero term of the series (121).