9.7: Analytical Mechanics of Charged Particles

The general Eq. (145) gives a full description of relativistic particle dynamics in electric and magnetic fields, just as the 2nd Newton law (1) does it in the non-relativistic limit. However, we know that in the latter case, the Lagrange formalism of analytical mechanics allows an easier solution of many problems.[61] We can expect that to be true in relativistic mechanics as well, so let us expand the analysis of Sec. 3 (valid only for free particles) to particles in the field.

For a free particle, our main result was Eq. (68), which may be rewritten as

$$\gamma \mathscr{L}=-m c^{2}, \tag{9.179}$$

with $\gamma \equiv\left(1-u^{2} / c^{2}\right)^{-1 / 2}$, showing that the product on the left-hand side is Lorentz-invariant. How can the electromagnetic field affect this relation? In non-relativistic electrostatics, we could write

$$\mathscr{L}=T-U=T-q \phi. \tag{9.180}$$

However, in relativity, the scalar potential ϕ is just one component of the potential 4-vector (116). The only way to get from this full 4-vector a Lorentz-invariant contribution to $\gamma \mathscr{L}$, that would be also proportional to the first power of the particle's velocity (to account for the magnetic component of the Lorentz force), is evidently

$$\gamma \mathscr{L}=-m c^{2}+\text{const} \times u^{\alpha} A_{\alpha}, \tag{9.181}$$

where u^{α} is the 4-velocity (63). To comply with Eq. (180), the constant factor should be equal to $(-qc)$, so that Eq. (181) becomes

$$\gamma \mathscr{L}=-m c^{2}-q u^{\alpha} A_{\alpha}, \tag{9.182}$$

and with account of Eq. (9.63), we get an extremely important equality:

$$\mathscr{L}=-\frac{m c^{2}}{\gamma}-q \phi+q \mathbf{u} \cdot \mathbf{A}, \qquad \text{Particle's Lagrangian function} \tag{9.183}$$

whose Cartesian form is

$$\mathscr{L}=-m c^{2}\left(1-\frac{u_{x}^{2}+u_{y}^{2}+u_{z}^{2}}{c^{2}}\right)^{1 / 2}-q \phi+q\left(u_{x} A_{x}+u_{y} A_{y}+u_{z} A_{z}\right). \tag{9.184}$$

Let us see whether this relation (which admittedly was derived by an educated guess rather than by a strict derivation) passes a natural sanity check. For the case of an unconstrained motion of a particle, we can select its three Cartesian coordinates $r_{j}(j=1,2,3)$ as the generalized coordinates, and its linear velocity components u_{j} as the corresponding generalized velocities. In this case, the Lagrange equations of motion are

$$\frac{d}{d t} \frac{\partial \mathscr{L}}{\partial u_{j}}-\frac{\partial \mathscr{L}}{\partial r_{j}}=0. \tag{9.185}$$

For example, for $r_{1}=x$, Eq. (184) yields

$$\frac{\partial \mathscr{L}}{\partial u_{x}}=\frac{m u_{x}}{\left(1-u^{2} / c^{2}\right)^{1 / 2}}+q A_{x} \equiv p_{x}+q A_{x}, \quad \frac{\partial \mathscr{L}}{\partial x}=-q \frac{\partial \phi}{\partial x}+q \mathbf{u} \cdot \frac{\partial \mathbf{A}}{\partial x}, \tag{9.186}$$

so that Eq. (185) takes the form

$$\frac{d p_{x}}{d t}=-q \frac{\partial \phi}{\partial x}+q \mathbf{u} \cdot \frac{\partial \mathbf{A}}{\partial x}-q \frac{d A_{x}}{d t}. \tag{9.187}$$

In the equations of motion, the field values have to be taken at the instant position of the particle, so that the last (full) derivative has components due to both the actual field's change (at a fixed point of space) and the particle's motion. Such addition is described by the so-called convective derivative[62]

$$\text{Convective derivative} \qquad \frac{d}{d t}=\frac{\partial}{\partial t}+\mathbf{u} \cdot \boldsymbol{\nabla}. \tag{9.188}$$

Spelling out both scalar products, we may group the terms remaining after cancellations as follows:

$$\frac{dp_x}{dt} = q\left[\left(-\frac{\partial\phi}{\partial x} - \frac{\partial A_x}{\partial t}\right) + u_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - u_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]. \tag{9.189}$$

But taking into account the relations (121) between the electric and magnetic fields and potentials, this expression is nothing more than

$$\frac{dp_x}{dt} = q\left(E_x + u_y B_z - u_z B_y\right) = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})_x, \tag{9.190}$$

i.e. the x-component of Eq. (144). Since other Cartesian coordinates participate in Eq. (184) in a similar way, it is evident that the Lagrangian equations of motion along other coordinates yield other components of the same vector equation of motion.

So, Eq. (183) does indeed give the correct Lagrangian function, and we can use it for further analysis, in particular to discuss the first of Eqs. (186). This relation shows that in the electromagnetic field, the generalized momentum corresponding to particle's coordinate x is not $p_x = m\gamma u_x$, but[63]

$$P_x \equiv \frac{\partial\mathscr{L}}{\partial u_x} = p_x + qA_x. \tag{9.191}$$

Thus, as was already discussed (at that point, without proof) in Sec. 6.4, the particle's motion in a magnetic field may be is described by two different momentum vectors: the kinetic momentum $\mathbf{p}$, defined by Eq. (70), and the canonical (or "conjugate") momentum[64]

$$\text{Particle's canonical momentum} \qquad \mathbf{P} = \mathbf{p} + q\mathbf{A}. \tag{9.192}$$

In order to facilitate discussion of this notion, let us generalize Eq. (72) for the Hamiltonian function $\mathscr{H}$ of a free particle to the case of a particle in the field:

$$\mathscr{H} \equiv \mathbf{P}\cdot\mathbf{u} - \mathscr{L} = (\mathbf{p} + q\mathbf{A})\cdot\mathbf{u} - \left(-\frac{mc^2}{\gamma} + q\mathbf{u}\cdot\mathbf{A} - q\phi\right) = \mathbf{p}\cdot\mathbf{u} + \frac{mc^2}{\gamma} + q\phi. \tag{9.193}$$

Merging the first two terms of the last expression exactly as it was done in Eq. (72), we get an extremely simple result,

$$\mathscr{H} = \gamma mc^2 + q\phi, \tag{9.194}$$

which may leave us wondering: where is the vector potential $\mathbf{A}$ here – and the magnetic field effects it has to describe? The resolution of this puzzle is easy: as we know from analytical mechanics,[65] for most applications, for example for an alternative derivation of the equations of motion, $\mathscr{H}$ has to be represented as a function of particle's generalized coordinates (in the case of unconstrained motion, these may be the Cartesian components of the vector $\mathbf{r}$ that serves as an argument for potentials $\mathbf{A}$ and ϕ), and the generalized momenta, i.e. the components of the vector $\mathbf{P}$ – generally, plus time. Hence, the factor γ in Eq. (194) has to be expressed via these variables. This may be done using the relation (192), $\gamma m\mathbf{u} = \mathbf{P} - q\mathbf{A}$: it is sufficient to notice that according to Eq. (193), the difference $(\mathscr{H} - q\phi)$ is equal to the right-hand side of Eq. (72), so that the generalization of Eq. (78) is[66]

$$(\mathscr{H} - q\phi)^2 = \left(mc^2\right)^2 + c^2(\mathbf{P} - q\mathbf{A})^2, \quad \text{giving } \mathscr{H} = mc^2\left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2} + q\phi, \tag{9.195}$$

$$\text{Particle's Hamiltonian}$$

It is straightforward to verify that the Hamilton equations of motion for three Cartesian coordinates of the particle, obtained in the regular way from this $\mathscr{H}$, may be merged into the same vector equation (144). In the non-relativistic limit, performing the expansion of the latter of Eqs. (195) into the Taylor series in p^2, and limiting it to two leading terms, we get the following generalization of Eq. (74):

$$\mathscr{H} \approx mc^2 + \frac{p^2}{2m} + q\phi, \quad \text{i.e. } \mathscr{H} - mc^2 \approx \frac{1}{2m}(\mathbf{P} - q\mathbf{A})^2 + U, \quad \text{with } U = q\phi. \tag{9.196}$$

These expressions for $\mathscr{H}$, and Eq. (183) for $\mathscr{L}$, give a clear view of the electromagnetic field effects' description in analytical mechanics. The electric part $q\mathbf{E}$ of the total Lorentz force can perform work on the particle, i.e. change its kinetic energy – see Eq. (148) and its discussion. As a result, the scalar potential ϕ, whose gradient gives a contribution to $\mathbf{E}$, may be directly associated with the potential energy $U = q\phi$ of the particle. On the contrary, the magnetic component $q\mathbf{u}\times\mathbf{B}$ of the Lorentz force is always

perpendicular to particle's velocity $\mathbf{u}$, and cannot perform a non-zero work on it, and as a result, cannot be described by a contribution to U. However, if $\mathbf{A}$ did not participate in the functions $\mathscr{L}$ and/or $\mathscr{H}$ at all, the analytical mechanics would be unable to describe effects of the magnetic field $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ on
the particle's motion. The relations (183) and (195)-(196) show the wonderful way in which physics (with some help from Mother Nature herself :-) solves this problem: the vector potential gives such contributions to the functions $\mathscr{L}$ and $\mathscr{H}$ that cannot be uniquely attributed to either kinetic or potential energy, but ensure both the Lagrange and Hamilton formalisms yield the correct equation of motion (144) – including the magnetic field effects.

I believe I still owe the reader some discussion of the physical sense of the canonical momentum $\mathbf{P}$. For that, let us consider a charged particle moving near a region of localized magnetic field $\mathbf{B}(\mathbf{r},t)$, but not entering this region (see Fig. 14), so that on its trajectory $\mathbf{B}\equiv\boldsymbol{\nabla}\times\mathbf{A}=0$.

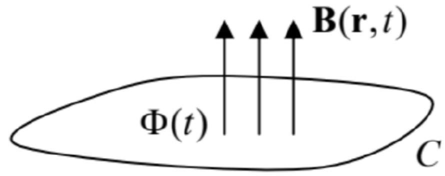

Fig. 9.14. Particle's motion around a localized magnetic field with a time-dependent flux.

If there is no electrostatic field (i.e. no other electric charges nearby), we may select such a local gauge that $\phi(\mathbf{r},t)=0$ and $\mathbf{A}=\mathbf{A}(t)$, so that Eq. (144) is reduced to

$$\frac{d\mathbf{p}}{dt}=q\mathbf{E}=-q\frac{d\mathbf{A}}{dt},\tag{9.197}$$

and Eq. (192) immediately gives

$$\frac{d\mathbf{P}}{dt}\equiv\frac{d\mathbf{p}}{dt}+q\frac{d\mathbf{A}}{dt}=0.\tag{9.198}$$

Hence, even if the magnetic field is changed in time, so that the induced electric field does accelerate the particle, its canonical momentum does not change. Hence $\mathbf{P}$ is a variable more stable to magnetic field changes than its kinetic counterpart $\mathbf{p}$. This conclusion may be criticized because it relies on a specific gauge, and generally $\mathbf{P}\equiv\mathbf{p}+q\mathbf{A}$ is not gauge–invariant, because the vector potential $\mathbf{A}$ is not.[67] However, as was already discussed in Sec. 5.3, the integral $\int\mathbf{A}\cdot d\mathbf{r}$ over a closed contour does not depend on the chosen gauge and is equal to the magnetic flux Φ through the area limited by the contour – see Eq. (5.65). So, integrating Eq. (197) over a closed trajectory of a particle (Fig. 14), and over the time of one orbit, we get

$$\Delta\oint_C\mathbf{p}\cdot d\mathbf{r}=-q\Delta\Phi,\quad\text{ so that }\Delta\oint_C\mathbf{P}\cdot d\mathbf{r}=0,\tag{9.199}$$

where $\Delta\Phi$ is the change of flux during that time. This gauge-invariant result confirms the above conclusion about the stability of the canonical momentum to magnetic field variations.

Generally, Eq. (199) is invalid if a particle moves inside a magnetic field and/or changes its trajectory at the field variation. However, if the field is almost uniform, i.e. its gradient is small in the sense of Eq. (177), this result is (approximately) applicable. Indeed, analytical mechanics[68] tells us that for any canonical coordinate-momentum pair $\{q_j,p_j\}$, the corresponding action variable,

$$J_j\equiv\frac{1}{2\pi}\oint p_jdq_j,\tag{9.200}$$

remains virtually constant at slow variations of motion conditions. According to Eq. (191), for a particle in a magnetic field, the generalized momentum corresponding to the Cartesian coordinate r_j is P_j rather than p_j. Thus forming the net action variable $J\equiv J_x+J_y+J_z$, we may write

$$2\pi J=\oint\mathbf{P}\cdot d\mathbf{r}=\oint\mathbf{p}\cdot d\mathbf{r}+q\Phi=\text{ const }.\tag{9.201}$$

Let us apply this relation to the motion of a non-relativistic particle in an almost uniform magnetic field, with a small longitudinal velocity, $\cdot u\|/u_\perp\rightarrow0$ – see Fig. 15.

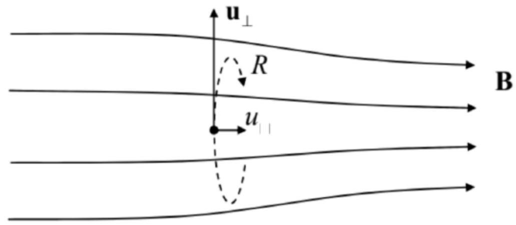

Fig. 9.15. Particle in a magnetic field with a small longitudinal gradient $\nabla B || \mathbf{B}$.

In this case, Φ in Eq. (201) is the flux encircled by a cyclotron orbit, equal to $\left(-\pi R^2 B\right)$, where R is its radius given by Eq. (153), and the negative sign accounts for the fact that in our case, the "correct" direction of the normal vector $\mathbf{n}$ in the definition of flux, $\Phi = \int \mathbf{B} \cdot \mathbf{n} d^2 r$, is antiparallel to the vector $\mathbf{B}$. At $u << c$, the kinetic momentum is just $p_\perp = m u_\perp$, while Eq. (153) yields

$$m u_\perp = qBR. \tag{9.202}$$

Plugging these relations into Eq. (201), we get

$$2\pi J = m u_\perp 2\pi R - q\pi R^2 B = m\frac{qRB}{m} 2\pi R - q\pi R^2 B \equiv (2-1) q\pi R^2 B \equiv -q\Phi. \tag{9.203}$$

This means that even if the circular orbit slowly moves through the magnetic field, the flux encircled by the cyclotron orbit should remain virtually constant. One manifestation of this effect is the result already mentioned at the end of Sec. 6: if a small gradient of the magnetic field is perpendicular to the field itself, then the particle orbit's drift direction is perpendicular to ∇B, so that Φ stays constant.

Now let us analyze the case of a small longitudinal gradient, $\nabla B || \mathbf{B}$ (Fig. 15). If a small initial longitudinal velocity $u||$ is directed toward the higher field region, the cyclotron orbit has to gradually shrink to keep Φ constant. Rewriting Eq. (202) as

$$m u_\perp = q\frac{\pi R^2 B}{\pi R} = q\frac{|\Phi|}{\pi R}, \tag{9.204}$$

we see that this reduction of R (at constant Φ) should increase the orbiting speed $u_\perp$. But since the magnetic field cannot perform any work on the particle, its kinetic energy,

$$\mathscr{E} = \frac{m}{2}\left(u_\parallel^2 + u_\perp^2\right), \tag{9.205}$$

should stay constant, so that the longitudinal velocity $u||$ has to decrease. Hence eventually the orbit's drift has to stop, and then the orbit has to start moving back toward the region of lower fields, being essentially repulsed from the high-field region. This effect is very important, in particular, for plasma confinement systems. In the simplest of such systems, two coaxial magnetic coils, inducing magnetic fields of the same direction (Fig. 16), naturally form a "magnetic bottle", which traps charged particles injected, with sufficiently low longitudinal velocities, into the region between the coils. More complex systems of this type, but working on the same basic principle, are the most essential components of the persisting large-scale efforts to achieve controllable nuclear fusion.[69]

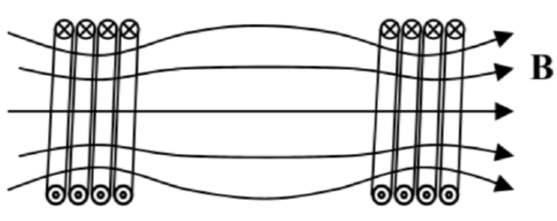

Fig. 9.16. A simple magnetic bottle (schematically).

Returning to the constancy of the magnetic flux encircled by free particles, it reminds us of the Meissner-Ochsenfeld effect, which was discussed in Sec. 6.4, and gives a motivation for a brief revisit of the electrodynamics of superconductivity. As was emphasized in that section, superconductivity is a substantially quantum phenomenon; nevertheless, the classical notion of the conjugate momentum $\mathbf{P}$ helps to understand its theoretical description. Indeed, the general rule of quantization of physical

https://phys.libretexts.org/@go/page/57041

systems[70] is that each canonical pair $\{q_j, p_j\}$ of a generalized coordinate q_j and the corresponding generalized momentum p_j is described by quantum-mechanical operators that obey the following commutation relation:

$$\left[\hat{q}_j, \hat{p}_{j'}\right] = i\hbar\delta_{jj'}. \tag{9.206}$$

According to Eq. (191), for the Cartesian coordinates r_j of a particle in the magnetic field, the corresponding generalized momenta are P_j, so that their operators should obey the following commutation relations:

$$\left[\hat{r}_j, \hat{P}_{j'}\right] = i\hbar\delta_{jj'}. \tag{9.207}$$

In the coordinate representation of quantum mechanics, the canonical operators of the Cartesian components of the linear momentum are described by the corresponding components of the vector operator $-i\hbar\nabla$. As a result, ignoring the rest energy mc^2 (which gives an inconsequential phase factor $\exp\left\{-imc^2t/\hbar\right\}$ in the wavefunction), we can use Eq. (196) to rewrite the usual non-relativistic Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{\mathscr{H}}\psi, \tag{9.208}$$

as follows:

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{\hat{p}^2}{2m} + U\right)\psi \equiv \left[\frac{1}{2m}(-i\hbar\nabla - q\mathbf{A})^2 + q\phi\right]\psi. \tag{9.209}$$

Thus, I believe I have finally delivered on my promise to justify the replacement (6.50), which had been used in Secs. 6.4 and 6.5 to discuss the electrodynamics of superconductors, including the Meissner-Ochsenfeld effect. The Schrödinger equation (209) may be also used as the basis for the quantum-mechanical description of other magnetic field phenomena, including the so-called Aharonov-Bohm and quantum Hall effects – see, e.g., QM Secs. 3.1-3.2.

Reference

[61] See, e.g., CM Sec. 2.2 and on.

[62] Alternatively called the "Lagrangian derivative"; for its (rather simple) derivation see, e.g., CM Sec. 8.3.

[63] With regrets, I have to use for the generalized momentum the same (very common) notation as was used earlier in the course for the electric polarization – which will not be discussed in the balance of these notes.

[64] In Gaussian units, Eq. (192) has the form $\mathbf{P} = \mathbf{p} + q\mathbf{A}/c$.

[65] See, e.g., CM Sec. 10.1.

[66] Alternatively, this relation may be obtained from the expression for the Lorentz-invariant norm, $p^\alpha p_\alpha = (mc)^2$, of the 4-momentum (75), $p^\alpha = \{\mathscr{E}/c, \mathbf{p}\} = \{(\mathscr{H} - q\phi)/c, \mathbf{P} - q\mathbf{A}\}$.

[67] In contrast, the kinetic momentum $\mathbf{p} = M\mathbf{u}$ is evidently gauge- (though not Lorentz-) invariant.

[68] See, e.g., CM Sec. 10.2.

[69] For further reading on this technology, the reader may be referred, for example, to the simple monograph by F. Chen, Introduction to Plasma Physics and Controllable Fusion, vol. 1, 2nd ed., Springer, 1984, and/or the graduate-level theoretical treatment by R. Hazeltine and J. Meiss, Plasma Confinement, Dover, 2003.

[70] See, e.g., CM Sec. 10.1.

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