7.9: Energy Loss Effects

The inevitable energy losses ("dissipation") in passive media lead, in two different situations, to two different effects. In a long transmission line fed by a constant wave source, the losses lead to a gradual attenuation of the wave, i.e. to a decrease of its amplitude, and hence its power $\mathscr{P}$, with the distance z from the source. In linear materials, the power losses $\mathscr{P}_{\text{loss}}$ are proportional to the time-averaged power $\mathscr{P}$ carried by the wave, so that the energy balance on a small segment dz takes the form

$$d\mathscr{P} = -\frac{d\mathscr{P}_{\text{loss}}}{dz}dz \equiv -\alpha\mathscr{P}dz. \tag{7.213}$$

The coefficient α participating in the last form of Eq. (213), and hence defined as

$$\alpha \equiv \frac{d\mathscr{P}_{\text{loss}}/dz}{\mathscr{P}}, \tag{7.214}$$

is called the attenuation constant.[85] Comparing the solution of Eq. (213),

$$\mathscr{P}(z) = \mathscr{P}(0)e^{-\alpha z} \qquad \text{Wave's attenuation} \tag{7.215}$$

with Eq. (29), where k is replaced with k_z, we see that α may be expressed as

$$\alpha = 2\,\text{Im}\,k_z, \tag{7.216}$$

where k_z is the component of the wave vector along the transmission line. In the most important limit when the losses are low in the sense $\alpha << |k_z| \approx \text{Re}\,k_z$, its effects on the field distribution along the line's cross-section are negligible, making the calculation of α rather straightforward. In particular, in this limit the contributions to attenuation from two major sources, the energy losses in the filling dielectric, and the skin effect-losses in conducting walls, are independent and additive.

The dielectric losses are especially simple to describe. Indeed, a review of our calculations in Secs. 5-7 shows that all of them remain valid if either $\varepsilon(\omega)$, or $\mu(\omega)$, or both, and hence $k(\omega)$, have small imaginary parts:

$$k'' = \omega\,\text{Im}\left[\varepsilon^{1/2}(\omega)\mu^{1/2}(\omega)\right] << k'. \qquad \text{Attenuation due to filling} \tag{7.218}$$

For dielectric waveguides, in particular optical fibers, these losses are the main attenuation mechanism. As was discussed in Sec. 7, in practical optical fibers $\kappa_t R >> 1$, i.e. most of the field propagates (as an evanescent wave) in the cladding, with a field distribution very close to the TEM wave. This is why Eq. (218) is approximately valid if it is applied to the cladding material alone. In waveguides with non-TEM waves, we can use the relations between k_z and k, derived in the previous sections, to re-calculate k'' into Im k_z. (Note that at this recalculation, the values of k_t have to be kept real, because they are just the eigenvalues of the Helmholtz equation (101), which does not include the filling media parameters.).

In transmission lines and waveguides and with metallic walls, higher energy losses may come from the skin effect. If the wavelength λ is much larger than δ_s, as it usually is,[86] the losses may be readily evaluated using Eq. (6.36):

$$\frac{d\mathscr{P}_{\text{loss}}}{dA} = H_{\text{wall}}^2\frac{\mu\omega\delta_s}{4}, \tag{7.219}$$

where H_{wall} is the real amplitude of the tangential component of the magnetic field at the wall's surface. The total power loss $\mathscr{P}_{\text{loss}}/dz$ per unit length of a waveguide, i.e. the right-hand side of Eq. (213), now may be calculated by the integration of this $d\mathscr{P}_{\text{loss}}/dA$ along the contour(s) limiting the cross-section of all conducting walls. Since our calculation is only valid for low losses, we may ignore their effect on the field distribution, so that the unperturbed distributions may be used both in Eq. (219), i.e. in the numerator of Eq. (214), and also for the calculation of the average propagating power, i.e. the denominator of Eq. (214) – as the integral of the Poynting vector over the cross-section of the waveguide.

Let us see how this approach works for the TEM mode in one of the simplest transmission lines, the coaxial cable (Fig. 20). As we already know from Sec. 5, in the coarse-grain approximation, implying negligible power loss, the TEM mode field distribution between the two conductors are the same as in statics, namely:

$$H_z = 0, \quad H_\rho = 0, \quad H_\varphi(\rho) = H_0\frac{a}{\rho}, \tag{7.220}$$

where H_0 is the field's amplitude on the surface of the inner conductor, and

$$E_z = 0, \quad E_\rho(\rho) = ZH_\varphi(\rho) = ZH_0\frac{a}{\rho}, \quad E_\varphi = 0, \quad \text{where } Z \equiv \left(\frac{\mu}{\varepsilon}\right)^{1/2}. \tag{7.221}$$

Neglecting the power losses for now, we may plug these expressions into Eq. (42) to calculate the time-averaged Poynting vector:

$$\bar{S} = \frac{Z|H_\varphi(\rho)|^2}{2} = \frac{Z|H_0|^2}{2}\left(\frac{a}{\rho}\right)^2, \tag{7.222}$$

and from it, the total wave power flow through the cross-section:

$$\mathscr{P} = \int_A \bar{S}d^2r = \frac{Z|H_0|^2a^2}{2}2\pi\int_a^b \frac{\rho d\rho}{\rho^2} = \pi Z|H_0|^2a^2\ln\frac{b}{a}. \tag{7.223}$$

Next, for the particular case of the coaxial cable (Fig. 20), the contours limiting the wall cross-section are circles of radii $\rho = a$ (where the surface field amplitude H_{walls} equals, in our notation, H_0), and $\rho = b$ (where, according to Eq. (214), the field is a factor of b/a lower). As a result, for the power loss per unit length, Eq. (219) yields

$$\frac{d\mathscr{P}_{\text{loss}}}{dz} = \oint_{C_a+C_b}\frac{d\mathscr{P}_{\text{loss}}}{dA}dl = \left(2\pi a|H_0|^2 + 2\pi b\left|H_0\frac{a}{b}\right|^2\right)\frac{\mu_0\omega\delta_s}{4} = \frac{\pi}{2}a\left(1+\frac{a}{b}\right)\mu\omega\delta_s|H_0|^2. \tag{7.224}$$

Note that at $a << b$, the losses in the inner conductor dominate, despite its smaller surface, because of the higher surface field.

Now we may plug Eqs. (223) and (224) into the definition (214) of α, to calculate the skin-effect contribution to the attenuation constant:

$$\alpha_{\text{skin}} \equiv \frac{d\mathscr{P}_{\text{loss}}/dz}{\mathscr{P}} = \frac{1}{2\ln(b/a)}\left(\frac{1}{a}+\frac{1}{b}\right)\frac{\mu\omega\delta_s}{Z} \equiv \frac{k\delta_s}{2\ln(b/a)}\left(\frac{1}{a}+\frac{1}{b}\right). \tag{7.225}$$

This result shows that the relative (dimensionless) attenuation, α/k, scales approximately as the ratio $\delta_s/\min[a,b]$, in a semi-quantitative agreement with Eq. (78).

Let us use this result to evaluate α for the standard TV cable RG-6/U, with copper conductors of diameters $2a = 1$ mm, $2b = 4.7$ mm, and $\varepsilon \approx 2.2\varepsilon_0$ and $\mu \approx \mu_0$. According to Eq. (6.33), for $f = 100$MHz (i.e. $\omega \approx 6.3\times10^8$ s^{-1}) the skin depth of pure copper at room temperature (with $\sigma \approx 6.0\times10^7$ S/m) is close to 6.5×10^{-6} m, while $k = \omega(\varepsilon\mu)^{1/2} = (\varepsilon/\varepsilon_0)^{1/2}(\omega/c) \approx 3.1$ m^{-1}. As a result, the attenuation is rather low: $\alpha_{\text{skin}} \approx 0.016$ m^{-1}, so that the attenuation length scale $l_d \equiv 1/\alpha$ is about 60 m. Hence the attenuation in a cable connecting a roof TV antenna to a TV set in the same house is not a big problem, though using a worse conductor, e.g., steel, would make the losses rather noticeable. (Hence the current worldwide shortage of copper.) However, the use of such cable in the X-band $(f \sim 10\text{GHz})$ is more problematic. Indeed, though the skin depth $\delta_s \propto \omega^{-1/2}$ decreases with frequency, the wavelength drops, i.e. k increases, even faster ($k \propto \omega$), so that the attenuation $\alpha_{\text{skin}} \propto \omega^{1/2}$ becomes close to 0.16 m^{-1}, i.e. l_d to ~6 m. This is why at such frequencies, it may be necessary to use rectangular waveguides, with their larger internal dimensions a, $b \sim 1/k$, and hence lower attenuation. Let me leave the calculation of this attenuation, using Eq. (219) and the results derived in Sec. 7, for the reader's exercise.

The effect of dissipation on free oscillations in resonators is different: here it leads to a gradual decay of the oscillating fields' energy U in time. A useful dimensionless measure of this decay, called the Q factor, is commonly defined by writing the following temporal analog of Eq. (213):[87]

$$dU = -\mathscr{P}_{\text{loss}}\,dt \equiv -\frac{\omega}{Q}Udt, \tag{7.226}$$

where ω in the eigenfrequency in the loss-free limit, and

$$\frac{\omega}{Q} \equiv \frac{\mathscr{P}_{\text{loss}}}{U} \qquad \text{Q-factor} \tag{7.227}$$

The solution of Eq. (226),

$$U(t) = U(0)e^{-t/\tau}, \quad \text{with } \tau \equiv \frac{Q}{\omega} \equiv \frac{Q/2\pi}{\omega/2\pi} = \frac{Q^{\mathcal{T}}}{2\pi}, \tag{7.228}$$

which is the temporal analog of Eq. (215), shows the physical meaning of the Q-factor: the characteristic time τ of the oscillation energy's decay is $(Q/2\pi)$ times longer than the oscillation period $\mathcal{T}=2\pi/\omega$. (Another useful interpretation of Q comes from the universal relation[88]

$$Q=\frac{\omega}{\Delta\omega},\tag{7.229}$$

where $\Delta\omega$ is the so-called FWHM [89] bandwidth of the resonance, namely the difference between the two values of the external signal frequency, one above and one below ω, at which the energy of the oscillations induced in the resonator by an input signal is twice lower than its resonance value.)

In the important particular case of resonators formed by the insertion of metallic walls into a TEM transmission line of small cross-section (with the linear size scale a much less than the wavelength λ), there is no need to calculate the Q-factor directly, provided that the line attenuation coefficient α is already known. In fact, as was discussed in Sec. 8 above, the standing waves in such a resonator, of the length given by Eq. (196): $l=p(\lambda/2)$ with $p=1,2,\ldots$, may be understood as an overlap of two TEM waves running in opposite directions, or in other words, a traveling wave plus its reflection from one of the ends, the whole roundtrip taking time $\Delta t=2l/\nu=p\lambda/\nu=2\pi p/\omega=p\mathcal{T}$. According to Eq. (215), at this distance, the wave's power drops by a factor of $\exp\{-2\alpha l\}=\exp\{-p\alpha\lambda\}$. On the other hand, the same decay may be viewed as taking place in time, and according to Eq. (228), results in the drop by a factor of $\exp\{-\Delta t/\tau\}=\exp\{-(p\mathcal{T})/(Q/\omega)\}=\exp\{-2\pi p/Q\}$. Comparing these two exponents, we get

$$Q\text{ vs.}\alpha\qquad\qquad Q=\frac{2\pi}{\alpha\lambda}=\frac{k}{\alpha}.\tag{7.230}$$

This simple relation neglects the losses at the wave reflection from the walls limiting the resonator length. Such approximation is indeed legitimate at $a<<\lambda$; if this relation is violated, or if we are dealing with more complex resonator modes (such as those based on the reflection of E or H waves), the Q-factor may be different from that given by Eq. (230), and needs to be calculated directly from Eq. (227). A substantial relief for such a direct calculation is that, just at the calculation of small attenuation in waveguides, in the low-loss limit $(Q>>1)$, both the numerator and denominator of the right-hand side of that formula may be calculated neglecting the effects of the power loss on the field distribution in the resonator. I am leaving such a calculation, for the simplest (rectangular and circular) resonators, for the reader's exercise.

To conclude this chapter, the last remark: in some resonators (including certain dielectric resonators and metallic resonators with holes in their walls), additional losses due to the wave radiation into the environment are also possible. In some simple cases (say, the Fabry-Pérot interferometer shown in Fig. 31) the calculation of these radiative losses is straightforward, but sometimes it requires more elaborated approaches that will be discussed in the next chapter.

Reference

[85] In engineering, attenuation is frequently measured in decibels per meter, abbreviated as db/m (the term not to be confused with dBm standing for decibel-milliwatt):

$$\alpha|_{\text{db/m}}\equiv 10\log_{10}\frac{\mathscr{P}(z)}{\mathscr{P}(z+1\text{ m})}=10\log_{10}e^{\alpha[1/\text{m}]}\equiv\frac{10}{\ln 10}\alpha\left[\text{m}^{-1}\right]\approx 4.34\alpha\left[\text{m}^{-1}\right].$$

[86] As follows from Eq. (78), which may be used for crude estimates even in cases of arbitrary wave incidence, this condition is necessary for low attenuation: $\alpha<<k$ only if $\mathscr{F}<<1$.

[87] As losses grow, the oscillation waveform deviates from the sinusoidal one, and the very notion of "oscillation frequency" becomes vague. As a result, the parameter Q is well defined only if it is much higher than 1.

[88] See, e.g., CM Sec. 5.1.

[89] This is the acronym for "Full Width at Half-Maximum".