10.6: Radiation's Back-action

An attentive and critically-minded reader could notice that so far our treatment of charged particle dynamics has never been fully self-consistent. Indeed, in Sec. 9.6 we have analyzed particle's motion in various external fields, ignoring those radiated by particle itself, while in Sec. 8.2 and earlier in this chapter these fields have been calculated (admittedly, just for a few simple cases), but, again, their back-action on the emitting particle has been ignored. Only in very few cases we have taken the back effects of the radiation implicitly, via the energy conservation arguments. However, even in these cases, the near-field effects, such as the first term in Eq. (19), which affect the moving particle most, have been ignored.

At the same time, it is clear that in sharp contrast with electrostatics, the interaction of a moving point charge with its own field cannot be always ignored. As the simplest example, if an electron is made to fly through a resonant cavity, thus inducing electromagnetic oscillations in it, and then is forced (say, by an appropriate static field) to return into the cavity before the oscillations have decayed, its motion will certainly be affected by the oscillating fields, just as if they had been induced by another source. There is no conceptual problem with applying the Maxwell theory to such "field-particle rendezvous" effects; moreover, it is the basis of the engineering design of such vacuum electron devices as klystrons, magnetrons, and free-electron lasers.

A problem arises only when no clear "rendezvous" points are enforced by boundary conditions, so that the most important self-field effects are at $R \equiv |\mathbf{r}-\mathbf{r}'| \rightarrow 0$, the most evident example being the charged particle's radiation into free space, described earlier in this chapter. We already know that such radiation takes away a part of the charge's kinetic energy, i.e. has to cause its deceleration. One should wonder, however, whether such self-action effects might be described in a more direct, non-perturbative way.

As the first attempt, let us try a phenomenological approach based on the already derived formulas for the radiation power $\mathscr{P}$. For the sake of simplicity, let us consider a non-relativistic point charge q in free space, so that $\mathscr{P}$ is described by Eq. (8.27), with the electric dipole moment's derivative over time equal to $q\mathbf{u}$:

$$\mathscr{P}=\frac{Z_0 q^2}{6\pi c^2}\dot{u}^2 \equiv \frac{2}{3c^3}\frac{q^2}{4\pi\varepsilon_0}\dot{u}^2. \tag{10.133}$$

The most naïve approach would be to write the equation of particle's motion in the form

$$m\dot{\mathbf{u}}=\mathbf{F}_{\text{ext}}+\mathbf{F}_{\text{self}}, \tag{10.134}$$

and try to calculate the radiation back-action force Fself by requiring its instant power, $-\mathbf{F}_{\text{self}}\,\mathbf{u}$, to be equal to $\mathscr{P}$. However, this approach (say, for a 1D motion) would give a very unnatural result,

$$F_{\text{self}} \propto \frac{\dot{u}^2}{u}, \tag{10.135}$$

that might diverge at some points of the particle's trajectory. This failure is clearly due to the retardation effect: as the reader may recall, Eq. (133) results from the analysis of radiation fields in the far-field zone, i.e. at large distances R from the particle, e.g., from the second term in Eq. (19), i.e. when the non-radiative first term (which is much larger at small distances, $R \rightarrow 0$) is ignored.

Before exploring the effects of this term, let us, however, make one more try at Eq. (133), considering its average effect on some periodic motion of the particle. (A possible argument for this step is that at the periodic motion, the retardation effects should be averaged out – just at the transfer from Eq. (8.27) to Eq. (8.28).) To calculate the average, let us write

$$\overline{\dot{u}^2} \equiv \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \dot{\mathbf{u}}\cdot\dot{\mathbf{u}}dt, \tag{10.136}$$

and carry out the integration on the right-hand side of this identity by parts over the motion period $\mathcal{T}$:

$$\overline{\mathscr{P}}=\frac{2}{3c^3}\frac{q^2}{4\pi\varepsilon_0}\overline{(\dot{\mathbf{u}})^2}=\frac{2}{3c^3}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{\mathcal{T}}\left(\dot{\mathbf{u}}\cdot\mathbf{u}\Big|_0^{\mathcal{T}}-\int_0^{\mathcal{T}}\ddot{\mathbf{u}}\cdot\mathbf{u}dt\right)=-\frac{1}{\tau}\int_0^{\mathcal{T}}\frac{2}{3c^3}\frac{q^2}{4\pi\varepsilon_0}\ddot{\mathbf{u}}\cdot\mathbf{u}dt. \tag{10.137}$$

One the other hand, the back-action force should give

$$\overline{\mathscr{P}}=-\frac{1}{\mathcal{T}}\int_0^{\mathcal{T}}\mathbf{F}_{\text{self}}\cdot\mathbf{u}dt. \tag{10.138}$$

These two averages coincide if[53]

$$\text{Abraham-Lorentz force} \qquad \mathbf{F}_{\text{self}} = \frac{2}{3c^3}\frac{q^2}{4\pi\varepsilon_0}\ddot{\mathbf{u}}. \tag{10.139}$$

This is the so-called Abraham-Lorentz force for self-action. Before going after a more serious derivation of this formula, let us estimate its scale, representing Eq. (139) as

$$\mathbf{F}_{\text{self}} = m\tau\ddot{\mathbf{u}}, \quad \text{with } \tau \equiv \frac{2}{3mc^3}\frac{q^2}{4\pi\varepsilon_0}, \tag{10.140}$$

where the constant τ evidently has the dimension of time. Recalling the definition (8.41) of the classical radius r_c of the particle, Eq. (140) for τ may be rewritten as

$$\tau = \frac{2}{3}\frac{r_c}{c}. \tag{10.141}$$

For the electron, τ is of the order of 10^{-23} s, so that the right-hand side of Eq. (139) is very small. This means that in most cases the Abrahams-Lorentz force is either negligible or leads to the same results as the perturbative treatments of energy loss we have used earlier in this chapter.

However, Eq. (140) brings some unpleasant surprises. For example, let us consider a 1D oscillator with the own frequency ω_0. For it, Eq. (134), with the back-action force given by Eq. (140), takes the form

$$m\ddot{x} + m\omega_0^2 x = m\tau\dddot{x}. \tag{10.142}$$

Looking for the solution to this linear differential equation in the usual exponential form, $x(t) \propto \exp\{\lambda t\}$, we get the following characteristic equation,

$$\lambda^2 + \omega_0^2 = \tau\lambda^3. \tag{10.143}$$

It may look like that for any "reasonable" value of $\omega_0 << 1/\tau \sim 10^{23}\ \text{s}^{-1}$, the right-hand side of this nonlinear algebraic equation may be treated as a perturbation. Indeed, looking for its solutions in the natural form $\lambda_\pm = \pm i\omega_0 + \lambda'$, with $|\lambda'| << \omega_0$, expanding both parts of Eq. (143) in the Taylor series in the small parameter λ', and keeping only the terms linear in λ', we get

$$\lambda' \approx -\frac{\omega_0^2\tau}{2}. \tag{10.144}$$

This means that the energy of free oscillations decreases in time as $\exp\{2\lambda' t\} = \exp\{-\omega_0^2\tau t\}$; this is exactly the radiative damping analyzed earlier. However, Eq. (143) is deceiving; it has the third root corresponding to unphysical, exponentially growing (so-called run-away) solutions. It is easiest to see this for a free particle, with $\omega_0 = 0$. Then Eq. (143) becomes very simple,

$$\lambda^2 = \tau\lambda^3, \tag{10.145}$$

and it is easy to find all its 3 roots explicitly: $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1/\tau$. While the first two roots correspond to the values $\lambda_\pm$ found earlier, the last one describes an exponential (and extremely rapid!) acceleration.

In order to remove this artifact, let us try to develop a self-consistent approach to the back-action effects, taking into account the near-field terms of particle fields. For that, we need to somehow overcome the divergence of Eqs. (10) and (19) at $R \to 0$. The most reasonable way to do this is to spread the particle's charge over a ball of radius a, with a spherically-symmetric (but not necessarily constant) density $\rho(r)$, and in the end of calculations trace the limit $a \to 0$.[54] Again sticking to the non-relativistic case (so that the magnetic component of the Lorentz force is not important), we should calculate

$$\mathbf{F}_{\text{self}} = \int_V \rho(\mathbf{r})\mathbf{E}(\mathbf{r},t)d^3r, \tag{10.146}$$

where the electric field is that of the charge itself, with the field of any elementary charge $dq = \rho(r)d^3r$ described by Eq. (19).

To enable an analytical calculation of the force, we need to make the assumption $a << r_c$, treat the ratio $R/r_c \sim a/r_c$ as a small parameter, and expand the resulting the right-hand side of Eq. (146) into the Taylor series in small R. This procedure yields

$$\mathbf{F}_{\text{self}} = -\frac{2}{3}\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^{n+2}n!}\frac{d^{n+1}\mathbf{u}}{dt^{n+1}}\int_V d^3r\int_V d^3r'\rho(r)R^{n-1}\rho(r'). \tag{10.147}$$

The distance R cancels only in the term with $n=1$,

$$\mathbf{F}_1 = \frac{2}{3c^3}\frac{\ddot{\mathbf{u}}}{4\pi\varepsilon_0}\int_V d^3r\int_V d^3r'\rho(r)\rho(r') \equiv \frac{2}{3c^3}\frac{q^2}{4\pi\varepsilon_0}\ddot{\mathbf{u}}, \tag{10.148}$$

showing that we have recovered (now in an apparently legitimate fashion) Eq. (139) for the Abrahams-Lorentz force. One could argue that in the limit $a \to 0$ the terms higher in $R \sim a$ (with $n > 1$) could be ignored. However, we have to notice that the main contribution into the series (147) is not described by Eq. (148) for $n=1$, but is given by the much larger term with $n=0$:

$$\mathbf{F}_0 = -\frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{\dot{\mathbf{u}}}{c^2}\int_V d^3r\int_V d^3r'\frac{\rho(r)\rho(r')}{R} \equiv -\frac{4}{3}\frac{\dot{\mathbf{u}}}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int_V d^3r\int_V d^3r'\frac{\rho(r)\rho(r')}{R} \equiv -\frac{4}{3c^2}\dot{\mathbf{u}}U, \tag{10.149}$$

where U is the electrostatic energy (1.59) of the static charge's self–interaction. This term may be interpreted as the inertial "force" [55] $(-m_{\text{ef}}\mathbf{a})$ with the following effective electromagnetic mass:

$$\text{Electro-magnetic mass} \qquad m_{\text{ef}} = \frac{4}{3}\frac{U}{c^2}. \tag{10.150}$$

This is the famous (or rather infamous :-) 4/3 problem that does not allow one to interpret the electron's mass as that of its electric field. An (admittedly, rather formal) resolution of this paradox is possible only in quantum electrodynamics with its renormalization techniques – beyond the framework of this course.

Note, however, that all these issues are only important for motions with frequencies of the order of $1/\tau \sim 10^{23}\ \text{s}^{-1}$, i.e. at energies $\mathscr{E} \sim \hbar/\tau \sim 10^8\,\text{eV}$, while other quantum electrodynamics effects may be observed at much lower frequencies, starting from $\sim 10^{10}\ \text{s}^{-1}$. Hence the 4/3 problem is by no means the only motivation for the transfer from classical to quantum electrodynamics. However, the reader should not think that their time spent on this course has been lost: quantum electrodynamics incorporates virtually all results of classical electrodynamics, and the basic transition to it is surprisingly straightforward.[56] So, I look forward to welcoming the reader to the next, quantum-mechanics part of this series.

Reference

53 Just for the reader's reference, this formula may be readily generalized to the relativistic case, in the 4-form:

$$F^{\alpha}_{\text{self}} = \frac{2}{3mc^3}\frac{q^2}{4\pi\varepsilon_0}\left[\frac{d^2p^\alpha}{d\tau^2} + \frac{p^\alpha}{(mc)^2}\left(\frac{dp_\beta}{d\tau}\frac{dp^\beta}{d\tau}\right)\right],$$

-the so-called Abraham-Lorentz-Dirac force.

54 Note: this operation cannot be interpreted as describing a quantum spread due to the finite extent of the point particle's wavefunction. In quantum mechanics, different parts of the wavefunction of the same charged particle do not interact with each other!

55 See, e.g., CM Sec. 4.6.

56 See, e.g., QM Secs. 9.1-9.4.