1.3: Scalar Potential and Electric Field Energy

One more help for solving electrostatics (and more complex) problems may be obtained from the notion of the electrostatic potential, which is just the electrostatic potential energy U of a probe point charge q placed into the field in question, normalized by its charge:

$$\phi \equiv \frac{U}{q}. \qquad \text{Electrostatic potential} \tag{1.31}$$

As we know from classical mechanics,[15] the notion of U (and hence ϕ) makes the most sense for the case of potential forces, for example those depending just on the particle's position. Eqs. (6) and (9) show that, in stationary situations, the electric field falls into this category. For such a field, the potential energy may be defined as a scalar function $U(\mathbf{r})$ that allows the force to be calculated as its gradient (with the opposite sign):

$$\mathbf{F} = -\nabla U. \tag{1.32}$$

Dividing both sides of this equation by the probe charge, and using Eqs. (6) and (31), we get[16]

$$\mathbf{E} = -\nabla\phi. \qquad \text{Electrostatic field as a greadient} \tag{1.33}$$

To calculate the scalar potential, let us start from the simplest case of a single point charge q placed at the origin. For it, Eq. (7) takes the simple form

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} q\frac{\mathbf{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} q\frac{\mathbf{n}_r}{r^2}. \tag{1.34}$$

It is straightforward to verify that the last fraction in the last form of Eq. (34) is equal to $-\nabla(1/r)$.[17] Hence, according to the definition (33), for this particular case

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \qquad \text{Potential of a point charge} \tag{1.35}$$

(In the Gaussian units, this result is spectacularly simple: $\phi = q/r$.) Note that we could add an arbitrary constant to this potential (and indeed to any other distribution of ϕ discussed below) without changing the field, but it is convenient to define the potential energy to approach zero at infinity.

In order to justify the introduction and the forthcoming exploration of U and ϕ, let me demonstrate (I hope, unnecessarily :-) how useful the notions are, on a very simple example. Let two similar charges q be launched from afar, with the same initial speed $v_0 << c$ each, straight toward each other (i.e. with the zero impact parameter) – see Fig. 5. Since, according to the Coulomb law, the charges repel each other with increasing force, they will stop at some minimum distance $r_{\min}$ from each other, and then fly back. We could of course find rmin directly from the Coulomb law. However, for that we would need to write the 2^{nd} Newton law for each particle (actually, due to the problem symmetry, they would be similar), then integrate them over time to find the particle velocity v as a function of distance, and then recover $r_{\min}$ from the requirement $v = 0$.

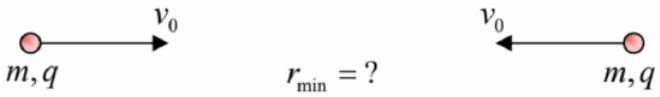

Fig. 1.5. A simple problem of charged particle motion.

The notion of potential allows this problem to be solved in one line. Indeed, in the field of potential forces, the system's total energy $\mathcal{E} = T + U \equiv T + q\phi$ is conserved. In our non-relativistic case $v << c$, the kinetic energy T is just $mv^2/2$. Hence, equating the total energy of two particles at the points $r = \infty$ and $r = r_{\min}$, and using Eq. (35) for ϕ, we get

$$2\frac{mv_0^2}{2} + 0 = 0 + \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r_{\min}}, \tag{1.36}$$

immediately giving us the final answer: $r_{\min} = q^2/4\pi\varepsilon_0 m v_0^2$. So, the notion of scalar potential is indeed very useful.

With this motivation, let us calculate ϕ for an arbitrary configuration of charges. For a single charge in an arbitrary position (say, $\mathbf{r}_{k'}$), $r \equiv |\mathbf{r}|$ in Eq. (35) should be evidently replaced with $|\mathbf{r} - \mathbf{r}_{k'}|$. Now, the linear superposition principle (3) allows for an easy generalization of this formula to the case of an arbitrary set of discrete charges,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\mathbf{r}_{k'}\neq\mathbf{r}}\frac{q_{k'}}{|\mathbf{r}-\mathbf{r}_{k'}|}. \tag{1.37}$$

Finally, using the same arguments as in Sec. 1, we can use this result to argue that in the case of an arbitrary continuous charge distribution

$$\text{Potential of a charge distribution} \qquad \phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{1.38}$$

Again, Dirac's delta function allows using the last equation to recover Eq. (37) for discrete charges as well, so that Eq. (38) may be considered as the general expression for the electrostatic potential.

For most practical calculations, using this expression and then applying Eq. (33) to the result, is preferable to using Eq. (9), because ϕ is a scalar, while $\mathbf{E}$ is a 3D vector, mathematically equivalent to three scalars. Still, this approach may lead to technical problems similar to those discussed in Sec. 2. For example, applying it to the spherically-symmetric distribution of charge (Fig. 2), we get the integral

$$\phi = \frac{1}{4\pi\varepsilon_0}2\pi\int_0^{\pi}\sin\theta'd\theta'\int_0^{\infty}r'^2dr'\frac{\rho(r')}{R}\cos\theta, \tag{1.39}$$

which is not much simpler than Eq. (11).

The situation may be much improved by recasting Eq. (38) into a differential form. For that, it is sufficient to plug the definition of ϕ, Eq. (33), into Eq. (27):

$$\nabla\cdot(-\nabla\phi) = \frac{\rho}{\varepsilon_0}. \tag{1.40}$$

The left-hand side of this equation is nothing else than the Laplace operator of ϕ (with the minus sign), so that we get the famous Poisson equation[18] for the electrostatic potential:

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}. \qquad \text{Poisson equation for } \phi \tag{1.41}$$

(In the Gaussian units, the Poisson equation is $\nabla^2\phi = -4\pi\rho$.) This differential equation is so convenient for applications that even its particular case for $\rho = 0$,

$$\nabla^2\phi = 0, \qquad \text{Laplace equation for } \phi \tag{1.42}$$

has earned a special name – the Laplace equation.[19]

In order to get a gut feeling of the Poisson equation's value as a problem-solving tool, let us return to the spherically-symmetric charge distribution (Fig. 2) with a constant charge density ρ. Using the symmetry, we can represent the potential as $\phi(r)$, and hence use the following simple expression for its Laplace operator:[20]

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right), \tag{1.43}$$

so that for the points inside the charged sphere $(r \le R)$ the Poisson equation yields

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{\rho}{\varepsilon_0}, \quad \text{i.e. } \frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -\frac{\rho}{\varepsilon_0}r^2. \tag{1.44}$$

1.3: Scalar Potential and Electric Field Energy

Integrating the last form of the equation over r once, with the natural boundary condition $d\phi/dr|_{r=0} = 0$ (because of the condition $E(0) = 0$, which has been discussed above), we get

$$\frac{d\phi}{dr}(r) = -\frac{\rho}{r^2\varepsilon_0}\int_0^r r'^2dr' = -\frac{\rho r}{3\varepsilon_0} \equiv -\frac{1}{4\pi\varepsilon_0}\frac{Qr}{R^3}. \tag{1.45}$$

Since this derivative is nothing more than $-E(r)$, in this formula we can readily recognize our previous result (22). Now we may like to carry out the second integration to calculate the potential itself:

$$\phi(r) = -\frac{Q}{4\pi\varepsilon_0 R^3}\int_0^r r'dr' + c_1 = -\frac{Qr^2}{8\pi\varepsilon_0 R^3} + c_1. \tag{1.46}$$

Before making any judgment on the integration constant c_1, let us solve the Poisson equation (in this case, just the Laplace equation) for the range outside the sphere $(r > R)$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0. \tag{1.47}$$

Its first integral,

$$\frac{d\phi}{dr}(r) = \frac{c_2}{r^2}, \tag{1.48}$$

also gives the electric field (with the minus sign). Now using Eq. (45) and requiring the field to be continuous at $r = R$, we get

$$\frac{c_2}{R^2} = -\frac{Q}{4\pi\varepsilon_0 R^2}, \quad \text{i.e. } \frac{d\phi}{dr}(r) = -\frac{Q}{4\pi\varepsilon_0 r^2}, \tag{1.49}$$

in an evident agreement with Eq. (19). Integrating this result again,

$$\phi(r) = -\frac{Q}{4\pi\varepsilon_0}\int\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0 r} + c_3, \quad \text{for } r > R, \tag{1.50}$$

we can select $c_3 = 0$, so that $\phi(\infty) = 0$, in accordance with the usual (though not compulsory) convention. Now we can finally determine the constant c_1 in Eq. (46) by requiring that this equation and Eq. (50) give the same value of ϕ at the boundary $r = R$. (According to Eq. (33), if the potential had a jump, the electric field at that point would be infinite.) The final answer may be represented as

$$\phi(r) = \frac{Q}{4\pi\varepsilon_0 R}\left(\frac{R^2 - r^2}{2R^2} + 1\right), \quad \text{for } r \le R \tag{1.51}$$

This calculation shows that using the Poisson equation to find the electrostatic potential distribution for highly symmetric problems may be more cumbersome than directly finding the electric field – say, from the Gauss law. However, we will repeatedly see below that if the electric charge distribution is not fixed in advance, using Eq. (41) may be the only practicable way to proceed.

Returning now to the general theory of electrostatic phenomena, let us calculate the potential energy U of an arbitrary system of point electric charges q_k. Despite the apparently simple relation (31) between U and ϕ, the result is not that straightforward. Indeed, let us assume that the charge distribution has a finite spatial extent, so that at large distances from it (formally, at $\mathbf{r} = \infty$) the electric field tends to zero, so that the electrostatic potential tends to a constant. Selecting this constant, for convenience, to equal zero, we may calculate U as a sum of the energy increments ΔU_k created by bringing the charges, one by one, from infinity to their final positions $\mathbf{r}_k$ – see Fig. 6.[21] According to the integral form of Eq. (32), such a contribution is

$$\Delta U_k = -\int_\infty^{\mathbf{r}_k}\mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = -q_k\int_\infty^{\mathbf{r}_k}\mathbf{E}(\mathbf{r})\cdot d\mathbf{r} \equiv q_k\phi(\mathbf{r}_k), \tag{1.52}$$

where $\mathbf{E}(\mathbf{r})$ is the total electric field, and $\phi(\mathbf{r})$ is the total electrostatic potential during this process, besides the field created by the very charge q_k that is being moved.

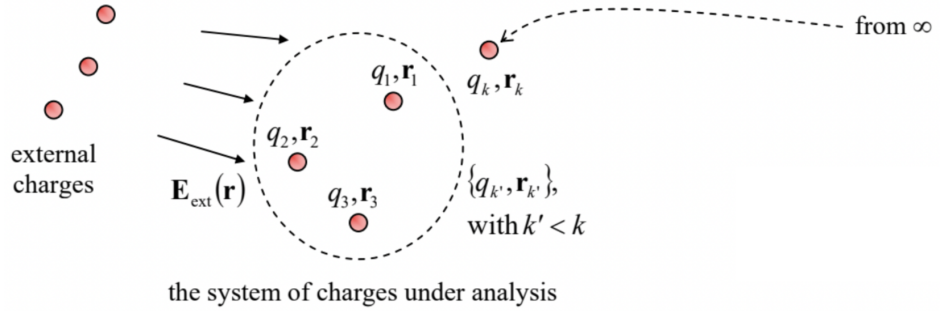

Fig. 1.6. Deriving Eqs. (55) and (60) for the potential energies of a system of several point charges.

This expression shows that the increment ΔU_k, and hence the total potential energy U, depend on the source of the electric field **E**. If the field is dominated by an external field $\mathbf{E}_{\text{ext}}$, induced by some external charges, not being a part of the charge configuration under our analysis (whose energy we are calculating, see Fig. 6), then the spatial distribution $\phi(\mathbf{r})$ is determined by this field, i.e. does not depend on how many charges we have already brought in, so that Eq. (52) is reduced to

$$\Delta U_k = q_k \phi_{\text{ext}}(\mathbf{r}_k), \qquad \text{where } \phi_{\text{ext}}(\mathbf{r}) \equiv -\int_{\infty}^{\mathbf{r}} E_{\text{ext}}(\mathbf{r}') \cdot d\mathbf{r}'. \tag{1.53}$$

Summing up these contributions, we get what is called the charge system's energy in the external field:[22]

$$U_{\text{ext}} \equiv \sum_k \Delta U_k = \sum_k q_k \phi_{\text{ext}}(\mathbf{r}_k). \tag{1.54}$$

Now repeating the argumentation that has led us to Eq. (9), we see that for a continuously distributed charge, this sum turns into an integral:

$$U_{\text{ext}} = \int \rho(\mathbf{r}) \phi_{\text{ext}}(\mathbf{r}) d^3 r. \qquad \text{Energy: external field} \tag{1.55}$$

(As was discussed above, using the delta-functional representation of point charges, we may always return from here to Eq. (54), so that Eq. (55) may be considered as a final, universal result.)

The result is different in the opposite limit, when the electric field **E(r)** is created only by the very charges whose energy we are calculating. In this case, $\phi(\mathbf{r}_k)$ in Eq. (52) is the potential created only by the charges with numbers $k' = 1, 2, \ldots, (k-1)$ already in place when the k^{th} charge is moved in (in Fig. 6, the charges inside the dashed boundary), and we may use the linear superposition principle to write

$$\Delta U_k = q_k \sum_{k'<k} \phi_{k'}(\mathbf{r}_k), \qquad \text{so that } U = \sum_k U_k = \sum_{\substack{k,k' \\ (k'<k)}} q_k \phi_{k'}(\mathbf{r}_k). \tag{1.56}$$

This result is so important that is worthy of rewriting it in several other forms. First, we may use Eq. (35) to represent Eq. (56) in a more symmetric form:

$$U = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{k,k' \\ (k'<k)}} \frac{q_k q_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}. \tag{1.57}$$

The expression under this sum is evidently symmetric with respect to the index swap, so that it may be extended into a different form,

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \sum_{\substack{k',k \\ (k' \neq k)}} \frac{q_k q_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}, \tag{1.58}$$

where the interaction between each couple of charges is described by two equal terms under the sum, and the front coefficient 1⁄2 is used to compensate for this double counting. The convenience of the last form is that it may be readily generalized to the continuous case:

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \int d^3 r \int d^3 r' \frac{\rho(\mathbf{r}) \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{1.59}$$

(As before, in this case the restriction expressed in the discrete charge case as $k \neq k'$ is not important, because if the charge density is a continuous function, the integral (59) does not diverge at point **r** = **r'**.)

To represent this result in one more form, let us notice that according to Eq. (38), the inner integral over r' in Eq. (59), divided by $4\pi\varepsilon_0$, is just the full electrostatic potential at point **r**, and hence

$$\text{Energy: charge interation} \qquad U = \frac{1}{2} \int \rho(\mathbf{r}) \phi(\mathbf{r}) d^3 r. \tag{1.60}$$

For the discrete charge case, this result is

$$U = \frac{1}{2}\sum_k q_k \phi(\mathbf{r}_k), \tag{1.61}$$

but here it is important to remember that here the "full" potential's value $\phi(\mathbf{r}_k)$ should exclude the (infinite) contribution from the point charge k itself. Comparing the last two formulas with Eqs. (54) and (55), we see that the electrostatic energy of charge interaction within the system, as expressed via the charge-by potential product, is twice less than that of the energy of charge interaction with a fixed ("external") field. This is evidently the result of the fact that in the case of mutual interaction of the charges, the electric field $\mathbf{E}$ in the basic Eq. (52) is proportional to the charge's magnitude, rather than constant.[23]

Now we are ready to address an important conceptual question: can we locate this interaction energy in space? Eqs. (58)-(61) seem to imply that non-zero contributions to U come only from the regions where the electric charges are located. However, one of the most beautiful features of physics is that sometimes completely different interpretations of the same mathematical result are possible. To get an alternative view of our current result, let us write Eq. (60) for a volume V so large that the electric field on the limiting surface S is negligible, and plug into it the charge density expressed from the Poisson equation (41):

$$U = -\frac{\varepsilon_0}{2}\int_V \phi\nabla^2\phi d^3r. \tag{1.62}$$

This expression may be integrated by parts as[24]

$$U = -\frac{\varepsilon_0}{2}\left[\oint_S \phi(\nabla\phi)_n d^2r - \int_V (\nabla\phi)^2 d^3r\right]. \tag{1.63}$$

According to our condition of negligible field $\mathbf{E} = -\nabla\phi$ at the surface, the first integral vanishes, and we get a very important formula

$$U = \frac{\varepsilon_0}{2}\int (\nabla\phi)^2 d^3r = \frac{\varepsilon_0}{2}\int E^2 d^3r. \tag{1.64}$$

This result, represented in the following equivalent form:[25]

$$U = \int u(\mathbf{r}) d^3r, \quad \text{with } u(\mathbf{r}) \equiv \frac{\varepsilon_0}{2}E^2(\mathbf{r}), \qquad \text{Energy: electric field} \tag{1.65}$$

certainly invites an interpretation very much different than Eq. (60): it is natural to interpret $u(\mathbf{r})$ as the spatial density of the electric field energy, which is continuously distributed over all the space where the field exists – rather than just its part where the charges are located.

Let us have a look at how these two alternative pictures work for our testbed problem, a uniformly charged sphere. If we start with Eq. (60), we may limit the integration by the sphere volume $(0 \le r \le R)$ where $\rho \ne 0$. Using Eq. (51), and the spherical symmetry of the problem $\left(d^3r = 4\pi r^2 dr\right)$, we get

$$U = \frac{1}{2}4\pi\int_0^R \rho\phi r^2 dr = \frac{1}{2}4\pi\rho\frac{Q}{4\pi\varepsilon_0 R}\int_0^R \left(\frac{R^2 - r^2}{2R^2} + 1\right) r^2 dr = \frac{6}{5}\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2R}. \tag{1.66}$$

On the other hand, if we use Eq. (65), we need to integrate the energy density everywhere, i.e. both inside and outside of the sphere:

$$U = \frac{\varepsilon_0}{2}4\pi\left(\int_0^R E^2 r^2 dr + \int_R^\infty E^2 r^2 dr\right). \tag{1.67}$$

Using Eqs. (19) and (22) for, respectively, the external and internal regions, we get

$$U = \frac{\varepsilon_0}{2}4\pi\left[\int_0^R \left(\frac{Qr}{4\pi\varepsilon_0}\right)^2 r^2 dr + \int_R^\infty \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 r^2 dr\right] = \left(\frac{1}{5} + 1\right)\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2R}. \tag{1.68}$$

This is (fortunately :-) the same answer as given by Eq. (66), but to some extent, Eq. (68) is more informative because it shows how exactly the electric field's energy is distributed between the interior and exterior of the charged sphere.[26]

We see that, as we could expect, within the realm of electrostatics, Eqs. (60) and (65) are equivalent. However, when we examine electrodynamics (in Chapter 6 and beyond), we will see that the latter equation is more general and that it is more adequate to

associate the electric energy with the field itself rather than its sources – in our current case, the electric charges.

Finally, let us calculate the potential energy of a system of charges in the general case when both the internal interaction of the charges, and their interaction with an external field are important. One might fancy that such a calculation should be very hard since, in both ultimate limits, when one of these interactions dominates, we have got different results. However, once again we get help from the almighty linear superposition principle: in the general case, for the total electric field we may write

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\rm int}(\mathbf{r}) + \mathbf{E}_{\rm ext}(\mathbf{r}), \quad \phi(\mathbf{r}) = \phi_{\rm int}(\mathbf{r}) + \phi_{\rm ext}(\mathbf{r}), \tag{1.69}$$

where the index "int" now marks the field induced by the charge system under analysis, i.e. the variables participating (without indices) in Eqs. (56)-(68). Now let us imagine that our system is being built up in the following way: first, the charges are brought together at $\mathbf{E}_{\rm ext} = 0$, giving the potential energy $U_{\rm int}$ expressed by Eq. (60), and then $\mathbf{E}_{\rm ext}$ is slowly increased. Evidently, the energy contribution from the latter process cannot depend on the internal interaction of the charges, and hence may be expressed in the form (55). As the result, the total potential energy[27] is the sum of these two components:

$$U = U_{\rm int} + U_{\rm ext} = \frac{1}{2}\int \rho(\mathbf{r})\phi_{\rm int}(\mathbf{r})d^3r + \int \rho(\mathbf{r})\phi_{\rm ext}(\mathbf{r})d^3r. \tag{1.70}$$

Now making the transition from the potentials to the fields, absolutely similar to that performed in Eqs. (62)-(65), we may rewrite this expression as

$$U = \int u(\mathbf{r})d^3r, \quad \text{with } u(\mathbf{r}) \equiv \frac{\varepsilon_0}{2}\left[E_{\rm int}^2(\mathbf{r}) + 2\mathbf{E}_{\rm int}(\mathbf{r})\cdot\mathbf{E}_{\rm ext}(\mathbf{r})\right]. \tag{1.71}$$

One might think that this result, more general than Eq. (65) and perhaps less familiar to the reader, is something entirely new; however, it is not. Indeed, let us add to, and subtract $E_{\rm ext}^2(\mathbf{r})$ from the sum in the brackets, and use Eq. (69) for the total electric field $\mathbf{E}(\mathbf{r})$; then Eq. (71) takes the form

$$U = \frac{\varepsilon_0}{2}\int E^2(\mathbf{r})d^3r - \frac{\varepsilon_0}{2}\int E_{\rm ext}^2(\mathbf{r})d^3r. \tag{1.72}$$

Hence, in the most important case when we are using the potential energy to analyze the statics and dynamics of a system of charges in a fixed external field, i.e. when the second term in Eq. (72) may be considered as a constant, we may still use for U an expression similar to the familiar Eq. (65), but with the field $\mathbf{E}(\mathbf{r})$ being the sum (69) of the internal and external fields.

Let us see how does this work in a very simple problem. A uniform external electric field $\mathbf{E}_{\rm ext}$ is applied normally to a very broad, plane layer that contains a very large and equal number of free electric charges of both signs – see Fig. 7. What is the equilibrium distribution of the charges over the layer?

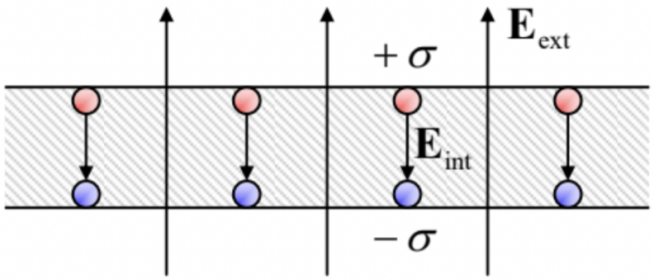

Fig. 1.7. A simple model of the electric field screening in a conductor. Here (and in all figures below) the red and blue colors are used to denote the opposite charge signs.

Since the equilibrium distribution should minimize the total potential energy of the system, Eq. (72) immediately gives the answer: the distribution should provide $\mathbf{E} \equiv \mathbf{E}_{\rm int} + \mathbf{E}_{\rm ext} = 0$ inside the whole layer[28] – the effect called the electric field screening. The only way to ensure this equality is to have enough free charges of opposite signs residing on the layer's surfaces to induce a uniform field $\mathbf{E}_{\rm int} = -\mathbf{E}_{\rm ext}$, exactly compensating the external field at each point inside the layer – see Fig. 7. According to Eq. (24), the areal density of these surface charges should equal $\pm\sigma$, with $\sigma = E_{\rm ext}/\varepsilon_0$. This is a rudimentary but reasonable model of the conductor's polarization – to be discussed in detail in the next chapter.

Reference

[15]See, e.g., CM Sec. 1.4.

[16]Eq. (28) could be also derived from this relation, because according to vector algebra, any gradient field has no curl – see, e.g., MA Eq. (11.1).

[17]This may be done either by Cartesian components or using the well-known expression $\nabla f = (df/dr)\mathbf{n}_r$ valid for any spherically-symmetric scalar function $f(r)$ – see, e.g., MA Eq. (10.8) for the particular case $\partial/\partial\theta = \partial/\partial\varphi = 0$.

[18]Named after Siméon Denis Poisson (1781-1840), also famous for the Poisson distribution – one of the central results of the probability theory – see, e.g., SM Sec. 5.2.

[19]After the famous mathematician (and astronomer) Pierre-Simon Laplace (1749-1827) who, together with Alexis Clairault, is credited for the development of the very concept of potential.

[20]See, e.g., MA Eq. (10.8) for $\partial/\partial\theta = \partial/\partial\varphi = 0$.

[21]Indeed, by the very definition of the potential energy of a system, it should not depend on the way we are arriving at its final configuration.

[22]An alternative, perhaps more accurate term for U_{ext} is the energy of the system's interaction with the external field.

[23]The nature of this additional factor 1⁄2 is absolutely the same as in the well-known formula $U = (1/2)\kappa x^2$ for the potential energy of an elastic spring providing the returning force $F = -\kappa x$, proportional to its displacement x from the equilibrium position.

[24]This transformation follows from the divergence theorem MA (12.2) applied to the vector function $\mathbf{f} = \phi\nabla\phi$, taking into account the differentiation rule MA Eq. (11.4a): $\nabla\cdot(\phi\nabla\phi) = (\nabla\phi)\cdot(\nabla\phi) + \phi\nabla\cdot(\nabla\phi) = (\nabla\phi)^2 + \phi\nabla^2\phi$.

[25]In the Gaussian units, the standard replacement $\varepsilon_0 \to 1/4\pi$ turns the last of Eqs. (65) into $u(\mathbf{r}) = E^2/8\pi$.

[26]Note that $U \to \infty$ at $R \to 0$. Such divergernce appears at the application of Eq. (65) to any point charge. Since it does not affect the force acting on the charge, the divergence does not create any technical diffeiculty for analysis of charge statics or non-relativistic dynamics, but it points to a possible conceptual problem of classical electrodynamics as the whole at describing point charges. This issue will be discussed at the very end of the course (Sec. 10.6).

[27]This total U (or rather its part dependent on our system of charges) is sometimes called the Gibbs potential energy of the system. (I will discuss this notion in detail in Sec. 3.5.)

[28]Note that the area-uniform distribution of the charge inside the layer does not affect the field (and hence its energy) outside it.