# 10.1: Liénard-Wiechert Potentials

A convenient starting point for the discussion of radiation by relativistic charges is provided by Eqs. (8.17) for the retarded potentials. In the free space, these formulas, with the integration variable changed from $\mathbf{r}'$ to $\mathbf{r}''$ for the clarity of what follows, are reduced to

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho\left(\mathbf{r}'',t-R/c\right)}{R}d^3r'',\quad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{j}\left(\mathbf{r}'',t-R/c\right)}{R}d^3r'',\quad \text{with}\,\mathbf{R}\equiv\mathbf{r}-\mathbf{r}''. \tag{10.1a}$$

As a reminder, Eqs. (1a) were derived from the Maxwell equations without any restrictions, and are very natural for situations with continuous distributions of the electric charge and/or current. However, for a single point charge, with

$$\rho(\mathbf{r},t)=q\delta\left(\mathbf{r}-\mathbf{r}'\right),\quad \mathbf{j}(\mathbf{r},t)=q\mathbf{u}\delta\left(\mathbf{r}-\mathbf{r}'\right),\quad \text{with}\,\mathbf{u}=\dot{\mathbf{r}}', \tag{10.1b}$$

where $\mathbf{r}'$ is the instantaneous position of the charge, it is more convenient to recast Eqs. (1a) into an explicit form that would not require integration in each particular case. Indeed, as Eqs. (1) show, the potentials at a given observation point $\{\mathbf{r},t\}$ are contributed by only one specific point $\{\mathbf{r}'(t_{\text{ret}}),t_{\text{ret}}\}$ of the particle's 4D trajectory (called its world line), which satisfies the following condition:

$$t_{\text{ret}}\equiv t-\frac{R_{\text{ret}}}{c}, \tag{10.2}$$

where $t_{\text{ret}}$ is called the retarded time, and $R_{\text{ret}}$ is the length of the following distance vector

$$\mathbf{R}_{\text{ret}}\equiv\mathbf{r}(t)-\mathbf{r}'\left(t_{\text{ret}}\right) \tag{10.3}$$

– physically, the distance covered by the electromagnetic wave from its emission to observation.

The reduction of Eqs. (1a) to such a simpler form, however, requires some care. Indeed, their naïve integration over $\mathbf{r}''$ would yield the following apparent but wrong results:

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\frac{q}{R_{\text{ret}}},\ \text{i.e.}\ \frac{\phi(\mathbf{r},t)}{c}=\frac{\mu_0}{4\pi}\frac{qc}{R_{\text{ret}}};\quad \mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\frac{q\mathbf{u}_{\text{ret}}}{R_{\text{ret}}},\qquad \text{(WRONG!)} \tag{10.4}$$

where $\mathbf{u}_{\text{ret}}$ is the particle's velocity at the retarded point $\mathbf{r}'(t_{\text{ret}})$. Eqs. (4) is a good example of how the relativity theory (even the special one :-) cannot be taken too lightly. Indeed, the strings (9.84)-(9.85), formed from the apparent potentials (4), would not obey the Lorentz transform rule (9.91), because according to Eqs. (2)-(3), the distance $R_{\text{ret}}$ also depends on the reference frame it is measured in.

In order to correct the error, we need, first of all, discuss the conditions (2)-(3). Combining them (by eliminating $R_{\text{ret}}$), we get the following equation for $t_{\text{ret}}$:

$$c\left(t-t_{\text{ret}}\right)=\left|\mathbf{r}(t)-\mathbf{r}'\left(t_{\text{ret}}\right)\right|.\qquad \text{Retarded time} \tag{10.5}$$

Figure 1 depicts the graphical solution of this self-consistency equation as the only[1] point of intersection of the light cone of the observation point (see Fig. 9.9 and its discussion) and the particle's world line.

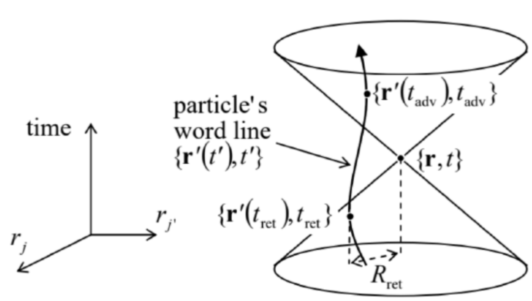


Fig. 10.1. Graphical solution of Eq. (5).

In Eq. (5), just as in Eqs. (1)-(3), all variables have to be measured in the same inertial ("lab") reference frame, in which the observation point $\mathbf{r}$ rests. Now let us write Eqs. (1) for a point charge in another inertial reference frame 0', whose velocity (as measured in the lab frame) coincides, at the moment $t' = t_{\text{ret}}$, with the velocity $\mathbf{u}_{\text{ret}}$ of the charge. In that frame, the charge rests, so that, as we know from the electro- and magnetostatics,

$$\phi' = \frac{q}{4\pi\varepsilon_0 R'}, \quad \mathbf{A}' = 0. \tag{10.6a}$$

(Remember that this $R'$ may not be equal to $R_{\text{ret}}$, because the latter distance is measured in the "lab" reference frame.) Let us use the identity $1/\varepsilon_0 \equiv \mu_0 c^2$ again to rewrite Eqs. (6a) in the form of components of a 4-vector similar in structure to Eq. (4):

$$\frac{\phi'}{c} = \frac{\mu_0}{4\pi} q \frac{c}{R'}, \quad \mathbf{A}' = 0. \tag{10.6b}$$

Now it is easy to guess the correct answer for the 4-potential for an arbitrary reference frame:

$$A^\alpha = \frac{\mu_0}{4\pi} q \frac{c u^\alpha}{u_\beta R^\beta}, \tag{10.7}$$

where (mostly as a reminder), $A^\alpha \equiv \{\phi/c, \mathbf{A}\}, u^\alpha \equiv \gamma\{c, \mathbf{u}\}$, and $R^\alpha$ is a 4-vector of the inter-event distance, formed similarly to that of a single event – cf. Eq. (9.48):

$$R^\alpha \equiv \{c(t-t'), \mathbf{R}'\} \equiv \{c(t-t'), \mathbf{r}-\mathbf{r}'\}. \tag{10.8}$$

Indeed, we needed the 4-vector $A^\alpha$ that would:

(i) obey the Lorentz transform,

(ii) have its spatial components $A_j$ scaling, at low velocity, as $u_j$, and

(iii) be reduced to the correct result (6) in the reference frame moving with the charge.

Eq. (7) evidently satisfies all these requirements, because the scalar product in its denominator is just

$$u_\beta R^\beta = \gamma\{c, -\mathbf{u}\} \cdot \{c(t-t'), \mathbf{R}\} \equiv \gamma\left[c^2(t-t') - \mathbf{u}\cdot\mathbf{R}\right] \equiv \gamma c(R - \boldsymbol{\beta}\cdot\mathbf{R}) \equiv \gamma c R(1-\boldsymbol{\beta}\cdot\mathbf{n}), \tag{10.9}$$

where $\mathbf{n} \equiv \mathbf{R}/R$ is a unit vector in the observer's direction, $\boldsymbol{\beta} \equiv \mathbf{u}/c$ is the normalized velocity of the particle, and $\gamma \equiv 1/\left(1-u^2/c^2\right)^{1/2}$. In the reference frame of the charge (in which $\boldsymbol{\beta} = 0$ and $\gamma = 1$), the expression (9) is reduced to $cR$, so that Eq. (7) is correctly reduced to Eq. (6b). Now let us spell out the components of Eq. (7) for the lab frame (in which $t' = t_{\text{ret}}$ and $R = R_{\text{ret}}$):

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{(R - \boldsymbol{\beta}\cdot\mathbf{R})_{\text{ret}}} = \frac{1}{4\pi\varepsilon_0} q \left[\frac{1}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}, \tag{10.10a}$$

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$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} q \left(\frac{\mathbf{u}}{R - \boldsymbol{\beta}\cdot\mathbf{R}}\right)_{\text{ret}} = \frac{\mu_0}{4\pi} qc \left[\frac{\boldsymbol{\beta}}{R(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}} \equiv \phi(\mathbf{r}, t) \frac{\mathbf{u}_{\text{ret}}}{c^2}. \tag{10.10b}$$

These formulas are called the Liénard-Wiechert potentials.[2] In the non-relativistic limit, they coincide with the naïve guess (4), but in the general case include an additional factor $1/(1-\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}}$. Its physical origin may be illuminated by one more formal calculation – whose result we will need anyway. Let us differentiate the geometric relation (5), rewritten as

$$R_{\text{ret}} = c(t - t_{\text{ret}}), \tag{10.11}$$

over $t_{\text{ret}}$ and then, independently, over $t$, assuming that **r** is fixed. For that, let us first differentiate, over $t_{\text{ret}}$, both sides of the identity $R_{\text{ret}}{}^2 = \mathbf{R}_{\text{ret}} \cdot \mathbf{R}_{\text{ret}}$:

$$2R_{\text{ret}} \frac{\partial R_{\text{ret}}}{\partial t_{\text{ret}}} = 2\mathbf{R}_{\text{ret}} \cdot \frac{\partial \mathbf{R}_{\text{ret}}}{\partial t_{\text{ret}}}. \tag{10.12}$$

If **r** is fixed, then $\partial\mathbf{R}_{\text{ret}}/\partial t_{\text{ret}} \equiv \partial(\mathbf{r}-\mathbf{r}')/\partial t_{\text{ret}} = -\partial\mathbf{r}'/\partial t_{\text{ret}} \equiv -\mathbf{u}_{\text{ret}}$, and Eq. (12) yields

$$\frac{\partial R_{\text{ret}}}{\partial t_{\text{ret}}} = \frac{\mathbf{R}_{\text{ret}}}{R_{\text{ret}}} \cdot \frac{\partial \mathbf{R}_{\text{ret}}}{\partial t_{\text{ret}}} = -(\mathbf{n}\cdot\mathbf{u})_{\text{ret}}. \tag{10.13}$$

Now let us differentiate the same $R_{\text{ret}}$ over $t$. On one hand, Eq. (11) yields

$$\frac{\partial R_{\text{ret}}}{\partial t} = c - c\frac{\partial t_{\text{ret}}}{\partial t}. \tag{10.14}$$

On the other hand, according to Eq. (5), at the partial differentiation over time, i.e. if **r** is fixed, $t_{\text{ret}}$ is a function of $t$ alone, so that (using Eq. (13) at the second step), we may write

$$\frac{\partial R_{\text{ret}}}{\partial t_{\text{ret}}} = \frac{\partial R_{\text{ret}}}{\partial t_{\text{ret}}}\frac{\partial t_{\text{ret}}}{\partial t} = -(\mathbf{n}\cdot\mathbf{u})_{\text{ret}}\frac{\partial t_{\text{ret}}}{\partial t}. \tag{10.15}$$

Now requiring Eqs. (14) and (15) to give the same result, we get:[3]

$$\frac{\partial t_{\text{ret}}}{\partial t} = \frac{c}{c-(\mathbf{n}\cdot\mathbf{u})_{\text{ret}}} = \left(\frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right)_{\text{ret}}. \tag{10.16}$$

This relation may be readily re-derived (and more clearly understood) for the simple particular case when the charge's velocity is directed straight toward the observation point. In this case, its vector **u** resides in the same space-time plane as the observation point's world line $\mathbf{r} = \text{const}$ – say, the plane $[x, t]$, shown in Fig. 2.

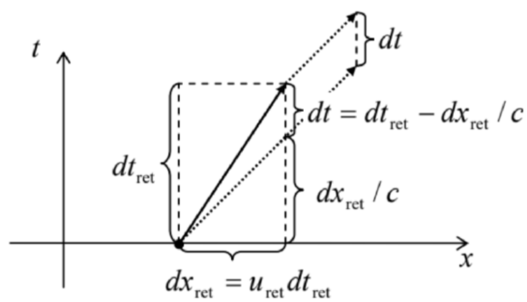


Fig. 10.2. Deriving Eq. (16) for the case $\boldsymbol{\beta}\cdot\mathbf{n} = \beta$.

Let us consider an elementary time interval $dt_{\text{ret}} \equiv dt'$, during which the particle would travel the space interval $dx_{\text{ret}} = u_{\text{ret}}dt_{\text{ret}}$. In Fig. 2, the corresponding segment of its world line is shown with a solid vector. The dotted vectors in this figure show the world lines of the radiation emitted by the particle in the beginning and at the end of this interval, and propagating with the speed of light $c$. As it follows from the drawing, the time interval $dt$ between the instants of the arrival of the radiation from these two points to any time-independent spatial point of observation is

$$dt = dt_{\text{ret}} - \frac{dx_{\text{ret}}}{c} = dt_{\text{ret}} - \frac{u_{\text{ret}}}{c}dt_{\text{ret}}, \quad \text{so that } \frac{dt_{\text{ret}}}{dt} = \frac{1}{1-u_{\text{ret}}/c} \equiv \frac{1}{1-\beta_{\text{ret}}}. \tag{10.17}$$

This expression coincides with Eq. (16), because in our particular case when the directions of the vectors $\boldsymbol{\beta} \equiv \mathbf{u}/c$ and $\mathbf{n} \equiv \mathbf{R}/R$ (both taken at time $t_{\text{ret}}$) coincide, and hence $(\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}} = \beta_{\text{ret}}$. Now the general Eq. (16) may be interpreted by saying that the particle's velocity in the transverse directions (normal to the vector **n**) is not important for this kinematic effect[4] – the fact almost evident from Fig. 1.

So, the additional factor in the Liénard-Wiechert potentials is just the derivative $\partial t_{\text{ret}}/\partial t$. The reason for its appearance in Eqs. (10) is usually interpreted along the following lines. Let the charge $q$ be spread along the direction of the vector $\mathbf{R}_{\text{ret}}$ (in Fig. 2, along the x-axis) by an infinitesimal speed-independent interval $\delta x_{\text{ret}}$, so that the linear density $\lambda$ of its charge is proportional to $1/\delta x_{\text{ret}}$. Then the time rate of charge's arrival at some spatial point is $\lambda u_{\text{ret}} = \lambda dx_{\text{ret}}/dt_{\text{ret}}$. However, the rate of radiation's arrival at the observation point scales as $1/dt$, so that due to the non-zero velocity $\mathbf{u}_{\text{ret}}$ of the particle, this rate differs from the charge arrival rate by the factor of $dt_{\text{ret}}/dt$, given by Eq. (16). (If the particle moves toward the observation point, $(\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}} > 0$, as shown in Fig. 2, this factor is larger than 1.) This radiation compression effect leads to the field change (at $(\boldsymbol{\beta}\cdot\mathbf{n})_{\text{ret}} > 0$, its enhancement) by the same factor (16) – as described by Eqs. (10).

So, the 4-vector formalism was very instrumental for the calculation of field potentials. It may be also used to calculate the fields **E** and **B** – by plugging Eq. (7) into Eq. (9.124) to calculate the field strength tensor. This calculation yields

$$F^{\alpha\beta} = \frac{\mu_0 q}{4\pi}\frac{1}{u_\gamma R^\gamma}\frac{d}{d\tau}\left[\frac{R^\alpha u^\beta - R^\beta u^\alpha}{u_\delta R^\delta}\right]. \tag{10.18}$$

Now using Eq. (9.125) to identify the elements of this tensor with the field components, we may bring the result to the following vector form:[5]

$$\mathbf{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}+\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3cR}\right]_{\text{ret}},\tag{10.19}$$

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$$\mathbf{B}=\frac{\mathbf{n}_{\text{ret}}\times\mathbf{E}}{c},\quad \text{i.e. } \mathbf{H}=\frac{\mathbf{n}_{\text{ret}}\times\mathbf{E}}{Z_0}.\tag{10.20}$$

Thus the magnetic and electric fields of a relativistic particle are always proportional and perpendicular to each other, and related just as in a plane wave – cf. Eq. (7.6), with the difference that now the vector nret may be a function of time. Superficially, this result contradicts the electro- and magnetostatics, because for a particle at rest, **B** should vanish while **E** stays finite. However, note that according to the Coulomb law for a point charge, in this case $\mathbf{E}=E\mathbf{n}_{\text{ret}}$, so that $\mathbf{B}\propto\mathbf{n}_{\text{ret}}\times\mathbf{E}\propto\mathbf{n}_{\text{ret}}\times\mathbf{n}_{\text{ret}}=0$. (Actually, in these relations, the subscript "ret" is unnecessary.)

As a sanity check, let us use Eq. (19) as an alternative way to find the electric field of a charge moving without acceleration, i.e. uniformly, along a straight line – see Fig. 9.11a reproduced, with minor changes, in Fig. 3. (This calculation will also illustrate the technical challenges of practical applications of the Liénard-Wiechert formulas for even simple cases.) In this case, the vector $\boldsymbol{\beta}$ does not change in time, so that the second term in Eq. (19) vanishes, and all we need to do is to spell out the Cartesian components of the first term.

Let us select the coordinate axes and the time origin in the way shown in Fig. 3, and make a clear distinction between the actual position, $\mathbf{r}'(t)=\{ut,0,0\}$ of the charged particle at the instant $t$ we are considering, and its position $\mathbf{r}'(t_{\text{ret}})$ at the retarded instant defined by Eq. (5), i.e. the moment when the particle's field had to be radiated to reach the observation point **r** at the given time $t$, propagating with the speed of light. In these coordinates

$$\boldsymbol{\beta}=\{\beta,0,0\},\quad \mathbf{r}=\{0,b,0\},\quad \mathbf{r}'(t_{\text{ret}})=\{ut_{\text{ret}},0,0\},\quad \mathbf{n}_{\text{ret}}=\{\cos\theta,\sin\theta,0\},\tag{10.21}$$

with $\cos\theta=-ut_{\text{ret}}/R_{\text{ret}}$, so that $[(\mathbf{n}-\boldsymbol{\beta})_x]_{\text{ret}}=-ut_{\text{ret}}/R_{\text{ret}}-\beta$, and Eq. (19) yields, in particular:

$$E_x=\frac{q}{4\pi\varepsilon_0}\frac{-ut_{\text{ret}}/R_{\text{ret}}-\beta}{\gamma^2\left[(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2\right]_{\text{ret}}}\equiv\frac{q}{4\pi\varepsilon_0}\frac{-ut_{\text{ret}}-\beta R_{\text{ret}}}{\gamma^2\left[(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^3\right]_{\text{ret}}}.\tag{10.22}$$

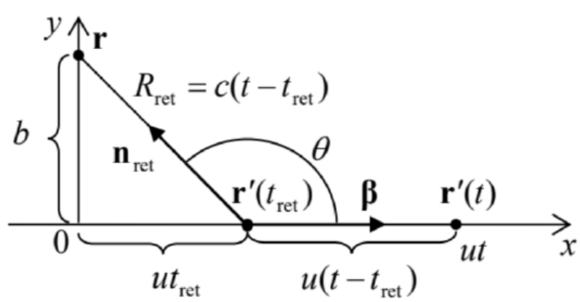

Fig. 10.3. The linearly moving charge problem.

But according to Eq. (5), the product $\beta R_{\text{ret}}$ may be represented as $\beta c(t-t_{\text{ret}})\equiv u(t-t_{\text{ret}})$. Plugging this expression into Eq. (22), we may eliminate the explicit dependence of $E_x$ on time $t_{\text{ret}}$:

$$E_x=\frac{q}{4\pi\varepsilon_0}\frac{-ut}{\gamma^2[(1-\boldsymbol{\beta}\cdot\mathbf{n})R]^3_{\text{ret}}}.\tag{10.23}$$

The only non-zero transverse component of the field also has a similar form:

$$E_y=\frac{q}{4\pi\varepsilon_0}\left[\frac{\sin\theta}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{\text{ret}}=\frac{q}{4\pi\varepsilon_0}\frac{b}{\gamma^2[(1-\boldsymbol{\beta}\cdot\mathbf{n})R]^3_{\text{ret}}},\tag{10.24}$$

while $E_z=0$. From Fig. 3, $\boldsymbol{\beta}-\mathbf{n}_{\text{ret}}=\beta\cos\theta=-\beta ut_{\text{ret}}/R_{\text{ret}}$, so that $(1-\boldsymbol{\beta}\cdot\mathbf{n})R_{\text{ret}}\equiv R_{\text{ret}}+\beta ut_{\text{ret}}$, and we may again use Eq. (5) to get $(1-\boldsymbol{\beta}\cdot\mathbf{n})R_{\text{ret}}=c(t-t_{\text{ret}})+\beta ut_{\text{ret}}\equiv ct-ct_{\text{ret}}/\gamma^2$. What remains is to calculate $t_{\text{ret}}$ from the self-consistency equation (5), whose square in our current case (Fig. 3) takes the form

$$R_{\text{ret}}^{2}\equiv b^{2}+(ut_{\text{ret}})^{2}=c^{2}\left(t-t_{\text{ret}}\right)^{2}. \tag{10.25}$$

This is a simple quadratic equation for tret, which (with the appropriate negative sign before the square root, to get $t_{\text{ret}}<t$) yields:

$$t_{\text{ret}}=\gamma^{2}t-\left[\left(\gamma^{2}t\right)^{2}-\gamma^{2}\left(t^{2}-b^{2}/c^{2}\right)\right]^{1/2}\equiv\gamma^{2}t-\frac{\gamma}{c}\left(u^{2}\gamma^{2}t^{2}+b^{2}\right)^{1/2}, \tag{10.26}$$

so that the only retarded-function combination that participates in Eqs. (23)-(24) is

$$[(1-\boldsymbol{\beta}\cdot\mathbf{n})R]_{\text{ret}}=\frac{c}{\gamma^{2}}\left(u^{2}\gamma^{2}t^{2}+b^{2}\right)^{1/2}, \tag{10.27}$$

and, finally, the electric field components are

$$E_{x}=-\frac{q}{4\pi\varepsilon_{0}}\frac{\gamma ut}{\left(b^{2}+\gamma^{2}u^{2}t^{2}\right)^{3/2}},\quad E_{y}=\frac{q}{4\pi\varepsilon_{0}}\frac{\gamma b}{\left(b^{2}+\gamma^{2}u^{2}t^{2}\right)^{3/2}},\quad E_{z}=0. \tag{10.28}$$

These are exactly Eqs. (9.139),[6] which had been obtained in Sec. 9.5 by much simpler means, without the necessity to solve the self-consistency equation (5). However, that alternative approach was essentially based on the inertial motion of the particle, and cannot be used in problems in which particle moves with acceleration. In such problems, the second term in Eq. (19), dropping with distance more slowly, as 1/Rret, and hence describing wave radiation, is essential and most important.

## Reference

[1] As Fig. 1 shows, there is always another, “advanced” point $\left\{\mathbf{r}'\left(t_{\text{adv}}\right), t_{\text{adv}}\right\}$ of the particle’s world line, with $t_{\text{adv}}>t$, which is also a solution of Eq. (5), but it does not fit Eqs. (1), because the observation, at the point $\left\{\mathbf{r}, t<t_{\text{adv}}\right\}$, of the field induced at the advanced point, would violate the causality principle.

[2] They were derived in 1898 by Alfred-Marie Liénard and (independently) in 1900 by Emil Wiechert.

[3] This relation may be used for an alternative derivation of Eqs. (10) directly from Eqs (1) – the exercise highly recommended to the reader.

[4] Note that this effect (linear in $\beta$) has nothing to do with the Lorentz time dilation (9.21), which is quadratic in $\beta$. (Indeed, all our arguments above referred to the same, lab frame.) Rather, it is close in nature to the Doppler effect.

[5] An alternative way of deriving these formulas (highly recommended to the reader as an exercise) is to plug Eqs. (10) into the general relations (9.121), and carry out the required temporal and spatial differentiations directly, using Eq. (16) and its spatial counterpart (which may be derived absolutely similarly):

$$\nabla t_{\text{ret}}=-\left[\frac{\mathbf{n}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}.$$

[6] A similar calculation of magnetic field components from Eq. (20) gives the results identical to Eqs. (9.140).