6.8: Finally, the Full Maxwell Equation System

This is a very special moment in this course: with the displacement current inclusion, i.e. with the replacement of Eq. (5.107) with Eq. (93), we have finally arrived at the full set of macroscopic Maxwell equations for time-dependent fields,[60]

$$\begin{array}{lll} \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0, & \nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\mathbf{j}, & (6.99a) \\ \nabla\cdot\mathbf{D}=\rho, & \nabla\cdot\mathbf{B}=0, & (6.99b) \end{array} \quad \text{Macroscopic Maxwell equations} \tag{6.8.1}$$

whose validity has been confirmed by an enormous body of experimental data. Indeed, despite numerous efforts, no other corrections (e.g., additional terms) to the Maxwell equations have been ever found, and these equations are still considered exact within the range of their validity, i.e. while the electric and magnetic fields may be considered classically. Moreover, even in quantum theory, these equations are believed to be strictly valid as relations between the Heisenberg operators of the electric and magnetic fields.[61] (Note that the microscopic Maxwell equations for the genuine fields $\mathbf{E}$ and $\mathbf{B}$ may be formally obtained from Eqs. (99) by the substitutions $\mathbf{D}=\varepsilon_0\mathbf{E}$ and $\mathbf{H}=\mathbf{B}/\mu_0$, and the simultaneous replacement of the stand-alone charge and current densities on their right-hand sides with the full ones.)

Perhaps the most striking feature of these equations is that, even in the absence of stand-alone charges and currents inside the region of our interest, when the equations become fully homogeneous,

$$\begin{array}{lll} \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, & \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}, & (6.100a) \\ \nabla\cdot\mathbf{D}=0, & \nabla\cdot\mathbf{B}=0, & (6.100b) \end{array} \tag{6.8.2}$$

they still describe something very non-trivial: electromagnetic waves, including light. The physics of the waves may be clearly seen from Eqs. (100a): according to the first of them, the change of the magnetic field in time creates a vortex-like (divergence-free) electric field. On the other hand, the second of Eqs. (100a) describes how the changing electric field, in turn, creates a vortex-like magnetic field. So-coupled electric and magnetic fields may propagate as waves – even very far from their sources.

We will carry out a detailed quantitative analysis of the waves in the next chapter, and here I will only use this notion to fulfill the promise given in Sec. 3, namely to establish the condition of validity of the quasistatic approximation (21). For simplicity, let us consider an electromagnetic wave with a time period T, velocity ν, and hence the wavelength[62] $\lambda=\nu T$ in a linear medium with $\mathbf{D}=\varepsilon\mathbf{E}, \mathbf{B}=\mu\mathbf{H}$, and $\mathbf{j}=0$ and $\rho=0$. Then the magnitude of the left-hand side of the first of Eqs. (100a) is of the order of $E/\lambda=E/\nu T$, while that of its right-hand side may be estimated as $B/T\sim\mu H/T$. Using similar estimates for the second of Eqs. (100a), we arrive at the following two requirements:[63]

$$\frac{E}{H}\sim\mu\nu\sim\frac{1}{\varepsilon\nu}. \tag{6.101}$$

To insure the compatibility of these two relations, the waves' speed should satisfy the estimate

$$\nu\sim\frac{1}{(\varepsilon\mu)^{1/2}}, \tag{6.102}$$

reduced to $\nu\sim 1/(\varepsilon_0\mu_0)^{1/2}\equiv c$ in free space, while the ratio of the electric and magnetic field amplitudes should be of the following order:

$$\frac{E}{H}\sim\mu\nu\sim\mu\frac{1}{(\varepsilon\mu)^{1/2}}\equiv\left(\frac{\mu}{\varepsilon}\right)^{1/2}. \tag{6.103}$$

(In the next chapter we will see that for plane electromagnetic waves, these results are exact.)

Now, let a system of a linear size $\sim a$ carry currents producing a certain magnetic field H. Then, according to Eqs. (100a), their magnetic field Faraday-induces the electric field of magnitude $E\sim\mu Ha/T$, whose displacement currents, in turn, produce an additional magnetic field with magnitude

$$H'\sim\frac{a\varepsilon}{\tau}E\sim\frac{a\varepsilon}{\tau}\frac{\mu a}{\tau}H\equiv\left(\frac{a\lambda}{\nu\tau\lambda}\right)^2 H\equiv\left(\frac{a}{\lambda}\right)^2 H. \tag{6.104}$$

Hence, the displacement current effects are negligible for a system of size $a<<\lambda$.[64]

In particular, the quasistatic picture of the skin effect, discussed in Sec. 3, is valid while the skin depth (33) remains much smaller than the corresponding wavelength,

$$\lambda=\nu\tau=\frac{2\pi\nu}{\omega}=\left(\frac{4\pi^2}{\varepsilon\mu\omega^2}\right)^{1/2}. \tag{6.105}$$

The wavelength decreases with the frequency as $1/\omega$, i.e. faster than $\delta_s\propto 1/\omega^{1/2}$, so that they become comparable at the crossover frequency

$$\omega_r=\frac{\sigma}{\varepsilon}\equiv\frac{\sigma}{\kappa\varepsilon_0}, \tag{6.106}$$

which is nothing else than the reciprocal charge relaxation time (4.10). As was discussed in Sec. 4.2, for good metals this frequency is extremely high (about $10^{18}\ s^{-1}$), so the validity of Eq. (33) is typically limited by the anomalous skin effect (which was briefly discussed in Sec. 3), rather than the wave effects.

Before going after the analysis of the full Maxwell equations for particular situations (that will be the main goal of the next chapters of this course), let us have a look at the energy balance they yield for a certain volume V, which may include both charged particles and the electromagnetic field. Since, according to Eq. (5.10), the magnetic field does no work on charged particles even if they move, the total power $\mathscr{P}$ being transferred from the field to the particles inside the volume is due to the electric field alone – see Eq. (4.38):

$$\mathscr{P}=\int_V \mathscr{I} d^3r,\quad \text{with } \mathscr{I}=\mathbf{j}\cdot\mathbf{E}, \tag{6.107}$$

Expressing $\mathbf{j}$ from the corresponding Maxwell equation of the system (99), we get

$$\mathscr{P}=\int_V\left[\mathbf{E}\cdot(\nabla\times\mathbf{H})-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right]d^3r. \tag{6.108}$$

Let us pause here for a second, and transform the divergence of $\mathbf{E}\times\mathbf{H}$, using the well-known vector algebra identity:[65]

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{H}). \tag{6.109}$$

The last term on the right-hand side of this equality is exactly the first term in the square brackets of Eq. (108), so that we may rewrite that formula as

$$\mathscr{P}=\int_V\left[-\nabla\cdot(\mathbf{E}\times\mathbf{H})+\mathbf{H}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right]d^3r. \tag{6.110}$$

However, according to the Maxwell equation for $\nabla\times\mathbf{E}$, this curl is equal to $-\partial\mathbf{B}/\partial t$, so that the second term in the square brackets of Eq. (110) equals $-\mathbf{H}\cdot\partial\mathbf{B}/\partial t$ and, according to Eq. (14), is just the (minus) time derivative of the magnetic energy per unit volume. Similarly, according to Eq. (3.76), the third term under the integral is the minus time derivative of the electric energy per unit volume. Finally, we can use the divergence theorem to transform the integral of the first term in the square brackets to a 2D integral over the surface S limiting the volume V. As a result, we get the so-called Poynting theorem[66] for the power balance in the system:

$$\text{Poynting theorem}\qquad \int_V\left(\mathscr{I}+\frac{\partial u}{\partial t}\right)d^3r+\oint_S S_n d^2r=0. \tag{6.111}$$

Here u is the density of the total (electric plus magnetic) energy of the electromagnetic field, with

$$\text{Field's energy variation}\qquad \delta u\equiv\mathbf{E}\cdot\delta\mathbf{D}+\mathbf{H}\cdot\delta\mathbf{B} \tag{6.112}$$

- just a sum of the expressions given by Eqs. (3.76) and (14). For the particular case of an isotropic, linear, and dispersion-free medium, with $\mathbf{D}(t)=\varepsilon\mathbf{E}(t),\mathbf{B}(t)=\mu\mathbf{H}(t)$, Eq. (112) yields

$$\text{Field's energy}\qquad u=\frac{\mathbf{E}\cdot\mathbf{D}}{2}+\frac{\mathbf{H}\cdot\mathbf{B}}{2}\equiv\frac{\varepsilon E^2}{2}+\frac{B^2}{2\mu}. \tag{6.113}$$

Another key notion participating in Eq. (111) is the Poynting vector, defined as[67]

$$\text{Poynting vector}\qquad \mathbf{S}\equiv\mathbf{E}\times\mathbf{H}. \tag{6.114}$$

The first integral in Eq. (111) is evidently the net change of the energy of the system (particles + field) per unit time, so that the second (surface) integral has to be the power flowing out from the system through the surface. As a result, it is tempting to interpret the Poynting vector **S** locally, as the power flow density at the given point. In many cases, such a local interpretation of vector **S** is legitimate; however, in other cases, it may lead to wrong conclusions. Indeed, let us consider the simple system shown in Fig. 11: a charged plane capacitor placed into a static and uniform external magnetic field, so that the electric and magnetic fields are mutually perpendicular.

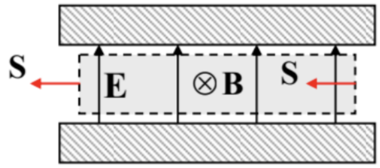

Fig. 6.11. The Poynting vector paradox.

In this static situation, with no charges moving, both $\mathscr{I}$ and $\partial/\partial t$ are equal to zero, and there should be no power flow in the system. However, Eq. (114) shows that the Poynting vector is not equal to zero inside the capacitor, being directed as the red arrows in Fig. 11 show. From the point of view of the only unambiguous corollary of the Maxwell equations, Eq. (111), there is no contradiction here, because the fluxes of the vector **S** through the side boundaries of the volume shaded in Fig. 11 are equal and opposite (and they are zero for other faces of this rectilinear volume), so that the total flux of the Poynting vector through the volume boundary equals zero, as it should. It is, however, useful to recall this example each time before giving a local interpretation of the vector **S**.

The paradox illustrated in Fig. 11 is closely related to the radiation recoil effects, due to the electromagnetic field's momentum – more exactly, it linear momentum. Indeed, acting as at the Poynting theorem derivation, it is straightforward to use the microscopic Maxwell equations[68] to prove that, neglecting the boundary effects, the vector sum of the mechanical linear momentum of the particles in an arbitrary volume, and the integral of the following vector,

$$\mathbf{g} \equiv \frac{\mathbf{S}}{c^2}, \qquad \text{Electro-magnetic field's momentum} \tag{6.115}$$

over the same volume, is conserved, enabling an interpretation of $\mathbf{g}$ as the density of the linear momentum of the electromagnetic field. (It will be more convenient for me to prove this relation, and discuss the related issues, in Sec. 9.8, using the 4-vector formalism of the special relativity.) Due to this conservation, if some static fields coupled to mechanical bodies are suddenly decoupled from them and are allowed to propagate in space, i.e. to change their local integral of $\mathbf{g}$, they give the bodies an opposite the equal and opposite impulse of force.

Finally, to complete our initial discussion of the Maxwell equations,[69] let us rewrite them in terms of potentials $\mathbf{A}$ and ϕ, because this is more convenient for the solution of some (though not all!) problems. Even when dealing with the system (99) of the more general Maxwell equations than discussed before, Eqs. (7) are still used for the definition of the potentials. It is straightforward to verify that with these definitions, the two homogeneous Maxwell equations (99b) are satisfied automatically. Plugging Eqs. (7) into the inhomogeneous equations (99a), and considering, for simplicity, a linear, uniform medium with frequency-independent ε and μ, we get

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{\rho}{\varepsilon}, \quad \nabla^2\mathbf{A} - \varepsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \varepsilon\mu\frac{\partial\phi}{\partial t}\right) = -\mu\mathbf{j}. \tag{6.116}$$

This is a more complex result than what we would like to get. However, let us select a special gauge, which is frequently called (especially for the free space case, when $\nu = c$) the Lorenz gauge condition[70]

$$\nabla\cdot\mathbf{A} + \varepsilon\mu\frac{\partial\phi}{\partial t} = 0, \qquad \text{Lorenz gauge condition} \tag{6.117}$$

which is a natural generalization of the Coulomb gauge (5.48) to time-dependent phenomena. With this condition, Eqs. (107) are reduced to a simpler, beautifully symmetric form:

$$\text{Potentials' dynamics} \qquad \nabla^2\phi - \frac{1}{\nu^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon}, \quad \nabla^2\mathbf{A} - \frac{1}{\nu^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{j}, \tag{6.118}$$

where $\nu^2 \equiv 1/\varepsilon\mu$. Note that these equations are essentially a set of 4 similar equations for 4 scalar functions (namely, ϕ and three Cartesian components of $\mathbf{A}$) and thus clearly invite the 4-component vector formalism of the relativity theory; it will be discussed in Chapter 9.[71]

If ϕ and $\mathbf{A}$) depend on just one spatial coordinate, say z, in a region without field sources: $\rho = 0$, $\mathbf{j} = 0$, Eqs. (118) are reduced to the well-known 1D wave equations

$$\frac{\partial^2 \phi}{\partial^2 z} - \frac{1}{\nu^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \quad \frac{\partial^2 \mathbf{A}}{\partial^2 z} - \frac{1}{\nu^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{6.119}$$

It is well known[72] that these equations describe waves, with arbitrary waveforms (including sinusoidal waves of any frequency), propagating with the same speed v in either of directions of the z-axis. According to the definitions of the constants ε_0 and μ_0, in the free space v is just the speed of light:

$$\nu = \frac{1}{(\varepsilon_0 \mu_0)^{1/2}} \equiv c. \tag{6.120}$$

Historically, the experimental observation of relatively low-frequency (GHz-scale) electromagnetic waves, with their speed equal to that of light, was the decisive proof (actually, a real triumph!) of the Maxwell theory and his prediction of such waves.[73] This was first accomplished in 1886 by Heinrich Rudolf Hertz, using the electronic circuits and antennas he had invented for this purpose.

Before proceeding to the detailed analysis of these waves in the following chapters, let me mention that the invariance of Eqs. (119) with respect to the wave propagation direction is not occasional; it is just a manifestation of one more general property of the Maxwell equations (99), called the Lorentz reciprocity. We have already met its simplest example, for time-independent electrostatic fields, in one of the problems of Chapter 1. Let us now consider a much more general case when two time-dependent electromagnetic fields, say $\{\mathbf{E}_1(\mathbf{r},t), \mathbf{H}_1(\mathbf{r},t)\}$ and $\{\mathbf{E}_2(\mathbf{r},\mathrm{t}), \mathbf{H}_2(\mathbf{r},t)\}$ are induced, respectively, by spatially-localized stand-alone currents $\mathbf{j}_1(\mathbf{r},t)$ and $\mathbf{j}_2(\mathbf{r},t)$. Then it may be proved[74] that if the medium is linear, and either isotropic or even anisotropic, but with symmetric tensors $\varepsilon_{jj'}$ and $\mu_{jj'}$, then for any volume V, limited by a closed surface S,

$$\int_V (\mathbf{j}_1 \cdot \mathbf{E}_2 - \mathbf{j}_2 \cdot \mathbf{E}_1)\, d^3r = \oint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1)\, d^2r. \tag{6.121}$$

This property implies, in particular, that the waves propagate similarly in two reciprocal directions even in situations much more general than the 1D case described by Eqs. (119). For some important practical applications (e.g., for low-noise amplifiers and detectors) such reciprocity is rather inconvenient. Fortunately, Eq. (121) may be violated in anisotropic media with asymmetric tensors $\varepsilon_{jj'}$ and/or $\mu_{jj'}$. The simplest, and most important case of such an anisotropy, the Faraday rotation of the wave polarization in plasma, will be discussed in the next chapter.

Reference

60 This vector form of the equations, magnificent in its symmetry and simplicity, was developed in 1884-85 by Oliver Heaviside, with substantial contributions by H. Lorentz. (The original Maxwell's result circa 1864 looked like a system of 20 equations for Cartesian components of the vector and scalar potentials.)

61 See, e.g., QM Chapter 9.

62 Let me hope the reader knows that the relation $\lambda = \nu T$ is universal, valid for waves of any nature – see, e.g., CM Chapter 6. (In the case of substantial dispersion, ν means the phase velocity.)

63 The fact that T has canceled, shows that these estimates are valid for waves of arbitrary frequency.

64 Let me emphasize that if this condition is not fulfilled, the lumped-circuit representation of the system (see Fig. 9 and its discussion) is typically inadequate – besides some special cases, to be discussed in the next chapter.

65 See, e.g., MA Eq. (11.7) with **f** = **E** and **g** = **H**.

66 It is named after John Henry Poynting for his work published in 1884, though this fact was independently discovered by O. Heaviside in 1885 in a simpler form, while a similar result for the intensity of mechanical elastic waves had been obtained earlier (in 1874) by Nikolay Alekseevich Umov – see, e.g., CM Sec. 7.7.

[67] Actually, an addition to **S** of the curl of an arbitrary vector function $\mathbf{f}(\mathbf{r}, t)$ does not change Eq. (111). Indeed, we may use the divergence theorem to transform the corresponding change of the surface integral in Eq. (111) to a volume integral of scalar function $\nabla \cdot (\nabla \times \mathbf{f})$ that equals zero at any point – see, e.g., MA Eq. (11.2).

[68] The situation with the macroscopic Maxwell equations is more complex, and is still a subject of some lingering discussions (usually called the Abraham-Minkowski controversy, despite contributions by many other scientists including A. Einstein), because of the ambiguity of momentum's division between its field and particle components – see, e.g., the review paper by R. Pfeiffer et al., Rev. Mod. Phys. **79**, 1197 (2007).

[69] We will return to their general discussion (in particular, to the analytical mechanics of the electromagnetic field, and its stress tensor) in Sec. 9.8, after we have got equipped with the special relativity theory.

[70] This condition, named after Ludwig Lorenz, should not be confused with the so-called Lorentz invariance condition of relativity, due to Hendrik Lorentz, to be discussed in Sec. 9.4. (Note the last names' spelling.)

[71] Here I have to mention in passing the so-called Hertz vector potentials Π_e and $\Pi_\mathbf{m}$ (whose introduction may be traced back at least to the 1904 work by E. Whittaker). They may be defined by the following relations:

$$\mathbf{A} = \mu \frac{\partial \mathbf{\Pi}_\mathrm{e}}{\partial t} + \mu \nabla \times \mathbf{\Pi}_\mathrm{m}, \quad \phi = -\frac{1}{\varepsilon} \nabla \cdot \mathbf{\Pi}_\mathrm{e},$$

which make the Lorentz gauge condition (117) automatically satisfied. These potentials are especially convenient for the solution of problems in which the electromagnetic field is induced by sources characterized by field-independent electric and magnetic polarizations $\mathbf{P}$ and $\mathbf{M}$ – rather than by field-independent charge and current densities ρ and $\mathbf{j}$. Indeed, it is straightforward to check that both $\Pi_\mathbf{e}$ and $\Pi_\mathbf{m}$ satisfy the equations similar to Eqs. (118), but with their right-hand sides equal to, respectively, $-\mathbf{P}$ and $-\mathbf{M}$. Unfortunately, I would not have time for a discussion of such problems and have to refer interested readers elsewhere – for example, to a classical text by J. Stratton, Electromagnetic Theory, Adams Press, 2008.

[72] See, e.g., CM Secs. 6.3-6.4 and 7.7-7.8.

[73] By that time, the speed of light (estimated very reasonably by Ole Rømer as early as 1676) has been experimentally measured, by Hippolyte Fizeau and then Léon Foucault, with an accuracy better than 1%.

[74] A warning: some proofs of Eq. (121) given in textbooks and online sites, are deficient.