LibreTexts

9.8: Analytical Mechanics of Electromagnetic Field

We have just seen that the analytical mechanics of a particle in an electromagnetic field may be used to get some important results. The same is true for the analytical mechanics of the field as such, and the field-particle system as a whole. For such a space-distributed system as the field, governed by local dynamics laws (Maxwell equations), we need to apply analytical mechanics to the local densities l and h of the Lagrangian and Hamiltonian functions, defined by relations

$$\mathscr{L}=\int l d^{3} r, \quad \mathscr{H}=\int h d^{3} r. \tag{9.210}$$

Let us start, as usual, from the Lagrange formalism. Some clues on the possible structure of the Lagrangian function density l may be obtained from that of the particle-field interaction in this formalism, discussed in the last section. As we have seen, for the case of a single particle, the interaction is described by the last two terms of Eq. (183):

$$\mathscr{L}_{\text{int}}=-q \phi-q \mathbf{u} \cdot \mathbf{A}. \tag{9.211}$$

It is virtually obvious that if the charge q is continuously distributed over some volume, we may represent this $\mathscr{L}_{\text{int}}$ as a volume integral of the following Lagrangian function density:

$$l_{\text{int}}=-\rho \phi+\mathbf{j} \cdot \mathbf{A} \equiv-j_{\alpha} A^{\alpha}. \qquad \text{Interaction Lagrangian density} \tag{9.212}$$

Notice that this density (in contrast to $\mathscr{L}_{\text{int}}$ itself!) is Lorentz-invariant. (This is due to the contraction of the longitudinal coordinate, and hence volume, at the Lorentz transform.) Hence we may expect the density of the field's part of the Lagrangian to be Lorentz-invariant as well. Moreover, in the view of the simple, local structure of the Maxwell equations (containing only the first spatial and temporal derivatives of the fields), l_{field} should be a simple function of the potential's 4-vector and its 4-derivative:

$$l_{\text{field}}=l_{\text{field}}\left(A^{\alpha}, \partial_{\alpha} A^{\beta}\right). \tag{9.213}$$

Also, the density should be selected in such a way that the 4-vector analog of the Lagrangian equation of motion,

$$\partial_{\alpha} \frac{\partial l_{\text{field}}}{\partial\left(\partial_{\alpha} A^{\beta}\right)}-\frac{\partial l_{\text{field}}}{\partial A^{\beta}}=0, \tag{9.214}$$

gave us the correct inhomogeneous Maxwell equations (127).[71] It is clear that the field part l_{field} of the total Lagrangian density l should be a scalar, and a quadratic form of the field strength, i.e. of $F^{\alpha \beta}$, so that the natural choice is

$$l_{\text{field}}=\text{const} \times F_{\alpha \beta} F^{\alpha \beta}. \tag{9.215}$$

with the implied summation over both indices. Indeed, adding to this expression the interaction Lagrangian (212),

$$l=l_{\text{field}}+l_{\text{int}}=\text{const} \times F_{\alpha \beta} F^{\alpha \beta}-j_{\alpha} A^{\alpha}, \tag{9.216}$$

and performing the differentiation (214), we see that the relations (214)-(215) indeed yield Eqs. (127), provided that the constant factor equals $\left(-1 / 4 \mu_{0}\right)$.[72] So, the field's Lagrangian density is

$$\text{Field's Lagrangian density} \qquad l_{\text{field}}=-\frac{1}{4 \mu_{0}} F_{\alpha \beta} F^{\alpha \beta}=\frac{1}{2 \mu_{0}}\left(\frac{E^{2}}{c^{2}}-B^{2}\right) \equiv \frac{\varepsilon_{0}}{2} E^{2}-\frac{B^{2}}{2 \mu_{0}} \equiv u_{\mathrm{e}}-u_{\mathrm{m}}, \tag{9.217}$$

where u_{e} is the electric field energy density (1.65), and u_{m} is the magnetic field energy density (5.57). Let me hope the reader agrees that Eq. (217) is a wonderful result because the Lagrangian function has a structure absolutely similar to the well-known expression $\mathscr{L}=T-U$ of the classical mechanics. So, for the field alone, the "potential" and "kinetic" energies are separable again.[73]

Now let us explore whether we can calculate the 4-form of the field's Hamiltonian function $\mathscr{H}$. In the generic analytical mechanics,

$$\mathscr{H}=\sum_{j} \frac{\partial \mathscr{L}}{\partial \dot{q}_{j}} \dot{q}_{j}-\mathscr{L}. \tag{9.218}$$

However, just as for the Lagrangian function, for a field we should find the spatial density h of the Hamiltonian, defined by the second of Eqs. (210), for which the natural 4-form of Eq. (218) is

$$h^{\alpha \beta}=\frac{\partial l}{\partial\left(\partial_{\alpha} A^{\gamma}\right)} \partial^{\beta} A^{\gamma}-g^{\alpha \beta} l. \tag{9.219}$$

Calculated for the field alone, i.e. using Eq. (217) for l_{field}, this definition yields

$$h_{\text{field}}^{\alpha\beta} = \theta^{\alpha\beta} - \tau_D^{\alpha\beta}, \tag{9.220}$$

where the tensor

$$\text{Symmetric energy-momentum tensor} \qquad \theta^{\alpha\beta} \equiv \frac{1}{\mu_0}\left(g^{\alpha\gamma} F_{\gamma\delta} F^{\delta\beta} + \frac{1}{4} g^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}\right), \tag{9.221}$$

is gauge-invariant, while the remaining term,

$$\tau_D^{\alpha\beta} \equiv \frac{1}{\mu_0} g^{\alpha\gamma} F_{\gamma\delta} \partial^\delta A^\beta, \tag{9.222}$$

is not, so that it cannot correspond to any measurable variables. Fortunately, it is straightforward to verify that the last tensor may be represented in the form

$$\tau_D^{\alpha\beta} = \frac{1}{\mu_0} \partial_\gamma \left(F^{\gamma\alpha} A^\beta\right), \tag{9.223}$$

and as a result, obeys the following relations:

$$\partial_\alpha \tau_D^{\alpha\beta} = 0, \qquad \int \tau_D^{0\beta} d^3 r = 0, \tag{9.224}$$

so it does not interfere with the conservation properties of the gauge-invariant, symmetric energy-momentum tensor (also called the symmetric stress tensor) $\theta^{\alpha\beta}$, to be discussed below.

Let us use Eqs. (125) to express the components of the latter tensor via the electric and magnetic fields. For $\alpha = \beta = 0$, we get

$$\theta^{00} = \frac{\varepsilon_0}{2} E^2 + \frac{B^2}{2\mu_0} = u_{\mathrm{e}} + u_{\mathrm{m}} \equiv u, \tag{9.225}$$

i.e. the expression for the total energy density u – see Eq. (6.113). The other 3 components of the same row/column turn out to be just the Cartesian components of the Poynting vector (6.114), divided by c:

$$\theta^{j0} = \frac{1}{\mu_0}\left(\frac{\mathbf{E}}{c} \times \mathbf{B}\right)_j = \left(\frac{\mathbf{E}}{c} \times \mathbf{H}\right)_j \equiv \frac{S_j}{c}, \qquad \text{for } j = 1, 2, 3. \tag{9.226}$$

The remaining 9 components $\theta_{jj'}$ of the tensor, with $j,\ j' = 1, 2, 3$, are usually represented as

$$\theta^{jj'} = -\tau_{jj'}^{(\mathrm{M})}, \tag{9.227}$$

where $\tau^{(\mathrm{M})}$ is the so-called Maxwell stress tensor:

$$\tau_{jj'}^{(\mathrm{M})} = \varepsilon_0\left(E_j E_{j'} - \frac{\delta_{jj'}}{2} E^2\right) + \frac{1}{\mu_0}\left(B_j B_{j'} - \frac{\delta_{jj'}}{2} B^2\right), \qquad \text{Maxwell stress tensor} \tag{9.228}$$

so that the whole symmetric energy-momentum tensor (221) may be conveniently represented in the following symbolic way:

$$\theta^{\alpha\beta} = \left(\begin{array}{c:ccc} u & \leftarrow & \mathbf{S}/c & \rightarrow \\ \hdashline \uparrow & & & \\ \mathbf{S} & & & \\ - & & -\tau_{jj'}^{(M)} & \\ c & & & \\ \downarrow & & & \end{array}\right). \tag{9.229}$$

The physical meaning of this tensor may be revealed in the following way. Considering Eq. (221) as the definition of the tensor $\theta^{\alpha\beta}$,[74] and using the 4-vector form of Maxwell equations given by Eqs. (127) and (129), it is straightforward to verify an extremely simple result for the 4-derivative of the symmetric tensor:

$$\partial_\alpha \theta^{\alpha\beta} = -F^{\beta\gamma} j_\gamma. \tag{9.230}$$

This expression is valid in the presence of electromagnetic field sources, e.g., for any system of charged particles and the fields they have created. Of these four equations (for four values of the index β), the temporal one (with $\beta = 0$) may be simply expressed via the energy density (225) and the Poynting vector (226):

$$\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=-\mathbf{j}\cdot\mathbf{E},\tag{9.231}$$

while three spatial equations (with $\beta=j=1,2,3$) may be represented in the form

$$\frac{\partial}{\partial t}\frac{S_j}{c^2}-\sum_{j'=1}^{3}\frac{\partial}{\partial r_{j'}}\tau_{jj'}^{(\mathrm{M})}=-(\rho\mathbf{E}+\mathbf{j}\times\mathbf{B})_j.\tag{9.232}$$

Integrated this expression over a volume V limited by surface S, with the account of the divergence theorem, Eq. (231) returns us to the Poynting theorem (6.111):

$$\int_V\left(\frac{\partial u}{\partial t}+\mathbf{j}\cdot\mathbf{E}\right)d^3r+\oint_S S_n d^2r=0,\tag{9.233}$$

while Eq. (232) yields[75]

$$\int_V\left[\frac{\partial}{\partial t}\frac{\mathbf{S}}{c^2}+\mathbf{f}\right]_j d^3r=\sum_{j'=1}^{3}\oint_S\tau_{jj'}^{(\mathrm{M})}dA_{j'},\qquad \text{with }\mathbf{f}\equiv\rho\mathbf{E}+\mathbf{j}\times\mathbf{B},\tag{9.234}$$

where $dA_j=n_jdA=n_jd^2r$ is the j^{th} component of the elementary area vector $d\mathbf{A}=\mathbf{n}dA=\mathbf{n}d^2\mathbf{r}$ that is normal to volume's surface, and directed out of the volume – see Fig. 17.[76]

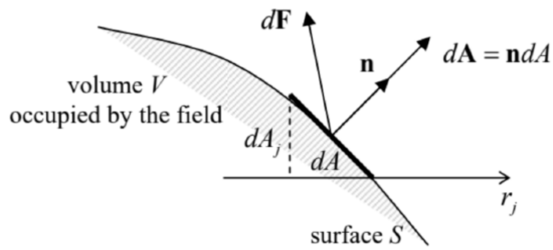

Fig. 9.17. The force $d\mathbf{F}$ exerted on a boundary element $d\mathbf{A}$ of the volume V occupied by the field.

Since, according to Eq. (5.10), the vector **f** in Eq. (234) is nothing other than the density of volume-distributed Lorentz forces exerted by the field on the charged particles, we can use the 2^{nd} Newton law, in its relativistic form (144), to rewrite Eq. (234), for a stationary volume V, as

$$\text{Field momentum's dynamics}\qquad \frac{d}{dt}\left[\int_V\frac{\mathbf{S}}{c^2}d^3r+\mathbf{p}_{\text{part}}\right]=\mathbf{F},\tag{9.235}$$

where $\mathbf{p}_{\text{part}}$ is the total mechanical (relativistic) momentum of all particles in the volume V, and the vector **F** is defined by its Cartesian components:

$$F_j=\sum_{j'=1}^{3}\oint_S\tau_{jj'}^{(\mathrm{M})}dA_{j'}.\qquad \text{Force via the Maxwell tensor}\tag{9.236}$$

Relations (235)-(236) are our main new results. The first of them shows that the vector

$$\mathbf{g}\equiv\frac{\mathbf{S}}{c^2},\tag{9.237}$$

already discussed in Sec. 6.8 without derivation, may be indeed interpreted as the density of momentum of the electromagnetic field (per unit volume). This classical relation is consistent with the quantum-mechanical picture of photons as ultra-relativistic particles, with the momentum's magnitude $\mathscr{E}/c$, because then the total flux of the momentum carried by photons through a unit normal area per unit time may be represented either as S_n/c or as g_nc. It also allows us to revisit the Poynting vector paradox that was discussed in Sec. 6.8 – see Fig. 611 and its discussion. As was emphasized in this discussion, the vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ in this case does not correspond to any measurable energy flow. However, the corresponding momentum of the field, equal to the integral of the density (237) over a volume of interest,[77] is not only real but may be measured by the recoil impulse it gives to the field sources – say, to a magnetic coil inducing the field **H**, or to the capacitor plates creating the field **E**.

Now let us turn to our second result, Eq. (236). It tells us that the 3×3-element Maxwell stress tensor complies with the general definition of the stress tensor[78] characterizing the force **F** exerted on the boundary of a volume, in our current case occupied by the electromagnetic field (Fig. 17). Let us use this important result to analyze two simple examples of static fields.

(i) Electrostatic field's effect on a perfect conductor. Since Eq. (235) has been derived for a free space region, we have to select volume V outside the conductor, but we may align one of its faces with the conductor's surface (Fig. 18).

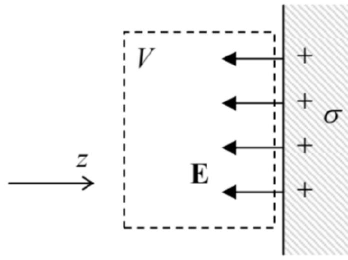

Fig. 9.18. The electrostatic field near a conductor's surface.

From Chapter 2, we know that the electrostatic field has to be perpendicular to the conductor's surface. Selecting the z-axis in this direction, we have $E_x = E_y = 0, E_z = \pm E$, so that only diagonal components of the tensor (228) are not equal to zero:

$$\tau_{xx}^{(\mathrm{M})} = \tau_{yy}^{(\mathrm{M})} = -\frac{\varepsilon_0}{2}E^2, \quad \tau_{zz}^{(\mathrm{M})} = \frac{\varepsilon_0}{2}E^2, \tag{9.238}$$

Since the elementary surface area vector has just one non-zero component, dA_z, according to Eq. (236), only the last component (that is positive regardless of the sign of E) gives a contribution to the surface force **F**. We see that the force exerted by the conductor (and eventually by external forces that hold the conductor in its equilibrium position) on the field is normal to the conductor and directed out of the field volume: $dF_z \geq 0$. Hence, by the 3rd Newton law, the force exerted by the field on the conductor's surface is directed toward the field-filled space:

$$\text{Electric field's pull} \qquad dF_{\text{surface}} = -dF_z = -\frac{\varepsilon_0}{2}E^2 dA \tag{9.239}$$

This important result could be obtained by simpler means as well. (Actually, this was the task of one of the problems given in Chapter 2.) For example, one could argue, quite convincingly, that the local relation between the force and the field should not depend on the global configuration creating the field, and thus consider the simplest configuration, a planar capacitor (see, e.g. Fig. 2.3) with surfaces of both plates charged by equal and opposite charges of density $\sigma = \pm\varepsilon_0 E$. According to the Coulomb law, the charges should attract each other, pulling each plate toward the field region, so that the Maxwell-tensor result gives the correct direction of the force. The force's magnitude given by Eq. (239) may be verified either by the direct integration of the Coulomb law or by the following simple reasoning. In the plane capacitor, the field $E_z = \sigma/\varepsilon_0$ is equally contributed by two surface charges; hence the field created by the negative charge of the counterpart plate (not shown in Fig. 18) is $E_- = -\sigma/2\varepsilon_0$, and the force it exerts of the elementary surface charge $dQ = \sigma dA$ of the positively charged plate is $dF_{\text{surface}} = EdQ = -\sigma^2 dA/2\varepsilon_0 = \varepsilon_0 E^2 dA/2$, in accordance with Eq. (239).[79]

Quantitatively, even for such a high electric field as $E = 3$ MV/m (close to electric breakdown threshold in the air), the "negative pressure" (dF/dA) given by Eq. (239) is of the order of 500 Pa $\left(\mathrm{N/m^2}\right)$, i.e. below one-thousandth of the ambient atmospheric pressure $\left(1 \text{ bar} \approx 10^5 \text{ Pa}\right)$. Still, this negative pressure may be substantial (above 1 bar) in some cases, for example in good dielectrics (such as high-quality $\mathrm{SiO_2}$, grown at high temperature, which is broadly used in integrated circuits), which can withstand electric fields up to $\sim 10^9$ V/m.

(ii) Static magnetic field's effect on its source[80] – say a solenoid's wall or a superconductor's surface (Fig. 19). With the choice of coordinates shown in that figure, we have $B_x = \pm B, B_y = B_z = 0$, so that the Maxwell stress tensor (228) is diagonal again:

$$\tau_{xx}^{(\mathrm{M})} = \frac{1}{2\mu_0}B^2, \quad \tau_{yy}^{(\mathrm{M})} = \tau_{zz}^{(\mathrm{M})} = -\frac{1}{2\mu_0}B^2. \tag{9.240}$$

However, since for this geometry, only dA_z differs from 0 in Eq. (236), the sign of the resulting force is opposite to that in electrostatics: $dF_z \leq 0$, and the force exerted by the magnetic field upon the conductor's surface,

$$\text{Magnetic field's push} \qquad dF_{\text{surface}} = -dF_z = \frac{1}{2\mu_0}B^2 dA, \tag{9.241}$$

corresponds to positive pressure. For good laboratory magnets $(B \sim 10 \text{ T})$, this pressure is of the order of $4 \times 10^7 \text{ Pa} \approx 400 \text{ bars}$, i.e. is very substantial, so the magnets require solid mechanical design.

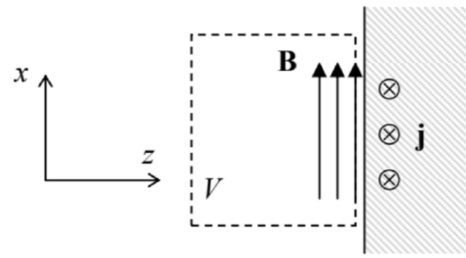

Fig. 9.19. The magnetostatic field near a current-carrying surface.

The direction of the force (241) could be also readily predicted using elementary magnetostatics arguments. Indeed, we can imagine the magnetic field volume limited by another, parallel wall with the opposite direction of surface current. According to the starting point of magnetostatics, Eq. (5.1), such surface currents of opposite directions have to repulse each other – doing that via the magnetic field.

Another explanation of the fundamental sign difference between the electric and magnetic field pressures may be provided using the electric circuit language. As we know from Chapter 2, the potential energy of the electric field stored in a capacitor may be represented in two equivalent forms,

$$U_{\mathrm{e}}=\frac{CV^2}{2}=\frac{Q^2}{2C}. \tag{9.242}$$

Similarly, the magnetic field energy of in an inductive coil is

$$U_{\mathrm{m}}=\frac{LI^2}{2}=\frac{\Phi^2}{2L}. \tag{9.243}$$

If we do not want to consider the work of external sources at a virtual change of the system dimensions, we should use the last forms of these relations, i.e. consider a galvanically detached capacitor $(Q=\text{const})$ and an externally-shorted inductance $(\Phi=\text{const})$.[81] Now if we let the electric field forces (239) to drag capacitor's plates in the direction they "want", i.e. toward each other, this would lead to a reduction of the capacitor thickness, and hence to an increase of its capacitance C, and hence to a decrease of U_{e}. Similarly, for a solenoid, allowing the positive pressure (241) to move its walls from each other would lead to an increase of the solenoid's volume, and hence of its inductance L, so that the potential energy U_{m} would be also reduced – as it should be. It is remarkable (actually, beautiful) how do the local field formulas (239) and (241) "know" about these global circumstances.

Finally, let us see whether the major results (237) and (241), obtained in this section, match each other. For that, let us return to the normal incidence of a plane, monochromatic wave from the free space upon the plane surface of a perfect conductor (see, e.g., Fig. 7.8 and its discussion), and use those results to calculate the time average of the pressure dF_{surface}/dA imposed by the wave on the surface. At elastic reflection from the conductor's surface, the electromagnetic field's momentum retains its amplitude but reverses its sign, so that the average momentum transferred to a unit area of the surface in a unit time (i.e. the average pressure) is

$$\overline{\frac{dF_{\text{surface}}}{dA}}=2c\overline{g_{\text{incident}}}=2c\frac{\overline{S_{\text{incident}}}}{c^2}=2c\frac{\overline{EH}}{c^2}=\frac{E_\omega H_\omega^*}{2}, \tag{9.244}$$

where E_ω and H_ω are complex amplitudes of the incident wave. Using the relation (7.7) between these amplitudes (for $\varepsilon=\varepsilon_0$ and $\mu=\mu_0$ giving $E_\omega=cB_\omega$), we get

$$\overline{\frac{dF_{\text{surface}}}{dA}}=\frac{1}{c}cB_\omega\frac{B_\omega^*}{\mu_0}\equiv\frac{|B_\omega|^2}{\mu_0}. \tag{9.245}$$

On the other hand, as was discussed in Sec. 7.3, at the surface of a perfect mirror the electric field vanishes while the magnetic field doubles, so that we can use Eq. (241) with $B\rightarrow B(t)=2\,\mathrm{Re}[B_\omega\exp\{-i\omega t\}]$. Averaging the pressure given by Eq. (241) over time, we get

$$\overline{\frac{dF_{\text{surface}}}{dA}}=\frac{1}{2\mu_0}\overline{\left(2\,\mathrm{Re}\left[B_\omega e^{-i\omega t}\right]\right)^2}=\frac{|B_\omega|^2}{\mu_0}, \tag{9.246}$$

i.e. the same result as Eq. (245).

For physics intuition development, it is useful to estimate the electromagnetic radiation pressure's magnitude. Even for a relatively high wave intensity S_n of $1\,\mathrm{kW/m^2}$ (close to that of the direct sunlight at the Earth's surface), the pressure $2cg_n=2S_n/c$ is somewhat below 10^{-5} Pa $\sim 10^{-10}\,\mathrm{bar}$. Still, this extremely small effect was experimentally observed (by P. Lebedev) as early as 1899, giving one more confirmation of Maxwell's theory. Currently, there are ongoing attempts to use the pressure of the Sun's light for propelling small spacecraft, e.g., the LightSail 2 satellite with a $32-\mathrm{m}^2$ sail, launched in 2019.

Reference

[71] Here the implicit summation over the index α plays the role similar to the convective derivative (188) in replacing the full derivative over time, in a way that reflects the symmetry of time and space in special relativity. I do not want to spend more time justifying Eq. (214), because of the reasons that will be clear imminently.

[72] In the Gaussian units, this coefficient is $(-1/16\pi)$.

[73] Since the Lagrange equations of motion are homogeneous, the simultaneous change of the signs of T and U does not change them. Thus, it is not important which of the two energy densities, u_e or u_m, we count as the potential, and which as the kinetic energy. (Actually, such duality of the two field energy components is typical for all analytical mechanics – see, e.g., the discussion in CM Sec. 2.2.)

[74] In this way, we are using Eq. (219) just as a useful guess, which has led us to the definition of $\theta^{\alpha\beta}$, and may leave its strict justification for more in-depth field theory courses.

[75] Just like the Poynting theorem (233), Eq. (234) may be obtained directly from the Maxwell equations, without resorting to the 4-vector formalism – see, e.g., Sec. 8.2.2 in D. Griffiths, Introduction to Electrodynamics, 3rd ed., Prentice-Hall, 1999. However, the derivation discussed above is superior because it shows the wonderful unity between the laws of conservation of energy and momentum.

[76] The same notions are used in the mechanical stress theory – see, e.g., CM Sec. 7.2.

[77] It is sometimes called the hidden momentum.

[78] See, e.g., CM Sec. 7.2.

[79] By the way, repeating these arguments for a plane capacitor filled with a linear dielectric, we may readily see that Eq. (239) may be generalized for this case by replacing ε_0 for ε. A similar replacement $(\mu_0 \rightarrow \mu)$ is valid for Eq. (241) in a linear magnetic medium.

[80] The causal relation is not important here. Especially in the case of a superconductor, the magnetic field may be induced by another source, with the surface supercurrent **j** just shielding the superconductor's bulk from its penetration – see Sec. 6.

[81] Of course, this condition may hold "forever" only for solenoids with superconducting wiring, but even in normal-metal solenoids with practicable inductances, the flux relaxation constants L/R may be rather large (practically, up to a few minutes), quite sufficient to carry out the force measurement.