6.1: Electromagnetic Induction

As Eqs. (5.36) show, in static situations $(\partial/\partial t = 0)$ the Maxwell equations describing the electric and magnetic fields are independent – coupled only implicitly, via the continuity equation (4.5) relating their right-hand sides ρ and $\mathbf{j}$. In dynamics, when the fields change in time, the situation in different.

Historically, the first discovered explicit coupling between the electric and magnetic fields was the effect of electromagnetic induction. Although the induction effect was discovered independently by Joseph Henry, it was a brilliant series of experiments by Michael Faraday, carried out mostly in 1831, that resulted in the first general formulation of the induction law. The summary of Faraday's numerous experiments has turned out to be very simple: if the magnetic flux, defined by Eq. (5.65),

$$\Phi \equiv \int_S B_n d^2 r, \tag{6.1}$$

through the surface S limited by a closed contour C, changes in time by whatever reason (e.g., either due to a change of the magnetic field $\mathbf{B}$ (as in Fig.1), or the contour's motion, or its deformation, or any combination of the above), it induces an additional, vortex-like electric field $\mathbf{E}_{\text{ind}}$ directed along the
contour – see Fig. 1.

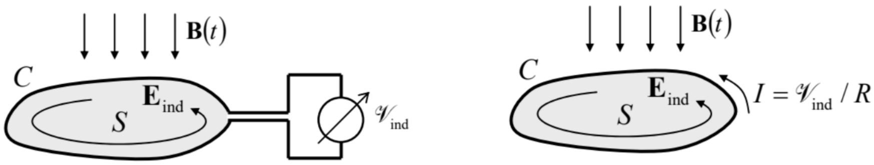

Fig. 6.1. Two simplest ways to observe the Faraday electromagnetic induction.

The exact distribution of $\mathbf{E}_{\text{ind}}$ in space depends on the system's details, but its integral along the contour C, called the inductive electromotive force (e.m.f.), obeys a very simple Faraday induction law:

$$\mathscr{V}_{\text{ind}} \equiv \oint_C \mathbf{E}_{\text{ind}} \cdot d\mathbf{r} = -\frac{d\Phi}{dt}. \qquad \text{Faraday induction law} \tag{6.2}$$

(In the Gaussian units, the right-hand side of this formula has an additional coefficient $1/c$.)

It is straightforward (and hence left for the reader's exercise) to show that this e.m.f. may be measured, for example, either inserting a voltmeter into a conducting loop following the contour C, or by measuring the small current $I = \mathscr{V}_{\text{ind}}/R$ it induces in a thin wire with a sufficiently large Ohmic resistance R,[2] whose shape follows that contour – see Fig. 1. (Actually, these methods are not entirely different, because a typical voltmeter measures voltage by the small Ohmic current it drives through a known high internal resistance of the device.) In the context of the latter approach, the minus sign in Eq. (2) may be described by the following Lenz rule: the magnetic field of the induced current I provides a partial compensation of the change of the original flux $\Phi(t)$ with time.[3]

In order to recast Eq. (2) in a differential form, more convenient in many cases, let us apply to the contour integral in it the same Stokes theorem that was repeatedly used in Chapter 5. The result is

$$\mathscr{V}_{\text{ind}} = \int_S (\nabla \times \mathbf{E}_{\text{ind}})_n d^2 r. \tag{6.3}$$

Now combining Eqs. (1)-(3), for a contour C whose shape does not change in time (so that the integration along it is interchangeable with the time derivative), we get

$$\int_S \left(\nabla \times \mathbf{E}_{\text{ind}} + \frac{\partial \mathbf{B}}{\partial t} \right)_n d^2 r = 0. \tag{6.4}$$

Since the induced electric field is an addition to the gradient field (1.33) created by electric charges, for the net field we may write $\mathbf{E} = \mathbf{E}_{\text{ind}} - \nabla\phi$. However, since the curl of any gradient field is zero,[4] $\nabla \times (\nabla\phi) = 0$, Eq. (4) remains valid even for the net field $\mathbf{E}$. Since this equation should be correct for any closed area S, we may conclude that

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \text{Faraday law: differential form} \tag{6.5}$$

at any point. This is the final (time-dependent) form of this Maxwell equation. Superficially, it may look that Eq. (5) is less general than Eq. (2); for example, it does not describe any electric field, and hence any e.m.f. in a moving loop, if the field $\mathbf{B}$ is constant in time, even if the magnetic flux (1) through the loop does change in time. However, this is not true; in Chapter 9 we will see that in the reference frame moving with the loop such e.m.f. does appear.[5]

Now let us reformulate Eq. (5) in terms of the vector potential $\mathbf{A}$. Since the induction effect does not alter the fundamental relation $\nabla \cdot \mathbf{B} = 0$, we still may represent the magnetic field as prescribed by Eq. (5.27), i.e. as $\mathbf{B} = \nabla \times \mathbf{A}$. Plugging this expression into Eq. (5), and changing the order of the temporal and spatial differentiation, we get

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0. \tag{6.6}$$

Hence we can use the same argumentation as in Sec. 1.3 (there applied to the vector $\mathbf{E}$ alone) to represent the expression in the parentheses as $-\nabla\phi$, so that we get

$$\text{Fields via potentials} \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{6.7}$$

It is very tempting to interpret the first term of the right-hand side of the expression for $\mathbf{E}$ as the one describing the electromagnetic induction alone, and the second term as representing a purely electrostatic field induced by electric charges. However, the separation of these two terms is, to a certain extent, conditional. Indeed, let us consider the gauge transformation already mentioned in Sec. 5.2,

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi, \tag{6.8}$$

that, as we already know, does not change the magnetic field. According to Eq. (7), to keep the full electric field intact (gauge-invariant) as well, the scalar electric potential has to be transformed simultaneously, as

$$\phi \rightarrow \phi - \frac{\partial \chi}{\partial t}, \tag{6.9}$$

leaving the choice of an addition to ϕ restricted only by the Laplace equation – since the full ϕ should satisfy the Poisson equation (1.41) with a gauge-invariant right-hand side. We will return to the discussion of the gauge invariance in Sec. 4.

Reference

[2] Such induced current is sometimes called the eddy current, though most often this term is reserved for the distributed currents induced by changing magnetic fields in bulk conductors – see Sec. 3 below.

[3] Let me also hope that the reader is familiar with the paradox arising at attempts to measure $\mathscr{V}_{\text{ind}}$ with a voltmeter without its insertion into the wire loop; if not, I would highly recommend them to solve Problem 2.

[4] See, e.g., MA Eq. (11.1).

[5] I have to admit that from the beginning of the course, I was carefully sweeping under the rug a very important question: in what exactly reference frame(s) all the equations of electrodynamics are valid? I promise to discuss this issue in detail later in the course (in Chapter 9), and for now would like to get away with a very short answer: all the formulas discussed so far are valid in any inertial reference frame, as defined in classical mechanics – see, e.g., CM Sec. 1.3; however, the fields $\mathbf{E}$ and $\mathbf{B}$ have to be measured in the same reference frame.