5.3: Magnetic Flux, Energy, and Inductance

Considering the currents flowing in a system as generalized coordinates, the magnetic forces (1) between them are their unique functions, and in this sense, the energy U of their magnetic interaction may be considered the potential energy of the system. The apparent (but somewhat deceptive) way to derive an expression for this energy is to use the analogy between Eq. (1) and its electrostatic analog, Eq. (2). Indeed, Eq. (2) may be transformed into Eq. (1) with just three replacements:

(i) $\rho(\mathbf{r})\rho'(\mathbf{r}')$ should be replaced with $\left[\mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}')\right]$,

(ii) ε_0 should be replaced with $1/\mu_0$, and

(iii) the sign before the double integral has to be replaced with the opposite one.

Hence we may avoid repeating the calculation made in Chapter 1, by making these replacements in Eq. (1.59), which gives the electrostatic potential energy of the system with $\rho(\mathbf{r})$ and $\rho'(\mathbf{r}')$ describing the same charge distribution, i.e. with $\rho'(\mathbf{r})=\rho(\mathbf{r})$, to get the following expression for the magnetic potential energy in the system with, similarly, $\mathbf{j}'(\mathbf{r})=\mathbf{j}(\mathbf{r})$:[27]

$$U_j=-\frac{\mu_0}{4\pi}\frac{1}{2}\int d^3r\int d^3r'\frac{\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{5.53}$$

However, this is not the unique, and even not the most convenient answer. Actually, Eq. (53) describes the proper energy of the system (whose minimum corresponds to its stable equilibrium), only in the case when the interacting currents are fixed – just as Eq. (1.59) is adequate when the interacting charges are fixed. Here comes a substantial difference between the electrostatics and the magnetostatics: due to the fundamental fact of charge conservation (already discussed in Secs. 1.1 and 4.1), keeping electric charges fixed does not require external work, while the maintenance of currents generally does. As a result, Eq. (53) describes the energy of the magnetic interaction plus of the system keeping the currents constant – or rather of its part depending on the system under our consideration.[28]

Now to exclude from U_j the contribution due to the interaction with the current-supporting system(s), i.e. calculate the potential energy U of our system as such, we need to know this contribution. The simplest way to do this is to use the Faraday induction law, which describes this interaction. This is why let me postpone the derivation until the beginning of the next chapter, and for now ask the reader to believe me that its account leads to an addition to U_j a term of a twice larger magnitude, so that the result is given by an expression similar to Eq. (53), but with the opposite sign:

$$U=\frac{\mu_0}{4\pi}\frac{1}{2}\int d^3r\int d^3r'\frac{\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},\qquad \text{Magnetic interaction energy} \tag{5.54}$$

I promise to prove this fact in Sec. 6.2 below, but actually, this sign dichotomy should not be quite surprising to the attentive reader, in the context of a similar duality of Eqs. (3.73) and (3.81) for the electrostatic energy.

Due to the importance of Eq. (54), let us rewrite it in several other forms, convenient for different applications. First of all, just as in electrostatics, it may be recast into a potential-based form. Indeed, with the definition (28) of the vector potential $\mathbf{A}(\mathbf{r})$, Eq. (54) becomes[29]

This formula, which is a clear magnetic analog of Eq. (1.60) of electrostatics, is very popular among field theorists, because it is very handy for their manipulations. However, for many calculations, it is more convenient to have a direct expression for energy via the magnetic field. Again, this may be done very similarly to what had been done for electrostatics in Sec. 1.3, i.e. by plugging into Eq. (55) the current density expressed from Eq. (35), and then transforming it as[30]

$$U=\frac{1}{2}\int\mathbf{j}\cdot\mathbf{A}d^3r=\frac{1}{2\mu_0}\int\mathbf{A}\cdot(\nabla\times\mathbf{B})d^3r=\frac{1}{2\mu_0}\int\mathbf{B}\cdot(\nabla\times\mathbf{A})d^3r-\frac{1}{2\mu_0}\int\nabla\cdot(\mathbf{A}\times\mathbf{B})d^3r. \tag{5.56}$$

Now using the divergence theorem, the second integral may be transformed into a surface integral of $(\mathbf{A}\times\mathbf{B})_n$. According to Eqs. (27)-(28) if the current distribution $\mathbf{j}(\mathbf{r})$ is localized, this vector product drops, at large distances, faster than $1/r^2$, so that if the integration volume is large enough, the surface integral is negligible. In the remaining first integral in Eq. (56), we may use Eq. (27) to rewrite $\nabla\times\mathbf{A}$ as $\mathbf{B}$. As a result, we get a very simple and fundamental formula.

$$U=\frac{1}{2\mu_0}\int B^2d^3r. \tag{5.57a}$$

Just as with the electric field, this expression may be interpreted as a volume integral of the magnetic energy density u:

$$\text{Magnetic field energy} \qquad U=\int u(\mathbf{r}) d^3 r, \quad \text{with } u(\mathbf{r}) \equiv \frac{1}{2 \mu_0} \mathbf{B}^2(\mathbf{r}), \tag{5.57b}$$

clearly similar to Eq. (1.65).[31] Again, the conceptual choice between the spatial localization of magnetic energy – either at the location of electric currents only, as implied by Eqs. (54) and (55), or in all regions where the magnetic field exists, as apparent from Eq. (57b), cannot be done within the framework of magnetostatics, and only the electrodynamics gives a decisive preference for the latter choice.

For the practically important case of currents flowing in several thin wires, Eq. (54) may be first integrated over the cross-section of each wire, just as was done at the derivation of Eq. (4). Again, since the integral of the current density over the k^{th} wire's cross-section is just the current I_k in the wire, and cannot change along its length, it may be taken from the remaining integrals, giving

$$U=\frac{\mu_0}{4 \pi} \frac{1}{2} \sum_{k, k'} I_k I_{k'} \oint_{l_k l_{k'}} \frac{d \mathbf{r}_k \cdot d \mathbf{r}_{k'}}{\left|\mathbf{r}_k-\mathbf{r}_{k'}\right|}, \tag{5.58}$$

where l_k is the full length of the k^{th} wire loop. Note that Eq. (58) is valid if all currents I_k are independent of each other, because the double sum counts each current pair twice, compensating the coefficient ½ in front of the sum. It is useful to decompose this relation as

$$U=\frac{1}{2} \sum_{k, k'} I_k I_{k'} L_{k k'}, \qquad \text{Mutual inductance coefficients} \tag{5.60}$$

The coefficient $L_{k k'}$ with $k \neq k'$, is called the mutual inductance between current the k^{th} and k'^{th} loops, while the diagonal coefficient $L_k \equiv L_{k k}$ is called the self-inductance (or just inductance) of the k^{th} loop.[32] From the symmetry of Eq. (60) with respect to the index swap, $k \leftrightarrow k'$, it evident that the matrix of coefficients $L_{k k'}$ is symmetric:[33]

$$L_{k k'}=L_{k' k}, \tag{5.61}$$

so that for the practically most important case of two interacting currents I_1 and I_2, Eq. (59) reads

$$U=\frac{1}{2} L_1 I_1^2+M I_1 I_2+\frac{1}{2} L_2 I_2^2, \tag{5.62}$$

where $M \equiv L_{12}=L_{21}$ is the mutual inductance coefficient.

These formulas clearly show the importance of the self- and mutual inductances, so I will demonstrate their calculation for at least a few basic geometries. Before doing that, however, let me recast Eq. (58) into one more form that may facilitate such calculations. Namely, let us notice that for the magnetic field induced by current I_k in a thin wire, Eq. (28) is reduced to

$$\mathbf{A}_k(\mathbf{r})=\frac{\mu_0}{4 \pi} I_k \int_{l'} \frac{d \mathbf{r}_k}{\left|\mathbf{r}-\mathbf{r}_k\right|}, \tag{5.63}$$

so that Eq. (58) may be rewritten as

$$U=\frac{1}{2} \sum_{k, k'} I_k \oint_{l_k} \mathbf{A}_{k'}\left(\mathbf{r}_k\right) \cdot d \mathbf{r}_{k'}. \tag{5.64}$$

But according to the same Stokes theorem that was used earlier in this chapter to derive the Ampère law, and Eq. (27), the integral in Eq. (64) is nothing else than the magnetic field's flux Φ (more frequently called just the magnetic flux) through a surface S limited by the contour l:

$$\text{Magnetic flux} \qquad \oint_l \mathbf{A}(\mathbf{r}) \cdot d \mathbf{r}=\int_S(\nabla \times A)_n d^2 r=\int_S B_n d^2 r \equiv \Phi \tag{5.65}$$

- in that particular case, the flux $\Phi_{k k'}$ of the field induced by the k'^{th} current through the loop of the k^{th} current.[34] As a result, Eq. (64) may be rewritten as

$$U=\frac{1}{2} \sum_{k, k'} I_k \Phi_{k k'}. \tag{5.66}$$

Comparing this expression with Eq. (59), we see that

$$\Phi_{k k'} \equiv \int_{S_k}\left(\mathbf{B}_{k'}\right)_n d^2 r=L_{k k'} I_{k'}, \tag{5.67}$$

This expression not only gives us one more means for calculating the coefficients $L_{kk'}$, but also shows their physical sense: the mutual inductance characterizes what part of the magnetic field (colloquially, "what fraction of field lines") induced by the current $I_{k'}$, pierces the k^{th} loop's area S_k – see Fig. 7.

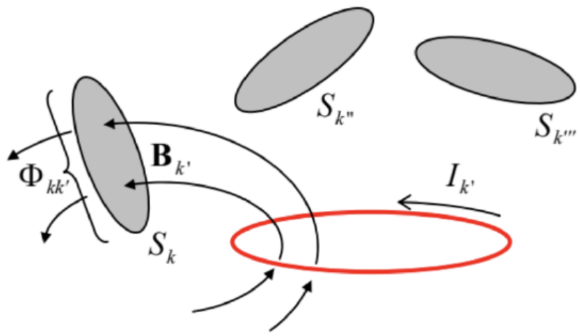

Fig. 5.7. The physical sense of the mutual inductance coefficient $L_{kk'} \equiv \Phi'_{kk'}/I_{k'}$ – schematically.

Due to the linear superposition principle, the total flux piercing the k^{th} loop may be represented as

$$\text{Magnetic flux from currents} \qquad \Phi_k \equiv \sum_{k'} \Phi_{kk'} = \sum_{k'} L_{kk'} I_{k'} \tag{5.68}$$

For example, for the system of two currents, this expression is reduced to a clear analog of Eqs. (2.19):

$$\begin{aligned} \Phi_1 &= L_1 I_1 + M I_2, \\ \Phi_2 &= M I_1 + L_2 I_2. \end{aligned} \tag{5.69}$$

For the even simpler case of a single current,

$$\Phi \text{ of a single current} \qquad \Phi = LI, \tag{5.70}$$

so that the magnetic energy of the current may be represented in several equivalent forms:

$$U = \frac{L}{2} I^2 = \frac{1}{2} I\Phi = \frac{1}{2L}\Phi^2. \qquad \text{U of a single current} \tag{5.71}$$

These relations, similar to Eqs. (2.14)-(2.15) of electrostatics, show that the self-inductance L of a current loop may be considered as a measure of the system's magnetic energy. However, as we will see in Sec. 6.1, this measure is adequate only if the flux I, rather than the current I, is fixed.

Now we are well equipped for the calculation of inductance coefficients for particular systems, having three options. The first one is to use Eq. (60) directly.[35] The second one is to calculate the magnetic field energy from Eq. (57) as the function of all currents I_k in the system, and then use Eq. (59) to find all coefficients $L_{kk'}$. For example, for a system with just one current, Eq. (71) yields

$$L = \frac{U}{I^2/2}. \tag{5.72}$$

Finally, if the system consists of thin wires, so that the loop areas Sk and hence fluxes $\Phi_{kk'}$ are well defined, we may calculate them from Eq. (65), and then use Eq. (67) to find the inductances.

Actually, the first two options may have technical advantages over the third one even for some thin-wire systems, in which the notion of magnetic flux is not quite apparent. As an important example, let us find the self-inductance of a long solenoid – see Fig. 6a again. We have already calculated the magnetic field inside it – see Eq. (40) – so that, due to the field uniformity, the magnetic flux piercing each wire turn is just

$$\Phi_1 = BA = \mu_0 n I A, \tag{5.73}$$

where A is the area of the solenoid's cross-section – for example πR^2 for a round solenoid, though Eq. (40), and hence Eq. (73) are valid for cross-sections of any shape. Comparing Eqs. (73) with Eq. (70), one might wrongly conclude that $L = \Phi_1/I = \mu_0 n A$ (WRONG!), i.e. that the solenoid's inductance is independent of its length. Actually, the magnetic flux Φ_1 pierces each wire turn, so that the total flux through the whole current loop, consisting of N turns, is

$$\Phi = N\Phi_1 = \mu_0 n^2 l A I, \tag{5.74}$$

and the correct expression for the long solenoid's self-inductance is

$$L=\frac{\Phi}{I}=\mu_0 n^2 lA\equiv\frac{\mu_0 N^2 A}{l},\qquad L \text{ of a solenoid} \tag{5.75}$$

i.e. the inductance scales as N^2, not as N.

Since this reasoning may seem not quite evident, it is prudent to verify this result by using Eq. (72), with the full magnetic energy inside the solenoid (neglecting minor fringe field contributions), given by Eq. (57) with $\mathbf{B}=\text{const}$ within the internal volume $V=lA$, and zero outside of it:

$$U=\frac{1}{2\mu_0}B^2Al=\frac{1}{2\mu_0}(\mu_0 nI)^2Al\equiv\mu_0 n^2 lA\frac{I^2}{2}. \tag{5.76}$$

Plugging this relation into Eq. (72) immediately confirms the result (75).

This approach becomes virtually inevitable for continuously distributed currents. As an example, let us calculate the self-inductance L of a long coaxial cable with the cross section shown in Fig. 8,[36] with the full current in the outer conductor equal and opposite to that (I) in the inner conductor.

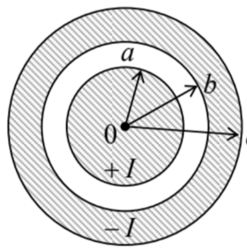

Fig. 5.8. The cross-section of a coaxial cable.

Let us assume that the current is uniformly distributed over the cross-sections of both conductors. (As we know from the previous chapter, this is indeed the case if both the internal and external conductors are made of a uniform resistive material.) First, we should calculate the radial distribution of the magnetic field – which has only one, azimuthal component, because of the axial symmetry of the problem. This distribution may be immediately found applying the Ampère law (37) to the circular contours of radii ρ within four different ranges:

$$2\pi\rho B=\mu_0 I|_{\text{piercing the contour}}=\mu_0 I\times\begin{cases}\rho^2/a^2, & \text{for } \rho<a,\\ 1, & \text{for } a<\rho<b,\\ \left(c^2-\rho^2\right)/\left(c^2-b^2\right), & \text{for } b<\rho<c,\\ 0, & \text{for } c<\rho.\end{cases} \tag{5.77}$$

Now, an easy integration yields the magnetic energy per unit length of the cable:

$$\begin{aligned}\frac{U}{l}&=\frac{1}{2\mu_0}\int B^2d^2r=\frac{\pi}{\mu_0}\int_0^\infty B^2\rho d\rho=\frac{\mu_0 I^2}{4\pi}\left[\int_0^a\left(\frac{\rho}{a^2}\right)^2\rho d\rho+\int_a^b\left(\frac{1}{\rho}\right)^2\rho d\rho+\int_b^c\left(\frac{c^2-\rho^2}{\rho\left(c^2-b^2\right)}\right)^2\rho d\rho\right]\\ &=\frac{\mu_0}{2\pi}\left[\ln\frac{b}{a}+\frac{c^2}{c^2-b^2}\left(\frac{c^2}{c^2-b^2}\ln\frac{c}{b}-\frac{1}{2}\right)\right]\frac{I^2}{2}.\end{aligned} \tag{5.78}$$

From here, and Eq. (72), we get the final answer:

$$\frac{L}{l}=\frac{\mu_0}{2\pi}\left[\ln\frac{b}{a}+\frac{c^2}{c^2-b^2}\left(\frac{c^2}{c^2-b^2}\ln\frac{c}{b}-\frac{1}{2}\right)\right]. \tag{5.79}$$

Note that for the particular case of a thin outer conductor, $c-b<<b$, this expression reduces to

$$\frac{L}{l}\approx\frac{\mu_0}{2\pi}\left(\ln\frac{b}{a}+\frac{1}{4}\right), \tag{5.80}$$

where the first term in the parentheses is due to the contribution of the magnetic field energy in the free space between the conductors. This distinction is important for some applications because in superconductor cables, as well as the normal-metal cables as high frequencies (to be discussed in the next chapter), the field does not penetrate the conductor's bulk, so that Eq. (80) is valid without the last term, 1/4, in the parentheses – for any $b<c$.

As the last example, let us calculate the mutual inductance between a long straight wire and a round wire loop adjacent to it (Fig. 9), neglecting the thickness of both wires.

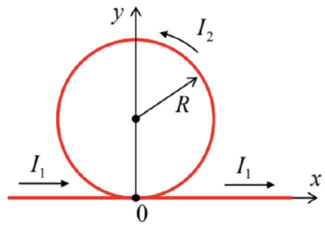

Fig. 5.9. An example of the mutual inductance calculation.

Here there is no problem with using the last of the approaches discussed above, based on the direct calculation of the magnetic flux. Indeed, as was discussed in Sec. 1, the field $\mathbf{B}_1$ induced by the current I_1 at any point of the round loop is normal to its plane – e.g., to the plane of drawing of Fig. 9. In the Cartesian coordinates shown in that figure, Eq. (20) reads $B_1 = \mu_0 I_1 / 2\pi y$, giving the following magnetic flux through the loop:

$$\Phi_{21} = \frac{\mu_0 I_1}{2\pi} \int_{-R}^{+R} dx \int_{R-(R^2-x^2)^{1/2}}^{R+(R^2-x^2)^{1/2}} \frac{dy}{y} = \frac{\mu_0 I_1}{\pi} \int_0^R \ln \frac{R+(R^2-x^2)^{1/2}}{R-(R^2-x^2)^{1/2}} dx \equiv \frac{\mu_0 I_1 R}{\pi} \int_0^1 \ln \frac{1+(1-\xi^2)^{1/2}}{1+(1-\xi^2)^{1/2}} d\xi. \tag{5.81}$$

This is a table integral equal to π,[37] so that $\Phi_{21} = \mu_0 I_1 R$, and the final answer for the mutual inductance $M \equiv L_{12} = L_{21} = \Phi_{21}/I_1$ is finite (and very simple):

$$M = \mu_0 R, \tag{5.82}$$

despite the magnetic field's divergence at the lowest point of the loop $(y=0)$.

Note that in contrast with the finite mutual inductance of this system, the self-inductances of both wires are formally infinite in the thin-wire limit – see, e.g., Eq. (80), which in the limit $b/a >> 1$ describes a thin straight wire. However, since this divergence is very weak (logarithmic), it is quenched by any deviation from this perfectly axial geometry. For example, a good estimate of the inductance of a wire of a large but finite length $l >> a$ may be obtained from Eq. (80) via the replacement of b with l:

$$L \sim \frac{\mu_0 l}{2\pi} \ln \frac{l}{a}. \tag{5.83}$$

(Note, however, that the exact result depends on where from/to the current flows beyond that segment.) A close estimate, with l replaced with $2\pi R$ in the front factor, and with R under the logarithm, is valid for the self-inductance of the round loop. A more exact calculation of this inductance, which would be
asymptotically correct in the limit $a << R$, is a very useful exercise, highly recommended to the reader.[38]

Reference

[27] Just as in electrostatics, for the interaction of two independent current distributions $\mathbf{j}(\mathbf{r})$ and $\mathbf{j}'(\mathbf{r}')$, the factor ½ should be dropped.

[28] In the terminology already used in Sec. 3.5 (see also a general discussion in CM Sec. 1.4.), U_j may be called the Gibbs potential energy of our magnetic system.

[29] This relation remains the same in the Gaussian units because in those units, both Eq. (28) and Eq. (54) should be stripped of their $\mu_0/4\pi$ coefficients.

[30] For that, we may use MA Eq. (11.7) with $\mathbf{f} = \mathbf{A}$ and $\mathbf{g} = \mathbf{B}$, giving $\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \nabla \cdot (\mathbf{A} \times \mathbf{B})$.

[31] The transfer to the Gaussian units in Eqs. (77)-(78) may be accomplished by the usual replacement $\mu_0 \to 4\pi$, thus giving, in particular, $u = B^2/8\pi$.

[32] As evident from Eq. (60), these coefficients depend only on the geometry of the system. Moreover, in the Gaussian units, in which Eq. (60) is valid without the factor $\mu_0/4\pi$, the inductance coefficients have the dimension of length (centimeters). The SI unit of

inductance is called the henry, abbreviated H – after Joseph Henry, who in particular discovered the effect of electromagnetic induction (see Sec. 6.1) independently of Michael Faraday.

[33] Note that the matrix of the mutual inductances $L_{jj'}$ is very much similar to the matrix of reciprocal capacitance coefficients $p_{kk'}$ – for example, compare Eq. (62) with Eq. (2.21).

[34] The SI unit of magnetic flux is called weber, abbreviated Wb – after Wilhelm Edward Weber (1804-1891), who in particular co-invented (with Carl Gauss) the electromagnetic telegraph. More importantly for this course, in 1856 he was the first (together with Rudolf Kohlrausch) to notice that the value of (in modern terms) $1/(\varepsilon_0\mu_0)^{1/2}$, derived from electrostatic and magnetostatic measurements, coincides with the independently measured speed of light c. This observation gave an important motivation for Maxwell's theory.

[35] Numerous applications of this Neumann formula to electrical engineering problems may be found, for example, in the classical text by F. Grover, Inductance Calculations, Dover, 1946.

[36] As a reminder, the mutual capacitance C between the conductors of such a system was calculated in Sec. 2.3.

[37] See, e.g., MA Eq. (6.13), with $a = 1$.

[38] It may be found, for example, just after Sec. 34 of L. Landau et al., Electrodynamics of Continuous Media, 2nd ed., Butterwort Heinemann, 1984.

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