9.5: The Maxwell Equations in the 4-form

This 4-vector formalism background is already sufficient to analyze the Lorentz transform of the electromagnetic field. Just to warm up, let us consider the continuity equation (4.5),

$$\frac{\partial \rho}{\partial t}+\nabla \cdot \mathbf{j}=0, \tag{9.110}$$

which expresses the electric charge conservation, and, as we already know, is compatible with the Maxwell equations. If we now define the contravariant and covariant 4-vectors of electric current as

$$\text{4-vector of electric current} \qquad j^{\alpha} \equiv\{\rho c, \mathbf{j}\}, \quad j_{\alpha} \equiv\{\rho c,-\mathbf{j}\}, \tag{9.111}$$

then Eq. (110) may be represented in the form

$$\text{Continuity equation: 4-form} \qquad \partial^{\alpha} j_{\alpha}=\partial_{\alpha} j^{\alpha}=0, \tag{9.112}$$

showing that the continuity equation is form-invariant[45] with respect to the Lorentz transform.

Of course, such a form's invariance of a relation does not mean that all component values of the 4-vectors participating in it are the same in both frames. For example, let us have some static charge density ρ in frame 0; then Eq. (97b), applied to the contravariant form of the 4-vector (111), reads

$$j^{\prime \alpha}=\frac{\partial x^{\prime \alpha}}{\partial x^{\beta}} j^{\beta}, \quad \text{with } j^{\beta}=\{\rho c, 0,0,0\}. \tag{9.113}$$

Using the particular form (98) of the reciprocal Lorentz matrix for the coordinate choice shown in Fig. 1, we see that this relation yields

$$\rho^{\prime}=\gamma \rho, \quad j_{x}^{\prime}=-\gamma \beta \rho c=-\gamma \nu \rho, \quad j_{y}^{\prime}=j_{z}^{\prime}=0. \tag{9.114}$$

Since the charge velocity, as observed from frame 0', is $(-\mathbf{v})$, the non-relativistic results would be $\rho^{\prime}=\rho$, $\mathbf{j}^{\prime}=-\mathbf{v} \rho$. The additional γ factor in the relativistic results is caused by the length contraction: $d x^{\prime}=d x / \gamma$, so that to keep the total charge $d Q=\rho d^{3} r=\rho d x d y d z$ inside the elementary volume $d^{3} r=d x d y d z$ intact, ρ (and hence j_{x}) should increase proportionally.

Next, at the end of Chapter 6 we have seen that Maxwell equations for the electromagnetic potentials ϕ and $\mathbf{A}$ may be represented in similar forms (6.118), under the Lorenz (again, not "Lorentz", please!) gauge condition (6.117). For free space, this condition takes the form

$$\nabla \cdot \mathbf{A}+\frac{1}{c^{2}} \frac{\partial \phi}{\partial t}=0. \tag{9.115}$$

This expression gives us a hint of how to form the 4-vector of the potentials:[46]

$$A^{\alpha} \equiv\left\{\frac{\phi}{c}, \mathbf{A}\right\}, \quad A_{\alpha} \equiv\left\{\frac{\phi}{c},-\mathbf{A}\right\}; \qquad \text{4-vector of potentials} \tag{9.116}$$

indeed, this vector satisfies Eq. (115) in its 4-form:

$$\partial^{\alpha} A_{\alpha}=\partial_{\alpha} . A^{\alpha}=0 \qquad \text{Lorenz gauge: 4-form} \tag{9.117}$$

Since this scalar product is Lorentz-invariant, and the derivatives (104)-(105) are legitimate 4-vectors, this implies that the 4-vector (116) is also legitimate, i.e. obeys the Lorentz transform formulas (97), (99). Even more convincing evidence of this fact may be obtained from the Maxwell equations (6.118) for the potentials. In free space, they may be rewritten as

$$\left[\frac{\partial^{2}}{\partial(c t)^{2}}-\nabla^{2}\right] \frac{\phi}{c}=\frac{\rho c}{\varepsilon_{0} c^{2}} \equiv \mu_{0}(\rho c), \quad\left[\frac{\partial^{2}}{\partial(c t)^{2}}-\nabla^{2}\right] \mathbf{A}=\mu_{0} \mathbf{j}. \tag{9.118}$$

Using the definition (116), these equations may be merged to one:[47]

$$\square A^{\alpha}=\mu_{0} j^{\alpha}, \qquad \text{Maxwell equation for 4-potential} \tag{9.119}$$

where $\square$ is the d' Alembert operator,[48] which may be represented as either of two scalar products,

$$\text{D'Alembert operator} \qquad \Box \equiv \frac{\partial^2}{\partial(ct)^2} - \nabla^2 = \partial^\beta \partial_\beta = \partial_\beta \partial^\beta. \tag{9.120}$$

and hence is Lorentz-invariant. Because of that, and the fact that the Lorentz transform changes both 4-vectors A^α and j^α in a similar way, Eq. (119) does not depend on the reference frame choice. Thus we have arrived at a key point of this chapter: we see that the Maxwell equations are indeed form-invariant with respect to the Lorentz transform. As a by-product, the 4-vector form (119) of these equations (for potentials) is extremely simple – and beautiful!

However, as we have seen in Chapter 7, for many applications the Maxwell equations for the field vectors are more convenient; so let us represent them in the 4-form as well. For that, we may express all Cartesian components of the usual (3D) field vector vectors (6.7),

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{9.121}$$

via those of the potential 4-vector A^α. For example,

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} = -c\left(\frac{\partial}{\partial x}\frac{\phi}{c} + \frac{\partial A_x}{\partial(ct)}\right) \equiv -c\left(\partial^0 A^1 - \partial^1 A^0\right), \tag{9.122}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \equiv -\left(\partial^2 A^3 - \partial^3 A^2\right). \tag{9.123}$$

Completing similar calculations for other field components (or just generating them by appropriate index shifts), we find that the following antisymmetric, contravariant field-strength tensor,

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{9.124}$$

may be expressed via the field components as follows:[49]

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \tag{9.125a}$$

Field-strength tensors

so that the covariant form of the tensor is

$$F_{\alpha\beta} \equiv g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{9.125b}$$

If Eq. (124) looks a bit too bulky, please note that as a reward, the pair of inhomogeneous Maxwell equations, i.e. two equations of the system (6.99), which in free space $(\mathbf{D} = \varepsilon_0 \mathbf{E}, \mathbf{B} = \mu_0 \mathbf{H})$ may be rewritten as

$$\nabla \cdot \frac{\mathbf{E}}{c} = \mu_0 c\rho, \quad \boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial}{\partial(ct)}\frac{\mathbf{E}}{c} = \mu_0 \mathbf{j}, \tag{9.126}$$

may now be expressed in a very simple (and manifestly form-invariant) way,

$$\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta, \qquad \text{Maxwell equation for tensor } F \tag{9.127}$$

which is comparable with Eq. (119) in its simplicity – and beauty. Somewhat counter-intuitively, the pair of homogeneous Maxwell equations of the system (6.99),

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{9.128}$$

look, in the 4-vector notation, a bit more complicated:[50]

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{9.129}$$

Note, however, that Eqs. (128) may be also represented in a much simpler 4-form,

$$\partial_\alpha G^{\alpha\beta} = 0, \tag{9.130}$$

using the so-called dual tensor

$$G^{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}, \tag{9.131}$$

which may be obtained from $F^{\alpha\beta}$, given by Eq. (125a), by the following replacements:

$$\frac{\mathbf{E}}{c} \rightarrow -\mathbf{B}, \quad \mathbf{B} \rightarrow \frac{\mathbf{E}}{c}. \tag{9.132}$$

Besides the proof of the form-invariance of the Maxwell equations with respect to the Lorentz transform, the 4-vector formalism allows us to achieve our initial goal: find out how do the electric and magnetic field components change at the transfer between (inertial!) reference frames. For that, let us apply to the tensor $F^{\alpha\beta}$ the reciprocal Lorentz transform described by the second of Eqs. (109). Generally, it gives, for each field component, a sum of 16 terms, but since (for our choice of coordinates, shown in Fig. 1) there are many zeros in the Lorentz transform matrix, and the diagonal components of $F^{\gamma\delta}$ equal zero as well, the calculations are rather doable. Let us calculate, for example, $E'_x \equiv -cF'^{01}$. The only non-zero terms on the right-hand side are

$$E'_x = -cF'^{01} = -c\left(\frac{\partial x'^0}{\partial x^1}\frac{\partial x'^1}{\partial x^0}F^{10} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^1}F^{01}\right) \equiv -c\gamma^2\left(\beta^2 - 1\right)\frac{E_x}{c} \equiv E_x. \tag{9.133}$$

Repeating the calculation for the other five components of the fields, we get very important relations

$$\begin{aligned} E'_x &= E_x, & B'_x &= B_x \\ E'_y &= \gamma\left(E_y - \nu B_z\right), & B'_y &= \gamma\left(B_y + \nu E_z/c^2\right), \\ E'_z &= \gamma\left(E_z + \nu B_y\right), & B'_z &= \gamma\left(B_z - \nu E_y/c^2\right), \end{aligned} \tag{9.134}$$

whose more compact "semi-vector" form is

$$\text{Lorentz transform of field components} \qquad \begin{aligned} \mathbf{E}'_{||} &= \mathbf{E}_{||}, & \mathbf{B}'_{||} &= \mathbf{B}_{||}, \\ \mathbf{E}'_{\perp} &= \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})_{\perp}, & \mathbf{B}'_{\perp} &= \gamma\left(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2\right)_{\perp}, \end{aligned} \tag{9.135}$$

where the indices $||$ and $\perp$ stand, respectively, for the field components parallel and perpendicular to the relative velocity $\mathbf{v}$ of the two reference frames. In the non-relativistic limit, the Lorentz factor γ tends to 1, and Eqs. (135) acquire an even simpler form

$$\mathbf{E}' \rightarrow \mathbf{E} + \mathbf{v}\times\mathbf{B}, \quad \mathbf{B}' \rightarrow \mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}. \tag{9.136}$$

Thus we see that the electric and magnetic fields are transformed to each other even in the first order of the ν/c ratio. For example, if we fly across the field lines of a uniform, static, purely electric field $\mathbf{E}$ (e.g., the one in a plane capacitor) we will see not only the electric field's renormalization (in the second order of the ν/c ratio), but also a non-zero dc magnetic field $\mathbf{B}'$ perpendicular to both the vector $\mathbf{E}$ and the vector $\mathbf{v}$, i.e. to the direction of our motion. This is of course what might be expected from the relativity principle: from the point of view of the moving observer (which is as legitimate as that of a stationary observer), the surface charges of the capacitor's plates, which create the field $\mathbf{E}$, move back creating the dc currents (114), which induce the magnetic field $\mathbf{B}'$. Similarly, motion across a magnetic field creates, from the point of view of the moving observer, an electric field.

This fact is very important philosophically. One may say there is no such thing in Mother Nature as an electric field (or a magnetic field) all by itself. Not only can the electric field induce the magnetic field (and vice versa) in dynamics, but even in an apparently static configuration, what exactly we measure depends on our speed relative to the field sources – hence the very appropriate term for the whole field we are studying: the electromagnetism.

Another simple but very important application of Eqs. (134)-(135) is the calculation of the fields created by a charged particle moving in free space by inertia, i.e. along a straight line with constant velocity $\mathbf{u}$, at the impact parameter[51] (the closest distance) b from the observer. Selecting the reference frame 0' to move with the particle in its origin, and the frame 0 to reside in the "lab" in that the fields $\mathbf{E}$ and $\mathbf{B}$ are measured, we can use the above formulas with $\mathbf{v}=\mathbf{u}$. In this case the fields $\mathbf{E}'$ and $\mathbf{B}'$ may be calculated from, respectively, electro- and magnetostatics:

$$\mathbf{E}'=\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}'}{r'^3},\quad \mathbf{B}'=0, \tag{9.137}$$

because in frame 0', the particle does not move. Selecting the coordinate axes so that at the measurement point $x=0, y=b, z=0$ (Fig. 11a), for this point we may write $x'=-ut', y'=b, z'=0$, so that $r'=\left(u^2t'^2+b^2\right)^{1/2}$, and the Cartesian components of the fields (137) are:

$$\begin{gathered} E_x'=-\frac{q}{4\pi\varepsilon_0}\frac{ut'}{\left(u^2t'^2+b^2\right)^{3/2}},\quad E_y'=\frac{q}{4\pi\varepsilon_0}\frac{b}{\left(u^2t'^2+b^2\right)^{3/2}},\quad E_z'=0, \\ B_x'=B_y'=B_z'=0. \end{gathered} \tag{9.138}$$

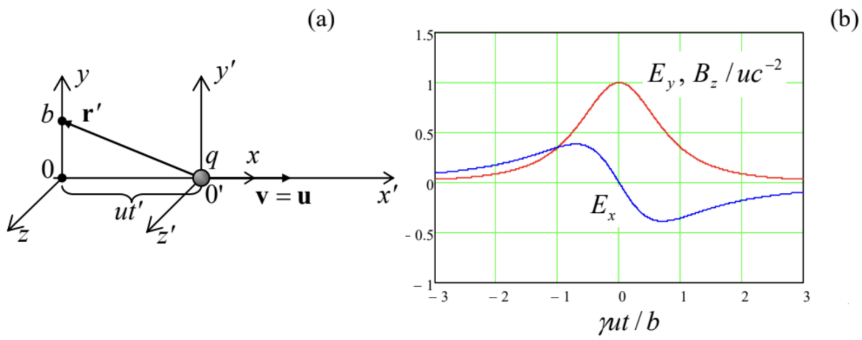

Fig. 9.11. The field pulses induced by a uniformly moving charge.

Now using the last of Eqs. (19b) with $x=0$, giving $t'=\gamma t$, and the relations reciprocal to Eqs. (134) for the field transform (it is evident that they are similar to the direct transform, with ν replaced with $-\nu=-u$), in the lab frame we get

$$E_x=E_x'=-\frac{q}{4\pi\varepsilon_0}\frac{u\gamma t}{\left(u^2\gamma^2t^2+b^2\right)^{3/2}},\quad E_y=\gamma E_y'=\frac{q}{4\pi\varepsilon_0}\frac{\gamma b}{\left(u^2\gamma^2t^2+b^2\right)^{3/2}},\quad E_z=0, \tag{9.139}$$

$$B_x=0,\quad B_y=0,\quad B_z=\frac{nu}{c^2}E_y'=\frac{u}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{\gamma b}{\left(u^2\gamma^2t^2+b^2\right)^{3/2}}\equiv\frac{u}{c^2}E_y. \tag{9.140}$$

These results,[52] plotted in Fig. 11b in the units of $\gamma q^2/4\pi\varepsilon_0 b^2$, reveal two major effects. First, the charge passage by the observer generates not only an electric field pulse but also a magnetic field pulse. This is natural, because, as was repeatedly discussed in Chapter 5, any charge motion is essentially an electric current.[53] Second, Eqs. (139)-(140) show that the pulse duration scale is

$$\Delta t=\frac{b}{nu}=\frac{b}{u}\left(1-\frac{u^2}{c^2}\right)^{1/2}, \tag{9.141}$$

i.e. shrinks to virtually zero as the charge's velocity u approaches the speed of light. This is of course a direct corollary of the relativistic length contraction: in the frame 0' moving with the charge, the longitudinal spread of its electric field at distance b from the motion line is of the order of $\Delta x'=b$. When observed from the lab frame 0, this interval, in accordance with Eq. (20), shrinks to $\Delta x=\Delta x'/\gamma=b/\gamma$, and hence so does the pulse duration scale $\Delta t=\Delta x/u=b/\gamma u$.

Reference

45 In some texts, the equations preserving their form at a transform are called "covariant", creating a possibliliy for confusion with the covariant vectors and tensors. On the other hand, calling such equations "invariant" would not distinguish them properly from invariant quantities, such as the scalar products of 4-vectors.

46 In the Gaussian units, the scalar potential should not be divided by c in this relation.

[47] In the Gaussian units, the coefficient μ_0 in Eq. (119) should be replaced, as usual, with $4\pi/c$.

[48] Named after Jean-Baptiste le Rond d'Alembert (1717-1783), who has made several pioneering contributions to the general theory of waves – see, e.g., CM Chapter 6. (Some older textbooks use notation $\Box^2$ for this operator.)

[49] In Gaussian units, this formula, as well as Eq. (131) for $G^{\alpha\beta}$, do not have the factors c in all the denominators.

[50] To be fair, note that just as Eq. (127), Eq. (129) this is also a set of four scalar equations – in the latter case with the indices α, β, and γ taking any three different values of the set $\{0,1,2,3\}$.

[51] This term is very popular in the theory of particle scattering – see, e.g., CM Sec. 3.7.

[52] In the next chapter, we will re-derive them in a different way.

[53] It is straightforward to use Eq. (140) and the linear superposition principle to calculate, for example, the magnetic field of a string of charges moving along the same line and separated by equal distances $\Delta x=a$ (so that the average current, as measured in frame 0, is qu/a), and to show that the time-average of the magnetic field is given by the familiar Eq. (5.20) of magnetostatics, with b instead of ρ.

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