5.8: Exercise Problems

5.1. Use the discussion in Sec. 1 to find an alternative solution of Problem 4.18.

5.2. A spin- $1/2$ is placed into an external magnetic field, with a time-independent orientation, its magnitude $\mathscr{B}(t)$ being an arbitrary function of time. Find explicit expressions for the Heisenberg operators and the expectation values of all three Cartesian components of the spin, as functions of time, in a coordinate system of your choice.

5.3. A two-level system is in the quantum state α described by the ket-vector $|\alpha\rangle=\alpha_{\uparrow}|\uparrow\rangle+\ \alpha_{\downarrow}\downarrow\downarrow$, with given (generally, complex) c-number coefficients $\alpha\uparrow\downarrow$. Prove that we can always select such a geometric c-number vector $\mathbf{c}=\{c_x,c_y,c_z\}$, that α is an eigenstate of $\mathbf{c}\cdot\hat{\boldsymbol{\sigma}}$, where $\hat{\boldsymbol{\sigma}}$ is the Pauli vector operator. Find all possible values of $\mathbf{c}$ satisfying this condition, and the second eigenstate (orthogonal to α) of the operator $\mathbf{c}\cdot\hat{\boldsymbol{\sigma}}$. Give a Bloch-sphere interpretation of your result.

5.4. * Analyze statistics of the spacing $S\equiv E_{+}-E$. between energy levels of a two-level system, assuming that all elements H_{jj} ' of its Hamiltonian matrix (2) are independent random numbers, with equal and constant probability densities within the energy interval of interest. Compare the result with that for a purely diagonal Hamiltonian matrix, with the similar probability distribution of its random diagonal elements.

5.5. For a periodic motion of a single particle in a confining potential $U(\mathbf{r})$, the virial theorem of non-relativistic classical mechanics 55 is reduced to the following equality:

$$\bar{T}=\frac{1}{2}\overline{\mathbf{r}\cdot\nabla U}, \tag{5.8.1}$$

where T is the particle's kinetic energy, and the top bar means averaging over the time period of motion. Prove the following quantum-mechanical version of the theorem for an arbitrary stationary quantum state, in the absence of spin effects:

$$\langle T\rangle=\frac{1}{2}\langle\mathbf{r}\cdot\nabla U\rangle, \tag{5.8.2}$$

where the angular brackets mean the expectation values of the observables.

Hint: Mimicking the proof of the classical virial theorem, consider the time evolution of the following operator:

$$\hat{G}\equiv\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} \tag{5.8.3}$$

5.6. Calculate, in the WKB approximation, the transparency $\mathscr{T}$ of the following saddle-shaped potential barrier:

$$U(x,y)=U_0\left(1+\frac{xy}{a^2}\right), \tag{5.8.4}$$

where $U_0>0$ and a are real constants, for tunneling of a $2\mathrm{D}$ particle with energy $E<U_0$.

5.7. Calculate the so-called Gamow factor [56] for the alpha decay of atomic nuclei, i.e. the exponential factor in the transparency of the potential barrier resulting from the following simple model for the alpha-particle's potential energy as a function of its distance from the nuclear center:

$$U(r)=\begin{cases}U_0<0, & \text{for } r<R,\\ \frac{ZZ'e^2}{4\pi\varepsilon_0 r}, & \text{for } R<r,\end{cases} \tag{5.8.5}$$

(where $Ze=2e>0$ is the charge of the particle, $Z'e>0$ is that of the nucleus after the decay, and R is the nucleus' radius), in the WKB approximation.

5.8. Use the WKB approximation to calculate the average time of ionization of a hydrogen atom, initially in its ground state, made metastable by application of an additional weak, uniform, timeindependent electric field $\mathscr{E}$. Formulate the conditions of validity of your result.

5.9. For a 1D harmonic oscillator with mass m and frequency ω_0, calculate:

(i) all matrix elements $\left\langle n\left|\hat{x}^3\right|n'\right\rangle$, and

(ii) the diagonal matrix elements $\left\langle n\left|\hat{x}^4\right|n\right\rangle$, where n and n ' are arbitrary Fock states.

5.10. Calculate the sum (over all $n > 0$) of the so-called oscillator strengths,

$$f_n \equiv \frac{2m}{\hbar^2}(E_n - E_0)\left|\langle n|\hat{x}|0\rangle\right|^2, \tag{5.8.6}$$

(i) for a 1D harmonic oscillator, and

(ii) for a 1D particle confined in an arbitrary stationary potential.

5.11. Prove the so-called Bethe sum rule,

$$\sum_{n'}(E_{n'} - E_n)\left|\left\langle n\left|e^{ik\hat{x}}\right|n'\right\rangle\right|^2 = \frac{\hbar^2 k^2}{2m}, \tag{5.8.7}$$

valid for a 1D particle moving in an arbitrary time-independent potential $U(x)$, and discuss its relation with the Thomas-Reiche-Kuhn sum rule whose derivation was the subject of the previous problem.

Hint: Calculate the expectation value, in a stationary state n, of the following double commutator,

$$\hat{D} \equiv \left[\left[\hat{H}, e^{ik\hat{x}}\right], e^{-ik\hat{x}}\right], \tag{5.8.8}$$

in two ways: first, just spelling out both commutators, and, second, using the commutation relations between operators $\hat{p}_x$ and $e^{ik\hat{x}}$, and compare the results.

5.12. Given Eq. (116), prove Eq. (117), using the hint given in the accompanying footnote.

5.13. Use Eqs. (116)-(117) to simplify the following operators:

(i) $\exp\{+ia\hat{x}\}\hat{p}_x \exp\{-ia\hat{x}\}$, and

(ii) $\exp\{+ia\hat{p}_x\}\hat{x}\exp\{-ia\hat{p}_x\}$,

where a is a c-number.

5.14. For a 1D harmonic oscillator, calculate:

(i) the expectation value of energy, and

(ii) the time evolution of the expectation values of the coordinate and momentum, provided that in the initial moment $(t = 0)$ it was in the state described by the following ket-vector:

$$|\alpha\rangle = \frac{1}{\sqrt{2}}(|31\rangle + |32\rangle), \tag{5.8.9}$$

where $|n\rangle$ are the ket-vectors of the stationary (Fock) states of the oscillator.

5.15. * Re-derive the London dispersion force's potential of the interaction of two isotropic 3D harmonic oscillators (already calculated in Problem 3.16), using the language of mutually-induced polarization.

5.16. An external force pulse $F(t)$, of a finite time duration τ, has been exerted on a 1D harmonic oscillator, initially in its ground state. Use the Heisenberg-picture equations of motion to calculate the expectation value of the oscillator's energy at the end of the pulse.

5.17. Use Eqs. (144)-(145) to calculate the uncertainties δx and δp for a harmonic oscillator in its squeezed ground state, and in particular, to prove Eqs. (143) for the case $\theta = 0$.

5.18. Calculate the energy of a harmonic oscillator in the squeezed ground state ζ.

5.19. * Prove that the squeezed ground state, described by Eqs. (142) and (144)-(145), may be sustained by a sinusoidal modulation of a harmonic oscillator's parameter, and calculate the squeezing factor r as a function of the parameter modulation depth, assuming that the depth is small, and the oscillator's damping is negligible.

5.20. Use Eqs. (148) to prove that the operators $\hat{L}_j$ and $\hat{L}^2$ commute with the Hamiltonian of a spinless particle placed in any central potential field.

5.21. Use Eqs. (149)-(150) and (153) to prove Eqs. (155).

5.22. Derive Eq. (164), using any of the prior formulas.

5.23. In the basis of common eigenstates of the operators $\hat{L}_z$ and $\hat{L}^2$, described by kets $|l, m\rangle$:

(i) calculate the matrix elements $\left\langle l, m_1 \left|\hat{L}_x\right| l, m_2\right\rangle$ and $\left\langle l, m_1 \left|\hat{L}_x^2\right| l, m_2\right\rangle$;

(ii) spell out your results for diagonal matrix elements (with $m_1 = m_2$) and their y-axis counterparts; and

(iii) calculate the diagonal matrix elements $\left\langle l, m \left|\hat{L}_x \hat{L}_y\right| l, m\right\rangle$ and $\left\langle l, m \left|\hat{L}_y \hat{L}_x\right| l, m\right\rangle$.

5.24. For the state described by the common eigenket $|l, m\rangle$ of the operators $\hat{L}_z$ and $\hat{L}^2$ in a reference frame $\{x, y, z\}$, calculate the expectation values $\langle L_z' \rangle$ and $\left\langle {L_z}^2 \right\rangle$ in the reference frame whose z '-axis forms angle θ with the z-axis.

5.25. Write down the matrices of the following angular momentum operators: $\hat{L}_x, \hat{L}_y, \hat{L}_z$, and $\hat{L}_\pm$, in the z-basis of the $\{l, m\}$ states with $l = 1$.

5.26. Calculate the angular factor of the orbital wavefunction of a particle with a definite value of L^2, equal to $6\hbar^2$, and the largest possible value of L_x. What is this value? 5.27. For the state with the wavefunction $\psi = Cxye^{-\lambda r}$, with a real, positive λ, calculate:

(i) the expectation values of the observables L_x, L_y, L_z, and L^2, and

(ii) the normalization constant C.

5.28. An angular state of a spinless particle is described by the following ket-vector:

$$|\alpha\rangle = \frac{1}{\sqrt{2}}(|l = 3, m = 0\rangle + |l = 3, m = 1\rangle). \tag{5.8.10}$$

Calculate the expectation values of the x - and y-components of its angular momentum. Is the result sensitive to a possible phase shift between the component eigenkets?

5.29. A particle is in a quantum state α with the orbital wavefunction proportional to the spherical harmonic $Y_1^1(\theta, \varphi)$. Find the angular dependence of the wavefunctions corresponding to the following ket-vectors:

(i) $\hat{L}_x|\alpha\rangle$,

(ii) $\hat{L}_y|\alpha\rangle$,

(iii) $\hat{L}_z|\alpha\rangle$,

(iv) $\hat{L}_+\hat{L}_-|\alpha\rangle$, and (v) $\hat{L}^2|\alpha\rangle$.

5.30. A charged, spinless 2D particle of mass m is trapped in a soft potential well $U(x, y) = m\omega_0^2\left(x^2 + y^2\right)/2$. Calculate its energy spectrum in the presence of a uniform magnetic field $\mathscr{B}$, normal to the $[x, y]$-plane of particle's motion.

5.31. Solve the previous problem for a spinless 3D particle, placed (in addition to a uniform magnetic field $\mathscr{B}$) into a spherically-symmetric potential well $U(\mathbf{r}) = m\omega_0^2 r^2/2$.

5.32. Calculate the spectrum of rotational energies of an axially-symmetric, rigid body.

5.33. Simplify the following double commutator:

$$\left[\hat{r}_j, \left[\hat{L}^2, \hat{r}_{j'}\right]\right]. \tag{5.8.11}$$

5.34. Prove the following commutation relation:

$$\left[\hat{L}^2, \left[\hat{L}^2, \hat{r}_j\right]\right] = 2\hbar^2\left(\hat{r}_j\hat{L}^2 + \hat{L}^2\hat{r}_j\right) \tag{5.8.12}$$

5.35. Use the commutation relation proved in the previous problem, and Eq. (148), to prove the orbital electric-dipole selection rules mentioned in Sec. 5.6 of the lecture notes.

5.36. Express the commutators listed in Eq. (179), $\left[\hat{J}^2, \hat{L}_z\right]$ and $\left[\hat{J}^2, \hat{S}_z\right]$, via $\hat{L}_j$ and $\hat{S}_j$.

5.37. Find the operator $\hat{T}_{\phi}$ describing a quantum state's rotation by angle ϕ about a certain axis, using the similarity of this operation with the shift of a Cartesian coordinate, discussed in Sec. 5. Then use this operator to calculate the probabilities of measurements of spin- $1/2$ components of a beam of particles with z-polarized spin, by a Stern-Gerlach instrument turned by angle θ within the $[z, x]$ plane, where y is the axis of particle propagation - see Fig. 4.1.57

5.38. The rotation ("angular translation") operator $\hat{T}_{\phi}$ analyzed in the previous problem, and the linear translation operator $\hat{T}_{X}$ discussed in Sec. 5 , have a similar structure:

$$\hat{T}_{\lambda}=\exp\{-i\hat{C}\lambda/\hbar\}, \tag{5.8.13}$$

where λ is a real c-number, characterizing the shift, and $\hat{C}$ is a Hermitian operator, which does not explicitly depend on time.

(i) Prove that such operators $\hat{T}_{\lambda}$ are unitary.

(ii) Prove that if the shift by λ, induced by the operator $\hat{T}_{\lambda}$, leaves the Hamiltonian of some system unchanged for any λ, then $\langle C\rangle$ is a constant of motion for any initial state of the system.

(iii) Discuss what the last conclusion means for the particular operators $\hat{T}_{X}$ and $\hat{T}_{\phi}$.

5.39. A particle with spin s is in a state with definite quantum numbers l and j. Prove that the observable $\mathbf{L}\cdot\mathbf{S}$ also has a definite value, and calculate it.

5.40. For a spin- $1/2$ particle in a state with definite quantum numbers l, m_l, and m_s, calculate the expectation value of the observable J^2, and the probabilities of all its possible values. Interpret your results in the terms of the Clebsh-Gordan coefficients (190).

5.41. Derive general recurrence relations for the Clebsh-Gordan coefficients.

Hint: Using the similarity of the commutation relations discussed in Sec. 7, write the relations similar to Eqs. (164) for other components of the angular momentum, and apply them to Eq. (170).

5.42. Use the recurrence relations derived in the previous problem to prove Eqs. (190) for the spin- $1/2$ Clebsh-Gordan coefficients.

5.43. A spin-1/2 particle is in a state with definite values of L^2, J^2, and J_z. Find all possible values of the observables S^2, S_z, and L_z, the probability of each listed value, and the expectation value for each of these observables.

5.44. Re-solve the Landau-level problem discussed in Sec. 3.2, for a spin-1/2 particle. Discuss the result for the particular case of an electron, with the g-factor equal to 2 .

5.45. In the Heisenberg picture of quantum dynamics, find an explicit relation between the operators of velocity $\hat{\mathbf{v}}\equiv d\hat{\mathbf{r}}/dt$ and acceleration $\hat{\mathbf{a}}\equiv d\hat{\mathbf{v}}/dt$ of a spin-1/2 particle with electric charge q, moving in an arbitrary external electromagnetic field. Compare the result with the corresponding classical expression.

Hint: For the orbital motion's description, you may use Eq. (3.26).

5.46. A byproduct of the solution of Problem 41 is the following relation for the spin operators (valid for any spin s):

$$\left\langle m_s\pm 1\left|\hat{S}_{\pm}\right|m_s\right\rangle=\hbar\left[\left(s\pm m_s+1\right)\left(s\mp m_s\right)\right]^{1/2}. \tag{5.8.14}$$

Use this result to spell out the matrices S_x, S_y, S_z, and S^2 of a particle with $s=1$, in the z-basis - defined as the basis in which the matrix S_z is diagonal.

5.47. * For a particle with an arbitrary spin s, moving in a spherically-symmetric field, find the ranges of the quantum numbers m_j and j that are necessary to describe, in the coupled-representation basis:

(i) all states with a definite quantum number l, and

(ii) a state with definite values of not only l, but also m_l and m_s.

Give an interpretation of your results in terms of the classical geometric vector diagram (Fig. 13).

5.48. A particle of mass m, with electric charge q and spin s, free to move along a plane ring of a radius R, is placed into a constant, uniform magnetic field $\mathscr{B}$, directed normally to the ring's plane. Calculate the energy spectrum of the system. Explore and interpret the particular form the result takes when the particle is an electron with the g-factor $g_{\mathrm{e}}=2$.

[55] See, e.g., CM Problem 1.12

[56] Named after G. Gamow, who made this calculation as early as in 1928.

[57] Note that the last task is just a particular case of Problem 4.18 (see also Problem 1).