7.5: The Heisenberg-Langevin Approach

The fluctuation-dissipation theorem offers a very simple and efficient, though limited approach to the analysis of the system of interest (s in Fig. 1). It is to write its Heisenberg equations (4.199) of motion of the relevant operators, which would now include the environmental force operator, and explore these equations using the Fourier transform and the Wiener-Khinchin theorem (112)-(113). This approach to classical equations of motion is commonly associated with the name of Langevin, 46 so that its extension to dynamics of Heisenberg-picture operators is frequently referred to as the HeisenbergLangevin (or "quantum Langevin", or "Langevin-Lax"47) approach to open system analysis.

Perhaps the best way to describe this method is to demonstrate how it works for the very important case of a 1D harmonic oscillator, so that the generalized coordinate x of Sec. 4 is just the oscillator's coordinate. For the sake of simplicity, let us assume that the environment provides the simple Ohmic dissipation described by Eq. (137) - which is a very good approximation in many cases. As we already know from Chapter 5 , the Heisenberg equations of motion for operators of coordinate and momentum of the oscillator, in the presence of an external force $F(t)$, are

$$\dot{\hat{x}}=\frac{\hat{p}}{m}, \quad \dot{\hat{p}}=-m \omega_{0}^{2} \hat{x}+\hat{F}, \tag{7.5.1}$$

so that using Eqs. (92) and (137), we get

$$\dot{\hat{x}}=\frac{\hat{p}}{m}, \quad \dot{\hat{p}}=-m \omega_{0}^{2} \hat{x}-\dot{\hat{x}}+\hat{\tilde{F}}(t). \tag{7.5.2}$$

Combining Eqs. (144), we may write their system as a single differential equation

$$m \ddot{\hat{x}}+\eta \dot{\hat{x}}+m \omega_{0}^{2} \hat{x}=\hat{\tilde{F}}(t), \tag{7.5.3}$$

which is similar to the well-known classical equation of motion of a damped oscillator under the effect of an external force. In the view of Eqs. (5.29) and (5.35), whose corollary the Ehrenfest theorem (5.36) is, this may look not surprising, but please note again that the approach discussed in the previous section justifies such quantitative description of the drag force in quantum mechanics - necessarily in parallel with the accompanying fluctuation force.

For the Fourier images of the operators, defined similarly to Eq. (115), Eq. (145) gives the following relation,

$$\hat{x}_{\omega}=\frac{F_{\omega}}{m\left(\omega_{0}^{2}-\omega^{2}\right)-i \eta \omega}, \tag{7.5.4}$$

which should be also well known to the reader from the classical theory of forced oscillations. 48 However, since these Fourier components are still Heisenberg-picture operators, and their "values" for different ω generally do not commute, we have to tread carefully. The best way to proceed is to write a copy of Eq. (146) for frequency $(-\omega')$, and then combine these equations to form a symmetrical combination similar to that used in Eq. (114). The result is

$$\frac{1}{2}\left\langle\hat{x}_{\omega} \hat{x}_{-\omega'}+\hat{x}_{-\omega'} \hat{x}_{\omega}\right\rangle=\frac{1}{\left|m\left(\omega_{0}^{2}-\omega^{2}\right)-i \eta \omega\right|^{2}} \frac{1}{2}\left\langle\hat{F}_{\omega} \hat{F}_{-\omega'}+\hat{F}_{-\omega'} \hat{F}_{\omega}\right\rangle . \tag{7.5.5}$$

Since the spectral density definition similar to Eq. (114) is valid for any observable, in particular for x, Eq. (147) allows us to relate the symmetrized spectral densities of coordinate and force:

$$S_{x}(\omega)=\frac{S_{F}(\omega)}{\left|m\left(\omega_{0}^{2}-\omega^{2}\right)-i \eta \omega\right|^{2}} \equiv \frac{S_{F}(\omega)}{m^{2}\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+(\eta \omega)^{2}}. \tag{7.5.6}$$

Now using an analog of Eq. (116) for x, we can calculate the coordinate's variance:

$$\left\langle x^{2}\right\rangle=K_{x}(0)=\int_{-\infty}^{+\infty} S_{x}(\omega) d \omega=2 \int_{0}^{+\infty} \frac{S_{F}(\omega) d \omega}{m^{2}\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+(\eta \omega)^{2}}, \tag{7.5.7}$$

where now, in contrast to the notation used in Sec. 4 , the sign $\langle\ldots\rangle$ means averaging over the usual statistical ensemble of many systems of interest - in our current case, of many harmonic oscillators.If the coupling to the environment is so weak that the drag coefficient η is small (in the sense that the oscillator's dimensionless Q-factor is large, $Q \equiv m \omega_{0} / \eta >> 1$), this integral is dominated by the resonance peak in a narrow vicinity, $|\omega-\omega_0| \equiv |\xi| << \omega_0$, of its resonance frequency, and we can take the relatively smooth function $S_F(\omega)$ out of the integral, thus reducing it to a table form: 49

$$\begin{aligned}\left\langle x^2\right\rangle \approx 2 S_F\left(\omega_0\right) \int_0^{+\infty} \frac{d \omega}{m^2\left(\omega_0^2-\omega^2\right)^2+(\eta \omega)^2} \approx 2 S_F\left(\omega_0\right) \int_{-\infty}^{+\infty} \frac{d \xi}{\left(2 m \omega_0 \xi\right)^2+\left(\eta \omega_0\right)^2} \\ \equiv 2 S_F\left(\omega_0\right) \frac{1}{\left(\eta \omega_0\right)^2} \int_{-\infty}^{+\infty} \frac{d \xi}{(2 m \xi / \eta)^2+1}=2 S_F\left(\omega_0\right) \frac{1}{\left(\eta \omega_0\right)^2} \frac{\pi \eta}{2 m}=\frac{\pi S_F\left(\omega_0\right)}{\eta m \omega_0^2} .\end{aligned}$$

With the account of the FDT (134) and of Eq. (138), this gives [50]

$$\left\langle x^2\right\rangle=\frac{\pi}{\eta m \omega_0^2} \frac{\hbar}{2 \pi} \eta \omega_0 \operatorname{coth} \frac{\hbar \omega_0}{2 k_B T} \equiv \frac{\hbar}{2 m \omega_0} \operatorname{coth} \frac{\hbar \omega_0}{2 k_B T} . \tag{7.5.8}$$

But this is exactly Eq. (48), which was derived in Sec. 2 from the Gibbs distribution, without any explicit account of the environment - though keeping it in mind by using the notion of the thermallyequilibrium ensemble. [51]

Notice that in the final form of Eq. (151) the coefficient η, which characterizes the oscillator-toenvironment interaction strength, has canceled! Does this mean that in Sec. 4 we toiled in vain? By no means. First of all, the result (150), augmented by the FDT (134), has an important conceptual value. For example, let us consider the low-temperature limit $k_{\mathrm{B}} T << \hbar \omega_0$ where Eq. (151) is reduced to

$$\left\langle x^2\right\rangle=\frac{\hbar}{2 m \omega_0} \equiv \frac{x_0^2}{2} . \tag{7.5.9}$$

Let us ask a naïve question: what exactly is the origin of this coordinate’s uncertainty? From the point of view of the usual quantum mechanics of absolutely closed (Hamiltonian) systems, there is no doubt: this non-vanishing variance of the coordinate is the result of the final spatial extension of the ground-state wavefunction (2.275), reflecting Heisenberg’s uncertainty relation - which in turn results from the fact that the operators of coordinate and momentum do not commute. However, from the point of view of the Heisenberg-Langevin equation (145), the variance (152) is an inalienable part of the oscillator’s response to the fluctuation force $\widetilde{F}(t)$ exerted by the environment at frequencies $\omega \approx \omega_0$. Though it is impossible to refute the former, absolutely legitimate point of view, in many applications it is easier to subscribe to the latter standpoint and treat the coordinate’s uncertainty as the result of the so-called quantum noise of the environment, which, in equilibrium, obeys the FTD (134). This notion has received numerous confirmations in experiments that did not include any oscillators with their own frequencies ω_0 close to the noise measurement frequency ω. [52]

The second advantage of the Heisenberg-Langevin approach is that it is possible to use Eq. (148) to calculate the (experimentally measurable!) distribution $S_x(\omega)$, i.e. decompose the fluctuations into their spectral components. This procedure is not restricted to the limit of small η (i.e. of large Q); for any damping, we may just plug the FDT (134) into Eq. (148). For example, let us have a look at the so-called quantum diffusion. A free 1D particle, moving in a viscous medium providing it with the Ohmic damping (137), may be considered as the particular case of a 1D harmonic oscillator (145), but with $\omega_0=0$, so that combining Eqs. (134) and (149), we get

$$\left\langle x^2\right\rangle=2 \int_0^{+\infty} \frac{S_F(\omega) d \omega}{\left(m \omega^2\right)^2+(\eta \omega)^2}=2 \eta \int_0^{+\infty} \frac{1}{\left(m \omega^2\right)^2+(\eta \omega)^2} \frac{\hbar \omega}{2 \pi} \operatorname{coth} \frac{\hbar \omega}{2 k_B T} d \omega . \tag{7.5.10}$$

This integral has two divergences. The first one, of the type $\int d \omega / \omega^2$ at the lower limit, is just a classical effect: according to Eq. (85), the particle’s displacement variance grows with time, so it cannot have a finite time-independent value that Eq. (153) tries to calculate. However, we still can use that result to single out the quantum effects on diffusion - say, by comparing it with a similar but purely classical case. These effects are prominent at high frequencies, especially if the quantum noise overcomes the thermal noise before the dynamic cut-off, i.e. if

$$\frac{k_{\mathrm{B}} T}{\hbar} << \frac{\eta}{m} . \tag{7.5.11}$$

In this case, there is a broad range of frequencies where the quantum noise gives a substantial contribution to the integral:

$$\left\langle x^2\right\rangle_{\mathrm{Q}} \approx 2 \eta \int_{k_{\mathrm{B}} T / \hbar}^{\eta / m} \frac{1}{(\eta \omega)^2} \frac{\hbar \omega}{2 \pi} d \omega \equiv \frac{\hbar}{\pi \eta} \int_{k_{\mathrm{B}} T / \hbar}^{\eta / m} \frac{d \omega}{\omega}=\frac{\hbar}{\pi \eta} \ln \frac{\hbar \eta}{m k_{\mathrm{B}} T} \sim \frac{\hbar}{\eta} \tag{7.5.12}$$

Formally, this contribution diverges at either $m \to 0$ or $T \to 0$, but this logarithmic (i.e. extremely weak) divergence is readily quenched by almost any change of the environment model at very high frequencies, where the "Ohmic" approximation (136) becomes unrealistic.

The Heisenberg-Langevin approach is very powerful, because its straightforward generalizations enable analyses of fluctuations in virtually arbitrary linear systems, i.e. the systems described by linear differential (or integro-differential) equations of motion, including those with many degrees of freedom, and distributed systems (continua), and such systems prevail in many fields of physics. However, this approach also its limitations. The main of them is that if the equations of motion of the Heisenberg operators are not linear, there is no linear relation, such as Eq. (146), between the Fourier images of the generalized forces and the generalized coordinates, and as the result, there is no simple relation, such as Eq. (148), between their spectral densities. In other words, if the Heisenberg equations of motion are nonlinear, there is no regular simple way to use them to calculate the statistical properties of the observables.

For example, let us return to the dephasing problem described by Eqs. (68)-(70), and assume that the deterministic and fluctuating parts of the effective force $-f$ exerted by the environment, are characterized by relations similar, respectively, to Eqs. (124) and (134). Now writing the Heisenberg equations of motion for the two remaining spin operators, and using the commutation relations between them, we get

$$\dot{\hat{\sigma}}_x = \frac{1}{i\hbar}\left[\hat{\sigma}_x, \hat{H}\right] = \frac{1}{i\hbar}\left[\hat{\sigma}_x, \left(c_z + \hat{f}\right)\hat{\sigma}_z\right] = -\frac{2}{\hbar}\hat{\sigma}_y\left(c_z + \hat{f}\right) = -\frac{2}{\hbar}\hat{\sigma}_y\left(c_z + \eta\dot{\hat{\sigma}}_z + \hat{\tilde{f}}\right), \tag{7.5.13}$$

and a similar equation for $\dot{\hat{\sigma}}_y$. Such nonlinear equations cannot be used to calculate the statistical properties of the Pauli operators in this system exactly - at least analytically.

For some calculations, this problem may be circumvented by linearization: if we are only interested in small fluctuations of the observables, their nonlinear Heisenberg equations of motion, such as Eq. (156), may be linearized with respect to small deviations of the operators about their (generally, time-dependent) deterministic "values", and then the resulting linear equations for the operator variations may be solved either as has been demonstrated above, or (if the deterministic "values" evolve in time) using their Fourier expansions. Sometimes such approach gives relatively simple and important results, [53] but for many other problems, this approach is insufficient, leaving a lot of space for alternative methods.

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[46] A 1908 work by Paul Langevin was the first systematic development of Einstein's ideas (1905) on the Brownian motion, using the random force language, as an alternative to Smoluchowski's approach using the probability density language - see Sec. 6 below.

[47] Indeed, perhaps the largest credit for the extension of the Langevin approach to quantum systems belongs to Melvin J. Lax, whose work in the early 1960s was motivated mostly by quantum electronics applications - see, e.g., his monograph M. Lax, Fluctuation and Coherent Phenomena in Classical and Quantum Physics, Gordon and Breach, 1968, and references therein.

[48] If necessary, see CM Sec. 5.1.

[49] See, e.g., MA Eq. (6.5a).

[50] Note that this calculation remains correct even if the dissipation's dispersion law deviates from the Ohmic model (138), provided that the drag coefficient η is replaced with its effective value $\operatorname{Im}\chi(\omega_0)/\omega_0$, because the effects of the environment are only felt, by the oscillator, at its oscillation frequency.

[51] By the way, the simplest way to calculate $S_F(\omega)$, i.e. to derive the FDT, is to require that Eqs. (48) and (150) give the same result for an oscillator with any eigenfrequency ω. This is exactly the approach used by H. Nyquist (for the classical case) - see also SM Sec. 5.5.

[52] See, for example, R. Koch et al., Phys. Lev. B 26, 74 (1982).

[53] For example, the formula used for processing the experimental results by R. Koch et al. (mentioned above), had been derived in this way. (This derivation will be suggested to the reader as an exercise.)
