

3.3: Scattering and Diffraction

The second class of quantum effects, which becomes richer in multi-dimensional spaces, is typically referred to as either diffraction or scattering - depending on the context. In classical physics, these two terms are used to describe very different effects. The term "diffraction" is used for the interference of the waves re-emitted by elementary components of extended objects, under the effect of a single incident wave ²⁸ On the other hand, the term "scattering" is used in classical mechanics to describe the result of the interaction of a beam of incident particles ²⁹ with such an extended object, called the scatterer - see Fig. 8 .

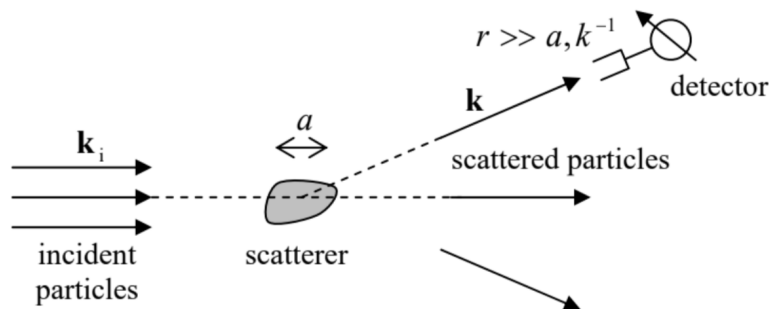


Fig. 3.8. Scattering (schematically).

Most commonly, the detector of the scattered particles is located at a large distance $r \gg a$ from the scatterer. In this case, the main observable independent of r is the flux (the number per unit time) of particles scattered in a certain direction, i.e. their flux per unit solid angle Ω . Since it is proportional to the incident flux of particles per unit area, the efficiency of scattering in a particular direction may be characterized by the ratio of these two fluxes. This ratio has is called the differential cross-section of the scatterer:

$$\frac{d\sigma}{d\Omega} \equiv \frac{\text{flux of scattered particles per unit solid angle}}{\text{flux of incident particles per unit area}}. \quad (3.3.1)$$

Such terminology and notation stem from the fact that the integral of $d\sigma/d\Omega$ over all scattering angles,

$$\sigma \equiv \oint \frac{d\sigma}{d\Omega} d\Omega = \frac{\text{total flux of scattered particles}}{\text{incident flux per per unit area}} \quad (3.3.2)$$

evidently having the dimensionality of area, has a simple interpretation as the total cross-section of scattering. For the simplest case when a solid object scatters all classical particles hitting its surface, but does not affect the particles flying by it, σ is just the geometric area of the scatterer, as observed from the direction of the incident particles. In classical mechanics, we first calculate the particle's scattering angle as a function of its impact parameter b and then average the result over all values of b , considered random. ³⁰

In quantum mechanics, due to the particle/wave duality, a relatively broad, parallel beam of incident particles of the same energy E may be fairly represented with a plane de Broglie wave (1.88):

$$\psi_i = |\psi_i| \exp\{i\mathbf{k}_i \cdot \mathbf{r}\}, \quad (3.3.3)$$

with the free-space wave number $k_i = k = (2mE)^{1/2}/\hbar$. As a result, the particle scattering becomes a synonym of the de Broglie wave diffraction, and (somewhat counter-intuitively) the description of the effect becomes simpler, excluding the notion of the impact parameter. Indeed, the wave (60) corresponds to a constant probability current density (1.49):

$$\mathbf{j}_i = |\psi_i|^2 \frac{\hbar}{m} \mathbf{k}_i, \quad (3.3.4)$$

which is exactly the flux of incident particles per unit area that is used in the denominator of Eq. (58), while the numerator of that fraction may be simply expressed via the probability current density \mathbf{j}_s of the scattered de Broglie waves:

$$\frac{d\sigma}{d\Omega} = \frac{j_s r^2}{j_i}, \quad \text{at } r \gg a \quad (3.3.5)$$

Hence our task is reduced to the calculation of j_s at sufficiently large distances r from the scatterer. For that, let us rewrite the stationary Schrödinger equation for the elastic scattering problem (when the energy E of the scattered particles is the same as that

of the incident particles) in the form

$$(E - \hat{H}_0) \psi = U(\mathbf{r})\psi, \quad \text{with } \hat{H}_0 \equiv -\frac{\hbar^2}{2m} \nabla^2, \text{ and } E = \frac{\hbar^2 k^2}{2m}, \quad (3.3.6)$$

where the potential energy $U(\mathbf{r})$ describes the effect of the scatterer. Looking for the solution of Eq. (62) in the natural form

$$\psi = \psi_i + \psi_s, \quad (3.3.7)$$

where ψ_i is the incident wave (60) and ψ_s has the sense of the scattered wave, and taking into account that the former wave satisfies the free-space Schrödinger equation

$$\hat{H}_0 \psi_i = E \psi_i, \quad (3.3.8)$$

we may reduce Eq. (63) to either of the following equivalent forms:

$$(E - \hat{H}_0) \psi_s = U(\mathbf{r}) (\psi_i + \psi_s), \quad (\nabla^2 + k^2) \psi_s = \frac{2m}{\hbar^2} U(\mathbf{r}) \psi \quad (3.3.9)$$

For applications, an integral form of this equation is more convenient. To derive it, we may look at the second of Eqs. (66) as a linear, inhomogeneous differential equation for the function ψ_s , thinking of its right-hand side as a known "source". The solution of such an equation obeys the linear superposition principle, i.e. we may represent it as the sum of the waves outcoming from all elementary volumes $d^3 r'$ of the scatterer. Mathematically, this sum may be expressed as either

$$\psi_s(\mathbf{r}) = \frac{2m}{\hbar^2} \int U(\mathbf{r}') \psi(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d^3 r', \quad (3.3.10)$$

or, equivalently, as ³¹

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \frac{2m}{\hbar^2} \int U(\mathbf{r}') \psi(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d^3 r', \quad (3.3.11)$$

where $G(\mathbf{r}, \mathbf{r}')$ is the spatial Green's function, defined as such an elementary, spherically-symmetric response of the 3D Helmholtz equation to a point source, i.e. the outward-propagating solution of the following equation ³²

$$(\nabla^2 + k^2) G = \delta(\mathbf{r} - \mathbf{r}'). \quad (3.3.12)$$

But we already know such solution of this equation - see Eq. (7) and its discussion:

$$G(\mathbf{r}, \mathbf{r}') = \frac{f_+}{R} e^{ikR}, \quad \text{where } \mathbf{R} \equiv \mathbf{r} - \mathbf{r}', \quad (3.3.13)$$

so that we need just to calculate the coefficient f_+ for Eq. (68). This can be done in several ways, for example by noticing that at $R \ll k^{-1}$, the second term on the left-hand side of Eq. (68) is negligible, so that it is reduced to the well-known Poisson equation with a delta-functional right-hand side, which describes, for example, the electrostatic potential induced by a point electric charge. Either recalling the Coulomb law or applying the Gauss theorem, ³³ we readily get the asymptote

$$G \rightarrow -\frac{1}{4\pi R}, \quad \text{at } kR \ll 1, \quad (3.3.14)$$

which is compatible with Eq. (69) only if $f_+ = -1/4\pi$, i.e. if

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi R} e^{ikR}. \quad (3.3.15)$$

Plugging this result into Eq. (67a), we get the following formal solution of Eq. (66):

Scattering problem: formal solution

$$\psi_s(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \int U(\mathbf{r}') \frac{\psi(\mathbf{r}')}{R} e^{ikR} d^3 r'. \quad (3.3.16)$$

Note that if the function $U(\mathbf{r})$ is smooth, the singularity in the denominator is integrable (i.e. not dangerous); indeed, the contribution of a sphere with some radius $R \rightarrow 0$, with the center at point \mathbf{r}' , into this integral scales as

$$\int_{R < R} \frac{d^3 R}{R} \equiv 4\pi \int_0^R \frac{R^2 dR}{R} \equiv 4\pi \int_0^R R dR = 2\pi R^2 \rightarrow 0. \quad (3.3.17)$$

So far, our result (72) is exact, but its apparent simplicity is deceiving, because the wavefunction ψ on its right-hand side generally includes not only the incident wave ψ_1 but also the scattered wave ψ_s - see Eq. (64). The most straightforward, and most common simplification of this problem, called the Born approximation,³⁴ is possible if the scattering potential $U(\mathbf{r})$ is in some sense small. (We will derive the quantitative condition of this smallness in a minute.) Since at $U(\mathbf{r}) = 0$ the scattering wave ψ_s has to disappear, at small but non-zero $U(\mathbf{r})$, $|\psi_s|$ has to be much smaller than $|\psi_1|$. In this case, on the righthand side of Eq. (73) we may ignore ψ_s in comparison with ψ_1 , getting

$$\psi_s(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} |\psi_1| \int U(\mathbf{r}') \frac{\exp\{i\mathbf{k}_i \cdot \mathbf{r}'\}}{R} e^{ikR} d^3 r' \quad (3.3.18)$$

Actually, Eq. (74) gives us even more than we wanted: it evaluates the scattered wave at any point, including those within of the scattering object, while to spell out Eq. (62), we only need to find the wave far from the scatterer, at $r \rightarrow \infty$. However, before going to that limit, we can use this general formula to find a quantitative criterion of the Born approximation's validity. For that, let us estimate the magnitude of the right-hand side of this equation for a scatterer of a linear size $\sim a$, and the potential magnitude's scale U_0 . The results are different in the following two limits:

(i) If $ka \ll 1$, then inside the scatterer (i.e., at distances $\Delta r' \sim a$), both $\exp\{i\mathbf{k} \cdot \mathbf{r}'\}$ and the second exponent under the integral in Eq. (74) change little, so that a crude but fair estimate of the solution's magnitude is

$$|\psi_s| \sim \frac{m}{2\pi\hbar^2} |\psi_1| U_0 a^2 \quad (3.3.19)$$

(ii) In the opposite limit $ka \gg 1$, the function under the integral is nearly periodic in one of the spatial directions (that of the scattered wave propagation), so that the net integral accumulates only on distances of the order of the de Broglie wavelength, $\sim k^{-1}$, and the integral is correspondingly smaller:

$$|\psi_s| \sim \frac{m}{2\pi\hbar^2} |\psi_1| U_0 \frac{a^2}{ka}. \quad (3.3.20)$$

These relations allow us to spell out the Born approximation's condition, $|\psi_s| \ll |\psi_1|$, as

$$U_0 < \frac{\hbar^2}{ma^2} \max[ka, 1]. \quad (3.3.21)$$

In the fraction on the right-hand side, we may readily recognize the scale of the kinetic (quantumconfinement) energy E_a of the particle inside a potential well of a size of the order of a , so that the Born approximation is valid essentially if the potential energy of particle's interaction with the scatterer is smaller than E_a . Note, however, that the estimates (75) and (76) are not valid in some special situations when the effects of scattering accumulate in some direction. This is frequently the case for small angles θ of scattering by extended objects, when $ka \gg 1$, but $ka\theta \lesssim 1$.

Now let us proceed to large distances $r \gg r' \sim a$, and simplify Eq. (74) using an approximation similar to the dipole expansion in electrodynamics.³⁵ Namely, in the denominator's R , we may ignore r' in comparison with the much larger r , but the exponents require more care, because even if $r' \sim a < r$, the product $kr' \sim ka$ may still be of the order of 1. In the first approximation in r' , we can take (Fig. 9a):

$$R \equiv |\mathbf{r} - \mathbf{r}'| \approx r - \mathbf{n}_r \cdot \mathbf{r}', \quad (3.3.22)$$

and since directions of the vectors \mathbf{k} and \mathbf{r} coincide, i.e. $\mathbf{k} = k\mathbf{n}_r$, we get

$$kR \approx kr - \mathbf{k} \cdot \mathbf{r}', \text{ so that } e^{ikR} \approx e^{ikr} e^{-i\mathbf{k} \cdot \mathbf{r}'} \quad (3.3.23)$$

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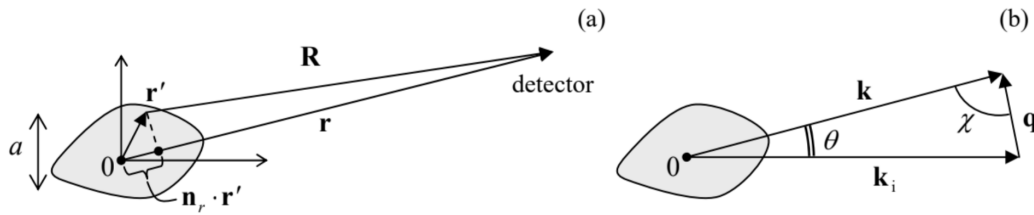


Fig. 3.9. (a) The long-range expansion of R , and (b) the definitions of \mathbf{q} , χ , and θ .

With this replacement, Eq. (74) yields

$$\psi_s(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \frac{|\psi_i|}{r} e^{ikr} \int U(\mathbf{r}') \exp\{-i(\mathbf{k} - \mathbf{k}_i) \cdot \mathbf{r}'\} d^3r'. \quad (3.3.24)$$

This relation is a particular case of a more general formula³⁶

$$\psi_s = |\psi_i| \frac{f(\mathbf{k}, \mathbf{k}_i)}{r} e^{ikr}, \quad (3.3.25)$$

where $f(\mathbf{k}, \mathbf{k}_i)$ is called the scattering function.³⁷ The physical sense of this function becomes clear from the calculation of the corresponding probability current density \mathbf{j}_s . For that, generally, we need to use Eq. (1.47) with the gradient operator having all spherical-coordinate components.³⁸ However, at $kr \gg 1$, the main contribution to $\nabla\psi_s$, proportional to $k \gg 1/r$, is provided by differentiating the factor e^{ikr} , which changes in the common direction of vectors \mathbf{r} and \mathbf{k} , so that

$$\nabla\psi_s \approx \mathbf{n}_r \frac{\partial}{\partial r} \psi_s \approx \mathbf{k} \psi_s, \quad \text{at } kr \gg 1, \quad (3.3.26)$$

and Eq. (1.47) yields

$$\mathbf{j}_s(\theta) \approx \frac{\hbar}{m} |\psi_i|^2 \frac{|f(\mathbf{k}, \mathbf{k}_i)|^2}{r^2} \mathbf{k}. \quad (3.3.27)$$

Plugging this expression, and also Eq. (61) into Eq. (62), for the differential cross-section we get simply

$$\frac{d\sigma}{d\Omega} = |f(\mathbf{k}, \mathbf{k}_i)|^2, \quad (3.3.28)$$

while the total cross-section is

$$\sigma = \oint |f(\mathbf{k}, \mathbf{k}_i)|^2 d\Omega, \quad (3.3.29)$$

so that the scattering function $f(\mathbf{k}, \mathbf{k}_i)$ gives us everything we need - and in fact more, because the function also contains information about the phase of the scattered wave.

According to Eq. (80), in the Born approximation the scattering function is reduced to the so-called Born integral

$$f(\mathbf{k}, \mathbf{k}_i) = -\frac{m}{2\pi\hbar^2} \int U(\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}} d^3r \quad (3.3.30)$$

where for the notation simplicity \mathbf{r}' is replaced with \mathbf{r} , and the following scattering vector is introduced:

$$\mathbf{q} \equiv \mathbf{k} - \mathbf{k}_i \quad (3.3.31)$$

with the length $q = 2k \sin(\theta/2)$, where θ is the scattering angle between the vectors \mathbf{k} and \mathbf{k}_i - see Fig. 9b. For the differential cross-section, Eqs. (84) and (86) yield³⁹

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2} \right)^2 \left| \int U(\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}} d^3r \right|^2. \quad (3.3.32)$$

This is the main result of this section; it may be further simplified for spherically-symmetric scatterers, with

$$U(\mathbf{r}) = U(r). \quad (3.3.33)$$

In this case, it is convenient to represent the exponent in the Born integral as $\exp\{-iqr' \cos \chi\}$, where χ is the angle between the vectors \mathbf{k} (i.e. the direction \mathbf{n}_r toward the detector) and \mathbf{q} (rather than the incident wave vector \mathbf{k}_i !) - see Fig. 9 b. Now, for a fixed \mathbf{q} , we can take this vector's direction for the polar axis of a spherical coordinate system, and reduce Eq. (86) to a 1D integral:

$$f(\mathbf{k}, \mathbf{k}_i) = -\frac{m}{2\pi\hbar^2} \int_0^\infty r^2 dr U(r) \int_0^{2\pi} d\varphi \int_0^\pi \sin \chi d\chi \exp\{-iqr' \cos \chi\} \quad (3.3.34)$$

$$\equiv -\frac{m}{2\pi\hbar^2} \int_0^\infty r^2 dr U(r) 2\pi \frac{2 \sin qr}{qr} \equiv -\frac{2m}{\hbar^2 q} \int_0^\infty U(r) \sin(qr) r dr. \quad (3.3.35)$$

As a simple example, let us use the Born approximation to analyze scattering on the following spherically-symmetric potential:

$$U(\mathbf{r}) = U_0 \exp\left\{-\frac{r^2}{2a^2}\right\}. \quad (3.3.36)$$

In this particular case, it is better to avoid the temptation to exploit the spherical symmetry by using Eq. (90), and instead, use the general Eq. (88), because it may be represented as a product of three similar Cartesian factors:

$$f(\mathbf{k}, \mathbf{k}_i) = -\frac{mU_0}{2\pi\hbar^2} I_x I_y I_z, \quad \text{with } I_x \equiv \int_{-\infty}^{+\infty} \exp\left\{-\left(\frac{x^2}{2a^2} + iq_x x\right)\right\} dx, \quad (3.3.37)$$

and similar integrals for I_y and I_z . From Chapter 2, we already know that the Gaussian integrals like I_x may be readily worked out by complementing the exponent to the full square, in our current case giving

$$I_x = (2\pi)^{1/2} a \exp\left\{-\frac{q_x^2 a^2}{2}\right\}, \text{ etc.,}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{mU_0}{2\pi\hbar^2} I_x I_y I_z\right)^2 = 2\pi a^2 \left(\frac{mU_0 a^2}{\hbar^2}\right)^2 e^{-q^2 a^2}.$$

Now, the total cross-section σ is an integral of $d\sigma/d\Omega$ over all directions of the vector \mathbf{k} . Since in our case the scattering intensity does not depend on the azimuthal angle φ , the only nontrivial integration is over the scattering angle θ - see Fig. 9 b :

$$\begin{aligned} \sigma &\equiv \oint \frac{d\sigma}{d\Omega} d\Omega = 2\pi \int_0^\pi \frac{d\sigma}{d\Omega} \sin \theta d\theta = 4\pi^2 a^2 \left(\frac{mU_0 a^2}{\hbar^2}\right)^2 \int_0^\pi \exp\left\{-\left(2k \sin \frac{\theta}{2}\right)^2 a^2\right\} \sin \theta d\theta \\ &\equiv 4\pi^2 a^2 \left(\frac{mU_0 a^2}{\hbar^2}\right)^2 \int_{\theta=0}^{2\theta\pi} \exp\{-2k^2 a^2 (1 - \cos \theta)\} d(1 - \cos \theta) = \frac{2\pi^2}{k^2} \left(\frac{mU_0 a^2}{\hbar^2}\right)^2 (1 - e^{-4k^2 a^2}) \end{aligned}$$

Let us analyze these results. In the low-energy limit, $ka \ll 1$ (and hence $qa \ll 1$ for any scattering angle), the scattered wave is virtually isotropic: $d\sigma/d\Omega \approx \text{const}$ - a very typical feature of a scalar-wave scattering⁴⁰ by small objects, in any approximation. Note that according to Eq. (77), the Born expression for σ , following from Eq. (94) in this limit,

$$\sigma = 8\pi^2 a^2 \left(\frac{mU_0 a^2}{\hbar^2}\right)^2, \quad (3.3.38)$$

is only valid if σ is much smaller than the scale a^2 of the physical cross-section of the scatterer. In the opposite, high-energy limit $ka \gg 1$, the scattering is dominated by small angles $\theta \approx q/k \sim 1/ka \sim \lambda/a$:

$$\frac{d\sigma}{d\Omega} \approx 2\pi a^2 \left(\frac{mU_0 a^2}{\hbar^2}\right)^2 \exp\{-k^2 a^2 \theta^2\}. \quad (3.3.39)$$

This is, again, very typical for diffraction. Note, however, that due to the smooth character of the Gaussian potential (91), the diffraction pattern (98) exhibits no oscillations of $d\sigma/d\Omega$ as a function of the diffraction angle θ .

Such oscillations naturally appear for scatterers with sharp borders. Indeed, let us consider a uniform spherical scatterer, described by the potential

$$U(\mathbf{r}) = \begin{cases} U_0, & \text{for } r < R \\ 0, & \text{otherwise.} \end{cases}$$

In this case, integration by parts of Eq. (90) readily yields

$$f(\mathbf{k}, \mathbf{k}_i) = \frac{2mU_0}{\hbar^2 q^3} (qR \cos qR - \sin qR), \quad \text{so that } \frac{d\sigma}{d\Omega} = \left(\frac{2mU_0}{\hbar^2 q^3} \right)^2 (qR \cos qR - \sin qR)^2. \quad (3.3.40)$$

According to this result, the scattered wave's intensity drops very fast with q , so that one needs a semilog plot (such as shown in Fig. 10) to reveal small diffraction fringes ⁴¹ with the n^{th} destructive interference (zero-intensity) point tending to $qR = \pi(n + 1/2)$ at $n \rightarrow \infty$. Since, as Fig. 9 b shows, q may only change from 0 to $2k$, these intensity minima are only observable at sufficiently large values of the parameter kR , when they correspond to real values of the scattering angle θ . (At $kR \gg 1$, approximately kR/π of these minima, i.e. "dark rings" of low scattering probability, are observable.) On the contrary, at $kR < 1$ all allowed values of qR are much smaller than 1, and in this limit, the differential cross-section does not depend on qR , i.e. the scattering by the sphere (as by any object in this limit) is isotropic.

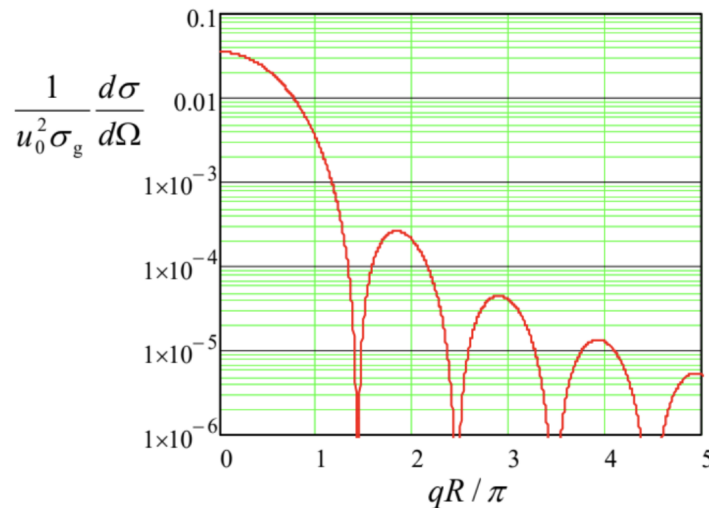


Fig. 3.10. The differential cross-section of the Born scattering of a particle by a "hard" (sharp-border) sphere (97), normalized to its geometric cross-section $\sigma_g \equiv \pi R^2$ and the square of the potential's magnitude parameter $u_0 \equiv U_0 / (\hbar^2 / 2mR^2)$, as a function of the normalized magnitude of the scattering vector \mathbf{q} .

This example shows that in quantum mechanics the notions of particle scattering and diffraction are essentially inseparable.

The Born approximation, while being very simple and used more than any other scattering theory, is not without substantial shortcomings, as becomes clear from the following example. It is not too difficult to prove the so-called optical theorem, strictly valid for an arbitrary scatterer:

$$\text{Im } f(\mathbf{k}_i, \mathbf{k}_i) = \frac{k}{4\pi} \sigma \quad (3.3.41)$$

However, Eq. (86) shows that in the Born approximation, the function f is purely real at $q = 0$ (i.e. for $\mathbf{k} = \mathbf{k}_i$), and hence cannot satisfy the optical theorem. Even more evidently, it cannot describe such a simple effect as a dark shadow ($\psi \approx 0$) cast by a virtually opaque object (say, with $U \gg \delta$). There are several ways to improve the Born approximation, while still keeping its general idea of an approximate treatment of U .

(i) Instead of the main assumption $\psi_s \propto U_0$, we may use a complete perturbation series:

$$\psi_s = \psi_1 + \psi_2 + \dots \quad (3.3.42)$$

with $\psi_n \propto U_0^n$, and find successive approximations ψ_n one by one. In the 1st approximation we return to the Born formula, but already the 2nd approximation yields

$$\text{Im } f_2(\mathbf{k}_i, \mathbf{k}_i) = \frac{k}{4\pi} \sigma_1, \quad (3.3.43)$$

where σ_1 is the total cross-section calculated in the 1st approximation, so that the optical theorem (99) is "almost satisfied".

(ii) As was mentioned above, the Born approximation does not work very well for the objects stretching along the direction (say, x) of the initial wave vector \mathbf{k}_i . This deficiency may be corrected by the so-called eikonal ⁴² approximation, which replaces the plane-wave representation (60) of the incident wave with a WKB-like exponent, though still in the 1st approximation in $U \rightarrow 0$:

$$\begin{aligned}\exp\{ik_0x\} &\rightarrow \exp\left\{\int_0^x k(\mathbf{r}') dx'\right\} \equiv \exp\left\{i \int_0^x \frac{\{2m[E - U(\mathbf{r}')]\}^{1/2}}{\hbar} dx'\right\} \\ &\approx \exp\left\{i \left[k_0x - \frac{m}{\hbar^2 k_0} \int_0^x U(\mathbf{r}') dx'\right]\right\}\end{aligned}$$

Results of this approach satisfy the optical theorem (99) already in the 1st approximation.

Another way toward quantitative results in the theory of scattering, beyond the Born approximation, may be pursued for spherically-symmetric potentials (89); I will discuss it in Sec. 8, after a general discussion of particle motion in such potentials in Sec. 7.

²⁸ The notion of interference is very close to diffraction, but the former term is typically reserved for the wave reemission by just a few components, such as two slits in the Young experiment - see Figs. 1 and 2. A detailed discussion of diffraction and interference of electromagnetic waves may be found in EM Secs. 8.3-8.8.

²⁹ In the context of classical waves, the term "scattering" is typically reserved for wave interaction with disordered sets of small objects - see, e.g., EM Sec. 8.3.

³⁰ See, e.g., CM Sec. 3.5.

³¹ This formula is sometimes called the Lipmann-Schwinger equation, though more frequently this term is reserved for either its operator form or the resulting equation for the spatial Fourier components of ψ and ψ_i . ³² Please notice both the similarity and difference between this Green's function and the propagator discussed in Sec. 2.1. In both cases, we use the linear superposition principle to solve wave equations, but while Eq. (67) gives the solution of the inhomogeneous equation (66), Eq. (2.44) does that for a homogeneous Schrödinger equation. In the latter case, the elementary wave sources are the elementary parts of the initial wavefunction, rather than of the equation's right-hand side as in our current problem.

³³ See, e.g., EM Sec. 1.2.

³⁴ Named after M. Born, who was the first to apply this approximation in quantum mechanics. However, the basic idea of this approach had been developed much earlier (in 1881) by Lord Rayleigh in the context of electromagnetic wave scattering - see, e.g., EM Sec. 8.3. Note also that the contents of that section repeat some aspects of our current discussion - perhaps regrettably but unavoidably so, because the Born approximation is a centerpiece of the theory of scattering/diffraction for both the electromagnetic waves and the de Broglie waves. Hence I felt I had to cover it in this course for the benefit of the readers who skipped the EM part of my series.

³⁵ See, e.g., EM Sec. 8.2.

³⁶ It is easy to prove that this form is an asymptotic form of any solution ψ_s of the scattering problem (even that beyond the Born approximation) at sufficiently large distances $r \gg a, k^{-1}$.

³⁷ Note that the function f has the dimension of length, and does not account for the incident wave. This is why sometimes a dimensionless function, $S = 1 + 2ikf$, is used instead. This function S is called the scattering matrix, because it may be considered a natural generalization of the 1D matrix S defined by Eq. (2.124), to higher dimensionality.

³⁸ See, e.g., MA Eq. (10.8).

³⁹ Note that according to Eq. (88), in the Born approximation the scattering intensity does not depend on the sign of the potential U , and also that scattering in a certain direction is completely determined by a specific Fourier component of the function $U(\mathbf{r})$, namely by its harmonic with the wave vector equal to the scattering vector \mathbf{q} .

⁴⁰ Note that this is only true for scalar (e.g., the de Broglie) waves, and different for vector ones, in particular the electromagnetic waves, where the intensity of the dipole radiation, and hence the scattering by small objects vanishes in the direction of the incident field's polarization - see, e.g., EM Eqs. (8.26) and (8.139).

⁴¹ Their physics is very similar to that of the Fraunhofer diffraction on a 1D scatterer - see, e.g., EM Sec. 8.4.

⁴² From the Greek word $\epsilon\iota\kappa\omicron\nu$, meaning "image". In our current context, this term is purely historic.

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