1.9: Exercise Problems

1.1. The actual postulate made by N. Bohr in his original 1913 paper was not directly Eq. (8), but the assumption that at quantum leaps between adjacent large (quasiclassical) orbits with $n >> 1$, the hydrogen atom either emits or absorbs energy $\Delta E = \hbar\omega$, where ω is its classical radiation frequency according to classical electrodynamics, equal to the angular velocity of electron's rotation. [66] Prove that this postulate is indeed compatible with Eqs. (7)-(8).

1.2. Use Eq. (53) to prove that the linear operators of quantum mechanics are commutative: $\hat{A}_2 + \hat{A}_1 = \hat{A}_1 + \hat{A}_2$, and associative: $\left(\hat{A}_1 + \hat{A}_2\right) + \hat{A}_3 = \hat{A}_1 + \left(\hat{A}_2 + \hat{A}_3\right)$. 1.3. Prove that for any time-independent Hamiltonian operator $\hat{H}$ and two arbitrary complex functions $f(\mathbf{r})$ and $g(\mathbf{r})$,

$$\int f(\mathbf{r})\hat{H}g(\mathbf{r})d^3r = \int \hat{H}f(\mathbf{r})g(\mathbf{r})d^3r. \tag{1.9.1}$$

1.4. Prove that the Schrödinger equation (25) with the Hamiltonian operator given by Eq. (41), is Galilean form-invariant, provided that the wavefunction is transformed as

$$\Psi'\left(\mathbf{r}', t'\right) = \Psi(\mathbf{r}, t)\exp\left\{-i\frac{m\mathbf{v}\cdot\mathbf{r}}{\hbar} + i\frac{mv^2t}{2\hbar}\right\}, \tag{1.9.2}$$

where the prime sign marks the variables measured in the reference frame 0 ' that moves, without rotation, with a constant velocity $\mathbf{v}$ relatively to the "lab" frame 0 . Give a physical interpretation of this transformation.

1.5. * Prove the so-called Hellmann-Feynman theorem: 67

$$\frac{\partial E_n}{\partial\lambda} = \left\langle\frac{\partial H}{\partial\lambda}\right\rangle_n, \tag{1.9.3}$$

where λ is some c-number parameter, on which the time-independent Hamiltonian $\hat{H}$, and hence its eigenenergies E_n, depend.

1.6. * Use Eqs. (73) and (74) to analyze the effect of phase locking of Josephson oscillations on the dc current flowing through a weak link between two superconductors (frequently called the Josephson junction), assuming that an external source applies to the junction a sinusoidal ac voltage with frequency ω and amplitude A.

1.7. Calculate $\langle x\rangle$, $\langle p_x\rangle$, δx, and δp_x for the eigenstate $\{n_x, n_y, n_z\}$ of a particle in a rectangular hard-wall box described by Eq. (77), and compare the product $\delta x\delta p_x$ with the Heisenberg's uncertainty relation.

1.8. Looking at the lower (red) line in Fig. 8, it seems plausible that the 1D ground-state function (84) of the simple potential well (77) may be well approximated with an inverted quadratic parabola:

$$X_{\text{trial}}(x) = Cx\left(a_x - x\right), \tag{1.9.4}$$

where C is a normalization constant. Explore how good this approximation is.

1.9. A particle placed in a hard-wall rectangular box with sides a_x, a_y, and a_z, is in its ground state. Calculate the average force acting on each face of the box. Can the forces be characterized by a certain pressure?

1.10. A 1D quantum particle was initially in the ground state of a very deep, rectangular potential well of width a :

$$U(x) = \begin{cases} 0, & \text{for } -a/2 < x < +a/2 \\ +\infty, & \text{otherwise} \end{cases}$$

At some instant, the well's width is abruptly increased to a new value $a' > a$, leaving the potential symmetric with respect to the point $x = 0$, and then left constant. Calculate the probability that after the change, the particle is still in the ground state of the system.

1.11. At $t = 0$, a 1D particle of mass m is placed into a hard-wall, flat-bottom potential well

$$U(x) = \begin{cases} 0, & \text{for } 0 < x < a \\ +\infty, & \text{otherwise} \end{cases}$$

in a 50/50 linear superposition of the lowest (ground) state and the first excited state. Calculate:

(i) the normalized wavefunction $\Psi(x, t)$ for arbitrary time $t \geq 0$, and

(ii) the time evolution of the expectation value $\langle x\rangle$ of the particle's coordinate.

1.12. Calculate the potential profiles $U(x)$ for that the following wavefunctions,

(i) $\Psi = c\exp\{-ax^2 - ibt\}$, and

(ii) $\Psi = c\exp\{-a|x| - ibt\}$

(with real coefficients $a > 0$ and b), satisfy the 1D Schrödinger equation for a particle with mass m. For each case, calculate $\langle x\rangle$, $\langle p_x\rangle$, δx, and δp_x, and compare the product $\delta x \delta p_x$ with the Heisenberg's uncertainty relation.

1.13. A 1D particle of mass m, moving in the field of a stationary potential $U(x)$, has the following eigenfunction

$$\psi(x) = \frac{C}{\cosh \kappa x}, \tag{1.9.5}$$

where C is the normalization constant, and κ is a real constant. Calculate the function $U(x)$ and the state's eigenenergy E.

1.14. Calculate the density dN/dE of traveling-wave quantum states in large rectangular potential wells of various dimensions: $d = 1, 2$, and 3.

1.15. * Use the finite-difference method with steps $a/2$ and $a/3$ to find as many eigenenergies as possible for a 1D particle in the infinitely deep, hard-wall 1D potential well of width a. Compare the results with each other, and with the exact formula. [68]

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[67] Despite this common name, H. Hellmann (in 1937) and R. Feynman (in 1939) were not the first ones in the long list of physicists who had (apparently, independently) discovered this equality. Indeed, it has been traced back to a 1922 paper by W. Pauli, and was carefully proved by P. Güttinger in 1931.

[68] You may like to start by reading about the finite-difference method - see, e.g., CM Sec. 8.5 or EM Sec. 2.11.
