9.2: Photon Absorption and Counting

As a matter of principle, the Casimir effect may be used to measure quantum effects in not only the free-space electromagnetic field but also that the field arriving from active sources - lasers, etc. However, usually such studies may be done by simpler detectors, in which the absorption of a photon by a single atom leads to its ionization. This ionization, i.e. the emission of a free electron, triggers an avalanche reaction (e.g., an electric discharge in a Geiger-type counter), which may be readily registered using appropriate electronic circuitry. In good photon counters, the first step, the "trigger" atom ionization, is the bottleneck of the whole process (the photon count), so that to analyze their statistics, it is sufficient to consider the field's interaction with just this atom.

Its ionization is a quantum transition from a discrete initial state of the atom to its final, ionized state with a continuous energy spectrum, induced by an external electromagnetic field. This is exactly the situation shown in Fig. 6.12, so we may apply to it the Golden Rule of quantum mechanics in the form (6.149), with the system a associated with the electromagnetic field, and system b with the trigger atom. The atom's size is typically much smaller than the radiation wavelength, so that the field-atom interaction may be adequately described in the electric dipole approximation (6.146)

$$\hat{H}_{\text{int}} = -\hat{\mathscr{E}} \cdot \hat{\mathbf{d}}, \tag{9.2.1}$$

where $\hat{\mathbf{d}}$ is the dipole moment's operator. Hence we may associate this operator with the operand $\hat{B}$ in Eqs. (6.145)-(6.149), while the electric field operator $\hat{E}$ is associated with the operand $\hat{A}$ in those relations. First, let us assume that our field consists of only one mode $\mathbf{e}_j(\mathbf{r})$ of frequency ω. Then we can keep only one term in the sum (16a), and drop the index j, so that Eq. (6.149) may be rewritten as

$$\begin{aligned}\Gamma &= \frac{2\pi}{\hbar}|\langle \text{fin}|\hat{E}(\mathbf{r},t)|\,\text{ini}\rangle|^2 \,|\,\left\langle \text{fin}\left|\hat{\mathbf{d}}(t)\cdot\mathbf{n}_e\right|\,\text{ini}\,\right\rangle\right|^2\rho_{\text{a}} \\ &= \frac{2\pi}{\hbar}\frac{\hbar\omega}{2}\left|\left\langle \text{fin}\,\left|\left[\hat{a}^\dagger(t)-\hat{a}(t)\right]e(\mathbf{r})\right|\,\text{ini}\,\right\rangle\right|^2 \,|\,\left\langle \text{fin}\left|\hat{\mathbf{d}}(t)\cdot\mathbf{n}_e\right|\,\text{ini}\,\right\rangle\right|^2\rho_{\text{a}}\end{aligned}$$

where $\mathbf{n}_e \equiv \mathbf{e}(\mathbf{r})/e(\mathbf{r})$ is the local direction of the vector $\mathbf{e}(\mathbf{r})$, symbols "ini" and "fin" denote the initial and final states of the corresponding system (the electromagnetic field in the first long bracket, and the atom in the second bracket), and the density ρ_{a} of the continuous atomic states should be calculated at its final energy $E_{\text{fin}} = E_{\text{ini}} + \hbar\omega$.

As a reminder, in the Heisenberg picture of quantum dynamics, the initial and final states are time-independent, while the creation-annihilation operators are functions of time. In the Golden Rule formula (25), as in any perturbative result, this time dependence has to be calculated ignoring the perturbation - in this case, the field-atom interaction. For the field's creation-annihilation operators, this dependence coincides with that of the usual 1D oscillator - see Eq. (5.141), in which ω_0 should be, in our current notation, replaced with ω :

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}, \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{+i\omega t}. \tag{9.2.2}$$

Hence Eq. (25) becomes

$$\Gamma = \pi\omega\,|\,\left\langle\,\text{fin}\,\left[\hat{a}^\dagger(0)e^{i\omega t}-\hat{a}(0)e^{-i\omega t}\right]e(\mathbf{r})\right|\,\text{ini}\,\right\rangle|^2\,|\,\left\langle\,\text{fin}\,\left|\hat{\mathbf{d}}(t)\cdot\mathbf{n}_e\right|\,\text{ini}\,\right\rangle\right|^2\rho_{\text{a}}. \tag{9.2.3}$$

Now let us multiply the first long bracket by $\exp\{i\omega t\}$, and the second one by $\exp\{-i\omega t\}$:

$$\Gamma = \pi\omega\left|\left\langle \text{fin}\,\left|\left[\hat{a}^\dagger(0)e^{2i\omega t}-\hat{a}(0)\right]e(\mathbf{r})\right|\,\text{ini}\,\right\rangle\right|^2\,|\,\left\langle \text{fin}\left|\hat{\mathbf{d}}(t)\cdot\mathbf{n}_e e^{-i\omega t}\right|\text{ini}\,\right\rangle\right|^2\rho_{\text{a}}. \tag{9.2.4}$$

This, mathematically equivalent form of the previous relation shows more clearly that at resonant photon absorption, only the annihilation operator gives a significant time-averaged contribution to the first bracket matrix element. (As a reminder, the quantum-mechanical Golden Rule for time-dependent perturbations is a result of averaging over a time interval much larger than $1/\omega - \text{see Sec. 6.6.}$) Similarly, according to Eq. (4.199), the Heisenberg operator of the dipole moment, corresponding to the increase of atom's energy by $\hbar\omega$, has the Fourier components that differ in frequency from ω only by $\sim\Gamma << \omega$, so that its time dependence virtually compensates the additional factor in the second bracket of Eq. $(27b)$, and this bracket also may have a substantial time average. Hence, in the first bracket we may neglect the fast-oscillating term, whose average over time interval $\sim 1/\Gamma$ is very close to zero. [16]

Now let us assume, first, that we use the same detector, characterized by the same matrix element of the quantum transition, i.e. the same second bracket in Eq. (27), and the same final state density ρ_{a}, for measurement of various electromagnetic fields - or just of the same field at different points $\mathbf{r}$. Then we are only interested in the behavior of the first, field-related bracket, and may write

$$\Gamma \propto\left|\langle\operatorname{fin}|\hat{a} e(\mathbf{r})| \text { ini }\rangle\right|^2 \equiv\langle\operatorname{fin}|\hat{a} e(\mathbf{r})| \text { ini }\rangle\langle\operatorname{fin}|\hat{a} e(\mathbf{r})| \text { ini }\rangle^* \equiv\left\langle\text { ini }\left|\hat{a}^{\dagger} e^*(\mathbf{r})\right| \operatorname{fin}\right\rangle\langle\operatorname{fin}|\hat{a} e(\mathbf{r})| \text { ini }\rangle, \tag{9.2.5}$$

where the creation-annihilation operators are implied to be taken at $t=0$, i.e. in the Schrödinger picture, and the initial and final states are those of the field alone. Second, let us now calculate the total rate of transitions to all available final states of the given mode $e(\mathbf{r})$. If such states formed a full and orthonormal set, we could use the closure relation (4.44), applied to the final states, to write

$$\Gamma \propto \sum_{\text {fin }}\left\langle\text { ini }\left|\hat{a}^{\dagger} e^*(\mathbf{r})\right| \text { fin }\right\rangle\langle\operatorname{fin}|\hat{a} e(\mathbf{r})| \text { ini }\rangle=\left\langle\text { ini }\left|\hat{a}^{\dagger} \hat{a}\right| \text { ini }\right\rangle e^*(\mathbf{r}) e(\mathbf{r})=\langle n\rangle_{\text {ini }}|e(\mathbf{r})|^2 \tag{9.2.6}$$

where, for a given field mode, $\langle n\rangle_{\text {ini }}$ is the expectation value of the operator $\hat{n} \equiv \hat{a}^{\dagger} \hat{a}$ for the initial state of the electromagnetic field. In the more realistic case of fields in relatively large volumes, $V \gg \lambda^3$, with their virtually continuous spectrum of final states, the middle equality in this relation is not strictly valid, but it is correct to a constant multiplier, [17] which we are currently not interested in. Note, however, that Eq. (29) may be substantially wrong for high- Q electromagnetic resonators ("cavities"), which may make just one (or a few) modes available for transitions. (Quantum electrodynamics of such cavities will be briefly discussed in Sec. 4 below.)

Let us apply Eq. (29) to several possible quantum states of the mode.

(i) First, as a sanity check, the ground initial state, $n=0$, gives no photon absorption at all. The interpretation is easy: the ground state field, cannot emit a photon that would ionize an atom in the counter. Again, this does not mean that the ground-state "motion" is not observable (if you still think so, please review the Casimir effect discussion in Sec. 1), just that it cannot ionize the trigger atom because it does not have any spare energy for doing that.

(ii) All other coherent states (Fock, Glauber, squeezed, etc.) of the field oscillator give the same counting rate, provided that their $\langle n\rangle_{\text {ini }}$ is the same. This result may be less evident if we apply Eq. (29) to the interference of two light beams from the same source - say, in the double-slit or the Braggscattering configurations. In this case, we may represent the spatial distribution of the field as a sum

$$e(\mathbf{r})=e_1(\mathbf{r})+e_2(\mathbf{r}). \tag{9.2.7}$$

Here each term describes one possible wave path, so that the operator product in Eq. (29) may be a rapidly changing function of the detector position. For this configuration, our result (29) means that the interference pattern (and its contrast) are independent of the particular state of the electromagnetic field's mode.

(iii) Surprisingly, the last statement is also valid for a classical mixture of the different eigenstates of the same field mode, for example for its thermal-equilibrium state. Indeed, in this case we need to average Eq. (29) over the corresponding classical ensemble, but it would only result in a different meaning of averaging n in that equation; the field part describing the interference pattern is not affected.

The last result may look a bit counter-intuitive because common sense tells us that the stochasticity associated with thermal equilibrium has to suppress the interference pattern contrast. These expectations are (partly :-) justified because a typical thermal source of radiation produces many field modes j, rather than one mode we have analyzed. These modes may have different wave numbers k_j and hence different field distribution functions $\mathbf{e}_j(\mathbf{r})$, resulting in shifted interference patterns. Their summation would indeed smear the interference, suppressing its contrast.

So the use of one photon detector is not the best way to distinguish different quantum states of an electromagnetic field mode. This task, however, may be achieved using the photon counting correlation technique shown in Fig. $2 \cdot$ [18]

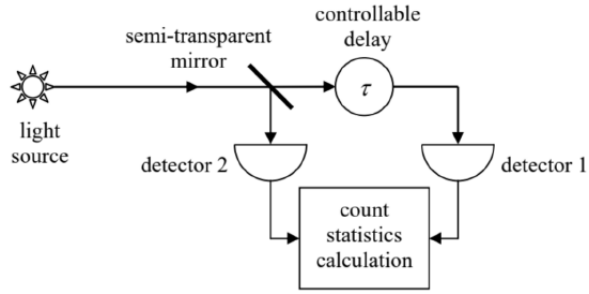

Fig. 9.2. Photon count correlation measurement.

In this experiment, the counter rate correlation may be characterized by the so-called secondorder correlation function of the counting rates,

$$g^{(2)}(\tau) \equiv \frac{\langle\Gamma_1(t)\Gamma_2(t-\tau)\rangle}{\langle\Gamma_1(t)\rangle\langle\Gamma_2(t)\rangle}, \tag{9.2.8}$$

where the averaging may be carried out either over many similar experiments, or over a relatively long time interval $t \gg \tau$, with usual field sources - due to their ergodicity. Using the normalized correlation function (31) is very convenient because the characteristics of both detectors and the beam splitter (e.g., a semi-transparent mirror, see Fig. 2) drop out from this fraction.

Very unexpectedly for the mid-1950s, Hanbury Brown and Twiss discovered that the correlation function depends on time delay τ in the way shown (schematically) with the solid line in Fig. 3. It is evident from Eq. (31) that if the counting events are completely independent, $g^{(2)}(\tau)$ should be equal to 1 - which is always the case in the limit $\tau \rightarrow \infty$. (As will be shown in the next section, the characteristic time of this approach is usually between 10^{-11} s and 10^{-8} s, so that for its measurement, the delay time control may be provided just by moving one of the detectors by a human-scale distance between a few millimeters to a few meters.) Hence, the observed behavior at $\tau \rightarrow 0$ corresponds to a positive correlation of detector counts at small time delays, i.e. to a higher probability of the nearly simultaneous arrival of photons to both counters. This counter-intuitive effect is called photon bunching.

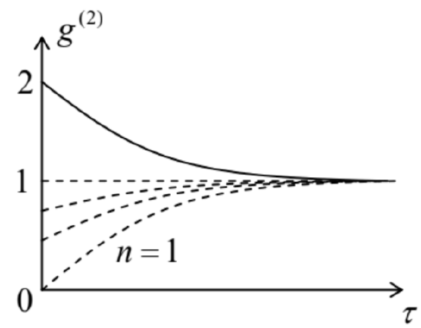

Fig. 9.3. Photon bunching (solid line) and antibunching for various n (dashed lines). The lines approach level $g^{(2)}=1$ at $\tau \rightarrow \infty$ (on the time scale depending on the light source).

Let us use our simple single-mode model to analyze this experiment. Now the elementary quantum process characterized by the numerator of Eq. (31), is the correlated, simultaneous ionization of two trigger atoms, at two spatial-temporal points $\{\mathbf{r}_1, t\}$ and $(\mathbf{r}_2, t-\tau\}$, by the same field mode, so that we need to make the following replacement in the first of Eqs. (25):

$$\hat{\mathscr{E}}(\mathbf{r}, t) \rightarrow \text{ const } \times \hat{\mathscr{E}}\left(\mathbf{r}_1, t\right) \hat{\mathscr{E}}\left(\mathbf{r}_2, t-\tau\right). \tag{9.2.9}$$

Repeating all the manipulations done above for the single-counter case, we get

$$\langle\Gamma_1(t)\Gamma_2(t-\tau)\rangle \propto \left\langle\text{ini}\left|\hat{a}(t)^{\dagger}\hat{a}(t-\tau)^{\dagger}\hat{a}(t-\tau)\hat{a}(t)\right|\text{ ini }\right\rangle e^*\left(\mathbf{r}_1\right) e^*\left(\mathbf{r}_2\right) e\left(\mathbf{r}_1\right) e\left(\mathbf{r}_2\right). \tag{9.2.10}$$

Plugging this expression, as well as Eq. (29) for single-counter rates, into Eq. (31), we see that the field distribution factors (as well as the detector-specific brackets and the density of states ρ_{a}) cancel, giving a very simple final expression:

$$g^{(2)}(\tau)=\frac{\left\langle\hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t-\tau)\hat{a}(t-\tau)\hat{a}(t)\right\rangle}{\left\langle\hat{a}^{\dagger}(t)\hat{a}(t)\right\rangle^2}, \tag{9.2.11}$$

where the averaging should be carried out, as before, over the initial state of the field.

Still, the calculation of this expression for arbitrary τ may be quite complex, because in many cases the relaxation of the correlation function to the asymptotic value $\mathrm{g}^{(2)}(\infty)$ is due to the interaction of the light source with the environment, and hence requires the open-system techniques that were discussed in Chapter 7 . However, the zero-delay value $\mathrm{g}^{(2)}(0)$ may be calculated straightforwardly, because the time arguments of all operators are equal, so that we may write

$$g^{(2)}(0)=\frac{\left\langle\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\right\rangle}{\left\langle\hat{a}^{\dagger}\hat{a}\right\rangle^{2}}\tag{9.2.12}$$

Let us evaluate this ratio for the simplest states of the field.

(i) The n^{th} Fock state. In this case, it is convenient to act with the annihilation operators upon the ket-vectors, and by the creation operators, upon the bra-vectors, using Eqs. (19):

$$g^{(2)}(0)=\frac{\left\langle n\left|\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\right|n\right\rangle}{\left\langle n\left|\hat{a}^{\dagger}\hat{a}\right|n\right\rangle^{2}}=\frac{\left\langle n-2\left|[n(n-1)]^{1/2}[n(n-1)]^{1/2}\right|n-2\right\rangle}{\left\langle n-1\left|n^{1/2}n^{1/2}\right|n-1\right\rangle^{2}}=\frac{n(n-1)}{n^{2}}\equiv 1-\frac{1}{n}.\tag{9.2.13}$$

We see that the correlation function at small delays is suppressed rather than enhanced - see the dashed lines in Fig. 3. This photon antibunching effect has a very simple handwaving explanation: a single photon emitted by the wave source may be absorbed by just one of the detectors. For the initial state $n=1$, this is the only option, and it is very natural that Eq. (36) predicts no simultaneous counts at $\tau=0$. Despite this theoretical simplicity, reliable observations of the antibunching have not been carried out until 1977,[19] due to the experimental difficulty of driving electromagnetic field oscillators into their Fock states - see Sec. 4 below.

(ii) The Glauber state α. A similar procedure, but now using Eq. (5.124) and its Hermitian conjugate, $\langle\alpha|\hat{a}^{\dagger}=\langle\alpha|\alpha^{*}$, yields

$$g^{(2)}(0)=\frac{\left\langle\alpha\left|\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\right|\alpha\right\rangle}{\left\langle\alpha\left|\hat{a}^{\dagger}\hat{a}\right|\alpha\right\rangle^{2}}=\frac{\alpha^{*}\alpha^{*}\alpha\alpha}{\left(\alpha^{*}\alpha\right)^{2}}\equiv 1,\tag{9.2.14}$$

for any parameter α. We see that the result is different from that for the Fock states, unless in the latter case $n\rightarrow\infty$. (We know that the Fock and Glauber properties should also coincide for the ground state, but at that state the correlation function's value is uncertain, because there are no photon counts at all.)

(iii) Classical mixture. From Chapter 7 , we know that such statistical ensembles cannot be described by single state vectors, and require the density matrix w for their description. Here, we may combine Eqs. (35) and (7.5) to write

$$g^{(2)}(0)=\frac{\operatorname{Tr}\left(\hat{w}\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\right)}{\left[\operatorname{Tr}\left(\hat{w}\hat{a}^{\dagger}\hat{a}\right)\right]^{2}}.\tag{9.2.15}$$

Spelling out this expression is easy for the field in thermal equilibrium at some temperature T, because its density matrix is diagonal in the basis of Fock states $n-$ see Eqs. (7.24):

$$w_{nn'}=W_{n}\delta_{nn'},\quad W_{n}=\exp\left\{-\frac{E_{n}}{k_{\mathrm{B}}T}\right\}/Z\equiv\lambda^{n}/\sum_{n=0}^{\infty}\lambda^{n},\quad\text{where }\lambda\equiv\exp\left\{-\frac{\hbar\omega}{k_{\mathrm{B}}T}\right\}.\tag{9.2.16}$$

So, for the operators in the numerator and denominator of Eq. (38) we also need just the diagonal terms of the operator products, which have already been calculated - see Eq. (36). As a result, we get

$$g^{(2)}(0)=\frac{\sum_{n=0}^{\infty}W_{n}n(n-1)}{\left(\sum_{n=0}^{\infty}W_{n}n\right)^{2}}=\frac{\sum_{n=0}^{\infty}\lambda^{n}n(n-1)\times\sum_{n=0}^{\infty}\lambda^{n}}{\left(\sum_{n=0}^{\infty}\lambda^{n}n\right)^{2}}.\tag{9.2.17}$$

One of the three series involved in this expression is just the usual geometric progression,

$$\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \tag{9.2.18}$$

and the remaining two series may be readily calculated by its differentiation over the parameter λ :

$$\sum_{n=0}^{\infty} \lambda^n n \equiv \lambda \sum_{n=0}^{\infty} \lambda^{n-1} n = \lambda \frac{d}{d\lambda} \sum_{n=0}^{\infty} \lambda^n = \lambda \frac{d}{d\lambda} \frac{1}{1-\lambda} = \frac{\lambda}{(1-\lambda)^2},$$

$$\sum_{n=0}^{\infty} \lambda^n n(n-1) \equiv \lambda^2 \sum_{n=0}^{\infty} \lambda^{n-2} n(n-1) = \lambda^2 \frac{d^2}{d^2\lambda} \left(\sum_{n=0}^{\infty} \lambda^n \right) = \lambda^2 \frac{d^2}{d\lambda^2} \frac{1}{1-\lambda} = \frac{2\lambda^2}{(1-\lambda)^3},$$

and for the correlation function we get an extremely simple result independent of the parameter λ and hence of temperature:

$$g^{(2)}(0) = \frac{\left[2\lambda^2/(1-\lambda)^3\right] \left[1/(1-\lambda)\right]}{\left[\lambda/(1-\lambda)^2\right]^2} \equiv 2. \tag{9.2.19}$$

This is exactly the photon bunching effect first observed by Hanbury Brown and Twiss – see Fig. 3. We see that in contrast to antibunching, this is an essentially classical (statistical) effect. Indeed, Eq. (43) allows a purely classical derivation. In the classical theory, the counting rate (of a single counter) is proportional to the wave intensity I, so that Eq. (31) with $\tau = 0$ is reduced to

$$g^{(2)}(0) = \frac{\langle I^2 \rangle}{\langle I \rangle^2}, \quad \text{with } I \propto \overline{E^2(t)} \propto E_\omega E_\omega^*. \tag{9.2.20}$$

For a sinusoidal field, the intensity is constant, and $g^{(2)}(0) = 1$. (This is also evident from Eq. (37), because the classical state may be considered as a Glauber state with $\alpha \to \infty$.) On the other hand, if the intensity fluctuates (either in time, or from one experiment to another), the averages in Eq. (44) should be calculated as

$$\langle I^k \rangle = \int_0^\infty w(I) I^k dI, \quad \text{with } \int_0^\infty w(I) dI = 1, \quad \text{and } k = 1, 2, \tag{9.2.21}$$

where $w(I)$ is the probability density. For classical statistics, the probability is an exponential function of the electromagnetic field energy, and hence its intensity:

$$w(I) = Ce^{-\beta I}, \text{ where } \beta \propto 1/k_\mathrm{B} T, \tag{9.2.22}$$

so that Eqs. (45) yield: [20]

$$\int_0^\infty C \exp\{-\beta I\} dI \equiv C/\beta = 1, \text{ and hence } C = \beta,$$

$$\langle I^k \rangle = \int_0^\infty w(I) I^k dI = C \int_0^\infty \exp\{-\beta I\} I^k dI = \frac{1}{\beta^k} \int_0^\infty \exp\{-\xi\} \xi^k d\xi = \begin{cases} 1/\beta, & \text{for } k = 1, \\ 2/\beta^2, & \text{for } k = 2. \end{cases}$$

Plugging these results into Eq. (44), we get $g^{(2)}(0) = 0$, in complete agreement with Eq. (43).

For some field states, including the squeezed ground states ζ discussed at the end of Sec. 5.5, values $g^{(2)}(0)$ may be even higher than 2 - the so-called super-bunching. Analyses of two cases of such super-bunching are offered for the reader's exercise - see the problem list in the chapter's end.

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[16] This is essentially the same rotating wave approximation (RWA), which was already used in Sec. 6.5 and beyond - see, e.g., the transition from Eq. (6.90) to the first of Eqs. (6.94).

[17] As the Golden Rule shows, this multiplier is proportional to the density ρ_f of the final states of the field.

[18] It was pioneered as early as the mid-1950s (i.e. before the advent of lasers), by Robert Hanbury Brown and Richard Twiss. Their second experiment was also remarkable for the rather unusual light source - the star Sirius! (Their work was an effort to improve astrophysics interferometry techniques.)

[19] By H. J. Kimble et al., Phys. Rev. Lett. 39, 691 (1977). For a detailed review of phonon antibunching, see, e.g., H. Paul, Rev. Mod. Phys. 54, 1061 (1982).

[20] See, e.g., MA Eq. (6.7c) with $n = 0$ and $n = 1$.