9.8: Exercise Problems

9.1. Prove the Casimir formula, given by Eq. (23), by calculating the net force $F=\mathcal{P}A$ exerted by the electromagnetic field, in its ground state, on two perfectly conducting parallel plates of area A, separated by a vacuum gap of width $t \lll A^{1/2}$.

Hint: Calculate the field energy in the gap volume with and without the account of the plate effect, and then apply the Euler-Maclaurin formula [63] to the difference between these two results.

9.2. Electromagnetic radiation by some single-mode quantum sources may have such a high degree of coherence that it is possible to observe the interference of waves from two independent sources with virtually the same frequency, incident on one detector.

(i) Generalize Eq. (29) to this case.

(ii) Use this generalized expression to show that incident waves in different Fock states do not create an interference pattern.

9.3. Calculate the zero-delay value $g^{(2)}(0)$ of the second-order correlation function of a singlemode electromagnetic field in the so-called Schrödinger-cat state 64 a coherent superposition of two Glauber states, with equal but sign-opposite parameters α, and a certain phase shift between them.

9.4. Calculate the zero-delay value $g^{(2)}(0)$ of the second-order correlation function of a singlemode electromagnetic field in the squeezed ground state ζ defined by Eq. (5.142).

9.5. Calculate the rate of spontaneous photon emission (into unrestricted free space) by a hydrogen atom, initially in the $2p$ state $(n=2, l=1)$ with $m=0$. Would the result be different for $m=\pm 1$? for the $2s$ state $(n=2, l=0, m=0)$? Discuss the relation between these quantum-mechanical results and those given by the classical theory of radiation for the simplest classical model of the atom.

9.6. An electron has been placed on the lowest excited level of a spherically-symmetric, quadratic potential well $U(\mathbf{r})=m_{\mathrm{e}}\omega^2 r^2/2$. Calculate the rate of its relaxation to the ground state, with the emission of a photon (into unrestricted free space). Compare the rate with that for a similar transition of the hydrogen atom, for the case when the radiation frequencies of these two systems are equal.

9.7. Derive an analog of Eq. (53) for the spontaneous photon emission into the free space, due to a change of the magnetic dipole moment $\mathbf{m}$ of a small-size system.

9.8. A spin- $1/2$ particle, with a gyromagnetic ratio γ, is in its orbital ground state in dc magnetic field $\mathscr{B}_0$. Calculate the rate of its spontaneous transition from the higher to the lower energy level, with the emission of a photon into the free space. Evaluate this rate for in an electron in a field of $10\ \mathrm{T}$, and discuss the implications of this result for laboratory experiments with electron spins.

9.9. Calculate the rate of spontaneous transitions between the two sublevels of the ground state of a hydrogen atom, formed as a result of its hyperfine splitting. Discuss the implications of the result for the width of the $21-\mathrm{cm}$ spectral line of hydrogen.

9.10. Find the eigenstates and eigenvalues of the Jaynes-Cummings Hamiltonian (78), and discuss their behavior near the resonance point $\omega=\Omega$.

9.11. Analyze the Purcell effect, mentioned in Secs. 3 and 4, quantitatively; in particular, calculate the so-called Purcell factor F_{P} defined as the ratio of the rate Γ_{s} of atom's spontaneous emission into a resonant cavity tuned exactly to the quantum transition frequency, to that into the free space.

9.12. Prove that the Klein-Gordon equation (84) may be rewritten in the form similar to the nonrelativistic Schrödinger equation (1.25), but for a two-component wavefunction, with the Hamiltonian represented (in the usual z-basis) by the following 2×2-matrix:

$$\mathrm{H}=-\left(\sigma_z+i\sigma_y\right)\frac{\hbar^2}{2m}\nabla^2+mc^2\sigma_z. \tag{9.8.1}$$

Use your solution to discuss the physical meaning of the wavefunction's components.

9.13. Calculate and discuss the energy spectrum of a relativistic, spinless, charged particle placed into an external uniform, time-independent magnetic field $\mathscr{R}$. Use the result to formulate the condition of validity of the non-relativistic theory in this situation.

9.14. Prove Eq. (91) for the energy spectrum of a hydrogen-like atom/ion, starting from the relativistic Schrödinger equation.

Hint: A mathematical analysis of Eq. (3.193) shows that its eigenvalues are given by Eq. (3.201), $\varepsilon_n = -1/2n^2$, with $n = l + 1 + n_r$, where $n_r = 0, 1, 2, \ldots$, even if the parameter l is not integer.

9.15. Derive a general expression for the differential cross-section of elastic scattering of a spinless relativistic particle by a static potential $U(\mathbf{r})$, in the Born approximation, and formulate the conditions of its validity. Use these results to calculate the differential cross-section of scattering of a particle with the electric charge $-e$ by the Coulomb electrostatic potential $\phi(\mathbf{r}) = Ze/4\pi\varepsilon_0 r$.

9.16. Starting from Eqs. (95)-(98), prove that the probability density w given by Eq. (101) and the probability current density $\mathbf{j}$ defined by Eq. (102) do indeed satisfy the continuity equation (1.52): $\partial w/\partial t + \nabla \cdot \mathbf{j} = 0$.

9.17. Calculate the commutator of the operator $\hat{L}^2$ and Dirac's Hamiltonian of a free particle. Compare the result with that for the non-relativistic Hamiltonian, and interpret the difference.

9.18. Calculate commutators of the operators $\hat{S}^2$ and $\hat{J}^2$ with Dirac's Hamiltonian (97), and give an interpretation of the results.

9.19. In the Heisenberg picture of quantum dynamics, derive an equation describing the time evolution of free electron's velocity in the Dirac theory. Solve the equation for the simplest state, with definite energy and momentum, and discuss the solution.

9.20. Calculate the eigenstates and eigenenergies of a relativistic spin- $1/2$ particle with charge q, placed into a uniform, time-independent external magnetic field $\mathscr{B}$. Compare the calculated energy spectrum with those following from the non-relativistic theory and the relativistic Schrödinger equation.

9.21. Following the discussion at the very end of Section 7, introduce quantum field operators $\hat{\psi}$ that would be related to the usual wavefunctions ψ just as the electromagnetic field operators (16) are related to the classical electromagnetic fields, and explore basic properties of these operators. (For this preliminary study, consider the fixed-time situation.)

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[63] See, e.g., MA Eq. (2.12a).

[64] Its name stems from the well-known Schrödinger cat paradox, which is (very briefly) discussed in Sec. 10.1.
