9.1: Electromagnetic Field Quantization

Classical physics gives us *[Math Processing Error]* the following general relativistic relation between the momentum *[Math Processing Error]* and energy *[Math Processing Error]* of a free particle with rest mass *[Math Processing Error]*, which may be simplified in two limits - non-relativistic and ultra-relativistic: *[Math Processing Error]* In both limits, the transfer from classical to quantum mechanics is easier than in the arbitrary case. Since all the previous part of this course was committed to the first, non-relativistic limit, I will now jump to a brief discussion of the ultra-relativistic limit *[Math Processing Error]*, for a particular but very important system - the electromagnetic field. Since the excitations of this field, called photons, are currently believed to have zero rest mass *[Math Processing Error]* the ultra-relativistic relation *[Math Processing Error]* is exactly valid for any photon energy *[Math Processing Error]*, and the quantization scheme is rather straightforward.

As usual, the quantization has to be based on the classical theory of the system - in this case, the Maxwell equations. As the simplest case, let us consider the electromagnetic field inside a finite freespace volume limited by ideal walls, which reflect incident waves perfectly. *[Math Processing Error]* Inside the volume, the Maxwell equations give a simple wave equation *[Math Processing Error]* for the electric field *[Math Processing Error]* and an absolutely similar equation for the magnetic field *[Math Processing Error]*. We may look for the general solution of Eq. (2) in the variable-separating form *[Math Processing Error]* Physically, each term of this sum is a standing wave whose spatial distribution and polarization ("mode") are described by the vector function *[Math Processing Error]*, while the temporal dynamics, by the function *[Math Processing Error]*. Plugging an arbitrary term of this sum into Eq. (2), and separating the variables exactly as we did, for example, in the Schrödinger equation in Sec. 1.5, we get *[Math Processing Error]* so that the spatial distribution of the mode satisfies the 3D Helmholtz equation: *[Math Processing Error]* The set of solutions of this equation, with appropriate boundary conditions, determines the set of the functions *[Math Processing Error]*, and simultaneously the spectrum of the wave number magnitudes *[Math Processing Error]*. The latter values determine the mode eigenfrequencies, following from Eq. (4): *[Math Processing Error]* There is a big philosophical difference between the quantum-mechanical approach to Eqs. (5) and (6), despite their single origin (4). The first (Helmholtz) equation may be rather difficult to solve in realistic geometries, *[Math Processing Error]* but it remains intact in the basic quantum electrodynamics, with the scalar components of the vector functions *[Math Processing Error]* still treated (at each point *[Math Processing Error]*) as *[Math Processing Error]*-numbers. In contrast, the classical Eq. (6) is readily solvable (giving sinusoidal oscillations with frequency *[Math Processing Error]*), but this is exactly where we can make the transfer to quantum mechanics, because we already know how to quantize a mechanical 1D harmonic oscillator, which in classics obeys the same equation.

As usual, we need to start with the appropriate Hamiltonian - the operator corresponding to the classical Hamiltonian function *[Math Processing Error]* of the proper set of generalized coordinates and momenta. The electromagnetic field's Hamiltonian function (which in this case coincides with the field's energy) is *[Math Processing Error]* *[Math Processing Error]* Let us represent the magnetic field in a form similar to Eq. (3), *[Math Processing Error]* *[Math Processing Error]* Since, according to the Maxwell equations, in our case the magnetic field satisfies the equation similar to Eq. (2), the time-dependent amplitude *[Math Processing Error]* of each of its modes *[Math Processing Error]* obeys an equation similar to Eq. (6), i.e. in the classical theory also changes in time sinusoidally, with the same frequency *[Math Processing Error]*. Plugging Eqs. (3) and (8) into Eq. (7), we may recast it as *[Math Processing Error]* Since the distribution of constant factors between two multiplication operands in each term of Eq. (3) is so far arbitrary, we may fix it by requiring the first integral in Eq. (9) to equal 1. It is straightforward to check that according to the Maxwell equations, which give a specific relation between vectors *[Math Processing Error]* and *[Math Processing Error]* this normalization makes the second integral in Eq. (9) equal 1 as well, and Eq. (9) becomes *[Math Processing Error]* Note that that *[Math Processing Error]* is the legitimate generalized momentum corresponding to the generalized coordinate *[Math Processing Error]*, because it is equal to *[Math Processing Error]*, where *[Math Processing Error]* is the Lagrangian function of the field - see EM Eq. (9.217): *[Math Processing Error]* Hence we can carry out the standard quantization procedure, namely declare *[Math Processing Error]*, and *[Math Processing Error]* the quantum-mechanical operators related exactly as in Eq. (10a), *[Math Processing Error]* We see that this Hamiltonian coincides with that of a 1D harmonic oscillator with the mass *[Math Processing Error]* formally equal to *[Math Processing Error]* and the eigenfrequency equal to *[Math Processing Error]*. However, in order to use Eq. (11) in the general Eq. (4.199) for the time evolution of Heisenberg-picture operators *[Math Processing Error]* and *[Math Processing Error]*, we need to know the commutation relation between these operators. To find them, let us calculate the Poisson bracket (4.204) for the functions *[Math Processing Error]*, and *[Math Processing Error]*, taking into account that in the classical Hamiltonian mechanics, all generalized coordinates *[Math Processing Error]* and the corresponding momenta *[Math Processing Error]* have to be considered independent arguments of *[Math Processing Error]*, only one term (with *[Math Processing Error]*) in only one of the sums (12) (namely, with *[Math Processing Error]*), gives a non-zero value *[Math Processing Error]*, so that

[Math Processing Error]

Hence, according to the general quantization rule (4.205), the commutation relation of the operators corresponding to [Math Processing Error], and [Math Processing Error], is [Math Processing Error] i.e. is exactly the same as for the usual Cartesian components of the radius-vector and momentum of a mechanical particle [Math Processing Error] see Eq. (2.14).

As the reader already knows, Eqs. (11) and (13) open for us several alternative ways to proceed:

(i) Use the Schrödinger-picture wave mechanics based on wavefunctions [Math Processing Error]. As we know from Sec. 2.9, this way is inconvenient for most tasks, because the eigenfunctions of the harmonic oscillator are rather clumsy.

(ii) A substantially better way (for the harmonic oscillator case) is to write the equations of the time evolution of the operators [Math Processing Error] and [Math Processing Error] in the Heisenberg picture of quantum dynamics.

(iii) An even more convenient approach is to use equations similar to Eqs. (5.65) to decompose the Heisenberg operators [Math Processing Error] and [Math Processing Error] into the creation-annihilation operators [Math Processing Error] and [Math Processing Error], and work with these operators.

In this chapter, I will mostly use the last route. Replacing [Math Processing Error] with [Math Processing Error], and [Math Processing Error] with [Math Processing Error], the last forms of Eqs. (5.65) become [Math Processing Error] Due to Eq. (13), the creation-annihilation operators obey the commutation similar to Eq. (5.68), [Math Processing Error] As a result, according to Eqs. (3) and (8), the quantum-mechanical operators of the electric and magnetic fields are sums over all field oscillators: [Math Processing Error] and Eq. (11) for the [Math Processing Error] mode's Hamiltonian becomes [Math Processing Error] absolutely similar to Eq. (5.72) for a mechanical oscillator.

Now comes a very important conceptual step. From Sec. [Math Processing Error] we know that the eigenfunctions (Fock states) [Math Processing Error] of the Hamiltonian (17) have energies [Math Processing Error] and, according to Eq. (5.89), the operators [Math Processing Error] and [Math Processing Error] act on the eigenkets of these partial states as [Math Processing Error] regardless of the quantum states of other modes. These rules coincide with the definitions (8.64) and (8.68) of bosonic creation-annihilation operators, and hence their action may be considered as the creation/annihilation of certain bosons. Such a "particle" (actually, an excitation, with energy [Math Processing Error], of an electromagnetic field oscillator) is exactly what is, strictly speaking, called a photon. Note immediately that according to Eq. (16), such an excitation does not change the spatial distribution of the [Math Processing Error] mode of the field. So, such a "global" photon is an excitation created simultaneously at all points of the field confinement region.

If this picture is too contrary to the intuitive image of a particle, please recall that in Chapter 2 , we discussed a similar situation with the fundamental solutions of the Schrödinger equation of a free non-relativistic particle: they represent sinusoidal de Broglie waves existing simultaneously in all points of the particle confinement region. The (partial :-) reconciliation with the classical picture of a moving particle might be obtained by using the linear superposition principle to assemble a quasi-localized wave packet, as a group of sinusoidal waves with close wave numbers. Very similarly, we may form a similar wave packet using a linear superposition of the "global" photons with close values of [Math Processing Error] (and hence [Math Processing Error]), to form a quasi-localized photon. An additional simplification here is that the dispersion relation for electromagnetic waves (at least in free space) is linear: [Math Processing Error] so that, according to Eq. (2.39a), the electromagnetic wave packets (i.e. space-localized photons) do not spread out during their propagation. Note also that due to the fundamental classical relations [Math Processing Error] for the linear momentum of the traveling electromagnetic wave packet of energy [Math Processing Error], propagating along the direction [Math Processing Error], and [Math Processing Error] for its angular momentum, [Math Processing Error] such photon may be prescribed the linear momentum [Math Processing Error] and the angular momentum [Math Processing Error], with the sign depending on the direction of its circular polarization ("helicity").

This electromagnetic field quantization scheme should look very straightforward, but it raises an important conceptual issue of the ground state energy. Indeed, Eq. (18) implies that the total groundstate (i.e., the lowest) energy of the field is [Math Processing Error] Since for any realistic model of the field-confining volume, either infinite or not, the density of electromagnetic field modes only grows with frequency, [Math Processing Error] this sum diverges on its upper limit, leading to infinite ground-state energy per unit volume. This infinite-energy paradox cannot be dismissed by declaring the ground-state energy of field oscillators unobservable, because this would contradict numerous experimental observations [Math Processing Error] starting perhaps from the famous Casimir effect. [Math Processing Error] The conceptually simplest implementation of this effect involves two parallel, perfectly conducting plates of area [Math Processing Error], separated by a vacuum gap of thickness [Math Processing Error] (Fig. 1).

Fig. 9.1. The simplest geometry of the Casimir effect manifestation.

Rather counter-intuitively, the plates attract each other with a force [Math Processing Error] proportional to the area [Math Processing Error] and rapidly increasing with the decrease of [Math Processing Error], even in the absence of any explicit electromagnetic field sources. The effect's explanation is that the energy of each electromagnetic field mode, including its ground-state energy, exerts average pressure, [Math Processing Error] on the walls constraining it to volume [Math Processing Error]. While the field's pressure on the external surfaces on the plates is due to the contributions (22) of all free-space modes, with arbitrary values of [Math Processing Error] (the [Math Processing Error]-component of the wave vector [Math Processing Error]), in the gap between the plates the spectrum of [Math Processing Error] is limited to the multiples of [Math Processing Error], so that the pressure on the internal surfaces is lower. This is why the net force exerted on the plates may be calculated as the sum of the contributions (22) from all "missing" low-frequency modes in the gap, with the minus sign. In the simplest model when the plates are made of an ideal conductor, which provides boundary conditions [Math Processing Error] on their surfaces, [Math Processing Error] such calculation is quite straightforward (and is hence left for the reader's exercise), and its result is [Math Processing Error] Note that for such calculation, the high-frequency divergence of Eq. (21) is not important, because it participates in the forces exerted on all surfaces of each plate, and cancels out from the net pressure. In this way, the Casimir effect not only confirms Eq. (21), but also teaches us an important lesson on how to deal with the divergences of such sums at [Math Processing Error]. The lesson is: just get accustomed to the idea that the divergence exists, and ignore this fact while you can, i.e. if the final result you are interested in is finite. However, for some more complex problems of quantum electrodynamics (and the quantum theory of any other fields), this simplest approach becomes impossible, and then more complex, renormalization techniques become necessary. For their study, I have to refer the reader to a quantum field theory course [Math Processing Error] see the references at the end of this chapter.

[Math Processing Error] Note that some material covered in this chapter is frequently taught as a part of the quantum field theory. I will focus on the most important results that may be obtained without starting the heavy engines of that theory.

[Math Processing Error] The described approach was pioneered by the same P. A. M. Dirac as early as [Math Processing Error]

[Math Processing Error] See, e.g., EM Chapter [Math Processing Error]

[Math Processing Error] By now this fact has been verified experimentally with an accuracy of at least [Math Processing Error] see [Math Processing Error]. Eidelman et al., Phys. Lett. B 592, 1 (2004).

[Math Processing Error] In the case of finite energy absorption in the walls, or in the wave propagation media (say, described by complex constants [Math Processing Error] and [Math Processing Error]), the system is not energy-conserving (Hamiltonian), i.e. interacts with some dissipative environment. Specific cases of such interaction will be considered in Sections 2 and 3 below.

[Math Processing Error] See, e.g., EM Eq. (7.3), for the particular case [Math Processing Error], so that [Math Processing Error].

[Math Processing Error] See, e.g., various problems discussed in EM Chapter 7, especially in Sec. 7.9.

[Math Processing Error] See, e.g., EM Sec. 9.8, in particular, Eq. (9.225). Here I am using SI units, with [Math Processing Error]; in the Gaussian units, the coefficients [Math Processing Error] and [Math Processing Error] disappear, but there is an additional common factor [Math Processing Error] in the equation for energy. However, if we modify the normalization conditions (see below) accordingly, all the subsequent results, starting from Eq. (10), look similar in any system of units.

[Math Processing Error] Here I am using the letter [Math Processing Error], instead of [Math Processing Error], for the generalized coordinate of the field oscillator, in order to emphasize the difference between the former variable, and one of the Cartesian coordinates, i.e. one of the arguments of the [Math Processing Error]-number functions [Math Processing Error] and [Math Processing Error].

[Math Processing Error] See, e.g., EM Eq. (7.6).

[Math Processing Error] Selecting a different normalization of the functions [Math Processing Error] and [Math Processing Error], we could readily arrange any value of [Math Processing Error], and the choice corresponding to [Math Processing Error] is the best one just for the notation simplicity.

[Math Processing Error] See, e.g., EM Sections *[Math Processing Error]* and *[Math Processing Error]*.

[Math Processing Error] See, e.g., Eq. (1.1), which is similar to Eq. (1.90) for the de Broglie waves, derived in Sec. 1.7.

[Math Processing Error] This effect was predicted in 1948 by Hendrik Casimir and Dirk Polder, and confirmed semi-quantitatively in experiments by M. Sparnaay, Nature 180,334 (1957). After this, and several other experiments, a decisive error bar reduction (to about *[Math Processing Error]*), providing a quantitative confirmation of the Casimir formula (23), was achieved by S. Lamoreaux, Phys. Rev. Lett. *[Math Processing Error]* (1997) and by U. Mohideen and A. Roy, Phys. Rev. Lett. 81, 004549 (1998). Note also that there are other experimental confirmations of the reality of the ground-state electromagnetic field, including, for example, the experiments by R. Koch et al. already discussed in Sec. *[Math Processing Error]*, and the recent spectacular direct observations by C. Riek et al., Science 350, 420 (2015).

[Math Processing Error] For realistic conductors, the reduction of *[Math Processing Error]* below *[Math Processing Error]* causes significant deviations from this simple model, and hence from Eq. (23). The reason is that for gaps so narrow, the depth of field penetration into the conductors (see, e.g., EM Sec. 6.2), at the important frequencies *[Math Processing Error]*, becomes comparable with *[Math Processing Error]*, and an adequate theory of the Casimir effect has to involve a certain model of the penetration. (It is curious that in-depth analyses of this problem, pioneered in 1956 by E. Lifshitz, have revealed a deep relation between the Casimir effect and the London dispersion force which was the subject of Problems *[Math Processing Error]*, and *[Math Processing Error]* - for a review see, e.g., either I. Dzhyaloshinskii et al., Sov. Phys. Uspekhi 4, 153 (1961), or K. Milton, The Casimir Effect, World Scientific, 2001. Recent experiments in the *[Math Processing Error]* range of *[Math Processing Error]*, with an accuracy better than *[Math Processing Error]*, have allowed not only to observe the effects of field penetration on the Casimir force, but even to make a selection between some approximate models of the penetration - see D. Garcia-Sanchez et al., Phys. Rev. Lett. 109, 027202 (2012).
