7.7: Exercise Problems

7.1. Calculate the density matrix of a two-level system whose Hamiltonian is described, in a certain basis, by the following matrix:

$$\mathrm{H}=\mathbf{c}\cdot\boldsymbol{\sigma}\equiv c_x\sigma_x+c_y\sigma_y+c_z\sigma_z, \tag{7.7.1}$$

where σ_k are the Pauli matrices and c_j are c-numbers, in thermodynamic equilibrium at temperature T

7.2. In the usual z-basis, spell out the density matrix of a spin- $1/2$ with gyromagnetic ratio γ.

(i) in the pure state with the spin definitely directed along the z-axis,

(ii) in the pure state with the spin definitely directed along the x-axis,

(iii) in thermal equilibrium at temperature T, in a magnetic field directed along the z-axis, and

(iv) in thermal equilibrium at temperature T, in a magnetic field directed along the x-axis.

7.3. Calculate the Wigner function of a harmonic oscillator in:

(i) in thermodynamic equilibrium at temperature T,

(ii) in the ground state, and

(ii) in the Glauber state with dimensionless complex amplitude α.

Discuss the relation between the first of the results and the Gibbs distribution.

7.4. Calculate the Wigner function of a harmonic oscillator, with mass m and frequency ω_0, in its first excited stationary state $(n=1)$.

7.5. * A harmonic oscillator is weakly coupled to an Ohmic environment.

(i) Use the rotating-wave approximation to write the reduced equations of motion for the Heisenberg operators of the complex amplitude of oscillations.

(ii) Calculate the expectation values of the correlators of the fluctuation force operators participating in these equations, and express them via the average number $\langle n\rangle$ of thermally-induced excitations in equilibrium, given by the second of Eqs. (26b).

7.6. Calculate the average potential energy of long-range electrostatic interaction between two similar isotropic, 3D harmonic oscillators, each with the electric dipole moment $\mathbf{d}=q\mathbf{s}$, where $\mathbf{s}$ is the oscillator's displacement from its equilibrium position, at arbitrary temperature T.

7.7. A semi-infinite string with mass μ per unit length is attached to a wall and stretched with a constant force (tension) \mathscr{ } \text { . Calculate the spectral density of the transverse force exerted on the wall, in } thermal equilibrium at temperature T.

7.8. * Calculate the low-frequency spectral density of small fluctuations of the voltage V across a Josephson junction, shunted with an Ohmic conductor, and biased with a dc external current $\bar{I}>I_{\mathrm{c}}$.

Hint: You may use Eqs. (1.73)-(1.74) to describe the junction's dynamics, and assume that the shunting conductor remains in thermal equilibrium.

7.9. Prove that in the interaction picture of quantum dynamics, the expectation value of an arbitrary observable A may be indeed calculated using Eq. (167).

7.10. Show that the quantum-mechanical Golden Rule (6.149) and the master equation (196) give the same results for the rate of spontaneous quantum transitions $n'\rightarrow n$ in a system with a discrete energy spectrum, weakly coupled to a low-temperature heat bath (with $k_{\mathrm{B}}T<<\hbar\omega_{nn}$).

Hint: You may start by establishing a relation between the function χ " (ω_{nn}), which participates in Eq. (196), and the density of states ρ_n, which participates in the Golden Rule formula, using the particular case of sinusoidal classical oscillations in the system of interest.

7.11. For a harmonic oscillator with weak Ohmic dissipation, use Eqs. (208)-(209) to find the time evolution of the expectation value $\langle E\rangle$ of oscillator's energy for an arbitrary initial state, and compare the result with that following from the Heisenberg-Langevin approach.

7.12. Derive Eq. (219) in an alternative way, using an expression dual to Eq. (4.244).

7.13. A particle in a system of two coupled potential wells (see, e.g., Fig. 7.4 in the lecture notes) is weakly coupled to an Ohmic environment.

(i) Derive equations describing the time evolution of the density matrix elements.

(ii) Solve these equations in the low-temperature limit, when the energy level splitting is much larger than $k_B T$, to calculate the time evolution of the probability of finding the particle in one of the wells, after it had been placed there at $t=0$.

7.14. * A spin-1/2 with gyromagnetic ratio γ is placed into the magnetic field $\mathscr{B}(t)=\mathscr{B}_0+\mathscr{B}(t)$ with an arbitrary but relatively small time-dependent component, and is also weakly coupled to a dissipative environment. Derive differential equations describing the time evolution of the expectation values of spin's Cartesian components, at arbitrary temperature.
