4.1: Motivation

As the reader could see from the previous chapters of these notes, wave mechanics gives many results of primary importance. Moreover, it is mostly sufficient for many applications, for example, solid-state electronics and device physics. However, in the course of our survey, we have filed several grievances about this approach. Let me briefly summarize these complaints:

(i) Attempts to analyze the temporal evolution of quantum systems, beyond the trivial time behavior of the stationary states, described by Eq. (1.62), run into technical difficulties. For example, we could derive Eq. (2.151) describing the metastable state's decay and Eq. (2.181) describing the quantum oscillations in coupled wells, only for the simplest potential profiles, though it is intuitively clear that such simple results should be common for all problems of this kind. Solving such problems for more complex potential profiles would entangle the time evolution analysis with the calculation of the spatial distribution of the evolving wavefunctions - which (as we could see in Secs. 2.9 and 3.6) may be rather complex even for simple time-independent potentials. Some separation of the spatial and temporal dependencies is possible using perturbation approaches (to be discussed in Chapter 6), but even those would lead, in the wavefunction language, to very cumbersome formulas.

(ii) The last statement can also be made concerning other issues that are conceptually addressable within the wave mechanics, e.g., the Feynman path integral approach, coupling to the environment, etc. Pursuing them in the wave mechanics language would lead to formulas so bulky that I had postponed their discussion until we would have a more compact formalism on hand.

(iii) In the discussion of several key problems (for example the harmonic oscillator and spherically-symmetric potentials), we have run into rather complicated eigenfunctions coexisting with very simple energy spectra – that infer some simple background physics. It is very important to get this physics revealed.

(iv) In the wave-mechanics postulates formulated in Sec. 1.2, the quantum mechanical operators of the coordinate and momentum are treated rather unequally - see Eqs. (1.26b). However, some key expressions, e.g., for the fundamental eigenfunction of a free particle,

$$\exp\left\{i\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right\} \tag{4.1.1}$$

or the harmonic oscillator's Hamiltonian,

$$\hat{H}=\frac{1}{2m}\hat{p}^2+\frac{m\omega_0^2}{2}\hat{r}^2, \tag{4.1.2}$$

just beg for a similar treatment of coordinates and momenta. However, the strongest motivation for a more general formalism comes from wave mechanics' conceptual inability to describe elementary particles' spins [1] and other internal quantum degrees of freedom, such as quark flavors or lepton numbers. In this context, let us review the basic facts on spin (which is very representative and experimentally the most accessible of all internal quantum numbers), to understand what a more general formalism has to explain - as a minimum.

Figure 1 shows the conceptual scheme of the simplest spin-revealing experiment, first conceived by Otto Stern in 1921 and implemented by Walther Gerlach in 1922. A collimated beam of electrons from a natural source, such as a heated cathode, is passed through a gap between the poles of a strong magnet, whose magnetic field $\mathscr{B}$, (in Fig. 1, directed along the z-axis) is nonuniform, so that both $\mathscr{B}_z$ and $d\mathscr{B}_z/dz$ are not equal to zero. The experiment shows that the beam splits into two beams of equal intensity.

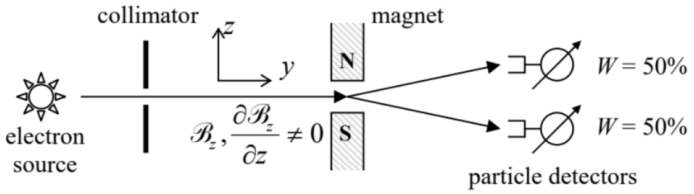

Fig. 4.1. The simplest Stern-Gerlach experiment.

This result may be semi-quantitatively explained on classical if somewhat phenomenological grounds, by assuming that each electron has an intrinsic, permanent magnetic dipole moment $\mathbf{m}$. Indeed, classical electrodynamics [2] tells us that the potential energy U of a magnetic dipole in an external magnetic field $\mathscr{B}$ is equal to $(-\mathbf{m}\cdot\mathscr{B})$, so that the force acting on the electron,

$$\mathbf{F}=-\nabla U=-\nabla(-\mathbf{m}\cdot\mathscr{B}), \tag{4.1.3}$$

has a non-zero vertical component

$$F_z = -\frac{\partial}{\partial z}(-m_z \cdot \mathscr{B}_z) \equiv m_z \frac{\partial \mathscr{B}_z}{\partial z}. \tag{4.1.4}$$

Hence if we further assume that electron's magnetic moment may take only two equally probable discrete values of $m_z = \pm\mu$ (though such discreteness does not follow from any classical model of the particle), this may explain the original Stern-Gerlach effect qualitatively. The quantitative explanation of the beam splitting angle requires the magnitude of μ to be equal (or very close) to the so-called *Bohr* magneton [3]

$$\mu_{\rm B} \equiv \frac{\hbar e}{2m_{\rm e}} \approx 0.9274 \times 10^{-23} \frac{\rm J}{\rm T}. \tag{4.1.5}$$

However, as we will see below, this value cannot be explained by any internal motion of the electron, say its rotation about the z-axis. More importantly, this semi-classical phenomenology cannot explain, even qualitatively, other experimental results, for example those of the set of multi-stage SternGerlach experiments shown in Fig. 2. In the first of the experiments, the electron beam is first passed through a magnetic field (and its gradient) oriented along the z-axis, just as in Fig. 1. Then one of the two resulting beams is absorbed (or removed from the setup in some other way), while the other one is passed through a similar but x-oriented field. The experiment shows that this beam is split again into two components of equal intensity. A classical explanation of this experiment would require an even more unnatural additional assumption that the initial electrons had random but discrete components of the magnetic moment simultaneously in two directions, z and x.

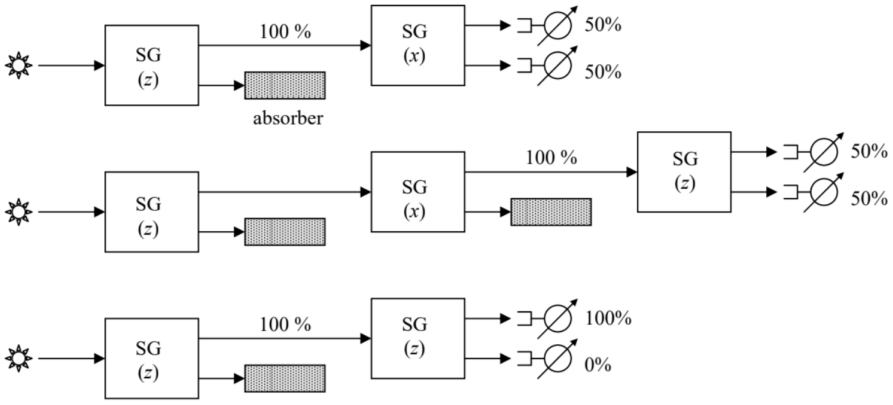

Fig. 4.2. Three multi-stage Stern-Gerlach experiments. The boxes SG (...) denote magnets similar to one shown in Fig. 1, with the field oriented in the indicated direction.

However, even this assumption cannot explain the results of the three-stage Stern-Gerlach experiment shown on the middle panel of Fig. 2. Here, the previous two-state setup is complemented with one more absorber and one more magnet, now with the z-orientation again. Completely counterintuitively, it again gives two beams of equal intensity, as if we have not yet filtered out the electrons with m_z corresponding to the lower beam, at the first z-stage. The only way to save the classical explanation here is to say that maybe, electrons somehow interact with the magnetic field so that the x polarized (non-absorbed) beam becomes spontaneously depolarized again somewhere between the two last stages. But any hope for such an explanation is ruined by the control experiment shown on the bottom panel of Fig. 2, whose results indicate that no such depolarization happens.

We will see below that all these (and many more) results find a natural explanation in the socalled matrix mechanics pioneered by Werner Heisenberg, Max Born, and Pascual Jordan in 1925. However, the matrix formalism is rather inconvenient for the solution of most problems discussed in Chapters 1-3, and for a short time, it was eclipsed by E. Schrödinger's wave mechanics, which had been put forward just a few months later. However, very soon Paul Adrien Maurice Dirac introduced a more general bra-ket formalism of quantum mechanics, which provides a generalization of both approaches and proves their equivalence. Let me describe it, begging for the reader's patience, because (in a contrast with my usual style), I will not be able to give particular examples of its application for a while - until all the basic notions of the formalism have been introduced.

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[1] To the best of my knowledge, the concept of spin as a measure of the internal rotation of a particle was first suggested by Ralph Kronig, then a 20-year-old student, in January 1925, a few months before two other students, G. Uhlenbeck and S. Goudsmit - to whom the idea is usually attributed. The concept was then accepted (rather reluctantly) and developed quantitatively by Wolfgang Pauli.

[2] See, e.g., EM Sec. 5.4, in particular Eq. (5.100).

[3] A good mnemonic rule is that it is close to $1\ \mathrm{K/T}$. In the Gaussian units, $\mu_\mathrm{B} \equiv \hbar e/2m_\mathrm{e}c \approx 0.9274 \times 10^{-20}\mathrm{erg/G}$.