10.3: Hidden Variables, the Bell Theorem, and Local Reality

Now we are ready to proceed to the discussion of the last, hardest group (iii) of the questions posed in Sec. 1, namely on the state of a quantum system just before its measurement. After a very important but inconclusive discussion of this issue by Albert Einstein and his collaborators on one side, and Niels Bohr on the other side, in the mid-1930s, such discussions have resumed in the 1950s. [26] They have led to a key contribution by John Stewart Bell in the early 1960s, summarized as so-called Bell's inequalities, and then to experimental work on better and better verification of these inequalities. (Besides that work, the recent progress, in my humble view, has been rather marginal.)

The central question may be formulated as follows: what had been the "real" state of a quantummechanical system just before a virtually perfect single-shot measurement was performed on it, and gave a certain, documented outcome? To be specific, let us focus again on the example of Stern-Gerlach measurements of spin-1/2 particles - because of their conceptual simplicity. [27] For a single-component system (in this case a single spin- $-\frac{2}{2}$ the answer to the posed question may look evident. Indeed, as we know, if the spin is in a pure (least-uncertain) state α, i.e. its ket-vector may be expressed in the form similar to Eq. (4),

$$|\alpha\rangle=\alpha_{\uparrow}|\uparrow\rangle+\alpha_{\downarrow}|\downarrow\rangle, \tag{10.3.1}$$

where, as usual, $\uparrow$ and $\downarrow$ denote the states with definite spin orientations along the z-axis, the probabilities of the corresponding outcomes of the z-oriented Stern-Gerlach experiment are $W_{\uparrow}=|\alpha\uparrow|^2$ and $W_{\downarrow}=|\alpha_{\downarrow}|^2$. Then it looks natural to suggest that if a particular experiment gave the outcome corresponding to the state $\uparrow$, the spin had been in that state just before the experiment. For a classical system such answer would be certainly correct, and the fact that the probability $W_{\uparrow}=|\alpha\uparrow|^2$, defined for the statistical ensemble of all experiments (regardless of their outcome), may be less than 1, would merely reflect our ignorance about the real state of this particular system before the measurement which just reveals the real situation.

However, as was first argued in the famous EPR paper published in 1935 by A. Einstein, B. Podolsky, and N. Rosen, such an answer becomes impossible in the case of an entangled quantum system, if only one of its components is measured with an instrument. The original EPR paper discussed thought experiments with a pair of 1D particles prepared in a quantum state in that both the sum of their momenta and the difference of their coordinates simultaneously have definite values: $p_1+p_2=0, x_1-x_2=a$.[28] However, usually this discussion is recast into an equivalent Stern-Gerlach experiment shown in Fig. 4a. [29] A source emits rare pairs of spin- $1/2$ particles, propagating in opposite directions. The particle spin states are random, but with the net spin of the pair definitely equal to zero. After the spatial separation of the particles has become sufficiently large (see below), the spin state of each of them is measured with a Stern-Gerlach detector, with one of them (in Fig. 1, SG_1) somewhat closer to the particle source, so it makes the measurement first, at a time $t_1<t_2$.

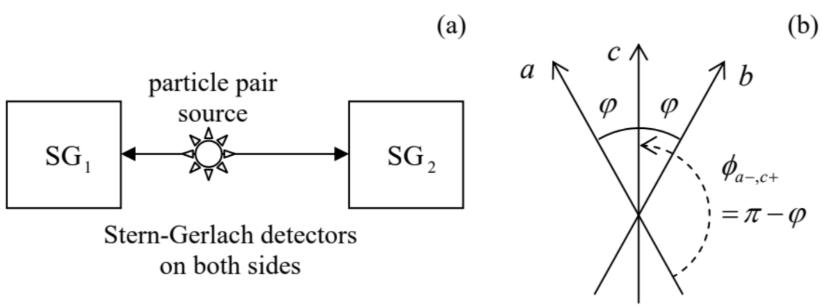


Fig. 10. 4. (a) General scheme of two-particle Stern-Gerlach experiments, and (b) the orientation of the detectors, assumed at Wigner's deviation of Bell's inequality (36).

First, let the detectors be oriented say along the same direction, say the z-axis. Evidently, the probability of each detector to give any of the values $s_z=\pm\hbar/2$ is 50%. However, if the first detector had given the result $S_z=-\hbar/2$, then even before the second detector's measurement, we know that the latter will give the result $S_z=+\hbar/2$ with the 100% probability. So far, this situation still allows for a classical interpretation, just as for the single-particle measurements: we may fancy that the second particle has a definite spin before the measurement, and the first measurement just removes our ignorance about that reality. In other words, the change of the probability of the outcome $S_z=+\hbar/2$ at the second detection from 50% to 100% is due to the statistical ensemble re-definition: the 50% probability of this detection belongs to the ensemble of all experiments, while the 100% probability, to the sub-ensemble of experiments with the $S_z=-\hbar/2$ outcome of the first experiment.However, let the source generate the spin pairs in the entangled, singlet state (8.18),

$$|s_{12}\rangle=\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle), \tag{10.3.2}$$

that certainly satisfies the above assumptions: the probability of each value of S_z of any particle is 50%, and the sum of both S_z is definitely zero, so that if the first detector's result is $S_z=-\hbar/2$, then the state of the remaining particle is $\uparrow$, with zero uncertainty. Now let us use Eqs. (4.123) to represent the same state (24) in a different form:

$$|s_{12}\rangle=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|\rightarrow\rangle+|\leftarrow\rangle)\frac{1}{\sqrt{2}}(|\rightarrow\rangle-|\leftarrow\rangle)-\frac{1}{\sqrt{2}}(|\rightarrow\rangle-|\leftarrow\rangle)\frac{1}{\sqrt{2}}(|\rightarrow\rangle+|\leftarrow\rangle)\right]. \tag{10.3.3}$$

Opening the parentheses (carefully, without swapping the ket-vector order, which encodes the particle numbers!), we get an expression similar to Eq. (24), but now for the x-basis:

$$|s_{12}\rangle=\frac{1}{\sqrt{2}}(|\rightarrow\leftarrow\rangle-|\leftarrow\rightarrow\rangle). \tag{10.3.4}$$

Hence if we use the first detector (closest to the particle source) to measure S_x rather than S_z, then after it had given a certain result (say, $S_x=-\hbar/2$), we know for sure, before the second particle spin's measurement, that its S_x component definitely equals $+\hbar/2$.

So, depending on the experiment performed on the first particle, the second particle, before its measurement, may be in one of two states - either with a definite component S_z or with a definite component S_x, in each case with zero uncertainty. Evidently, this situation cannot be interpreted in classical terms if the particles do not interact during the measurements. A. Einstein was deeply unhappy with such situation because it did not satisfy what, in his view, was the general requirement to any theory, which nowadays is called the local reality. His definition of this requirement was as follows: "The real factual situation of system 2 is independent of what is done with system 1 that is spatially separated from the former". (Here the term "spatially separated" is not defined, but from the context, it is clear that Einstein meant the detector separation by a superluminal interval, i.e. by distance

$$|\mathbf{r}_1-\mathbf{r}_2|>c\,|t_1-t_2|\,, \tag{10.3.5}$$

where the measurement time difference on the right-hand side includes the measurement duration.) In Einstein's view, since quantum mechanics did not satisfy the local reality condition, it could not be considered a complete theory of Nature.

This situation naturally raises the question of whether something (usually called hidden variables) may be added to the quantum-mechanical description to enable it to satisfy the local reality requirement. The first definite statement in this regard was John von Neumann's "proof" 30 (first famous, then infamous :-) that such variables cannot be introduced; for a while, his work satisfied the quantum mechanics practitioners, who apparently did not pay much attention. [31] A major new contribution to the problem was made only in the 1960s by J. Bell. [32] First of all, he has found an elementary (in his words, "foolish") error in von Neumann's logic, which voids his "proof". Second, he has demonstrated that Einstein's local reality condition is incompatible with conclusions of quantum mechanics - that had been, by that time, confirmed by too many experiments to be seriously questioned. Let me describe a particular version of the Bell's result (suggested by E. Wigner), using the same EPR pair experiment (Fig. 4a), in that each SG detector may be oriented in any of 3 directions: a, b, or c - see Fig. 4b. As we already know from Chapter 4, if a fully-polarized beam of spin-1/2 particles is passed through a Stern-Gerlach apparatus forming angle ϕ with the polarization axis, the probabilities of two alternative outcomes of the experiment are

$$W\left(\phi_{+}\right)=\cos^2\frac{\phi}{2}, \quad W\left(\phi_{-}\right)=\sin^2\frac{\phi}{2}. \tag{10.3.6}$$

Let us use this formula to calculate all joint probabilities of measurement outcomes, starting from the detectors 1 and 2 oriented, respectively, in the directions a and c. Since the angle between the negative direction of the a-axis and the positive direction of the c-axis is $\phi_{a-c+c}=\pi-\varphi$ (see the dashed arrow in Fig. 4b), we get

$$W\left(a_{+}\wedge c_{+}\right)\equiv W\left(a_{+}\right)W\left(c_{+}\mid a_{+}\right)=W\left(a_{+}\right)W\left(\phi_{a-c+}\right)=\frac{1}{2}\cos^2\frac{\pi-\varphi}{2}\equiv\frac{1}{2}\sin^2\frac{\varphi}{2}, \tag{10.3.7}$$

where $W(x\wedge y)$ is the joint probability of both outcomes x and y, while $W(x\mid y)$ is the conditional probability of the outcome x, provided that the outcome y has happened. (The first equality in Eq. (29) is the well-known identity of the probability theory.) Absolutely similarly,

$$W\left(c_{+} \wedge b_{+}\right) \equiv W\left(c_{+}\right) W\left(b_{+} \mid c_{+}\right)=\frac{1}{2} \sin ^{2} \frac{\varphi}{2},$$

$$W\left(a_{+} \wedge b_{+}\right) \equiv W\left(a_{+}\right) W\left(b_{+} \mid a_{+}\right)=\frac{1}{2} \cos ^{2} \frac{\pi-2 \varphi}{2} \equiv \frac{1}{2} \sin ^{2} \varphi.$$

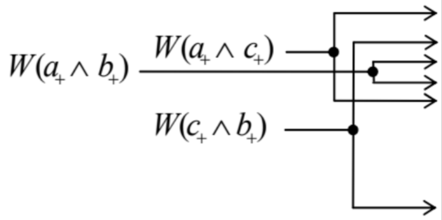
Now note that for any angle φ smaller than $\pi / 2$ (as in the case shown in Fig. 4b), trigonometry gives

$$\frac{1}{2} \sin ^{2} \varphi \geq \frac{1}{2} \sin ^{2} \frac{\varphi}{2}+\frac{1}{2} \sin ^{2} \frac{\varphi}{2} \equiv \sin ^{2} \frac{\varphi}{2}. \tag{10.3.8}$$

(For example, for $\varphi \rightarrow 0$ the left-hand side of this inequality tends to $\varphi^{2} / 2$, while the right-hand side, to $\left.\varphi^{2} / 4 .\right)$ Hence the quantum-mechanical result gives, in particular,

$$W\left(a_{+} \wedge b_{+}\right) \geq W\left(a_{+} \wedge c_{+}\right)+W\left(c_{+} \wedge b_{+}\right), \quad \text { for }|\varphi| \leq \pi / 2. \tag{10.3.9}$$

On the other hand, we can get a different inequality for these probabilities without calculating them from any particular theory, but using the local reality assumption. For that, let us prescribe some probability to each of $2^{3}=8$ possible outcomes of a set of three spin measurements. (Due to zero net spin of particle pairs, the probabilities of the sets shown in both columns of the table have to be equal.)



Detector 1	Detector 2	Probability
$a_+ \wedge b_+ \wedge c_+$	$a_- \wedge b_- \wedge c_-$	W_1
$a_+ \wedge b_+ \wedge c_-$	$a_- \wedge b_- \wedge c_+$	W_2
$a_+ \wedge b_- \wedge c_+$	$a_- \wedge b_+ \wedge c_-$	W_3
$a_+ \wedge b_- \wedge c_-$	$a_- \wedge b_+ \wedge c_+$	W_4
$a_- \wedge b_+ \wedge c_+$	$a_+ \wedge b_- \wedge c_-$	W_5
$a_- \wedge b_+ \wedge c_-$	$a_+ \wedge b_- \wedge c_+$	W_6
$a_- \wedge b_- \wedge c_+$	$a_+ \wedge b_+ \wedge c_-$	W_7
$a_- \wedge b_- \wedge c_-$	$a_+ \wedge b_+ \wedge c_+$	W_8

From the local-reality point of view, these measurement options are independent, so we may write (see the arrows on the left of the table):

$$W\left(a_{+} \wedge c_{+}\right)=W_{2}+W_{4}, \quad W\left(c_{+} \wedge b_{+}\right)=W_{3}+W_{7}, \quad W\left(a_{+} \wedge b_{+}\right)=W_{3}+W_{4}. \tag{10.3.10}$$

On the other hand, since no probability may be negative (by its very definition), we may always write

$$W_{3}+W_{4} \leq\left(W_{2}+W_{4}\right)+\left(W_{3}+W_{7}\right). \tag{10.3.11}$$

Plugging into this inequality the values of these two parentheses, given by Eq. (34), we get

$$W\left(a_{+} \wedge b_{+}\right) \leq W\left(a_{+} \wedge c_{+}\right)+W\left(c_{+} \wedge b_{+}\right). \tag{10.3.12}$$

This is the Bell's inequality, which has to be satisfied by any local-reality theory; it directly contradicts the quantum-mechanical result (33) - opening the issue to direct experimental testing. Such tests were started in the late 1960 s, but the first results were vulnerable to two criticisms:

(i) The detectors were not fast enough and not far enough to have the relation (27) satisfied. This is why, as a matter of principle, there was a chance that information on the first measurement outcome had been transferred (by some, mostly implausible) means to particles before the second measurement the so-called locality loophole.

(ii) The particle/photon detection efficiencies were too low to have sufficiently small error bars for both parts of the inequality - the detection loophole.

Gradually, these loopholes have been closed. [33] As expected, substantial violations of the Bell inequalities (36) (or their equivalent forms) have been proved, essentially rejecting any possibility to reconcile quantum mechanics with Einstein's local reality requirement.

[26] See, e.g., J. Wheeler and W. Zurek (eds.), Quantum Theory and Measurement, Princeton U. Press, 1983.

[27] As was discussed in Sec. 1, the Stern-Gerlach-type experiments may be readily made virtually perfect, provided that we do not care about the evolution of the system after the single-shot measurement.

[28] This is possible because the corresponding operators commute: $[\hat{p}_1+\hat{p}_2, \hat{x}_1-\hat{x}_2]=[\hat{p}_1, \hat{x}_1]-[\hat{p}_2, \hat{x}_2]=0$

[29] Another equivalent but experimentally more convenient (and as a result, frequently used) technique is the degenerate parametric excitation of entangled optical photon pairs - see, e.g., the publications cited at the end of this section.

[30] In his very early book J. von Neumann, Mathematische Grundlagen der Quantenmechanik [Mathematical Foundations of Quantum Mechanics], Springer, 1932. (The first English translation was published only in 1955.)

[31] Perhaps it would not satisfy A. Einstein, but reportedly he did not know about the von Neumann's publication before signing the EPR paper.

[32] See, e. g., either J. Bell, Rev. Mod. Phys. $\mathbf{38}, 447$ (1966) or J. Bell, Foundations of Physics 12, 158 (1982).

[33] Important milestones in that way were the experiments by A. Aspect et al., Phys. Rev. Lett. 49, 91 (1982) and M. Rowe et al., Nature $\mathbf{409}, \mathbf{791}$ (2001). Detailed reviews of the experimental situation were given, for example, by M. Genovese, Phys. Repts. $\mathbf{413}, 319$ (2005) and A. Aspect, Physics $\mathbf{8}, 123$ (2015); see also the later paper by J. Handsteiner et al., Phys. Rev. Lett. 118, 060401 (2017). Presently, a high-fidelity demonstration of the Bell inequality violation has become a standard test in virtually every experiment with entangled qubits used for quantum encryption research - see Sec. 8.5, in particular the paper by J. Lin cited there.