4.7: Coordinate and Momentum Representations

Now let me show that in application to the orbital motion of a particle, the bra-ket formalism naturally reduces to the notions and postulates of wave mechanics, which were discussed in Chapter 1. For that, we first have to modify some of the above formulas for the case of a basis with a continuous spectrum of eigenvalues. In that case, it is more appropriate to replace discrete indices, such as j, j ', etc. broadly used above, with the corresponding eigenvalue - just as it was done earlier for functions of the wave vector - see, e.g., Eqs. (1.88), (2.20), etc. For example, the key Eq. (68), defining the eigenkets and eigenvalues of an operator, may be conveniently rewritten in the form

$$\hat{A}\left|a_{A}\right\rangle=A\left|a_{A}\right\rangle . \tag{4.7.1}$$

More substantially, all sums over such continuous eigenstate sets should be replaced with integrals. For example, for a full and orthonormal set of the continuous eigenstates $\left|a_{A}\right\rangle$, the closure relation (44) should be replaced with

$$\int d A\left|a_{A}\right\rangle\left\langle a_{A}\right|=\hat{I}, \tag{4.7.2}$$

where the integral is over the whole interval of possible eigenvalues of the observable A.[44] Applying this relation to the ket-vector of an arbitrary state α, we get the following replacement of Eq. (37):

$$|\alpha\rangle \equiv \hat{I}|\alpha\rangle=\int d A\left|a_{A}\right\rangle\left\langle a_{A} \mid \alpha\right\rangle=\int d A\left\langle a_{A} \mid \alpha\right\rangle\left|a_{A}\right\rangle . \tag{4.7.3}$$

For the particular case when $|\alpha\rangle=\left|a_{A}\right\rangle$, this relation requires that

$$\left\langle a_{A} \mid a_{A^{\prime}}\right\rangle=\delta\left(A-A^{\prime}\right); \tag{4.7.4}$$

this formula replaces the orthonormality condition (38).

According to Eq. (221), in the continuous case the bracket $\left\langle a_{A} \mid \alpha\right\rangle$ still the role of probability amplitude, i.e. a complex c-number whose modulus squared determines the state a_{A} 's probability - see the last form of Eq. (120). However, for a continuous observable, the probability to find the system exactly in a particular state is infinitesimal; instead, we should speak about the probability $d W=w(A) d A$ of finding the observable within a small interval $d A<<A$ near the value A, with probability density $w(A) \propto\left|\left\langle a_{A} \mid \alpha\right\rangle\right|^{2}$. The coefficient in this relation may be found by making a similar change from the summation to integration in the normalization condition (121):

$$\int d A\left\langle\alpha \mid a_{A}\right\rangle\left\langle a_{A} \mid \alpha\right\rangle=1. \tag{4.7.5}$$

Since the total probability of the system to be in some state should be equal to $\int w(A) d A$, this means that

$$w(A)=\left\langle\alpha \mid a_{A}\right\rangle\left\langle a_{A} \mid \alpha\right\rangle=\left|\left\langle\alpha \mid a_{A}\right\rangle\right|^{2}. \tag{4.7.6}$$

Now let us see how we can calculate the expectation values of continuous observables, i.e. their ensemble averages. If we speak about the same observable A whose eigenstates are used as the continuous basis (or any compatible observable), everything is simple. Indeed, inserting Eq. (224) into the general statistical relation

$$\langle A\rangle=\int w(A) A d A, \tag{4.7.7}$$

which is just the obvious continuous version of Eq. (1.37), we get

$$\langle A\rangle=\int\left\langle\alpha \mid a_{A}\right\rangle A\left\langle a_{A} \mid \alpha\right\rangle d A. \tag{4.7.8}$$

Inserting a delta-function to represent this expression as a formally double integral,

$$\langle A\rangle=\int d A \int d A^{\prime}\left\langle\alpha \mid a_{A}\right\rangle A \delta\left(A-A^{\prime}\right)\left\langle a_{A^{\prime}} \mid \alpha\right\rangle, \tag{4.7.9}$$

and using the continuous-spectrum version of Eq. (98),

$$\left\langle a_{A}|\hat{A}| a_{A^{\prime}}\right\rangle=A \delta\left(A-A^{\prime}\right), \tag{4.7.10}$$

we may write

$$\langle A\rangle=\int dA \int dA^{\prime}\left\langle\alpha \mid a_A\right\rangle\left\langle a_A|\hat{A}| a_{A^{\prime}}\right\rangle\left\langle a_{A^{\prime}} \mid \alpha\right\rangle \equiv\langle\alpha|\hat{A}| \alpha\rangle, \tag{4.7.11}$$

so that Eq. (4.125) remains valid in the continuous-spectrum case without any changes.

The situation is a bit more complicated for the expectation values of an operator that does not commute with the basis-generating operator, because its matrix in that basis may not be diagonal. We will consider (and overcome :-) this technical difficulty very soon, but otherwise we are ready for a discussion of the relation between the bra-ket formalism and the wave mechanics. (For the notation simplicity I will discuss its 1D version; its generalization to 2D and 3D cases is straightforward.)

Let us postulate the (intuitively almost evident) existence of a quantum state basis, whose ketvectors will be called $|x\rangle$, corresponding to a certain definite value x of the particle's coordinate. Writing the following trivial identity:

$$x|x\rangle=x|x\rangle \tag{4.7.12}$$

and comparing this relation with Eq. (219), we see that they do not contradict each other if we assume that x on the left-hand side of this relation is the (Hermitian) operator $\hat{x}$ of particle's coordinate, whose action on a ket- (or bra-) vector is just its multiplication by the c-number x. (This looks like a proof, but is actually a separate, independent postulate, no matter how plausible.) Hence we may consider vectors $|x\rangle$ as the eigenstates of the operator $\hat{x}$. Let me hope that the reader will excuse me if I do not pursue here a strict proof that this set is full and orthogonal, [45] so that we may apply to them Eq. (222):

$$\left\langle x \mid x^{\prime}\right\rangle=\delta\left(x-x^{\prime}\right). \tag{4.7.13}$$

Using this basis is called the coordinate representation - the term which was already used at the end of Sec. 1.1, but without explanation.

In the basis of the x-states, the inner product $\left\langle a_A \mid \alpha(t)\right\rangle$ becomes $\langle x \mid \alpha(t)\rangle$, and Eq. (223) takes the following form:

$$w(x, t)=\langle\alpha(t) \mid x\rangle\langle x \mid \alpha(t)\rangle \equiv\langle x \mid \alpha(t)\rangle^*\langle x \mid \alpha(t)\rangle \tag{4.7.14}$$

Comparing this formula with the basic postulate (1.22) of wave mechanics, we see that they coincide if the wavefunction of a time-dependent state α is identified with that short bracket: [46]

$$\Psi_\alpha(x, t) \equiv\langle x \mid \alpha(t)\rangle. \tag{4.7.15}$$

This key formula provides the desired connection between the bra-ket formalism and the wave mechanics, and should not be too surprising for the (thoughtful :-) reader. Indeed, Eq. (45) shows that any inner product of two state vectors describing two states is a measure of their coincidence - just as the scalar product of two geometric vectors is; the orthonormality condition (38) is a particular manifestation of this fact. In this language, the particular value (233) of a wavefunction Ψ_α at some point x and moment t characterizes "how much of a particular coordinate x " does the state α contain at time t. (Of course, this informal language is too crude to reflect the fact that $\Psi_\alpha(x, t)$ is a complex function, which has not only a modulus but also an argument - the quantum-mechanical phase.)

Now let us rewrite the most important formulas of the bra-ket formalism in the wave mechanics notation. Inner-multiplying both parts of Eq. (219) by the ket-vector $\langle x|$, and then inserting into the lefthand side of that relation the identity operator in the form (220) for coordinate $x^{\prime}$, we get

$$\int d x^{\prime}\left\langle x|\hat{A}| x^{\prime}\right\rangle\left\langle x^{\prime} \mid a_A\right\rangle=A\left\langle x \mid a_A\right\rangle, \tag{4.7.16}$$

i.e., using the wavefunction's definition (233),

$$\int d x^{\prime}\left\langle x|\hat{A}| x^{\prime}\right\rangle \Psi_A\left(x^{\prime}\right)=A \Psi_A(x), \tag{4.7.17}$$

where, for the notation brevity, the time dependence of the wavefunction is just implied (with the capital Ψ serving as a reminder of this fact), and will be restored when needed.

For a general operator, we would have to stop here, because if it does not commute with the coordinate operator, its matrix in the x-basis is not diagonal, and the integral on the left-hand side of Eq. (235) cannot be worked out explicitly. However, virtually all quantum-mechanical operators discussed in this course [47] are (space-) local: they depend on only one spatial coordinate, say x. For such operators, the left-hand side of Eq. (235) may be further transformed as

$$\int\left\langle x|\hat{A}| x^{\prime}\right\rangle \Psi\left(x^{\prime}\right) d x^{\prime}=\int\left\langle x \mid x^{\prime}\right\rangle \hat{A} \Psi\left(x^{\prime}\right) d x^{\prime} \equiv \hat{A} \int \delta\left(x-x^{\prime}\right) \Psi\left(x^{\prime}\right) d x^{\prime}=\hat{A} \Psi(x). \tag{4.7.18}$$

The first step in this transformation may appear as elementary as the last two, with the ket-vector $|x$ ' $\rangle$ swapped with the operator depending only on x; however, due to the delta-functional character of the bracket (231), this step is, in fact, an additional postulate, so that the second equality in Eq. (236) essentially defines the coordinate representation of the local operator, whose explicit form still needs to be determined.

Let us consider, for example, the 1D version of the Hamiltonian (1.41),

$$\hat{H}=\frac{\hat{p}_x^2}{2m}+U(\hat{x}), \tag{4.7.19}$$

which was the basis of all our discussions in Chapter 2. Its potential-energy part U (which may be timedependent as well) commutes with the operator $\hat{x}$, i.e. its matrix in the x-basis has to be diagonal. For such an operator, the transformation (236) is indeed trivial, and its coordinate representation is given merely by the c-number function $U(x)$.The situation the momentum operator $\hat{p}_x$ (and hence the kinetic energy $\hat{p}_x^2/2m$), not commuting with $\hat{x}$, is less evident. Let me show that its coordinate representation is given by the 1D version of Eq. (1.26), if we postulate that the commutation relation (2.14),

$$[\hat{x},\hat{p}]=i\hbar\hat{I}, \quad \text{ i.e. } \hat{x}\hat{p}_x-\hat{p}_x\hat{x}=i\hbar\hat{I}, \tag{4.7.20}$$

is valid in any representation. [48] For that, let us consider the following matrix element, $\langle x|\hat{x}\hat{p}_x-\hat{p}_x\hat{x}|x'\rangle$. On one hand, we may use Eq. (238), and then Eq. (231), to write

$$\langle x|\hat{x}\hat{p}_x-\hat{p}_x\hat{x}|x'\rangle=\left\langle x|i\hbar\hat{I}|x'\right\rangle=i\hbar\langle x\mid x'\rangle=i\hbar\delta(x-x'). \tag{4.7.21}$$

On the other hand, since $\hat{x}|x'\rangle=x'|x'\rangle$ and $\langle x|\hat{x}=\langle x|x$, we may represent the same matrix element as

$$\langle x|\hat{x}\hat{p}_x-\hat{p}_x\hat{x}|x'\rangle=\langle x|x\hat{p}_x-\hat{p}_x x'|x'\rangle=(x-x')\langle x|\hat{p}_x|x'\rangle. \tag{4.7.22}$$

Comparing Eqs. (239) and (240), we get

$$\langle x|\hat{p}_x|x'\rangle=i\hbar\frac{\delta(x-x')}{x-x'}. \tag{4.7.23}$$

As it follows from the definition of the delta function, [49] all expressions involving it acquire final sense only at their integration, in our current case, at that described by Eq. (236). Plugging Eq. (241) into the left-hand side of that relation, we get

$$\int\langle x|\hat{p}_x|x'\rangle\Psi(x')\,dx'=i\hbar\int\frac{\delta(x-x')}{x-x'}\Psi(x')\,dx'. \tag{4.7.24}$$

Since the right-hand-part integral is contributed only by an infinitesimal vicinity of the point $x'=x$, we may calculate it by expanding the continuous wavefunction $\Psi(x')$ into the Taylor series in small $(x'-x)$, and keeping only two leading terms of the series, so that Eq. (242) is reduced to

$$\int\langle x|\hat{p}_x|x'\rangle\Psi(x')\,dx'=i\hbar\left[\Psi(x)\int\frac{\delta(x-x')}{x-x'}dx'-\int\delta(x-x')\left.\frac{\partial\Psi(x')}{\partial x'}\right|_{x'=x}dx'\right]. \tag{4.7.25}$$

Since the delta function may be always understood as an even function of its argument, in our case of $(x-x')$, the first term on the right-hand side is proportional to an integral of an odd function in symmetric limits and is equal to zero, and we get [50]

$$\int\langle x|\hat{p}_x|x'\rangle\Psi(x')\,dx'=-i\hbar\frac{\partial\Psi}{\partial x} \tag{4.7.26}$$

Comparing this expression with the right-hand side of Eq. (236), we see that in the coordinate representation we indeed get the $1\mathrm{D}$ version of Eq. (1.26), which was used so much in Chapter 2, [51]

$$\hat{p}_x=-i\hbar\frac{\partial}{\partial x}. \tag{4.7.27}$$

It is straightforward to show (and is virtually evident) that the coordinate representation of any operator function $f(\hat{p}_x)$ is

$$f\left(-i\hbar\frac{\partial}{\partial x}\right) \tag{4.7.28}$$

In particular, this pertains to the kinetic energy operator in Eq. (237), so the coordinate representation of this Hamiltonian also takes the very familiar form:

$$\hat{H}=\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2+U(x,t)\equiv-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x,t). \tag{4.7.29}$$

Now returning to the discussion of the general Eq. (235), and comparing its last form with that of Eq. (236), we see that for a local operator in the coordinate representation, the eigenproblem (219) takes the form

$$\hat{A}\Psi_A(x) = A\Psi_A(x), \tag{4.7.30}$$

even if the operator $\hat{A}$ does not commute with the operator $\hat{x}$. The most important case of this coordinate-representation form of the eigenproblem (68) is the familiar Eq. (1.60) for the eigenvalues E_n of the energy of a system with a time-independent Hamiltonian.

The operator locality also simplifies the expression for its expectation value. Indeed, plugging the closure relation in the form (231) into the general Eq. (125) twice (written in the first case for x and in the second case for x '), we get

$$\langle A\rangle = \int dx \int dx' \langle\alpha(t) \mid x\rangle \left\langle x|\hat{A}|x'\right\rangle \langle x' \mid \alpha(t)\rangle = \int dx \int dx' \Psi_\alpha^*(x,t) \left\langle x|\hat{A}|x'\right\rangle \Psi_\alpha\left(x',t\right). \tag{4.7.31}$$

Now, Eq. (236) reduces this result to just

$$\langle A\rangle = \int dx \int dx^\psi \Psi_\alpha^*(x,t)\hat{A}\Psi_\alpha(x,t)\delta\left(x-x'\right) \equiv \int \Psi_\alpha^*(x,t)\hat{A}\Psi_\alpha(x,t)dx. \tag{4.7.32}$$

i.e. to Eq. (1.23), which had to be postulated in Chapter 1 .

Finally, let us discuss the time evolution of the wavefunction, in the Schrödinger picture. For that, we may use Eq. (233) to calculate the (partial) time derivative of the wavefunction of some state α :

$$i\hbar\frac{\partial\Psi_\alpha}{\partial t} = i\hbar\frac{\partial}{\partial t}\langle x \mid \alpha(t)\rangle. \tag{4.7.33}$$

Since the coordinate operator $\hat{x}$ does not depend on time explicitly, its eigenstates x are stationary, and we can swap the time derivative and the time-independent bra-vector $\langle x|$. Now using the Schrödingerpicture equation (158), and then inserting the identity operator in the continuous form (220) of the closure relation, written for the coordinate eigenstates,

$$\int dx' \left|x'\right\rangle \left\langle x'\right| = \hat{I}, \tag{4.7.34}$$

we may continue to develop the right-hand side of Eq. (251) as

$$\left\langle x\left|i\hbar\frac{\partial}{\partial t}\right|\alpha(t)\right\rangle = \langle x|\hat{H}|\alpha(t)\rangle = \int dx' \left\langle x|\hat{H}|x'\right\rangle \langle x' \mid \alpha(t)\rangle = \int dx' \left\langle x|\hat{H}|x'\right\rangle \Psi_\alpha\left(x'\right), \tag{4.7.35}$$

If the Hamiltonian operator is local, we may apply Eq. (236) to the last expression, to get the familiar form (1.28) of the Schrödinger equation:

$$i\hbar\frac{\partial\Psi_\alpha}{\partial t} = \hat{H}\Psi_\alpha, \tag{4.7.36}$$

in which the coordinate representation of the operator $\hat{H}$ is implied.

So, for the local operators that obey Eq. (236), we have been able to derive all the basic notions and postulates of the wave mechanics from the bra-ket formalism. Moreover, the formalism has allowed us to get the very useful equation (248) for an arbitrary local operator, which will be repeatedly used below. (In the first three chapters of this course, we have only used its particular case (1.60) for the Hamiltonian operator.)

Now let me deliver on my promise to develop a more balanced view at the monochromatic de Broglie waves (1), which would be more respectful to the evident $\mathbf{r} \leftrightarrow \mathbf{p}$ symmetry of the coordinate and momentum. Let us do this for the 1D case when the wave may be represented as

$$\psi_p(x) = a_p \exp\left\{i\frac{px}{\hbar}\right\}, \quad \text{for all } -\infty < x < +\infty \tag{4.7.37}$$

(For the sake of brevity, from this point to the end of the section, I am dropping the index x in the notation of the momentum - just as it was done in Chapter 2.) Let us have a good look at this function. Since it satisfies Eq. (248) for the 1D momentum operator (245),

$$\hat{p}\psi_p = p\psi_p, \tag{4.7.38}$$

ψ_p is an eigenfunction of that operator. But this means that we can also write Eq. (219) for the corresponding ket-vector:

$$\hat{p}|p\rangle = p|p\rangle, \tag{4.7.39}$$

and according to Eq. (233), the wavefunction (255) may be represented as

$$\psi_p(x) = \langle x \mid p\rangle. \tag{4.7.40}$$

This expression is quite remarkable in its $x \leftrightarrow p$ symmetry - which may be pursued further on. Before doing that, however, we have to discuss the normalization of such wavefunctions. Indeed, in this case, the probability density $w(x)(18)$ is constant, so that its integral

$$\int_{-\infty}^{+\infty} w(x) dx = \int_{-\infty}^{+\infty} \psi_p(x) \psi_p^*(x) dx \tag{4.7.41}$$

diverges if $a_p \neq 0$. Earlier in the course, we discussed two ways to avoid this divergence. One is to use a very large but finite integration volume - see Eq. (1.31). Another way is to work with wave packets of the type (2.20), possibly of a very large length and hence a very narrow spread of the momentum values. Then the integral (259) may be required to equal 1 without any conceptual problem.

However, both these methods, convenient for the solution of many particular problems, violate the $x \leftrightarrow p$ symmetry and hence are inconvenient for our current conceptual discussion. Instead, let us continue to identify the eigenvectors $\langle p|$ and $|p\rangle$ of the momentum with the bra- and ket-vectors $\langle a_A|$ and $|a_A\rangle$ of the general theory described at the beginning of this section. Then the normalization condition (222) becomes

$$\langle p \mid p'\rangle = \delta\left(p - p'\right). \tag{4.7.42}$$

Inserting the identity operator in the form (252), with the integration variable x ' replaced by x, into the left-hand side of this equation, and using Eq. (258), we can translate this normalization rule to the wavefunction language:

$$\int dx \langle p \mid x\rangle \langle x \mid p'\rangle \equiv \int dx \psi_p^*(x) \psi_{p'}(x) = \delta\left(p - p'\right) \tag{4.7.43}$$

For the wavefunction (255), this requirement turns into the following condition:

$$a_p^* a_{p'} \int_{-\infty}^{+\infty} \exp\left\{i \frac{\left(p - p'\right) x}{\hbar}\right\} dx \equiv \left|a_p\right|^2 2\pi\hbar\delta\left(p - p'\right) = \delta\left(p - p'\right) \tag{4.7.44}$$

so that, finally, $a_p = e^{i\phi}/(2\pi\hbar)^{1/2}$, where ϕ is an arbitrary (real) phase, and Eq. (255) becomes [52]

$$\psi_p(x) = \langle x \mid p\rangle = \frac{1}{(2\pi\hbar)^{1/2}} \exp\left\{i\left(\frac{px}{\hbar} + \phi\right)\right\}, \tag{4.7.45}$$

Now let us represent an arbitrary wavefunction $\psi(x)$ as a wave packet of the type (2.20), based on the wavefunctions (263), taking $\phi = 0$ for the notation brevity, because the phase may be incorporated into the (generally, complex) envelope function $\varphi(p)$:

$$\psi(x) = \frac{1}{(2\pi\hbar)^{1/2}} \int \varphi(p) \exp\left\{i \frac{px}{\hbar}\right\} dp \tag{4.7.46}$$

From the mathematical point of view, this is just a 1D Fourier spatial transform, and its reciprocal is

$$\varphi(p) \equiv \frac{1}{(2\pi\hbar)^{1/2}} \int \psi(x) \exp\left\{-i \frac{px}{\hbar}\right\} dx \tag{4.7.47}$$

These expressions are completely symmetric, and represent the same wave packet; this is why the functions $\psi(x)$ and $\varphi(p)$ are frequently called the reciprocal representations of a quantum state of the particle: respectively, its coordinate $(x-)$ and momentum $(p-)$ representations. Using Eq. (258), and Eq. (263) with $\phi = 0$, they may be recast into simpler forms,

$$\psi(x) = \int \varphi(p) \langle x \mid p\rangle dp, \quad \varphi(p) = \int \psi(x) \langle p \mid x\rangle dx, \tag{4.7.48}$$

in which the inner products satisfy the basic postulate (14) of the bra-ket formalism:

$$\langle p \mid x\rangle = \frac{1}{(2\pi\hbar)^{1/2}} \exp\left\{-i \frac{px}{\hbar}\right\} = \langle x \mid p\rangle^*. \tag{4.7.49}$$

Next, we already know that in the x-representation, i.e. in the usual wave mechanics, the coordinate operator $\hat{x}$ is reduced to the multiplication by x, and the momentum operator is proportional to the partial derivative over the coordinate:

$$\left.\hat{x}\right|_{\text{in } x} = x, \quad \left.\hat{p}\right|_{\text{in } x} = -i\hbar \frac{\partial}{\partial x}. \tag{4.7.50}$$

It is natural to guess that in the p-representation, the expressions for operators would be reciprocal:

$$\left.\hat{x}\right|_{\text{in } p}=+i\hbar\frac{\partial}{\partial p},\quad \left.\hat{p}\right|_{\text{in } p}=p, \tag{4.7.51}$$

with the only difference of one sign, which is due to the opposite signs of the Fourier exponents in Eqs. (264) and (265). The proof of Eqs. (269) is straightforward; for example, acting by the momentum operator on the arbitrary wavefunction (264), we get

$$\hat{p}\psi(x)=-i\hbar\frac{\partial}{\partial x}\psi(x)=\frac{1}{(2\pi\hbar)^{1/2}}\int\varphi(p)\left(-i\hbar\frac{\partial}{\partial x}\exp\left\{i\frac{px}{\hbar}\right\}\right)dp=\frac{1}{(2\pi\hbar)^{1/2}}\int p\varphi(p)\exp\left\{i\frac{px}{\hbar}\right\}dp, \tag{4.7.52}$$

and similarly for the operator $\hat{x}$ acting on the function $\varphi(p)$. Comparing the final form of Eq. (270) with the initial Eq. (264), we see that the action of the operators (268) on the wavefunction ψ (i.e. the state's x-representation) gives the same results as the action of the operators (269) on the function φ (i.e. its p representation).

It is illuminating to have one more, different look at this coordinate-momentum duality. For that, notice that according to Eqs. (82)-(84), we may consider the bracket $\langle x \mid p\rangle$ as an element of the (infinitesize) matrix U_{xp} of the unitary transform from the x-basis to the p-basis. Let us use this fact to derive the general operator transform rule that would be a continuous version of Eq. (92). Say, we want to calculate the general matrix element of some operator, known in the x-representation, in the p representation:

$$\left\langle p|\hat{A}|p'\right\rangle. \tag{4.7.53}$$

Inserting two identity operators (252), written for x and x ', into this bracket, and then using Eq. (258) and its complex conjugate, and also Eq. (236) (again, valid only for space-local operators!), we get

$$\begin{aligned}\left\langle p|\hat{A}|p'\right\rangle&=\int dx\int dx'\langle p\mid x\rangle\left\langle x|\hat{A}|x'\right\rangle\langle x'\mid p'\rangle=\int dx\int dx'\psi_p^*(x)\left\langle x|\hat{A}|x'\right\rangle\psi_{p'}(x')\\&=\frac{1}{2\pi\hbar}\int dx\int dx'\exp\left\{-i\frac{px}{\hbar}\right\}\delta(x-x')\hat{A}\exp\left\{+i\frac{p'x'}{\hbar}\right\}=\frac{1}{2\pi\hbar}\int dx\exp\left\{-i\frac{px}{\hbar}\right\}\hat{A}\exp\left\{+i\frac{p'x}{\hbar}\right\}.\end{aligned}$$

As a sanity check, for the momentum operator itself, this relation yields:

$$\langle p|\hat{p}|p'\rangle=\frac{1}{2\pi\hbar}\int dx\exp\left\{-i\frac{px}{\hbar}\right\}\left(-i\hbar\frac{\partial}{\partial x}\right)\exp\left\{i\frac{p'x}{\hbar}\right\}=\frac{p'}{2\pi\hbar}\int_{-\infty}^{+\infty}\exp\left\{i\frac{(p'-p)x}{\hbar}\right\}dx=p'\delta(p'-p). \tag{4.7.54}$$

Due to Eq. (257), this result is equivalent to the second of Eqs. (269).

From a thoughtful reader, I anticipate the following natural question: why is the momentum representation used much less frequently than the coordinate representation - i.e. wave mechanics? The answer is purely practical: besides the important special case of the 1D harmonic oscillator (to be revisited in Sec. 5.4), in most systems the orbital-motion Hamiltonian (237) is not $x\leftrightarrow p$ symmetric, with the potential energy $U(\mathbf{r})$ typically being a more complex function than the kinetic energy $p^2/2m$. Because of that, it is easier to analyze such systems treating such potential energy operator just a $c-$ number multiplier, as it is in the coordinate representation - as it was done in Chapters 1-3.

The most significant exception from this practice is the motion in a periodic potential in presence of a coordinate-independent external force $\mathbf{F}(t)$. As was discussed in Secs. 2.7 and 3.4, in such periodic systems the eigenenergies $E_n(\mathbf{q})$, playing the role of the effective kinetic energy of the particle, may be rather involved functions of its quasimomentum $\hbar\mathbf{q}$, while its effective potential energy $U_{\text{ef}}=-\mathbf{F}(t)\cdot\mathbf{r}$, due to the additional force $\mathbf{F}(t)$, is a very simple function of coordinates. This is why detailed analyses of the quantum effects briefly discussed in Sec. 2.8 (the Bloch oscillations, etc.) and also such statistical phenomena as drift, diffusion, etc.,[53] in solid-state theory are typically based on the momentum (or rather quasimomentum) representation.

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[44] The generalization to cases when the eigenvalue spectrum consists of both a continuum interval plus some set of discrete values, is straightforward, though leads to somewhat cumbersome formulas.

[45] Such proof is rather involved mathematically, but physically this fact should be evident.

[46] I do not quite like expressions like $\langle x \mid \Psi\rangle$ used in some papers and even textbooks. Of course, one is free to replace α with any other letter (Ψ including) to denote a quantum state, but then it is better not to use the same letter to denote the wavefunction, i.e. an inner product of two state vectors, to avoid confusion.

[47] The only substantial exception is the statistical operator $\hat{w}(x,x')$, to be discussed in Chapter 7.

[48] Another possible approach to the wave mechanics axiomatics is to derive Eg. (238) by postulating the form, $\hat{\mathcal{T}}_X=\exp\{-i\hat{p}_xX/\hbar\}$, of the operator that shifts any wavefunction by distance X along the axis x. In my approach, this expression will be derived when we need it (in Sec. 5.5), while Eq. (238) is postulated.

[49] If necessary, please revisit MA Sec. 14.

[50] One more useful expression of this type, which may be proved similarly, is $(\partial/\partial x)\delta\left(x-x'\right)=\delta\left(x-x'\right)\partial/\partial x'$,

[51] This means, in particular, that in the sense of Eq. (236), the operator of differentiation is local, despite the fact that its action on a function f may be interpreted as the limit of the fraction $\Delta f/\Delta x$, involving two points. (In some axiomatic systems, local operators are defined as arbitrary polynomials of functions and their derivatives.)

[52] Repeating such calculation for each Cartesian component of a plane monochromatic wave of arbitrary dimensionality d, we get $\psi_{\mathrm{p}}=(2\pi\hbar)^{-d/2}\exp\{i(\mathbf{p}\cdot\mathbf{r}/\hbar+\varphi)\}$.

[53] In this series, a brief discussion of these effects may be found in SM Chapter 6 .

[54] See, e.g., MA Eqs. (13.1) and (13.2).

[55] See, e.g. MA Eq. (2.9).