8.1: Distinguishable and Indistinguishable Particles

The importance of quantum systems of many similar particles is probably self-evident; just the very fact that most atoms include several/many electrons is sufficient to attract our attention. There are also important systems where the total number of electrons is much higher than in one atom; for example, a cubic centimeter of a typical metal houses $\sim 10^{23}$ conduction electrons that cannot be attributed to particular atoms, and have to be considered as common parts of the system as the whole. Though quantum mechanics offers virtually no exact analytical results for systems of substantially interacting particles, [1] it reveals very important new quantum effects even in the simplest cases when particles do not interact, and least explicitly (directly).

If non-interacting particles are either different from each other by their nature, or physically similar but still distinguishable because of other reasons, everything is simple - at least, conceptually. Then, as was already discussed in Sec. 6.7, a system of two particles, 1 and 2, each in a pure quantum state, may be described by a state vector which is a direct product,

$$|\alpha\rangle=|\beta\rangle_1 \otimes\left|\beta^{\prime}\right\rangle_2, \tag{8.1.1}$$

Distinguish- of single-particle vectors, describing their states β and β ’ defined in different Hilbert spaces. (Below, I able will frequently use, for such direct product, the following convenient shorthand:

$$\begin{aligned} &\text{able particles will frequently use, for such direct product, the following conve} \\ &|\alpha\rangle=\left|\beta \beta^{\prime}\right\rangle \end{aligned}$$

in which the particle’s number is coded by the state symbol’s position.) Hence the permuted state

$$\hat{\mathcal{P}}\left|\beta \beta^{\prime}\right\rangle \equiv\left|\beta^{\prime} \beta\right\rangle \equiv\left|\beta^{\prime}\right\rangle_1 \otimes|\beta\rangle_2, \tag{8.1.2}$$

where $\hat{P}$ is the permutation operator defined by Eq. (2), is clearly different from the initial one.

This operator may be also used for states of systems of identical particles. In physics, the last term may be used to describe:

(i) the "really elementary" particles like electrons, which (at least at this stage of development of physics) are considered as structure-less entities, and hence are all identical;

(ii) any objects (e.g., hadrons or mesons) that may be considered as a system of "more elementary" particles (e.g., quarks and gluons), but are placed in the same internal quantum state - most simply, though not necessarily, in the ground state. [2]

It is important to note that identical particles still may be distinguishable - say by their clear spatial separation. Such systems of similar but distinguishable particles (or subsystems) are broadly discussed nowadays in the context of quantum computing and encryption - see Sec. 5 below. This is why it is insufficient to use the term "identical particles" if we want to say that they are genuinely indistinguishable, so I below I will use the latter term, despite it being rather unpleasant grammatically.

It turns out that for a quantitative description of systems of indistinguishable particles we need to use, instead of direct products of the type (1), linear combinations of such products, for example of $\left|\beta \beta^{\prime}\right\rangle$ and $\left|\beta^{\prime} \beta\right\rangle$. [3] To see this, let us discuss the properties of the permutation operator defined by Eq. (2). Consider an observable A, and a system of eigenstates of its operator:

$$\hat{A}\left|a_j\right\rangle=A_j\left|a_j\right\rangle . \tag{8.1.3}$$

If the particles are indistinguishable, the observable’s expectation value should not be affected by their permutation. Hence the operators $\hat{A}$ and $\hat{\mathcal{P}}$ have to commute and share their eigenstates. This is why the eigenstates of the operator $\hat{\mathcal{P}}$ are so important: in particular, they include the eigenstates of the Hamiltonian, i.e. the stationary states of a system of indistinguishable particles.Let us have a look at the action of the permutation operator squared, on an elementary ket-vector product:

$$\hat{\mathcal{P}}^2\left|\beta \beta^{\prime}\right\rangle=\hat{\mathcal{P}}\left(\hat{\mathcal{P}}\left|\beta \beta^{\prime}\right\rangle\right)=\hat{\mathcal{P}}\left|\beta^{\prime} \beta\right\rangle=\left|\beta \beta^{\prime}\right\rangle, \tag{8.1.4}$$

i.e. $\hat{P}^2$ brings the state back to its original form. Since any pure state of a two-particle system may be represented as a linear combination of such products, this result does not depend on the state, and may be represented as the following operator relation:

$$\hat{P}^2=\hat{I} . \tag{8.1.5}$$

Now let us find the possible eigenvalues $\mathcal{P}_j$ of the permutation operator. Acting by both sides of Eq. (5) on any of eigenstates $\left|\alpha_j\right\rangle$ of the permutation operator, we get a very simple equation for its eigenvalues:

$$\mathcal{P}_j^2 = 1, \tag{8.1.6}$$

with two possible solutions:

$$\mathcal{P}_j = \pm 1. \tag{8.1.7}$$

Let us find the eigenstates of the permutation operator in the simplest case when each of the component particles can be only in one of two single-particle states - say, β and β '. Evidently, none of the simple products $|\beta\beta'\rangle$ and $|\beta'\beta\rangle$, taken alone, does qualify for the eigenstate - unless the states β and β' are identical. This is why let us try their linear combination

$$|\alpha_j\rangle = a\,|\beta\beta'\rangle + b\,|\beta'\beta\rangle\,, \tag{8.1.8}$$

so that

$$\hat{\mathcal{P}}\,|\alpha_j\rangle = \mathcal{P}_j\,|\alpha_j\rangle = a\,|\beta'\beta\rangle + b\,|\beta\beta'\rangle\,. \tag{8.1.9}$$

For the case $\mathcal{P}_j = +1$ we have to require the states (8) and (9) to be the same, so that $a = b$, giving the socalled symmetric eigenstate [4]

$$|\alpha_+\rangle = \frac{1}{\sqrt{2}}(|\beta\beta'\rangle + |\beta'\beta\rangle)\,, \tag{8.1.10}$$

where the front coefficient guarantees the orthonormality of the two-particle state vectors, provided that the single-particle vectors are orthonormal. Similarly, for $\mathcal{P}_j = -1$ we get $a = -b$, i.e. an antisymmetric eigenstate

$$|\alpha_-\rangle = \frac{1}{\sqrt{2}}(|\beta\beta'\rangle - |\beta'\beta\rangle) \tag{8.1.11}$$

These are the simplest (two-particle, two-state) examples of entangled states, defined as multiparticle system states whose vectors cannot be factored into a direct product (1) of single-particle vectors.

So far, our math does not preclude either sign of $\mathcal{P}_i$, in particular the possibility that the sign would depend on the state (i.e. on the index j). Here, however, comes in a crucial fact: all indistinguishable particles fall into two groups: [5]

(i) bosons, particles with integer spin s, for whose states $\mathcal{P}_j = +1$, and

(ii) fermions, particles with half-integer spin, with $\mathcal{P}_j = -1$.

In the non-relativistic theory we are discussing now, this key fact should be considered as an experimental one. (The relativistic quantum theory, whose elements will be discussed in Chapter 9 , offers proof that the half-integer-spin particles cannot be bosons and the integer-spin ones cannot be fermions.) However, our discussion of spin in Sec. 5.7 enables the following handwaving interpretation of the difference between these two particle species. In the free space, the permutation of particles 1 and 2 may be viewed as a result of their pair's common rotation by angle $\phi = \pm\pi$ about a properly selected z-axis. As we have seen in Sec. 5.7, at the rotation by this angle, the state vector $|\beta\rangle$ of a particle with a definite quantum number m_s acquires an extra factor $\exp\{\pm i m_s \pi\}$. As we know, the quantum number m_s ranges from $-s$ to $+s$, in unit steps. As a result, for bosons, with integer s, m_s can take only integer values, so that $\exp\{\pm i m_s \pi\} = \pm 1$, so that the product of two such factors in the state product $|\beta\beta'\rangle$ is equal to $+1$. On the contrary, for the fermions with their half-integer s, all m_s are half-integer as well, so that $\exp\{\pm i m_s \pi\} = \pm i$ so that the product of two such factors in vector $|\beta\beta'\rangle$ is equal to $(\pm i)^2 = -1$.

The most impressive corollaries of Eqs. (10) and (11) are for the case when the partial states of the two particles are the same: $\beta = \beta$ '. The corresponding Bose state α_+, defined by Eq. (10), is possible; in particular, at sufficiently low temperatures, a set of non-interacting Bose particles condenses on the ground state - the so-called Bose-Einstein condensate ("BEC"). [6] The most fascinating feature of the condensates is that their dynamics is governed by quantum mechanical laws, which may show up in the behavior of their observables with virtually no quantum uncertainties [7] — see, e.g., Eqs. (1.73)-(1.74).

On the other hand, if we take $\beta = \beta'$ in Eq. (11), we see that state α becomes the null-state, i.e. cannot exist at all. This is the mathematical expression of the Pauli exclusion principle: [8] two indistinguishable fermions cannot be placed into the same quantum state. (As will be discussed below, this is true for systems with more than two fermions as well.) Probably, the key importance of this principle is self-evident: if it was not valid for electrons (that are fermions), all electrons of each atom would condense on in their ground ($1s$-like) state, and all the usual chemistry (and biochemistry, and biology, including dear us!) would not exist. The Pauli principle makes fermions implicitly interacting even if they do not interact directly, i.e. in the usual sense of this word.

[1] As was emphasized in Sec.7.3, for such systems of similar particles the powerful methods discussed in the last chapter, based on the separation of the whole Universe into a "system of our interest" and its "environment", typically do not work well - mostly because the quantum state of the "particle of interest" may be substantially correlated (in particular, entangled) with those of similar particles forming its "environment" - see below.

[2] Note that from this point of view, even complex atoms or molecules, in the same internal quantum state, may be considered on the same footing as the "really elementary" particles. For example, the already mentioned recent spectacular interference experiments by R. Lopes et al., which require particle identity, were carried out with couples of $^4\mathrm{He}$ atoms in the same internal quantum state.

[3] A very legitimate question is why, in this situation, we need to introduce the particles' numbers to start with. A partial answer is that in this approach, it is much simpler to derive (or guess) the system Hamiltonians from the correspondence principle - see, e.g., Eq. (27) below. Later in this chapter, we will discuss an alternative approach (the so-called "second quantization"), in which particle numbering is avoided. While that approach is more logical, writing adequate Hamiltonians (which, in particular, would avoid spurious self-interaction of the particles) within it is more challenging $-$ see Sec. 3 below.

[4] As in many situations we have met earlier, the kets given by Eqs. (10) and (11) may be multiplied by $\exp\{i\varphi\}$ with an arbitrary real phase φ. However, until we discuss coherent superpositions of various states α, there is no good motivation for taking the phase different from 0 ; that would only clutter the notation.

[5] Sometimes this fact is described as having two different "statistics": the Bose-Einstein statistics of bosons and Fermi-Dirac statistics of fermions, because their statistical distributions in thermal equilibrium are indeed different - see, e.g., SM Sec. 2.8. However, this difference is actually deeper: we are dealing with two different quantum mechanics.

[6] For a quantitative discussion of the Bose-Einstein condensation see, e.g., SM Sec. 3.4. Examples of such condensates include superfluids like helium, Cooper-pair condensates in superconductors, and BECs of weakly interacting atoms.

[7] For example, for a coherent condensate of $N \gg 1$ particles, Heisenberg's uncertainty relation takes the form $\delta x \delta p = \delta x \delta(Nmv) \geq \hbar/2$, so that its coordinate x and velocity v may be measured simultaneously with much higher precision than those of a single particle.

[8] It was first formulated for electrons by Wolfgang Pauli in 1925, on the background of less general rules suggested by Gilbert Lewis (1916), Irving Langmuir (1919), Niels Bohr (1922), and Edmund Stoner (1924) for the explanation of experimental spectroscopic data.