4.8: Non-Linear Waves

Tutorial 4.8: Dispersion, Friction, Dissipation, and Nonlinearity

In simulation 4.6 we derived the **linear wave equation**, $\partial^2 y/\partial x^2 = 1/v^2 \partial^2 y/\partial t^2$ for an elastic string by considering forces acting on a small section of the string. The right side of the equation is basically the vertical acceleration of a piece of the string and the left side is the force. Constants like the tension and mass per unit length appear in the speed, *v* which is constant for a linear system. But what happens if other forces acts on the string? Some additional forces cause the dispersion we saw in simulations 3.8 and 3.9. Friction, dissipation and nonlinearity cause other behavior as we will see in this simulation.

> **Note**
>
> The applet below actually simulates a long chain of masses coupled by springs, as in the previous simulation. Mathematically we know that in the continuum limit of small masses that are very close together we get the linear wave equation as shown in simulation 4.5. As long as the waves on the chain are smooth and change gradually, the chain of masses approximates the continuous string. New forces acting on the chain will result in new terms being added to the wave equation as shown below. It should be kept in mind, however, that because the underlying model is discrete, the simulation may fail to accurately represent an elastic string. Except for friction and dissipation the total energy should remain approximately constant (there may be small fluctuations). If the energy starts to drift significantly this is a sign that the numerical calculations of the simulation are failing and the simulation no longer represents the elastic string (or any real system).

Non-Linear Waves

Questions:

> **Exercise 4.8.1**
>
> Test the simulation with dispersion, friction, dissipation and nonlinearity set to zero and the Gaussian pulse as the initial condition. Is energy conserved?

Suppose each point on the string had an additional force acting on it that was proportional to the amplitude of the string at that point. In other words springs attached to each point on the string. The wave equation would now look like $\partial^2 y/\partial x^2 + \alpha y = 1/v^2 \partial^2 y/\partial t^2$. This term leads to **dispersion**, a phenomena we examined previously.

> **Exercise 4.8.2**
>
> Turn on a small amount of dispersion with all the other terms still zero. Play the Gaussian initial contions. What happens to the pulse over time? How does this compare with what you saw in simulation 3.9?

> **Exercise 4.8.3**
>
> Find a dispersion relation for the dispersion equation by substituting $A\exp(-(kx-\omega t)^2)$ into the equation and solving for ω. You should be able to show that for this equation $\omega = (\alpha + v^2 k)^{1/2}$.

> **Exercise 4.8.4**
>
> Recall from simulation 2.3 that the group velocity of a wave packet is given by $v_{\text{group}} = \partial\omega(k)/\partial k$ but the phase velocity is $v_{\text{phase}} = \omega(k)/k$. The phase and group velocity are different and are dependent on the wavelength. Explain what this means physically. How does this explain what you saw in question 4.7.2? (Hint: Review simulation 2.3.)

Suppose our string was trying to vibrate in a medium like water or a dense gas. This would introduce a friction force that would be proportional to velocity; $-\eta\partial y/\partial t$ where η is the **coefficent of friction**. Now the wave equation would look like $\partial^2 y/\partial x^2 - \eta\partial y/\partial t = 1/v^2 \partial^2 y/\partial t^2$. We expect this force to gradually transfer energy to the surrounding liquid or gas.

Exercise 4.8.5

With the other parameters set to zero, turn on a small amount of friction and play the Gaussian initial condition. What happens? What happens to the energy?

For real strings there is also internal friction. If you have ever taken a metal coat hanger and bent it back and forth many times you know that the place where the bending occurs gets hot. This internal friciton, called **dissipation**, also gradually transfers the wave energy in a string into random thermal motion of the atoms in the string. For the wave equation this force can be represented by $-\gamma\partial y/\partial x$.

Exercise 4.8.6

With the other parameters set to zero, turn on a small amount of dissipation and play the Gaussian initial condition. What happens? What happens to the energy? You will notice that this is slightly different from the friction case. This is because a Gaussian pulse is not a solution to the wave equation with this additinal term in it. So not only is energy transfered into the string from the pulse, the pulse changes shape.

As a final example of adding external forces to a string we consider a force represented by $\pm\beta\partial^2y^2/\partial x^2$. This is a **nonlinear** term and it has the opposite effect of dispersion; nonlinear terms cause a wave packet to steepen instead of spread out.

Exercise 4.8.7

With the other parameters set to zero, turn on a small amount of nonlinearity and play the Gaussian initial condition. What happens? What happens for negative values of nonlinearity?

Note

Energy should be approximately conserved - if the energy change significantly the numerical calculations in the simulation are begining to fail.

Exercise 4.8.8

You may wonder what makes an equation nonlinear. In simulation 4.5 you showed that $y(x,t)=y_1(x,t)+y_2(x,t)$ was a solution to the linear wave equation as long as $y_1(x,t)$ and $y_2(x,t)$ are also solutions. Try this with the equation $\partial^2y/\partial x^2\pm\beta\partial^2y^2/\partial x^2=1/v^2\partial^2y/\partial t^2$. Is the sum of two solutions also a solution? Note that this means superposition does not work for nonlinear systems; we cannot construct a Fourier series of sines and cosines in order to make a wave pulse.

Exercise 4.8.9

It is also the case that trigonometric functions (sine, cosine and exponential) are generally not solutions to nonlinear equations. What happens to a sine wave initial condition over time with a small amount of nonlinearity?

Exercise 4.8.10

As further evidence that trigonometric functions are not solutions, try $A\exp(i(kx-\omega t))$ as a possible solution to the nonlinear equation by substituting it into the equation in question $4.8.7$. Is it a solution? Explain. Is it a solution to the equation with friction but no nonlinear term? What about the dissipation and dispersion cases?

Although an exponential (plane) wave is not a solution to the nonlinear equation you should have been able to arrive at this expression in the previous question: $\omega^2=k^2v^2\pm4\beta k^2A\exp(i(kx-\omega t))$. We can get an approximate dispersion relation from this expression if we use the Taylor series expansion for the exponent: $\exp\theta\approx(1+\theta+\theta^2/2!+\ldots)$ and keep only the first term. this gives us the aproximate **dispersion relation** for this equation as $\omega=k(v^2\pm4\beta A)^{1/2}$.

Exercise 4.8.11

The group velocity of a wave packet is given by $v_{\text{group}} = \partial\omega(k)/\partial k$. Find an expression for the group velocity from the approximate expression of the dispersion relation for a nonlinear string.

Exercise 4.8.12

Notice that the group and phase velocity are dependent on the amplitude, A. For the plus sign in the dispersion relation, should taller waves travel faster or slower than short waves? What about for the minus sign?

Exercise 4.8.13

Because taller waves travel faster (for the plus sign case), a collection of waves made of several different frequencies will gradually pile up as the taller waves catch up with the slower, lower amplitude waves. Repeat question 4.7.6 and comment on what you see.

Note

For this simple nonlinear force the simulation cannot represent a true breaking wave like at the beach. It is also the case that the numerical calculations will fail once the wave gets steep. But it is nonlinearity that causes waves to break. Water waves interact with the ocean floor as they move into shallow water. These interactions are nonlinear and cause a wave packet to steepen and then break as the taller waves move faster than small amplitude waves.