1.5: Simple Harmonic Motion and Resonance

Tutorial 1.5: Simple Harmonic Motion and Resonance

The following simulation shows a driven, damped harmonic oscillator; a $1\ \mathrm{kg}$ mass on a spring with spring constant $2\ \mathrm{N/m}$. The amplitude of the motion is graphed versus time. The initial position of the mass, x_0, can be adjusted by dragging the mass to a starting position. The driving frequency, f, can be adjusted so we expect that for one particular frequency we will see the amplitude of the motion to be very large; in other words, resonance will occur.

Several other parameters can also be adjusted. b is the amount of friction in $\mathrm{Ns/m}$ (this could be air resistance or sliding friction or friction in the spring itself); v_0 in $\mathrm{m/s}$ is the initial velocity of the mass, and F_0 is the magnitude of the driving force in newtons.

In this (and many other simulations we will use) it is easier to write $\omega = 2\pi f$ where ω is the **angular frequency** in radians per second instead of having to write $2\pi f$ everywhere.

Simple Harmonic Motion II

Questions:

Exercise 1.5.1

Start with the default parameters, drag the ball to the maximum starting position ($10\ \mathrm{m}$) and hit play. How does this motion compare with simple harmonic motion (in the last chapter)?

Exercise 1.5.2

Reset the simulation, change the friction parameter, b to $0.5\ \mathrm{Ns/m}$, drag the ball to the maximum starting position ($10\ \mathrm{m}$) and hit play. (You can also click the button at the top for under damped motion.) What happens? This motion is called **damped harmonic motion**.

Exercise 1.5.3

Try different values for b. How is the behavior of the mass for small values of b (less than one) different than for values larger than $2\ \mathrm{Ns/m}$ (be sure to use the same starting position each time)?

If the mass oscillates at least once before stopping the damping is called **under damped** motion. If the mass never quite gets back to the equilibrium position the motion is called **over damped**. The case where there is just enough damping so that an oscillation does not occur (the mass just barely makes it back to equilibrium) is called **critically damped** motion.

Exercise 1.5.4

Click the buttons on the top for under, over and critically damped motion. Describe the differences between these three cases.

Exercise 1.5.5

Reset the simulation, set the friction parameter, b to $1.0\ \mathrm{Ns/m}$ and the magnitude of the driving force, F_0 to $1.0\ \mathrm{N}$. The angular frequency should be $1.0\ \mathrm{rad/s}$ also. Drag the ball to the maximum starting position and hit play. In this and the next few questions you will need to wait $10\ \mathrm{s}$ or so for the motion to stabilize. Describe the stable motion after the initial oscillations for this case. This motion is called **driven, damped harmonic motion**.

Exercise 1.5.6

Experiment with different amounts of force, keeping friction and angular frequency equal to one. Also start from the same position each time. What is the effect of larger values of force amplitude, F_0 on the final, stable motion of the mass? HINT: Use a screen shot of the graph to compare results.

Exercise 1.5.7

With damping, b set to $0.2\ \mathrm{Ns/m}$ and F_0 set to $1.0\ \mathrm{N}$ try several values of the angular driving frequency, ω. Start with a value of $1.1\ \mathrm{rad/s}$ and increase by 0.1 each time until you get to $1.8\ \mathrm{rad/s}$. In each case, wait until the animation ends and measure the amplitude by clicking on the top of the curve near the end. The second number in the yellow box is the amplitude in m. Write down the amplitude for each driving frequency. Which driving frequency ended up giving the largest amplitude?

Resonance is defined to occur when a vibrating system is driven with a frequency that causes the largest amplitude. Generally this occurs when the driving frequency equals the natural frequency. An example of resonance is when a wine glass is driven by sound waves with a frequency equal to the natural frequency of the glass. If the amplitude becomes large enough the glass will break. There are many other examples of resonance. Musical instruments depend on the phenomena of resonance to produce fixed pitches, tuning a radio to a particular chanel depends on selecting a resonance frequency and Magnetic Resonance Imagining (MRI) in the medical world uses resonance to form images of the inside of the human body.

As we saw in the previous chapter the **natural frequency**, written as f_0 is given by the stiffness of the spring, κ, and the mass; $f_0 = (\kappa/m)^{1/2}/2\pi$. In this simulation the mass is $1\ \mathrm{kg}$ and the spring constant is $2\ \mathrm{N/m}$ so $f_0 = 0.225\ \mathrm{Hz}$ and the natural angular frequency, $\omega_0 = 2\pi f$ equals $1.41\ \mathrm{rad/s}$. In the previous question you should have seen the maximum amplitude for a driving frequency of $1.41\ \mathrm{rad/s}$. In other words a driving frequency, ω of $1.41\ \mathrm{rad/s}$ leads to resonance (maximum amplitude) because it equals the natural frequency, ω_0.

Exercise 1.5.8

Use a calculator to find the natural frequency for a spring and mass system with $m = 2.0\ \mathrm{kg}$ and $\kappa = 5.0\ \mathrm{N/m}$. What do you expect the resonance frequency to be for this case?

The mathematical formula that describes damped harmonic motion is $Ae^{-\gamma t}\cos(\omega t + \varphi)$ where $\gamma = b/2m$. Notice that this is the same cosine function for simple harmonic motion but the amplitude, A, is multiplied by an exponentially decreasing function of time, $e^{-\gamma t}$. So we expect the oscillation of a damped harmonic oscillator to be an up and down cosine function with an amplitude that decreases over time.

Exercise 1.5.9

Check to see if the formula $Ae^{-\gamma t}\cos(\omega t + \varphi)$ really does describe the behavior of damped harmonic motion in the simulation. To do this, use a graphing calculator (or go to meta calculator and chose graphing calculator or use desmos calculator) and plot $y = 10 * \exp(-.2 * x) * \cos(1. * x)$. This is the case $A = 10\ \mathrm{m}$; $b = 0.4\ \mathrm{Ns/m}$; $m = 1.0\ \mathrm{kg}$; and $\omega = 1.0\ \mathrm{rad/s}$. (You can cut and paste the equation into the online calculator). How does this graph compare with the simulation for these same parameters (note: you are only interested in the positive x values)?

The mathematical formula that describes driven, damped harmonic motion is $A_0\cos(\omega t + \varphi)$ where $A_0 = (F_0/m)/((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2)^{1/2}$. In this case the amplitude, A_0 does not change over time but it is dependent on the driving frequency, ω.

Exercise 1.5.10

What happens to the amplitude, A_0, when $\omega = \omega_0$ (assume all the other factors are constant numbers)? Is there any other combination of ω and ω_0 that gives a larger amplitude?

Exercise 1.5.11

Verify your previous answer by making a plot of $A_0 = (F_0/m)/((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2)^{1/2}$ for a range of driving frequency. To do this, use a graphing calculator to graph $y = 1.0/((1.4 \wedge 2 - x \wedge 2) \wedge 2 + 4 * .04 * 1.4 \wedge 2) \wedge .5$. Where does this graph have a maximum for positive values of x?

Exercise 1.5.12

Based on the previous two questions, what is the resonance frequency of this system and how does this compare with the natural frequency?