4.6: Wave Equation

Tutorial 4.6: The Linear Wave Equation

In mechanics the subject of kinematics is the description of the motion of objects (velocity, acceleration, etc.) while the subject of dynamics describes the physical situation and the forces that give rise to the motion (Newton's laws). So far we have looked at the kinematics of waves (descriptions of their shapes and how they move). In this simulation we look at the dynamics of waves; the physical situations and laws give rise to waves.

We start with a string that has a standing wave on it and look at the forces acting on each end of a small segment of the string due to the neighboring sections. For visualization purposes the string is shown as a series of masses but the physical system is a continuous string. Although the derivation is for a string, similar results occur in many other systems. The ends of the section of string we are interested in are marked with red dots in the simulation. The tension acting on each end is shown with a vector (in red) and its components (green and blue). The horizontal forces cancel (the string segment does not move to the left or right) but there is a net force in the vertical direction (the left and right vertical components are not the same). Remember from simulation four that there is a transverse acceleration (and therefore a transverse force) that changes over time at each point on the string.

Forces on a Segment

Questions:

Exercise 4.6.1

Before playing the simulation, use the protractor and measure the angle of the tension at each end of the segment (the protractor can be dragged and also the arrow tip can be moved). Are they the same angles?

Exercise 4.6.2

Play the simulation. Based on what you know about the acceleration of a point on a sine wave, explain why the magnitude and direction of the net force on the segment acts the way it does. (Hint: Review simulation four, question seven.)

The tension on segment is T_1 from the neigboring string on the left and T_2 from the string to the right. If θ_1 is the angle the tension makes with the horizontal on the left end of the segment and θ_2 is the angle with the horizontal on the right end then the total y force is $F=-T_1\sin\theta_1+T_2\sin\theta_2$.

If the wave is of low amplitude the angles are small and we have $\sin\theta\approx\tan\theta$ which is the slope of the string at that point. Slope is given by the derivative of the position, $\tan\theta=\partial y/\partial x$, so we can replace the total force on the segment by $F=T(-\partial y/\partial x_1+\partial y/\partial x_2)$. The segment has a length of Δx. Multiply and divide by this to get $F=T\Delta x(-\partial y/\partial x_1+\partial y/\partial x_2)/\Delta x$. If we now let Δx become very small this reduces to a second derivative: $F=T\Delta x\partial^2 y/\partial x^2$.

We also know from $F=ma$ the force is proportional to acceleration, $a=\partial^2 y/\partial t^2$. For a mass per length given by μ the mass of the segment is $\mu\Delta x$ and we have $F=\Delta x\mu\partial^2 y/\partial t^2$.

Setting these two equations for force equal to each other we have the wave equation for a string: $T\partial^2 y/\partial x^2=\mu\partial^2 y/\partial t^2$ (the Δx cancels). Recall from simulation two that $v=(T/\mu)^{1/2}$ for waves on a string so we can also write the **linear wave equation** as $\partial^2 y/\partial x^2=1/v^2\partial^2 y/\partial t^2$. Although we started with and string and applied $F=ma$, this equations turns up in many other physical situations. The same equation holds for sound waves in gas, liquids and solids and for electromagnetic waves; only the velocity is different, as noted in simulation three.

Exercise 4.6.3

Verify that $y(x,t)=A\sin(kx-\omega t)$ is a solution to the linear wave equation. Do this by taking two x derivatives (the left side of the equation) and two time derivatives (the right side of the equation) and substituting your answers for the terms in the linear wave equation. Cancel similar terms on both sides. You should be able to show that, in order for $y(x,t)=A\sin(kx-\omega t)$ to be a solution you must have $v=\omega/k$.

Exercise 4.6.4

Verify that $A\exp(-(kx-\omega t)^2)$ is a solution to the linear wave equation (this is the Gaussian wave pulse in simulation 2.4). Do this by taking two x derivatives (the left side of the equation) and two time derivatives (the right side of the equation) and substituting your answers for the terms in the linear wave equation. Cancel similar terms on both sides. You should be able to again show that $v=\omega/k$ if this is a solution to the linear wave equation.

In several simulations we added waves together. This means that if $y_1(x,t)$ is a solution and $y_2(x,t)$ is a solution we have assumed that $y(x,t)=y_1(x,t)+y_2(x,t)$ is also a solution to the linear wave equation.

Exercise 4.6.5

Prove the previous statement is true by substituting $y(x,t)=y_1(x,t)+y_2(x,t)$ into the linear wave equation. Separate out terms to find two wave equations, one for $y_1(x,t)$ and a second for $y_2(x,t)$ which are equal to each other. This proves the law of superposition; when two waves arrive at the same point at the same time we can simply add their amplitudes to find the resulting wave.