3.9: Dispersion of Fourier Components

Tutorial 3.9: Dispersion II

In simulation 2.5 on the Fourier series we found that complicated periodic wave forms can always be constructed from sine and/or cosine functions of different frequencies and wavelengths. In the previous simulation we found that different wavelengths may travel at different speeds depending on the medium. So what happens to a complex wave shape if it travels through a medium where the individual components have different speeds?

We can write a Fourier series for a square wave moving in time and space as $y(x,t) = \sum_{n=1} A_n \sin(nkx - n\omega t)$ where n is the number of the harmonic or mode ($n = 1$ for the fundamental, 2 for the second harmonic etc.), A_n is the amplitude of harmonic n, k is the wave vector and ω is the angular frequency. Recall from simulation 2.3 that the group velocity of a combination of waves is $v_{\text{group}} = \partial\omega(k)/\partial k$. The dependency of ω on k is called the **dispersion relation**. In a vacuum or medium with no dispersion we expect each component of the series to have the the same speed, $v = \omega/k$, so the dispersion relation is $\omega(k) = kv$ and the group velocity is v, the same as the individual components. In this case the square wave would not change shape as it travels.

In real life, however, it is often the case that the angular frequency, $\omega(k)$, is *not* a linear function of the wave vector, k in which case the individual components of the Fourier series travel at different speeds. If different frequencies of a wave travel at different speeds the effect is called **dispersion**. As we saw in the previous simulation, dispersion causes the separation of colors by prisms, water droplets, etc. In this simulation we explore a different aspect of dispersion.

Dispersion

Questions:

Exercise 3.9.1

The simulation starts with the first four components of the Fourier series for a traveling square wave with no dispersion. Play the simulation and describe what happens to the shape as time goes on.

Exercise 3.9.2

Given that the speed of a sine wave is $v = \omega/k$, what are the speeds of the first four components of the square wave: $y(x,t) = \sin(1*x - 1*t) + \sin(3*x - 3*t)/3 + \sin(5*x - 5*t)/5 + \sin(7*x - 7*t)/7$

Exercise 3.9.3

What would be the fifth term in the Fourier series of a square wave? Add your answer to the first four terms and see if the shape is closer to a square wave. It would require an infinite number of terms to create a perfect square wave but we can get as close as we like by adding as many terms as necessary.

Exercise 3.9.4

Click 'reset' and then change the angular frequency of the second term from 3 to 2.95 and hit enter. This will cause the second term to have a slightly different speed. What is this new speed for the second term? How does the initial shape compare with the initial shape in question $3.9.1$ (if you reset before entering the change they should be identical)?

Exercise 3.9.5

Now play the simulation for the wave in the previous question. What happens to the shape of the square wave in this case as time goes on?

Exercise 3.9.6

Reset to the original case and change the angular frequency of the third term from 5 to 4.95. What effect does this have on the behavior of the wave?

Exercise 3.9.7

Based on the previous two questions, explain what would happen to a digital signal (which is basically a series of square waves) traveling down a cable (either wires or optical fiber) where there is a small amount of dispersion.

Exercise 3.9.8

All cables (fiber optical or metal) have some dispersion. Why is there a limit to how long a cable can be before a signal traveling on it has to pass through a relay (where the signal is amplified and 'cleaned up')?