# 4.7: Oscillator Chain

## Tutorial 4.7: Chains of Oscillating Masses

In this simulation we examine waves that occur on chains of masses with mass $M$ coupled together with elastic, Hooke's law forces ($f = -\kappa x$ where $\kappa$ is the spring constant and $x$ is the amount the spring stretches). The masses are constrained to only move up and down so that the stretching depends only on the difference in the $y$ locations of the masses. In this case the force on mass number $i$ due to its neighbors at $i+1$ and $i-1$ is $F_i = -\kappa[(u_{i+1} - y_i) - (y_1 - y_{i-1})]$ . The masses on each end of the chain are fixed and there is a small amount of friction in the system so that eventually the oscillations will die off.

As a first approximation, atoms in a solid can be imagined to be coupled by spring-like forces to their neighbors so this simulation models a one dimensional solid. In the limit as the distance between the masses becomes very small this model turns into the model of an elastic string.

The simulation opens a sine wave with $32$ masses. The number of masses can be changed and you can also grab a single mass, move it to a new location and start the simulation with this new configuration. The other buttons set up initial conditions for different numbers of masses. Each of these special cases are an example of a **Normal Mode** of the system.

## Coupled Oscillator Chain

### Questions:

#### Exercise 4.7.1

Try the various pre-set configurations using the buttons below the simulation. How many normal modes are available for three masses (one moving mass) on the chain? For four masses (two in motion)? For five masses?

#### Exercise 4.7.2

Use the pause and step buttons to measure the frequency of each mode. What are they? Are they the same?

#### Exercise 4.7.3

Sketch the possible modes for four moving masses (six masses total). How many modes are there?

A normal mode is a special configuration (state) where every particle moves sinusoidally with the same angular frequency $\omega_m$ where $m$ is an integer. The $m$-th mode, $\Phi_m$, of the oscillator chain of length $L$ with $N$ masses is given by $\Phi_m(x,t) = \sin(m\pi x/L)\cos(\omega_m t + \varphi)$. The angular frequency of each particle is given by $\omega_m^2 = (4\kappa/M)\sin^2(m\pi/2N)$.

It turns out that any possible type of vibration can be described mathematically by a sum of the normal modes with appropriate amplitudes. This is equivalent to the statement we investigated in simulations 2.5 and 3.9; complicated periodic waves can always be described by a Fourier series of sines and cosines. The difference for masses on a string is that there are only a finite number of modes available. In a continuous system there are an infinite number of modes.

#### Exercise 4.7.4

You can also grab and change the position of the masses in the simulation. Try this starting with the sine wave initial condition. Describe what you did and what you see.

In simulation 3.9 we saw the linear dispersion relation, $\omega(k) = kv$ which tells us that the angular frequency is proportional to the wave vector, $k$. If the speed of the wave $v$ is independent of frequency (i.e. there is no dispersion) then a plot of $\omega$ versus $k$ is a straight line. Since $\omega(k)$ is a continous function there is no limitation on the value of wavelengths, as long as the proportionality holds.

But on a string of masses you cannot have wavelengths that are shorter than the distance between the masses (there is nothing there to vibrate). So the dispersion relation for a mass string cannot be the same as the linear dispersion relation. As shown above, the dispersion relation for masses separated by a distance $a$ each with mass $M$ conected to its neighbor by a spring with spring constant $\kappa$ is given by $\omega(k) = 2(\kappa/M)^{1/2} \sin(ka/2)$. (Careful! $\kappa$ is the spring constant, **not** the wave vector, $k = 2\pi/\lambda$.)

### Exercise 4.7.5

Try various values of $n$ for the chain with the sine wave as initial condition. What can you conclude from your experiements? How is the wavelength limited by the number of masses?

### Exercise 4.7.6

Make a plot of $\omega(k) = kv$ and $\omega(k) = 2(\kappa/m)^{1/2} \sin(ka/2)$ versus $k$ on the same graph. Use $v = 10$, $\kappa = 2$, $m = 1$ and $a = 0.1$. How do the graphs differ? For what wavelengths (small or large) do they give about the same result? Why do they overlap for large values of wavelength?

### Exercise 4.7.7

Recall from simulation 2.3 that the group velocity of a wave packet is given by $v_{\text{group}} = \partial\omega(k)/\partial k$. Find an expression for the group velocity of a wave on a chain of masses.

### Exercise 4.7.8

Notice that the group velocity for a wavepacket traveling on a chain of masses is dependent on the wavelength. Based on what you learned about dispersion in simulation 3.9, what do you expect to happen to a wavepacket as it travels down a chain of masses connected by springs?

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