Waves: An Interactive Tutorial (Forinash and Christian)

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This text was compiled on 04/15/2025

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Acknowledgements

There are a great many people and institutions that have contributed to our efforts, and we take great pleasure in acknowledging their support and their interest.

The Open Source Physics project has benefited tremendously from collaborations with U.S. and non-U.S. universities. In particular, we give special thanks and recognition to the Easy JavaScript Simulations developers Francisco Esquembre and Felix Garcia at the University of Murcia, Spain. We would also like to thank the AAPT- ComPADRE team headed by Bruce Mason, Lyle Barbato, and Matt Riggsbee.

W.C. would like to thank his Davidson College colleague and collaborator Mario Belloni for his support and many contributions to the Open Source Physics project. He would also like to thank the numerous students who have worked with him over the years developing programs for use in undergraduate physics education. In particular, he would like to single out Drew Glassman for his work on the JavaScript adaptations of the original Java-based Physlet material.

K.F. would like to thank Indiana University Southeast for a summer fellowship which was used for the preparation of the initial version of this tutorial.

Preface

Waves: An Interactive Tutorial is a set of 33 exercises designed to teach the fundamentals of wave dynamics. It starts with very simple wave properties and ends with an examination of nonlinear wave behavior. The emphasis here is on the properties of waves which are difficult to illustrate in a static textbook figure. Simulations are not a substitute for laboratory work. However they allow for visualization of processes that cannot normally be seen (for example electric and magnetic fields). They allow for visualization of process that are too fast (for example waves) to follow in real time or too small to see (for example thermodynamics at the molecular scale). They allow manipulation of processes which might be dangerous (collisions) or hard to experiment with (waves). They also allow for easy repetition. For all of these reasons, simulations are an excellent way to introduce students to the complex phenomena of waves.

The tutorial may be used in conjunction with a text or as a stand-alone introduction to waves. Exposure to calculus and basic physics is assumed in the latter sections. The material was constructed in such a way that the student is required to actively work their way through the material instead of passively reading printed text at a low comprehension level. The objective is to replace the traditional textbook for the course (but not necessarily replace the entire course) with computer aided instruction that requires active learning on the part of the student. The goal was to create a guided text that integrates the strengths of printed, static textbooks and the interactive dynamics of the Internet to engage the student in actively learning the physics of waves.

Play the above simulation and observe the propagation of the wave along a beaded string. How can this motion be expressed mathematically? What would happen if the string is attached to a wall? What happens if the mass of the beads is increased? We will develop answers to these and related questions in this tutorial but what is as important is the mental image gained from a visualization of the phenomena.

Interactive Engagement

We have known for sometime that active teaching methods work better than lecturing in the classroom. (See R.R. Hake, "Interactive-engagement vs traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses", *Am. J. Phys*., Jan. (1998)). Many instructors now use collaborative group work, lecture demonstration, Socratic dialog and other interactive techniques in their classroom. But what about the textbook? Students now have access to electronic textbooks for sale or rent but these are still mostly static (pdf) files. All that has been done is to speed up access to the same material that existed in hard copy. Except for the introduction of color pictures, not much has changed since Guttenberg’s time.

Physics textbooks are notoriously hard to read. In Student Evaluation of Teaching (SET) score results, the perception of the physics textbook ALWAYS receives the lowest score, regardless of which text is chosen or who teaches the class. When (if?) students read the text, what do they get out of it? Do they actively engage in the subject or do they just look at the words and pictures? Some research evidence hints that, at least for physics, students don't get much from a traditional textbook but rather depend on the instructor to interpret for them because they do not know how to read a physics book. But what if the textbook required answering questions and manipulating computer simulations as the student worked through a topic? The goal of this tutorial is to explore the possibility that a textbook which required the student to be engaged would lead to better student understanding of the material; to make a better textbook, an *active learning textbook.*

In the education world the Internet and course management technology is still too often used as a one way communication tool, simply making it a bit more convenient for students in a conventional course to receive the same or similar material which previously was handed out in class. But the same technology can be used as a two way communication tool where students are actively engaged in manipulating course material, controlling both simulations and live experiments, collaborating with other students and interacting with the instructor. Many of these activities can be done in a laboratory setting as an integral part of a regularly scheduled course, performed asynchronously as homework assignments or completed as a component of a new hybrid kind of distance learning course.

CHAPTER OVERVIEW

1: Basic Properties

The fundamental properties of waves, such as frequency, amplitude, wavelength, and motion direction, are quite different from those of rigid objects studied in mechanics. These tutorials explore the basics of wave motion.

1.1: Sine Wave

Tutorial 1.1: Sine Waves

Imagine a perfect, smooth wave out on the ocean far enough from shore so that it has not started to break (complications involved in describing real waves will be discussed later in this tutorial). If we take a snapshot picture of this wave at a single instant in time and measure the distance in meters from one peak to the next we have measured the **wavelength**, λ of the wave.

If instead we watch a floating cork at a single location in space and measure the time in seconds between arriving peaks we have measured the **period**, T of the wave. We could also measure the number of times the cork bobs up, down and back up per second which would be the **frequency** in hertz or cycles per second. The period and frequency are inverses of each other: $f = 1/T$.

The height of the wave at any location and time, measured from the middle, or equilibrium position is the **displacement**. The maximum displacement is called the **amplitude**.

As a first approximation, water waves, electromagnetic waves and many other kinds of waves can be modeled by the mathematical functions sine or cosine or some combination of them. For a wave traveling along the x axis the mathematical description of the displacement of a wave at location x and time t can be written as $y(x,t) = A\sin(kx - \omega t + \varphi)$ where A is the **amplitude** (maximum height measured from the middle of the wave). Here we have used the **wavenumber** $k = 2\pi/\lambda$ (measured in radians per meter), the **angular frequency**, $\omega = 2\pi f$ (measured in radians per sec), and φ, the **phase** angle (in radians) which are often easier to use mathematically.

In this and all of the following exercises the angles will be in radians (Hint: Set your calculator to radian mode to avoid problems later on!)

With the following physlet you can explore different values of amplitude, wavenumber, wavelength, angular frequency, frequency, period and phase. Make any changes you wish in the $y(x,t) =$ window, hit enter or return and then the play button, ▸, to see the wave in action. The initial values are amplitude, $A = 1.2\text{ m}$; wavenumber, $k = 2.0\text{ rad/m}$; angular frequency, $\omega = 0.8\text{ rad/s}$; and phase angle, $\varphi = 10.0\text{ radians}$. Clicking in on the graph shows the coordinates of the cursor in a yellow box in the lower left corner.

Sine Wave Simulation

Questions:

Exercise 1.1.1

Stop the simulation, double the maximum amplitude (from 1.2 m to 2.4 m) enter and play. What effect does maximum amplitude, A, have on the wave?

Exercise 1.1.2

Stop the simulation, double the wave number, k, (from 2.0 rad/m to 4.0 rad/m) enter and play. What effect does the wave number have on the wave?

Exercise 1.1.3

Stop the simulation, double the angular frequency, ω, (from 0.8 rad/s to 1.6 rad/s) enter and play. What effect does the angular frequency have on the wave?

Exercise 1.1.4

Stop the simulation, double the phase, φ, (from 10.0 to 20.0 radians) enter and play. Try several different values for the phase. What effect does the phase have on the wave?

Exercise 1.1.5

Go back to the original wave by clicking the reload button, . Pause the wave and measure the wavelength, λ, on the graph (find the x location of two successive peaks or troughs using the cursor; the wavelength is the x distance between peaks or troughs). Calculate the wavenumber, k, from this wavelength. How does your value for wavenumber compare with the wavenumber in the equation?

Exercise 1.1.6

Now start the original wave in motion with the play button. Use the time numbers in the lower panel to find the period, T, of the wave (the time from when one peak passes a point until the next peak passes the same point). To get an accurate number you can use the step buttons. From the period you measure, calculate the angular frequency, ω. How does your value for angular frequency compare with the angular frequency in the equation?

Exercise 1.1.7

Go back to the original wave using the reload button. Change the minus sign in the equation between kx and ωt to a plus sign and click 'play'. What does changing this sign do to the wave?

Exercise 1.1.8

Now change the plus sign in front of the phase to a minus sign, enter and play. Try several values of phase (you may want to use the pause button to be sure you can tell what is happening). What does changing this sign do to the wave?

Exercise 1.1.9

In general the power transmitted by a wave, measured in watts, is proportional to the amplitude squared. What happens to the power if the amplitude is doubled? What happens to the power if the amplitude is cut in half?

1.2: Speed of a Wave

Tutorial 1.2: Speed of a Wave

There are three different velocities involved with describing a wave, one of which will be introduced here. The **velocity** of the wave, v, is a constant determined by the properties of the medium in which the wave is moving. The velocity is a vector which gives the forward **speed** of the wave and the direction the wave is traveling. For now we will not worry about direction since the waves being discussed will all be assumed to travel along the x-axis. The speed of a sine wave is given by $v = \lambda/T$ in meters per second where wavelength and period for a sine wave were defined in the previous exercise.

In this simulation the original wave will remain in the window so that as you make changes to $f(x,t)$ you can see how the new wave (in red) compares to the original ($g(x,t)$, in blue).

Wave Speed Simulation

Questions:

Exercise 1.2.1

Determine the speed of the wave in the simulation using $v = \lambda/T$ where wavelength and period are determined from the simulation as you did in the previous exercise using the mouse to find the wavelength and the time to find the period. What is the forward speed of this wave?

Exercise 1.2.2

The speed of this wave is also given mathematically by $v = \omega/k$ since $\omega = 2\pi f = 2\pi/T$ and $k = 2\pi/\lambda$. What is the speed of this wave based on the values of ω and k in the equation? Does this match the speed you got from the simulation?

Exercise 1.2.3

Reload the initial conditions and experiment with values of the wavenumber both smaller and larger than $2.0\ \text{rad/m}$ keeping the angular frequency fixed. How does the wavenumber change the speed of the wave?

Exercise 1.2.4

Reload the initial conditions and experiment with values of the angular frequency both smaller and larger than $0.8\ \text{rad/s}$ keeping the wavenumber fixed. How does the angular frequency change the speed of the wave?

This simulation is *misleading* in one important way. In the simulation you can set any combination of angular frequency and wavenumber you choose and so have any speed you want for the wave. But for mechanical and acoustic waves the speed is determined by the medium in which the wave travels. As we will see, for these waves it is often the case that $v = \omega/k$ so that the angular frequency and wavenumber are inversely proportional with v a constant. Examples:

- For sound waves in a fluid (for example air) the speed is determined by $v = (B/\rho)^{1/2}$ where B is the bulk modulus or compressibility of the fluid in newtons per meter squared and ρ is the density in kilograms per cubic meter.
- For waves in a solid the speed is determined by $v = (Y/\rho)^{1/2}$ where Y is Young's modulus or stiffness in newtons per meter squared and and ρ is the density in kilograms per meter cubed.
- For waves on a string the speed is determined by $v = (T/\mu)^{1/2}$ where T is the tension in the string in newtons and μ is the mass per length in kilograms per meter.
- Although electromagnetic waves do not need a medium to travel (they can travel through a vacuum) their speed in a vacuum, $c = (1/\mu_0\varepsilon_0)^{1/2}$ is governed by two physical constants, the permeability μ_0 and the permittivity, ε_0.
- If an electromagnetic wave enters a medium (such as light going through air, or water or glass) the speed changes because the permittivity and/or permeability may be different. The amount the speed changes is given by the **index of refraction** $n = c/v$ where c is the speed of light and v is the speed in the medium.

As we will see later, it is the case that speed can sometimes depend on the frequency of the wave, a phenomenon know as dispersion.

Exercise 1.2.5

Does sound travel slightly faster on a hot day or a cool day? Does sound travel faster or slower if the humidity is high?

Exercise 1.2.6

Density is relatively easy to measure. What would be a clever way to measure Young's modulus, Y for a solid?

Exercise 1.2.7

Do waves on a string travel faster or slower if the string is tighter? Do waves on a string travel faster or slower if the string is thicker?

Exercise 1.2.8

In general the index of refraction is larger than one. What does this tell you about the speed of light in glass?

Exercise 1.2.9

Reload the initial conditions with the 'reload' button. For a wavenumber of $4.0\ \mathrm{rad/m}$ experiment to find the correct angular frequency which gives the original speed of the wave you found in Exercises $1.2.1$ and $1.2.2$ (you should be able to see from the simulation when the new wave is traveling at the same speed as the original).

Exercise 1.2.10

Calculate the wavenumber which gives the speed of the original wave for angular frequencies of $0.4,\ 0.6,\ 1.0$,and $1.2\ \mathrm{rad/s}$ using the relationship in Exercise $1.2.2$. Check your answers with the simulation if you are in doubt.

1.3: Transverse Waves

Tutorial 1.3: Transverse Waves

Transverse waves are the kind of wave you usually think of when you think of a wave. The motion of the material constituting the wave is up and down so that as the wave moves forward the material moves perpendicular (or **transverse**) to the direction the wave moves. Examples of transverse waves include waves on a string and electromagnetic waves. Water waves can be approximately transverse in some cases.

The following simulation shows a graph of the motion of one location, the red circle, on a string which has a transverse wave on it. The vertical location of points on the string (represented by the circles) as a function of horizontal location along the x-axis and time is again described mathematically by $y(x,t) = A\sin(kx-\omega t+\varphi)$. Notice that, while the wave moves forward along the string, the red circle does not (in fact none of the circles move forward).

Transverse Wave

Questions:

Exercise 1.3.1

Play the animation (lower left button). From the graph, what are the amplitude and period of the motion of the red dot?

Exercise 1.3.2

Clicking on the lower panel gives the mouse location (in the yellow box) which in this case are the x and y location of points on the wave. Use these numbers to determine the wavelength of the wave (this is easiest to do with the animation paused or finished).

Exercise 1.3.3

From the period and wavelength that you just measured, calculate the forward speed of the wave (as you did in the previous simulation).

A *second* velocity associated with a wave is how fast the material of the wave moves up and down at a single location (the vertical speed of the circles in the simulation). This velocity, the **transverse velocity**, is not a constant but is a function of location and time (different places on the wave move upward or downward at different speeds at different times). Since velocity is the rate of change of position, this second speed (in the *y* direction) is given by the derivative of the amplitude with respect to time:

$$v(x,t) = \partial y(x,t)/\partial t = -A\omega\cos(kx-\omega t+\varphi)$$

Notice that the maximum speed of a section of the wave at location x and time t will be given by $v_{\max} = A\omega$. We use a partial derivative here because $y(x,t)$ is a function of two variables.

Exercise 1.3.4

Click on the 'Velocity' button and then 'play'. The upper graph now gives the speed of the red circle in the y-direction as a function of time. What is the maximum speed (approximately) of the red circle according to the graph? How does this compare with the speed of the wave which you found in $1.3.3$; are they the same or different?

Exercise 1.3.5

Where is the red dot (relative to the rest position before the wave passes) when the maximum transverse velocity occurs? Where is it when the transverse velocity is approximately zero?

Exercise 1.3.6

Using $v_{\max}$ from the graph, the amplitude from $1.3.1$ and $v_{\max} = A\omega$, what is the angular frequency? How does this compare with the value calculated from the period?

Since points on the wave change their transverse velocity over time there must also be a vertical or **transverse acceleration**. Since acceleration is the time rate of change of velocity we have $a(x,t) = \partial v(x,t)/\partial t = -A\omega^2 \sin(kx - \omega t + \varphi)$ where the **maximum acceleration** is $a_{\max} = A\omega^2$.

Exercise 1.3.7

Calculate the maximum acceleration of the red circle. What are the units of this acceleration if amplitude is in meters and angular frequency is in radians per second?

Exercise 1.3.8

Based on the equation for acceleration, where will the red circle be when the acceleration is a maximum? Where will it be when the acceleration is approximately zero?

Exercise 1.3.9

Carefully state the relationship between positon, velocity and acceleration. When the position is zero (equilibrium position of the red dot) what are the velocity and acceleration? When the position is a maximum, what are the velocity and acceleration?

Exercise 1.3.10

State in your own words the difference between wave speed and transverse speed of a wave.

1.4: Simple Harmonic Motion

Tutorial 1.4: Simple Harmonic Motion I

The following is a simulation of a mass on a spring. The graph shows the y location of the mass. The force acting on the mass in this case is called a **Hooke's Law** force: $F=-\kappa y$ where κ is called the **spring constant**, in $\mathrm{N/m}$ indicating the stiffness of the spring and y is the location of the mass from some equilibrium position. (Careful! κ is **not** the same as the wavenumber, $k=2\pi/\lambda$.)

Simple Harmonic Motion

Questions:

Exercise 1.4.1

Drag the mass to some initial location using the mouse. Click 'play' button (lower left) to see the motion and the graph of the mass location. How is the graph you see here similar to the graph you saw for the motion of the red circle in the transverse wave case?

Exercise 1.4.2

Clicking on the graph shows the coordinates of the mouse in a yellow box at the lower left. Determine the period, frequency and angular frequency of this motion from the values on the graph (Hint: Frequency in Hz is the inverse of period; angular frequency is $\omega=2\pi f$).

Exercise 1.4.3

Check the 'show velocity' box to see graphs of both position and velocity. Where is the mass when the velocity is a maximum? Where is the mass when the velocity becomes zero?

Exercise 1.4.4

Try different spring constant values, κ, between $0.5\ \mathrm{N/m}$ to $5.0\ \mathrm{N/m}$, reseting and releasing the mass at the same point each time. What is the relationship between spring constant and frequency?

Exercise 1.4.5

For the same spring constant try releasing the ball at several different amplitudes. What is the relationship between amplitude and frequency?

For a given spring the frequency is determined by the mass hanging on it and the stiffness of the spring; $f=(\kappa/m)^{1/2}/2\pi$.

Exercise 1.4.6

Measure the frequency for the case of a spring constant, κ equal to 2.0. What must be the mass on the end of this spring? Verify that you get the same mass by measuring the fequency for several different spring constants (this is equivalent to hanging the same mass on several different springs).

Exercise 1.4.7

What happens to the period of oscillation of a spring-mass system if the mass is doubled?

In the previous simulation (1.3: Transverse Waves) the red circle moved up and down as the result of a transverse wave traveling horizontally along the string of particles. The equation for the motion of the entire string is $y(x,t)=A\sin(kx-\omega t+\varphi)$. If we were to assume that the red particle was located at $x=0$ the equation describing the motion of just the red circle would be

$y(t) = A\sin(-\omega t + \varphi)$ where A is the maximum amplitude, ω is the angular velocity and φ is the phase. This is also the formula for **simple harmonic motion** which describes the location of a mass on a spring as a function of time. In other words the motion of each point on a transverse wave is exactly the same as if each of those points was undergoing simple harmonic motion but with a slightly different phase from its neighbor.

Exercise 1.4.8

Using $y(t) = A\sin(-\omega t + \varphi)$ and a calculator, what is the location of the mass when $t = 0$ and $\varphi = 0$? What is the location of the mass when $t = 0$ and $\varphi = \pi$ (don't forget we are in radian mode)? Explain what the phase angle, φ tells you about the intial position ($t = 0$) of the mass on the spring.

Exercise 1.4.9

If $y(t) = A\sin(-\omega t + \varphi)$ is the location of the mass on the spring and the time derivative of location is velocity, then the velocity of the mass is given by $v(t) = A\omega\cos(-\omega t + \varphi)$. Check the 'show velocity' box and run a simulation with an initial amplitude of 6.0 m and a spring constant of 2. From the graph find the angular frequency and calculate the maximum amplitude $v_{\text{max}} = A\omega$. How does this number compare with the maximum value on the velocity graph; are they the same?

Exercise 1.4.10

Write down an expression for the acceleration of a mass on a spring, based on the position given by $y(t) = A\sin(-\omega t + \varphi)$ and the fact that acceleration is the second derivative of position with respect to time.

1.5: Simple Harmonic Motion and Resonance

Tutorial 1.5: Simple Harmonic Motion and Resonance

The following simulation shows a driven, damped harmonic oscillator; a $1\ \mathrm{kg}$ mass on a spring with spring constant $2\ \mathrm{N/m}$. The amplitude of the motion is graphed versus time. The initial position of the mass, x_0, can be adjusted by dragging the mass to a starting position. The driving frequency, f, can be adjusted so we expect that for one particular frequency we will see the amplitude of the motion to be very large; in other words, resonance will occur.

Several other parameters can also be adjusted. b is the amount of friction in $\mathrm{Ns/m}$ (this could be air resistance or sliding friction or friction in the spring itself); v_0 in $\mathrm{m/s}$ is the initial velocity of the mass, and F_0 is the magnitude of the driving force in newtons.

In this (and many other simulations we will use) it is easier to write $\omega = 2\pi f$ where ω is the **angular frequency** in radians per second instead of having to write $2\pi f$ everywhere.

Simple Harmonic Motion II

Questions:

Exercise 1.5.1

Start with the default parameters, drag the ball to the maximum starting position ($10\ \mathrm{m}$) and hit play. How does this motion compare with simple harmonic motion (in the last chapter)?

Exercise 1.5.2

Reset the simulation, change the friction parameter, b to $0.5\ \mathrm{Ns/m}$, drag the ball to the maximum starting position ($10\ \mathrm{m}$) and hit play. (You can also click the button at the top for under damped motion.) What happens? This motion is called **damped harmonic motion**.

Exercise 1.5.3

Try different values for b. How is the behavior of the mass for small values of b (less than one) different than for values larger than $2\ \mathrm{Ns/m}$ (be sure to use the same starting position each time)?

If the mass oscillates at least once before stopping the damping is called **under damped** motion. If the mass never quite gets back to the equilibrium position the motion is called **over damped**. The case where there is just enough damping so that an oscillation does not occur (the mass just barely makes it back to equilibrium) is called **critically damped** motion.

Exercise 1.5.4

Click the buttons on the top for under, over and critically damped motion. Describe the differences between these three cases.

Exercise 1.5.5

Reset the simulation, set the friction parameter, b to $1.0\ \mathrm{Ns/m}$ and the magnitude of the driving force, F_0 to $1.0\ \mathrm{N}$. The angular frequency should be $1.0\ \mathrm{rad/s}$ also. Drag the ball to the maximum starting position and hit play. In this and the next few questions you will need to wait $10\ \mathrm{s}$ or so for the motion to stabilize. Describe the stable motion after the initial oscillations for this case. This motion is called **driven, damped harmonic motion**.

Exercise 1.5.6

Experiment with different amounts of force, keeping friction and angular frequency equal to one. Also start from the same position each time. What is the effect of larger values of force amplitude, F_0 on the final, stable motion of the mass? HINT: Use a screen shot of the graph to compare results.

Exercise 1.5.7

With damping, b set to $0.2\ \mathrm{Ns/m}$ and F_0 set to $1.0\ \mathrm{N}$ try several values of the angular driving frequency, ω. Start with a value of $1.1\ \mathrm{rad/s}$ and increase by 0.1 each time until you get to $1.8\ \mathrm{rad/s}$. In each case, wait until the animation ends and measure the amplitude by clicking on the top of the curve near the end. The second number in the yellow box is the amplitude in m. Write down the amplitude for each driving frequency. Which driving frequency ended up giving the largest amplitude?

Resonance is defined to occur when a vibrating system is driven with a frequency that causes the largest amplitude. Generally this occurs when the driving frequency equals the natural frequency. An example of resonance is when a wine glass is driven by sound waves with a frequency equal to the natural frequency of the glass. If the amplitude becomes large enough the glass will break. There are many other examples of resonance. Musical instruments depend on the phenomena of resonance to produce fixed pitches, tuning a radio to a particular chanel depends on selecting a resonance frequency and Magnetic Resonance Imagining (MRI) in the medical world uses resonance to form images of the inside of the human body.

As we saw in the previous chapter the **natural frequency**, written as f_0 is given by the stiffness of the spring, κ, and the mass; $f_0 = (\kappa/m)^{1/2}/2\pi$. In this simulation the mass is $1\ \mathrm{kg}$ and the spring constant is $2\ \mathrm{N/m}$ so $f_0 = 0.225\ \mathrm{Hz}$ and the natural angular frequency, $\omega_0 = 2\pi f$ equals $1.41\ \mathrm{rad/s}$. In the previous question you should have seen the maximum amplitude for a driving frequency of $1.41\ \mathrm{rad/s}$. In other words a driving frequency, ω of $1.41\ \mathrm{rad/s}$ leads to resonance (maximum amplitude) because it equals the natural frequency, ω_0.

Exercise 1.5.8

Use a calculator to find the natural frequency for a spring and mass system with $m = 2.0\ \mathrm{kg}$ and $\kappa = 5.0\ \mathrm{N/m}$. What do you expect the resonance frequency to be for this case?

The mathematical formula that describes damped harmonic motion is $Ae^{-\gamma t}\cos(\omega t + \varphi)$ where $\gamma = b/2m$. Notice that this is the same cosine function for simple harmonic motion but the amplitude, A, is multiplied by an exponentially decreasing function of time, $e^{-\gamma t}$. So we expect the oscillation of a damped harmonic oscillator to be an up and down cosine function with an amplitude that decreases over time.

Exercise 1.5.9

Check to see if the formula $Ae^{-\gamma t}\cos(\omega t + \varphi)$ really does describe the behavior of damped harmonic motion in the simulation. To do this, use a graphing calculator (or go to meta calculator and chose graphing calculator or use desmos calculator) and plot $y = 10 * \exp(-.2 * x) * \cos(1. * x)$. This is the case $A = 10\ \mathrm{m}$; $b = 0.4\ \mathrm{Ns/m}$; $m = 1.0\ \mathrm{kg}$; and $\omega = 1.0\ \mathrm{rad/s}$. (You can cut and paste the equation into the online calculator). How does this graph compare with the simulation for these same parameters (note: you are only interested in the positive x values)?

The mathematical formula that describes driven, damped harmonic motion is $A_0\cos(\omega t + \varphi)$ where $A_0 = (F_0/m)/((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2)^{1/2}$. In this case the amplitude, A_0 does not change over time but it is dependent on the driving frequency, ω.

Exercise 1.5.10

What happens to the amplitude, A_0, when $\omega = \omega_0$ (assume all the other factors are constant numbers)? Is there any other combination of ω and ω_0 that gives a larger amplitude?

Exercise 1.5.11

Verify your previous answer by making a plot of $A_0 = (F_0/m)/((\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2)^{1/2}$ for a range of driving frequency. To do this, use a graphing calculator to graph $y = 1.0/((1.4 \wedge 2 - x \wedge 2) \wedge 2 + 4 * .04 * 1.4 \wedge 2) \wedge .5$. Where does this graph have a maximum for positive values of x?

Exercise $1.5.12$

Based on the previous two questions, what is the resonance frequency of this system and how does this compare with the natural frequency?

1.6: Longitudinal Waves

In comparing simulations on transverse waves (Tutorial 1.3) with vertical harmonic motion (Tutorial 1.4) we discovered that particles in a transverse wave move up with simple harmonic motion. In the previous exercise (Tutorial 1.5) we saw that harmonic motion can also occur in the horizontal direction. Can we also have a wave moving horizontally where the particles move with harmonic motion in the horizontal direction?

YES! **Longitudinal waves** are waves where the motion of the material in the wave is back and forth in the same direction that the wave moves. Sound waves (in air and in solids) are examples of longitudinal waves. When a tuning fork or stereo speaker vibrates it moves back and forth creating regions of compressed air (where the pressure is slightly higher) and regions in between where the air has a lower pressure (called a rarefaction). These compressions and rarefactions move out away from the tuning fork or speaker at the speed of sound. When they reach your ear they cause your eardrum to vibrate, sending signals through the rest of the ear to the brain.

Longitudinal waves can be described with the same mathematical functions as transverse waves: $y(x,t) = A\sin(kx - \omega t + \varphi)$ where now $y(x,t)$ is the *horizontal* (or longitudinal) displacement from equilibrium at location x and time t instead of the vertical displacement from equilibrium. As was the case for transverse waves the forward velocity of a longitudinal wave is given by $v = \lambda/T = \omega/k$.

The following simulation shows a graph of the longitudinal motion of one molecule, the red circle, in a collection of molecules which has a longitudinal wave passing through it, much like sound passing through air. A vertical line marks the equilibrium location of the red circle. Random thermal motions are not shown.

Longitudinal Waves

Questions:

Exercise 1.6.1

Click on 'Position' and then 'play'. Left clicking on the upper panel gives the time and amplitude of points on the graph in the yellow box. Do any of the circles travel all the way across the simulation to the other side? Explain.

Exercise 1.6.2

Left clicking on the upper panel gives the time and amplitude of points on the graph in the yellow box. Determine the maximum amplitude and the period of oscillation from the graph.

Exercise 1.6.3

Left clicking on the lower panel gives the x and y locations of points on the wave in a yellow box. Pause and step the animation until the red circle at its equilibrium position. Find the wavelength of the wave using the mouse by finding the distance between one place where the circles are clumped together to the next location (or from two successive locations where the circles are furthest apart). What is the wavelength?

Exercise 1.6.4

From the period and wavelength find the speed of this wave (Hint: The same equations work for both longitudinal and transverse waves).

For sound the frequency (inversely proportional to the wavelenght) tells us something about the **pitch** of the sound. There are other aspects of pitch percption which inolve other physical features of the wave but the main component of pitch is the frequency.

Exercise 1.6.5

What is the frequency of the wave in the simulation?

Exercise 1.6.6

Write an equation of the form $y(x,t) = A\sin(kx - \omega t + \varphi)$, filling in the values of A, k and ω for this wave. Assume the phase angle is zero.

Notice that the circles in the simulation move back and forth with a variable speed around an equilibrium position while the wave moves only in one direction with a constant speed. The velocity of the individual particles is given as before by the derivative of the amplitude: $v(x,t) = \partial y(x,t)/\partial t = -A\omega\cos(kx - \omega t + \varphi)$.

Exercise 1.6.7

Click on 'Velocity' and then 'play'. The upper graph now gives the velocity of the red circle as a function of time. What is the maximum velocity (approximately) of the red dot according to the graph? How does this compare with the velocity of the wave which you found in 1.6.4? How does it compare with $v_{\max} = A\omega$?

Exercise 1.6.8

In your own words, explain the difference between wave speed and particle velocity for a longitudinal wave.

Exercise 1.6.9

Where is the red dot relative to the vertical line when the maximum velocity occurs? Where is it when the velocity is approximately zero? What is the relationship between position and velocity.

Exercise 1.6.10

Take a derivative of velocity to find an expression for acceleration of particles in the material (the red dot). Show that the maximum acceleration is given by $a_{\max} = A\omega^2$.

Exercise 1.6.11

Calculate the maximum acceleration of the red dot using $a_{\max} = A\omega^2$. If amplitude is in meters and angular frequency in radians per second, what are the units of this acceleration?

As a sound wave moves through the air the air molecules do not move forward at the speed of sound but rather, oscillate back and forth as harmonic oscillators in the same general location while the sound wave passes (see question one). For sound waves the displacement amplitude (distance from the equilibrium location) tells us something about the pressure of the air at that location. Pressure is measured in pascals ($1\text{ Pa} = 1\text{ N/m}^2$) and pressure squared is proportional to the **intensity** of the sound wave, measured in W/m^2.

The relationship between sound intensity, I measured in watts per meter squared and **loudness**, or sound intensity level (SIL) measured in decibels, is given by $SIL = 10\log(I/I_0)$. Here log is the logarithm and $I_0 = 10^{-12}\text{ W/m}^2$ is a reference sound intensity at about the threashold of human hearing.

Exercise 1.6.12

For the following sound intensities, what is the equivalent SIL in decibels: Jet engринge, 100 W/m^2; pain threshold, 1 W/m^2; vacuum cleaner, 10^{-4} W/m^2; converstaion, 10^{-6} W/m^2; the rustle of leaves, 10^{-11} W/m^2.

1.7: Water Waves

Tutorial 1.7: Water Waves

Many real, physical waves are combinations of three kinds of wave motion; transverse, longitudinal and torsional (which we have not yet discussed). The following is a more accurate simulation of water waves (but it still does not show breaking wave behavior which will come later). The dots are locations of water molecules or small objects floating in the water.

Water Waves

Questions:

Exercise 1.7.1

What two types of wave motion are represented in the simulation?

Exercise 1.7.2

Are the wavelengths and periods the same for both types of motion (Hint: Use 'pause' and hold the left mouse button down to make measurements)?

Exercise 1.7.3

Determine the wavelength, period and speed of the wave (use the 'pause' and 'step' buttons to measure the length of time it take a peak to pass a given location in the simulation).

Exercise 1.7.4

How does the motion at the top of the water compare with the motion at the bottom?

Exercise 1.7.5

Describe the overall motion of one of the red dots; what path does the dot follow?

Exercise 1.7.6

Earthquakes can produce several different kinds of waves, each traveling at a different speed. Search for a reliable source and find a definition for P-waves, S-waves, Rayleigh waves and Love waves. Be sure to include comments on their speeds and which ones are more destructive.

1.8: Two-Dimensional Waves

Tutorial 1.8: Two-Dimensional Waves

So far these simulations have only shown one dimensional waves. Even though there is motion perpendicular to the direction the wave travels for a transverse wave, the function describing the wave is a function of only one spacial variable, x. Waves can exist in two or three dimensions, however. One example is a **plane wave** where the wave front or crest of the wave makes a line (in two dimensions) or a plane (in three dimensions). **Circular waves** (in two dimensions) and **spherical waves** (in three dimensions) also exist. The present simulation shows plane and circular waves in two dimensions.

The simulation below starts by showing you a plane wave in two dimensions traveling in the $x-y$ plane, in the x direction, viewed from above. In these simulations the amplitude (in the z direction, towards you) is represented in greyscale. When the wave has a positive amplitude the color is white, when the amplitude is zero the color is light gray and when the amplitude is negative the color is black. The wavelength λ is in centimeters and the period, T is in seconds.

Two-Dimensional Waves

Questions:

Exercise 1.8.1

Click on the 'play' button with the plane wave selected. Experiment with wavelength and period. Is the simulation accurate in representing a fixed speed for a real wave? How do you know? Consult 1.2 on wave speed if you are unsure of your answer.

Exercise 1.8.2

Can this representation be used to describe longitudinal waves as well as transverse waves? Why or why not?

Exercise 1.8.3

The answer to the previous question is 'yes'. What would the white, grey and black colors represent if this was a representation of a longitudinal wave?

Keep in mind the wave shown is really a function of just two variables, x and t. For a wave traveling in the x direction the amplitude is the same for any value of y so the equation describing it is exactly the same as the one dimensional wave we have seen before: $z(x,y,t) = z(x,t) = A\sin(kx - \omega t + \varphi)$. This kind of wave is called a **plane wave**.

The equation for a plane wave traveling in an *arbitrary* direction in the $x-y$ plane is given by $z(x,y,t) = A\sin(k_1 x + k_2 y - \omega t + \varphi)$ where $z(x,y,t)$ is the height of the wave at location (x,y) at time t. Now the wavenumber k has become a *wave vector* with components $k_1 = k\cos\theta$ and $k_2 = k\sin\theta$ where θ is angle between the direction the wave is traveling and the x axis. This equation can also be writen using a vector dot product as $z(x,y,t) = A\sin(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi)$ where $\mathbf{r}$ and $\mathbf{k}$ are vectors with components in the x, y and (for waves moving in three dimensions) z directions. The vector $\mathbf{r}$ gives the direction the wave is traveling and the wave vector $\mathbf{k}$ has components of the wavelength in each direction.

Exercise 1.8.4

For what angle, θ, does the equation of a plane wave in an arbitrary direction become the same as the equation for a wave in the x direction?

Exercise 1.8.5

Now select the circular wave button. Suppose this circular wave simulation represents waves in a pan of water. Given a pan of water and other equipment of your choice in the laboratory, how could you create these waves? Explain the proceedure in detail.

Exercise 1.8.6

What is happening to the curvature of the waves as they move away from the source? Is it becoming more or less like a plane wave? Explain why it is reasonable to think of light from the sun as plane waves when the light arrives at the earth, even though they are actually spherical waves moving outwards in all directions from the sun.

Exercise 1.8.7

The circular wave simulation is unphysical in one sense because the amplitude of the circular wave does not change as the diameter gets bigger. Why is this unrealistic? Would you expect the amplitude of a circular wave to be the same once it has spread out over a very large distance from the source? Explain.

As noted previously the **Intensity** of a wave is defined to be the power per meter squared and is measured in $\mathrm{W/m^2}$. Power is energy transmitted per time and the energy of a wave is proportional to the amplitude squared so intensity will also be porportional to amplitude squared.

Exercise 1.8.8

In the simulation, does the intensity of the circular wave change as you get further from the source? How is this different in the case of real waves (for example sound) as you get further from the source?

Many waves spread out in three dimensions from what is essentially a point source, for example a small lamp sends out light in nearly all directions. The surface of a sphere, centered on the point source would catch all of the waves from the source. This imaginary sphere has a surface area of $4\pi r^2$ where r is the distance from the source. So at a distance of r the energy that left the source is spread over area $4\pi r^2$. This means the energy per area is $E/4\pi r^2$ and so decreases as $1/r^2$. We can thus conclude that the intensity of a spherical wave is inversely proportional to distance squared.

Exercise 1.8.9

Suppose the intensity of a spherical wave is $80\ \mathrm{W/m^2}$ at a distance of one meter. What is the intensity at $2\ \mathrm{m}$? $4\ \mathrm{m}$? $1/4\ \mathrm{m}$?

Exercise 1.8.10

If the amplitude squared is proportional to intensity, by what factor does the amplidue of the wave in the previous question change in going from $1\ \mathrm{m}$ to $2\ \mathrm{m}$? How much change is there in going from $1\ \mathrm{m}$ to $1/4\ \mathrm{m}$?

CHAPTER OVERVIEW

2: Combining Waves

Some of the most important properties of waves arise due to superposition, the idea that when two waves meet their displacements simply add. These tutorials explore the consequences of superposition.

2.1: Adding Two Linear Waves (Superposition)

Tutorial 2.1: Adding Two Linear Waves (Superposition)

The waves we have been discussing so far and the ones that are most often seen in everyday life, such as light and sound, are for the most part **linear waves**. Linear waves have the property, called **superposition**, that their amplitudes add linearly if they arrive at the same point at the same time. This gives rise to several interesting phenomena in nature which we will investigate in this and the next few simulations.

The simulation shows the function $f(x,t)$ in red, $g(x,t)$ in the blue and $u(x,t) = f(x,t) + g(x,t)$ in grey. Initial wavenumber and angular velocity are equal to one. The check boxes on the lower right determine which functions are visible. You can enter your own functions for $f(x,t)$ and $g(x,t)$ using the same notations used for spreadsheets and calculators.

Wave Superposition

Questions:

Exercise 2.1.1

Uncheck the $g(x,t)$ function so that just $f(x,t)$ is showing. What is the amplitude (half the height from highest point to lowest point) of $f(x,t)$? Now measure the amplitude of $g(x,t)$ (they should be the same). Now find the amplitude of the sum, $u(x,t)$. How does the amplitude of the sum compare with the amplitude of $f(x,t)$ or $g(x,t)$? This is an example of **constructive interference**; the two waves add to give a wave with and amplitude which is the sum of the two amplitudes.

Exercise 2.1.2

Start the simulation (button on lower left). How does the wavelength, frequency and speed of $f(x,t)$ or $g(x,t)$ compare with the wavelength, frequency and speed of $u(x,t)$?

Exercise 2.1.3

Change $f(x,t)$ to have a phase of π (the simulation reads *pi* as π; type or cut and paste $2.5 * \sin(x - t + pi)$ for $f(x,t)$ and press return or enter). Run the simulation. What happens to the amplitude of the sum of the two waves, $u(x,t)$? This is an example of **destructive interference**. Write a definition for destructive interference in your own words.

Exercise 2.1.4

Experiment with cases in between total destructive and total constructive interference by changing the phase of $f(x,t)$ to be $1/2\pi$, $1/3\pi$,and $1/4\pi$. Stop the simulation each time and record the amplitude of the sum compared to the amplitude of $f(x,t)$ or $g(x,t)$.

Exercise 2.1.5

Click the reset button (fourth button on lower left) and then change the amplitude of $f(x,t)$ from 2.0 to 3.0 and the amplitude of $g(x,t)$ from 2.0 to 1.0 (Hit Return or Enter to update the values). What is the amplitude of $f(x,t) + g(x,t)$ in this case? How does this amplitude compare to the original case?

Exercise 2.1.6

Go back to the original functions but change one of the minus signs to a plus sign (so now $f(x,t) = 2.0 * \sin(x+t)$ and $g(x,t) = 2.0 * \sin(x-t)$. The sum $u(x,t)$ is called a **standing wave** in this case (an example would be the waves on a guitar string). Describe the behavior of $u(x,t)$. How does the period and wavelength of the combined wave compare to the period and wavelength of two components? How is the maximum amplitude of the sum related to the amplitudes of the two components? What can you say about the speed of the sum?

Exercise 2.1.7

For standing waves on a string a **node** is a location where there is no motion and an **anti-node** is a location where there is maximum motion. For the standing wave in the previous exercise, how many nodes are there? How many anti-nodes?

Exercise 2.1.8

Use trigonometric identities to show that the sum of $f(x,t) = A\sin(kx+\omega t)$ and $g(x,t) = A\sin(kx-\omega t)$ equals $2A\cos(\omega t)\sin(kx)$. We can interpret this as a time dependent amplitude, $2A\cos(\omega t)$, multiplying a sine wave which is fixed in space. What happens to the amplitude as time increases? What fixes the location of the maximums and minimums of the standing wave (Hint: $k = 2\pi/\lambda$)?

Exercise 2.1.9

Notice that the standing wave has zero amplitude on both ends in the simulation. This means that only certian wavelengths will "fit" on a given length. See if you can adjust x min and x max so that you have a wave with more nodes and anti nodes that fits on a longer string with the amplitude still zero on the ends. (Hint: $6.28 = 2\pi$.)

Exercise 2.1.10

Now enter the following functions: $f(x,t) = 2.0 * \sin(x-t)$ and $g(x,t) = 2.0 * \sin(1.1 * x - 1.1 * t)$ (you can cut and paste instead of typing). Watch the just the sum $u(x,t) = f(x,t) + g(x,t)$ for a while and describe what happens (it changes slowly). Two waves with slightly different frequencies added together give rise to the phenomena of **beats**. Now turn $f(x,t)$ and $g(x,t)$ on. Are these waves still traveling at the same speed as $u(x,t)$? Find the beat frequency the following way: Stop the simulation when the two source waves exactly cancel ($f(x,t) + g(x,t)$ is a straight line) and record the time (use the step buttons if you overshoot). Start the simulation and stop it again the next time the waves cancel. Record the new time and subtract to get the elapsed time. The beat frequency is $1/$(time elapsed). How does this compare to the frequency of $f(x,t)$ subtracted from the frequency of $g(x,t)$?

2.2: Interference

Tutorial 2.2: Interference: Ripple Tank Simulation

This simulation shows a top view of a source making waves on the surface of a tank of water (imagine tapping the surface of a pond with the end of a stick at regular intervals). The white circles coming from the spot represents the wave crests with troughs in between. Two sources can be seen at the same time and the separation between them and the wavelength of both can be adjusted. The wavelength, λ, and distance between sources, d, are in the same arbitrary distance units (meters, cm, $\mu\mathrm{m}$, etc.). Speed is a unitless parameter that controls the rate at which the simulation is refreshed.

Interference

Questions:

Exercise 2.2.1

After looking at the left and right waves to verify they are the same, click on 'Both' to see them together. Waves from each source will cancel in some places (destructive interference) but add in other places (constructive interference). How many lines of constructive interference do you see?

Exercise 2.2.2

With two sources change the wavelength of the sources. How does the number of destructive interference lines change with wavelength? Write a statement about the relationship between wavelength and the number of interference lines.

Exercise 2.2.3

With two sources and a wavelength of 2.0, change the separation between the sources. How does the number of destructive interference lines change with separation? Write a statement about the relationship between source separation and the number of interference lines.

Exercise 2.2.4

Reset the simulation to the original values and click 'Both'. Suppose instead of a ripple tank this simulation represented two light sources (which have the same wavelength and start off in phase- for example laser light from from a single source shining through two small openings). The light starts at the middle of the simulation and reaches a screen at the top edge of the simulation. How many bright spots would be seen on the screen in the simulation for the this case?

Exercise 2.2.5

If the simulation represented a double slit light source, changing the wavelength would be equivalent to changing the color. Describe the difference in the location of the bright spots for the color represented by wavelength equal to 1 compared to the location of the spots for the color represented by wavelength equal to 4. Do they occur in the same location on the screen (at the top edge of the simulation)?

Exercise 2.2.6

What would be the result on the screen of shining light which was a mixture of two colors through a double slit?

The formula for double slit interference is given by $d\sin\theta_{\text{bright}} = m\lambda$ where $m = 0,\ \pm1,\ \pm2,\ \pm3\ldots$ for the case of constructive interference. For destructive interference $d\sin\theta_{\text{dark}} = (m + 1/2)\lambda$ where $m = 0,\ \pm1,\ \pm2,\ \pm3\ldots$ In both cases d is the distance between the center of the openings (the separation of the sources), the angle θ is the angle from the central maximum out to a minimum or maximum and m numbers the maximums (or minimums) starting from the center ($m = 0$).

Note

In order to actually measure this effect for light the slits must be similar to the wavelength of light; in other words, very small and close together with the screen quite a distance away. The light must also be coherent (have the same phase as is the case for laser light).

Exercise 2.2.7

For $600\ \mathrm{nm}$ light and a separation of $0.01\ \mathrm{mm}$, what is the angle (in radians) to the first maximum? What is the color of this light?

Exercise 2.2.8

For the previous question, how far away would the screen have to be in order to have a 2mm separation between the first and second maximum?

2.3: Group Velocity

Tutorial 2.3: Group Velocity versus Phase Velocity

In previous simulations the forward speed of a wave moving in the x-direction was determined by $v = \omega/k$ (see simulation three on wave speed). But what value do we use for speed if we add two waves together, each of which has a different value of $v = \omega/k$? In cases where several waves add together to form a single wave shape (called the **envelope**) we can quantify the speed with two numbers, the **group velocity** of the combined wave and the **phase velocity**.

In simulation 2.2 the waves that we added had the same speed so the places where there were destructive or constructive interference (beats for example) were fixed relative to the envelope. In the following simulation the component waves travel at different speeds so there will be internal oscillations in the envelope. The speed of these internal oscillations relative to the envelope is called the phase velocity and is given by $v_{\text{phase}} = \omega_{\text{ave}}/k_{\text{ave}}$.

The group velocity is velocity of the envelope. For two waves group velocity is defined by $v_{\text{group}} = \Delta\omega/\Delta k$ where $\Delta\omega = \omega_1 - \omega_2$ and $\Delta k = k_1 - k_2$. This expression for group velocity is the slope of a frequency versus wavenumber graph. In the case of adding many waves, each with its own angular velocity and wave vector, ω and k become continuous variables and we define the group velocity as a partial derivative; $v_{\text{group}} = \partial\omega(k)/\partial k$. Angular velocity as a function of the wave vector ($\omega(k)$, called the dispersion relation) will be examined in a later simulation.

Since this simulation has waves traveling in the x-direction only we will talk about **group speed** and **phase speed**.

Group and Phase Velocity

Questions:

Exercise 2.3.1

The simulation starts with two identical waves. The bottom graph shows the sum of the two top graphs. Change both the wave number (k_1) and the angular frequency (ω_1) for the first wave to 8.0, click 'set values' and 'play'. Describe what you see. Notice that the envelope moves to the right at the same speed as the components.

Exercise 2.3.2

Two numbers appear in the bottom of the simulation window. These show the phase speed and the group speed. What are the group and phase speed for the case $k_1 = 8.0 \text{ rad/m}$, $\omega_1 = 8.0 \text{ rad/s}$ and $k_2 = 8.4 \text{ rad/m}$, $\omega_2 = 8.4 \text{ rad/s}$?

Exercise 2.3.3

What are the group and phase speeds for the case $k_1 = 8.0 \text{ rad/m}$, $\omega_1 = 8.4 \text{ rad/s}$ with $k_2 = 8.4 \text{ rad/m}$, $\omega_2 = 8.4 \text{ rad/s}$? Describe what you see.

Exercise 2.3.4

What are the group and phase speeds for the case $k_1 = 8.8 \text{ rad/m}$, $\omega_1 = 8.0 \text{ rad/s}$ with $k_2 = 8.4 \text{ rad/m}$, $\omega_2 = 8.4 \text{ rad/s}$? Describe what you see.

Exercise 2.3.5

For $k_1 = 8.0 \text{ rad/m}$, $k_2 = 8.4 \text{ rad/m}$, $\omega_2 = 8.4 \text{ rad/s}$ try several values of ω_1 between 8.4 and 9.0 rad/s. What can you conclude about the group velocity as ω_1 gets larger?

Exercise 2.3.6

For $k_1 = 8.0 \text{ rad/m}$, $k_2 = 8.4 \text{ rad/m}$, $\omega_2 = 8.4 \text{ rad/s}$ try several values of ω_1 between 8.4 and 7.6 rad/s. What can you conclude about the group velocity as ω_1 gets smaller?

You will have noticed by now that the phase velocity can be greater than the group velocity. In the case of electromagnetic waves the phase velocity can be greater than $c = 3 \times 10^9\ \mathrm{m/s}$, the speed of light. This does not break the rules of special relativity because information is transmitted at the group velocity, which is never greater than c.

Exercise 2.3.7

Based on the previous few questions, what is the general rule for when group velocity is larger than the phase velocity? Under what conditions is phase velocity greater than the group velocity?

Exercise 2.3.8

For some material the wave speed is fixed by the medium in which the wave is traveling (see simulation 1.2). In these cases the ratio of ω/k is always the same number although ω and k might be different for different waves. For $k_2 = 8.4\ \mathrm{rad/m}$, $\omega_2 = 8.4\ \mathrm{rad/s}$ try several values of ω_1 and k_1 such that the ratio ω_1/k_1 is always equal to one (the same as ω_2/k_2). What can you conclude about the group velocity as compared to the phase velocity in cases where all the components travel at the same speed?

2.4: Other Wave Functions

Tutorial 2.4: Other Wave Functions

So far all the waves we have encountered were described mathematically by sine and cosine functions. In general however, any function of x and t which has these variables in the form $x - vt$ will be a traveling wave with speed v. Notice that our sine wave function can also be written in this form: $y(x,t) = A\sin(k(x - vt) + \varphi)$ where as before $v = \omega/k$. This simulation will allow us to investigate other functions which are also waves.

Other Wave Functions

Questions:

Exercise 2.4.1

The initial equation shown is called a Gaussian function (or bell curve). Experiment by changing the constants in the equation (click 'reset' each time, change the function and hit enter or return).

a. What does the number in front of the exponent do?
b. What does the number in front of the variable t do?
c. What does the number after the plus sign do?
d. What does the first number in the parentheses after 'exp' do?
e. Identify which of these numbers is amplitude, speed, width, and initial location.

Exercise 2.4.2

Reload the initial wave and experiment with the signs in the equation.

a. What happens if you change the minus sign between the x and the $-3 * t$ to a plus sign?
b. What happens if you change the other minus sign in front of the 2 to a plus sign? (Think about what function you are dealing with here- does the result make sense?)
c. What happens if you place a minus sign in front of the original function?

Exercise 2.4.3

For $y(x,t)$ delete the original function, type (or copy and paste) the function $2.0/((x - 3.0 * t) \wedge 2 + 1)$ into the function window and run the simulation. Experiment by changing the numbers (reset each time to set the new values and hit enter to load the new function). In this case a single number still governs the speed but amplitude and width both depend on two numbers. Which number is the speed? Which two numbers determine the width and amplitude?

Exercise 2.4.4

Create your own traveling wave. The only requirement is that x, t and speed appear in the equation with the relationship $(x - vt)$. You may have to adjust parameters to be visible in the screen. What is your equation and what did you learn from this exercise?

Exercise 2.4.5

For $y(x,t)$ delete the original function, type (or copy and paste) the function $1 * \exp(-3 * (x - 2 * t + 5) \wedge 2) + 2 * \exp(-2 * (x + 1.2 * t - 5) \wedge 2)$ and run the simulation. This is a collision of two Gausian pulses traveling in opposite directions. What happens when they collide? How is does the amplitude at the moment they overlap compare to the amplitude of the two separate pulses (use the 'pause' and 'step' buttons to confirm your answer)?

Exercise 2.4.6

Reload the previous case of a collision but change one pulse to have a negative amplitude. What happens in this case? How is does the amplitude at the moment they overlap compare to the amplitude of the two separate pulses?

Exercise 2.4.7

Explain the connection between superposition (Chapter 2.1) and the answer to the previous two questions. Give a general definition of superposition based on your observations.

2.5: Fourier Analysis and Synthesis

2.5: Fourier Series with Sound

As we saw in the previous simulation, waves may have very complicated shapes which don't resemble a sine wave. However the French mathematician **Jean Baptiste Joseph Fourier** showed that *any periodic function* can be formed from an infinite sum of sines and cosines. This is very convenient because it means that everything we know about sines and cosines applies to a periodic function of any shape. Although the sum is infinite in theory in many cases using just a few terms may be close enough to provide a good approximation.

Fourier analysis is the process of mathematically breaking down a complex wave into a sum of of sines and cosines. **Fourier synthesis** is the process of building a particular wave shape by adding sines and cosines. Fourier analysis and synthesis can be done for any type of wave, not just the sound waves shown in this simulation.

This simulation shows the sum of up to 8 harmonics of a sine wave with fundamental frequency f_1. The Run buttons shows a slow motion animation of the resulting wave where time t is in milliseconds and distance x is in meters. A **harmonic** is a sine wave that has a frequency which is a whole number multiple of the frequency of the original sine wave. The first harmonic (set with the slider A1) is called the **fundamental**. So if the fundamental is $f_1 = 200\text{ Hz}$ the second harmonic is $f_2 = 400\text{ Hz}$, the third harmonic is $f_3 = 600\text{ Hz}$, etc. We know $v = \lambda f$ so for a fixed speed, doubling the frequency means the wavelength of the second harmonic is half that of the fundamental. Conversely, holding your finger down in the middle of a guitar string and plucking one side cuts the wavelength in half which doubles the frequency being played. For standing waves on a string fixed at each end (such as a guitar string) each harmonic is also called a **normal mode** of vibration.

Fourier Series with Sound

Note

The Fourier Series and Sound JavaScript Model uses the HTML 5 Web Audio API. This API is still under development and may not be supported on all platforms. Press the Reset button to reinitialize the simulation if the sound does not play when the simulations is first loaded.

Questions:

Exercise $2.5.1$

Try adjusting the slider A1 to different values. What does this slider do?

Exercise $2.5.2$

You may have noticed the amplitude shows up in the graph on the right. You can also see the magnitude of the amplitude by holding the mouse the mouse button down and moving the mouse to the top of one of the peaks on the graph on the left. Does it match the value of A1 set by the slider? Use the mouse to find the wavelength (distance between peaks on the left graph), what is the wavelength of this wave?

Exercise $2.5.3$

Use the 'reset' button and move slider A2 (the second harmonic). What is the wavelength of the second harmonic? How does this wavelength compare to the wavelength of the fundamental? This is a 400 Hz sine wave.

Exercise $2.5.4$

The pitch of a sound wave is determined by the fundamental frequency. Turn on the velocity (344 m/s for sound at room temperature). With only amplitude A1 showing, run the simulation. Find the period by measuring the time (in 10^{-3} sec) between when one peak passes the origin and when the next peak passes (use the step button to get an accurate time measurement). What is the frequency of the fundamental?

Exercise 2.5.5

Now find the period of a wave with several harmonics. What is the period of the combination (the time between successive highest peaks)? Although the wave looks more complicated it has the same fundamental frequency and therefor the same pitch. The additional, smaller peaks are due to the harmonics and give a sound its *timbre*. A trumpet and trombone playing the same note have the same fundamental frequency but sound different because of the number and amount of harmonics present.

The graph on the right is called a **Fourier spectrum** and is a short hand way of showing how much of each harmonic is present in the graph on the left. Fourier series usually include sine and cosine functions and can represent periodic functions in time or space or both. In this simulation we only have combinations of sine waves. The **Fourier series** for the wave function showing in the left graph is given by $y(t) = \sum_{n=1} A_n \sin(n2\pi x/\lambda - n2\pi ft)$. Here t is time, n is the number of the harmonic or mode ($n = 1$ for the fundamental, 2 for the second harmonic etc.), A_n is the amplitude of harmonic or mode number n and f is the fundamental frequency ($f = 1/T$).

Exercise 2.5.6

To get the exact shape of an arbitrary periodic function we would need an infinite number of terms in the Fourier series but in this simulation we can only add a maximum of 10 terms. Try the following combination of harmonics (you can type the amplitudes into the boxes next the sliders to get exact values): $\mathrm{A1} = 1.0$, $\mathrm{A2} = 0$, $\mathrm{A3} = 0.333 (= 1/3)$, $\mathrm{A4} = 0$, $\mathrm{A5} = 0.20 (= 1/5)$, $\mathrm{A6} = 0$, $\mathrm{A7} = 0.143 (= 1/7)$, $\mathrm{A8} = 0$. What is the approximate shape of this wave?

Exercise 2.5.7

Reset the simulation and try the following combination of harmonics (you can type the amplitudes into the boxes next the sliders to get exact values): $\mathrm{A1} = 1.0$, $\mathrm{A2} = -0.5$, $\mathrm{A3} = 0.333$, $\mathrm{A4} = -0.25$, $\mathrm{A5} = 0.20$, $\mathrm{A6} = -0.166$, $\mathrm{A7} = 0.143$, $\mathrm{A8} = -0.083$. What is the approximate shape of this wave?

If you could hear the notes represented by the waves in exercise $2.5.6$ and $2.5.7$, what would be the same for both? What would be different? What is the difference in sound between the two?

Exercise 2.5.8

Suppose a clarinet and a trumpet both play the same note (have the same fundamental frequency). Why is it that you can still tell them apart, even though they are playing the same note?

Exercise 2.5.9

Find a definition of timbre and write it in your own words. Based on your answers to the above questions, what causes timbre?

Exercise 2.5.10

Suppose you wanted to build an electronic instrument which added waves together to imitate other instruments (this is how some musical synthesizers works). What would you need to know about the sound a trumpet makes in order to reconstruct that sound? (Hint: think about the information contained in the graph on the right.)

CHAPTER OVERVIEW

3: External Interactions

The details of wave properties in materials are determined by how the waves interact with their surroundings. Barriers, mirrors, edges, and material types all influence the physics of objects.

3.10: Diffraction

Tutorial 3.10: Diffraction

When waves pass through an opening or past an object of roughly the same size as the wavelength, the direction the waves travel will appear to bend. This is called **diffraction**. The simulation shows what happens to a planewave light source (below the simulation, not shown) as it passes through an opening. The wavelength of the waves and the size of the opening are in the same arbitrary units (meters, cm, μm, etc.) and can be adjusted.

Diffraction

Questions:

Exercise 3.10.1

Leave the opening width fixed and experiment with the wavelength of the waves. Describe what yo see.

You should have noticed that for a long wavelength the opening basically becomes a point source of waves. The waves on the other side of the opening move outward in all directions. But once the wavelength is much smaller than the opening the waves do not bend as much and appear to be more like plane wave all headed in the same direction.

Exercise 3.10.2

Light, with it's very small wavelength, passes through a doorway without bending because the door is much larger than the wavelength. Sound, however, is a wave with wavelengths close to the size of the opening of a door. Explain why we can hear noise through a doorway to another room even though the source (a person, radio, TV. etc.) is not in our direct line of sight.

Exercise 3.10.3

All optical instruments (telescopes, microscopes, even radio teloscopes which look at radio waves instead of light waves) have openings to allow light in. This means diffraction will be a problem for that instrument for some sizes of waves. If you want to reduce the effects of diffraction for a particular instrument, would you want to try to use longer or shorter wavelengths? (Hint: Electron microscopes can provide much higher magnification because electron waves can be much smaller than light.)

Exercise 3.10.4

Reset the simulation and leave the wavelength fixed while changing the size of the opening. Describe what you see. How does the opening size affect the diffraction pattern?

Diffraction can also be explained as a type of interference resulting from a path difference from multiple sources. Recall in the ripple tank simulation of two sources (simulation 2.2) waves from the source on the left must travel a longer path to get to a point at the top right of the simulation than waves from the source on the right. This path difference changes depending on how far to the right we look resulting in spots of destructive and constructive interference along the top. For a single opening instead of two separate sources we can imagine a row of many sources filling up the single opening. Again there will be a path difference from the different sources but the pattern will look different because there are now many sources lined up next to each other.

The formula for the location of destructive interference in the case of single slit diffration is given by $\sin\theta_{\text{dark}} = m\lambda/a$ where a is the opening size and $m = 0,\ \pm1,\ \pm2,\ \pm3\ldots$ where a is the width of the opening and θ is the angle to each successive dark spot, labeled with the number m.

Exercise 3.10.5

For $600\ \text{nm}$ light and an opening of $0.01\ \text{mm}$, what is the angle (in radians) to the first minimum?

Exercise $3.10.6$

For the previous question, how far away would the screen have to be in order to have a 2 mm separation between the central maxium ($\theta = 0$) and the first minimum?

Exercise $3.10.7$

Red light has a longer wavelength than green light. Which color bends the least when going through a small opening, red or green?

Exercise $3.10.8$

What would be the result of shining white light through a small opening?

3.1: Mirrors

Tutorial 3.1: Reflection I

In many cases plane waves will travel in a straight line, reflecting off of objects and surfaces at the same angle that they strike the surface. This is called the **law of reflection** and is true for sound waves as well as light as long as the surface is smooth. In these cases we can treat the wave as a straight line or ray which is perpendicular to the wave front. **Geometric** (or ray) **optics** is the study of light in cases where the wave travels in a straight line except for reflection and refraction (treated later).

By smooth we mean that the roughness of the surface has variations that are smaller than the wavelength of the wave. If the surface is smooth enough a group of parallel rays will all reflect in the same direction. This is called **specular reflection**. When the surface is rough relative to the wavelength, although individual rays still obey the law of reflection they end up not being parallel after being reflected. This is called **diffuse reflection**.

For light, with very small wavelengths, this means the surface must be very smooth for specular reflection. For larger electromagnetic waves, such as microwaves, digital TV signals and radio waves, specular reflection occurs even when the surface is relatively rough. This is why mirrors are used in optical telescopes but radio telescopes and Direct TV dishes do not need to be as smooth as a mirror.

The simulation below allows for a brief exploration of specular reflection. The mirror in each case in in the center and everything to the right is behind the mirror (virtual). These examples are for light reflecting off of mirrors, the same behavior occurs for any wave (sound for example) if the surface is smooth relative to the wavelength. The units of height, h, and distance, d, are arbitrary (cm, inches, etc.).

Mirrors

Questions:

Exercise 3.1.1

The simulation starts with a flat mirror, an object on the left (a candle) and the candle's image (shown on the right). Three real rays (white arrows) are shown leaving the object and reflecting off the mirror. Carefully describe each ray. What does the parallel ray do when it reflects off the mirror? What does the ray that goes to the center of the mirror do when it reflects?

Exercise 3.1.2

The small protractor can be dragged to different locations and the arrow tip can be moved to change the angle. Use it to measure the incident and reflected angles (they are measured from the perpendicular, not the surface of the mirror). What are the incident and reflected angles for the ray going from the object to the center of the mirror? Is the law of reflection obeyed?

Exercise 3.1.3

Our eyes and brain do not percieve the rays as reflecting. Instead, our brains percieve the reflected rays as coming from behind the mirror (the magenta arrows) from an imaginary or **virtual** image behind the mirror. The object can be moved using the mouse. Does the size of the image change if the object is moved back and forth?

Exercise 3.1.4

Now choose the convex mirror button. Move the object back and forth. Describe what happens to the image as the object moves back and forth. What happens to a parallel ray when it reflects? What happens to a ray going to the center of the mirror? The red dot on the right is the focal point of the mirror and is $1/2$ the radius of curvature of the mirror. What happens to a ray that starts on the left and heads towards the focal point on the other side; how does it reflect?

Exercise 3.1.5

Use the protractor to measure the incident and reflected angles for the ray striking the center of the mirror for the convex case. Are the equal? The other rays also obey the law of reflection but the mirror surface at those locations is not perpendicular to the x-axis. Place the protractor at the location where the top ray strikes the mirror and rotate the arrow to point directly away from the radius of curvature (turquoise dot). This is the direction of a perpendicular to the mirror and should fall precisely between the incoming and outgoing rays. What is the angle?

Exercise 3.1.6

For the flat and convex mirror, does the size of the object change the laws of reflection? Describe what you see.

Exercise 3.1.7

Now click on the concave mirror button. Slide the object back and forth, moving it from very far away from the mirror to very close. Describe where the image is, its size relative to the object and orientation (Hint: There are three different cases; Closer than the focus; further away than the focus but closer than twice the focal length; further away than twice the focal length.)

Images appearing behind the mirror (to the right in the simulation) are **virtual images**; we can only see them by looking into the mirror. Images appearing on the left side, in front of the mirror are **real images**. These can be seen in the mirror and can also be projected onto a screen.

Exercise 3.1.8

For the flat mirror were the images real or virtual? Are the images for the convex mirror real or imaginary? How do you know? In which cases were the images real and virtual in the concave mirror?

Exercise 3.1.9

Describe the three reflected rays for the case of a virtual image in the concave mirror case. Do they obey the law of reflection? Explain.

Exercise 3.1.10

Describe the three reflected rays for the case of a real image in the concave mirror case. Do they obey the law of reflection? Do the laws of reflection change depending on the size of the object in the concave case? Explain.

Exercise 3.1.11

Go back to the flat mirror and use the mouse to find the distance from the mirror to the object and from the mirror to the image. How are these related?

For a flat mirror the image height and image distance are the the same as the object's. For a curved mirror the relation between the distance to the object, s and to the image s' are related to the focal length, f which is equal to one half the radius of curvature of the mirror. The relationship is $1/f = 1/s + 1/s'$ where the focal length is postive for a concave mirror and negative for a convex mirror. The distance to the image will be positive if the image is real and negative if virtual.

Exercise 3.1.12

Use the mouse to find the distance to the object and to the image for several different distances in the convex mirror case. Use these values to find the focal length, f (don't for get the sign convention given in the previous paragraph).

3.2: Collisions with Boundaries

Tutorial 3.2: Collisions with boundaries

In the previous simulation we did not take the wave nature of reflected waves into account; the waves were assumed to be exactly the same after reflection. However the phase of the wave may be different after reflection, depending on the surface from which they reflect. The example below is for strings but a similar effect occurs when light or sound reflect off of different types of surfaces or boundaries.

Waves reflect from a boundary in two basic ways depending on whether the boundary is "hard" or "soft". In the case of waves on a string a "hard" boundary is where the string is firmly attached and a "soft" boundary is when the end of the string can slide up and down. The string in this animation is simulated as a row of individual masses connected by invisible springs.

In the case of strings a boundary where the end is free is called a **free boundary condition**. If the end is fixed it is called a **rigid** or **fixed boundary condition**. A third possible boundary is **circular boundary conditions** which is the case if the right end of the string loops around to smoothly connect to the left end. For circular boundary conditions a pulse moving the the right would re-appear at the left after it leaves the right hand side of the simulation. Fixed and free boundaries can also occur for sound waves in a tube and determine the resonant frequencies for the tube.

Reflection from Boundaries

Questions:

Exercise 3.2.1

Run the simulation to see how a Gaussian pulse reflects off the two different boundaries. How is a pulse reflected from a fixed boundary different from one reflected from a free boundary?

Exercise 3.2.2

Now check the sine wave check box to see what happens when a sine wave hits the two types of boundaries. What is the end result in these cases? (Hint: Go back to simulation 2.1 and add two identical waves moving in opposite directions.)

Exercise 3.2.3

Although the reflecting sine waves in both cases interacts with the incoming wave to form a standing waves there is a slight difference between the two. Which case has a node at the boundary and which has an anti-node at the boundary? (Nodes and anti-nodes were defined in simulation 2.1.)

A wave reflected from a stiff or fixed boundary is said to have a **phase shift** of 180° (or π radians). This means a pulse will invert itself on reflection and the first anti-node of a standing wave will occur 180° from the boundary. If the boundary is soft the first anti-node occurs at the boundary. As we will see shortly, other kinds of waves also experience a phase shift on reflection from some kinds of boundaries. For example when light in air reflects from a material that is more optically dense (such as a glass) there is a phase change of 180° but when light in glass reflects from a glass/air boundary there is no phase change since the light is going from a more optically dense material (glass) to a less optically dense material (air).

3.3: Standing Waves on a String and in a Tube

Tutorial 3.3: Standing waves

In simulation 2.1, Question 2.1.6 you created a standing wave from two identical waves moving in opposite directions. For standing waves on a string the ends are fixed and there are nodes at the ends of the string. This limits the wavelengths that are possible which in turn determines the frequencies (recall that $v = f\lambda$ and the speed is fixed by the mass, tension and length of the string). The lowest frequency is called the **fundamental** or first harmonic. Higher frequencies are all multiples of the the fundamental and are called **harmonics**. Sometimes the term **overtone** is used to indicate harmonics greater than the fundamental (this can be confusing because the second harmonic is the first overtone, etc.). Harmonics are always multiples of the fundamental frequency but the term overtone can be used for other frequencies which are not necessarily multiples of the fundamental. The various harmonics are also called the **normal modes** of the string, a subject we will come back to later.

Standing waves determine the notes a musical instrument, such as a guitar or piano can play. The available frequencies are determined by the length of the string and the speed of the wave on the string which in turn is determined by the tension and density (thickness) of the string.

Standing waves also occur for sound waves enclosed in tubes. It is these standing waves that determine the frequencies that a wind instrument can play. At the center of a tube open on both ends the air cannot easily move so the fundamental frequency has a displacement node at that point. Because the pressure fluctuates the most at that location there is a **pressure anti-node** there. At the open ends of the tube the air can move more freely so a **displacement anti-node** occurs which is also a pressure node (the moving air prevents much pressure change).

Standing Waves on a String and Tube

Questions:

Exercise 3.3.1

Play the standing wave simulation for the case of the fundamental. The length of the string is 3.14 m. What is the wavelength of the fundamental?

Exercise 3.3.2

Describe the fundamental of the tube simulation (bottom). Where do the dots (representing air molecules) move the most? Where do they form a displacement node? Assuming the tube is the same length as the string, what is the fundamental wavelength of the tube open on both ends?

Exercise 3.3.3

Use the time in the simulations to find the period and calculate the frequency of the fundamental for both simulations. For a musical instrument this would be the frequency of the tone being sounded by the instrument when it plays its lowest note.

Exercise 3.3.4

What is the wave speed of each of the component waves making up the fundamental (the speed determined by $v = f\lambda$)?

Exercise 3.3.5

Now click the box for a tube closed at one end. What is the wavelength of the fundamental for a tube closed at one end? How is this different for the case of the tube opened at both ends?

Exercise 3.3.6

Reset the simulation and look at the second harmonic for the string and tube open on both ends. What is the wavelength and frequency of the second harmonic/first overtone for the string and tube opened on both ends? What is the speed of the component waves?

Exercise 3.3.7

Try the third and fourth harmonics for the string and tube opened on both ends. What are the wavelengths and frequencies of these waves? What are the speeds of the component waves?

Exercise 3.3.8

The formula for the wavelength as a function of the length of the string or open tube is given by $\lambda = 2L/n$ where n is a whole number and L is the length of the string. Verify this relationship with the numbers you got in the previous questions.

Exercise 3.3.9

Now check the box for a pipe with closed end simulation and examine the harmonics. Describe the difference in the node and anti-node pattern. What are the wavelengths for these cases? The forumula for the frequencies of a tube closed on one end are given by $\lambda = 4L/n$ where n is an odd whole number. Verify this relationship with the numbers you got in the previous questions.

Exercise 3.3.10

Flutes are basically pipes with openings on both ends but clarinets, trumpets and trombones are basically tubes that are closed on one end. Why does this make a difference in the frequencies each instrument plays?

Exercise 3.3.11

Pressure anti-nodes occur at places where the air is not moving (displacement nodes). What would be the effect of cutting a hole in the tube at the location of a pressure anti-node? Would the standing wavelength be affected? (This is the basis behind using finger holes in wind instruments to play different frequencies.)

Exercise 3.3.12

Based on what you learned about reflection from boundaries in simulation 3.2, explain what is going on at the closed end where the two waves that make up the standing wave reflect.

3.4: Refraction

Tutorial 3.4: Refraction

When waves pass from one medium to another they often interact with the medium and change their wavelength. This means their speed also changes. Electromagnetic waves may be briefly absorbed by the atoms of substance (depending on the wavelength of and the type of atom) so that the total time taken to traverse the material is longer than if the wave were in free space, even though they travel at the usual speed of light between atoms.

A wave that changes speed as it crosses the boundary of between two materials will also change direction if it crosses the boundary at an angle other than perpendicular. This is because the part of the wavefront that gets to the boundary first slows down first. The bending of a wave due to changes in speed as it crosses a boundary is called **refraction**.

The ratio of the speed of light in a material to the speed in a vacuum ($c = 3.0 \times 10^9 \text{ m/s}$) is called the index of refraction; $n = c/v$ where v is the speed of light in the medium. In this simulation we will investigate the effects of a change in the speed of a wave as it moves from one material to another. Although our example is for light, the same behavior can be demonstrated with other waves. For example, much of what we know about the interior of the earth, the sun and other planets comes from tracking earthquake waves when they refract as they pass through layers of material that have different densities.

The relationship between the index of refraction and the change in the direction angle of the a ray as it goes across a medium is given by **Snell's law**: $n_1 \sin\theta_1 = n_2 \sin\theta_2$ where n_1 and θ_1 are values measured on side of the boundary and the subscript 2 is for the material on the other side of the boundary.

Refraction

Questions:

The simulation shows a ray of light passing through a medium with index of refraction n surrounded by air on both sides. You can move the protractor around and use it to measure angles by dragging the tip of the arrow.

Exercise 3.4.1

Set the index of refraction to 1.40. Change the angle of the rays from the source. The **angle of incidence** θ is the angle the ray makes with a perpendicular to the surface (not the angle between the ray and the surface). Use the protractor to measure the angle of incidence (put the bottom left edge of the protractor where the ray enters the glass and use the slider to line up the arrow with the ray). Move the protractor to the boundary and line up the arrow with the rays inside the material to get the **refracted angle** (the angle inside the medium).

Exercise 3.4.2

The index of refraction of air is approximately one (light travels about as fast in air as it does in a vacume). Use Snell's law to calculate the index of refraction in the medium. ($n_1 = 1$; $\theta_1 =$ the incident angle; θ_2 is the refracted angle; you are solving for n_2).

Note

It is very difficult to get the protractor arrow to line up exactly with the rays so your answer may be off a little.

Exercise 3.4.3

For the same angle you used in the previous two questions, move the protracter once again to the right edge of the material and measure the angle the ray leaves the material. How does this compare with the incident angle on the left?

Exercise 3.4.4

Choose a different angle and use Snell's law to find the index of refraction outside the medium on the right. This time the incident angle will be inside the material at the right boundary, and n_1 will be the index inside the material (your answer to question two).

Exercise 3.4.5

Drag the source inside the medium and try different angles. What do you notice?

When light goes from a material where the index is higher to a lower index there are angles for which the rays cannot leave the material. The angle at which a ray starts to reflect off the boundary instead of passing through is called the **angle of total internal reflection**. You can see this if you look at the surface of the water in a pool from under water; at a certian angle you see a reflection of the bottom of the pool instead of objects above the water. Total internal reflection is also why light remains in a fiber optic cable instead of being absorbed by the coating.

Exercise 3.4.6

Find the angle for total internal reflection two ways. First experiment with the angle of incidence (source inside the medium) for a fixed index of refraction of 1.40. At what incident angle does the ray reflect? You can also find the angle using $n_1 \sin\theta_1 = n_2 \sin\theta_2$ where n_1 is the index inside the medium and $\theta_2 = 90^\circ$. Do your two values match?

3.5: Lenses

Tutorial 3.5: Refraction: Lenses

You may have noticed in the last simulation, a wave that passes all the way through a piece of material with parallel sides (for instance light through a flat slab of glass) leaves the material at the same angle that it entered. The wave un-bends when it exits the material by the same amount that it bent when entering. This is only true if the sides of the material are parallel, however. In the simulation below we have convex and concave lenses where the sides of the glass are *not* parallel (except near the center). In this case parallel rays of light end up exiting in different directions. This is the basis for any optical device that uses lenses, for example cameras, binoculars, microscopes, glasses, eyes of various animals, etc.

At each surface the waves obey the law of refraction (Snell's law) but the result is that parallel rays that enter are not parallel when they exit. Although our example is for light it should be kept in mind that the same behavior occurs for other types of waves when they enter a medium where their speed is different.

For this simulation we use the thin lens approximation which assumes the lens thickness is small compared to the curvature of the lens. This allows us to approximate the bending as if it occurs all at once at the middle line of the lens (instead of some bending at each surface which in fact is what happens).

The object (a candle) in the simulation can be moved using the mouse. White arrows show where the light rays actually travel. The purple lines are imaginary extensions of the real light rays. Our brain/vision system assumes light always travels in a straight line and does not bend. For light that does bend due to refraction, our brain interprets the light as following the purple paths and constructs and image based on this information. The units of height, h, and distance, d, are arbitrary (cm, inches, etc.)

Lenses

Questions:

Exercise 3.5.1

The definition of the focal length of a converging lens is the point where rays initially parallel to the axis meet after passing through the lens. These are marked by a red circle. Why is there a focal point on each side of the lens? Does it make any difference which way light travels through a thin lens?

Exercise 3.5.2

Drag the object back and forth. Describe what you see. What two things are different about the image if the object is closer than the focal length, as compared to when it is further away from the focal length?

Exercise 3.5.3

Use the slider to change the height of the object. How does the height of the image compare to the object height? Does the height of the object change any of your conclusions from the previous question? Explain.

Exercise 3.5.4

For all cases a one ray goes straight through the center of the lens. Why is that? (Hint: Read the introduction.)

Exercise 3.5.5

Carefully describe the other two rays. What happens to a ray that enters the lens parallel to the horizontal axis? What happens to a ray that goes throught the focus (if the object is further away from the focus)? What happens to a ray that appears to come from the focus (if the object is closer than the focus)?

The previous two questions are about the rules for drawing light rays for a converging lens:

1. Rays parallel to the axis bend as if coming from the focus;

2. Rays going through the focus (or coming from the focus if the object is closer to the focus) bend to exit the lens parallel to the axis; and
3. Rays through the center go straight through without bending. Using these three rules, it is possible to determine where the image will be and how big it will be for any converging lens.

Exercise 3.5.6

Now choose the diverging lens case and experiment. How is it different from the converging case? How does the image size compare with the object size? Is there any case where the image is bigger than the object?

Exercise 3.5.7

One of the rules for drawing rays for a diverging lens is the same as for a converging lens. Which one?

Exercise 3.5.8

Carefully state what happens to a ray that is parallel to the axis when it exits a diverging lens. Also describe what happens to a ray that starts from the object and heads towards the focus on the opposite side. How are these rules different from the converging lens case?

Exercise 3.5.9

As in the case of mirrors, some images from lenses are real (can be projected onto a screen) while others are virtual (are only seen by looking through the lens). For lenses, real images appear inverted and on the other side of the lens. Which cases above had real images and which had virtual images?

Exercise 3.5.10

Your eye has a single lens which projects a real image onto your retina. The retina turns the image into nerve impulses which go to the brain to be interpreted. What is the orientation of this image? Is this surprising?

3.6: Path Difference and Interference

Tutorial 3.6: Interference due to Path Difference

In simulation 2.1 we saw constructive and destructive interference which was the result of adding waves that had different phases. How can two identical waves end up out of phase with each other? If two waves travel a different distance they can end up arriving out of phase. This is in fact what causes the interference patterns seen in simulation 2.2 where there were two point sources. Waves arriving at points along a center line between the sources travel the same distance and so arrive in phase. For points not on the central line the waves may arrive in phase or out of phase, depending on the distance they travel.

In this simulation the top two waves are identical but start at different locations. The bottom graph shows the sum of the two waves. Depending on the path difference, D, the two waves may end up exactly in phase (leading to constructive interference), exactly out of phase (destructive interference) or something in between. The wavelength can also be changed by changing the wavenumber, k. The units of $k = 2\pi/\lambda$ and distance, D, are arbitrary.

Path Difference

Questions:

Exercise 3.6.1

Start the simulation with the step thickness, D, equal to zero. Are the waves in phase? Slowly increase the Step Thickness until the the waves are exactly out of phase. What step thickness causes this?

Path differences can occur in a number of different ways. In the Ripple Tank simulation of the double slit experiment (tutorial 2.2) the distance to a point on the screen is different for each source (except for the center of the screen) so the light experiences a path difference. If the path difference is a whole number of wavelengths ($1,\ 2,\ 3\ldots$) there is constructive interference and a bright spot for that color appears on the screen. If the path difference is $1/2,\ 3/2,\ 5/2$etc. of a wavelength then a dark spot appears due to destructive interference.

Exercise 3.6.2

Based on your findings in question $3.6.1$, what is the wavelength (in arbitrary units) of the waves in the simulation? (Recall that $k = 2\pi/\lambda$.)

Exercise 3.6.3

Reset the simulation and slowly increase the step hickness to find the first three thicknesses that cause destructive interference. Verify that destructive interference occurs at step thicknesses given by $1/2\lambda,\ 3/2\lambda$,and $5/2\lambda$.

Path differences can also occur due to reflection from a surface that has multiple layers. In this case the waves that hit the deeper surface travel *twice* the distance between the surfaces before recombining (*twice* the thickness in the simulations above for a wave that comes in from the right and reflects back to the right).

Exercise 3.6.4

Imagine this simulation now represents **monochromatic** (single wavelength) light reflecting off a surface with two levels. Only the reflected waves (going to the right) are show (the incoming waves coming from the right are not shown). Now the path difference is twice the depth of the step. How would this change the results? In which case would there be no reflection from the surface? In which case would there be constructive interference?

The formula for **constructive interference due to a path difference** is given by $\delta = (m + 1/2)\lambda/n$ where n is the index of refraction of the medium in which the wave is traveling, λ is the wavelength, δ is the path difference and $m = 0,\ 1,\ 2,\ 3\ldots$. For a *reflected wave* in the simulation $\delta = 2X\ \text{Step Thickness}$ is the actual path difference; a wave reflected off the upper surface must travel an extra distance equal to twice the Step Thickness to catch up with a wave reflected off the lower surface.

Exercise 3.6.5

Reset the simulation and enter 1.57 for the Step Thickness (this is the case of half a wavelength path difference so the waves cancel). Increase the wave vector, $k = 2\pi/\lambda$, until you find the next wavelength that experiences destructive interference (don't change the Step Thickness). What is the wave vector k and wave length of this wave?

Exercise 3.6.6

For light, changing the wavelength changes the color. Can the same Step Thickness cause destructive interference for all wavelengths? Explain.

Exercise 3.6.7

A music CD has information stored on it in the form of tiny divots blasted into the surface with a laser. Suppose you see constructive interference for red light (wavelength of 650 nm). What is the minimum ($m = 0$) depth of the divots? (Hint: The path difference is twice the divot depth.)

Exercise 3.6.8

Explain why you only see one particular color when looking at a small region of a CD at a fixed angle. What happens to the other colors?

Exercise 3.6.9

If you look at the colors being reflected from a CD you will notice that the color changes depending on the angle. How does the path difference change as you look at the divots at different angles? (Hint: Imagine the waves in the simulation coming in at different angles instead of horizontally. Now the path difference is the hypotenuse of a triangle, one side of which is the Step height.)

Exercise 3.6.10

Suppose you wanted to make a "stealth" jet plane which was non-reflective to a particular wavelength of radar. Describe one way you might try to do this by modifying the surface of the plane.

Exercise 3.6.11

Some insect wings and the feathers of some birds (for example peacocks) exhibit a feature known as **iridescence**. From a fixed angle only one color of reflected light can be seen. Explain this phenomena given the fact that insect wings and feathers consist of overlapping layers causing the surface to be multi-layered.

Exercise 3.6.12

Soap bubbles show different colors at different places on the bubble. So do oil slicks on water. In both cases light reflects off the upper and lower surfaces of the layer of soap or oil. Explain the different colors in terms of path difference (Hint: draw a picture where the wall of the soap bubble is nearly the same thickness as one wavelength and explain why the path difference is twice the thickness of the soap).

There is one other detail needed to explain the soap bubble and oil slick color phenomena completely. The light reflecting from the top surface is going from a "soft" medium (air) to a "stiff" medium (soap) but the light reflecting from the bottom layer of the soap is going from a "stiff" medium (soap) to a "soft" medium (air inside the bubble).

Exercise 3.6.13

From what you learned about reflection and transmission of waves from "stiff" and "soft" boundaries in simulation 3.2, what happens to a wave reflected from the top layer of a soap bubble which does not occur at the bottom layer?

Exercise 3.6.14

If the path difference for a particular thickness of soap film was just right for destructive interference but there was a 180° phase change for the top reflected wave but not the bottom reflected wave, what would happen to that color?

3.7: Impedance

Tutorial 3.7: Impedance: Collisions with boundaries

Generally the term **impedance** refers to how easily oscillating energy is transferred from one location to another. There are many kinds of impedance; mechanical, electrical, acoustic and wave impedances all have definitions specific to their use in different fields. Here we will briefly investigate the mechanical energy transferred from one medium to another.

In many cases a wave colliding with a boundary will partially reflect and be partially transmitted. The kind of wave reflected and the amount of energy transmitted depend on the properties of the material on either side of the boundary. This animation again simulates a string as a row of individual masses connected by invisible springs. In this case the mass of the string is different on the left as compared with the right.

Impedance

Questions:

Exercise 3.7.1

Run the simulation and describe what happens when a pulse goes from a light string to a heavy string.

Exercise 3.7.2

Click the 'Heavy to Light' button, run the simulation and describe what happens when a pulse goes from a heavy string to a light string.

Exercise 3.7.3

In which case is the reflected pulse inverted? Based on what you learned about reflections from soft and hard boundaries in the previous simulation, explain this result.

Exercise 3.7.4

In which case is the reflected pulse larger than the transmitted pulse?

Exercise 3.7.5

In which case is the reflected pulse faster? Based on what you learned about how physical properties determine the speed of a wave, explain this result.

In this simulation the change in mass of the string affects the amount of reflected and transmitted energy and the speeds of the waves. If the string had the same mass on both sides there would be no reflection. A similar thing happens when two electrical devices are hooked together and one device is sending energy to the second one (for example signals going from a stereo system to a set of speakers). In this case the impedance is determined by the components (resistors, capacitors, inductors, etc. in the circuits) and is frequency dependent. If the electrical impedance of the two devices is different, some energy is reflected rather than being transmitted to the second device.

Exercise 3.7.6

You are buying a stero system and a set of speakers. The stereo has an output imedance of 10 ohms. What impedance speakers do you need to buy to get the loudest sound?

One way to try to **match impedances** is to gradually change the medium between two different values. This is the purpose of the bell on brass and woodwind instruments. An instrument such as a trumpet would not produce as much sound if it ended abruptly with no flared bell on the end. This is because there is an impedance missmatch for waves inside the instrument (where the pressure is constrained by the sides of the instrument) and the pressure outside.

Exercise 3.7.7

Why are flutes softer than trombones?

The bell on an instrument also affects the frequencies produced. If there was no impedance missmatch at all there would be no reflected wave going back into the instrument from the bell region. This reflected wave is needed to set up a standing wave (much like the standing wave in on a string) in the instrument so that a given pitch is produded.

Exercise 3.7.8

At the mouthpiece of a trumpet the wave reflects from a "hard" surface. At the other end (the bell) the wave reflects at a boundary that is going from stiffer (inside the horn) to softer; a "soft" boundary. Based on this and simulation 3.2, which end has a dispacement node and which end has an anti-node?

Note

The closed end at the mouthpiece doesn't allow the movement of air but that means there is a large fluctuation of pressure. So an air displacement node occurs at the same location as a pressure anti-node and vice versa.

3.8: Dispersion of Light

Tutorial 3.8: Dispersion I

In the previous simulations on refraction we assumed that all wavelengths bend by the same amount. This is not true; the index of refraction changes slightly for different wavelengths. So if we start with several different wavelengths (different colors) we expect there may be some situations in which the colors will separate. If the sides of the medium are parallel, each color unbends by the same amount that it bent going into the medium so all the colors are again going in the same direction. However if the sides are not parallel, such as a prism or lens, there will be a separation of color. This is in fact how a prism and water droplets separate colors and why good camera lenses (which compensate for this effect by using compound lenses) are expensive.

The change of wave speed as a function of wavelength is called **dispersion** and occurs for all types of waves. For example, longer wavelength surface waves on the ocean travel faster than shorter wavelength waves. There is not much dispersion for sound waves in air but acoustic waves in solids do experience significant dispersion. The simulation below is for visible light passing through a prism. You can choose the color and see what the index is for that wavelength. A different manifestation of dispersion is shown in the next simulation.

Dispersion of Light

Questions:

Exercise 3.8.1

Use the slider at the bottom of the simulation to try different wavelengths. Which visible wavelength is bent the most? Which the least? Note that the wavelength is given in nanometers (nm).

Exercise 3.8.2

What would would you see on the right if the source were a white light composed of all wavelengths?

Exercise 3.8.3

The speed of light is $c = 3 \times 10^9 \text{ m/s}$ and the index of refraction is $n = v/c$ where v is the speed in the medium. Using the index given in the simulation for the chosen wavelength, what are the maximum and minimum speeds for colors in the visible spectrum?

Exercise 3.8.4

For one of the wavelenghts use the protractor to measure the incident and refracted angles (as you did in tutorial 3.4) for the exiting ray on the right. Calculate the index of the prism for that color using Snell's law. Don't forget that the angles are measured from a perpendicular to the surface (you will have to correct for the fact that the prism sides are slanted at 60 degrees). What is your answer? Do you get the index of refraction shown in the simulation for that color?

3.9: Dispersion of Fourier Components

Tutorial 3.9: Dispersion II

In simulation 2.5 on the Fourier series we found that complicated periodic wave forms can always be constructed from sine and/or cosine functions of different frequencies and wavelengths. In the previous simulation we found that different wavelengths may travel at different speeds depending on the medium. So what happens to a complex wave shape if it travels through a medium where the individual components have different speeds?

We can write a Fourier series for a square wave moving in time and space as $y(x,t) = \sum_{n=1} A_n \sin(nkx - n\omega t)$ where n is the number of the harmonic or mode ($n = 1$ for the fundamental, 2 for the second harmonic etc.), A_n is the amplitude of harmonic n, k is the wave vector and ω is the angular frequency. Recall from simulation 2.3 that the group velocity of a combination of waves is $v_{\text{group}} = \partial\omega(k)/\partial k$. The dependency of ω on k is called the **dispersion relation**. In a vacuum or medium with no dispersion we expect each component of the series to have the the same speed, $v = \omega/k$, so the dispersion relation is $\omega(k) = kv$ and the group velocity is v, the same as the individual components. In this case the square wave would not change shape as it travels.

In real life, however, it is often the case that the angular frequency, $\omega(k)$, is *not* a linear function of the wave vector, k in which case the individual components of the Fourier series travel at different speeds. If different frequencies of a wave travel at different speeds the effect is called **dispersion**. As we saw in the previous simulation, dispersion causes the separation of colors by prisms, water droplets, etc. In this simulation we explore a different aspect of dispersion.

Dispersion

Questions:

Exercise 3.9.1

The simulation starts with the first four components of the Fourier series for a traveling square wave with no dispersion. Play the simulation and describe what happens to the shape as time goes on.

Exercise 3.9.2

Given that the speed of a sine wave is $v = \omega/k$, what are the speeds of the first four components of the square wave: $y(x,t) = \sin(1*x - 1*t) + \sin(3*x - 3*t)/3 + \sin(5*x - 5*t)/5 + \sin(7*x - 7*t)/7$

Exercise 3.9.3

What would be the fifth term in the Fourier series of a square wave? Add your answer to the first four terms and see if the shape is closer to a square wave. It would require an infinite number of terms to create a perfect square wave but we can get as close as we like by adding as many terms as necessary.

Exercise 3.9.4

Click 'reset' and then change the angular frequency of the second term from 3 to 2.95 and hit enter. This will cause the second term to have a slightly different speed. What is this new speed for the second term? How does the initial shape compare with the initial shape in question $3.9.1$ (if you reset before entering the change they should be identical)?

Exercise 3.9.5

Now play the simulation for the wave in the previous question. What happens to the shape of the square wave in this case as time goes on?

Exercise 3.9.6

Reset to the original case and change the angular frequency of the third term from 5 to 4.95. What effect does this have on the behavior of the wave?

Exercise 3.9.7

Based on the previous two questions, explain what would happen to a digital signal (which is basically a series of square waves) traveling down a cable (either wires or optical fiber) where there is a small amount of dispersion.

Exercise 3.9.8

All cables (fiber optical or metal) have some dispersion. Why is there a limit to how long a cable can be before a signal traveling on it has to pass through a relay (where the signal is amplified and 'cleaned up')?

CHAPTER OVERVIEW

4: Applications

These materials consider various applications of the wave properties studied in previous sections.

4.1: Doppler Effect

Tutorial 4.1: The Doppler Effect

If either the source or the receiver of a wave are in motion the apparent wavelength and frequency of the received wave change. This is apparent shift in frequency of a moving source or observer is called the **Doppler Effect**. The speed of the wave is *not* affected by the motion of the source or receiver and neither is the amplitude. This simulation looks at the Doppler effect for sound; the black circle is the source and the red circle is the receiver. The time is measured in centiseconds (10^{-2} s), distances are in meters. The speed of sound is $345 \mathrm{~m/s}$. A similar effect occurs for light (speed $= 3 \times 10^8 \mathrm{~m/s}$) but in that case the source and receiver cannot travel faster than the wave speed (the speed of light).

Doppler Effect

Questions:

Exercise 4.1.1

Click the play button to see a stationary source and receiver (Animation 1). Reset and use the pause and step buttons to verify that the period at the receiver (time elapsed from when one wave reaches the receiver until the next one reaches it) is $0.5 \times 10^{-2} \mathrm{~s}$. What is the frequency of this wave?

Exercise 4.1.2

After there are several waves in the simulation pause it and use the mouse to find the wavelength (distance between two successive crests). What are the wavelength and speed of the wave (wavelength/period)?

Exercise 4.1.3

Now look at Animation 2, where the receiver is moving. Use the step button above to find the period (time between crests) *as measured by the moving receiver* when it is on the right of the source (moving towards the source). What is the frequency at the receiver if it is moving towards the source?

Exercise 4.1.4

When the receiver gets to the left of the source (moving away from the source) pause the simulation and measure the period. What is the frequency at the receiver if it is moving away from the source?

Exercise 4.1.5

Now look at Animation 3 which shows the source moving but the receiver stationary. Again find the frequency while the source is on the left, moving towards the receiver and the frequency when it is on the right moving away.

Animation 4 shows the effects of a moving source and a moving observer at the same time. The equation for the Doppler shift with both a moving source and observer is given by $f' = f(v \pm v_0)/(v \mp v_s)$ where f' is the received frequency, f is the original frequency, v is the speed of the wave, v_0 is the speed of the observer and v_s is the speed of the source. The upper signs in the equation are used if either the observer or source is moving towards each other and the lower signs are used if the either object is moving away from the other (so if the observer is moving towards the source but the source is moving away from the observer the equation to use is $f' = f(v + v_0)/(v + v_s)$).

Exercise 4.1.6

For the case of the moving receiver and stationary source ($v_s = 0$) use the original frequency you found in question 3.10.1, the shifted frequency (f') you found in question 3.10.3 and the speed of sound you found in 3.10.2 to find the speed of the observer.

Exercise 4.1.7

Animation 5 shows a source moving faster than the speed of the sound wave. In this case all of the wave crests arrive together forming a shock wave or "sonic boom". Why can this not happen in the case of light from a moving light source?

Electromagnetic waves will also undergo a Doppler shift except that the relative velocity between the source and observer can never be larger than the speed of light and the formula for calculating the shift is slightly different. For electromagnetic waves we have $f' = f((c+v)/(c-v))^{1/2}$ where v is the relative speed between the observer and source (positive if they are approaching and negative if they are moving away from each other) and c is the speed of light.

Exercise 4.1.8

As you can see from the question 3.10.6, if the speed of the wave is known and the original and received frequencies are know the speed of the source or observer can be found. Explain how you could determine the speed of a car or thunderstorm by bouncing radio or microwaves off of them. (Police radar and thunderstorm tracking both use the Doppler Effect.)

Exercise 4.1.9

If a car goes past with its radio blaring we easily hear the Doppler shift for sound as the car passes (the sound appears to shift from a pitch which is too high to one which is too low).

Note

We are talking about the change in pitch, *not* the change in volume. But if a car goes past with its lights on we do not notice the Doppler shift for light (the color does not seem to shift towards the red frequencies). Explain why this is so. (Hint: Try plugging in some numbers for a car speed in the equation for the Doppler shift for light).

Exercise 4.1.10

If an astronomer notices that the spectrum of colors coming from a star are all shifted towards the red end of the spectrum (the frequencies are lower than they should be) what can she conclude about the motion of the star relative to the earth? (This is one of the pieces of evidence that the universe is expanding; nearly all the stars and galaxies around us are moving away from us.)

4.2: EM Waves from an Accelerating Charge

Tutorial 4.2: Accelerated Charge Radiation

Electric charges have electric fields. The simulation first shows a moving positive charge and the electric field around it in two dimensions. If the charge is accelerated there will be a disturbance in the field. This is the origin of electromagnetic waves. Note that the energy carried by the disturbance comes from the input energy needed to accelerate the charge.

Accelerating Charge

Questions:

Exercise 4.2.1

Run the simulation. Change the speed, v, and describe what you see. How does the *change* in speed (acceleration) affect the initial disturbance of the field (try changing the speed slowly versus rapidly)?

Exercise 4.2.2

What happens at constant speed? Is there still a disturbance? What happens if you suddenly slow the charge down?

Exercise 4.2.3

Now click the Animation 2 button. Describe what you see. Explain what you would notice about the field over time if you were measuring it at a point along the x-axis and compare that with what you would measure at a distant point along the y-axis.

A charge oscillating in the y-direction will produce an electromagnetic wave traveling in the x-direction as seen in Animation 2. For directions other than along the x-axis the wave has a lower amplitude (smaller variation from equilibrium), dropping to a zero amplitude along the y-direction. This configuration is called a **dipole antenna**. (To be technically correct a single wire with an oscillating charge is a monopole antenna. A dipole is created from two wires with opposite polarities, one in the x-direction the other in the $-x$-direction but in the present context we can ignore this subtlety.) Dipole antennas emit the strongest signal in a direction perpendicular to the antenna as Animation 2 shows (remember, the field is the same strength in both directions but the *change* in the field is zero in y-direction, largest along the x-axis). FM radio, AM radio, TV, cell phone, WiFi and short wave radio sending antennas are dipole or approximately dipole antennas.

Exercise 4.2.4

Why are sending antennas usually oriented vertically?

Note

Short wave antennas are sometimes oriented horizontally so that the signal can bounce off the ionosphere and return to earth a large distance away.

Exercise 4.2.5

Try different oscillation speeds for Animation 2. If you were measuring the field on the x-axis, how would the frequency of the wave compare with the frequency of oscillation of the charge in the antenna?

Exercise 4.2.6

Now try Animation 3. How would the amplitude of the wave at points along the x-axis compare with the amplitude of the wave along the y-axis for this case?

4.3: Antenna

Tutorial 4.3: Antenna

In the previous simulation you saw the electromagnetic waves created by an oscillating electric charge. In this simulation we look at only the wave traveling in the x-direction and its effect on a second charge inside a receiving antenna. Only the y-component of the *change* in the electric field is shown (so an oscillation frequency of zero will show nothing, because there is only a constant electric field).

The simplest type of receiving antenna can be approximated by free charges (electrons) constrained by a metal wire. For a receiving antenna oriented in the y-direction, an oscillating field traveling in the x-direction will cause the charges in the receiver to oscillate in the y-direction with the same frequency as the wave (the charges cannot move in the x-direction because they are confined to the wire). This oscillating current can then be analyzed by electronic circuitry to extract the transmitted signal. A general rule of thumb is that for strongest reception, the receiving antenna should be roughly the same length as the wavelength of the wave it is trying to receive.

For simplicity this simulation has oscillating positive charges in the sending and receiving antennas. (Electrons feel a force in the opposite direction of the applied field.) Time is measured in microseconds (10^{-6} s).

Antenna

Questions:

Exercise 4.3.1

Run the simulation and describe what you see. Does the receiving antenna charge start oscillating immediately? Why not?

Exercise 4.3.2

In the previous simulation you learned that sending antennas give the strongest signal perpendicular to the antenna and so are usually mounted vertically. This is so that the signal is equal in all horizontal directions where the receiving antennas will be located. Why are receiving antennas generally oriented vertically?

Exercise 4.3.3

Try different oscillation frequencies. How does the frequency of the sending charge compare with the frequency of oscillation of the charge in the receiving antenna?

Exercise 4.3.4

Use the step button to find the time lapse between when the source charge starts to oscillate and when the receiver starts oscillating. If the receiving antenna is $1.6 \times 10^3\ \mathrm{m}$ away, what is the speed of the wave?

Exercise 4.3.5

Repeat the previous exercise with different oscillation frequencies. Does changing the oscillation frequency change the speed the wave travels in the x-direction? What does change if the oscillation speed changes? (Hint: Recall that $v = \lambda f$ where v is the speed of the wave.)

Exercise 4.3.6

What do you notice about the amplitude of the wave as it travels away from the antenna? Explain.

Exercise 4.3.7

Why are cell phone antennas small (for modern cell phones they are hidden inside the phone itself) but FM and AM radio antennas and TV antennas typically around a meter in length? Why are short wave radio antennas much larger?

The simulation is incomplete in one sense because we know that a moving charge also creates a magnetic field. The Biot-Savart law and Ampere's law tell us that for positive charge flowing in one direction (for example upwards in the translation case) a magnetic field will be formed in a circle around the electric flow in a right hand sense (if your thumb points in the direction of positive charge flow, your curled fingers give the direction of the magnetic field).

Exercise 4.3.8

What direction will the magnetic field point in the vicinity of the sending antenna if the charge is moving upward? (Hint: Describe the field to the left and right the charge flow as well as behind the screen and in front of the computer screen.)

4.4: Electromagnetic Plane Waves

As we have seen previously, if you are far enough from a source of spherical waves, the waves flatten out into waves that can be approximated by plane waves. In the following simulation a plane electromagnetic wave is traveling in the y-direction. It's time dependent magnitude is initially in the x-direction and is given by $E(x,t) = E_x \sin(ky - \omega t + \varphi)$ where E_x is the maximum electric field in the x-direction and $E_x = 1.0\ \mathrm{N/C}$ in this simulation. The graph below shows the electric field, $E(y = \text{slider position}, t)$ in the $x - z$ plane at the location of the magenta square. The location of the square can be changed using the slider. You can also change the viewer's perspective of the wave (top graph) by grabbing the wave with the mouse and rotating the box.

Electromagnetic Waves

Questions:

Exercise 4.4.1

Play the simulation. Grab the box on the top with the mouse to look at the wave from several orientations. Describe what is happening to the electric field in the $x - z$ plane at the location of the magenta box.

Exercise 4.4.2

Time in nanoseconds ($\times\ 10^9$ sec) is shown on the graph. Step through the simulation from a time when the electric field is a maximum in the plus-x direction until it is again a maximum in the plus-x direction. What is the period of this wave? What is the frequency? What part of the electromagnetic spectrum is this?

Exercise 4.4.3

Pause the simulation and use the slider to slowly move the magenta square back and forth. Describe what you see in the lower graph for various locations of the magenta square.

Exercise 4.4.4

The slider shows the position of the magenta square in a small window to the left of the slider. Slide the square from a position of maximum electric field to the next position of maximum electric field to find the wavelength of this wave. Does it match a calculation based on $v = \lambda / T$?

Exercise 4.4.5

Reset the simulation and click on the 'show B' box. The magnetic field is in red and is measured in tesla, $T = \mathrm{Vs/m^2}$, or gauss, where $1\ \mathrm{G} = 10^{-4}\ \mathrm{T}$. Play the simulation and look at it from several different angles. What is the relationship between the magnetic and electric field? How are they oriented? Do the have the same amplitude? The same wavelength? The same period?

The simulation represents the electric component of a wave that is **polarized** in the x-direction. In other words, the components of the electric field point in either the plus or minus x-direction and the E_z component is initially zero.

Exercise 4.4.6

Reset the simulation and add an z-component of the electric field using the slider. Play the simulation and rotate it to see what this new wave looks like. It is still polarized but no longer in the x-direction. Describe what is different about this wave than the initial case.

Exercise 4.4.7

What is the direction of polarization of this wave for the maximum z-component available using the slider?

Exercise 4.4.8

What would be the direction of polarization if the y-component was zero (instead of $1.0\ \mathrm{N/C}$) and the z-component was $1.0\ \mathrm{N/C}$?

To be consistent with Maxwell's equations, the cross product $\mathbf{E}\times\mathbf{B}$ is a vector in the direction of motion of the wave where E and B are the magnitudes of the electric and magnetic fields which are related by $E/B=c$ where c is the speed of light. In the simulation this would make B too small to be visible using the same scales so in this sense the simulation is misleading; the scales for E and B are not the same.

Exercise 4.4.9

Add the magnetic field for the previous case of a polarized wave. Play the simulation, rotate the view using the mouse, pause and slide the square. What is the relationship between the magnetic and electric field in this case? How are they oriented? Is it true that $\mathbf{E}\times\mathbf{B}$ is a vector in the direction of motion? Is this true at all times and all slider positions? Why is the magnetic field perpendicular to the electric field (Hint: Think about Ampere's law)?

4.5: Polarization

Tutorial 4.5: Polarization

In the last simulation a **polarized wave** was defined to be an electromagnetic wave that has its electric field confined to change in only one direction. In this simulation we will further investigate polarized waves. In the simulation the small graph on the upper right (which can be expanded with the Phasor checkbox) shows the electric field component[s] for a plane wave traveling straight towards you in the $+y$ direction and the resultant field, $E(y,t) = E_{\max}\sin(ky - \omega t)$, in blue. The two other graphs plot the E_x and E_z components of the electric field. The magnitude of E_x is fixed at $6.0\ \mathrm{N/C}$. The magnetic component, always perpendicular to the electric component, is not shown. In all cases the components are sinusoidal (the time component of the field is shown for $\omega = 1$ and a fixed location of $y = 0$).

Polarization

Questions:

Exercise 4.5.1

Play the simulation. Describe what you see. The graph on the right, which can be enlarged with the phasor check box, is what was happening in red box in the initial case of the previous simulation; an electric field oscillating in the x-direction. What is the maxium field in the present case? Is this a polarized wave?

The Poynting vector is the vector cross product of the electric and magnetic field vectors: $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ where μ_0 is a constant called the permeability (recall from simulation three that the speed of an electromagnetic wave is given by $c = (1/\mu_0\varepsilon_0)^{1/2}$ where μ_0 is the permeability and ε_0 is the the permittivity. The magnitude of the vector gives the intensity of an electromagnetic wave in $\mathrm{W/m^2}$ and the vector points in the direction that the wave is traveling. Since the magnetic field amplitude is proportional to the electric field amplitude, the Poynting vector (the intensity) is proportional to electric field amplitude squared.

Exercise 4.5.2

If the Poynting vector points out of the screen towards you, what direction does the magnetic field point that corresponds to the electric field vector shown in the simulation initially?

Exercise 4.5.3

The x-component of the electric field is fixed at $6\ \mathrm{N/C}$. Use the slider to choose a value of $6\ \mathrm{N/C}$ for the value for the z-component of electric field and play the simulation. Describe what you see. How is this case similar to that of question 4.2.7? Is this a polarized wave?

Exercise 4.5.4

Try some other values for E_z. Describe the case for $E_x = 6\ \mathrm{N/C}$ and $E_z = 4\ \mathrm{N/C}$. Which way does the electric field vector point? These are all polarized waves with different orientations.

The phase, φ, determines the phase between E_x and E_z. For zero phase the two vector components start out at zero at the same time and increase together, in phase.

Exercise 4.5.5

For $E_z = 6\ \mathrm{N/C}$ choose a phase difference of $1\ \pi$ radians (use the slider to set the phase to 1.0). What do you observe?

Exercise 4.5.6

Reset the simulation, use $E_z = 6 \text{ N/C}$ and try $0.5\ \pi$ radians. This case is called **circularly polarized** light. Note that the x- and y-components are still sine waves but the total electric vector (blue arrow) has a fixed magnitude. Describe what you see.

If the phase angle φ is not a whole number or half a whole number times π the light is **elliptically** polarized.

Exercise 4.5.7

Try other values for the phase with the maximum amplitudes the same. Describe what you see. What can you conclude about whole numbers of π radians for a phase difference? What about half whole numbers? What about values in between?

Exercise 4.5.8

In your own words, describe how an elliptically polarized electromagnetic wave looks as it probagates through space. (Recall that the wave is travelling in the y-direction which is out of the screen towards you in this case.)

Cirularly and elliptically polarized waves can be **right-circularly polarized** or **left-circularly polarized** depending on the sign of the phase angle.

Exercise 4.5.9

Try some negative values for the phase. What is the difference between negative and positive values of phase? In which case does the polarization rotate clockwise as as the wave probagates forward?

Exercise 4.5.10

Explain the difference between left and right circular polarization.

4.6: Wave Equation

Tutorial 4.6: The Linear Wave Equation

In mechanics the subject of kinematics is the description of the motion of objects (velocity, acceleration, etc.) while the subject of dynamics describes the physical situation and the forces that give rise to the motion (Newton's laws). So far we have looked at the kinematics of waves (descriptions of their shapes and how they move). In this simulation we look at the dynamics of waves; the physical situations and laws give rise to waves.

We start with a string that has a standing wave on it and look at the forces acting on each end of a small segment of the string due to the neighboring sections. For visualization purposes the string is shown as a series of masses but the physical system is a continuous string. Although the derivation is for a string, similar results occur in many other systems. The ends of the section of string we are interested in are marked with red dots in the simulation. The tension acting on each end is shown with a vector (in red) and its components (green and blue). The horizontal forces cancel (the string segment does not move to the left or right) but there is a net force in the vertical direction (the left and right vertical components are not the same). Remember from simulation four that there is a transverse acceleration (and therefore a transverse force) that changes over time at each point on the string.

Forces on a Segment

Questions:

Exercise 4.6.1

Before playing the simulation, use the protractor and measure the angle of the tension at each end of the segment (the protractor can be dragged and also the arrow tip can be moved). Are they the same angles?

Exercise 4.6.2

Play the simulation. Based on what you know about the acceleration of a point on a sine wave, explain why the magnitude and direction of the net force on the segment acts the way it does. (Hint: Review simulation four, question seven.)

The tension on segment is T_1 from the neigboring string on the left and T_2 from the string to the right. If θ_1 is the angle the tension makes with the horizontal on the left end of the segment and θ_2 is the angle with the horizontal on the right end then the total y force is $F=-T_1\sin\theta_1+T_2\sin\theta_2$.

If the wave is of low amplitude the angles are small and we have $\sin\theta\approx\tan\theta$ which is the slope of the string at that point. Slope is given by the derivative of the position, $\tan\theta=\partial y/\partial x$, so we can replace the total force on the segment by $F=T(-\partial y/\partial x_1+\partial y/\partial x_2)$. The segment has a length of Δx. Multiply and divide by this to get $F=T\Delta x(-\partial y/\partial x_1+\partial y/\partial x_2)/\Delta x$. If we now let Δx become very small this reduces to a second derivative: $F=T\Delta x\partial^2 y/\partial x^2$.

We also know from $F=ma$ the force is proportional to acceleration, $a=\partial^2 y/\partial t^2$. For a mass per length given by μ the mass of the segment is $\mu\Delta x$ and we have $F=\Delta x\mu\partial^2 y/\partial t^2$.

Setting these two equations for force equal to each other we have the wave equation for a string: $T\partial^2 y/\partial x^2=\mu\partial^2 y/\partial t^2$ (the Δx cancels). Recall from simulation two that $v=(T/\mu)^{1/2}$ for waves on a string so we can also write the **linear wave equation** as $\partial^2 y/\partial x^2=1/v^2\partial^2 y/\partial t^2$. Although we started with and string and applied $F=ma$, this equations turns up in many other physical situations. The same equation holds for sound waves in gas, liquids and solids and for electromagnetic waves; only the velocity is different, as noted in simulation three.

Exercise 4.6.3

Verify that $y(x,t)=A\sin(kx-\omega t)$ is a solution to the linear wave equation. Do this by taking two x derivatives (the left side of the equation) and two time derivatives (the right side of the equation) and substituting your answers for the terms in the linear wave equation. Cancel similar terms on both sides. You should be able to show that, in order for $y(x,t)=A\sin(kx-\omega t)$ to be a solution you must have $v=\omega/k$.

Exercise 4.6.4

Verify that $A\exp(-(kx-\omega t)^2)$ is a solution to the linear wave equation (this is the Gaussian wave pulse in simulation 2.4). Do this by taking two x derivatives (the left side of the equation) and two time derivatives (the right side of the equation) and substituting your answers for the terms in the linear wave equation. Cancel similar terms on both sides. You should be able to again show that $v=\omega/k$ if this is a solution to the linear wave equation.

In several simulations we added waves together. This means that if $y_1(x,t)$ is a solution and $y_2(x,t)$ is a solution we have assumed that $y(x,t)=y_1(x,t)+y_2(x,t)$ is also a solution to the linear wave equation.

Exercise 4.6.5

Prove the previous statement is true by substituting $y(x,t)=y_1(x,t)+y_2(x,t)$ into the linear wave equation. Separate out terms to find two wave equations, one for $y_1(x,t)$ and a second for $y_2(x,t)$ which are equal to each other. This proves the law of superposition; when two waves arrive at the same point at the same time we can simply add their amplitudes to find the resulting wave.

4.7: Oscillator Chain

Tutorial 4.7: Chains of Oscillating Masses

In this simulation we examine waves that occur on chains of masses with mass M coupled together with elastic, Hooke's law forces ($f = -\kappa x$ where κ is the spring constant and x is the amount the spring stretches). The masses are constrained to only move up and down so that the stretching depends only on the difference in the y locations of the masses. In this case the force on mass number i due to its neighbors at $i+1$ and $i-1$ is $F_i = -\kappa[(u_{i+1} - y_i) - (y_1 - y_{i-1})]$. The masses on each end of the chain are fixed and there is a small amount of friction in the system so that eventually the oscillations will die off.

As a first approximation, atoms in a solid can be imagined to be coupled by spring-like forces to their neighbors so this simulation models a one dimensional solid. In the limit as the distance between the masses becomes very small this model turns into the model of an elastic string.

The simulation opens a sine wave with 32 masses. The number of masses can be changed and you can also grab a single mass, move it to a new location and start the simulation with this new configuration. The other buttons set up initial conditions for different numbers of masses. Each of these special cases are an example of a **Normal Mode** of the system.

Coupled Oscillator Chain

Questions:

Exercise 4.7.1

Try the various pre-set configurations using the buttons below the simulation. How many normal modes are available for three masses (one moving mass) on the chain? For four masses (two in motion)? For five masses?

Exercise 4.7.2

Use the pause and step buttons to measure the frequency of each mode. What are they? Are they the same?

Exercise 4.7.3

Sketch the possible modes for four moving masses (six masses total). How many modes are there?

A normal mode is a special configuration (state) where every particle moves sinusoidally with the same angular frequency ω_m where m is an integer. The m-th mode, Φ_m, of the oscillator chain of length L with N masses is given by $\Phi_m(x,t) = \sin(m\pi x/L)\cos(\omega_m t + \varphi)$. The angular frequency of each particle is given by $\omega_m^2 = (4\kappa/M)\sin^2(m\pi/2N)$.

It turns out that any possible type of vibration can be described mathematically by a sum of the normal modes with appropriate amplitudes. This is equivalent to the statement we investigated in simulations 2.5 and 3.9; complicated periodic waves can always be described by a Fourier series of sines and cosines. The difference for masses on a string is that there are only a finite number of modes available. In a continuous system there are an infinite number of modes.

Exercise 4.7.4

You can also grab and change the position of the masses in the simulation. Try this starting with the sine wave initial condition. Describe what you did and what you see.

In simulation 3.9 we saw the linear dispersion relation, $\omega(k) = kv$ which tells us that the angular frequency is proportional to the wave vector, k. If the speed of the wave v is independent of frequency (i.e. there is no dispersion) then a plot of ω versus k is a straight line. Since $\omega(k)$ is a continous function there is no limitation on the value of wavelengths, as long as the proportionality holds.

But on a string of masses you cannot have wavelengths that are shorter than the distance between the masses (there is nothing there to vibrate). So the dispersion relation for a mass string cannot be the same as the linear dispersion relation. As shown above, the dispersion relation for masses separated by a distance a each with mass M conected to its neighbor by a spring with spring constant κ is given by $\omega(k) = 2(\kappa/M)^{1/2}\sin(ka/2)$. (Careful! κ is the spring constant, **not** the wave vector, $k = 2\pi/\lambda$.)

Exercise 4.7.5

Try various values of n for the chain with the sine wave as initial condition. What can you conclude from your experiements? How is the wavelength limited by the number of masses?

Exercise 4.7.6

Make a plot of $\omega(k) = kv$ and $\omega(k) = 2(\kappa/m)^{1/2}\sin(ka/2)$ versus k on the same graph. Use $v = 10$, $\kappa = 2$, $m = 1$ and $a = 0.1$. How do the graphs differ? For what wavelengths (small or large) do they give about the same result? Why do they overlap for large values of wavelength?

Exercise 4.7.7

Recall from simulation 2.3 that the group velocity of a wave packet is given by $v_{\text{group}} = \partial\omega(k)/\partial k$. Find an expression for the group velocity of a wave on a chain of masses.

Exercise 4.7.8

Notice that the group velocity for a wavepacket traveling on a chain of masses is dependent on the wavelength. Based on what you learned about dispersion in simulation 3.9, what do you expect to happen to a wavepacket as it travels down a chain of masses connected by springs?

4.8: Non-Linear Waves

Tutorial 4.8: Dispersion, Friction, Dissipation, and Nonlinearity

In simulation 4.6 we derived the **linear wave equation**, $\partial^2 y/\partial x^2 = 1/v^2 \partial^2 y/\partial t^2$ for an elastic string by considering forces acting on a small section of the string. The right side of the equation is basically the vertical acceleration of a piece of the string and the left side is the force. Constants like the tension and mass per unit length appear in the speed, v which is constant for a linear system. But what happens if other forces acts on the string? Some additional forces cause the dispersion we saw in simulations 3.8 and 3.9. Friction, dissipation and nonlinearity cause other behavior as we will see in this simulation.

Note

The applet below actually simulates a long chain of masses coupled by springs, as in the previous simulation. Mathematically we know that in the continuum limit of small masses that are very close together we get the linear wave equation as shown in simulation 4.5. As long as the waves on the chain are smooth and change gradually, the chain of masses approximates the continuous string. New forces acting on the chain will result in new terms being added to the wave equation as shown below. It should be kept in mind, however, that because the underlying model is discrete, the simulation may fail to accurately represent an elastic string. Except for friction and dissipation the total energy should remain approximately constant (there may be small fluctuations). If the energy starts to drift significantly this is a sign that the numerical calculations of the simulation are failing and the simulation no longer represents the elastic string (or any real system).

Non-Linear Waves

Questions:

Exercise 4.8.1

Test the simulation with dispersion, friction, dissipation and nonlinearity set to zero and the Gaussian pulse as the initial condition. Is energy conserved?

Suppose each point on the string had an additional force acting on it that was proportional to the amplitude of the string at that point. In other words springs attached to each point on the string. The wave equation would now look like $\partial^2 y/\partial x^2 + \alpha y = 1/v^2 \partial^2 y/\partial t^2$. This term leads to **dispersion**, a phenomena we examined previously.

Exercise 4.8.2

Turn on a small amount of dispersion with all the other terms still zero. Play the Gaussian initial contions. What happens to the pulse over time? How does this compare with what you saw in simulation 3.9?

Exercise 4.8.3

Find a dispersion relation for the dispersion equation by substituting $A\exp(-(kx-\omega t)^2)$ into the equation and solving for ω. You should be able to show that for this equation $\omega = (\alpha + v^2 k)^{1/2}$.

Exercise 4.8.4

Recall from simulation 2.3 that the group velocity of a wave packet is given by $v_{\text{group}} = \partial\omega(k)/\partial k$ but the phase velocity is $v_{\text{phase}} = \omega(k)/k$. The phase and group velocity are different and are dependent on the wavelength. Explain what this means physically. How does this explain what you saw in question 4.7.2? (Hint: Review simulation 2.3.)

Suppose our string was trying to vibrate in a medium like water or a dense gas. This would introduce a friction force that would be proportional to velocity; $-\eta\partial y/\partial t$ where η is the **coefficent of friction**. Now the wave equation would look like $\partial^2 y/\partial x^2 - \eta\partial y/\partial t = 1/v^2 \partial^2 y/\partial t^2$. We expect this force to gradually transfer energy to the surrounding liquid or gas.

Exercise 4.8.5

With the other parameters set to zero, turn on a small amount of friction and play the Gaussian initial condition. What happens? What happens to the energy?

For real strings there is also internal friction. If you have ever taken a metal coat hanger and bent it back and forth many times you know that the place where the bending occurs gets hot. This internal friciton, called **dissipation**, also gradually transfers the wave energy in a string into random thermal motion of the atoms in the string. For the wave equation this force can be represented by $-\gamma\partial y/\partial x$.

Exercise 4.8.6

With the other parameters set to zero, turn on a small amount of dissipation and play the Gaussian initial condition. What happens? What happens to the energy? You will notice that this is slightly different from the friction case. This is because a Gaussian pulse is not a solution to the wave equation with this additinal term in it. So not only is energy transfered into the string from the pulse, the pulse changes shape.

As a final example of adding external forces to a string we consider a force represented by $\pm\beta\partial^2 y^2/\partial x^2$. This is a **nonlinear** term and it has the opposite effect of dispersion; nonlinear terms cause a wave packet to steepen instead of spread out.

Exercise 4.8.7

With the other parameters set to zero, turn on a small amount of nonlinearity and play the Gaussian initial condition. What happens? What happens for negative values of nonlinearity?

Note

Energy should be approximately conserved - if the energy change significantly the numerical calculations in the simulation are begining to fail.

Exercise 4.8.8

You may wonder what makes an equation nonlinear. In simulation 4.5 you showed that $y(x,t)=y_1(x,t)+y_2(x,t)$ was a solution to the linear wave equation as long as $y_1(x,t)$ and $y_2(x,t)$ are also solutions. Try this with the equation $\partial^2 y/\partial x^2 \pm \beta\partial^2 y^2/\partial x^2 = 1/v^2\partial^2 y/\partial t^2$. Is the sum of two solutions also a solution? Note that this means superposition does not work for nonlinear systems; we cannot construct a Fourier series of sines and cosines in order to make a wave pulse.

Exercise 4.8.9

It is also the case that trigonometric functions (sine, cosine and exponential) are generally not solutions to nonlinear equations. What happens to a sine wave initial condition over time with a small amount of nonlinearity?

Exercise 4.8.10

As further evidence that trigonometric functions are not solutions, try $A\exp(i(kx-\omega t))$ as a possible solution to the nonlinear equation by substituting it into the equation in question $4.8.7$. Is it a solution? Explain. Is it a solution to the equation with friction but no nonlinear term? What about the dissipation and dispersion cases?

Although an exponential (plane) wave is not a solution to the nonlinear equation you should have been able to arrive at this expression in the previous question: $\omega^2 = k^2v^2 \pm 4\beta k^2 A\exp(i(kx-\omega t))$. We can get an approximate dispersion relation from this expression if we use the Taylor series expansion for the exponent: $\exp\theta \approx (1+\theta+\theta^2/2!+\ldots)$ and keep only the first term. this gives us the aproximate **dispersion relation** for this equation as $\omega = k(v^2 \pm 4\beta A)^{1/2}$.

Exercise 4.8.11

The group velocity of a wave packet is given by $v_{\text{group}} = \partial\omega(k)/\partial k$. Find an expression for the group velocity from the approximate expression of the dispersion relation for a nonlinear string.

Exercise 4.8.12

Notice that the group and phase velocity are dependent on the amplitude, A. For the plus sign in the dispersion relation, should taller waves travel faster or slower than short waves? What about for the minus sign?

Exercise 4.8.13

Because taller waves travel faster (for the plus sign case), a collection of waves made of several different frequencies will gradually pile up as the taller waves catch up with the slower, lower amplitude waves. Repeat question 4.7.6 and comment on what you see.

Note

For this simple nonlinear force the simulation cannot represent a true breaking wave like at the beach. It is also the case that the numerical calculations will fail once the wave gets steep. But it is nonlinearity that causes waves to break. Water waves interact with the ocean floor as they move into shallow water. These interactions are nonlinear and cause a wave packet to steepen and then break as the taller waves move faster than small amplitude waves.

4.9: Solitons

Tutorial 4.9: Solitons

As we have seen, if other forces are present in the system (such as nonlinear springs or friction) the wave equation must be modified to account for those forces with the result that simple trigonometric functions are no longer solutions. It would be logical to assume that there may be *no* stable solutions if other forces act. However there are a few special cases where the effects of dispersion and dissipation (which tend to make a wave pulse spread out) are exactly compensated for by a nonlinear force (which, as we have seen, tends to cause steepening of a wave). In this case there may be a special wave pulse shape that can travel and maintain its shape called a **soliton**.

Solitons have been used to model many different physical phenomena, for example tidal bores (last two pictures), electro-chemical pulses moving along nerve fibers, domain barriers between regions of different magnetic orientation in a metal, light pulses in optical cables, ion acoustic waves in plasmas, sound waves in a crystal, the great red spot on Jupiter and many other phenomena.

In this simulation we present the special case of the Sine Gordon wave equation. The Sine Gordon equation is $\partial^2 y/\partial x^2 = 1/v^2 \partial^2 y/\partial t^2 + \zeta \sin(y)$ where $\sin(y)$ acts as both a nonlinear term and a dispersion term. The strength of the nonlinear term is given by ζ. Sines, cosines and exponentials are not solutions to this equation but there is an exact solution called a **kink** given by

$$y(x,t) = 4\tan^{-1}\frac{\exp \pm\zeta(x - x_0 - vt)}{(1-v^2)^{1/2}}. \tag{4.9.1}$$

Here x_0 is the initial center location of the kink and v is the speed. There are only a few other known solutions and, because the equation is nonlinear, we cannot find new solutions by adding known solutions as can be done in the linear case with a Fourier series.

Note that, in order to make the simulation run on a web page a certain amount of accuracy had to be sacrificed in the simulation. The energy should remain perfectly constant in all cases; non-constant energy indicates the numerical calculations are failing. It is also the case that we are approximating a continuous system with a set of discrete masses which makes the simulation less accurate.

Sine Gordon Soliton

Questions:

Exercise 4.9.1

With the nonlinear coefficient set to zero, first check to see if the sine wave and Gaussian pulse initial conditions are solutions. They should be since for $\zeta = 0$ the equation reduces to the linear wave equation. What happens to these solutions if you turn on a small amount ($\zeta = 2$) of nonlinearity?

Exercise 4.9.2

Try the kink initial condition with zero nonlinearity. You will notice that a kink is not a solution to the linear wave equation (energy is not conserved). Reset the simulation and try various choices of values of the nonlinear parameter and speeds for the kink initial condition. What effect does ζ have on the shape and speed of the kink?

Exercise 4.9.3

On a graphing calculator make a plot of the kink solution for $t = 0$ (use the plus sign in the solution). What is the value of the solution as x goes to ∞? What happens as x goes to $-\infty$? How does your answer change if you use the minus sign in the solution?

Solitons that have different values at $\pm\infty$ are called **topological** solitons. One way to think about what is going on is to imagine a series of parallel troughs or ditches (running horizontally in the simulation) with a hump in between. One physical system represented by the Sine Gordon equation is a stretched string which starts out on one end in one trough but at the location of the kink goes over the hump to lie in the neighboring trough. As the kink moves the string changes from one trough to another.

Another physical model of a Sine Gordon kink is a 360 degree twist in a chain of masses connected by springs. The location of the kink is where the twist is located.

Exercise $4.9.4$

Using the physical analogy of the elastic string in a series of troughs, what is the difference between the solution with the plus sign (the kink) and the solution with the minus sign (called an anti-kink)?

Exercise $4.9.5$

Show that the kink solution as given above really is a solution to the Sine Gordon equation. [Substitute the solution into the equation, take derivatives, etc. You will need several trig identities.]

Exercise $4.9.6$

In our trough analogy, we might ask what happens if the string goes over into the next trough but then sometime further away comes back to the original trough. Or imagine starting a kink from one end and an anti-kink from the other so that the string starts and ends in the same trough but is located in a second trough in the middle. If they are moving and collide in the middle what happens? Try the collision case for both small amounts of nonlinearity ($\zeta = 2$) and different speeds. Describe the results (there are several possible outcomes, depending on the nonlinear parameter and speed including a **breather** solution where a kink/anti-kink pair fluctuate together in place)?

Note

If the energy is not at least approximately conserved, the simulation has probably failed to show an accurate result; try a smaller amount of nonlinearity.

Exercise $4.9.7$

In our trough analogy we can imagine more than one neighboring trough. In fact the sine term in the Sine Gordon equation makes an infinite number of troughs available for the elastic string. Experiment with the two-kink initial condition for various amounts of nonlinearity and speeds. What happens to two kinks when they collide? This is a general property of solitons; they maintain their shape after collision. This is *not* the same thing as linear superposition; solitons often have different speeds and positions after collisions whereas colliding pulses in a linear system are exactly the same afterwards.

Exercise $4.9.8$

Find at least two interesting YouTube videos on solitons and discuss what you see (be sure they are really about solitons and not something else). Hint: Search for soliton, soliton wave, Korteweg de Vries soliton, Kline Gordon soliton, Phi-4 soliton, nonlinear schrodinger equation, or Boussinesq soliton. Be sure you understand what you are looking at before writing a description.

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