# 19.3: The Hydrogen Atom

The hydrogen atom consists of an electron and a proton bound together by the attractive electrostatic force between the negative and positive charges of these particles. Our experience with the one-dimensional particle in a box shows that a spatially restricted particle takes on only discrete values of the total energy. This conclusion carries over to arbitrary attractive potentials and three dimensions.

The energy of the ground state can be qualitatively understood in terms of the uncertainty principle. A particle restricted to a region of size a by an attractive force will have a momentum equal at least to the uncertainty in the momentum predicted by the uncertainty principle: $p \approx \hbar/a$. This corresponds to a kinetic energy $\mathrm{K}=\mathrm{mv}^2/2=\mathrm{p}^2/(2\ \mathrm{m}) \approx \hbar^2/\left(2\mathrm{ma}^2\right)$. For the particle in a box there is no potential energy, so the kinetic energy equals the total energy. Comparison of this estimate with the computed ground state energy of a particle in a box of length $a$, $\mathrm{E}_1=\hbar^2\Pi^2/\left(2\mathrm{ma}^2\right)$, shows that the estimate differs from the exact value by only a numerical factor $\pi^2$.

We can make an estimate of the ground state energy of the hydrogen atom using the same technique if we can somehow take into account the potential energy of this atom. Classically, an electron with charge $-e$ moving in a circular orbit of radius a around a proton with charge $e$ at speed $v$ must have the centripetal acceleration multiplied by the mass equal to the attractive electrostatic force, $\mathrm{mv}^2/\mathrm{a}=\mathrm{e}^2/\left(4\pi\epsilon_0\mathrm{a}^2\right)$, where $m$ is the electron mass. (The proton is so much more massive than the electron that we can assume it to be stationary.) Multiplication of this equation by $\mathrm{a}/2$ results in

$$K=\frac{mv^2}{2}=\frac{p^2}{2m}=\frac{e^2}{8\pi\epsilon_0 a}=-\frac{U}{2} \tag{19.3.1}$$

where $U$ is the (negative) potential energy of the electron and $K$ is its kinetic energy. Solving for $U$, we find that $U=-2K$. The total energy $E$ is therefore related to the kinetic energy by

$$E=K+U=K-2K=-K \quad \text{(hydrogen atom)} \tag{19.3.2}$$

Since the total energy is negative in this case, and since $U=0$ when the electron is infinitely far from the proton, we can define a binding energy that is equal to minus the total energy:

$$E_B \equiv -E=K=-U/2 \quad \text{(virial theorem)} \tag{19.3.3}$$

The binding energy is the minimum additional energy that needs to be added to the electron to make the total energy zero, and thus to remove it to infinity. Equation (19.3.3) is called the virial theorem, and it is even true for non-circular orbits if the energies are properly averaged over the entire trajectory.

Proceeding as before, we assume that the momentum of the electron is $p \approx \hbar/a$ and substitute this into equation (19.3.1). Solving this for $a \equiv a_0$ yields an estimate of the radius of the hydrogen atom:

$$a_0=\frac{4\pi\epsilon_0\hbar^2}{e^2 m}=\left(\frac{4\pi\epsilon_0\hbar c}{e^2}\right)\left(\frac{\hbar}{mc}\right) \tag{19.3.4}$$

This result was first obtained by the Danish physicist Niels Bohr, using another method, in an early attempt to understand the quantum nature of matter.

The grouping of terms by the large parentheses in equation (19.3.4) is significant. The dimensionless quantity

$$\alpha=\frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \quad \text{(fine structure constant)} \tag{19.3.5}$$

is called the *fine structure constant* for historical reasons. However, it is actually a fundamental measure of the strength of the electromagnetic interaction. The Bohr radius can be written in terms of the fine structure constant as

$$a_0=\frac{\hbar}{\alpha mc}=5.29\times 10^{-11}\ \mathrm{m} \quad \text{(Bohr radius).} \tag{19.3.6}$$

The binding energy predicted by equations (19.3.1) and (19.3.3) is

$$E_B=-\frac{U}{2}=\frac{e^2}{8\pi\epsilon_0 a_0}=\alpha\frac{\hbar c}{2a_0}=\frac{\alpha^2 mc^2}{2}=13.6\mathrm{eV} \tag{19.3.7}$$

The binding energy between the electron and the proton is thus proportional to the electron rest energy multiplied by the square of the fine structure constant.

The above estimated binding energy turns out to be precisely the ground state binding energy of the hydrogen atom. The energy levels of the hydrogen atom turn out to be

$$E_n = -\frac{E_B}{n^2} = -\frac{\alpha^2 mc^2}{2n^2}, \quad n = 1, 2, 3, \ldots \quad \text{(hydrogen energy levels)} \tag{19.3.8}$$

where $n$ is called the *principal quantum number* of the hydrogen atom.