

19.1: Fermions and Bosons

Review of Angular Momentum in Quantum Mechanics

As we learned earlier, angular momentum is quantized in quantum mechanics. We can simultaneously measure only the magnitude of the angular momentum vector and one component, usually taken to be the z component. Measurement of the other two components simultaneously with the z component is forbidden by the uncertainty principle.

The magnitude of the orbital angular momentum of an object can take on the values $|\mathbf{L}| = [l(l+1)]^{1/2}\hbar$ where $l = 0, 1, 2, \dots$. The z component can likewise equal $L_z = m\hbar$ where $m = -l, -l+1, \dots, l$.

Particles can have an intrinsic spin angular momentum as well as an orbital angular momentum. The possible values for the magnitude of the spin angular momentum are $|\mathbf{S}| = [s(s+1)]^{1/2}\hbar$ and the z component of the spin angular momentum $S_z = m_s\hbar$ where $m_s = -s, -s+1, \dots, s$. Spin differs from orbital angular momentum in that the spin can take on half-integer as well as integer values: $s = 0, 1/2, 1, 3/2, \dots$ are possible spin quantum numbers.

Spin is an intrinsic, unchangeable quantity for an elementary particle. Particles with half-integer spins, $s = 1/2, 3/2, 5/2, \dots$, are called **fermions**, while particles with integer spins, $s = 0, 1, 2, \dots$ are called **bosons**. Fermions can only be created or destroyed in particle-antiparticle pairs, whereas bosons can be created or destroyed singly.

Two-Particle Wave Functions

We learned in quantum mechanics that a particle is represented by a wave, $\Psi(x, y, z, t)$, the absolute square of which gives the relative probability of finding the particle at some point in spacetime. If we have two particles, then we must ask a more complicated question: What is the relative probability of finding particle 1 at point x_1 and particle 2 at point x_2 ? This probability can be represented as the absolute square of a joint wave function $\psi(x_1, x_2)$, i. e., a single wave function that represents both particles. If the particles are not identical (say, one is a proton and the other is a neutron) and if they are not interacting with each other via some force, then the above wave function can be broken into the product of the wave functions for the individual particles:

$$\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2) \quad (\text{non-interacting dissimilar particles}) \quad (19.1.1)$$

In this case the probability of finding particle 1 at x_1 and particle 2 at x_2 is just the absolute square of the joint wave amplitude: $P(x_1, x_2) = P_1(x_1)P_2(x_2)$. This is consistent with classical probability theory.

The situation in quantum mechanics when the two particles are identical is quite different. If $P(x_1, x_2)$ is, say, the probability of finding one electron at x_1 and another electron at x_2 , then since we can't tell the difference between one electron and another, the probability distribution cannot change if we switch the electrons. In other words, we must have $P(x_1, x_2) = P(x_2, x_1)$. There are two obvious ways to make this happen: Either $\Psi(x_1, x_2) = \Psi(x_2, x_1)$ or $\Psi(x_1, x_2) = -\Psi(x_2, x_1)$.

It turns out that the wave function for two identical fermions is antisymmetric to the exchange of particles whereas for two identical bosons it is symmetric. In the special case of two non-interacting particles, we can construct the joint wave function with the correct symmetry from the wave functions for the individual particles as follows:

$$\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1) \quad (\text{non-interacting fermions}) \quad (19.1.2)$$

for fermions

$$\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1) \quad (\text{non-interacting bosons}) \quad (19.1.3)$$

for bosons.

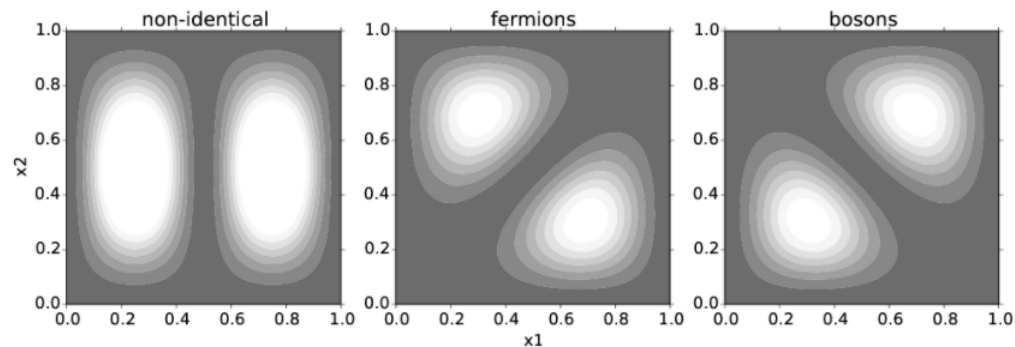


Figure 19.1.1:: Joint probability distributions for two particles, one in the ground state and one in the first excited state of a one-dimensional box. Left panel: non-identical particles. Middle panel: identical fermions. Right panel: identical bosons. The curved lines are contours of constant probability. The lighter shading shows where the probability is large.

Figure 19.1.1: shows the joint probability distribution for two particles in different energy states in an infinite square well: $P(x_1, x_2) = |\Psi(x_1, x_2)|^2$. Three different cases are shown: non-identical particles, identical fermions, and identical bosons. Notice that the probability of finding two fermions at the same point in space, i. e., along the diagonal dotted line in the center panel of Figure 19.1.1, is zero. This follows immediately from equation (19.1.2), which shows that $\Psi(x_1, x_2) = 0$ for fermions if $x_1 = x_2$. Notice also that if two fermions are in the same energy level (say, the ground state of the one-dimensional box) so that $\Psi_1(x) = \Psi_2(x)$, then $\Psi(x_1, x_2) = 0$ everywhere. This demonstrates that the two fermions cannot occupy the same state. This result is called the Pauli exclusion principle.

On the other hand, bosons tend to cluster together. Figure 19.1.1: shows that the highest probability in the joint distribution occurs along the line $x_1 = x_2$, i. e., when the particles are colocated. This tendency is accentuated when more particles are added to the system. When there are a large number of bosons, this tendency creates what is called a Bose-Einstein condensate in which most or all of the particles are in the ground state. Bose-Einstein condensation is responsible for such phenomena as superconductivity in metals and superfluidity in liquid helium at low temperatures.

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