17.1: The Capacitor and Ampère's Law

We first discuss a device that is commonly used in electronics, called the capacitor. We then introduce a new mathematical idea called the **circulation** of a vector field around a loop. Finally, we use this idea to investigate Ampère's law.

Capacitor

The **capacitor** is an electronic device for storing charge. The simplest type is the parallel plate capacitor, illustrated in Figure 17.1.1:. This consists of two conducting plates of area S separated by distance d, with the plate separation being much smaller than the plate dimensions. Positive charge q resides on one plate, while negative charge -q resides on the other.

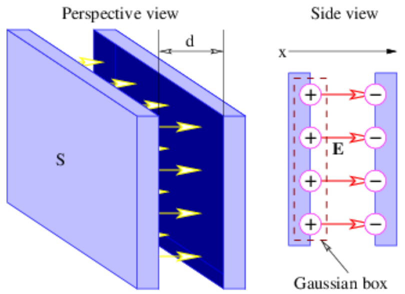

Figure 17.1.1:: Two views of a parallel plate capacitor.

The electric field between the plates is $E = \sigma/\epsilon_0$, where the charge per unit area on the inside of the left plate in Figure 17.1.1: is $\sigma = q/S$.. The density on the right plate is just -σ. All charge is assumed to reside on the inside surfaces and thus contributes to the electric field crossing the gap between the plates.

The above formula for the electric field comes from applying Gauss's law to the sheet of charge on the positive plate. The factor of 1\2 present in the equation for an isolated sheet of charge is absent here because all of the electric flux exits the Gaussian surface on the right side — the left side of the Gaussian box is inside the conductor where the electric field is zero, at least in a static situation.

There is no vector potential in this case, so the electric field is related solely to the scalar potential ϕ. Integrating $E_x = -\partial\phi/\partial x$ across the gap between the conducting plates, we find that the potential difference between the plates is $\Delta\phi = E_x d = qd\left(\epsilon_0 S\right)$ since E_x is known to be constant in this case. This equation indicates that the potential difference $\Delta\phi$ is proportional to the charge q on the left plate of the capacitor in Figure 17.1.1:. The constant of proportionality is $d/\left(\epsilon_0 S\right)$, and the inverse of this constant is called the **capacitance** :

$$C = \frac{\epsilon_0 S}{d} \quad \text{(parallel plate capacitor)}. \tag{17.1.1}$$

The relationship between potential difference, charge, and capacitance is thus

$$\Delta\phi = q/C \quad \text{or} \quad C = q/\Delta\phi \tag{17.1.2}$$

The equation for the capacitance of the illustrated parallel plates contains just a fundamental constant (ϵ_0) and geometrical factors (area of plates, spacing between them), and represents the amount of charge the parallel plate capacitor can store per unit potential difference between the plates. A word about signs: The higher potential is always on the plate of the capacitor that has the positive charge.

Note that Equation 17.1.1 is valid only for a parallel plate capacitor. Capacitors come in many different geometries and the formula for the capacitance of a capacitor with a different geometry will differ from this equation. However, Equation 17.1.2 is valid for **any** capacitor.

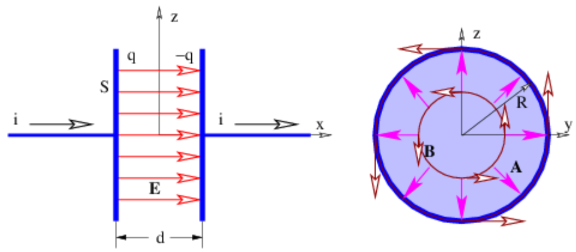

Figure 17.1.2:: Parallel plate capacitor with circular plates in a circuit with current i flowing into the left plate and out of the right plate. The magnetic field that occurs when the charge on the capacitor is increasing with time is shown at right as vectors tangent to circles. The radially outward vectors represent the vector potential giving rise to this magnetic field in the region where $x > 0$. The vector potential points radially inward for $x < 0$. The y axis is into the page in the left panel while the x axis is out of the page in the right panel.

We now show that a capacitor that is charging or discharging has a magnetic field between the plates. Figure $17.1.2$ shows a parallel plate capacitor with a current i flowing into the left plate and out of the right plate. This current is necessarily accompanied by an electric field that is changing with time: $E_x = q/(\epsilon_0 S) = it/(\epsilon_0 S)$. Such an electric field can be derived from a scalar potential that is a function of time: ϕ = -$itx/(\epsilon_0 S)$. However, the Lorenz condition

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{17.1.3}$$

demands that some component of the vector potential A be non-zero under these circumstances, since $\partial\phi/\partial t$ is non-zero.

How much can we infer about the vector potential from the geometry of the capacitor and Equation 17.1.3? Substituting $\phi = -itx/(\epsilon_0 S)$ into this equation results in

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{ix}{\epsilon_0 c^2 S} \tag{17.1.4}$$

which suggests a number of different possibilities for A. For instance, $\mathbf{A} = \left(0, ixy/\left(\epsilon_0 c^2 S\right), 0\right)$ and $\mathbf{A} = \left[0, 0, ixz\left(\epsilon_0 c^2 S\right)\right]$ both satisfy Equation 17.1.4. However, neither of these trial choices is satisfactory by itself, as they are not consistent with the cylindrical symmetry of the capacitor about the x axis.

A choice of vector potential that is consistent with the shape of the capacitor and satisfies the Lorenz condition is obtained by combining these two trial solutions:

$$\mathbf{A} = \left[0, ixy/\left(2\epsilon_0 c^2 S\right), ixz/\left(2\epsilon_0 c^2 S\right)\right] \tag{17.1.5}$$

This vector potential leads to the magnetic field

$$\mathbf{B} = \left[0, -iz/\left(2\epsilon_0 c^2 S\right), iy/\left(2\epsilon_0 c^2 S\right)\right] \tag{17.1.6}$$

These fields are illustrated in the right-hand panel of Figure $17.1.2$.

Circulation of a Vector Field

We have already seen one example of the circulation of a vector field, though we didn't label it as such. In chapter 15 we computed the work done on a charge by the electric field as it moves around a closed loop in the context of the electric generator and Faraday's law. The work done per unit charge, or the EMF, is an example of the *circulation* of a field, in this case the electric field, Γ_E. Faraday's law can be restated as

$$\Gamma_E = -\frac{d\Phi_B}{dt} \quad (\text{ Faraday's law }) \tag{17.1.7}$$

In the simple case of a circular loop with the field directed along the loop, the circulation is just the magnitude of the field multiplied by the circumference of the loop, as illustrated in the left panel of Figure $17.1.3$. In more complicated cases in which the field points in a direction other than the direction of the loop, just the component in the direction of traversal around the loop enters the circulation. If this component varies as one progresses around the loop, the calculation must be broken into pieces. The total circulation is then obtained by adding up the contributions from segments of the loop in which the value of the field component parallel to the motion around the loop is constant. An example of this type is the calculation of the EMF around a square loop of wire in an electric generator. Another is illustrated in the right panel of Figure $17.1.3$.

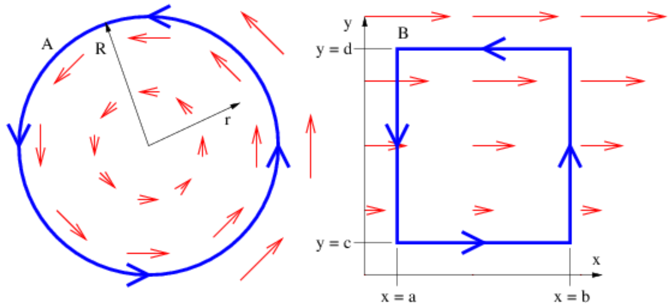

Figure 17.1.3:: Two examples of circulation paths in a vector field.

Ampère's Law

The magnetic circulation Γ_B around the periphery of the capacitor in the right panel of Figure 17.1.2 is easily computed by taking the magnitude of **B** in equation (17.1.6). The magnitude of the magnetic field on the inside of the capacitor is just $B = ir/\left(2\epsilon_0 c^2 S\right)$, since $r = \left(y^2 + z^2\right)^{1/2}$ in Figure 17.1.2. Thus, at the periphery of the capacitor, $r = R$, and $B = iR/\left(2\epsilon_0 c^2 S\right)$ there. The area of the capacitor plates is $S = \pi R^2$ and $\epsilon_0 c^2 = 1/\mu_0$, as we discussed previously. Thus, the magnetic field is $B = \mu_0 i/(2\pi R)$ at the periphery. If the periphery is traversed in the counter-clockwise direction, the magnetic circulation around the capacitor is $\Gamma_B = 2\Pi RB = \mu_0 i$.

Let us now compute the magnetic circulation around a wire carrying a current. The magnetic field a distance r from a straight wire carrying a current i is $B = \mu_0 i/(2\Pi r)$. The magnetic field points in the direction of a circle concentric with the wire. The magnetic circulation around the wire is thus $\Gamma_B = 2\Pi r B = \mu_0 i$.

Notice that the magnetic circulation is found to be the same around the wire and around the periphery of the capacitor. Furthermore, this circulation depends only on the current in the wire and the constant μ_0.

One further item needs to be calculated, namely the electric flux across the gap between the capacitor plates. This is just the electric field $E = \sigma/\epsilon_0$ multiplied by the area S, or $\Phi_E = S\sigma/\epsilon_0 = q/\epsilon_0$. The current into the capacitor is the time rate of change on the capacitor, so $i = dq/dt = \epsilon_0\, d\Phi_E/dt$.

We are now in a position to understand Ampère's law:

$$\Gamma_B = \mu_0 \left(i + \epsilon_0 \frac{d\Phi_E}{dt}\right) \quad (\text{ Ampère's law }). \tag{17.1.8}$$

This states that the magnetic circulation around a loop equals the sum of two contributions, (1) μ_0 multiplied by the electric current through the loop and (2) $\mu_0 \epsilon_0$ multiplied by the time rate of change of the electric flux through the loop. In the above example the first term dominates when the loop is around the wire, while the second term acts when the loop is around the gap between the capacitor plates.

Ampère actually formulated an incomplete version of the law named after him — he included only the first term containing the current. The Scottish physicist James Clerk Maxwell added the second term, based primarily on theoretical reasoning. Maxwell's additional term solved a serious internal inconsistency in electromagnetic theory — in our terms, the Lorenz condition **requires** a magnetic field to exist if the scalar potential ϕ is time-dependent. This magnetic field is only predicted by Ampère's law if Maxwell's term is included. The quantity $\epsilon_0 d\Phi_E/dt$ was called the **displacement current** by Maxwell since it has the dimensions of current and is numerically equal to the current entering the capacitor. However, it isn't really a current — it is just an electric flux that changes with time!

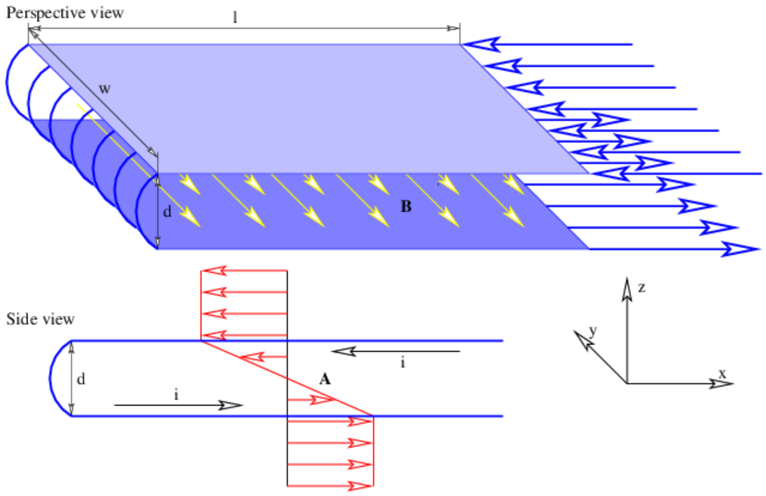

Figure 17.1.4:: Magnetic field and vector potential for two parallel plates carrying equal currents in opposite directions. This is an example of an inductor.

Gauss's law for electricity and magnetism, Faraday's law, and Ampère's law are collectively called **Maxwell's equations**. Together they form the basis for electromagnetism as it developed historically. However, our formulation of electromagnetism in terms of the four-potential, the dispersion relation for free electromagnetic waves, the Lorenz condition, and Coulomb's law, is precisely equivalent to Maxwell's equations, and is much closer to the modern approach to electromagnetism.
