15.2: Electric and Magnetic Fields and Forces

Electric and magnetic fields manifest themselves observationally by the forces that they cause. These vector quantities are related to the scalar and vector potentials as follows:

$$\mathbf{E}=-\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right)-\frac{\partial\mathbf{A}}{\partial t}\quad(\text{ electric field })\tag{15.2.1}$$

$$\mathbf{B}\equiv\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z},\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x},\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\quad(\text{ magnetic field })\tag{15.2.2}$$

Note that arbitrary scalar and vector constants may be added respectively to ϕ and A without changing either the electric or magnetic fields, since the latter are functions only of space and time derivatives of the former. This is a simple example of the concept of **gauge invariance** in action. We will see later that not just a constant, but any time-independent vector function $A\,'(x,y,z)$ may be added to A with similar null results, as long as $\partial A_x'/\partial y=\partial A_y'/\partial x$ etc. Gauge invariance is an important part of gauge theory, but a full understanding depends on more sophisticated mathematics than currently at our disposal.

By comparison of equations (15.2.1) and (15.2.2) with the general expression for force in gauge theory, we find that the electromagnetic force on a particle with charge q is

$$\begin{aligned}\mathbf{F}_{em}&=-\left(\frac{\partial U}{\partial x},\frac{\partial U}{\partial y},\frac{\partial U}{\partial z}\right)-\frac{\partial\mathbf{Q}}{\partial t}+\mathbf{v}\times\mathbf{P}\\&=-q\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right)-q\frac{\partial\mathbf{A}}{\partial t}+q\mathbf{v}\times\mathbf{B}\\&=q\mathbf{E}+q\mathbf{v}\times\mathbf{B}\qquad(\text{Lorentz force})\end{aligned}\tag{15.2.3}$$

where v is the velocity of the particle and where we have used equations (15.2) and (15.2.1). For historical reasons this is called the **Lorentz force**.