13.6: Kepler's Laws

Johannes Kepler, using data compiled by Tycho Brahe, inferred three laws governing the motions of planets in the solar system:

1. Planets move in elliptical orbits with the sun at one focus.
2. Equal areas are swept out in equal times by the line connecting the sun and the planet.
3. The square of the period of revolution of the planet around the sun is proportional to the cube of the semi-major axis of the ellipse.

These laws were instrumental in the development of modern mechanics and the universal law of gravitation by Isaac Newton.

Showing that the first law is consistent with Newtonian mechanics is mathematically more difficult than we can undertake in this course. However, the second law turns out to be a simple consequence of the conservation of angular momentum. Figure $13.6.7$: shows an elliptical orbit with the area swept out as a planet moves from position 1 to position 2. We estimate this area as $dA = Rdx/2$, where we have ignored the small unshaded part of the area to the right of the shaded triangle. The distance traveled by the planet in time dt is ds, so the magnitude of the velocity is $v = ds/dt$. However, in computing the angular momentum, we need the tangential component of the velocity, i. e., the component normal to the radius vector R. This is simply $v_t = dx/dt$. The angular momentum is $L = mRv_t = mRdx/dt$, where m is the mass of the planet. Combining this with the formula for dA results in

$$\frac{dA}{dt} = \frac{L}{2m} \tag{13.6.1}$$

Since gravitation is a central force, angular momentum is conserved, which means that dA/dt is constant. Thus, we have shown that conservation of angular momentum is equivalent to Kepler's second law.

Kepler's third law turns out to be a consequence of the universal law of gravitation. We can prove this for circular orbits. We know that a planet moving in a circular orbit around the sun is accelerating toward the sun with the centripetal acceleration $a = v^2/R$, where v is the speed of the planet's motion in its orbit and R is the orbit's radius. This acceleration is caused by the gravitational force, so we can equate the force divided by the planetary mass to a, resulting in

$$\frac{v^2}{R} = \frac{GM}{R^2} \tag{13.6.2}$$

where M is the mass of the sun. This may be solved for v :

$$v = \left(\frac{GM}{R}\right)^{1/2} \tag{13.6.3}$$

Eliminating v in favor of the period of revolution $T = 2\pi R/v$ results in

$$T^2 = \frac{4\pi^2 R^3}{GM} \tag{13.6.4}$$

This agrees with Kepler's third law since the semi-major axis of a circle is simply the radius R.