14.1: Potential Momentum

For a free, non-relativistic particle of mass m, the total energy E equals the kinetic energy K and is related to the momentum $\mathbf{\Pi}$ of the particle by

$$E=K=\frac{|\Pi|^{2}}{2m} \quad \text{(free, non-relativistic).} \tag{14.1.1}$$

(Note that we have ignored the contribution of the rest energy to the total energy here.) In the non-relativistic case, the momentum is $\mathbf{\Pi}=\boldsymbol{m}\mathbf{v}$ where $\boldsymbol{v}$ is the particle velocity.

If the particle is not free, but is subject to forces associated with a potential energy U(x,y,z), then equation (14.1.1)must be modified to account for the contribution of U to the total energy:

$$E-U=K=\frac{|\mathbf{\Pi}|^{2}}{2m} \text{ (non-free, non-relativistic).} \tag{14.1.2}$$

The force on the particle is related to the potential energy by

$$\mathbf{F}=-\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \tag{14.1.3}$$

For a free, relativistic particle, we have

$$E=\left(|\mathbf{\Pi}|^{2} c^{2}+m^{2} c^{4}\right)^{1/2} \quad \text{(free, relativistic).} \tag{14.1.4}$$

The obvious way to add forces to the relativistic case is by rewriting equation (14.1.4) with a potential energy, in analogy with equation (14.1.2):

$$E-U=\left(|\mathbf{\Pi}|^{2} c^{2}+m^{2} c^{4}\right)^{1/2} \quad \text{(incomplete!).} \tag{14.1.5}$$

Unfortunately, equation (14.1.5) is incomplete, because we have subtracted U from the energy E without subtracting a corresponding term from the momentum $\mathbf{\Pi}$ as well. However, $\underline{\Pi}=(\mathbf{\Pi}, \mathrm{E}/\mathrm{c})$ is a four-vector, so an equation with something subtracted from just one of the components of this four-vector is not relativistically invariant. In other words, equation (14.1.5) doesn't obey the principle of relativity, and therefore cannot be correct!

How can we fix this problem? One way is to define a new four-vector with U/c being its timelike part and some new vector $\mathbf{Q}$ being its spacelike part:

$$\underline{Q} \equiv(\mathbf{Q}, U/c) \quad \text{(potential four-momentum).} \tag{14.1.6}$$

We then subtract $\boldsymbol{Q}$ from the momentum $\mathbf{\Pi}$. When we do this, equation (14.1.5) becomes

$$E-U=\left(|\mathbf{\Pi}-\mathbf{Q}|^{2} c^{2}+m^{2} c^{4}\right)^{1/2} \quad \text{(non-free, relativistic).} \tag{14.1.7}$$

The quantity $\mathbf{Q}$ is called the potential momentum and $\underline{\mathrm{Q}}$ is the potential four-momentum.

Some additional terminology is useful. We define

$$\mathbf{p} \equiv \mathbf{\Pi}-\mathbf{Q} \quad \text{(kinetic momentum)} \tag{14.1.8}$$

as the kinetic momentum for reasons discussed below. In order to avoid confusion, we rename $\mathbf{\Pi}$ the total momentum.[1] Thus, the total momentum equals the kinetic plus the potential momentum, in analogy with energy.

So far, we have shown that the introduction of a potential momentum complements the potential energy so as to make the energy-momentum relationship for a particle relativistically invariant. However, we as yet have no idea what causes potential momentum nor what it does to the affected particle. We shall put off answering the former question and address only the latter at this point. A hint comes from the corresponding behavior of energy. The total energy of a particle is related to the quantum mechanical frequency ω of the particle, and the total momentum is related to its wave vector $\mathbf{k}$:

$$E=\hbar \omega \quad \mathbf{\Pi}=\hbar \mathbf{k} \tag{14.1.9}$$

However, the *kinetic energy* and the *kinetic momentum* are related to the particle's velocity **v**:

$$E-U=\frac{mc^2}{\left(1-v^2/c^2\right)^{1/2}} \quad \mathbf{p}=\Pi-\mathbf{Q}=\frac{m\mathbf{v}}{\left(1-v^2/c^2\right)^{1/2}} \tag{14.1.10}$$

where $v = |\mathbf{v}|$.

The relationship between kinetic momentum and velocity can be proven by dividing equation (14.1.7) by $\hbar$ to obtain a dispersion relation and then computing the group velocity, which we equate to the particle velocity. However, we will not do this here.