23.7: Problems

1. Compute an approximate value for $N^{\mathrm{N}}/\mathrm{N}!$ using the Stirling approximation. (This gives the essence of $_{\Delta\mathcal{N}}$ for N harmonic oscillators.) From this show that $\ln(\Delta\mathcal{N}) \propto N$.
2. States of a pair of distinguishable dice (e. g., one is red, the other is green):
   1. List all of the possible states of a pair of dice, i. e., all the possible combinations of face-up numbers.
   2. Given that each of the dice has six faces, does the total number of states equal that given by equation (23.16)?
3. There are $N!/[M!(N-M)!]$ ways of arranging N pennies with M heads up. Verify this for 2, 3, and 4 pennies. (Note that by definition 0! = 1.)
4. Suppose we have N pennies on a shaking table that bounces the pennies around, flipping them over at random. The pennies are weighted so that the gravitational potential energy of a penny is zero with tails up and U with heads up.
   1. If M heads are up, what is the total energy E?
   2. How many "states", $\Delta\mathcal{N}$, are there with M heads up? Hint: Compute this directly from the statement of the previous problem, not by computing $d\mathcal{N}/dE$ as we did for N harmonic oscillators.
   3. Compute the entropy of the system as a function of E and N. Hint: You will need to use the Stirling approximation to do this part.
   4. Compute the temperature as a function of E and N.
   5. Invert the temperature equation derived in the previous step to obtain E as a function of T and N. To understand this result, approximate it in the low and high limits, i. e., $\mathrm{k_B T/U} \ll 1$ and $\mathrm{k_B}T/\mathrm{U} \gg 1$. Try to think of an explanation of the behavior of the pennies in these limits that would make sense to (say) an 8th grade student. In particular, how is the intensity of the shaking of the table related to the "temperature"? Hint: In the low temperature limit note that $\exp(U/k_B T) \gg 1$, while in the high temperature limit $\exp(\cup/k_B T) \approx 1$.
5. Suppose that two systems, A and B, have available states $\Delta\mathcal{N}_A = E_A^X$ and $\Delta\mathcal{N}_B = E_B^Y$, where $E = E_A + E_B = 2$. . Compute and plot $\Delta\mathcal{N} = \Delta\mathcal{N}_A \Delta\mathcal{N}_B$ as a function of $\mathcal{E}_A$ over the range 0 < $\mathcal{E}_A$ < 2 for:
   1. $X = Y = 1$;
   2. $X = Y = 5$;
   3. $X = Y = 25$;
   4. $X = 2; Y = 8$ — explain the position of the peak in terms of the values of X and Y .

   How does the width of the peak change as X and Y get larger? Explain the consequences of this result for the reliability of the second law of thermodynamics as a function of the number of particles in each system.
6. Suppose we have a system of mass M in which $\mathrm{k_B}T = \mathrm{AE}^{1/2}$, where T is the temperature, E is the internal energy, k_B is Boltzmann's constant, and A is a constant.
   1. Derive a formula for the entropy of the system as a function of internal energy. Hint: Remember the thermodynamic definition of temperature.
   2. Compute the specific heat of this system.