7.5: Mass, Momentum, and Energy

In this section we relate the classical ideas of mass, momentum and energy to what we have done so far. Historically, these connections were first made by Max Planck and Louis de Broglie with help from Albert Einstein. Bragg diffraction of electrons is invoked as an experimental test of the Planck and de Broglie relations.

Technically, we don't need the ideas of mass, momentum, and energy to do physics – the notions of wavenumber, frequency, and group velocity are sufficient to describe and explain all observed phenomena. However, mass, momentum, and energy are so firmly embedded in physics that one couldn't talk to other physicists without an understanding of these quantities!

Planck, Einstein, and de Broglie

Max Planck was the first to develop a theory explaining the energy density of electromagnetic radiation in a box at a fixed temperature. Albert Einstein extended Planck's ideas by postulating that the energy of electromagnetic radiation is quantized into chunks called photons. The energy E of a photon is related to the frequency of the electromagnetic radiation by the equation

$$E=hf=\hbar\omega \quad \text{(Planck-Einstein relation)} \tag{7.5.1}$$

where f is the rotational frequency of the associated electromagnetic wave and ω is its angular frequency. The constant $\mathrm{h}=6.63\times10^{-34}\ \mathrm{kg\ m^2\ s^{-1}}$ is called Planck's constant. The related constant $\hbar=h/2\pi=1.06\times10^{-34}\ \mathrm{kg\ m^2\ s^{-1}}$ is also referred to as Planck's constant, but to avoid confusion with the original constant, we will generally refer to it as "h bar".

Notice that a new physical dimension has appeared, namely mass, with the unit kilogram, abbreviated "kg". The physical meaning of mass is much like our intuitive understanding of the concept, i. e., as a measure of the resistance of an object to its velocity being changed. The precise scientific meaning will emerge shortly.

Einstein showed that Planck's idea could be used to explain the emission of electrons which occurs when light impinges on the surface of a metal. This emission, which is called the photoelectric effect, can only occur when electrons are supplied with a certain minimum energy $\mathrm{E_B}$ required to break them loose from the metal. Experiment shows that this emission occurs only when the frequency of the light exceeds a certain minimum value. This value turns out to equal $\omega_{\min}=\mathrm{E_B}/\hbar$, which suggests that electrons gain energy by absorbing a single photon. If the photon energy, $\hbar\omega$, exceeds $\mathrm{E_B}$, then electrons are emitted, otherwise they are not. It is much more difficult to explain the photoelectric effect from the classical theory of light. The value of $\mathrm{E_B}$, called the binding energy or work function, is different for different metals.

Louis de Broglie proposed that Planck's energy-frequency relationship be extended to all kinds of particles. In addition he hypothesized that the momentum Π of the particle and the wave vector $\mathbf{k}$ of the corresponding wave were similarly related:

$$\mathbf{\Pi}=\hbar\mathbf{k} \quad \text{(de Broglie relation)} \tag{7.5.2}$$

Note that this can also be written in scalar form in terms of the wavelength as $\Pi=\mathrm{h}/\lambda$. (We use Π rather than the more common p for momentum, because as we shall see, there are two different kinds of momentum, one related to the wave number, the other related to the velocity of a particle. In many cases they are equal, but there are certain important situations in which they are not.)

De Broglie's hypothesis was inspired by the fact that wave frequency and wave number are components of the same four-vector according to the theory of relativity, and are therefore closely related to each other. Thus, if the energy of a particle is related to the frequency of the corresponding wave, then there ought to be some similar quantity which is correspondingly related to the wave number. It turns out that the momentum is the appropriate quantity. The physical meaning of momentum will become clear as we proceed.

We will also find that the rest frequency, μ, of a particle is related to its mass, m:

$$E_{rest}\equiv mc^2=\hbar\mu. \tag{7.5.3}$$

The quantity $\mathrm{E_{rest}}$ is called the rest energy of the particle.

From our perspective, energy, momentum, and rest energy are just scaled versions of frequency, wave vector, and rest frequency, with a scaling factor $\hbar$. We can therefore define a four-momentum as a scaled version of the wave four-vector:

$$\underline{\Pi}=\hbar\underline{k}. \tag{7.5.4}$$

The spacelike component of $\underline{\Pi}$ is just Π, while the time like part is $\mathrm{E/c}$.

Planck, Einstein, and de Broglie had extensive backgrounds in classical mechanics, in which the concepts of energy, momentum, and mass have precise meaning. In this text we do not presuppose such a background. Perhaps the best strategy at this point is to think of these quantities as scaled versions of frequency, wave number, and rest frequency, where the scale factor is $\hbar$. The significance of these quantities to classical mechanics will emerge bit by bit.

Particle Quantities

Let us now recapitulate what we know about relativistic waves, and how this knowledge translates into knowledge about the mass, energy, and momentum of particles. In the following equations, the form on the left is expressed in wave terms, i. e., in terms of frequency, wave number, and rest frequency. The form on the right is the identical equation expressed in terms of energy, momentum, and mass. Since the latter variables are just scaled forms of the former, the two forms of each equation are equivalent.

We begin with the dispersion relation for relativistic waves:

$$\omega^2 = k^2c^2 + \mu^2 \quad E^2 = \Pi^2c^2 + m^2c^4 \tag{7.5.5}$$

Calculation of the group velocity,$\mathrm{u_g} = \mathrm{d}\omega/\mathrm{dk}$, from the dispersion relation yields

$$u_g = \frac{c^2k}{\omega} \quad u_g = \frac{c^2\Pi}{E} \tag{7.5.6}$$

These two sets of equations represent what we know about relativistic waves, and what this knowledge tells us about the relationships between the mass, energy, and momentum of relativistic particles. When in doubt, refer back to these equations, as they work in all cases, including for particles with zero mass!

It is useful to turn equations (7.5.5) and (7.5.6) around so as to express the frequency as a function of rest frequency and group velocity,

$$\omega = \frac{\mu}{\left(1 - u_g^2/c^2\right)^{1/2}} \quad E = \frac{mc^2}{\left(1 - u_g^2/c^2\right)^{1/2}} \tag{7.5.7}$$

and the wavenumber as a similar function of these quantities:

$$k = \frac{\mu u_g/c^2}{\left(1 - u_g^2/c^2\right)^{1/2}} \quad \Pi = \frac{mu_g}{\left(1 - u_g^2/c^2\right)^{1/2}} \tag{7.5.8}$$

Note that equations (7.5.7) and (7.5.8) work only for particles with non-zero mass! For zero mass particles both the numerators and denominators of equations (7.5.7) and (7.5.8) are zero, making these equations undefined, and you need to use equations (7.5.5) and (7.5.6) with $\mathrm{m} = 0$ and $\mu = 0$ instead.

The quantity $\omega - \mu$ indicates how much the frequency exceeds the rest frequency. Notice that if $\omega = \mu$, then from equation (7.5.5) k = 0. Thus, positive values of $\omega_k \equiv \omega - \mu$ indicate $|\mathrm{k}| > 0$, which means that the particle is moving according to equation (7.5.6). Let us call ω_k the kinetic frequency:

$$\omega_k = \left[\frac{1}{\left(1 - u_g^2/c^2\right)^{1/2}} - 1\right]\mu \quad K = \left[\frac{1}{\left(1 - u_g^2/c^2\right)^{1/2}} - 1\right]mc^2 \tag{7.5.9}$$

We call K the *kinetic energy* for similar reasons. Again, equation (7.5.9) only works for particles with non-zero mass. For zero mass particles the kinetic energy equals the total energy.

Note that the results of this section are valid only for free particles, i. e., particles to which no force is applied. Force in classical and quantum mechanics is treated in the next chapter.

Non-Relativistic Limits

When the mass is non-zero and the group velocity is much less than the speed of light, it is useful to compute approximate forms of the above equations valid in this limit. Using the approximation $(1+\epsilon)^X \approx 1 + x\epsilon$, we find that the dispersion relation becomes

$$\omega = \mu + \frac{k^2c^2}{2\mu} \quad E = mc^2 + \frac{\Pi^2}{2m}, \tag{7.5.10}$$

and the group velocity equation takes the approximate form

$$u_g = \frac{c^2 k}{\mu} \quad u_g = \frac{\Pi}{m} \tag{7.5.11}$$

The non-relativistic limits for equations (7.5.7) and (7.5.8) become

$$\omega = \mu + \frac{\mu u_g^2}{2c^2} \quad E = mc^2 + \frac{m u_g^2}{2} \tag{7.5.12}$$

and

$$k = \mu u_g / c^2 \quad \Pi = m u_g \tag{7.5.13}$$

while the approximate kinetic energy equation is

$$\omega_k = \frac{\mu u_g^2}{2c^2} \quad K = \frac{m u_g^2}{2} \tag{7.5.14}$$

Just a reminder — the equations in this section are not valid for massless particles!

Experimental Test

How can we test the above predictions against experiment? The key point is to be able to relate the wave aspects to the particle aspects of a quantum mechanical wave-particle. Equation (7.5.8), or equation (7.5.13) in the non-relativistic case, relates a particle's wave number k to its velocity $\mathrm{u_g}$. Both of these quantities can be measured in a Bragg's law experiment with electrons. In this experiment electrons are fired at a crystal with known atomic dimensions at a known speed, which we identify with the group velocity $\mathrm{u_g}$. The Bragg angle which yields constructive interference can be used to calculate the wavelength of the corresponding electron wave, and hence the wave number and momentum. If the momentum is plotted against group velocity in the non-relativistic case, a straight line should be found, the slope of which is the particle's mass. In the fully relativistic case one needs to plot momentum versus $\mathrm{u_g}/\left(1-\mathrm{u_g^2}/\mathrm{c^2}\right)^{1/2}$. Again, a straight line indicates agreement with the theory and the slope of the line is the particle's mass. This particular experiment is difficult to do, but the corresponding theories verify in many other experiments.