12.2: Analysis Using Newton's Laws

The acceleration of the mass at any time is given by Newton's second law:

$$a=\frac{d^2x}{dt^2}=\frac{F}{M}=-\frac{kx}{M} \tag{12.2.1}$$

An equation of this type is known as a differential equation since it involves a derivative of the dependent variable x. Equations of this type are generally more difficult to solve than algebraic equations, as there are no universal techniques for solving all forms of such equations. In fact, it is fair to say that the solutions of most differential equations were originally obtained by guessing!

We already have the basis on which to make an intelligent guess for the solution to equation (12.2.1) since we know that the mass oscillates back and forth with a period that is independent of the amplitude of the oscillation. A function which might fill the bill is the cosine function. Let us try substituting $x=\cos(\omega t)$, where ω is a constant, into this equation. The second derivative of x with respect to t is $-\omega^2\cos(\omega t)$, so performing this substitution results in

$$-\omega^2\cos(\omega t)=-\frac{k}{M}\cos(\omega t) \tag{12.2.2}$$

Notice that the cosine function cancels out, leaving us with $-\omega^2=-k/M$. The guess thus works if we set

$$\omega=\left(\frac{k}{M}\right)^{1/2} \tag{12.2.3}$$

The constant ω is the angular oscillation frequency for the oscillator, from which we infer the period of oscillation to be $\mathrm{T}=2\Pi(\mathrm{Mk})^{1/2}$. This agrees with the earlier approximate result of equation 12.5, except that the approximation has a numerical factor of 8 rather than $2\Pi\approx 6$. Thus, the earlier guess is only off by about 30%!

It is easy to show that $x=B\cos(\omega t)$ is also a solution of equation (12.2.1), where B is any constant and $\omega=(\mathrm{k}/\mathrm{M})^{1/2}$. This confirms that the oscillation frequency and period are independent of amplitude. Furthermore, the sine function is equally valid as a solution: $x=A\sin(\omega t)$, where A is another constant. In fact, the most general possible solution is just a combination of these two, i. e.,

$$x=A\sin(\omega t)+B\cos(\omega t)=C\cos(\omega t-\phi) \tag{12.2.4}$$

The values of A and B depend on the position and velocity of the mass at time t = 0. The right side of equation (12.2.4) shows an alternate way of writing the general harmonic oscillator solution that uses a cosine function with a phase factor ϕ.