6.5: Accelerated Reference Frames

Referring back to the forces being felt by the occupant of a car, it is clear that the forces associated with accelerations are directed opposite the accelerations and proportional to their magnitudes. For instance, when accelerating away from a stoplight, the acceleration is forward and the perceived force is backward. When turning a corner, the acceleration is toward the corner while the perceived force is away from the corner. Such forces are called *inertial forces*.

The origin of these forces can be understood by determining how acceleration changes when one observes it from a reference frame which is itself accelerated. Suppose that the primed reference frame is accelerating to the right with acceleration A relative to the unprimed frame. The position x' in the primed frame can be related to the position x in the unprimed frame by

$$x = x' + X \tag{6.5.1}$$

where X is the position of the origin of the primed frame in the unprimed frame. Taking the second time derivative, we see that

$$a = a' + A \tag{6.5.2}$$

where $\mathrm{a} = \mathrm{d}^2\mathrm{x}/\mathrm{dt}^2$ is the acceleration in the unprimed frame and $a' = d^2x'/dt^2$ is the acceleration according to an observer in the primed frame.

We now substitute this into equation (6.9) and move the term involving A to the left side:

$$F - mA = ma' \tag{6.5.3}$$

This shows that Newton's second law represented by equation (6.9) is not valid in an accelerated reference frame, because the total force F and the acceleration a' in this frame don't balance as they do in the unaccelerated frame; the additional term $-mA$ messes up this balance.

We can fix this problem by considering $-mA$ to be a type of force, in which case we can include it as a part of the total force F. This is the inertial force which we mentioned above. Thus, to summarize, we can make Newton's second law work when objects are observed from accelerated reference frames if we include as part of the total force an inertial force which is equal to $-mA$, A being the acceleration of the reference frame of the observer and m the mass of the object being observed.

The right panel of Figure $6.5.2$: shows the inertial force observed in the reference frame of an object moving in circular motion at constant speed. In the case of circular motion the inertial force is called the centrifugal force. It points away from the center of the circle and just balances the tension in the string. This makes the total force on the object zero in its own reference frame, which is necessary since the object cannot move (or accelerate) in this frame.

General relativity says that gravity is nothing more than an inertial force. This was called the equivalence principle by Einstein. Since the gravitational force on the Earth points downward, it follows that we must be constantly accelerating upward as we stand on the surface of the Earth! The obvious problem with this interpretation of gravity is that we don't appear to be moving away from the center of the Earth, which would seem to be a natural consequence of such an acceleration. However, relativity has a surprise in store for us here.

It follows from the above considerations that something can be learned about general relativity by examining the properties of accelerated reference frames. In particular, we can gain insight into the above apparent paradox. Equation (6.17) shows that the velocity of an object undergoing constant intrinsic acceleration a (note that we have dropped the "prime" from a to simplify the notation) is

$$v = \frac{dx}{dt} = \frac{at}{\left[1 + (at/c)^2\right]^{1/2}} \tag{6.5.4}$$

where t is the time and c is the speed of light. A function $x(t)$ which satisfies equation (6.5.4) is

$$x(t) = \left(c^2/a\right)\left[1 + (at/c)^2\right]^{1/2} \tag{6.5.5}$$

(Verify this by differentiating it.) The interval OB in Figure $6.5.6$: is of length $x(0) = c^2/A$.

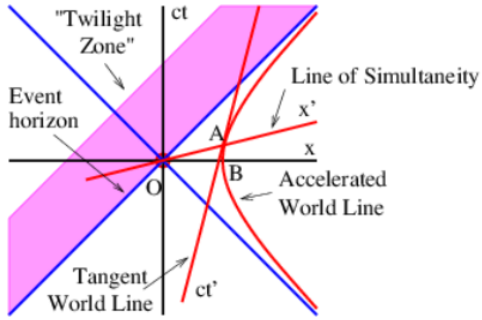

Figure 6.5.6:: Spacetime diagram showing the world line of the origin of a reference frame undergoing constant acceleration.

The slanted line OA is a line of simultaneity associated with the unaccelerated world line tangent to the accelerated world line at point A. This line of simultaneity goes through the origin, as is shown in Figure 6.5.6:. To demonstrate this, multiply equations (6.5.4) and (6.5.5) together and solve for v/c:

$$v/c = ct/x. \tag{6.5.6}$$

From Figure 6.5.6: we see that ct/x is the slope of the line OA, where (x, ct) are the coordinates of event A. Equation (6.23) shows that this line is indeed the desired line of simultaneity, since its slope is the inverse of the slope of the world line, c/v. Since there is nothing special about the event A, we infer that all lines of simultaneity associated with the accelerated world line pass through the origin.

We now inquire about the length of the invariant interval OA in Figure 6.5.6:. Recalling that $I^2 = x^2 - c^2t^2$ and using equation (6.5.5), we find that the length of OA is

$$I = \left(x^2 - c^2t^2\right)^{1/2} = \left(c^4/a^2\right)^{1/2} = c^2/a \tag{6.5.7}$$

which is the same as the length of the interval OB. By extension, *all* events on the accelerated world line are the same invariant interval from the origin. Recalling that the interval along a line of simultaneity is the distance in the associated reference frame, we reach the astonishing conclusion that even though the object associated with the curved world line in Figure 6.5.6: is accelerating away from the origin, it always remains the same distance (in its own frame) from the origin.

The analogy between this problem and the apparent paradox in which one remains a fixed distance from the center of the earth while accelerating away from it is not perfect. In particular, the earth case depends on the existence of the earth's mass.