8.4: Mechanics and Geometrical Optics

Louis de Broglie made an analogy between matter waves and light waves, pointing out that wave packets of light change their velocity as the result of spatial variations in the index of refraction of the medium in which they are traveling. This behavior comes about because the dispersion relation for light traveling through a medium with index of refraction n is $\omega = \mathrm{kc}/n$, so that the group velocity, $u_g = d\omega/dk = c/n$. Thus, when n increases, u_g decreases, and vice versa.

In this section we pursue de Broglie's analogy to see if we can come up with a theory of matter waves which gives the same results as classical mechanics in the geometrical optics limit of these waves. The dispersion relation for free matter waves is $\omega = \left(k^2c^2 + \mu^2\right)^{1/2}$. In the non-relativistic limit $k^2c^2 \ll \mu^2$. As done previously, we use $(1+\epsilon)^n \approx 1 + n\epsilon$ for small ϵ. In the non-relativistic limit, the dispersion relation for free waves thus becomes

$$\omega = \mu\left(1 + k^2c^2/\mu^2\right)^{1/2} \approx \mu + k^2c^2/(2\mu) \tag{8.4.1}$$

The above equation can be transformed into the total energy equation for a free, non-relativistic particle, $\mathrm{E} = \mathrm{mc}^2 + \mathrm{K}$, where mc^2 is the rest energy and K is the kinetic energy, by multiplying by $\hbar$. We convert the free particle energy equation into the equation for a particle subject to a conservative force by adding the potential energy U the right side. The analogous change to equation (8.4.1) is to add $\mathrm{S} = \mathrm{U}/\hbar$ to the right side, resulting in a modified dispersion relation:

$$\omega = S(x) + k^2c^2/(2\mu) \tag{8.4.2}$$

(Since the rest energy is just a constant, we have absorbed it into S.) This gives us the dispersion relation for one-dimensional matter waves subject to a spatially varying potential energy. The quantity S, which we see is just a scaled potential energy, plays a role for matter waves which is analogous to the role played by a spatially variable index of refraction for light waves.

Let us now imagine that all parts of the wave governed by this dispersion relation oscillate in phase. The only way this can happen is if ω is constant, i. e., it takes on the same value in all parts of the wave.

If ω is constant, the only way S can vary with x in equation (8.4.2) is if the wavenumber varies in a compensating way. Thus, constant frequency and spatially varying S together imply that $\mathrm{k} = \mathrm{k}(\mathrm{x})$. Solving equation (8.4.2) for k yields

$$k(x) = \pm\left[\frac{2\mu[\omega - S(x)]}{c^2}\right]^{1/2} \tag{8.4.3}$$

Since ω is constant, the wavenumber becomes smaller and the wavelength larger as the wave moves into a region of increased *S*.

In the geometrical optics limit, we assume that *S* doesn't change much over one wavelength so that the wave remains reasonably sinusoidal in shape with approximately constant wavenumber over a few wavelengths. However, over distances of many wavelengths the wavenumber and amplitude of the wave are allowed to vary considerably.

The group velocity calculated from the dispersion relation given by equation (8.4.2) is

$$u_g = \frac{d\omega}{dk} = \frac{kc^2}{\mu} = \pm\left(\frac{2c^2(\omega - S)}{\mu}\right)^{1/2} \tag{8.4.4}$$

where k is eliminated in the last step with the help of equation (8.4.3). The resulting equation tells us how the group velocity varies as a matter wave traverses a region of slowly varying S. Thus, as S increases, u_g decreases and vice versa.

We can now calculate the acceleration of a wave packet resulting from the spatial variation in S. We assume that x(t) represents the position of the wave packet, so that $\mathrm{u_g} = \mathrm{dx}/\mathrm{dt}$. Using the chain rule $\mathrm{du_g}/\mathrm{dt} = (\mathrm{du_g}/\mathrm{dx})(\mathrm{dx}/\mathrm{dt}) = (\mathrm{du_g}/\mathrm{dx})\,\mathrm{u_g}$, we find

$$a = \frac{du_g}{dt} = \frac{du_g}{dx}u_g = \frac{du_g^2/2}{dx} = -\frac{c^2}{\mu}\frac{dS}{dx} = -\frac{\hbar}{m}\frac{dS}{dx} \tag{8.4.5}$$

The group velocity is eliminated in favor of S by squaring equation (8.4.4) and substituting the result into equation (8.4.5).

Recalling that $\mathrm{U} = \hbar\mathrm{S}$, equation (8.4.5) becomes

$$a = -\frac{1}{m}\frac{dU}{dx} = \frac{F}{m} \tag{8.4.6}$$

which is just Newton's second law! Thus, the geometrical optics approach to particle motion is completely equivalent to the classical mechanics of a particle moving under the influence of a conservative force, at least in one dimension. We therefore have two ways of solving for the motion of a particle subject to a potential energy $U(x)$. We can apply the principles of classical mechanics to get the force and the acceleration of the particle, from which we can derive the motion. Alternatively, we can apply the principles of geometrical optics to compute the spatially variable velocity of the wave packet using equation (8.4.4). The results are completely equivalent, though the methods are conceptually very different.