9.2: Symmetry and Quantum Mechanics

The idea of symmetry plays a huge role in physics. We have already used symmetry arguments in the theory of relativity — applying the principle of relativity to obtain the dispersion relation for relativistic matter waves is just such an argument. In this section we begin to explore how symmetry can be used to increase our understanding of quantum mechanics.

Particle

For our first example we take the case of a free particle in quantum mechanics, i. e., a particle subject to no force. The wave function for a free particle of definite momentum Π and energy E is given by

$$\psi=\exp[i(kx-\omega t)]=\exp[i(\Pi x-Et)/\hbar] \quad \text{(free particle)} \tag{9.2.1}$$

For this wave function $|\Psi|^2=1$ everywhere, so the probability of finding the particle anywhere in space and time is uniform. This contrasts with the probability distribution which arises if we assume a free particle to have the wave function $\Psi=\cos[(\Pi\mathrm{x}-\mathrm{Et})/\hbar]$. In this case $|\Psi|^2=\cos^2[(\Pi x-Et)/\hbar]$, which varies with position and time, and is inconsistent with a uniform probability distribution.

Symmetry and Definiteness

Quantum mechanics is a probabilistic theory, in the sense that the predictions it makes tell us, for instance, the probability of finding a particle somewhere in space. If we know nothing about a particle's previous history, and if there are no physical constraints that would make it more likely for a particle to be at one point along the x axis than any another, then the probability distribution must be P(x) = constant.

This is an example of a symmetry argument. Expressed more formally, it states that if the above conditions apply, then the probability distribution ought to be subject to the condition P(x + D) = P(x) for any constant value of D. The only possible P(x) in this case is P = constant. In the language of physics, if there is nothing that gives the particle a higher probability of being at one point rather than another, then the probability is independent of position and the system is invariant under displacement in the x direction.

The above argument doesn't suffice for quantum mechanics, since as we have learned, the fundamental quantity describing a particle is not the probability distribution, but the wave function $\Psi(\mathrm{x})$. Thus, the wave function rather than the probability distribution ought to be the quantity which is invariant under displacement, i. e., $\psi(x+D)=\Psi(x)$.

This condition turns out to be too restrictive, because it implies that $\Psi(\mathrm{x})=\text{ constant }$, whereas we know that a one-dimensional plane wave, which describes a particle with a uniform probability of being found anywhere along the x axis, has the form $\psi(\mathrm{x})=\exp(\mathrm{ikx})$. (For simplicity we temporarily ignore the time dependence.) If we make the substitution x → x + D in a plane wave, we get $\exp[\mathrm{ik}(\mathrm{x}+\mathrm{D})]=\exp(\mathrm{ikx})\exp(\mathrm{ikD})$. The wave function is thus technically not invariant under displacement, in that the displaced wave function is multiplied by the factor $\exp(\mathrm{ikD})$. However, the probability distribution of the displaced wave function still equals one everywhere, so there is no change in what we observe. Thus, in determining invariance under displacement, we are allowed to ignore changes in the wave function which consist only of multiplying it by a complex constant with an absolute value of one. Such a multiplicative constant is called a phase factor.

It is easy to convince oneself by trial and error or by more sophisticated means that the only form of wave function $\psi(\mathrm{x})$ which satisfies the condition $\Psi(\mathrm{x}+\mathrm{D})=\Psi(\mathrm{x})\times(\text{ phase factor })$ is $\psi(\mathrm{x})=\mathrm{A}\exp(\mathrm{ikx})$ where A is a (possibly complex) constant. This is just in the form of a complex exponential plane wave with wave number k. Thus, not only is the complex exponential wave function invariant under displacements in the manner defined above, it is the only wave function which is invariant to displacements. Furthermore, the phase factor which appears for a displacement D of such a plane wave takes the form $\exp(\mathrm{iC})=\exp(\mathrm{ikD})$, where k is the wave number of the plane wave.

As an experiment, let us see if a wave packet is invariant under displacement. Let's define a wave packet consisting of two plane waves:

$$\psi(x)=\exp(ik_1x)+\exp(ik_2x) \tag{9.2.2}$$

Making the substitution $x\to x+D$ in this case results in

$$\begin{aligned}\psi(x+D) &= \exp[ik_1(x+D)]+\exp[ik_2(x+D)] \\ &= \exp(ik_1x)\exp(ik_1D)+\exp(ik_2x)\exp(ik_2D) \\ &\neq [\exp(ik_1x)+\exp(ik_2x)]\times(\text{ phase factor }) \end{aligned} \tag{9.2.3}$$

The impossibility of writing $\psi(x+D)=\psi(x)\times(\text{ phase factor })$ lends plausibility to the assertion that a single complex exponential is the only possible form of the wave function that is invariant under displacement.

Notice that the wave packet does not have definite wavenumber, and hence, momentum. In particular, the wave packet is a sum of complex exponentials with wavenumbers k_1 and k_2, which means that the associated particle can have either momentum $\Pi_1=\hbar k_1$ or $\Pi_2=\hbar k_2$. This makes sense from the point of view of the uncertainty principle – for a single plane wave the uncertainty in position is complete and the uncertainty in momentum is zero. For a wave packet the uncertainty in position is reduced and the uncertainty in the momentum is non-zero. As we have seen, this idea can be carried further: A definite value of momentum must be associated with a completely indefinite probability distribution in position, i. e., with P = constant. This corresponds to a wave function which has the form of a complex exponential plane wave. However, such a plane wave is invariant under displacement D, except for the multiplicative phase factor $\exp(ikD)$, , which has no physical consequences since it disappears when the probability distribution is obtained. Thus, we see that invariance under displacement of the wave function and a definite value of the momentum are linked, in that each implies the other:

$$\text{invariance under displacement} \Longleftrightarrow \text{ definite momentum} \tag{9.2.4}$$

The idea of potential energy was introduced in the previous chapter. In particular, we found that if the total energy is constant, then the momentum cannot be constant in the presence of spatially varying potential energy. This means that the wavenumber, and hence the wavelength of the oscillations in the wave function also vary with position. The spatial inhomogeneity of the potential energy gives rise to spatial inhomogeneity in the wave function, and hence an indefinite momentum.

The above argument can be extended to other variables besides momentum. In particular since the time dependence of a complex exponential plane wave is $\exp(-i\omega t)=\exp(-\mathrm{iEt}/\hbar)$, where E is the total energy, we have by analogy with the above argument that

$$\text{invariance under time shift} \Longleftrightarrow \text{ definite energy} \tag{9.2.5}$$

Thus, invariance of the wave function under a displacement in time implies a definite value of the energy of the associated particle.

In the previous chapter we assumed that the frequency (and hence the energy) was definite and constant for a particle passing through a region of variable potential energy. We now see that this assumption is justified only if the potential energy doesn't change with time. This is because a time-varying potential energy eliminates the possibility of invariance under time shift.

Invariance

We have seen a few examples of invariance in quantum mechanics. It is now time to define this concept more precisely. A quantum mechanical wave function is said to be invariant under some transformation if the transformed wave function is observationally indistinguishable from the original.

In the above examples, the transformation is accomplished by replacing x by $x+D$ in the case of displacement in space and similarly by replacing t by $\mathrm{t}+\mathrm{T}$ for displacement in time. However, the idea of a transformation is much more general; other examples will be discussed as they arise.

The idea of "observationally indistinguishable" can be tricky. For example, if some transformation results in a new wave function which is the old wave function times a constant phase factor, then the new wave function is observationally indistinguishable from the old one. This is because physical measurements capture phase differences between different parts of wave functions (think of how interferometers work), but not absolute phases. The constant phase factor disappears in this difference calculation. However, if the multiplicative phase factor created by some transformation is a function of position, then the phase difference between different parts of a wave function changes as a result of the transformation. The wave function is not invariant under this transformation.

Compatible Variables

We already know that definite values of certain pairs of variables cannot be obtained simultaneously in quantum mechanics. For instance, the indefiniteness of position and momentum are related by the uncertainty principle — a definite value of position implies an indefinite value of the momentum and vice versa. If definite values of two variables can be simultaneously obtained, then we call these variables *compatible*. If not, the variables are *incompatible*.

If the wave function of a particle is invariant under the displacements associated with both variables, then the variables are compatible. For instance, the complex exponential plane wave associated with a free particle is invariant under displacements in both space and time. Since momentum is associated with space displacements and energy with time displacements, the momentum and energy are compatible variables for a free particle.

Compatibility and Conservation

Variables which are compatible with the energy have a special status. The wave function which corresponds to a definite value of such a variable is invariant to displacements in time. In other words, the wave function doesn't change under this displacement except for a trivial phase factor. Thus, if the wave function is also invariant to some other transformation at a particular time, it is invariant to that transformation for all time. The variable associated with that transformation therefore retains its definite value for all time — i. e., it is *conserved*.

For example, the plane wave implies a definite value of energy, and is thus invariant under time displacements. At time $t = 0$, it is also invariant under x displacements, which corresponds to the fact that it represents a particle with a known value of momentum. However, since momentum and energy are compatible for a free particle, the wave function will represent the *same* value of momentum at all other times. In other words, if the momentum is definite at $t = 0$, it will be definite at all later times, and furthermore will have the same value. This is how the conservation of momentum (and by extension, the conservation of any other variable compatible with energy) is expressed in quantum mechanics.

Symmetries and Variables

In modern quantum physics, the discovery of new symmetries leads to new dynamical variables. In the problems we show how that comes about for the symmetries of parity $(x \rightarrow -x)$, time reversal $\mathrm{t} \rightarrow -\mathrm{t})$, and charge conjugation (the interchange of particles with antiparticles). One of the key examples of this was the development of the quark theory of matter, which came from the observation that the interchange of certain groups of elementary particles left the universe approximately unchanged, meaning that the universe was (approximately) symmetric under these interchanges.