7.6: Heisenberg Uncertainty Principle

Classically, we consider the location of a particle to be a knowable piece of information. In quantum mechanics the position of a particle is well known if the wave packet representing it is small in size. However, quantum mechanics imposes a price on accurately knowing the position of a particle in terms of the future predictability of its position. This is because a small wave packet, which corresponds to accurate knowledge of the corresponding particle's position, implies the superposition of plane waves corresponding to a broad distribution of wavenumbers. This translates into a large uncertainty in the wavenumber, and hence the momentum of the particle. In contrast, a broad wave packet corresponds to a narrower distribution of wavenumbers, and correspondingly less uncertainty in the momentum.

Referring back to chapters 1 and 2, recall that both the longitudinal (along the direction of motion) and transverse (normal to the direction of motion) dimensions of a wave packet, $\Delta \mathrm{x}_{\mathrm{L}}$ and $\Delta \mathrm{x}_{\mathrm{T}}$, can be related to the spread of longitudinal and transverse wavenumbers, $\Delta \mathrm{k}_{\mathrm{L}}$ and $\Delta \mathrm{k}_{\mathrm{T}}$: :

$$\Delta k_{L} \Delta x_{L} \approx 1 \tag{7.6.1}$$

$$\Delta k_{T} \Delta x_{T} \approx 1 \tag{7.6.2}$$

We have omitted numerical constants which are of order unity in these approximate relations so as to show their essential similarity.

The above equations can be interpreted in the following way. Since the absolute square of the wave function represents the probability of finding a particle, $\Delta \mathrm{x}_{\mathrm{L}}$ and $\Delta \mathrm{x}_{\mathrm{T}}$ represent the uncertainty in the particle's position. Similarly, $\Delta \mathrm{k}_{\mathrm{L}}$ and $\Delta \mathrm{k}_{\mathrm{T}}$ represent the uncertainty in the particle's longitudinal and transverse wave vector components. This latter uncertainty leads to uncertainty in the particle's future motion — larger or smaller longitudinal k results respectively in larger or smaller particle speed, while uncertainty in the transverse wavenumber results in uncertainty in the particle's direction of motion. Thus uncertainties in any component of **k** result in uncertainties in the corresponding component of the particle's velocity, and hence in its future position.

The equations (7.6.1) and (7.6.2) show that uncertainty in the present and future positions of a particle are complimentary. If the present position is accurately known due to the small size of the associated wave packet, then the future position is not very predictable, because the wave packet disperses rapidly. On the other hand, a broad-scale initial wave packet means that the present position is poorly known, but the uncertainty in position, poor as it is, doesn't rapidly increase with time, since the wave packet has a small uncertainty in wave vector and thus disperses slowly. This is a statement of the *Heisenberg uncertainty principle*.

The uncertainty principle also applies between frequency and time:

$$\Delta \omega \Delta t \approx 1 \tag{7.6.3}$$

This shows up in the beat frequency equation $1 / \mathrm{T}_{\text{beat}} = \Delta \mathrm{f} = \Delta \omega / 2 \pi$. The beat period T_{beat} may be thought of as the size of a "wave packet in time". The beat frequency equation may be rewritten as $\Delta \omega \mathrm{T}_{\text{beat}} = 2 \pi$, which is the same as equation (7.6.3) if the factor of 2π is ignored and T_{beat} is identified with Δt.

The above forms of the uncertainty principle are not relativistically invariant. A useful invariant form may be obtained by transforming to the coordinate system in which a particle is stationary. In this reference frame the time t becomes the proper time τ associated with the particle. Furthermore, the frequency ω becomes the rest frequency μ. The uncertainty principle thus becomes

$$\Delta \mu \Delta \tau \approx 1 \tag{7.6.4}$$

in this reference frame. However, since $\Delta \mu$ and $\Delta \mathrm{T}$ are relativistic invariants, this expression of the uncertainty principle is valid in any reference frame.

It is more common to express the uncertainty principle in terms of the mass, momentum, and energy by multiplying equations (7.6.1) - (7.6.4) by $\hbar$. Lumping the momentum equations, we find

$$\Delta \Pi \Delta x \approx \hbar \tag{7.6.5}$$

$$\Delta E \Delta t \approx \hbar \tag{7.6.6}$$

and

$$\Delta\left(m c^{2}\right) \Delta \tau \approx \hbar \tag{7.6.7}$$

Classical mechanics is the realm of quantum mechanics in which the dimensions of the system of interest are much larger than the wavelengths of the waves corresponding to the particles constituting the system. In this case the uncertainties induced by the uncertainty principle are unimportant. This limit is analogous to the geometrical optics limit for light. Thus, we can say that classical mechanics is the geometrical optics limit of quantum mechanics.
