12.4: Complex Exponential Solutions

Complex exponential functions of the form $\mathrm{x} = \exp(\pm \mathrm{i}\omega \mathrm{t})$ also constitute solutions to the free harmonic oscillator governed by equation (12.2.1). This makes sense, as the complex exponential is the sum of sines and cosines. However, for the frictionless harmonic oscillator, the exponential solutions provide no particular advantage over sines and cosines. Furthermore, oscillator displacements are real, not complex quantities.

The superposition principle solves the problem of complex versus real solutions. For an equation like (12.2.1) which has real coefficients, if $\exp(i\omega t)$ is a solution, then so is $\exp(-\mathrm{i}\omega \mathrm{t})$, so the superposition of these two solutions is also a solution. Furthermore

$$\exp(i\omega t) + \exp(-i\omega t) = 2\cos(\omega t) = 2\,\mathrm{Re}[\exp(i\omega t)] \tag{12.4.1}$$

This shows a shortcut for getting the physical part of a complex exponential solution to equations like the harmonic oscillator equation; simply take the real part.

Complex exponential solutions come into their own for more complicated equations. For instance, suppose the force on the mass in the mass-spring system takes the form

$$F = -kx - b\frac{dx}{dt} \tag{12.4.2}$$

The term containing b represents a frictional damping effect on the harmonic oscillator and the governing differential equation becomes

$$\frac{d^2x}{dt^2} + \frac{b}{M}\frac{dx}{dt} + \frac{k}{M}x = 0 \tag{12.4.3}$$

Trying the exponential function $\exp(\sigma t)$ in this equation results in

$$\sigma = \frac{1}{2}\left[-\frac{b}{M} \pm \left(\frac{b^2}{M^2} - \frac{4k}{M}\right)^{1/2}\right] = -\beta \pm i\left(\omega_0^2 - \beta^2\right)^{1/2} \tag{12.4.4}$$

where we have set

$$\beta = \frac{b}{2M} \quad \omega_0 = \left(\frac{k}{M}\right)^{1/2} \tag{12.4.5}$$

The quantity $\omega \equiv \left(\omega_0^2 - \beta^2\right)^{1/2}$ is the actual frequency of oscillation of the damped oscillator, which one can see is less than the oscillation frequency ω_0 that occurs with the damping turned off. The physical solution to the damped oscillator is thus

$$x(t) = \mathrm{Re}[\exp(\sigma t)] = \mathrm{Re}[\exp(i\omega t)\exp(-\beta t)] = \cos(\omega t)\exp(-\beta t) \tag{12.4.6}$$

as long as $\omega_0^2 > \beta^2$. Notice that this solution is in the form of an oscillation $\cos(\omega t)$ multiplied by a decaying exponential $\exp(-\beta t)$. This confirms that the b term decreases the amplitude of the oscillation with time.