9.1: Math Tutorial — Complex Waves

Until now we have represented quantum mechanical plane waves by sine and cosine functions, just as with other types of waves. However, plane matter waves cannot be truly represented by sines and cosines. We need instead mathematical functions in which the wave displacement is complex rather than real. This requires the introduction of a bit of new mathematics, which we tackle first. Using our new mathematical tool, we are then able to explore two crucially important ideas in quantum mechanics; (1) the relationship between symmetry and conservation laws, and (2) the dynamics of spatially confined waves.

A complex number z is the sum of a real number and an imaginary number. An imaginary number is just a real number multiplied by $i \equiv (-1)^{1/2}$. Thus, we can write $z = a + ib$ for any complex z, where a and b are real. The quantities a and b are the real and imaginary parts of z, sometimes written $Re(z)$ and $Im(z)$.

Quantum mechanics requires wave functions to be complex, i. e., to possess real and imaginary parts. Plane waves in quantum mechanics actually take the form $\Psi = \exp[\mathrm{i}(\mathrm{kx} - \omega \mathrm{t})]$ rather than, say, $\cos(kx - \omega t)$. The reason for this is the need to distinguish between waves with positive and negative frequencies. If we replace k and ω with $-\mathrm{k}$ and $-\omega$ in the cosine form, we get

$$\cos(-kx + \omega t) = \cos[-(kx - \omega t)] = \cos(kx - \omega t). \tag{9.1.1}$$

In other words, changing the sign of k and ω results in no change in a wave expressed as a cosine function. The two quantum mechanical states, one with wavenumber and frequency k and ω and the other with $-\mathrm{k}$ and $-\omega$, yield indistinguishable wave functions and therefore would represent physically indistinguishable states. The cosine form is thus insufficiently flexible to represent quantum mechanical waves. On the other hand, if we replace k and ω with their negatives in the complex exponential form of a plane wave we get $\psi = \exp[-\mathrm{i}(\mathrm{kx} - \omega \mathrm{t})]$, which is different from $\exp[\mathrm{i}(\mathrm{kx} - \omega \mathrm{t})]$. These two wave functions are distinguishable and thus correspond to distinct physical states.

It is not immediately obvious that a complex exponential function provides the oscillatory behavior needed to represent a plane wave. However, the complex exponential can be expressed in terms of sines and cosines using Euler's equation:

$$\exp(i\phi) = \cos(\phi) + i\sin(\phi) \quad \text{(Euler's equation)}. \tag{9.1.2}$$

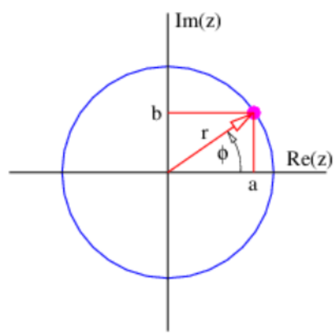

Figure 9.1.1:: Graphical representation of a complex number z as a point in the complex plane. The horizontal and vertical Cartesian components give the real and imaginary parts of z respectively.

If we define $r = \left(a^2 + b^2\right)^{1/2}$ and $\phi = \tan^{-1}(b/a)$, then an alternate way of expressing a complex number is z = r exp(iϕ), which by Euler's equation equals $r\cos(\phi) + \mathrm{i}r\sin(\phi)$. Comparison shows that $\mathrm{a} = r\cos(\phi)$ and $\mathrm{b} = r\sin(\phi)$. Thus, a complex number can be thought of as a point in the a-b plane with Cartesian coordinates a and b and polar coordinates r and ϕ. The a-b plane is called the complex plane.

We now see how the complex wave function represents an oscillation. If $\Psi = \exp[\mathrm{i}(\mathrm{kx} - \omega \mathrm{t})]$, the complex function $\psi(x, t)$ moves round and round the unit circle in the complex plane as x and t change, as illustrated in Figure 9.1.1:. This contrasts with the back and forth oscillation along the horizontal axis of the complex plane represented by $\cos(\mathrm{kx} - \omega \mathrm{t})$.

We will not present a formal proof of Euler's equation — you will eventually see it in your calculus course. However, it may be helpful to note that the ϕ derivatives of exp(iϕ) and $\cos(\phi) + \mathrm{i}\sin(\phi)$ have the same behavior:

$$\frac{d}{d\phi}\exp(i\phi) = i\exp(i\phi) \tag{9.1.3}$$

$$\frac{d}{d\phi}[\cos(\phi) + i\sin(\phi)] = -\sin(\phi) + i\cos(\phi) \tag{9.1.4}$$

$$= i[\cos(\phi) + i\sin(\phi)]. \tag{9.1.5}$$

(In the second of these equations we have replaced the minus sign in front of the sine function by i^2 and then extracted a common factor of i.) The ϕ derivative of both of these functions thus yields the function back again times i. This is a strong hint that $\exp(i\phi)$ and $\cos(\phi) + i\sin(\phi)$ are different ways of representing the same function.

We indicate the complex conjugate of a complex number z by a superscripted asterisk, i. e., z^*. It is obtained by replacing i by -i. Thus, $(a+ib)^* = a - ib$. The absolute square of a complex number is the number times its complex conjugate:

$$|z|^2 = |a+ib|^2 \equiv (a+ib)(a-ib) = a^2 + b^2 = r^2 \tag{9.1.6}$$

Notice that the absolute square of a complex exponential function is one:

$$|\exp(i\phi)|^2 = \exp(i\phi)\exp(-i\phi) = \exp(i\phi - i\phi) = \exp(0) = 1 \tag{9.1.7}$$

In quantum mechanics the absolute square of the wave function at any point expresses the relative probability of finding the associated particle at that point. Thus, the probability of finding a particle represented by a plane wave is uniform in space. Contrast this with the relative probability associated with a sine wave: $|\sin(kx-\omega t)|^2 = \sin^2(kx-\omega t)$. This varies from zero to one, depending on the phase of the wave. The “waviness” in a complex exponential plane wave resides in the phase rather than in the magnitude of the wave function.

One more piece of mathematics is needed. The complex conjugate of Euler’s equation is

$$\exp(-i\phi) = \cos(\phi) - i\sin(\phi) \tag{9.1.8}$$

Taking the sum and the difference of this with the original Euler’s equation results in the expression of the sine and cosine in terms of complex exponentials:

$$\cos(\phi) = \frac{\exp(i\phi) + \exp(-i\phi)}{2} \quad \sin(\phi) = \frac{\exp(i\phi) - \exp(-i\phi)}{2i} \tag{9.1.9}$$

We aren’t used to having complex numbers show up in physical theories and it is hard to imagine how we would measure such a number. However, everything observable comes from taking the absolute square of a wave function, so we deal only with real numbers in experiments.

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