5.2: Math Tutorial – Four-Vectors

Also shown in Figure 5.2.1: is a spacetime vector or four-vector which represents the frequency and wavenumber of the wave, which we refer to as the wave four-vector. It is called a four-vector because it has 3 spacelike components and one timelike component when there are 3 space dimensions. In the case shown, there is only a single space dimension. The spacelike component of the wave four-vector is just k (or **k** when there are 3 space dimensions), while the timelike component is ω/c. The c is in the denominator to give the timelike component the same dimensions as the spacelike component. From Figure 5.2.1: it is clear that the slope of the line representing the four-vector is ω/ck, which is just the inverse of the slope of the wave fronts.

Let us define some terminology. We indicate a four-vector by underlining and write the components in the following way: $\underline{k}=(k, \omega/c)$, where k is the wave four-vector, k is its spacelike component, and ω/c is its timelike component. For three space dimensions, where we have a wave vector rather than just a wavenumber, we write $\underline{\mathrm{k}}=(\mathbf{k}, \omega/\mathrm{c})$.

Another example of a four-vector is simply the position vector in spacetime, $\underline{x}=(x, ct)$, or $\underline{x}=(\mathbf{x}, \mathrm{ct})$ in three space dimensions. The c multiplies the timelike component in this case, because that is what is needed to give it the same dimensions as the spacelike component.

In three dimensions we define a vector as a quantity with magnitude and direction. Extending this to spacetime, a four-vector is a quantity with magnitude and direction in spacetime. Implicit in this definition is the notion that the vector's magnitude is a quantity independent of coordinate system or reference frame. We have seen that the invariant interval in spacetime from the origin to the point $(\mathrm{x}, \mathrm{ct})$ is $I=\left(x^{2}-c^{2} t^{2}\right)^{1/2}$ so it makes sense to identify this as the magnitude of the position vector. This leads to a way of defining a dot product of four-vectors. Given two four-vectors $\underline{A}=(\mathbf{A}, A_{t})$ and $\underline{B}=(\mathbf{B}, B_{t})$, the dot product is

$$\underline{A} \cdot \underline{B}=\mathbf{A} \cdot \mathbf{B}-A_{t} B_{t} \quad \text{(dot product in spacetime)}. \tag{5.2.1}$$

This is consistent with the definition of invariant interval if we set $\underline{A}=\underline{B}=\underline{x}$, since then $\underline{x} \cdot \underline{x}=x^{2}-c^{2} t^{2}=I^{2}$

In the odd geometry of spacetime it is not obvious what "perpendicular" means. We therefore define two four-vectors $\underline{\mathrm{A}}$ and $\underline{\mathrm{B}}$ to be perpendicular if their dot product is zero, $\underline{A} \cdot \underline{B}=0$, in analogy with ordinary vectors.

The dot product of two four-vectors is a scalar result, i. e., its value is independent of coordinate system. This can be used to our advantage on occasion. For instance, consider the dot product of a four-vector $\underline{\mathrm{A}}$ which resolves into (A_{x}, A_{t}) in the unprimed frame. Let us further suppose that the spacelike component is zero in some primed frame, so that the components in this frame are $(0, A_{t}^{\prime})$. The fact that the dot product is independent of coordinate system means that

$$\underline{A} \cdot \underline{A}=A_{x}^{2}-A_{t}^{2}=-A_{t}^{\prime 2} \tag{5.2.2}$$

This constitutes an extension of the spacetime Pythagorean theorem to four-vectors other than the position four-vector. Thus, for instance, the wavenumber for some wave may be zero in the primed frame, which means that the wavenumber and frequency in the unprimed frame are related to the frequency in the primed frame by

$$k^{2}-\omega^{2}/c^{2}=-\omega^{\prime 2}/c^{2} \tag{5.2.3}$$