5.6: Addition of Velocities

Figure $5.6.4$: shows the world line of a moving object from the point of view of two different reference frames, with the primed frame (left panel) moving to the right at speed U relative to the unprimed frame (right panel). The goal is to calculate the velocity of the object relative to the unprimed frame, v, assuming its velocity in the primed frame, v' is known. The classical result is simply

$$v = U + v' \quad \text{(classical result).} \tag{5.6.1}$$

However, this is inconsistent with the speed of light being constant in all reference frames, since if we substitute c for v′, this formula predicts that the speed of light in the unprimed frame is U + c.

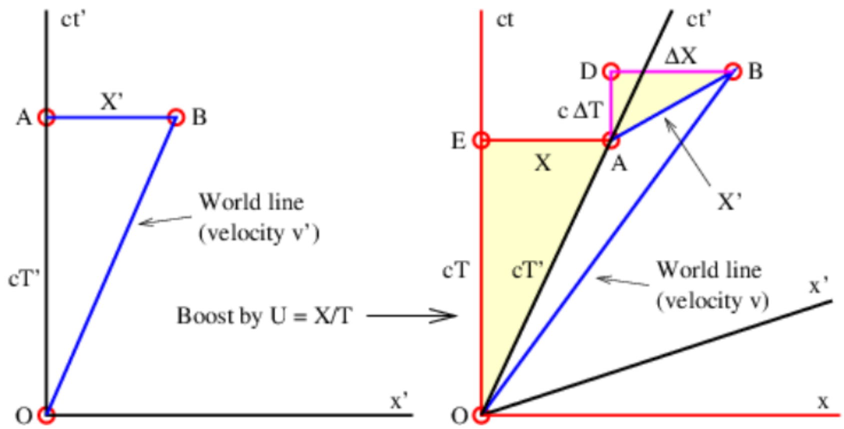

Figure $5.6.4$:: Definition sketch for relativistic velocity addition. The two panels show the world line of a moving object relative to two different reference frames moving at velocity U with respect to each other. The velocity of the world line in the left panel is v' while its velocity in the right panel is $\mathcal{V}$.

We can use the geometry of Figure $5.6.4$: to come up with the correct relativistic formula. From the right panel of this figure we infer that

$$\frac{v}{c} = \frac{X + \Delta X}{cT + c\Delta T} = \frac{X/(cT) + \Delta X/(cT)}{1 + \Delta T/T} \tag{5.6.2}$$

This follows from the fact that the slope of the world line of the object in this frame is c/v. The slope is calculated as the ratio of the rise, $c(T + \Delta T)$ to the run, $X + \Delta X$.

From the left panel of Figure $5.6.4$: we similarly see that

$$\frac{v'}{c} = \frac{X'}{cT'} \tag{5.6.3}$$

However, we can apply our calculations of Lorentz contraction and time dilation from the previous chapter to triangles ABD and OAE in the right panel. The slope of AB is U/c because AB is horizontal in the left panel, so $\mathrm{X}' = \Delta\mathrm{X}\left(1 - \mathrm{U}^2/\mathrm{c}^2\right)^{1/2}$. Similarly, the slope of OA is c/U since OA is vertical in the left panel, and $\mathrm{T}' = \mathrm{T}\left(1 - \mathrm{U}^2/\mathrm{c}^2\right)^{1/2}$. Substituting these formulas into the equation for v'/c yields

$$\frac{v'}{c} = \frac{\Delta X}{cT} \tag{5.6.4}$$

Again using what we know about the triangles ABD and OAE, we see that

$$\frac{U}{c} = \frac{c\Delta T}{\Delta X} = \frac{X}{cT} \tag{5.6.5}$$

Finally, we calculate $\Delta T/T$ by noticing that

$$\frac{\Delta T}{T} = \frac{\Delta T}{T}\frac{c\Delta X}{c\Delta X} = \left(\frac{c\Delta T}{\Delta X}\right)\left(\frac{\Delta X}{cT}\right) = \frac{U}{c}\frac{v'}{c} \tag{5.6.6}$$

Substituting equations (5.6.4), (5.6.5), and (5.6.6) into equation (5.6.2) and simplifying yields the relativistic velocity addition formula:

$$v=\frac{U+v'}{1+Uv'/c^2} \quad \text{(special relativity)} \tag{5.6.7}$$

Notice how this equation behaves in various limits. If $|Uv'| \ll c^2$, the denominator of equation (5.6.7) is nearly unity, and the special relativistic formula reduces to the classical case. On the other hand, if $\mathrm{v}'=\mathrm{c}$, then equation (5.6.7) reduces to $v=c$. In other words, if the object in question is moving at the speed of light in one reference frame, it is moving at the speed of light in all reference frames, i. e., for all possible values of U. Thus, we have found a velocity addition formula that 1) reduces to the classical formula for low velocities and 2) gives the observed results for very high velocities as well.

Equation (5.6.7) is valid even if v' is negative, i. e., if the object is moving to the right less rapidly than the primed reference frame, or even if it is moving to the left in the unprimed frame.