3.14.1.2: Equal Temperament

Seems simple enough; start with a note at a given frequency, apply the multiplication rules to get the other notes in the scale. So multiplying by 2 gives a note an octave higher, multiplying by $3/2$ a note in between, and so forth. What could go wrong?

We already know the rules for the Pythagorean scale do not give exactly the same frequencies as the rules for the Ptolemaic (or Just) Scale. Let's stick with Pythagoras and decide we want to fill in the notes between C (at the modern day frequency of $261.63\ \mathrm{Hz}$) and C an octave above ($261.63\ \mathrm{Hz} \times 2 = 523.15\ \mathrm{Hz}$). We start with $261.63\ \mathrm{Hz}$ and multiply by $3/2$ to get $392.44\ \mathrm{Hz}$ which is G. We take G ($392.44\ \mathrm{Hz}$) and multiply by $3/2$ again to get $588.67\ \mathrm{Hz}$ which is an octave above so we divide by 2 to get $294.33\ \mathrm{Hz}$ which is D. We can get A at $441.49\ \mathrm{Hz}$ by multiplying $294.33\ \mathrm{Hz}$ by $3/2$. Multiplying by $3/2$ and dividing by 2 gives $331.12\ \mathrm{Hz}$ which is E. Multiplying by $3/2$ gives $496.69\ \mathrm{Hz}$ which is B. Repeating this procedure (multiplying by $3/2$ and dividing by 2 if it is outside the octave) two more times gives $\mathrm{F}^{\#}$ (F sharp) and $\mathrm{C}^{\#}$ (C sharp). So far so good.

Now we run into a problem. Because if we use this $\mathrm{C}^{\#}$ at $279.39\ \mathrm{Hz}$ and multiply by $3/2$ we get $419.08\ \mathrm{Hz}$ *which is not a note on the scale!* In fact it is so close to A^{b} (A flat) at $412.42\ \mathrm{Hz}$ that we probably would mistake it for A^{b}. Likewise if we do the procedure again we get $314.31\ \mathrm{Hz}$ which is very close to E^{b} at $310.08\ \mathrm{Hz}$ but is not on this scale. In other words, the Pythagorean system of generating notes finds an infinite set of notes between C at $261.63\ \mathrm{Hz}$ and the C an octave above. The Pythagorean solution was to stop after five notes (a pentatonic scale) and not use any more. But the idea of using the simple mathematical rule of $2/3$ for generating more notes where all the notes harmonize doesn't work. The dilemma is that ratios of $3:2$ sound good to the ear but using a ratio of $3:2$ does not result in a closed, limited number of notes.

As music became more complicated, musicians started to want to switch keys during the piece of music. When a musician changes key they are essentially starting with a different note (say B at $496.67\ \mathrm{Hz}$ in the Pythagorean scale) and using an octave based on that new starting point. But starting on a different note and applying the rule to generate the scale results in new notes which are not close to previous notes (it almost works in the Pythagorean scale but not in the Just temperament). If you have a musical instrument, say a piano, set up so that the notes correspond to a Just scale based on F (the key of F) for example, changing to a key of B isn't possible unless you add extra strings to the piano because there are new frequencies in the key of B. The new notes also do not harmonize with the old notes. To maintain the Just scale requires a different instrument, one tuned differently for each key you want to play in (or many extra strings).

This problem gets worse when you try to generate notes an octave above or below the initial octave using the rules; the higher octave doesn't harmonize with the lower octave if they are generated using the rules. Notes an octave above should be twice the frequency of the note in the lower octave but using the formula to generate the note doesn't give a frequency twice that of the lower octave. For singing and even violins, which can make any desired frequency (within a given range), this isn't a problem because the performer can just shift the note a little so that it sounds right. But for the piano, organ, flute and stringed instruments with frets this is a big problem because the notes are determined by the construction of the instrument and cannot be changed without re-tuning it. So it appears there is a choice between making instruments so they harmonize well in one octave and scale but not another; or have different instruments for each octave and each scale; or re-tune the instrument every time you want to change scales or octaves.

A scale system that tries to correct for the problem of being able to play in any key is called a ***temperament***. The chosen starting note of a scale is called the ***tonic*** and a musical ***fifth*** consists of the tonic and the fifth note of the scale.

Historically several scale adjustments generically known as ***well-tempered scales*** were proposed and eventually the ***even-tempered scale*** won out (see Well versus Even Temperament). In this scale the frequencies between octaves are chosen to be equally spaced in which case the ratio between notes is 1.059463 (the notes can be generated by multiplying or dividing by either 2 and/or 1.059463). This divides an octave into twelve notes. Another way to write this mathematically is $f = f_o \times (2^{1/2})^n$ where f_o is generally chosen to be $440\ \mathrm{Hz}$ in modern times. $2^{1/2} = 1.059463094\ldots$and each value of n generates a new note.

Johann Sebastian Bach was the musician who managed to get the world to pay attention to alternative scales by writing music (this was in the 1700's) specifically to sound good for instruments tuned a certain way. Eventually most western cultures adopted the even-tempered scale (some other cultures did not which is why, for example, music from India often sounds very different than western music). Because the ratios between frequencies are not exactly $3:2$ or $5:4$ you can often hear chords where there is beating and dissonance. The chords also do not sound as pure as Pythagorean tuning because they are not exactly harmonic however the greater flexibility to change keys and to (almost) harmonize over several octaves makes equal temperament a useful compromise. Below is a diagram of the frequencies of various scale systems (temperaments).

Frequencies of Various Scale Systems									
Pythagorean		Just Major		Mean-tone		Werckmeister		Equal-tempered	
Note	Hz	*Note*	Hz	*Note*	Hz	*Note*	Hz	*Note*	Hz
C	523.25	C	523.25	C	523.25	C	523.25	C	523.25
B	496.67	B	490.55	B	489.03	B	491.67	B	493.88
B^b	465.12	B^b	470.93	B^b	468.02	$B^b/A^\#$	465.12	$B^b/A^\#$	466.16
A	441.49	A	436.05	A	437.41	A	437.05	A	440.00
A^b	413.42	A^b	418.60	A^b	418.60	$A^b/G^\#$	413.42	$A^b/G^\#$	415.30
G	392.44	G	392.44	G	391.21	G	391.16	G	392.00
$F^\#$	372.52	$F^\#$	367.92	$F^\#$	365.62	$G^b/F^\#$	367.51	$G^b/F^\#$	369.99
F	348.83	F	348.83	F	349.92	F	348.83	F	349.23
E	331.11	E	326.03	E	327.03	E	327.76	E	329.62
E^b	310.08	E^b	313.96	E^b	312.98	$E^b/D^\#$	310.08	$E^b/D^\#$	311.13
D	294.33	D	294.33	D	292.50	D	292.37	D	293.66
$C^\#$	279.39	$C^\#$	272.54	$C^\#$	273.37	$D^b/C^\#$	275.62	$D^b/C^\#$	277.18
C	261.63	C	261.63	C	261.63	C	261.63	C	261.63

Table $3.14.1.2.1$: Frequencies of various scales, based on $C_4 = 261.626$ Hz.

Modern musical scales in western culture are different in one other way from older classical music. At the time of Bach the scales were based on the note A being about $415\ \mathrm{Hz}$. In Handel's time the frequency of A was $422.5\ \mathrm{Hz}$ and today it is $440\ \mathrm{Hz}$. This gradual shift upward was made possible by stronger materials for pianos and guitars which could withstand the greater tension in tighter strings.

As mentioned previously, the constraints of making a piano with a wide range of notes but is not too large requires using strings that are dense and under a great deal of tension. This makes them slightly nonlinear, meaning that the overtones are not exactly harmonic. If the piano was tuned in a precise mathematical way (using any of the scale systems) these an-harmonic overtones would not sound well when two notes an octave apart are played. What is generally done is to tune the piano by ear so that it sounds correct, starting with low notes and working up to higher pitches. The result of this process is that, for a modern piano, low notes are slightly flat (lower) while high notes are slightly sharp (higher). The figure below (called a ***Railsback curve***) shows the difference from perfect tuning of a modern piano.

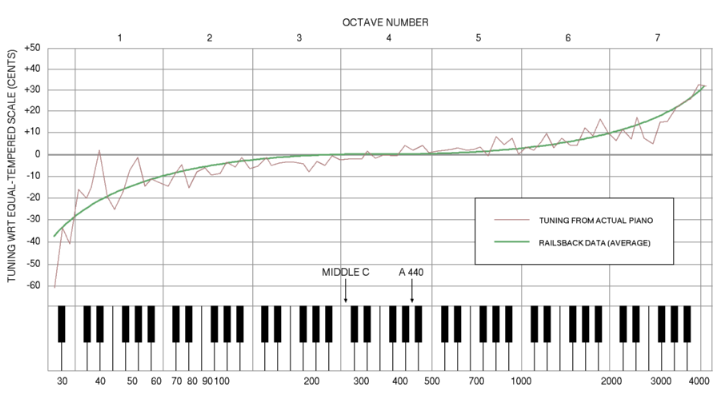

Figure $3.14.1.2.1$