

## 9.1.4.4: The Del-operator

### The $\nabla$ -operator

In cartesian coordinates  $(x, y, z)$ :

$$\vec{\nabla} = \frac{\partial}{\partial x} \vec{e}_x + \frac{\partial}{\partial y} \vec{e}_y + \frac{\partial}{\partial z} \vec{e}_z, \quad \text{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x} \vec{e}_x + \frac{\partial f}{\partial y} \vec{e}_y + \frac{\partial f}{\partial z} \vec{e}_z \quad (9.1.4.4.1)$$

$$\text{div } \vec{a} = \vec{\nabla} \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}, \quad \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (9.1.4.4.2)$$

$$\text{rot } \vec{a} = \vec{\nabla} \times \vec{a} = \left( \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) \vec{e}_x + \left( \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) \vec{e}_y + \left( \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \vec{e}_z \quad (9.1.4.4.3)$$

In cylinder coordinates  $(r, \varphi, z)$  holds:

$$\vec{\nabla} = \frac{\partial}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \varphi} \vec{e}_\varphi + \frac{\partial}{\partial z} \vec{e}_z, \quad \text{grad} f = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi + \frac{\partial f}{\partial z} \vec{e}_z \quad (9.1.4.4.4)$$

$$\text{div } \vec{a} = \frac{\partial a_r}{\partial r} + \frac{a_r}{r} + \frac{1}{r} \frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}, \quad \nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \quad (9.1.4.4.5)$$

$$\text{rot } \vec{a} = \left( \frac{1}{r} \frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z} \right) \vec{e}_r + \left( \frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \right) \vec{e}_\varphi + \left( \frac{\partial a_\varphi}{\partial r} + \frac{a_\varphi}{r} - \frac{1}{r} \frac{\partial a_r}{\partial \varphi} \right) \vec{e}_z \quad (9.1.4.4.6)$$

In spherical coordinates  $(r, \theta, \varphi)$ :

$$\begin{aligned} \vec{\nabla} &= \frac{\partial}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \vec{e}_\varphi \\ \text{grad} f &= \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi \\ \text{div } \vec{a} &= \frac{\partial a_r}{\partial r} + \frac{2a_r}{r} + \frac{1}{r} \frac{\partial a_\theta}{\partial \theta} + \frac{a_\theta}{r \tan \theta} + \frac{1}{r \sin \theta} \frac{\partial a_\varphi}{\partial \varphi} \\ \text{rot } \vec{a} &= \left( \frac{1}{r} \frac{\partial a_\varphi}{\partial \theta} + \frac{a_\theta}{r \tan \theta} - \frac{1}{r \sin \theta} \frac{\partial a_\theta}{\partial \varphi} \right) \vec{e}_r + \left( \frac{1}{r \sin \theta} \frac{\partial a_r}{\partial \varphi} - \frac{\partial a_\varphi}{\partial r} - \frac{a_\varphi}{r} \right) \vec{e}_\theta + \\ &\quad \left( \frac{\partial a_\theta}{\partial r} + \frac{a_\theta}{r} - \frac{1}{r} \frac{\partial a_r}{\partial \theta} \right) \vec{e}_\varphi \\ \nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \tan \theta} \frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} \end{aligned}$$

General orthonormal curvilinear coordinates  $(u, v, w)$  can be obtained from cartesian coordinates by the transformation  $\vec{x} = \vec{x}(u, v, w)$ . The unit vectors are then given by:

$$\vec{e}_u = \frac{1}{h_1} \frac{\partial \vec{x}}{\partial u}, \quad \vec{e}_v = \frac{1}{h_2} \frac{\partial \vec{x}}{\partial v}, \quad \vec{e}_w = \frac{1}{h_3} \frac{\partial \vec{x}}{\partial w} \quad (9.1.4.4.7)$$

where the factors  $h_i$  set the norm to 1. Then holds:

$$\begin{aligned}
 \text{grad} f &= \frac{1}{h_1} \frac{\partial f}{\partial u} \vec{e}_u + \frac{1}{h_2} \frac{\partial f}{\partial v} \vec{e}_v + \frac{1}{h_3} \frac{\partial f}{\partial w} \vec{e}_w \\
 \text{div } \vec{a} &= \frac{1}{h_1 h_2 h_3} \left( \frac{\partial}{\partial u} (h_2 h_3 a_u) + \frac{\partial}{\partial v} (h_3 h_1 a_v) + \frac{\partial}{\partial w} (h_1 h_2 a_w) \right) \\
 \text{rot } \vec{a} &= \frac{1}{h_2 h_3} \left( \frac{\partial (h_3 a_w)}{\partial v} - \frac{\partial (h_2 a_v)}{\partial w} \right) \vec{e}_u + \frac{1}{h_3 h_1} \left( \frac{\partial (h_1 a_u)}{\partial w} - \frac{\partial (h_3 a_w)}{\partial u} \right) \vec{e}_v + \\
 &\quad \frac{1}{h_1 h_2} \left( \frac{\partial (h_2 a_v)}{\partial u} - \frac{\partial (h_1 a_u)}{\partial v} \right) \vec{e}_w \\
 \nabla^2 f &= \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial w} \right) \right]
 \end{aligned}$$

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