

9.1.2: Oscillations

Harmonic oscillation

The general form of a harmonic oscillation is: $\Psi(t) = \hat{\Psi}e^{i(\omega t \pm \varphi)} \equiv \hat{\Psi} \cos(\omega t \pm \varphi)$,

where $\hat{\Psi}$ is the *amplitude*. A superposition of several harmonic oscillations with the same frequency results in another harmonic oscillation:

$$\sum_i \hat{\Psi}_i \cos(\alpha_i \pm \omega t) = \hat{\Phi} \cos(\beta \pm \omega t) \quad (9.1.2.1)$$

with:

$$\tan(\beta) = \frac{\sum_i \hat{\Psi}_i \sin(\alpha_i)}{\sum_i \hat{\Psi}_i \cos(\alpha_i)} \quad \text{and} \quad \hat{\Phi}^2 = \sum_i \hat{\Psi}_i^2 + 2 \sum_{j>i} \sum_i \hat{\Psi}_i \hat{\Psi}_j \cos(\alpha_i - \alpha_j) \quad (9.1.2.2)$$

For harmonic oscillations: $\int x(t)dt = \frac{x(t)}{i\omega}$ and $\frac{d^n x(t)}{dt^n} = (i\omega)^n x(t)$.

Mechanic oscillation

For a spring with constant C and damping k which is connected to a mass M , to which a periodic force $F(t) = \hat{F} \cos(\omega t)$ is applied the equation of motion is $m\ddot{x} = F(t) - k\dot{x} - Cx$. With complex amplitudes, this becomes $-m\omega^2 x = F - Cx - ik\omega x$. With $\omega_0^2 = C/m$ it follows that:

$$x = \frac{F}{m(\omega_0^2 - \omega^2) + ik\omega}, \quad \text{and for the velocity: } \dot{x} = \frac{F}{i\sqrt{Cm}\delta + k} \quad (9.1.2.3)$$

where $\delta = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}$. The quantity $Z = F/\dot{x}$ is called the *impedance* of the system. The *quality* of the system is given by $Q = \frac{\sqrt{Cm}}{k}$.

The frequency with minimal $|Z|$ is called the *velocity resonance frequency*. This is equal to ω_0 . In the *resonance curve* $|Z|/\sqrt{Cm}$ is plotted against ω/ω_0 . The width of this curve is characterized by the points where $|Z(\omega)| = |Z(\omega_0)|\sqrt{2}$. At these points: $R = X$ and $\delta = \pm Q^{-1}$, and the width is $2\Delta\omega_B = \omega_0/Q$.

The *stiffness* of an oscillating system is given by F/x . The *amplitude resonance frequency* ω_A is the frequency where $i\omega Z$ is a minimum. This is the case for $\omega_A = \omega_0 \sqrt{1 - \frac{1}{2}Q^2}$.

The *damping frequency* ω_D is a measure for the time in which an oscillating system comes to rest. It is given by $\omega_D = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}$. A weak damped oscillation ($k^2 < 4mC$) dies out after $T_D = 2\pi/\omega_D$. For a *critically damped* oscillation ($k^2 = 4mC$) $\omega_D = 0$. A strong damped oscillation ($k^2 > 4mC$) decays like (if $k^2 \gg 4mC$) $x(t) \approx x_0 \exp(-t/\tau)$.

Electric oscillations

The *impedance* is given by: $Z = R + iX$. The phase angle is $\varphi := \arctan(X/R)$. The impedance of a resistor is R , of a capacitor $1/i\omega C$ and of a self inductor $i\omega L$. The quality of a coil is $Q = \omega L/R$. The total impedance in case several elements are connected is given by:

1. Series connection: $V = IZ$,

$$Z_{\text{tot}} = \sum_i Z_i, \quad L_{\text{tot}} = \sum_i L_i, \quad \frac{1}{C_{\text{tot}}} = \sum_i \frac{1}{C_i}, \quad Q = \frac{Z_0}{R}, \quad Z = R(1 + iQ\delta) \quad (9.1.2.4)$$

2. Parallel connection: $V = IZ$,

$$\frac{1}{Z_{\text{tot}}} = \sum_i \frac{1}{Z_i}, \quad \frac{1}{L_{\text{tot}}} = \sum_i \frac{1}{L_i}, \quad C_{\text{tot}} = \sum_i C_i, \quad Q = \frac{R}{Z_0}, \quad Z = \frac{R}{1 + iQ\delta} \quad (9.1.2.5)$$

Here, $Z_0 = \sqrt{\frac{L}{C}}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$.

The power from a source is given by $P(t) = V(t) \cdot I(t)$, so $\langle P \rangle_t = \hat{V}_{\text{eff}} \hat{I}_{\text{eff}} \cos(\Delta\phi)$
 $= \frac{1}{2} \hat{V} \hat{I} \cos(\phi_v - \phi_i) = \frac{1}{2} \hat{I}^2 \text{Re}(Z) = \frac{1}{2} \hat{V}^2 \text{Re}(1/Z)$, where $\cos(\Delta\phi)$ is the work factor.

Waves in long conductors

If cables are used for signal transfer, e.g. coax cables then: $Z_0 = \sqrt{\frac{dL}{dx} \frac{dx}{dC}}$.

The transmission velocity is given by $v = \sqrt{\frac{dx}{dL} \frac{dx}{dC}}$.

Coupled conductors and transformers

For two coils enclosing each others flux if Φ_{12} is the part of the flux originating from I_2 through coil 2 which is enclosed by coil 1, then $\Phi_{12} = M_{12}I_2$, $\Phi_{21} = M_{21}I_1$. The coefficients of mutual induction M_{ij} is given by:

$$M_{12} = M_{21} := M = k\sqrt{L_1 L_2} = \frac{N_1 \Phi_1}{I_2} = \frac{N_2 \Phi_2}{I_1} \sim N_1 N_2 \quad (9.1.2.6)$$

where $0 \leq k \leq 1$ is the *coupling factor*. For a transformer $k \approx 1$. At full load:

$$\frac{V_1}{V_2} = \frac{I_2}{I_1} = -\frac{i\omega M}{i\omega L_2 + R_{\text{load}}} \approx -\sqrt{\frac{L_1}{L_2}} = -\frac{N_1}{N_2} \quad (9.1.2.7)$$

Pendulums

The oscillation time $T = 1/f$, for different types of pendulums is given by:

- Oscillating spring: $T = 2\pi\sqrt{m/C}$ if the spring force is given by $F = C \cdot \Delta l$.
- Physical pendulum: $T = 2\pi\sqrt{I/\tau}$ with τ the moment of force and I the moment of inertia.
- Torsion pendulum: $T = 2\pi\sqrt{I/\kappa}$ where $\kappa = \frac{2lm}{\pi r^4 \Delta\varphi}$ is the constant of torsion and I the moment of inertia.
- Mathematical pendulum: $T = 2\pi\sqrt{l/g}$ with g the acceleration of gravity and l the length of the pendulum.

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