

9.1.1: Mechanics

Point-kinetics in a fixed coordinate system

Definitions

The position \vec{r} , the velocity \vec{v} and the acceleration \vec{a} are defined by: $\vec{r} = (x, y, z)$, $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$, $\vec{a} = (\ddot{x}, \ddot{y}, \ddot{z})$. The following holds:

$$s(t) = s_0 + \int |\vec{v}(t)| dt; \quad \vec{r}(t) = \vec{r}_0 + \int \vec{v}(t) dt; \quad \vec{v}(t) = \vec{v}_0 + \int \vec{a}(t) dt \quad (9.1.1.1)$$

When the acceleration is constant this gives: $v(t) = v_0 + at$ and $s(t) = s_0 + v_0 t + \frac{1}{2} at^2$.

For the unit vectors in a direction \perp to the orbit \vec{e}_t and parallel to it \vec{e}_n :

$$\vec{e}_t = \frac{\vec{v}}{|\vec{v}|} = \frac{d\vec{r}}{ds} \quad \dot{\vec{e}}_t = \frac{v}{\rho} \vec{e}_n; \quad \vec{e}_n = \frac{\dot{\vec{e}}_t}{|\dot{\vec{e}}_t|} \quad (9.1.1.2)$$

For the curvature k and the radius of curvature ρ :

$$\vec{k} = \frac{d\vec{e}_t}{ds} = \frac{d^2\vec{r}}{ds^2} = \left| \frac{d\varphi}{ds} \right|; \quad \rho = \frac{1}{|k|} \quad (9.1.1.3)$$

Polar coordinates

Polar coordinates are defined by: $x = r \cos(\theta)$, $y = r \sin(\theta)$. So, for the unit coordinate vectors: $\dot{\vec{e}}_r = \dot{\theta} \vec{e}_\theta$, $\dot{\vec{e}}_\theta = -\dot{\theta} \vec{e}_r$.

The velocity and the acceleration are derived from:

$$\vec{r} = r \vec{e}_r, \quad \vec{v} = \dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta, \quad \vec{a} = (\ddot{r} - r \dot{\theta}^2) \vec{e}_r + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \vec{e}_\theta \quad (9.1.1.4)$$

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Relative motion

For the motion of a point D w.r.t. a point Q: $\vec{r}_D = \vec{r}_Q + \frac{\vec{\omega} \times \vec{v}_Q}{\omega^2}$ with $\vec{QD} = \vec{r}_D - \vec{r}_Q$ and $\omega = \dot{\theta}$.

Further a prime on a symbol $\alpha = \dot{\theta}$ means that the quantity is defined in a moving system of coordinates. In a moving system: $\vec{v} = \vec{v}_Q + \vec{v}' + \vec{\omega} \times \vec{r}'$ and $\vec{a} = \vec{a}_Q + \vec{a}' + \dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}')$ with $\vec{\omega} \times (\vec{\omega} \times \vec{r}') = -\omega^2 \vec{r}'_n$.

Point-dynamics in a fixed coordinate system

Force, (angular) momentum and energy

Newton's 2nd law connects the force on an object and the resulting acceleration of the object where the *momentum* is given by $\vec{p} = m\vec{v}$:

$$\vec{F}(\vec{r}, \vec{v}, t) = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m \frac{d\vec{v}}{dt} + \vec{v} \frac{dm}{dt} \stackrel{m=\text{const}}{=} m\vec{a} \quad (9.1.1.5)$$

Newton's 3rd law is given by: $\vec{F}_{\text{action}} = -\vec{F}_{\text{reaction}}$.

For the power P : $P = \dot{W} = \vec{F} \cdot \vec{v}$. For the total energy W , the kinetic energy T and the potential energy U : $W = T + U$; $\dot{T} = -\dot{U}$ with $T = \frac{1}{2} mv^2$.

The kick \vec{S} is given by: $\vec{S} = \Delta\vec{p} = \int \vec{F} dt$

The work A , delivered by a force, is $A = \int_1^2 \vec{F} \cdot d\vec{s} = \int_1^2 F \cos(\alpha) ds$

The torque $\vec{\tau}$ is related to the angular momentum \vec{L} : $\vec{\tau} = \dot{\vec{L}} = \vec{r} \times \vec{F}$; and $\vec{L} = \vec{r} \times \vec{p} = m\vec{v} \times \vec{r}$, $|\vec{L}| = mr^2\omega$. The following equation is valid:

$$\tau = -\frac{\partial U}{\partial \theta} \quad (9.1.1.6)$$

Hence, the conditions for a mechanical equilibrium are: $\sum \vec{F}_i = 0$ and $\sum \vec{\tau}_i = 0$.

The *force of friction* is usually proportional to the force perpendicular to the surface, except when the motion starts, when a threshold has to be overcome: $F_{\text{fric}} = f \cdot F_{\text{norm}} \cdot \vec{e}_t$.

Conservative force fields

A conservative force can be written as the gradient of a potential: $\vec{F}_{\text{cons}} = -\vec{\nabla}U$. From this follows that $\nabla \times \vec{F} = \vec{0}$. For such a force field also:

$$\oint \vec{F} \cdot d\vec{s} = 0 \Rightarrow U = U_0 - \int_{r_0}^{r_1} \vec{F} \cdot d\vec{s} \quad (9.1.1.7)$$

So the work delivered by a conservative force field depends not on the trajectory covered but only on the starting and ending points of the motion.

Gravitation

The *Newtonian law of gravitation* is (in GRT one also uses κ instead of G):

$$\vec{F}_g = -G \frac{m_1 m_2}{r^2} \vec{e}_r \quad (9.1.1.8)$$

The gravitational potential is then given by $V = -Gm/r$. From *Gauss' law* it then follows: $\nabla^2 V = 4\pi G\rho$.

Orbital equations

If $V = V(r)$ one can derive from the equations of *Lagrange* for ϕ the conservation of angular momentum:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial V}{\partial \phi} = 0 \Rightarrow \frac{d}{dt}(mr^2\dot{\phi}) = 0 \Rightarrow L_z = mr^2\dot{\phi} = \text{constant} \quad (9.1.1.9)$$

For the radial position as a function of time it can be found that:

$$\left(\frac{dr}{dt}\right)^2 = \frac{2(W-V)}{m} - \frac{L^2}{m^2 r^2} \quad (9.1.1.10)$$

The angular equation is then:

$$\phi - \phi_0 = \int_0^r \left[\frac{mr^2}{L} \sqrt{\frac{2(W-V)}{m} - \frac{L^2}{m^2 r^2}} \right]^{-1} dr \stackrel{r=r_{\text{field}}}{=} \arccos\left(1 + \frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{r_0} + km/L_z^2}\right) \quad (9.1.1.11)$$

If $F = F(r)$: $L = \text{constant}$, if F is conservative: $W = \text{constant}$, if $\vec{F} \perp \vec{v}$ then $\Delta T = 0$ and $U = 0$.

Kepler's orbital equations

In a force field $F = kr^{-2}$, the orbits are conic sections with the origin of the force in one of the foci (*Kepler's 1st law*). The equation of the orbit is:

$$r(\theta) = \frac{\ell}{1 + \varepsilon \cos(\theta - \theta_0)}, \text{ or } x^2 + y^2 = (\ell - \varepsilon x)^2 \quad (9.1.1.12)$$

with

$$\ell = \frac{L^2}{G\mu^2 M_{\text{tot}}}; \quad \varepsilon^2 = 1 + \frac{2WL^2}{G^2\mu^3 M_{\text{tot}}^2} = 1 - \frac{\ell}{a}; \quad a = \frac{\ell}{1 - \varepsilon^2} = \frac{k}{2W} \quad (9.1.1.13)$$

a is half the length of the long axis of the elliptical orbit in case the orbit is closed. Half the length of the short axis is $b = \sqrt{a\ell}$. ε is the *excentricity* of the orbit. Orbits with an equal ε are of equal shape. Now, five types of orbits are possible:

1. $k < 0$ and $\varepsilon = 0$: a circle.
2. $k < 0$ and $0 < \varepsilon < 1$: an ellipse.
3. $k < 0$ and $\varepsilon = 1$: a parabola.
4. $k < 0$ and $\varepsilon > 1$: a hyperbola, curved towards the centre of force.
5. $k > 0$ and $\varepsilon > 1$: a hyperbola, curved away from the centre of force.

Other combinations are not possible: the total energy in a repulsive force field is always positive so $\varepsilon > 1$.

If the surface between the orbit covered between t_1 and t_2 and the focus C around which the planet moves is $A(t_1, t_2)$, Kepler's 2nd law is

$$A(t_1, t_2) = \frac{L_C}{2m} (t_2 - t_1) \quad (9.1.1.14)$$

Kepler's 3rd law is, with T the period and M_{tot} the total mass of the system is:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_{\text{tot}}} \quad (9.1.1.15)$$

The virial theorem

The virial theorem for one particle is:

$$\langle m\vec{v} \cdot \vec{r} \rangle = 0 \Rightarrow \langle T \rangle = -\frac{1}{2} \langle \vec{F} \cdot \vec{r} \rangle = \frac{1}{2} \left\langle r \frac{dU}{dr} \right\rangle = \frac{1}{2} n \langle U \rangle \text{ if } U = -\frac{k}{r^n} \quad (9.1.1.16)$$

The virial theorem for a collection of particles is:

$$\langle T \rangle = -\frac{1}{2} \left\langle \sum_{\text{particles}} \vec{F}_i \cdot \vec{r}_i + \sum_{\text{pairs}} \vec{F}_{ij} \cdot \vec{r}_{ij} \right\rangle \quad (9.1.1.17)$$

These propositions can also be written as: $2E_{\text{kin}} + E_{\text{pot}} = 0$.

Point dynamics in a moving coordinate system

Fictitious forces

The total force in a moving coordinate system can be found by subtracting the fictitious forces from the forces working in the reference frame: $\vec{F}' = \vec{F} - \vec{F}_{\text{app}}$. The different fictitious forces are:

1. Transformation of the origin: $F_{\text{or}} = -m\vec{a}_a$
2. Rotation: $\vec{F}_{\alpha} = -m\vec{\alpha} \times \vec{r}'$
3. Coriolis force: $F_{\text{cor}} = -2m\vec{\omega} \times \vec{v}$
4. Centrifugal force: $\vec{F}_{\text{cf}} = m\omega^2 \vec{r}_n' = -\vec{F}_{\text{cp}}$; $\vec{F}_{\text{cp}} = -\frac{mv^2}{r} \vec{e}_r$

Tensor notation

Transformation of the Newtonian equations of motion to $x^\alpha = x^\alpha(x)$ gives:

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d\bar{x}^\beta}{dt}; \quad (9.1.1.18)$$

The chain rule gives:

$$\frac{d}{dt} \frac{dx^\alpha}{dt} = \frac{d^2 x^\alpha}{dt^2} = \frac{d}{dt} \left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d\bar{x}^\beta}{dt} \right) = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d^2 \bar{x}^\beta}{dt^2} + \frac{d\bar{x}^\beta}{dt} \frac{d}{dt} \left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta} \right) \quad (9.1.1.19)$$

so:

$$\frac{d}{dt} \frac{\partial x^\alpha}{\partial \bar{x}^\beta} = \frac{\partial}{\partial \bar{x}^\gamma} \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d\bar{x}^\gamma}{dt} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{d\bar{x}^\gamma}{dt} \quad (9.1.1.20)$$

This leads to:

$$\frac{d^2 x^\alpha}{dt^2} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{d^2 \bar{x}^\beta}{dt^2} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{d\bar{x}^\gamma}{dt} \left(\frac{d\bar{x}^\beta}{dt} \right) \quad (9.1.1.21)$$

Hence the Newtonian equation of motion

$$m \frac{d^2 x^\alpha}{dt^2} = F^\alpha \quad (9.1.1.22)$$

will be transformed into:

$$m \left\{ \frac{d^2 x^\alpha}{dt^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} \right\} = F^\alpha \quad (9.1.1.23)$$

The apparent forces are projected from the origin to the side affected by $\Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt}$.

Dynamics of masspoint collections

The centre of mass

The velocity w.r.t. the centre of mass \vec{R} is given by $\vec{v} - \dot{\vec{R}}$. The coordinates of the centre of mass are given by:

$$\vec{r}_m = \frac{\sum m_i \vec{r}_i}{\sum m_i} \quad (9.1.1.24)$$

In a 2-particle system, the coordinates of the centre of mass are given by:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad (9.1.1.25)$$

With $\vec{r} = \vec{r}_1 - \vec{r}_2$, the kinetic energy becomes: $\frac{1}{2} M_{\text{tot}} \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2$, with the *reduced mass* μ given by:

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad (9.1.1.26)$$

The motion of the centre of mass and relative to it can be separated:

$$\dot{\vec{L}}_{\text{outside}} = \vec{\tau}_{\text{outside}}; \quad \dot{\vec{L}}_{\text{inside}} = \vec{\tau}_{\text{inside}} \quad (9.1.1.27)$$

$$\vec{p} = m\vec{v}_m; \quad \vec{F}_{\text{ext}} = m\vec{a}_m; \quad \vec{F}_{12} = \mu\vec{u} \quad (9.1.1.28)$$

Collisions

With collisions, where B are the coordinates of the collision and C an arbitrary other position: $\vec{p} = m\vec{v}_m$ is constant, and $T = \frac{1}{2} m \vec{v}_m^2$ is constant. The changes in the *relative velocities* can be derived from: $\vec{S} = \Delta\vec{p} = \mu(\vec{v}_{\text{aft}} - \vec{v}_{\text{before}})$. Further $\Delta\vec{L}_C = \vec{CB} \times \vec{S}$, $\vec{p} \parallel \vec{S} = \text{constant}$ and \vec{L} w.r.t. B is constant.

Dynamics of rigid bodies

Moment of Inertia

The angular momentum in a moving coordinate system is given by:

$$\vec{L}' = I\vec{\omega} + \vec{L}'_n \quad (9.1.1.29)$$

where I is the *moment of inertia* with respect to a central axis, which is given by:

$$I = \sum_i m_i \vec{r}_i^2 ; \quad T' = W_{\text{rot}} = \frac{1}{2} \omega I_{ij} \vec{e}_i \vec{e}_j = \frac{1}{2} I \omega^2 \quad (9.1.1.30)$$

or, in the continuous case:

$$I = \frac{m}{V} \int r_n'^2 dV = \int r_n'^2 dm \quad (9.1.1.31)$$

Further:

$$L_i = I^{ij} \omega_j ; \quad I_{ii} = I_i ; \quad I_{ij} = I_{ji} = - \sum_k m_k x_i' x_j' \quad (9.1.1.32)$$

Steiner's theorem is: $I_{\text{w.r.t.D}} = I_{\text{w.r.t.C}} + m(DM)^2$ if axis C \parallel axis D.

Object	I	Object	I
Hollow cylinder	$I = mR^2$	Massive cylinder	$I = \frac{1}{2} mR^2$
Disc, axis in plane disc through m	$I = \frac{1}{4} mR^2$	Dumbbell	$I = \frac{1}{2} \mu R^2$
Hollow sphere	$I = \frac{2}{3} mR^2$	Massive sphere	$I = \frac{2}{5} mR^2$
Bar, axis \perp through c.o.m.	$I = \frac{1}{2} ml^2$	Bar, axis \perp through end	$I = \frac{1}{3} ml^2$
Rectangle, axis \perp plane thr. c.o.m.	$I = \frac{1}{2} m(a^2 + b^2)$	Rectangle, axis $\parallel b$ thr. m	$I = ma^2$

Principal axes

Each rigid body has (at least) 3 principal axes which stand \perp to each other. For a principal axis:

$$\frac{\partial I}{\partial \omega_x} = \frac{\partial I}{\partial \omega_y} = \frac{\partial I}{\partial \omega_z} = 0 \quad \text{so} \quad L'_n = 0 \quad (9.1.1.33)$$

The following holds: $\dot{\omega}_k = -a_{ijk} \omega_i \omega_j$ with $a_{ijk} = \frac{I_i - I_j}{I_k}$ if $I_1 \leq I_2 \leq I_3$.

Time dependence

For the torque $\vec{\tau}$:

$$\vec{\tau}' = I \ddot{\theta} ; \quad \frac{d'' \vec{L}'}{dt} = \vec{\tau}' - \vec{\omega} \times \vec{L}' \quad (9.1.1.34)$$

The torque \vec{T} is defined by: $\vec{T} = \vec{F} \times \vec{d}$.

Variational Calculus, Hamilton and Lagrange mechanics

Variational Calculus

Starting with:

$$\delta \int_a^b \mathcal{L}(q, \dot{q}, t) dt = 0 \quad \text{where} \quad \delta(a) = \delta(b) = 0 \quad \text{and} \quad \delta \left(\frac{du}{dx} \right) = \frac{d}{dx} (\delta u) \quad (9.1.1.35)$$

the equations of Lagrange can be derived:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \quad (9.1.1.36)$$

When there are additional conditions applying to the variational problem $\delta J(u) = 0$ of the type $K(u) = \text{constant}$, the new problem becomes: $\delta J(u) - \lambda \delta K(u) = 0$.

Hamilton mechanics

The *Lagrangian* is given by: $\mathcal{L} = \sum T(\dot{q}_i) - V(q_i)$. The *Hamiltonian* is given by: $H = \sum \dot{q}_i p_i - \mathcal{L}$. In two dimensions: $\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$.

If the coordinates used are *canonical* the *Hamilton equations* are the equations of motion for the system:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (9.1.1.37)$$

Coordinates are canonical if the following holds: $\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$, $\{q_i, p_j\} = \delta_{ij}$ where $\{, \}$ is the *Poisson bracket*:

$$\{A, B\} = \sum_i \left[\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right] \quad (9.1.1.38)$$

The *Hamiltonian* of an harmonic oscillator is given by $H(x, p) = p^2/2m + \frac{1}{2}m\omega^2 x^2$. With new coordinates (θ, I) , obtained by the canonical transform $x = \sqrt{2I/m\omega} \cos(\theta)$ and $p = -\sqrt{2Im\omega} \sin(\theta)$, with inverse $\theta = \arctan(-p/m\omega x)$ and $I = p^2/2m\omega + \frac{1}{2}m\omega x^2$ it follows: $H(\theta, I) = \omega I$.

The *Hamiltonian* of a charged particle with charge q in an external electromagnetic field is given by:

$$H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 + qV \quad (9.1.1.39)$$

This *Hamiltonian* can be derived from the *Hamiltonian* of a free particle $H = p^2/2m$ with the transform $\vec{p} \rightarrow \vec{p} - q\vec{A}$ and $H \rightarrow H - qV$. This is elegant from a relativistic point of view: it is equivalent to the transformation of the momentum 4-vector $p^\alpha \rightarrow p^\alpha - qA^\alpha$. A gauge transform on the potentials A^α corresponds with a canonical transform, which make the *Hamilton equations* the equations of motion for the system.

Motion near equilibrium, linearization

For natural systems near equilibrium the following equations are valid:

$$\left(\frac{\partial V}{\partial q_i} \right)_0 = 0; \quad V(q) = V(0) + V_{ik} q_i q_k \text{ with } V_{ik} = \left(\frac{\partial^2 V}{\partial q_i \partial q_k} \right)_0 \quad (9.1.1.40)$$

With $T = \frac{1}{2}(M_{ik}\dot{q}_i\dot{q}_k)$ one obtains the set of equations $M\ddot{q} + Vq = 0$. If $q_i(t) = a_i \exp(i\omega t)$ is substituted, this set of equations has solutions if $\det(V - \omega^2 M) = 0$. This leads to the eigenfrequencies of the problem: $\omega_k^2 = \frac{a_k^T V a_k}{a_k^T M a_k}$. If the equilibrium is stable: $\forall k$ that $\omega_k^2 > 0$. The general solution is a superposition of eigenvibrations.

Phase space, Liouville's equation

In phase space:

$$\nabla = \left(\sum_i \frac{\partial}{\partial q_i}, \sum_i \frac{\partial}{\partial p_i} \right) \text{ so } \nabla \cdot \vec{v} = \sum_i \left(\frac{\partial}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \quad (9.1.1.41)$$

If the equation of continuity, $\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0$ holds, this can be written as:

$$\{\rho, H\} + \frac{\partial \rho}{\partial t} = 0 \quad (9.1.1.42)$$

For an arbitrary quantity A :

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t} \quad (9.1.1.43)$$

Liouville's theorem can then be written as:

$$\frac{d\rho}{dt} = 0; \quad \text{or } \int \rho dq = \text{constant} \quad (9.1.1.44)$$

Generating functions

Starting with the coordinate transformation:

$$\begin{cases} Q_i = Q_i(q_i, p_i, t) \\ P_i = P_i(q_i, p_i, t) \end{cases} \quad (9.1.1.45)$$

one can derive the following *Hamilton equations* with the new *Hamiltonian K*:

$$\frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i}; \quad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i} \quad (9.1.1.46)$$

Now, a distinction between 4 cases can be made:

1. If $p_i \dot{q}_i - H = P_i \dot{Q}_i - K(P_i, Q_i, t) - \frac{dF_1(q_i, Q_i, t)}{dt}$, the coordinates follow from:

$$p_i = \frac{\partial F_1}{\partial q_i}; \quad P_i = -\frac{\partial F_1}{\partial Q_i}; \quad K = H + \frac{\partial F_1}{\partial t} \quad (9.1.1.47)$$

2. If $p_i \dot{q}_i - H = -\dot{P}_i Q_i - K(P_i, Q_i, t) + \frac{dF_2(q_i, P_i, t)}{dt}$, the coordinates follow from:

$$p_i = \frac{\partial F_2}{\partial q_i}; \quad Q_i = \frac{\partial F_2}{\partial P_i}; \quad K = H + \frac{\partial F_2}{\partial t} \quad (9.1.1.48)$$

3. If $-\dot{p}_i q_i - H = P_i \dot{Q}_i - K(P_i, Q_i, t) + \frac{dF_3(p_i, Q_i, t)}{dt}$, the coordinates follow from:

$$q_i = -\frac{\partial F_3}{\partial p_i}; \quad P_i = -\frac{\partial F_3}{\partial Q_i}; \quad K = H + \frac{\partial F_3}{\partial t} \quad (9.1.1.49)$$

4. If $-\dot{p}_i q_i - H = -\dot{P}_i Q_i - K(P_i, Q_i, t) + \frac{dF_4(p_i, P_i, t)}{dt}$, the coordinates follow from:

$$q_i = -\frac{\partial F_4}{\partial p_i}; \quad Q_i = \frac{\partial F_4}{\partial P_i}; \quad K = H + \frac{\partial F_4}{\partial t} \quad (9.1.1.50)$$

The functions F_1 , F_2 , F_3 and F_4 are called *generating functions*.

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