

9.1.3: Waves

The wave equation

The general form of the wave equation is: $\square u = 0$, or:

$$\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad (9.1.3.1)$$

where u is the disturbance and v the *propagation velocity*. In general $v = f\lambda$ holds. By definition $k\lambda = 2\pi$ and $\omega = 2\pi f$.

In principle, there are two types of waves:

1. Longitudinal waves: for these $\vec{k} \parallel \vec{v} \parallel \vec{u}$ holds.
2. Transversal waves: for these $\vec{k} \parallel \vec{v} \perp \vec{u}$ holds.

The *phase velocity* is given by $v_{\text{ph}} = \omega/k$. The *group velocity* is given by:

$$v_g = \frac{d\omega}{dk} = v_{\text{ph}} + k \frac{dv_{\text{ph}}}{dk} = v_{\text{ph}} \left(1 - \frac{k}{n} \frac{dn}{dk} \right) \quad (9.1.3.2)$$

where n is the refractive index of the medium. If v_{ph} does not depend on ω then: $v_{\text{ph}} = v_g$. In a dispersive medium it is possible that $v_g > v_{\text{ph}}$ or $v_g < v_{\text{ph}}$, and $v_g \cdot v_f = c^2$. If one wants to transfer information with a wave, e.g. by modulation of an EM wave, the information travels with the velocity at which a change in the electromagnetic field propagates. This velocity is often almost equal to the group velocity.

For some media, the propagation velocity follows from:

- Pressure waves in a liquid or gas: $v = \sqrt{\kappa/\rho}$, where κ is the modulus of compression.
- Further for pressure waves in a gas: $v = \sqrt{\gamma p/\rho} = \sqrt{\gamma RT/M}$.
- Pressure waves in a thin solid bar with diameter $\ll \lambda$: $v = \sqrt{E/\rho}$
- Waves in a string: $v = \sqrt{F_{\text{span}} l/m}$
- Surface waves on a liquid: $v = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\rho\lambda} \right) \tanh\left(\frac{2\pi h}{\lambda} \right)}$

where h is the depth of the liquid and γ the surface tension. If $h \ll \lambda$ then $v \approx \sqrt{gh}$ holds.

Solutions of the wave equation

Plane waves

In n dimensions a harmonic plane wave is defined by:

$$u(\vec{x}, t) = 2^n \hat{u} \cos(\omega t) \sum_{i=1}^n \sin(k_i x_i) \quad (9.1.3.3)$$

The equation for a harmonic traveling plane wave is: $u(\vec{x}, t) = \hat{u} \cos(\vec{k} \cdot \vec{x} \pm \omega t + \varphi)$

If waves reflect at the end of a spring this will result in a change in phase. A fixed end imposes a phase change of $\pi/2$ to the reflected wave, with boundary condition $u(l) = 0$. A loose end yields no change in the phase of the reflected wave, with boundary condition $(\partial u / \partial x)_l = 0$.

If an observer is moving w.r.t. the wave with a velocity v_{obs} , they will observe a change in frequency: the *Doppler effect*. This is given by: $\frac{f}{f_0} = \frac{v_f - v_{\text{obs}}}{v_f}$.

Spherical waves

When the situation is spherically symmetric, the homogeneous wave equation is given by:

$$\frac{1}{v^2} \frac{\partial^2 (ru)}{\partial t^2} - \frac{\partial^2 (ru)}{\partial r^2} = 0 \quad (9.1.3.4)$$

with a general solution:

$$u(r, t) = C_1 \frac{f(r - vt)}{r} + C_2 \frac{g(r + vt)}{r} \quad (9.1.3.5)$$

Cylindrical waves

When the situation has a cylindrical symmetry, the homogeneous wave equation becomes:

$$\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = 0 \quad (9.1.3.6)$$

This is a Bessel equation, with solutions that can be written as Hankel functions. For sufficient large values of r these are approximated by:

$$u(r, t) = \frac{\hat{u}}{\sqrt{r}} \cos(k(r \pm vt)) \quad (9.1.3.7)$$

The general solution in one dimension

Starting from the equation:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \sum_{m=0}^N \left(b_m \frac{\partial^m}{\partial x^m} \right) u(x, t) \quad (9.1.3.8)$$

where $b_m \in \mathbb{R}$. Substituting $u(x, t) = Ae^{i(kx - \omega t)}$ gives two solutions $\omega_j = \omega_j(k)$ as dispersion relations. The general solution is given by:

$$u(x, t) = \int_{-\infty}^{\infty} \left(a(k) e^{i(kx - \omega_1(k)t)} + b(k) e^{i(kx - \omega_2(k)t)} \right) dk \quad (9.1.3.9)$$

Because in general the frequencies ω_j are non-linear in k there is dispersion and the solution cannot be written any more as a sum of functions depending only on $x \pm vt$: the wave front transforms.

The stationary phase method

Usually the *Fourier integrals* of the previous section cannot be calculated exactly. If $\omega_j(k) \in \mathbb{R}$ the stationary phase method can be applied. Assuming that $a(k)$ is only a slowly varying function of k , one can state that the parts of the k -axis where the phase of $kx - \omega(k)t$ changes rapidly will give no net contribution to the integral because the exponent oscillates rapidly there. The only areas contributing significantly to the integral are areas with a stationary phase, determined by $\frac{d}{dk}(kx - \omega(k)t) = 0$. Now the following approximation is possible:

$$\int_{-\infty}^{\infty} a(k) e^{i(kx - \omega(k)t)} dk \approx \sum_{i=1}^N \sqrt{\frac{2\pi}{\frac{d^2 \omega(k_i)}{dk_i^2}}} \exp \left[-i \frac{1}{4} \pi + i(k_i x - \omega(k_i)t) \right] \quad (9.1.3.10)$$

Green functions for the initial-value problem

This method is preferable if the solutions deviate strongly from the stationary solutions, like point-like excitations. Starting with the wave equation in one dimension, with $\nabla^2 = \partial^2 / \partial x^2$ if $Q(x, x', t)$ is the solution with initial values $Q(x, x', 0) = \delta(x - x')$ and $\frac{\partial Q(x, x', 0)}{\partial t} = 0$, and $P(x, x', t)$ the solution with initial values $P(x, x', 0) = 0$ and $\frac{\partial P(x, x', 0)}{\partial t} = \delta(x - x')$, then the solution of the wave equation with arbitrary initial conditions $f(x) = u(x, 0)$ and $g(x) = \frac{\partial u(x, 0)}{\partial t}$ is given by:

$$u(x, t) = \int_{-\infty}^{\infty} f(x') Q(x, x', t) dx' + \int_{-\infty}^{\infty} g(x') P(x, x', t) dx' \quad (9.1.3.11)$$

P and Q are called the *propagators*. They are defined by:

$$Q(x, x', t) = \frac{1}{2} [\delta(x - x' - vt) + \delta(x - x' + vt)]$$

$$P(x, x', t) = \begin{cases} \frac{1}{2v} & \text{if } |x - x'| < vt \\ 0 & \text{if } |x - x'| > vt \end{cases}$$

Further the relation: $Q(x, x', t) = \frac{\partial P(x, x', t)}{\partial t}$ holds.

Waveguides and resonating cavities

The boundary conditions for a perfect conductor can be derived from *Maxwell's equations*. If \vec{n} is a unit vector \perp the surface, pointing from 1 to 2, and \vec{K} is a surface current density, then:

$$\begin{aligned} \vec{n} \cdot (\vec{D}_2 - \vec{D}_1) &= \sigma & \vec{n} \times (\vec{E}_2 - \vec{E}_1) &= 0 \\ \vec{n} \cdot (\vec{B}_2 - \vec{B}_1) &= 0 & \vec{n} \times (\vec{H}_2 - \vec{H}_1) &= \vec{K} \end{aligned} \quad (9.1.3.12)$$

In a waveguide because of the cylindrical symmetry: $\vec{E}(\vec{x}, t) = \vec{\mathcal{E}}(x, y)e^{i(kz - \omega t)}$ and $\vec{B}(\vec{x}, t) = \vec{\mathcal{B}}(x, y)e^{i(kz - \omega t)}$ holds. From this one can now deduce that, if \mathcal{B}_z and \mathcal{E}_z are not $\equiv 0$:

$$\begin{aligned} \mathcal{B}_x &= \frac{i}{\varepsilon\mu\omega^2 - k^2} \left(k \frac{\partial \mathcal{B}_z}{\partial x} - \varepsilon\mu\omega \frac{\partial \mathcal{E}_z}{\partial y} \right) & \mathcal{B}_y &= \frac{i}{\varepsilon\mu\omega^2 - k^2} \left(k \frac{\partial \mathcal{B}_z}{\partial y} + \varepsilon\mu\omega \frac{\partial \mathcal{E}_z}{\partial x} \right) \\ \mathcal{E}_x &= \frac{i}{\varepsilon\mu\omega^2 - k^2} \left(k \frac{\partial \mathcal{E}_z}{\partial x} + \varepsilon\mu\omega \frac{\partial \mathcal{B}_z}{\partial y} \right) & \mathcal{E}_y &= \frac{i}{\varepsilon\mu\omega^2 - k^2} \left(k \frac{\partial \mathcal{E}_z}{\partial y} - \varepsilon\mu\omega \frac{\partial \mathcal{B}_z}{\partial x} \right) \end{aligned} \quad (9.1.3.13)$$

Now one can distinguish between three cases:

1. $B_z \equiv 0$: the Transverse Magnetic Modes (TM). Boundary condition: $\mathcal{E}_z|_{\text{surf}} = 0$.
2. $E_z \equiv 0$: the Transverse Electric Modes (TE). Boundary condition: $\frac{\partial \mathcal{B}_z}{\partial n}|_{\text{surf}} = 0$.

For the TE and TM modes this results in an eigenvalue problem for \mathcal{E}_z resp. \mathcal{B}_z with boundary conditions:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = -\gamma^2 \psi \text{ with eigenvalues } \gamma^2 := \varepsilon\mu\omega^2 - k^2 \quad (9.1.3.14)$$

This has a discrete solution ψ_ℓ with eigenvalue $\gamma_\ell^2: k = \sqrt{\varepsilon\mu\omega^2 - \gamma_\ell^2}$. For $\omega < \omega_\ell$, k is imaginary and the wave is damped. Therefore, ω_ℓ is called the *cut-off frequency*. In rectangular conductors the following expression can be found for the cut-off frequency for modes $\text{TE}_{m,n}$ or $\text{TM}_{m,n}$:

$$\lambda_\ell = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}} \quad (9.1.3.15)$$

3. E_z and B_z are zero everywhere for the Transversal Electro-Magnetic Modes (TEM). Then: $k = \pm\omega\sqrt{\varepsilon\mu}$ and $v_f = v_g$, just as if there were no waveguide. Further $k \in \mathbb{R}$, so no cut-off frequency exists.

In a rectangular, three dimensional resonating cavity with edges a , b and c the possible wave numbers are given by: $k_x = \frac{n_1\pi}{a}$, $k_y = \frac{n_2\pi}{b}$, $k_z = \frac{n_3\pi}{c}$ This results in the possible frequencies $f = vk/2\pi$ in the cavity:

$$f = \frac{v}{2} \sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}} \quad (9.1.3.16)$$

For a cubic cavity, with $a = b = c$, the possible number of oscillating modes N_L for longitudinal waves is given by:

$$N_L = \frac{4\pi a^3 f^3}{3v^3} \quad (9.1.3.17)$$

Because transverse waves have two possible polarizations $N_T = 2N_L$ holds for them.

Non-linear wave equations

The *Van der Pol* equation is given by:

$$\frac{d^2x}{dt^2} - \varepsilon\omega_0(1 - \beta x^2)\frac{dx}{dt} + \omega_0^2x = 0 \quad (9.1.3.18)$$

βx^2 can be ignored for very small values of the amplitude. Substitution of $x \sim e^{i\omega t}$ gives: $\omega = \frac{1}{2}\omega_0(i\varepsilon \pm 2\sqrt{1 - \frac{1}{2}\varepsilon^2})$. The lowest-order instabilities grow as $\frac{1}{2}\varepsilon\omega_0$. While x is growing, the 2nd term becomes larger and which limits the growth. Oscillations on a time scale $\sim \omega_0^{-1}$ can exist. If x is expanded as $x = x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \dots$ and this is substituted one obtains, additional periodic, *secular terms* $\sim \varepsilon t$. If it is assumed that there exist timescales τ_n , $0 \leq \tau \leq N$ with $\partial\tau_n/\partial t = \varepsilon^n$ and if the secular terms are put to 0 one obtains:

$$\frac{d}{dt} \left\{ \frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \frac{1}{2} \omega_0^2 x^2 \right\} = \varepsilon\omega_0(1 - \beta x^2) \left(\frac{dx}{dt} \right)^2 \quad (9.1.3.19)$$

This is an energy equation. Energy is conserved if the left-hand side is 0. If $x^2 > 1/\beta$, the right-hand side changes sign and an increase in energy changes into a decrease of energy. This mechanism limits the growth of oscillations.

The *Korteweg-De Vries* equation is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \underbrace{au \frac{\partial u}{\partial x}}_{\text{non-lin}} + \underbrace{b^2 \frac{\partial^3 u}{\partial x^3}}_{\text{dispersive}} = 0 \quad (9.1.3.20)$$

This equation is for example a model for ion-acoustic waves in a plasma. For this equation, soliton solutions of the following form exist:

$$u(x - ct) = \frac{-d}{\cosh^2(e(x - ct))} \quad (9.1.3.21)$$

with $c = 1 + \frac{1}{3}ad$ and $e^2 = ad/(12b^2)$.

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