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A Physics Formulary

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9.1.1: Mechanics

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Definitions

The position $\vec{r}$, the velocity $\vec{v}$ and the acceleration $\vec{a}$ are defined by: $\vec{r}=(x,y,z)$, $\vec{v}=(\dot{x},\dot{y},\dot{z})$, $\vec{a}=(\ddot{x},\ddot{y},\ddot{z})$. The following holds:

$$s(t)=s_0+\int|\vec{v}(t)|dt\,;\quad \vec{r}(t)=\vec{r}_0+\int\vec{v}(t)dt\,;\quad \vec{v}(t)=\vec{v}_0+\int\vec{a}(t)dt \tag{9.1.1.1}$$

When the acceleration is constant this gives: $v(t)=v_0+at$ and $s(t)=s_0+v_0t+\frac{1}{2}at^2$.

For the unit vectors in a direction $\perp$ to the orbit $\vec{e}_{\rm t}$ and parallel to it $\vec{e}_{\rm n}$:

$$\vec{e}_{\rm t}=\frac{\vec{v}}{|\vec{v}|}=\frac{d\vec{r}}{ds}\quad \dot{\vec{e}}_{\rm t}=\frac{v}{\rho}\vec{e}_{\rm n}\,;\quad \vec{e}_{\rm n}=\frac{\dot{\vec{e}}_{\rm t}}{|\dot{\vec{e}}_{\rm t}|} \tag{9.1.1.2}$$

For the *curvature* k and the *radius of curvature* ρ:

$$\vec{k}=\frac{d\vec{e}_{\rm t}}{ds}=\frac{d^2\vec{r}}{ds^2}=\left|\frac{d\varphi}{ds}\right|\;;\quad \rho=\frac{1}{|k|} \tag{9.1.1.3}$$

Polar coordinates

Polar coordinates are defined by: $x=r\cos(\theta)$, $y=r\sin(\theta)$. So, for the unit coordinate vectors: $\dot{\vec{e}}_r=\dot{\theta}\vec{e}_\theta$, $\dot{\vec{e}}_\theta=-\dot{\theta}\vec{e}_r$

The velocity and the acceleration are derived from:

$$\vec{r}=r\vec{e}_r\,,\quad \vec{v}=\dot{r}\vec{e}_r+r\dot{\theta}\vec{e}_\theta\,,\quad \vec{a}=(\ddot{r}-r\dot{\theta}^2)\vec{e}_r+(2\dot{r}\dot{\theta}+r\ddot{\theta})\vec{e}_\theta \tag{9.1.1.4}$$

.

Relative motion

For the motion of a point D w.r.t. a point Q: $\vec{r}_{\rm D}=\vec{r}_{\rm Q}+\dfrac{\vec{\omega}\times\vec{v}_{\rm Q}}{\omega^2}$ with $\overrightarrow{\rm QD}=\vec{r}_{\rm D}-\vec{r}_{\rm Q}$ and $\omega=\dot{\theta}$.

Further a prime on a symbol $\alpha=\ddot{\theta}\,'$ means that the quantity is defined in a moving system of coordinates. In a moving system: $\vec{v}=\vec{v}_Q+\vec{v}\,'+\vec{\omega}\times\vec{r}\,'$ and $\vec{a}=\vec{a}_Q+\vec{a}\,'+\vec{\alpha}\times\vec{r}\,'+2\vec{\omega}\times\vec{v}\,'+\vec{\omega}\times(\vec{\omega}\times\vec{r}\,')$ with $\vec{\omega}\times(\vec{\omega}\times\vec{r}\,')=-\omega^2\vec{r}\,'_n$

Point-dynamics in a fixed coordinate system

Force, (angular)momentum and energy

Newton's 2nd law connects the force on an object and the resulting acceleration of the object where the *momentum* is given by $\vec{p}=m\vec{v}$:

$$\vec{F}(\vec{r},\vec{v},t)=\frac{d\vec{p}}{dt}=\frac{d(m\vec{v})}{dt}=m\frac{d\vec{v}}{dt}+\vec{v}\frac{dm}{dt}\overset{m={\rm const}}{=}m\vec{a} \tag{9.1.1.5}$$

Newton's 3rd law is given by: $\vec{F}_{\rm action}=-\vec{F}_{\rm reaction}$.

For the power P: $P=\dot{W}=\vec{F}\cdot\vec{v}$. For the total energy W, the kinetic energy T and the potential energy U: $W=T+U\,;\quad \dot{T}=-\dot{U}$ with $T=\frac{1}{2}mv^2$.

The *kick* $\vec{S}$ is given by: $\vec{S}=\Delta\vec{p}=\int\vec{F}dt$

The work A, delivered by a force, is $A=\int_1^2\vec{F}\cdot d\vec{s}=\int_1^2F\cos(\alpha)ds$

The torque $\vec{\tau}$ is related to the angular momentum $\vec{L}$: $\vec{\tau}=\dot{\vec{L}}=\vec{r}\times\vec{F}$; and $\vec{L}=\vec{r}\times\vec{p}=m\vec{v}\times\vec{r}$, $|\vec{L}|=mr^2\omega$. The following equation is valid:

$$\tau=-\frac{\partial U}{\partial\theta} \tag{9.1.1.6}$$

Hence, the conditions for a mechanical equilibrium are: $\sum\vec{F}_i=0$ and $\sum\vec{\tau}_i=0$.

The *force of friction* is usually proportional to the force perpendicular to the surface, except when the motion starts, when a threshold has to be overcome: $F_{\text{fric}}=f\cdot F_{\text{norm}}\cdot\vec{e}_{\text{t}}$.

Conservative force fields

A conservative force can be written as the gradient of a potential: $\vec{F}_{\text{cons}}=-\vec{\nabla}U$. From this follows that $\nabla\times\vec{F}=\vec{0}$. For such a force field also:

$$\oint\vec{F}\cdot d\vec{s}=0\;\Rightarrow\;U=U_0-\int\limits_{r_0}^{r_1}\vec{F}\cdot d\vec{s} \tag{9.1.1.7}$$

So the work delivered by a conservative force field depends not on the trajectory covered but only on the starting and ending points of the motion.

Gravitation

The *Newtonian law of gravitation* is (in GRT one also uses κ instead of G):

$$\vec{F}_{\text{g}}=-G\frac{m_1m_2}{r^2}\vec{e}_r \tag{9.1.1.8}$$

The gravitational potential is then given by $V=-Gm/r$. From *Gauss' law* it then follows: $\nabla^2V=4\pi G\varrho$.

Orbital equations

If $V=V(r)$ one can derive from the equations of *Lagrange* for ϕ the conservation of angular momentum:

$$\frac{\partial\mathcal{L}}{\partial\phi}=\frac{\partial V}{\partial\phi}=0\Rightarrow\frac{d}{dt}(mr^2\phi)=0\Rightarrow L_z=mr^2\phi=\text{constant} \tag{9.1.1.9}$$

For the radial position as a function of time it can be found that:

$$\left(\frac{dr}{dt}\right)^2=\frac{2(W-V)}{m}-\frac{L^2}{m^2r^2} \tag{9.1.1.10}$$

The angular equation is then:

$$\phi-\phi_0=\int\limits_0^r\left[\frac{mr^2}{L}\sqrt{\frac{2(W-V)}{m}-\frac{L^2}{m^2r^2}}\right]^{-1}dr\overset{r^{-2}\text{field}}{=}\arccos\left(1+\frac{\frac{1}{r}-\frac{1}{r_0}}{\frac{1}{r_0}+km/L_z^2}\right) \tag{9.1.1.11}$$

If $F=F(r)$: $L=$ constant, if F is conservative: $W=$ constant, if $\vec{F}\perp\vec{v}$ then $\Delta T=0$ and $U=0$.

Kepler's orbital equations

In a force field $F=kr^{-2}$, the orbits are conic sections with the origin of the force in one of the foci (*Kepler's 1st law*). The equation of the orbit is:

$$r(\theta)=\frac{\ell}{1+\varepsilon\cos(\theta-\theta_0)}\ ,\ \text{or:~~}x^2+y^2=(\ell-\varepsilon x)^2 \tag{9.1.1.12}$$

with

$$\ell=\frac{L^2}{G\mu^2M_{\text{tot}}}\ ;\ \ \varepsilon^2=1+\frac{2WL^2}{G^2\mu^3M_{\text{tot}}^2}=1-\frac{\ell}{a}\ ;\ \ a=\frac{\ell}{1-\varepsilon^2}=\frac{k}{2W} \tag{9.1.1.13}$$

a is half the length of the long axis of the elliptical orbit in case the orbit is closed. Half the length of the short axis is $b = \sqrt{a\ell}$. ε is the *excentricity* of the orbit. Orbits with an equal ε are of equal shape. Now, five types of orbits are possible:

1. $k < 0$ and $\varepsilon = 0$: a circle.
2. $k < 0$ and $0 < \varepsilon < 1$: an ellipse.
3. $k < 0$ and $\varepsilon = 1$: a parabola.
4. $k < 0$ and $\varepsilon > 1$: a hyperbola, curved towards the centre of force.
5. $k > 0$ and $\varepsilon > 1$: a hyperbola, curved away from the centre of force.

Other combinations are not possible: the total energy in a repulsive force field is always positive so $\varepsilon > 1$.

If the surface between the orbit covered between t_1 and t_2 and the focus C around which the planet moves is $A(t_1, t_2)$, Kepler's 2nd law is

$$A(t_1, t_2) = \frac{L_C}{2m}(t_2 - t_1) \tag{9.1.1.14}$$

Kepler's 3rd law is, with T the period and M_{tot} the total mass of the system is:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_{\text{tot}}} \tag{9.1.1.15}$$

The virial theorem

The virial theorem for one particle is:

$$\langle m\vec{v} \cdot \vec{r} \rangle = 0 \Rightarrow \langle T \rangle = -\frac{1}{2}\left\langle \vec{F} \cdot \vec{r} \right\rangle = \frac{1}{2}\left\langle r\frac{dU}{dr} \right\rangle = \frac{1}{2}n\langle U \rangle \text{ if } U = -\frac{k}{r^n} \tag{9.1.1.16}$$

The virial theorem for a collection of particles is:

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_{\text{particles}} \vec{F}_i \cdot \vec{r}_i + \sum_{\text{pairs}} \vec{F}_{ij} \cdot \vec{r}_{ij} \right\rangle \tag{9.1.1.17}$$

These propositions can also be written as: $2E_{\text{kin}} + E_{\text{pot}} = 0$.

Point dynamics in a moving coordinate system

Fictitious forces

The total force in a moving coordinate system can be found by subtracting the fictitious forces from the forces working in the reference frame: $\vec{F}\,' = \vec{F} - \vec{F}_{\text{app}}$. The different fictictous forces are:

1. Transformation of the origin: $F_{\text{or}} = -m\vec{a}_a$
2. Rotation: $\vec{F}_\alpha = -m\vec{\alpha} \times \vec{r}\,'$
3. Coriolis force: $F_{\text{cor}} = -2m\vec{\omega} \times \vec{v}$
4. Centrifugal force: $\vec{F}_{\text{cf}} = m\omega^2\vec{r}_n\,' = -\vec{F}_{\text{cp}}$; $\vec{F}_{\text{cp}} = -\frac{mv^2}{r}\vec{e}_r$

Tensor notation

Transformation of the Newtonian equations of motion to $x^\alpha = x^\alpha(x)$ gives:

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\beta}{dt}; \tag{9.1.1.18}$$

The chain rule gives:

$$\frac{d}{dt}\frac{dx^\alpha}{dt} = \frac{d^2x^\alpha}{dt^2} = \frac{d}{dt}\left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\beta}{dt}\right) = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d^2\bar{x}^\beta}{dt^2} + \frac{d\bar{x}^\beta}{dt}\frac{d}{dt}\left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right) \tag{9.1.1.19}$$

so:

$$\frac{d}{dt}\frac{\partial x^\alpha}{\partial \bar{x}^\beta}=\frac{\partial}{\partial \bar{x}^\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\gamma}{dt}=\frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{d\bar{x}^\gamma}{dt} \tag{9.1.1.20}$$

This leads to:

$$\frac{d^2 x^\alpha}{dt^2}=\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d^2\bar{x}^\beta}{dt^2}+\frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{d\bar{x}^\gamma}{dt}\left(\frac{d\bar{x}^\beta}{dt}\right) \tag{9.1.1.21}$$

Hence the Newtonian equation of motion

$$m\frac{d^2 x^\alpha}{dt^2}=F^\alpha \tag{9.1.1.22}$$

will be transformed into:

$$m\left\{\frac{d^2 x^\alpha}{dt^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}\right\}=F^\alpha \tag{9.1.1.23}$$

The apparent forces are projected from the origin to the side affected by $\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}$.

Dynamics of masspoint collections

The centre of mass

The velocity w.r.t. the centre of mass $\vec{R}$ is given by $\vec{v}-\dot{\vec{R}}$. The coordinates of the centre of mass are given by:

$$\vec{r}_{\rm m}=\frac{\sum m_i\vec{r}_i}{\sum m_i} \tag{9.1.1.24}$$

In a 2-particle system, the coordinates of the centre of mass are given by:

$$\vec{R}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2} \tag{9.1.1.25}$$

With $\vec{r}=\vec{r}_1-\vec{r}_2$, the kinetic energy becomes: $frac12T=M_{\rm tot}\dot{R}^2+frac12\mu\dot{r}^2$, with the *reduced mass* μ given by:

$$\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2} \tag{9.1.1.26}$$

The motion of the centre of mass and relative to it can be separated:

$$\dot{\vec{L}}_{\rm outside}=\vec{\tau}_{\rm outside}\ ;\quad \dot{\vec{L}}_{\rm inside}=\vec{\tau}_{\rm inside} \tag{9.1.1.27}$$

$$\vec{p}=m\vec{v}_{\rm m}\ ;\quad \vec{F}_{\rm ext}=m\vec{a}_{\rm m}\ ;\quad \vec{F}_{12}=\mu\vec{u} \tag{9.1.1.28}$$

Collisions

With collisions, where B are the coordinates of the collision and C an arbitrary other position: $\vec{p}=m\vec{v}_{\rm m}$ is constant, and $T=\frac{1}{2}m\vec{v}_{\rm m}^{\,2}$ is constant. The changes in the *relative velocities* can be derived from: $\vec{S}=\Delta\vec{p}=\mu(\vec{v}_{\rm aft}-\vec{v}_{\rm before})$. Further $\Delta\vec{L}_{\rm C}=\overrightarrow{\rm CB}\times\vec{S}$, $\vec{p}\ \parallel\ \vec{S}=$ constant and $\vec{L}$ w.r.t. B is constant.

Dynamics of rigid bodies

Moment of Inertia

The angular momentum in a moving coordinate system is given by:

$$\vec{L}'=I\vec{\omega}+\vec{L}'_n \tag{9.1.1.29}$$

where I is the *moment of inertia* with respect to a central axis, which is given by:

$$I=\sum_i m_i \vec{r}_i^{\ 2}\ ;\quad T'=W_{\text{rot}}=\frac{1}{2}\omega I_{ij}\vec{e}_i\vec{e}_j=\frac{1}{2}I\omega^2 \tag{9.1.1.30}$$

or, in the continuous case:

$$I=\frac{m}{V}\int r'^2_n dV=\int r'^2_n dm \tag{9.1.1.31}$$

Further:

$$L_i=I^{ij}\omega_j\ ;\quad I_{ii}=I_i\ ;\quad I_{ij}=I_{ji}=-\sum_k m_k x'_i x'_j \tag{9.1.1.32}$$

Steiner's theorem is: $I_{\text{w.r.t.D}}=I_{\text{w.r.t.C}}+m(DM)^2$ if axis C || axis D.

Object	I	Object	I
Hollow cylinder	$I=mR^2$	Massive cylinder	$I=\frac{1}{2}mR^2$
Disc, axis in plane disc through m	$I=\frac{1}{4}mR^2$	Dumbbell	$I=\frac{1}{2}\mu R^2$
Hollow sphere	$I=\frac{2}{3}mR^2$	Massive sphere	$I=\frac{2}{5}mR^2$
Bar, axis $\perp$ through c.o.m.	$I=\frac{1}{2}ml^2$	Bar, axis $\perp$ through end	$I=\frac{1}{3}ml^2$
Rectangle, axis $\perp$ plane thr. c.o.m.	$I=\frac{1}{2}m(a^2+b^2)$	Rectangle, axis $\|b$ thr. m	$I=ma^2$

Principal axes

Each rigid body has (at least) 3 principal axes which stand $\perp$ to each other. For a principal axis:

$$\frac{\partial I}{\partial\omega_x}=\frac{\partial I}{\partial\omega_y}=\frac{\partial I}{\partial\omega_z}=0\ \text{ so }\ L'_n=0 \tag{9.1.1.33}$$

The following holds: $\dot{\omega}_k=-a_{ijk}\omega_i\omega_j$ with $a_{ijk}=\dfrac{I_i-I_j}{I_k}$ if $I_1\le I_2\le I_3$.

Time dependence

For the torque $\vec{\tau}$:

$$\vec{\tau}\,'=I\ddot{\theta}\ ;\quad \frac{d''\vec{L}'}{dt}=\vec{\tau}\,'-\vec{\omega}\times\vec{L}' \tag{9.1.1.34}$$

The *torque* $\vec{T}$ is defined by: $\vec{T}=\vec{F}\times\vec{d}$.

Variational Calculus, Hamilton and Lagrange mechanics

Variational Calculus

Starting with:

$$\delta\int_a^b \mathcal{L}(q,\dot{q},t)dt=0\ \text{ where }\ \delta(a)=\delta(b)=0\text{~~and~~}\delta\left(\frac{du}{dx}\right)=\frac{d}{dx}(\delta u) \tag{9.1.1.35}$$

the equations of *Lagrange* can be derived:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}=\frac{\partial\mathcal{L}}{\partial q_i} \tag{9.1.1.36}$$

When there are additional conditions applying to the variational problem $\delta J(u)=0$ of the type $K(u)=$constant, the new problem becomes: $\delta J(u)-\lambda\delta K(u)=0$.

Hamilton mechanics

The *Lagrangian* is given by: $\mathcal{L}=\sum T(\dot{q}_i)-V(q_i)$. The *Hamiltonian* is given by: $H=\sum \dot{q}_i p_i-\mathcal{L}$. In two dimensions: $\mathcal{L}=T-U=frac12m(\dot{r}^2+r^2\dot{\phi}^2)-U(r,\phi)$.

If the coordinates used are *canonical* the *Hamilton equations* are the equations of motion for the system:

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}\ ;\quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i} \tag{9.1.1.37}$$

Coordinates are canonical if the following holds: $\{q_i,q_j\}=0,\ \{p_i,p_j\}=0,\ \{q_i,p_j\}=\delta_{ij}$ where $\{,\}$ is the *Poisson bracket*:

$$\{A,B\}=\sum_i\left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right] \tag{9.1.1.38}$$

The *Hamiltonian* of an harmonic oscillator is given by $H(x,p)=p^2/2m+\frac{1}{2}m\omega^2x^2$. With new coordinates (θ,I), obtained by the canonical transform $x=\sqrt{2I/m\omega}\cos(\theta)$ and $p=-\sqrt{2Im\omega}\sin(\theta)$, with inverse $\theta=\arctan(-p/m\omega x)$ and $I=p^2/2m\omega+\frac{1}{2}m\omega x^2$ it follows: $H(\theta,I)=\omega I$.

The *Hamiltonian* of a charged particle with charge *q* in an external electromagnetic field is given by:

$$H=\frac{1}{2m}\left(\vec{p}-q\vec{A}\right)^2+qV \tag{9.1.1.39}$$

This *Hamiltonian* can be derived from the *Hamiltonian* of a free particle $H=p^2/2m$ with the transform $\vec{p}\to\vec{p}-q\vec{A}$ and $H\to H-qV$. This is elegant from a relativistic point of view: it is equivalent to the transformation of the momentum 4-vector $p^\alpha\to p^\alpha-qA^\alpha$. A gauge transform on the potentials A^α corresponds with a canonical transform, which make the *Hamilton equations* the equations of motion for the system.

Motion near equilibrium, linearization

For natural systems near equilibrium the following equations are valid:

$$\left(\frac{\partial V}{\partial q_i}\right)_0=0\ ;\quad V(q)=V(0)+V_{ik}q_iq_k\tilde{}\tilde{}\text{with}\tilde{}\tilde{}V_{ik}=\left(\frac{\partial^2 V}{\partial q_i\partial q_k}\right)_0 \tag{9.1.1.40}$$

With $T=\frac{1}{2}(M_{ik}\dot{q}_i\dot{q}_k)$ one obtains the set of equations $M\ddot{q}+Vq=0$. If $q_i(t)=a_i\exp(i\omega t)$ is substituted, this set of equations has solutions if $\det(V-\omega^2M)=0$. This leads to the eigenfrequencies of the problem: $\omega_k^2=\dfrac{a_k^{\mathrm{T}}Va_k}{a_k^{\mathrm{T}}Ma_k}$. If the equilibrium is stable: $\forall k$ that $\omega_k^2>0$. The general solution is a superposition of eigenvibrations.

Phase space, Liouville's equation

In phase space:

$$\nabla=\left(\sum_i\frac{\partial}{\partial q_i},\sum_i\frac{\partial}{\partial p_i}\right)\tilde{}\tilde{}\text{so}\tilde{}\tilde{}\nabla\cdot\vec{v}=\sum_i\left(\frac{\partial}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{9.1.1.41}$$

If the equation of continuity, $\partial_t\varrho+\nabla\cdot(\varrho\vec{v})=0$ holds, this can be written as:

$$\{\varrho,H\}+\frac{\partial\varrho}{\partial t}=0 \tag{9.1.1.42}$$

For an arbitrary quantity A :

$$\frac{dA}{dt}=\{A,H\}+\frac{\partial A}{\partial t} \tag{9.1.1.43}$$

Liouville's theorem can then be written as:

$$\frac{d\varrho}{dt}=0\ ;\quad \text{or:}\tilde{}\int pdq=\text{constant} \tag{9.1.1.44}$$

Generating functions

Starting with the coordinate transformation:

$$\begin{cases} Q_i = Q_i(q_i, p_i, t) \\ P_i = P_i(q_i, p_i, t) \end{cases} \tag{9.1.1.45}$$

one can derive the following *Hamilton equations* with the new *Hamiltonian* K:

$$\frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i} \; ; \quad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i} \tag{9.1.1.46}$$

Now, a distinction between 4 cases can be made:

1. If $p_i\dot{q}_i - H = P_iQ_i - K(P_i, Q_i, t) - \dfrac{dF_1(q_i, Q_i, t)}{dt}$, the coordinates follow from:

$$p_i = \frac{\partial F_1}{\partial q_i} \; ; \quad P_i = -\frac{\partial F_1}{\partial Q_i} \; ; \quad K = H + \frac{\partial F_1}{\partial t} \tag{9.1.1.47}$$

2. If $p_i\dot{q}_i - H = -\dot{P}_iQ_i - K(P_i, Q_i, t) + \dfrac{dF_2(q_i, P_i, t)}{dt}$, the coordinates follow from:

$$p_i = \frac{\partial F_2}{\partial q_i} \; ; \quad Q_i = \frac{\partial F_2}{\partial P_i} \; ; \quad K = H + \frac{\partial F_2}{\partial t} \tag{9.1.1.48}$$

3. If $-\dot{p}_iq_i - H = P_i\dot{Q}_i - K(P_i, Q_i, t) + \dfrac{dF_3(p_i, Q_i, t)}{dt}$, the coordinates follow from:

$$q_i = -\frac{\partial F_3}{\partial p_i} \; ; \quad P_i = -\frac{\partial F_3}{\partial Q_i} \; ; \quad K = H + \frac{\partial F_3}{\partial t} \tag{9.1.1.49}$$

4. If $-\dot{p}_iq_i - H = -P_iQ_i - K(P_i, Q_i, t) + \dfrac{dF_4(p_i, P_i, t)}{dt}$, the coordinates follow from:

$$q_i = -\frac{\partial F_4}{\partial p_i} \; ; \quad Q_i = \frac{\partial F_4}{\partial P_i} \; ; \quad K = H + \frac{\partial F_4}{\partial t} \tag{9.1.1.50}$$

The functions F_1, F_2, F_3 and F_4 are called *generating functions*.

9.1.2: Oscillations

Harmonic oscillation

The general form of a harmonic oscillation is: $\Psi(t)=\hat{\Psi}e^{i(\omega t\pm\varphi)}\equiv\hat{\Psi}\cos(\omega t\pm\varphi)$,

where $\hat{\Psi}$ is the *amplitude*. A superposition of several harmonic oscillations *with the same frequency* results in another harmonic oscillation:

$$\sum_i \hat{\Psi}_i\cos(\alpha_i\pm\omega t)=\hat{\Phi}\cos(\beta\pm\omega t) \tag{9.1.2.1}$$

with:

$$\tan(\beta)=\frac{\sum\limits_i \hat{\Psi}_i\sin(\alpha_i)}{\sum\limits_i \hat{\Psi}_i\cos(\alpha_i)} \quad\text{and}\quad \hat{\Phi}^2=\sum_i \hat{\Psi}_i^2+2\sum_{j>i}\sum_i \hat{\Psi}_i\hat{\Psi}_j\cos(\alpha_i-\alpha_j) \tag{9.1.2.2}$$

For harmonic oscillations: $\int x(t)dt=\frac{x(t)}{i\omega}$ and $\frac{d^n x(t)}{dt^n}=(i\omega)^n x(t)$.

Mechanic oscillation

For a spring with constant C and damping k which is connected to a mass M, to which a periodic force $F(t)=\hat{F}\cos(\omega t)$ is applied the equation of motion is $m\ddot{x}=F(t)-k\dot{x}-Cx$. With complex amplitudes, this becomes $-m\omega^2 x=F-Cx-ik\omega x$. With $\omega_0^2=C/m$ it follows that:

$$x=\frac{F}{m(\omega_0^2-\omega^2)+ik\omega}\ \text{, and for the velocity: }\ \dot{x}=\frac{F}{i\sqrt{Cm}\delta+k} \tag{9.1.2.3}$$

where $\delta=\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}$. The quantity $Z=F/\dot{x}$ is called the *impedance* of the system. The *quality* of the system is given by $Q=\frac{\sqrt{Cm}}{k}$.

The frequency with minimal $|Z|$ is called the *velocity resonance frequency*. This is equal to ω_0. In the *resonance curve* $|Z|/\sqrt{Cm}$ is plotted against ω/ω_0. The width of this curve is characterized by the points where $|Z(\omega)|=|Z(\omega_0)|\sqrt{2}$. At these points: $R=X$ and $\delta=\pm Q^{-1}$, and the width is $2\Delta\omega_B=\omega_0/Q$.

The *stiffness* of an oscillating system is given by F/x. The *amplitude resonance frequency* ω_A is the frequency where $i\omega Z$ is a minimum. This is the case for $\omega_A=\omega_0\sqrt{1-\frac{1}{2}Q^2}$.

The *damping frequency* ω_D is a measure for the time in which an oscillating system comes to rest. It is given by $\omega_D=\omega_0\sqrt{1-\frac{1}{4Q^2}}$. A weak damped oscillation $(k^2<4mC)$ dies out after $T_D=2\pi/\omega_D$. For a *critically damped* oscillation $(k^2=4mC)$ $\omega_D=0$. A strong damped oscillation $(k^2>4mC)$ decays like (if $k^2\gg 4mC$) $x(t)\approx x_0\exp(-t/\tau)$.

Electric oscillations

The *impedance* is given by: $Z=R+iX$. The phase angle is $\varphi:=\arctan(X/R)$. The impedance of a resistor is R, of a capacitor $1/i\omega C$ and of a self inductor $i\omega L$. The quality of a coil is $Q=\omega L/R$. The total impedance in case several elements are connected is given by:

1. Series connection: $V=IZ$,

$$Z_{\text{tot}}=\sum_i Z_i\ ,\quad L_{\text{tot}}=\sum_i L_i\ ,\quad \frac{1}{C_{\text{tot}}}=\sum_i\frac{1}{C_i}\ ,\quad Q=\frac{Z_0}{R}\ ,\quad Z=R(1+iQ\delta) \tag{9.1.2.4}$$

2. Parallel connection: $V=IZ$,

$$\frac{1}{Z_{\text{tot}}}=\sum_i\frac{1}{Z_i}\ ,\quad \frac{1}{L_{\text{tot}}}=\sum_i\frac{1}{L_i}\ ,\quad C_{\text{tot}}=\sum_i C_i\ ,\quad Q=\frac{R}{Z_0}\ ,\quad Z=\frac{R}{1+iQ\delta} \tag{9.1.2.5}$$

Here, $Z_0=\sqrt{\frac{L}{C}}$ and $\omega_0=\frac{1}{\sqrt{LC}}$.

The power from a source is given by $P(t)=V(t)\cdot I(t)$, so $\langle P\rangle_t=\hat{V}_{\text{eff}}\hat{I}_{\text{eff}}\cos(\Delta\phi)$ $=\frac{1}{2}\hat{V}\hat{I}\,\cos(\phi_v-\phi_i)=\frac{1}{2}\hat{I}^2\text{Re}(Z)=\frac{1}{2}\hat{V}^2\text{Re}(1/Z)$, where $\cos(\Delta\phi)$ is the work factor.

Waves in long conductors

If cables are used for signal transfer, e.g. coax cables then: $Z_0=\sqrt{\frac{dL}{dx}\frac{dx}{dC}}$.

The transmission velocity is given by $v=\sqrt{\frac{dx}{dL}\frac{dx}{dC}}$.

Coupled conductors and transformers

For two coils enclosing each others flux if Φ_{12} is the part of the flux originating from I_2 through coil 2 which is enclosed by coil 1, then $\Phi_{12}=M_{12}I_2$, $\Phi_{21}=M_{21}I_1$. The coefficients of mutual induction M_{ij} is given by:

$$M_{12}=M_{21}:=M=k\sqrt{L_1L_2}=\frac{N_1\Phi_1}{I_2}=\frac{N_2\Phi_2}{I_1}\sim N_1N_2 \tag{9.1.2.6}$$

where $0\leq k\leq 1$ is the *coupling factor*. For a transformer $k\approx 1$. At full load:

$$\frac{V_1}{V_2}=\frac{I_2}{I_1}=-\frac{i\omega M}{i\omega L_2+R_{\text{load}}}\approx-\sqrt{\frac{L_1}{L_2}}=-\frac{N_1}{N_2} \tag{9.1.2.7}$$

Pendulums

The oscillation time $T=1/f$, for different types of pendulums is given by:

- Oscillating spring: $T=2\pi\sqrt{m/C}$ if the spring force is given by $F=C\cdot\Delta l$.
- Physical pendulum: $T=2\pi\sqrt{I/\tau}$ with τ the moment of force and I the moment of inertia.
- Torsion pendulum: $T=2\pi\sqrt{I/\kappa}$ where $\kappa=\frac{2lm}{\pi r^4\Delta\varphi}$ is the constant of torsion and I the moment of inertia.
- Mathematical pendulum: $T=2\pi\sqrt{l/g}$ with g the acceleration of gravity and l the length of the pendulum.

9.1.3: Waves

The wave equation

The general form of the wave equation is: $\Box u = 0$, or:

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{9.1.3.1}$$

where u is the disturbance and v the *propagation velocity*. In general $v = f\lambda$ holds. By definition $k\lambda = 2\pi$ and $\omega = 2\pi f$.

In principle, there are two types of waves:

1. Longitudinal waves: for these $\vec{k} \parallel \vec{v} \parallel \vec{u}$ holds.
2. Transversal waves: for these $\vec{k} \parallel \vec{v} \perp \vec{u}$. holds

The *phase velocity* is given by $v_{\mathrm{ph}} = \omega/k$. The *group velocity* is given by:

$$v_{\mathrm{g}} = \frac{d\omega}{dk} = v_{\mathrm{ph}} + k\frac{dv_{\mathrm{ph}}}{dk} = v_{\mathrm{ph}}\left(1 - \frac{k}{n}\frac{dn}{dk}\right) \tag{9.1.3.2}$$

where n is the refractive index of the medium. If v_{ph} does not depend on ω then: $v_{\mathrm{ph}} = v_{\mathrm{g}}$. In a dispersive medium it is possible that $v_{\mathrm{g}} > v_{\mathrm{ph}}$ or $v_{\mathrm{g}} < v_{\mathrm{ph}}$, and $v_{\mathrm{g}} \cdot v_{\mathrm{f}} = c^2$. If one wants to transfer information with a wave, e.g. by modulation of an EM wave, the information travels with the velocity at which a change in the electromagnetic field propagates. This velocity is often almost equal to the group velocity.

For some media, the propagation velocity follows from:

- Pressure waves in a liquid or gas: $v = \sqrt{\kappa/\varrho}$, where κ is the modulus of compression.
- Further for pressure waves in a gas: $v = \sqrt{\gamma p/\varrho} = \sqrt{\gamma RT/M}$.
- Pressure waves in a thin solid bar with diameter $<< \lambda$: $v = \sqrt{E/\varrho}$
- Waves in a string: $v = \sqrt{F_{\mathrm{span}} l/m}$
- Surface waves on a liquid: $v = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\varrho\lambda}\right)\tanh\left(\frac{2\pi h}{\lambda}\right)}$

  where h is the depth of the liquid and γ the surface tension. If $h \ll \lambda$ then $v \approx \sqrt{gh}$ holds.

Solutions of the wave equation

Plane waves

In n dimensions a harmonic plane wave is defined by:

$$u(\vec{x}, t) = 2^n \hat{u}\cos(\omega t)\sum_{i=1}^{n}\sin(k_i x_i) \tag{9.1.3.3}$$

The equation for a harmonic traveling plane wave is: $u(\vec{x}, t) = \hat{u}\cos(\vec{k}\cdot\vec{x} \pm \omega t + \varphi)$

If waves reflect at the end of a spring this will result in a change in phase. A fixed end imposes a phase change of $\pi/2$ to the reflected wave, with boundary condition $u(l) = 0$. A loose end yields no change in the phase of the reflected wave, with boundary condition $(\partial u/\partial x)_l = 0$.

If an observer is moving w.r.t. the wave with a velocity v_{obs}, they will observe a change in frequency: the *Doppler effect*. This is given by: $\frac{f}{f_0} = \frac{v_{\mathrm{f}} - v_{\mathrm{obs}}}{v_{\mathrm{f}}}$.

Spherical waves

When the situation is spherically symmetric, the homogeneous wave equation is given by:

$$\frac{1}{v^2}\frac{\partial^2(ru)}{\partial t^2} - \frac{\partial^2(ru)}{\partial r^2} = 0 \tag{9.1.3.4}$$

with a general solution:

$$u(r,t)=C_1\frac{f(r-vt)}{r}+C_2\frac{g(r+vt)}{r} \tag{9.1.3.5}$$

Cylindrical waves

When the situation has a cylindrical symmetry, the homogeneous wave equation becomes:

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=0 \tag{9.1.3.6}$$

This is a Bessel equation, with solutions that can be written as Hankel functions. For sufficient large values of r these are approximated by:

$$u(r,t)=\frac{\hat{u}}{\sqrt{r}}\cos(k(r\pm vt)) \tag{9.1.3.7}$$

The general solution in one dimension

Starting from the equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\sum_{m=0}^{N}\left(b_m\frac{\partial^m}{\partial x^m}\right)u(x,t) \tag{9.1.3.8}$$

where $b_m\in\mathbb{R}$. Substituting $u(x,t)=A\mathrm{e}^{i(kx-\omega t)}$ gives two solutions $\omega_j=\omega_j(k)$ as dispersion relations. The general solution is given by:

$$u(x,t)=\int_{-\infty}^{\infty}\left(a(k)\mathrm{e}^{i(kx-\omega_1(k)t)}+b(k)\mathrm{e}^{i(kx-\omega_2(k)t)}\right)dk \tag{9.1.3.9}$$

Because in general the frequencies ω_j are non-linear in k there is dispersion and the solution cannot be written any more as a sum of functions depending only on $x\pm vt$: the wave front transforms.

The stationary phase method

Usually the *Fourier integrals* of the previous section cannot be calculated exactly. If $\omega_j(k)\in\mathbb{R}$ the stationary phase method can be applied. Assuming that $a(k)$ is only a slowly varying function of k, one can state that the parts of the k-axis where the phase of $kx-\omega(k)t$ changes rapidly will give no net contribution to the integral because the exponent oscillates rapidly there. The only areas contributing significantly to the integral are areas with a stationary phase, determined by $\frac{d}{dk}(kx-\omega(k)t)=0$. Now the following approximation is possible:

$$\int_{-\infty}^{\infty}a(k)\mathrm{e}^{i(kx-\omega(k)t)}dk\approx\sum_{i=1}^{N}\sqrt{\frac{2\pi}{\frac{d^2\omega(k_i)}{dk_i^2}}}\exp\left[-i\tfrac{1}{4}\pi+i(k_ix-\omega(k_i)t)\right] \tag{9.1.3.10}$$

Green functions for the initial-value problem

This method is preferable if the solutions deviate strongly from the stationary solutions, like point-like excitations. Starting with the wave equation in one dimension, with $\nabla^2=\partial^2/\partial x^2$ if $Q(x,x',t)$ is the solution with initial values $Q(x,x',0)=\delta(x-x')$ and $\frac{\partial Q(x,x',0)}{\partial t}=0$, and $P(x,x',t)$ the solution with initial values $P(x,x',0)=0$ and $\frac{\partial P(x,x',0)}{\partial t}=\delta(x-x')$, then the solution of the wave equation with arbitrary initial conditions $f(x)=u(x,0)$ and $g(x)=\frac{\partial u(x,0)}{\partial t}$ is given by:

$$u(x,t)=\int_{-\infty}^{\infty}f(x')Q(x,x',t)dx'+\int_{-\infty}^{\infty}g(x')P(x,x',t)dx' \tag{9.1.3.11}$$

P and Q are called the *propagators*. They are defined by:

$$Q(x, x', t) = \frac{1}{2}[\delta(x - x' - vt) + \delta(x - x' + vt)]$$

$$P(x, x', t) = \begin{cases} \dfrac{1}{2v} & \text{if } |x - x'| < vt \\ 0 & \text{if } |x - x'| > vt \end{cases}$$

Further the relation: $Q(x, x', t) = \dfrac{\partial P(x, x', t)}{\partial t}$ holds.

Waveguides and resonating cavities

The boundary conditions for a perfect conductor can be derived from *Maxwell's equations*. If $\vec{n}$ is a unit vector $\perp$ the surface, pointing from 1 to 2, and $\vec{K}$ is a surface current density, then:

$$\begin{aligned} \vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma \qquad & \vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \\ \vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0 \qquad & \vec{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K} \end{aligned} \tag{9.1.3.12}$$

In a waveguide because of the cylindrical symmetry: $\vec{E}(\vec{x}, t) = \vec{\mathcal{E}}(x, y)e^{i(kz-\omega t)}$ and $\vec{B}(\vec{x}, t) = \vec{\mathcal{B}}(x, y)e^{i(kz-\omega t)}$ holds. From this one can now deduce that, if $\mathcal{B}_z$ and $\mathcal{E}_z$ are not $\equiv 0$:

$$\begin{aligned} \mathcal{B}_x = \frac{i}{\varepsilon\mu\omega^2 - k^2}\left(k\frac{\partial \mathcal{B}_z}{\partial x} - \varepsilon\mu\omega\frac{\partial \mathcal{E}_z}{\partial y}\right) \qquad & \mathcal{B}_y = \frac{i}{\varepsilon\mu\omega^2 - k^2}\left(k\frac{\partial \mathcal{B}_z}{\partial y} + \varepsilon\mu\omega\frac{\partial \mathcal{E}_z}{\partial x}\right) \\ \mathcal{E}_x = \frac{i}{\varepsilon\mu\omega^2 - k^2}\left(k\frac{\partial \mathcal{E}_z}{\partial x} + \varepsilon\mu\omega\frac{\partial \mathcal{B}_z}{\partial y}\right) \qquad & \mathcal{E}_y = \frac{i}{\varepsilon\mu\omega^2 - k^2}\left(k\frac{\partial \mathcal{E}_z}{\partial y} - \varepsilon\mu\omega\frac{\partial \mathcal{B}_z}{\partial x}\right) \end{aligned} \tag{9.1.3.13}$$

Now one can distinguish between three cases:

1. $B_z \equiv 0$: the Transverse Magnetic Modes (TM). Boundary condition: $\mathcal{E}_z|_{\text{surf}} = 0$.
2. $E_z \equiv 0$: the Transverse Electric Modes (TM). Boundary condition: $\left.\dfrac{\partial \mathcal{B}_z}{\partial n}\right|_{surf} = 0$.

   For the TE and TM modes this results in an eigenvalue problem for $\mathcal{E}_z$ resp. $\mathcal{B}_z$ with boundary conditions:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\gamma^2\psi \text{ with eigenvalues } \gamma^2 := \varepsilon\mu\omega^2 - k^2 \tag{9.1.3.14}$$

   This has a discrete solution ψ_ℓ with eigenvalue γ_ℓ^2: $k = \sqrt{\varepsilon\mu\omega^2 - \gamma_\ell^2}$. For $\omega < \omega_\ell$, k is imaginary and the wave is damped. Therefore, ω_ℓ is called the *cut-off frequency*. In rectangular conductors the following expression can be found for the cut-off frequency for modes $\text{TE}_{m,n}$ or $\text{TM}_{m,n}$:

$$\lambda_\ell = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}} \tag{9.1.3.15}$$

3. E_z and B_z are zero everywhere for the Transversal Eectro-Magnetic Modes (TEM). Then: $k = \pm\omega\sqrt{\varepsilon\mu}$ and $v_{\text{f}} = v_{\text{g}}$, just as if there were no waveguide. Further $k \in \mathbb{R}$, so no cut-off frequency exists.

In a rectangular, three dimensional resonating cavity with edges a, b and c the possible wave numbers are given by: $k_x = \dfrac{n_1\pi}{a}$, $k_y = \dfrac{n_2\pi}{b}$, $k_z = \dfrac{n_3\pi}{c}$ This results in the possible frequencies $f = vk/2\pi$ in the cavity:

$$f = \frac{v}{2}\sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}} \tag{9.1.3.16}$$

For a cubic cavity, with $a = b = c$, the possible number of oscillating modes N_{L} for longitudinal waves is given by:

$$N_{\text{L}} = \frac{4\pi a^3 f^3}{3v^3} \tag{9.1.3.17}$$

Because transverse waves have two possible polarizations $N_{\text{T}} = 2N_{\text{L}}$ holds for them.

Non-linear wave equations

The *Van der Pol* equation is given by:

$$\frac{d^2x}{dt^2} - \varepsilon\omega_0(1-\beta x^2)\frac{dx}{dt} + \omega_0^2 x = 0 \tag{9.1.3.18}$$

βx^2 can be ignored for very small values of the amplitude. Substitution of $x \sim e^{i\omega t}$ gives: $\omega = \frac{1}{2}\omega_0(i\varepsilon \pm 2\sqrt{1-\frac{1}{2}\varepsilon^2})$. The lowest-order instabilities grow as $\frac{1}{2}\varepsilon\omega_0$. While x is growing, the 2nd term becomes larger and which limits the growth. Oscillations on a time scale $\sim \omega_0^{-1}$ can exist. If x is expanded as $x = x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \cdots$ and this is substituted one obtains, additional periodic, *secular terms* $\sim \varepsilon t$. If it is assumed that there exist timescales τ_n, $0 \le \tau \le N$ with $\partial\tau_n/\partial t = \varepsilon^n$ and if the secular terms are put to 0 one obtains:

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\omega_0^2 x^2\right\} = \varepsilon\omega_0(1-\beta x^2)\left(\frac{dx}{dt}\right)^2 \tag{9.1.3.19}$$

This is an energy equation. Energy is conserved if the left-hand side is 0. If $x^2 > 1/\beta$, the right-hand side changes sign and an increase in energy changes into a decrease of energy. This mechanism limits the growth of oscillations.

The *Korteweg-De Vries* equation is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \underbrace{au\frac{\partial u}{\partial x}}_{\text{non-lin}} + \underbrace{b^2\frac{\partial^3 u}{\partial x^3}}_{\text{dispersive}} = 0 \tag{9.1.3.20}$$

This equation is for example a model for ion-acoustic waves in a plasma. For this equation, soliton solutions of the following form exist:

$$u(x-ct) = \frac{-d}{\cosh^2(e(x-ct))} \tag{9.1.3.21}$$

with $c = 1 + \frac{1}{3}ad$ and $e^2 = ad/(12b^2)$.

CHAPTER OVERVIEW

9.1.4: Physical Constants, Units, Del Operator

9.1.4.1: Physical Constants

Name	Symbol	Value	Unit
Number π	π	3.14159265358979323846	
Number e	e	2.71828182845904523536	
Euler's constant		0.577215664901532860606	
Elementary charge	e	$1.60217733 \cdot 10^{-19}$	C
Gravitational constant	G, κ	$6.67259 \cdot 10^{-11}$	$\mathrm{m^3 kg^{-1} s^{-2}}$
Fine-structure constant	$\alpha = e^2/2hc\varepsilon_0$	$\approx 1/137$	
Speed of light in vacuum	c	$2.99792458 \cdot 10^{8}$	m/s (def)
Permittivity of the vacuum	ε_0	$8.854187 \cdot 10^{-12}$	F/m
Permeability of the vacuum	μ_0	$4\pi \cdot 10^{-7}$	H/m
$(4\pi\varepsilon_0)^{-1}$		$8.9876 \cdot 10^{9}$	$\mathrm{Nm^2C^{-2}}$
Planck's constant	h	$6.6260755 \cdot 10^{-34}$	Js
Dirac's constant	$\hbar = h/2\pi$	$1.0545727 \cdot 10^{-34}$	Js
Bohr magneton	$\mu_\mathrm{B} = e\hbar/2m_\mathrm{e}$	$9.2741 \cdot 10^{-24}$	$\mathrm{Am^2}$
Bohr radius	a_0	0.52918	Å
Rydberg's constant	Ry	13.595	eV
Electron Compton wavelength	$\lambda_\mathrm{Ce} = h/m_\mathrm{e}c$	$2.2463 \cdot 10^{-12}$	m
Proton Compton wavelength	$\lambda_\mathrm{Cp} = h/m_\mathrm{p}c$	$1.3214 \cdot 10^{-15}$	m
Reduced mass of the H-atom	μ_H	$9.1045755 \cdot 10^{-31}$	kg
Stefan-Boltzmann's constant	σ	$5.67032 \cdot 10^{-8}$	$\mathrm{Wm^{-2}K^{-4}}$
Wien's constant	k_W	$2.8978 \cdot 10^{-3}$	mK
Molar gasconstant	R	8.31441	$\mathrm{J{\cdot}mol^{-1}{\cdot}K^{-1}}$
Avogadro's constant	N_A	$6.0221367 \cdot 10^{23}$	$\mathrm{mol^{-1}}$
Boltzmann's constant	$k = R/N_\mathrm{A}$	$1.380658 \cdot 10^{-23}$	J/K
Electron mass	m_e	$9.1093897 \cdot 10^{-31}$	kg
Proton mass	m_p	$1.6726231 \cdot 10^{-27}$	kg
Neutron mass	m_n	$1.674954 \cdot 10^{-27}$	kg
Elementary mass unit	$m_\mathrm{u} = \frac{1}{12}m({}^{12}_{6}\mathrm{C})$	$1.6605656 \cdot 10^{-27}$	kg
Nuclear magneton	μ_N	$5.0508 \cdot 10^{-27}$	J/T
Diameter of the Sun	$D_\odot$	$1392 \cdot 10^{6}$	m

Name	Symbol	Value	Unit
Mass of the Sun	$M_{\odot}$	$1.989 \cdot 10^{30}$	kg
Rotational period of the Sun	$T_{\odot}$	25.38	days
Radius of Earth	R_{A}	$6.378 \cdot 10^{6}$	m
Mass of Earth	M_{A}	$5.976 \cdot 10^{24}$	kg
Rotational period of Earth	T_{A}	23.96	hours
Earth orbital period	Tropical year	365.24219879	days
Astronomical unit	AU	$1.4959787066 \cdot 10^{11}$	m
Light year	lj	$9.4605 \cdot 10^{15}$	m
Parsec	pc	$3.0857 \cdot 10^{16}$	m
Hubble constant	H	$\approx (75 \pm 25)$	km·s^{-1}·Mpc^{-1}

9.1.4.2: Prefixes for Powers of 10

yotta	Y	10^{24}	giga	G	10^{9}	deci	d	10^{-1}	pico	p	10^{-12}
zetta	Z	10^{21}	mega	M	10^{6}	centi	c	10^{-2}	femto	f	10^{-15}
exa	E	10^{18}	kilo	k	10^{3}	milli	m	10^{-3}	atto	a	10^{-18}
peta	P	10^{15}	hecto	h	10^{2}	micro	μ	10^{-6}	zepto	z	10^{-21}
tera	T	10^{12}	deca	da	10	nano	n	10^{-9}	yocto	y	10^{-24}

9.1.4.3: SI Units

SI Base Units

Quantity	Unit	Sym.
Length	metre	m
Mass	kilogram	kg
Time	second	s
Therm. temp.	kelvin	K
Electr. current	ampere	A
Luminous intens.	candela	cd
Amount of subst.	mol	mol
Plane angle	radian	rad
solid angle	sterradian	sr

SI Derivative Units

Quantity	**Unit**	**Sym.**	**Derivation**
Frequency	hertz	Hz	s^{-1}
Force	newton	N	$kg \cdot m \cdot s^{-2}$
Pressure	pascal	Pa	$N \cdot m^{-2}$
Energy	joule	J	$N \cdot m$
Power	watt	W	$J \cdot s^{-1}$
Charge	coulomb	C	$A \cdot s$
El. Potential	volt	V	$W \cdot A^{-1}$
El. Capacitance	farad	F	$C \cdot V^{-1}$
El. Resistance	ohm	Ω	$V \cdot A^{-1}$
El. Conductance	siemens	S	$A \cdot V^{-1}$
Mag. flux	weber	Wb	$V \cdot s$
Mag. flux density	tesla	T	$Wb \cdot m^{-2}$

Inductance	henry	H	$\mathrm{Wb} \cdot \mathrm{A}^{-1}$
Luminous flux	lumen	lm	$\mathrm{cd} \cdot \mathrm{sr}$
Illuminance	lux	lx	$\mathrm{lm} \cdot \mathrm{m}^{-2}$
Activity	bequerel	Bq	s^{-1}
Absorbed dose	gray	Gy	$\mathrm{J} \cdot \mathrm{kg}^{-1}$
Dose equivalent	sievert	Sv	$\mathrm{J} \cdot \mathrm{kg}^{-1}$

9.1.4.4: The Del-operator

The ∇-operator

In cartesian coordinates (x, y, z) :

$$\vec{\nabla}=\frac{\partial}{\partial x}\vec{e}_x+\frac{\partial}{\partial y}\vec{e}_y+\frac{\partial}{\partial z}\vec{e}_z\ ,\ \ \mathrm{grad}f=\vec{\nabla}f=\frac{\partial f}{\partial x}\vec{e}_x+\frac{\partial f}{\partial y}\vec{e}_y+\frac{\partial f}{\partial z}\vec{e}_z \tag{9.1.4.4.1}$$

$$\mathrm{div}\,\vec{a}=\vec{\nabla}\cdot\vec{a}=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\ ,\ \ \nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} \tag{9.1.4.4.2}$$

$$\mathrm{rot}\,\vec{a}=\vec{\nabla}\times\vec{a}=\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\vec{e}_x+\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)\vec{e}_y+\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)\vec{e}_z \tag{9.1.4.4.3}$$

In cylinder coordinates (r, φ, z) holds:

$$\vec{\nabla}=\frac{\partial}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial}{\partial\varphi}\vec{e}_\varphi+\frac{\partial}{\partial z}\vec{e}_z\ ,\ \ \mathrm{grad}f=\frac{\partial f}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial f}{\partial\varphi}\vec{e}_\varphi+\frac{\partial f}{\partial z}\vec{e}_z \tag{9.1.4.4.4}$$

$$\mathrm{div}\,\vec{a}=\frac{\partial a_r}{\partial r}+\frac{a_r}{r}+\frac{1}{r}\frac{\partial a_\varphi}{\partial\varphi}+\frac{\partial a_z}{\partial z}\ ,\ \ \nabla^2 f=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial^2 f}{\partial z^2} \tag{9.1.4.4.5}$$

$$\mathrm{rot}\,\vec{a}=\left(\frac{1}{r}\frac{\partial a_z}{\partial\varphi}-\frac{\partial a_\varphi}{\partial z}\right)\vec{e}_r+\left(\frac{\partial a_r}{\partial z}-\frac{\partial a_z}{\partial r}\right)\vec{e}_\varphi+\left(\frac{\partial a_\varphi}{\partial r}+\frac{a_\varphi}{r}-\frac{1}{r}\frac{\partial a_r}{\partial\varphi}\right)\vec{e}_z \tag{9.1.4.4.6}$$

In spherical coordinates (r, θ, φ):

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e}_\varphi\\
\mathrm{grad}f &= \frac{\partial f}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial f}{\partial\theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\vec{e}_\varphi\\
\mathrm{div}\,\vec{a} &= \frac{\partial a_r}{\partial r}+\frac{2a_r}{r}+\frac{1}{r}\frac{\partial a_\theta}{\partial\theta}+\frac{a_\theta}{r\tan\theta}+\frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial\varphi}\\
\mathrm{rot}\,\vec{a} &= \left(\frac{1}{r}\frac{\partial a_\varphi}{\partial\theta}+\frac{a_\theta}{r\tan\theta}-\frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial\varphi}\right)\vec{e}_r+\left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\varphi}-\frac{\partial a_\varphi}{\partial r}-\frac{a_\varphi}{r}\right)\vec{e}_\theta+\\
&\quad\left(\frac{\partial a_\theta}{\partial r}+\frac{a_\theta}{r}-\frac{1}{r}\frac{\partial a_r}{\partial\theta}\right)\vec{e}_\varphi\\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r^2\tan\theta}\frac{\partial f}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}
\end{aligned}$$

General orthonormal curvelinear coordinates (u, v, w) can be obtained from cartesian coordinates by the transformation $\vec{x}=\vec{x}(u, v, w)$. The unit vectors are then given by:

$$\vec{e}_u=\frac{1}{h_1}\frac{\partial\vec{x}}{\partial u}\ ,\ \ \vec{e}_v=\frac{1}{h_2}\frac{\partial\vec{x}}{\partial v}\ ,\ \ \vec{e}_w=\frac{1}{h_3}\frac{\partial\vec{x}}{\partial w} \tag{9.1.4.4.7}$$

where the factors h_i set the norm to 1. Then holds:

$$\begin{aligned}
\operatorname{grad} f = & \frac{1}{h_1}\frac{\partial f}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial f}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial f}{\partial w}\vec{e}_w \\
\operatorname{div} \vec{a} = & \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u}(h_2h_3a_u) + \frac{\partial}{\partial v}(h_3h_1a_v) + \frac{\partial}{\partial w}(h_1h_2a_w)\right) \\
\operatorname{rot} \vec{a} = & \frac{1}{h_2h_3}\left(\frac{\partial(h_3a_w)}{\partial v} - \frac{\partial(h_2a_v)}{\partial w}\right)\vec{e}_u + \frac{1}{h_3h_1}\left(\frac{\partial(h_1a_u)}{\partial w} - \frac{\partial(h_3a_w)}{\partial u}\right)\vec{e}_v + \\
& \frac{1}{h_1h_2}\left(\frac{\partial(h_2a_v)}{\partial u} - \frac{\partial(h_1a_u)}{\partial v}\right)\vec{e}_w \\
\nabla^2 f = & \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)\right]
\end{aligned}$$

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Glossary

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