10.3: Electromagnetic fields

Learning Objectives

- Explain electromagnetic field tensor

The electric field

Section 10.1 showed that relativity requires magnetic forces to exist, and section 10.2 gave us a peek at what this implies about or how electric and magnetic fields transform. To understand this on a more general basis, let's explicitly list some assumptions about the electric field and see how they lead to the existence and properties of a magnetic field:

1. *Definition of the electric field*: In the frame of reference of an inertial observer o, take some standard, charged test particle, release it at rest, and observe the force F_o (section 4.5) acting on the particle. (The timelike component of this force vanishes.) Then the electric field three-vector E in frame o is defined by $F_o = qE$, where we fix our system of units by taking some arbitrary value for the charge q of the test particle.
2. *Definition of electric charge*: For charges other than the standard test charge, we take Gauss's law to be our definition of electric charge.
3. Charge is Lorentz invariant (section 1.3).
4. Fields must have transformation laws (section 10.2). Many times already in our study of relativity, we've followed the strategy of taking a Galilean vector and trying to redefine it as a four-dimensional vector in relativity. Let's try to do this with the electric field. Then we would have no other obvious thing to try than to change its definition to $F = qE$, where $F = ma$ is the relativistic force vector (section 4.5), so that the electric field three-vector was just the spacelike part of E. Because $a \cdot v = 0$ for a material particle, this would imply that E was orthogonal to o for any observer o. But this is impossible, since then a spacetime displacement vector s along the direction of E would be a vector of simultaneity for all observers, and we know that this isn't possible in relativity.

The magnetic field

Our situation is very similar to the one encountered in section 9.1, where we found that knowledge of the charge density in one frame was insufficient to tell us the charge density in other frames. There was missing information, which turned out to be the current density. The problems we've encountered in defining the transformation properties of the electric field suggest a similar "missing-information" situation, and it seems likely that the missing information is the magnetic field. How should we modify the assumptions above to allow for the existence of a magnetic field in addition to the electric one? What properties could this additional field have? How would we define or measure it?

One way of imagining a new type of field would be if, in addition to charge q, particles had some other characteristic, call it r, and there was then be some entirely separate field defined by their action on a particle with this "r-ness." But going down this road leads us to unrelated phenomena such as the the strong nuclear interaction.

The electromagnetic field tensor

The nature of the contradiction arrived at in the above section is such that our additional field is closely linked to the electric one, and therefore we expect it to act on charge, not on r-ness. Without inventing something new like r-ness, the only other available property of the test particle is its state of motion, characterized by its velocity vector v. Now the simplest rule we could imagine for determining the force on a test particle would be a linear one, which would look like matrix multiplication:

$$F = q\mathcal{F}v \tag{10.3.1}$$

or in index notation,

$$F^a = q\mathcal{F}^a{}_b\, v^b \tag{10.3.2}$$

Although the form $\mathcal{F}^a{}_b$ with one upper and one lower index occurs naturally in this expression, we'll find it more convenient from now on to work with the upper-upper form $\mathcal{F}^{ab}$. $\mathcal{F}$ would be 4×4, so it would have 16 elements:

$$\begin{pmatrix} \mathcal{F}^{tt} & \mathcal{F}^{tx} & \mathcal{F}^{ty} & \mathcal{F}^{tz} \\ \mathcal{F}^{xt} & \mathcal{F}^{xx} & \mathcal{F}^{xy} & \mathcal{F}^{xz} \\ \mathcal{F}^{yt} & \mathcal{F}^{yx} & \mathcal{F}^{yy} & \mathcal{F}^{yz} \\ \mathcal{F}^{zt} & \mathcal{F}^{zx} & \mathcal{F}^{zy} & \mathcal{F}^{zz} \end{pmatrix} \tag{10.3.3}$$

Presumably these 16 numbers would encode the information about the electric field, as well as some additional information about the field or fields we were missing.

But these are not 16 numbers that we can choose freely and independently. For example, consider a charged particle that is instantaneously at rest in a certain observer's frame, with $v = (1, 0, 0, 0)$. (In this situation, the four-force equals the force measured by the observer.) The work done by a force is positive if the force is in the same direction as the motion, negative if in the opposite direction, and zero if there is no motion. Therefore the power $P = dW/dt$ in this example should be zero. Power is the timelike component of the force vector, which forces us to take $\mathcal{F}^{tt} = 0$.

More generally, consider the kinematical constraint $a \cdot v = 0$. When we require $a \cdot v = 0$ for any v, not just this one, we end up with the constraint that F must be antisymmetric, meaning that when we transpose it, the result is another matrix that looks just like the original one, but with all the signs flipped:

$$\begin{pmatrix} 0 & \mathcal{F}^{tx} & \mathcal{F}^{ty} & \mathcal{F}^{tz} \\ -\mathcal{F}^{tx} & 0 & \mathcal{F}^{xy} & \mathcal{F}^{xz} \\ -\mathcal{F}^{ty} & -\mathcal{F}^{xy} & 0 & \mathcal{F}^{yz} \\ -\mathcal{F}^{tz} & -\mathcal{F}^{xz} & -\mathcal{F}^{yz} & 0 \end{pmatrix} \tag{10.3.4}$$

Each element equals minus the corresponding element across the main diagonal from it, and antisymmetry also requires that the main diagonal itself be zero. In terms of the concept of degrees of freedom introduced in section 3.5, we are down to 6 degrees of freedom rather than 16. We now relabel the elements of the matrix and follow up with a justification of the relabeling. The result is the following rank-2 tensor:

$$\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{10.3.5}$$

We'll call this the electromagnetic field tensor. The labeling of the left column simply expresses the definition of the electric field, which is expressed in terms of the velocity $\vec{v} = (1, 0, 0, 0)$ of a particle at rest. The top row then follows from antisymmetry. For an arbitrary velocity vector, writing out the matrix multiplication $\mathcal{F}^{\mu} = q\mathcal{F}^{\mu}{}_{\nu} v^{\nu}$ results in expressions such as $F^x = \gamma q(E_x + u_y B_z - u_z B_y)$ (problem , p.~). Taking into account the difference of a factor of γ between the four-force and the force measured by an observer, we end up with the familiar Lorentz force law,

$$\vec{F}_o = q(\vec{E} + \vec{u} \times \vec{B}) \tag{10.3.6}$$

where $\vec{B}$ is the magnetic field. This is expressed in units where $c = 1$, so that the electric and magnetic field have the same units. In units with $c \neq 1$, the magnetic components of the electromagnetic field matrix should be multiplied by c.

Thus starting only from the assumptions above, we deduce that the electric field must be accompanied by a magnetic field.

Example 10.3.1: Parity properties of E and B

In Example 9.2.3, we saw that under the parity transformation $(t, x, y, z) \to (t, -x, -y, -z)$, any rank-2 tensor expressed in Minkowski coordinates changes the signs of its components according to the same rule:

$$\begin{pmatrix} \text{no flip} & \text{flip} & \text{flip} & \text{flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \end{pmatrix} \tag{10.3.7}$$

Since this holds for the electromagnetic field tensor F, we find that under parity, $E \to -E$ and $B \to B$. For example, a capacitor seen in a mirror has its electric field pointing the opposite way, but there is no change in the magnetic field of a current loop, since the location of each current element is flipped to the other side of the loop, but its direction of flow is also reversed, so that the picture as a whole remains unchanged.

What about gravity?

A funny puzzle pops up if we go back and think about the assumptions above that went into all this. Those assumptions were so general that it almost seems as though the only possible behavior for fields is the behavior of electric and magnetic fields. But other fields do behave differently. How did the assumptions fail in the case of gravity, for example? Gauss's law (assumption 2) certainly holds for gravity. But the source of gravitational fields isn't charge, it's mass-energy, and mass-energy isn't a Lorentz invariant, contrary to assumption 3. Furthermore, assumption 1 entailed that our field could be defined in terms of forces measured by an inertial observer, but for an inertial observer gravity doesn't exist (section 5.2).