6.4: Duality

Learning Objectives

- Explain the concept of duality

Duality in 3+1 dimensions

In our original $0+1$-dimensional example of the cuckoo clock and the earth, we had duality: the measurements $c \rightarrow e = 24$ and $e \rightarrow c = 1/24$ really provided the same information, and it didn't matter whether we made our scalar out of covector $c \rightarrow$ and vector $\rightarrow e$ or covector $e \rightarrow$ and vector $\rightarrow c$. All these quantities were simply clock rates, which could be described either by their frequencies (covectors) or their periods (vectors).

To generalize this to $3+1$ dimensions, we need to use the metric — a piece of machinery that we have never had to employ since the beginning of the chapter. Given a vector $\rightarrow r$, suppose we knew how to produce its covector version $r \rightarrow$. Then we could hook up the plumbing to form $r \rightarrow r$, which is just a number. What number could it be? The only reasonable possibility is the squared magnitude of r, which we calculate using the metric as $r^2 = g(r, r)$. Since we can think of covectors as functions that take vectors to real numbers, clearly $r \rightarrow$ should be the function f defined by $f(x) = g(r, x)$.

Example 6.4.1: Finding the dual of a given vector

Given the vector $\rightarrow v = (3, 4)$ in $1+1$-dimensional Minkowski coordinates, find the covector $v \rightarrow$, i.e., it is dual.

Our goal is to write out an explicit expression for the covector in component form,

$$v \rightarrow = (a, b) \tag{6.4.1}$$

To define these components, we have to have some basis in mind, consisting of one time like observer-vector o and one spacelike vector of simultaneity s. Since we're doing this in Minkowski coordinates (section 1.2), let's notate these as $\rightarrow \hat{t}$ and $\rightarrow \hat{x}$, where the hats indicate that these are unit vectors in the sense that $\hat{t}^2 = 1$ and $\hat{x}^2 = -1$. Writing $v \rightarrow$ in terms of a and b means that we're identifying $v \rightarrow$ with the function f defined by $f(x) = g(v, x)$. Therefore

$$f(\rightarrow \hat{t}) = a \text{ and } f(\rightarrow \hat{x}) = b \tag{6.4.2}$$

or

$$g(\rightarrow v, \rightarrow \hat{t}) = 3 = a \quad \text{and} \quad g(\rightarrow v, \rightarrow \hat{x}) = -4 = b \tag{6.4.3}$$

The result of the formidable, fancy-looking calculation in Example $6.4.1$ was simply to take the vector $(3, 4)$ and flip the sign of its spacelike component to give the its dual, the covector $(3, -4)$. Looking back at why this happened, it was because we were using Minkowski coordinates, and in Minkowski coordinates the form of the metric is

$$g(p, q) = (+1)p_t q_t + (-1)p_x q_x + \dots \tag{6.4.4}$$

Therefore, we can always find duals in this way, provided that

1. we're using Minkowski coordinates, and
2. the signature of the metric is, as assumed throughout this book, $+---$, not $-+++$.

Example 6.4.2: Going both ways

Assume Minkowski coordinates and signature $+---$. Given the vector

$$\rightarrow e = (8, 7) \tag{6.4.5}$$

and the covector

$$f \rightarrow = (1, 2) \tag{6.4.6}$$

find $e \rightarrow$ and $\rightarrow f$.

Solution

By the rule established above, we can find $e \rightarrow$ simply by flipping the sign of the 7,

$$e \rightarrow = (8, -7) \tag{6.4.7}$$

To find $\rightarrow f$, we need to ask what vector (a, b), if we flipped the sign of b, would give us $(a, -b) = (1, 2)$. Obviously this is

$$\rightarrow f = (1, -2) \tag{6.4.8}$$

In other words, flipping the sign of the spacelike part of a vector is also the recipe for changing covectors into vectors.

Example $6.4.2$ shows that in Minkowski coordinates, the operation of changing a covector to the corresponding vector is the same as that of changing a vector to its covector. Thus, the dual of a dual is the same thing you started with. In this respect, duality is similar to arithmetic operations such as $x \rightarrow -x$ and $x \rightarrow 1/x$. That is, the duality is a self-inverse operation — it undoes itself, like getting two sex-change operations in a row, or switching political parties twice in a country that has a two-party system. Birdtracks notation makes this self-inverse property look obvious, since duality means switching a inward arrow to an outward one or vice versa, and clearly doing two such switches gives back the original notation. This property was established in Example $6.4.2$ by using Minkowski coordinates and assuming the signature to be $+---$, but it holds without these assumptions.

In the general case where the coordinates may not be Minkowski, the above analysis plays out as follows. Covectors and vectors are represented by row and column vectors. The metric can be specified by a matrix g so that the inner product of column vectors p and q is given by $p^T gq$, where T represents the transpose. Rerunning the same logic with these additional complications, we find that the dual of a vector q is $(gq)^T$, while the dual of a covector ω is $(\omega g^{-1})^T$, where $g{-}1$ is the inverse of the matrix g.

Change of basis

We saw in Section 6.2 that in $0+1$ dimensions, vectors and covectors has opposite scaling properties under a change of units, so that switching our base unit from hours to minutes caused our frequency covectors to go up by a factor of 60, while our time vectors went down by the same factor. This behavior was necessary in order to keep scalar products the same. In more than one dimension, the notion of changing units is replaced with that of a change of basis. In linear algebra, row vectors and column vectors act like covectors and vectors; they are dual to each other. Let B be a matrix made of column vectors, representing a basis for the column-vector space. Then a change of basis for a row vector r is expressed as $r' = rB$, while the same change of basis for a column vector c is $c' = B^{-1}c$. We then find that the scalar product is unaffected by the change of basis, since $r'c' = rBB^{-1}c = rc$.

In the important special case where B is a Lorentz transformation, this means that covectors transform under the inverse transformation, which can be found by flipping the sign of v. This fact will be important in the following section.