4.5: Force

Learning Objectives

- Explain force and relativity

Force is a concept that is seldom needed in relativity, and that's why this section is optional.

Four-force

By analogy with Newtonian mechanics, we define a relativistic force vector

$$F = m \cdot a \tag{4.5.1}$$

where a is the acceleration four-vector (section 3.5) and m is the mass of a particle that has that acceleration as a result of the force F. This is equivalent to

$$F = \frac{\mathrm{d}p}{\mathrm{d}\tau} \tag{4.5.2}$$

where p is the mass of the particle and τ its proper time. Since the timelike part of p is the particle's mass-energy, the timelike component of the force is related to the power expended by the force. These definitions only work for massive particles, since for a massless particle we can't define a or τ. F has been defined in terms of Lorentz invariants and four-vectors, and therefore it transforms as a god-fearing four-vector itself.

The force measured by an observer

The trouble with all this is that F isn't what we actually measure when we measure a force, except if we happen to be in a frame of reference that momentarily coincides with the rest frame of the particle. As with velocity and acceleration (section 3.7), we have a four-vector that has simple, standard transformation properties, but a different F_o, which is what is actually measured by the observer o. It's defined as

$$F_o = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{4.5.3}$$

with a dt in the denominator rather than a $d\tau$. In other words, it measures the rate of transfer of momentum according to the observer, whose time coordinate is t, not τ — unless the observer happens to be moving along with the particle. Unlike the three-vectors v_o and a_o, whose timelike components are zero by definition according to observer o, F_o usually has a nonvanishing timelike component, which is the rate of change of the particle's mass-energy, i.e., the power. We can refer to the spacelike part of F_o as the three-force.

The following two examples show that an object moving at relativistic speeds has less inertia in the transverse direction than in the longitudinal one. A corollary is that the three-acceleration need not be parallel to the three-force.

Example 4.5.1: Circular motion

For a particle in uniform circular motion, γ is constant, and we have

$$F_o = \frac{\mathrm{d}}{\mathrm{d}t}(m\gamma v) = m\gamma\frac{\mathrm{d}v}{\mathrm{d}t} \tag{4.5.4}$$

The particle's mass-energy is constant, so the timelike component of F_o does happen to be zero in this example. In terms of the three-vectors v_o and a_o defined in section 3.7, we have

$$F_o = m\gamma\frac{\mathrm{d}v_o}{\mathrm{d}t} = m\gamma a_o \tag{4.5.5}$$

which is greater than the Newtonian value by the factor γ. As a practical example, in a cathode ray tube(CRT) such as the tube in an old-fashioned oscilloscope or television, a beam of electrons is accelerated upto relativistic speed. To paint a picture on the screen, the beam has to be steered by transverse forces, and since the deflection angles are small, the world-line of the beam is approximately that of uniform circular motion. The force required to deflect the beam is greater by a factor of γ than would have been expected according to Newton's laws.

Example 4.5.2: Linear motion

For accelerated linear motion in the x direction, ignoring y and z, we have a velocity vector

$$v=\frac{\mathrm{d}r}{\mathrm{d}\tau} \tag{4.5.6}$$

whose x component is γv. Then

$$\begin{aligned}
F_{o,x} &= m\frac{\mathrm{d}(\gamma v)}{\mathrm{d}t} \\
&= m\frac{\mathrm{d}(\gamma)}{\mathrm{d}t}v+m\gamma\frac{\mathrm{d}v}{\mathrm{d}t} \\
&= m\frac{\mathrm{d}\gamma}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}t}+m\gamma a \\
&= m\left(v^2\gamma^3 a+\gamma a\right) \\
&= ma\gamma^3
\end{aligned}$$

The particle's apparent inertia is increased by a factor of γ^3 due to relativity.

The results of above two examples can be combined as follows:

$$F_o=m\gamma a_{o,\perp}+m\gamma^3 a_{o,\|} \tag{4.5.7}$$

where the subscripts $\perp$ and $\|$ refer to the parts of a_o perpendicular and parallel to v_o.

Transformation of the force measured by an observer

Define a frame of reference o for the inertial frame of reference of an observer who does happen to be moving along with the particle at a particular instant in time. Then t is the same as τ, and F_o the same as F. In this frame, the particle is momentarily at rest, so the work being done on it vanishes, and the timelike components of F_o and F are both zero.

Suppose we do a Lorentz transformation from o to a new frame o', and suppose the boost is parallel to F_o and F (which are both purely spatial in frame o). Call this direction x. Then $dp=(dp_t,dp_x)=(0,dp_x)$ transforms to $dp'=(-\gamma v dp_x,\gamma dp_x)$, so that $F_{o',x}=dp'_x/dt'=(\gamma dp_x)/(\gamma dt)=F_{o,x}$. The two factors of γ cancel, and we find that $F_{o',x}=F_{o,x}$.

Now let's do the case where the boost is in the y direction, perpendicular to the force. The Lorentz transformation doesn't change dp_y, so

$$\begin{aligned}
F_{o',y} &= \frac{\mathrm{d}p'_y}{\mathrm{d}t'} \\
&= \frac{\mathrm{d}p_y}{(\gamma\mathrm{d}t)} \\
&= \frac{F_{o',y}}{\gamma}
\end{aligned}$$

The summary of our results is as follows. Let F_o be the force acting on a particle, as measured in a frame instantaneously comoving with the particle. Then in a frame of reference moving relative to this one, we have

$$F_{o',\|}=F_{o,\|} \tag{4.5.8}$$

and

$$F_{o',\perp}=\frac{F_{o,\perp}}{\gamma} \tag{4.5.9}$$

where $\|$ indicates the direction parallel to the relative velocity of the two frames, and $\perp$ a direction perpendicular to it.

Work

Consider the one-dimensional version of the three-force, $F = dp/dt$. An advantantage of this quantity is that it allows us to use the Newtonian form of the (one-dimensional) work-kinetic energy relation $dE/dx = F$ without correction. Proof:

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{\mathrm{d}E}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}E}{\mathrm{d}p}\frac{F}{v} \tag{4.5.10}$$

By implicit differentiation of the definition of mass, we find that $dE/dp = p/E$, and this in turn equals v by the identity proved in Example 4.3.2. This leads to the claimed result, which is valid for both massless and material particles.
