6.2: Phase

Learning Objectives

- Explain why phase is a scalar

Phase is a scalar

In section 1.3, we defined a (Lorentz) invariant as a quantity that was unchanged under rotations and Lorentz boosts. A measurement such as $c \to e = 24$ is an invariant because it is simply a count. We've counted the number of periods. In fact, a count is not just invariant under rotations and boosts but under any well-behaved change of coordinates — the technical condition being that each coordinate in each set is a differentiable function of each coordinate in the other set. Such a change of coordinates is called a *diffeomorphism*. For example, a uniform scaling of the coordinates $(t, x, y, z) \to (kt, kx, ky, kz)$, which is analogous to a change of units,[1] is all right as long as k is nonzero. A quantity that stays the same under any *diffeomorphism* is called a *scalar*. Since a Lorentz transformation is a *diffeomorphism*, every *scalar* is a Lorentz invariant. Not every Lorentz invariant is a *scalar*.

Example 6.2.1: The determinant of the metric

Minkowski coordinates can be defined as coordinates in which the metric has the standard form $g = diag(1, -1, -1, -1)$. If we rescale these coordinates according to $(t, x, y, z) \to (kt, kx, ky, kz)$, then the metric changes according to $g \to k^{-2} g$. To keep track of how "un-Minkowski" this scaling is, we could use the determinant of the metric $det(g) = -k^{-8}$. This determinant tells us how many coordinate-grid boxes fit in a unit volume, and it is of interest in a more general context than this example of uniform rescaling, e.g., it serves a similar function when converting from Cartesian coordinates to polar coordinates.

Under a Lorentz transformation or a rotation, the metric retains its standard form, and therefore $det(g)$ is Lorentz invariant. Another way of seeing this is that spacetime volume is Lorentz invariant, so that a Lorentz transformation doesn't change how many coordinate-grid boxes fit in a unit volume.

But although $det(g)$ is a Lorentz invariant, it is not a scalar, because it changes under the transformation described above.

In birdtracks notation, any expression that has no external arrows at all represents a scalar. Since the expression $c \to e = 24$ has no external arrows, only internal ones, it represents a scalar. Another way of describing this measurement is as a phase. If we prefer to measure the phase φ in units of cycles, then we have

$$\varphi = c \to e \tag{6.2.1}$$

If we like radians, we can use

$$\varphi = 2\pi c \to e \tag{6.2.2}$$

Scaling

A convenient way of summarizing all of our categories of variables is by their behavior when we rescale our coordinates. If we switch our time unit from hours to minutes, the number of apples in a bowl is unchanged, the earth's period of rotation gets 60 times bigger, and the frequency of the cuckoo clock changes by a factor of $1/60$. In other words, a quantity u under rescaling of coordinates by a factor α becomes $\alpha^p u$, where the exponents -1, 0, and $+1$ correspond to covectors, scalars, and vectors, respectively. We can therefore see that these distinctions are of interest even in one dimension, contrary to what one would have expected from the freshman-physics concept of a vector as something transforming in a certain way under rotations.

In section 1.3, we defined an invariant as a quantity that did not change under rotations or Lorentz boosts, i.e., one that was independent of the frame of reference. For a scalar we have the even more restrictive condition that it must not change under any change of coordinates. For example, area in $1+1$-dimensional spacetime is an invariant, but it's not a scalar; it changes when we rescale our coordinates.

References

[1] The appropriate relativistic way of defining a change of units is subject to some ambiguity. See section 9.6