4.E: Dynamics (Exercises)

Q1

Criticize the following reasoning.

> *Temperature is a measure of the energy per atom. In nonrelativistic physics, there is a minimum temperature, which corresponds to zero energy per atom, but no maximum. In relativity, there should be a maximum temperature, which would be the temperature at which all the atoms are moving at c.*

Q2

In an old-fashioned cathode ray tube (CRT) television, electrons are accelerated through a voltage difference that is typically about $20\ kV$. At what fraction of the speed of light are the electrons moving?

Q3

In nuclear beta decay, an electron or antielectron is typically emitted with an energy on the order of $1\ MeV$. In alpha decay, the alpha particle typically has an energy of about $5\ MeV$. In each case, do a rough estimate of whether the particle is nonrelativistic, relativistic, or ultrarelativistic.

Q4

Suppose that the starship Enterprise from Star Trek has a mass of $8.0\times 10^7\ kg$, about the same as the Queen Elizabeth 2. Compute the kinetic energy it would have to have if it was moving at half the speed of light. Compare with the total energy content of the world's nuclear arsenals, which is about $10^{21}\ J$.

Q5

Cosmic-ray neutrinos may be the fastest material particles in the universe. In 2013 the IceCube neutrino detector in Antarctica detected two neutrinos,[1] dubbed Bert and Ernie, after the Sesame Street characters, with energies in the neighborhood of $1\ PeV = 10^{15}\ eV$. The higher energy was Ernie's $1.14\pm 0.17\ PeV$. It is not known what type of neutrino he was, nor do we have exact masses for neutrinos, but let's assume $m = 1\ eV$. Find Ernie's rapidity.

Q6

Science fiction stories often depict spaceships traveling through solar systems at relativistic speeds. Interplanetary space contains a significant number of tiny dust particles, and such a ship would sweep these dust particles out of a large volume of space, impacting them at high speeds. A 1975 experiment aboard the Skylab space station measured the frequency of impacts from such objects and found that a square meter of exposed surface experienced an impact from a particle with a mass of $\sim 10^{-15}\ kg$ about every few hours. A relativistic object, sweeping through space much more rapidly, would experience such impacts at rates of more like one every few seconds. (Larger particles are significantly more rare, with the frequency falling off as something like m^{-8}.) These particles didn't damage Skylab, because at relative velocities of $\sim 10^4\ m/s$ their kinetic energies were on the order of microjoules. At relativistic speeds it would be a different story. Real-world spacecraft are lightweight and rather fragile, so there would probably be serious consequences from any impact having a kinetic energy of about $10^2\ J$ (comparable to a bullet from a small handgun).

a. Find the speed at which a starship could cruise through a solar system if frequent $10^2\ J$ collisions were acceptable, assuming no object with a mass of more than $10^{-15}\ kg$. Express your result relative to c.
b. Find the speed under the more conservative parameters of $10\ J$ and $10^{-14}\ kg$.

Q7

Example 3.5.1 derives the equation

$$x = \frac{1}{a}\cosh a\tau \tag{4.E.1}$$

for a particle moving with constant acceleration. (Note that a constant of integration was taken to be zero, so that $x \neq 0$ at $\tau = 0$.)

a. Rewrite this equation in metric units by inserting the necessary factors of c.
b. If we had a rocket ship capable of accelerating indefinitely at g, how much proper time would be needed in order to travel the distance $\Delta x = 27,000$ light-years to the galactic center? (This will be a flyby, so the ship accelerates all the way rather than decelerating to stop at its destination.) Answer: 11 years
c. An observer at rest relative to the galaxy explains the surprisingly short time calculated in part b as being due to the time dilation experienced by the traveler. How does the traveler explain it?

Q8

Show, as claimed in section 4.7, that if tachyons exist, then it is possible to have two tachyons whose momentum vectors add up to zero.

Q9

a. A free neutron (as opposed to a neutron bound into an atomic nucleus) is unstable, and undergoes spontaneous radioactive decay into a proton, an electron, and an antineutrino. The masses of the particles involved are as follows:

neutron	1.67495×10^{-27} kg
proton	1.67265×10^{-27} kg
electron	0.00091×10^{-27} kg
antineutrino	\(< 10^{-35} kg

Find the energy released in the decay of a free neutron.

b. Neutrons and protons make up essentially all of the mass of the ordinary matter around us. We observe that the universe around us has no free neutrons, but lots of free protons (the nuclei of hydrogen, which is the element that 90% of the universe is made of). We find neutrons only inside nuclei along with other neutrons and protons, not on their own. If there are processes that can convert neutrons into protons, we might imagine that there could also be proton-to-neutron conversions, and indeed such a process does occur sometimes in nuclei that contain both neutrons and protons: a proton can decay into a neutron, a positron, and a neutrino. A positron is a particle with the same properties as an electron, except that its electrical charge is positive. A neutrino, like an antineutrino, has negligible mass. Although such a process can occur within a nucleus, explain why it cannot happen to a free proton. (If it could, hydrogen would be radioactive, and you wouldn't exist!)

Q10

a. Find a relativistic equation for the velocity of an object in terms of its mass and momentum (eliminating γ).
b. Show that your result is approximately the same as the classical value, p/m, at low velocities.
c. Show that very large momenta result in speeds close to the speed of light.

Q11

Expand the equation for relativistic kinetic energy $K = m(\gamma - 1)$ in a Taylor series, and find the first two nonvanishing terms. Show that the first term is the nonrelativistic expression.

Q12

Expand the equation $p = m\gamma v$ in a Taylor series, and find the first two nonvanishing terms. Show that the first term is the classical expression.

Q13

An atom in an excited state emits a photon, ending up in a lower state. The initial state has mass m_1, the final one m_2. To a very good approximation, we expect the energy E of the photon to equal $m_1 - m_2$. However, conservation of momentum dictates that the atom must recoil from the emission, and therefore it carries away a small amount of kinetic energy that is not available to the photon. Find the exact energy of the photon, in the frame in which the atom was initially at rest.

Q14

The following are the three most common ways in which gamma rays interact with matter:

- **Photoelectric effect**: The gamma ray hits an electron, is annihilated, and gives all of its energy to the electron.
- **Compton scattering**: The gamma ray bounces off of an electron, exiting in some direction with some amount of energy.
- **Pair production**: The gamma ray is annihilated, creating an electron and a positron.

Example 4.3.5 shows that pair production can't occur in a vacuum due to conservation of the energy-momentum four-vector. What about the other two processes? Can the photoelectric effect occur without the presence of some third particle such as an atomic nucleus? Can Compton scattering happen without a third particle?

Q15

This problem assumes you know some basic quantum physics. The point of this problem is to estimate whether or not a neutron or proton in an an atomic nucleus is highly relativistic. Nuclei typically have diameters of a few fm ($1\ fm = 10^{-15}\ m$). Take a neutron or proton to be a particle in a box of this size. In the ground state, half a wavelength would fit in the box. Use the de Broglie relation to estimate its typical momentum and thus its typical speed. How relativistic is it?

Q16

Show, as claimed in Example 4.3.6, that if a massless particle were to decay, Lorentz invariance requires that the timescale τ for the process be proportional to the particle's energy. What units would the constant of proportionality have?

Q17

Derive the equation

$$T = \sqrt{\frac{3\pi}{G\rho}} \tag{4.E.2}$$

given in section 4.6, for the period of a rotating, spherical object that results in zero apparent gravity at its surface.

Q18

Neutrinos with energies of $\sim 1\ MeV$ (the typical energy scale of nuclear physics) make up a significant part of the matter in our universe. If a neutrino and an antineutrino annihilate each other, the product is two back-to-back photons whose energies are equal in the center-of-mass frame. Should astronomers be able to detect these photons by selecting only those with the correct energy?

Q19

In a certain frame of reference, a gamma ray with energy E_1 is moving to the right, while a second gamma way with energy E_2 flies off to the left.

a. Find the mass of the system.
b. Find the velocity of the center-of-mass frame, i.e., the frame of reference in which the total momentum is zero.

Q20

In section 4.5 we proved the work-energy relation $dE/dx = F$ in the context of relativity. Recapitulate the derivation in the context of pure Newtonian mechanics.

Q21

Section 4.3 discusses the possibility that the photon has a small but nonzero mass m. One of the consequences is that the electric field of an infinite, uniformly charged plane is $E_x = \pm 2\pi k\sigma exp(-\mu|x|)$, where k is the Coulomb constant, σ is the charge per unit area, $\mu = mc/\hbar$, and x is the distance from the plane. When $m = 0$, we recover the result of standard electromagnetism. The purpose of this problem is to analyze a laboratory experiment that can put an upper bound on m.

Consider a rectangular, hollow, conducting box with charge placed on it. If $m = 0$, then Gauss's law holds, and the field inside is exactly zero. We now consider the possibility that $m > 0$. We make the box very thin in the x direction, with sides localed at $x = \pm a$. We refer to these two sides as the "plates." The box's extent in the y and z directions is much greater than a, so that the density of charge σ on each of the two plates is nearly constant as long as we stay away from the fringing fields at the edges. Consider a point located at $x=b$, with $0<b<a$, and far from the edges. Show that there is a nonvanishing interior field, which can be measured in this experiment by the fractional difference in electric potential

$$\frac{V(a)-V(b)}{V(a)} \approx \frac{1}{2} m^2\left(a^2-b^2\right)+\cdots \tag{4.E.3}$$

where ... indicates higher-order terms.

Remark: The experiment is more practical when carried out using a spherical geometry, since there are no fringing fields to worry about. The analysis comes out the same except that the factor of $1/2$ becomes $1/6$. Experiments of this type were first carried out by Cavendish in 1722, and then with a seris of order-of-magnitude improvements in precision by Plimpton in 1936 and Williams in 1971.

Q22

Potassium 40 is the strongest source of naturally occurring beta radioactivity in our environment. It decays according to

$$^{40}\mathrm{K} \rightarrow {}^{40}\mathrm{Ca}+e^{-}+\bar{v} \tag{4.E.4}$$

The energy released in the decay is $1.33\ MeV$. The energy is shared randomly among the products, subject to the constraint imposed by conservation of energy-momentum, which dictates that very little of the energy is carried by the recoiling calcium nucleus. Determine the maximum energy of the calcium, and compare with the typical energy of a chemical bond, which is a few eV. If the potassium is part of a molecule, do we expect the molecule to survive? Carry out the calculation first by assuming that the electron is ultrarelativistic, then without the approximation, and comment on the how good the approximation is.

Q23

The products of a certain radioactive decay are a massive particle and a gamma ray.

a. Show that, in the center of mass frame, the energy of the gamma is less than the mass-energy of the massive particle.
b. Show that the opposite inequality holds if we compare the *kinetic* energy of the massive particle to the energy of the gamma.
c. Suppose someone tells you that a certain massive particle has a mode of radioactive decay in which it disappears, and the only product is a gamma ray — no residual massive particle exists. Use the result of part a to show that this is impossible, and then see if you can find a simpler argument to demonstrate the same thing.

References

[1] arxiv.org/abs/1304.5356