10.5: Invariants

Learning Objectives

- Invariants and electromagnetic field

We've seen cases before in which an invariant can be formed from a rank-1 tensor. The square of the proper time corresponding to a timelike spacetime displacement $\vec{r}$ is $\vec{r} \cdot \vec{r}$ or, in the index notation introduced in section , $r^a r_a$. From the momentum tensor we can construct the square of the mass $p^a p_a$.

There are good reasons to believe that something similar can be done with the electromagnetic field tensor, since electromagnetic fields have certain properties that are preserved when we switch frames. Specifically, an electromagnetic wave consists of electric and magnetic fields that are equal in magnitude and perpendicular to one another. An electromagnetic wave that is a valid solution to Maxwell's equations in one frame should also be a valid wave in another frame. It can be shown that the following two quantities are invariants:

$$P = B^2 - E^2 \tag{10.5.1}$$

and

$$Q = \vec{E} \cdot \vec{B} \tag{10.5.2}$$

The fact that these are written as vector dot products of three-vectors shows that they are invariant under rotation, but we also want to show that they are relativistic scalars, i.e., invariant under boosts as well. To prove this, we can write them both in tensor notation. The first invariant can be expressed as $P = \frac{1}{2}\mathcal{F}^{ab}\mathcal{F}_{ab}$, while the second equals $Q = \frac{1}{4}\epsilon^{abcd}\mathcal{F}_{ab}\mathcal{F}_{cd}$, where $\epsilon^{\kappa\lambda\mu\nu}$ is the Levi-Civita tensor.

A field for which both $P = Q = 0$ is called a null field. An electromagnetic plane wave is a null field, and although this is easily verifiable from the definitions of P and Q, there is a deeper reason why this should be true, and this reason applies not just to electromagnetic waves but to other types of waves, such as gravitational waves. Consider any relativistic scalar s that is a continuous function of the electromagnetic field tensor $\mathcal{F}$, i.e., a continuous function of $\mathcal{F}$'s components. We want s to vanish when $\mathcal{F} = 0$. Given an electromagnetic plane wave, we can do a Lorentz boost parallel to the wave's direction of propagation. Under such a boost the wave suffers a Doppler shift in its wavelength and frequency, but in addition to that, the transformation equations in section 10.4 imply that the intensity of the fields is reduced at any given point. Thus in the limit of an indefinite process of acceleration, $\mathcal{F} \to 0$, and therefore $s \to 0$ as well. But since s is a scalar, its value is independent of our frame of reference, and so it must be zero in all frames.

P and Q are a complete set of invariants for the electromagnetic field, meaning that the only other electromagnetic invariants are those that either can be determined from P and Q or depend on the derivatives of the fields, not just their values. To see that P and Q are complete in this sense, we can break the possibilities down into cases, according to whether P and Q are zero or nonzero, positive or negative. As a representative example, consider the case where $P < 0$ and $Q > 0$. First we rotate our frame of reference so that $\vec{E}$ is along the x axis, and $\vec{B}$ lies in the x-y plane. Next we do a boost along the z axis in order to eliminate the y component of $\vec{B}$; the field transformation equations in section 10.4 make this possible because $|\vec{E}| > |\vec{B}|$. The result is that we have found a frame of reference in which $\vec{E}$ and $\vec{B}$ both lie along the positive x axis. The only frame-independent information that there is to know is the information available in this frame, and that consists of only two positive real numbers, E_x and B_x, which can be determined from the values of P and Q.

Example 10.5.1: A static null FIeld

Although an electromagnetic plane wave is a null field, the converse is not true. For example, we can create a static null field out of a static, uniform electric field and a static, uniform magnetic field, with the two fields perpendicular to one another.

Example 10.5.2: Another invariant?

Let Π be the squared magnitude of the Poynting vector, $\Pi = (\vec{E} \times \vec{B}) \cdot (\vec{E} \times \vec{B})$. Since Π can be expressed in terms of dot products and scalar products, it is guaranteed to be invariant under rotations. However, it is not a relativistic invariant. For example, if we do a Lorentz boost parallel to the direction of an electromagnetic wave, the intensity of the wave changes, and so does Π.

Example 10.5.3: A non-null invariant for electromagnetic waves?

The quantity $Q^{-1} = \frac{1}{\vec{E} \cdot \vec{B}}$ is clearly an invariant, and it doesn't vanish for an electromagnetic plane wave — in fact, it is infinite for a plane wave. Does this contradict our proof that any invariant must vanish for a plane wave? No, because we only proved this in the case where the invariant is defined as a continuous function of F. Our function $Q-1$ is a discontinuous function of F when $F = 0$. Such discontinuous invariants tend not to be very interesting. For suppose we try to measure Q^{-1}, and the thing we're measuring happens to be an electromagnetic wave. Our measurements of the fields will probably be statistically consistent with zero, and therefore the error bars on our measurement of Q^{-1} will likely be infinitely large.