9.6: Units of Measurement for Tensors

Learning Objectives

- Analyzing units for relativity

Analyzing units, also known as dimensional analysis, is one of the first things we learn in freshman physics. It's a useful way of checking our math, and it seems as though it ought to be straightforward to extend the technique to relativity. It certainly can be done, but it isn't quite as trivial as might be imagined. We'll see below that different authors prefer differing systems, and clashes occur between some of the notational systems in use.

One of our most common jobs is to change from one set of units to another, but in relativity it becomes nontrivial to define what we mean by the notion that our units of measurement change or don't change. We could, e.g., appeal to an atomic standard, but Dicke[1] points out that this could be problematic. Imagine, he says, that

> *you are told by a space traveller that a hydrogen atom on Sirius has the same diameter as one on the earth. A few moments' thought will convince you that the statement is either a definition or else meaningless.*

(Some related ideas about the numerical value of c were discussed in section 1.1.)

To start with, we note that abstract index notation is more convenient than concrete index notation for these purposes. As noted in section 7.5, concrete index notation assigns different units to different components of a tensor if we use coordinates, such as spherical coordinates (t, r, θ, φ), that don't all have units of length. In abstract index notation, a symbol like v^i stands for the whole vector, not for one of its components. Since abstract index notation does not even offer us a notation for components, if we want to apply dimensional analysis we must define a system in which units are attributed to a tensor as a whole. Suppose we write down the abstract-index form of the equation for proper time:

$$ds^2 = g_{ab} dx^a dx^a \tag{9.6.1}$$

In abstract index notation, dx^a doesn't mean an infinitesimal change in a particular coordinate, it means an infinitesimal displacement vector[2] This equation has one quantity on the left and three factors on the right. Suppose we assign these parts of the equation units $[ds] = L^\sigma$, $[g_{ab}] = L^{2\gamma}$, and $[dx^a] = [dx^b] = L^\xi$, where square brackets mean "the units of" and L stands for units of length. We then have $\sigma = \gamma + \xi$. Due to the ambiguities referred to above, we can pick any values we like for these three constants, as long as they obey this rule. I find $(\sigma, \gamma, \xi) = (1, 0, 1)$ to be natural and convenient, but Dicke, in the above-referenced paper, likes $(1, 1, 0)$, while the mathematician Terry Tao advocates $(0, \mp 1, \pm 1)$.

Suppose we raise and lower indices to form a tensor with r upper indices and s lower indices We refer to this as a tensor of rank (r, s). (We don't count contracted indices, e.g., uava is a rank-$(0, 0)$ scalar.) Since the metric is the tool we use for raising and lowering indices, and the units of the lower-index form of the metric are $L^{2\gamma}$, it follows that the units vary in proportion to $L^{\gamma(s-r)}$. In general, you can assign a physical quantity units L^u that are a product of two factors, a "kinematical" or purely geometrical factor L^k, where $k = \gamma(s - r)$, and a dynamical factor $L^d \ldots$, which can depend on what kind of quantity it is, and where the ...indicates that if your system of units has more than just one base unit, those can be in there as well. Dicke uses units with $\hbar = c = 1$, for example, so there is only one base unit, and mass has units of inverse length and $d_{mass} = -1$. In general relativity it would be more common to use units in which $G = c = 1$, which instead give $d_{mass} = +1$.

Example 9.6.1: The units of momentum

Consider the equation

$$p^a = m v^a \tag{9.6.2}$$

for the momentum of a material particle. Suppose we use special relativistic units in which $c = 1$, but because gravity isn't incorporated into the theory, G plays no special role, and it is natural to use a system of units in which there is a base unit of mass M.

The kinematic units check out, because $k_p = k_m + k_v$:

$$\gamma(-1) = \gamma(0) + \gamma(-1) \tag{9.6.3}$$

This is merely a matter of counting indices, and was guaranteed to check out as long as the indices were written in a grammatical way on both sides of the equation. What this check is essentially telling us is that if we were to establish Minkowski coordinates in a neighborhood of some point, and do a change of coordinates $(t, x, y, z) \rightarrow (\alpha t, \alpha x, \alpha y, \alpha z)$, then the quantities on both sides of the equation would vary under the tensor transformation laws according to the same exponent of α. For example, if we changed from meters to centimeters, the equation would still remain valid.

For the dynamical units, suppose that we use $(\sigma, \gamma, \xi) = (1, 0, 1)$, so that an infinitesimal displacement dx^a has units of length L, as does proper time ds. These two quantities are purely kinematic, so we don't assign them any dynamical units, and therefore the velocity vector $v^a = \frac{dx^a}{ds}$ also has no dynamical units. Our choice of a system of units gives $[m] = M$. We require that the equation $p^a = mv^a$ have dynamical units that check out, so:

$$M = 1 \cdot M \tag{9.6.4}$$

We must also assign units of mass to the momentum.

A system almost identical to this one, but with different terminology, is given by Schouten.[3]

For practical purposes in checking the units of an equation, we can see from example $9.6.1$ that worrying about the kinematic units is a waste of time as long as we have checked that the indices are grammatical. We can therefore give a simplified method that suffices for checking the units of any equation in abstract index notation.

1. We assign a tensor the same units that one of its concrete components would have if we were to adopt (local) Minkowski coordinates, in the system with $(\sigma, \gamma, \xi) = (1, 0, 1)$. These are the units we would automatically have imputed to it after learning special relativity but before learning about tensors or fancy coordinate transformations. Since $\gamma = 0$, the positions of the indices do not affect the result.
2. The units of a sum are the same as the units of the terms.
3. The units of a tensor product are the product of the units of the factors.

References

[1] "Mach's principle and invariance under transformation of units," Phys Rev 125 (1962) 2163

[2] For a modern and rigorous development of differential geometry along these lines, see Nowik and Katz, arxiv.org/abs/1405.0984.

[3] Tensor Analysis for Physicists, ch. VI