7.4: Summary of Transformation Laws

Learning Objectives

- Transformation laws summarized

Having worked through one example in detail, let's progress from the specific to the general. In the Einstein concrete index notation, let coordinates (x^0, x^1, x^2, x^3) be transformed to new coordinates (x'^0, x'^1, x'^2, x'^3). Then vectors transform according to the rule

$$v'^{\mu} = v^{\kappa} \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \tag{7.4.1}$$

where the Einstein summation convention implies a sum over the repeated index κ. By the same reasoning as in section 6.4, the transformation for a covector ω is

$$\omega'_{\mu} = \omega_{\kappa} \frac{\partial x^{\kappa}}{\partial x'^{\mu}} \tag{7.4.2}$$

Note the inversion of the partial derivative in one equation compared to the other. Because these equations describe a change from one coordinate system to another, they clearly depend on the coordinate system, so we use Greek indices rather than the Latin ones that would indicate a coordinate-independent abstract index equation.

The letter μ in these equations always appears as an index referring to the new coordinates, κ to the old ones. For this reason, we can get away with dropping the primes and writing, e.g.,

$$v^{\mu} = v^{\kappa} \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \tag{7.4.3}$$

rather than v', counting on context to show that v^{μ} is the vector expressed in the new coordinates, v^{κ} in the old ones. This becomes especially natural if we start working in a specific coordinate system where the coordinates have names. For example, if we transform from coordinates (t, x, y, z) to (a, b, c, d), then it is clear that v^t is expressed in one system and v^c in the other.

In Equation $7.4.2$, μ appears as a subscript on the left side of the equation, but as a superscript on the right. This would appear to violate the grammatical rules given in section 6.7, but the interpretation here is that in expressions of the form $\partial / \partial x^i$ and $\partial / \partial x_i$, the superscripts and subscripts should be understood as being turned upside-down. Similarly, Equation $7.4.1$ appears to have the implied sum over κ written ungrammatically, with both κ's appearing as superscripts. Normally we only have implied sums in which the index appears once as a superscript and once as a subscript. With our new rule for interpreting indices on the bottom of derivatives, the implied sum is seen to be written correctly. This rule is similar to the one for analyzing the units of derivatives written in Leibniz notation, with, e.g., d^2x/dt^2 having units of meters per second squared. That is, the flipping of the indices like this is required for consistency so that everything will work out properly when we change our units of measurement, causing all our vector components to be rescaled.

Example 7.4.1: The identity transformation

In the case of the identity transformation $x'^{\mu} = x^{\mu}$, Equation $7.4.1$ clearly gives $v' = v$, since all the mixed partial derivatives $\partial x'^{\mu} / \partial x^{\kappa}$ with $\mu \neq \kappa$ are zero, and all the derivatives for $\kappa = \mu$ equal 1.

In Equation $7.4.2$, it is tempting to write

$$\frac{\partial^{\kappa}}{\partial x'^{\mu}} = \frac{1}{\frac{\partial x'^{\mu}}{\partial x^{\kappa}}} \quad \text{wrong!} \tag{7.4.4}$$

but this would give infinite results for the mixed terms! Only in the case of functions of a single variable is it possible to flip derivatives in this way; it doesn't work for partial derivatives. To evaluate these partial derivatives, we have to invert the transformation (which in this example is trivial to accomplish) and then take the partial derivatives.

Example 7.4.2: Polar coordinates

None of the techniques discussed here are particular to relativity. For example, consider the transformation from polar coordinates (r,θ) in the plane to Cartesian coordinates

$$x = rcos\theta \tag{7.4.5}$$

$$y = rsin\theta \tag{7.4.6}$$

A bug sits on the edge of a phonograph turntable, at $(r,\theta) = (1,0)$. The turntable rotates clockwise, giving the bug a velocity vector $v^\kappa = (v^r, v^\theta) = (0,-1)$, i.e., the angular velocity is one radian per second in the negative (counterclockwise)direction. Let's find the bug's velocity vector in Cartesian coordinates. The transformation law for vectors gives:

$$v^x = v^\kappa \frac{\partial x}{\partial x^\kappa} \tag{7.4.7}$$

Expanding the implied sum over the repeated index κ, we have

$$\begin{aligned} v^x &= v^r \frac{\partial x}{\partial r} + v^\theta \frac{\partial x}{\partial \theta} \\ &= (0)\frac{\partial x}{\partial r} + (-1)\frac{\partial x}{\partial \theta} \\ &= -r\sin\theta \\ &= 0 \end{aligned}$$

For the y component,

$$\begin{aligned} v^y &= v^r \frac{\partial y}{\partial r} + v^\theta \frac{\partial y}{\partial \theta} \\ &= (0)\frac{\partial y}{\partial r} + (-1)\frac{\partial y}{\partial \theta} \\ &= -r\sin\theta \\ &= -1 \end{aligned}$$