

9.2: The Stress-Energy Tensor

Learning Objectives

- Explain conservation and energy momentum
- Explain stress-energy tensor

Conservation and flux of energy-momentum

A particle such as an electron has a charge, but it also has a mass. We can't define a relativistic mass flux because flux is defined by addition, but mass isn't additive in relativity. Mass-energy is additive, but unlike charge it isn't an invariant. Mass-energy is part of the energy-momentum four vector $p = (E, p^x, p^y, p^z)$. We then have sixteen different fluxes we can define. For example, we could replay the description in section 9.1 of the three-surface S perpendicular to the x direction, but now we would be interested in a quantity such as the z component of momentum. We then have a measure of the density of flux of p^z in the x direction, which we notate as T^{zx} . The matrix T is called the *stress-energy tensor*, and it is an object of central importance in relativity. (The reason for the odd name will become more clear in a moment.) In general relativity, it is the source of gravitational fields.

The stress-energy tensor is related to physical measurements as follows. Let o be the future-directed, normalized velocity vector of an observer; let s express a spatial direction according to this observer, i.e., it points in a direction of simultaneity and is normalized with $s \cdot s = -1$; and let S be a three-volume covector, directed toward the future (i.e., $o^a S_a > 0$). Then measurements by this observer come out as follows:

$$T^{ab} o_a S_b = \text{mass-energy inside the three-volume } S \quad (9.2.1)$$

$$T^{ab} s_a S_b = \text{momentum in the direction } s, \text{ inside } S \quad (9.2.2)$$

The stress-energy tensor allows us to express conservation of energy-momentum as

$$\frac{\partial T^{ab}}{\partial x^a} = 0 \quad (9.2.3)$$

This *local* conservation of energy-momentum is all we get in general relativity. As discussed in section 4.3, there is no such global law in curved spacetime. However, we will show in section 9.3 that in the special case of flat spacetime, i.e., special relativity, we do have such a global conservation law.

Symmetry of the stress-energy tensor

The stress-energy tensor is a symmetric matrix. For example, let's say we have some nonrelativistic particles. If we have a nonzero T^{tx} , it represents a flux of mass-energy (p^t) through a three-surface perpendicular to x . This means that mass is moving in the x direction. But if mass is moving in the x direction, then we have some x momentum p^x . Therefore we must also have a T^{xt} , since this momentum is carried by the particles, whose world-lines pass through a hypersurface of simultaneity.

Dust

The simplest example of a stress-energy tensor would be a cloud of particles, all at rest in a certain frame of reference, described in Minkowski coordinates:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (9.2.4)$$

where we now use ρ to indicate the density of mass-energy, not charge as in section 9.1. This could be the stress-energy tensor of a stack of oranges at the grocery store, the atoms in a hunk of copper, or the galaxies in some small neighborhood of the universe. Relativists refer to this type of matter, in which the velocities are negligible, as "dust." The nonvanishing component T^{tt} indicates that for a three-surface S perpendicular to the t axis, particles with mass-energy $E = P^t$ are crossing that surface from the past to the future. Conservation of energy-momentum is satisfied, since all the elements of this T are constant, so all the partial derivatives vanish.

Rank-2 tensors and their transformation law

Suppose we were to look at this cloud in a different frame of reference. Some or all of the timelike row $T^{t\nu}$ and timelike column $T^{\mu t}$ would fill in because of the existence of momentum, but let's just focus for the moment on the change in the mass-energy density represented by T^{tt} . It will increase for two reasons. First, the kinetic energy of each particle is now nonzero; its mass-energy increases from m to $m\gamma$. But in addition, the volume occupied by the cloud has been reduced by $1/\gamma$ due to length contraction. We've picked up two factors of gamma, so the result is $\rho \rightarrow \rho\gamma^2$. This is different from the transformation behavior of a vector. When a vector is purely timelike in one frame, transformation to another frame raises its timelike component only by a factor of γ , not γ^2 . This tells us that a matrix like T transforms differently than a vector (section 7.2). The general rule is that if we transform from coordinates x to x' , then:

$$T'^{\mu\nu} = T^{\kappa\lambda} \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \frac{\partial x'^{\nu}}{\partial x^{\lambda}} \quad (9.2.5)$$

An object that transforms in this standard way is called a rank-2 tensor. The 2 is because it has two indices. Vectors and covectors have rank 1, invariants rank 0. In section 7.3, we developed a method of transforming the metric from one set of coordinates to another; we now see that technique as an application of the more general rule given in equation 9.2.5. Considered as a tensor, the metric is symmetric, $g_{ab} = g_{ba}$. In most of the example's we've been considering, the metric tensor is diagonal, but when it has off-diagonal elements, each of these is one half the corresponding coefficient in the expression for ds , as in the following example.

Example 9.2.1: An non-diagonal metric tensor

The answer to problem Q2 in chapter 7 was the metric

$$ds^2 = dx^2 + dy^2 + 2\cos\phi dx dy \quad (9.2.6)$$

Writing this in terms of the metric tensor, we have

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^{\mu} dx^{\nu} \\ &= g_{xx} dx^2 + g_{xy} dx dy + g_{yx} dy dx + g_{yy} dy^2 \\ &= g_{xx} dx^2 + 2g_{xy} dx dy + g_{yy} dy^2 \end{aligned}$$

Therefore we have $g_{xy} = \cos\phi$, not $g_{xy} = 2\cos\phi$.

Example 9.2.2: Dust in a different frame

We start with the stress-energy tensor of the cloud of particles, in the rest frame of the particles.

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (9.2.7)$$

Under a boost by ν in the x direction, the tensor transformation law gives

$$T'^{\mu'\nu'} = \begin{pmatrix} \gamma^2 \rho & \gamma^2 \nu \rho & 0 & 0 \\ \gamma^2 \nu \rho & \gamma^2 \nu^2 \rho & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (9.2.8)$$

The over-all factor of γ^2 arises for the reasons previously described.

Example 9.2.3: Parity

The parity transformation is a change of coordinates that looks like this:

$$t' = t \quad (9.2.9)$$

$$x' = -x \quad (9.2.10)$$

$$y' = -y \quad (9.2.11)$$

$$z' = -z \quad (9.2.12)$$

It turns right-handed screws into left-handed ones, but leaves the arrow of time unchanged. Under this transformation, the tensor transformation law tells us that some of the components of the stress-energy tensor will flip their signs, while others will stay the same:

$$\begin{pmatrix} \text{no flip} & \text{flip} & \text{flip} & \text{flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \\ \text{flip} & \text{no flip} & \text{no flip} & \text{no flip} \end{pmatrix} \quad (9.2.13)$$

Everything here was based solely on the fact that T was a rank 2 tensor expressed in Minkowski coordinates, and therefore the same parity properties hold for other rank-2 tensors as well.

Pressure

The stress-energy tensor carries information about pressure. For example, T^{xx} is the flux in the x direction of x -momentum. This is simply the pressure, P , that would be exerted on a surface with its normal in the x direction. Negative pressure is tension, and this is the origin of the term “tensor,” coined by Levi-Civita.

Example 9.2.4: Pressure as a source of gravitational fields

Because the stress-energy tensor is the source of gravitational fields in general relativity, we can see that the gravitational field of an object should be influenced not just by its mass-energy but by its internal stresses. The very early universe was dominated by photons rather than by matter, and photons have a much higher ratio of momentum to mass-energy than matter, so the importance of the pressure components in the stress-energy tensor was much greater in that era. In the universe today, the largest pressures are those found inside atomic nuclei. Inside a heavy nucleus, the electromagnetic pressure can be as high as 10^{33} Pa ! If general relativity’s description of pressure as a source of gravitational fields were wrong, then we would see anomalous effects in the gravitational forces exerted by heavy elements compared to light ones. Such effects have been searched for both in the laboratory¹ and in lunar laser ranging experiments,² with results that agreed with general relativity’s predictions.

A Perfect Fluid

The cloud in Example 9.2.2 had a stress-energy tensor in its own rest frame that was isotropic, i.e., symmetric with respect to the x , y , and z directions. The tensor became anisotropic when we switched out of this frame. If a physical system has a frame in which its stress-energy tensor is isotropic, i.e., of the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \quad (9.2.14)$$

we call it a perfect fluid in equilibrium. Although it may contain moving particles, this special frame is the one in which their momenta cancel out. In other cases, the pressure need not be isotropic, and the stress exerted by the fluid need not be perpendicular to the surface on which it acts. The space-space components of T would then be the classical stress tensor, whose diagonal elements are the anisotropic pressure, and whose off-diagonal elements are the shear stress. This is the reason for calling T the stress-energy tensor.

The perfect fluid form of the stress-energy tensor is extremely important and common. For example, cosmologists find that it is a nearly perfect description of the universe on large scales.

We discussed in section ?? the ideas of converting back and forth between vectors and their corresponding covectors, and of notating this as the raising and lowering indices. We can do the same thing with the two indices of a rank-2 tensor, so that the stress-energy tensor can be expressed in four different ways: T^{ab} , T^{ab} , T^a_b , and T_a^b , but the symmetry of T means that there is no interesting distinction between the final two of these. In special relativity, the distinctions among the various forms are not

especially fascinating. We can always cover all of spacetime with Minkowski coordinates, so that the form of the metric is simply a diagonal matrix with elements ± 1 on the diagonal. As with a rank-1 tensor, raising and lowering indices on a rank-2 tensor just flips some components and leaves others alone. The methods for raising and lowering don't need to be deduced or memorized, since they follow uniquely from the grammar of index notation, e.g., $T^a{}_b = g_{bc}T^{ac}$. But there is the potential for a lot of confusion with all the signs, and in addition there is the fact that some people use a $+- - -$ signature while others use $-+++$. Since perfect fluids are so important, I'll demonstrate how all of this works out in that case.

For a perfect fluid, we can write the stress-energy tensor in the coordinate-independent form

$$T^{ab} = (\rho + P)o^a o^b - (o^c o_c)Pg^{ab} \quad (9.2.15)$$

where o represents the velocity vector of an observer in the fluid's rest frame, and $o^c o_c = o^2 = o \cdot o$ equals 1 for our $+- - -$ signature or -1 for the signature $-+++$. For ease of writing, let's abbreviate the signature factor as $s = o^c o_c$.

Suppose that the metric is diagonal, but its components are varying, $g_{\alpha\beta} = \text{diag}(sA^2, -sB^2, \dots)$. The properly normalized velocity vector of an observer at (coordinate-)rest is $o^\alpha = (A^{-1}, 0, 0, 0)$. Lowering the index gives $o_\alpha = (sA, 0, 0, 0)$. The various forms of the stress-energy tensor then look like the following:

$$T_{00} = A^2 \rho \quad T_{11} = B^2 P \quad (9.2.16)$$

$$T^0{}_0 = s\rho \quad T^1{}_1 = -sP \quad (9.2.17)$$

$$T^{00} = A^{-2} \rho \quad T^{11} = B^{-2} P \quad (9.2.18)$$

Which of these forms is the "real" one, e.g., which form of the 00 component is the one that the observer o actually measures when she sticks a shovel in the ground, pulls out a certain volume of dirt, weighs it, and determines ρ ? The answer is that the index notation is so slick and well designed that all of them are equally "real," and we don't need to memorize which actually corresponds to measurements. When she does this measurement with the shovel, she could say that she is measuring the quantity $T^{ab}o_a o_b$. But because all of the a 's and b 's are paired off, this expression is a rank-0 tensor. That means that $T^{ab}o_a o_b$, $T_{ab}o^a o^b$, and $T^a{}_b o_a o^b$ are all the same number. If, for example, we have coordinates in which the metric is diagonal and has elements ± 1 , then in all these expressions the differing signs of the o 's are exactly compensated for by the signs of the T 's.

Example 9.2.5: A rope under tension

As a real-world example in which the pressure is not isotropic, consider a rope that is moving inertially but under tension, i.e., equal forces at its ends cancel out so that the rope doesn't accelerate. Tension is the same as negative pressure. If the rope lies along the x axis and its fibers are only capable of supporting tension along that axis, then the rope's stress-energy tensor will be of the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (9.2.19)$$

where P is negative and equals minus the tension per unit cross-sectional area. Conservation of energy-momentum is expressed as (Equation 9.2.3) $\frac{\partial T^{ab}}{\partial x^a} = 0$. Converting the abstract indices to concrete ones, we have

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0 \quad (9.2.20)$$

where there is an implied sum over μ , and the equation must hold both in the case where ν is a label for t and the one where it refers to x . In the first case, we have

$$\frac{\partial T^{tt}}{\partial t} + \frac{\partial T^{xt}}{\partial x} = 0 \quad (9.2.21)$$

which is a statement of conservation of energy, energy being the timelike component of the energy-momentum. The first term is zero because ρ is constant by virtue of our assumption that the rope was uniform. The second term is zero because $T^{xt} = 0$. Therefore conservation of energy is satisfied. This came about automatically because by writing down a time-independent

expression for the stress-energy, we were dictating a static equilibrium. When ν stands for x , we get an equation that requires the x component of momentum to be conserved,

$$\frac{\partial T^{tx}}{\partial t} + \frac{\partial T^{xx}}{\partial x} = 0 \quad (9.2.22)$$

This simply says

$$\frac{\partial P}{\partial x} = 0 \quad (9.2.23)$$

meaning that the tension in the rope is constant along its length.

Example 9.2.6: A rope supporting its own weight

A variation on Example 9.2.5 is one in which the rope is hanging and supports its own weight. Although gravity is involved, we can solve this problem without general relativity, by exploiting the equivalence principle (section 5.2). As discussed in section 5.1, an inertial frame in relativity is one that is free-falling. We define an inertial frame of reference o , corresponding to an observer free-falling past the rope, and a noninertial frame o' at rest relative to the rope.

Since the rope is hanging in static equilibrium, observer o' sees a stress-energy tensor that has no time-dependence. The off-diagonal components vanish in this frame, since there is no momentum. The stress-energy tensor is

$$T^{\mu'\nu'} = \begin{pmatrix} \rho & 0 \\ 0 & P \end{pmatrix} \quad (9.2.24)$$

where the components involving y and z are zero and not shown, and P is negative as in Example 9.2.5. We could try to apply the conservation of energy condition to this stress-energy tensor as in example 8, but that would be a mistake. As discussed in section 7.5, rates of change can only be measured by taking partial derivatives with respect to the coordinates if the coordinates are Minkowski, i.e., in an inertial frame. Therefore we need to transform this stress-energy tensor into the inertial frame o .

For simplicity, we restrict ourselves to the Newtonian approximation, so that the change of coordinates between the two frames is

$$t \approx t' \quad (9.2.25)$$

$$x \approx x' + \frac{1}{2}at^2 \quad (9.2.26)$$

where $a > 0$ if the free-falling observer falls in the negative x direction, i.e., positive x is up. That is, if a point on the rope at a fixed x' is marked with a spot of paint, then free-falling observer o sees the spot moving up, to larger values of x , at $t > 0$. Applying the tensor transformation law, we find

$$T^{\mu\nu} = \begin{pmatrix} \rho & \rho at \\ \rho at & P + \rho a^2 t^2 \end{pmatrix} \quad (9.2.27)$$

As in Example 9.2.5, conservation of energy is trivially satisfied. Conservation of momentum gives

$$\frac{\partial T^{tx}}{\partial t} + \frac{\partial T^{xx}}{\partial x} = 0 \quad (9.2.28)$$

Integrating this with respect to x , we have

$$P = -\rho ax + \text{constant} \quad (9.2.29)$$

Let the cross-sectional area of the rope be A , and let $\mu = \rho A$ be the mass per unit length and $T = -PA$ the tension. We then find

$$T = \mu ax + \text{constant} \quad (9.2.30)$$

Conservation of momentum requires that the tension vary along the length of the rope, just as we expect from Newton's laws: a section of the rope higher up has more weight below it to support.

Energy conditions

The result of Example 9.2.6 could cause something scary to happen. If we walk up to a clothesline under tension and give it a quick karate chop, we will observe wave pulses propagating away from the chop in both directions, at velocities $v = \pm\sqrt{T/\mu}$. But the result of the example is that this expression increases without limit as x gets larger and larger. At some point, v will exceed the speed of light. (Of course any real rope would break long before this much tension was achieved.) Two things led to the problematic result:

1. we assumed there was no constraint on the possible stress-energy tensor in the rest frame of the rope; and
2. we used a Newtonian approximation to change from this frame to the free-falling frame.

In reality, we don't know of any material so stiff that vibrations propagate in it faster than c . In fact, all ordinary materials are made of atoms, atoms are bound to each other by electromagnetic forces, and therefore no material made of atoms can transmit vibrations faster than the speed of an electromagnetic wave, c .

Based on these conditions, we therefore expect there to be certain constraints on the stress-energy tensor of any ordinary form of matter. For example, we don't expect to find any rope whose stress-energy tensor looks like this:

$$T^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (9.2.31)$$

because here the tensile stress $+2$ is greater than the mass density 1 , which would lead to $|v| = \sqrt{2/1} > 1$. Constraints of this kind are called energy conditions. Hypothetical forms of matter that violate them are referred to as exotic matter; if they exist, they are not made of atoms. This particular example violates the an energy condition known as the dominant energy condition, which requires $\rho > 0$ and $|P| > \rho$. There are about five energy conditions that are commonly used, and a detailed discussion of them is more appropriate for a general relativity text. The common ideas that recur in many of them are:

1. that energy density is never negative in any frame of reference, and
2. that there is never a flux of energy propagating at a speed greater than c .

An energy condition that is particularly simple to express is the trace energy condition (**TEC**),

$$T^a{}_a \geq 0 \quad (9.2.32)$$

where we have to have one upper index and one lower index in order to obey the grammatical rules of index notation. In Minkowski coordinates (t, x, y, z) , this becomes $T^\mu{}_\mu \geq 0$, with the implied sum over μ expanding to give

$$T^t{}_t + T^x{}_x + T^y{}_y + T^z{}_z \geq 0 \quad (9.2.33)$$

The left-hand side of this relation, the sum of the main-diagonal elements of a matrix, is called the trace of the matrix, hence the name of this energy condition. Since this book uses the signature $+- - -$ for the metric, raising the second index changes this to

$$T^{tt} - T^{xx} - T^{yy} - T^{zz} \geq 0 \quad (9.2.34)$$

In Example 9.2.2, we computed the stress-energy tensor of a cloud of dust, in a frame moving at velocity v relative to the cloud's rest frame. The result was

$$T^{\mu'\nu'} = \begin{pmatrix} \gamma^2\rho & \gamma^2\nu\rho & 0 & 0 \\ \gamma^2\nu\rho & \gamma^2\nu^2\rho & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (9.2.35)$$

In this example, the trace energy condition is satisfied precisely under the condition $|v| \leq 1$, which can be interpreted as a statement that according to the **TEC**, the mass-energy of the cloud can never be transported at a speed greater than c in any frame.

References

¹ Kreuzer, Phys. Rev. 169 (1968) 1007. Described in section 3.7.3 of Will, “The Confrontation between General Relativity and Experiment,” [relativity. livingreviews.org/Articles/lrr-2006-3/](http://livingreviews.org/Articles/lrr-2006-3/).

² Bartlett and van Buren, Phys. Rev. Lett. 57 (1986) 21, also described in Will.

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