

7.1: An Example - Accelerated Coordinates

Learning Objectives

- Explain how to generalize the rules from chapter 6 to any change of coordinates
- How to find the form of the metric expressed in non-Minkowski coordinates.

In your previous study of physics, you've seen many examples where one coordinate system makes life easier than another. For a block being pushed up an inclined plane, the most convenient choice may be to tilt the x and y axes. To find the moment of inertia of a disk we use cylindrical coordinates. The same is true in relativity. Minkowski coordinates are not always the most convenient. In Chapter 6 we learned to classify physical quantities as covectors, scalars, and vectors, and we learned rules for how these three types of quantities transformed in two special changes of coordinates:

1. When we rescale all coordinates by a factor α , the components of vectors, scalars, and covectors scale by α^p , where $p = +1, 0$, and -1 , respectively.
2. Under a boost, the three cases require respectively the Lorentz transformation, no transformation, and the inverse Lorentz transformation.

In this chapter we'll learn how to generalize this to any change of coordinates,¹ and also how to find the form of the metric expressed in non-Minkowski coordinates.

Let's start with a concrete example that has some physical interest. In Section 5.2, we saw that we could have "gravity without gravity:" an experiment carried out in a uniform gravitational field can be interpreted as an experiment in flat spacetime (so that special relativity applies), but with the measurements expressed in the accelerated frame of the earth's surface. In the Pound-Rebka experiment, all of the results could have been expressed in an inertial (free-falling) frame of reference, using Minkowski coordinates, but this would have been extremely inconvenient, because, for example, they didn't want to drop their expensive atomic clocks and take the readings before the clocks hit the floor and were destroyed.

Since this is "gravity without gravity," we don't actually need a planet cluttering up the picture. Imagine a universe consisting of limitless, empty, flat spacetime. Describe it initially using Minkowski coordinates (t, x, y, z) . Now suppose we want to find a new set of coordinates (T, X, Y, Z) that correspond to the frame of reference of an observer aboard a spaceship accelerating in the x direction with a constant acceleration.

The Galilean answer would be $X = x - \frac{1}{2}at^2$. But this is unsatisfactory from a relativistic point of view for several reasons. At $t = c/a$ the observer would be moving at the speed of light, but relativity doesn't allow frames of reference moving at c (Section 3.4). At $t > c/a$, the observer's motion would be faster than c , but this is impossible in $3 + 1$ dimensions (Section 3.8).

These problems are related to the fact that the observer's proper acceleration, i.e., the reading on an accelerometer aboard the ship, isn't constant if $x = \frac{1}{2}at^2$. We saw in Example 3.5.2 that constant proper acceleration is described by $x = \frac{1}{a} \cosh a\tau$, $t = \frac{1}{a} \sinh a\tau$, where τ is the proper time. For this motion, the velocity only approaches c asymptotically. This suggests the following for the relationship between the two sets of coordinates:

$$t = X \sinh T \quad (7.1.1)$$

$$x = X \cosh T \quad (7.1.2)$$

$$y = Y \quad (7.1.3)$$

$$z = Z \quad (7.1.4)$$

For example, if the ship follows a world-line $(T, X) = (\tau, 1)$, then its motion in the unaccelerated frame is $(t, x) = (\sinh \tau, \cosh \tau)$, which is of the desired form with $a = 1$.

The (T, X, Y, Z) coordinates, called *Rindler coordinates*, have many, but not all, of the properties we would like for an accelerated frame. Ideally, we'd like to have all of the following:

1. the proper acceleration is constant for any world-line of constant (X, Y, Z)
2. the proper acceleration is the same for all such world-lines, i.e., the fictitious gravitational field is uniform
3. the description of the accelerated frame is just a change of coordinates, i.e., we're just talking about the flat spacetime of special relativity, with events renamed.

It turns out that we can pick two out of three of these, but it's not possible to satisfy all three at the same time. Rindler coordinates satisfy conditions 1 and 3, but not 2. This is because the proper acceleration of a world-line of constant (X, Y, Z) can easily be shown to be $1/X$, which depends on X . Thus we don't speak of Rindler coordinates as "the" coordinates of an accelerated observer.

Rindler coordinates have the property that if a rod extends along the X -axis, and external forces are applied to it in just such a way that every point on the rod has constant X , then it accelerates along its own length without any stress.

The diagonals are event horizons. Their intersection lies along every constant- T line; cf. Example 1.4.7.

References

¹ We do require the change of coordinates to be smooth in the sense defined on p. 127, i.e., it should be a diffeomorphism.

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