4.6: Two Applications

Learning Objectives

- The Stefan-Boltzmann law
- Degenerate matter

The Stefan-Boltzmann law

In 1818, Dulong and Petit analyzed experimental data to find the empirical and totally incorrect law

$$P \propto \exp[T/(13.5K)] \tag{4.6.1}$$

relating the temperature T of a body to the power (P) it emits as electromagnetic radiation. (To see that it must be wrong, note that it doesn't vanish at absolute zero.) It was accepted until 1884, when Boltzmann corrected a systematic error in their analysis of the data, and offered a theoretical argument for the correct Stefan-Boltzmann law:

$$P \propto T^4. \tag{4.6.2}$$

This law is extremely important in a variety of applications including global warming, stellar structure, cosmology, and warming your hands by the glow of a fire. Modern physics students usually come across it as a corollary in the story of the development of the quantum theory by Planck, Einstein, et al., but as we will see below, it is a purely classical result, depending only on relativity and thermodynamics.

Consider an insulated cubical box of volume V containing radiation in thermal equilibrium. We let it expand uniformly with constant entropy, so that all three sides grow by the same factor a. (This is exactly what happens in cosmological expansion.) Boltzmann's clever idea was that the radiation could be treated like the working fluid in a heat engine.

By the relativistic relation between momentum and energy, the energy and momentum of a ray of light are equal (in natural units). Therefore if we lived in a one-dimensional world, the pressure p exerted by our radiation on the walls of its one-dimensional vessel would equal its energy density ρ. Because we live in a three-dimensional world, and the momenta along the three axes are in equilibrium, we have instead $p = \rho/3$. This is called the equation of state of the radiation. In cosmology, other components of the universe, such as galaxies, have equations of state with some factor other than $1/3$ in front.

As the box expands, the pressure of the radiation on the walls does work W. By conservation of energy, we have

$$dU + dW = 0 \tag{4.6.3}$$

where U is the energy of the radiation. Substituting $U = \rho V$ and $dW = pdV$, we obtain

$$d(\rho V) + pdV = 0 \tag{4.6.4}$$

Applying the product rule, separating variables, and integrating, we find

$$\rho \propto a^{-4}. \tag{4.6.5}$$

Here the exponent 4 is simply the number of spatial dimensions plus one. Exactly the same relation held in the early universe, which was dominated by radiation rather than matter.

Because there is no heat transfer, the entropy is constant. Entropy can be interpreted as a measure of the number of accessible states, and because state-counting doesn't depend on scaling, the occupied modes of vibration stay the same. Thus, the wavelengths simply grow in proportion to a. (A more formal and rigorous version of this argument is called the adiabatic theorem, proved by Born and Fock in 1928.) Although this is a classical argument, we can save some work at this point by appealing to quantum mechanics for a shortcut. Since a photon has an energy $1/\lambda$, we have

$$U \propto \frac{1}{\lambda} \propto \frac{1}{a} \tag{4.6.6}$$

The temperature of the radiation is proportional to the average energy per degree of freedom, so we have

$$T \propto \frac{1}{a} \tag{4.6.7}$$

as well.

Therefore $\rho \propto T^4$. This is equivalent to the Stefan-Boltzmann result, because light rays travel at the fixed speed c, and therefore the flux of radiation is proportional to the energy density. Even though this final proportionality is classical in nature, the value of the proportionality constant depends on Planck's constant, and is quantum-mechanical.

Degenerate matter

The properties of the momentum vector have surprising implications for matter subject to extreme pressure, as in a star that uses up all its fuel for nuclear fusion and collapses. These implications were initially considered too exotic to be taken seriously by astronomers.

An ordinary, smallish star such as our own sun has enough hydrogen to sustain fusion reactions for billions of years, maintaining an equilibrium between its gravity and the pressure of its gases. When the hydrogen is used up, it has to begin fusing heavier elements. This leads to a period of relatively rapid fluctuations in structure. Nuclear fusion proceeds up until the formation of elements as heavy as oxygen ($Z=8$), but the temperatures are not high enough to overcome the strong electrical repulsion of these nuclei to create even heavier ones. Some matter is blown off, but finally nuclear reactions cease and the star collapses under the pull of its own gravity.

To understand what happens in such a collapse, we have to understand the behavior of gases under very high pressures. In general, a surface area A within a gas is subject to collisions in a time t from the n particles occupying the volume

$$V = Avt \tag{4.6.8}$$

where v is the typical velocity of the particles. The resulting pressure is given by

$$P \sim \frac{npv}{V} \tag{4.6.9}$$

where p is the typical momentum.

Nondegenerate gas

In an ordinary gas such as air, the particles are nonrelativistic, so

$$v = p/m \tag{4.6.10}$$

and the thermal energy per particle is

$$\frac{p^2}{2m} \sim kT \tag{4.6.11}$$

so the pressure is

$$P \sim \frac{nkT}{V} \tag{4.6.12}$$

Nonrelativistic, degenerate gas

When a fermionic gas is subject to extreme pressure, the dominant effects creating pressure are quantum-mechanical. Because of the Pauli exclusion principle, the volume available to each particle is $\sim V/n$, so its wavelength is no more than $(\sim V/n)^{1/3}$, leading to

$$p = \frac{h}{\lambda} \sim h\left(\frac{n}{V}\right)^{1/3} \tag{4.6.13}$$

If the speeds of the particles are still nonrelativistic, then $v = p/m$ still holds, so the pressure becomes

$$P \sim \left(\frac{h^2}{m}\right)\left(\frac{n}{V}\right)^{5/3} \tag{4.6.14}$$

Relativistic, degenerate gas

If the compression is strong enough to cause highly relativistic motion for the particles, then $v \approx c$, and the result is

$$P \sim hc\left(\frac{n}{V}\right)^{4/3} \tag{4.6.15}$$

As a star with the mass of our sun collapses, it reaches a point at which the electrons begin to behave as a degenerate gas, and the collapse stops. The resulting object is called a white dwarf. A white dwarf should be an extremely compact body, about the size of the Earth. Because of its small surface area, it should emit very little light. In 1910, before the theoretical predictions had been made, Russell, Pickering, and Fleming discovered that 40 Eridani B had these characteristics. Russell recalled: "*I knew enough about it, even in these paleozoic days, to realize at once that there was an extreme inconsistency between what we would then have called 'possible' values of the surface brightness and density. I must have shown that I was not only puzzled but crestfallen, at this exception to what looked like a very pretty rule of stellar characteristics; but Pickering smiled upon me, and said: 'It is just these exceptions that lead to an advance in our knowledge,' and so the white dwarfs entered the realm of study!*"

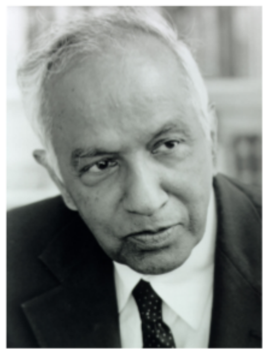

Figure 4.6.1: Subrahmanyan Chandrasekhar (1910-1995)

S. Chandrasekhar showed in that 1930's that there was an upper limit to the mass of a white dwarf. We will recapitulate his calculation briefly in condensed order-of-magnitude form. The pressure at the core of the star is

$$P \sim \rho g r \sim \frac{GM^2}{r^4} \tag{4.6.16}$$

where M is the total mass of the star. The star contains roughly equal numbers of neutrons, protons, and electrons, so $M = Knm$, where m is the mass of the electron, n is the number of electrons, and $K \approx 4000$. For stars near the limit, the electrons are relativistic. Setting the pressure at the core equal to the degeneracy pressure of a relativistic gas, we find that the Chandrasekhar limit is $\sim \left(\frac{hc}{G}\right)^{3/2}(Km)^{-2} = 6M\odot$. A less sloppy calculation gives something more like $1.4M\odot$.

What happens to a star whose mass is above the Chandrasekhar limit? As nuclear fusion reactions flicker out, the core of the star becomes a white dwarf, but once fusion ceases completely this cannot be an equilibrium state. Now consider the nuclear reactions

$$n \rightarrow p + e^- + \bar{v} \tag{4.6.17}$$

$$p + e^- \rightarrow n + v \tag{4.6.18}$$

which happen due to the weak nuclear force. The first of these releases $0.8\ MeV$, and has a half-life of 14 minutes. This explains why free neutrons are not observed in significant numbers in our universe, e.g., in cosmic rays. The second reaction requires an input of $0.8\ MeV$ of energy, so a free hydrogen atom is stable. The white dwarf contains fairly heavy nuclei, not individual protons, but similar considerations would seem to apply. A nucleus can absorb an electron and convert a proton into a neutron, and in this context the process is called electron capture. Ordinarily this process will only occur if the nucleus is neutron-deficient; once it reaches a neutron-to-proton ratio that optimizes its binding energy, neutron capture cannot proceed without a source of energy to make the reaction go. In the environment of a white dwarf, however, there is such a source. The annihilation of an electron opens up a hole in the "Fermi sea." There is now an state into which another electron is allowed to drop without violating the exclusion principle, and the effect cascades upward. In a star with a mass above the Chandrasekhar limit, this process runs to completion, with every proton being converted into a neutron. The result is a neutron star, which is essentially an atomic nucleus (with $Z = 0$) with the mass of a star!

Observational evidence for the existence of neutron stars came in 1967 with the detection by Bell and Hewish at Cambridge of a mysterious radio signal with a period of 1.3373011seconds. The signal's observability was synchronized with the rotation of the earth relative to the stars, rather than with legal clock time or the earth's rotation relative to the sun. This led to the conclusion that its origin was in space rather than on earth, and Bell and Hewish originally dubbed it LGM-1 for "little green men." The discovery of a second signal, from a different direction in the sky, convinced them that it was not actually an artificial signal being generated by aliens. Bell published the observation as an appendix to her PhD thesis, and it was soon interpreted as a signal from a neutron star. Neutron stars can be highly magnetized, and because of this magnetization they may emit a directional beam of electromagnetic radiation that sweeps across the sky once per rotational period — the "lighthouse effect." If the earth lies in the plane of the beam, a periodic signal can be detected, and the star is referred to as a pulsar. It is fairly easy to see that the short period of rotation makes it difficult to explain a pulsar as any kind of less exotic rotating object. In the approximation of Newtonian mechanics, a spherical body of density ρ, rotating with a period $T=\sqrt{\frac{3\pi}{G\rho}}$, has zero apparent gravity at its equator, since gravity is just strong enough to accelerate an object so that it follows a circular trajectory above a fixed point on the surface. In reality, astronomical bodies of planetary size and greater are held together by their own gravity, so we have $T\gtrsim\frac{1}{\sqrt{G\rho}}$ for any body that does not fly apart spontaneously due to its own rotation. In the case of the Bell-Hewish pulsar, this implies $\rho\gtrsim 10^{10}\ kg/m^3$, which is far larger than the density of normal matter, and also $10-100$ times greater than the typical density of a white dwarf near the Chandrasekhar limit.

An upper limit on the mass of a neutron star can be found in a manner entirely analogous to the calculation of the Chandrasekhar limit. The only difference is that the mass of a neutron is much greater than the mass of an electron, and the neutrons are the only particles present, so there is no factor of K. Assuming the more precise result of $1.4M\odot$ for the Chandrasekhar limit rather than our sloppy one, and ignoring the interaction of the neutrons via the strong nuclear force, we can infer an upper limit on the mass of a neutron star:

$$1.4M\odot\left(\frac{Km_e}{m_n}\right)^2\approx 5M\odot \tag{4.6.19}$$

The theoretical uncertainties in such an estimate are fairly large. Tolman, Oppenheimer, and Volkoff originally estimated it in 1939 as $0.7M\odot$, whereas modern estimates are more in the range of $1.5M\odot$ to $3M\odot$. These are significantly lower than our crude estimate of $5M\odot$, mainly because the attractive nature of the strong nuclear force tends to pull the star toward collapse. Unambiguous results are presently impossible because of uncertainties in extrapolating the behavior of the strong force from the regime of ordinary nuclei, where it has been relatively well parametrized, into the exotic environment of a neutron star, where the density is significantly different and no protons are present. There are a variety of effects that may be difficult to anticipate or to calculate. For example, Brown and Bethe found in 1994[1] that it might be possible for the mass limit to be drastically revised because of the process $e^-\to K^-+v_e$, which is impossible in free space due to conservation of energy, but might be possible in a neutron star. Observationally, nearly all neutron stars seem to lie in a surprisingly small range of mass, between $1.3M\odot$ and $1.45M\odot$, but in 2010 a neutron star with a mass of $1.97\pm0.04M\odot$ was discovered, ruling out most neutron-star models that included exotic matter.[2]

For stars with masses above the Tolman-Oppenheimer-Volkoff limit, it seems likely, both on theoretical and observational grounds, we end up with a black hole: an object with an event horizon that cuts its interior off from the rest of the universe.

References

[1] H.A. Bethe and G.E. Brown, "Observational constraints on the maximum neutron star mass," Astrophys. J. 445 (1995) L129. G.E. Brown and H.A. Bethe, "A Scenario for a Large Number of Low-Mass Black Holes in the Galaxy," Astrophys. J. 423 (1994) 659. Both papers are available at adsabs.harvard.edu.

[2] Demorest et al., arxiv.org/abs/1010.5788v1.