10.9: Exercises

Exercise 10.9.1

Find the relationship between the coefficients $\{\alpha_n, \beta_m\}$ in the sine/cosine Fourier series and the coefficients f_n in the complex exponential Fourier series:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{2\pi nx}{a}\right) + \sum_{m=0}^{\infty} \beta_m \cos\left(\frac{2\pi mx}{a}\right) && (10.9.1) \\ &= \sum_{n=-\infty}^{\infty} f_n \exp\left(\frac{2\pi inx}{a}\right). && (10.9.2) \end{aligned}$$

Exercise 10.9.2

Consider the triangular wave

$$f(x) = \begin{cases} -x, & -a/2 \le x < 0, \\ x, & 0 \le x < a/2 \end{cases} \tag{10.9.3}$$

a. Derive the Fourier series expansion.

b. Plot the Fourier series numerically, and show that it converges to the triangular wave as the number of terms increases.

Exercise 10.9.3

A periodic function $f(x)$ (with period a) is written as a complex Fourier series with coefficients $\{f_0, f_{\pm 1}, f_{\pm 2}, \ldots\}$. Determine the relationship(s) between the Fourier coefficients under each of the following scenarios:

a. $f(x)$ is real for all x.

b. $f(x) = f(-x)$ for all x

c. $f(x) = f(-x)^*$ for all x.

Answer

The Fourier coefficients are given by

$$f_n = \frac{1}{a} \int_{-a/2}^{a/2} dx\; e^{-ik_n x}\, f(x), \quad \text{where } k_n = \frac{2\pi n}{a}. \tag{10.9.4}$$

First, consider the case where $f(x)$ is real. Take the complex conjugate of both sides:

$$\begin{aligned} f_n^* &= \frac{1}{a} \int_{-a/2}^{a/2} dx\; \left(e^{-ik_n x}\, f(x)\right)^* && (10.9.5) \\ &= \frac{1}{a} \int_{-a/2}^{a/2} dx\; e^{ik_n x}\, f(x)^* && (10.9.6) \\ &= \frac{1}{a} \int_{-a/2}^{a/2} dx\; e^{ik_n x}\, f(x) && (10.9.7) \\ &= f_{-n}. && (10.9.8) \end{aligned}$$

Hence,

$$f_n = f_{-n}^*. \tag{10.9.9}$$

For the second case, $f(x) = f(-x)$, perform a change of variables $x = -u$ in the Fourier integral:

$$f_n = \frac{1}{a}\int_{-a/2}^{a/2} du\; e^{ik_n u}\, f(u) \tag{10.9.10}$$

$$= f_{-n}. \tag{10.9.11}$$

For $f(x) = f(-x)^*$, the same change of variables gives

$$f_n = f_n^*. \tag{10.9.12}$$

Exercise 10.9.4

Prove the properties of the Fourier transform listed in Section 10.4.

Exercise 10.9.5

Find the Fourier transform of $f(x) = \sin(\kappa x)/x$.

Exercise 10.9.6

Prove that if $f(x)$ is a real function, then its Fourier transform satisfies $F(k) = F(-k)^*$.

Exercise 10.9.7

Prove that

$$\delta(ax) = \frac{1}{a}\,\delta(x), \tag{10.9.13}$$

where a is any nonzero real number.

Answer

From the definition of the delta function as the narrow-peak limit of a Gaussian wavepacket:

$$\delta(ax) = \lim_{\gamma\to 0}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikax}\, e^{-\gamma k^2}. \tag{10.9.14}$$

Perform a change of variables $k = q/a$ and $\gamma = \gamma'\, a^2$:

$$\delta(ax) = \lim_{\gamma'\to 0}\int_{-\infty}^{\infty}\frac{1}{a}\frac{dq}{2\pi}\, e^{iqx}\, e^{-\gamma' q^2} \tag{10.9.15}$$

$$= \frac{1}{a}\delta(x). \tag{10.9.16}$$

Exercise 10.9.8

Calculate

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; x^2\, \delta\left(\sqrt{x^2+y^2} - a\right), \tag{10.9.17}$$

where a is a real number.

Answer

Perform a change of variables from Cartesian coordinates (x, y) to polar coordinates (r, ϕ):

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, x^2\, \delta\left(\sqrt{x^2+y^2}-a\right) = \int_0^{\infty} dr \int_0^{2\pi} r d\phi \cdot r^2 \cos^2\phi\; \delta(r-a) \tag{10.9.18}$$

$$= \left(\int_0^{\infty} dr\, r^3\, \delta(r-a)\right)\left(\int_0^{2\pi} d\phi\, \cos^2\phi\right) \tag{10.9.19}$$

$$= \begin{cases} \pi a^3, & a \geq 0 \\ 0, & a < 0. \end{cases} \tag{10.9.20}$$