3.7: Exercises

Exercise 3.7.1

Consider the step function

$$\Theta(x)=\begin{cases}1, & \text{for } x \geq 0\\ 0, & \text{otherwise.}\end{cases} \tag{3.7.1}$$

Write down an expression for the antiderivative of $\Theta(x)$, and sketch its graph.

Exercise 3.7.2

Show that

$$\int_0^{2\pi} dx\,[\sin(x)]^2 = \int_0^{2\pi} dx\,[\cos(x)]^2 = \pi. \tag{3.7.2}$$

Exercise 3.7.3

Calculate the following definite integrals:

a. $\int_0^{\pi} dx\; x^2 \sin(2x)$
b. $\int_1^{\alpha} dx\; x \ln(x)$
c. $\int_0^{\infty} dx\; e^{-\gamma x}\,\cos(x)$
d. $\int_0^{\infty} dx\; e^{-\gamma x}\, x\cos(x)$
e. $\int_{-\infty}^{\infty} dx\; e^{-\gamma|x|}$
f. $\int_{-\infty}^{\infty} dx\; e^{-|x+1|}\,\sin(x)$

Exercise 3.7.4

By differentiating under the integral, solve

$$\int_0^1 dx\;\frac{x^2-1}{\ln(x)}. \tag{3.7.3}$$

Hint: replace x^2 in the numerator with x^γ.

Answer

Let us define

$$I(\gamma)=\int_0^1 \frac{x^\gamma-1}{\ln(x)}, \tag{3.7.4}$$

so that $I(2)$ is our desired integral. To take the derivative, first note that

$$\frac{d}{d\gamma}(x^\gamma)=\ln(x)\,x^\gamma, \tag{3.7.5}$$

which can be proven using the generalized definition of the power operation. Thus,

$$\frac{d}{d\gamma}I(\gamma) = \int_0^1 \frac{\ln(x)x^\gamma}{\ln(x)} \tag{3.7.6}$$
$$= \int_0^1 x^\gamma \tag{3.7.7}$$
$$= \frac{1}{1+\gamma}. \tag{3.7.8}$$

This can be integrated straightforwardly:

$$I(\gamma) = \int \frac{d\gamma}{1+\gamma} = \ln(1+\gamma) + c, \tag{3.7.9}$$

where c is a constant of integration, which we now have to determine. Referring to the original definition of $I(\gamma)$, observe that $I(0) = \int_0^1 (1-1)/\ln(x) = 0$. This implies that $c = 0$. Therefore, the answer is

$$I(2) = \ln(3). \tag{3.7.10}$$

Exercise 3.7.5

Let $f(x,y)$ be a function that depends on two inputs x and y, and define

$$I(x) = \int_0^x f(x,y)dy. \tag{3.7.11}$$

Prove that

$$\frac{dI}{dx} = f(x,y) + \int_0^x \frac{\partial f}{\partial x}(x,y)\; dy. \tag{3.7.12}$$

Exercise 3.7.6

Consider the ordinary differential equation

$$\frac{dy}{dt} = -\gamma y(t) + f(t), \tag{3.7.13}$$

where $\gamma > 0$ and $f(t)$ is some function of t. The solution can be written in the form

$$y(t) = y(0) + \int_0^t dt'\, e^{-\gamma(t-t')}\, g(t'). \tag{3.7.14}$$

Find the appropriate function g, in terms of f and $y(0)$.

Answer

We are provided with the following ansatz for the solution to the differential equation:

$$y(t) = y(0) + \int_0^t dt' e^{-\gamma(t-t')} g(t'). \tag{3.7.15}$$

First, note that when $t = 0$, the integral's range shrinks to zero, so the result is $y(0)$, as expected. In order to determine the appropriate function g, we perform a derivative in t. The tricky part is that t appears in two places: in the upper range of the integral, as well as in the integrand. So when we take the derivative, there should be two distinct terms (see problem 3.7.5):

$$\frac{dy}{dt} = \left[e^{-\gamma(t-t')}\, g(t')\right]_{t'=t} + \int_0^t dt'\, (-\gamma)\, e^{-\gamma(t-t')}\, g(t') \tag{3.7.16}$$
$$= g(t) - \gamma[y(t) - y(0)]. \tag{3.7.17}$$

In the last step, we again made use of the ansatz for $y(t)$. Finally, comparing this with the original differential equation for $y(t)$, we find that

$$g(t)-\gamma[y(t)-y(0)]=-\gamma y(t)+f(t) \quad \Rightarrow \quad g(t)=f(t)-\gamma y(0). \tag{3.7.18}$$

Hence, the solution to the differential equation is

$$\begin{aligned} y(t) &= y(0)+\int_0^t dt'\, e^{-\gamma(t-t')}\,[f(t')-\gamma y(0)] && (3.7.19)\\ &= y(0)\,e^{-\gamma t}+\int_0^t dt'\, e^{-\gamma(t-t')}\, f(t'). && (3.7.20) \end{aligned}$$