3.1: Basic Properties of Definite Integrals

The value of a definite integral depends only on the integrand, and the two integration bounds. The variable which is integrated over is a **dummy variable**, which means that changing the symbol does not affect the value of the overall expression:

$$\int_a^b dx\ f(x) = \int_a^b dy\ f(y). \tag{3.1.1}$$

Since the value of the integral does not depend on the dummy variable, it is nonsensical to write something like

$$\frac{d}{dx}\left[\int_a^b dx\ f(x)\right]. \quad \text{(Nonsense expression!)} \tag{3.1.2}$$

Since an integral is defined as the limiting form of a sum, it can be algebraically manipulated in the same way as a summation expression. For instance, an integral of a linear combination is equal to a linear combination of two integrals *with the same bounds*:

$$\int_a^b dx\ \Big[c_1\, f_1(x) + c_2\, f_2(x)\Big] = c_1 \int_a^b dx\ f_1(x)\ +\ c_2 \int_a^b dx\ f_2(x). \tag{3.1.3}$$

This is analogous to how the summation of a linear combination is equal to the linear combination of separate summations:

$$\sum_{n=p}^{q} \Big[c_1 A_n + c_2 B_n\Big] = c_1 \sum_{n=p}^{q} A_n + c_2 \sum_{n=p}^{q} B_n. \tag{3.1.4}$$

For a similar reason, multiple integrals can be manipulated like multiple summations. If we have a double integral where the integrals have *independent* bounds, we can swap the order of the integrals:

$$\int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2\ \ f(x_1, x_2) = \int_{a_2}^{b_2} dx_2 \int_{a_1}^{b_1} dx_1\ \ f(x_1, x_2). \tag{3.1.5}$$

This is analogous to how we can swap the order of two independent summations. Note, however, that this manipulation is invalid if the integration bounds are not independent. For instance, if the upper or lower bound of the inner integral depends on the integration variable of the outer integral, we can't swap the two integrals:

$$\int_{a_1}^{b_1} dx_1 \int_{a_1}^{x_1} dx_2\ \ f(x_1, x_2) \neq \int_{a_1}^{x_1} dx_2 \int_{a_1}^{b_1} dx_1\ \ f(x_1, x_2). \quad \text{(Nonsense expression!)} \tag{3.1.6}$$

Note

Note that the expression on the right is nonsensical: x_1 is meant to be a dummy variable, yet it exists outside of any integration sign.