4.6: Trajectories in the Complex Plane

If we have a function $z(t)$ which takes a real input t and outputs a complex number z, it is often useful to plot a curve in the complex plane called the "parametric trajectory" of z. Each point on this curve indicates the value of z for a particular value of t. We will give a few examples below.

First, consider

$$z(t)=e^{i\omega t}, \quad \omega\in\mathbb{R}. \tag{4.6.1}$$

The trajectory is a circle in the complex plane, centered at the origin and with radius 1:

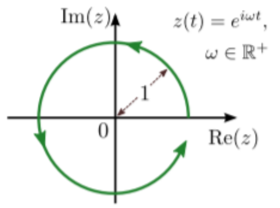

Figure 4.6.1

To see why, observe that the function has the form $z(t)=r(t)\,e^{i\theta(t)}$, which has magnitude $r(t)=1$, and argument $\theta(t)=\omega t$ varying proportionally with t. If ω is positive, the argument increases with t, so the trajectory is counter-clockwise. If ω is negative, the trajectory is clockwise.

Next, consider

$$z(t)=e^{(\gamma+i\omega)t}, \tag{4.6.2}$$

where $\gamma,\omega\in\mathbb{R}$. For $\gamma=0$, this reduces to the previous example. For $\gamma\neq 0$, the trajectory is a spiral:

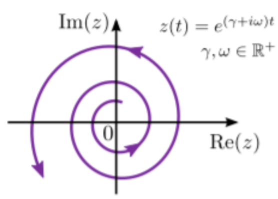

Figure 4.6.2

To see this, we again observe that this function can be written in the form

$$z(t)=r(t)\;e^{i\theta(t)}, \tag{4.6.3}$$

where $r(t)=e^{\gamma t}$ and $\theta=\omega t$. The argument varies proportionally with t, so the trajectory loops around the origin. The magnitude increases with t if γ is positive, and decreases with t if γ is negative. Thus, for instance, if γ and ω are both positive, then the trajectory is an anticlockwise spiral moving outwards from the origin. Try checking how the trajectory behaves when the signs of γ and/or ω are flipping.

Finally, consider

$$z(t)=\frac{1}{\alpha t+\beta}, \quad \alpha,\beta\in\mathbb{C}. \tag{4.6.4}$$

This trajectory is a circle which passes through the origin, as shown below:

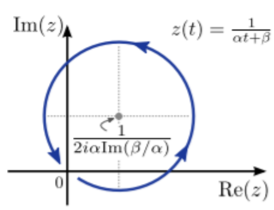

Figure 4.6.3

Showing this requires a bit of ingenuity, and is left as an exercise. This is an example of something called a Möbius transformation.