5.4: Stating the Solution in Terms of Initial Conditions

The general solution for the complex damped harmonic oscillator equation, Eq. (5.3.3), contains two undetermined parameters which are the complex amplitudes of the "clockwise" and "counterclockwise" complex oscillations:

$$z(t) = \psi_+ e^{-i\omega_+ t} + \psi_- e^{-i\omega_- t}, \quad \text{where } \omega_\pm = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{5.4.1}$$

However, mechanics problems are often expressed in terms of an **initial value problem**, specifying the state of the system at some initial time $t = 0$. In other words, given $z(0) \equiv x_0$ and $\dot{z}(0) \equiv v_0$, what is $z(t)$ in terms of x_0 and v_0?

We can solve the initial-value problem by finding $z(0)$ and $\dot{z}(0)$ in terms of the above general solution for $z(t)$. The results are

$$z(0) = \psi_+ + \psi_- = x_0 \tag{5.4.2}$$

$$\dot{z}(0) = -i\omega_+\psi_+ - i\omega_-\psi_- = v_0. \tag{5.4.3}$$

These two equations can be combined into a 2x2 matrix equation:

$$\begin{bmatrix} 1 & 1 \\ -i\omega_+ & -i\omega_- \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}. \tag{5.4.4}$$

So long as the system is not at the critical point (i.e., $\omega_+ \neq \omega_-$), the matrix is non-singular, and we can invert it to obtain $\psi_\pm$:

$$\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \frac{1}{i(\omega_+ - \omega_-)} \begin{bmatrix} -i\omega_- x_0 - v_0 \\ i\omega_+ x_0 + v_0 \end{bmatrix}. \tag{5.4.5}$$

We can plug these coefficients back into the general solution. After some algebra, the result simplifies to

$$z(t) = e^{-\gamma t}\left[x_0 \cos(\Omega t) + \frac{\gamma x_0 + v_0}{\Omega}\sin(\Omega t)\right], \quad \text{where } \Omega \equiv \sqrt{\omega_0^2 - \gamma^2}. \tag{5.4.6}$$

For the under-damped case, Ω is real, and this solution is consistent with the one found in Section 5.3, except that it is now explicitly expressed in terms our initial conditions x_0 and v_0. As for the over-damped case, we can perform the replacement

$$\Omega \rightarrow i\Gamma = i\sqrt{\gamma^2 - \omega_0^2}. \tag{5.4.7}$$

Then, using the relationships between trigonometric and hyperbolic functions discussed in Section 4.5, the solution can be re-written as

$$z(t) = e^{-\gamma t}\left[x_0 \cosh(\Gamma t) + \frac{\gamma x_0 + v_0}{i\Gamma}\, i\sinh(\Gamma t)\right] \tag{5.4.8}$$

$$= \left(\frac{x_0}{2} + \frac{\gamma x_0 + v_0}{2\Gamma}\right) e^{-(\gamma - \Gamma)t} + \left(\frac{x_0}{2} - \frac{\gamma x_0 + v_0}{2\Gamma}\right) e^{-(\gamma + \Gamma)t}, \tag{5.4.9}$$

which is consistent with the solution found in Section 5.3.

In either case, so long as we plug in real values for x_0 and v_0, the solution is guaranteed to be real for all t. That's to be expected, since the real solution is also one of the specific solutions for the complex harmonic oscillator equation.