1.4: Non-Natural Powers

Having defined the exponential and logarithm, we have the tools needed to address the issue raised earlier, i.e. how to define non-natural power operations. First, observe that

$$\text{For }\; y \in \mathbb{N}, \quad \ln(x^y) = \underbrace{\ln(x)\ln(x)\cdots\ln(x)}_{y\text{ times}} = y\ln(x). \tag{1.4.1}$$

Hence, by applying the exponential to each side of the above equation,

$$x^y = \exp[y\ln(x)] \quad \text{for }\; y \in \mathbb{N}. \tag{1.4.2}$$

We can generalize the above equation so that it holds for any positive x and real y, not just $y \in \mathbb{N}$. In other words, we treat this as our *definition* of the power operation for non-natural powers:

$$x^y \equiv \exp[y\ln(x)] \quad \text{for }\; x \in \mathbb{R}^+,\; y \notin \mathbb{N}. \tag{1.4.3}$$

By this definition, the power operation always gives a positive result. And for $y \in \mathbb{N}$, the result of the formula is consistent with the standard definition based on multiplying x by itself y times.

This generalization of the power operation leads to several important consequences:

1. The zeroth power yield unity:

$$x^0 = 1 \;\text{ for }\; x \in \mathbb{R}^+. \tag{1.4.4}$$

2. Negative powers are reciprocals:

$$x^{-y} = \exp[-y\ln(x)] = \exp[-\ln(x^y)] = \frac{1}{x^y}. \tag{1.4.5}$$

3. The output of the exponential function is equivalent to a power operation:

$$\exp(y) = e^y \tag{1.4.6}$$

   where

$$e \equiv \exp(1) = 2.718281828459\ldots \tag{1.4.7}$$

   (This follows by plugging in $x = e$ and using the fact that $\ln(e) = 1$.)

4. For $x \le 0$, the meaning of x^y for non-natural y is ill-defined, since the logarithm does not accept negative inputs.