2.4: Partial Derivatives

So far, we have focused on functions which take a single input. Functions can also take multiple inputs; for instance, a function $f(x,y)$ maps two input numbers, x and y, and outputs a number. In general, the inputs are allowed to vary independently of one another. The **partial derivative** of such a function is its derivative with respect to one of its inputs, keeping the others fixed. For example,

$$f(x,y) = \sin(2x-3y^2) \tag{2.4.1}$$

has partial derivatives

$$\frac{\partial f}{\partial x} = 2\cos(2x-3y^2), \tag{2.4.2}$$

$$\frac{\partial f}{\partial y} = -6\cos(2x-3y^2). \tag{2.4.3}$$

Change of variables

We saw in Section 2.1 that single-variable functions obey a derivative composition rule,

$$\frac{d}{dx}\,f\big(g(x)\big) = g'(x)\,f'\big(g(x)\big). \tag{2.4.4}$$

This composition rule has a important generalization for partial derivatives, which is related to the physical concept of a **change of coordinates**. Suppose a function $f(x,y)$ takes two inputs x and y, and we wish to express them using a different coordinate system denoted by u and v. In general, each coordinate in the old system depends on *both* coordinates in the new system:

$$x = x(u,v), \quad y = y(u,v). \tag{2.4.5}$$

Expressed in the new coordinates, the function is

$$F(u,v) \equiv f\big(x(u,v), y(u,v)\big). \tag{2.4.6}$$

It can be shown that the transformed function's partial derivatives obey the composition rule

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \tag{2.4.7}$$

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \tag{2.4.8}$$

On the right-hand side of these equations, the partial derivatives are to be expressed in terms of the new coordinates (u,v). For example,

$$\frac{\partial f}{\partial x} = \left.\frac{\partial f}{\partial x}\right|_{x=x(u,v),\; y=y(u,v)} \tag{2.4.9}$$

The generalization of this rule to more than two inputs is straightforward. For a function $f(x_1,\ldots,x_N)$, a change of coordinates $x_i = x_i(u_1,\ldots,u_N)$ involves the composition

$$F(u_1,\ldots,u_N) = f\big(x_1(u_1,\ldots,u_N),\ldots\big), \quad \frac{\partial F}{\partial u_i} = \sum_{j=1}^{N}\frac{\partial x_j}{\partial u_i}\frac{\partial f}{\partial x_j}. \tag{2.4.10}$$

Example 2.4.1

In two dimensions, Cartesian and polar coordinates are related by

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{2.4.11}$$

Given a function $f(x,y)$, we can re-write it in polar coordinates as $F(r,\theta)$. The partial derivatives are related by

$$\begin{aligned} \frac{\partial F}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta. \qquad (2.4.12) \\ \frac{\partial F}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial f}{\partial x} r\,\sin\theta + \frac{\partial f}{\partial y} r\cos\theta. \qquad (2.4.13) \end{aligned}$$

Partial differential equations

A **partial differential equation** is a differential equation involving multiple partial derivatives (as opposed to an ordinary differential equation, which involves derivatives with respect to a single variable). An example of a partial differential equation encountered in physics is Laplace's equation,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \qquad (2.4.14)$$

which describes the electrostatic potential $\Phi(x, y, z)$ at position (x, y, z), in the absence of any electric charges.

Partial differential equations are considerably harder to solve than ordinary differential equations. In particular, their boundary conditions are more complicated to specify: whereas each boundary condition for an ordinary differential equation consists of a single number (e.g., the value of $f(x)$ at some point $x = x_0$), each boundary condition for a partial differential equation consists of a *function* (e.g., the values of $\Phi(x, y, z)$ along some curve $g(x, y, z) = 0$).