3.6: Differentiating Under the Integral Sign

In the previous section, we noted that if an integrand contains a parameter (denoted γ) which is independent of the integration variable (denoted x), then the definite integral can itself be regarded as a function of γ. It can then be shown that taking the derivative of the definite integral with respect to γ is equivalent to taking the *partial derivative* of the integrand:

$$\frac{d}{d\gamma}\left[\int_a^b dx\; f(x,\gamma)\right] = \int_a^b dx\; \frac{\partial f}{\partial \gamma}(x,\gamma). \tag{3.6.1}$$

This operation, called **differentiating under the integral sign**, was first used by Leibniz, one of the inventors of calculus. It can be applied as a technique for solving integrals, popularized by Richard Feynman in his book *Surely You're Joking, Mr. Feynman!*.

Here is the method. Given a definite integral I_0:

1. Come up with a way to generalize the integrand, by introducing a parameter γ, such that the generalized integral becomes a function $I(\gamma)$ which reduces to the original integral I_0 for a particular parameter value, say $\gamma = \gamma_0$.
2. Differentiate under the integral sign. If you have chosen the generalization right, the resulting integral will be easier to solve, so...
3. Solve the integral to obtain $I'(\gamma)$.
4. Integrate I' over γ to obtain the desired integral $I(\gamma)$, and evaluate it at γ_0 to obtain the desired integral I_0.

An example is helpful for demonstrating this procedure. Consider the integral

$$\int_0^\infty dx\; \frac{\sin(x)}{x}. \tag{3.6.2}$$

First, (i) we generalize the integral as follows (we'll soon see why):

$$I(\gamma) = \int_0^\infty dx\; \frac{\sin(x)}{x}\, e^{-\gamma x}. \tag{3.6.3}$$

The desired integral is $I(0)$. Next, (ii) differentiating under the integral gives

$$I'(\gamma) = -\int_0^\infty dx\;\; \sin(x)\, e^{-\gamma x}. \tag{3.6.4}$$

Taking the partial derivative of the integrand with respect to γ brought down a factor of $-x$, cancelling out the troublesome denominator. Now, (iii) we solve the new integral, which can be done by integrating by parts twice:

$$\begin{aligned} I'(\gamma) &= \left[\cos(x)\, e^{-\gamma x}\right]_0^\infty + \gamma \int_0^\infty dx\;\; \cos(x)\, e^{-\gamma x} && (3.6.5)\\ &= -1 + \gamma\left[\sin(x)\, e^{-\gamma x}\right]_0^\infty + \gamma^2 \int_0^\infty dx\;\; \sin(x)\, e^{-\gamma x} && (3.6.6)\\ &= -1 - \gamma^2 I'(\gamma). && (3.6.7)\end{aligned}$$

Hence,

$$I'(\gamma) = -\frac{1}{1+\gamma^2}. \tag{3.6.8}$$

Finally, (iv) we need to integrate this over γ. But we already saw how to do this particular integral in Section 3.4, and the result is

$$I(\gamma) = A - \tan^{-1}(\gamma), \tag{3.6.9}$$

where A is a constant of integration. When $\gamma \to \infty$, the integral must vanish, which implies that $A = \tan^{-1}(+\infty) = \pi/2$. Finally, we arrive at the result

$$\int_0^\infty dx\; \frac{\sin(x)}{x} = I(0) = \frac{\pi}{2}. \tag{3.6.10}$$

When we discuss contour integration in Chapter 9, we will see a more straightforward way to do this integral.

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