10.3: Fourier Transforms for Time-Domain Functions

So far, we have been dealing with functions of a spatial coordinate x. Of course, mathematical relations don't care about what kind of physical variable we are dealing with, so the same equations could be applied to functions of time t. However, there is a important difference in *convention*. When dealing with functions of the time coordinate t, it is customary to use a different sign convention in the Fourier relations!

The Fourier relations for a function of time, $f(t)$, are:

Definition: Fourier relations

$$\left\{\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} dt\; e^{i\omega t}\, f(t) \\ f(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, F(\omega). \end{aligned}\right. \tag{10.3.1}$$

Compared to the Fourier relations previously given in Eq. (10.2.9), the signs of the $\pm i\omega t$ exponents are flipped.

There's a good reason for this difference in sign convention: it arises from the need to describe propagating waves, which vary with both space *and* time. As discussed in Chapter 5, a propagating plane wave can be described by a wavefunction of the form

$$f(x,t) = A e^{i(kx-\omega t)}, \tag{10.3.2}$$

where k is the wave-number and ω is the angular frequency. We write the plane wave function this way so that positive k indicates forward propagation in space (i.e., in the $+x$ direction), and positive ω indicates forward propagation in time (i.e., in the $+t$ direction). This requires the kx and ωt terms in the exponent to have opposite signs. Thus, when t increases by some amount, a corresponding *increase* in x leaves the exponent unchanged.

As we have seen, the inverse Fourier transform relation describes how a wave-form is broken up into a superposition of elementary waves. For a wavefunction $f(x,t)$, the superposition is given in terms of plane waves:

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\;\; e^{i(kx-\omega t)}\, F(k,\omega). \tag{10.3.3}$$

To be consistent with this, we need to treat space and time variables with oppositely-signed exponents:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, F(k) \tag{10.3.4}$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, F(\omega). \tag{10.3.5}$$

The other equations follow similarly.