6.3: Complex Solutions to the Wave Equation

It is much easier to deal with the wave equation if we promote it into a complex PDE by letting $f(x,t)$ take on complex values. However, x and t will remain real. We will also take the wave speed v to be real, for now.

From any complex solution to the wave equation, we can take the real part to get a solution to the real PDE, thanks to linearity (see Section 4.1):

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\mathrm{Re}\left[f(x,t)\right]=\mathrm{Re}\left[\left(\frac{\partial^2}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)f(x,t)\right]=0. \tag{6.3.1}$$

There exists a nice set of complex solutions to the wave equation, called **complex travelling waves**, which take the form

$$f(x,t)=A\,e^{i(kx-\omega t)}\quad\text{where}\;\;\left|\frac{\omega}{k}\right|=v. \tag{6.3.2}$$

It can be verified by direct substitution that this satisfies the PDE. The complex constant A is called the **complex amplitude** of the wave. Consider what happens if we take the real part of the above solution:

$$\begin{aligned}\mathrm{Re}\Big\{A\,e^{i(kx-\omega t)}\Big\}&=\mathrm{Re}\Big\{|A|\,e^{i\arg[A]}\,e^{i(kx-\omega t)}\Big\} &(6.3.3)\\ &=\big|A\big|\,\mathrm{Re}\Big\{e^{i\arg[A]}\,e^{i(kx-\omega t)}\Big\} &(6.3.4)\\ &=\big|A\big|\;\cos\big(kx-\omega t+\arg[A]\big) &(6.3.5)\end{aligned}$$

Comparing this to Eq. (6.2.1), we see that $|A|$ serves as the amplitude of the real wave, while $\arg(A)$ serves as the phase factor ϕ. Mathematically, the complex solution is more succinct than the real solution: a single complex parameter A combines the roles of two parameters in the real solution.

The complex representation also makes wave superpositions easier to handle. As an example, consider the superposition of two counter-propagating waves of equal amplitude and frequency, with arbitrary phases. Using complex traveling waves, we can calculate the superposition with a few lines of algebra:

$$\begin{aligned}f(x,t)&=\big|A\big|\,e^{i(kx-\omega t+\phi_1)}+\big|A\big|\,e^{i(-kx-\omega t+\phi_2)} &(6.3.6)\\ &=\big|A\big|\;\Big(e^{i(kx+\phi_1)}+e^{-i(kx-\phi_2)}\Big)\;e^{-i\omega t} &(6.3.7)\\ &=\big|A\big|\;\Big(e^{i[kx+(\phi_1-\phi_2)/2]}+e^{-i[kx+(\phi_1-\phi_2)/2]}\Big)\;e^{i(\phi_1+\phi_2)/2}\,e^{-i\omega t} &(6.3.8)\\ &=2\big|A\big|\;\cos[kx+(\phi_1-\phi_2)/2]\;e^{-i[\omega t-(\phi_1+\phi_2)/2]} &(6.3.9)\end{aligned}$$

Taking the real part yields Eq. (6.2.5), without the need for tedious manipulations of trigonometric formulas.