1.5: 1.5 Trigonometric Functions

Another extremely important group of functions are the fundamental trignonometric functions $\sin$, $\cos$, and $\tan$. These can be defined in terms of the geometric ratios of the sides of right-angled triangles, as shown below:

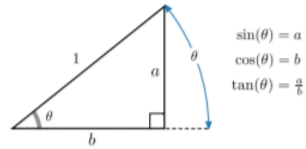

Figure 1.5.1

If we use this basic definition, the domain is $\theta \in [0,\, \pi/2)$, where the input angle θ is given in radians.

We can generalize the definition using the following scheme, which allows for negative values of a and/or b:

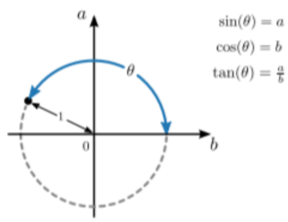

Figure 1.5.2

With this, the domain is extended to $\theta \in [0,\, 2\pi)$. We can further extend the domain to all real numbers, $\theta \in \mathbb{R}$, by treating input values modulo 2π as equivalent; in other words, $f(\theta + 2\pi) = f(\theta)$. With this generalization, the trigonometric functions become many-to-one functions.

According to the Pythagorean theorem,

$$\big[\sin(\theta)\big]^2 + \big[\cos(\theta)\big]^2 = 1. \tag{1.5.1}$$

Using this, we can go on to prove a variety of identities, like the addition identities

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \tag{1.5.2}$$

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \tag{1.5.3}$$

As you may recall, the trigonometric proofs for these identities involve drawing complicated triangle diagrams, cleverly applying the Pythagorean formula, etc. There are two problems with such proofs: (i) they require some ingenuity in the construction of the triangle diagrams, and (ii) it may not be obvious whether the proofs work if the angles lie outside $[0, \pi/2)$.

Happily, there is a solution to both problems. As we'll soon see, such trigonometric identities can be proven algebraically by using complex numbers.