1.8: Exercises

Exercise 1.8.1

An alternative definition of the exponential function is the limiting expression

$$\exp(x) \equiv \lim_{n\rightarrow\infty} \left(1+\frac{x}{n}\right)^n. \tag{1.8.1}$$

Prove that this is equivalent to the definition in terms of an infinite series,

$$\exp(x) \equiv 1+\sum_{n=1}^\infty \frac{x^n}{n!}. \tag{1.8.2}$$

Exercise 1.8.2

Prove that

$$\exp(x+y) = \exp(x)\,\exp(y) \tag{1.8.3}$$

using the definition of the exponential as an infinite series. Your proof must avoid using the concept of "raising to the power" of a non-natural number; this is to avoid circular logic, since this feature of the exponential function can be used in the generalized definition of the power operation (Section 1.4).

Answer

To prove that $\exp(x+y) = \exp(x)\,\exp(y)$, we employ the infinite series formula

$$\exp(x) = \sum_{n=0}^\infty \frac{x^n}{n!}. \tag{1.8.4}$$

Here, for notational convenience, we let the sum start from $n=0$, so that the leading term 1 in the definition of the exponential is grouped with the rest of the sum as its first term. This relies on the understanding that $0! \equiv 1$, and that $x^0 = 1$ (the latter is consistent with the generalized definition of the power operation; but to avoid circular logic, treat this as the *definition* of x^0 just for the sake of this proof). We begin by substituting the series formula into the right-hand side of our target equation:

$$\exp(x)\,\exp(x) = \left(\sum_{n=0}^\infty \frac{x^n}{n!}\right)\;\left(\sum_{m=0}^\infty \frac{y^m}{m!}\right). \tag{1.8.5}$$

Note that we use the symbol n for the first sum, and the symbol m for the second sum; n and m are bound variables, whose terms run over the values specified by the summation signs. The actual choice of symbol used in either sum is unimportant, except that *we must not use the same symbol for both sums*, because the two variables belong to distinct sums. In other words:

$$\exp(x)\,\exp(x) \ne \left(\sum_{n=0}^\infty \frac{x^n}{n!}\right)\;\left(\sum_{n=0}^\infty \frac{y^n}{n!}\right). \quad \text{(Nonsense expression!)} \tag{1.8.6}$$

Next, we make use of the fact that the product of two series can be written as a double sum:

$$\exp(x)\,\exp(x) = \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{x^n}{n!}\,\frac{y^m}{m!}. \tag{1.8.7}$$

Here, we are summing over all possible pair-wise combinations of n and m, which is precisely what happens when one expands the product of two series according to the usual rules of algebra. The next step is to perform a *change of variables* on m and n. In the above expression, we are summing over all non-negative integer m and n; however, the bound variable n can be re-expressed in terms of a newly-defined variable,

$$N = m + n. \tag{1.8.8}$$

In the original double sum, n and m both run from 0 to $+\infty$, so it follows that their sum N runs from 0 to $+\infty$. For each given value of N, we can write $n = N - m$, and moreover the allowed values of m would only go from 0 to N (it can't exceed N, otherwise n would be negative). In this way, the double sum is converted to

$$\exp(x)\exp(x) = \sum_{N=0}^{\infty}\sum_{m=0}^{N} \frac{x^{N-m}}{(N-m)!}\frac{y^m}{m!} \tag{1.8.9}$$

Note that after this change of variables, the two summation signs are no longer interchangeable. In the $\sum_{m=0}^{N}$ sign, the variable N appears in the upper limit, so this needs to be written to the right of $\sum_{N=0}^{\infty}$. One sum is thus "encapsulated" inside the other; we could write the algebraic expression more rigorously like this:

$$\exp(x)\exp(x) = \sum_{N=0}^{\infty}\left(\sum_{m=0}^{N} \frac{x^{N-m}}{(N-m)!}\frac{y^m}{m!}\right). \tag{1.8.10}$$

Finally, we use the binomial theorem to simplify the inner sum:

$$\exp(x)\exp(x) = \sum_{N=0}^{\infty}\frac{(x+y)^N}{N!}, \quad \text{since} \quad (x+y)^N = \sum_{m=0}^{N}\frac{N!}{m!(N-m)!}x^{N-m}\,y^m. \tag{1.8.11}$$

Referring again to the series definition of the exponential, we obtain the desired result:

$$\exp(x)\exp(x) = \exp(x+y) \tag{1.8.12}$$

Exercise 1.8.3

One of the most important features of the exponential function $\exp(x)$ is that it becomes large *extremely* quickly with increasing x. To illustrate this behavior, consider the graph shown in Section 1.2, which plots the exponential up to $x = 4$. On your screen or page, the height of the graph should be around 4 cm. Suppose we keep to the same resolution, and plot up to $x = 10$; how high would the graph be? What if we plot up to $x = 20$?

Exercise 1.8.4

Prove that $\exp(x) = e^x$.

Answer

The definition of non-natural powers is

$$a^b = \exp[b\ln(a)]. \tag{1.8.13}$$

Let $a = \exp(1) = e$ and $b = x$. Then

$${}^{x} = \exp\Big[x\ln\Big(\exp(1)\Big)\Big]. \tag{1.8.14}$$

Since the logarithm is the inverse of the exponential function, $\ln(\exp(1)) = 1$. Hence,

$$e^x = \exp(x). \tag{1.8.15}$$

Exercise 1.8.5

A "non-natural" logarithm of base c is defined as

$$\log_c(x) = y \quad \text{where} \quad c^y = x. \tag{1.8.16}$$

Derive an expression for the non-natural logarithm in terms of the natural logarithm.

Exercise 1.8.6

Prove, using trigonometry, that

$$\sin(\theta_1+\theta_2)=\sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2). \tag{1.8.17}$$

You may assume that $\theta_1, \theta_2 \in [0, \pi/2]$.

Exercise 1.8.7

Prove that

$$\cos(3x) = 4[\cos(x)]^3 - 3\cos(x) \tag{1.8.18}$$

$$\sin(3x) = 3\sin(x) - 4[\sin(x)]^3. \tag{1.8.19}$$