10.4: Basic Properties of the Fourier Transform

The Fourier transform has several important properties. These can all be derived from the definition of the Fourier transform; the proofs are left as exercises.

1. The Fourier transform is linear: if we have two functions $f(x)$ and $g(x)$, whose Fourier transforms are $F(k)$ and $G(k)$ respectively, then for any constants $a, b \in \mathbb{C}$,

$$a f(x) + b g(x) \quad \stackrel{\mathrm{FT}}{\longrightarrow} \quad a F(k) + b G(k). \tag{10.4.1}$$

2. Performing a coordinate translation on a function causes its Fourier transform to be multiplied by a phase factor:

$$f(x+b) \quad \stackrel{\mathrm{FT}}{\longrightarrow} \quad e^{ikb}\, F(k). \tag{10.4.2}$$

   As a consequence, translations leave the Fourier spectrum $|F(k)|^2$ unchanged.

3. If the Fourier transform of $f(x)$ is $F(k)$, then

$$f^*(x) \quad \stackrel{\mathrm{FT}}{\longrightarrow} \quad F^*(-k). \tag{10.4.3}$$

   As a consequence, the Fourier transform of a real function must satisfy the symmetry relation $F(k) = F^*(-k)$, meaning that the Fourier spectrum is symmetric about the origin in k-space: $\big|\,F(k)\,\big|^2 = \big|\,F(-k)\,\big|^2$.

4. When you take the derivative of a function, that is equivalent to multiplying its Fourier transform by a factor of ik:

$$\frac{d}{dx} f(x) \quad \stackrel{\mathrm{FT}}{\longrightarrow} \quad ikF(k). \tag{10.4.4}$$

   For functions of time, because of the difference in sign convention discussed in Section 10.3, there is an extra minus sign:

$$\frac{d}{dt} f(t) \quad \stackrel{\mathrm{FT}}{\longrightarrow} \quad -i\omega F(\omega). \tag{10.4.5}$$