# 4.7: Why Complex Numbers?

Here are some questions that might have occurred to you:

- If we extend the concept of numbers to complex numbers, why stop here? Why not extend the concept further, and formulate other abstract number systems that are even more complicated than complex numbers?
- Integers and real numbers have intuitive connections to the phenomena we experience in everyday life, such as the counting of discrete objects, or measuring lengths and weights. Complex numbers, however, seem like completely abstract concepts. Why should we study them?

As we have seen, complex numbers are appealing mathematical objects because they can be manipulated via the same rules of algebra as real numbers. We can add, subtract, multiply, and divide them (apart from division by zero), without running into any logical inconsistencies. One limitation is that complex numbers have no ordering (Section 4.4), so complex algebra only involves equations, not inequality relations.

One very important feature possessed by complex numbers and not real numbers is that the complex numbers are *algebraically closed*. This means that all complex polynomial equations have solutions in $\mathbb{C}$. The set of real numbers, $\mathbb{R}$, lacks this property: there are certain real algebraic equations, like $x^2+1=0$ , which have no solution in $\mathbb{R}$. The "closure" property of $\mathbb{C}$ is called the Fundamental Theorem of Algebra, which gives an idea of its importance. As a consequence, $\mathbb{C}$ cannot be generalized to a more complicated number system via the same route used to extend $\mathbb{R}$ into $\mathbb{C}$.

There do exist number systems more complicated than the complex numbers, which are formulated not by algebraic extension but by discarding one or more of the usual rules of algebra. The quaternions are a system of four-component numbers obeying an algebra that is *non-commutative* (i.e., $ab = ba$ is not generally true). The octonions are an even more complicated system of eight-component numbers which are not only non-commutative but also non-associative (i.e., $(ab)c = a(bc)$ is not generally true). These and other still-more-complicated number systems have a few applications in physics and other fields, but are overall much less important than $\mathbb{C}$.

One big reason that complex numbers have proven to be so important and useful is that it's easy to formulate a version of calculus for them. The study of smooth complex functions, and their derivatives and integrals, is called **complex analysis**. We will discuss this subject extensively later in the course. We shall see that complex analysis has important implications for the *real* calculus; for example, many real integrals can be easily solved by first generalizing them into complex integrals. By contrast, since quaternions and octonions are not commutative, the concept of "derivative" is tricky to define for these number systems, making it harder to formulate a useful calculus with them.