4.1: Complex Algebra

Any **complex number** z can be written as

$$z = x + iy, \tag{4.1.1}$$

where x and y are real numbers that are respectively called the **real part** and the **imaginary part** of z. The real and imaginary parts are also denoted as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, where Re and Im can be regarded as functions mapping a complex number to a real number.

The set of complex numbers is denoted by $\mathbb{C}$. We can define algebraic operations on complex numbers (addition, subtraction, products, etc.) by following the usual rules of algebra and setting $i^2 = -1$ whenever it shows up.

Example 4.1.1

Let $z = x + iy$, where $x, y \in \mathbb{R}$.
What are the real and imaginary parts of z^2?

$$\begin{align} z^2 &= (x+iy)^2 \tag{4.1.2}\\ &= x^2 + 2x(iy) + (iy)^2 \tag{4.1.3}\\ &= x^2 - y^2 + 2ixy \tag{4.1.4}\end{align}$$

Hence,

$$\mathrm{Re}(z^2) = x^2 - y^2, \quad \mathrm{Im}(z^2) = 2xy. \tag{4.1.5}$$

We can also perform power operations on complex numbers, with one caveat: for now, we'll only consider *integer* powers like z^2 or $z^{-1} = 1/z$. Non-integer powers, such as $z^{1/3}$, introduce vexatious complications which we'll postpone for now (we will figure out how to deal with them when studying branch points and branch cuts in Chapter 7).

Another useful fact: real coefficients (and *only* real coefficients) can be freely moved into or out of $\mathrm{Re}(\cdots)$ and $\mathrm{Im}(\cdots)$ operations:

$$\left\{\begin{array}{l}\mathrm{Re}(\alpha z + \beta z') = \alpha\,\mathrm{Re}(z) + \beta\,\mathrm{Re}(z') \\ \mathrm{Im}(\alpha z + \beta z') = \alpha\,\mathrm{Im}(z) + \beta\,\mathrm{Im}(z')\end{array}\right. \qquad \text{for } \alpha, \beta \in \mathbb{R}. \tag{4.1.6}$$

As a consequence, if we have a complex function of a real variable, the derivative of that function can be calculated from the derivatives of the real and imaginary parts, as shown in the following example:

Example 4.1.2

If $z(t)$ is a complex function of a real input t, then

$$\mathrm{Re}\left[\frac{dz}{dt}\right] = \frac{d}{dt}\mathrm{Re}\left[z(t)\right], \quad \text{and} \quad \mathrm{Im}\left[\frac{dz}{dt}\right] = \frac{d}{dt}\mathrm{Im}\left[z(t)\right]. \tag{4.1.7}$$

This can be proven using the definition of the derivative:

$$\begin{align} \mathrm{Re}\left[\frac{dz}{dt}\right] &= \mathrm{Re}\left[\lim_{\delta t\to 0}\frac{z(t+\delta t) - z(t)}{\delta t}\right] \tag{4.1.8}\\ &= \lim_{\delta t\to 0}\left[\frac{\mathrm{Re}[z(t+\delta t)] - \mathrm{Re}[z(t)]}{\delta t}\right] \tag{4.1.9}\\ &= \frac{d}{dt}\mathrm{Re}\left[z(t)\right]. \tag{4.1.10}\end{align}$$

The $\mathrm{Im}[\cdots]$ case works out similarly. Note that the infinitesimal quantity δt is real; otherwise, this wouldn't work.