4.3: Euler's Formula

Euler's formula is an extremely important result which states that

$$e^{iz} = \cos(z) + i\sin(z). \tag{4.3.1}$$

To prove this, recall the definition of the exponential from Chapter 1:

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots \tag{4.3.2}$$

Previously, we assumed the input to the exponential to be a real number. But since complex numbers can be added and multiplied using the same rules of algebra as real numbers, we can adopt the same series formula as the definition of the **complex exponential**, a function that takes complex inputs and gives complex outputs. When the input happens to be real, the complex exponential gives the same result as the original real exponential.

Plugging iz as the input to the complex exponential function gives

$$\begin{aligned}\exp(iz) &= 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \cdots && (4.3.3)\\ &= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!} + \cdots && (4.3.4)\\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right). && (4.3.5)\end{aligned}$$

Now, compare the two terms in parentheses to the series expansions for the cosine and sine functions from Chapter 2. We can define the **complex cosine** and **complex sine** functions using the corresponding complex series:

$$\begin{aligned}\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots && (4.3.6)\\ \sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots && (4.3.7)\end{aligned}$$

These are perfect matches for the real and imaginary parts of Eq. (4.3.5)! Hence, we have proven Eq. (4.3.1).

One important consequence of Euler's formula is that

$$\left|e^{i\theta}\right| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1 \qquad \text{for } \theta \in \mathbb{R}. \tag{4.3.8}$$

Another consequence is that

$$e^{i\pi} = -1, \tag{4.3.9}$$

which is a formula that relates two transcendental constants $e = 2.7182818285\ldots$ and $\pi = 3.141592654\ldots$, by means of the imaginary unit. (We saw a different relationship between these two constants when solving the Gaussian integral in Chapter 3.)