3.3: Integration by Parts

If the integrand consists of two factors, and you know the antiderivative of one of the factors, you can **integrate by parts** by shifting the derivative onto the other factor:

$$\int_a^b dx\; f(x)\, \frac{dg}{dx} \;=\; \Big[\,f(x)\,g(x)\,\Big]_a^b - \int_a^b \frac{df}{dx}\, g(x). \tag{3.3.1}$$

The first term on the right hand side is a constant denoting $[f(a)g(a) - f(b)g(b)]$. Hopefully, the integral in the second term is easier to solve than the original integral.

Judicious use of integration by parts is a key step for solving many integrals. For example, consider

$$\int_a^b dx\; x\, e^{\gamma x}. \tag{3.3.2}$$

The integrand consists of two factors, x and $e^{\gamma x}$; we happen to know the antiderivative of both factors. Integrating by parts lets us replace one of these factors with its antiderivative, while applying an additional derivative on the other factor. The smart thing to do is to apply the derivative on the x factor, and the antiderivative on the $e^{\gamma x}$:

$$\int_a^b dx\; x\, e^{\gamma x} \;=\; \left[\,x\,\frac{e^{\gamma x}}{\gamma}\,\right]_a^b - \int_a^b dx\; \frac{e^{\gamma x}}{\gamma} \tag{3.3.3}$$

$$= \left[\,x\,\frac{e^{\gamma x}}{\gamma} - \frac{e^{\gamma x}}{\gamma^2}\,\right]_a^b. \tag{3.3.4}$$

Whenever we finish doing an integral, it is good practice to double-check the result by making sure the dimensions match up. Note that γ has units of inverse x, so the integral on the left-hand side has units of x^2. The solution on the right hand side has two terms, with units x/γ and $1/\gamma^2$; both of these are equivalent to units of x^2, which is what we need!