10.8: Multi-Dimensional Fourier Transforms

When studying problems such as wave propagation, we often deal with Fourier transforms of several variables. This is conceptually straightforward. For a function $f(x_1, x_2, \ldots, x_d)$ which depends on d independent spatial coordinates $x_1, x_2, \ldots x_d$, we can Fourier transform each coordinate individually:

$$F(k_1, k_2, \ldots, k_d) = \int_{-\infty}^{\infty} dx_1 \; e^{-ik_1x_1} \; \int_{-\infty}^{\infty} dx_2 \; e^{-ik_2x_2} \; \cdots \; \int_{-\infty}^{\infty} dx_d \; e^{-ik_dx_d} \; f(x_1, x_2, \ldots, x_N) \tag{10.8.1}$$

Each coordinate gets Fourier-transformed into its own independent k variable, so the result is also a function of d independent variables.

We can express the multi-dimensional Fourier transform more compactly using vector notation. If $\vec{x}$ is a d-dimensional coordinate vector, the Fourier-transformed coordinates can be written as $\vec{k}$, and the Fourier transform is

$$F(\vec{k}) = \int d^d x \;\; \exp\left(-i\,\vec{k}\cdot\vec{x}\right) \, f(\vec{x}), \tag{10.8.2}$$

where $\int d^d x$ denotes an integral over the entire d-dimensional space, and $\vec{k}\cdot\vec{x}$ is the usual dot product of two vectors. The inverse Fourier transform is

$$f(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} \; \exp\left(i\,\vec{k}\cdot\vec{x}\right) \, F\left(\vec{k}\right). \tag{10.8.3}$$

The delta function, which we introduced in Section 10.7, can also be defined in d-dimensional space, as the Fourier transform of a plane wave:

$$\delta^d(\vec{x}-\vec{x}\,') = \int \frac{d^d k}{(2\pi)^d} \; \exp\left[i\vec{k}\cdot(\vec{x}-\vec{x}\,')\right]. \tag{10.8.4}$$

Note that δ^d has the dimensions of $[x]^{-d}$. The multi-dimensional delta function has a “filtering” property similar to the one-dimensional delta function. For any $f(x_1, \ldots, x_d)$,

$$\int d^d x \;\, \delta^d(\vec{x}-\vec{x}\,') \, f(\vec{x}) = f(\vec{x}\,'). \tag{10.8.5}$$