2.5: Exercises

Exercise 2.5.1

Show that if a function is differentiable, then it is also continuous.

Exercise 2.5.2

Prove that the derivative of $\ln(x)$ is $1/x$.

Answer

If $y = \ln(x)$, it follows from the definition of the logarithm that

$$\exp(y) = x. \tag{2.5.1}$$

Taking d/dx on both sides, and using the product rule, gives

$$\frac{dy}{dx}\exp(y) = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\exp(y)} = \frac{1}{x}. \tag{2.5.2}$$

Exercise 2.5.3

Using the definition of non-natural powers, prove that

$$\frac{d}{dx}[x^y] = yx^{y-1}, \quad \text{for} \;\; x \in \mathbb{R}^+, \;\; y \notin \mathbb{N}. \tag{2.5.3}$$

Exercise 2.5.4

Consider $f(x) = \tanh(\alpha x)$.

a. Sketch $f(x)$ versus x, for two cases: (i) $\alpha = 1$ and (ii) $\alpha \gg 1$.
b. Sketch the derivative function $f'(x)$ for the two cases, based on your sketches in part (A) (i.e., without evaluating the derivative directly).
c. Evaluate the derivative function, and verify that the result matches your sketches in part (B).

Exercise 2.5.5

Prove geometrically that the derivatives of the sine and cosine functions are:

$$\frac{d}{dx}\sin(x) = \cos(x), \quad \frac{d}{dx}\cos(x) = -\sin(x). \tag{2.5.4}$$

Hence, derive their Taylor series, Eqs. (2.2.5) and (2.2.6).

Exercise 2.5.6

For each of the following functions, derive the Taylor series around $x = 0$:

a. $f(x) = \ln[\alpha\cos(x)]$, to the first 3 non-vanishing terms.
b. $f(x) = \cos[\pi\exp(x)]$, to the first 4 non-vanishing terms.
c. $f(x) = \dfrac{1}{\sqrt{1 \pm x}}$, to the first 4 non-vanishing terms. Keep track of the signs (i.e., $\pm$ versus $\mp$).

Exercise 2.5.7

For each of the following functions, sketch the graph and state the domains over which the function is differentiable:

a. $f(x) = |\sin(x)|$

b. $f(x) = [\tan(x)]^2$

c. $f(x) = \dfrac{1}{1-x^2}$

Exercise 2.5.8

Let $\vec{v}(x)$ be a *vectorial* function which takes an input x (a number), and gives an output value $\vec{v}$ that is a 2-component vector. The derivative of this vectorial function is defined in terms of the derivatives of each vector component:

$$\vec{v}(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix} \Rightarrow \frac{d\vec{v}}{dx} = \begin{bmatrix} dv_1/dx \\ dv_2/dx \end{bmatrix}. \tag{2.5.5}$$

Now suppose $\vec{v}(x)$ obeys the vectorial differential equation

$$\frac{d\vec{v}}{dx} = \mathbf{A}\vec{v}, \tag{2.5.6}$$

where

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{2.5.7}$$

is a matrix that has two distinct real eigenvectors with real eigenvalues.

a. How many independent numbers do we need to specify for the general solution?

b. Let $\vec{u}$ be one of the eigenvectors of $\mathbf{A}$, with eigenvalue λ:

$$\mathbf{A}\vec{u} = \lambda\vec{u}. \tag{2.5.8}$$

Show that $\vec{v}(x) = \vec{u}\, e^{\lambda x}$ is a specific solution to the vectorial differential equation. Hence, find the general solution.

Answer

For an ordinary differential equation for a scalar (one-component) function of order n, the general solution must contain n independent variables. In this case, $\vec{v}$ is a two-component function, so it requires $2n$ indpendent variables. The differential equation

$$\frac{d\vec{v}}{dx} = \mathbf{A}\vec{v} \tag{2.5.9}$$

has order $n = 1$, so a total of 2 independent variables is required for the general solution.

Let u be an eigenvector of $\mathbf{A}$ with eigenvalue λ, and suppose that $\vec{v}(x) = \vec{u}\, e^{\lambda x}$ (note that $\vec{u}$ itself does not depend on x). Then

$$\begin{aligned} \frac{d\vec{v}}{dx} &= \vec{u}\frac{d}{dx}\left(e^{\lambda x}\right) && (2.5.10) \\ &= \lambda\,\vec{u}\, e^{\lambda x} && (2.5.11) \\ &= (\mathbf{A}\vec{u})\, e^{\lambda x} && (2.5.12) \\ &= \mathbf{A}\left(\vec{u}e^{\lambda x}\right) && (2.5.13) \\ &= \mathbf{A}\vec{v}(x). && (2.5.14) \end{aligned}$$

Hence, $\vec{v}(x)$ satisfies the desired differential equation.

Let $\vec{u}_1$ and $\vec{u}_2$ be the eigenvectors of $\mathbf{A}$, with eigenvalues λ_1 and λ_2. The general solutions will be

$$\vec{v}(x) = c_1\vec{u}_1 e^{\lambda_1 x} + c_2\vec{u}_2 e^{\lambda_2 x}, \tag{2.5.15}$$

where c_1 and c_2 are independent variables.