11.2: Space-Time Green's Functions

The Green’s function method can also be used for studying waves. For simplicity, we will restrict the following discussion to waves propagating through a uniform medium. Also, we will just consider 1D space; the generalization to higher spatial dimensions is straightforward.

As discussed in Chapter 6, wave propagation can be modeled by the wave equation

$$\left[\frac{\partial^2}{\partial x^2} - \left(\frac{1}{c}\right)^2 \frac{\partial^2}{\partial t^2}\right] \psi(x,t) = 0, \tag{11.2.1}$$

where $\psi(x,t)$ is a complex wavefunction and c is the wave speed. Henceforth, to simplify the equations, we will set $c=1$. (You can reverse this simplification by replacing all instances of t with ct, and ω with ω/c, in the subsequent formulas.)

The wave equation describes how waves propagate *after* they have already been created. To describe how the waves are generated in the first place, we must modify the wave equation by introducing a term on the right-hand side, called a **source**:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right] \psi(x,t) = f(x,t). \tag{11.2.2}$$

The source term turns the wave equation into an inhomogeneous partial differential equation, similar to the driving force for the driven harmonic oscillator.

Time-domain Green’s function

The wave equation’s **time-domain Green’s function** is defined by setting the source term to delta functions in both space and time:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right] G(x,x';t-t') = \delta(x-x')\,\delta(t-t'). \tag{11.2.3}$$

As can be seen, G is a function of two spatial variables, x and x', as well as two temporal variables t and t'. It corresponds to the wave generated by a pulse

$$f(x,t) = \delta(x-x')\,\delta(t-t'). \tag{11.2.4}$$

The differential operator in the Green’s function equation only involves x and t, so we can regard x' and t' as parameters specifying where the pulse is localized in space and time. This Green’s function ought to depend on the time variables only in the combination $t-t'$, as we saw in our earlier discussion of the harmonic oscillator Green’s function (see Section 11.1). To emphasize this, we have written it as $G(x,x';t-t')$.

The Green’s function describes how a source localized at a space-time point influences the wavefunction at other positions and times. Once we have found the Green’s function, it can be used to construct solutions for arbitrary sources:

$$\psi(x,t) = \int dx' \int_{-\infty}^{\infty} dt'\; G(x,x';t-t')\, f(x',t'). \tag{11.2.5}$$

Frequency-domain Green’s function

The **frequency-domain Green’s function** is obtained by Fourier transforming the time-domain Green’s function in the $t-t'$ coordinate:

$$G(x,x';\omega) = \int_{-\infty}^{\infty} d\tau\; e^{i\omega\tau}\, G(x,x';\tau). \tag{11.2.6}$$

It obeys the differential equation

$$\left[\frac{\partial^2}{\partial x^2} + \omega^2\right] G(x,x';\omega) = \delta(x-x'). \tag{11.2.7}$$

Just as we can write the time-domain solution to the wave equation in terms of the time-domain Green’s function, we can do the same for the frequency-domain solution:

$$\Psi(x,\omega) = \int dx'\; G(x,x';\omega)\, F(x',\omega), \tag{11.2.8}$$

where

$$\Psi(x,\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\,\psi(x,t), \quad F(x,\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\, f(x,t). \tag{11.2.9}$$

Outgoing boundary conditions

So far, we have not specified the boundary conditions along x. There are several possible choices of boundary conditions, corresponding to different physical scenarios. For example, if the waves are trapped within a finite domain $x \in (x_a, x_b)$, with reflecting walls, we would impose **Dirichlet boundary conditions**: $G(x,x';\omega) = 0$ for $x, x' = (x_a \text{ or } x_b)$.

We will focus on the interesting case of an unbounded spatial domain: $x \in (-\infty, \infty)$. This describes, for example, a loudspeaker emitting sound waves into an infinite empty space. The relevant boundary conditions for this case are called **outgoing boundary conditions**. The Green's function should correspond to a left-moving wave for x to the left of the source, and to a right-moving wave for x to the right of the source.

We can guess the form of the Green's function obeying these boundary conditions:

$$G(x,x';\omega) = \begin{cases} A\, e^{-i\omega(x-x')}, & x \le x', \\ B\, e^{i\omega(x-x')}, & x \ge x' \end{cases} \quad \text{for some } \; A, B \in \mathbb{C}. \tag{11.2.10}$$

It is straightforward to verify that this formula for $G(x,x',\omega)$ satisfies the wave equation in both the regions $x < x'$ and $x > x'$, as well as satisfying outgoing boundary conditions. To determine the A and B coefficients, note that $G(x,x')$ should be continuous at $x = x'$, so $A = B$. Then, integrating the Green's function equation across x' gives

$$\lim_{\epsilon\to 0} \int_{x'-\epsilon}^{x'+\epsilon} \left[\frac{\partial^2}{\partial x^2} + \omega^2\right] G(x-x') \;= \lim_{\epsilon\to 0}\int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x') \tag{11.2.11}$$

$$= \lim_{\epsilon\to 0}\left\{ \frac{\partial G}{dx}(x,x')\bigg|_{x=x'+\epsilon} - \frac{\partial G}{\partial x}(x,x')\bigg|_{x=x'-\epsilon}\right\} \;= i\omega(B+A) = 1. \tag{11.2.12}$$

Combining these two equations gives $A = B = 1/2i\omega$. Hence,

$$G(x,x';\omega) = \frac{e^{i\omega|x-x'|}}{2i\omega}. \tag{11.2.13}$$