7.3: The Cauchy-Riemann Equations

The **Cauchy-Riemann equations** are a pair of real partial differential equations that provide an alternative way to understand complex derivatives. Their importance comes from the following two theorems.

Theorem 7.3.1

Let f be a complex function that can be written as $f(z=x+iy)=u(x,y)+iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are real functions of two real inputs. If f is complex-differentiable at a given $z=x+iy$, then $u(x,y)$ and $v(x,y)$ have valid first-order partial derivatives (i.e., they are real-differentiable in both the x and y directions), and these derivatives satisfy

$$\left\{\begin{array}{l}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{array}\right. \quad \text{Cauchy-Riemann equations} \tag{7.3.1}$$

Conversely,

Theorem 7.3.2

Let $u(x,y)$ and $v(x,y)$ be real functions whose first-order partial derivatives exist and are continuous at (x,y), and satisfy the Cauchy-Riemann equations. Then the function $f(z=x+iy)=u(x,y)+iv(x,y)$ is complex-differentiable at $z=x+iy$.

Proof

We will now prove the theorem, which states that f being complex-differentiable implies the Cauchy-Riemann equations. The proof of the converse is left as an exercise.

Suppose the function f is complex-differentiable at some point z. Following from the definition of complex differentiability, there exists a derivative $f'(z)$ defined as

$$f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}, \tag{7.3.2}$$

whose value is independent of the argument that we take for the infinitesimal δz. If we take this to be real, i.e. $\delta z=\delta x\in\mathbb{R}$, the expression for the derivative can be written as

$$\begin{aligned} f'(z) &= \lim_{\delta x\to 0}\frac{f(x+\delta x+iy)-f(x+iy)}{\delta x} && (7.3.3)\\ &= \lim_{\delta x\to 0}\frac{[u(x+\delta x,y)+iv(x+\delta x,y)]-[u(x,y)+iv(x,y)]}{\delta x} && (7.3.4)\\ &= \lim_{\delta x\to 0}\frac{[u(x+\delta x,y)-u(x,y)]+i\,[v(x+\delta x,y)-v(x,y)]}{\delta x} && (7.3.5)\\ &= \left[\lim_{\delta x\to 0}\frac{u(x+\delta x,y)-u(x,y)}{\delta x}\right]+i\left[\lim_{\delta x\to 0}\frac{v(x+\delta x,y)-v(x,y)}{\delta x}\right] && (7.3.6)\end{aligned}$$

On the last line, the quantities in square brackets are the real partial derivatives of u and v (with respect to x). Therefore those partial derivatives are well-defined, and

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}. \tag{7.3.7}$$

On the other hand, we could also take an infinitesimal displacement in the imaginary direction, by setting $\delta z=i\delta y$ where $\delta y\in\mathbb{R}$. Then the expression for the derivative is

$$f'(z) = \lim_{\delta y \to 0} \frac{f(x+iy+i\delta y) - f(x+iy)}{i\delta y} \tag{7.3.8}$$

$$= \lim_{\delta y \to 0} \frac{[u(x,y+\delta y) + iv(x,y+\delta y)] - [u(x,y) + iv(x,y)]}{i\delta y} \tag{7.3.9}$$

$$= \lim_{\delta y \to 0} \frac{[u(x,y+\delta y) - u(x,y)] + i\,[v(x,y+\delta y) - v(x,y)]}{i\delta y} \tag{7.3.10}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{7.3.11}$$

Since $f(z)$ is complex-differentiable, these two expressions must be equal, so

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{7.3.12}$$

Noting that u and v are real functions, we can take the real and imaginary parts of the above equation separately. This yields a pair of real equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{7.3.13}$$

These are precisely the Cauchy-Riemann equations. As a corollary, we also obtain a set of convenient expressions for the complex derivative of $f(z)$:

$$\mathrm{Re}\,[f'(z)] = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{7.3.14}$$

$$\mathrm{Im}\,[f'(z)] = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{7.3.15}$$

Interpretation of the Cauchy-Rieman equations

The central message of the Cauchy-Riemann equations is that when dealing with analytic functions, the real and imaginary parts of complex numbers cannot be regarded as independent quantities, but are closely intertwined. There are two complementary ways to think about this:

- For an analytic function $f(z)$, the real and imaginary parts of the input z do not independently affect the output value. If I tell you how the function varies in the x direction, by giving you $\partial u/\partial x$ and $\partial v/\partial x$, then you can work out how the function varies in the y direction, by using the Cauchy-Riemann equations to find $\partial u/\partial y$ and $\partial v/\partial y$.
- Similarly, for the complex outputs of $f(z)$, the real and imaginary parts cannot be regarded as independent. If I tell you how the real part of the output varies, by giving you $\partial u/\partial x$ and $\partial u/\partial y$, then you can work out how the imaginary part of the output varies, by using the Cauchy-Riemann equations to find $\partial v/\partial x$ and $\partial v/\partial y$.

These constraints have profound implications for the mathematical discipline of complex analysis, one of the most important being Cauchy's integral theorem, which we will encounter when studying contour integration in Chapter 9.

Consequences of the Cauchy-Rieman equations

Often, the easiest way to prove that a function is analytic in a given domain is to prove that the Cauchy-Riemann equations are satisfied.

Example 7.3.1

We can use the Cauchy-Riemann equations to prove that the function

$$f(z) = 1/z \tag{7.3.16}$$

is analytic everywhere, except at $z = 0$. Let us write the function as

$$f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}. \tag{7.3.17}$$

Hence the real and imaginary component functions are

$$u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = -\frac{y}{x^2+y^2}. \tag{7.3.18}$$

Except at $x = y = 0$, these functions have well-defined and continuous partial derivatives satisfying

$$\frac{\partial u}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y} \tag{7.3.19}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} = -\frac{\partial u}{\partial y}. \tag{7.3.20}$$

More generally, we can use the Cauchy-Riemann equations to prove the following facts about analytic functions:

- *Compositions of analytic functions are analytic*.
  If $f(z)$ is analytic in $D \subset \mathbb{C}$ and $g(z)$ is analytic in the range of f, then $g(f(z))$ is analytic in D.
- *Reciprocals of analytic functions are analytic, except at singularities*.
  If $f(z)$ is analytic in $D \subset \mathbb{C}$, then $1/f(z)$ is analytic everywhere in D except where $f(z) = 0$.

The proofs for these can be obtained via the Cauchy-Riemann equations, and are left as exercises.