7.2: Analytic Functions

If a function $f(z)$ is complex-differentiable for all points z in some domain $D \subset \mathbb{C}$, then $f(z)$ is said to be **analytic** in D.

The concepts of analyticity and complex-differentiability are closely related. It's mainly a matter of terminology: we speak of a function being complex-differentiable *at a given point*, and we speak of a function being analytic *in a given domain*.

Example 7.2.1

As shown in the preceding section, $f(z) = z$ is complex-differentiable for any point $z \in \mathbb{C}$. Thence, $f(z) = z$ is analytic in $\mathbb{C}$.

A function's domain of analyticity is often described spatially, in terms of the complex plane. For example, we might say that a function is analytic "everywhere in the complex plane", which means the entire domain $\mathbb{C}$. Or we might say that a function is analytic "in the upper half of the complex plane", meaning for all z such that $\mathrm{Im}(z) > 0$.

Common analytic functions

There is an important class of functions which are analytic over the entire complex plane, or most of the complex plane. These are functions generated from algebraic formulas which do not contain z^*, and involve z in some "simple" combination of operations like addition, multiplication, and integer powers.

For example, we have seen that the function $f(z) = z$ is analytic in $\mathbb{C}$. Likewise, $f(z) = \alpha z + \beta$, where α, β are complex constants, is analytic everywhere in $\mathbb{C}$. This can be proven in a similar fashion:

$$\begin{aligned} f'(z) &= \lim_{\delta z \to 0} \frac{[\alpha\,(z+\delta z)+\beta] - [\alpha z + \beta]}{\delta z} && (7.2.1) \\ &= \lim_{\delta z \to 0} \frac{\alpha \delta z}{\delta z} && (7.2.2) \\ &= \alpha. && (7.2.3) \end{aligned}$$

We can also show that $f(z) = z^n$, with $n \in \mathbb{N}$, is analytic everywhere in $\mathbb{C}$:

$$\begin{aligned} f'(z) &= \lim_{\delta z \to 0} \frac{(z+\delta z)^n - z^n}{\delta z} && (7.2.4) \\ &= \lim_{\delta z \to 0} \frac{(z^n + n z^{n-1}\delta z + \cdots) - z^n}{\delta z} && (7.2.5) \\ &= n z^{n-1}. && (7.2.6) \end{aligned}$$

Note that these derivatives have exactly the same algebraic formulas as the corresponding real derivatives. This is no coincidence: to derive the complex derivatives, we take the same series of algebra steps used for deriving the real derivatives.

From the discussion so far, it is evident that complex polynomials are analytic everywhere in $\mathbb{C}$. Likewise, functions that are defined in terms of power series, including the complex exponential and complex sines and cosines, are analytic everywhere in $\mathbb{C}$. Functions involving reciprocals (negative integer powers), such as $f(z) = z^{-1}$ or $f(z) = z^{-2}$, are analytic everywhere *except* at points where $f(z)$ becomes singular (i.e., the denominator goes to zero). (We will prove this in Section 7.3.)

More generally, whenever a function involves z in some combination of integer polynomials, reciprocals, or functions with power series expansions—and does not involve z^* in an irreducible way—then the function is analytic everywhere except at the singular points. Moreover, the formula for the complex derivative is the same as the corresponding formula for real derivatives.

Example 7.2.2

The function

$$f(z) = \frac{1}{\cos(z)} \tag{7.2.7}$$

is analytic everywhere in $\mathbb{C}$, except for values of z such that $\cos(z) = 0$. With a bit of work (try it!), one can show that these z occur at isolated points along the real line, at $z = (m + 1/2)\pi$ where $m \in \mathbb{Z}$, and nowhere else in the complex plane. The complex derivative is

$$f'(z)=\frac{\sin(z)}{[\cos(z)]^2}. \tag{7.2.8}$$

The easiest way to prove these statements is to use the Cauchy-Riemann equations, which are discussed in Section 7.3.

One proviso should be kept in mind. For non-integer powers, z^a where $a \notin \mathbb{Z}$, the situation is more complicated because the operation is multi-valued. We'll postpone the discussion of these special operations until Chapter 8.