9.3: Poles

In the previous section, we referred to situations where $f(z)$ is non-analytic at discrete points. “Discrete”, in this context, means that each point of non-analyticity is surrounded by a finite region over which $f(z)$ is analytic, isolating it from other points of non-analyticity. Such situations commonly arise from functions of the form

$$f(z) \approx \frac{A}{(z-z_0)^n}, \quad \text{where } \; n \in \{1,2,3,\ldots\}. \tag{9.3.1}$$

For $z = z_0$, the function is non-analytic because its value is singular. Such a function is said to have a **pole** at z_0. The integer n is called the **order** of the pole.

Residue of a simple pole

Poles of order 1 are called **simple poles**, and they are of special interest. Near a simple pole, the function has the form

$$f(z) \approx \frac{A}{z-z_0}. \tag{9.3.2}$$

In this case, the complex numerator A is called the **residue** of the pole (so-called because it’s what’s left-over if we take away the singular factor corresponding to the pole.) The residue of a function at a point z_0 is commonly denoted $\mathrm{Res}[f(z_0)]$. Note that if a function is analytic at z_0, then $\mathrm{Res}[f(z_0)] = 0$.

Example 9.3.1

Consider the function

$$f(z) = \frac{5}{i-3z}. \tag{9.3.3}$$

To find the pole and residue, divide the numerator and denominator by -3:

$$f(z) = \frac{-5/3}{z-i/3}. \tag{9.3.4}$$

Thus, there is a simple pole at $z = i/3$ with residue $-5/3$.

Example 9.3.2

Consider the function

$$f(z) = \frac{z}{z^2+1}. \tag{9.3.5}$$

To find the poles and residues, we factorize the denominator:

$$f(z) = \frac{z}{(z+i)(z-i)}. \tag{9.3.6}$$

Hence, there are two simple poles, at $z = \pm i$.

To find the residue at $z = i$, we separate the divergent part to obtain

$$f(z) = \frac{\left(\frac{z}{z+i}\right)}{z-i} \tag{9.3.7}$$

$$\Rightarrow \quad \mathrm{Res}\big[\, f(z) \big]_{z=i} = \left[\frac{z}{z+i}\right]_{z=i} = 1/2. \tag{9.3.8}$$

Similarly, for the other pole,

$$\mathrm{Res}\big[\, f(z) \big]_{z=-i} = \left[\frac{z}{z-i}\right]_{z=-i} = 1/2. \tag{9.3.9}$$

The residue theorem

In Section 9.1, we used contour parameterization to calculate

$$\oint_{\Gamma} \frac{dz}{z} = 2\pi i, \tag{9.3.10}$$

where Γ is a counter-clockwise circular loop centered on the origin. This holds for any (non-zero) loop radius. By combining this with the results of Section 9.2, we can obtain the **residue theorem**:

Theorem 9.3.1

For any analytic function $f(z)$ with a simple pole at z_0,

$$\oint_{\Gamma[z_0]} dz\, f(z) = \pm 2\pi i \,\mathrm{Res}\big[\, f(z)\,\big]_{z=z_0}, \tag{9.3.11}$$

where $\Gamma[z_0]$ denotes an infinitesimal loop around z_0. The $+$ sign holds for a counter-clockwise loop, and the $-$ sign for a clockwise loop.

By combining the residue theorem with the results of the last few sections, we arrive at a technique for integrating a function $f(z)$ over a loop Γ, called the **calculus of residues**:

1. Identify the poles of $f(z)$ in the domain enclosed by Γ.
2. Check that these are all simple poles, and that $f(z)$ has no other non-analytic behaviors (e.g. branch cuts) in the enclosed region.
3. Calculate the residue, $\mathrm{Res}[f(z_n)]$, at each pole z_n.
4. The value of the loop integral is

$$\oint_{\Gamma} dz\, f(z) = \pm 2\pi i \sum_n \mathrm{Res}\big[\, f(z)\,\big]_{z=z_n}. \tag{9.3.12}$$

   The plus sign holds if Γ is counter-clockwise, and the minus sign if it is clockwise.

Example of the calculus of residues

Consider

$$f(z) = \frac{1}{z^2+1}. \tag{9.3.13}$$

This can be re-written as

$$f(z) = \frac{1}{(z+i)\,(z-i)}. \tag{9.3.14}$$

By inspection, we can identify two poles: one at $+i$, with residue $1/2i$, and the other at $-i$, with residue $-1/2i$. The function is analytic everywhere else.

Suppose we integrate $f(z)$ around a counter-clockwise contour Γ_1 that encloses only the pole at $+i$, as indicated by the blue curve in the figure below:

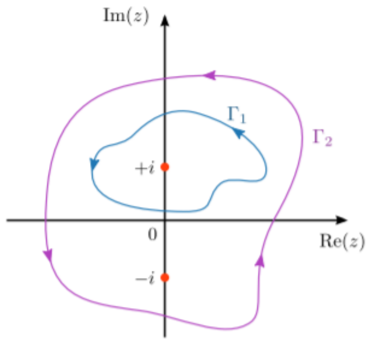

Figure 9.3.1

According to the residue theorem, the result is

$$\oint_{\Gamma_1} dz\, f(z) = 2\pi i \,\mathrm{Res}\big[\, f(z)\,\big]_{z=i} \tag{9.3.15}$$

$$= 2\pi i \cdot \frac{1}{2i} \tag{9.3.16}$$

$$= \pi. \tag{9.3.17}$$

On the other hand, suppose we integrate around a contour Γ_2 that encloses *both* poles, as shown by the purple curve. Then the result is

$$\oint_{\Gamma_2} dz\, f(z) = 2\pi i \cdot \left[\frac{1}{2i} - \frac{1}{2i}\right] = 0. \tag{9.3.18}$$