8.1: Non-Integer Powers as Multi-Valued Operations

Given a complex number in its polar representation, $z = r\exp[i\theta]$, raising to the power of p could be handled this way:

$$z^p = \left(re^{i\theta}\right)^p = r^p e^{ip\theta}. \tag{8.1.1}$$

Let's take a closer look at the complex exponential term $e^{ip\theta}$. Since $\theta = \arg(z)$ is an angle, we can change it by any integer multiple of 2π without altering the value of z. Taking this fact into account, we can re-write the above equation more carefully as

$$z^p = \left(r\, e^{i(\theta+2\pi n)}\right)^p = \left(r^p e^{ip\theta}\right) e^{2\pi inp} \quad \text{where}\;\; n\in\mathbb{Z}. \tag{8.1.2}$$

Thus, there is an ambiguous factor of $\exp(2\pi inp)$, where n is any integer. If p is an integer, there is no problem, since $2\pi np$ will be an integer multiple of 2π, so z^p has the same value regardless of n:

$$z^p = r^p e^{ip\theta}\;\;\text{unambiguously}\quad (\text{if}\, p\in\mathbb{Z}). \tag{8.1.3}$$

But if p is not an integer, there is no unique answer, since $\exp(2\pi inp)$ has different values for different n. In that case, "raising to the power of p" is a **multi-valued operation**. It cannot be treated as a function in the usual sense, since functions must have unambiguous outputs (see Chapter 0).

Roots of unity

Let's take a closer look at the problematic exponential term,

$$\exp(2\pi inp),\quad n\in\mathbb{Z}. \tag{8.1.4}$$

If p is irrational, $2\pi np$ never repeats itself modulo 2π. Thus, z^p has an infinite set of values, one for each integer n.

More interesting is the case of a non-integer *rational* power, which can be written as $p = P/Q$ where P and Q are integers with no common divisor. It can be proven using modular arithmetic (though we will not go into the details) that $2\pi n\,(P/Q)$ has exactly Q unique values modulo 2π:

$$2\pi n\left(\frac{P}{Q}\right) = 2\pi\times\left\{0,\, \frac{1}{Q},\, \frac{2}{Q},\, \dots,\, \frac{(Q-1)}{Q}\right\}\quad (\text{modulo }2\pi). \tag{8.1.5}$$

This set of values is independent of the numerator P, which merely affects the sequence in which the numbers are generated. We can clarify this using a few simple examples:

Example 8.1.1

Consider the complex square root operation, $z^{1/2}$. If we write z in its polar respresentation,

$$z = re^{i\theta}, \tag{8.1.6}$$

then

$$z^{1/2} = \left[r\, e^{i(\theta+2\pi n)}\right]^{1/2} = r^{1/2}\, e^{i\theta/2}\, e^{i\pi n},\quad n\in\mathbb{Z}. \tag{8.1.7}$$

The factor of $e^{i\pi n}$ has two possible values: $+1$ for even n, and -1 for odd n. Hence,

$$z^{1/2} = r^{1/2}\, e^{i\theta/2}\;\times\;\{1,-1\}\,. \tag{8.1.8}$$

Example 8.1.2

Consider the cube root operation $z^{1/3}$. Again, we write z in its polar representation, and obtain

$$z^{1/3} = r^{1/3}\, e^{i\theta/3}\, e^{2\pi in/3},\quad n\in\mathbb{Z}. \tag{8.1.9}$$

The factor of $\exp(2\pi in/3)$ has the following values for different n:

n	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$e^{2\pi in/3}$	$\cdots$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1	$e^{2\pi i/3}$	$\cdots$

From the pattern, we see that there are three possible values of the exponential factor:

$$e^{2\pi in/3} = \left\{1, e^{2\pi i/3}, e^{-2\pi i/3}\right\}. \tag{8.1.10}$$

Therefore, the cube root operation has three distinct values:

$$z^{1/3} = r^{1/3}\, e^{i\theta/3} \;\times\; \left\{1, e^{2\pi i/3}, e^{-2\pi i/3}\right\}. \tag{8.1.11}$$

Example 8.1.3

Consider the operation $z^{2/3}$. Again, writing z in its polar representation,

$$z^{2/3} = r^{2/3}\, e^{2i\theta/3}\, e^{4\pi in/3}, \quad n\in\mathbb{Z}. \tag{8.1.12}$$

The factor of $\exp(4\pi in/3)$ has the following values for different n:

n	$\cdots$	-2	-1	0	1	2	3	4	$\cdots$
$e^{4\pi in/3}$	$\cdots$	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	1	$e^{-2\pi i/3}$	$\cdots$

Hence, there are three possible values of this exponential factor,

$$e^{2\pi in(2/3)} = \left\{1, e^{2\pi i/3}, e^{-2\pi i/3}\right\}. \tag{8.1.13}$$

Note that this is the exact same set we obtained for $e^{2\pi in/3}$ in the previous example, in agreement with the earlier assertion that the numerator P has no effect on the set of values. Thus,

$$z^{2/3} = r^{2/3}\, e^{2i\theta/3} \;\times\; \left\{1, e^{2\pi i/3}, e^{-2\pi i/3}\right\}. \tag{8.1.14}$$

From the above examples, we deduce the following expression for rational powers:

$$z^{P/Q} = r^{P/Q}\; e^{i\theta\,(P/Q)} \;\times \left\{1,\; e^{2\pi i\cdot(1/Q)},\; e^{2\pi i\cdot(2/Q)},\; \dots,\, e^{2\pi i\cdot[(1-Q)/Q]}\right\}. \tag{8.1.15}$$

The quantities in the curly brackets are called the **roots of unity**. In the complex plane, they sit at Q evenly-spaced points on the unit circle, with 1 as one of the values:

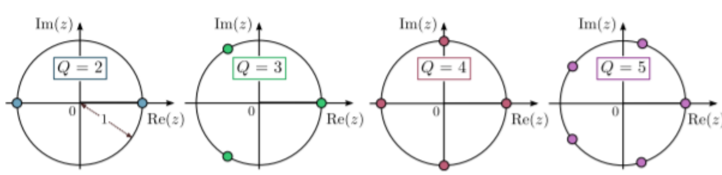

Figure 8.1.1

Complex logarithms

Here is another way to think about non-integer powers. Recall what it means to raise a number to, say, the power of 5: we simply multiply the number by itself five times. What about raising a number to a non-integer power p? For the real case, we used the following definition based on a combination of exponential and logarithm functions:

$$x^p \equiv \exp\left[\,p\ln(x)\right]. \tag{8.1.16}$$

This definition relies on the fact that, for real inputs, the logarithm is a well-defined function. That, in turn, comes from the definition of the logarithm as the inverse of the exponential function. Since the real exponential is one-to-one, its inverse is also one-to-one.

The complex exponential, however, is many-to-one, since changing its input by any multiple of $2\pi i$ yields the same output:

$$\exp(z+2\pi in) = \exp(z)\cdot e^{2\pi in} = \exp(z) \quad \forall\, n \in \mathbb{Z}. \tag{8.1.17}$$

The inverse of the complex exponential is the **complex logarithm**. Since the complex exponential is many-to-one, the complex logarithm does not have a unique output. Instead, $\ln(z)$ refers to an infinite discrete set of values, separated by integer multiples of $2\pi i$. We can express this state of affairs in the following way:

$$\ln(z) = \Big[\ln(z)\Big]_{\text{p.v.}} \; + \; 2\pi i n, \quad n \in \mathbb{Z}. \tag{8.1.18}$$

Here, $[\ln(z)]_{\text{p.v.}}$ denotes the **principal value** of $\ln(z)$, which refers to a reference value of the logarithm operation (which we'll define later). Do not think of the principal value as the "actual" result of the $\ln(z)$ operation! There are multiple values, each equally legitimate; the principal value is merely one of these possible results.

Plugging Eq. (8.1.18) into the formula $z^p \equiv \exp[p\ln(z)]$ gives

$$z^p = \exp\Big\{p\Big([\ln(z)]_{\text{p.v.}} + 2\pi in\Big)\Big\} \tag{8.1.19}$$

$$= \exp\Big\{p[\ln(z)]_{\text{p.v.}}\Big\} \times e^{2\pi inp}, \quad n \in \mathbb{Z}. \tag{8.1.20}$$

The final factor, which is responsible for the multi-valuedness, are the roots of unity found in Section 8.1.