7.4: Exercises

Exercise 7.4.1

For each of the following functions $f(z)$, find the real and imaginary component functions $u(x,y)$ and $v(x,y)$, and hence verify whether they satisfy the Cauchy-Riemann equations.

a. $f(z) = z$
b. $f(z) = z^2$
c. $f(z) = |z|$
d. $f(z) = |z|^2$
e. $f(z) = \exp(z)$
f. $f(z) = \cos(z)$
g. $f(z) = 1/z$

Exercise 7.4.2

Suppose a function $f(z)$ is well-defined and obeys the Cauchy-Riemann equations at a point z, and the partial derivatives in the Cauchy-Riemann equations are continuous at that point. Show that the function is complex differentiable at that point. Hint: consider an arbitary displacement $\Delta z = \Delta x + i\Delta y$.

Exercise 7.4.3

Prove that products of analytic functions are analytic: if $f(z)$ and $g(z)$ are analytic in $D \subset \mathbb{C}$, then $f(z)g(z)$ is analytic in D.

Answer

We will use the Cauchy-Riemann equations. Decompose z, f, and g into real and imaginary parts as follows: $z = x + iy$, $f = u + iv$, and $g = p + iq$. Since $f(z)$ and $g(z)$ are analytic in D, they satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{7.4.1}$$

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \quad -\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \tag{7.4.2}$$

This holds for all $z \in D$. Next, expand the product $f(z)\,g(z)$ into real and imaginary parts:

$$f(z)\,g(z) = A(x,y) + iB(x,y), \quad \text{where} \; \left\{ \begin{array}{l} A = up - vq \\ B = uq + vp. \end{array} \right. \tag{7.4.3}$$

Our goal is to prove that A and B satisfy the Cauchy-Riemann equations for $x + iy \in D$, which would then imply that fg is analytic in D. Using the product rule for derivatives:

$$\frac{\partial A}{\partial x} = \frac{\partial u}{\partial x}p + u\frac{\partial p}{\partial x} - \frac{\partial v}{\partial x}q - v\frac{\partial q}{\partial x} \tag{7.4.4}$$

$$= \frac{\partial v}{\partial y}p + u\frac{\partial q}{\partial y} + \frac{\partial u}{\partial y}q + v\frac{\partial p}{\partial y} \tag{7.4.5}$$

$$\frac{\partial B}{\partial y} = \frac{\partial u}{\partial y}q + u\frac{\partial q}{\partial y} + \frac{\partial v}{\partial y}p + v\frac{\partial p}{\partial y}. \tag{7.4.6}$$

By direct comparison, we see that the two expressions are equal. Similarly,

$$\frac{\partial A}{\partial y} = \frac{\partial u}{\partial y}p + u\frac{\partial p}{\partial y} - \frac{\partial v}{\partial y}q - v\frac{\partial q}{\partial y} \tag{7.4.7}$$

$$= -\frac{\partial v}{\partial x}p - u\frac{\partial q}{\partial x} - \frac{\partial u}{\partial x}q - v\frac{\partial p}{\partial x} \tag{7.4.8}$$

$$\frac{\partial B}{\partial x} = \frac{\partial u}{\partial x}q + u\frac{\partial q}{\partial x} + \frac{\partial v}{\partial x}p + v\frac{\partial p}{\partial x}. \tag{7.4.9}$$

These two are the negatives of each other. Q.E.D.

Exercise 7.4.4

Prove that compositions of analytic functions are analytic: if $f(z)$ is analytic in $D \subset \mathbb{C}$ and $g(z)$ is analytic in the range of f, then $g(f(z))$ is analytic in D.

Exercise 7.4.5

Prove that reciprocals of analytic functions are analytic away from poles: if $f(z)$ is analytic in $D \subset \mathbb{C}$, then $1/f(z)$ is analytic everywhere in D except where $f(z) = 0$.

Exercise 7.4.6

Show that if $f(z = x+iy) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann equations, then the real functions u and v each obey Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{7.4.10}$$

(Such functions are called “harmonic functions”.)

Exercise 7.4.7

We can write the real and imaginary parts of a function in terms of polar coordinates: $f(z) = u(r,\theta) + iv(r,\theta)$, where $z = re^{i\theta}$. Show that the Cauchy-Riemann equations can be re-written in polar form as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{7.4.11}$$