6.6: Exercises

Exercise 6.6.1

Consider the 1D wave equation in a enclosed box of length L and uniform refractive index $n \in \mathbb{R}$. The walls of the box are at $x=-L/2$ and $x=L/2$, and the wavefunction goes to zero at these points: $\psi(\pm L/2)=0$ (i.e., Dirichlet boundary conditions). Show that $\psi(x)=0$ for all x, *except* for certain discrete values of the frequency ω. Find these frequencies, and the corresponding non-zero solutions $\psi(x)$.

Exercise 6.6.2

As discussed in Section 6.5, a harmonic traveling wave in an energy-nonconserving medium is described by

$$\left[\frac{d^2}{dx^2}+n^2\left(\frac{\omega}{c}\right)^2\right]\psi(x)=0, \tag{6.6.1}$$

where n is a complex number. (As usual, ω and c are assumed to be positive real numbers.) Show that the relative sign of $\mathrm{Re}(n)$ and $\mathrm{Im}(n)$ determines whether the wave experiences amplification or dissipation, and that the result does not depend of the wave's propagation direction.

Answer

Writing $n=n'+in''$, where n' and n'' are real, the travelling wave solutions are

$$\psi(x)=A\exp\left[\pm i(n'+in'')\frac{\omega}{c}x\right]. \tag{6.6.2}$$

The magnitude and argument are:

$$|\psi(x)|=|A|\exp\left[\mp n''\frac{\omega}{c}x\right] \tag{6.6.3}$$

$$\arg\left[\psi(x)\right]=\arg(A)\pm n'\frac{\omega}{c}x. \tag{6.6.4}$$

The wave's propagation direction is determined by the argument: if the argument increases with x then it is right-moving, and if the argument decreases with x it is left-moving. Moreover, the wave is said to experience amplification if its amplitude grows along the propagation direction, and damping if its amplitude decreases along the propagation direction.

Consider the upper choice of sign (i.e., $+$ for the $\pm$ symbol and $-$ for the $\mp$ symbol). From the magnitude, we see that the wave's amplitude decreases with x if $n''>0$, and increases with x if $n''<0$. From the argument, the wave is right-moving if $n'>0$, and left-moving if $n'<0$. Hence, the wave is damped if $n'n''>0$ and amplified if $n'n''<0$.

(For example, consider the case $n'<0$ and $n''<0$. The amplitude increases with x but the wave is moving in the $-x$ direction; this means the amplitude grows in the direction opposite to the propagation direction, so the wave is damped.)

For the lower choice of sign, we see from the magnitdue that the amplitude increases with x if $n''>0$, and decreases with x if $n''<0$. From the argument, we see that the wave is left-moving if $n'>0$ and right-moving if $n'<0$. Hence, the wave is damped if $n'n''>0$ and amplified if $n'n''<0$, exactly the same as in the previous case.

Hence, whether the wave is amplified or damped only depends on the relative signs of n' and n'', and is independent of the direction of propagation.

Exercise 6.6.3

When the refractive index is complex, can the real part of the complex wavefunction be regarded as the solution to the same wave equation? If not, derive a real differential equation whose solution is the real part of Eq. (6.5.6).