10.6: Common Fourier Transforms

To accumulate more intuition about Fourier transforms, let us examine the Fourier transforms of some interesting functions. We will just state the results; the calculations are left as exercises.

Damped waves

We saw in Section 10.2 that an exponentially decay function with decay constant $\eta \in \mathbb{R}^+$ has the following Fourier transform:

$$f(x)=\begin{cases} e^{-\eta x}, & x \geq 0 \\ 0, & x<0, \end{cases} \quad \overset{\text{FT}}{\longrightarrow} \quad F(k)=\frac{-i}{k-i\eta}. \tag{10.6.1}$$

Observe that $F(k)$ is given by a simple algebraic formula. If we "extend" the domain of k to complex values, $F(k)$ corresponds to an analytic function with a simple pole in the upper half of the complex plane, at $k=i\eta$.

Next, consider a decaying wave with wave-number $q \in \mathbb{R}$ and decay constant $\eta \in \mathbb{R}^+$. The Fourier transform is a function with a simple pole at $q+i\eta$:

$$f(x)=\begin{cases} e^{i(q+i\eta)x}, & x \geq 0 \\ 0, & x<0. \end{cases} \quad \overset{\text{FT}}{\longrightarrow} \quad F(k)=\frac{-i}{k-(q+i\eta)}. \tag{10.6.2}$$

On the other hand, consider a wave that grows exponentially with x for $x<0$, and is zero for $x>0$. The Fourier transform is a function with a simple pole in the lower half-plane:

$$f(x)=\begin{cases} 0, & x \geq 0 \\ e^{i(q-i\eta)x}, & x<0. \end{cases} \quad \overset{\text{FT}}{\longrightarrow} \quad F(k)=\frac{i}{k-(q-i\eta)}. \tag{10.6.3}$$

From these examples, we see that oscillations and amplification/decay in $f(x)$ are related to the existence of poles in the algebraic expression for $F(k)$. The real part of the pole position gives the wave-number of the oscillation, and the distance from the pole to the real axis gives the amplification or decay constant. A decaying signal produces a pole in the upper half-plane, while a signal that is increasing exponentially with x produces a pole in the lower half-plane. In both cases, if we plot the Fourier spectrum of $|F(k)|^2$ versus real k, the result is a Lorentzian peak centered at $k=q$, with width 2η.

Gaussian wave-packets

Consider a function with a decay envelope given by a Gaussian function:

$$f(x)=e^{iqx}\, e^{-\gamma x^2}, \quad \text{where } q \in \mathbb{C},\ \gamma \in \mathbb{R}. \tag{10.6.4}$$

This is called a **Gaussian wave-packet**. The width of the envelope is usually characterized by the Gaussian function's **standard deviation**, which is where the curve reaches $e^{-1/2}$ times its peak value. In this case, the standard deviation is $\Delta x=1/\sqrt{2\gamma}$.

We will show that $f(x)$ has the following Fourier transform:

$$F(k)=\sqrt{\frac{\pi}{\gamma}}\, e^{-\frac{(k-q)^2}{4\gamma}}. \tag{10.6.5}$$

To derive this result, we perform the Fourier integral as follows:

$$F(k)=\int_{-\infty}^{\infty} dx\, e^{-ikx}\, f(x) \tag{10.6.6}$$

$$=\int_{-\infty}^{\infty} dx\, \exp\left\{-i(k-q)x-\gamma x^2\right\}. \tag{10.6.7}$$

In the integrand, the expression inside the exponential is quadratic in x. We complete the square:

$$F(k)=\int_{-\infty}^{\infty} dx\, \exp\left\{-\gamma\left(x+\frac{i(k-q)}{2\gamma}\right)^2+\gamma\left(\frac{i(k-q)}{2\gamma}\right)^2\right\} \tag{10.6.8}$$

$$=\exp\left\{-\frac{(k-q)^2}{4\gamma}\right\}\int_{-\infty}^{\infty} dx\, \exp\left\{-\gamma\left(x+\frac{i(k-q)}{2\gamma}\right)^2\right\}. \tag{10.6.9}$$

The remaining integral is the Gaussian integral with a constant imaginary shift in x. By shifting the integration variable, one can show that this is equal the standard Gaussian integral, $\sqrt{\pi/\gamma}$; the details are left as an exercise. We thus arrive at the result stated above.

The Fourier spectrum, $|F(k)|^2$, is a Gaussian function with standard deviation

$$\Delta k = \frac{1}{\sqrt{2(1/2\gamma)}} = \sqrt{\gamma}. \tag{10.6.10}$$

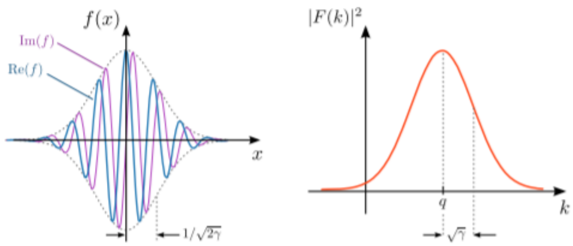

Figure 10.6.1

Once again, the Fourier spectrum is peaked at a value of k corresponding to the wave-number of the underlying sinusoidal wave in $f(x)$, and a stronger (weaker) decay in $f(x)$ leads to a broader (narrower) Fourier spectrum. These features can be observed in the plot above.