8.3: Aside- The Meaning of "Infinity" for Complex Numbers

When talking about $z = \infty$, we are referring to something called **complex infinity**, which can be regarded as a complex number with infinite magnitude and *undefined* argument.

The fact that the argument is undefined may seem strange, but actually we already know of another complex number with this feature: $z = 0$ has zero magnitude and undefined argument. These two special complex numbers are the reciprocals of each other: $1/\infty = 0$ and $1/0 = \infty$.

The complex ∞ behaves differently from the familiar concept of infinity associated with real numbers. For real numbers, positive infinity ($+\infty$) is distinct from negative infinity ($-\infty$). But this doesn't hold for complex numbers, since complex numbers occupy a two-dimensional plane rather than a line. Thus, for complex numbers it does not make sense to define "positive infinity" and "negative infinity" as distinct entities. Instead, we work with a single complex ∞.

From this discussion, we can see why z^p is has a branch point at $z = \infty$. For any finite and nonzero z, we can write $z = re^{i\theta}$, where r is a positive number. The z^p operation then yields a set of complex numbers of the form $r^p\, e^{ip\theta} \times \{\text{root of unity}\}$. For each number in this set, the magnitude goes to infinity as $r \rightarrow \infty$. In this limit, the argument (i.e., the choice of root of unity) becomes irrelevant, and the result is simply ∞.

By similar reasoning, one can prove that $\ln(z)$ has branch points at $z = 0$ and $z = \infty$. This is left as an exercise.

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