2.1: Properties of Derivatives

Rules for limit expressions

Since derivatives are defined using limit expressions, let us review the rules governing limits.

First, the limit of a linear superposition is equal to the linear superposition of limits. Given two constants a_1 and a_2 and two functions f_1 and f_2,

$$\lim_{x\rightarrow c} \big[a_1\, f_1(x) \;+\; a_2\, f_2(x)\big] = a_1 \lim_{x\rightarrow c} f_1(x) \;+\; a_2 \lim_{x\rightarrow c} f_2(x). \tag{2.1.1}$$

Second, limits obey a product rule and a quotient rule:

$$\lim_{x\rightarrow c} \big[\; f_1(x) \; f_2(x) \;\big] = \Big[\lim_{x\rightarrow c} f_1(x)\Big] \; \Big[\lim_{x\rightarrow c} f_2(x)\Big] \tag{2.1.2}$$

$$\lim_{x\rightarrow c} \left[\; \frac{f_1(x)}{f_2(x)} \;\right] = \frac{\lim_{x\rightarrow c} \; f_1(x)}{\lim_{x\rightarrow c} \; f_2(x)}. \tag{2.1.3}$$

As a special exception, the product rule and quotient rule are inapplicable if they result in $0 \times \infty$, ∞/∞, or $0/0$, which are undefined. As an example of why such combinations are problematic, consider this:

$$\lim_{x\rightarrow 0} \; x = \lim_{x\rightarrow 0} \Big[x^2 \; \frac{1}{x}\;\Big] \overset{?}{=} \lim_{x\rightarrow 0} \Big[x^2\Big] \; \lim_{x\rightarrow 0} \Big[\frac{1}{x}\Big] = 0 \,\times\, \infty \;\;\; (??) \tag{2.1.4}$$

In fact, the limit expression has the value of 0; it was not correct to apply the product rule in the second step.

Composition rules for derivatives

Using the rules for limit expressions, we can derive the elementary composition rules for derivatives:

$$\frac{d}{dx}\big[\,\alpha\, f(x) + \beta\, g(x)\,\big] \;= \alpha\, f'(x) + \beta\, g'(x) \qquad\qquad \text{(linearity)} \tag{2.1.5}$$

$$\frac{d}{dx}\big[\,f(x)\, g(x)\,\big] \;= f(x)\, g'(x) + f'(x)\, g(x) \qquad \text{(product rule)} \tag{2.1.6}$$

$$\frac{d}{dx}\big[\,f(g(x))\,\big] \;= f'(g(x))\, g'(x) \qquad\qquad \text{(chain rule)} \tag{2.1.7}$$

These can all be proven by direct substitution into the definition of the derivative, and taking appropriate orders of limits. With the aid of these rules, we can prove various standard results, such as the “power rule” for derivatives:

$$\frac{d}{dx}\big[x^n\big] = n x^{n-1}, \;\;\; n \in \mathbb{N}. \tag{2.1.8}$$

The linearity of the derivative operation implies that derivatives “commute” with sums, i.e. you can move them to the left or right of summation signs. This is a very useful feature. For example, we can use it to prove that the exponential is its own derivative, as follows:

$$\frac{d}{dx}[\exp(x)] = \frac{d}{dx}\sum_{n=0}^\infty \frac{x^n}{n!} \tag{2.1.9}$$

$$= \sum_{n=0}^\infty \frac{d}{dx} \frac{x^n}{n!} \tag{2.1.10}$$

$$= \sum_{n=1}^\infty \frac{x^{n-1}}{(n-1)!} \tag{2.1.11}$$

$$= \exp(x). \tag{2.1.12}$$

Derivatives also commute with limits. For example, we can use this on the alternative definition of the exponential function from Exercise 1 of Chapter 1:

$$\frac{d}{dx}[\exp(x)] = \frac{d}{dx}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \tag{2.1.13}$$

$$= \lim_{n\to\infty}\frac{d}{dx}\left(1+\frac{x}{n}\right)^n \tag{2.1.14}$$

$$= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} \tag{2.1.15}$$

$$= \exp(x). \tag{2.1.16}$$