10.5: Fourier Transforms of Differential Equations

The Fourier transform is a useful tool for solving many differential equations. As an example, consider a damped harmonic oscillator subjected to an additional driving force $f(t)$. This force has an arbitrary time dependence, and is not necessarily harmonic. The equation of motion is

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x(t) = \frac{f(t)}{m}. \tag{10.5.1}$$

To solve for $x(t)$, we first take the Fourier transform of both sides of the above equation. The result is

$$-\omega^2 X(\omega) - 2i\gamma\omega X(\omega) + \omega_0^2 X(\omega) = \frac{F(\omega)}{m}, \tag{10.5.2}$$

where $X(\omega)$ and $F(\omega)$ are the Fourier transforms of $x(t)$ and $f(t)$ respectively. To obtain the left-hand side of this equation, we used the properties of the Fourier transform described in Section 10.4, specifically linearity (1) and the Fourier transforms of derivatives (4). Note also that we are using the convention for time-domain functions introduced in Section 10.3.

The Fourier transform has turned our ordinary differential equation into an algebraic equation which can be easily solved:

$$X(\omega) = \frac{F(\omega)/m}{-\omega^2 - 2i\gamma\omega + \omega_0^2} \tag{10.5.3}$$

Knowing $X(\omega)$, we can use the inverse Fourier transform to obtain $x(t)$:

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \frac{e^{-i\omega t}\, F(\omega)/m}{-\omega^2 - 2i\gamma\omega + \omega_0^2}, \quad \text{where} \quad F(\omega) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} f(t). \tag{10.5.4}$$

To summarize, the solution procedure for the driven harmonic oscillator equation consists of (i) using the Fourier transform on $f(t)$ to obtain $F(\omega)$, (ii) using the above equation to find $X(\omega)$ algebraically, and (iii) performing an inverse Fourier transform to obtain $x(t)$. This is the basis for the Green's function method, a method for systematically solving differential equations that will be discussed in the next chapter.