10.7: The Delta Function

What happens when we feed the Fourier relations into one another? Plugging the Fourier transform into the inverse Fourier transform, we get

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} F(k) && (10.7.1)\\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \int_{-\infty}^{\infty} dx' e^{-ikx'} f(x') && (10.7.2)\\ &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} e^{-ikx'} f(x') && (10.7.3)\\ &= \int_{-\infty}^{\infty} dx'\; \delta(x-x')\, f(x'), && (10.7.4)\end{aligned}$$

In the last step, we have introduced

$$\delta(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}, \tag{10.7.5}$$

which is called the **delta function**. According to the above equations, the delta function acts as a kind of filter: when we multiply it by any function $f(x')$ and integrate over x', the result is the value of that function at a particular point x.

But here's a problem: the above integral definition of the delta function is non-convergent; in particular, the integrand does not vanish at $\pm\infty$. We can get around this by thinking of the delta function as a limiting case of a convergent integral. Specifically, let's take

$$\delta(x-x') = \lim_{\gamma\rightarrow 0} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}\, e^{-\gamma k^2}. \tag{10.7.6}$$

For $\gamma \rightarrow 0$, the "regulator" $\exp(-\gamma k^2)$ which we have inserted into the integrand goes to one, so that the integrand goes back to what we had before; on the other hand, for $\gamma > 0$ the regulator ensures that the integrand vanishes at the end-points so that the integral is well-defined. But the expression on the right is the Fourier transform for a Gaussian wave-packet (see Section 10.6), so

$$\delta(x-x') = \lim_{\gamma\rightarrow 0} \frac{1}{\sqrt{4\pi\gamma}}\, e^{-\frac{(x-x')^2}{4\gamma}}. \tag{10.7.7}$$

This is a Gaussian function of width $\sqrt{2\gamma}$ and area 1. Hence, the delta function can be regarded as the limit of a Gaussian function as its width goes to zero while keeping the area under the curve fixed at unity (which means the height of the peak goes to infinity).

The most important feature of the delta function is it acts like a filter. Whenever it shows up in an integral, it picks out the value of the rest of the integrand evaluated where the delta function is centered:

$$\int_{-\infty}^{\infty} dx\; \delta(x-x_0)\, f(x) = f(x_0). \tag{10.7.8}$$

Intuitively, we can understand this behavior from the above definition of the delta function as the zero-width limit of a Gaussian. When we multiply a function $f(x)$ with a narrow Gaussian centered at x_0, the product will approach zero almost everywhere, because the Gaussian goes to zero. The product is non-zero only in the vicinity of $x = x_0$, where the Gaussian peaks. And because the area under the delta function is unity, integrating that product over all x simply gives the value of the other function at the point x_0.

Note

In physics, the delta function is commonly used to represent the density distributions of **point particles**. For instance, the distribution of mass within an object can be represented by a mass density function. Assuming one-dimensional space for simplicity, we define the mass density $\rho(x)$ as the mass per unit length at position x. By this definition,

$$M = \int_{-\infty}^{\infty} \rho(x)\, dx \tag{10.7.9}$$

is the total mass of the object. Now suppose the mass is distributed among N point particles, which are located at distinct positions x_1, x_2, …, x_N, and have masses m_1, m_2, … m_N. To describe this situation, we can write the mass density function as

$$\rho(x) = \sum_{j=1}^{N} m_j \, \delta(x - x_j). \tag{10.7.10}$$

The reason for this is that if we integrate $\rho(x)$ around the vicinity of the j-th particle, the result is just the mass of that single particle, thanks to the features of the delta function:

$$\begin{aligned} \lim_{\varepsilon\rightarrow 0^+} \int_{x_j-\varepsilon}^{x_j+\varepsilon} \rho(x)\, dx &= \sum_{i=1}^N m_i \left[\lim_{\varepsilon\rightarrow 0^+} \int_{x_j-\varepsilon}^{x_j+\varepsilon} \delta(x-x_i)\, dx\right] && (10.7.11) \\ &= \sum_{i=1}^N m_i \; \delta_{ij} && (10.7.12) \\ &= m_j. && (10.7.13) \end{aligned}$$

Likewise, integrating $\rho(x)$ over all space gives the total mass $m_1 + m_2 + \cdots + m_N$.