6.2: Real Solutions to the Wave Equation

We first consider real solutions to the wave equation. One family of solutions are **travelling waves** of the form

$$f(x,t) = f_0\,\cos\big(kx - \omega t + \phi\big), \quad \text{where } \left|\frac{\omega}{k}\right| = v. \tag{6.2.1}$$

By direct substitution, we can verify that this satisfies the PDE. We call f_0 the **amplitude** of the wave, ϕ the **phase**, ω the (angular) **frequency**, and k the **wavenumber**. By convention, ω is taken to be a positive real number. However, k can be either positive or negative, and its sign determines the direction of propagation of the wave; the magnitude of the wavenumber is inversely related to the wavelength λ by $\lambda = 2\pi/|k|$.

As t increases, the wave moves to the right if k is positive, whereas it moves to the left if k is negative. Here's one way to reason out why this is the case. Consider introducing a small change in time, δt, into the function $\cos(kx - \omega t + \phi)$. If, together with this time shift, we change x by $\delta x = (\omega/k)\,\delta t$, then the change in the kx term and the change in the ωt term cancel, leaving the value of the cosine unchanged:

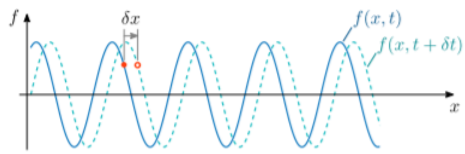

Figure 6.2.1

This implies that the wave shifts by $\delta x = (\omega/k)\,\delta t$ during the time interval δt. Hence, the wave velocity is

$$\text{velocity} = \frac{\delta x}{\delta t} = \frac{(\omega/k)\,\delta t}{\delta t} = \frac{\omega}{k}. \tag{6.2.2}$$

As previously noted, ω is conventionally taken to be a positive real number. Hence, positive k implies that the wave is right-moving (positive velocity), and negative k implies the wave is left-moving (negative velocity). Moreover, the wave speed is the absolute value of the velocity, which is precisely equal to the constant v:

$$\text{speed } = \left|\frac{\delta x}{\delta t}\right| = \frac{\omega}{|k|} = v. \tag{6.2.3}$$

Standing waves

Suppose we have two traveling wave solutions, with equal amplitude and frequency, moving in opposite directions:

$$f(x,t) = f_0\,\cos(kx - \omega t + \phi_1) + f_0\cos(-kx - \omega t + \phi_2). \tag{6.2.4}$$

Here, we denote $k = \omega/c$. Such a superposition is also a solution to the wave equation, called a **standing wave**. It can be re-written in a variable-separated form (i.e., as the product of a function of x and a function of t):

$$f(x,t) = 2f_0\,\cos\big[kx + (\phi_1 - \phi_2)/2\big]\,\cos\big[\omega t - (\phi_1 + \phi_2)/2\big]. \tag{6.2.5}$$

This can be proven using the trigonometric addition formulas, but the proof is tedious.