11.5: Exercises

Exercise 11.5.1

Find the time-domain Green's function of the critically-damped harmonic oscillator ($\gamma = \omega_0$).

Exercise 11.5.2

Consider an overdamped harmonic oscillator ($\gamma > \omega_0$) subjected to a *random* driving force $f(t)$, which fluctuates between random values, which can be either positive or negative, at each time t. The random force satisfies

$$\langle f(t)\rangle = 0 \quad \text{and} \quad \langle f(t) f(t')\rangle = A\,\delta(t-t'), \tag{11.5.1}$$

where $\langle\cdots\rangle$ denotes an average taken over many realizations of the random force and A is some constant. Using the causal Green's function, find the correlation function $\langle x(t_1)\,x(t_2)\rangle$ and the mean squared deviation $\left\langle [x(t+\Delta t) - x(t)]^2\right\rangle$.

Answer

For the over-damped oscillator, the Green's function is

$$G(t,t') = \Theta(t-t')\,\frac{e^{-\gamma(t-t')}}{\Gamma}\sinh\left[\Gamma(t-t')\right], \quad \text{where } \Gamma = \sqrt{\gamma^2 - \omega_0^2}. \tag{11.5.2}$$

Hence, the response to the force f is

$$x(t) = \frac{1}{m\Gamma}\int_{-\infty}^{t} dt'\; e^{-\gamma(t-t')}\,\sinh\left[\Gamma(t-t')\right] f(t'). \tag{11.5.3}$$

From this, we get the following expression for the desired correlation function:

$$\begin{aligned}\langle x(t_1)\,x(t_2)\rangle = \frac{1}{m^2\Gamma^2}\int_{-\infty}^{t_1} dt' \int_{-\infty}^{t_2} dt''\; e^{-\gamma(t_1-t')}\, e^{-\gamma(t_2-t'')} \\ \times \sinh\left[\Gamma(t_1-t')\right] \sinh\left[\Gamma(t_2-t'')\right]\, \langle f(t') f(t'')\rangle.\end{aligned} \tag{11.5.4}$$

Note that the $\langle\cdots\rangle$ can be shifted inside the integrals, because it represents taking the mean over independent sample trajectories. Now, without loss of generality, let us take

$$t_1 \geq t_2. \tag{11.5.5}$$

Since $\langle f(t') f(t'')\rangle = A\delta(t'-t'')$ which vanishes for $t' \neq t''$, the double integral only receives contributions from values of t' not exceeding t_2 (which is the upper limit of the range for t''). Thus, we revise $\int^{t_1} dt'$ into $\int^{t_2} dt'$. The delta function then reduces the double integral into a single integral, which can be solved and simplified with a bit of tedious algebra:

$$\langle x(t_1)\,x(t_2)\rangle = \frac{A}{m^2\Gamma^2} e^{-\gamma(t_1+t_2)} \int_{-\infty}^{t_2} dt' e^{2\gamma t'} \sinh\left[\Gamma(t'-t_1)\right]\, \sinh\left[\Gamma(t'-t_2)\right] \tag{11.5.6}$$

$$\begin{aligned}= \frac{A}{8m^2\Gamma^2} e^{-\gamma(t_1+t_2)} \Bigg[\frac{e^{-\Gamma t_1} e^{(2\gamma+\Gamma)t_2}}{\gamma+\Gamma} + \frac{e^{\Gamma t_1} e^{(2\gamma-\Gamma)t_2}}{\gamma-\Gamma} \\ - \frac{e^{-\Gamma t_1} e^{(\Gamma+2\gamma)t_2} + e^{\Gamma t_1} e^{(-\Gamma+2\gamma)t_2}}{\gamma}\Bigg]\end{aligned} \tag{11.5.7}$$

$$= \frac{A}{8m^2\Gamma\gamma}\left[\frac{e^{-(\gamma-\Gamma)(t_1-t_2)}}{\gamma-\Gamma} - \frac{e^{-(\gamma+\Gamma)(t_1-t_2)}}{\gamma+\Gamma}\right]. \tag{11.5.8}$$

Hence,

$$\left\langle [x(t+\Delta t) - x(t)]^2\right\rangle = 2\left[\left\langle x(t)^2\right\rangle - \langle x(t+\Delta t) x(t)\rangle\right] \tag{11.5.9}$$

$$= \frac{A}{4m^2\Gamma\gamma}\left[\frac{1-e^{-(\gamma-\Gamma)\Delta t}}{\gamma-\Gamma} - \frac{1-e^{-(\gamma+\Gamma)\Delta t}}{\gamma+\Gamma}\right]. \tag{11.5.10}$$