4.2: Conjugates and Magnitudes

For each complex number $z = x + iy$, its **complex conjugate** is a complex number whose imaginary part has the sign flipped:

$$z^* = x - iy. \tag{4.2.1}$$

Conjugation obeys two important properties:

$$(z_1 + z_2)^* = z_1^* + z_2^* \tag{4.2.2}$$

$$(z_1 z_2)^* = z_1^* z_2^*. \tag{4.2.3}$$

Example 4.2.1

Let us prove that $(z_1 z_2)^* = z_1^* z_2^*$. First, let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then,

$$\begin{aligned}
(z_1 z_2)^* &= [(x_1 + iy_1)(x_2 + iy_2)]^* && (4.2.4)\\
&= [(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)]^* && (4.2.5)\\
&= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2) && (4.2.6)\\
&= (x_1 - iy_1)(x_2 - iy_2) && (4.2.7)\\
&= z_1^* z_2^* && (4.2.8)
\end{aligned}$$

For a complex number $z = x + iy$, the **magnitude** of the complex number is

$$|z| = \sqrt{x^2 + y^2}. \tag{4.2.9}$$

This is a non-negative real number. A complex number and its conjugate have the same magnitude: $|z| = |z^*|$. Also, we can show that complex magnitudes have the property

$$|z_1 z_2| = |z_1|\,|z_2|. \tag{4.2.10}$$

This property is similar to the “absolute value” operation for real numbers, hence the similar notation.

As a corollary, taking a power of a complex number raises its magnitude by the same power:

$$|z^n| = |z|^n \quad \text{for} \;\; n \in \mathbb{Z}. \tag{4.2.11}$$