11.4: Looking Ahead

Green's functions are widely used in the study of acoustic and electromagnetic waves, which is a vast topic covered in advanced courses in theoretical physics, electrical engineering, and mechanical engineering. Here, we give a brief sketch of some future directions of study.

So far, we have focused our attentions on the simplest case of an infinite one-dimensional uniform medium. Most practical applications are concerned with three spatial dimensions and non-uniform media. For such cases, the wave equation's frequency-domain Green's function can be generalized to

$$\left[\nabla^2 + n^2(\vec{r}) \left(\frac{\omega}{c}\right)^2\right] G(\vec{r}, \vec{r}\,'; \omega) = \delta^3(\vec{r} - \vec{r}\,'), \tag{11.4.1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the three-dimensional Laplacian operator, and $n(\vec{r})$ is a space-dependent refractive index. On the right-hand side of this equation is the three-dimensional delta function, which describes a point source located at position $\vec{r}\,'$ in the three-dimensional space.

When $n = 1$, the above equation is similar to the frequency-domain Green's function derived in Section 11.2, except that the problem is three-dimensional rather than one-dimensional. Again assuming outgoing boundary conditions, the Green's function in three dimensions can be found using contour integrals similar to those we have previously covered; the result is

$$G(\vec{r}, \vec{r}\,'; \omega) = -\frac{e^{i(\omega/c)|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}. \tag{11.4.2}$$

Like the 1D Green's function derived in Eq. (11.2.13), this depends on $|\vec{r}-\vec{r}\,'|$, and thence describes waves that are emitted isotropically from the source at $\vec{r}\,'$. However, the magnitude of G now decreases to zero with distance, due to the $|\vec{r}-\vec{r}\,'|$ in the denominator. This matches our everyday experience that the sound emitted from a point source grows fainter with distance, which is because the energy carried by the outgoing wave is spread out over a larger area with increasing distance from the source. This is unlike waves in one-dimensional space, which do not become weaker with distance.

When $n(\vec{r})$ is not a constant but varies with position $\vec{r}$, then the waves emitted by the source do not radiate outwards in a simple way. The variations in the refractive index cause the waves to scatter in complicated ways. In most situations, the exact solution for the Green's function cannot be obtained analytically, but must be computed using specialized numerical methods.

For electromagnetic waves, there is another important complication coming from the fact that electromagnetic fields are described by vectors (i.e., the electric field vector and the magnetic field vector), not scalars. The propagation of electromagnetic waves is therefore described by a vectorial wave equation, not the scalar wave equation that we have looked at so far. Moreover, electromagnetic waves are not generated by scalar sources, but by vector sources (typically, electrical currents). The corresponding Green's function is not a scalar function, but a multi-component entity called a **dyadic Green's function**, which describes the vector waves emitted by a vector source.

Finally, even though we have dealt so far with classical (non-quantum) waves, the Green's function concept extends to the theory of quantum mechanics. In quantum field theory, which is the principal theoretical framework used in fundamental physics, calculations typically involve quantum mechanical generalizations of the Green's functions we have studied above, whose values are no longer simple numbers but rather quantum mechanical operators.