8.5: Exercises

Exercise 8.5.1

Find the values of $(i)^i$.

Answer

We can write i in polar coordinates as $\exp(i\pi/2)$. Hence,

$$\begin{aligned}(i)^i &= \exp\left\{i\ln\left[\exp(i\pi/2)\right]\right\} && (8.5.1)\\ &= \exp\left\{i\left[\frac{i\pi}{2}+2\pi i n\right]\right\}, \quad n\in\mathbb{Z} && (8.5.2)\\ &= \exp\left[-2\pi\left(n+\frac{1}{4}\right)\right], \quad n\in\mathbb{Z}. && (8.5.3)\end{aligned}$$

Exercise 8.5.2

Prove that $\ln(z)$ has branch points at $z=0$ and $z=\infty$.

Answer

Let $z=r\exp(i\theta)$, where $r>0$. The values of the logarithm are

$$\ln(z)=\ln(r)+i(\theta+2\pi n), \quad n\in\mathrm{Z}. \tag{8.5.4}$$

For each n, note that the first term is the real part and the second term is the imaginary part of a complex number w_n. The logarithm in the first term can be taken to be the real logarithm.

For $z\rightarrow 0$, we have $r\rightarrow 0$ and hence $\ln(r)\rightarrow -\infty$. This implies that w_n lies infinitely far to the left of the origin on the complex plane. Therefore, $w_n\rightarrow\infty$ (referring to the complex infinity) regardless of the value of n. Likewise, for $z\rightarrow\infty$, we have $r\rightarrow\infty$ and hence $\ln(r)\rightarrow+\infty$. This implies that w_n lies infinitely far to the right of the origin on the complex plane, so $w_n\rightarrow\infty$ regardless of the value of n. Therefore, 0 and ∞ are both branch points of the complex logarithm.

Exercise 8.5.3

For each of the following multi-valued functions, find all the possible function values, at the specified z:

a. $z^{1/3}$ at $z=1$.

b. $z^{3/5}$ at $z=i$.

c. $\ln(z+i)$ at $z=1$.

d. $\cos^{-1}(z)$ at $z=i$

Exercise 8.5.4

For the square root operation $z^{1/2}$, choose a branch cut. Then show that both the branch functions $f_{\pm}(z)$ are analytic over all of $\mathbb{C}$ excluding the branch cut.

Exercise 8.5.5

Consider $f(z)=\ln(z+a)-\ln(z-a)$. For simplicity, let a be a positive real number. As discussed in Section 8.4, we can write this as

$$f(z)=\ln\left|\frac{z+a}{z-a}\right|+i(\theta_+-\theta_-), \qquad \theta_\pm\equiv\arg(z\pm a). \tag{8.5.5}$$

Suppose we represent the arguments as $\theta_+ \in (-\pi, \pi)$ and $\theta_- \in (-\pi, \pi)$. Explain why this implies a branch cut consisting of a straight line joining a with $-a$. Using this representation, calculate the change in $f(z)$ over an infinitesimal loop encircling $z = a$ or $z = -a$. Calculate also the change in $f(z)$ over a loop of radius $R \gg a$ encircling the origin (and thus enclosing both branch points).