2.2: Taylor Series

A function is **infinitely differentiable** at a point x_0 if all orders of derivatives (i.e., the first derivative, the second derivative, etc.) are well-defined at x_0. If a function is infinitely differentiable at x_0, then near that point it can be expanded in a **Taylor series**:

$$f(x) \leftrightarrow \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left[\frac{d^n f}{dx^n}\right](x_0) \tag{2.2.1}$$

$$= f(x_0) + (x-x_0)\, f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0) + \cdots \tag{2.2.2}$$

Here, the "zeroth derivative" refers to the function itself. The Taylor series can be derived by assuming that $f(x)$ can be written as a general polynomial involving terms of the form $(x-x_0)^n$, and then using the definition of the derivative to find the series coefficients.

Many common encountered functions have Taylor series that are exact (i.e., the series is convergent and equal to the value of the function itself) over some portion of their domain. But beware: it is possible for a function to have a divergent Taylor series, or a Taylor series that converges to a different value than the function itself! The conditions under which this happens is a complicated topic that we will not delve into.

Here are some useful Taylor series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{for } |x| < 1 \tag{2.2.3}$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \qquad \text{for } |x| < 1 \tag{2.2.4}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{2.2.5}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{2.2.6}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \tag{2.2.7}$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \tag{2.2.8}$$

The last four Taylor series, (2.2.5)-(2.2.8), converge to the value of the function for all $x \in \mathbb{R}$.