# 3.5: The Gaussian Integral

Here's a famous integral:

$$\int_{-\infty}^{\infty} e^{-\gamma x^2}\, dx. \tag{3.5.1}$$

The integrand is called a **Gaussian**, or **bell curve**, and is plotted below. The larger the value of $\gamma$, the more narrowly-peaked the curve.

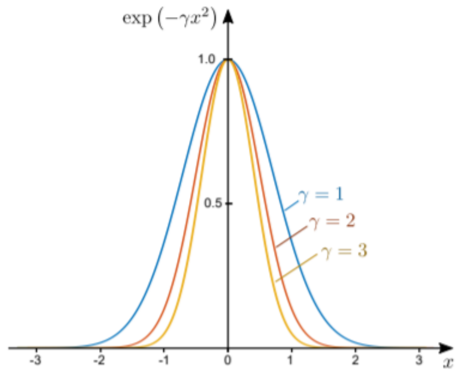


Figure 3.5.1

The integral was solved by Gauss in a brilliant way. Let $I(\gamma)$ denote the value of the integral. Then $I^2$ is just two independent copies of the integral, multiplied together:

$$I^2(\gamma) = \left[\int_{-\infty}^{\infty} dx\; e^{-\gamma x^2}\right] \times \left[\int_{-\infty}^{\infty} dy\; e^{-\gamma y^2}\right]. \tag{3.5.2}$$

Note that in the second copy of the integral, we have changed the "dummy" label $x$ (the integration variable) into $y$, to avoid ambiguity. Now, this becomes a two-dimensional integral, taken over the entire 2D plane:

$$I^2(\gamma) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; e^{-\gamma(x^2+y^2)}. \tag{3.5.3}$$

Next, change from Cartesian to polar coordinates:

$$I^2(\gamma) = \int_0^{\infty} dr\, r \int_0^{2\pi} d\phi\; e^{-\gamma r^2} = \left[\int_0^{\infty} dr\, r\, e^{-\gamma r^2}\right] \times \left[\int_0^{2\pi} d\phi\right] = \frac{1}{2\gamma}\cdot 2\pi. \tag{3.5.4}$$

By taking the square root, we arrive at the result

$$I(\gamma) = \int_{-\infty}^{\infty} dx\; e^{-\gamma x^2} = \sqrt{\frac{\pi}{\gamma}}. \tag{3.5.5}$$