11.3: Causality and the Time-Domain Green's Function

Let us try converting Eq. (11.2.13) into a time-domain Green's function by using the inverse Fourier transform:

$$\begin{aligned} G(x, x'; t-t') &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega(t-t')} \, G(x, x'; \omega) \\ &= \int_{-\infty}^{\infty} d\omega \, \frac{e^{i\omega[|x-x'|-(t-t')]}}{4\pi i\omega} \qquad (?!?) \end{aligned} \tag{11.3.1}$$

There is a problem on the last line: the integral runs over the real-ω line, yet the integrand has a pole at $\omega = 0$, on the real axis, making the integral ill-defined.

To resolve this, we redefine $G(x, x'; \omega)$ as an integral over a *deformed* contour Γ:

$$G(x, x'; t-t') \equiv \int_\Gamma d\omega \, \frac{e^{i\omega[|x-x'|-(t-t')]}}{4\pi i\omega}. \tag{11.3.2}$$

We will choose the deformed contour in a very specific way, which turns out to be the choice that satisfies causality. As shown in the left subplot of the figure below, it runs along the real axis, but skips *above* the pole at the origin.

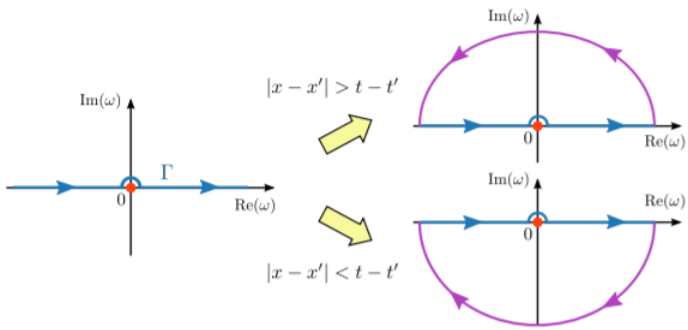

Figure 11.3.1

The integral can be solved by either closing the contour in the upper half-plane, or in the lower half-plane. If we close the contour above, then the loop contour does not enclose the pole, and hence $G(x, x'; t-t') = 0$. According to Jordan's lemma, we must do this if the exponent in the integrand obeys

$$|x-x'| - (t-t') > 0 \quad \Rightarrow \quad |x-x'| > t-t'. \tag{11.3.3}$$

This inequality is satisfied in two cases: either (i) $t < t'$ (in which case the inequality is satisfied for all x, x' because $|x-x'|$ is strictly non-negative), or (ii) $t > t'$ but the value of $t-t'$ is smaller than $|x-x'|$. To understand the physical meaning of these two cases, recall that $G(x, x'; t-t')$ represents the field at position x and time t resulting from a pulse at the space-time point (x', t'). Thus, case (i) corresponds to times occurring before the pulse, and case (ii) corresponds to times occurring after the pulse but too far away from the pulse location for a wave to reach in time.

For the other case, $|x-x'| - (t-t') < 0$, the residue theorem gives

$$G(x, x'; t-t') = -1/2. \tag{11.3.4}$$

The space-time diagram below summarizes the above results:

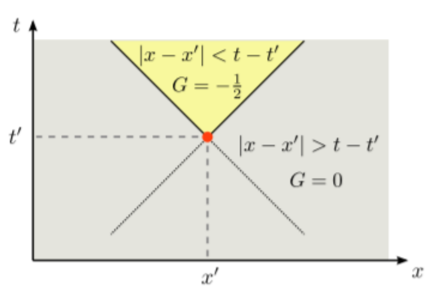

Figure 11.3.2

The resulting time-domain wavefunctions can be written as

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' \left[-\frac{1}{2}\,\Theta(t-t'-|x-x'|)\right] f(x',t'), \tag{11.3.5}$$

where Θ denotes the unit step function. In other words, the wavefunction at each space-time point (x,t) receives equal contribution from the sources $f(x',t')$ at space-time points (x',t') lying within the “past light cone”.