1.7: Continuity

Continuity

Continuity is an important concept in the theory of real functions. A continuous function is one whose output $f(x)$ does not undergo abrupt jumps when x changes by tiny amounts. A function can be continuous over its entire domain, or only a subset of its domain. For example, $\sin(x)$ is continuous for all x, whereas $f(x) = 1/x$ is discontinuous at $x = 0$. Another function that is discontinuous at $x = 0$ is the step function

$$\Theta(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{1.7.1}$$

Mathematicians have even come up with functions that are discontinuous everywhere in their domain, but we won't be dealing with such cases.

The rigorous definition of continuity is as follows:

Definition: Word

A function f is continuous at a point x_0 if, for any $\epsilon > 0$, we can find a $\delta > 0$ such that setting x closer to x_0 than a distance of δ brings $f(x)$ closer to $f(x_0)$ than the specified distance ϵ.

That's a very complicated sentence, and it may be easier to understand using this illustration:

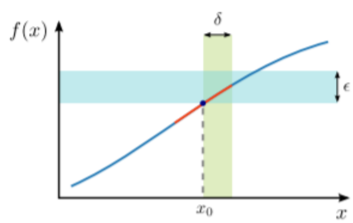

Figure 1.7.1

A counter-example, with a function that has a discontinuity at some x_0, is shown below:

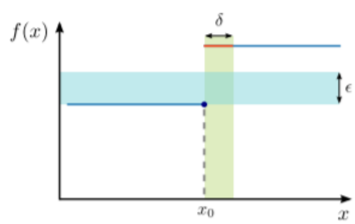

Figure 1.7.2

If we choose ϵ smaller than the gap, then no matter what value of $\delta > 0$ we try, any choice of $0 < x < \delta$ will give a value of $f(x)$ that's further than ϵ from $f(x_0)$. Hence, the continuity condition is violated for sufficiently small choices of $\epsilon = 1/2$, and we say that f is **discontinuous** at x_0.