6.5: Harmonic Waves

We are often interested in waves undergoing **harmonic oscillation**, i.e. varying sinusoidally with a constant frequency ω everywhere in space. Such waves can be described by wavefunctions of the form

$$f(x,y,z,t) = \psi(x,y,z)\, e^{-i\omega t}. \tag{6.5.1}$$

By writing the wavefunction in this form, we are performing a separation of variables between $\vec{r}$ and t. This is a common method for simplifying PDEs, and is justified by the linearity of the wave equation. If we can find harmonic solutions for each frequency ω, we can linearly combine them to construct more general solutions that are non-harmonic.

By direct substitution into Eq. (6.4.1), we can show that $\psi(x)$ obeys

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{v}\right)^2\right] \psi(x,y,z) = 0. \tag{6.5.2}$$

This is related to the original time-dependent wave equation by the replacement of $\partial/\partial t$ with $-i\omega$.

Waves in complex media

So far, our discussion has been limited to waves propagating in a uniform, energy-conserving medium with a fixed wave speed v. There are two important generalizations of this scenario: (i) non-uniform media, in which the wave speed varies with position, and (ii) energy non-conserving media, in which the waves lose or gain energy as they propagate. To capture these phenomena, we replace the constant v by

$$v = \frac{c}{n}, \tag{6.5.3}$$

where n is called the **refractive index**, and the constant c is the wave speed in the limit $n = 1$. In the case of electromagnetic waves, c is the speed of light in a vacuum.

If the refractive index is now allowed to vary with position, the wave equation in the harmonic representation becomes

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + n^2(x,y,z)\left(\frac{\omega}{c}\right)^2\right] \psi(x,y,z) = 0. \tag{6.5.4}$$

Wave amplification and attenuation

By allowing the refractive index n to be *complex*, the wave equation can describe the phenomena of **wave amplification** (which is also called **gain**) and **wave attenuation** (also called **loss**). Amplified and attenuated waves occur in many different contexts in physics; for example, the amplification of light waves is the underlying basis for the laser.

To study these phenomena, let us go back to one-dimensional space and the simple scenario of a position-independent refractive index. For harmonic waves, the wave equation reduces to

$$\left[\frac{d^2}{dx^2} + n^2\left(\frac{\omega}{c}\right)^2\right] \psi(x) = 0. \tag{6.5.5}$$

We now let n be complex, while keeping ω and c as positive real numbers. The solutions to the ODE have the form

$$\psi(x) = A \exp\left(\pm\frac{in\omega}{c}x\right), \quad \text{where} \;\; A \in \mathbb{C}. \tag{6.5.6}$$

Let us write the complex refractive index as

$$n = n' + in'', \quad \text{where} \;\; n', n'' \in \mathbb{R}. \tag{6.5.7}$$

Then

$$\psi(x) = A \exp[\pm i n'(\omega/c)x] \; \exp[\mp n''(\omega/c)x]. \tag{6.5.8}$$

The first exponential factor describes the oscillation of the wavefunction, with the $\pm$ sign determining whether the harmonic wave is moving to the right or to the left. The second exponential describes the amplification or attenuation of the wave. If $n'' \neq 0$, the amplitude varies exponentially with x. Thus, depending on the signs of the various parameters, the wave might grow exponentially along its direction of propagation, which corresponds to amplification, or decrease exponentially along its direction of propagation, which corresponds to damping.