5.5: Exercises

Exercise 5.5.1

In Section 5.2, we encountered the complex frequencies

$$\omega_{\pm} = -i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}. \tag{5.5.1}$$

For fixed ω_0 and $\omega_0 > \gamma$ (under-damping), prove that $\omega_{\pm}$ lie along a circular arc in the complex plane.

Exercise 5.5.2

Derive the general solution for the critically damped harmonic oscillator, Eq. (5.3.16), by following these steps:

a. Consider the complex ODE, in the under-damped regime $\omega_0 > \gamma$. We saw in Section 5.3 that the general solution has the form

$$z(t) = \psi_+ \exp\left[\left(-\gamma - i\sqrt{\omega_0^2 - \gamma^2}\right) t\right] \;+\; \psi_- \exp\left[\left(-\gamma + i\sqrt{\omega_0^2 - \gamma^2}\right) t\right] \tag{5.5.2}$$

for some complex parameters ψ_+ and ψ_-. Define the positive parameter $\varepsilon = \sqrt{\omega_0^2 - \gamma^2}$. Re-write $z(t)$ in terms of γ and ε (i.e., eliminating ω_0).

b. The expression for $z(t)$ is presently parameterized by the independent parameters ψ_+, ψ_-, ε, and γ. We are free to re-define the parameters, by taking

$$\alpha = \psi_+ + \psi_- \tag{5.5.3}$$
$$\beta = -i\varepsilon(\psi_+ - \psi_-). \tag{5.5.4}$$

Using these equations, express $z(t)$ using a new set of independent complex parameters, one of which is ε. Explicitly identify the other independent parameters, and state whether they are real or complex.

c. Expand the exponentials in $z(t)$ in terms of the parameter ε. Then show that in the limit $\varepsilon \to 0$, $z(t)$ reduces to the critically-damped general solution (5.3.16).

Exercise 5.5.3

Repeat the above derivation for the critically-damped solution, but starting from the over-damped regime $\gamma > \omega_0$.

Exercise 5.5.4

Let $z(t)$ be a complex function of a real input t, which obeys the differential equation

$$\frac{dz}{dt} = -i\,(\omega_1 - i\gamma)\; z(t), \tag{5.5.5}$$

where ω_1 and γ are real. Find the general solution for $z(t)$, and hence show that $z(t)$ satisfies the damped oscillator equation

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] z(t) = 0 \tag{5.5.6}$$

for some ω_0^2. Finally, show that this harmonic oscillator is always under-damped.

Answer

The general solution is

$$z(t) = A\exp[-i(\omega_1 - i\gamma)t]. \tag{5.5.7}$$

It can be verified by direct substitution that this is a solution to the differential equation. It contains one free parameter, and the differential equation is first-order, so it must be a general solution. Next,

$$\frac{d^2 z}{dt^2} + 2\gamma \frac{dz}{dt} = (-i)^2 (\omega_1 - i\gamma)^2 z(t) - 2i\gamma(\omega_1 - i\gamma) z(t) \tag{5.5.8}$$

$$= \left[-\omega_1^2 + \gamma^2 + 2i\gamma\omega_1 - 2i\gamma\omega_1 - 2\gamma^2)\right] z(t) \tag{5.5.9}$$

$$= -\left(\omega_1^2 + \gamma^2\right) z(t). \tag{5.5.10}$$

Hence, $z(t)$ obeys a damped harmonic oscillator equation with $\omega_0^2 = \omega_1^2 + \gamma^2$. This expression for the natural frequency ensures that $\omega_0^2 > \gamma^2$ (assuming the parameters γ and ω_1 are both real); hence, the harmonic oscillator is always under-damped.