4.8: Exercises

Exercise 4.8.1

Let $z = x + iy$, where $x, y \in \mathbb{R}$. For each of the following expressions, find (i) the real part, (ii) the imaginary part, (iii) the magnitude, and (iv) the complex argument, in terms of x and y:

a. z^2

b. $1/z$

c. $\exp(z)$

d. $\exp(iz)$

e. $\cos(z)$

Exercise 4.8.2

Prove that $|z_1 z_2| = |z_1|\,|z_2|$, by using (i) the polar representation, and (ii) the Cartesian representation.

Answer

Using the polar representation: let $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$. Then

$$\begin{align} |z_1\, z_2| &= \left|r_1 e^{i\theta_1}\, r_2 e^{i\theta_2}\right| \tag{4.8.1} \\ &= \left|(r_1 r_2)\, e^{i(\theta_1+\theta_2)}\right| \tag{4.8.2} \\ &= r_1 r_2 \tag{4.8.3} \\ &= |z_1|\,|z_2|. \tag{4.8.4} \end{align}$$

Using the Cartesian representation: let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. For convenience, we evaluate the squared magnitude:

$$\begin{align} |z_1\, z_2|^2 &= |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)|^2 \tag{4.8.5} \\ &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \tag{4.8.6} \\ &= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 \tag{4.8.7} \\ &= \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) \tag{4.8.8} \\ &= |z_1|^2\,|z_2|^2. \tag{4.8.9} \end{align}$$

Exercise 4.8.3

Prove that $(z_1 z_2)^* = z_1^* z_2^*$, by using (i) the polar representation, and (ii) the Cartesian representation.

Answer

Using the polar representation: let $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$. Then

$$\begin{align} (z_1\, z_2)^* &= \left((r_1 r_2)\, e^{i(\theta_1+\theta_2)}\right)^* \tag{4.8.10} \\ &= (r_1 r_2)\, e^{-i(\theta_1+\theta_2)} \tag{4.8.11} \\ &= \left(r_1 e^{-i\theta_1}\right)\left(r_2 e^{-i\theta_2}\right) \tag{4.8.12} \\ &= z_1^*\, z_2^*. \tag{4.8.13} \end{align}$$

Using the Cartesian representation: let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

$$
\begin{aligned}
(z_1\, z_2)^* &= [(x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)]^* && (4.8.14)\\
&= (x_1x_2 - y_1y_2) - i\,(x_1y_2 + x_2y_1) && (4.8.15)\\
&= (x_1 - iy_1)\,(x_2 - iy_y) && (4.8.16)\\
&= z_1^*\, z_2^*. && (4.8.17)
\end{aligned}
$$

Exercise 4.8.4

Identify the problem with this chain of equations:

$$-1 = i \cdot i = \sqrt{-1}\,\sqrt{-1} = \sqrt{-1\cdot -1} = \sqrt{1} = 1. \tag{4.8.18}$$

Answer

The problem arises in this part of the chain: $i \cdot i = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)}$. The square root is a non-integer power, and non-integer powers are not allowed to take part in standard complex algebra equations in the same way as addition, subtraction, multiplication, division, and integer powers.

As discussed in Chapter 8, square roots and other non-integer powers have multiple values. The definition of the imaginary unit is often written as $i = \sqrt{-1}$, but this is misleading. Actually, $\sqrt{-1}$ has two legitimate values; one of these values is (by definition) i, while the other value is $-i$.

Exercise 4.8.5

With the aid of Euler's formula, prove that

$$
\begin{aligned}
\cos(3x) &= 4[\cos(x)]^3 - 3\cos(x) && (4.8.19)\\
\sin(3x) &= 3\sin(x) - 4[\sin(x)]^3 && (4.8.20)
\end{aligned}
$$

Exercise 4.8.6

For $z_1, z_2 \in \mathbb{C}$ and $\theta \in \mathbb{R}$, show that $\mathrm{Re}\left[z_1 e^{i\theta} + z_2 e^{-i\theta}\right] = A\cos(\theta) + B\sin(\theta)$, for some $A, B \in \mathbb{R}$. Find explicit expressions for A and B in terms of z_1 and z_2.

Exercise 4.8.7

In Section 4.4, we saw that the conjugation operation corresponds to a reflection about the real axis. What operation corresponds to a reflection about the imaginary axis?

Exercise 4.8.8

Consider the complex function of a real variable $z(t) = 1/(\alpha t + \beta)$, where $\alpha, \beta \in \mathbb{C}$ and $t \in \mathbb{R}$.

a. For $\alpha = 1$ and $\beta = i$, show that $z(t)$ can be re-expressed as $z(s) = (1 + e^{is})/(2i)$, where $s \in (-\pi, \pi)$. Hint: find a real mapping $t(s)$.

b. Hence, show that the trajectory for arbitrary complex values of α, β has the form of a circle.

Exercise 4.8.9

With the help of a computer plotting program, generate complex trajectories for the following functions (for real inputs $t \in \mathbb{R}$). Explain their key features, including the directions of the trajectories:

a. $z(t) = \left[1 + \dfrac{\cos(\beta t)}{2}\right]\exp(it)$, for $\beta = 10$ and for $\beta = \sqrt{5}$.

b. $z(t) = -it \pm \sqrt{1 - t^2}$.

c. $z(t) = ae^{it} + be^{-it}$, for $a = 1, b = -2$ and for $a = 1, b = 2$.