3.9: Exercises

Exercises

Exercise 3.9.1

Let $\mathscr{H}_A$ and $\mathscr{H}_B$ denote single-particle Hilbert spaces with well-defined inner products. That is to say, for all vectors $|\mu\rangle, |\mu'\rangle, |\mu''\rangle \in \mathscr{H}_A$, that Hilbert space's inner product satisfies the inner product axioms

a. $\langle\mu|\mu'\rangle = \langle\mu'|\mu\rangle^*$

b. $\langle\mu|\mu\rangle \in \mathbb{R}_0^+$, and $\langle\mu|\mu\rangle = 0$ if and only if $|\mu\rangle = 0$.

c. $\langle\mu|\,\big(|\mu'\rangle + |\mu''\rangle\big) = \langle\mu|\mu'\rangle + \langle\mu|\mu''\rangle$

d. $\langle\mu|\,\big(c|\mu'\rangle\big) = c\langle\mu|\mu'\rangle$ for all $c \in \mathbb{C}$,

and likewise for vectors from $\mathscr{H}_B$ with that Hilbert space's inner product.

In Section 3.1, we defined a tensor product space $\mathscr{H}_A \otimes \mathscr{H}_B$ as the space spanned by the basis vectors $\{|\mu\rangle \otimes |\nu\rangle\}$, where the $|\mu\rangle$'s are basis vectors for $\mathscr{H}_A$ and the $|\nu\rangle$'s are basis vectors for $\mathscr{H}_B$. Prove that we can define an inner product using

$$\Big(\langle\mu| \otimes \langle\nu|\Big)\Big(|\mu'\rangle \otimes |\nu'\rangle\Big) \;\equiv\; \langle\mu|\mu'\rangle\,\langle\nu|\nu'\rangle = \delta_{\mu\mu'}\delta_{\nu\nu'} \tag{3.9.1}$$

which satisfies the inner product axioms.

Exercise 3.9.2

Consider the density operator

$$\hat{\rho} = \frac{1}{2}|+z\rangle\langle+z| + \frac{1}{2}|+x\rangle\langle+x| \tag{3.9.2}$$

where $|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$. This can be viewed as an equal-probability sum of two different pure states. However, the density matrix can also be written as

$$\hat{\rho} \;=\; p_1\,|\psi_1\rangle\langle\psi_1| \;+\; p_2\,|\psi_2\rangle\langle\psi_2| \tag{3.9.3}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are the eigenvectors of $\hat{\rho}$. Show that p_1 and p_2 are *not* 1/2.

Exercise 3.9.3

Consider two distinguishable particles, A and B. The 2D Hilbert space of A is spanned by $\{|m\rangle, |n\rangle\}$, and the 3D Hilbert space of B is spanned by $\{|p\rangle, |q\rangle, |r\rangle\}$. The two-particle state is

$$|\psi\rangle = \frac{1}{3}\,|m\rangle|p\rangle + \frac{1}{\sqrt{6}}\,|m\rangle|q\rangle + \frac{1}{\sqrt{18}}\,|m\rangle|r\rangle + \frac{\sqrt{2}}{3}\,|n\rangle|p\rangle + \frac{1}{\sqrt{3}}\,|n\rangle|q\rangle + \frac{1}{3}\,|n\rangle|r\rangle. \tag{3.9.4}$$

Find the entanglement entropy.

Further Reading

[1] Bransden & Joachain, §14.1—14.4, §17.1–17.5

[2] Sakurai, §3.9

[3] A. Einstein, B. Podolsky, and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Physical Review **47**, 777 (1935).

[4] J. S. Bell, *On the Einstein-Podolsky-Rosen paradox*, Physics **1**, 195 (1964).

[5] N. D. Mermin, *Bringing home the atomic world: Quantum mysteries for anybody*, American Journal of Physics **49**, 940 (1981).

[6] A. Aspect, *Bell's inequality test: more ideal than ever*, Nature (News and Views) **398**, 189 (1999).

[7] A. K. Ekert, *Quantum Cryptography Based on Bell's Theorem*, Physical Review Letters **67**, 661 (1991).

[8] H. Everett, III, *The Theory of the Universal Wave Function* (PhD thesis), Princeton University (1956)

[9] A. Albrecht, *Following a "collapsing" wave function*, Physical Review D 48, 3768 (1993).

[10] A. Edelman and N. R. Rao, *Random matrix theory*, Acta Numerica **14**, 233 (2005).