3.1: Quantum States of Multi-Particle Systems

So far, we have studied quantum mechanical systems consisting of single particles. The next important step is to look at systems of more than one particle. We shall see that the postulates of quantum mechanics, when applied to multi-particle systems, give rise to interesting and counterintuitive phenomena such as **quantum entanglement**.

Suppose we have two particles labeled A and B. If each individual particle is treated as a quantum system, the postulates of quantum mechanics require that its state be described by a vector in a Hilbert space. Let $\mathscr{H}_A$ and $\mathscr{H}_B$ denote the respective single-particle Hilbert spaces. Then the Hilbert space for the combined system of two particles is

$$\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B. \tag{3.1.1}$$

The symbol $\otimes$ refers to a **tensor product**, a mathematical operation that combines two Hilbert spaces to form another Hilbert space. It is most easily understood in terms of explicit basis vectors: let $\mathscr{H}_A$ be spanned by a basis $\{|\mu_1\rangle, |\mu_2\rangle, |\mu_3\rangle, \ldots\}$, and $\mathscr{H}_B$ be spanned by $\{|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle, \ldots\}$. Then $\mathscr{H}_A \otimes \mathscr{H}_B$ is a space spanned by basis vectors consisting of *pairwise combinations* of basis vectors drawn from the $\mathscr{H}_A$ and $\mathscr{H}_B$ bases:

$$\Big\{ \; |\mu_i\rangle \otimes |\nu_j\rangle \;\; \text{for all} \;\; |\mu_i\rangle, \; |\nu_j\rangle \; \Big\}. \tag{3.1.2}$$

Thus, if $\mathscr{H}_A$ has dimension d_A and $\mathscr{H}_B$ has dimension d_B, then $\mathscr{H}_A \otimes \mathscr{H}_B$ has dimension $d_A d_B$. Any two-particle state can be written as a superposition of these basis vectors:

$$|\psi\rangle = \sum_{ij} c_{ij} \; |\mu_i\rangle \otimes |\nu_j\rangle. \tag{3.1.3}$$

The inner product between the tensor product basis states is defined as follows:

$$\Big(|\mu_i\rangle \otimes |\nu_j\rangle \; , \; |\mu_p\rangle \otimes |\nu_q\rangle \Big) \;\equiv\; \Big(\langle\mu_i| \otimes \langle\nu_j| \Big) \Big(|\mu_p\rangle \otimes |\nu_q\rangle \Big) \;\equiv\; \langle\mu_i|\mu_p\rangle \, \langle\nu_j|\nu_q\rangle = \delta_{ip}\delta_{jq}. \tag{3.1.4}$$

In other words, the inner product is performed "slot-by-slot". We calculate the inner product for A, calculate the inner product for B, and then multiply the two resulting numbers. You can check that this satisfies all the formal requirements for an inner product in linear algebra (see Exercise 3.9.1).

For example, suppose $\mathscr{H}_A$ and $\mathscr{H}_B$ are both 2D Hilbert spaces describing spin-$1/2$ degrees of freedom. Each space can be spanned by an orthonormal basis $\{\, |+z\rangle, \, |-z\rangle \,\}$, representing "spin-up" and "spin-down". Then the tensor product space $\mathscr{H}$ is a 4D space spanned by

$$\Big\{ \; |+z\rangle \otimes |+z\rangle \, , \; |+z\rangle \otimes |-z\rangle \, , \; |-z\rangle \otimes |+z\rangle \, , \; |-z\rangle \otimes |-z\rangle \; \Big\}. \tag{3.1.5}$$

We now make an important observation. If A is in state $|\mu\rangle$ and B is in state $|\nu\rangle$, then the state of the combined system is fully specified: $|\mu\rangle \otimes |\nu\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$. But the reverse is not generally true! There exist states of the combined system that *cannot* be expressed in terms of definite states of the individual particles. For example, consider the following quantum state of two spin-$1/2$ particles:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \Big(|+z\rangle \otimes |-z\rangle \; - \; |-z\rangle \otimes |+z\rangle \Big). \tag{3.1.6}$$

This state is constructed from two of the four basis states in (3.1.5), and you can check that the factor of $1/\sqrt{2}$ ensures the normalization $\langle\psi|\psi\rangle = 1$ with the inner product rule (3.1.4). It is evident from looking at Equation (3.1.6) that neither A nor B possesses a definite $|+z\rangle$ or $|-z\rangle$ state. Moreover, we shall show (in Section 3.7) that there's *no* choice of basis that allows this state to be expressed in terms of definite individual-particle states; i.e.,

$$|\psi\rangle \ne |\psi_A\rangle \otimes |\psi_B\rangle \quad \text{for any} \;\; |\psi_A\rangle \in \mathscr{H}_A, \; |\psi_B\rangle \in \mathscr{H}_B. \tag{3.1.7}$$

In such a situation, the two particles are said to be **entangled**.

It is cumbersome to keep writing $\otimes$ symbols, so we will henceforth omit the $\otimes$ in cases where the tensor product is obvious. For instance,

$$\frac{1}{\sqrt{2}} \Big(|+z\rangle \otimes |-z\rangle \; - \; |-z\rangle \otimes |+z\rangle \Big) \;\equiv\; \frac{1}{\sqrt{2}} \Big(|+z\rangle |-z\rangle \; - \; |-z\rangle |+z\rangle \Big). \tag{3.1.8}$$

For systems of more than two particles, quantum states can be defined using multiple tensor products. Suppose a quantum system contains N particles described by the individual Hilbert spaces $\{\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_N\}$ having dimensionality $\{d_1, \ldots, d_N\}$. Then the overall system is described by the Hilbert space

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_N, \tag{3.1.9}$$

which has dimensionality $d = d_1 d_2 \cdots d_N$. The dimensionality scales exponentially with the number of particles! For instance, if each particle has a 2D Hilbert space, a 20-particle system has a Hilbert space with $2^{20} = 1\,048\,576$ dimensions. Thus, even in quantum systems with a modest number of particles, the quantum state can carry huge amounts of information. This is one of the motivations behind the active research field of quantum computing.

Finally, a proviso: although we refer to subsystems like A and B as "particles" for narrative convenience, they need not be actual particles. All this formalism applies to general subsystems—i.e., subsets of a large quantum system's degrees of freedom. For instance, if a quantum system has a position eigenbasis for 3D space, the x, y, and z coordinates are distinct degrees of freedom, so each position eigenstate is really a tensor product:

$$|\, \mathbf{r} = (x, y, z) \,\rangle \;\equiv\; |x\rangle \, |y\rangle \, |z\rangle. \tag{3.1.10}$$

Also, if the subsystems really *are* particles, we are going to assume for now that the particles are distinguishable. There are other complications that arise if the particles are "identical", which will be the subject of the next chapter (if you're unsure what this means, just read on).