5.3: Quantizing The Electromagnetic Field

Previously (Section 4.4), we have gone through the process of quantizing a scalar boson field. The classical field is decomposed into normal modes, and each mode is quantized by assigning it an independent set of creation and annihilation operators. By comparing the oscillator energies in the classical and quantum regimes, we can derive the Hermitian operator corresponding to the classical field variable, expressed using the creation and annihilation operators. We will use the same approach, with only minor adjustments, to quantize the electromagnetic field.

First, consider a "source-free" electromagnetic field—i.e., with no electric charges and currents. Without sources, Maxwell's equations (in SI units, and in a vacuum) reduce to:

$$\nabla\cdot\mathbf{E} = 0 \tag{5.3.1}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{5.3.2}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{5.3.3}$$

$$\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}. \tag{5.3.4}$$

Once again, we introduce the scalar potential Φ and vector potential $\mathbf{A}$:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} \tag{5.3.5}$$

$$\mathbf{B} = \nabla\times\mathbf{A}. \tag{5.3.6}$$

With these relations, Equations (5.3.2) and (5.3.3) are satisfied automatically via vector identities. The two remaining equations, (5.3.1) and (5.3.4), become:

$$\nabla^2\Phi = -\frac{\partial}{\partial t}\nabla\cdot\mathbf{A} \tag{5.3.7}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = \nabla\left[\frac{1}{c^2}\frac{\partial}{\partial t}\Phi + \nabla\cdot\mathbf{A}\right]. \tag{5.3.8}$$

In the next step, we choose a convenient gauge called the **Coulomb gauge**:

$$\Phi = 0, \quad \nabla\cdot\mathbf{A} = 0. \tag{5.3.9}$$

(To see that we can always make such a gauge choice, suppose we start out with a scalar potential Φ_0 and vector potential $\mathbf{A}_0$ not satisfying (5.3.9). Perform a gauge transformation with a gauge field $\Lambda(\mathbf{r},t) = -\int^t dt'\;\Phi_0(\mathbf{r},t')$. The new scalar potential is $\Phi = \Phi_0 + \dot{\Lambda} = 0$; moreover, the new vector potential satisfies

$$\nabla\cdot\mathbf{A} = \nabla\cdot\mathbf{A}_0 - \nabla^2\Lambda = \nabla\cdot\mathbf{A}_0 + \int^t dt'\;\nabla^2\Phi_0(\mathbf{r},t'). \tag{5.3.10}$$

Upon using Equation (5.3.7), we find that $\nabla\cdot\mathbf{A} = 0$.)

In the Coulomb gauge, Equation (5.3.7) is automatically satisfied. The sole remaining equation, (5.3.8), simplifies to

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = 0. \tag{5.3.11}$$

This has plane-wave solutions of the form

$$\mathbf{A}(\mathbf{r},t) = \left(\mathcal{A}\;e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \text{c. c.}\right)\mathbf{e}, \tag{5.3.12}$$

where $\mathcal{A}$ is a complex number (the **mode amplitude**) that specifies the magnitude and phase of the plane wave, $\mathbf{e}$ is a real unit vector (the **polarization vector**) that specifies which direction the vector potential points along, and "c.c." denotes the complex conjugate of the first term. Referring to Equation (5.3.11), the angular frequency ω must satisfy

$$\omega = c|\mathbf{k}|. \tag{5.3.13}$$

Moreover, since $\nabla\cdot\mathbf{A} = 0$, it must be the case that

$$\mathbf{k}\cdot\mathbf{e}=0. \tag{5.3.14}$$

In other words, the polarization vector is perpendicular to the propagation direction. For any given $\mathbf{k}$, we can choose (arbitrarily) two orthogonal polarization vectors.

Now suppose we put the electromagnetic field in a box of volume $V=L^3$, with periodic boundary conditions (we will take $L\to\infty$ at the end). The $\mathbf{k}$ vectors form a discrete set:

$$k_j=\frac{2\pi n_j}{L},\quad n_j\in\mathbf{Z},\quad \text{for}\;\; j=1,2,3. \tag{5.3.15}$$

Then the vector potential field can be decomposed as a superposition of plane waves,

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\lambda}\left(\mathcal{A}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}+\text{c. c.}\right)\mathbf{e}_{\mathbf{k}\lambda},\quad\text{where}\quad \omega_{\mathbf{k}}=c|\mathbf{k}|. \tag{5.3.16}$$

Here, λ is a two-fold polarization degree of freedom indexing the two possible orthogonal polarization vectors for each $\mathbf{k}$. (We won't need to specify how exactly these polarization vectors are defined, so long as the definition is used consistently.)

To convert the classical field theory into a quantum field theory, for each $(\mathbf{k},\lambda)$ we define an independent set of creation and annihilation operators:

$$\left[\hat{a}_{\mathbf{k}\lambda},\hat{a}^{\dagger}_{\mathbf{k}'\lambda'}\right]=\delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'},\quad \left[\hat{a}_{\mathbf{k}\lambda},\hat{a}_{\mathbf{k}'\lambda'}\right]=\left[\hat{a}^{\dagger}_{\mathbf{k}\lambda},\hat{a}^{\dagger}_{\mathbf{k}'\lambda'}\right]=0. \tag{5.3.17}$$

Then the Hamiltonian for the electromagnetic field is

$$\hat{H}=\sum_{\mathbf{k}\lambda}\hbar\omega_{\mathbf{k}}\,\hat{a}^{\dagger}_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda},\quad\text{where}\quad \omega_{\mathbf{k}}=c|\mathbf{k}|. \tag{5.3.18}$$

The vector potential is now promoted into a Hermitian operator in the Heisenberg picture:

$$\hat{\mathbf{A}}(\mathbf{r},t)=\sum_{\mathbf{k}\lambda}\mathcal{C}_{\mathbf{k}\lambda}\left(\hat{a}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}+\text{h. c.}\right)\mathbf{e}_{\mathbf{k}\lambda}. \tag{5.3.19}$$

Here, $\mathcal{C}_{\mathbf{k}\lambda}$ is a constant to be determined, and "h.c." denotes the Hermitian conjugate. The creation and annihilation operators in this equation are Schrödinger picture ($t=0$) operators. The particles they create/annihilate are **photons**—elementary particles of light.

To find $\mathcal{C}_{\mathbf{k}\lambda}$, we compare the quantum and classical energies. Suppose the electromagnetic field is in a coherent state $|\alpha\rangle$ such that for any $\mathbf{k}$ and λ,

$$\hat{a}_{\mathbf{k}\lambda}|\alpha\rangle=\alpha_{\mathbf{k}\lambda}|\alpha\rangle \tag{5.3.20}$$

for some $\alpha_{\mathbf{k}\lambda}\in\mathbb{C}$. From this and Equation (5.3.19), we identify the corresponding classical field

$$\mathbf{A}(\mathbf{r},t)=\sum_{\mathbf{k}\lambda}\left(\mathcal{A}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}+\text{c. c.}\right)\mathbf{e}_{\mathbf{k}\lambda},\quad\text{where}\quad \mathcal{C}_{\mathbf{k}\lambda}\alpha_{\mathbf{k}\lambda}=\mathcal{A}_{\mathbf{k}\lambda}. \tag{5.3.21}$$

For each $\mathbf{k}$ and λ, Equations (5.3.5)–(5.3.6) give the electric and magnetic fields

$$\mathbf{E}_{\mathbf{k}\lambda}=\left(i\omega_{\mathbf{k}}\mathcal{A}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}+\text{c. c.}\right)\mathbf{e}_{\mathbf{k}\lambda} \tag{5.3.22}$$

$$\mathbf{B}_{\mathbf{k}\lambda}=\left(i\mathcal{A}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}+\text{c. c.}\right)\mathbf{k}\times\mathbf{e}_{\mathbf{k}\lambda}. \tag{5.3.23}$$

In the classical theory of electromagnetism, Poynting's theorem tells us that the total energy carried by a classical plane electromagnetic wave is

$$\begin{aligned}E&=\int_V d^3r\;\frac{\epsilon_0}{2}\left(|\mathbf{E}_{\mathbf{k}\lambda}|^2+c^2|\mathbf{B}_{\mathbf{k}\lambda}|^2\right)\\&=2\,\epsilon_0\,\omega_{\mathbf{k}}^2\,|\mathcal{A}_{\mathbf{k}\lambda}|^2\,V.\end{aligned} \tag{5.3.24}$$

Here, V is the volume of the enclosing box, and we have used the fact that terms like $e^{2i\mathbf{k}\cdot\mathbf{r}}$ vanish when integrated over $\mathbf{r}$. Hence, we make the correspondence

$$2\,\epsilon_0\,\omega_{\mathbf{k}}^2\,\left|\mathcal{C}_{\mathbf{k}\lambda}\alpha_{\mathbf{k}\lambda}\right|^2 V = \hbar\omega_{\mathbf{k}}|\alpha_{\mathbf{k}\lambda}|^2 \quad\Rightarrow\quad \mathcal{C}_{\mathbf{k}\lambda} = \sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}. \tag{5.3.25}$$

We thus arrive at the result

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}\lambda}\hbar\omega_{\mathbf{k}}\,\hat{a}_{\mathbf{k}\lambda}^\dagger\hat{a}_{\mathbf{k}\lambda} \\ \hat{\mathbf{A}}(\mathbf{r},t) &= \sum_{\mathbf{k}\lambda}\sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}\left(\hat{a}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \mathrm{h.\,c.}\right)\mathbf{e}_{\mathbf{k}\lambda} \\ \omega_{\mathbf{k}} &= c|\mathbf{k}|, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}'\lambda'}^\dagger\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}'\lambda'}\right] = 0. \end{aligned} \tag{5.3.26}$$

To describe infinite free space rather than a finite-volume box, we take the $L\rightarrow\infty$ limit and re-normalize the creation and annihilation operators by the replacement

$$\hat{a}_{\mathbf{k}\lambda} \rightarrow \sqrt{\frac{(2\pi)^3}{V}}\;\hat{a}_{\mathbf{k}\lambda}. \tag{5.3.27}$$

Then the sums over $\mathbf{k}$ become integrals over the infinite three-dimensional space:

$$\begin{aligned} \hat{H} &= \int d^3k\sum_{\lambda}\hbar\omega_{\mathbf{k}}\,\hat{a}_{\mathbf{k}\lambda}^\dagger\hat{a}_{\mathbf{k}\lambda} \\ \hat{\mathbf{A}}(\mathbf{r},t) &= \int d^3k\sum_{\lambda}\sqrt{\frac{\hbar}{16\pi^3\epsilon_0\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k}\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \mathrm{h.\,c.}\right)\mathbf{e}_{\mathbf{k}\lambda} \\ \omega_{\mathbf{k}} &= c|\mathbf{k}|, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}'\lambda'}^\dagger\right] = \delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}, \quad \left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}_{\mathbf{k}'\lambda'}\right] = 0. \end{aligned} \tag{5.3.28}$$