4.5: Exercises

Exercises

Exercise 4.5.1

Consider a system of two identical particles. Each single-particle Hilbert space $\mathscr{H}^{(1)}$ is spanned by a basis $\{|\mu_i\rangle\}$. The exchange operator is defined on $\mathscr{H}^{(2)} = \mathscr{H}^{(1)} \otimes \mathscr{H}^{(1)}$ by

$$P\left(\sum_{ij} \psi_{ij} |\mu_i\rangle |\mu_j\rangle\right) \equiv \sum_{ij} \psi_{ij} |\mu_j\rangle |\mu_i\rangle. \tag{4.5.1}$$

Prove that $\hat{P}$ is linear, unitary, and Hermitian. Moreover, prove that the operation is basis-independent: i.e., given any other basis $\{\nu_j\}$ that spans $\mathscr{H}^{(1)}$,

$$P\left(\sum_{ij} \varphi_{ij} |\nu_i\rangle |\nu_j\rangle\right) = \sum_{ij} \varphi_{ij} |\nu_j\rangle |\nu_i\rangle. \tag{4.5.2}$$

Exercise 4.5.2

Prove that the exchange operator commutes with the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m_e}\left(\nabla_1^2 + \nabla_2^2\right) + \frac{e^2}{4\pi\varepsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|}. \tag{4.5.3}$$

Exercise 4.5.3

An N-boson state can be written as

$$|\phi_1, \phi_2, \ldots, \phi_N\rangle = \mathcal{N} \sum_p \left(|\phi_{p(1)}\rangle |\phi_{p(2)}\rangle |\phi_{p(3)}\rangle \cdots |\phi_{p(N)}\rangle\right). \tag{4.5.4}$$

Prove that the normalization constant is

$$\mathcal{N} = \sqrt{\frac{1}{N! \prod_\mu n_\mu!}}, \tag{4.5.5}$$

where n_μ denotes the number of particles occupying the single-particle state μ.

Exercise 4.5.4

$\mathscr{H}_S^{(N)}$ and $\mathscr{H}_A^{(N)}$ denote the Hilbert spaces of N-particle states that are totally symmetric and totally antisymmetric under exchange, respectively. Prove that

$$\begin{aligned} \dim\left(\mathscr{H}_S^{(N)}\right) &= \frac{(d+N-1)!}{N!(d-1)!}, \\ \dim\left(\mathscr{H}_A^{(N)}\right) &= \frac{d!}{N!(d-N)!}. \end{aligned} \tag{4.5.6}$$

Exercise 4.5.5

Prove that for boson creation and annihilation operators, $[\hat{a}_\mu, \hat{a}_\nu] = [\hat{a}_\mu^\dagger, \hat{a}_\nu^\dagger] = 0$.

Exercise 4.5.6

Let $\hat{A}_1$ be an observable (Hermitian operator) for single-particle states. Given a single-particle basis $\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots\}$, define the bosonic multi-particle observable

$$\hat{A} = \sum_{\mu\nu} a_\mu^\dagger \, \langle\varphi_\mu|\hat{A}_1|\varphi_\nu\rangle \, a_\nu, \tag{4.5.7}$$

where $a_\mu^\dagger$ and a_μ are creation and annihilation operators satisfying the usual bosonic commutation relations, $[a_\mu, a_\nu] = 0$ and $[a_\mu, a_\nu^\dagger] = \delta_{\mu\nu}$. Prove that $\hat{A}$ commutes with the total number operator:

$$\left[\hat{A}, \sum_\mu a_\mu^\dagger a_\mu\right] = 0. \tag{4.5.8}$$

Next, repeat the proof for a fermionic multi-particle observable

$$\hat{A} = \sum_{\mu\nu} c_\mu^\dagger \, \langle\varphi_\mu|\hat{A}_1|\varphi_\nu\rangle \, c_\nu, \tag{4.5.9}$$

where $c_\mu^\dagger$ and c_μ are creation and annihilation operators satisfying the fermionic anticommutation relations, $\{c_\mu, c_\nu\} = 0$ and $\{c_\mu, c_\nu^\dagger\} = \delta_{\mu\nu}$. In this case, prove that

$$\left[\hat{A}, \sum_\mu c_\mu^\dagger c_\mu\right] = 0. \tag{4.5.10}$$

Further Reading

[1] Bransden & Joachain, §10.1–10.5

[2] Sakurai, §6

[3] J. M. Leinaas and J. Myrheim, *On the Theory of Identical Particles*, Nuovo Cimento B **37**, 1 (1977).

[4] F. Wilczek, *The Persistence of Ether*, Physics Today **52**, 11 (1999).