# 4.4: Quantum Field Theory

## Field operators

So far, we have been agnostic about the nature of the single-particle states $\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots\}$ used to construct the creation and annihilation operators. Let us now consider the special case where these quantum states are representable by wavefunctions. Let $|\mathbf{r}\rangle$ denote a position eigenstate for a $d$-dimensional space. A single-particle state $|\varphi_\mu\rangle$ has a wavefunction

$$\varphi_\mu(\mathbf{r}) = \langle \mathbf{r}|\varphi_\mu\rangle. \tag{4.4.1}$$

Due to the completeness and orthonormality of the basis, these wavefunctions satisfy

$$\begin{aligned}\int d^d r\; \varphi_\mu^*(\mathbf{r})\,\varphi_\nu(\mathbf{r}) &= \langle\varphi_\mu| \left(\int d^d r\,|\mathbf{r}\rangle\langle\mathbf{r}|\right)|\varphi_\nu\rangle = \delta_{\mu\nu},\\ \sum_\mu \varphi_\mu^*(\mathbf{r})\varphi_\mu(\mathbf{r}') &= \langle\mathbf{r}'|\left(\sum_\mu |\varphi_\mu\rangle\langle\varphi_\mu|\right)|\mathbf{r}\rangle = \delta^d(\mathbf{r}-\mathbf{r}').\end{aligned} \tag{4.4.2}$$

We can use the wavefunctions and the creation/annihilation operators to construct a new and interesting set of operators. For simplicity, suppose the particles are bosons, and let

$$\hat{\psi}(\mathbf{r}) = \sum_\mu \varphi_\mu(\mathbf{r})\,\hat{a}_\mu, \qquad \hat{\psi}^\dagger(\mathbf{r}) = \sum_\mu \varphi_\mu^*(\mathbf{r})\,\hat{a}_\mu^\dagger. \tag{4.4.3}$$

Using the aforementioned wavefunction properties, we can derive the inverse relations

$$\hat{a}_\mu = \int d^d r\; \varphi_\mu^*(\mathbf{r})\,\hat{\psi}(\mathbf{r}), \qquad \hat{a}_\mu^\dagger = \int d^d r\; \varphi_\mu(\mathbf{r})\,\hat{\psi}^\dagger(\mathbf{r}). \tag{4.4.4}$$

From the commutation relations for the bosonic $a_\mu$ and $a_\mu^\dagger$ operators, we can show that

$$\left[\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')\right] = \left[\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\right] = 0, \quad \left[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\right] = \delta^d(\mathbf{r}-\mathbf{r}'). \tag{4.4.5}$$

In the original commutation relations, the operators for different single-particle states commute; now, the operators for different *positions* commute. A straightforward interpretion for the operators $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$ is that they respectively create and annihilate one particle at a point $\mathbf{r}$ (rather than one particle in a given eigenstate).

It is important to note that $\mathbf{r}$ here does not play the role of an observable. It is an *index*, in the sense that each $\mathbf{r}$ is associated with distinct $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ operators. These $\mathbf{r}$-dependent operators serve to generalize the classical concept of a **field**. In a classical field theory, each point $\mathbf{r}$ is assigned a set of numbers corresponding to physical quantities, such as the electric field components $E_x(\mathbf{r})$, $E_y(\mathbf{r})$, and $E_z(\mathbf{r})$. In the present case, each $\mathbf{r}$ is assigned a set of quantum operators. This kind of quantum theory is called a **quantum field theory**.

We can use the $\hat{\psi}(\mathbf{r})$ and $\hat{\psi}^\dagger(\mathbf{r})$ operators to write second quantized observables in a way that is independent of the choice of single-particle basis wavefunctions. As discussed in the previous section, given a Hermitian single-particle operator $\hat{A}_1$ we can define a multi-particle observable $\hat{A} = \sum_{\mu\nu} \hat{a}_\mu^\dagger A_{\mu\nu} \hat{a}_\nu$, where $A_{\mu\nu} = \langle\varphi_\mu|\hat{A}_1|\varphi_\nu\rangle$. This multi-particle observable can be re-written as

$$\hat{A} = \int d^d r\, d^d r'\; \hat{\psi}^\dagger(\mathbf{r})\;\langle\mathbf{r}|\hat{A}_1|\mathbf{r}'\rangle\;\hat{\psi}(\mathbf{r}'), \tag{4.4.6}$$

which makes no explicit reference to the single-particle basis states.

For example, consider the familiar single-particle Hamiltonian describing a particle in a potential $V(\mathbf{r})$:

$$\hat{H}_1 = \hat{T}_1 + \hat{V}_1, \quad \hat{T}_1 = \frac{|\hat{\mathbf{p}}|^2}{2m}, \quad \hat{V}_1 = V(\hat{\mathbf{r}}), \tag{4.4.7}$$

where $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ are position and momentum operators (single-particle observables). The corresponding second quantized operators for the kinetic energy and potential energy are

$$\begin{aligned}\hat{T} &= \frac{\hbar^2}{2m}\int d^dr\, d^dr'\, \hat{\psi}^\dagger(\mathbf{r}) \left(\int \frac{d^dk}{(2\pi)^d}\, |\mathbf{k}|^2\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right)\, \hat{\psi}(\mathbf{r}') \\ &= \frac{\hbar^2}{2m}\int d^dr\, \nabla\hat{\psi}^\dagger(\mathbf{r})\cdot\nabla\hat{\psi}(\mathbf{r}) \\ \hat{V} &= \int d^dr\, \hat{\psi}^\dagger(\mathbf{r})\, V(\mathbf{r})\, \hat{\psi}(\mathbf{r}).\end{aligned} \tag{4.4.8}$$

(In going from the first to the second line, we performed integrations by parts.) This result is strongly reminiscent of the expression for the expected kinetic and potential energies in single-particle quantum mechanics:

$$\langle T\rangle = \frac{\hbar^2}{2m}\int d^dr |\nabla\psi(\mathbf{r})|^2, \quad \langle V\rangle = \int d^dr\, V(\mathbf{r})\, |\psi(\mathbf{r})|^2, \tag{4.4.9}$$

where $\psi(\mathbf{r})$ is the single-particle wavefunction.

How are the particle creation and annihilation operators related to the classical notion of “the value of a field at point $\mathbf{r}$”, like an electric field $\mathbf{E}(\mathbf{r})$ or magnetic field $\mathbf{B}(\mathbf{r})$? Field variables are measurable quantities, and should be described by Hermitian operators. As we have just seen, Hermitian operators corresponding to the kinetic and potential energy can be constructed via *products* of $\hat{\psi}^\dagger(\mathbf{r})$ with $\hat{\psi}(\mathbf{r})$. But there is another type of Hermitian operator that we can construct by taking *linear combinations* of of $\hat{\psi}^\dagger(\mathbf{r})$ with $\hat{\psi}(\mathbf{r})$. One example is

$$\psi(\mathbf{r}) + \psi(\mathbf{r})^\dagger. \tag{4.4.10}$$

Other possible Hermitian operators have the form

$$F(\mathbf{r}) = \int d^dr' \Big(f(\mathbf{r},\mathbf{r}')\hat{\psi}(\mathbf{r}) + f^*(\mathbf{r},\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r}')\Big), \tag{4.4.11}$$

where $f(\mathbf{r},\mathbf{r}')$ is some complex function. As we shall see, it is this type of Hermitian operator that corresponds to the classical notion of a field variable like an electric or magnetic field.

In the next two sections, we will try to get a better understanding of the relationship between classical fields and *bosonic* quantum fields. (For fermionic quantum fields, the situation is more complicated; they cannot be related to classical fields of the sort we are familiar with, for reasons that lie outside the scope of this course.)

## Revisiting the harmonic oscillator

Before delving into the links between classical fields and bosonic quantum fields, it is first necessary to revisit the harmonic oscillator, to see how the concept of a **mode of oscillation** carries over from classical to quantum mechanics.

A classical harmonic oscillator is described by the Hamiltonian

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2, \tag{4.4.12}$$

where $x$ is the “position” of the oscillator, which we call the **oscillator variable**; $p$ is the corresponding momentum variable; $m$ is the mass; and $\omega$ is the natural frequency of oscillation. We know that the classical equation of motion has the general form

$$x(t) = \mathcal{A}\, e^{-i\omega t} + \mathcal{A}^*\, e^{i\omega t}. \tag{4.4.13}$$

This describes an oscillation of frequency $\omega$. It is parameterized by the **mode amplitude** $\mathcal{A}$, a complex number that determines the magnitude and phase of the oscillation.

For the quantum harmonic oscillator, $x$ and $p$ are replaced by the Hermitian operators $\hat{x}$ and $\hat{p}$. From these, the operators $\hat{a}$ and $\hat{a}^\dagger$ can be defined:

$$\left\{\begin{aligned}\hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right).\end{aligned}\right. \quad\Leftrightarrow\quad \left\{\begin{aligned}\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \\ \hat{p} &= -i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^\dagger\right).\end{aligned}\right. \tag{4.4.14}$$

We can then show that

$$\left[\hat{a}\,,\,\hat{a}^\dagger\right] = 1, \quad \hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{4.4.15}$$

and from these the energy spectrum of the quantum harmonic oscillator can be derived. These facts should have been covered in an earlier course.

Here, we are interested in how the creation and annihilation operators relate to the *dynamics* of the quantum harmonic oscillator. In the Heisenberg picture, with $t = 0$ as the reference time, we define the time-dependent operator

$$\hat{x}(t) = \hat{U}^\dagger(t)\,\hat{x}\,\hat{U}(t), \quad \hat{U}(t) \equiv \exp\left(-\frac{i}{\hbar}\hat{H}t\right). \tag{4.4.16}$$

We will adopt the convention that all operators written with an explicit time dependence are Heisenberg picture operators, while operators without an explicit time dependence are Schrödinger picture operators; hence, $\hat{x} \equiv \hat{x}(0)$. The Heisenberg picture creation and annihilation operators, $\hat{a}^\dagger(t)$ and $\hat{a}(t)$, are related to $\hat{x}(t)$ by

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}(t) + \hat{a}^\dagger(t)\right). \tag{4.4.17}$$

The Heisenberg equation for the annihilation operator is

$$\begin{aligned} \frac{d\hat{a}(t)}{dt} &= \frac{i}{\hbar}\left[\hat{H}, \hat{a}(t)\right] \\ &= \frac{i}{\hbar}\hat{U}^\dagger(t)\left[\hat{H}, \hat{a}\right]\hat{U}(t) \\ &= \frac{i}{\hbar}\hat{U}^\dagger(t)\Big(-\hbar\omega\hat{a}\Big)\hat{U}(t) \\ &= -i\omega\hat{a}(t). \end{aligned} \tag{4.4.18}$$

Hence, the solution for this differential equation is

$$\hat{a}(t) = \hat{a}\,e^{-i\omega t}, \tag{4.4.19}$$

and Equation (4.4.17) becomes

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}\,e^{-i\omega t} + \hat{a}^\dagger\,e^{i\omega t}\right). \tag{4.4.20}$$

This has exactly the same form as the classical oscillatory solution (4.4.13)! Comparing the two, we see that $\hat{a}$ times the scale factor $\sqrt{\hbar/2m\omega}$ plays the role of the mode amplitude $\mathcal{A}$.

Now, suppose we come at things from the opposite end. Let's say we start with creation and annihilation operators satisfying Equation (4.4.15), from which Equations (4.4.18)–(4.4.19) follow. Using the creation and annihilation operators, we would like to construct an observable that corresponds to a classical oscillator variable. A natural Hermitian ansatz is

$$\hat{x}(t) = \mathcal{C}\left(\hat{a}\,e^{-i\omega t} + \hat{a}^\dagger\,e^{i\omega t}\right), \tag{4.4.21}$$

where $\mathcal{C}$ is a constant that is conventionally taken to be real.

How might $\mathcal{C}$ be chosen? A convenient way is to study the behavior of the oscillator variable *in the classical limit*. The classical limit of a quantum harmonic oscillator is described by a **coherent state**. The details of how this state is defined need not concern us for now (see Appendix E). The most important things to know are that (i) it can be denoted by $|\alpha\rangle$ where $\alpha \in \mathbb{C}$, (ii) it is an eigenstate of the annihilation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{4.4.22}$$

And (iii) its energy expectation value is

$$\langle E\rangle = \langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right) \overset{|\alpha|^2\to\infty}{\longrightarrow} \hbar\omega|\alpha|^2. \tag{4.4.23}$$

When the system is in a coherent state, we can effectively substitute the $\hat{a}$ and $\hat{a}^\dagger$ operators in Equation (4.4.21) with the complex numbers $\alpha$ and $\alpha^*$, which gives a classical trajectory

$$x_{\text{classical}}(t) = \mathcal{C}\left(\alpha\, e^{-i\omega t} + \alpha^*\, e^{i\omega t}\right). \tag{4.4.24}$$

This trajectory has amplitude $2\mathcal{C}|\alpha|$. At maximum displacement, the classical momentum is zero, so the total energy of the classical oscillator must be

$$E_{\text{classical}} = \frac{1}{2} m\omega^2 \Big(2\mathcal{C}|\alpha|\Big)^2 = 2m\omega^2\mathcal{C}^2|\alpha|^2. \tag{4.4.25}$$

Equating the classical energy (4.4.25) to the coherent state energy (4.4.23) gives

$$\mathcal{C} = \sqrt{\frac{\hbar}{2m\omega}}, \tag{4.4.26}$$

which is precisely the scale factor found in Equation (4.4.20).

## A scalar boson field

We now have the tools available to understand the connection between a very simple classical field and its quantum counterpart. Consider a classical scalar field variable $f(x,t)$, defined in one spatial dimension, whose classical equation of motion is the wave equation:

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2}. \tag{4.4.27}$$

The constant $c$ is a wave speed. This sort of classical field arises in many physical contexts, including the propagation of sound through air, in which case $c$ is the speed of sound.

For simplicity, let us first assume that the field is defined within a finite interval of length $L$, with periodic boundary conditions: $f(x,t) \equiv f(x+L,t)$. Solutions to the wave equation can be described by the following ansatz:

$$f(x,t) = \sum_n \left(\mathcal{A}_n\, \varphi_n(x)\, e^{-i\omega_n t} + \mathcal{A}_n^*\, \varphi_n^*(x)\, e^{i\omega_n t}\right). \tag{4.4.28}$$

This ansatz describes a superposition of **normal modes**. Each normal mode (labelled $n$) varies harmonically in time with a mode frequency $\omega_n$, and varies in space according to a complex mode profile $\varphi_n(x)$; its overall magnitude and phase is specified by the mode amplitude $\mathcal{A}_n$. The mode profiles are normalized according to some fixed convention, e.g.

$$\int_0^L dx\, |\varphi_n(x)|^2 = 1. \tag{4.4.29}$$

Substituting Equation (4.4.28) into Equation (4.4.27), and using the periodic boundary conditions, gives

$$\varphi_n(x) = \frac{1}{\sqrt{L}} \exp(ik_n x), \quad \omega_n = ck_n = \frac{2\pi cn}{L}, \quad n \in \mathbb{Z}. \tag{4.4.30}$$

These mode profiles are orthonormal:

$$\int_0^L dx\, \varphi_m^*(x)\, \varphi_n(x) = \delta_{mn}. \tag{4.4.31}$$

Each normal mode carries energy. By analogy with the classical harmonic oscillator—see Equations (4.4.24)–(4.4.25)—we assume that the energy density (i.e., energy per unit length) is proportional to the square of the field variable. Let it have the form

$$U(x) = 2\rho \sum_n |\mathcal{A}_n|^2\, |\varphi_n(x)|^2, \tag{4.4.32}$$

where $\rho$ is some parameter that has to be derived from the underlying physical context. For example, for acoustic modes, $\rho$ is the mass density of the underlying acoustic medium; in the next chapter, we will see a concrete example involving the energy density of an electromagnetic mode. From Equation (4.4.32), the total energy is

$$E=\int_{0}^{L} dx\; U(x)=2\rho\sum_{n}|\mathcal{A}_{n}|^{2}. \tag{4.4.33}$$

To quantize the classical field, we treat each normal mode as an independent oscillator, with creation and annihilation operators $\hat{a}_n^\dagger$ and $\hat{a}_n$ satisfying

$$\left[\,\hat{a}_{m}\,,\,\hat{a}_{n}^{\dagger}\right]=\delta_{mn},\quad \left[\,\hat{a}_{m}\,,\,\hat{a}_{n}\right]=\left[\,\hat{a}_{m}^{\dagger}\,,\,\hat{a}_{n}^{\dagger}\right]=0. \tag{4.4.34}$$

We then take the Hamiltonian to be that of a set of independent harmonic oscillators:

$$\hat{H}=\sum_{n}\hbar\omega_{n}\hat{a}_{n}^{\dagger}\hat{a}_{n}+E_{0}, \tag{4.4.35}$$

where $E_0$ is the ground-state energy. Just like in the previous section, we can define a Heisenberg-picture annihilation operator, and solving its Heisenberg equation yields

$$\hat{a}_{n}(t)=\hat{a}_{n}e^{-i\omega_{n}t}. \tag{4.4.36}$$

We then define a Schrödinger picture Hermitian operator of the form

$$\hat{f}\,(x)=\sum_{n}\mathcal{C}_{n}\left(\hat{a}_{n}\varphi_{n}(x)+\hat{a}_{n}^{\dagger}\varphi_{n}^{*}(x)\right), \tag{4.4.37}$$

where $\mathcal{C}_n$ is a real constant (one for each normal mode). The corresponding Heisenberg picture operator is

$$\hat{f}\,(x,t)=\sum_{n}\mathcal{C}_{n}\left(\hat{a}_{n}\varphi_{n}(x)\,e^{-i\omega_{n}t}+\hat{a}_{n}^{\dagger}\varphi_{n}^{*}(x)\,e^{i\omega_{n}t}\right), \tag{4.4.38}$$

which is the quantum version of the classical solution (4.4.28).

To determine the $\mathcal{C}_n$ scale factors, we consider the classical limit. The procedure is a straightforward generalization of the harmonic oscillator case discussed in Section 4.4. We introduce a state $|\alpha\rangle$ that is a coherent state for all the normal modes; i.e., for any given $n$,

$$\hat{a}_{n}|\alpha\rangle=\alpha_{n}|\alpha\rangle \tag{4.4.39}$$

for some $\alpha_n\in\mathbb{C}$. The energy expectation value is

$$\langle E\rangle=\sum_{n}\hbar\omega_{n}|\alpha_{n}|^{2}. \tag{4.4.40}$$

In the coherent state, the $\hat{a}_n$ and $\hat{a}_n^\dagger$ operators in Equation (4.4.38) can be replaced with $\alpha_n$ and $\alpha_n^*$ respectively. Hence, we identify $\mathcal{C}_n\alpha_n$ as the classical mode amplitude $\mathcal{A}_n$ in Equation (4.4.28). In order for the classical energy (4.4.33) to match the coherent state energy (4.4.40), we need

$$2\rho|\mathcal{A}_{n}|^{2}=2\rho|\mathcal{C}_{n}\alpha_{n}|^{2}=\hbar\omega_{n}|\alpha_{n}|^{2}\quad\Rightarrow\quad\mathcal{C}_{n}=\sqrt{\frac{\hbar\omega_{n}}{2\rho}}. \tag{4.4.41}$$

Hence, the appropriate field operator is

$$\hat{f}\,(x,t)=\sum_{n}\sqrt{\frac{\hbar\omega_{n}}{2\rho}}\left(\hat{a}_{n}\varphi_{n}(x)\,e^{-i\omega_{n}t}+\hat{a}_{n}^{\dagger}\varphi_{n}^{*}(x)\,e^{i\omega_{n}t}\right). \tag{4.4.42}$$

Returning to the Schrödinger picture, and using the explicit mode profiles from Equation (4.4.30), we get

$$\hat{f}\,(x)=\sum_{n}\sqrt{\frac{\hbar\omega_{n}}{2\rho L}}\left(\hat{a}_{n}e^{ik_{n}x}+\hat{a}_{n}^{\dagger}e^{-ik_{n}x}\right). \tag{4.4.43}$$

Finally, if we are interested in the infinite-$L$ limit, we can convert the sum over $n$ into an integral. The result is

$$\hat{f}\,(x)=\int dk\,\sqrt{\frac{\hbar\omega(k)}{4\pi\rho}}\left(\hat{a}(k)\,e^{ikx}+\hat{a}^{\dagger}(k)\,e^{-ikx}\right), \tag{4.4.44}$$

where $\hat{a}(k)$ denotes a rescaled annihilation operator defined by $\hat{a}_n \rightarrow \sqrt{2\pi/L}\,\hat{a}(k)$, satisfying

$$\left[\hat{a}(k)\,,\,\hat{a}^\dagger(k')\right] = \delta(k-k'). \tag{4.4.45}$$

## Looking ahead

In the next chapter, we will use these ideas to formulate a quantum theory of electromagnetism. This is a bosonic quantum field theory in which the creation and annihilation operators act upon particles called **photons**—the elementary particles of light. Linear combinations of these photon operators can be used to define Hermitian field operators that correspond to the classical electromagnetic field variables. In the classical limit, the quantum field theory reduces to Maxwell's theory of the electromagnetic field.

It is hard to overstate the importance of quantum field theories in physics. At a fundamental level, all elementary particles currently known to humanity can be described using a quantum field theory called the Standard Model. These particles are roughly divided into two categories. The first consists of "force-carrying" particles: photons (which carry the electromagnetic force), gluons (which carry the strong nuclear force), and the $W/Z$ bosons (which carry the weak nuclear force); these particles are excitations of bosonic quantum fields, similar to the one described in the previous section. The second category consists of "particles of matter", such as electrons, quarks, and neutrinos; these are excitations of *fermionic* quantum fields, whose creation and annihilation operators obey anticommutation relations.

As Wilczek (1999) has pointed out, the modern picture of fundamental physics bears a striking resemblance to the old idea of "luminiferous ether": a medium filling all of space and time, whose vibrations are physically-observable light waves. The key difference, as we now understand, is that the ether is not a classical medium, but one obeying the rules of quantum mechanics. (Another difference, which we have not discussed so far, is that modern field theories can be made compatible with relativity.)

It is quite compelling to think of fields, not individual particles, as the fundamental objects in the universe. This point of view "explains", in a sense, why all particles of the same type have the same properties (e.g., why all electrons in the universe have exactly the same mass). The particles themselves are not fundamental; they are excitations of deeper, more fundamental entities—quantum fields!