# 4.1: Particle Exchange Symmetry

In the previous chapter, we discussed how the principles of quantum mechanics apply to systems of multiple particles. That discussion omitted an important feature of multi-particle systems, namely the fact that particles of the same type are fundamentally indistinguishable from each other. As it turns out, indistinguishability imposes a strong constraint on the form of the multi-particle quantum states, and looking into this will ultimately lead us to a fundamental re-interpretation of what "particles" are.

Suppose we have two particles of the same type, e.g. two electrons. It is a fact of Nature that all electrons have identical physical properties: the same mass, same charge, same total spin, etc. As a consequence, the single-particle Hilbert spaces of the two electrons must be mathematically identical. Let us denote this space by $\mathscr{H}^{(1)}$. For a two-electron system, the Hilbert space is a tensor product of two single-electron Hilbert spaces, denoted by

$$\mathscr{H}^{(2)}=\mathscr{H}^{(1)}\otimes\mathscr{H}^{(1)}. \tag{4.1.1}$$

Moreover, any Hamiltonian must affect the two electrons in a symmetrical way. An example of such a Hamiltonian is

$$\hat{H}=\frac{1}{2m_e}\Big(|\hat{\mathbf{p}}_1|^2+|\hat{\mathbf{p}}_2|^2\Big)+\frac{e^2}{4\pi\varepsilon_0|\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2|}, \tag{4.1.2}$$

consisting of the non-relativistic kinetic energies and the Coulomb potential energy. Operators $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{r}}_1$ act on electron 1, while $\hat{\mathbf{p}}_2$ and $\hat{\mathbf{r}}_2$ act on electron 2.

Evidently, this Hamiltonian is invariant under an interchange of the operators acting on the two electrons (i.e., $\hat{\mathbf{p}}_1\leftrightarrow\hat{\mathbf{p}}_2$ and $\hat{\mathbf{r}}_1\leftrightarrow\hat{\mathbf{r}}_2$ ). This can be regarded as a kind of symmetry, called **exchange symmetry**. As we know, symmetries of quantum systems can be represented by unitary operators that commute with the Hamiltonian. Exchange symmetry is represented by an operator $\hat{P}$, defined as follows: let $\{|\mu\rangle\}$ be a basis for the single-electron Hilbert space $\mathscr{H}^{(1)}$; then $\hat{P}$ interchanges the basis vectors for the two electrons:

$$\begin{aligned}\hat{P}\Big(\sum_{\mu\nu}\psi_{\mu\nu}|\mu\rangle|\nu\rangle\Big) &\equiv \sum_{\mu\nu}\psi_{\mu\nu}|\nu\rangle|\mu\rangle \\ &=\sum_{\mu\nu}\psi_{\nu\mu}|\nu\rangle|\mu\rangle \quad (\text{interchanging } \mu\leftrightarrow\nu \text{ in the double sum})\end{aligned} \tag{4.1.3}$$

The exchange operator has the following properties:

1. $\hat{P}^2=\hat{I}$ , where $\hat{I}$ is the identity operator.
2. $\hat{P}$ is linear, unitary, and Hermitian (see Exercise 4.5.1).
3. The effect of $\hat{P}$ does not depend on the choice of basis (see Exercise 4.5.1).
4. $\hat{P}$ commutes with the above Hamiltonian $\hat{H}$; more generally, it commutes with any two-particle operator built out of symmetrical combinations of single-particle operators (see Exercise 4.5.2).

According to Noether's theorem, any symmetry implies a conservation law. In the case of exchange symmetry, $\hat{P}$ is both Hermitian *and* unitary, so we can take the conserved quantity to be the eigenvalue of $\hat{P}$ itself. We call this eigenvalue, $p$, the **exchange parity**. Given that $\hat{P}^2=\hat{I}$ , there are just two possibilities:

$$\hat{P}|\psi\rangle=p|\psi\rangle \quad\Rightarrow\quad p=\begin{cases}+1 & (\text{"symmetric state''}),\ \text{ or}\\ -1 & (\text{"antisymmetric state''}).\end{cases} \tag{4.1.4}$$

Since $\hat{P}$ commutes with $\hat{H}$, if the system starts out in an eigenstate of $\hat{P}$ with parity $p$, it retains the same parity for all subsequent times.

The concept of exchange parity generalizes to systems of more than two particles. Given $N$ particles, we can define a set of exchange operators $\hat{P}_{ij}$, where $i,j\in\{1,2,\ldots,N\}$ and $i\neq j$, such that $\hat{P}_{ij}$ exchanges particle $i$ and particle $j$. If the particles are identical, the Hamiltonian must commute with *all* the exchange operators, so the parities ($\pm1$) are individually conserved.

We now invoke the following postulates:

1. A multi-particle state of identical particles is an eigenstate of every $\hat{P}_{ij}$.

2. For each $\hat{P}_{ij}$, the exchange parity $p_{ij}$ has the same value: i.e., all $+1$ or all $-1$.
3. The exchange parity $p_{ij}$ is determined solely by the type of particle involved.

Do *not* think of these as statements as being derived from more fundamental facts! Rather, they are hypotheses about the way particles behave—facts about Nature that physicists have managed to deduce through examining a wide assortment of empirical evidence. Our task, for now, shall be to explore the consequences of these hypotheses.

Particles that have symmetric states ($p_{ij} = +1$) are called **bosons**. It turns out that the elementary particles that “carry” the fundamental forces are all bosons: these are the photons (elementary particles of light, which carry the electromagnetic force), gluons (elementary particles that carry the strong nuclear force, responsible for binding protons and neutrons together), and $W$ and $Z$ bosons (particles that carry the weak nuclear force responsible for beta decay). Other bosons include particles that carry non-fundamental forces, such as phonons (particles of sound), as well as certain composite particles such as alpha particles (helium-4 nuclei).

Particles that have antisymmetric states ($p_{ij} = -1$) are called **fermions**. All the elementary particles of “matter” are fermions: electrons, muons, tauons, quarks, neutrinos, and their anti-particles (positrons, anti-neutrinos, etc.). Certain composite particles are also fermions, including protons and neutrons, which are each composed of three quarks.

By the way, one might question whether particle indistinguishability invalidates the concept of assigning single-particle states to (say) the “first slot” or “second slot” in a tensor product. It seems unsatisfactory that our mathematical framework allows us to write down a state like $|\mu\rangle|\nu\rangle$ (where $\mu \neq \nu$), which is physically impossible since it is not symmetric or antisymmetric, and then uses such states to define a “particle exchange” operation that has no physical meaning. To get around this, Leinaas and Myrheim (1977) have developed an interesting formulation of particle indistinguishability that avoids the concept of particle exchange. In this view, in a multi-particle wavefunction the coordinates $(\mathbf{r}_1, \dots, \mathbf{r}_N)$ are not to be regarded as an ordinary vector, but as a mathematical object in which interchanging entries leaves the object invariant. Bosonic or fermionic states can then be constructed by carefully analyzing the topological structure of wavefunctions defined on such configuration spaces. For more details, the interested reader is referred to the paper by Leinaas and Myrheim. In this course, however, we will adopt the usual formulation based on particle exchange.

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