1.4: Scattering in 2D and 3D

We now wish to consider scattering experiments in spatial dimension $d \geq 2$, which have a new and important feature. For $d = 1$, the particle can only scatter forward or backward, but for $d \geq 2$ it can be scattered to the side.

Far from the scatterer, where $V(\mathbf{r}) \rightarrow 0$, the scattered wavefunction $\psi_s(\mathbf{r})$ satisfies

$$-\frac{\hbar^2}{2m}\nabla^2\psi_s(\mathbf{r}) = E\psi_s(\mathbf{r}), \tag{1.4.1}$$

where ∇^2 denotes the d-dimensional Laplacian. Let $E = \hbar^2k^2/2m$, where $k \in \mathbf{R}^+$ is the wave-number in free space. Then the above equation can be written as

$$\left[\nabla^2 + k^2\right]\psi_s(\mathbf{r}) = 0, \tag{1.4.2}$$

which is the **Helmholtz equation** in d-dimensional space.

One set of elementary solutions to the Helmholtz equation are the plane waves

$$\Big\{\exp(i\mathbf{k}\cdot\mathbf{r}), \quad \text{where} \;\; |\mathbf{k}| = k\Big\}. \tag{1.4.3}$$

But we're looking for an outgoing solution, and a plane wave can't be said to be "outgoing".

Therefore, we turn to curvilinear coordinates. In 2D, we use the polar coordinates (r, ϕ). We will skip the mathematical details of how to solve the 2D Helmholtz equation in these coordinates; the result is that the general solution can be written as a linear combination

$$\psi(\mathbf{r}) = \sum_{\pm}\sum_{m=-\infty}^{\infty} c_m^{\pm}\,\Psi_m^{\pm}(r,\phi), \quad \text{where} \quad \Psi_m^{\pm}(r,\phi) = H_m^{\pm}(kr)\,e^{im\phi}. \tag{1.4.4}$$

This is a superposition of circular waves $\Psi_m^{\pm}(r,\phi)$, with coefficients $c_m^{\pm} \in \mathbb{C}$. Each circular wave is a solution to the 2D Helmholtz equation with angular momentum quantum number $m \in \mathbb{Z}$. Its r-dependence is given by $H_m^{\pm}$, called a **Hankel function** of the "first kind" $(+)$ or "second kind" $(-)$. Some Hankel functions of the first kind are plotted below:

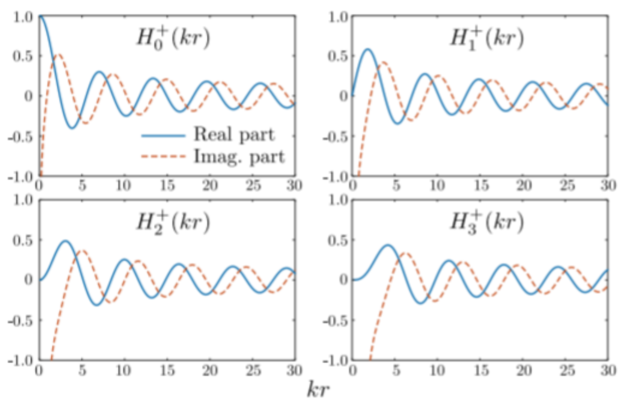

Figure 1.4.1

The H_m^- functions are the complex conjugates of H_m^+. For large values of the input,

$$H_m^{\pm}(kr) \overset{r\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi kr}}\exp\left[\pm i\left(kr - \frac{(m+\frac{1}{2})\pi}{2}\right)\right] \sim r^{-1/2}e^{\pm ikr}. \tag{1.4.5}$$

Therefore, the $\pm$ index specifies whether the circular wave is an **outgoing wave** directed outward from the origin $(+)$, or an **incoming wave** directed toward the origin $(-)$.

The 3D case is treated similarly. We use spherical coordinates (r, θ, ϕ), and the solutions of the 3D Helmholtz equation are superpositions of incoming and outgoing spherical waves:

$$\psi(\mathbf{r})=\sum_{\pm}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell m}^{\pm}\,\Psi_{\ell m}^{\pm}(r,\theta,\phi)\quad\text{where}\quad \Psi_{\ell m}^{\pm}(r,\theta,\phi)=h_{\ell}^{\pm}(kr)\,Y_{\ell m}(\theta,\phi). \tag{1.4.6}$$

The $c_{\ell m}^{\pm}$ factors are complex coefficients. Each $h_{\ell}^{\pm}$ is a **spherical Hankel function**, and each $Y_{\ell m}$ is a **spherical harmonic**. The ℓ and m indices specify the angular momentum of the spherical wave. For large inputs, the spherical Hankel functions have the limiting form

$$h_{\ell}^{\pm}(kr)\overset{r\rightarrow\infty}{\longrightarrow}\pm\frac{\exp\left[\pm i\left(kr-\frac{\ell\pi}{2}\right)\right]}{ikr}. \tag{1.4.7}$$

Hence, the $\pm$ index specifies whether the spherical wave is outgoing $(+)$ or incoming $(-)$. More discussion about these spherical waves can be found in Appendix A.

It is now clear what we need to do to get a scattered wavefunction $\psi_s(\mathbf{r})$ that is outgoing at infinity. We take a superposition with only outgoing $(+)$ wave components:

$$\psi_s(\mathbf{r})=\begin{cases}\displaystyle\sum_m c_m^{+}\,H_m^{+}(kr)\,e^{im\phi}, & d=2\\ \displaystyle\sum_{\ell m} c_{\ell m}^{+}\,h_{\ell}^{+}(kr)\,Y_{\ell m}(\theta,\phi), & d=3.\end{cases} \tag{1.4.8}$$

For large r, the outgoing wavefunction has the r-dependence

$$\psi_s(\mathbf{r})\;\overset{r\rightarrow\infty}{\sim}\; r^{\frac{1-d}{2}}\,\exp(ikr). \tag{1.4.9}$$

For $d>1$, the magnitude of the wavefunction decreases with distance from the origin. This is as expected, because with increasing r each outgoing wave spreads out over a wider area. The probability current density is $\mathbf{J}=(\hbar/m)\mathrm{Im}\left[\psi_s^*\nabla\psi_s\right]$, and its r-component is

$$\begin{aligned}J_r\;&\overset{r\rightarrow\infty}{\sim}\;\mathrm{Im}\left[r^{\frac{1-d}{2}}\,e^{-ikr}\,\frac{\partial}{\partial r}\left(r^{\frac{1-d}{2}}\,e^{ikr}\right)\right]\\ &=\;\mathrm{Im}\left[\frac{1-d}{2}\,r^{-d}+ikr^{1-d}\right]\\ &=\;k\,r^{1-d}.\end{aligned} \tag{1.4.10}$$

In d dimensions, the area of a wave-front scales as r^{d-1}, so the probability flux goes as $J_r\,r^{d-1}\sim k$, which is positive and independent of r. This describes a constant probability flux flowing outward from the origin. Note that if we plug $d=1$ into the above formula, we find that J_r scales as r^0 (i.e., a constant), consistent with the results of the previous section: waves in 1D do not spread out with distance as there is no transverse dimension.
