4.2: Symmetric and Antisymmetric States

Bosons

A state of N bosons must be symmetric under every possible exchange operator:

$$\hat{P}_{ij}\,|\psi\rangle = |\psi\rangle \quad \forall\, i, j \in \{1, \ldots, N\}, \quad i \neq j. \tag{4.2.1}$$

There is a standard way to construct multi-particle states obeying this symmetry condition. First, consider a two-boson system ($N = 2$). If both bosons occupy the same single-particle state, $|\mu\rangle \in \mathscr{H}^{(1)}$, the two-boson state is simply

$$|\mu, \mu\rangle = |\mu\rangle|\mu\rangle. \tag{4.2.2}$$

This evidently satisfies the required symmetry condition (4.2.1). Next, suppose the two bosons occupy *different* single-particle states, $|\mu\rangle$ and $|\nu\rangle$, which are orthonormal vectors in $\mathscr{H}^{(1)}$. It would be wrong to write the two-boson state as $|\mu\rangle|\nu\rangle$, because the particles would not be symmetric under exchange. Instead, we construct the multi-particle state

$$|\mu, \nu\rangle = \frac{1}{\sqrt{2}}\Big(|\mu\rangle|\nu\rangle + |\nu\rangle|\mu\rangle\Big). \tag{4.2.3}$$

This has the appropriate exchange symmetry:

$$\hat{P}_{12}\,|\mu, \nu\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|\nu\rangle|\mu\rangle + |\mu\rangle|\nu\rangle\Big) \;=\; |\mu, \nu\rangle. \tag{4.2.4}$$

The $1/\sqrt{2}$ factor in Equation (4.2.3) ensures that the state is normalized (check for yourself that this is true—it requires $|\mu\rangle$ and $|\nu\rangle$ to be orthonormal to work out).

The above construction can be generalized to arbitrary numbers of bosons. Suppose we have N bosons occupying single-particle states enumerated by

$$|\phi_1\rangle,\; |\phi_2\rangle,\; |\phi_3\rangle,\; \ldots,\; |\phi_N\rangle. \tag{4.2.5}$$

Each of the states $|\phi_j\rangle$ is drawn from an orthonormal basis set $\{|\mu\rangle\}$ for $\mathscr{H}^{(1)}$. We use the ϕ labels to indicate that the listed states can overlap. For example, we could have $|\phi_1\rangle = |\phi_2\rangle = |\mu\rangle$, meaning that the single-particle state $|\mu\rangle$ is occupied by two particles.

The N-boson state can now be written as

$$|\phi_1, \phi_2, \ldots, \phi_N\rangle = \mathcal{N} \sum_p \Big(|\phi_{p(1)}\rangle|\phi_{p(2)}\rangle|\phi_{p(3)}\rangle \cdots |\phi_{p(N)}\rangle\Big). \tag{4.2.6}$$

The sum is taken over each of the $N!$ permutations acting on $\{1, 2, \ldots, N\}$. For each permutation p, we let $p(j)$ denote the integer that j is permuted into.

The prefactor $\mathcal{N}$ is a normalization constant, and it can be shown that its appropriate value is

$$\mathcal{N} = \sqrt{\frac{1}{N!\, n_a! \, n_b! \cdots}}, \tag{4.2.7}$$

where n_μ denotes the number of particles in each distinct state $|\varphi_\mu\rangle$, and $N = n_\alpha + n_\beta + \cdots$ is the total number of particles. The proof of this is left as an exercise (Exercise 4.5.3).

To see that the above N-particle state is symmetric under exchange, apply an arbitrary exchange operator $\hat{P}_{ij}$:

$$\begin{aligned} \hat{P}_{ij}|\phi_1, \phi_2, \ldots, \phi_N\rangle &= \mathcal{N} \sum_p \hat{P}_{ij} \Big(\cdots |\phi_{p(i)}\rangle \cdots |\phi_{p(j)}\rangle \cdots\Big) \\ &= \mathcal{N} \sum_p \Big(\cdots |\phi_{p(j)}\rangle \cdots |\phi_{p(i)}\rangle \cdots\Big). \end{aligned} \tag{4.2.8}$$

In each term of the sum, two states i and j are interchanged. Since the sum runs through all permutations of the states, the result is the same with or without the exchange, so we still end up with $|\phi_1, \phi_2, \ldots, \phi_N\rangle$. Therefore, the multi-particle state is symmetric under every possible exchange operation.

Example 4.2.1

A three-boson system has two particles in a state $|\mu\rangle$, and one particle in a different state $|\nu\rangle$. To express the three-particle state, define $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$ where $|\phi_1\rangle = |\phi_2\rangle = |\mu\rangle$ and $|\phi_3\rangle = |\nu\rangle$. Then

$$\begin{aligned}|\phi_1,\phi_2,\phi_3\rangle &= \frac{1}{\sqrt{12}}\Big(|\phi_1\rangle|\phi_2\rangle|\phi_3\rangle + |\phi_2\rangle|\phi_3\rangle|\phi_1\rangle + |\phi_3\rangle|\phi_1\rangle|\phi_2\rangle \\ &\qquad + |\phi_1\rangle|\phi_3\rangle|\phi_2\rangle + |\phi_3\rangle|\phi_2\rangle|\phi_1\rangle + |\phi_2\rangle|\phi_1\rangle|\phi_3\rangle\Big) \\ &= \frac{1}{\sqrt{3}}\Big(|\mu\rangle|\mu\rangle|\nu\rangle + |\mu\rangle|\nu\rangle|\mu\rangle + |\nu\rangle|\mu\rangle|\mu\rangle\Big).\end{aligned} \tag{4.2.9}$$

The exchange symmetry operators have the expected effects:

$$\begin{aligned}\hat{P}_{12}|\phi_1,\phi_2,\phi_3\rangle &= \frac{1}{\sqrt{3}}\Big(|\mu\rangle|\mu\rangle|\nu\rangle + |\nu\rangle|\mu\rangle|\mu\rangle + |\mu\rangle|\nu\rangle|\mu\rangle\Big) = |\phi_1,\phi_2,\phi_3\rangle \\ \hat{P}_{23}|\phi_1,\phi_2,\phi_3\rangle &= \frac{1}{\sqrt{3}}\Big(|\mu\rangle|\nu\rangle|\mu\rangle + |\mu\rangle|\mu\rangle|\nu\rangle + |\nu\rangle|\mu\rangle|\mu\rangle\Big) = |\phi_1,\phi_2,\phi_3\rangle \\ \hat{P}_{13}|\phi_1,\phi_2,\phi_3\rangle &= \frac{1}{\sqrt{3}}\Big(|\nu\rangle|\mu\rangle|\mu\rangle + |\mu\rangle|\nu\rangle|\mu\rangle + |\mu\rangle|\mu\rangle|\nu\rangle\Big) = |\phi_1,\phi_2,\phi_3\rangle.\end{aligned} \tag{4.2.10}$$

Fermions

A state of N fermions must be antisymmetric under every possible exchange operator:

$$\hat{P}_{ij}\,|\psi\rangle = -|\psi\rangle \quad \forall\, i,j \in \{1,\ldots,N\},\; i \neq j. \tag{4.2.11}$$

Similar to the bosonic case, we can explicitly construct multi-fermion states based on the occupancy of single-particle state.

First consider $N=2$, with the fermions occupying the single-particle states $|\mu\rangle$ and $|\nu\rangle$ (which, once again, we assume to be orthonormal). The appropriate two-particle state is

$$|\mu,\nu\rangle = \frac{1}{\sqrt{2}}\Big(|\mu\rangle|\nu\rangle - |\nu\rangle|\mu\rangle\Big). \tag{4.2.12}$$

We can easily check that this is antisymmetric:

$$\hat{P}_{12}|\mu,\nu\rangle = \frac{1}{\sqrt{2}}\Big(|\nu\rangle|\mu\rangle - |\mu\rangle|\nu\rangle\Big) = -|\mu,\nu\rangle. \tag{4.2.13}$$

Note that if $|\mu\rangle$ and $|\nu\rangle$ are the same single-particle state, Equation (4.2.12) doesn't work, since the two terms would cancel to give the zero vector, which is not a valid quantum state. This is a manifestation of the **Pauli exclusion principle**, which states that two fermions cannot occupy the same single-particle state. Thus, each single-particle state is either unoccupied or occupied by one fermion.

For general N, let the occupied single-particle states be $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_N\rangle$, each drawn from some orthonormal basis $\{|\mu\rangle\}$ for $\mathscr{H}^{(1)}$, and each distinct. Then the appropriate N-fermion state is

$$|\phi_1,\ldots,\phi_N\rangle = \frac{1}{\sqrt{N!}}\sum_p s(p)\,|\phi_{p(1)}\rangle|\phi_{p(2)}\rangle\cdots|\phi_{p(N)}\rangle. \tag{4.2.14}$$

It is up to you to verify that the $1/\sqrt{N!}$ prefactor is the right normalization constant. The sum is taken over every permutation p of the sequence $\{1,2,\ldots,N\}$, and each term in the sum has a coefficient $s(p)$ denoting the **parity of the permutation**. The parity of any permutation p is defined as $+1$ if p is constructed from an even number of transpositions (i.e., exchanges of adjacent elements) starting from the sequence $\{1,2,\ldots,N\}$, and -1 if p involves an odd number of transpositions.

Let's look at a couple of concrete examples.

Example 4.2.2

For $N=2$, the sequence $\{1,2\}$ has two permutations:

$$\begin{aligned} p_1 : \{1,2\} &\to \{1,2\}, \quad s(p_1) = +1 \\ p_2 : \{1,2\} &\to \{2,1\}, \quad s(p_2) = -1. \end{aligned} \tag{4.2.15}$$

Plugging these into Equation (4.2.14) yields the previously-discussed two-fermion state (4.2.12).

Example 4.2.3

For $N=3$, the sequence $\{1,2,3\}$ has $3! = 6$ permutations:

$$\begin{aligned} p_1 : \{1,2,3\} &\to \{1,2,3\}, \quad s(p_1) = +1 \\ p_2 : \{1,2,3\} &\to \{2,1,3\}, \quad s(p_2) = -1 \\ p_3 : \{1,2,3\} &\to \{2,3,1\}, \quad s(p_3) = +1 \\ p_4 : \{1,2,3\} &\to \{3,2,1\}, \quad s(p_4) = -1 \\ p_5 : \{1,2,3\} &\to \{3,1,2\}, \quad s(p_5) = +1 \\ p_6 : \{1,2,3\} &\to \{1,3,2\}, \quad s(p_6) = -1. \end{aligned} \tag{4.2.16}$$

The permutations can be generated by consecutive transpositions of elements. Each time we perform a transposition, the sign of $s(p)$ is reversed. Hence, the three-fermion state is

$$\begin{aligned} |\phi_1, \phi_2, \phi_3\rangle = \frac{1}{\sqrt{6}} \Big(& |\phi_1\rangle|\phi_2\rangle|\phi_3\rangle - |\phi_2\rangle|\phi_1\rangle|\phi_3\rangle \\ & + |\phi_2\rangle|\phi_3\rangle|\phi_1\rangle - |\phi_3\rangle|\phi_2\rangle|\phi_1\rangle \\ & + |\phi_2\rangle|\phi_3\rangle|\phi_1\rangle - |\phi_1\rangle|\phi_3\rangle|\phi_2\rangle \Big). \end{aligned} \tag{4.2.17}$$

We now see why Equation (4.2.14) describes the N-fermion state. Let us apply $\hat{P}_{ij}$ to it:

$$\begin{aligned} \hat{P}_{ij}|\phi_1, \dots, \phi_N\rangle &= \frac{1}{\sqrt{N!}} \sum_p s(p)\, \hat{P}_{ij} \Big[\cdots |\phi_{p(i)}\rangle \cdots |\phi_{p(j)}\rangle \cdots \Big] \\ &= \frac{1}{\sqrt{N!}} \sum_p s(p) \Big[\cdots |\phi_{p(j)}\rangle \cdots |\phi_{p(i)}\rangle \cdots \Big]. \end{aligned} \tag{4.2.18}$$

Within each term in the above sum, the single-particle states for $p(i)$ and $p(j)$ have exchanged places. The resulting term must be an exact match for another term in the original expression for $|\phi_1, \dots, \phi_N\rangle$, since the sum runs over all possible permutations, except for one difference: the coefficient $s(p)$ must have an *opposite* sign, since the two permutations are related by an exchange. It follows that $\hat{P}_{ij}|\phi_1, \dots, \phi_N\rangle = -|\phi_1, \dots, \phi_N\rangle$ for any choice of $i \neq j$.

Distinguishing particles

When studying the phenomenon of entanglement in the previous chapter, we implicitly assumed that the particles are distinguishable. For example, in the EPR thought experiment, we started with the two-particle state

$$|\psi_{\text{EPR}}\rangle = \frac{1}{\sqrt{2}} \Big(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle \Big), \tag{4.2.19}$$

which appears to be antisymmetric. Does this mean that we cannot prepare $|\psi_{\text{EPR}}\rangle$ using photons (which are bosons)? More disturbingly, we discussed how measuring $\hat{S}_z$ on particle A, and obtaining the result $+\hbar/2$, causes the two-particle state to collapse into $|+z\rangle|-z\rangle$, which is neither symmetric nor antisymmetric. Is this result invalidated if the particles are identical?

The answer to each question is no. The confusion arises because the particle exchange symmetry has to involve an exchange of *all* the degrees of freedom of each particle, and Equation (4.2.19) only shows the spin degree of freedom.

To unpack the above statement, let us suppose the two particles in the EPR experiment are identical bosons. We have focused on each particle's spin degree of freedom, but they must also have a position degree of freedom—that's how we can have a particle at

Alpha Centauri (A) and another at Betelgeuse (B). If we explicitly account for this position degree of freedom, the single-particle Hilbert space should be

$$\mathscr{H}^{(1)} = \mathscr{H}_{\text{spin}} \otimes \mathscr{H}_{\text{position}}. \tag{4.2.20}$$

For simplicity, let us treat position as a twofold degree of freedom, treating $\mathscr{H}_{\text{position}}$ as a 2D space spanned by the basis $\{|A\rangle, |B\rangle\}$.

Now consider the state we previously denoted by $|+z\rangle|-z\rangle$, which refers to a spin-up particle at A and a spin-down particle at B. In our previous notation, it was implicitly assumed that A refers to the left-hand slot of the tensor product, and B refers to the right-hand slot. If we account for the position degrees of freedom, the state is written as

$$|+z, A;\ -z, B\rangle = \frac{1}{\sqrt{2}}\Big(|+z\rangle|A\rangle|-z\rangle|B\rangle + |-z\rangle|B\rangle|+z\rangle|A\rangle\Big), \tag{4.2.21}$$

where the kets are written in the following order:

$$\Big[(\text{spin } 1) \otimes (\text{position } 1)\Big] \otimes \Big[(\text{spin } 2) \otimes (\text{position } 2)\Big]. \tag{4.2.22}$$

The exchange operator $\hat{P}_{12}$ swaps the two particles' Hilbert spaces—which includes both the position *and* the spin part. Hence, Equation (4.2.21) is explicitly symmetric:

$$\begin{aligned}\hat{P}_{12}\,|+z, A\,;\,-z, B\rangle &= \frac{1}{\sqrt{2}}\Big(|-z\rangle|B\rangle|+z\rangle|A\rangle + |+z\rangle|A\rangle|-z\rangle|B\rangle\Big)\\ &= |+z, A\,;\,-z, B\rangle.\end{aligned} \tag{4.2.23}$$

Likewise, if there is a spin-down particle at A and a spin-up particle at B, the bosonic two-particle state is

$$|-z, A\,;\,+z, B\rangle = \frac{1}{\sqrt{2}}\Big(|-z\rangle|A\rangle|+z\rangle|B\rangle + |+z\rangle|B\rangle|-z\rangle|A\rangle\Big). \tag{4.2.24}$$

Using Equations (4.2.21) and (4.2.24), we can rewrite the EPR singlet state (4.2.19) as

$$\begin{aligned}|\psi_{\text{EPR}}\rangle &= \frac{1}{\sqrt{2}}\Big(|+z, A\,;\,-z, B\rangle - |-z, A\,;\,+z, B\rangle\Big)\\ &= \frac{1}{2}\Big(\,|+z\rangle|A\rangle|-z\rangle|B\rangle + |-z\rangle|B\rangle|+z\rangle|A\rangle\\ &\qquad - |-z\rangle|A\rangle|+z\rangle|B\rangle - |+z\rangle|B\rangle|-z\rangle|A\rangle\Big).\end{aligned} \tag{4.2.25}$$

This state looks like a mess, but it turns out that we can clarify it with some careful re-ordering. Instead of the ordering (4.2.22), order by spins and then positions:

$$\Big[(\text{spin } 1) \otimes (\text{spin } 2)\Big] \otimes \Big[(\text{position } 1) \otimes (\text{position } 2)\Big] \tag{4.2.26}$$

Then Equation (4.2.25) can be rewritten as

$$|\psi_{\text{EPR}}\rangle = \frac{1}{\sqrt{2}}\Big(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle\Big) \otimes \frac{1}{\sqrt{2}}\Big(|A\rangle|B\rangle - |B\rangle|A\rangle\Big). \tag{4.2.27}$$

Evidently, even though the spin degrees of freedom form an antisymmetric combination, as described by Equation (4.2.19), the position degrees of freedom in Equation (4.2.27) also have an antisymmetric form, and this allows the two-particle state to meet the bosonic symmetry condition.

Suppose we perform a measurement on $|\psi_{\text{EPR}}\rangle$, and find that the particle at position A has spin $+z$. As usual, a measurement outcome can be associated with a projection operator. Using the ordering (4.2.22), we can write the relevant projection operator as

$$\hat{\Pi} = \Big(|+z\rangle\langle+z| \otimes |A\rangle\langle A|\Big) \otimes \Big(\hat{I} \otimes \hat{I}\Big) + \Big(\hat{I} \otimes \hat{I}\Big) \otimes \Big(|+z\rangle\langle+z| \otimes |A\rangle\langle A|\Big). \tag{4.2.28}$$

This accounts for the fact that the observed phenomenon—spin $+z$ at position A—may refer to either particle. Applying $\hat{\Pi}$ to the EPR state (4.2.25) yields

$$|\psi'\rangle = \frac{1}{2}\Big(|+z\rangle|A\rangle|-z\rangle|B\rangle + |-z\rangle|B\rangle|+z\rangle|A\rangle\Big). \tag{4.2.29}$$

Apart from a change in normalization, this is precisely the fermionic state $|+z, A; -z, B\rangle$ defined in Equation (4.2.21). In our earlier notation, this state was simply written as $|+z\rangle|-z\rangle$. This goes to show that particle exchange symmetry is fully compatible with the concepts of partial measurements, entanglement, etc., discussed in the previous chapter.