1.3: Scattering From a 1D Delta-Function Potential

We are now ready to solve a simple scattering problem. Consider a 1D space with spatial coordinate denoted by x, and a scattering potential that consists of a "spike" at $x=0$:

$$V(x)=\frac{\hbar^2\gamma}{2m}\,\delta(x). \tag{1.3.1}$$

The form of the prefactor $\hbar^2\gamma/2m$ is chosen for later convenience; the parameter γ, which has units of $[1/x]$, controls the strength of the scattering potential.

Figure 1.3.1

If you are disturbed by the idea of a delta function potential, just regard it as the limiting case of a family of increasingly tall and narrow gaussian functions centered at $x=0$. For each non-singular potential, the applicability of the Schrödinger wave equation implies that the wavefunction $\psi(x)$ is continuous and has well-defined first and second derivatives. In the delta function limit, however, these conditions are relaxed: $\psi(x)$ remains continuous, but at $x=0$ the first derivative becomes discontinuous and the second derivative blows up. To see this, we integrate the Schrödinger wave equation over an infinitesimal range around $x=0$:

$$\begin{aligned}\lim_{\varepsilon\to0^+}\int_{-\varepsilon}^{+\varepsilon}dx\,\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{\hbar^2\gamma}{2m}\delta(x)\right]\psi(x) &= \lim_{\varepsilon\to0^+}\int_{-\varepsilon}^{+\varepsilon}dx\;E\psi(x)\\ =\lim_{\varepsilon\to0^+}\left\{-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\right]_{-\varepsilon}^{+\varepsilon}\right\}+\frac{\hbar^2\gamma}{2m}\psi(0) &= 0\end{aligned} \tag{1.3.2}$$

Hence,

$$\lim_{\varepsilon\to0^+}\left\{\left.\frac{d\psi}{dx}\right|_{x=+\varepsilon}-\left.\frac{d\psi}{dx}\right|_{x=-\varepsilon}\right\}=\gamma\,\psi(0). \tag{1.3.3}$$

To proceed, consider a particle incident from the left, with energy E. This is described by an incident state proportional to a momentum eigenstate $|k\rangle$, where $k=\sqrt{2mE/\hbar^2}>0$. We said "proportional", not "equal", for it is conventional to adopt the normalization

$$|\psi_i\rangle=\sqrt{2\pi}\Psi_i|k\rangle\quad\Leftrightarrow\quad\psi_i(x)=\langle x|\psi\rangle=\Psi_i\,e^{ikx}. \tag{1.3.4}$$

The complex constant Ψ_i is called the "incident amplitude." Plugging this into the Schrödinger wave equation gives

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{\hbar^2\gamma}{2m}\delta(x)\right]\left(\Psi_i\,e^{ikx}+\psi_s(x)\right)=E\left(\Psi_i\,e^{ikx}+\psi_s(x)\right). \tag{1.3.5}$$

Taking $E=\hbar^2k^2/2m$, and doing a bit of algebra, simplifies this to

$$\left[\frac{d^2}{dx^2}+k^2\right]\psi_s(x)=\gamma\delta(x)\left(\Psi_i\,e^{ikx}+\psi_s(x)\right), \tag{1.3.6}$$

which is an inhomogenous ordinary differential equation for $\psi_s(x)$, with the potential on the right hand side acting as a "driving term".

To find the solution, consider the two regions $x<0$ and $x>0$. Since $\delta(x)\to0$ for $x\neq0$, the equation in each half-space reduces to

$$\left[\frac{d^2}{dx^2}+k^2\right]\psi_s(x)=0. \tag{1.3.7}$$

This is the **Helmholtz equation**, whose general solution may be written as

$$\psi_s(x) = \Psi_i \left(f_1\, e^{ikx} + f_2\, e^{-ikx}\right). \tag{1.3.8}$$

Here, f_1 and f_2 are complex numbers that can take on different values in the two different regions $x < 0$ and $x > 0$.

We want $\psi_s(x)$ to describe an **outgoing wave**, moving away from the scatterer towards infinity. So it should be purely left-moving for $x < 0$, and purely right-moving for $x > 0$. To achieve this, let $f_1 = 0$ for $x < 0$, and $f_2 = 0$ for $x > 0$, so that $\psi_s(x)$ has the form

$$\psi_s(x) = \Psi_i \times \begin{cases} f_-\, e^{-ikx}, & x < 0 \\ f_+\, e^{ikx}, & x > 0. \end{cases} \tag{1.3.9}$$

The complex numbers f_- and f_+ are called **scattering amplitudes**. They describe the magnitude and phase of the wavefunction scattered backwards into the $x < 0$ region, and scattered forward into the $x > 0$ region, respectively.

Recall from the discussion at the beginning of this section that $\psi(x)$ must be continuous everywhere, including at $x = 0$. Since $\psi_i(x)$ is continuous, $\psi_s(x)$ must be as well, so $f_- = f_+$. Moreover, we showed in Equation (1.3.3) that the first derivative of $\psi(x)$ is discontinuous at the scatterer. Plugging (1.3.3) into our expression for $\psi(x)$, at $x = 0$, gives

$$\Psi_i \left[ik(1 + f_\pm) - ik(1 - f_\pm)\right] = \Psi(1 + f_\pm)\gamma. \tag{1.3.10}$$

Hence, we obtain

$$f_+ = f_- = -\frac{\gamma}{\gamma - 2ik}. \tag{1.3.11}$$

For now, let us focus on the magnitude of the scattering amplitude (in the next chapter, we will see that the phase also contains useful information). The quantity $|f_\pm|^2$ describes the overall strength of the scattering process:

$$|f_\pm|^2 = \left[1 + \frac{8mE}{(\hbar\gamma)^2}\right]^{-1}. \tag{1.3.12}$$

Its dependence on E is plotted below:

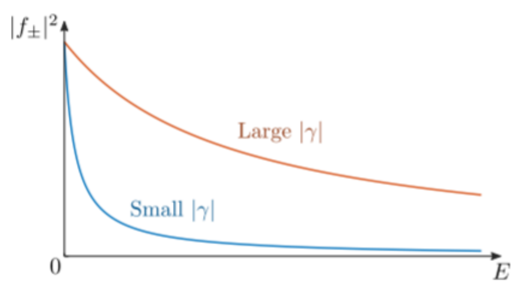

Figure 1.3.2

There are several notable features in this plot. First, for fixed potential strength γ, the scattering strength decreases monotonically with E—i.e., higher-energy particles are scattered less easily. Second, for given E, the scattering strength increases with $|\gamma|$, with the limit $|f|^2 \to 1$ as $|\gamma| \to \infty$. Third, an attractive potential ($\gamma < 0$) and a repulsive potential ($\gamma > 0$) are equally effective at scattering the particle.