1.8: Scattering Amplitudes in 3D

The propagator can now be plugged into the scattering problem posed in Sections 1.5-1.6:

$$\begin{aligned}\psi_i(\mathbf{r}) &= \Psi_i \, e^{i\mathbf{k}\cdot\mathbf{r}}, \\ \psi_s(\mathbf{r}) &= \langle \mathbf{r}|\hat{G}_0\hat{V}|\psi\rangle \;\overset{r\rightarrow\infty}{\longrightarrow}\; \Psi_i \, r^{\frac{1-d}{2}} \, e^{ikr} \, f(\mathbf{k}_i \rightarrow k\hat{\mathbf{r}}).\end{aligned} \tag{1.8.1}$$

Our goal is to determine the scattering amplitude f. We will focus on the 3D case; the 1D and 2D cases are handled in a similar way.

In the $r \rightarrow \infty$ limit, the propagator can be simplified using the Taylor expansion

$$|\mathbf{r}-\mathbf{r}'| = r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \cdots, \tag{1.8.2}$$

where $\hat{\mathbf{r}}$ denotes the unit vector pointing parallel to $\mathbf{r}$. (This is the same "large-r" expansion used in deriving the electric dipole moment in classical electromagnetism.) Applying this to the 3D outgoing propagator gives, to lowest order,

$$\langle \mathbf{r}|\hat{G}_0|\mathbf{r}'\rangle \overset{r\rightarrow\infty}{\approx} -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\exp(-ik\,\hat{\mathbf{r}}\cdot\mathbf{r}') \tag{1.8.3}$$

Hence, the scattered wavefunction is

$$\begin{aligned}\psi_s(\mathbf{r}) \;&=\; \int d^3r' \;\langle \mathbf{r}|\hat{G}_0|\mathbf{r}'\rangle\, V(\mathbf{r}')\,\psi(\mathbf{r}') \\ &\overset{r\rightarrow\infty}{\approx}\; -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d^3r' \exp(-ik\,\hat{\mathbf{r}}\cdot\mathbf{r}')\,V(\mathbf{r}')\,\psi(\mathbf{r}') \\ &=\; -\frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\,(2\pi)^{3/2}\,\langle \mathbf{k}_f|\hat{V}|\psi\rangle, \quad \text{where}\;\; \mathbf{k}_f \equiv k\hat{\mathbf{r}}.\end{aligned} \tag{1.8.4}$$

We can combine this with the Green's function relation from Section 1.6,

$$|\psi\rangle = \left(\hat{I} + \hat{G}\hat{V}\right)|\psi_i\rangle. \tag{1.8.5}$$

This yields

$$\begin{aligned}\psi_s(\mathbf{r}) \;&\overset{r\rightarrow\infty}{\longrightarrow}\; -\frac{2m}{\hbar^2}\frac{e^{ikr}}{r}\sqrt{\frac{\pi}{2}}\,\langle \mathbf{k}_f|\hat{V}+\hat{V}\hat{G}\hat{V}|\psi_i\rangle \\ &= -\frac{2m}{\hbar^2}\,\Psi_i\,\frac{e^{ikr}}{r}\;2\pi^2\,\langle \mathbf{k}_f|\hat{V}+\hat{V}\hat{G}\hat{V}|\mathbf{k}_i\rangle.\end{aligned} \tag{1.8.6}$$

This can be compared to the earlier definition of the scattering amplitude,

$$\psi_s(\mathbf{r}) \;\overset{r\rightarrow\infty}{\longrightarrow}\; \Psi_i\,\frac{e^{ikr}}{r} f(\mathbf{k}_i \rightarrow \mathbf{k}_f). \tag{1.8.7}$$

Hence, we find that

Definition: Scattering

$$\begin{aligned}f(\mathbf{k}_i \rightarrow \mathbf{k}_f) \;&= -\frac{2m}{\hbar^2}\cdot 2\pi^2\,\langle \mathbf{k}_f|\hat{V}+\hat{V}\hat{G}\hat{V}|\mathbf{k}_i\rangle \\ &= -\frac{2m}{\hbar^2}\cdot 2\pi^2\,\langle \mathbf{k}_f|\hat{V}+\hat{V}\hat{G}_0\hat{V}+\hat{V}\hat{G}_0\hat{V}\hat{G}_0\hat{V}+\cdots|\mathbf{k}_i\rangle,\end{aligned} \tag{1.8.8}$$

subject to the elasticity constraint $|\mathbf{k}_i| = |\mathbf{k}_f|$. In deriving the last line, we used the Born series formula (1.6.11).

This result is the culmination of the numerous definitions and derivations from the preceding sections. On the left side is the scattering amplitude, the fundamental quantity of interest in scattering experiments. The right side contains quantities that are known to us, or that can be calculated: the initial and final momenta, the scattering potential, and the Green's function. Although this result was derived for the 3D case, very similar formulas hold for other dimensions, but with the $2\pi^2$ factor replaced with other numerical factors.