3.2: Partial Measurements

Let us recall how measurements work in single-particle quantum theory. Each observable Q is described by some Hermitian operator $\hat{Q}$, which has an eigenbasis $\{|q_i\rangle\}$ such that

$$\hat{Q}|q_i\rangle = q_i|q_i\rangle. \tag{3.2.1}$$

For simplicity, let the eigenvalues $\{q_i\}$ be non-degenerate. Suppose a particle initially has quantum state $|\psi\rangle$. This can always be expanded in terms of the eigenbasis of $\hat{Q}$:

$$|\psi\rangle = \sum_i \psi_i\,|q_i\rangle, \quad \text{where} \quad \text{and}\ \psi_i = \langle q_i|\psi\rangle. \tag{3.2.2}$$

The **measurement postulate of quantum mechanics** states that if we measure Q, then (i) the probability of obtaining the measurement outcome q_i is $P_i = |\psi_i|^2$, the absolute square of the coefficient of $|q_i\rangle$ in the basis expansion; and (ii) upon obtaining this outcome, the system instantly "collapses" into state $|q_i\rangle$.

Mathematically, these two rules can be summarized using the projection operator

$$\hat{\Pi}(q_i) = |q_i\rangle\langle q_i|. \tag{3.2.3}$$

Applying this operator to $|\psi\rangle$ gives the non-normalized state vector

$$|\psi'\rangle = |q_i\rangle\langle q_i|\psi\rangle. \tag{3.2.4}$$

From this, we glean two pieces of information:

1. The probability of obtaining this outcome is $\langle\psi'|\psi'\rangle = |\langle q_i|\psi\rangle|^2$.
2. The post-collapse state is obtained by the re-normalization $|\psi'\rangle \rightarrow |q_i\rangle$.

For multi-particle systems, there is a new complication: what if a measurement is performed on just one particle?

Consider a system of two particles A and B, with two-particle Hilbert space $\mathscr{H}_A \otimes \mathscr{H}_B$. We perform a measurement on particle A, corresponding to a Hermitian operator $\hat{Q}_A$ that acts upon $\mathscr{H}_A$ and has eigenvectors $\{|\mu\rangle \mid \mu = 1, 2, \ldots\}$ (i.e., the eigenvectors are enumerated by some index μ). We can write any state $|\psi\rangle$ using the eigenbasis of $\hat{Q}_A$ for the $\mathscr{H}_A$ part, and an arbitrary basis $\{|\nu\rangle\}$ for the $\mathscr{H}_B$ part:

$$\begin{aligned} |\psi\rangle &= \sum_{\mu\nu} \psi_{\mu\nu}\,|\mu\rangle|\nu\rangle \\ &= \sum_{\mu} |\mu\rangle|\varphi_\mu\rangle, \quad \text{where} \quad |\varphi_\mu\rangle \equiv \sum_\nu \psi_{\mu\nu}\,|\nu\rangle \ \in\ \mathscr{H}_B. \end{aligned} \tag{3.2.5}$$

Unlike the single-particle case, the "coefficient" of $|\mu_i\rangle$ in this basis expansion is not a complex number, but a vector in $\mathscr{H}_B$.

Proceeding by analogy, the probability of obtaining the outcome labelled by μ should be the "absolute square" of this "coefficient", $\langle\varphi_\mu|\varphi_\mu\rangle$. Let us define the partial projector

$$\hat{\Pi}(\mu) \;=\; |\mu\rangle\langle\mu| \otimes \hat{I}\,. \tag{3.2.6}$$

The A slot of this operator contains a projector, $|\mu\rangle\langle\mu|$, while the B slot leaves the $\mathscr{H}_B$ part of the two-particle space unchanged. Applying the partial projector to the state given in Equation $(3.2.5)$ gives

$$|\psi'\rangle \;=\; \hat{\Pi}(\mu)\,|\psi\rangle \;=\; |\mu\rangle|\varphi_\mu\rangle. \tag{3.2.7}$$

Now we follow the same measurement rules as before. The outcome probability is

$$P_\mu = \langle\psi'|\psi'\rangle = \langle\mu|\mu\rangle\,\langle\varphi_\mu|\varphi_\mu\rangle = \sum_\nu |\psi_{\mu\nu}|^2. \tag{3.2.8}$$

The post-measurement collapsed state is obtained by the re-normalization

$$|\psi'\rangle \;\rightarrow\; \frac{1}{\sqrt{\sum_{\nu'} |\psi_{\mu\nu'}|^2}} \;\sum_\nu \psi_{\mu\nu} |\mu\rangle |\nu\rangle. \tag{3.2.9}$$

Example 3.2.1

A system of two spin-$1/2$ particles is in the "singlet state"

$$|\psi\rangle = \frac{1}{\sqrt{2}} \Big(|{+z}\rangle|{-z}\rangle \,-\, |{-z}\rangle|{+z}\rangle\Big). \tag{3.2.10}$$

For each particle, $|{+z}\rangle$ and $|{-z}\rangle$ denote eigenstates of the operator $\hat{S}_z$, with eigenvalues $+\hbar/2$ and $-\hbar/2$ respectively. Suppose we measure S_z on particle A. What are the probabilities of the possible outcomes, and the associated post-collapse states?

- **First outcome: $+\hbar/2$.**
  - The partial projector is $|{+z}\rangle\langle{+z}| \otimes \hat{I}$.
  - Applying the projection to $|\psi\rangle$ yields $|\psi'\rangle = (1/\sqrt{2})\, |{+z}\rangle|{-z}\rangle$.
  - The outcome probability is $P_+ = \langle\psi'|\psi'\rangle = \dfrac{1}{2}$.
  - The post-collapse state is $\dfrac{1}{\sqrt{P_+}} |\psi'\rangle = |{+z}\rangle|{-z}\rangle$
- **Second outcome: $-\hbar/2$.**
  - The partial projector is $|{-z}\rangle\langle{-z}| \otimes \hat{I}$.
  - Applying the projection to $|\psi\rangle$ yields $|\psi'\rangle = (1/\sqrt{2})\, |{-z}\rangle|{+z}\rangle$.
  - The outcome probability is $P_- = \langle\psi'|\psi'\rangle = \dfrac{1}{2}$.
  - The post-collapse state is $= \dfrac{1}{\sqrt{P_-}} |\psi'\rangle = |{-z}\rangle|{+z}\rangle$.

The two possible outcomes, $+\hbar/2$ and $-\hbar/2$, occur with equal probability. In either case, the two-particle state collapses so that A is in the observed spin eigenstate, and B has the opposite spin. After the collapse, the two-particle state is no longer entangled.