

## 11.2: Spectral Resolution and Range

In the previous section, we showed how a continuous Fourier integral is converted into a DFT. This process involved two distinct approximations. Firstly, the Fourier integral is *truncated* from its original range,  $x \in (-\infty, \infty)$ , to a finite interval of length  $N\Delta x$ . Secondly, the integral is *discretized* by reducing the continuous variable  $x$  to a set of discrete points  $\{x_0, \dots, x_{N-1}\}$ . Both of these approximations have important consequences for the accuracy of our numerical Fourier spectrum, which we will examine in turn.

### 11.2.1 Spectral Resolution

The truncation of the Fourier integral limits the **spectral resolution** of the Fourier spectrum. To see this, suppose we perform truncation without discretization, by taking a continuous Fourier integral and truncating it to a finite range  $x \in [0, X]$ :

$$F(k) \approx \int_0^X dx e^{-ikx} f(x). \quad (11.2.1)$$

Consider a harmonic function  $f(x) = e^{ik_0 x}$ . The exact Fourier transform can be shown to be a [delta function](#),  $F(k) = 2\pi \delta(k - k_0)$ , i.e. an infinitely sharp peak centered at  $k = k_0$ . With the above truncation, however, the resulting integral is

$$F(k) \approx \int_0^X dx e^{-i(k-k_0)x} = \frac{2 \sin[(k-k_0)X/2]}{k-k_0} \cdot e^{-i(k-k_0)X/2}. \quad (11.2.2)$$

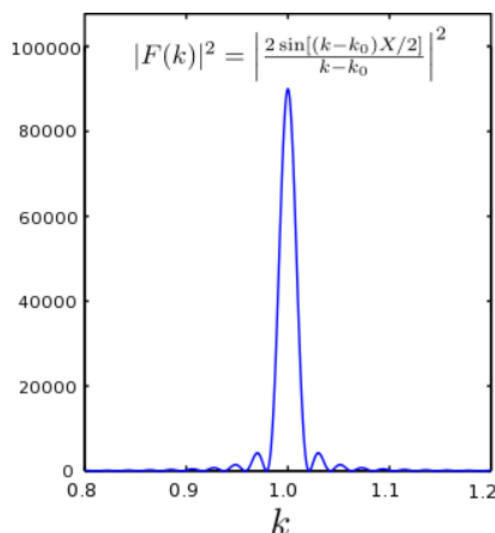


Figure 11.2.1: Fourier power spectrum from a truncated Fourier transform of  $f(x) = \exp(ik_0 x)$ , with  $k_0 = 1$  and sampling interval  $k \in [0, 300]$  (about 48 periods).

For  $X \rightarrow \infty$ , the above formula approaches a delta function (an infinitesimally-thin peak) centered at  $k = k_0$ . But for finite  $X$ , the plot of  $|F(k)|^2$  versus  $\omega$  behaves as shown in Fig 11.2.1. Evidently, truncating the Fourier integral has "smeared out" the Fourier spectrum, broadening the infinitesimally-thin delta function peak into a finite-width peak. The peak width,  $\Delta k \sim 1/X$ , limits the "resolution" of our Fourier analysis.

In the discretized Fourier transform, the truncation of the Fourier integral has essentially the same effect. As discussed in the previous section, the DFT is defined at  $k_n \equiv 2\pi n/X$ ; hence, the resolution of the Fourier spectrum is  $\Delta k = 2\pi/X$ .

### 11.2.2 Spectral Range

The other approximation which we made in going from a continuous Fourier transform to the DFT involved sampling  $f(x)$  at discrete values of  $x$ . This discretization has the effect of limiting the **spectral range**. To see this, let us look again at the DFT formula, which is dimensionless:

$$F_n = \sum_{m=0}^{N-1} e^{-2\pi i \frac{mn}{N}} f_m. \quad (11.2.3)$$

Normally, we consider only the indices  $n = 0, 1, \dots, N-1$ . However, if we replace  $n$  with  $n + N$  in the right-hand side, the result would be the same:

$$F_{n+N} = \sum_{m=0}^{N-1} e^{-2\pi i \frac{m(n+N)}{N}} f_m = \sum_{m=0}^{N-1} e^{-2\pi i \frac{mn}{N} - 2\pi i m} f_m = F_n. \quad (11.2.4)$$

We can hence regard  $F$  as a periodic discrete function of  $n$ , with period  $N$ . Next, consider how the DFT is related to the physical  $x$  and  $k$  variables. Taking  $x_0 = 0$  for simplicity,

$$F(k_n) = \sum_{m=0}^{N-1} e^{-2\pi i \frac{mn}{N}} f(x_m) = \sum_{m=0}^{N-1} e^{-ik_n \cdot m\Delta x} f(x_m). \quad (11.2.5)$$

If we perform the replacement

$$k_n \rightarrow k_n + K, \quad \text{where } K \equiv \frac{2\pi}{\Delta x}, \quad (11.2.6)$$

then evidently  $F(k_n)$  is left unchanged. Indeed, we could add any integer multiple of  $K$  without altering the result. This means that the DFT spectrum is only defined under  $k$  modulo  $K$ , by contrast with the continuous Fourier transform which is defined over the entire interval  $-\infty < k < \infty$ .

The default definition of the DFT gives the integer indices  $n = 0, 1, \dots, N-1$ , which corresponds to  $0 \leq k \lesssim K$ . However, when plotting the DFT spectrum, we usually adjust the range of  $k$  to  $-K/2 \lesssim k \lesssim K/2$ . This is done by taking the "upper half" of the DFT spectrum,  $K/2 \lesssim k \lesssim K$ , and translating it via the replacement  $k \rightarrow k - K$ . Due to the periodicity of the DFT, the upper half of the DFT spectrum becomes the negative  $k$  part of the spectrum. In terms of the integer indices  $n$ , the process is depicted in the figure below:

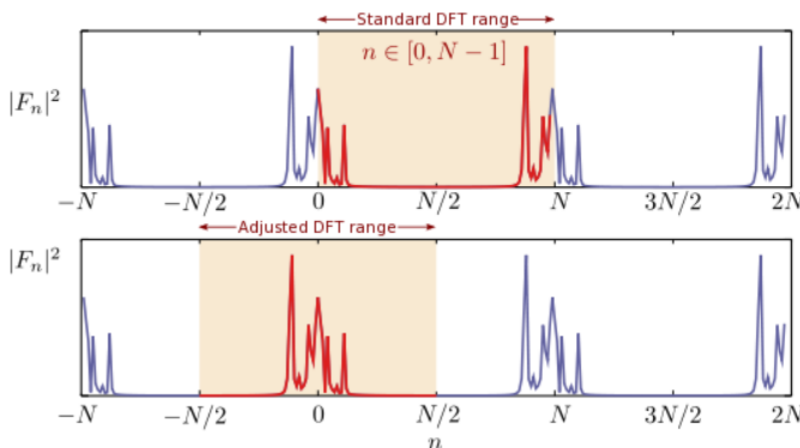


Figure 11.2.2: A DFT spectrum,  $F_n$ , is periodic with period  $N$ . By default, the DFT is reported in the spectral range  $n \in [0, N-1]$  (red curve in the upper plot). To relate this to the continuous Fourier transform, we re-center the spectrum at  $n = 0$ , which is equivalent to translating the upper half of the spectrum to negative values (red curve in the lower plot).

The reason for this adjustment is that, intuitively, the discretized Fourier spectrum contains information about the "low-frequency" part of the spectrum,  $|k| < K/2$ , including both positive and negative values of  $k$ . On the other hand, the discretized Fourier spectrum lacks information about the "high frequency" part of the spectrum, which correspond to harmonics with periods shorter than the discretization step  $\Delta x$ . Hence, it makes sense to "center" our Fourier spectrum around the origin. It can then be shown that as the discretization step approaches zero (and hence  $K = 2\pi/\Delta x \rightarrow \infty$ ), the  $|k| \ll K$  part of the adjusted DFT spectrum converges to the exact (continuous) Fourier spectrum.

The corollary to the above discussion is that if we have a function which has no frequency components larger than  $k_{\max}$ , then it is sufficient to use a sampling interval  $\Delta x = \pi/k_{\max}$ . This is called the [Nyquist-Shannon sampling theorem](#).

### 11.2.3 Summary of Spectral Relations

The results of the previous sections can be summarized in this way:

- The total range of  $x$ , which is denoted by  $X$ , limits the resolution of the spectrum to  $\Delta k = 2\pi/X$ .

- The resolution of  $x$ , which is the discretization step  $\Delta x$ , limits the range of the spectrum to  $K = 2\pi/\Delta x$ .

These relations are easy to remember, because the "interval length" in the one domain places a limit on the "discretization step" in the other domain. It is very important to keep these relations in mind when working with discrete Fourier transforms! For example, a common mistake that people make is to try to improve the resolution of a Fourier spectrum by increasing the number of discretization steps,  $N$ , while keeping the total interval  $X$  fixed. This doesn't work; it leaves the spectral resolution unchanged! In order to improve the spectral resolution, one has to increase the total interval instead.

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