

## 6.1: Basic Facts about Eigenvalue Problems

Even if a matrix  $\mathbf{A}$  is real, its eigenvectors and eigenvalues can be complex. For example,

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (1+i) \begin{bmatrix} 1 \\ i \end{bmatrix}. \quad (6.1.1)$$

Eigenvectors are not uniquely defined. Given an eigenvector  $\vec{x}$ , any nonzero complex multiple of that vector is also an eigenvector of the same matrix, with the same eigenvalue. We can reduce this ambiguity by **normalizing** eigenvectors to a fixed unit length:

$$\sum_{n=0}^{N-1} |x_n|^2 = 1. \quad (6.1.2)$$

Note, however, that even after normalization, there is still an inherent ambiguity in the overall complex phase. Multiplying a normalized eigenvector by any phase factor  $e^{i\phi}$  gives another normalized eigenvector with the same eigenvalue.

### 6.1.1 Matrix Diagonalization

Most matrices are **diagonalizable**, meaning that their eigenvectors span the  $N$ -dimensional complex space (where  $N$  is the matrix size). Matrices which are not diagonalizable are called **defective**. Many classes of matrices that are relevant to physics (such as Hermitian matrices) are always diagonalizable; i.e., never defective.

The reason for the term "diagonalizable" is as follows. A diagonalizable  $N \times N$  matrix  $\mathbf{A}$  has eigenvectors that span the  $N$ -dimensional space, meaning that we can choose  $N$  linearly independent eigenvectors,  $\{\vec{x}_0, \vec{x}_1, \dots, \vec{x}_{N-1}\}$ , with eigenvalues  $\{\lambda_0, \lambda_1, \dots, \lambda_{N-1}\}$ . We refer to such a set of  $N$  eigenvalues as the "eigenvalues of  $\mathbf{A}$ ". If we group the eigenvectors into an  $N \times N$  matrix

$$\mathbf{Q} = [\vec{x}_0, \vec{x}_1, \dots, \vec{x}_{N-1}], \quad (6.1.3)$$

then, since the eigenvectors are linearly independent,  $\mathbf{Q}$  is guaranteed to be invertible. Using the eigenvalue equation, we can then show that

$$\mathbf{Q}^{-1} \mathbf{A} \mathbf{Q} = \begin{bmatrix} \lambda_0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N-1} \end{bmatrix}. \quad (6.1.4)$$

In other words, there exists a **similarity transformation** which converts  $\mathbf{A}$  into a diagonal matrix. The  $N$  numbers along the diagonal are precisely the eigenvalues of  $\mathbf{A}$ .

### 6.1.2 The Characteristic Polynomial

One of the most important consequences of diagonalizability is that the determinant of a diagonalizable matrix  $\mathbf{A}$  is the product of its eigenvalues:

$$\det(\mathbf{A}) = \prod_{n=0}^{N-1} \lambda_n \quad (6.1.5)$$

This can be proven by taking the determinant of the similarity transformation equation, and using (i) the property of the determinant that  $\det(\mathbf{UV}) = \det(\mathbf{U})\det(\mathbf{V})$ , and (ii) the fact that the determinant of a diagonal matrix is the product of the elements along the diagonal.

In particular, the determinant of  $\mathbf{A}$  is zero if one of its eigenvalues is zero. This fact can be further applied to the following rearrangement of the eigenvalue equation:

$$(\mathbf{A} - \lambda \mathbf{I}) \vec{x} = 0, \quad (6.1.6)$$

where  $\mathbf{I}$  is the  $N \times N$  identity matrix. This says that the matrix  $\mathbf{A} - \lambda \mathbf{I}$  has an eigenvalue of zero, meaning that for any eigenvalue  $\lambda$ ,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \quad (6.1.7)$$

The left-hand side of the above equation is a polynomial in the variable  $\lambda$ , of degree  $N$ . This is called the **characteristic polynomial** of the matrix  $\mathbf{A}$ . Its roots are eigenvalues of  $\mathbf{A}$ , and vice versa.

For  $2 \times 2$  matrices, the standard way of calculating the eigenvalues is to find the roots of the characteristic polynomial. However, this is not a reliable method for finding the eigenvalues of larger matrices. There is a well-known and important result in mathematics, known as [Abel's impossibility theorem](#), which states that polynomials of degree 5 and higher have no general algebraic solution. (By comparison, degree-2 polynomials have a general algebraic solution, which is the familiar quadratic formula, and similar formulas exist for [degree-3](#) and [degree-4](#) polynomials.) A matrix of size  $N \geq 5$  has a characteristic polynomial of degree  $N \geq 5$ , and Abel's impossibility theorem tells us that we can't calculate the roots of that characteristic polynomial by ordinary arithmetic.

In fact, Abel's impossibility theorem leads to an even stronger conclusion: there is no general algebraic method for finding the eigenvalues of a matrix of size  $N \geq 5$ , whether using the characteristic polynomial *or any other method*. For suppose we had such a method for finding the eigenvalues of a matrix. Then, for any polynomial equation of degree  $N \geq 5$ , of the form

$$a_0 + a_1 \lambda + \cdots + a_{N-1} \lambda^{N-1} + \lambda^N = 0, \quad (6.1.8)$$

we can construct an  $N \times N$  "companion matrix" of the form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix}. \quad (6.1.9)$$

As you can check for yourself, each root  $\lambda$  of the polynomial is also an eigenvalue of the companion matrix, with corresponding eigenvector

$$\vec{x} = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{N-1} \end{bmatrix}. \quad (6.1.10)$$

Hence, if there exists a general algebraic method for finding the eigenvalues of a large matrix, that would allow us to find solve polynomial equations of high degree. Abel's impossibility theorem tells us that no such solution method can exist.

This might seem like a terrible problem, but in fact there's a way around it, as we'll shortly see.

### 6.1.3 Hermitian Matrices

A **Hermitian** matrix  $\mathbf{H}$  is a matrix which has the property

$$\mathbf{H}^\dagger = \mathbf{H}, \quad (6.1.11)$$

where  $\mathbf{H}^\dagger$  denotes the "Hermitian conjugate", which is matrix transposition accompanied by complex conjugation:

$$\mathbf{H}^\dagger \equiv (\mathbf{H}^T)^*, \quad \text{i. e. } (H^\dagger)_{ij} = H_{ji}^*. \quad (6.1.12)$$

Hermitian matrices have the nice property that all their eigenvalues are real. This can be easily proven using index notation:

$$\sum_j H_{ij} x_j = \lambda x_i \Rightarrow \sum_j x_j^* H_{ji} = \lambda^* x_i^* \quad (6.1.13)$$

$$\Rightarrow \sum_{ij} x_i^* H_{ij} x_j = \lambda \sum_i |x_i|^2 = \lambda^* \sum_j |x_j|^2 \quad (6.1.14)$$

$$\Rightarrow \lambda = \lambda^*. \quad (6.1.15)$$

In quantum mechanics, Hermitian matrices play a special role: they represent measurement operators, and their eigenvalues (which are restricted to the real numbers) are the set of possible measurement outcomes.

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