

## 12.1: The Simplest Markov Chain- The Coin-Flipping Game

### 12.1.1 Game Description

Before giving the general description of a Markov chain, let us study a few specific examples of simple Markov chains. One of the simplest is a "coin-flip" game. Suppose we have a coin which can be in one of two "states": heads (H) or tails (T). At each step, we flip the coin, producing a new state which is H or T with equal probability. In this way, we generate a sequence like "HTTHTHTTHH..." If we run the game again, we would generate another different sequence, like "HTTTTHHTTH..." Each of these sequences is a Markov chain.

This process can be visualized using a "state diagram":

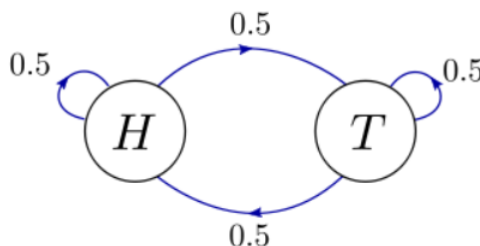


Figure 12.1.1: State diagram for a fair coin-flipping game.

Here, the two circles represent the two possible states of the system, "H" and "T", at any step in the coin-flip game. The arrows indicate the possible states that the system could transition into, during the next step of the game. Attached to each arrow is a number giving the probability of that transition. For example, if we are in the state "H", there are two possible states we could transition into during the next step: "T" (with probability 0.5), or "H" (with probability 0.5). By conservation of probability, the transition probabilities coming out of each state must sum up to one.

Next, suppose the coin-flipping game is unfair. The coin might be heavier on one side, so that it is overall more likely to land on H than T. It might also be slightly more likely to land on the same face that it was flipped from ([real coins actually do behave this way](#)). The resulting state diagram can look like this:

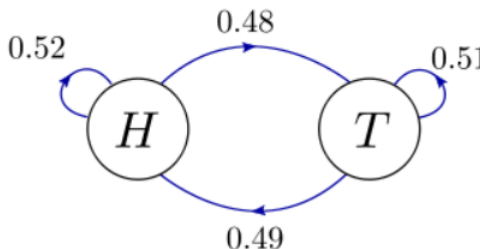


Figure 12.1.2: State diagram for a (particular) unfair coin-flipping game.

Notice that the individual transition probabilities are no longer 0.5, reflecting the aforementioned unfair effects. However, the transition probabilities coming out of each state still sum to 1 ( $0.52 + 0.48$  coming out of "H", and  $0.51 + 0.49$  coming out of "T").

### 12.1.2 State Probabilities

If we play the above unfair game many times, H and T will tend to occur with slightly different probabilities, not exactly equal to 0.5. At a given step, let  $p_H$  denote the probability to be in state H, and  $p_T$  the probability to be in state T. Let  $P(T|H)$  denote the transition probability for going from H to T, during the next step; and similarly for the other three possible transitions. According to [Bayes' rule](#), we can write the probability to get H on the next step as

$$p'_H = P(H|H)p_H + P(H|T)p_T. \quad (12.1.1)$$

Similarly, the probability to get T on the next step is

$$p'_T = P(T|H)p_H + P(T|T)p_T. \quad (12.1.2)$$

We can combine these into a single matrix equation:

$$\begin{bmatrix} p'_H \\ p'_T \end{bmatrix} = \begin{bmatrix} P(H|H) & P(H|T) \\ P(T|H) & P(T|T) \end{bmatrix} \begin{bmatrix} p_H \\ p_T \end{bmatrix} = \begin{bmatrix} 0.52 & 0.49 \\ 0.48 & 0.51 \end{bmatrix} \begin{bmatrix} p_H \\ p_T \end{bmatrix}. \quad (12.1.3)$$

The matrix of transition probabilities is called the *transition matrix*. At the beginning of the game, we can specify the coin state to be (say) H, so that  $p_H = 1$  and  $p_T = 0$ . If we multiply the vector of state probabilities by the transition matrix, that gives the state probabilities for the next step. Multiplying by the transition matrix  $K$  times gives the state probabilities after  $K$  steps.

After a large number of steps, the states probabilities might converge to a "stationary" distribution, such that they no longer change significantly on subsequent steps. Let these stationary probabilities be denoted by  $\{\pi_H, \pi_T\}$ . According to the above equation for transition probabilities, the stationary probabilities must satisfy

$$\begin{bmatrix} \pi_H \\ \pi_T \end{bmatrix} = \begin{bmatrix} P(H|H) & P(H|T) \\ P(T|H) & P(T|T) \end{bmatrix} \begin{bmatrix} \pi_H \\ \pi_T \end{bmatrix}. \quad (12.1.4)$$

This system of linear equations can be solved by brute force (we'll discuss a more systematic approach later). The result is

$$\pi_H = \frac{P(H|T)}{P(T|H) + P(H|T)}, \quad \pi_T = \frac{P(T|H)}{P(T|H) + P(H|T)} \quad (\text{stationary distribution}). \quad (12.1.5)$$

Plugging in the numerical values for the transition probabilities, we end up with  $\pi_H = 0.50515$ ,  $\pi_T = 0.49485$

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