

## 10.1: Example- Equations of Motion in Classical Mechanics

The above standard formulation of the initial-value problem can be used to describe a very large class of time-dependent ODEs found in physics. For example, suppose we have a classical mechanical particle with position  $\vec{r}$ , subject to an arbitrary external space-and-time-dependent force  $\vec{f}(\vec{r}, t)$  and a friction force  $-\lambda d\vec{r}/dt$  (where  $\lambda$  is a damping coefficient). Newton's second law gives the following equation of motion:

$$m \frac{d^2 \vec{r}}{dt^2} = -\lambda \frac{d\vec{r}}{dt} + \vec{f}(\vec{r}, t). \quad (10.1.1)$$

This is a second-order ODE, whereas the standard initial-value problem involves a first-order ODE. However, we can turn it into a first-order ODE with the following trick. Define the velocity vector

$$\vec{v} = \frac{d\vec{r}}{dt}, \quad (10.1.2)$$

and define the state vector by combining the position and velocity vectors:

$$\vec{y} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix}. \quad (10.1.3)$$

Then the equation of motion takes the form

$$\frac{d\vec{y}}{dt} = \frac{d}{dt} \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} \vec{v} \\ -(\lambda/m)\vec{v} + \vec{f}(\vec{r}, t)/m \end{bmatrix}, \quad (10.1.4)$$

which is a first-order ODE, as desired. The quantity on the right-hand side is the derivative function  $\vec{F}(\vec{y}, t)$  for the initial-value problem. Its dependence on  $\vec{r}$  and  $\vec{v}$  is simply regarded as a dependence on the upper and lower portions of the state vector  $\vec{y}$ . In particular, note that the derivative function does not need to be linear, since  $\vec{f}$  can have any arbitrary nonlinear dependence on  $\vec{r}$ , e.g. it could depend on the quantity  $|\vec{r}|$ .

The "initial state",  $\vec{y}(t_0)$ , requires us to specify both the initial position and velocity of the particle, which is consistent with the fact that the original equation of motion was a second-order equation, requiring two sets of initial values to fully specify a solution. In a similar manner, ODEs of higher order can be converted into first-order form, by defining the higher derivatives as state variables and increasing the size of the state vector.

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