

9.1: Mid-Point Rule

The simplest numerical integration method is called the **mid-point rule**. Consider a definite 1D integral

$$\mathcal{I} = \int_a^b f(x) dx. \quad (9.1.1)$$

Let us divide the range $a \leq x \leq b$ into a set of N segments of equal width, as shown in Fig. 9.1.1 for the case of $N = 5$. The mid-points of these segments are a set of N discrete points $\{x_0, \dots, x_{N-1}\}$, where

$$x_n = a + \left(n + \frac{1}{2}\right) \Delta x, \quad \Delta x \equiv \frac{b-a}{N}. \quad (9.1.2)$$

We then estimate the integral as

$$\mathcal{I}^{(\text{mp})} = \Delta x \sum_{n=0}^{N-1} f(x_n) \xrightarrow{N \rightarrow \infty} \mathcal{I}. \quad (9.1.3)$$

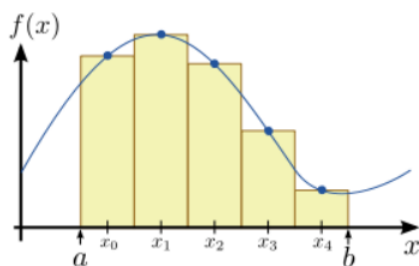


Figure 9.1.1: Computing a definite integral using the mid-point rule.

The principle behind this formula is very simple to understand. As shown in Fig. 9.1.1, I_N represents the area enclosed by a sequence of rectangles, where the height of each rectangle is equal to the value of $f(x)$ at its mid-point. As $N \rightarrow \infty$, the spacing between rectangles goes to zero; hence, the total area enclosed by the rectangles becomes equal to the area under the curve of $f(x)$.

9.1.1 Numerical Error for the Mid-Point Rule

Let us estimate the numerical error resulting from this approximation. To do this, consider one of the individual segments, which is centered at x_n with length $\Delta x = (b-a)/N$. Let us define the integral over this segment as

$$\Delta \mathcal{I}_n \equiv \int_{x_n - \Delta x/2}^{x_n + \Delta x/2} f(x) dx. \quad (9.1.4)$$

Now, consider the Taylor expansion of $f(x)$ in the vicinity of x_n :

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2}(x - x_n)^2 + \frac{f'''(x_n)}{6}(x - x_n)^3 + \dots \quad (9.1.5)$$

If we integrate both sides of this equation over the segment, the result is

$$\Delta \mathcal{I}_n = f(x_n) \Delta x + f'(x_n) \int_{x_n - \Delta x/2}^{x_n + \Delta x/2} (x - x_n) dx \quad (9.1.6)$$

$$+ \frac{f''(x_n)}{2} \int_{x_n - \Delta x/2}^{x_n + \Delta x/2} (x - x_n)^2 dx \quad (9.1.7)$$

$$+ \dots \quad (9.1.8)$$

On the right hand side, every other term involves an integrand which is odd around x_n . Such terms integrate to zero. From the remaining terms, we find the following series for the integral of $f(x)$ over the segment:

$$\Delta \mathcal{I}_n = f(x_n) \Delta x + \frac{f''(x_n) \Delta x^3}{24} + O(\Delta x^5). \quad (9.1.9)$$

By comparison, the estimation provided by the mid-point rule is simply

$$\Delta \mathcal{I}_n^{\text{mp}} = f(x_n) \Delta x \quad (9.1.10)$$

This is simply the first term in the exact series. The remaining terms constitute the numerical error in the mid-point rule integration, over this segment. We denote this error as

$$\mathcal{E}_n = |\Delta \mathcal{I}_n - \Delta \mathcal{I}_n^{\text{mp}}| \sim \frac{|f''(x_n)|}{24} \Delta x^3 \sim O\left(\frac{1}{N^3}\right). \quad (9.1.11)$$

The last step comes about because, by our definition, $\Delta x \sim O(1/N)$.

Now, consider the integral over the entire integration range, which consists of N such segments. In general, there is no guarantee that the numerical errors of each segment will cancel out, so the total error should be N times the error from each segment. Hence, for the mid-point rule,

$$\mathcal{E}_{\text{total}} \sim O\left(\frac{1}{N^2}\right). \quad (9.1.12)$$

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