3.E: Differential Geometry (Exercises)

1. Consider a spacetime that is locally exactly like the standard Lorentzian spacetime described in ch. 2, but that has a global structure differing in the following way from the one we have implicitly assumed. This spacetime has global property G: Let two material particles have world-lines that coincide at event A, with some nonzero relative velocity; then there may be some event B in the future light-cone of A at which the particles' world-lines coincide again. This sounds like a description of something that we would expect to happen in curved spacetime, but let's see whether that is necessary. We want to know whether this violates the flat-space properties L1-L5 in Appendix C, if those properties are taken as local.
   a. Demonstrate that it does not violate them, by using a model in which space "wraps around" like a cylinder.
   b. Now consider the possibility of interpreting L1-L5 as *global* statements. Do spacetimes with property G always violate L3 if L3 is taken globally?
2. Usually in relativity we pick units in which c = 1. Suppose, however, that we want to use SI units. The convention is that coordinates are written with upper indices, so that, fixing the usual Cartesian coordinates in 1+1 dimensions of spacetime, an infinitesimal displacement between two events is notated (ds^t, ds^x). In SI units, the two components of this vector have different units, which may seem strange but is perfectly legal. Describe the form of the metric, including the units of its elements. Describe the lower-index vector ds_a.
3. (a) Explain why the following expressions ain't got good grammar: U_{aa}, $x_a y^a$, $p^a - q^a$. (Recall our notational convention that Latin indices represent abstract indices, so that it would not make sense, for example, to interpret U_{aa} as U's ath diagonal element rather than as an implied sum.)
   (b) Which of these could also be nonsense in terms of units?
4. Suppose that a mountaineer describes her location using coordinates (θ, ϕ, h), representing colatitude, longitude, and altitude. Infer the units of the components of ds^a and of the elements of g_{ab} and g^{ab}. Given that the units of mechanical work should be newton-meters (example 5), infer the components of a force vector F_a and its upper-index version F^a.
5. Generalize Figure 2.1.8 (2) to three dimensions.
6. Suppose you have a collection of pencils, some of which have been sharpened more times than others so that they they're shorter. You toss them all on the floor in random orientations, and you're then allowed to slide them around but not to rotate them. Someone asks you to make up a definition of whether or not a given set of three pencils "cancels." If all pencils are treated equally (i.e., order doesn't matter), and if we respect the rotational invariance of Euclidean geometry, then you will be forced to reinvent vector addition and define cancellation of pencils **p**, **q**, and **r** as **p** + **q** + **r** = 0. Do something similar with "pencil" replaced by "an oriented pairs of lines as in Figure 2.1.8 (2).
7. Describe the quantity $g^a{}_a$. (Note the repeated index.)
8. Example 17 discusses the discontinuity that would result if one attempted to define a time coordinate for the GPS system that was synchronized globally according to observers in the rotating frame, in the sense that neighboring observers could verify the synchronization by exchanging electromagnetic signals. Calculate this discontinuity at the equator, and estimate the resulting error in position that would be experienced by GPS users.
9. Resolve the following paradox.
   Equation [3] claims to give the metric obtained by an observer on the surface of a rotating disk. This metric is shown to lead to a non-Euclidean value for the ratio of the circumference of a circle to its radius, so the metric is clearly non-Euclidean. Therefore a local observer should be able to detect violations of the Pythagorean theorem.
   And yet this metric was originally derived by a series of changes of coordinates, starting from the Euclidean metric in polar coordinates, as derived in example 8. Section 3.4 argued that the intrinsic measurements available in relativity are not capable of detecting an arbitrary smooth, one-to-one change of coordinates. This contradicts our earlier conclusion that there are locally detectable violations of the Pythagorean theorem.
10. This problem deals with properties of the metric [3].
    a. A pulse of collimated light is emitted from the center of the disk in a certain direction. Does the spatial track of the pulse form a geodesic of this metric?
    b. Characterize the behavior of the geodesics near $r = \frac{1}{\omega}$.
    c. An observer at rest with respect to the surface of the disk proposes to verify the non-Euclidean nature of the metric by doing local tests in which right triangles are formed out of laser beams, and violations of the Pythagorean theorem are detected. Will this work?

11. In the early decades of relativity, many physicists were in the habit of speaking as if the Lorentz transformation described what an observer would actually "see" optically, e.g., with an eye or a camera. This is not the case, because there is an additional effect due to optical aberration: observers in different states of motion disagree about the direction from which a light ray originated. This is analogous to the situation in which a person driving in a convertible observes raindrops falling from the sky at an angle, even if an observer on the sidewalk sees them as falling vertically. In 1959, Terrell and Penrose independently provided correct analyses,[17] showing that in reality an object may appear contracted, expanded, or rotated, depending on whether it is approaching the observer, passing by, or receding. The case of a sphere is especially interesting. Consider the following four cases:
    A. The sphere is not rotating. The sphere's center is at rest. The observer is moving in a straight line.
    B. The sphere is not rotating, but its center is moving in a straight line. The observer is at rest.
    C. The sphere is at rest and not rotating. The observer moves around it in a circle whose center coincides with that of the sphere.
    D. The sphere is rotating, with its center at rest. The observer is at rest.

    Penrose showed that in case A, the outline of the sphere is still seen to be a circle, although regions on the sphere's surface appear distorted.
    What can we say about the generalization to cases B, C, and D?
12. This problem involves a relativistic particle of mass m which is also a wave, as described by quantum mechanics. Let $c = 1$ and $\hbar = 1$ throughout. Starting from the de Broglie relations $E = \omega$ and $p = k$, where k is the wavenumber, find the dispersion relation connecting ω to k. Calculate the group velocity, and verify that it is consistent with the usual relations $p = m\gamma v$ and $E = m\gamma$ for $m > 0$. What goes wrong if you instead try to associate v with the phase velocity?

References

[17] James Terrell, "Invisibility of the Lorentz Contraction," Physical Review 116 (1959) 1045. Roger Penrose, "The Apparent Shape of a Relativistically Moving Sphere," Proceedings of the Cambridge Philosophical Society 55 (1959) 139.