3.1: Introduction to Differential Geometry

General relativity is described mathematically in the language of *differential geometry*. Let's take those two terms in reverse order.

The *geometry* of spacetime is non-Euclidean, not just in the sense that the 3+1-dimensional geometry of Lorentz frames is different than that of 4 interchangeable Euclidean dimensions, but also in the sense that parallels do not behave in the way described by E5 or A1-A3. In a Lorentz frame, which describes space without any gravitational fields, particles whose world-lines are initially parallel will continue along their parallel world-lines forever. But in the presence of gravitational fields, initially parallel world-lines of free-falling particles will in general diverge, approach, or even cross. Thus, neither the existence nor the uniqueness of parallels can be assumed. We can't describe this lack of parallelism as arising from the curvature of the world-lines, because we're using the world-lines of free-falling particles as our definition of a "straight" line. Instead, we describe the effect as coming from the curvature of spacetime itself. The Lorentzian geometry is a description of the case in which this curvature is negligible.

What about the word *differential*? The equivalence principle states that even in the presence of gravitational fields, local Lorentz frames exist. How local is "local?" If we use a microscope to zoom in on smaller and smaller regions of spacetime, the Lorentzian approximation becomes better and better. Suppose we want to do experiments in a laboratory, and we want to ensure that when we compare some physically observable quantity against predictions made based on the Lorentz geometry, the resulting discrepancy will not be too large. If the acceptable error is ϵ, then we should be able to get the error down that low if we're willing to make the size of our laboratory no bigger than δ. This is clearly very similar to the Weierstrass style of defining limits and derivatives in calculus. In calculus, the idea expressed by differentiation is that every smooth curve can be approximated locally by a line; in general relativity, the equivalence principle tells us that curved spacetime can be approximated locally by flat spacetime. But consider that no practitioner of calculus habitually solves problems by filling sheets of scratch paper with epsilons and deltas. Instead, she uses the Leibniz notation, in which dy and dx are interpreted as infinitesimally small numbers. You may be inclined, based on your previous training, to dismiss infinitesimals as neither rigorous nor necessary. In 1966, Abraham Robinson demonstrated that concerns about rigor had been unfounded; we'll come back to this point in section 3.3. Although it is true that any calculation written using infinitesimals can also be carried out using limits, the following example shows how much more well suited the infinitesimal language is to differential geometry.

Example 1: Areas on a sphere

The area of a region S in the Cartesian plane can be calculated as $\int_S$ dA, where dA = dx dy is the area of an infinitesimal rectangle of width dx and height dy. A curved surface such as a sphere does not admit a global Cartesian coordinate system in which the constant coordinate curves are both uniformly spaced and perpendicular to one another. For example, lines of longitude on the earth's surface grow closer together as one moves away from the equator. Letting θ be the angle with respect to the pole, and ϕ the azimuthal angle, the approximately rectangular patch bounded by $\theta, \theta+d\theta, \phi$, and $\phi+d\phi$ has width r sin θ dθ and height r dϕ, giving dA = r^2 sin θ dθ dϕ. If you look at the corresponding derivation in an elementary calculus textbook that strictly eschews infinitesimals, the technique is to start from scratch with Riemann sums. This is extremely laborious, and moreover must be carried out again for every new case. In differential geometry, the curvature of the space varies from one point to the next, and clearly we don't want to reinvent the wheel with Riemann sums an infinite number of times, once at each point in space.