4.E: Tensors (Exercises)

1. Describe the four-velocity of a photon.
2. The Large Hadron Collider is designed to accelerate protons to energies of 7 TeV. Find $1 - v$ for such a proton.
3. Prove that an electron in a vacuum cannot absorb a photon. (This is the reason that the ability of materials to absorb gammarays is strongly dependent on atomic number Z. The case of $Z = 0$ corresponds to the vacuum.)
4. (a) For an object moving in a circle at constant speed, the dot product of the classical three-vectors **v** and **a** is zero. Give an interpretation in terms of the work-kinetic energy theorem. (b) In the case of relativistic four-vectors, $v^i a_i = 0$ for *any* world-line. Give a similar interpretation. *Hint*: find the rate of change of the four-velocity's squared magnitude.
5. Starting from coordinates (t, x) having a Lorentzian metric g, transform the metric tensor into reflected coordinates $(t', x') = (t, -x)$, and verify that g' is the same as g.
6. Starting from coordinates (t, x) having a Lorentzian metric g, transform the metric tensor into Lorentz-boosted coordinates (t', x'), and verify that g' is the same as g.
7. Verify the transformation of the metric given in example 19.
8. A skeptic claims that the Hafele-Keating experiment can only be explained correctly by relativity in a frame in which the earth's axis is at rest. Prove mathematically that this is incorrect. Does it matter whether the frame is inertial?
9. Assume the metric g = diag(+1, +1, +1). Which of the following correctly expresses the noncommutative property of ordinary matrix multiplication?
10. Example 10 introduced the Dirac sea, whose existence is implied by the two roots of the relativistic relation $E = \pm\sqrt{p^2 + m^2}$. Prove that a Lorentz boost will never transform a positive-energy state into a negative-energy state.
11. In section 4.2, we found the relativistic Doppler shift in 1+1 dimensions. Extend this to 3+1 dimensions, and check your result against the one given by Einstein in Appendix A.
12. Estimate the energy contained in the electric field of an electron, if the electron's radius is r. Classically (i.e., assuming relativity but no quantum mechanics), this energy contributes to the electron's rest mass, so it must be less than the rest mass. Estimate the resulting lower limit on r, which is known as the classical electron radius.
13. For gamma-rays in the MeV range, the most frequent mode of interaction with matter is Compton scattering, in which the photon is scattered by an electron without being absorbed. Only part of the gamma's energy is deposited, and the amount is related to the angle of scattering. Use conservation of four-momentum to show that in the case of scattering at 180 degrees, the scattered photon has energy $E' = E/(1+2E/m)$, where m is the mass of the electron.
14. Derive the equation $T = \sqrt{\frac{3\pi}{G\rho}}$ given in section 4.4 for the period of a rotating, spherical object that results in zero apparent gravity at its surface.
15. Section 4.4 presented an estimate of the upper limit on the mass of a white dwarf. Check the self-consistency of the solution in the following respects: (1) Why is it valid to ignore the contribution of the nuclei to the degeneracy pressure? (2) Although the electrons are ultrarelativistic, spacetime is approximated as being flat. As suggested in example 14, a reasonable order-ofmagnitude check on this result is that we should have $\frac{M}{r} << \frac{c^2}{G}$.
16. The laws of physics in our universe imply that for bodies with a certain range of masses, a neutron star is the unique equilibrium state. Suppose we knew of the existence of neutron stars, but didn't know the mass of the neutron. Infer upper and lower bounds on the mass of the neutron.
17. Example 20 briefly introduced the electromagnetic potential four-vector F_{ij}, and this implicitly defines the transformation properties of the electric and magnetic fields under a Lorentz boost **v**. To lowest order in **v**, this transformation is given by

$$\begin{aligned} \mathbf{E}' &\approx \mathbf{E} + \mathbf{v} \times \mathbf{B} \qquad \textit{and} \\ \mathbf{B}' &\approx \mathbf{B} - \mathbf{v} \times \mathbf{E}. \end{aligned}$$

    I'm not a historian of science, but apparently ca. 1905 people like Hertz believed that these were the *exact* transformations of the field.[21] Show that this can't be the case, because performing two such transformations in a row does not in general result in a transformation of the same form.
18. We know of massive particles, whose velocity vectors always lie inside the future light cone, and massless particles, whose velocities lie on it. In principle, we could have a third class of particles, called tachyons, with spacelike velocity vectors. Tachyons would have $m^2 < 0$, i.e., their masses would have to be imaginary. Show that it is possible to pick momentum four-vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ for a pair of tachyons such that $\mathbf{p}_1 + \mathbf{p}_2 = 0$. This implies that the vacuum would be unstable with respect to spontaneous creation of tachyon-antitachyon pairs.

References

[21] Montigny and Rousseaux, arxiv.org/abs/physics/0512200.
