

8.8: Mach's Principle Revisited

The Brans-Dicke Theory

Mach himself never succeeded in stating his ideas in the form of a precisely testable physical theory, and we've seen that to the extent that Einstein's hopes and intuition had been formed by Mach's ideas, he often felt that his own theory of gravity came up short. The reader has so far encountered Mach's principle in the context of certain thought experiments that are obviously impossible to realize, involving a hypothetical universe that is empty except for certain apparatus (e.g., [section 3.6](#)). It would be easy, then, to get an impression of Mach's principle as one of those theories that is "not even wrong," i.e., so ill-defined that it cannot even be falsified by experiment, any more than Christianity can be.

But in 1961, Robert Dicke and his student Carl Brans came up with a theory of gravity that made testable predictions, and that was specifically designed to be more Machian than general relativity. Their paper³² is extremely readable, even for the non-specialist. In this theory, the seemingly foolproof operational definition of a Lorentz frame given in [section 1.5](#) fails. On the first page, Brans and Dicke propose one of those seemingly foolish thought experiments about a nearly empty universe:

The imperfect expression of [Mach's ideas] in general relativity can be seen by considering the case of a space empty except for a lone experimenter in his laboratory. [...] The observer would, according to general relativity, observe normal behavior of his apparatus in accordance with the usual laws of physics. However, also according to general relativity, the experimenter could set his laboratory rotating by leaning out a window and firing his 22-caliber rifle tangentially. Thereafter the delicate gyroscope in the laboratory would continue to point in a direction nearly fixed relative to the direction of motion of the rapidly receding bullet. The gyroscope would rotate relative to the walls of the laboratory. Thus, from the point of view of Mach, the tiny, almost massless, very distant bullet seems to be more important than the massive, nearby walls of the laboratory in determining inertial coordinate frames and the orientation of the gyroscope.

They then proceed to construct a mathematical and more Machian theory of gravity. From the Machian point of view, the correct local definition of an inertial frame must be determined relative to the bulk of the matter in the universe. We want to retain the Lorentzian local character of spacetime, so this influence can't be transmitted via instantaneous action at a distance. It must propagate via some physical field, at a speed less than or equal to c . It is implausible that this field would be the gravitational field as described by general relativity. Suppose we divide the cosmos up into a series of concentric spherical shells centered on our galaxy. In Newtonian mechanics, the gravitational field obeys Gauss's law, so the field of such a shell vanishes identically on the interior. In relativity, the corresponding statement is Birkhoff's theorem, which states that the Schwarzschild metric is the unique spherically symmetric solution to the vacuum field equations. Given this solution in the exterior universe, we can set a boundary condition at the outside surface of the shell, use the Einstein field equations to extend the solution through it, and find a unique solution on the interior, which is simply a flat space.

Since the Machian effect can't be carried by the gravitational field, Brans and Dicke took up an idea earlier proposed by Pascual Jordan³³ of hypothesizing an auxiliary field ϕ . The fact that such a field has never been detected directly suggests that it has no mass or charge. If it is massless, it must propagate at exactly c , and this also makes sense because if it were to propagate at speeds less than c , there would be no obvious physical parameter that would determine that speed. How many tensor indices should it have? Since Mach's principle tries to give an account of inertia, and inertial mass is a scalar,³⁴ ϕ should presumably be a scalar (quantized by a spin-zero particle). Theories of this type are called tensor-scalar theories, because they use a scalar field in addition to the metric tensor.

Notes

³³ Jordan was a member of the Nazi *Sturmabteilung* or "brown shirts" who nevertheless ran afoul of the Nazis for his close professional relationships with Jews.

³⁴ A limit of 5×10^{-23} has been placed on the anisotropy of the inertial mass of the proton: R.W.P. Drever, "A search for anisotropy of inertial mass using a free precession technique," *Philosophical Magazine*, 6:687 (1961) 683.

The wave equation for a massless scalar field, in the absence of sources, is simply $\nabla_i \nabla^i \phi = 0$. The solutions of this wave equation fall off as $\phi \sim \frac{1}{r}$. This is gentler than the $\frac{1}{r^2}$ variation of the gravitational field, so results like Newton's shell theorem and Birkhoff's theorem no longer apply. If a spherical shell of mass acts as a source of ϕ , then ϕ can be nonzero and varying inside the shell. The ϕ that you experience right now as you read this book should be a sum of wavelets originating from all the masses whose world-lines intersected the surface of your past light-cone. In a static universe, this sum would diverge linearly, so a self-consistency requirement for Brans-Dicke gravity is that it should produce cosmological solutions that avoid such a divergence, e.g., ones that begin with Big Bangs.

Masses are the sources of the field ϕ . How should they couple to it? Since ϕ is a scalar, we need to construct a scalar as its source, and the only reasonable scalar that can play this role is the trace of the stress-energy tensor, T^i_i . As discussed in [example 11](#), this vanishes for light, so the only sources of ϕ are material particles.³⁵ Even so, the Brans-Dicke theory retains a form of the equivalence principle. As discussed in [section 1.5](#) and [problem 7](#), the equivalence principle is a statement about the results of local experiments, and ϕ at any given location in the universe is dominated by contributions from matter lying at cosmological distances. Objects of different composition will have differing fractions of their mass that arise from internal electromagnetic fields. Two such objects will still follow identical geodesics, since their own effect on the local value of ϕ is negligible. This is unlike the behavior of electrically charged objects, which experience significant back-reaction effects in curved space ([problem 7](#)). However, the strongest form of the equivalence principle requires that all experiments in free-falling laboratories produce identical results, no matter where and when they are carried out. Brans-Dicke gravity violates this, because such experiments could detect differences between the value of ϕ at different locations — but of course this is part and parcel of the purpose of the theory.

Note

This leads to an exception to the statement above that all Brans-Dicke spacetimes are expected to look like Big Bang cosmologies. Any solution of the GR field equations containing nothing but vacuum and electromagnetic fields (known as an “electrovac” solution) is also a valid Brans-Dicke spacetime. In such a spacetime, a constant ϕ can be set arbitrarily. Such a spacetime is in some sense not generic for Brans-Dicke gravity.

We now need to see how to connect ϕ to the local notion of inertia so as to produce an effect of the kind that would tend to fulfill Mach's principle. In Mach's original formulation, this would entail some kind of local rescaling of all inertial masses, but Brans and Dicke point out that in a theory of gravity, this is equivalent to scaling the Newtonian gravitational constant G down by the same factor. The latter turns out to be a better approach. For one thing, it has a natural interpretation in terms of units. Since ϕ 's amplitude falls off as $\frac{1}{r}$, we can write $\phi \sim \frac{\sum m_i}{r}$, where the sum is over the past light cone. If we then make the identification of ϕ with $\frac{1}{G}$ (or $\frac{c^2}{G}$ in a system where $c \neq 1$), the units work out properly, and the coupling constant between matter and ϕ can be unitless. If this coupling constant, notated $\frac{1}{\omega}$, were not unitless, then the theory's predictive value would be weakened, because there would be no way to know what value to pick for it. For a unitless constant, however, there is a reasonable way to guess what it should be: “in any sensible theory,” Brans and Dicke write, “ ω must be of the general order of magnitude of unity.” This is, of course, assuming that the Brans-Dicke theory was correct. In general, there are other reasonable values to pick for a unitless number, including zero and infinity. The limit of $\omega \rightarrow \infty$ recovers the special case of general relativity. Thus Mach's principle, which once seemed too vague to be empirically falsifiable, comes down to measuring a specific number, ω , which quantifies how non-Machian our universe is.³⁶

Note

Another good technical reason for thinking of ϕ as relating to the gravitational constant is that general relativity has a standard prescription for describing fields on a background of curved spacetime. The vacuum field equations of general relativity can be derived from the principle of least action, and although the details are beyond the scope of this book (see, e.g., Wald, *General Relativity*, appendix E), the general idea is that we define a Lagrangian density \mathcal{L}_G that depends on the Ricci scalar curvature, and then extremize its integral over all possible histories of the evolution of the gravitational field. If we want to describe some other field, such as matter, light, or ϕ , we simply take the special-relativistic Lagrangian \mathcal{L}_M for that field, change all the derivatives to covariant derivatives, and form the sum $(\frac{1}{G})\mathcal{L}_G + \mathcal{L}_M$. In the Brans-Dicke theory, we have three pieces, $(\frac{1}{G})\mathcal{L}_G + \mathcal{L}_M + \mathcal{L}_\phi$, where \mathcal{L}_M is for matter and \mathcal{L}_ϕ for ϕ . If we were to interpret ϕ as a rescaling of inertia, then we would have to have ϕ appearing as a fudge factor modifying all the inner workings of \mathcal{L}_M . If, on the other hand, we think of ϕ as changing the value of the gravitational constant G , then the necessary modification is extremely simple. Brans and Dicke

introduce one further modification to \mathcal{L}_ϕ so that the coupling constant ω between matter and ϕ can be unitless. This modification has no effect on the wave equation of ϕ in flat spacetime.

Predictions of the Brans-Dicke Theory

Returning to the example of the spherical shell of mass, we can see based on considerations of units that the value of ϕ inside should be $\sim \frac{m}{r}$, where m is the total mass of the shell and r is its radius. There may be a unitless factor out in front, which will depend on ω , but for $\omega \sim 1$ we expect this constant to be of order 1. Solving the nasty set of field equations that result from their Lagrangian, Brans and Dicke indeed found $\phi \approx \left[\frac{2}{3+2\omega} \right] \left(\frac{m}{r} \right)$, where the constant in square brackets is of order unity if ω is of order unity. In the limit of $\omega \rightarrow \infty$, $\phi = 0$, and the shell has no physical effect on its interior, as predicted by general relativity.

Brans and Dicke were also able to calculate cosmological models, and in a typical model with a nearly spatially flat universe, they found ϕ would vary according to

$$\phi = 8\pi \frac{4+3\omega}{6+4\omega} \rho_o t_o^2 \left(\frac{t}{t_o} \right)^{\frac{2}{4+\omega}}, \quad (8.8.1)$$

where ρ_o is the density of matter in the universe at time $t = t_o$. When the density of matter is small, G is large, which has the same observational consequences as the disappearance of inertia; this is exactly what one expects according to Mach's principle. For $\omega \rightarrow \infty$, the gravitational "constant" $G = \frac{1}{\phi}$ really is constant.

Returning to the thought experiment involving the 22-caliber rifle fired out the window, we find that in this imaginary universe, with a very small density of matter, G should be very large. This causes a frame-dragging effect from the laboratory on the gyroscope, one much stronger than we would see in our universe. Brans and Dicke calculated this effect for a laboratory consisting of a spherical shell, and although technical difficulties prevented the reliable extrapolation of their result to $\rho_o \rightarrow 0$, the trend was that as ρ_o became small, the frame-dragging effect would get stronger and stronger, presumably eventually forcing the gyroscope to precess in lock-step with the laboratory. There would thus be no way to determine, once the bullet was far away, that the laboratory was rotating at all — in perfect agreement with Mach's principle.

Hints of Empirical Support

Only six years after the publication of the Brans-Dicke theory, Dicke himself, along with H.M. Goldenberg³⁷ carried out a measurement that seemed to support the theory empirically. Fifty years before, one of the first empirical tests of general relativity, which it had seemed to pass with flying colors, was the anomalous perihelion precession of Mercury. The word "anomalous," which is often left out in descriptions of this test, is required because there are many nonrelativistic reasons why Mercury's orbit precesses, including interactions with the other planets and the sun's oblate shape. It is only when these other effects are subtracted out that one sees the general-relativistic effect calculated in [section 6.2](#). The sun's oblateness is difficult to measure optically, so the original analysis of the data had proceeded by determining the sun's rotational period by observing sunspots, and then assuming that the sun's bulge was the one found for a rotating fluid in static equilibrium. The result was an assumed oblateness of about 1×10^{-5} . But we know that the sun's dynamics are more complicated than this, since it has convection currents and magnetic fields. Dicke, who was already a renowned experimentalist, set out to determine the oblateness by direct optical measurements, and the result was $(5.0 \pm 0.7) \times 10^{-5}$, which, although still very small, was enough to put the observed perihelion precession out of agreement with general relativity by about 8%. The perihelion precession predicted by Brans-Dicke gravity differs from the general relativistic result by a factor of $\frac{4+3\omega}{6+3\omega}$. The data therefore appeared to require $\omega \approx 6 \pm 1$, which would be inconsistent with general relativity.

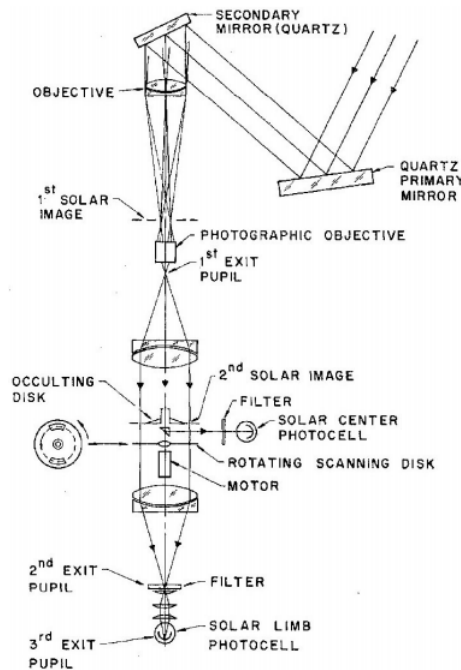


Figure 8.8.1 - The apparatus used by Dicke and Goldenberg to measure the oblateness of the sun was essentially a telescope with a disk inserted in order to black out most of the light from the sun.

Mach's Principle is False

The trouble with the solar oblateness measurements was that they were subject to a large number of possible systematic errors, and for this reason it was desirable to find a more reliable test of Brans-Dicke gravity. Not until about 1990 did a consensus arise, based on measurements of oscillations of the solar surface, that the pre-Dicke value was correct. In the interim, the confusion had the salutary effect of stimulating a renaissance of theoretical and experimental work in general relativity. Often if one doesn't have an alternative theory, one has no reasonable basis on which to design and interpret experiments to test the original theory.

Currently, the best bound on ω is based on measurements³⁸ of the propagation of radio signals between earth and the Cassini-Huygens space probe in 2003, which require $\omega > 4 \times 10^4$. This is so much greater than unity that it is reasonable to take Brans and Dicke at their word that "in any sensible theory, ω must be of the general order of magnitude of unity." Brans-Dicke fails this test, and is no longer a "sensible" candidate for a theory of gravity. We can now see that Mach's principle, far from being a fuzzy piece of philosophical navel-gazing, is a testable hypothesis. It has been tested and found to be false, in the following sense. Brans-Dicke gravity is about as natural a formal implementation of Mach's principle as could be hoped for, and it gives us a number ω that parametrizes how Machian the universe is. The empirical value of ω is so large that it shows our universe to be essentially as non-Machian as general relativity.

References

³² C. Brans and R. H. Dicke, "Mach's Principle and a Relativistic Theory of Gravitation," *Physical Review* 124 (1961) 925

³⁷ Dicke and Goldenberg, "Solar Oblateness and General Relativity," *Physical Review Letters* 18 (1967) 313

³⁸ Bertotti, Iess, and Tortora, "A test of general relativity using radio links with the Cassini spacecraft," *Nature* 425 (2003) 374

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