1.4: Ordered Geometry

Let's try to put what we've learned into a general geometrical context.

Euclid's familiar geometry of two-dimensional space has the following axioms,[6] which are expressed in terms of operations that can be carried out with a compass and unmarked straightedge:

- **E1** Two points determine a line.
- **E2** Line segments can be extended.
- **E3** A unique circle can be constructed given any point as its center and any line segment as its radius.
- **E4** All right angles are equal to one another.
- **E5** *Parallel postulate:* Given a line and a point not on the line, no more than one line can be drawn through the point and parallel to the given line.[7]

> **Note**
>
> This is a form known as Playfair's axiom, rather than the version of the postulate originally given by Euclid.

The modern style in mathematics is to consider this type of axiomatic system as a self-contained sandbox, with the axioms, and any theorems proved from them, being true or false only in relation to one another. Euclid and his contemporaries, however, believed them to be empirical facts about physical reality. For example, they considered the fifth postulate to be less obvious than the first four, because in order to verify physically that two lines were parallel, one would theoretically have to extend them to an infinite distance and make sure that they never crossed. In the first 28 theorems of the *Elements*, Euclid restricts himself entirely to propositions that can be proved based on the more secure first four postulates. The more general geometry defined by omitting the parallel postulate is known as *absolute geometry*.

What kind of geometry is likely to be applicable to general relativity? We can see immediately that Euclidean geometry, or even absolute geometry, would be far too specialized. We have in mind the description of events that are points in both space and time. Confining ourselves for ease of visualization to one dimension worth of space, we can certainly construct a plane described by coordinates (t, x), but imposing Euclid's postulates on this plane results in physical nonsense. Space and time are physically distinguishable from one another. But postulates 3 and 4 describe a geometry in which distances measured along non-parallel axes are comparable, and figures may be freely rotated without affecting the truth or falsehood of statements about them; this is only appropriate for a physical description of different spacelike directions, as in an (x,y) plane whose two axes are indistinguishable.

We need to throw most of the specialized apparatus of Euclidean geometry overboard. Once we've stripped our geometry to a bare minimum, then we can go back and build up a different set of equipment that will be better suited to relativity.

The stripped-down geometry we want is called *ordered geometry*, and was developed by Moritz Pasch around 1882. As suggested by the parable of Alice and Betty, ordered geometry does not have any global, all-encompassing system of measurement. When Betty goes on her trip, she traces out a particular path through the space of events, and Alice, staying at home, traces another. Although events play out in cause-and-effect order along each of these paths, we do not expect to be able to measure times along paths A and B and have them come out the same. This is how ordered geometry works: points can be put in a definite order along any particular line, but not along different lines. Of the four primitive concepts used in Euclid's E1-E5 — point, line, circle, and angle — only the non-metrical notions of point (i.e., event) and line are relevant in ordered geometry. In a geometry without measurement, there is no concept of measuring distance (hence no compasses or circles), or of measuring angles. The notation [ABC] indicates that event B lies on a line segment joining A and C, and is strictly between them.

The axioms of ordered geometry are as follows:[8]

- **O1** Two events determine a line.
- **O2** Line segments can be extended: given A and B, there is at least one event such that [ABC] is true.
- **O3** Lines don't wrap around: if [ABC] is true, then [BCA] is false.
- **O4** Betweenness: For any three distinct events A, B, and C lying on the same line, we can determine whether or not B is between A and C (and by statement 3, this ordering is unique except for a possible over-all reversal to form [CBA]).

Note

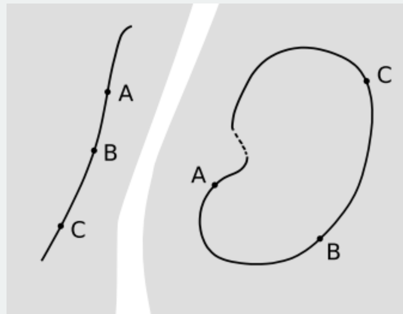

Figure 1.4.1 - Axioms O2 (left) and O3 (right).

The axioms are summarized for convenient reference. This is meant to be an informal, readable summary of the system, pitched to the same level of looseness as Euclid’s E1-E5. Modern mathematicians have found that systems like these actually need quite a bit more technical machinery to be perfectly rigorous, so if you look up an axiomatization of ordered geometry, or a modern axiomatization of Euclidean geometry, you’ll typically find a much more lengthy list of axioms than the ones presented here. The axioms I’m omitting take care of details like making sure that there are more than two points in the universe, and that curves can’t cut through one another without intersecting. The classic, beautifully written book on these topics is H.S.M. Coxeter’s *Introduction to Geometry*, which is “introductory” in the sense that it’s the kind of book a college math major might use in a first upper-division course in geometry.

O1-O2 express the same ideas as Euclid's E1-E2. Not all lines in the system will correspond physically to chains of causality; we could have a line segment that describes a snapshot of a steel chain, and O3-O4 then say that the order of the links is well defined. But O3 and O4 also have clear physical significance for lines describing causality. O3 forbids time travel paradoxes, like going back in time and killing our own grandmother as a child; The figure above illustrates why a violation of O3 is referred to as a closed timelike curve. O4 says that events are guaranteed to have a well-defined cause-and-effect order only if they lie on the same line. This is completely different from the attitude expressed in Newton's famous statement: “Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external ...”

If you're dismayed by the austerity of a system of geometry without any notion of measurement, you may be more appalled to learn that even a system as weak as ordered geometry makes some statements that are too strong to be completely correct as a foundation for relativity. For example, if an observer falls into a black hole, at some point he will reach a central point of infinite density, called a singularity. At this point, his chain of cause and effect terminates, violating \O2). It is also an open question whether O3's prohibition on time-loops actually holds in general relativity; this is Stephen Hawking's playfully named chronology protection conjecture. We’ll also see that in general relativity O1 is almost always true, but there are exceptions.

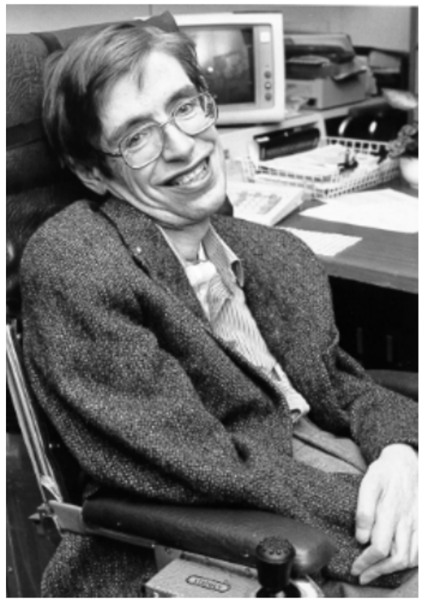
Figure 1.4.2 - Stephen Hawking (1942-).