5.E: Curvature (Exercises)

1. Example 6 discussed some examples in electrostatics where the charge density on the surface of a conductor depends on the Gaussian curvature, when the curvature is positive. In the case of a knife-edge formed by two half-planes at an exterior angle $\beta > \pi$, there is a standard result[29] that the charge density at the edge blows up to infinity as $R^{\frac{\pi}{\beta-1}}$. Does this match up with the hypothesis that Gaussian curvature determines the charge density?
2. Show that for polar coordinates in a Euclidean plane, $\Gamma^{r}_{\phi\phi} = -r$ and $\Gamma^{\phi}_{r\phi} = \frac{1}{r}$.
3. In 1+1 dimensions, let the metric be $ds^2 = \frac{1}{t}\,dt^2 - t\,d\theta^2$, where θ is an angle running around the circle. Calculate all the nonvanishing Christoffel symbols by hand. These will be used in example 4 where we investigate some further properties of this interesting spacetime.
4. Partial derivatives commute with partial derivatives. Covariant derivatives don't commute with covariant derivatives. Do covariant derivatives commute with partial derivatives?
5. Show that if the differential equation is satisfied for one affine parameter λ, then it is also satisfied for any other affine parameter $\lambda' = a\lambda + b$, where a and b are constants.
6. Equation [2] gives a flat-spacetime metric in rotating polar coordinates. (a) Verify by explicit computation that this metric represents a flat spacetime. (b) Re-express the metric in rotating Cartesian coordinates, and check your answer by verifying that the Riemann tensor vanishes.
7. The purpose of this problem is to explore the difficulties inherent in finding anything in general relativity that represents a uniform gravitational field g. In example 11, we found, based on elementary arguments about the equivalence principle and photons in elevators, that gravitational time dilation must be given by e^{Φ}, where $\Phi = gz$ is the gravitational potential. This results in a metric

   $$ds^2 = e^{2gz}dt^2 - dz^2. \tag{[1]}$$

   On the other hand, example 19 derived the metric

   $$ds^2 = (1+gz)^2 dt^2 - dz^2. \tag{[2]}$$

   by transforming from a Lorentz frame to a frame whose origin moves with constant proper acceleration g. (These are known as Rindler coordinates.) Prove the following facts. None of the calculations are so complex as to require symbolic math software, so you might want to perform them by hand first, and then check yourself on a computer.

   a. The metrics [1] and [2] are approximately consistent with one another for z near 0.
   b. When a test particle is released from rest in either of these metrics, its initial proper acceleration is g.
   c. The two metrics are not exactly equivalent to one another under any change of coordinates.
   d. Both spacetimes are uniform in the sense that the curvature is constant. (In both cases, this can be proved without an explicit computation of the Riemann tensor.)

Note

The incompatibility between [1] and [2] can be interpreted as showing that general relativity does not admit any spacetime that has all the global properties we would like for a uniform gravitational field. This is related to Bell's spaceship paradox (example 15). Some further properties of the metric [1] are analyzed in section 7.5.

8. In a topological space T, the complement of a subset U is defined as the set of all points in T that are not members of U. A set whose complement is open is referred to as closed. On the real line, give (a) one example of a closed set and (b) one example of a set that is neither open nor closed. (c) Give an example of an inequality that defines an open set on the rational number line, but a closed set on the real line.
9. Prove that a double cone (e.g., the surface r = z in cylindrical coordinates) is not a manifold.
10. Prove that a torus is a manifold.
11. Prove that a sphere is not homeomorphic to a torus.
12. Curvature on a Riemannian space in 2 dimensions is a topic that goes back to Gauss and has a simple interpretation: the only intrinsic measure of curvature is a single number, the Gaussian curvature. What about 1+1 dimensions? The simplest metrics I can think of are of the form $ds^2 = dt^2 - f(t)dx^2$. (Something like $ds^2 = f(t)dt^2 - dx^2$ is obviously equivalent to Minkowski space under a change of coordinates, while $ds^2 = f(x)dt^2 - dx^2$ is the same as the original example except that we've swapped x and t.)

Playing around with simple examples, one stumbles across the seemingly mysterious fact that the metric $ds^2 = dt^2 - t^2 dx^2$ is flat, while $ds^2 = dt^2 - t dx^2$ is not. This seems to require some simple explanation. Consider the metric $ds^2 = dt^2 - t^p dx^2$.

a. Calculate the Christoffel symbols by hand.
b. Use a computer algebra system such as Maxima to show that the Ricci tensor vanishes only when $p = 2$.

Note

The explanation is that in the case $p = 2$, the x coordinate is expanding in proportion to the t coordinate. This can be interpreted as a situation in which our length scale is defined by a lattice of test particles that expands inertially. Since their motion is inertial, no gravitational fields are required in order to explain the observed change in the length scale; cf. the Milne universe.

References

[29] Jackson, *Classical Electrodynamics*