

7.6: The Uniform Gravitational Field Revisited

This section gives a somewhat exotic example. It is not necessary to read it in order to understand the later material.

In problem 7, we made a wish list of desired properties for a uniform gravitational field, and found that they could not all be satisfied at once. That is, there is no global solution to the Einstein field equations that uniquely and satisfactorily embodies all of our Newtonian ideas about a uniform field. We now revisit this question in the light of our new knowledge.

The 1+1-dimensional metric

$$ds^2 = e^{2gz} dt^2 - dz^2 \quad (7.6.1)$$

is the one that uniquely satisfies our expectations based on the equivalence principle (example 11), and it is a vacuum solution. We might logically try to generalize this to 3+1 dimensions as follows:

$$ds^2 = e^{2gz} dt^2 - dx^2 - dy^2 - dz^2. \quad (7.6.2)$$

But a funny thing happens now — simply by slapping on the two new Cartesian axes x and y , it turns out that we have made our vacuum solution into a non-vacuum solution, and not only that, but the resulting stress-energy tensor is unphysical (problem 8).

One way to proceed would be to relax our insistence on making the spacetime one that exactly embodies the equivalence principle's requirements for a uniform field.¹³ This can be done by taking $g_{tt} = e^{2\Phi}$, where Φ is not necessarily equal to $2gz$. By requiring that the metric be a 3+1 vacuum solution, we arrive at a differential equation whose solution is $\Phi = \ln(z + k_1) + k_2$, which recovers the flat-space metric that we found in example 19 by applying a change of coordinates to the Lorentz metric.

Note

Thanks to physicsforums.com user Mentz114 for suggesting this approach and demonstrating the following calculation

What if we want to carry out the generalization from 1+1 to 3+1 without violating the equivalence principle? For physical motivation in how to get past this obstacle, consider the following argument made by Born in 1920.¹⁴ Take a frame of reference tied to a rotating disk, as in the example from which Einstein originally took much of the motivation for creating a geometrical theory of gravity (section 3.5). Clocks near the edge of the disk run slowly, and by the equivalence principle, an observer on the disk interprets this as a gravitational time dilation. But this is not the only relativistic effect seen by such an observer. Her rulers are also Lorentz contracted as seen by a non-rotating observer, and she interprets this as evidence of a non-Euclidean spatial geometry. There are some physical differences between the rotating disk and our default conception of a uniform field, specifically in the question of whether the metric should be static (i.e., lacking in cross-terms between the space and time variables). But even so, these considerations make it natural to hypothesize that the correct 3+1-dimensional metric should have transverse spatial coefficients that decrease with height.

With this motivation, let's consider a metric of the form

$$ds^2 = e^{2z} dt^2 - e^{-2jz} dx^2 - e^{-2kz} dy^2 - dz^2, \quad (7.6.3)$$

where j and k are constants, and I've taken $g = 1$ for convenience.¹⁵ The following Maxima code calculates the scalar curvature and the Einstein tensor:

```

1  load(ctensor);
2  ct_coords:[t,x,y,z];
3  lg:matrix([exp(2*z),0,0,0],
4            [0,-exp(-2*j*z),0,0],
5            [0,0,-exp(-2*k*z),0],
6            [0,0,0,-1]
7  );
8  cmetric();
9  scurvature();
10 leinstein(true);

```

Note

A metric of this general form is referred to as a Kasner metric. One usually sees it written with a logarithmic change of variables, so that z appears in the base rather than in the exponent.

The output from line 9 shows that the scalar curvature is constant, which is a necessary condition for any spacetime that we want to think of as representing a uniform field. Inspecting the Einstein tensor output by line 10, we find that in order to get G_{xx} and G_{yy} to vanish, we need j and k to be $\frac{1 \pm \sqrt{3}i}{2}$. By trial and error, we find that assigning the complex-conjugate values to j and k makes G_{tt} and G_{zz} vanish as well, so that we have a vacuum solution. This solution is, unfortunately, complex, so it is not of any obvious value as a physically meaningful result. Since the field equations are nonlinear, we can't use the usual trick of forming real-valued superpositions of the complex solutions. We could try simply taking the real part of the metric. This gives $g_{xx} = e^{-z} \cos \sqrt{3}z$ and $g_{yy} = e^{-z} \sin \sqrt{3}z$, and is unsatisfactory because the metric becomes degenerate (has a zero determinant) at $z = \frac{n\pi}{2\sqrt{3}}$, where n is an integer.

It turns out, however, that there is a very similar solution, found by Petrov in 1962,¹⁶ that is real-valued. The Petrov metric, which describes a spacetime with cylindrical symmetry, is:

$$ds^2 = -dr^2 - e^{-2r} dz^2 + e^r [2 \sin \sqrt{3}r d\phi dt - \cos \sqrt{3}r (d\phi^2 - dt^2)] \quad (7.6.4)$$

Note that it has many features in common with the complex oscillatory solution we found above. There are transverse length contractions that decay and oscillate in exactly the same way. The presence of the $d\phi dt$ term tells us that this is a non-static, rotating solution — exactly like the one that Einstein and Born had in mind in their prototypical example! We typically obtain this type of effect due to frame dragging by some rotating massive body (see section 4.5), and the Petrov solution can indeed be interpreted as the spacetime that exists in the vacuum on the exterior of an infinite, rigidly rotating cylinder of “dust” (see example 13).

The complicated Petrov metric might seem like the furthest possible thing from a uniform gravitational field, but in fact it is about the closest thing general relativity provides to such a field. We first note that the metric has Killing vectors ∂_z , ∂_ϕ , and ∂_r , so it has at least three out of the four translation symmetries we expect from a uniform field. By analogy with electromagnetism, we would expect this symmetry to be absent in the radial direction, since by Gauss's law the electric field of a line of charge falls off like $\frac{1}{r}$. But surprisingly, the Petrov metric is also uniform radially. It is possible to give the fourth killing vector explicitly (it is $\partial_r + z\partial_z + (\frac{1}{2})(\sqrt{3}t - \phi)\partial_\phi - (\frac{1}{2})(\sqrt{3}\phi + t)\partial_t$), but it is perhaps more transparent to check that it represents a field of constant strength (problem 4).

For insight into this surprising result, recall that in our attempt at constructing the Cartesian version of this metric, we ran into the problem that the metric became degenerate at $z = \frac{n\pi}{2\sqrt{3}}$. The presence of the $d\phi dt$ term prevents this from happening in Petrov's cylindrical version; two of the metric's diagonal components can vanish at certain values of r , but the presence of the off-diagonal component prevents the determinant from going to zero. (The determinant is in fact equal to -1 everywhere.) What is happening physically is that although the labeling of the ϕ and t coordinates suggests a time and an azimuthal angle, these two coordinates are in fact treated completely symmetrically. At values of r where the cosine factor equals 1, the metric is diagonal, and has signature $(t, \phi, r, z) = (+, -, -, -)$, but when the cosine equals -1 , this becomes $(-, +, -, -)$, so that ϕ is now the timelike coordinate. This

perfect symmetry between ϕ and t is an extreme example of frame-dragging, and is produced because of the specially chosen rate of rotation of the dust cylinder, such that the velocity of the dust at the outer surface is exactly c (or approaches it).

Classically, we would expect that a test particle released close enough to the cylinder would be pulled in by the gravitational attraction and destroyed on impact, while a particle released farther away would fly off due to the centrifugal force, escaping and eventually approaching a constant velocity. Neither of these would be anything like the experience of a test particle released in a uniform field. But consider a particle released at rest in the rotating frame at a radius r_1 for which $\cos \sqrt{3}r_1 = 1$, so that t is the timelike coordinate. The particle accelerates (let's say outward), but at some point it arrives at an r_2 where the cosine equals zero, and the $\phi - t$ part of the metric is purely of the form $d\phi dt$. At this location, we can define local coordinates $u = \phi - t$ and $v = \phi + t$, so that the metric depends only on $du^2 - dv^2$. One of the coordinates, say u , is now the timelike one. Since our particle is material, its world-line must be timelike, so it is swept along in the $-\phi$ direction. Gibbons and Gielen show that the particle will now come back inward, and continue forever by oscillating back and forth between two radii at which the cosine vanishes.

Closed Timelike Curves

This oscillation still doesn't sound like the motion of a particle in a uniform field, but another strange thing happens, as we can see by taking another look at the values of r at which the cosine vanishes. At such a value of r , construct a curve of the form ($t = \text{constant}$, $r = \text{constant}$, ϕ , $z = \text{constant}$). This is a closed curve, and its proper length is zero, i.e., it is lightlike. This violates causality.

A photon could travel around this path and arrive at its starting point at the same time when it was emitted. Something similarly weird happens to the test particle described above: whereas it seems to fall sometimes up and sometimes down, in fact it is always falling down — but sometimes it achieves this by falling up while moving backward in time!

Although the Petrov metric violates causality, Gibbons and Gielen have shown that it satisfies the chronology protection conjecture: “In the context of causality violation we have shown that one cannot create CTCs [closed timelike curves] by spinning up a cylinder beyond its critical angular velocity by shooting in particles on timelike or null curves.”

We have an exact vacuum solution to the Einstein field equations that violates causality. This raises troublesome questions about the logical self-consistency of general relativity. A very readable and entertaining overview of these issues is given in the final chapter of Kip Thorne's *Black Holes and Time Warps: Einstein's Outrageous Legacy*. In a toy model constructed by Thorne's students, involving a billiard ball and a wormhole, it turned out that there always seemed to be self-consistent solutions to the ball's equations of motion, but they were not unique, and they often involved disquieting possibilities in which the ball went back in time and collided with its earlier self. Among other things, this seems to lead to a violation of conservation of mass-energy, since no mass was put into the system to create extra copies of the ball. This would then be an example of the fact that, as discussed in section 4.5, general relativity does not admit global conservation laws. However, there is also an argument that the mouths of the wormhole change in mass in such a way as to preserve conservation of energy.¹⁷

References

¹⁴ Max Born, Einstein's *Theory of Relativity*, 1920. In the 1962 Dover edition, the relevant passage is on p. 320

¹⁶ Petrov, in *Recent Developments in General Relativity*, 1962, Pergamon, p. 383. For a presentation that is freely accessible online, see Gibbons and Gielen, “The Petrov and Kaigorodov-Ozsváth Solutions: Spacetime as a Group Manifold,” arxiv.org/abs/0802.4082.

¹⁷ <http://golem.ph.utexas.edu/string/archives/000550.html>

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