

## 7.4: Static and Stationary Spacetimes (Part 1)

### Stationary Spacetimes

When we set out to describe a generic spacetime, the Alice in Wonderland quality of the experience is partly because coordinate invariance allows our time and distance scales to be arbitrarily rescaled, but also partly because the landscape can change from one moment to the next. The situation is drastically simplified when the spacetime has a timelike Killing vector. Such a spacetime is said to be stationary. Two examples are flat spacetime and the spacetime surrounding the rotating earth (in which there is a frame-dragging effect). Non-examples include the solar system, cosmological models, gravitational waves, and a cloud of matter undergoing gravitational collapse.

Can Alice determine, by traveling around her spacetime and carrying out observations, whether it is stationary? If it's not, then she might be able to prove it. For example, suppose she visits a certain region and finds that the *Kretschmann invariant*  $R^{abcd}R_{abcd}$  varies with time in her frame of reference. Maybe this is because an asteroid is coming her way, in which case she could readjust her velocity vector to match that of the asteroid. Even if she can't see the asteroid, she can still try to find a velocity that makes her local geometry stop changing in this particular way. If the spacetime is truly stationary, then she can always “tune in” to the right velocity vector in this way by searching systematically. If this procedure ever fails, then she has proved that her spacetime is not stationary.

#### Exercise 7.4.1

Why is the timelike nature of the Killing vector important in this story?

Proving that a spacetime is stationary is harder. This is partly just because spacetime is infinite, so it will take an infinite amount of time to check everywhere. We aren't inclined to worry too much about this limitation on our geometrical knowledge, which is of a type that has been familiar since thousands of years ago, when it upset the ancient Greeks that the parallel postulate could only be checked by following lines out to an infinite distance. But there is a new type of limitation as well. The Schwarzschild spacetime is not stationary according to our definition. In the coordinates used in section 6.2,  $\partial_t$  is a Killing vector, but is only timelike for  $r > 2m$ ; for  $r < 2m$  it is spacelike. Although the solution describes a black hole that is going to sit around forever without changing, no observer can ever verify that fact, because once she strays inside the horizon she must follow a timelike world-line, which will end her program of observation within some finite time.

### Isolated Systems

#### Asymptotic flatness

This unfortunate feature of our definition of stationarity — its empirical unverifiability — is something that in general we just have to live with. But there is an alternative in the special case of an isolated system, such as our galaxy or a black hole. It may be a good approximation to ignore distant matter, modeling such a system with a spacetime that is asymptotically flat. The notion of asymptotic flatness was introduced informally in section 4.5. Formulating the definition of this term rigorously and in a coordinate-invariant way involves a large amount of technical machinery, since we are not guaranteed to be presented in advance with a special, physically significant set of coordinates that would lead directly to a quantitative way of defining words like “nearby.” The essential idea is that a spacetime is asymptotically flat if it is possible to perform a conformal transformation in such a way that the result has idealized regions at infinity  $i^0$ ,  $\mathcal{I}^+$  and  $\mathcal{I}^-$  (but not  $i^+$  and  $i^-$ ) that look like those of Minkowski space. The reader who wants to see the full machinery presented can find presentations in various places, such as Hawking and Ellis, ch. 11 of Wald, or the online review article “Conformal Infinity” at [livingreviews.org](http://livingreviews.org).

#### Asymptotically stationary spacetimes

In the case of an asymptotically flat spacetime, we say that it is also asymptotically stationary if it has a Killing vector that becomes timelike far away. Some authors (e.g., Ludvigsen) define “stationary” to mean what I'm calling “asymptotically stationary,” others (Hawking and Ellis) define it the same way I do, and still others (Carroll) are not self-consistent. The Schwarzschild spacetime is asymptotically stationary, but not stationary.

## A Stationary Field with No Other Symmetries

Consider the most general stationary case, in which the only Killing vector is the timelike one. The only ambiguity in the choice of this vector is a rescaling; its direction is fixed. At any given point in space, we therefore have a notion of being at rest, which is to have a velocity vector parallel to the Killing vector. An observer at rest detects no time-dependence in quantities such as tidal forces.

Points in space thus have a permanent identity. The gravitational field, which the equivalence principle tells us is normally an elusive, frame-dependent concept, now becomes more concrete: it is the proper acceleration required in order to stay in one place. We can therefore use phrases like “a stationary field,” without the usual caveats about the coordinate-dependent meaning of “field.”

Space can be sprinkled with identical clocks, all at rest. Furthermore, we can compare the rates of these clocks, and even compensate for mismatched rates, by the following procedure. Since the spacetime is stationary, experiments are reproducible. If we send a photon or a material particle from a point A in space to a point B, then identical particles emitted at later times will follow identical trajectories. The time lag between the arrival of two such particles tells an observer at B the amount of time at B that corresponds to a certain interval at A. If we wish, we can adjust all the clocks so that their rates are matched. An example of such rate-matching is the GPS satellite system, in which the satellites’ clocks are tuned to 10.22999999543 MHz, matching the ground-based clocks at 10.23 MHz. (Strictly speaking, this example is out of place in this subsection, since the earth’s field has an additional azimuthal symmetry.)

It is tempting to conclude that this type of spacetime comes equipped with a naturally preferred time coordinate that is unique up to a global affine transformation  $t \rightarrow at + b$ . But to construct such a time coordinate, we would have to match not just the rates of the clocks, but also their phases. The best method relativity allows for doing this is Einstein synchronization (Appendix 1), which involves trading a photon back and forth between clocks A and B and adjusting the clocks so that they agree that each clock gets the photon at the mid-point in time between its arrivals at the other clock. The trouble is that for a general stationary spacetime, this procedure is not transitive: synchronization of A with B, and of B with C, does not guarantee agreement between A with C. This is because the time it takes a photon to travel clockwise around triangle ABCA may be different from the time it takes for the counterclockwise itinerary ACBA. In other words, we may have a Sagnac effect, which is generally interpreted as a sign of rotation. Such an effect will occur, for example, in the field of the rotating earth, and it cannot be eliminated by choosing a frame that rotates along with the earth, because the surrounding space experiences a frame-dragging effect, which falls off gradually with distance.

Although a stationary spacetime does not have a uniquely preferred time, it does prefer some time coordinates over others. In a stationary spacetime, it is always possible to find a “nice”  $t$  such that the metric can be expressed without any  $t$ -dependence in its components.

## A Stationary Field with Additional Symmetries

Most of the results given above for a stationary field with no other symmetries also hold in the special case where additional symmetries are present. The main difference is that we can make linear combinations of a particular timelike Killing vector with the other Killing vectors, so the timelike Killing vector is not unique. This means that there is no preferred notion of being at rest. For example, in a flat spacetime we cannot define an observer to be at rest if she observes no change in the local observables over time, because that is true for any inertial observer. Since there is no preferred rest frame, we can’t define the gravitational field in terms of that frame, and there is no longer any preferred definition of the gravitational field.

## Static Spacetimes

In addition to synchronizing all clocks to the same frequency, we might also like to be able to match all their phases using Einstein synchronization, which requires transitivity. Transitivity is framedependent. For example, flat spacetime allows transitivity if we use the usual coordinates. However, if we change into a rotating frame of reference, transitivity fails (see section 3.5). If coordinates exist in which a particular spacetime has transitivity, then that spacetime is said to be static. In these coordinates, the metric is diagonalized, and since there are no space-time cross-terms like  $dx dt$  in the metric, such a spacetime is invariant under time reversal. Roughly speaking, a static spacetime is one in which there is no rotation.

## Birkhoff’s Theorem

Birkhoff’s theorem, proved below, states that in the case of spherical symmetry, the vacuum field equations have a solution, the Schwarzschild spacetime, which is unique up to a choice of coordinates and the value of  $m$ . Let’s enumerate the assumptions that

went into our derivation of the Schwarzschild metric in section 6.2. These were: (1) the vacuum field equations, (2) spherical symmetry, (3) asymptotic staticity, (4) a certain choice of coordinates, and (5)  $\Lambda = 0$ . Birkhoff's theorem says that the assumption of staticity was not necessary. That is, even if the mass distribution contracts and expands over time, the exterior solution is still the Schwarzschild solution. Birkhoff's theorem holds because gravitational waves are transverse, not longitudinal (see section 9.2), so the mass distribution's radial throbbing cannot generate a gravitational wave.

### Proof of Birkhoff's theorem

Spherical symmetry guarantees that we can introduce coordinates  $r$  and  $t$  such that the surfaces of constant  $r$  and  $t$  have the structure of a sphere with radius  $r$ . On one such surface we can introduce colatitude and longitude coordinates  $\theta$  and  $\phi$ . The  $(\theta, \phi)$  coordinates can be extended in a natural way to other values of  $r$  by choosing the radial lines to lie in the direction of the covariant derivative vector<sup>8</sup>  $\nabla_a r$ , and this ensures that the metric will not have any nonvanishing terms in  $dr d\theta$  or  $dr d\phi$ , which could only arise if our choice had broken the symmetry between positive and negative values of  $d\theta$  and  $d\phi$ . Just as we were free to choose any way of threading lines of constant  $(\theta, \phi, t)$  between spheres of different radii, we can also choose how to thread lines of constant  $(\theta, \phi, r)$  between different times, and this can be done so as to keep the metric free of any time-space cross-terms such as  $d\theta dt$ . The metric can therefore be written in the form<sup>9</sup>

$$ds^2 = h(t, r)dt^2 - k(t, r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (7.4.1)$$

This has to be a solution of the vacuum field equations,  $R_{ab} = 0$ , and in particular a quick calculation with Maxima shows that  $R_{rt} = -\frac{\partial_r k}{k^2 r}$ , so  $k$  must be independent of time. With this restriction, we find

$$\nabla_{\{r\}} \nabla_{\{r\}} h = -\frac{\partial_r h}{h} h = -\frac{1}{r^2} \frac{\partial_r h}{h} r^2 = 0,$$

and since  $k$  is time-independent,  $\frac{\partial_r h}{h}$  is also time-independent. This means that for a particular time  $t_0$ , the function  $f(r) = h(t_0, r)$  has some universal shape set by a differential equation, with the only possible ambiguity being an over-all scaling that depends on  $t_0$ . But since  $h$  is the timetime component of the metric, this scaling corresponds physically to a situation in which every clock, all over the universe, speeds up and slows down in unison. General relativity is coordinate-independent, so this has no observable effects, and we can absorb it into a redefinition of  $t$  that will cause  $h$  to be time-independent. Thus the metric can be expressed in the time-independent diagonal form

$$ds^2 = h(r)dt^2 - k(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (7.4.2)$$

We have already solved the field equations for a metric of this form and found as a solution the Schwarzschild spacetime.<sup>10</sup> Since the metric's components are all independent of  $t$ ,  $\partial_t$  is a Killing vector, and it is timelike for large  $r$ , so the Schwarzschild spacetime is asymptotically static.

<sup>8</sup> It may seem backwards to start talking about the covariant derivative of a particular coordinate before a complete coordinate system has even been introduced. But (excluding the trivial case of a flat spacetime),  $r$  is not just an arbitrary coordinate, it is something that an observer at a certain point in spacetime can determine by mapping out a surface of geometrically identical points, and then determining that surface's radius of curvature. Another worry is that it is possible for  $\nabla_a r$  to misbehave on certain surfaces, such as the event horizon of the Schwarzschild spacetime, but we can simply require that radial lines remain continuous as they pass through these surfaces.

<sup>9</sup> On the same surfaces referred to in the preceding footnote, the functions  $h$  and  $k$  may go to 0 or  $\infty$ . These turn out to be nothing more serious than coordinate singularities.

<sup>10</sup> The Schwarzschild *spacetime* is the uniquely defined geometry found by removing the coordinate singularities from this form of the Schwarzschild *metric*.