10.2: Appendix A (Part 2)

§ 4. Physical Meaning of the Equations Obtained in Respect to Moving Rigid Bodies and Moving Clocks

We envisage a rigid sphere[26] of radius R, at rest relative to the moving system k, and with its centre at the origin of coordinates of k. The equation of the surface of this sphere moving relative to the system K with velocity v is

$$\xi^2+\eta^2+\zeta^2=R^2. \tag{10.2.1}$$

Note

That is, a body possessing spherical form when examined at rest.—AS

The equation of this surface expressed in x, y, z at the time t = 0 is

$$\frac{x^2}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}+y^2+z^2=R^2. \tag{10.2.2}$$

A rigid body which, measured in a state of rest, has the form of a sphere, therefore has in a state of motion—viewed from the stationary system—the form of an ellipsoid of revolution with the axes

$$R\sqrt{1-\frac{v^2}{c^2}}, R, R. \tag{10.2.3}$$

Thus, whereas the Y and Z dimensions of the sphere (and therefore of every rigid body of no matter what form) do not appear modified by the motion, the X dimension appears shortened in the ratio $1 : sqrt1-\frac{v^2}{c^2}$, i.e., the greater the value of v, the greater the shortening. For v = c all moving objects—viewed from the “stationary” system—shrivel up into plane figures.[27] For velocities greater than that of light our deliberations become meaningless; we shall, however, find in what follows, that the velocity of light in our theory plays the part, physically, of an infinitely great velocity.

Note

That is, a body possessing spherical form when examined at rest.—AS

It is clear that the same results hold good of bodies at rest in the “stationary” system, viewed from a system in uniform motion.

Further, we imagine one of the clocks which are qualified to mark the time t when at rest relative to the stationary system, and the time τ when at rest relative to the moving system, to be located at the origin of the coordinates of k, and so adjusted that it marks the time τ. What is the rate of this clock, when viewed from the stationary system?

Between the quantities x, t, and τ, which refer to the position of the clock, we have, evidently, x = vt and

$$\tau=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t-\frac{vx}{c^2}\right). \tag{10.2.4}$$

Therefore,

$$\tau=t\sqrt{1-\frac{v^2}{c^2}}=t-\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)t \tag{10.2.5}$$

whence it follows that the time marked by the clock (viewed in the stationary system) is slow by $1-\sqrt{1-\frac{v^2}{c^2}}$ seconds per second, or—neglecting magnitudes of fourth and higher order—by $\frac{1}{2}\frac{v^2}{c^2}$.

From this there ensues the following peculiar consequence. If at the points A and B of K there are stationary clocks which, viewed in the stationary system, are synchronous; and if the clock at A is moved with the velocity v along the line AB to B, then on its arrival at B the two clocks no longer synchronize, but the clock moved from A to B lags behind the other which has remained at B by $\frac{1}{2}\frac{tv^2}{c^2}$ (up to magnitudes of fourth and higher order), t being the time occupied in the journey from A to B.

It is at once apparent that this result still holds good if the clock moves from A to B in any polygonal line, and also when the points A and B coincide.

If we assume that the result proved for a polygonal line is also valid for a continuously curved line, we arrive at this result: If one of two synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t seconds, then by the clock which has remained at rest the travelled clock on its arrival at A will be $\frac{1}{2}\frac{tv^2}{c^2}$ second slow. Thence we conclude that a spring-clock at the equator must go more slowly, by a very small amount, than a precisely similar clock situated at one of the poles under otherwise identical conditions.[28]

> Note
>
> Einstein specifies a spring-clock ("unruhuhr") because the effective gravitational field is weaker at the equator than at the poles, so a pendulum clock at the equator would run more slowly by about two parts per thousand than one at the north pole, for nonrelativistic reasons. This would completely mask any relativistic effect, which he expected to be on the order of $\frac{v^2}{c^2}$, or about 10^{-13}. In any case, it later turned out that Einstein was mistaken about this example. There is also a gravitational time dilation that cancels the kinematic effect. See example 10. The two clocks would actually agree.—BC

§ 5. The Composition of Velocities

In the system k moving along the axis of X of the system K with velocity v, let a point move in accordance with the equations

$$\xi = w_\xi \tau, \quad \eta = w_\eta \tau, \quad \zeta = 0, \tag{10.2.6}$$

where w_ξ and w_η denote constants.

Required: the motion of the point relative to the system K. If with the help of the equations of transformation developed in § 3 we introduce the quantities x, y, z, t into the equations of motion of the point, we obtain

$$\begin{aligned} x &= \frac{w_\xi + v}{1 + \frac{vw_\xi}{c^2}} t, \\ y &= \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{vw_\xi}{c^2}} w_\eta t, \\ z &= 0. \end{aligned}$$

Thus the law of the parallelogram of velocities is valid according to our theory only to a first approximation. We set[29]

$$\begin{aligned} V^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \\ w^2 &= w_\xi^2 + w_\eta^2, \\ a &= \tan^{-1}\frac{w_\eta}{w_\xi}, \end{aligned}$$

a is then to be looked upon as the angle between the velocities v and w. After a simple calculation we obtain

$$V = \frac{\sqrt{(v^2 + w^2 + 2vw\cos a) - (vw\sin\frac{a}{c})^2}}{1 + vw\cos\frac{a}{c^2}}. \tag{10.2.7}$$

> Note
>
> This equation was incorrectly given in Einstein's original paper and the 1923 English translation as a = $\tan^{-1}\frac{w_y}{w_x}$.—JW

It is worthy of remark that v and w enter into the expression for the resultant velocity in a symmetrical manner. If w also has the direction of the axis of X, we get

$$V=\frac{v+w}{1+\frac{vw}{c^2}}. \tag{10.2.8}$$

It follows from this equation that from a composition of two velocities which are less than c, there always results a velocity less than c. For if we set v = c − κ, w = c − λ, κ and λ being positive and less than c, then

$$V=c\frac{2c-\kappa-\lambda}{2c-\kappa-\lambda+\frac{\kappa\lambda}{c}}<c. \tag{10.2.9}$$

It follows, further, that the velocity of light c cannot be altered by composition with a velocity less than that of light. For this case we obtain

$$V=\frac{c+w}{1+\frac{w}{c}}=c. \tag{10.2.10}$$

We might also have obtained the formula for V, for the case when v and w have the same direction, by compounding two transformations in accordance with § 3. If in addition to the systems K and k figuring in § 3 we introduce still another system of coordinates k' moving parallel to k, its initial point moving on the axis of Ξ[30] with the velocity w, we obtain equations between the quantities x, y, z, t and the corresponding quantities of k', which differ from the equations found in § 3 only in that the place of “v” is taken by the quantity

$$\frac{v+w}{1+\frac{vw}{c^2}}; \tag{10.2.11}$$

from which we see that such parallel transformations—necessarily—form a group.

Note

“X” in the 1923 English translation.—JW

We have now deduced the requisite laws of the theory of kinematics corresponding to our two principles, and we proceed to show their application to electrodynamics.[31]

Note

The remainder of the paper is not given here, but can be obtained from John Walker’s web site at www.fourmilab.ch.—BC

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