4.4: The Tensor Transformation Laws

We may wish to represent a vector in more than one coordinate system, and to convert back and forth between the two representations. In general relativity, the transformation of the coordinates need not be linear, as in the Lorentz transformations; it can be any smooth, one-to-one function. For simplicity, however, we start by considering the one-dimensional case, and by assuming the coordinates are related in an affine manner,

$$x'^{\mu} = ax^{\mu} + b. \tag{4.4.1}$$

The addition of the constant b is merely a change in the choice of origin, so it has no effect on the components of the vector, but the dilation by the factor a gives a change in scale, which results in $v'^{\mu} = av^{\mu}$ for a contravariant vector. In the special case where v is an infinitesimal displacement, this is consistent with the result found by implicit differentiation of the coordinate transformation. For a contravariant vector, $v'_{\mu} = \frac{1}{a} v_{\mu}$. Generalizing to more than one dimension, and to a possibly nonlinear transformation, we have

$$v'^{\mu} = v^{\kappa} \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \tag{4.4.2}$$

$$v'_{\mu} = v_{\kappa} \frac{\partial x'^{\kappa}}{\partial x^{\mu}} \tag{4.4.3}$$

Note the inversion of the partial derivative in one equation compared to the other. Because these equations describe a change from one coordinate system to another, they clearly depend on the coordinate system, so we use Greek indices rather than the Latin ones that would indicate a coordinate-independent equation. Note that the letter μ in these equations always appears as an index referring to the new coordinates, κ to the old ones. For this reason, we can get away with dropping the primes and writing, e.g., $v^{\mu} = \frac{v^{\kappa} \partial x'^{\mu}}{\partial x^{\kappa}}$ rather than v' counting on context to show that v^{μ} is the vector expressed in the new coordinates,v^{κ} in the old ones. This becomes especially natural if we start working in a specific coordinate system where the coordinates have names. For example, if we transform from coordinates (t, x, y, z) to (a, b, c, d), then it is clear that v^{t} is expressed in one system and v^{c} in the other.

Exercise 4.4.1

Recall that the gauge transformations allowed in general relativity are not just any coordinate transformations; they must be (1) smooth and (2) one-to-one. Relate both of these requirements to the features of the vector transformation laws above.

In Equation 4.4.3, μ appears as a subscript on the left side of the equation, but as a superscript on the right. This would appear to violate our rules of notation, but the interpretation here is that in expressions of the form $\frac{\partial}{\partial x^{i}}$ and $\frac{\partial}{\partial x_{i}}$, the superscripts and subscripts should be understood as being turned upside-down. Similarly, Equation 4.4.2 appears to have the implied sum over κ written ungrammatically, with both κ's appearing as superscripts. Normally we only have implied sums in which the index appears once as a superscript and once as a subscript. With our new rule for interpreting indices on the bottom of derivatives, the implied sum is seen to be written correctly. This rule is similar to the one for analyzing the units of derivatives written in Leibniz notation, with, e.g., $\frac{d^{2}x}{dt^{2}}$ having units of meters per second squared. That is, the flipping of the indices like this is required for consistency so that everything will work out properly when we change our units of measurement, causing all our vector components to be rescaled.

A quantity v that transforms according to Equations 4.4.2 or 4.4.3 is referred to as a rank-1 tensor, which is the same thing as a vector.

Example 17: The identity transformation

In the case of the identity transformation $x'^{\mu} = x^{\mu}$, Equation 4.4.2 clearly gives v' = v, since all the mixed partial derivatives $\frac{\partial x'^{\mu}}{\partial x^{\kappa}}$ with $\mu \neq \kappa$ are zero, and all the derivatives for $\kappa = \mu$ equal 1.

In Equation 4.4.3, it is tempting to write

$$\frac{\partial x^{\kappa}}{\partial x'^{\mu}} = \frac{1}{\frac{\partial x'^{\mu}}{\partial x^{\kappa}}} \quad (wrong!), \tag{4.4.4}$$

but this would give infinite results for the mixed terms! Only in the case of functions of a single variable is it possible to flip derivatives in this way; it doesn't work for partial derivatives. To evaluate these partial derivatives, we have to invert the transformation (which in this example is trivial to accomplish) and then take the partial derivatives.

The metric is a rank-2 tensor, and transforms analogously:

$$g_{\mu\nu} = g_{\kappa\lambda} \frac{\partial x^\kappa}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} \tag{4.4.5}$$

writing g rather than g' on the left, because context makes the distinction clear).

Exercise 4.4.2

Self-check: Write the similar expressions for $g^{\mu\nu}, g^\mu_\nu$, and g^ν_μ, which are entirely determined by the grammatical rules for writing superscripts and subscripts. Interpret the case of a rank-0 tensor.

Example 18: An accelerated coordinate system?

Let's see the effect on Lorentzian metric g of the transformation

$$t' = t \qquad x' = x + \frac{1}{2}at^2. \tag{4.4.6}$$

The inverse transformation is

$$t = t' \qquad x = x' - \frac{1}{2}at'^2. \tag{4.4.7}$$

The tensor transformation law gives

$$g'_{t't'} = 1 - (at')^2 \tag{4.4.8}$$

$$g'_{x'x'} = -1 \tag{4.4.9}$$

$$g'_{x't'} = -at'. \tag{4.4.10}$$

Clearly something bad happens at $at' = \pm 1$, when the relative velocity surpasses the speed of light: the t' component of the metric vanishes and then reverses its sign. This would be physically unreasonable if we viewed this as a transformation from observer A's Lorentzian frame into the accelerating reference frame of observer B aboard a spaceship who feels a constant acceleration. Several things prevent such an interpretation:

1. B cannot exceed the speed of light.
2. Even before B gets to the speed of light, the coordinate t' cannot correspond to B's proper time, which is dilated.
3. Due to time dilation, A and B do not agree on the rate at which B is accelerating. If B measures her own acceleration to be a', A will judge it to be a < a', with a → 0 as B approaches the speed of light.

There is nothing invalid about the coordinate system (t', x'), but neither does it have any physically interesting interpretation.

Example 19: Physically meaningful constant acceleration

To make a more physically meaningful version of example 18, we need to use the result of example 4. The somewhat messy derivation of the coordinate transformation is given by Semay.[11] The result is

$$t' = \left(x + \frac{1}{a}\right) \sinh at \tag{4.4.11}$$

$$x' = \left(x + \frac{1}{a}\right) \cosh at \tag{4.4.12}$$

Applying the tensor transformation law gives (problem 7):

$$g'_{t't'} = (1 + ax')^2 \tag{4.4.13}$$

$$g'_{x'x'} = -1 \tag{4.4.14}$$

Unlike the result of example 18, this one never misbehaves. The closely related topic of a uniform gravitational field in general relativity is considered in problem 7.

[11] arxiv.org/abs/physics/0601179

Example 20: Accurate timing signals

The relation between the potential **A** and the fields **E** and **B** given in section 4.2 can be written in manifestly covariant form as

$$F_{ij} = \partial_{[i} A_{j]} \tag{4.4.15}$$

where F, called the electromagnetic tensor, is an antisymmetric rank-two tensor whose six independent components correspond in a certain way with the components of the **E** and **B** three-vectors. If F vanishes completely at a certain point in spacetime, then the linear form of the tensor transformation laws guarantees that it will vanish in all coordinate systems, not just one. The GPS system takes advantage of this fact in the transmission of timing signals from the satellites to the users. The electromagnetic wave is modulated so that the bits it transmits are represented by phase reversals of the wave. At these phase reversals, F vanishes, and this vanishing holds true regardless of the motion of the user's unit or its position in the earth's gravitational field. Cf. problem 17.

Example 21: Momentum wants a lower index

In example 5, we saw that once we arbitrarily chose to write ruler measurements in Euclidean three-space as Δx^a rather than Δx_a, it became natural to think of the Newtonian force threevector as "wanting" to be notated with a lower index. We can do something similar with the momentum 3- or 4-vector. The Lagrangian is a relativistic scalar, and in Lagrangian mechanics momentum is defined by $p_a = \frac{\partial L}{\partial v^a}$. The upper index in the denominator on the right becomes a lower index on the left by the same reasoning as was employed in the notation of the tensor transformation laws. Newton's second law shows that this is consistent with the result of example 5.