6.E: Vacuum Solutions (Exercises)

1. Show that in geometrized units, power is unitless. Find the equivalent in watts of a power that equals 1 in geometrized units.
2. The metric of coordinates (θ, ϕ) on the unit sphere is $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. (a) Show that there is a singular point at which $g^{ab} \rightarrow \infty$. (b) Verify directly that the scalar curvature R = R^a_a constructed from the trace of the Ricci tensor is never infinite. (c) Prove that the singularity is a coordinate singularity.
3. (a) Space probes in our solar system often use a slingshot maneuver. In the simplest case, the probe is scattered gravitationally through an angle of 180 degrees by a planet. Show that in some other frame such as the rest frame of the sun, in which the planet has speed u toward the incoming probe, the maneuver adds 2u to the speed of the probe. (b) Suppose that we replace the planet with a black hole, and the space probe with a light ray. Why doesn't this accelerate the ray to a speed greater than c?
4. An observer outside a black hole's event horizon can never observe a test particle falling past the event horizon and later hitting the singularity. We could therefore wonder whether general relativity's predictions about the interior of a black hole, and the singularity in particular, are even a testable scientific theory. However, the observer could herself fall into the black hole. The question is then whether she would reach the singularity within a finite proper time; if so, then it is observable to her. The purpose of this problem is to prove that this is so, using the techniques of section 6.2. Suppose for simplicity that the observer starts at rest far away from the black hole, and falls directly inward toward it. (a) In the notation of section 6.2, what are the values of E and L in this case? (b) Find the function r(s), i.e., the observer's Schwarzschild radial coordinate as a function of her proper time, and show that she does reach the singularity in finite proper time.
5. The curve given parametrically by ($\cos^3$ t, $\sin^3$ t) is called an astroid. The arc length along this curve is given by s = $(\frac{3}{2})\sin^2$ t, and its curvature by k = $-(\frac{2}{3})\csc^2$ t. By rotating this astroid about the x axis, we form a surface of revolution that can be described by coordinates (t, ϕ), where ϕ is the angle of rotation. (a) Find the metric on this surface. (b) Identify any singularities, and classify them as coordinate or intrinsic singularities.
6. (a) Section 3.5 gave a flat-spacetime metric in rotating polar coordinates,

   $$ds^2 = (1-\omega^2 r^2)dt^2 - dr^2 - r^2 d\theta'^2 - 2\omega r^2 d\theta' dt. \tag{6.E.1}$$

   Identify the two values of r at which singularities occur, and classify them as coordinate or non-coordinate singularities.
   (b) The corresponding spatial metric was found to be

   $$ds^2 = -dr^2 - \frac{r^2}{1-\omega^2 r^2} d\theta'^2. \tag{6.E.2}$$

   Identify the two values of r at which singularities occur, and classify them as coordinate or non-coordinate singularities.
   (c) Consider the following argument, which is intended to provide an answer to part b without any computation. In two dimensions, there is only one measure of curvature, which is equivalent (up to a constant of proportionality) to the Gaussian curvature. The Gaussian curvature is proportional to the angular deficit ϵ of a triangle. Since the angular deficit of a triangle in a space with negative curvature satisfies the inequality $-\pi < \epsilon < 0$, we conclude that the Gaussian curvature can never be infinite. Since there is only one measure of curvature in a two-dimensional space, this means that there is no non-coordinate singularity. Is this argument correct, and is the claimed result consistent with your answers to part b?
7. The first experimental verification of gravitational redshifts was a measurement in 1925 by W.S. Adams of the spectrum of light emitted from the surface of the white dwarf star Sirius B. Sirius B has a mass of $0.98 M_\odot$ and a radius of 5.9×10^6 m. Find the redshift.
8. Show that, as claimed in section 6.3, applying the change of coordinates t' = t−2m ln(r−2m) to the Schwarzschild metric results in a metric for which g_{rr} and $g_{t't'}$ never blow up, but that $g^{t't'}$ does blow up.
9. Use the geodesic equation to show that, in the case of a circular orbit in a Schwarzschild metric, $\frac{d^2t}{ds^2}$ = 0. Explain why this makes sense.
10. Verify by direct calculation, as asserted in section 6.4, that the Riemann tensor vanishes for the metric $ds^2 = -t\, dt^2 - d\ell^2$, where $d\ell^2 = dx^2 + dy^2 + dz^2$.
11. Suppose someone proposes that the vacuum field equation of general relativity isn't $R_{ab} = 0$ but rather $R_{ab} = k$, where k is some constant that describes an innate tendency of spacetime to have tidal distortions. Explain why this is not a good proposal.
12. Prove, as claimed in section 6.3, that in 2+1 dimensions, with a vanishing cosmological constant, there is no nontrivial Schwarzschild metric.
13. In section 6.2, I argued that there is no way to define a time-reversal operation in general relativity so that it applies to all spacetimes. Why can't we define it by picking some arbitrary spacelike surface that covers the whole universe, flipping the

velocity of every particle on that surface, and evolving a new version of the spacetime backward and forward from that surface using the field equations?