7.2: Spherical Symmetry

A little more work is required if we want to link the existence of Killing vectors to the existence of a specific symmetry such as spherical symmetry. When we talk about spherical symmetry in the context of Newtonian gravity or Maxwell's equations, we may say, "The fields only depend on r," implicitly assuming that there is an r coordinate that has a definite meaning for a given choice of origin. But coordinates in relativity are not guaranteed to have any particular physical interpretation such as distance from a particular origin. The origin may not even exist as part of the spacetime, as in the Schwarzschild metric, which has a singularity at the center. Another possibility is that the origin may not be unique, as on a Euclidean two-sphere like the earth's surface, where a circle centered on the north pole is also a circle centered on the south pole; this can also occur in certain cosmological spacetimes that describe a universe that wraps around on itself spatially.

We therefore define spherical symmetry as follows. A spacetime S is spherically symmetric if we can write it as a union $S = \cup s_{r,t}$ of nonintersecting subsets $s_{r,t}$, where each s has the structure of a twosphere, and the real numbers r and t have no preassigned physical interpretation, but $s_{r,t}$ is required to vary smoothly as a function of them. By "has the structure of a two-sphere," we mean that no intrinsic measurement on s will produce any result different from the result we would have obtained on some two-sphere. A two-sphere has only two intrinsic properties:

1. it is spacelike, i.e., locally its geometry is approximately that of the Euclidean plane;
2. it has a constant positive curvature.

If we like, we can require that the parameter r be the corresponding radius of curvature, in which case t is some timelike coordinate.

To link this definition to Killing vectors, we note that condition 2 is equivalent to the following alternative condition: (2') The set s should have three Killing vectors (which by condition 1 are both spacelike), and it should be possible to choose these Killing vectors such that algebraically they act the same as the ones constructed explicitly in example 4 in section 7.1. As an example of such an algebraic property, Figure $7.2.1$ shows that rotations are noncommutative.

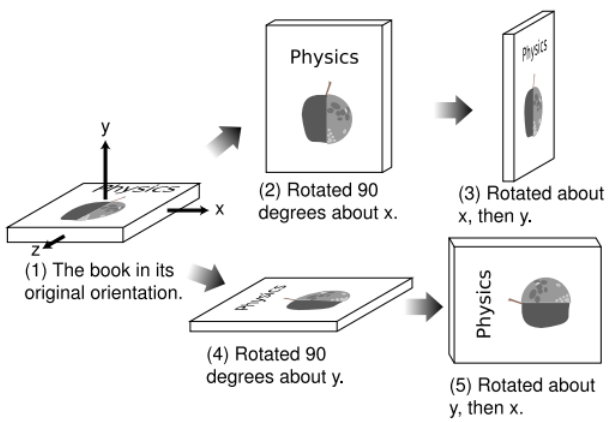

Figure $7.2.1$: Performing the rotations in one order gives one result, 3, while reversing the order gives a different result, 5.

Example 7: A cylinder is not a sphere

- Show that a cylinder does not have the structure of a two-sphere.
- The cylinder passes condition 1. It fails condition 2 because its Gaussian curvature is zero. Alternatively, it fails condition 2' because it has only two independent Killing vectors (example 3).

Example 8: A plane is not a sphere

- Show that the Euclidean plane does not have the structure of a two-sphere.
- Condition 2 is violated because the Gaussian curvature is zero. Or if we wish, the plane violates 2' because ∂_x and ∂_y commute, but none of the Killing vectors of a 2-sphere commute.