7.E: Symmetries (Exercises)

1. Example 3 gave the Killing vectors ∂_z and ∂_ϕ of a cylinder. If we express these instead as two linearly independent Killing vectors that are linear combinations of these two, what is the geometrical interpretation?
2. Section 7.4 told the story of Alice trying to find evidence that her spacetime is not stationary, and also listed the following examples of spacetimes that were not stationary: (a) the solar system, (b) cosmological models, (c) gravitational waves propagating at the speed of light, and (d) a cloud of matter undergoing gravitational collapse. For each of these, show that it is possible for Alice to accomplish her mission.
3. If a spacetime has a certain symmetry, then we expect that symmetry to be detectable in the behavior of curvature scalars such as the scalar curvature $R = R^a{}_a$ and the Kretchmann invariant $k = R^{abcd}R_{abcd}$.
   a. Show that the metric
   $$ds^2 = e^{2gz}dt^2 - dx^2 - dy^2 - dz^2 \tag{7.E.1}$$
   from section 7.5 has constant values of R = 1/2 and k = 1/4. Note that Maxima’s ctensor package has built-in functions for these; you have to call the `lriemann` and `uriemann` before calling them.
   b. Similarly, show that the Petrov metric
   $$ds^2 = -dr^2 - e^{-2r}dz^2 + e^r[2\sin\sqrt{3}rd\phi dt - \cos\sqrt{3}r(d\phi^2 - dt^2] \tag{7.E.2}$$
   has R = 0 and k = 0.
4. Section 7.5 presented the Petrov metric. The purpose of this problem is to verify that the gravitational field it represents does not fall off with distance. For simplicity, let’s restrict our attention to a particle released at an r such that $\cos\sqrt{3}r = 1$, so that t is the timelike coordinate. Let the particle be released at rest in the sense that initially it has $\dot{z} = \dot{r} = \dot{\phi} = 0$, where dots represent differentiation with respect to the particle’s proper time. Show that the magnitude of the proper acceleration is independent of r.
5. The idea that a frame is “rotating” in general relativity can be formalized by saying that the frame is stationary but not static. Suppose someone says that any rotation must have a center. Give a counterexample.