8.E: Sources in General Relativity (Exercise)

1. Verify, as claimed in section 8.1, that the electromagnetic pressure inside a medium-weight atomic nucleus is on the order of 10^{33} Pa.
2. Is the Big Bang singularity removable by the coordinate transformation $t \rightarrow \frac{1}{t}$?
3. Verify the claim made in section 8.2 that a is a linear function of time in the case of the Milne universe, and that k = −1.
4. Examples 16 and 18 discussed ropes with cosmological lengths. Reexamine these examples in the case of the Milne universe.
5. (a) Show that the Friedmann equations are symmetric under time reversal. (b) The spontaneous breaking of this symmetry in perpetually expanding solutions was discussed in section 8.2. Use the definition of a manifold to show that this symmetry cannot be restored by gluing together an expanding solution and a contracting one "back to back" to create a single solution on a single, connected manifold.
6. The Einstein field equations are

$$G_{ab} = 8\pi T_{ab} + \Lambda g_{ab}, \tag{8.E.1}$$

and when it is possible to adopt a frame of reference in which the local mass-energy is at rest on average, we can interpret the stress-energy tensor as

$$T^{\mu}_{\nu} = diag(-\rho, P, P, P), \tag{8.E.2}$$

where ρ is the mass-energy density and P is the pressure. Fix some point as the origin of a local Lorentzian coordinate system. Analyze the properties of these relations under a reflection such as $x \rightarrow -x$ or $t \rightarrow -t$.
7. (a) Show that a positive cosmological constant violates the strong energy condition in a vacuum. In applying the definition of the strong energy condition, treat the cosmological constant as a form of matter, i.e., "roll in" the cosmological constant term to the stress-energy term in the field equations. (b) Comment on how this affects the results of the following paper: Hawking and Ellis, "The Cosmic Black-Body Radiation and the Existence of Singularities in Our Universe," Astrophysical Journal, 152 (1968) 25, articles.adsabs.harvard.edu/f...pJ...152...25H.
8. In problem 7 in chapter 5, we analyzed the properties of the metric

$$ds^2 = e^{2gz} dt^2 - dz^2. \tag{8.E.3}$$

   a. In that problem we found that this metric had the same properties at all points in space. Verify in particular that it has the same scalar curvature R at all points in space.
   b. Show that this is a vacuum solution in the two-dimensional (t, z) space.
   c. Suppose we try to generalize this metric to four dimensions as

$$ds^2 = e^{2gz} dt^2 - dx^2 - dy^2 - dz^2. \tag{8.E.4}$$

   Show that this requires an Einstein tensor with unphysical properties.
9. Consider the following proposal for defeating relativity's prohibition on velocities greater than c. Suppose we make a chain billions of light-years long and attach one end of the chain to a particular galaxy. At its other end, the chain is free, and it sweeps past the local galaxies at a very high speed. This speed is proportional to the length of the chain, so by making the chain long enough, we can make the speed exceed c.
Debunk this proposal in the special case of the Milne universe.
10. Make a rigorous definition of the volume V of the *observable* universe. Suppose someone asks whether V depends on the observer's state of motion. Does this question have a well-defined answer? If so, what is it? Can we calculate V 's observer-dependence by applying a Lorentz contraction?
11. For a perfect fluid, we have $P = w\rho$, where w is a constant. The cases w = 0 and $w = \frac{1}{3}$ correspond, respectively, to dust and radiation. Show that for a flat universe with $\Lambda = 0$ dominated by a single component that is a perfect fluid, the solution to the Friedmann equations is of the form $a \propto t^{\delta}$, and determine the exponent δ. Check your result in the dust case against the one in section 8.2, then find the exponent in the radiation case. Although the w = −1 case corresponds to a cosmological constant, show that the solution is not of this form for w = −1.
12. Apply the result of problem 11 to generalize the result of example 22 for the size of the observable universe. What is the result in the case of the radiation-dominated universe?
13. The Kantowski-Sachs metric is

$$ds^2 = dt^2 - \Lambda^{-1}(d\theta^2 + \sin^2\theta d\phi^2) - e^{2\sqrt{\Lambda}t}dz^2. \tag{8.E.5}$$

It describes a universe with the spatial topology of a 3-cylinder. Use a computer algebra system such as Maxima to verify the following facts.

a. Any world-line of the form (t, θ, ϕ, z) = (λ, constants) is a geodesic parametrized by proper time. (If using Maxima, you will find that the function cgeodesic() saves time here.)
b. If two such geodesics are separated only in the z direction, the distance between them along a surface of fixed t increases exponentially with t, while geodesics separated only in θ and ϕ do not recede from one another.
c. There are no matter fields, only a cosmological constant Λ.
d. The Ricci scalar R = -4Λ (+ $- --$ signature) equals $\frac{1}{3}$ of the value for the de Sitter vacuum-dominated cosmology (sec. 8.2), the factor of 3 occurring because there is expansion along only one axis rather than three.
e. The vacuum-dominated cosmology found by de Sitter and presented in the text was supposed to be the unique cosmological solution of this type. Why is the Kantowski-Sachs metric not a counterexample?

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