4.1: Lorentz Scalars

A Lorentz scalar is a quantity that remains invariant under both spatial rotations and Lorentz boosts. Mass is a Lorentz scalar.[1] Electric charge is also a Lorentz scalar, as demonstrated to extremely high precision by experiments measuring the electrical neutrality of atoms and molecules to a relative precision of better than 10^{-20}; the electron in a hydrogen atom has typically velocities of about 1/100, and those in heavier elements such as uranium are highly relativistic, so any violation of Lorentz invariance would give the atoms a nonvanishing net electric charge.

Note

Some older books define mass as transforming according to $m \rightarrow \gamma m$, which can be made to give a self-consistent theory, but is ugly.

The time measured by a clock traveling along a particular world-line from one event to another is something that all observers will agree upon; they will simply note the mismatch with their own clocks. It is therefore a Lorentz scalar. This clock-time as measured by a clock attached to the moving body in question is often referred to as proper time, “proper” being used here in the somewhat archaic sense of “own” or “self,” as in “The Vatican does not lie within Italy proper.” Proper time, which we notate τ , can only be defined for timelike world-lines, since a lightlike or spacelike world-line isn’t possible for a material clock.

More generally, when we express a metric as $ds^2 = \ldots$, the quantity ds is a Lorentz scalar. In the special case of a timelike world-line, ds and $d\tau$ are the same thing. (In books that use a $- + ++$ metric, one has $ds = -d\tau$.)

Even more generally, affine parameters, which exist independent of any metric at all, are scalars. As a trivial example, if τ is a particular object’s proper time, then τ is a valid affine parameter, but so is $2\tau + 7$. Less trivially, a photon’s proper time is always zero, but one can still define an affine parameter along its trajectory. We will need such an affine parameter, for example, in section 6.2, when we calculate the deflection of light rays by the sun, one of the early classic experimental tests of general relativity.

Another example of a Lorentz scalar is the pressure of a perfect fluid, which is often assumed as a description of matter in cosmological models.

Example 1: Infinitesimals and the clock “postulate”

At the beginning of chapter 3, I motivated the use of infinitesimals as useful tools for doing differential geometry in curved spacetime. Even in the context of special relativity, however, infinitesimals can be useful. One way of expressing the proper time accumulated on a moving clock is

$$\begin{aligned} s &= \int ds \\ &= \int \sqrt{g_{ij}dx^i dx^j} \\ &= \int \sqrt{1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}\, dt, \end{aligned}$$

which only contains an explicit dependence on the clock’s velocity, not its acceleration. This is an example of the clock “postulate” referred to in the remark at the end of homework problem 1. Note that the clock postulate only applies in the limit of a small clock. This is represented in the above equation by the use of infinitesimal quantities like dx.