

5.13: Units in General Relativity

Analyzing units, also known as [dimensional analysis](#), is one of the first things we learn in freshman physics. It's a useful way of checking our math, and it seems as though it ought to be straightforward to extend the technique to relativity. It certainly can be done, but it isn't quite as trivial as might be imagined, and it leads to some surprising new physical ideas.

One of our most common jobs is to change from one set of units to another, but in relativity it becomes nontrivial to define what we mean by the notion that our units of measurement change or don't change. We could, e.g., appeal to an atomic standard, but Dicke²⁴ points out that this could be problematic. Imagine, he says, that

... you are told by a space traveller that a hydrogen atom on Sirius has the same diameter as one on the earth. A few moments' thought will convince you that the statement is either a definition or else meaningless.

To start with, we note that *abstract index notation* is more convenient than *concrete index notation* for these purposes. Concrete index notation assigns different units to different components of a tensor if we use coordinates, such as spherical coordinates (t, r, θ, ϕ) , that don't all have units of length. In abstract index notation, a symbol like v^i stands for the whole vector, not for one of its components.

In concrete index notation, it also doesn't necessarily make sense to talk about rescaling. E.g., for polar coordinates in the Euclidean plane, the transformation $(r, \theta) \rightarrow (2r, 2\theta)$ doesn't have any interesting interpretation, and can't even be applied globally. In abstract index notation, we can say $v^i \rightarrow 2v^i$, and this simply means that the vector v^i has been scaled up by a factor of 2.

Since abstract index notation does not even offer us a notation for components, if we want to apply dimensional analysis we must define a system in which units are attributed to a tensor as a whole. Suppose we write down the abstract-index form of the equation for proper time:

$$ds^2 = g_{ab} dx^a dx^b \quad (5.13.1)$$

In abstract index notation, dx^a doesn't mean an infinitesimal change in a particular coordinate, it means an infinitesimal displacement vector.²⁵ This equation has one quantity on the left and three factors on the right. Suppose we assign these parts of the equation units $[ds] = L^\sigma$, $[g_{ab}] = L^{2\gamma}$, and $[dx^a] = [dx^b] = L^\xi$, where square brackets mean "the units of" and L stands for units of length. We then have $\sigma = \gamma + \xi$. Due to the ambiguities referred to above, we can pick any values we like for these three constants, as long as they obey this rule. I find $(\sigma, \gamma, \xi) = (1, 0, 1)$ to be natural and convenient, but Dicke, in the above-referenced paper, likes $(1, 1, 0)$, while the mathematician Terry Tao advocates $(0, \mp 1, \pm 1)$.

Suppose we raise and lower indices to form a tensor with r upper indices and s lower indices. We refer to this as a tensor of rank (r, s) . (We don't count contracted indices, e.g., $u^a v_a$ is a rank-(0, 0) scalar.) Since the metric is the tool we use for raising and lowering indices, and the units of the lower-index form of the metric are $L^{2\gamma}$, it follows that the units vary in proportion to $L^{\gamma(s-r)}$. In general, you can assign a physical quantity units L^u that are a product of two factors, a "kinematical" or purely geometrical factor L^k , where $k = \gamma(s - r)$, and a dynamical factor $L^d \dots$, which can depend on what kind of quantity it is, and where the \dots indicates that if your system of units has more than just one base unit, those can be in there as well. Dicke uses units with $\hbar = c = 1$, for example, so there is only one base unit, and mass has units of inverse length and $d_{\text{mass}} = -1$. In general relativity it would be more common to use units in which $G = c = 1$, which instead give $d_{\text{mass}} = +1$.

Note

For a modern and rigorous development of differential geometry along these lines, see Nowik and Katz, arxiv.org/abs/1405.0984.

Example 24: The units of momentum

Consider the equation

$$p^a = m v^a \quad (5.13.2)$$

for the momentum of a material particle. Suppose we use specialrelativistic units in which $c = 1$, but because gravity isn't incorporated into the theory, G plays no special role, and it is natural to use a system of units in which there is a base unit of

mass M .

The kinematic units check out, because $k_p = k_m + k_v$:

$$\gamma(-1) = \gamma(0) + \gamma(-1) \quad (5.13.3)$$

This is merely a matter of counting indices, and was guaranteed to check out as long as the indices were written in a grammatical way on both sides of the equation. What this check is essentially telling us is that if we were to establish Minkowski coordinates in a neighborhood of some point, and do a change of coordinates $(t, x, y, z) \rightarrow (\alpha t, \alpha x, \alpha y, \alpha z)$, then the quantities on both sides of the equation would vary under the tensor transformation laws according to the same exponent of α . For example, if we changed from meters to centimeters, the equation would still remain valid.

For the dynamical units, suppose that we use $(\sigma, \gamma, \xi) = (1, 0, 1)$, so that an infinitesimal displacement dx^a has units of length L , as does proper time ds . These two quantities are purely kinematic, so we don't assign them any dynamical units, and therefore the velocity vector $v^a = \frac{dx^a}{ds}$ also has no dynamical units. Our choice of a system of units gives $[m] = M$. We require that the equation $p^a = mv^a$ have dynamical units that check out, so:

$$M = 1 \cdot M \quad (5.13.4)$$

We must also assign units of mass to the momentum.

A system almost identical to this one, but with different terminology, is given by Schouten.²⁶

For practical purposes in checking the units of an equation, we can see from example 24 that worrying about the kinematic units is a waste of time as long as we have checked that the indices are grammatical. We can therefore give a simplified method that suffices for checking the units of any equation in abstract index notation.

1. We assign a tensor the same units that one of its concrete components *would* have if we were to adopt (local) Minkowski coordinates, in the system with $(\sigma, \gamma, \xi) = (1, 0, 1)$. These are the units we would automatically have imputed to it after learning special relativity but before learning about tensors or fancy coordinate transformations. Since $\gamma = 0$, the positions of the indices do not affect the result.
2. The units of a sum are the same as the units of the terms.
3. The units of a tensor product are the product of the units of the factors.

Our splitting of units into kinematic and dynamical parts can be understood as arising naturally from the following geometrical and physical considerations. In [section 3.2](#), we introduced the notion of a *connection*, which is a rule that relates tensors living in one local region of spacetime to those in another region, depending on the path used for parallel transport. The connection is embodied concretely in the Christoffel symbols, and we need it in order to define sensible derivatives of vectors, because otherwise we lack the information needed in order to tell whether a vector is in fact constant, and only changing its components due to the way the coordinate system is defined. The connection and the metric embody a lot of the same geometrical information. If we know the metric, we can always find the connection ([section 5.9](#)).

We might then naturally ask whether it is possible to go in the other direction. Given the connection, can we find the metric? But this is clearly not true, because the connection doesn't carry any information about units of measurement, while the metric does. In fact, if the metric g results in a certain connection Γ , then so will the metric $\Omega^2 g$, where Ω is a real constant.²⁷ One way of thinking about the transformation $g \rightarrow \Omega^2 g$ is that in the expression $ds^2 = g_{ab} dx^a dx^a$ for proper time, we scale up any clock reading s by a factor of Ω . This helps to explain Dicke's preference for the convention $(\sigma, \gamma, \xi) = (1, 1, 0)$, according to which the units are attributed to ds and g , while vectors are considered to be unitless. A further advantage of this system is that it can be adapted to concrete index notation, because we simply declare coordinates to be unitless names for points.

Note

If we multiplied g by a negative constant, then we would change the signature, e.g., from $----$ to $-+++$. Changing the signature would be particularly goofy in the context of Riemannian geometry, where it is customary to have a positive-definite metric.

The following table summarizes the factors by which various quantities change under rescaling of the lower-index metric and rescaling of local Minkowski coordinates x^μ . As above, r is the number of upper indices and s the number of lower indices. Entries

in lighter text follow from the more general rule. A curvature monomial of order p is an expression formed from the multiplication of p curvature tensors, possibly with contracted indices.

	$g_{ab} \rightarrow \Omega^2 g_{ab}$ (5.13.5)	$x^\mu \rightarrow \alpha x^\mu$ (5.13.6)
g	Ω^{s-r} (5.13.7)	α^{s-r} (5.13.8)
tensor density of rank (r, s) and weight w		α^{2w+r-s} (5.13.9)
Γ_{bc}^a (5.13.10)	1	α^{-1} (5.13.11)
curvature monomial of order p	Ω^{s-r-2p} (5.13.12)	α^{s-r} (5.13.13)

It makes sense that rescaling the metric doesn't change the Christoffel symbols, because it doesn't change the connection or the coordinates, and therefore shouldn't change the geodesic equation. Verifying the other entries in the table is a good exercise.

Example 25: A change of signature

Suppose that we change the signature of a metric from $+ - - -$ to $- + ++$ or vice versa. Although the notation Ω^2 was intended to imply that the signature of the metric would not be changed, nothing goes wrong in the logic if we take $\Omega^2 = -1$. According to the table, the lower-index form of the metric, with $(r, s) = (0, 2)$ changes by a factor of -1 , which is what we set out to do. A curvature polynomial of order p changes by a factor of $(-1)^p$. As a specific example, a cosmological model dominated by the cosmological constant (section 8.2) has Ricci scalar $R = -12\Lambda$ in the $+ - - -$ signature used in this book, but $R = +12\Lambda$ in the $- + ++$ signature.

Example 26: Curvature scalars for the Gödel metric

The Ricci scalar $R = R^a_a$ is a curvature monomial of order 1. Because it is a relativistic scalar, its value is invariant under a change of coordinates. A scalar constructed in this way from a curvature tensor is called a curvature scalar. In the system described above, it is a curvature monomial of order 1, and it is a tensor of rank $(0, 0)$. It is a pure tensor, i.e., it is a tensor density in only the trivial sense, having weight $w = 0$.

The Kretschmann invariant $K = R^{abcd}R_{abcd}$, discussed in more detail in section 6.3, is a curvature monomial of order 2, with properties that are otherwise similar to the ones listed above for the Ricci scalar.

To have a specific example to talk about, let us consider the metric

$$ds^2 = dt^2 - dx^2 - dy^2 + \frac{1}{2}e^{2x}dz^2 - 2e^x dz dt. \quad (5.13.14)$$

This is the historically and philosophically important Gödel metric, discussed in section 8.2. A calculation using Maxima gives $R = 1$ ($+ - - -$ signature) and $K = 3$. (The fact that both of these are constant shows that the spacetime is highly symmetric, although this is not manifest when the metric is expressed in these coordinates.) Suppose that we recalibrate our clocks to use different units, changing the metric above according to $ds^2 \rightarrow \Omega^2 ds^2$. Then application of the rules given in the table tells us that $R = \Omega^{-2}$ and $K = 3\Omega^{-4}$.

To round out our discussion of this approach, we state more precisely the relationship between the metric and the connection. Given a metric, there is a unique torsion-free connection. Given a torsion-free connection, there may or may not exist a metric that gives rise to that connection. If such a metric does exist, then except in exceptional cases that metric is unique up to a nonzero multiplicative constant. The reason for the uniqueness of the metric up to a constant factor is as follows. Suppose we fix the metric at one point on our manifold. Then by using the connection we can paralleltransport the metric tensor to other points on the manifold, so that defining it at one point has the effect of defining it everywhere. But there may be a lack of consistency, because parallel transport is path-dependent. In particular, if we transport the metric around a closed loop, we want to recover the original

metric. This consistency requirement is usually enough to rule out any freedom in defining the metric beyond a global scaling factor. A more complete treatment of this problem is given by Schmidt.²⁸

An interesting exceptional case is flat spacetime. Because there is no curvature, parallel transport around a closed loop never changes the metric, so the consistency requirement is automatically satisfied, and we our freedom in choosing a metric is greater than just the ability to scale by a constant. In particular, some authors choose not to use natural units, so that instead of $g = \text{diag}(1, -1, -1, -1)$ in Cartesian coordinates, one has $g = \text{diag}(c^2, -1, -1, -1)$. In an approach where a change of units is represented by a change of coordinates, this change in the metric could be represented by $(t, x, y, z) \rightarrow (\frac{t}{c}, x, y, z)$. But in the convention followed by Dicke, we would take the coordinates to be immutable labels for points, and these would actually be physically different metrics, with different light cones.

A similar example in a Riemannian context is the Euclidean plane, in which the (trivial) connection is consistent any metric of the form given in [example 9](#).

Finally, we note that it can be of interest to generalize the transformation $g \rightarrow \Omega^2 g$ so that Ω can vary from point to point. This is called a conformal transformation. Conformal transformations can be used for a variety of purposes, including nontrivial physics (as in the Dicke paper) and techniques for visualization ([section 7.3](#)).

References

²⁴ “Mach’s principle and invariance under transformation of units,” Phys Rev 125 (1962) 2163

²⁶ Tensor Analysis for Physicists, ch. VI

²⁸ projecteuclid.org/euclid.cmp/1103858479

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