

2.4: Relativistic Properties of Lorentz Geometry (Part 1)

We now want to pin down the properties of the Lorentz geometry that are left unspecified by the affine treatment. We need some further input from experiments in order to show us how to proceed. We take the following as empirical facts about flat spacetime:⁵

- **L1** *Spacetime is homogeneous and isotropic.* No time or place has special properties that make it distinguishable from other points, nor is one direction in space distinguishable from another.⁶
- **L2** *Inertial frames of reference* exist. These are frames in which particles move at constant velocity if not subject to any forces. We can construct such a frame by using a particular particle, which is not subject to any forces, as a reference point.
- **L3** *Equivalence of inertial frames:* If a frame is in constant-velocity translational motion relative to an inertial frame, then it is also an inertial frame. No experiment can distinguish one preferred inertial frame from all the others.
- **L4** *Causality:* There exist events 1 and 2 such that $t_1 < t_2$ in all frames.
- **L5** *Relativity of time:* There exist events 1 and 2 and frames of reference (t, x) and (t', x') such that $t_1 < t_2$, but $t'_1 > t'_2$.

Note

These facts are summarized for convenience in [Appendix C](#).

Note

For the experimental evidence on isotropy, see http://www.edu-observatory.org/physi...tropy_of_space.

L4 makes it possible to have an event 1 that causes an event 2, with all observers agreeing on which caused which. L5 is supported by the experimental evidence in [section 1.2](#); if L5 were false, then space and time could work as imagined by Galileo and Newton.

Define affine parameters t and x for time and position, and construct a (t, x) plane. Axiom L1 guarantees that spacetime is flat, allowing us to do this; if spacetime had, for example, a curvature like that of a sphere, then the axioms of affine geometry would fail, and it would be impossible to lay out such a global grid of parallels. Although affine geometry treats all directions symmetrically, we're going beyond the affine aspects of the space, and t does play a different role than x here, as shown, for example, by L4 and L5.

In the (t, x) plane, consider a rectangle with one corner at the origin O . We can imagine its right and left edges as representing the world-lines of two objects that are both initially at rest in this frame; they remain at rest (L2), so the right and left edges are parallel.

How do we know that this is a rectangle and not some other kind of parallelogram? In purely affine geometry, there is no notion of perpendicularity, so this distinction is meaningless. But implicit in the existence of inertial frames (L2) is the assumption that spacetime has some additional structure that allows a particular observer to decide what events he considers to be simultaneous ([example 7](#)). He then considers his own world-line, i.e., his t axis, to be perpendicular to a proposed x axis if points on the x axis are simultaneous to him.

We now define a second frame of reference such that the origins of the two frames coincide, but they are in motion relative to one another with velocity v . The transformation L from the first frame to the second is referred to as a Lorentz boost with velocity v . L depends on v . By equivalence of inertial frames (L3), an observer in the new frame considers his own t axis to be perpendicular to his own x , even though they don't look that way in figure a. Thus, although we assume some notion of perpendicularity, we do not assume that it looks the same as the Euclidean one.

By homogeneity of spacetime (L1), L must be linear, so the original rectangle will be transformed into a parallelogram in the new frame; this is also consistent with L3, which requires that the world-lines on the right and left edges remain parallel. The left edge has inverse slope v . By L5 (no simultaneity), the top and bottom edges are no longer horizontal.

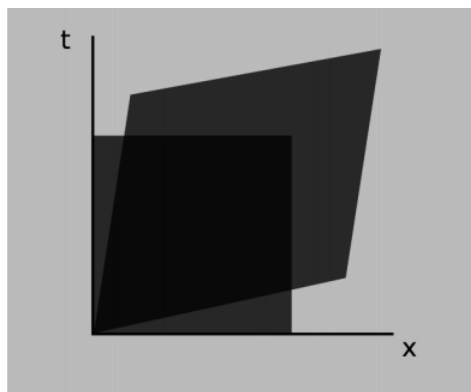


Figure 2.4.1 - Two objects at rest have world-lines that define a rectangle. In a second frame of reference in motion relative to the first one, the rectangle becomes a parallelogram.

For simplicity, let the original rectangle have unit area. Then the area of the new parallelogram is still 1, by the following argument. Let the new area be A , which is a function of v . By isotropy of spacetime (L1), $A(v) = A(-v)$. Furthermore, the function $A(v)$ must have some universal form for all geometrical figures, not just for a figure that is initially a particular rectangle; this follows because of our definition of affine area in terms of a dissection by a two-dimensional lattice, which we can choose to be a lattice of squares. Applying boosts $+v$ and $-v$ one after another results in a transformation back into our original frame of reference, and since A is universal for all shapes, it doesn't matter that the second transformation starts from a parallelogram rather than a square. Scaling the area once by $A(v)$ and again by $A(-v)$ must therefore give back the original square with its original unit area, $A(v)A(-v) = 1$, and since $A(v) = A(-v)$, $A(v) = \pm 1$ for any value of v . Since $A(0) = 1$, we must have $A(v) = 1$ for all v . The argument is independent of the shape of the region, so we conclude that all areas are preserved by Lorentz boosts. (See section 4.6 for further interpretation of A .)

If we consider a boost by an infinitesimal velocity dv , then the vanishing change in area comes from the sum of the areas of the four infinitesimally thin slivers where the rectangle lies either outside the parallelogram (call this negative area) or inside it (positive). (We don't worry about what happens near the corners, because such effects are of order dv^2 .) In other words, area flows around in the $x - t$ plane, and the flows in and out of the rectangle must cancel. Let v be positive; the flow at the sides of the rectangle is then to the right. The flows through the top and bottom cannot be in opposite directions (one up, one down) while maintaining the parallelism of the opposite sides, so we have the following three possible cases:

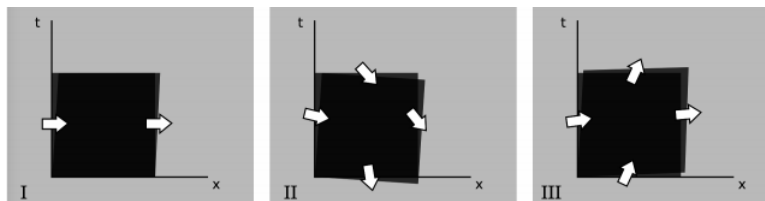


Figure 2.4.2 - Flows of area: (I) a shear that preserves simultaneity, (II) a rotation, (III) upward flow at all edges.

- I. There is no flow through the top and bottom. This case corresponds to Galilean relativity, in which the rectangle shears horizontally under a boost, and simultaneity is preserved, violating L5.
- II. Area flows downward at both the top and the bottom. The flow is clockwise at both the positive t axis and the positive x axis. This makes it plausible that the flow is clockwise everywhere in the (t, x) plane, and the proof is straightforward.⁷ As v increases, a particular element of area flows continually clockwise. This violates L4, because two events with a cause and effect relationship could be time-reversed by a Lorentz boost.
- III. Area flows upward at both the top and the bottom.

proof

By linearity of L , the flow is clockwise at the negative axes as well. Also by linearity, the handedness of the flow is the same at all points on a ray extending out from the origin in the direction θ . If the flow were counterclockwise somewhere, then it would have to switch handedness twice in that quadrant, at θ_1 and θ_2 . But by writing out the vector cross product $\mathbf{r} \times d\mathbf{r}$, where $d\mathbf{r}$ is the displacement caused by $L(dv)$, we find that it depends on $\sin(2\theta + \delta)$, which does not oscillate rapidly enough to have two zeroes in the same quadrant.

Only case III is possible, and given case III, there must be at least one point P in the first quadrant where area flows neither clockwise nor counterclockwise.⁸ The boost simply increases P's distance from the origin by some factor. By the linearity of the transformation, the entire line running through O and P is simply rescaled. This special line's inverse slope, which has units of velocity, apparently has some special significance, so we give it a name, c . We'll see later that c is the maximum speed of cause and effect whose existence we inferred in [section 1.3](#). Any world-line with a velocity equal to c retains the same velocity as judged by moving observers, and by isotropy the same must be true for $-c$.

Note

This follows from the fact that, as shown in the preceding footnote, the handedness of the flow depends only on θ .

For convenience, let's adopt time and space units in which $c = 1$, and let the original rectangle be a unit square. The upper right tip of the parallelogram must slide along the line through the origin with slope $+1$, and similarly the parallelogram's other diagonal must have a slope of -1 . Since these diagonals bisected one another on the original square, and since bisection is an affine property that is preserved when we change frames of reference, the parallelogram must be equilateral.

We can now determine the complete form of the Lorentz transformation. Let unit square PQRS, as described above, be transformed to parallelogram P'Q'R'S' in the new coordinate system (x' , t'). Let the t' coordinate of R' be γ , interpreted as the ratio between the time elapsed on a clock moving from P' to R' and the corresponding time as measured by a clock that is at rest in the (x' , t') frame. By the definition of v , R' has coordinates $(v\gamma, \gamma)$, and the other geometrical facts established above place Q' symmetrically on the other side of the diagonal, at $(\gamma, v\gamma)$. Computing the cross product of vectors P'R' and P'Q', we find the area of P'Q'R'S' to be $\gamma^2(1 - v^2)$, and setting this equal to 1 gives

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \quad (2.4.1)$$

The result for the transformation L, a Lorentz boost along the x axis with velocity v , is:

$$t' = \gamma t + v\gamma x \quad (2.4.2)$$

$$x' = v\gamma t + \gamma x \quad (2.4.3)$$

The symmetry of P'Q'R'S' with respect to reflection across the diagonal indicates that the time and space dimensions are treated symmetrically, although they are not entirely interchangeable as they would have been in case II.

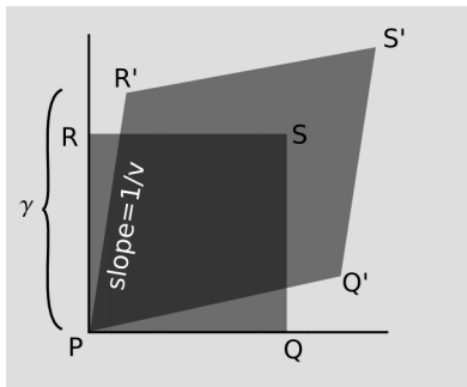


Figure 2.4.3 - Unit square PQRS is Lorentz-boosted to the parallelogram P'Q'R'S'.

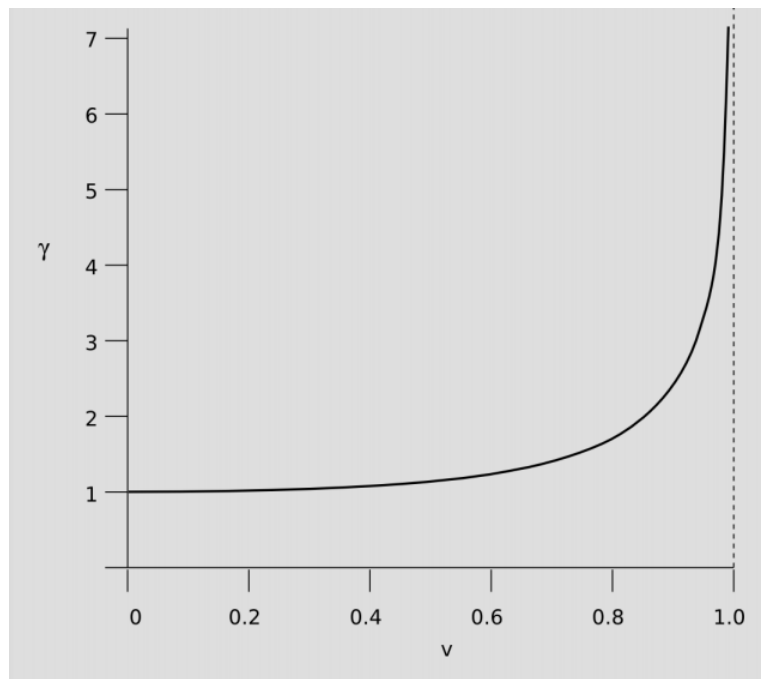


Figure 2.4.4 - The behavior of the γ factor.

A measuring rod, unlike a clock, sweeps out a two-dimensional strip on an $x - t$ graph. As in Galilean relativity, the two observers disagree on the positions of events at the two ends of their rods, but in addition they disagree on the simultaneity of such events. Calculation shows that a moving rod appears contracted by a factor γ .

In summary, a clock runs fastest according to an observer who is at rest relative to the clock, and a measuring rod likewise appears longest in its own rest frame.

The lack of a universal notion of simultaneity has a similarly symmetric interpretation. In prerelativistic physics, points in space have no fixed identity. A brass plaque commemorating a Civil War battle is not at the same location as the battle, according to an observer who perceives the Earth has having been hurtling through space for the intervening centuries. By symmetry, points in time have no fixed identity either.

In everyday life, we don't notice relativistic effects like time dilation, so apparently $\gamma \approx 1$, and $v \ll 1$, i.e., the speed c must be very large when expressed in meters per second. By setting c equal to 1, we have chosen a the distance unit that is extremely long in proportion to the time unit. This is an example of the correspondence principle, which states that when a new physical theory, such as relativity, replaces an old one, such as Galilean relativity, it must remain "backward-compatible" with all the experiments that verified the old theory; that is, it must agree with the old theory in the appropriate limit. Despite my coyness, you probably know that the speed of light is also equal to c . It is important to emphasize, however, that light plays no special role in relativity, nor was it necessary to assume the constancy of the speed of light in order to derive the Lorentz transformation; we will in fact prove that photons [must](#) travel at c , and that this [must](#) be true for any massless particle.

On the other hand, Einstein did originally develop relativity based on a different set of assumptions than our L1-L5. His treatment, given in his 1905 paper "On the electrodynamics of moving bodies," is reproduced [later](#). It starts from the following two postulates:

- **P1** The principle of relativity: "... the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest."
- **P2** "... light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body."

Einstein's P1 is essentially the same as our L3 (equivalence of inertial frames). He implicitly assumes something equivalent to our L1 (homogeneity and isotropy of spacetime). In his system, our L5 (relativity of time) is a theorem proved from the axioms P1-P2, whereas in our system, his P2 is a theorem proved from the axioms L1-L5.

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