1.6: Distances as Determined by Standard Candles

For now we move on to the measurement of distances, something we'll also need for the derivation of Hubble's Law and its generalization valid for very large distances. One way to measure a distance is called the "standard candle" method. Assume we have an object with luminosity L where luminosity is the energy per unit time leaving the object. Measuring the flux (energy/unit time/unit area) will give us a way to figure out the distance to the object assuming it is emitting isotropically. The further away it is, the weaker the flux will be.

To determine the relationship between luminosity, flux and distance we need to figure out the area over which the energy gets spread, and thus the area of a sphere.

As a reminder, the invariant distance equation in a homogeneous and isotropic Universe can be written as:

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{1.6.1}$$

Box 1.6.1

Calculate the area of a sphere ignoring effects of expansion, in 5 steps.

Exercise 6.1.1: According to the invariant distance equation, what is the distance between (t, r, θ, ϕ) and $(t, r, \theta + d\theta, \phi)$?

Answer

Constant t, r, and ϕ so we have

$$ds = a(t) r\, d\theta$$

Exercise 6.1.2: What is the distance between (t, r, θ, ϕ) and $(t, r, \theta, \phi + d\phi)$?

Answer

Similar to 7.1.1 above, we now have constant t, r, and θ, which gives

$$ds = a(t) r \sin\theta\, d\phi$$

Exercise 6.1.3: What is the area of a rectangle formed with those two lengths?

Answer

This is simply $a^2(t) r^2 \sin\theta\, d\theta d\phi$.

Exercise 6.1.4: What is the area of a sphere at coordinate value r with center at the origin?

Answer

Using the result from 7.1.3 above, we now just integrate over θ and ϕ, so the area is

$$A = \int_0^{2\pi} \int_0^{\pi} a(t) r sin\theta d\theta d\phi$$

$$= \int_0^{2\pi} 2a^2(t) r^2 d\phi$$

$$= 4\pi a^2(t) r^2$$

Exercise 6.1.5: Neglecting effects due to expansion (the changing of $a(t)$), how are luminosity and flux related for an observer at the origin and an object at coordinate distance $r = d$? You should find that $F = L/(4\pi d^2 a^2)$.

Answer

We know that Luminosity = (Flux)×(Surface Area). Making the appropriate substitutions we do indeed find that $F = L/(4\pi d^2 a^2)$.

Box 1.6.2

Now let us include effects of expansion. There are two distinct effects here:

Exercise 6.2.1: Convince yourself that the rate of photon arrival is slower than the rate of photon departure by a factor of $1+z$. (Hint: recall the arguments we made in chapter 6 about how the time in between emission of pulses is related to the time in between reception of pulses.)

Answer

As we found in Chapter 6, the rate of arrival of the wave crests will be slower than the rate of emission by the factor of $a(t_r)/a(t_e)$. The same argument applies to the rate of arrival of photons. We also saw that wavelength would be stretched out by a factor $1+z \equiv \frac{\lambda_r}{\lambda_e} = a(t_r)/a(t_e)$. Therefore the rate of arrival of photons will be slowed down by a factor of $1+z$.

Exercise 6.2.2: Convince yourself that the energy of each photon decreases by a factor of $1+z$.

Answer

The relationship between photon energy and wavelength is $E = \frac{hc}{\lambda}$. Substituting this into the definition of redshift, we find that

$$E_r/E_e = \frac{1}{1+z}$$

Each of these two effects reduces the flux by a factor of $1+z$ so the effect of expansion is to alter the flux-luminosity-distance relationship so that:

$$F = \frac{L}{4\pi d^2 a^2 (1+z)^2} \tag{1.6.2}$$

The presence of a here in this result raises a question, which we address next.

Now that the universe is expanding, what value of a should we include in Equation 1.6.2? To answer this, recall that $4\pi d^2 a^2$ is the area of the sphere over which the luminosity is spread, so we can determine power *per unit area*. We should therefore use the value of a at the time the measurement is being made. If we assume the measurement is being made in the current epoch (that we often just simply call "today") and we choose the common convention of normalizing the scale factor so that it's value is one today then we get:

$$F = \frac{L}{4\pi d^2 (1+z)^2} \tag{1.6.3}$$

In Minkowski spacetime (a non-expanding homogeneous and isotropic spacetime) we have the relationship $F = L/4\pi d^2$ where d is the distance between the observer and the source. This relationship motivates the definition of what is called "luminosity distance", d_{lum}, defined implicitly via:

$$F = \frac{L}{4\pi d_{\text{lum}}^2} \tag{1.6.4}$$

Box 1.6.3

Exercise 6.3.1: Make an explicit definition of d_{lum} by solving Equation 1.6.4 for d_{lum}.

Answer

Solving Equation 7.4 for d_{lum} we get

$$d_{\text{lum}} = \sqrt{\frac{L}{4\pi F}}$$

Exercise 6.3.2: In an FRW spacetime, for an observer at the origin, what is the luminosity distance to an object at coordinate distance $r = d$ with redshift z?

Answer

Substituting in Equation 7.3 to our result above we get

$$d_{\text{lum}}^2 = \frac{L}{4\pi}\frac{4\pi d^2(1+z)^2}{L} \quad \Longrightarrow \quad d_{\text{lum}} = d(1+z)$$

So now we know how to infer a particular kind of distance from an observation of a standard "candle." If a source of light is considered a standard candle, that means we know its luminosity. We can measure flux because it's a local property (how much energy per unit time per unit area is flowing past us right here and now). With L and F known we can calculate the luminosity distance. In the next chapter we work out the relationship of luminosity distance and redshift with the history of the scale factor, $a(t)$.

Summary

1. A common convention, that we will adopt, is to normalize the scale factor so that it is equal to one today. If we refer to the current epoch as $t = t_0$ then our normalization choice can be written as $a(t_0) = 1$.
2. The flux, F, from an isotropic emitter with luminosity, L, a coordinate distance d away from an observer observing today, is given by

   $$F = \frac{L}{4\pi d^2(1+z)^2}$$

   where z is the redshift of the light from the source due to the expansion.
3. The luminosity distance, d_{lum} from here and now to a source is, by definition, the distance that gives the Euclidean, non-expanding result:

   $$F = \frac{L}{4\pi d_{\text{lum}}^2}.$$

   Given summary item (2) above, we see that $d_{\text{lum}} = d \times (1+z)$.