1.36: The Simplest Expanding Spacetime

In this chapter we begin our exploration of physics in an expanding spacetime. We begin with a reminder about how coordinates themselves are meaningless, and that physical meaning comes from the expression for the invariant distance. We start with a spacetime with just one spatial dimension that is not expanding: a 1+1-dimensional Minkowski spacetime. I expect you are familiar with such a spacetime from your prior study of special relativity, and from the previous chapter. We then generalize it slightly to describe a spacetime with one spatial dimension that is *expanding*. With additional assumptions we then calculate the age of this spacetime as well as the "past horizon."

We can label spacetimes with coordinates; for example, we could label every point in a 1+1-dimensional space with a t value and an x value. These coordinates are just labels, with no physical meaning, until we also say something about the "invariant distance" between infinitesimally separated pairs of points. For example, in a 1+1-dimensional Minkowski space with which you are familiar, it is possible to do this labeling such that the square of the invariant distance between t, x and $t+dt, x+dx$ is given by:

$$ds^2 = -c^2 dt^2 + dx^2. \tag{1.36.1}$$

Note

The physical interpretation of ds^2 is as follows:

1. For time-like separations ($ds^2 < 0$), the time elapsed on a clock that freely falls (travels with no acceleration) between the two space-time points is $\sqrt{-ds^2/c^2}$; and
2. For space-like separations ($ds^2 > 0$), the length of a ruler with an end on each of the two space-time points, at rest in the frame in which the two events are simultaneous, is $\sqrt{ds^2}$.

Box 1.36.1

Exercise 5.1.1: For the spacetime specified by Equation 1.36.1. On a plot of x vs. t (what we call a spacetime diagram) draw the trajectory of a particle that is not moving, one that is moving slowly, and then of one that is moving at the speed of light. Place the x-coordinate on the horizontal axis, as is the usual convention.

The invariant distance rule above (Equation 1.36.1) is for a *static spacetime*. Our universe is expanding. We can make a simple alteration of the invariant distance equation to describe an expanding universe:

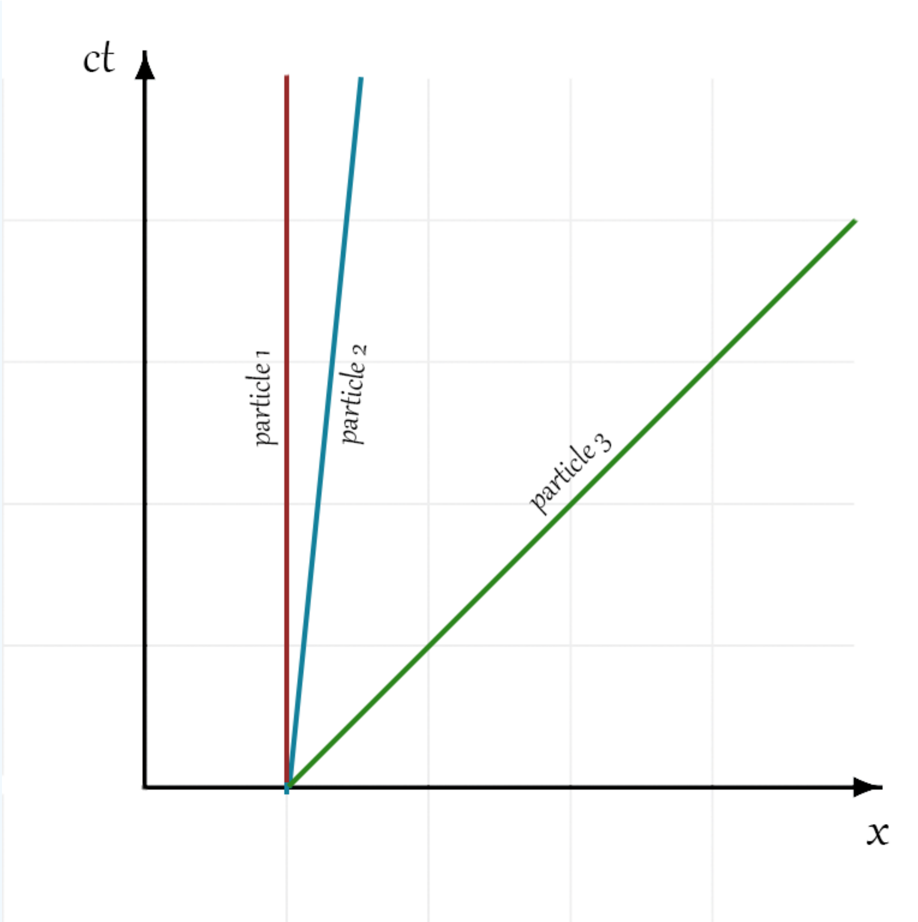
$$ds^2 = -c^2 dt^2 + a^2(t) dx^2 \tag{1.36.2}$$

with $a(t)$ a function of time. If $\dot{a} > 0$ the universe is expanding. If $\dot{a} < 0$ it is contracting. We call $a(t)$ the "scale factor."

Box 1.36.2

Exercise 5.2.1: Imagine a very small ruler instantaneously at rest in the x, t coordinate system of Equation 1.36.2 at time $t = t_1$, with one end at location $x = x_1$ and its other end at $x = x_1 + dx_1$. How long is the ruler?

Answer



We cheated a bit here and made a x vs. ct plot so that a particle moving at the speed of light has a slope of 1.

Box 1.36.3

Exercise 5.3.1: How much time elapses on a clock on a trajectory of constant x, from $t=t_1$ to $t=t_2$ for a spacetime and coordinate system with invariant distances given by Equation 1.36.2?

Answer

"at time $t=t_1$" so $dt=0$, so $ds=a(t)dx$. Since the ruler is at rest in the given coordinate system its length is indeed given by ds at time t_1. Therefore the length of the ruler is $ds=a(t_1)dx_1$.

Box 1.36.4

Exercise 5.4.1: Still assuming Equation 1.36.2, draw the paths through spacetime of a pair of particles that are separated from each other and that are not "moving" -- that is, their x coordinate values are not changing over time. Assume $a(t)$ is an increasing function of time. What do you notice about the distance between them and how it evolves over time? Be careful not to confuse "distance between them" with the difference in the values of their spatial coordinates.

Answer

"constant x" so $dx=0$, and then $ds=cdt$. Therefore the time elapsed on the clock is

$$\int \frac{1}{c}\sqrt{-ds^2} = \int_{t_1}^{t_2} dt = t_2 - t_1.$$

Exercise 5.4.2: Now, add in the trajectory of a light ray passing from one of these particles to the other. While sketching it out, remember that $a(t)dx$ is the distance traversed (as measured by an observer at rest in the x, t coordinate system) as the time coordinate changes by dt, which is the time elapsed as measured by an observer at rest in the x, t coordinate system. In this x vs. t diagram, does light travel in a straight line?

You should have seen in the box above that light does not travel on a straight line in this expanding spacetime as labeled with the t, x coordinates. This is kind of annoying. Often one can choose better coordinates to describe a problem in a simpler manner. For example, for a problem with spherical symmetry one can switch from Cartesian coordinates to spherical coordinates. We will do a similar thing here, introducing a coordinate called "conformal time."

The conformal time, τ, is defined via $d\tau = dt/a(t)$. In a conformal time diagram, for the expanding spacetime with which we have been working, light trajectories are straight lines. We will find that this is a very useful property.

Box 1.36.5

Exercise 5.5.1: Given a spacetime described by Equation 1.36.2, work out the invariant distance specified for τ, x labeling instead of t, x labeling. You should find $ds^2 = a^2(\tau)[-c^2 d\tau^2 + dx^2]$ where by $a(\tau)$ we just mean $a(t(\tau))$.

Answer

No solution available yet

Note that, for an observer at rest in the given coordinate system, and given our physical interpretation of the invariant distance, the equation for the invariant distance can always be written schematically as

$$ds^2 = -c^2(\text{infinitesimal time elapsed})^2 + (\text{infinitesimal spatial distance traversed})^2$$

where by "infinitesimal time elapsed" we mean as measured by a clock that is not moving in the given coordinate system and by "infinitesimal spatial distance traveled" we mean as measured by a ruler that is not moving in the given coordinate system. All observers see light traveling at the speed of light so for the path of a photon we have (infinitesimal spatial distance traversed) = $c\times$(infinitesimal time elapsed). Putting this together we can conclude that light rays travel on trajectories with $ds^2 = 0$.

Box 1.36.6

Exercise 5.6.1: Draw how light rays move on a plot of x vs. τ. Start from $ds^2 = 0$ to find the relationship between $d\tau$ and dx, then draw a trajectory consistent with that relationship.

Answer

Substituting in $dt = a(t)d\eta$ to Equation 5.2 and factoring out $a^2(t)$ gives us

$$ds^2 = -c^2 a^2(t)[d\eta^2 + dx^2].$$

Assuming a one-to-one correspondence between t and η (which one would have in an expanding universe given definition of $d\eta$) we can use $a(\eta) \equiv a(t(\eta))$ in its place and write

$$ds^2 = a^2(\eta)[-c^2 d\eta^2 + dx^2]$$

An interesting question to ask about an expanding spacetime is whether the universe ever had, in the past, the scale factor equal to zero. As this would render everything currently in the observable universe all with zero separation between them, this would be quite an extreme situation. Just to get some practice working things out in an expanding spacetime, practice that will be useful later, let's assume $\dot{a} = \kappa/a$ for κ some positive constant and see if such a universe ever had $a = 0$. Let us call the time since $a = 0$, Δt. We can then write

$$\Delta t = \int dt = \int_0^{a(t)} da/\dot{a} = \int_0^{a(t)} da(a/\kappa) = a^2(t)/(2\kappa). \tag{1.36.3}$$

Since the integral converged, we find that with the assumption given, namely $\dot{a} \propto 1/a$, the answer is yes, a finite time in the past the scale factor had the value 0. This is the singularity of the big bang. In such spacetimes we usually choose to call the zero point of time ($t=0$), the time when $a=0$. [Note that this Δt is the time that would elapse on a stationary clock; i.e., a clock with a fixed spatial coordinate.]

Also note that we made progress with this calculation by replacing dt with $dt=da/\dot{a}$. This is a trick we will use many times to calculate a variety of things.

Another question we can ask is, "how far has light traveled since the beginning." It's interesting because nothing travels faster than the speed of light, so this tells us what the maximum distance is that any signal can propagate. We call this distance the "past horizon." Let's once again assume, for definiteness, $\dot{a}=\kappa/a$ and calculate how far light can travel. We know that for light $ds^2=0$ so we have $c^2dt^2=a^2(t)dx^2$ and therefore $cdt/a(t)=dx$ so we can write

$$\Delta x=\int dx=\int cdt/a=c\int_0^{a(t)} da/(a\dot{a})=\frac{c}{\kappa}\int_0^{a(t)} da=\frac{c}{\kappa}a(t) \tag{1.36.4}$$

(where you'll note we used the same trick again to convert an integral over time to an integral over the scale factor). Therefore we know the coordinate distance that light has traveled, Δx. That coordinate distance corresponds to a physical distance, at time t, of $a(t)\Delta x=\frac{c}{\kappa}a^2(t)$.

HOMEWORK Problems

Problem 1.36.1

Derive the phenomenon of Lorentz contraction using the invariance of the invariant distance. [Do not assume an expanding universe; assume $ds^2=-c^2dt^2+dx^2$]. The trick to doing this is careful choice of the two events (points in spacetime) for which to calculate their invariant distance. Imagine a ruler moving with respect to an observer at speed v, with the ruler oriented so that it is parallel to the relative velocity. Take event 1 to be when/where the front end of the ruler is at the same spacetime location as the observer, and event 2 to be when/where the back end of the ruler is at the same spacetime location as the observer. By calculating the invariant distance in the observer's rest frame and the ruler's rest frame you should find that the length of the ruler as determined by the observer is $L'=L/\gamma$ where L is the length of the ruler in its rest frame.

Problem 1.36.2

Assume that the scale factore evolves via $\dot{a}=\kappa a$ for κ a positive constant. (Note that this is a *different* assumption than the previous $\dot{a}=\kappa/a$). Show that in this spacetime the universe never has $a=0$. Do so by showing that the amount of time between $a=0$ and any finite a is infinite; i.e., show that the appropriate definite integral does not converge.

Problem 1.36.3

Assume $ds^2=-c^2dt^2+a^2(t)dx^2$ and once again that $\dot{a}=\kappa a$ for κ a positive constant. Our universe appears to be moving asymptotically toward such a case (although except with a 3-dimensional space instead of a 1-dimensional space). Determine what we call the "future horizon." If a light signal is sent out at time t_1 from x_1, in the positive x direction, to what value of x_2 will it get given an infinite amount of time? The distance between x_1 and x_2 at time t_1, $a(t_1)(x_2-x_1)$, is called the future horizon.