1.14: Pressure and Energy Density Evolution

There is a sense in which energy is conserved in general relativity. We say it is *locally* conserved, which effectively means that in a sufficiently small region of spacetime, the change in energy is equal to the flux of energy across the boundary of the region, including that via any work being done on the region. From this principle, or from the Einstein equations themselves, someone with some skill in general relativity can derive for a homogeneous expanding universe that:

$$dE = -P\,dV \tag{1.14.1}$$

where E is the energy in some volume V. This *looks* a lot like conservation of energy as we are used to seeing it. Indeed it is the first law of thermodynamics, in the special case of no heat flow across the boundary. If you have a gas in a volume V and you squeeze it down by dV, the work you do ($-PdV$), increases the kinetic energy (and hence total energy) of the gas particles by

$$dE = -P\,dV. \tag{1.14.2}$$

Since dV is negative, this is an increase in energy, assuming $P > 0$.

But the simplicity of the result is deceptive. As soon as one starts to ask some obvious questions about it, things can become confusing. In a homogeneous expanding universe, how is the work being done? There are no pressure gradients to push things around. Then is it somehow being done by gravitational potential energy? That is a reasonable guess, since the expansion of space is a gravitational effect.

These questions implicitly assume energy is globally conserved, when that is not actually the case in general relativity. We can, however, use our Newtonian intuition to guide us about how the gas will behave given that the region it occupies is expanding. If the volume slowly increases that is containing a gas (with $P > 0$), then the energy of that gas will decrease no matter if it's because of the expansion of space or the expansion of the walls that contain the gas.

From $dE = -PdV$ we can derive how energy density evolves as the scale factor evolves. Gas comoving with the expansion and in a region with comoving volume V_c, occupies a physical volume of a^3V_c. The energy content of this homogeneous gas is $\rho c^2 a^3 V_c$. Thus Equation 1.14.1 leads to

$$a^3c^2d\rho + 3\rho c^2 a^2 da = -3Pa^2da \tag{1.14.3}$$

or

$$a\frac{d\rho}{da} = -3(P/c^2 + \rho) \tag{1.14.4}$$

Box 1.14.1

Exercise 14.1.1: Use Equation 1.14.4 to find $P(\rho)$ for non-relativistic matter, given that $\rho \propto a^{-3}$.

Exercise 14.1.2: Use Equation 1.14.4 to find $P(\rho)$ for relativistic matter, given that $\rho \propto a^{-4}$.

Exercise 14.1.3: Use Equation 1.14.4 to find $P(\rho)$ for a cosmological constant, given that $\rho \propto a^0$.

Answer

The equation is

$$a\frac{d\rho}{da} = -3(P/c^2 + \rho)$$

Plugging in $\rho \propto a^n$ we get $n\rho = -3(P/c^2 + \rho)$. Solving for P for $n = -3, -4, 0$ we find $P = 0$, $P = \rho c^2/3$, and $P = -\rho c^2$ respectively.

For Exercise 15.1.1 you should find that $P = 0$. This might be surprising since non-relativistic matter in general has non-zero pressure. Remember though that our $\rho \propto a^{-3}$ result came from neglecting all kinetic energy of the gas, because it was so small compared to the kinetic energy. Of course it is not exactly zero so the pressure is also not exactly zero. For 15.1.2 you should find $P(\rho) = \frac{1}{3}\rho c^2$.

Perhaps most surprisingly, for 15.1.3 you should find that the pressure is *negative*. More precisely, you should find that $P(\rho) = -\rho c^2$. How can this be? What does it mean to have negative pressure? We will explore these questions in a homework problem.

We now have a closed set of equations that we can use to solve for the evolution of the scale factor. Given the mass-energy density of all the components of the universe at one time and their equations of state $P(\rho)$, and given the expansion rate at that same time, we can find how these densities evolve and how the scale factor does as well. We summarize the key equations here, as well as writing them out with explicit sums over components for the first time.

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi g \Sigma_i \rho_i}{3} - k/a^2 \tag{1.14.5}$$

and

$$a\frac{d\rho_i}{da} = -3(P_i/c^2 + \rho_i) \tag{1.14.6}$$

where the subscript i enumerates the different contributors to the energy density.

Definitions of some prevalent notation

Densities are often expressed in units of the critical density, with the critical density defined as the total density of a zero-mean-curvature universe with expansion parameter H. Since the Friedmann equation is $H^2 = 8\pi G\rho_{\text{total}}/3 - k/a^2$ this means that the critical density, ρ_{c}, is such that $H^2 = 8\pi G\rho_{\text{c}}/3$. More explicitly

$$\rho_{\text{c}} \equiv 3H^2/(8\pi G). \tag{1.14.7}$$

The symbol used for the density of component i in units of the critical density is $\Omega_i \equiv \rho_{i,0}/\rho_{\text{c},0}$. Often when cosmologists write densities out in terms of Ω they implicitly mean the value in the current epoch. Ideally, we would use a 0 subscript in such cases, in order to explicitly denote the current epoch, but we don't usually do that. So, for example, the Friedmann equation for a universe with pressureless matter, a cosmological constant, radiation, and zero mean curvaturecan be rewritten as

$$H^2(a) = H_0^2\left[\Omega_{\text{m}}a^{-3} + \Omega_\Lambda + \Omega_{\text{rad}}a^{-4}\right]. \tag{1.14.8}$$

We also sometimes use $\Omega_{\text{k}} \equiv -k/H_0^2$.

Box 1.14.2

Exercise 14.2.1: Show that $\Sigma_i\Omega_i + \Omega_k = 1$.

Answer

We have $H^2 = 8\pi G\rho/3 - k/a^2$, $\Omega_i \equiv \rho_{i,0}/\rho_c$, and the critical density today, ρ_c defined indirectly via $H_0^2 = 8\pi G\rho_c/3$. Recall that ρ in the Friedmann equation is the total density so $\rho = \Sigma_i\rho_i$.

Let's take the Friedmann equation, evaluated today (so $H_0^2 = 8\pi G\rho_0/3 - k$ and divide each term by either H_0^2 or $8\pi G\rho_c/3$. We can divide by either because they are equal. We get

$$1 = \Sigma_i\rho_{i,0}/\rho_c - k/H_0^2 = \Sigma_i\Omega_i + \Omega_k \tag{1.14.9}$$

if we also use the given definition of $\Omega_k \equiv -k/H_0^2$.

Another notational convenience sometimes uses is to define h as a way of quantifying the Hubble constant. This "little h" is defined such that

$$H_0 = 100h \text{ km/sec/Mpc}. \tag{1.14.10}$$

Saying "$h = 0.72$" is the same as saying $H_0 =$ 72 km/sec/Mpc. I mention this notation because it leads to yet another common way of talking about densities. Sometimes we run across use of, for example, $\Omega_{\text{m}}h^2$, or even ω_{m} which is the same thing by definition. Why would we do this? It's a convenient way of writing out the density, but with no actual dependence on the critical density. In a homework problem you will show how to take a density in units of grams per cubic centimeter and find out what this corresponds to in terms of Ωh^2. You will see it has no dependence on the value of the expansion rate because the h^2 in Ωh^2 cancels the h^2 in the critical density.

Homework

14.1: Imagine you have a box of volume V full of a substance with energy density ρc^2. Outside the box the energy density is zero. Imagine that if you expanded the box by some amount dV that the energy density inside would not drop, but would stay constant.

A) By how much would the energy inside the box increase if this expansion occurred?

B) Imagine you are pulling on the walls of the box to make this increase in volume happen. Would it be hard to do? Would it require work? How much work? Articulate why ascribing $P < 0$ to the material inside the box makes sense.

14.2: For a substance with $P = w\rho c^2$ with w a constant, find n such that $\rho \propto a^{-n}$. Explicitly write out how ρ depends on a for these cases: $w = 0, 1/3, -1, +1$. For example, for $w = 0$ write, "For $w = 0$ we find $\rho \propto a^{-3}$."

14.3: According to multiple lines of argument (one of which we will learn about when we study big bang nucleosynthesis), the mean density of baryonic matter in the universe (matter whose mass comes from protons and neutrons and nuclei made out of protons and neutrons) is such that $\Omega_b h^2$ is about 0.022. What is the mean density of baryons in the universe in grams per cubic centimeter? If $H_0 = 72$ km/sec/Mpc, what is Ω_b?
