1.11: Particle Kinematics in an Expanding Universe - Newtonian Analysis

Imagine laying down a grid that gives a spatial coordinate everywhere in space, a grid that expands uniformly, as the uniform fluid filling space also expands uniformly. Let's call those coordinates $\vec{x}$. As the expansion occurs, we consider the origin of the grid ($\vec{x}=0$) to be at rest. The physical distance from the origin to a point labeled by $\vec{x}$ is $\vec{y}=a\vec{x}$. We call $\vec{x}$ a *comoving* coordinate system. Everywhere the fluid of uniform density has $\dot{\vec{x}}=0$ and $\dot{\vec{y}}=\dot{a}\vec{x}$. Everywhere, there is a local rest frame, defined by the rest frame of the local part of the grid. Of course this rest frame is different depending on where you are -- after all, every part of the grid is moving away from every other part of the grid.

Now let's consider a particle that is moving with respect to the comoving grid, so that its velocity with respect to the origin has two different contributions. Following from $\vec{y}=a\vec{x}$ and the chain rule we have:

$$\dot{\vec{y}}=\dot{a}\vec{x}+a\dot{\vec{x}} \tag{1.11.1}$$

where the first term on the left-hand side is what we get from the expansion alone, and the second term arises from the particle's motion with respect to the local rest frame. We call this latter term the *peculiar* velocity, $\vec{v}_{\text{pec}}$.

We want to understand how an observer, at rest in their local rest frame, will observe the evolution of peculiar velocities of free particles. In a Newtonian analysis, the local rest frame will be an inertial frame (one in which Newton's laws of motion apply) only if $\dot{a}$ is a constant. With $\dot{a}$ constant, as long as $\vec{y}$ is an inertial frame, then the local frame at $\vec{x}$ will be inertial as well since they only differ by a constant velocity $\dot{a}\vec{x}$. So we will perform our analysis assuming $\ddot{a}=0$. The more general result is an interesting one that we will comment upon later.

Let's first think visually and qualitatively about how peculiar velocities of free particles will evolve. Imagine a particle that is moving away from the origin faster than its Hubble velocity. It will soon have moved out to larger comoving radial distance, thereby increasing the Hubble velocity. Since it's total velocity is constant (it is a free particle), its peculiar velocity must have decreased.

Box 1.11.1

Exercise 12.1.1: Argue similarly what happens to the magnitude of the peculiar velocity if it starts out negative; i.e., if the particle is decreasing its comoving separation from the origin.

Answer

If the peculiar velocity is negative then the particle, over time, ends up with a smaller distance from the origin then it would have had if its peculiar velocity had been zero. In other words, it starts falling behind. Falling behind naturally brings its motion to be more similar to the surrounding fluid, as the fluid closer to the origin is moving more slowly. This improved agreement between its velocity and that of the surrounding fluid flow can be interpreted as a reduction in the magnitude of the peculiar velocity.

From this qualitative analysis, we expect to find that peculiar velocities decrease over time as a result of the expansion. Let's now do the calculation. Starting from Equation 1.11.1, and recalling our assumption that $\ddot{a}=0$ we get:

$$\ddot{\vec{y}}=a\ddot{\vec{x}}+\dot{a}\dot{\vec{x}}+\dot{a}\dot{\vec{x}} \tag{1.11.2}$$

From $\vec{v}_{\text{pec}}=a\dot{\vec{x}}$ one can easily show the above can be rewritten as:

$$\ddot{\vec{y}}=\dot{\vec{v}}_{\text{pec}}+(\dot{a}/a)\vec{v}_{\text{pec}} \tag{1.11.3}$$

Since this is a free particle we have $\ddot{\vec{y}}=0$ and therefore find:

$$\dot{\vec{v}}_{\text{pec}}=-(\dot{a}/a)\vec{v}_{\text{pec}} \tag{1.11.4}$$

which has the solution $\vec{v}_{\text{pec}}\propto 1/a$. We do indeed find that peculiar velocities of free particles decrease as the universe expands.

Box 1.11.2

Exercise 12.2.1: Fill in the steps in the above derivation.

Answer

Let’s look at just one of the three spatial components of the velocity. Notationally, we’ll drop the subscript “pec” to make room for the subscript “x” indicating we are talking about the x-component of the velocity. For the other components it is the same solution.

From $dv_x/dt = -(da/dt)/av_x$ we can rearrange and cancel terms to get $dv_x/v_x = -da/a$, which we can integrate up:

$$\int_{v_{x,i}}^{v_x} \frac{dv'_x}{v'_x} = -\int_{a_i}^{a} \frac{da'}{a'}$$

where the i subscript indicates “initial” and the integrals are easily done to get

$$\ln(v_x/v_{x,i}) = -\ln(a/a_i) = \ln(a_i/a)$$

which can be solved to find

$$v_x = v_{x,i}(a_i/a).$$

The same goes for the other components of the peculiar velocity. So we have what we wanted to show, that the peculiar velocity reduces with expansion as $1/a$.

While to get this result ($\vec{v}_{\text{pec}} \propto 1/a$) from our Newtonian analysis we had to make the assumption $\ddot{a} = 0$, in general relativity the result holds even without the assumption. Let's look at this more closely to see what it means.

If we did the above derivation *without* the $\ddot{a} = 0$ assumption we would find instead:

$$\ddot{\vec{y}} = \dot{\vec{v}}_{\text{pec}} + (\dot{a}/a)v_{\text{pec}} + \ddot{a}\vec{x} \tag{1.11.5}$$

This additional term makes perfect sense in the Newtonian theory. If the scale factor is accelerating, then a particle at a fixed value of $\vec{x}$ will be accelerating at a rate $\ddot{a}\vec{x}$. Due to this acceleration, the local rest frame around $\vec{x}$ is an accelerating frame, not an inertial one. If no force is applied (so $\ddot{y} = 0$), a particle at $\vec{x}$ will obey $\dot{\vec{v}}_{\text{pec}} = -\ddot{a}\vec{x} - (\dot{a}/a)v_{\text{pec}}$. From this one can see that if the peculiar velocity is initially zero, it will become negative (headed toward the origin), as the local grid accelerates away from it.

While in the Newtonian theory a free particle has $\ddot{\vec{y}} = 0$, in the Einstein theory the local fluid rest frame *is* a locally inertial frame even in the presence of $\ddot{a} \neq 0$. So a free particle, rather than obeying $\ddot{\vec{y}} = 0$ everywhere, will obey $\ddot{\vec{y}} - \ddot{a}x = 0$ everywhere. Thus even with the addition of this extra term we still find $\dot{\vec{v}}_{\text{pec}} + (\dot{a}/a)v_{\text{pec}} = 0$ and therefore $\vec{v}_{\text{pec}} \propto 1/a$ still holds.

You can actually evaluate for yourselves, using tools we've already presented, the above claim that a particle with fixed value of x is not accelerating even in the presence of $\ddot{a} \neq 0$. By not accelerating, we mean that an accelerometer on this same path through spacetime (constant x) will register zero. Another way of putting it: no force would be required to keep the particle on the path. All you need is to know that in general relativity objects in free fall (those experiencing no acceleration) from event A to event B follow the path that maximizes the proper time. Now you can check that the path from (x_1, t_1) (that we'll call event A) to (x_1, t_2) (that we'll call event B) with x always equal to x_1 maximizes $\int_A^B \sqrt{-ds^2/c^2}$ even if $\ddot{a} = 0$. To do so, you can use the calculus of variations result we described in chapter 1.

In the above we assumed non-relativistic speeds. We will also want to know how particles traveling at speeds near the speed of light, or even at the speed of light, are affected by expansion. You have actually already shown how particles traveling at the speed of light are affected by the expansion, because you have worked out how light redshifts. You'll recall that:

$$\lambda_{\text{received}}/\lambda_{\text{emitted}} = a_{\text{received}}/a_{\text{emitted}} \tag{1.11.6}$$

i.e., $\lambda \propto a$. From the fact that the momentum of a photon p is inversely proportional to its wavelength we have:

$$p \propto 1/a \tag{1.11.7}$$

Note that for non-relativistic particles we also have $p \propto 1/a$ (since $v \propto 1/a$ and $p = mv$). It turns out that this is a generally correct result for free particles in an expanding universe, their peculiar momentum decreases as $1/a$.
