1.7: The Distance-Redshift Relation

For a given scale factor history, $a(t)$, one can work out a relationship between luminosity distance and redshift. This will be useful to us because it indicates how we can *infer* $a(t)$ from measurements of luminosity distance and redshift, over a range of redshifts.

Recall that for light world lines (paths through spacetime), $ds^2 = 0$. For a radial trajectory (one with $d\phi = d\theta = 0$) we thus have $c^2dt^2 = a^2(t)dr^2/(1-kr^2)$. Taking the square root, and choosing the sign so that the photon is headed toward the origin ($dr/dt < 0$) we have:

$$cdt = -\frac{a(t)dr}{\sqrt{1-kr^2}} \tag{1.7.1}$$

Assuming $a(t)$ we could do the integrals on both sides and find out how long it takes for the light to go from $r = d$ to the observer at the origin. But that time interval is not something we can measure, so we'd have a prediction following from the assumed $a(t)$ but no way to confirm it (at least not from what we've developed so far in our exposition here.) What we *can* measure is redshift, which as we've seen depends on the scale factor at the time of emission, so instead we swap out the dt for da and integrate over da. Since $dt = da/(da/dt)$ we get:

$$\frac{cda}{a\dot{a}} = -\frac{dr}{\sqrt{1-kr^2}} \tag{1.7.2}$$

or

$$c\int_{a_e}^{1}\frac{da}{a\dot{a}} = -\int_{d}^{0}\frac{dr}{\sqrt{1-kr^2}} = \int_{0}^{d}\frac{dr}{\sqrt{1-kr^2}} \tag{1.7.3}$$

It is conventional, and we will later find it convenient, to define the Hubble parameter $H \equiv \dot{a}/a$. This is a generalization of the Hubble constant, $H_0 = H(t_0)$ where t_0 is the time today. With this definition we can write:

$$c\int_{a_e}^{1}\frac{da}{a^2H} = \int_{0}^{d}\frac{dr}{\sqrt{1-kr^2}} \tag{1.7.4}$$

Let's work out the consequences of the above in a simple case valid for short travel times and a Euclidean geometry. Putting it more precisely, let's assume $k = 0$ and take $a(t)$ as given by its first order Taylor expansion about the current epoch so that:

$$a(t) = 1 + (t - t_0)\dot{a}|_{t_0} \tag{1.7.5}$$

where for the last term we've indicated it's to be evaluated at time $t = t_0$ (consistent with our assumption of a Taylor expansion). Note that truncating this Taylor expansion to first order means that $da/dt = \dot{a}|_{t_0}$ is a constant. Since the scale factor is unity today (by convention) we also have $\dot{a} = H_0$ and $H \equiv \dot{a}/a = H_0/a$.

Box 1.7.1

Exercise 7.1.1: Plugging $H = H_0/a$ into Equation 1.7.4 one can now do the integral on the left-hand side. The right-hand side could not be easier (since we are assuming $k = 0$). Check that you find:

$$\frac{c}{H_0}[\ln(1) - \ln(a_e)] = d$$

Answer

$$c\int_{a_e}^{1}\frac{da}{a^2H} = \int_{0}^{d}dr = d$$

$H = \frac{\dot{a}}{a}$ and $\dot{a} =$ constant, so we have $H = \frac{H_0}{a}$. Now,

$$d = \frac{c}{H_0}\int_{a_e}^{1}\frac{da}{a} = \frac{c}{H_0}\left[\ln(1) - \ln(a_e)\right]$$

Box 1.7.2

Exercise 7.2.1: Relate a_e to the redshift z, and take advantage of $\ln(1+x) = x$ to first order in x to derive $cz = H_0 d$ to first order in z. How is d here related to luminosity distance? Simplify your result, again assuming $z << 1$. You should find $cz = H_0 d_{\text{lum}}$. Finally, if z is replaced with $z = v/c$ we get Hubble's Law.

Answer

$$\frac{c}{H_0}\left[-\ln(a_e)\right] = d$$

$$d = \frac{c}{H_0}\ln\left(\frac{1}{a_e}\right)$$

$$d = \frac{c}{H_0}\ln(1+z)$$

$$d \approx \frac{c}{H_0}z$$

Now multiply both sides by $(1+z)$,

$$d \times (1+z) = \frac{c}{H_0}z(1+z)$$

$$d_{\text{lum}} = \frac{c}{H_0}z(1+z)$$

Since $z << 1$ we can simplify our result to $cz = H_0 d_{\text{lum}}$.

Summary

1. The Hubble parameter is $H \equiv \frac{\dot{a}}{a}$. What we call "the Hubble constant", H_0 is the Hubble parameter evaluated today, $H_0 = H(t_0)$.
2. Luminosity distance and redshift are two things we can measure. The relationship depends on $a(t)$ and the curvature k. In principle, if we measure distances and redshifts for objects at a variety of distances we could then infer $a(t)$ and k. The general relationship between redshift and luminosity distance is contained in these equations:

$$c\int_{a_e}^{1}\frac{da}{a^2 H} = \int_0^d \frac{dr}{\sqrt{1-kr^2}} \tag{1.7.6}$$

and

$$d_{\text{lum}} = d(1+z) \tag{1.7.7}$$

with $1+z = 1/a_e$.

3) For small redshifts, the above reduces to $cz = H_0 d$ for $k=0$, (and for non-zero k: $cz = H_0 \int_0^d \frac{dr}{\sqrt{1-kr^2}}$). If one sets $v = cz$ (which makes sense for a Newtonian interpretation of the redshift), then we arrive at Hubble's Law $v = H_0 d$.

HOMEWORK Problems

Problem 1.7.1

Assume the Hubble parameter varies with scale factor as $H = H_0 a^{-3/2}$ and that $k=0$. As we will see in subsequent chapters this is what one gets (when $k=0$) for a universe filled with non-relativistic matter and nothing else. Note that we are using our convention that the scale factor today is unity; i.e., $a(t_0) = 1$ (and further note that we will not continue to give this reminder). Show that the luminosity distance is related to redshift via:

$$d_{\text{lum}} = \frac{2c}{H_0}\left[1 - \sqrt{\frac{1}{1+z}}\right] \times (1+z)$$

Problem 1.7.2

Show that to first order in z the above relationship reduces to $cz = H_0 d_{\text{lum}}$; i.e., Hubble's Law.

Problem 1.7.3

Assume the Hubble parameter varies with scale factor as $H = H_0 a^{-1}$ and that $k < 0$. As we will see in subsequent chapters this is what one gets for a universe filled with nothing. Show that

$$d_{\text{lum}} = \frac{1+z}{\sqrt{|k|}} \sinh\left[\sqrt{|k|}\frac{c}{H_0}\ln(1+z)\right].$$

Problem 1.7.4

Use appropriate Taylor expansions to show, once again, that to first order in z the result in 1.7.3 reduces to $cz = H_0 d_{\text{lum}}$.

Problem 1.7.5

Make a qualitative sketch, on the same graph, of d_{lum} vs. z for the universe model in problem 1.7.1 and for the universe model in problem 1.7.3. Assume the same value of H_0 for each. At low z the two curves should be coincident. I just want to see, from your drawings, which one starts to have d_{lum} grow more rapidly with z once z gets big enough that the Taylor series approximations break down. It would be sufficient to look at behavior as $z \to \infty$. To do so, you will want to use $\sinh(x) = (e^x - e^{-x})/2 \to e^x/2$ for large x. Be sure to label your curves.