1.13: Energy and Momentum Conservation

The lack of energy conservation in an expanding universe is quite surprising to people with any training in physics and therefore merits some discussion, which we present here in this chapter. The student could skip this chapter and proceed to 15 without serious harm. If, subsequently, the lack of energy conservation becomes too troubling, know that this chapter is here for you.

We begin by reminding the reader of the deep connection between symmetries of the action and conserved quantities. For example, if the action (time-integral of the Lagrangian) is invariant under time translations (and hence a symmetry of the action) then energy is conserved. Likewise, if spatial translations do not change the action, then momentum is conserved.

To review, let us consider a single particle moving in one dimension in a possibly space and time-dependent external potential with Lagrangian:

$$L(x, \dot{x}, t)=\frac{1}{2} m \dot{x}^{2}-V(x, t) \tag{1.13.1}$$

The action is given by integrating along a trajectory between two points fixed in space and time, points 1 and 2:

$$S=\int_{1}^{2} L(x, \dot{x}, t) d t \tag{1.13.2}$$

Although it is not obvious from the notation, it depends on an assumed $x(t)$. Invariance of the action under time translation means that if we send t to $t+\delta t$, the resulting change in S, δS, will be zero. It should be clear that this will be the case as long as L has no explicit time dependence; i.e., if $\partial L / \partial t=0$. Likewise, invariance of the action under spatial translation means that if we send x to $x+\delta x$, the resulting change in S, δS, will be zero. This will be the case if $\partial L / \partial x=0$.

From the Euler-Lagrange equations one can now see directly that space-translation invariance leads to momentum conservation. By definition, the momentum conjugate to x is $p=\partial L / \partial \dot{x}$. For our particular Lagrangian, that gives $p=m \dot{x}$ as expected. The Euler Lagrange equations are:

$$\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x} \tag{1.13.3}$$

which for our Lagrangian becomes:

$$\frac{d}{d t} p=-\frac{\partial V(x, t)}{\partial x} \tag{1.13.4}$$

So we see that p does not change with time if the potential has no dependence on x. Of course! If the potential has no dependence on x then it is not applying any force to the particle.

Seeing the consequences of time-translation invariance takes a little bit more work. However, from the Euler-Lagrange equations one can derive:

$$\frac{d}{d t}(\dot{x} \partial L / \partial \dot{x}-L)=-\partial L / \partial t \tag{1.13.5}$$

so we see that the quantity after $d / d t$ on the left-hand side is conserved if L has no explicit time dependence. As long as the kinetic energy is quadratic in the velocity, and the potential does not depend on the velocity, then that quantity is equal to the total energy. Thus, given these conditions, energy conservation follows from time translation invariance.

For our Lagrangian the above equation becomes:

$$\frac{d}{d t}\left(\frac{1}{2} m \dot{x}^{2}+V(x, t)\right)=\partial V(x, t) / \partial t \tag{1.13.6}$$

Thus we see total energy is conserved if the potential energy function does not have a dependence on t. Note that this is not a statement that the potential energy of the particle cannot change with time (which would be $d V / d t=0$) but is instead a statement about the functional form of $V(x, t)$, that it cannot have an explicit dependence on time. For example, $V(x)=1 / 2 k x^{2}$, has no explicit time dependence ($\partial V / \partial t=0$) even though x will change with time (so $d V / d t=k x \dot{x} \neq 0$), so for this potential, energy is conserved. However, if the spring constant were to change with time so $V(x)=1 / 2 k(t) x^{2}$ then energy would not be conserved.

Imagine the behavior of a mass attached to a spring with time-dependent spring constant. Can you see how energy would not be conserved?

Let's now consider how this works in an expanding universe. Consider a free non-relativistic particle of mass m. If we adopt the point of view that the comoving grid everywhere specifies local rest, then we will naturally define kinetic energy based on departures from local rest.This leads us to write the velocity of the particle in terms of $\dot{x}$ so that the velocity is $a\dot{x}$ and the Lagrangian is $L = 1/2ma^2\dot{x}^2$.

One might be tempted to use the physical coordinates $y = ax$ and write $v = \dot{y}$ and $L = 1/2m\dot{y}^2$ and therefore $m\dot{y}$ is conserved. However, this Lagrangian would give kinetic energy to a particle that is stationary with respect to the local rest frame, as long as it is not at the origin. Further, in a universe with \(\ddot a \ne 0\) we run into the same problem as discussed in chapter 12 that the coordinate system is locally inertial near the origin but not globally inertial. The comoving coordinate system, in contrast, is globally inertial.

Let's therefore return to $L = 1/2ma^2\dot{x}^2$ as our free-particle Lagrangian. Note that for this Lagrangian there is no explicit dependence on x so the momentum conjugate to x will be conserved. That momentum is $p = \partial L/\partial\dot{x} = ma^2\dot{x}$ $= amv$ which is a times the usual momentum. Since amv is conserved, we find $mv \propto 1/a$.

Thus, in an expanding homogeneous universe, spatial homogeneity; i.e., translational invariance, leads to a conserved quantity which is the usual momentum divided by the scale factor. Thus we see once again that free particles in an expanding universe have a momentum that decreases as $1/a$.

What about energy, the conserved quantity associated with time translation invariance? We do not have time translation invariance because of the expansion! Consequently, energy is not conserved as one can see explicitly for our free particle from $dE/dt = -\partial L/\partial t = -ma\dot{a}\dot{x}^2 \neq 0$. Of course this result is consistent with our finding that the momentum of the free particle decreases as the universe expands.

I wrote at the beginning of this chapter that the lack of energy conservation in the expanding universe comes as a surprise to many. I ran up against this when I wrote an article for Physics Today about cosmic inflation. In an initial draft I wrote, about the patch that eventually becomes our observable universe, "The total mass-energy in this patch was about 10^4 Joules, the caloric content of two diet Cokes." This line, which I loved, was then removed by the editor! I was very disappointed! The editor explained he was skeptical it was true and was afraid the readers would be as well. I tried to address the skepticism with this suggested replacement:

"The total mass-energy in this patch was about 10^4 Joules, the caloric content of two diet Cokes. (Surprising to most physicists is that there is no global conservation of energy in general relativity, a fact that allows for the existence today of regular Coke, too---as well as the other $\sim 10^{70}$ Joules of mass-energy in the currently observable universe.)"

which had the added benefit of further play on the Coke theme! Fortunately, the editor was open minded and we settled on this for the final published version:

"The total mass-energy in that patch was about 10^4 Joules, the caloric content of two diet Cokes. Today's universe includes regular Cokes and about 10^{70} joules of mass energy from other sources, a result compatible with the perhaps surprising fact that energy is not conserved in general relativity."

I also ran up against it a previous time I taught this class, when students pointed out to me that the textbook we were using for the course explicitly claimed energy was conserved. I pointed it out to the author, who quickly agreed with me it was an error. He was in fact quite unhappy he was finding this out just a little too late to keep the error from propagating into the next edition of his textbook.