1.20: Equilibrium Particle Abundances

At sufficiently high temperatures and densities, reactions that create and destroy particles can become sufficiently rapid that an equilibrium abundance is achieved. In this chapter we assume that such reaction rates are sufficiently high and work out the resulting abundances as a function of the key controlling parameter $\frac{mc^2}{k_BT}$. We will thus see how equilibrium abundance changes as the universe expands and cools. We will do so for the specific case of a fermionic particle (χ) and its anti-particle ($\bar{\chi}$) with non-zero mass (rest mass) m, and $g=2$, but the generalization to zero rest mass, bosons, and/or arbitrary g is trivial.

We make a few additional assumptions:

1. $n_\chi = n_{\bar{\chi}}$ initially (perhaps because these number densities are zero initially),
2. Any production or destruction of χ includes a production or destruction (respectively) of $\bar{\chi}$.
3. The reactions $\chi + \bar{\chi} \leftrightharpoons 2\gamma$ are fast.
4. Reactions that create and destroy photons are fast, such as $e^- + p^+ \rightarrow e^- + p^+ + \gamma$.

Assumption (4) allows us to determine that the photons have zero chemical potential; i.e., $\mu_\gamma = 0$.

Assumption (3) provides us with a constraint on μ_χ and $\mu_{\bar{\chi}}$: $\mu_\chi + \mu_{\bar{\chi}} = 2\mu_\gamma = 0$.

Assumption (2) ensures that the initial equality in Assumption (1) persists over time, providing us with another constraint on the chemical potentials. We can find this constraint with the following argument. The number density of a particle in kinetic equilibrium is determined entirely by its number of internal degrees of freedom, g, its mass, the temperature, and the chemical potential. The particle and antiparticle are able to exchange kinetic energy (with themselves, as well as other particles) and so share the same temperature. They also have the same mass and number of internal degrees of freedom. Therefore, the only thing left that affects the number density that they conceivably do not have in common is their chemical potentials. Since their number densities are equal, their chemical potentials must be equal.

So we simultaneously have $\mu_\chi = \mu_{\bar{\chi}}$ and $\mu_\chi + \mu_{\bar{\chi}} = 0$. The only solution is $\mu_\chi = \mu_{\bar{\chi}} = 0$.

We now have all the parameters of the phase space distribution function pinned down except for the temperature, T, so we are now ready to calculate the number density as a function of T. We have for fermions with zero chemical potential:

$$n_\chi = n_{\bar{\chi}} = \frac{g}{h^3}\int d^3p\left[\exp\left(\frac{E(p)}{k_BT}\right)+1\right]^{-1} \tag{1.20.1}$$

which we will now examine in the relativistic and then non-relativistic limits.

Relativistic Limit

Assuming $E(p) = pc$ we find

$$n_\chi = \frac{4\pi g}{h^3}\left(\frac{k_BT}{c}\right)^3\int_0^\infty x^2dx\left[e^x+1\right]^{-1}. \tag{1.20.2}$$

The integral can be numerically evaluated, or looked up in an integral table, with the result that it is $\frac{3}{2}\times\zeta(3)\simeq\frac{3}{2}\times 1.202$, where ζ is the Riemann zeta function and $\zeta(3) = \Sigma_1^\infty \frac{1}{n^3}$. We thus find

$$n_\chi = 6\pi\zeta(3)g\left(\frac{k_BT}{hc}\right)^3. \tag{1.20.3}$$

Box 1.20.1

Exercise 20.1.1: Derive

$$n_\chi = \frac{4\pi g}{h^3}\left(\frac{k_BT}{c}\right)^3\int_0^\infty x^2dx\left[e^x+1\right]^{-1} \tag{1.20.4}$$

from

$$n_\chi = \frac{g}{h^3}\int d^3p\left[\exp\left(\frac{E(p)}{k_BT}\right)+1\right]^{-1} \tag{1.20.5}$$

and $E(p) = pc$. Use the transformation to spherical momentum coordinates to rewrite $\int d^3p$ as $4\pi \int dpp^2$ and then transform the integration variable via $x = pc/(k_BT)$.

It is often helpful to look at the comoving number density $\equiv a^3 n$ because this quantity will be fixed as expansion occurs unless there is net creation or destruction of particles. Examining the comoving number density allows us to highlight changes that are due to effects other than simple dilution due to increased volume. Assuming $T \propto 1/a$ we find the comoving number density is *independent of temperature*, since T^3a^3 is independent of temperature. Even though particles are rapidly being created and destroyed, the net result is that the number density has the same dependence on the scale factor, $n_\chi \propto a^{-3}$ that would be the case if there were no creation or destruction.

Non-relativistic Limit

In the non-relativisitic limit the kinetic contribution to the square of the energy p^2c^2 is much less than the rest-mass contribution to the square of the energy m^2c^4. So we have

$$E = \sqrt{m^2c^4 + p^2c^2} = mc^2\sqrt{1 + p^2/(m^2c^2)} \simeq mc^2(1 + p^2/(2m^2c^2)) = mc^2 + p^2/(2m) \tag{1.20.6}$$

and therefore

$$n_\chi = \frac{4\pi g}{h^3}\int_0^\infty p^2dp\left[\exp[\frac{mc^2 + p^2/(2m)}{k_BT}] + 1\right]^{-1}. \tag{1.20.7}$$

In the non-relativistic regime $mc^2 >> k_BT$ (since in the non-relativistic regime k_BT is a typical particle kinetic energy), so we can neglect the +1 in the phase-space distribution function, which allows us to pull exp$[mc^2/(k_BT)]$ out of the integral, and make the variable substitution $x = p/\sqrt{2mk_BT}$ so that

$$n_\chi = \frac{4\pi g}{h^3}\exp\left(\frac{-mc^2}{k_BT}\right)(2mk_BT)^{3/2}\int_0^\infty x^2dxe^{-x^2}. \tag{1.20.8}$$

The integral is $\sqrt{\pi}/4$. Using that and with some rearranging we get

$$n_\chi = \frac{(2\pi)^{3/2}g}{h^3c^3}(k_BT)^3\exp\left(\frac{-mc^2}{k_BT}\right)\left(\frac{mc^2}{k_BT}\right)^{3/2}. \tag{1.20.9}$$

Evaluation of the comoving number density brings in a factor of a^3 that cancels out the first T-dependent factor and we get:

$$a^3n_\chi \propto \exp\left(\frac{-mc^2}{k_BT}\right)\left(\frac{mc^2}{k_BT}\right)^{3/2}. \tag{1.20.10}$$

We thus see that the abundance of the particles and antiparticles (recall $n_\chi = n_{\bar{\chi}}$) is controlled by $mc^2/(k_BT)$ and is exponentially suppressed when this quantity is much greater than 1.

Box 1.20.2

Exercise 20.2.1: Make a log-log sketch of $mc^2/(k_BT)$ vs. a^3n_χ assuming the transition between the two regimes (relativistic and non-relativistic) is as smooth as possible. Explicitly identify $mc^2/(k_BT) = 1$ on the x-axis and the relativistic and non-relativistic regimes. Indicate which direction along the x-axis (left or right) corresponds to increasing time and scale factor.