1.15: Distance and Magnitude

Distances

We have the invariant distance equation for a homogeneous and isotropic universe (an FRW spacetime):

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right]. \tag{1.15.1}$$

Here we introduce several distance definitions, and how they are related to the coordinate system that leads to the above invariant distance expresson.

Luminosity distance: By definition of luminosity distance d_L,

$$F = \frac{L}{4\pi d_L^2} \tag{1.15.2}$$

which is the relationship we expect in a Euclidean geometry with no expansion, assuming an isotropic emitter. We also calculated the relationship between flux and luminosity in an FRW spacetime and found

$$F = \frac{L}{4\pi r^2(1+z)^2} \tag{1.15.3}$$

so we conclude that in an FRW spacetime, $d_L = r(1+z)$.

Angular diameter distance: By definition of angular-diameter distance, d_A,

$$\ell = \theta d_A \tag{1.15.4}$$

where θ is the angle subtended by an arc of a circle with length ℓ, as it would be measured with measuring tape. By the angle subtended, we mean the angle between two light rays, one coming from one end of the arc, and the other from the other end of the arc. If we place ourselves in the center of the coordinate system we can work out what this means in terms of coordinates. Place the observer at the spatial origin $r=0$ and at time equals today. Place one end of the arc at $r=d, \theta=0, \phi=0$ and the other at $r=d, \theta=\alpha,\ \phi=0$. Light will travel from both of these points to the origin along purely radial paths; i.e., with no change in θ or ϕ. So the angle they subtend upon arrival is α. We can use the invariant distance expression to work out that $\ell = a\alpha d$ where a is the scale factor at the time the light we are receiving today is emitted from the object. Thus $d_A = \ell/\theta = ad\alpha/\alpha = ad$ where d is the radial coordinate separation between the object and the observer.

Comoving angular diameter distance: This is simply the angular diameter distance divided by the scale factor. We will reserve D_A for comoving angular diameter distance. The comoving angular diameter distance between $r=0$ and $r=d$ is $D_A = d$.

Box 1.15.1

Exercise 15.1.1: In an FRW spacetime, how are D_A and d_L related?

Box 1.15.2

Exercise 15.2.1: What is the comoving distance, ℓ, from the origin to some point with radial coordinate value r, along a path of constant θ and ϕ?

Curvature integrals: Although we've made use of a first order Taylor expansion to analytically solve the above integral, the exact integral does have an analytic solution. For $k>0$, $\ell = (1/\sqrt{k})\sin^{-1}(\sqrt{k}r)$. For $k<0$, $\ell = (1/\sqrt{-k})\sinh^{-1}(\sqrt{-k}r)$.

To work out how the comoving angular diameter distance D_A is related to the scale factor at the time light was emitted, a, we look at how light travels from coordinate value r to the origin. Light has $ds^2=0$, and from that we get

$$\int_0^r \frac{dr}{\sqrt{1-kr^2}} = \int_t^{t_0} cdt/a = c\int_a^1 da/(a^2H),$$

$$(1/\sqrt{-k})\sinh^{-1}(\sqrt{-k}r) = c\int_a^1 da/(a^2H)$$

$$r = (1/\sqrt{-k})\sinh\left(\sqrt{-k}c\int_a^1 da/(a^2H)\right) \tag{1.15.5}$$

$$D_A = (1/\sqrt{-k})\sinh\left(\sqrt{-k}c\int_a^1 da/(a^2H)\right)$$

where, except for the first line, we have assumed $k<0$. I leave it to the student to work out the $k>0$ case. The $k=0$ case should also be clear.

Box 1.15.3

Exercise 15.3.1: In calculating D_A vs. a, what are the two different ways curvature makes a difference?

Box 1.15.4

We have defined the density parameters $\Omega_x = \rho_{x,0}/\rho_{c,0}$ where $\rho_c \equiv 3H^2/(8\pi G)$ is the critical density, defined to be the total density for which the curvature, k is zero.

Exercise 15.4.1: Using the Friedmann equation, convince yourself that if $\rho=\rho_c$, then $k=0$.

With this notation we can write

$$H^2(a) = H_0^2\left(\Omega_\Lambda + \Omega_m a^{-3} + \Omega_K a^{-2}\right) \tag{1.15.6}$$

where $\Omega_K \equiv -k/(H_0^2)$.

Exercise 15.4.2: Show that Eq. 1.15.6 can be derived from the Friedmann equation and the fact that $\rho_\Lambda \propto a^0$ and $\rho_m \propto a^{-3}$.

Exercise 15.4.3: Further, show that $\Omega_\Lambda + \Omega_m + \Omega_K = 1$.

Apparent and Absolute Magnitudes and the Distance Modulus

Magnitudes are absurd but useful if you want to use data from astronomers. They are a means of expressing luminosity and flux.

Luminosity: The luminosity of an object, L, is its power output. Usually its total electromagnetic power output, sometimes referred to as *bolometric* luminosity. Typical units for luminosity are ergs/sec (10^7 erg = 1 Joule, 1 Watt = 1 Joule/sec) or solar luminosity, L_{Sun}. The Sun, by definition has a luminosity of one solar luminosity and $L_{\text{Sun}} = 3.826\times 10^{33}$ erg/sec = more than 10^{24} 100 Watt light bulbs. (You can remember this if you remember it's about as luminous as 7 Avogadro's number of 100 Watt light bulbs).

Flux: The flux, F, from an object is not an intrinsic property of the object, but also depends on the distance to the object. It is the amount of energy passing through a unit area, per unit of time. For an isotropic emitter in a non-expanding, Euclidean three-dimensional space, $F = L/(4\pi d^2)$ where d is the distance between source and observer. This equation just follows from energy conservation; note that the total power flowing through a spherical shell of radius d completely surrounding the emitter at its center is $4\pi d^2 \times F = L$.

Spectral Flux density: We usually do not measure the total flux from an object, but instead measure the flux in a manner that depends on how the flux is spread out in frequency. Thus a useful concept is the spectral flux density, S, that quantifies how much flux there is per unit frequency. The units of spectral flux density are erg/s/m^2/Hz. Where Hz is the unit of frequency called Hertz, equal to 1/s.

Apparent magnitude: Astronomers often use apparent magnitude, m, instead of flux. The apparent magnitude has a logarithmic dependence on flux; the reason for this is historical, and is fundamentally due to the logarithmic sensitivity of our eyes to flux. Not only is it logarithmic instead of linear, but *brighter* objects have *smaller* magnitudes. This is because the Greeks defined the brightest stars as stars of the first magnitude, and next brightest as stars of the 2nd magnitude, down to the stars we could just barely see at all, which are stars of the 6th magnitude. This ancient system, updated with precise definitions related to flux is still in use today (otherwise I would not bother telling you about it). One way of relating apparent magnitude to flux is the following:

$$m=M_{\text{Sun}}-2.5\log_{10}\left(\frac{F}{F_{\text{Sun10}}}\right) \tag{1.15.7}$$

where $M_{\text{Sun}}=4.76$ is the absolute magnitude of the Sun (see next definition) and F_{Sun10} is the flux we would get from the Sun if it were 10pc away. Since $L_{\text{Sun}}=3.826\times 10^{33}$ erg/sec and 1pc = 3.0856×10^{18} cm we get $F_{\text{Sun10}}=3.198\times 10^{-7}$ erg/cm^2/sec.

Note that because of the -2.5 factor in front of the $\log_{10}$, if the flux increases by a factor of 10, the apparent magnitude decreases by -2.5. Conversely, if the magnitude increases by 1, the flux decreases by a factor of $10^{1/2.5}=10^{0.4}$.

Absolute Magnitude: The absolute magnitude of an object, denoted by M, is another way of expressing its luminosity. One way of defining it is via:

$$M=M_{\text{Sun}}-2.5\log_{10}\left(\frac{L}{L_{\text{Sun}}}\right). \tag{1.15.8}$$

Putting this together with the apparent magnitude-flux relationship above, one can show that this means $M=m$ for an object at 10pc.

Distance Modulus: The distance modulus is defined as $\mu\equiv m-M$. Note that as a difference between apparent and absolute magnitudes, this is equal to a log of the ratio of flux and luminosity. By plugging in the definitions above of m and M one finds

$$\mu=5\log_{10}\left(\frac{d_L}{10\text{pc}}\right)=5\log_{10}\left(\frac{d_L}{1\text{pc}}\right)-5 \tag{1.15.9}$$
