1.4: Spatially Homogeneous and Isotropic Spacetimes

The Cosmological Principle

The Einstein field equations are extremely difficult to solve in generality. The first attempts at solving these equations for the universe as a whole thus involved extreme idealization. In the immediate years after Einstein presented his theory of general realtivity, several people used what you might call the most spherical cow approximation of all time: they approximated the whole universe as completely homogeneous; i.e., absolutely the same everywhere. The 'cosmological principle' is simply the assertion that the universe is homogeneous (invariant under translations) and isotropic (invariant under rotation). Model spacetimes with this high degree of symmetry are still of interest to us today because, as discussed in the Overview, on large scales and at early times, the universe is in fact very close to being homogeneous and isotropic.

Three-dimensional Homogeneous and Isotropic Spaces

So far we have worked with spacetimes with just one spatial dimension. But to get to Hubble's law we need to know how to measure distances. And for our particular method of measuring distances, we are going to need to work with more than one spatial dimension, as we will explain later. So the time has come to think about additional spatial dimensions. There appear to be three spatial dimensions, so let's start there.

In Minkowski space, the square of the invariant distance, ds^2, between spacetime point (t, x, y, z) and another one at $(t+dt, x+dx, y+dy, z+dz)$ is given by:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{1.4.1}$$

In spherical coordinates the above expression for the invariant distance becomes:

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.4.2}$$

The above is a special case valid for static (non-expanding) spacetimes with Euclidean spatial geometries. The invariant distance for *any* spatially homogeneous and isotropic Universe can be written as:

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{1.4.3}$$

Such spacetimes are known as Friedmann-Robertson-Walker (FRW) models, or sometimes just Robertson-Walker, and sometimes Friedmann-Robertson-Walker-Lemaitre models.

Derivation

We can construct the homogeneous and isotropic three-dimensional space and derive its invariant distance rule, at least for the case of $k > 0$, by embedding it in a 4-dimensional Euclidean space. In a 4-dimensional Euclidean space we can have a coordinate system consisting of three dimensions x, y, z, that are all orthogonal to each other, and a fourth we will call w that is orthogonal to each of the x, y, and z directions. Impossible as this is to visualize, we can describe it mathematically. The distance between w, x, y, z and $w+dw, x+dx, y+dy, z+dz$ is given by

$$d\ell^2 = dw^2 + dx^2 + dy^2 + dz^2. \tag{1.4.4}$$

In this 4-dimensional space, we construct a three-dimensional subspace that is the set of points all the same distance, R, from a common center. Let's center it on the origin so our subspace satisfies this constraint:

$$w^2 + x^2 + y^2 + z^2 = R^2. \tag{1.4.5}$$

This subspace is homogeneous (all points are the same) and isotropic (all directions are the same). You can see that this is true by imagining it's two-dimensional analog, a sphere, which is the set of all points satisfying $x^2 + y^2 + z^2 = R^2$.

It will be helpful at this point to swap out the Cartesian x, y, z for the spherical coordinate system r, θ, ϕ so we have

$$d\ell^2 = dw^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.4.6}$$

and our constraint equation can be written as

$$w^2 + r^2 = R^2. \tag{1.4.7}$$

From this new version of the constraint equation, we can see that if r changes by some amount then we will necessarily have to have a change in w in order to continue to satisfy the constraint. The exact relationship between differential changes you can easily work out to be $2wdw + 2rdr = 0$ (because changing r by dr ends up changing r^2 by $2rdr$ and likewise for w and dw and since R is fixed $dR^2 = 0$). Using this relationship to eliminate dw^2 from our invariant distance expression, and using the constraint equation to eliminate w^2 in favor of r^2 and R^2 we get

$$d\ell^2 = \frac{dr^2}{1 - r^2/R^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.4.8}$$

We see that our subspace has an invariant distance expression of the form we were intending to derive, and it is exactly the one introduced above if we make the identification $R^2 = 1/k$.

HOMEWORK Problems

Problem 1.4.1

Re-do the above derivation leading to $d\ell^2 = dw^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ but for a 2-dimensional space instead of a 3-dimensional space. Instead of spherical coordinates (r, θ, ϕ), use cylindrical coordinates (r, ϕ).
