

## 2.1: Non-Euclidean Geometry

We said previously that the metric defines the rules of geometry. In 3D Euclidean space, those rules include:

1. the length of the hypotenuse of a right triangle is determined by the Pythagorean theorem
2. the three angles of a triangle add up to 180 degrees
3. parallel lines never cross.

How could those rules could ever *not* apply? Let's consider an *operational definition* for a straight line given two thumbtacks and a piece of string. (An operational definition is a definition that arises from specifying a procedure.) One way to create a straight line between two points with these materials is to stretch the string between the points such that it is taut, then place a thumbtack at each of those points.

### ? Exercise 2.1.1

Imagine applying the operational definition of a straight line to a globe. Do the three rules of geometry for Euclidean space listed above apply?

#### Answer

Suppose we apply the straight line procedure to the North Pole and to a point on the equator with 0 degrees longitude; a string connecting those two points would be curved.

Now pick a third point, say 50 degrees longitude on the equator, and create two more straight lines using the first two points to create a triangle.

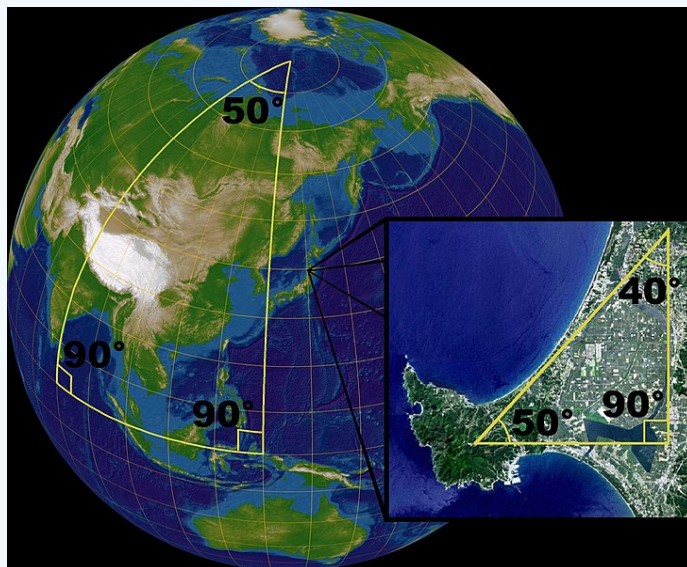


Figure 2.1.1: Three straight lines on the surface of a globe. On a small enough portion of the globe, the rules of geometry are Euclidean. (Image credit: [Wikipedia](#))

Since both lines connecting the equator to the North Pole are perpendicular to the equator, they are parallel. But they also intersect, so it is not true that parallel lines never cross.

Lastly, the angles in the triangle add up to 230 degrees, not 180 degrees. It also shouldn't be too surprising to you that the Pythagorean theorem doesn't not apply to such a triangle.

Note that, on a small enough patch of the earth, the rules of Euclidean geometry *do* apply, as shown in the inset of Figure 2.1.1. In principle, then, we can think of a large, curved patch of earth as consisting of many small patches (where Euclidean geometry applies) stitched together.

A space in which the rules of Euclidean space don't apply is called non-Euclidean. The reason for bringing this up is because our modern understanding of gravity is that particles subject to gravity exhibit curved motion not because there is a force acting on

them but because spacetime is non-Euclidean. Let's use some thought experiments to see why.

### ? Exercise 2.1.2

Can you tell whether this student is in orbit, in a free-falling vessel, or is in deep space far from any source of gravity? Assuming there are no windows, what experiments could the student do to find out? Does the answer depend on the size of the vessel or how long the experiment lasts for?

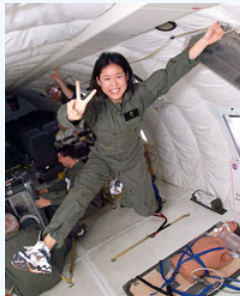
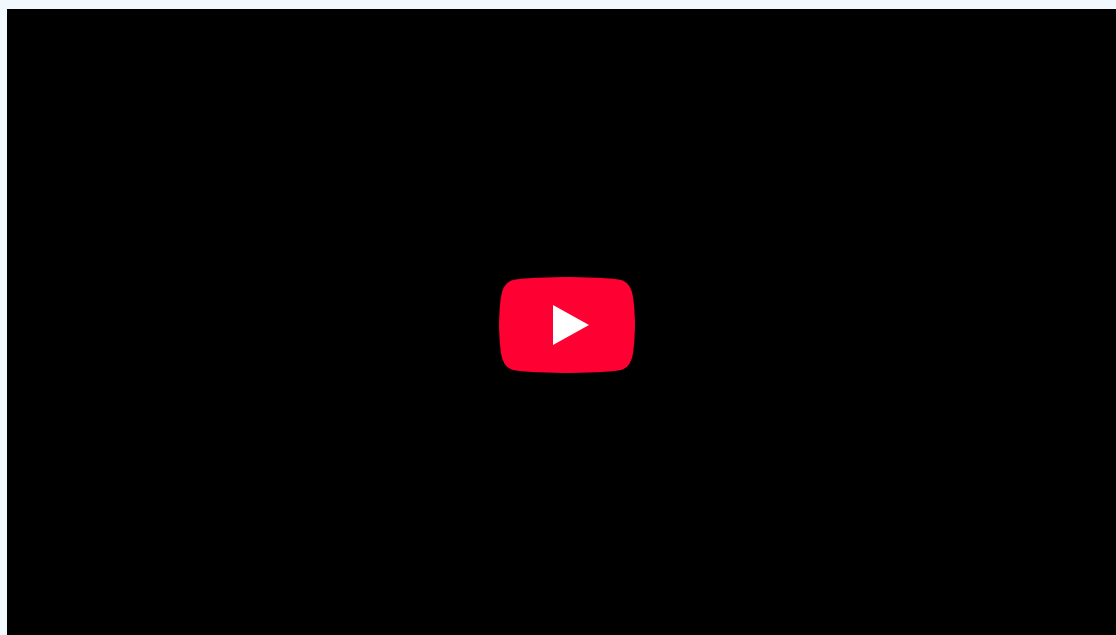


Figure 2.1.2: A student floats inside of a vessel. (Image credit: [nasa.gov](https://www.nasa.gov))

### Answer

We know that people float inside of vessels in the absence of gravity. You may also know that the same is true for people in the International Space Station, which orbits the earth. In the first case, the people are not subject to gravity at all. In the second case, the people are *only* subject to gravity. Furthermore, if you were in a freely falling elevator, you would no longer be pulled to the floor and would feel weightless, which is another situation in which you are *only* subject to gravity. The video below shows this effect using a pendulum inside a sealed box that drops 22 m. Without gravity, you would expect a pendulum to swing in a full circle since there is no restoring force, and that is exactly what happens when the box is dropped.





There *are* experiments you can do to differentiate between the three situations (orbit, free fall, and deep space). Consider releasing two balls from rest, one above the other. Since gravity is stronger the closer you are to earth, the distance between the two balls will increase with time (if you wait long enough or if the initial height difference is large enough) in the free-fall situation but not the other two. Alternatively, if you release two balls from rest from the same height but separated horizontally, the distance between the two balls will decrease with time (if you wait long enough or the initial horizontal separation is large enough) only in the free-fall situation. This is because the balls fall radially toward the center of the earth, which makes their paths converge.

The previous experiments can distinguish between free-fall and either orbit or deep space but can't distinguish between orbit and deep space. In the free-fall situation, if the vessel is small *and* the time the experiment is carried out for is small, then there is no way to distinguish between orbit, free-fall, and deep space.

### ? Exercise 2.1.3

Recall that an **inertial reference frame** is one in which Newton's First Law applies, i.e. an object initially at rest stays at rest and an object in motion maintains its velocity as long as the net force acting on it is zero. By definition, a reference frame in deep space is an inertial reference frame. Now consider a reference frame attached to a person floating inside the International Space Station.

- Is the reference frame inertial?
- Is a reference frame inside of a freely falling elevator inertial?

#### Answer

In the previous exercise, we determined that there are no experiments that could distinguish between being in orbit (only subject to gravity) and being in deep space (but subject to no gravity). That means that a reference frame attached to a person floating inside the International Space Station, just like a reference frame attached to a particle that is subject to no forces, must be inertial.

The freely falling elevator is more subtle. The previous exercise showed that large separation distances and/or large time intervals allow us to perform experiments to distinguish the freely falling frame from the orbiting or deep space frames. If we limit ourselves to small separations and small time intervals, though, then we lose the ability to distinguish. So one single reference frame inside of a freely falling elevator is not inertial over large distances or large time intervals, but you can create a *series* of inertial reference frames and then stitch them together to cover a large distance and/or time interval. Within each inertial reference frame there is no evidence of gravity (i.e. objects released from rest don't accelerate).

You may be used to thinking of inertial reference frames as reference frames with no net force (and hence no acceleration), yet here we see that an orbiting reference frame and a freely falling reference frame can be inertial. So perhaps gravity isn't a force at all. And if that's the case, then a reference frame attached to a person standing still on the earth's surface is *not* inertial because the net force on that person would *not* be zero.

### ? Exercise 2.1.4

Consider two particles in an inertial reference frame that are initially moving in the same direction and parallel to one another in deep space. Neglect the gravitational effects that they have on each other (that is, the mutual attraction). Describe the paths that the particles take.

Now consider two particles that are released from rest from the same height above the earth's surface. Neglect the gravitational effects that they have on each other but not the gravitational effect from the earth. Describe the paths that the particles take.

#### Answer

In the deep space example, the particles travel in straight lines, and the distance between them doesn't change.

In the free fall example, the particles will move slightly closer together because they are both falling radially toward the center of the earth. In fact, if the surface of the earth didn't get in the way, eventually the particles' paths would intersect.

Initially parallel lines that intersect may sound familiar; that was one of the rules of geometry that we discovered in the non-Euclidean space on the surface of a sphere!

The takeaway from the previous exercises is that gravity can be thought of not as a force, but rather as a changing of the rules of geometry. Albert Einstein's great contribution was in formulating the theory, called **The General Theory of Relativity** (General Relativity, or GR, for short), that relates mass to the rules of geometry. The equations relating mass to geometry are called the **Einstein Field Equations** (EFE for short). In natural units, they are written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (2.1.1)$$

where  $G_{\mu\nu}$  is a complicated quantity involving the *generalized* metric  $g_{\mu\nu}$  ( $\eta_{\mu\nu}$  is the symbol specifically reserved for Euclidean metrics),  $\Lambda$  is a constant called the **cosmological constant**, and  $T_{\mu\nu}$  is a quantity that represents the matter and energy distribution in the space. In short, the metric that defines the rules of geometry appears on the left hand side of the equation while the matter and energy content appears on the right hand side of the equation.

In practice, you input a matter and/or energy distribution on the right side and you solve for the metric. That, along with other things we will introduce later, allows you to determine the worldlines of particles. The problem is that solving the Einstein Field Equations is *extremely* difficult because they are a set of ten coupled, non-linear, second order, partial differential equations. For that reason, in this book we will typically just tell you what the metric corresponding to a particular matter/energy distribution is and focus on what we can learn from there.

There are ways to make it easier to solve the EFE. Specifically, you can assume very simple matter distributions. In this and following chapters we will look at some of these special cases:

Assumption	EFE result
spacetime is empty	Special Relativity
there is a single point mass	Schwarzschild geometry (i.e. black hole)
there is a single rotating point mass	Kerr geometry
a faraway mass oscillates	gravitational waves
spacetime is filled with a uniform fluid	cosmology

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