2.6: Global vs. Local Coordinates

In the previous section we saw that the slope of a light cone on an r-t slice is given by

$$\frac{dt}{dr} = \pm \frac{1}{1 - \frac{2M}{r}}.$$

The reciprocal, $\frac{dr}{dt}$, should describe the speed of light that is traveling radially. There appears to be a problem, though, since this describes a speed of light that is r-dependent, which violates one of the premises of relativity. It turns out that there really is no problem, however, since we saw in the previous sections that the t- and r-coordinates don't mean what we think they mean. We call these coordinates **global coordinates** because they are part of a coordinate system that embraces a large portion of curved spacetime. But global coordinates can't be trusted to tell us about measurable quantities.

Definition: Global Coordinate System

A global coordinate system embraces a large portion of spacetime that is curved in general. Global coordinate separations don't necessarily correspond to measurable quantities.

In many regions of curved spacetimes, it is possible to construct a **local inertial reference frame** (LIRF) where coordinate separations do correspond to measurable quantities and where we can use the rules of Special Relativity. By superimposing an LIRF on a portion of a global coordinate system, it is possible to translate between the two.

Definition: Local Inertial Reference Frame (LIRF)

A local inertial reference frame is a patch of curved spacetime that is small enough to be considered flat. The rules of Special Relativity can be used on an LIRF.

As an example of this coordinate translation, consider Fig. 2.6.1, which shows a locally flat Cartesian coordinate system superimposed on a Schwarzschild coordinate system.

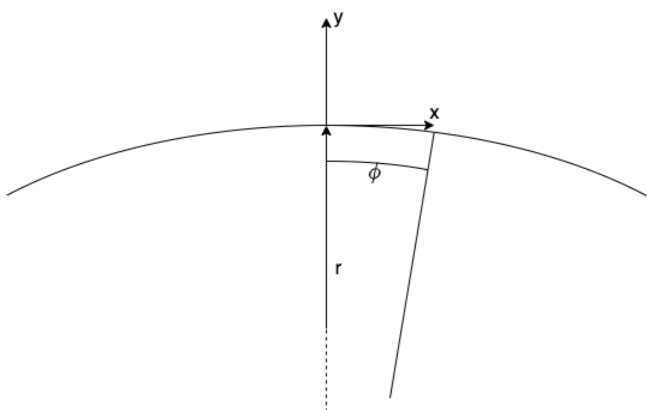

Figure 2.6.1: A locally flat Cartesian coordinate system is constructed on a curved portion of Schwarzschild spacetime. (Copyright; author via source)

For simplicity, let's ignore the θ-direction and place this Cartesian coordinate system on the equator ($\theta = 90^\circ$) such that the x-direction is east and the y-direction points away from the center. The r-direction matches up with the y-direction, and the ϕ-direction matches up with the x-direction. The Schwarzschild spacetime interval is

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1 - \frac{2M}{r}} dr^2 - r^2 d\phi^2$$

and the LIRF spacetime interval is

$$d\tau^2 = dt^2_{\text{LIRF}} - dx^2_{\text{LIRF}} - dy^2_{\text{LIRF}}.$$

By matching corresponding terms that represent the same direction, we get

$$\begin{aligned} dt_{\text{LIRF}} &= \sqrt{1-\frac{2M}{r}}\,dt \\ dx_{\text{LIRF}} &= r\,d\phi \\ dy_{\text{LIRF}} &= \frac{1}{\sqrt{1-\frac{2M}{r}}}dr \end{aligned}.$$

Now let's return to the question of the speed of light. For a beam of light traveling radially, the speed in the LIRF is $\frac{dy_{\text{LIRF}}}{dt_{\text{LIRF}}}$. We can use the local-to-global coordinate transformations along with the expression for $\frac{dr}{dt}$ in global coordinates to prove that the speed of light is 1 (see Box 2.6.1).

Box 2.6.1

Use the local-to-global coordinate transformations along with the expression for $\frac{dr}{dt}$ in global coordinates to prove that $c=\pm 1$.
