5.1: The Kerr Metric

The Schwarzschild metric assumes that the object at the center is completely stationary. Almost all spherical objects in space, however, spin. This is even true for black holes, which form when a star collapses. Since the progenitor star spins, by conservation of angular momentum the resulting black hole must spin as well. This spinning destroys the symmetry, which in turn means that we need a different metric to describe spinning, spherical objects. As with the Schwarzschild metric, though, we still need to make assumptions. They are:

1. the universe is empty except for a spherical mass
2. the mass itself is spherically symmetric, but it spins in the ϕ direction
3. the properties of the central mass do not depend on t.

In addition, we know that the resulting metric should reduce to the Schwarzschild metric when the spin is zero. If you put these assumptions and conditions into the Einstein Field Equations, you get

$$\begin{aligned} d\tau^2 &= \left(1-\frac{2Mr}{r^2+a^2\cos^2\theta}\right)dt^2 + \frac{4Mra\sin^2\theta}{r^2+a^2\cos^2\theta}dt\,d\phi - \left(\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2}\right)dr^2 \\ &\quad -\left(r^2+a^2\cos^2\theta\right)d\theta^2 - \left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{5.1.1}$$

where

$$a=\frac{J}{M} \tag{5.1.2}$$

is the angular momentum J per unit mass of the spinning object. The metric in Equation 5.1.1 is called the **Kerr metric**.

Definition: Kerr Metric

The Kerr metric is a metric for a spinning, spherical mass.

Note

Previously we used L to denote the angular momentum of a particle in an orbit. Here we use J to call out the fact that this is a different type of angular momentum. That is, it's the angular momentum of the spherical mass in the center.

Exercise 5.1.1

Verify that the Kerr metric reduces to the Schwarzschild metric when the central mass is not spinning.

Answer

If the central mass is not spinning, then $a=0$.

$$\begin{aligned} d\tau^2 &= \left(1-\frac{2Mr}{r^2}\right)dt^2 - \left(\frac{r^2}{r^2-2Mr}\right)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 && \text{set } a=0 \text{ in Eq. 5.1.1} \\ d\tau^2 &= \left(1-\frac{2M}{r}\right)dt^2 - \frac{r^2}{r^2\left(1-\frac{2M}{r}\right)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 && \text{simplify} \\ d\tau^2 &= \left(1-\frac{2M}{r}\right)dt^2 - \frac{1}{1-\frac{2M}{r}}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 && \text{simplify for final answer} \end{aligned}$$

Exercise 5.1.2

Estimate the value of a for Earth. Express your answer in terms of M. (Recall that $J=I\omega$, where $I=\frac{2}{5}MR^2$ for a uniform sphere.)

Answer

Substituting $I=\frac{2}{5}MR^2$, we get $a=\frac{J}{M}=\frac{2}{5}R^2\omega$. The angular speed of earth is 2π radians per day, which we can convert to radians per second. We can then use the speed of light as a conversion factor to partially cancel out the units from the R^2.

$$\omega=2\pi\frac{\text{rad}}{\text{day}}\times\frac{1\text{ day}}{24\text{ h}}\times\frac{1\text{ h}}{3600\text{ s}}\times\frac{1\text{ s}}{3\times 10^8\text{ m}}=2.42\times 10^{-13}\text{ m}^{-1}$$

Now we can substitute $R=6.37\times 10^6\text{ m}$ to find a.

$$a=\frac{2}{5}\left(6.37\times 10^6\text{ m}\right)^2\left(2.42\times 10^{-13}\text{ m}^{-1}\right)=3.93\text{ m}$$

In an earlier exercise, we calculated the mass of earth in meters at 0.0044 m, which means that

$$a=3.93\text{ m}\times\frac{M}{0.0044\text{ m}}=893M.$$

Exercise 5.1.3

Just like the Schwarzschild metric, the Kerr metric has problematic r-values for which some metric components are either zero or undefined. What are those r-values?

Answer

The coefficient of the dt^2 term is zero when $r^2+a^2\cos^2\theta=2Mr$. Let's solve this for r.

$$\begin{aligned}
r^2+a^2\cos^2\theta &= 2Mr \\
r^2-2Mr+a^2\cos^2\theta &= 0 && \text{set up as quadratic} \\
r &= \frac{2M\pm\sqrt{4M^2-4a^2\cos^2\theta}}{2} && \text{use quadratic equation} \\
r &= M\pm\sqrt{M^2-a^2\cos^2\theta} && \text{simplify}
\end{aligned}$$

Here we can see that r is only real-valued if $a^2\cos^2\theta\leq M^2$, which in turn implies that there is a maximum spin for the black hole (since imaginary metric components are a no-no).

Another place to check is $r^2-2Mr+a^2=0$. That is similar to the previous case except that it is missing the $\cos^2\theta$, so the result is $r=M\pm\sqrt{M^2-a^2}$. Here again we see that there is a limit on the spin.

We will talk more about these special r-values later. The big takeaway is that there are *two* horizons instead of one when the central mass spins.