5.2: Constants of Motion

When studying the Schwarzschild metric, we used the geodesic equation to determine constants of motion, which in turn gave us hints about how particles move. Let's repeat the process for the Kerr metric. As a reminder, the geodesic equation is

$$0=\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^{\nu}}{d\tau}\right)-\frac{1}{2}\frac{dg_{\alpha\beta}}{dx^{\mu}}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}.$$

Also, for the sake of simplicity, for the remainder of this section will we look only at motion in the equatorial plane ($\theta=90^{\circ}$).

You can show (see Box 5.2.1) that setting $\mu=t$ leads to

$$\frac{E_{\mathrm{Kerr}}}{m}=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}+\frac{2Ma}{r}\frac{d\phi}{d\tau} \tag{5.2.1}$$

and that setting $\mu=\phi$ (see Box 5.2.2) leads to

$$\frac{L_{\mathrm{Kerr}}}{m}=\left(r^2+a^2+\frac{2Ma^2}{r}\right)\frac{d\phi}{d\tau}-\frac{2Ma}{r}\frac{dt}{d\tau}. \tag{5.2.2}$$

Using these to determine the exact motion of particles is mathematically challenging. We can, however, use them to make some broad claims. The most interesting claim that I am going to make is that particles released from rest can end up moving *around* the central mass; this is called *frame dragging*. You will demonstrate the plausibility of this claim in the following exercises.

? Exercise 5.2.1

A rock is released from rest very far away from a spinning black hole. Argue that $\frac{E_{\mathrm{Kerr}}}{m}=1$ and that $\frac{L_{\mathrm{Kerr}}}{m}=0$.

Answer

Very far away, $\lim\limits_{r\to\infty}1-\frac{2M}{r}=1$, $\lim\limits_{r\to\infty}\frac{2Ma}{r}=0$, and $\lim\limits_{r\to\infty}\frac{dt}{d\tau}=1$. If the rock is released from rest, then $\frac{d\phi}{dt}=0$ at that moment. We then end up with

$$\frac{E_{\mathrm{Kerr}}}{m}=(1-0)\,(1)+0=1$$

and

$$\frac{L_{\mathrm{Kerr}}}{m}=\left(r^2+a^2+0\right)(0)-(0)(1)=0.$$

? Exercise 5.2.2

Eqs. 5.2.1 and 5.2.2 can be combined to yield

$$\frac{d\phi}{d\tau}=\frac{\frac{2Ma}{r}\frac{E_{\mathrm{Kerr}}}{m}-\left(1-\frac{2M}{r}\right)\frac{L_{\mathrm{Kerr}}}{m}}{\frac{4M^2a^a}{r}+\left(1-\frac{2M}{r}\right)\left(r^2+a^2+\frac{2Ma^2}{r}\right)}. \tag{5.2.3}$$

Argue that a rock released from rest from very far away from a spinning black hole will inevitably end up moving in the ϕ direction.

Answer

By substituting $\frac{E_{\mathrm{Kerr}}}{m}=1$ and $\frac{L_{\mathrm{Kerr}}}{m}=0$, we get

$$\frac{d\phi}{d\tau}=\frac{2Ma/r}{\frac{4M^2a^2}{r^2}+\left(1-\frac{2M}{r}\right)\left(r^2+a^2+\frac{2Ma^2}{r}\right)}.$$

As long as $r > 2M$, then the denominator will be positive and $\frac{d\phi}{d\tau}$ will have the same sign as a. In other words, the rock will always end up moving around the black hole in the same direction as the spin of the black hole.

Box 5.2.1

Set $\mu = 0$ in the geodesic equation and use it to show that $\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2Ma}{r}\frac{d\phi}{d\tau}$ is constant. (If you are wondering how we make the leap to calling it energy specifically, just set $a = 0$ to see that it reduces to the energy per unit mass for the Schwarzschild metric.)

Box 5.2.2

Set $\mu = \phi$ in the geodesic equation and use it to show that $\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\frac{d\phi}{d\tau} - \frac{2Ma}{r}\frac{dt}{d\tau}$ is a constant. (As before, simply set $a = 0$ to see how we justify calling this angular momentum per unit mass.)

Box 5.2.3

Equation 5.2.1 can be written as $\frac{E_{\text{Kerr}}}{m} = g_{tt}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau}$, and Equation 5.2.2 can be written as $\frac{L_{\text{Kerr}}}{m} = g_{t\phi}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau}$. Solve the latter for $\frac{dt}{d\tau}$, then substitute into the former. Solve for $\frac{d\phi}{d\tau}$ using algebra, then substitute the appropriate expressions for g_{tt} and $g_{t\phi}$ to prove Equation 5.2.3.
