7.5: The Friedmann Equation

Up until now, we haven't talked about how to determine the scale factor function $a(t)$. In 1922, Alexander Friedmann combined the FLRW metric with the Einstein Field equations and discovered a way to determine $a(t)$. We now call it the **Friedmann Equation**

$$H^2(t) = \frac{8\pi \rho_{\text{tot}}(t)}{3} - \frac{K}{a^2(t)} \tag{7.5.1}$$

where the **Hubble Parameter** $H(t)$ is defined as

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \tag{7.5.2}$$

and where ρ_{tot} is the mass-energy density of the universe. The constant K determines whether the geometry of the universe is spherical ($K > 0$), flat ($K = 0$), or saddle-like ($K < 0$).

Note

The "dot" over the $a(t)$ in Equation 7.5.2 is a shorthand notation for a derivative with respect to time.

By setting $K = 0$, we can determine an expression for the **critical density** $\rho_{\text{crit}}(t)$. A density greater than the critical density will produce a universe with a spherical geometry, and anything less will produce a universe with a saddle-like geometry.

Definition: Critical Density

The critical density is the density of the stuff in the universe that would produce a universe with flat geometry.

Setting $K = 0$ in Equation 7.5.1 in solving for $\rho(t)$ yields

$$\rho_{\text{crit}}(t) = \frac{3H^2(t)}{8\pi}. \tag{7.5.3}$$

As you can see, the critical density changes with time. The critical density *today* is written as

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi}. \tag{7.5.4}$$

Exercise 7.5.1

Use the conversion 1 pc = 3.1×10^{16} m and 1 kg = 7.42×10^{-28} m to determine the value of the critical density *today* in $\frac{\text{kg}}{\text{m}^3}$. How many hydrogen atoms per cubic meter is that equivalent to?

Answer

Let's start by converting the Hubble constant.

$$H_0 = 70 \frac{\text{km/s}}{\text{Mpc}} \times \frac{1000 \text{ m}}{1 \text{ km}} \times \frac{1 \text{ Mpc}}{10^6 \text{ pc}} \times \frac{1 \text{ pc}}{3.1 \times 10^{16} \text{ m}} = 2.26 \times 10^{-18} \frac{1}{\text{s}}$$

Then we square it.

$$H_0^2 = 5.1 \times 10^{-36} \frac{1}{\text{s}^2}$$

Now we need to figure out how to get this in units of $\frac{\text{kg}}{\text{m}^3}$. We can start by using the speed of light to convert seconds to meters.

$$H_0^2 = 5.1 \times 10^{-36} \frac{1}{\text{s}^2} \times \left(\frac{1 \text{ s}}{3 \times 10^8 \text{ m}} \right)^2 = 5.67 \times 10^{-53} \frac{1}{\text{m}^2}$$

Now we can write $\frac{1}{\mathrm{m}^2}$ as $\frac{\mathrm{m}}{\mathrm{m}^3}$ and convert the top to kilograms.

$$H_0^2 = 5.67 \times 10^{-53} \frac{\mathrm{m}}{\mathrm{m}^3} \times \frac{1 \text{ kg}}{7.42 \times 10^{-28} \text{ m}} = 7.6 \times 10^{-26} \frac{\mathrm{kg}}{\mathrm{m}^3}$$

Therefore the critical density is

$$\rho_{\mathrm{crit},0} = \frac{3}{8\pi}\left(7.6 \times 10^{-26} \frac{\mathrm{kg}}{\mathrm{m}^3}\right) = 9.1 \times 10^{-27} \frac{\mathrm{kg}}{\mathrm{m}^3}.$$

The mass of a hydrogen atom is about 1.67×10^{-27} kg, so the critical density is equivalent to a little over five hydrogen atoms per cubic meter of space. That is a very tiny number, especially considering the fact that the density of everything around us is *much* bigger than that. On the other hand, space is *really* big. As we will find out later, it appears as the universe as a whole has a density extremely close to the critical density. That goes to show you just how much empty space there is in the universe.