3.4: Local Inertial Reference Frames

The constants of motion E and L from a previous section are expressed in global coordinates, which means that they don't correspond to anything directly measurable. To turn them into something measurable, we have to use a global-to-local coordinate transformation. We did this back in Section 2.6. As a reminder the results were

$$\begin{aligned} dt_{\text{LIRF}} &= \sqrt{1-\frac{2M}{r}}\,dt \\ dx_{\text{LIRF}} &= r\,d\phi \\ dy_{\text{LIRF}} &= \frac{1}{\sqrt{1-\frac{2M}{r}}}dr \end{aligned}.$$

After applying the transformation, we get that the energy in the LIRF is

$$\frac{E_{\text{LIRF}}}{m} = \gamma = \frac{1}{\sqrt{1-\frac{2M}{r}}}\frac{E}{m}, \tag{3.4.1}$$

where $\gamma = \frac{1}{\sqrt{1-v_{\text{LIRF}}^2}}$ (recall that Special Relativity rules in the LIRF, so we can use the Special Relativity definition of energy). What this allows us to do is take a known value of E and use it to determine the speed at any given r-coordinate.

Exercise 3.4.1

Is E_{LIRF} a constant of motion? Is it an invariant?

Answer

Given that E_{LIRF} depends on speed, you may think that it is not a constant of motion. Recall, however, that an LIRF exists only for an infinitesimal amount of time, which means that the speed is effectively constant and therefore that E_{LIRF} is a constant of motion. It is not an invariant because it has different values in different reference frames.

Exercise 3.4.2

According to Equation 3.4.1, does speed increase, decrease, or remain the same as an object approaches the event horizon?

Answer

Let's start by solving Equation 3.4.1for the speed.

$$\begin{aligned} \frac{1}{\sqrt{1-v_{\text{LIRF}}^2}} &= \frac{1}{\sqrt{1-\frac{2M}{r}}}\frac{E}{m} \\ \sqrt{1-v_{\text{LIRF}}^2} &= \sqrt{1-\frac{2M}{r}}\frac{m}{E} && \text{take reciprocal of both sides} \\ 1-v_{\text{LIRF}}^2 &= \left(1-\frac{2M}{r}\right)\left(\frac{m}{E}\right)^2 && \text{square both sides} \\ v_{\text{LIRF}}^2 &= 1-\left(1-\frac{2M}{r}\right)\left(\frac{m}{E}\right)^2 && \text{isolate } v_{\text{LIRF}}^2 \\ v_{\text{LIRF}} &= \sqrt{1-\left(1-\frac{2M}{r}\right)\left(\frac{m}{E}\right)^2} && \text{final answer} \end{aligned}$$

As $r\to 2M$, $v_{\text{LIRF}}\to 1$. In other words, the object approaches the speed of light! This might seem like a contradiction since in global coordinates we saw objects seemed to slow down at the event horizon. But there is no reason that someone hovering just outside the event horizon of a black hole needs to make the same observations as someone who is very far away. As weird as it seems, there is no contradiction. Welcome to General Relativity!

Exercise 3.4.3

While nobody actually measures E, anyone in an LIRF can determine it. Explain how.

Answer

Equation 3.4.1provides a prescription for how to do it. Anyone in a LIRF can take local measurements of speed. Given that they know the r-value of their reference frame, they can simply plug in to determine E. Even though observers at different r-coordinates will have different measurements of the energy in their LIRF, they will all agree on the global energy E.

Exercise 3.4.4

Suppose a small stone of mass m is released from rest very far from a black hole of mass M. What is the global energy E of the stone? Determine v_{LIRF} for the stone as a function of r.

Answer

The global energy is given by $E=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}m$. Very far from the black hole, $dt=d\tau$ and $\frac{2M}{r}\to 0$, which means that $E=m$. We can plug this into Equation 3.4.1to determine v_{LIRF}. In fact, we already derived v_{LIRF} as a function of r in a previous exercise, so all we have to do is substitute $E=m$. As a reminder, the result was

$$v_{\text{LIRF}}=\sqrt{1-\left(1-\frac{2M}{r}\right)\left(\frac{m}{E}\right)^2}.$$

Substituting $E=m$, we get

$$v_{\text{LIRF}}=\sqrt{\frac{2M}{r}}. \tag{3.4.2}$$

We see that the stone accelerates as it approaches the event horizon.

Box 3.4.1

Use the definitions of the global-to-local coordinate transformations along with $\frac{E}{m}=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$ and $\frac{E_{\text{LIRF}}}{m}=\frac{dt_{\text{LIRF}}}{d\tau}$ to derive Equation 3.4.1.

Box 3.4.2

In one of the exercises, we showed how to determine v_{LIRF} as a function of r for a small stone released from rest infinitely far away from a non-spinning black hole of mass M. Let's repeat the process for a stone that is released from rest from a position that is *not* infinitely far away. That is, the stone is released from some radius $r=r_0$, where r_0 is a finite number.

a) Write an expression for the global map energy per unit mass $\frac{E}{m}$ in terms of r_0.
b) Use your previous answer as well as the fact that $\frac{E}{m}$ is a constant of motion to determine an expression for v_{LIRF} in terms of r. That is, write a function that could be used to determine the LIRF velocity at *any* r for a stone that is released from rest at r_0. Your answer should only contain the variables r, r_0, and M. (You can check your answer by taking the limit $r_0\to\infty$.)

Box 3.4.3

Now let's repeat the process from the previous Box, except this time for a stone that is *thrown inward* at a speed v_0 from infinitely far away.

a) Write an expression for the global map energy per unit mass $\frac{E}{m}$ in terms of v_0.
b) Use your previous answer as well as the fact that $\frac{E}{m}$ is a constant of motion to determine an expression for v_{LIRF} in

terms of r. Your answer should only contain the variables r, v_0, and M. (You can check your answer by taking the limit $v_0 \to 0$.)

Box 3.4.4

We can use the LIRF transformations to determine the speed of circular orbits in General Relativity. It can be shown (though we won't do so here) that

$$v_{\text{LIRF}} = \left[1 - \left(\frac{mr}{L}\right)^2\right]^{-1/2}. \tag{3.4.3}$$

We also saw in a previous section that, for a circular orbit,

$$\left(\frac{L}{m}\right)^2 = \frac{Mr^2}{r-3M}.$$

Substitute this into Equation 3.4.3 to show that

$$v_{\text{LIRF}} = \sqrt{\frac{\frac{M}{r}}{1-\frac{2M}{r}}} \tag{3.4.4}$$

then use it to determine v_{LIRF} at the minimum stable orbit ($r = 6M$) and the minimum unstable circular orbit ($r = 3M$).