1.5: Four-Momentum

In Newtonian mechanics, position in space can be indicated with a three-dimensional vector. In Special Relativity, however, events are indicated using *four* coordinates: $\mathbf{x}=(t, x, y, z)$.

The **four-velocity** is defined as

$$\mathbf{u}=\left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right). \tag{1.5.1}$$

Definition: Four-velocity

Four-velocity is defined by Equation 1.5.1. It is the four-dimensional relativistic counterpart to velocity.

Note

So far we have been writing Δ to refer to coordinate differences. When we instead use "d" as in Equation 1.5.1, we are specifically dealing with infinitesimally small coordinate differences. This is similar to how *average* velocity $v_{\text{avg}}=\frac{\Delta x}{\Delta t}$ becomes *instantaneous* velocity $v=\frac{dx}{dt}$ when we make the time interval infinitesimally small.

If we multiply the four-velocity by the mass of the particle, we get the **four-momentum**.

$$\mathbf{p}=\left(m\frac{dt}{d\tau}, m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\right) \tag{1.5.2}$$

Definition: Four-momentum

Four-momentum is defined by Equation 1.5.2. It is the four-dimensional relativistic counterpart to momentum.

You may recognize the spatial components as looking a lot like how momentum is defined in Newtonian mechanics: $\vec{p}=m\vec{v}$ (I will use **bold** to indicate a four-vector and an arrow to indicate a three-vector). So what is the meaning of the time-component of the four-momentum? It turns out that it is the energy, which can be shown by looking at the low-velocity limit (see Box 1.5.5). This allows us to write the four-momentum as

$$\mathbf{p}=(E, p^x, p^y, p^z), \tag{1.5.3}$$

where E is the energy. Using the spacetime interval, we can derive (see Box 1.5.1)

$$E=m\gamma \tag{1.5.4}$$

and (see Box 1.5.2)

$$\vec{p}=\gamma m\vec{v} \tag{1.5.5}$$

where

$$\gamma=\frac{1}{\sqrt{1-v^2}}. \tag{1.5.6}$$

We can also show that energy and momentum are related (see Box 1.5.3), as shown in Equation 1.5.7.

$$m^2=E^2-p^2 \tag{1.5.7}$$

As in Newtonian mechanics, momentum and energy are **constants of motion** in relativity, which makes them helpful for predicting motion.

Definition: Constant of Motion

A constant of motion is a quantity that has the same value at all points along a particular worldline.

Note

A constant of motion is distinct from an invariant. Constants of motion keep the same value within a particular reference frame but can be different in different reference frames. An invariant has the same value in all reference frames.

Box 1.5.1

Divide the infinitesimal spacetime interval $d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$ by dt^2, then use it along with the definition $E = m\frac{dt}{d\tau}$ to show that $E = m\gamma$.

Box 1.5.2

Use the fact that $\frac{dx}{d\tau}$ can be written as $\frac{dx}{dt}\frac{dt}{d\tau}$ along with the definition $\vec{p} = \left(m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\right)$ to show that $\vec{p} = m\gamma\vec{v}$.

Box 1.5.3

Divide the infinitesimal spacetime interval $d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$ by dt^2, then use $E = m\gamma$ and $\vec{p} = m\gamma\vec{v}$ to show that $m^2 = E^2 - p^2$.

Box 1.5.4

The equation $m^2 = E^2 - p^2$ from the previous box is written in natural units, which implies that energy and momentum have units of mass. One way to restore the equation to SI units is to figure out how many factors of c (in m/s) you need to insert in each term in order to get the units to all match. Do this.

Box 1.5.5

The total energy of a particle in relativity can be thought of as the sum of its kinetic energy and its *rest energy*. In natural units, the rest energy is *m*. (In non-natural units, the rest energy is mc^2, which might look familiar.) This means that the kinetic energy can be written as $KE = E - m = m(\gamma - 1)$, which provides a way for us to confirm that $m\frac{dt}{d\tau}$ represents energy. To do that, we need to take the low-speed limit of the kinetic energy and show that it is equal to the Newtonian expression $KE = \frac{1}{2}mv^2$.

The derivation requires using the *binomial approximation*, which states that $(1+x)^n \approx 1 + nx$ if x is very small. Use this approximation to show that the relativistic expression $KE = m(\gamma - 1)$ reduces to the Newtonian expression $KE = \frac{1}{2}mv^2$ in the speed is very small.
