3.3: Effective Potential

In the previous section, we applied the geodesic equation to three of the four Schwarzschild coordinates. Applying it to the r-coordinate yields a result that is complicated enough to warrant its own section. The result is

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right). \tag{3.3.1}$$

Now let's look at this term by term. On the left, there is something that looks like a velocity squared. Next is a term that is constant and represents the total energy. Finally, there is a term that depends only on position and other constants. This is very similar to a familiar equation from Newtonian Mechanics:

$$KE = \frac{1}{2}mv^2 = E_{\text{total}} - PE \tag{3.3.2}$$

If you make a graph of potential energy, then you can use the graph to come to conclusions about the motion. For example, let's graph $PE = \frac{1}{2}kx^2$. The total energy is a constant, which is a horizontal line on the graph. The kinetic energy is then given by the difference of the two curves, as shown in Fig. 3.3.1.

Figure 3.3.1: A graph of potential energy $PE = \frac{1}{2}kx^2$. The kinetic energy is the difference between the total energy (horizontal line) and the potential energy. (Copyright; author via source)

Since kinetic energy must always be greater than or equal to zero, we see that the object is forced to oscillate between the points of intersection (called **turning points**) of the two curves. Furthermore, the smaller the total energy the smaller the range of oscillation.

The point, though, is that we can obtain qualitative information about the velocity by plotting the other two parts. We can repeat the process with Equation 3.3.1 by plotting $\frac{E}{m}$ (as a horizontal line) and

$$\frac{PE_{\text{eff}}(r)}{m}=\sqrt{\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right)}, \tag{3.3.3}$$

where $PE_{\text{eff}}(r)$ is called the **effective potential energy**.

Definition: Effective Potential Energy

An effective potential energy is not technically a potential energy but can still be used to obtain qualitative information about motion.

Fig. 3.3.2 plots this effective potential energy for $\frac{L}{m}=3.75M$ with dimensionless axes.

Figure $3.3.2$: Schwarzschild effective potential energy for $\frac{L}{m}=3.75M$. The difference between the two curves tells us about the radial component of the velocity, $\frac{dr}{d\tau}$. (Copyright; author via source)

Exercise $3.3.1$

Describe the motion depicted in Fig. 3.3.2. How would the motion be different if $\frac{E}{m}=0.98$?

Answer

There are three turning points (i.e. points of intersection) on the graph, so the motion depends on which region the particle is in. If $6.5 \leq \frac{r}{M} \leq 15.5$, then the particle is in a trapped orbit. We know this because the particle has angular momentum, which mean it is orbiting around, but there is a minimum and a maximum r-value to that orbit (given by the turning points). This could, for example, be an elliptical orbit, though we can't say for sure that it is *exactly* an ellipse. The other possibility is if $\frac{r}{M} \leq 3.5$, in which case there is a maximum orbital r-value but no minimum. That particle may swing around a bit, but it is doomed to fall to the event horizon.

If the total energy were $\frac{E}{m} = 0.98$, then there would only be one turning point instead of three (it would intersect way off to the right). In that case, the orbit would have maximim r-value but no minimum, which again means that it is doomed to fall to the event horizon.

Exercise 3.3.2

Fig. 3.3.2 shows the effective potential energy for $\frac{L}{m} = 0$. Describe the resulting motion. Is that what you would expect?

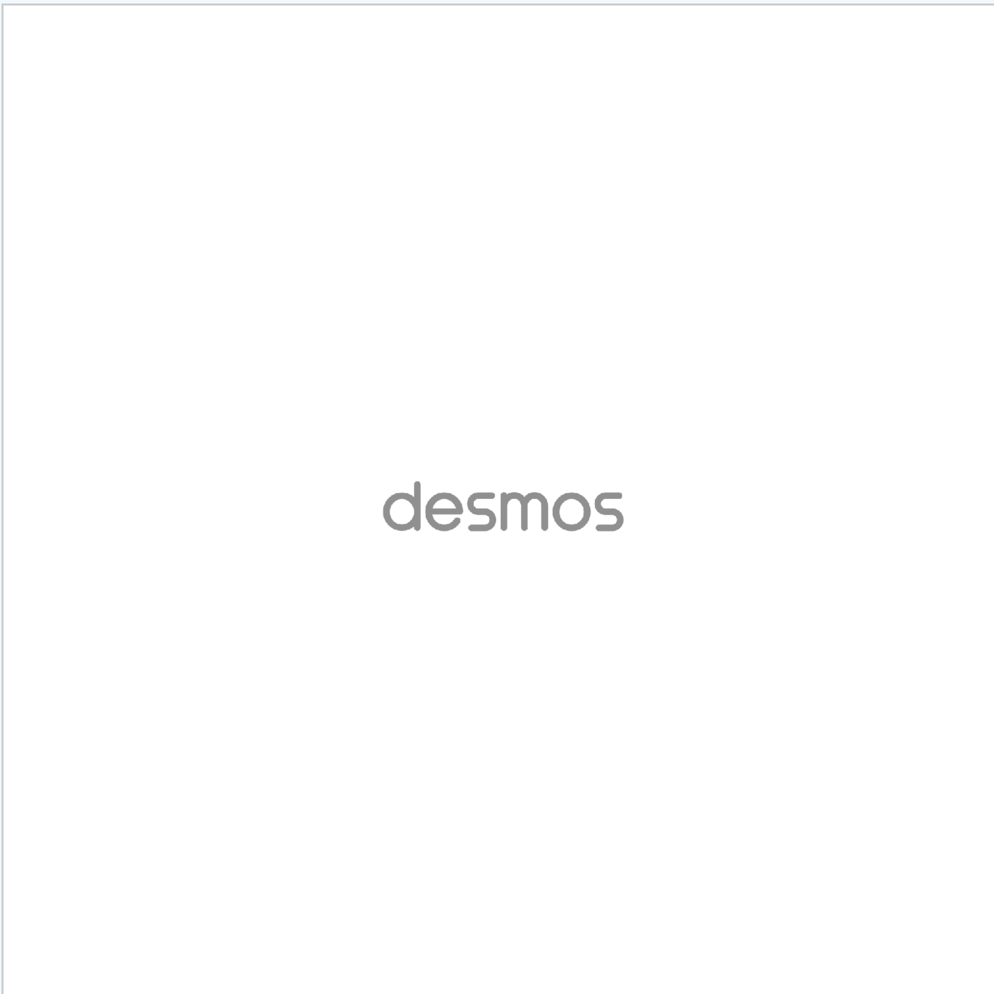

Figure 3.3.3: Schwarzschild effective potential energy for $\frac{L}{m} = 0$. (Copyright; author via source)

Answer

Zero angular momentum corresponds to a particle moving radially. The fact that there is only one turning point makes sense, since it shows that a particle initially moving radially away will reach a maximum r-value then fall back toward the event horizon. Note that the particle never passes the event horizon, which is consistent with what we learned from looking at light cones.

Box 3.3.1

Circular orbits are possible wherever the effective potential energy has zero slope. Take the derivative of the effective potential energy with respect to r, then set it equal to zero. Use it to show that

$$r^2 - \frac{L^2}{Mm^2}r + 3\frac{L^2}{m^2} = 0.$$

Box 3.3.2

Solve the quadratic equation from Box 3.3.1 to show that

$$r = \frac{L^2}{2m^2M}\left[1 \pm \sqrt{1 - \frac{12M^2m^2}{L^2}}\right].$$

Box 3.3.3

Using the result from the previous box, what is the minimum value of $\frac{L}{m}$ for which you get a real number for r? Express your answer in terms of M.

Box 3.3.4

Use the minimum value of $\frac{L}{m}$ from the previous box to determine the r-value of the orbit (in terms of M). This is called the **Innermost Stable Circular Orbit** (ISCO).