4.2: Effective Potential

The previous section discussed how ϕ evolves with t for a photon, but in order to get a full picture of the motion we also need $r(t)$. You can use the Schwarzschild metric along with the definition of the impact parameter to show that (see Box 4.2.1)

$$\frac{1}{b^2} = \left[\frac{1}{b\left(1-\frac{2M}{r}\right)}\frac{dr}{dt}\right]^2 + \frac{1-\frac{2M}{r}}{r^2}. \tag{4.2.1}$$

This establishes a relationship, albeit a complicated one, between r and $\frac{dr}{dt}$. In *principle*, we can use it along with the angular equation from the previous section to completely determine the motion of a photon in Schwarzschild spacetime. In *practice*, however, this is basically impossible to do by hand, which means we must either use a computer or use the qualitative *effective potential energy* technique that we discussed earlier. Fortunately, Equation 4.2.1 is already in a form where the left hand side is a constant, and the right hand side is a sum of one term that involves speed and one term that involves position.

$$\begin{gathered}\frac{1}{b^2} \Rightarrow \text{total energy} \\ \left[\frac{1}{b\left(1-\frac{2M}{r}\right)}\frac{dr}{dt}\right]^2 \Rightarrow \text{radial "kinetic energy"} \\ \frac{1-\frac{2M}{r}}{r^2} \Rightarrow \text{effective potential energy}\end{gathered}$$

Figure $4.2.1$ plots the effective potential for a photon in Schwarzschild spacetime.

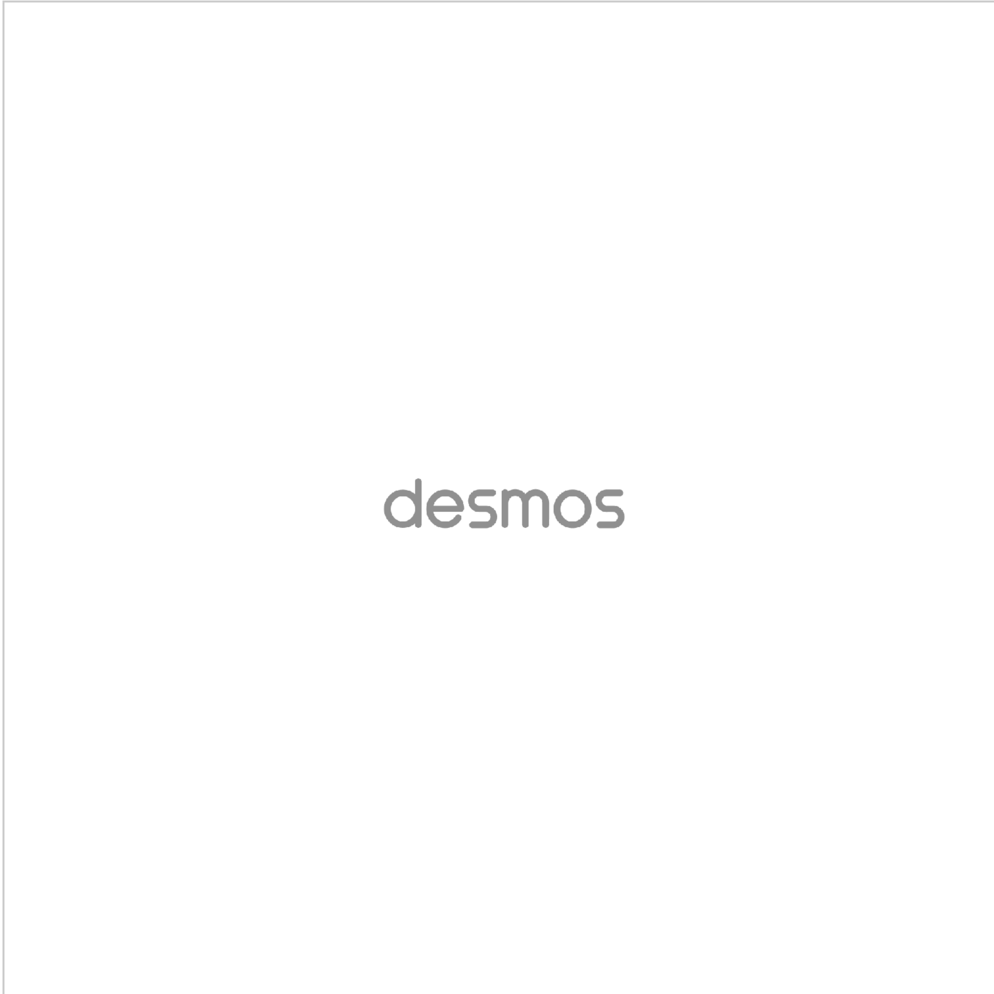

Figure 4.2.1: Effective potential for a photon in Schwarzschild spacetime. (Copyright; author via source)

Exercise 4.2.1

Describe the various photon paths according to Figure 4.2.1.

Answer

The photon paths depend on the impact parameter. Figure 4.2.2shows three possible classes of paths.

Figure 4.2.2: Photon paths in Schwarzschild spacetime can be broken into three different classes. (Copyright; author via source)

The upper-most horizontal line is for photons with a small impact parameter. These photons come in from large r-values and go straight to $r=0$; they are called **plunge orbits**. The lower-most horizontal line is for photons with a large impact parameter, and they divide into two groups. One group (the part of the line that is above and to the right of the red curve) comes in from large r-values, swings around the central mass at $r=0$, then escapes back to large r-values; these orbits are called **bounce orbits**. The second group (above and to the left of the red curve) originates near the black hole and then plunges in; these orbits are called **trapped orbits**. The third class of paths is the middle horizontal line, which touches the very top of the red curve. If a photon comes into existence at that exact intersection point, it will follow a circular orbit. This ciarcular orbit is *unstable*, though, so it won't last for very long.

Note that this graph can't show the "swing around" part of orbits, but recall that ϕ also changes with time according to the previous section.

Box 4.2.1

Start with the Schwarzschild metric for a photon in the equatorial plane ($\theta=90^{\circ}$):

$$0=\left(1-\frac{2M}{r}\right)dt^2-\frac{1}{1-\dfrac{2M}{r}}dr^2-r^2d\phi^2.$$

Divide both sides by $\left(1-\frac{2M}{r}\right)dt^2$, then use the definition of the impact parameter $b=\frac{r^2}{1-\frac{2M}{r}}\frac{d\phi}{dt}$ and rearrange to prove Equation 4.2.1.

Box 4.2.2

There exists an unstable circular orbit at the maximum of the effective potential energy function $PE_{\text{eff}}(r)=\frac{1-\frac{2M}{r}}{r^2}$. Use calculus to show that the r-coordinate of this circular orbit is $r=3M$.

Box 4.2.3

In Box 4.2.2, you showed that an unstable circular orbit occurs at $r=3M$. Determine the value of the effective potential energy at that r-coordinate, then use that to show that the impact parameter that leads to a circular orbit is $b=\sqrt{27}M$. This is called the **critical impact parameter** because it separates photon orbits that plunge into the black hole from photon orbits that swing around and escape.