2.4: The Schwarzschild r-coordinate

To see what the r-coordinate means in the Schwarzschild metric, consider two events with different r-coordinates (r_a and r_b) but the same t-coordinate, θ-coordinate, and ϕ-coordinate. This is essentially like measuring the distance between the edges of two concentric circles. Because this is a spacelike interval, we will use the proper length σ instead of the proper time τ.

$$d\sigma^2 = \frac{1}{1-\frac{2M}{r}}dr^2$$

By taking the square root and integrating, we can find the proper length between the two circles.

$$\int_{\sigma_a}^{\sigma_b} d\sigma = \pm\int_{r_a}^{r_b} \frac{1}{\sqrt{1-\frac{2M}{r}}}dr$$

$$\Delta\sigma = \pm\int_{r_a}^{r_b} \frac{1}{\sqrt{1-\frac{2M}{r}}}dr$$

Note that r is *not* a constant, so there is no way around performing the integral. For this course, I don't mind if you use an integral calculator (I like to use Desmos.com, where you can type *int* to get the integral sign.

Exercise 2.4.1

What is the proper length between two circles with $r = 3M$ and $r = 4M$, where $M = 1477\text{ m}$ (the mass of the sun), using the Schwarzschild metric?

Answer

Here is an example of how to type this into Desmos.

$$\Delta\sigma = \pm\int_{r=3M}^{r=4M} \frac{1}{\sqrt{1-\frac{2M}{r}}}dr = 2278\text{ m}$$

Notice that this answer is significantly larger than simply the difference $4M - 3M = 1477\text{ m}$. For this reason, it is not a good idea to refer to r as the "radius".

Box 2.4.1

A black hole has a mass $M = 5$ km, a little more than three times the mass of our Sun. Two concentric spherical shells surround this black hole. The lower shell has r-coordinate r_L; the higher shell has r-coordinate $r_H = r_L + \Delta r$. Assume that $\Delta r = 1$ km and consider the following cases.

i. $r_L = 50$ km
ii. $r_L = 15$ km
iii. $r_L = 10.1$ km
iv. $r_L = 10.01$ km
v. $r_L = 10.001$ km

a. For each case, calculate the radial proper length $\Delta\sigma$ between shells.
b. The predictions of general relativity are always more accurate than Newtonian predictions, but they come at the cost of being more complicated to calculate. In some of the above cases, for example, you may have found the difference from the Newtonian result to be small enough that invoking general relativity is not worth it. Suppose you had a really long list of r_L values and wanted to expedite the process of calculating ruler distance by using the Newtonian result as long as it didn't differ from the more accurate prediction of general relativity by more than 1%. Approximately what value of r_L should you

use as the dividing line? Explain your process clearly. (You may not be able to solve it analytically, in which case you will have to think of another way.)