3.1: The Geodesic Equation

The previous chapter dealt with the rules of geometry in Schwarzschild spacetime. If we want to look at *motion*, we need to look beyond the metric to something called the **geodesic equation**.

Definition: Geodesic

A geodesic is the worldline that results from applying the Principle of Maximal Aging. That is, it is the inertial worldline that connects two events.

The geodesic equation is a way of determining the geodesic that connects two events. You can think of it as an extension of Newton's First Law to curved spacetimes.

$$0=\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^{\nu}}{d\tau}\right)-\frac{1}{2}\frac{dg_{\alpha\beta}}{dx^{\mu}}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} \tag{3.1.1}$$

Let's see how this works by applying it to $g_{\mu\nu}=\eta_{\mu\nu}$ (i.e. the Special Relativity metric). As a reminder,

$$\eta_{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

The geodesic equation involves summing over ν, α, and β, while μ is a free index. Note also that the metric is zero if $\mu\neq\nu$ or $\alpha\neq\beta$. Let's work out the result for each μ.

$$\begin{aligned}
\mu=t:\ 0&=\frac{d}{d\tau}\left(g_{tt}\frac{dx^t}{d\tau}\right)-\frac{1}{2}\left(\frac{dg_{tt}}{dx^t}\frac{dx^t}{d\tau}\frac{dx^t}{d\tau}+\frac{dg_{xx}}{dx^t}\frac{dx^x}{d\tau}\frac{dx^x}{d\tau}+\frac{dg_{yy}}{dx^t}\frac{dx^y}{d\tau}\frac{dx^y}{d\tau}+\frac{dg_{zz}}{dx^t}\frac{dx^z}{d\tau}\frac{dx^z}{d\tau}\right) && \text{write out sum}\\
0&=g_{tt}\frac{d^2t}{d\tau^2}-\frac{1}{2}\left(\cancelto{0}{\frac{dg_{tt}}{dt}}\left(\frac{dt}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{xx}}{dt}}\left(\frac{dx}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{yy}}{dt}}\left(\frac{dy}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{zz}}{dt}}\left(\frac{dz}{d\tau}\right)^2\right) && \text{metric is constant}\\
0&=\frac{d^2t}{d\tau^2} && \text{simplify}\\
\mu=x:\ 0&=\frac{d}{d\tau}\left(g_{xx}\frac{dx^x}{d\tau}\right)-\frac{1}{2}\left(\frac{dg_{tt}}{dx^x}\frac{dx^t}{d\tau}\frac{dx^t}{d\tau}+\frac{dg_{xx}}{dx^x}\frac{dx^x}{d\tau}\frac{dx^x}{d\tau}+\frac{dg_{yy}}{dx^x}\frac{dx^y}{d\tau}\frac{dx^y}{d\tau}+\frac{dg_{zz}}{dx^x}\frac{dx^z}{d\tau}\frac{dx^z}{d\tau}\right) && \text{write out sum}\\
0&=g_{xx}\frac{d^2x}{d\tau^2}-\frac{1}{2}\left(\cancelto{0}{\frac{dg_{tt}}{dx}}\left(\frac{dt}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{xx}}{dx}}\left(\frac{dx}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{yy}}{dx}}\left(\frac{dy}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{zz}}{dx}}\left(\frac{dz}{d\tau}\right)^2\right) && \text{metric is constant}\\
0&=-\frac{d^2x}{d\tau^2} && \text{simplify}
\end{aligned}$$

The process works similarly for $\mu=y$ and $\mu=z$. The result is that the second derivative of each component with respect to proper time is zero. In other words, there is no acceleration, which is exactly what we expect for Special Relativity.

While the geodesic equation looks complicated, in practice it isn't so bad if you are careful about writing out all of the terms and noticing when you can leave terms out because they are zero.