6.2: Gravitational Wave Metric

Gravitational waves were predicted by General Relativity long before they were actually detected. How? Well, a lot of the theoretical predictions of General Relativity came about just from playing around with the Einstein Field Equations. You can propose a matter distribution and see what metric is consistent with that, or you can propose a metric and see what that implies. The metric corresponding to a gravitational plane wave traveling in the z-direction, for example, could be written as

$$d\tau^2 = dt^2 - (1+h)dx^2 - (1-h)dy^2 - dz^2, \tag{6.2.1}$$

where $h \ll 1$. This is very similar to the metric from Special Relativity, except that g_{xx} and g_{yy} are very slightly different from 1 (recall that they are exactly equal to 1 in Special Relativity). If you put Equation 6.2.1 into the Einstein Field Equations, you find that h is an oscillating function (the proof is beyond the scope of this course). Fig. 6.2.1 shows what the coordinate grid for such a metric could look like. If you start with a ring of particles in the plane of the page, then a gravitational wave traveling perpendicular to the page will cause the ring to stretch in one direction while squeezing in the other direction. As the wave continues to move, the part that was stretched becomes squeezed while the part that was squeezed becomes stretched.

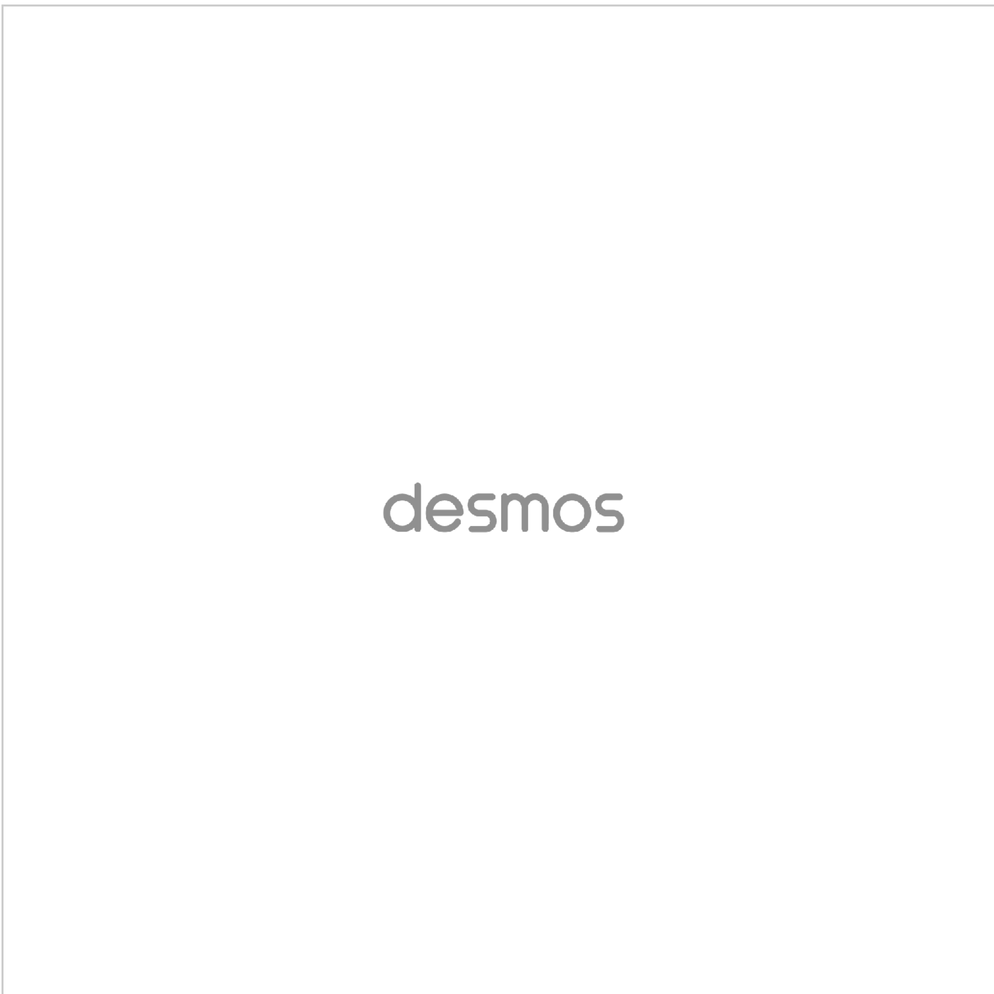

Figure $6.2.1$: This animation shows what happens to a ring of particles when a gravitational wave passes by. The wave is traveling perpendicular to the page. (Image credit)

Notice that the coordinate locations of the particles in Fig. 6.2.1 do not change. Nevertheless, the distances between the particles change with time. This is the result of the fact that the coordinates in Equation 6.2.1 are *global coordinates*, where coordinate separations don't necessarily correspond to anything measurable. To get measurable quantities, like distance, we need to superimpose a *local inertial reference frame* (LIRF) on top of the global coordinate system. Then we use the fact that, for any LIRF,

$$d\tau^2 = dt^2_{\text{LIRF}} - dx^2_{\text{LIRF}} - dy^2_{\text{LIRF}} - dz^2_{\text{LIRF}}.$$

By laying the axes of the LIRF along the axes of the global coordinate system, you can relate the two. The result is

$$\Delta t_{\text{LIRF}} = \Delta t \tag{6.2.2}$$

$$\Delta x_{\text{LIRF}} = \sqrt{1+h}\,\Delta x \qquad \approx \left(1+\frac{h}{2}\right)\Delta x \qquad (h \ll 1) \tag{6.2.3}$$

$$\Delta y_{\text{LIRF}} = \sqrt{1-h}\,\Delta y \qquad \approx \left(1-\frac{h}{2}\right)\Delta y \qquad (h \ll 1) \tag{6.2.4}$$

$$\Delta z_{\text{LIRF}} = \Delta z \tag{6.2.5}$$

where we have used the binomial approximation $(1+x)^n \approx 1+nx \;\text{ if } x \ll 1$.

Note

Large intervals (Δ's) come from adding up (i.e. integrating) many infinitesimally small intervals (d's). If all of the small intervals are the same size (i.e. the integrand is constant with respect to the integration variable), then you can freely switch between d's and Δ's as we did in going from the gravitational wave metric to the LIRF coordinate transformations.

According to these equations, measured distances between events on the x- or y-axes are either slightly bigger or slightly smaller than their respective global coordinate separations, with h representing the fractional change, or **gravitational wave strain**.

Definition: Gravitational Wave Strain

Gravitational wave strain is the amount that distances are stretched or compressed by a passing gravitational wave, relative to the original length. It is a dimensionless number.

Exercise 6.2.1

The Laser Interferometer Gravitational Wave Observatory (LIGO) works by firing a laser beam at a mirror that is 4 km away. Suppose a gravitational wave with $h = 10^{-21}$ passes by during that time. What is the maximum change in laser-to-mirror distance caused by the gravitational wave? Does the answer depend on the orientation of the gravitational wave?

Answer

In a local inertial reference frame, the separation of two points along, say, the x-axis is $\Delta x_{\text{LIRF}} = \left(1+\frac{h}{2}\right)\Delta x = \Delta x + \frac{h}{2}\Delta x$. In this case, $\Delta x = 4$ km and $\frac{h}{2}\Delta x = \left(\frac{10^{-21}}{2}\right)(4000\text{ m}) = 2\times 10^{-18}\text{ m}$ is the *change* in length. That is approximately 1/1000th of the width of the nucleus of an atom! (These are actually realistic numbers.)

The answer does depend on orientation of the wave. Our calculation assumes that the wave travels exactly in the z-direction (perpendicular to earth) and that one of the "stretchy" axes of the wave is parallel to the x-axis. In general, then, the actual change in length will be smaller than what we calculated.