3.2: Constants of Motion

By applying the geodesic equation to the Schwarzschild metric, we can see what an inertial worldline looks like. Recall that, for the Schwarzschild metric,

$$g_{\mu\nu} = \begin{cases} 1-\frac{2M}{r} & \mu=\nu=t \\ -\frac{1}{1-\frac{2M}{r}} & \mu=\nu=r \\ r^2 & \mu=\nu=\theta \\ r^2\sin^2\theta & \mu=\nu=\phi \end{cases}$$

and the geodesic equation is

$$0 = \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\frac{dg_{\alpha\beta}}{dx^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}.$$

Let's apply the geodesic equation for $\mu = t$.

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) - \frac{1}{2}\left(\frac{dg_{tt}}{dt}\left(\frac{dt}{d\tau}\right)^2 + \frac{dg_{rr}}{dt}\left(\frac{dr}{d\tau}\right)^2 + \frac{dg_{\theta\theta}}{dt}\left(\frac{d\theta}{d\tau}\right)^2 + \frac{dg_{\phi\phi}}{dt}\left(\frac{d\phi}{d\tau}\right)^2\right) && \text{write out sum} \\ 0 &= \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) - \frac{1}{2}\left(\cancelto{0}{\frac{dg_{tt}}{dt}}\left(\frac{dt}{d\tau}\right)^2 + \cancelto{0}{\frac{dg_{rr}}{dt}}\left(\frac{dr}{d\tau}\right)^2 + \cancelto{0}{\frac{dg_{\theta\theta}}{dt}}\left(\frac{d\theta}{d\tau}\right)^2 + \cancelto{0}{\frac{dg_{\phi\phi}}{dt}}\left(\frac{d\phi}{d\tau}\right)^2\right) && \text{metric has no t-dependence} \\ 0 &= \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) && \text{simplify} \\ g_{tt}\frac{dt}{d\tau} &= \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = \text{constant} && \text{zero derivative means constant} \end{aligned}$$

What we find from here is that this particular quantity is a constant of motion. In particular, it is the *energy per unit mass*.

$$\frac{E}{m} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \tag{3.2.1}$$

? Exercise 3.2.1

Any time we derive a new result in General Relativity, we should be able to show that the result reduces to something from Special Relativity in the appropriate limits. Do this for Equation 3.2.1.

Answer

We should recover the energy per unit mass from Special Relativity in the limits $M \to 0$ or $r \to \infty$. In either case, we get $\frac{E}{m} = \frac{dt}{d\tau}$, which was how we defined energy per unit mass in Special Relativity.

? Exercise 3.2.2

According to Equation 3.2.1, what happens to $\frac{dt}{d\tau}$ as a particle approaches the event horizon from outside. What does that physically mean?

Answer

As $r \to 2M$ from the outside, the quantity $1-\frac{2M}{r}$ gets smaller and smaller, which means that $\frac{dt}{d\tau}$ must get larger and larger in order for $\frac{E}{m}$ to remain constant. What this means physically is that for every tick on the wristwatch of the infalling particle, a much *larger* amount of time passes on the wristwatch of someone who is very far away (recall that *t* in Schwarzschild coordinates is *faraway time*). As $r \to 2M$, $\frac{dt}{d\tau} \to \infty$, which means that an infinite amount of time passes on the wristwatch of the faraway observer for every clock tick on the wristwatch of the infalling particle.

By applying the geodesic equation to $\mu = \phi$, we get another constant of motion: the **angular momentum per unit mass**.

$$\frac{L}{m} = r^2\sin^2\theta\frac{d\phi}{d\tau} \tag{3.2.2}$$

Exercise 3.2.3

Consider an inertial particle orbiting in the equatorial plane (i.e. $\theta = 90^\circ$) of a black hole at a constant r-coordinate. Describe the motion according to Equation 3.2.2. What if the motion is elliptical?

Answer

If r is a constant, then $\frac{d\phi}{d\tau}$ is a constant as well. In other words, the particle moves in a plane at a constant angular speed around the black hole. If the motion is elliptical, then r varies. In order for $\frac{L}{m}$ to remain constant, the angular speed must be larger when r is small and smaller when r is large. This is exactly what happens in elliptical orbits according to Newtonian mechanics.

Exercise 3.2.4

Argue that applying the geodesic equation to $\mu = \theta$ wouldn't yield any information about orbits not already contained in Equation 3.2.2.

Answer

Given that the Schwarschild metric is spherically symmetric, orbits in the θ direction can't be any different qualitatively than orbits in the ϕ direction. In fact, any orbital plane can be the equatorial plane just by rotating the sphere.

Box 3.2.1

Apply the geodesic equation to the Schwarzschild metric with $\mu = \phi$ in order to derive Equation 3.2.2.