2.3: The Schwarzschild t-coordinate

In General Relativity, we have to be careful not to assume that coordinates mean what we think they mean. Consider, for example, the t-coordinate in the Schwarzschild spacetime interval

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1-\frac{2M}{r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

To see what the t-coordinate means, consider two events with a t-coordinate separation of dt that occur at the same location (i.e. $dr = d\theta = d\phi = 0$). The spacetime interval reduces to

$$d\tau = \pm\sqrt{1-\frac{2M}{r}}dt.$$

Recall that τ, the proper time, is the time as measured by the wristwatch of someone who is present at both events. So in general, the proper time is *not* the same as the t-coordinate separation.

Exercise 2.3.1

Two flashes of light occur at the same location a distance of $r = 4M$ from the center of a spherical object with mass M. An observer at that location measures the time between the flashes to be 1 second. What is the t-coordinate separation of those flashes?

Answer

As shown before, the spacetime interval for two events at the same location around a spherical mass reduces to

$$d\tau = \pm\sqrt{1-\frac{2M}{r}}dt.$$

The 1 second given in the problem is the proper time, but 1 second is not infinitesimal. A non-infinitesimal interval, however, is the sum of many infinitesimal intervals, which is an integral. We will take the bounds of the integral to go from τ_i to τ_f on the left and t_i to t_f on the right.

$$\begin{aligned}
\int_{\tau_i}^{\tau_f} d\tau &= \pm\int_{t_i}^{t_f}\sqrt{1-\frac{2M}{r}}dt && \text{set up integral}\\
\int_{\tau_i}^{\tau_f} d\tau &= \pm\sqrt{1-\frac{2M}{r}}\int_{t_i}^{t_f} dt && \sqrt{1-\frac{2M}{r}} \text{ is a constant with respect to t}\\
\tau_f - \tau_i &= \pm\sqrt{1-\frac{2M}{r}}(t_f - t_i) && \text{perform integral}\\
t_f - t_i &= \pm\frac{1}{\sqrt{1-\frac{2M}{r}}}(\tau_f - \tau_i) && \text{solve for t-coordinate separation}\\
\Delta t &= \pm\frac{1}{\sqrt{1-\frac{2M}{4M}}}(1 \text{ s}) && \text{substitute}\\
\Delta t &= 1.41 \text{ s} && \text{simplify}
\end{aligned}$$

Here we choose the positive solution, and we see that the t-coordinate separation is larger than the proper time.

Exercise 2.3.2

Is there any place where the t-coordinate separation between events that occur at the same location is the same as the proper time?

Answer

If you look at $d\tau = \pm\sqrt{1-\frac{2M}{r}}\,dt$, you will see that $d\tau = dt$ only if $1-\frac{2M}{r} = 1$, but that only happens in the limit $r \to \infty$. In other words, the t-coordinate separation between two events is only equal to the proper time if those events occur infinitely far away from the spherical mass. For this reason, we can think of the t-coordinate as representing the **faraway time**.

Exercise 2.3.3

An astronaut with a wristwatch floats at a constant location from a black hole just outside of $r = 2M$. Given two events at the astronaut's location, we can say that $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{2M}{r}}}$. Using this, describe how the time interval measured on the watch of someone very far away compares to the time measured by the astronaut's wristwatch.

Answer

If r is just barely bigger than $2M$, then $\frac{dt}{d\tau}$ will be a huge number. In other words, the time between two events measured by the faraway observer (dt) will be much larger than the time between those two events as measured by the astronaut ($d\tau$).