1.2: The Spacetime Interval

In Newtonian mechanics, the length of an object is invariant. That is, if you use a different coordinate system to mark locations, the result you get for the length does not change.

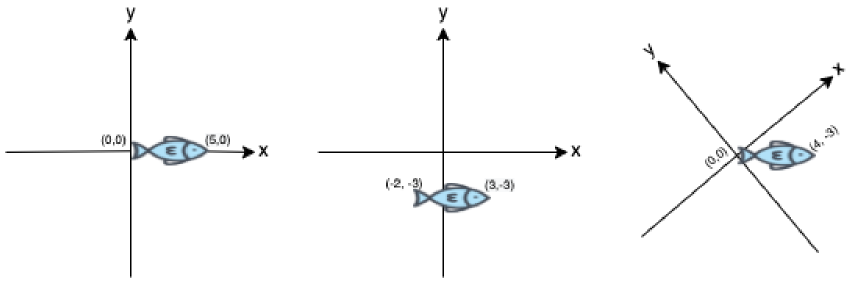

Figure 1.2.1: The length of the fish is 5 units regardless of the choice of coordinate system.

In Special Relativity, however, it is a combination of distance and time that is invariant from one coordinate system to another. We call this the **spacetime interval**.

Definition: Spacetime Interval

The spacetime interval is a combination of distance and time that is **invariant**.

Definition: Invariant

An invariant is a quantity that has the same value for all observers.

For constant velocity motion in **flat spacetime** (we will discuss what that means later), the spacetime interval is

$$c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 . , \tag{1.2.1}$$

where $c = 3 \times 10^8$ m/s is the speed of light. The coordinates t, x, y, and z mark an **event** that depends on the **reference frame**. The variable τ, called the **proper time**, is the same in all reference frames.

Definition: Event

An event is something that has both a location and a time.

Definition: Reference Frame

A reference frame is a coordinate system, usually imagined to be associated with a particular observer. Reference frames can move with respect to one another. A reference frame in which Newton's First Law holds (i.e. an object at rest stays at rest and an object in motion maintains it's velocity) is called an **inertial reference frame**.

Definition: Proper Time

The proper time is an invariant quantity that can be determined from the spacetime interval. Another name I like to use for proper time is **wristwatch time***, because it is the time interval measured by an inertial observer who is present at both events.

*This term is borrowed from *Exploring Black Holes* by Edwin Taylor and John Wheeler.

Exercise 1.2.1

Two events are separated by 50 ns and 12 m in Joo-Won's reference frame. The events are separated by 4 m in Fawn's reference frame. What is the time interval between the events in Fawn's reference frame?

Answer

We know that both reference frames will agree on the proper time between the events, which we can calculate using the spacetime interval. To simplify the calculation, we will orient the reference frame so that the events are on the x-axis. In Joo-Won's reference frame, this yields

$$\begin{aligned} c^2\Delta\tau^2 &= c^2\Delta t_J^2 - \Delta x_J^2 && \text{start with spacetime interval} \\ \Delta\tau &= \sqrt{\frac{c^2\Delta t_J^2 - \Delta x_J^2}{c^2}} && \text{solve for } \tau \\ \Delta\tau &= \sqrt{\frac{\left(3\times 10^8\ \mathrm{m/s}\right)^2 \left(50\times 10^{-9}\ \mathrm{s}\right)^2 - (12\ \mathrm{m})^2}{\left(3\times 10^8\ \mathrm{m/s}\right)^2}} && \text{substitute numbers} \\ \Delta\tau &= 30\ \mathrm{ns} && \text{solve for } \tau \end{aligned}$$

Now we can use the proper time to determine Δt in Fawn's reference frame.

$$\begin{aligned} c^2\Delta\tau^2 &= c^2\Delta t_F^2 - \Delta x_F^2 && \text{start with spacetime interval} \\ \Delta t_F &= \sqrt{\frac{c^2\Delta\tau^2 + \Delta x_F^2}{c^2}} && \text{solve for } t_F \\ \Delta t_F &= \sqrt{\frac{\left(3\times 10^8\ \mathrm{m/s}\right)^2 \left(30\times 10^{-9}\ \mathrm{s}\right)^2 + (4\ \mathrm{m})^2}{\left(3\times 10^8\ \mathrm{m/s}\right)^2}} && \text{substitute numbers} \\ \Delta t_F &= 32.8\ \mathrm{ns} && \text{final answer} \end{aligned}$$

Notice that $\Delta\tau$ is only appreciably different from Δt if Δt is small and/or if Δx is large. In other words, relativistic effects are only noticeable if $\frac{\Delta x}{\Delta t}$ is large (specifically, close to the speed of light).

You may have noticed that proper time is only a real number if $\frac{\Delta x}{\Delta t} \leq c$. What does it mean for the proper time to be imaginary? Since proper time is the time measured on the wristwatch of an inertial observer who is present at both events, an imaginary proper time means that it is impossible for an observer to be present at both events. In other words, nothing could move fast enough to be present at both events. Not even light. And if light isn't fast enough to be present at both events, then there can be no causal connection between them. As an analogy, imagine that a house is broken into at midnight. If Marcus left a party across town at 11:59 pm, then he couldn't have possibly perpetrated the crime. If Gerry left the same party at 11 pm, then he *could have* perpetrated the crime.

There are three different categories we can use to describe pairs of events: timelike, spacelike, and lightlike.

Definition: Timelike, Spacelike, Lightlike

Events with a timelike separation can be causally connected while events with a spacelike separation are causally disconnected. Events with a lightlike separation are exactly far enough away from each other that light could be present at both events (they are still causally connected).

$$\begin{aligned} \text{timelike interval: } \Delta\tau^2 &> 0 \\ \text{spacelike interval: } \Delta\tau^2 &< 0 \\ \text{lightlike interval: } \Delta\tau^2 &= 0 \end{aligned}$$

It may seem odd that $\Delta\tau^2$ could be negative, since that would lead to an imaginary value for the proper time. Nevertheless, spacelike intervals do exist. Any time you want to measure the length of something for example, you mark the locations of the endpoints at the same time, which is necessarily going to yield a negative $\Delta\tau^2$. For this reason, we will define

$$\Delta\sigma^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2, \tag{1.2.2}$$

where σ is called the **proper length**. Notice that that only differences are that the signs of everything on the right hand side have been switched, and there is no c^2 on the left hand side.

Definition: Proper Length

The proper length is an invariant quantity that represents the physical distance between two points that are marked at the same time.

Box 1.2.1

In most real-world scenarios, the difference between $\Delta\tau$ and Δt is actually very small. It is so small, in fact, that if you were to calculate Δt and $\Delta\tau$ separately and then subtract them, some calculators will display an answer of zero! There is a trick that we can use, however, to get around this.

$$\begin{aligned}
c^2\Delta\tau^2 &= c^2\Delta t^2 - \Delta x^2 - \cancelto{0}{\Delta y^2} - \cancelto{0}{\Delta z^2} && \text{assume one-dimensional motion} \\
\Delta\tau^2 - \Delta t^2 &= -\frac{\Delta x^2}{c^2} && \text{get } \Delta\tau \text{ and } \Delta t \text{ on the same side} \\
(\Delta\tau - \Delta t)(\Delta\tau + \Delta t) &= -\frac{\Delta x^2}{c^2} && \text{factor left hand side} \\
\Delta\tau - \Delta t &= -\frac{\Delta x^2}{c^2(\Delta\tau + \Delta t)} && \text{isolate the difference between the two times} \\
\Delta\tau - \Delta t &\approx -\frac{\Delta x^2}{2c^2\Delta t} && \text{use the approximation } \Delta\tau \approx \Delta t
\end{aligned}$$

That last step may seem like cheating, but if you consider an airplane ride that lasts several hours, for example, the difference between $\Delta\tau$ and Δt is only on the order of nanoseconds (so it really is a perfectly fine approximation).

Suppose that a clock on an airplane in New York City is synchronized with a clock in Madrid. The airplane then flies from New York City to Madrid. Calculate an approximation of the difference in clock readings between the two clocks when the airplane lands in Madrid. Which clock is ahead? Make your assumptions clear and include your sources. (You may neglect the curvature and rotation of earth in your calculation, as well as any General Relativistic effects; we will discuss those later.)
