1.3: Natural Units

The speed of light c shows up a lot in relativity. For this reason, we will usually choose to work in a system of units for which $c=1$. This is equivalent to expressing time in seconds, for example, and distance in light-seconds (or years and light-years). The definition of the spacetime interval then becomes

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \qquad (c=1). \tag{1.3.1}$$

This may seem weird at first, but if you think about it, the only reason the speed of light in meters per second is so large is because of the way that a meter is defined. If we define a particular unit of length to be the distance that light travels in one second (i.e. a light-second), then the speed of light can be written as 1 light-second/second. We call this system of units **natural units**. The advantage of natural units is that it allows you to drop every instance of c from equations; the downside is that if you want numbers that are not in natural units, then you need to figure out how to reinstate all of the factors of c.

Recall that unit conversions involve multiplying by the number 1, only you write the number 1 in a clever way, like $\frac{1\text{ m}}{100\text{ cm}}$. If we define $c=1$, then, we can use the speed of light as a conversion factor. This allows us, for example, to write times in meters, distance in seconds, and speeds as unitless. We will practice this with some examples.

Exercise 1.3.1

If it takes you 10 minutes to drive to work, how many meters of time do you travel?

Answer

We start by converting minutes to seconds, then we use the speed of light as a conversion factor to end up with meters.

$$10\text{ minutes} \times \frac{60\text{ s}}{1\text{ min}} \times 3\times 10^8\ \frac{\text{m}}{\text{s}} = 1.8\times 10^{11}\text{ m}$$

Exercise 1.3.2

A spaceship travels a distance of 3 light-years at a constant speed of 0.6 (in natural units) relative to earth. How long does the trip take as measured in a reference frame on earth, and how long does the trip take as measured by a crew member inside the spaceship?

Answer

Since the 3 light-years is a distance as measured in the earth reference frame, we can use the definition of velocity to determine the time as measured in the earth reference frame. In the following, I will use subscripts to indicate the reference frame even though the measurements apply to the spaceship.

$$\begin{aligned}
v_{\text{earth}} &= \frac{\Delta x_{\text{earth}}}{\Delta t_{\text{earth}}} && \text{definition of velocity}\\
\Delta t_{\text{earth}} &= \frac{\Delta x_{\text{earth}}}{v_{\text{earth}}} && \text{solve for } \Delta t_{\text{earth}}\\
\Delta t_{\text{earth}} &= \frac{3\text{ light-years}}{0.6} && \text{substitute numbers}\\
\Delta t_{\text{earth}} &= 5\text{ years} && \text{final answer}
\end{aligned}$$

Now we can use the definition of the spacetime interval to determine the time interval measured by a crew member inside the spaceship. In this reference frame, the coordinate system is carried along with the ship, which means that $\Delta x_{\text{spaceship}} = 0$ and therefore $\Delta\tau = \Delta t_{\text{spaceship}}$. Since $\Delta\tau$ is invariant, we can find it using the numbers as measured in the earth's reference frame.

$$
\begin{aligned}
\Delta\tau^2 &= \Delta t^2_{\text{earth}} - \Delta x^2_{\text{earth}} = \Delta t^2_{\text{spaceship}} - \Delta x^2_{\text{spaceship}} && \text{spacetime interval is invariant} \\
\Delta\tau^2 &= \Delta t^2_{\text{earth}} - \Delta x^2_{\text{earth}} = \Delta t^2_{\text{spaceship}} && \text{substitute } \Delta x_{\text{spaceship}} = 0 \\
\Delta t_{\text{spaceship}} &= \sqrt{\Delta t^2_{\text{earth}} - \Delta x^2_{\text{earth}}} && \text{solve for } \Delta t_{\text{spaceship}} \\
\Delta t_{\text{spaceship}} &= \sqrt{(5\text{ years})^2 - (3\text{ light-years})^2} && \text{substitute numbers} \\
\Delta t_{\text{spaceship}} &= 4\text{ years} && \text{final answer}
\end{aligned}
$$

Box 1.3.1

The introduction of this chapter discussed muons that arise due to cosmic ray interactions in the atmosphere approximately 15000 m above the surface of earth. Suppose that they travel at a speed of 0.994 and have a half-life of 1.52 microseconds. That is, a large batch of muons decays such that the population gets cut in half every 1.52 microseconds.

a) In a non-relativistic world, how many muon half-lives would it take for one of these muons to reach the ground? What does that tell you about the percentage of muons that should reach the ground in a non-relativistic world?

b) In a relativistic world, how many muon half-lives would it take for one of these muons to reach the ground? What does that tell you about the percentage of muons that should reach the ground in a relativistic world?
