Book: Introduction to General Relativity

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About this Book

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In my experience, textbook treatments of General Relativity are either highly mathematical with little discussion of concepts or highly conceptual with little discussion of the mathematics. The former is usually designed for graduate students or advanced undergraduates while the latter is designed for a general audience. Very few resources exist that are appropriate for students in their second or third year, and that is the audience that I wanted to reach when I decided to offer my own course in General Relativity.

One excellent resource that I found was *Exploring Black Holes*, the first edition of which was written by Edwin F. Taylor and John Archibald Wheeler but is unfortunately out of print. The second edition is much more comprehensive, and is freely available on Edwin Taylor's website. This particular treatment requires only calculus and avoids differential equations by simply stating the metric and focusing on what we can learn from the metric. I used that book the first two times I taught the course, and it very much inspired the structure and style of this book. I decided to write my own book primarily for two reasons. First, I wanted to introduce four-vectors and index notation. Second, I found the freely available edition of *Exploring Black Holes* to have a little bit too much information to fit into a one-semester course.

For anyone who is already familiar with General Relativity, here are some things you will find:

- metrics
- constants of motion
- effective potential
- four-vectors
- index notation
- the geodesic equation

Here are some things you will *not* find in this book:

- Ricci tensor and its contractions
- Christoffel symbols
- killing vectors
- how to solve the Einstein Field Equations
- stress-energy tensor

One of my goals in writing this book was to keep things concise. This required cutting some content and leaving many derivations as an exercise to the reader.

I have divided the problems in this book into two main types: Exercises and Boxes. Exercises are either conceptual in nature or are meant to test fluency in basic skills. They are intended to be completed in class by the students if possible. All Exercises include answers so that students can review them after class. Boxes are primarily scaffolded derivations and real-world applications that are intended to be done outside of class.

Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing****.*

CHAPTER OVERVIEW

1: Special Relativity

Thumbnail: Artist concept of Gravity Probe B orbiting the Earth to measure space-time, a four-dimensional description of the universe including height, width, length, and time. (Public Domain; NASA).

1.1: A Need for a New Model

In the world of Newtonian Mechanics, it was assumed that measured lengths and time intervals were **observer-independent**. A meter stick, for example, is one meter long for everybody, and something that lasts for one second will last one second for everybody. The discovery that the speed of light is the same regardless of the observer's state of motion, however, changed that. Let's use some thought experiments to see why.

? Exercise 1.1.1

Diana is inside of a train car with glass walls that moves with a constant velocity relative to the ground. She puts a laser on the ground pointing straight up at a mirror on the ceiling. The laser sends a short pulse that reflects off the mirror on the ceiling and back to the floor.

Phaedra is outside watching the train car pass by.

What path does the light pulse follow from Diana's point of view? What path does the light pulse follow from Phaedra's point of view? Given that everyone observes the speed of light to be the same, do Diana and Phaedra agree on how long the light pulse takes to return to the floor? If not, is one of them wrong?

Answer

Diana sees the light pulse travel straight up and then straight back down. Phaedra sees the light travel a diagonal path up and then back down, which is a longer distance.

Since both Diana and Phaedra observe the same speed of light, the time it takes for the light pulse to reach the ground as measured by Phaedra must be greater than that measured by Diana.

Is one of them wrong? As long as neither Diana nor Phaedra made a measurement mistake, there is no way to judge whose conclusion is valid. We simply have to accept that the two observers disagree on how much time passed.

? Exercise 1.1.2

Lionel is on a skateboard that moves with a constant velocity relative to the ground. He has installed a laser at each end of the skateboard; when he presses a button, the lasers simultaneously fire pulses toward the center of the skateboard. At the center of the skateboard, there is a device that will set off fireworks if the two lasers hit it at the same time.

Amit watches Lionel from the ground.

If Lionel presses the button, will the fireworks go off? Consider the situation as viewed by both Lionel and Amit.

Answer

You may think at first that Lionel would observe a faster speed of light for the light traveling from the front of the skateboard than the back. That would be true for everything except for light. In this case, both light pulses travel toward Lionel at the same speed, which means they would reach the center of the skateboard at the same time. We can conclude that the fireworks will go off.

Things look different from Amit's point of view. The light pulses travel in opposite directions at the same speed, but while they are doing that the skateboard moves toward one of them. It seems, then, that the device in the center of the skateboard will not register the two pulses as arriving simultaneously, and the fireworks will not go off.

So which is it, do the fireworks go off or not? Given that Lionel and Amit can meet up afterward and check to see if the fireworks went off, there must be an objective reality. We must conclude either that one of them was wrong or that one of our conclusions was incorrect because we made a faulty assumption. In the previous exercise, we discussed how as long as neither Diana nor Phaedra made a measurement mistake, we can't say that one of them is wrong. The same reasoning applies here, which means that we must have made a faulty assumption.

It turns out that there is a hidden assumption we made, which is that what is simultaneous for Lionel (the firing of the lasers) is also simultaneous for Amit. If Amit saw the laser pulse from the front of the skateboard fire *after* the one from the back, then it is plausible that the two could arrive at the center at the same time.

Exercise 1.1.3

A fish flies past you. Devise a method for measuring the length of the fish *while it is moving*. If someone who is moving along with the fish watches you apply this method, will they agree with the result you come up with? Why or why not?

Answer

The key to making the measurement is to mark the locations of the ends of the fish at the same time, and then use a ruler to measure the distance between those locations. If you don't mark them at the same time, the fish will have moved in the meantime and the measurement will be wrong.

In the previous exercise, however, we found that two events that are simultaneous for one observer may not be simultaneous for another observer. So an observer moving along with the fish will not agree with your measurement of the length.

The previous thought experiments demonstrate that measurements of time intervals, lengths, and simultaneity are all relative. This is the main idea of the **Special Theory of Relativity**, where the word *special* here refers to the fact that the theory applies to constant velocity motion, which is a special case type of motion.

Definition: The Special Theory of Relativity

The Special Theory of Relativity relates time intervals and lengths measured by different observers in the special case of constant velocity motion. For brevity, I will often refer to it as simply Special Relativity.

Note

Some people without a solid understanding of Special Relativity think that it implies that *everything* is relative, but that is not the case. In fact, it is the exist of quantities that are *not* relative that allows us to make predictions.

1.2: The Spacetime Interval

In Newtonian mechanics, the length of an object is invariant. That is, if you use a different coordinate system to mark locations, the result you get for the length does not change.

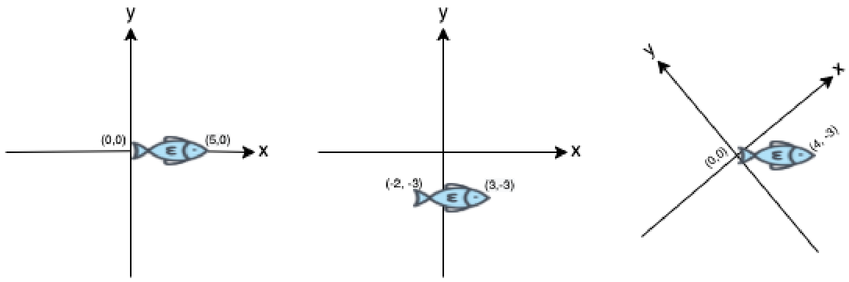


Figure 1.2.1: The length of the fish is 5 units regardless of the choice of coordinate system.

In Special Relativity, however, it is a combination of distance and time that is invariant from one coordinate system to another. We call this the **spacetime interval**.

Definition: Spacetime Interval

The spacetime interval is a combination of distance and time that is **invariant**.

Definition: Invariant

An invariant is a quantity that has the same value for all observers.

For constant velocity motion in **flat spacetime** (we will discuss what that means later), the spacetime interval is

$$c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2., \tag{1.2.1}$$

where $c = 3\times 10^8$ m/s is the speed of light. The coordinates t, x, y, and z mark an **event** that depends on the **reference frame**. The variable τ, called the **proper time**, is the same in all reference frames.

Definition: Event

An event is something that has both a location and a time.

Definition: Reference Frame

A reference frame is a coordinate system, usually imagined to be associated with a particular observer. Reference frames can move with respect to one another. A reference frame in which Newton's First Law holds (i.e. an object at rest stays at rest and an object in motion maintains it's velocity) is called an **inertial reference frame**.

Definition: Proper Time

The proper time is an invariant quantity that can be determined from the spacetime interval. Another name I like to use for proper time is **wristwatch time***, because it is the time interval measured by an inertial observer who is present at both events.

*This term is borrowed from *Exploring Black Holes* by Edwin Taylor and John Wheeler.

❓ Exercise 1.2.1

Two events are separated by 50 ns and 12 m in Joo-Won's reference frame. The events are separated by 4 m in Fawn's reference frame. What is the time interval between the events in Fawn's reference frame?

Answer

We know that both reference frames will agree on the proper time between the events, which we can calculate using the spacetime interval. To simplify the calculation, we will orient the reference frame so that the events are on the x-axis. In Joo-Won's reference frame, this yields

$$\begin{aligned} c^2\Delta\tau^2 &= c^2\Delta t_J^2 - \Delta x_J^2 && \text{start with spacetime interval} \\ \Delta\tau &= \sqrt{\frac{c^2\Delta t_J^2 - \Delta x_J^2}{c^2}} && \text{solve for } \tau \\ \Delta\tau &= \sqrt{\frac{\left(3\times10^8\ \mathrm{m/s}\right)^2\left(50\times10^{-9}\ \mathrm{s}\right)^2 - (12\ \mathrm{m})^2}{\left(3\times10^8\ \mathrm{m/s}\right)^2}} && \text{substitute numbers} \\ \Delta\tau &= 30\ \mathrm{ns} && \text{solve for } \tau \end{aligned}$$

Now we can use the proper time to determine Δt in Fawn's reference frame.

$$\begin{aligned} c^2\Delta\tau^2 &= c^2\Delta t_F^2 - \Delta x_F^2 && \text{start with spacetime interval} \\ \Delta t_F &= \sqrt{\frac{c^2\Delta\tau^2 + \Delta x_F^2}{c^2}} && \text{solve for } t_F \\ \Delta t_F &= \sqrt{\frac{\left(3\times10^8\ \mathrm{m/s}\right)^2\left(30\times10^{-9}\ \mathrm{s}\right)^2 + (4\ \mathrm{m})^2}{\left(3\times10^8\ \mathrm{m/s}\right)^2}} && \text{substitute numbers} \\ \Delta t_F &= 32.8\ \mathrm{ns} && \text{final answer} \end{aligned}$$

Notice that $\Delta\tau$ is only appreciably different from Δt if Δt is small and/or if Δx is large. In other words, relativistic effects are only noticeable if $\frac{\Delta x}{\Delta t}$ is large (specifically, close to the speed of light).

You may have noticed that proper time is only a real number if $\frac{\Delta x}{\Delta t} \leq c$. What does it mean for the proper time to be imaginary? Since proper time is the time measured on the wristwatch of an inertial observer who is present at both events, an imaginary proper time means that it is impossible for an observer to be present at both events. In other words, nothing could move fast enough to be present at both events. Not even light. And if light isn't fast enough to be present at both events, then there can be no causal connection between them. As an analogy, imagine that a house is broken into at midnight. If Marcus left a party across town at 11:59 pm, then he couldn't have possibly perpetrated the crime. If Gerry left the same party at 11 pm, then he *could have* perpetrated the crime.

There are three different categories we can use to describe pairs of events: timelike, spacelike, and lightlike.

Definition: Timelike, Spacelike, Lightlike

Events with a timelike separation can be causally connected while events with a spacelike separation are causally disconnected. Events with a lightlike separation are exactly far enough away from each other that light could be present at both events (they are still causally connected).

$$\begin{aligned} \text{timelike interval: } \Delta\tau^2 > 0 \\ \text{spacelike interval: } \Delta\tau^2 < 0 \\ \text{lightlike interval: } \Delta\tau^2 = 0 \end{aligned}$$

It may seem odd that $\Delta\tau^2$ could be negative, since that would lead to an imaginary value for the proper time. Nevertheless, spacelike intervals do exist. Any time you want to measure the length of something for example, you mark the locations of the endpoints at the same time, which is necessarily going to yield a negative $\Delta\tau^2$. For this reason, we will define

$$\Delta\sigma^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2, \tag{1.2.2}$$

where σ is called the **proper length**. Notice that that only differences are that the signs of everything on the right hand side have been switched, and there is no c^2 on the left hand side.

Definition: Proper Length

The proper length is an invariant quantity that represents the physical distance between two points that are marked at the same time.

Box 1.2.1

In most real-world scenarios, the difference between $\Delta\tau$ and Δt is actually very small. It is so small, in fact, that if you were to calculate Δt and $\Delta\tau$ separately and then subtract them, some calculators will display an answer of zero! There is a trick that we can use, however, to get around this.

$$\begin{aligned}
c^2\Delta\tau^2 &= c^2\Delta t^2 - \Delta x^2 - \cancelto{0}{\Delta y^2} - \cancelto{0}{\Delta z^2} && \text{assume one-dimensional motion} \\
\Delta\tau^2 - \Delta t^2 &= -\frac{\Delta x^2}{c^2} && \text{get } \Delta\tau \text{ and } \Delta t \text{ on the same side} \\
(\Delta\tau - \Delta t)(\Delta\tau + \Delta t) &= -\frac{\Delta x^2}{c^2} && \text{factor left hand side} \\
\Delta\tau - \Delta t &= -\frac{\Delta x^2}{c^2(\Delta\tau + \Delta t)} && \text{isolate the difference between the two times} \\
\Delta\tau - \Delta t &\approx -\frac{\Delta x^2}{2c^2\Delta t} && \text{use the approximation } \Delta\tau \approx \Delta t
\end{aligned}$$

That last step may seem like cheating, but if you consider an airplane ride that lasts several hours, for example, the difference between $\Delta\tau$ and Δt is only on the order of nanoseconds (so it really is a perfectly fine approximation).

Suppose that a clock on an airplane in New York City is synchronized with a clock in Madrid. The airplane then flies from New York City to Madrid. Calculate an approximation of the difference in clock readings between the two clocks when the airplane lands in Madrid. Which clock is ahead? Make your assumptions clear and include your sources. (You may neglect the curvature and rotation of earth in your calculation, as well as any General Relativistic effects; we will discuss those later.)

1.3: Natural Units

The speed of light c shows up a lot in relativity. For this reason, we will usually choose to work in a system of units for which $c = 1$. This is equivalent to expressing time in seconds, for example, and distance in light-seconds (or years and light-years). The definition of the spacetime interval then becomes

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \qquad (c = 1). \tag{1.3.1}$$

This may seem weird at first, but if you think about it, the only reason the speed of light in meters per second is so large is because of the way that a meter is defined. If we define a particular unit of length to be the distance that light travels in one second (i.e. a light-second), then the speed of light can be written as 1 light-second/second. We call this system of units **natural units**. The advantage of natural units is that it allows you to drop every instance of c from equations; the downside is that if you want numbers that are not in natural units, then you need to figure out how to reinstate all of the factors of c.

Recall that unit conversions involve multiplying by the number 1, only you write the number 1 in a clever way, like $\frac{1 \text{ m}}{100 \text{ cm}}$. If we define $c = 1$, then, we can use the speed of light as a conversion factor. This allows us, for example, to write times in meters, distance in seconds, and speeds as unitless. We will practice this with some examples.

Exercise 1.3.1

If it takes you 10 minutes to drive to work, how many meters of time do you travel?

Answer

We start by converting minutes to seconds, then we use the speed of light as a conversion factor to end up with meters.

$$10 \text{ minutes} \times \frac{60 \text{ s}}{1 \text{ min}} \times 3 \times 10^8 \ \frac{\text{m}}{\text{s}} = 1.8 \times 10^{11} \text{ m}$$

Exercise 1.3.2

A spaceship travels a distance of 3 light-years at a constant speed of 0.6 (in natural units) relative to earth. How long does the trip take as measured in a reference frame on earth, and how long does the trip take as measured by a crew member inside the spaceship?

Answer

Since the 3 light-years is a distance as measured in the earth reference frame, we can use the definition of velocity to determine the time as measured in the earth reference frame. In the following, I will use subscripts to indicate the reference frame even though the measurements apply to the spaceship.

$$\begin{aligned} v_{\text{earth}} &= \frac{\Delta x_{\text{earth}}}{\Delta t_{\text{earth}}} && \text{definition of velocity} \\ \Delta t_{\text{earth}} &= \frac{\Delta x_{\text{earth}}}{v_{\text{earth}}} && \text{solve for } \Delta t_{\text{earth}} \\ \Delta t_{\text{earth}} &= \frac{3 \text{ light-years}}{0.6} && \text{substitute numbers} \\ \Delta t_{\text{earth}} &= 5 \text{ years} && \text{final answer} \end{aligned}$$

Now we can use the definition of the spacetime interval to determine the time interval measured by a crew member inside the spaceship. In this reference frame, the coordinate system is carried along with the ship, which means that $\Delta x_{\text{spaceship}} = 0$ and therefore $\Delta\tau = \Delta t_{\text{spaceship}}$. Since $\Delta\tau$ is invariant, we can find it using the numbers as measured in the earth's reference frame.

$$\begin{aligned}
\Delta\tau^2 &= \Delta t^2_{\text{earth}} - \Delta x^2_{\text{earth}} = \Delta t^2_{\text{spaceship}} - \Delta x^2_{\text{spaceship}} && \text{spacetime interval is invariant} \\
\Delta\tau^2 &= \Delta t^2_{\text{earth}} - \Delta x^2_{\text{earth}} = \Delta t^2_{\text{spaceship}} && \text{substitute } \Delta x_{\text{spaceship}} = 0 \\
\Delta t_{\text{spaceship}} &= \sqrt{\Delta t^2_{\text{earth}} - \Delta x^2_{\text{earth}}} && \text{solve for } \Delta t_{\text{spaceship}} \\
\Delta t_{\text{spaceship}} &= \sqrt{(5 \text{ years})^2 - (3 \text{ light-years})^2} && \text{substitute numbers} \\
\Delta t_{\text{spaceship}} &= 4 \text{ years} && \text{final answer}
\end{aligned}$$

Box 1.3.1

The introduction of this chapter discussed muons that arise due to cosmic ray interactions in the atmosphere approximately 15000 m above the surface of earth. Suppose that they travel at a speed of 0.994 and have a half-life of 1.52 microseconds. That is, a large batch of muons decays such that the population gets cut in half every 1.52 microseconds.

a) In a non-relativistic world, how many muon half-lives would it take for one of these muons to reach the ground? What does that tell you about the percentage of muons that should reach the ground in a non-relativistic world?

b) In a relativistic world, how many muon half-lives would it take for one of these muons to reach the ground? What does that tell you about the percentage of muons that should reach the ground in a relativistic world?

1.4: Spacetime Diagrams

A **spacetime diagram** is used for plotting and visualizing events and **worldlines**.

> **Definition: Worldline**
>
> A worldline is a path through spacetime.

By convention, time is put on the vertical axis of spacetime diagrams, which means that the slope is not equal to the velocity like it is in typical position vs. time graphs. Instead, the slope is equal to the reciprocal of the velocity. An example spacetime diagram is shown in Fig. 1.4.1.

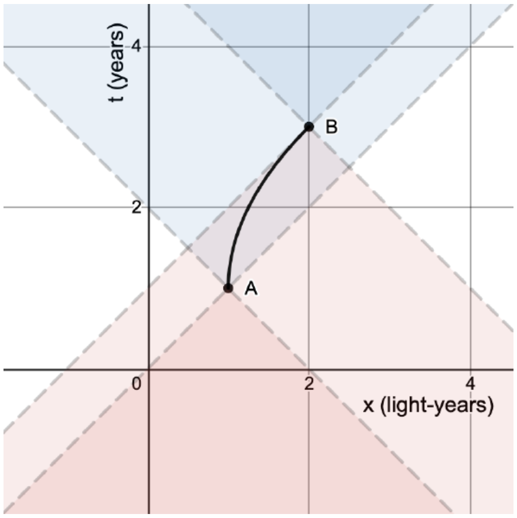


Figure 1.4.1: A and B are events. The dashed lines are light cones at A and B. The solid line is an example of a wordline connecting A and B.

The diagram shows two events, labeled A and B, that are connected by a wordline (solid line) that shows an object that starts at rest and accelerates close to the speed of light. Each event has a pair of lines (dashed lines) that form **light cones**.

> **Definition: Light Cone**
>
> A light cone is formed by the set of all possible wordlines of light that intersect a particular event, and it divides the spacetime into past, future, and *elsewhere* (causally disconnected).

The blue region above each event represents the set of all causally-connected events that exist in the event's future. The red region below each event represents the set of all causally-connected events that exist in the event's past. Events outside of the light cones are far enough away that even light could not be present at both.

> **Exercise 1.4.1**
>
> For each of the events in Figure 1.4.2, indicate whether the other events are in the past, future, or elsewhere regions.

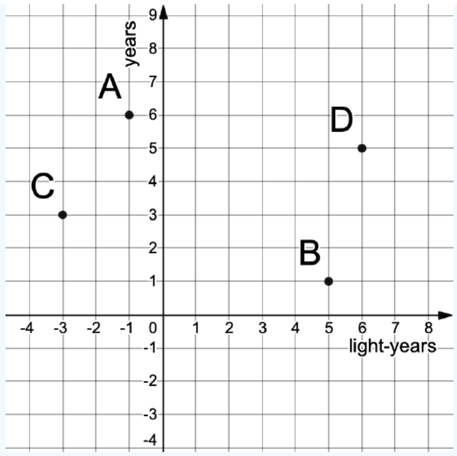


Figure 1.4.2: A spacetime diagram with four events indicated.

Answer

For each event, you can draw a pair of lines at 45 degree angles to divide up the space. Events within the light cone and above are in the future, within the light cone and below are in the past, and others are elsewhere.

- Event A: C in the past, all others elsewhere
- Event B: D in the future, all others elsewhere
- Event C: A in the future, all others elsewhere
- Event D: B in the past, all others elsewhere

Exercise 1.4.2

Events A and B are connected by three different wordlines, as shown in Fig. 1.4.3. Calculate the proper time along each worldine (AB, ACB, and ADB).

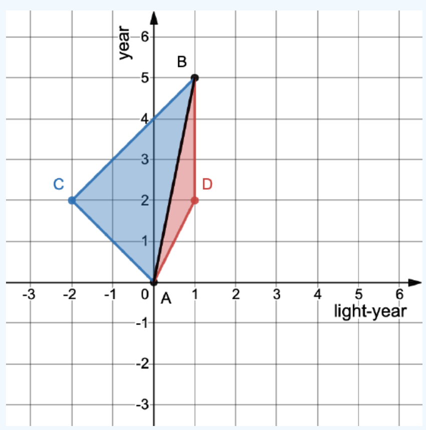


Figure 1.4.3: Events A and B are connected by three different worldlines.

Answer

We can use the definition of the spacetime interval to calculate the proper time. For worldlines ACB and ADB, we must break the calculation into two separate segments.

For worldline AB:

$$\begin{aligned}
\Delta\tau_{\mathrm{AB}}^2 &= \Delta t_{\mathrm{AB}}^2 - \Delta x_{\mathrm{AB}}^2 && \text{definition of spacetime interval} \\
\Delta\tau_{\mathrm{AB}} &= \sqrt{\Delta t_{\mathrm{AB}}^2 - \Delta x_{\mathrm{AB}}^2} && \text{solve for } \Delta\tau_{\mathrm{AB}} \\
\Delta\tau_{\mathrm{AB}} &= \sqrt{(5 \text{ year})^2 - (1 \text{ light-year})^2} && \text{substitute numbers} \\
\Delta\tau_{\mathrm{AB}} &= 4.90 \text{ years} && \text{final answer}
\end{aligned}$$

For wordline ACB:

$$\begin{aligned}
\Delta\tau_{\mathrm{AC}}^2 &= \Delta t_{\mathrm{AC}}^2 - \Delta x_{\mathrm{AC}}^2 && \text{definition of spacetime interval} \\
\Delta\tau_{\mathrm{AC}} &= \sqrt{\Delta t_{\mathrm{AC}}^2 - \Delta x_{\mathrm{AC}}^2} && \text{solve for } \Delta\tau_{\mathrm{AC}} \\
\Delta\tau_{\mathrm{AC}} &= \sqrt{(2 \text{ year})^2 - (-2 \text{ light-year})^2} && \text{substitute numbers} \\
\Delta\tau_{\mathrm{AC}} &= 0 \text{ years} && \text{final answer for AC}
\end{aligned}$$

$$\begin{aligned}
\Delta\tau_{\mathrm{CB}}^2 &= \Delta t_{\mathrm{CB}}^2 - \Delta x_{\mathrm{CB}}^2 && \text{definition of spacetime interval} \\
\Delta\tau_{\mathrm{CB}} &= \sqrt{\Delta t_{\mathrm{CB}}^2 - \Delta x_{\mathrm{CB}}^2} && \text{solve for } \Delta\tau_{\mathrm{CB}} \\
\Delta\tau_{\mathrm{CB}} &= \sqrt{(3 \text{ year})^2 - (3 \text{ light-year})^2} && \text{substitute numbers} \\
\Delta\tau_{\mathrm{CB}} &= 0 \text{ years} && \text{final answer for CB}
\end{aligned}$$

$$\Delta\tau_{\mathrm{ACB}} = \Delta\tau_{\mathrm{AC}} + \Delta\tau_{\mathrm{CB}} = 0 \text{ years}$$

For worldline ADB:

$$\begin{aligned}
\Delta\tau_{\mathrm{AD}}^2 &= \Delta t_{\mathrm{AD}}^2 - \Delta x_{\mathrm{AD}}^2 && \text{definition of spacetime interval} \\
\Delta\tau_{\mathrm{AD}} &= \sqrt{\Delta t_{\mathrm{AD}}^2 - \Delta x_{\mathrm{AD}}^2} && \text{solve for } \Delta\tau_{\mathrm{AD}} \\
\Delta\tau_{\mathrm{AD}} &= \sqrt{(2 \text{ year})^2 - (1 \text{ light-year})^2} && \text{substitute numbers} \\
\Delta\tau_{\mathrm{AD}} &= 1.73 \text{ years} && \text{final answer for AD}
\end{aligned}$$

$$\begin{aligned}
\Delta\tau_{\mathrm{DB}}^2 &= \Delta t_{\mathrm{DB}}^2 - \Delta x_{\mathrm{DB}}^2 && \text{definition of spacetime interval} \\
\Delta\tau_{\mathrm{DB}} &= \sqrt{\Delta t_{\mathrm{DB}}^2 - \Delta x_{\mathrm{DB}}^2} && \text{solve for } \Delta\tau_{\mathrm{DB}} \\
\Delta\tau_{\mathrm{DB}} &= \sqrt{(3 \text{ year})^2 - (0 \text{ light-year})^2} && \text{substitute numbers} \\
\Delta\tau_{\mathrm{CB}} &= 3 \text{ years} && \text{final answer for DB}
\end{aligned}$$

$$\Delta\tau_{\mathrm{ADB}} = \Delta\tau_{\mathrm{AD}} + \Delta\tau_{\mathrm{DB}} = 4.73 \text{ years}$$

In the previous exercise, note that the wordline connecting A and B directly was the one with the greatest proper time. That is also the path that a free particle would follow (i.e. the inertial path). This fact turns out to be true generally, not just for this problem. We call this phenomenon **The Principle of Maximal Aging**.

Definition: The Principle of Maximal Aging*

The wordline that a free particle would take between two events is the one with the greatest proper time compared to all other wordlines.

**This terminology is borrowed from Exploring Black Holes by Edwin Taylor and John Wheeler.*

1.5: Four-Momentum

In Newtonian mechanics, position in space can be indicated with a three-dimensional vector. In Special Relativity, however, events are indicated using *four* coordinates: $\mathbf{x} = (t, x, y, z)$.

The **four-velocity** is defined as

$$\mathbf{u} = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right). \tag{1.5.1}$$

Definition: Four-velocity

Four-velocity is defined by Equation 1.5.1. It is the four-dimensional relativistic counterpart to velocity.

Note

So far we have been writing Δ to refer to coordinate differences. When we instead use "d" as in Equation 1.5.1, we are specifically dealing with infinitesimally small coordinate differences. This is similar to how *average* velocity $v_{\text{avg}} = \frac{\Delta x}{\Delta t}$ becomes *instantaneous* velocity $v = \frac{dx}{dt}$ when we make the time interval infinitesimally small.

If we multiply the four-velocity by the mass of the particle, we get the **four-momentum**.

$$\mathbf{p} = \left(m\frac{dt}{d\tau}, m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\right) \tag{1.5.2}$$

Definition: Four-momentum

Four-momentum is defined by Equation 1.5.2. It is the four-dimensional relativistic counterpart to momentum.

You may recognize the spatial components as looking a lot like how momentum is defined in Newtonian mechanics: $\vec{p} = m\vec{v}$ (I will use **bold** to indicate a four-vector and an arrow to indicate a three-vector). So what is the meaning of the time-component of the four-momentum? It turns out that it is the energy, which can be shown by looking at the low-velocity limit (see Box 1.5.5). This allows us to write the four-momentum as

$$\mathbf{p} = (E, p^x, p^y, p^z)\,, \tag{1.5.3}$$

where E is the energy. Using the spacetime interval, we can derive (see Box 1.5.1)

$$E = m\gamma \tag{1.5.4}$$

and (see Box 1.5.2)

$$\vec{p} = \gamma m\vec{v} \tag{1.5.5}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}. \tag{1.5.6}$$

We can also show that energy and momentum are related (see Box 1.5.3), as shown in Equation 1.5.7.

$$m^2 = E^2 - p^2 \tag{1.5.7}$$

As in Newtonian mechanics, momentum and energy are **constants of motion** in relativity, which makes them helpful for predicting motion.

Definition: Constant of Motion

A constant of motion is a quantity that has the same value at all points along a particular worldline.

Note

A constant of motion is distinct from an invariant. Constants of motion keep the same value within a particular reference frame but can be different in different reference frames. An invariant has the same value in all reference frames.

Box 1.5.1

Divide the infinitesimal spacetime interval $d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$ by dt^2, then use it along with the definition $E = m\frac{dt}{d\tau}$ to show that $E = m\gamma$.

Box 1.5.2

Use the fact that $\frac{dx}{d\tau}$ can be written as $\frac{dx}{dt}\frac{dt}{d\tau}$ along with the definition $\vec{p} = \left(m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\right)$ to show that $\vec{p} = m\gamma\vec{v}$.

Box 1.5.3

Divide the infinitesimal spacetime interval $d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$ by dt^2, then use $E = m\gamma$ and $\vec{p} = m\gamma\vec{v}$ to show that $m^2 = E^2 - p^2$.

Box 1.5.4

The equation $m^2 = E^2 - p^2$ from the previous box is written in natural units, which implies that energy and momentum have units of mass. One way to restore the equation to SI units is to figure out how many factors of c (in m/s) you need to insert in each term in order to get the units to all match. Do this.

Box 1.5.5

The total energy of a particle in relativity can be thought of as the sum of its kinetic energy and its *rest energy*. In natural units, the rest energy is m. (In non-natural units, the rest energy is mc^2, which might look familiar.) This means that the kinetic energy can be written as $KE = E - m = m(\gamma - 1)$, which provides a way for us to confirm that $m\frac{dt}{d\tau}$ represents energy. To do that, we need to take the low-speed limit of the kinetic energy and show that it is equal to the Newtonian expression $KE = \frac{1}{2}mv^2$.

The derivation requires using the *binomial approximation*, which states that $(1+x)^n \approx 1 + nx$ if x is very small. Use this approximation to show that the relativistic expression $KE = m(\gamma - 1)$ reduces to the Newtonian expression $KE = \frac{1}{2}mv^2$ in the speed is very small.

1.6: Index Notation

You may be familiar with something called a **dot product**, which is a way of multiplying two vectors together. By definition, the dot product of two vectors $\vec{a}$ and $\vec{b}$ is usually written as

$$\vec{a}\cdot\vec{b}=a_x b_x+a_y b_y+a_z b_z.$$

If you are familiar with linear algebra, you may know that the previous expression can also be written as

$$\vec{a}\cdot\vec{b}=\begin{pmatrix}a_x & a_y & a_z\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}b_x\\b_y\\b_z\end{pmatrix}.$$

If you aren't familiar with matrix multiplication in linear algebra, you may instead think of it as

$$\vec{a}\cdot\vec{b}=\sum_{i,j=1}^{3}\eta_{ij}a^i b^j,$$

where

$$\eta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

is called the **metric** of the space and where the indices 1-3 represent the spatial components. If you take the dot product of a vector with itself, then you end up with the Pythagorean Theorem, which means that the metric essentially tells you how to find the length of a line. More generally, the metric defines the rules of geometry.

Definition: Metric

The metric is a function or matrix that can be used to determine the distance between two points. It can be thought of as defining the rules of geometry.

What is the relevance of all this? Recall that the spacetime interval is defined as

$$\Delta\tau^2=\Delta t^2-\Delta x^2-\Delta y^2-\Delta z^2,$$

which suggests that it can also be written as

$$\Delta\tau^2=\begin{pmatrix}\Delta t & \Delta x & \Delta y & \Delta z\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\Delta t\\\Delta x\\\Delta y\\\Delta z\end{pmatrix}=\sum_{\mu,\nu=0}^{3}\eta_{\mu\nu}\Delta x^\mu\Delta x^\nu, \tag{1.6.1}$$

where

$$\eta_{\mu\nu}=\begin{cases}1, & \mu=\nu=0\\ -1, & \mu=\nu=1,2,3\\ 0, & \mu\neq\nu\end{cases} \tag{1.6.2}$$

and where $x^0=t$, $x^1=x$, $x^2=y$, and $x^3=z$ (note that these are superscripts, not exponents).

Note

Some texts use the opposite sign convention for the metric, where the time component is negative and the spatial components are positive. Both sign conventions work, but some equations involving the metric will look different depending on which sign convention you are using.

For the remainder of this book, we will assume the following conventions:

1. If an index appears both "downstairs" and "upstairs," we can drop the summation symbol and assume the summation.
2. Roman letters such as *i* and *j* are for spatial components only.
3. Greek letters such as μ and ν are for all four spacetime components.

With this convention, the spacetime interval is

$$\Delta\tau^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu. \tag{1.6.3}$$

The following rules and definitions will also be useful to us.

1. Any index that is not summed over is called a **free index**.
2. The free indices on both sides of an equation must be the same.
3. Indices can be renamed.
4. The same index can't appear downstairs more than once or upstairs more the once.
5. Any index can be lowered using the metric (by definition). For example, $x_\mu = \eta_{\mu\nu}x^\nu$.
6. $u_\mu u^\mu = 1$, where u^μ is the four-velocity. (See Box 1.6.1)
7. The **inverse metric** $\eta_{\mu\nu}$ is defined by $\eta_{\alpha\mu}\eta^{\mu\nu} = \mathbf{I}$, where **I** is the **identity matrix** (1's on the diagonal and 0's everywhere else). (See Box 1.6.2)

Exercise 1.6.1

How is a vector with downstairs index different from its upstairs counterpart?

Answer

One of our rules is that $x_\mu = \eta_{\mu\nu}x^\nu$. Note that the sum only occurs over ν, since that is the only index that appears both downstairs and upstairs. The index μ is called a **free index** because it can take on any value 0-3 (i.e. t, x, y, or z).

$$x_\mu = \eta_{\mu t}x^t + \eta_{\mu x}x^x + \eta_{\mu y}x^y + \eta_{\mu z}x^z$$

The result depends on the value of μ. Let's check each one.

$$\begin{aligned} x_t &= \eta_{tt}x^t + \eta_{tx}x^x + \eta_{ty}x^y + \eta_{tz}x^z & &= x^t \\ x_x &= \eta_{xt}x^t + \eta_{xx}x^x + \eta_{xy}x^y + \eta_{xz}x^z & &= -x^x \\ x_y &= \eta_{yt}x^t + \eta_{yx}x^x + \eta_{yy}x^y + \eta_{yz}x^z & &= -x^y \\ x_z &= \eta_{zt}x^t + \eta_{zx}x^x + \eta_{zy}x^y + \eta_{zz}x^z & &= -x^z \end{aligned}$$

Therefore $x_\mu = \begin{pmatrix} t & -x & -y & -z \end{pmatrix}$.

Note that x_μ is exactly the same as x^μ except that the signs of the spatial indices have been reversed. There is also nothing special about x^μ; we could replace x^μ with any four-vector and the result would be that the vector with downstairs index has the same components but with the signs of the spatial components reversed.

Exercise 1.6.2

For each part below, indicate the indices that are free indices (or say "none").

a) $a_{\mu\nu}b^\mu c^\nu$

b) $a_{\mu\nu}b^\nu_\alpha$

c) $a_{\alpha\mu}a_{\beta\nu}b^{\mu\nu}c_\sigma + a_{\sigma\mu}a_{\alpha\nu}b^{\mu\nu}c_\beta + a_{\beta\mu}a_{\sigma\nu}b^{\mu\nu}c_\alpha$

d) $a^{\mu\nu}b_{\mu\alpha}b_{\nu\beta}c^\alpha c^\beta$

Answer

a) none

b) μ, α

c) α, β, and σ

d) none

? Exercise 1.6.3

Which of the following violate the index rules?

a) $a^{\mu}=b^{\alpha}c^{\mu}_{\beta}$

b) $a^{\mu}b_{\mu\nu}c^{\nu}+m^{2}=0$

c) $a^{\mu}b_{\mu\nu}=m^{2}$

c) $a_{\alpha\beta}=a_{\mu\nu}b^{\mu}_{\alpha}b^{\nu\beta}$

Answer

a) Violates. Left side has on free index while right side has three.

b) No violation. The first time has no free indices, so it can be added to a scalar.

c) Violates. The left side has a free index while the right side does not.

d) Violates. α and β are free indices on both sides, but β is downstairs on one side and upstairs on the other.

? Box 1.6.1

Prove that $u_{\mu}u^{\mu}=1$, where **u** is the four-velocity

? Box 1.6.2

Show that the inverse metric $\eta^{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$. (Note that both indices are upstairs.)

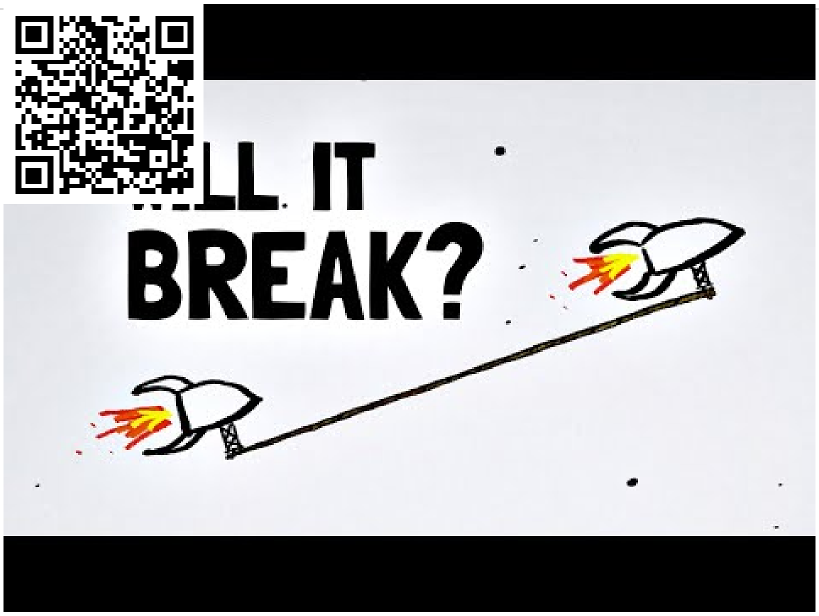
1.0: A Muon Anomaly

Created about 15000 meters above the Earth's surface when cosmic rays strike the upper atmosphere and lasting a mere microsecond or two, muons shouldn't be able to reach the ground. At least not according to classical mechanics. But particle detectors show that muons strike every inch of the Earth's surface, passing through almost everything in their paths. In fact, according to the DOE, muons may reach a mile or more beneath the Earth's surface.

How can we explain this?

Because of time dilation postulated by Special Relativity, the muon's speed (very close to speed of light) causes the muon to survive for longer than we would expect from classical mechanics, allowing the muon to travel a much farther distance than we might have expected.

1.7: Video Resources



CHAPTER OVERVIEW

2: Schwarzschild Geometry

Thumbnail: The supermassive black hole at the core of supergiant elliptical galaxy Messier 87, with a mass about 7 billion times that of the Sun, as depicted in the first false-colour image in radio waves released by the Event Horizon Telescope (10 April 2019). Visible are the crescent-shaped emission ring and central shadow,[19] which are gravitationally magnified views of the black hole's photon ring and the photon capture zone of its event horizon. The crescent shape arises from the black hole's rotation and relativistic beaming; the shadow is about 2.6 times the diameter of the event horizon. (CC BY 4.0; Event Horizon Telescope)

2.1: Non-Euclidean Geometry

We said previously that the metric defines the rules of geometry. In 3D Euclidean space, those rules include:

1. the length of the hypotenuse of a right triangle is determined by the Pythagorean theorem
2. the three angles of a triangle add up to 180 degrees
3. parallel lines never cross.

How could those rules could ever *not* apply? Let's consider an *operational definition* for a straight line given two thumbtacks and a piece of string. (An operational definition is a definition that arises from specifying a procedure.) One way to create a straight line between two points with these materials is to stretch the string between the points such that it is taut, then place a thumbtack at each of those points.

Exercise 2.1.1

Imagine applying the operational definition of a straight line to a globe. Do the three rules of geometry for Euclidean space listed above apply?

Answer

Suppose we apply the straight line procedure to the North Pole and to a point on the equator with 0 degrees longitude; a string connecting those two points would be curved.

Now pick a third point, say 50 degrees longitude on the equator, and create two more straight lines using the first two points to create a triangle.

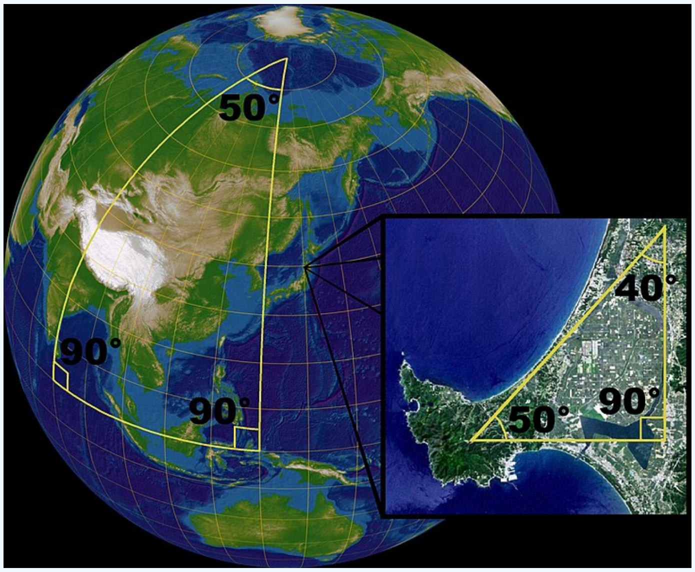


Figure 2.1.1: Three straight lines on the surface of a globe. On a small enough portion of the globe, the rules of geometry are Euclidean. (Image credit: Wikipedia)

Since both lines connecting the equator to the North Pole are perpendicular to the equator, they are parallel. But they also intersect, so it is not true that parallel lines never cross.

Lastly, the angles in the triangle add up to 230 degrees, not 180 degrees. It also shouldn't be too surprising to you that the Pythagorean theorem doesn't not apply to such a triangle.

Note that, on a small enough patch of the earth, the rules of Euclidean geometry *do* apply, as shown in the inset of Figure 2.1.1. In principle, then, we can think of a large, curved patch of earth as consisting of many small patches (where Euclidean geometry applies) stitched together.

A space in which the rules of Euclidean space don't apply is called non-Euclidean. The reason for bringing this up is because our modern understanding of gravity is that particles subject to gravity exhibit curved motion not because there is a force acting on them but because spacetime is non-Euclidean. Let's use some thought experiments to see why.

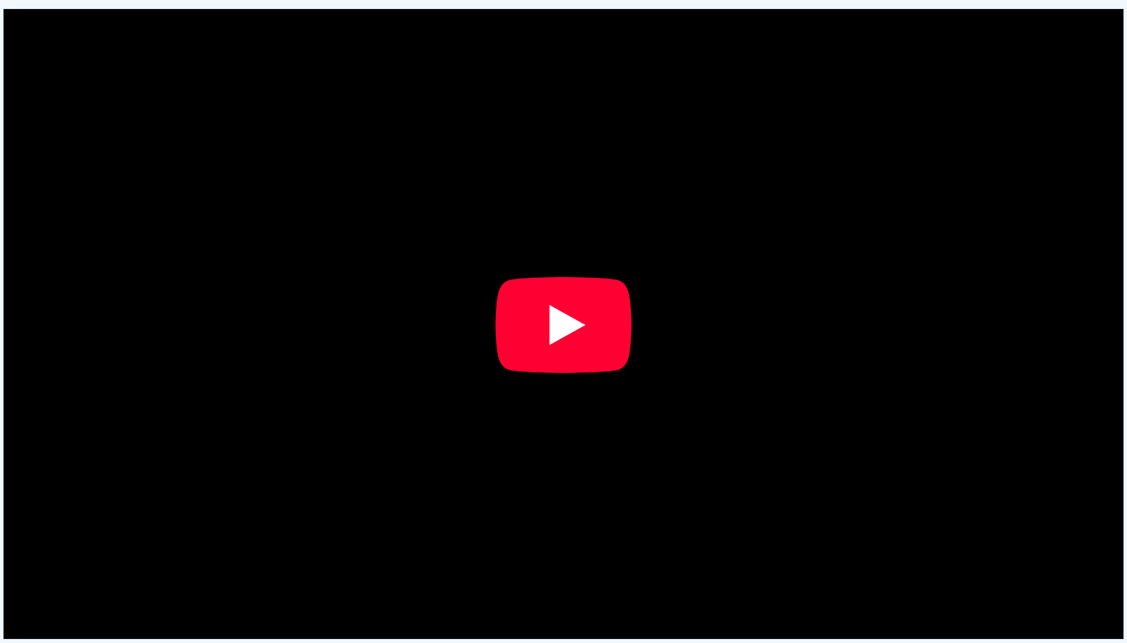
Exercise 2.1.2

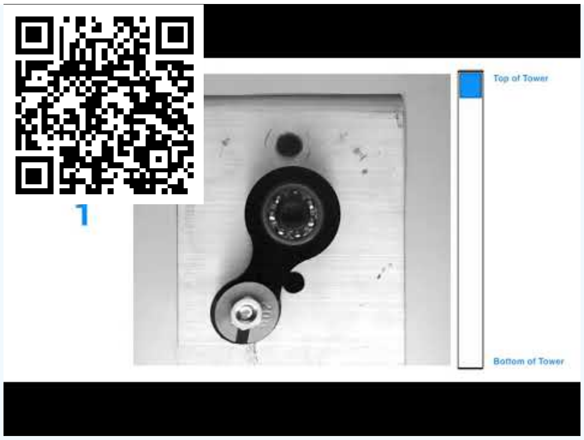
Can you tell whether this student is in orbit, in a free-falling vessel, or is in deep space far from any source of gravity? Assuming there are no windows, what experiments could the student do to find out? Does the answer depend on the size of the vessel or how long the experiment lasts for?

Figure 2.1.2: A student floats inside of a vessel. (Image credit: nasa.gov)

Answer

We know that people float inside of vessels in the absence of gravity. You may also know that the same is true for people in the International Space Station, which orbits the earth. In the first case, the people are not subject to gravity at all. In the second case, the people are *only* subject to gravity. Furthermore, if you were in a freely falling elevator, you would no longer be pulled to the floor and would feel weightless, which is another situation in which you are *only* subject to gravity. The video below shows this effect using a pendulum inside a sealed box that drops 22 m. Without gravity, you would expect a pendulum to swing in a full circle since there is no restoring force, and that is exactly what happens when the box is dropped.





There *are* experiments you can do to differentiate between the three situations (orbit, free fall, and deep space). Consider releasing two balls from rest, one above the other. Since gravity is stronger the closer you are to earth, the distance between the two balls will increase with time (if you wait long enough or if the initial height difference is large enough) in the free-fall situation but not the other two. Alternatively, if you release two balls from rest from the same height but separated horizontally, the distance between the two balls will decrease with time (if you wait long enough or the initial horizontal separation is large enough) only in the free-fall situation. This is because the balls fall radially toward the center of the earth, which makes their paths converge.

The previous experiments can distinguish between free-fall and either orbit or deep space but can't distinguish between orbit and deep space. In the free-fall situation, if the vessel is small *and* the time the experiment is carried out for is small, then there is no way to distinguish between orbit, free-fall, and deep space.

? Exercise 2.1.3

Recall that an **inertial reference frame** is one in which Newton's First Law applies, i.e. an object initially at rest stays at rest and an object in motion maintains its velocity as long as the net force acting on it it zero. By definition, a reference frame in deep space is an inertial reference frame. Now consider a reference frame attached to a person floating inside the International Space Station.

- Is the reference frame inertial?
- Is a reference frame inside of a freely falling elevator inertial?

Answer

In the previous exercise, we determined that there are no experiments that could distinguish between being in orbit (only subject to gravity) and being in deep space (but subject to no gravity). That means that a reference frame attached to a person floating inside the International Space Station, just like a reference frame attached to a particle that is subject to no forces, must be inertial.

The freely falling elevator is more subtle. The previous exercise showed that large separation distances and/or large time intervals allow us to perform experiments to distinguish the freely falling frame from the orbiting or deep space frames. If we limit ourselves to small separations and small time intervals, though, then we lose the ability to distinguish. So one single reference frame inside of a freely falling elevator is not inertial over large distances or large time intervals, but you can create a *series* of inertial reference frames and then stitch them together to cover a large distance and/or time interval. Within each inertial reference frame there is no evidence of gravity (i.e. objects released from rest don't accelerate).

You may be used to thinking of inertial reference frames as reference frames with no net force (and hence no acceleration), yet here we see that an orbiting reference frame and a freely falling reference frame can be inertial. So perhaps gravity isn't a force at all. And if that's the case, then a reference frame attached to a person standing still on the earth's surface is *not* inertial because the net force on that person would *not* be zero.

Exercise 2.1.4

Consider two particles in an inertial reference frame that are initially moving in the same direction and parallel to one another in deep space. Neglect the gravitational effects that they have on each other (that is, the mutual attraction). Describe the paths that the particles take.

Now consider two particles that are released from rest from the same height above the earth's surface. Neglect the gravitational effects that they have on each other but not the gravitational effect from the earth. Describe the paths that the particles take.

Answer

In the deep space example, the particles travel in straight lines, and the distance between them doesn't change.

In the free fall example, the particles will move slightly closer together because they are both falling radially toward the center of the earth. In fact, if the surface of the earth didn't get in the way, eventually the particles' paths would intersect.

Initially parallel lines that intersect may sound familiar; that was one of the rules of geometry that we discovered in the non-Euclidean space on the surface of a sphere!

The takeaway from the previous exercises is that gravity can be thought of not as a force, but rather as a changing of the rules of geometry. Albert Einstein's great contribution was in formulating the theory, called **The General Theory of Relativity** (General Relativity, or GR, for short), that relates mass to the rules of geometry. The equations relating mass to geometry are called the **Einstein Field Equations** (EFE for short). In natural units, they are written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.1.1}$$

where $G_{\mu\nu}$ is a complicated quantity involving the *generalized* metric $g_{\mu\nu}$ ($\eta_{\mu\nu}$ is the symbol specifically reserved for Euclidean metrics), Λ is a constant called the **cosmological constant**, and $T_{\mu\nu}$ is a quantity that represents the matter and energy distribution in the space. In short, the metric that defines the rules of geometry appears on the left hand side of the equation while the matter and energy content appears on the right hand side of the equation.

In practice, you input a matter and/or energy distribution on the right side and you solve for the metric. That, along with other things we will introduce later, allows you to determine the wordlines of particles. The problem is that solving the Einstein Field Equations is *extremely* difficult because they are a set of ten coupled, non-linear, second order, partial differential equations. For that reason, in this book we will typically just tell you what the metric corresponding to a particular matter/energy distribution is and focus on what we can learn from there.

There are ways to make it easier to solve the EFE. Specifically, you can assume very simple matter distributions. In this and following chapters we will look at some of these special cases:

Assumption	EFE result
spacetime is empty	Special Relativity
there is a single point mass	Schwarzschild geometry (i.e. black hole)
there is a single rotating point mass	Kerr geometry
a faraway mass oscillates	gravitational waves
spacetime is filled with a uniform fluid	cosmology

2.2: The Schwarzschild Metric

If you look out into space, you will find a lot of objects that are close to being spheres and that are very far away from other large objects. As an approximation, we can treat them as perfect spheres that are completely isolated. The most sensible coordinates to use for a sphere are spherical coordinates,

$$x^{\mu} = (t \quad r \quad \theta \quad \phi), \tag{2.2.1}$$

where ϕ is the **azimuthal angle** (i.e. longitude, $0 \leq \phi \leq 2\pi$), and θ is the **polar angle** (i.e. latitude, but measured from the north pole instead of the equator, $0 \leq \theta \leq \pi$), as shown in Figure 2.2.1.

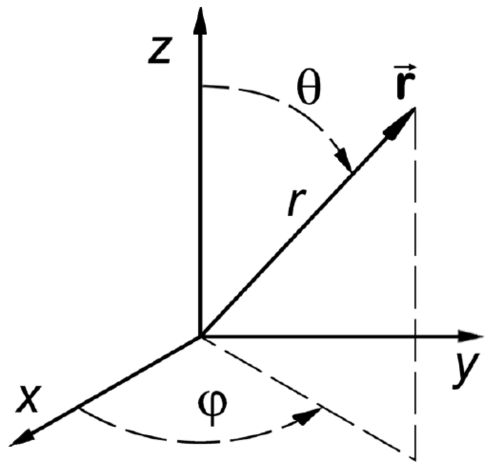


Figure 2.2.1: Spherical coordinates. (Image credit: Wikipedia)

Using this approximation in the EFE yields

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2M}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \tag{2.2.2}$$

where M is the mass of the sphere. This is referred to as the **Schwarzschild metric**.

Note

Even though this metric is different from the one in special relativity, the index notation rules still apply. That is, the index of a vector can be lowered using the metric (Box 2.2.1), the inverse metric $g^{\mu\nu}$ is defined by $g_{\alpha\mu}g^{\mu\nu} = \mathbf{I}$ (Box 2.2.2), and $u_{\mu}u^{\mu} = 1$ (Box 2.2.3).

Definition: Schwarzschild metric

The Schwarzschild metric is the metric for a (non-spinning) spherically symmetric mass in an otherwise empty spacetime.

The spacetime interval that goes along with Equation 2.2.2 is

$$d\tau^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{1}{1-\frac{2M}{r}}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{2.2.3}$$

You may notice that, in order for the units to work out, the mass M must have the same units as r! This is the result of using natural units, but this time setting Newton's Gravitational constant $G = 1$, which results in the following conversion between mass and distance.

$$1 \text{ kg} = 7.426 \times 10^{-28} \text{ m}$$

Exercise 2.2.1

What are the masses of the earth and sun in meters?

Answer

earth:

$$5.97 \times 10^{24}\ \mathrm{kg} \times \frac{7.426 \times 10^{-28}\ \mathrm{m}}{1\ \mathrm{kg}} = 0.0044\ \mathrm{m}$$

sun:

$$1.99 \times 10^{30}\ \mathrm{kg} \times \frac{7.426 \times 10^{-28}\ \mathrm{m}}{1\ \mathrm{kg}} = 1477\ \mathrm{m}$$

Box 2.2.1

Show that for a four-vector **a**, the vector with downstairs index using the Schwarzschild metric is $a_t = \left(1 - \frac{2M}{r}\right) a^t$, $a_r = -\frac{1}{1-\frac{2M}{r}} a^r$, $a_\theta = -r^2 a^\theta$, and $a_\phi = -r^2 \sin^2\theta a^\phi$.

Box 2.2.2

Show that the inverse Schwarzschild metric $g^{\mu\nu} = \begin{pmatrix} \frac{1}{1-\frac{2M}{r}} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}$.

Box 2.2.3

Using the Schwarzschild metric, show that $u_\mu u^\mu = 1$.

2.3: The Schwarzschild t-coordinate

In General Relativity, we have to be careful not to assume that coordinates mean what we think they mean. Consider, for example, the t-coordinate in the Schwarzschild spacetime interval

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1 - \frac{2M}{r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2.$$

To see what the t-coordinate means, consider two events with a t-coordinate separation of dt that occur at the same location (i.e. $dr = d\theta = d\phi = 0$). The spacetime interval reduces to

$$d\tau = \pm\sqrt{1 - \frac{2M}{r}}\, dt.$$

Recall that τ, the proper time, is the time as measured by the wristwatch of someone who is present at both events. So in general, the proper time is *not* the same as the t-coordinate separation.

Exercise 2.3.1

Two flashes of light occur at the same location a distance of $r = 4M$ from the center of a spherical object with mass M. An observer at that location measures the time between the flashes to be 1 second. What is the t-coordinate separation of those flashes?

Answer

As shown before, the spacetime interval for two events at the same location around a spherical mass reduces to

$$d\tau = \pm\sqrt{1 - \frac{2M}{r}}\, dt.$$

The 1 second given in the problem is the proper time, but 1 second is not infinitesimal. A non-infinitesimal interval, however, is the sum of many infinitesimal intervals, which is an integral. We will take the bounds of the integral to go from τ_i to τ_f on the left and t_i to t_f on the right.

$$\begin{aligned}
\int_{\tau_i}^{\tau_f} d\tau &= \pm \int_{t_i}^{t_f} \sqrt{1 - \frac{2M}{r}}\, dt && \text{set up integral} \\
\int_{\tau_i}^{\tau_f} d\tau &= \pm\sqrt{1 - \frac{2M}{r}} \int_{t_i}^{t_f} dt && \sqrt{1 - \frac{2M}{r}} \text{ is a constant with respect to t} \\
\tau_f - \tau_i &= \pm\sqrt{1 - \frac{2M}{r}}\,(t_f - t_i) && \text{perform integral} \\
t_f - t_i &= \pm\frac{1}{\sqrt{1 - \frac{2M}{r}}}(\tau_f - \tau_i) && \text{solve for t-coordinate separation} \\
\Delta t &= \pm\frac{1}{\sqrt{1 - \frac{2M}{4M}}}(1 \text{ s}) && \text{substitute} \\
\Delta t &= 1.41 \text{ s} && \text{simplify}
\end{aligned}$$

Here we choose the positive solution, and we see that the t-coordinate separation is larger than the proper time.

Exercise 2.3.2

Is there any place where the t-coordinate separation between events that occur at the same location is the same as the proper time?

Answer

If you look at $d\tau = \pm\sqrt{1-\frac{2M}{r}}\,dt$, you will see that $d\tau = dt$ only if $1-\frac{2M}{r} = 1$, but that only happens in the limit $r \to \infty$. In other words, the t-coordinate separation between two events is only equal to the proper time if those events occur infinitely far away from the spherical mass. For this reason, we can think of the t-coordinate as representing the **faraway time**.

Exercise 2.3.3

An astronaut with a wristwatch floats at a constant location from a black hole just outside of $r = 2M$. Given two events at the astronaut's location, we can say that $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{2M}{r}}}$. Using this, describe how the time interval measured on the watch of someone very far away compares to the time measured by the astronaut's wristwatch.

Answer

If r is just barely bigger than $2M$, then $\frac{dt}{d\tau}$ will be a huge number. In other words, the time between two events measured by the faraway observer (dt) will be much larger than the time between those two events as measured by the astronaut ($d\tau$).

2.4: The Schwarzschild r-coordinate

To see what the r-coordinate means in the Schwarzschild metric, consider two events with different r-coordinates (r_a and r_b) but the same t-coordinate, θ-coordinate, and ϕ-coordinate. This is essentially like measuring the distance between the edges of two concentric circles. Because this is a spacelike interval, we will use the proper length σ instead of the proper time τ.

$$d\sigma^2 = \frac{1}{1-\frac{2M}{r}} dr^2$$

By taking the square root and integrating, we can find the proper length between the two circles.

$$\int_{\sigma_a}^{\sigma_b} d\sigma = \pm \int_{r_a}^{r_b} \frac{1}{\sqrt{1-\frac{2M}{r}}} dr$$

$$\Delta\sigma = \pm \int_{r_a}^{r_b} \frac{1}{\sqrt{1-\frac{2M}{r}}} dr$$

Note that r is *not* a constant, so there is no way around performing the integral. For this course, I don't mind if you use an integral calculator (I like to use Desmos.com, where you can type *int* to get the integral sign.

? Exercise 2.4.1

What is the proper length between two circles with $r = 3M$ and $r = 4M$, where $M = 1477$ m (the mass of the sun), using the Schwarzschild metric?

Answer

Here is an example of how to type this into Desmos.

$$\Delta\sigma = \pm \int_{r=3M}^{r=4M} \frac{1}{\sqrt{1-\frac{2M}{r}}} dr = 2278 \text{ m}$$

Notice that this answer is significantly larger than simply the difference $4M - 3M = 1477$ m. For this reason, it is not a good idea to refer to r as the "radius".

Box 2.4.1

A black hole has a mass $M = 5$ km, a little more than three times the mass of our Sun. Two concentric spherical shells surround this black hole. The lower shell has r-coordinate r_L; the higher shell has r-coordinate $r_H = r_L + \Delta r$. Assume that $\Delta r = 1$ km and consider the following cases.

i. $r_L = 50$ km
ii. $r_L = 15$ km
iii. $r_L = 10.1$ km
iv. $r_L = 10.01$ km
v. $r_L = 10.001$ km

a. For each case, calculate the radial proper length $\Delta\sigma$ between shells.
b. The predictions of general relativity are always more accurate than Newtonian predictions, but they come at the cost of being more complicated to calculate. In some of the above cases, for example, you may have found the difference from the Newtonian result to be small enough that invoking general relativity is not worth it. Suppose you had a really long list of r_L values and wanted to expedite the process of calculating ruler distance by using the Newtonian result as long as it didn't differ from the more accurate prediction of general relativity by more than 1%. Approximately what value of r_L should you

use as the dividing line? Explain your process clearly. (You may not be able to solve it analytically, in which case you will have to think of another way.)

2.5: Spacetime Diagrams

In Special Relativity, we constructed light cones using lines at 45 degree angles. One way to understand that is to consider the fact that $d\tau^2 = 0$ for light, which we can use to solve for the slope $\frac{dt}{dx}$.

$$\begin{aligned} \cancelto{0}{d\tau^2} &= dt^2 - dx^2 && \text{lightlike interval} \\ \frac{dt}{dx} &= \pm 1 && \text{solve for } \frac{dt}{dx} \end{aligned}$$

Here we see that the slope is +1 or -1 if we use natural units. We can repeat the same process to determine how to sketch light cones on a t-r slice (that is, $d\theta = d\phi = 0$) using the Schwarzschild metric. The result is

$$\frac{dt}{dr} = \pm \frac{1}{1 - \frac{2M}{r}}.$$

The slopes of the light cones depend on r, as shown in Figure $2.5.1$.

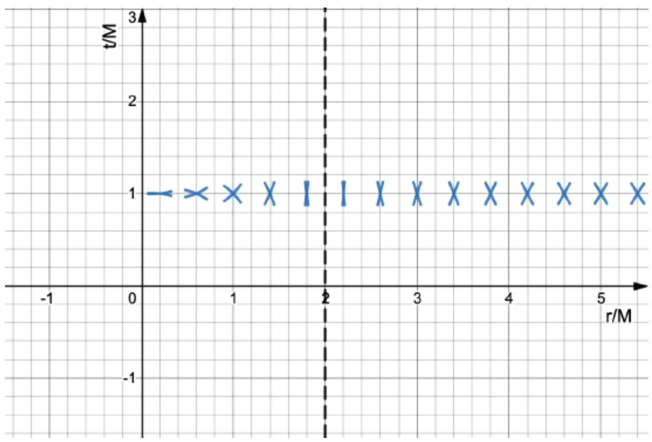


Figure $2.5.1$: Examples of Schwarzschild light cones. The light cones look like those from Special Relativity for large values of r but vertically squeeze as $r \to 2M$. Below $r = 2M$, the light cones squeeze horizontally as $r \to 0$. Note that the axes have been normalized by the mass M. (Copyright; author via source)

For large values of r, the light cones look like those from Special Relativity. As we approach $r = 2M$ from the outside, the light cones squeeze in the vertical direction, eventually becoming vertical lines right at $r = 2M$. Below $r = 2M$, the light cones squeeze horizontally.

Exercise $2.5.1$

What does the squeezing of the light cones say about the possible futures of particles near $r = 2M$ on the outside?

Answer

If we look at the light cone that is closest to $r = 2M$, we see that there is a very limited range of r-values where the particle could be a short time later. That limited range of r-values gets even more limited the closer you get to $r = 2M$. The result is that the particle cannot pass $r = 2M$.

Exercise $2.5.2$

Light cones divide spacetime into past, future, and elsewhere. In Special Relativity, we saw that past was below, future was above, and elsewhere was on either side. Which regions in Fig. 2.5.1 are elsewhere for $r < 2M$?

Answer

Let's start by rewriting the Schwarzschild spacetime interval on a t-r slice (i.e. $d\theta = d\phi = 0$).

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1 - \frac{2M}{r}} dr^2 - r^2 \cancelto{0}{d\theta^2} - r^2 \sin^2\theta \, \cancelto{0}{d\phi^2}$$

Now consider one of the light cones of Fig. 2.5.1. If we take a small step either directly up or down from one of the vertices, then $dr = 0$. The spacetime interval reduces to

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2.$$

If $r < 2M$, then $d\tau^2 < 0$ and the interval is spacelike, which means that the regions directly above and below the light cone are *elsewhere*.

If we were instead to take a small step either directly left or right from one of the vertices, then $dt = 0$. The spacetime interval reduces to

$$d\tau^2 = -\frac{1}{1 - \frac{2M}{r}} dr^2.$$

If $r < 2M$, then $d\tau^2 > 0$ and the interval is timelike, which means that the regions to the left and right are the future and past.

The bottom line is that the future does not require moving forward in *t*, but it does require moving to smaller values or r. Inside of $r = 2M$, the roles of the t-coordinate and r-coordinate effectively switch!

The previous examples demonstrate that there is something weird and special about $r = 2M$. Particles on the outside apparently cannot penetrate to the inside, and particles on the inside inevitably end up at $r = 0$. You can also see how the spacetime interval diverges (i.e. becomes undefined) at $r = 2M$. We call this special location the **event horizon**.

Definition: Event Horizon

In Schwarzschild coordinates, the event horizon is the set of all locations where $r = 2M$. It is one of two locations where the spacetime interval diverges.

For all stars and planets, $r = 2M$ lies well below the surface, which means that this "point of no return" is inaccessible. Black holes, on the other hand, have all of their mass concentrated within the event horizon, which means that the event horizon is accessible.

Note

While the event horizon is a special location, there is nothing physically located there.

There is one more place where the spacetime interval diverges: $r = 0$. This is called the **singularity**.

Definition: Singularity

The singularity of a black hole is its center, where $r = 0$.

It is difficult to say much about the singularity with confidence. The equations of General Relativity rely on the ability to patch together locally flat references frames, but spacetime is so warped at the singularity that it is not possible to construct a locally flat reference frame there. Is the entire mass of the black hole concentrated at that one single point? Is the singularity a portal to another universe? Is the singularity even a point, as opposed to a small chunk? These are difficult questions to answer.

2.6: Global vs. Local Coordinates

In the previous section we saw that the slope of a light cone on an r-t slice is given by

$$\frac{dt}{dr} = \pm\frac{1}{1-\frac{2M}{r}}.$$

The reciprocal, $\frac{dr}{dt}$, should describe the speed of light that is traveling radially. There appears to be a problem, though, since this describes a speed of light that is r-dependent, which violates one of the premises of relativity. It turns out that there really is no problem, however, since we saw in the previous sections that the t- and r-coordinates don't mean what we think they mean. We call these coordinates **global coordinates** because they are part of a coordinate system that embraces a large portion of curved spacetime. But global coordinates can't be trusted to tell us about measurable quantities.

Definition: Global Coordinate System

A global coordinate system embraces a large portion of spacetime that is curved in general. Global coordinate separations don't necessarily correspond to measurable quantities.

In many regions of curved spacetimes, it is possible to construct a **local inertial reference frame** (LIRF) where coordinate separations do correspond to measurable quantities and where we can use the rules of Special Relativity. By superimposing an LIRF on a portion of a global coordinate system, it is possible to translate between the two.

Definition: Local Inertial Reference Frame (LIRF)

A local inertial reference frame is a patch of curved spacetime that is small enough to be considered flat. The rules of Special Relativity can be used on an LIRF.

As an example of this coordinate translation, consider Fig. 2.6.1, which shows a locally flat Cartesian coordinate system superimposed on a Schwarzschild coordinate system.

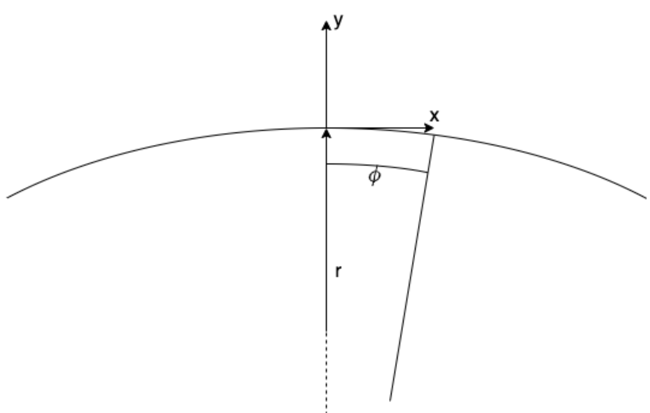


Figure 2.6.1: A locally flat Cartesian coordinate system is constructed on a curved portion of Schwarzschild spacetime. (Copyright; author via source)

For simplicity, let's ignore the θ-direction and place this Cartesian coordinate system on the equator ($\theta = 90^\circ$) such that the x-direction is east and the y-direction points away from the center. The r-direction matches up with the y-direction, and the ϕ-direction matches up with the x-direction. The Schwarzschild spacetime interval is

$$d\tau^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{1}{1-\frac{2M}{r}}dr^2 - r^2 d\phi^2$$

and the LIRF spacetime interval is

$$d\tau^2 = dt_{\mathrm{LIRF}}^2 - dx_{\mathrm{LIRF}}^2 - dy_{\mathrm{LIRF}}^2.$$

By matching corresponding terms that represent the same direction, we get

$$\begin{aligned} dt_{\mathrm{LIRF}} &= \sqrt{1-\frac{2M}{r}}\,dt \\ dx_{\mathrm{LIRF}} &= r\,d\phi \\ dy_{\mathrm{LIRF}} &= \frac{1}{\sqrt{1-\frac{2M}{r}}}dr \end{aligned}.$$

Now let's return to the question of the speed of light. For a beam of light traveling radially, the speed in the LIRF is $\frac{dy_{\mathrm{LIRF}}}{dt_{\mathrm{LIRF}}}$. We can use the local-to-global coordinate transformations along with the expression for $\frac{dr}{dt}$ in global coordinates to prove that the speed of light is 1 (see Box 2.6.1).

Box 2.6.1

Use the local-to-global coordinate transformations along with the expression for $\frac{dr}{dt}$ in global coordinates to prove that $c = \pm 1$.

2.7: Black Hole Formation

A black hole forms when all of its mass is concentrated inside of the event horizon, which can happen when a star undergoes gravitational collapse. Under normal circumstances, the gravitational attraction of the gas molecules in a star is held at bay by an outward thermal pressure. When a star runs out of fuel, however, the equilibrium is destroyed and the gas will collapse. The resulting object is called a **stellar remnant**, and it can take one of three forms.

1. White Dwarf: a star that is held stable by **electron degeneracy pressure**. White dwarves can have a mass similar to our sun but compressed to a size comparable to earth.
2. Neutron star: a star in which protons and electrons have fused together to form an object that consists only of neutrons. It is held stable by **neutron degeneracy pressure**. Neutrons stars can have a mass similar to our sun but compressed to the size of a city.
3. Black Hole: an object where gravity overcomes neutron degeneracy pressure and collapses all of the mass into the central singularity.

Exactly which of these stellar remnants is formed during a collapse depends on a number of factors, including the mass of the original star, the mass of the remnant (some of the gas can recoil after the implosion), and whether the star was accreting mass from another star.

2.8: The Global Positioning System

The Global Positioning System uses the positions of satellites along with very accurate clocks to determine the location of a GPS receiver. Figure $2.8.1$ shows a depiction of how this works. First, a GPS satellite sends out a signal that indicates the satellite's location as well as the time that the signal was sent (Time of Transmission, or ToT). A GPS receiver receives the signal and takes note of the time (Time of Arrival, or ToA). The receiver can then use the difference between ToA and ToT along with the speed of light to determine how far away the satellite is. The problem is that the receiver doesn't know its orientation relative to the satellite, so the location is constrained only to a sphere of a known size surrounding the satellite that sent the signal. If the receiver receives another signal from a second satellite, then the location of the receiver is constrained to be somewhere on the intersection of the two spheres (the intersection of two spheres is a circle). A third satellite constrains the position to the intersection of a sphere and a circle, which is two points. A fourth satellite constrains the position to a single point.

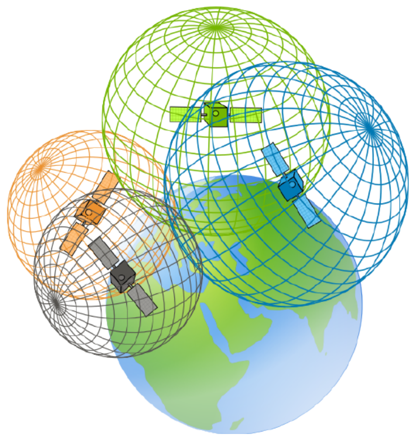


Figure $2.8.1$: GPS satellites send signals that carry information about their location and the time at which the signal was sent, which allows a GPS receiver to determine how far away it is from the satellite. The direction is not known, however, so the location of the receiver is constrained to a sphere. Additional satellite signals further constrain the receiver position to where the spheres intersect. (Image credit: freesvg.org)

Note

In order for GPS to work, the satellites have to know their own positions. This can be achieved by tracking from ground stations using techniques such as radar and signal doppler. They also follow very predictable paths in their orbits.

In order for the positioning to be accurate, the clocks in the GPS satellites have to be accurate as well. If the clock is off by one nanosecond, then the position will be off by about one foot (the speed of light is approximately 1 ft/ns). What's even worse is that if that clock inaccuracy is due to a difference in clock rate (rather than being a fixed offset), then the positioning will get more and more inaccurate over time.

This is where relativity comes in, because the clock rate on a satellite is going to be different from the clock rate on the ground for two reasons:

1. the satellite moves relative to a clock on the ground, and
2. the satellite and ground clock are at different r-values.

The first effect is from Special Relativity while the second is from General Relativity.

Suppose a GPS satellite has a circular orbit with an r-value of r_s, and the signal from that satellite is received by a GPS receiver with an r-value of r_r. For simplicity, let's assume that both the satellite and receiver are at a constant $\theta = 90^\circ$ (i.e. the equator). The Schwarzschild metric can then be rewritten as (see Box 2.81.)

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right) - v^2. \tag{2.8.1}$$

This shows how the clock rate from either one of the clocks differs from the rate of the t-coordinate. What we are interested in, though, is the rate of the satellite clock relative to the receiver clock, $\frac{d\tau_s}{d\tau_r}$. We can do this simply by taking a ratio (then taking the square root). The result is

$$\frac{d\tau_s}{d\tau_r} = \sqrt{\frac{\left(\frac{d\tau_r}{dt}\right)^2}{\left(\frac{d\tau_r}{dt}\right)^2}} = \frac{\sqrt{1 - \frac{2M}{r_s} - v_s^2}}{\sqrt{1 - \frac{2M}{r_r} - v_r^2}}. \tag{2.8.2}$$

At this point it should be noted that $\frac{2M}{r}$ and v are *very* small for both the satellite and the receiver, which allows us to rewrite Equation 2.8.2 as (see Box 2.8.1)

$$\frac{d\tau_s}{d\tau_r} \approx 1 - \frac{M}{r_s} - \frac{1}{2}v_s^2 + \frac{M}{r_r} + \frac{1}{2}v_r^2. \tag{2.8.3}$$

There are then a few more steps, outlined below.

$$\begin{aligned}
\frac{\Delta\tau_s}{\Delta\tau_r} &\approx 1 - \frac{M}{r_s} - \frac{1}{2}v_s^2 + \frac{M}{r_r} + \frac{1}{2}v_r^2 && \text{since ratio is constant, it applies to any size interval} \\
\Delta\tau_s &= \left(1 - \frac{M}{r_s} - \frac{1}{2}v_s^2 + \frac{M}{r_r} + \frac{1}{2}v_r^2\right)\Delta\tau_r && \text{write satellite time in terms of receiver time} \\
\Delta\tau_s - \Delta\tau_r &= \left(1 - \frac{M}{r_s} - \frac{1}{2}v_s^2 + \frac{M}{r_r} + \frac{1}{2}v_r^2\right)\Delta\tau_r - \Delta\tau_r && \text{we want the } \textit{difference} \text{ in clock readings} \\
\Delta\tau_s - \Delta\tau_r &= \left(- \frac{M}{r_s} - \frac{1}{2}v_s^2 + \frac{M}{r_r} + \frac{1}{2}v_r^2\right)\Delta\tau_r && \text{simplify for final answer} \qquad (2.8.4)
\end{aligned}$$

All that remains is to determine the speeds and r-values of the receiver and satellite. The r-value of the receiver is simply the radius of the earth, and the speed can be determined using the fact that it takes one day for the receiver to travel around the circumference of the earth. According to the FAA, GPS satellites orbit 20,200 km above the surface of earth (remember that to get the r-value you need to add that to the radius of the earth) with a 12-hour period, which you can use to determine the speed. You should find that the clocks would drift by tens of microseconds per day (see Box 2.8.2). This would lead to extreme inaccuracy is determining position, so real GPS satellites add a correction to fix the problem.

Box 2.8.1

a. Derive Equation 2.8.2.
b. Use the binomial approximation $(1+x)^n \approx 1+nx$, which applies whenever $x \ll 1$, to the numerator and denominator of Equation 2.8.2 (do each one separately). You should end up with the product of two binomials. Then expand the product and drop any terms that are the product of two tiny things (for example, if $\frac{2M}{r_s}$ is very small, then $\left(\frac{2M}{r_s}\right)^2$ is *very very small*, so it can be neglected). The result of all of this should be Equation 2.8.3.

Box 2.8.2

a. Calculate the *dimensionless* speed of a ground-based GPS receiver.
b. Calculate the *dimensionless* speed of a GPS satellite with a 12-hour period that is 20,200 km above the ground.
c. Use Equation 2.8.4 to calculate the time difference between the GPS satellite and the ground-based receiver after one full day. Which clock is ahead?

2.9: Video Resources

CHAPTER OVERVIEW

3: Schwarzschild Orbits

Thumbnail: The overall geometry of the universe is determined by whether the Omega cosmological parameter is less than, equal to or greater than 1. Shown from top to bottom are a closed universe with positive curvature, a hyperbolic universe with negative curvature and a flat universe with zero curvature. (Public Domain; NASA).

3.1: The Geodesic Equation

The previous chapter dealt with the rules of geometry in Schwarzschild spacetime. If we want to look at *motion*, we need to look beyond the metric to something called the **geodesic equation**.

> **Definition: Geodesic**
>
> A geodesic is the worldline that results from applying the Principle of Maximal Aging. That is, it is the inertial worldline that connects two events.

The geodesic equation is a way of determining the geodesic that connects two events. You can think of it as an extension of Newton's First Law to curved spacetimes.

$$0=\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)-\frac{1}{2}\frac{dg_{\alpha\beta}}{dx^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\tag{3.1.1}$$

Let's see how this works by applying it to $g_{\mu\nu}=\eta_{\mu\nu}$ (i.e. the Special Relativity metric). As a reminder,

$$\eta_{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

The geodesic equation involves summing over ν, α, and β, while μ is a free index. Note also that the metric is zero if $\mu\neq\nu$ or $\alpha\neq\beta$. Let's work out the result for each μ.

$$\begin{aligned}
\mu=t:\ 0&=\frac{d}{d\tau}\left(g_{tt}\frac{dx^t}{d\tau}\right)-\frac{1}{2}\left(\frac{dg_{tt}}{dx^t}\frac{dx^t}{d\tau}\frac{dx^t}{d\tau}+\frac{dg_{xx}}{dx^t}\frac{dx^x}{d\tau}\frac{dx^x}{d\tau}+\frac{dg_{yy}}{dx^t}\frac{dx^y}{d\tau}\frac{dx^y}{d\tau}+\frac{dg_{zz}}{dx^t}\frac{dx^z}{d\tau}\frac{dx^z}{d\tau}\right) && \text{write out sum}\\
0&=g_{tt}\frac{d^2t}{d\tau^2}-\frac{1}{2}\left(\cancelto{0}{\frac{dg_{tt}}{dt}}\left(\frac{dt}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{xx}}{dt}}\left(\frac{dx}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{yy}}{dt}}\left(\frac{dy}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{zz}}{dt}}\left(\frac{dz}{d\tau}\right)^2\right) && \text{metric is constant}\\
0&=\frac{d^2t}{d\tau^2} && \text{simplify}\\
\mu=x:\ 0&=\frac{d}{d\tau}\left(g_{xx}\frac{dx^x}{d\tau}\right)-\frac{1}{2}\left(\frac{dg_{tt}}{dx^x}\frac{dx^t}{d\tau}\frac{dx^t}{d\tau}+\frac{dg_{xx}}{dx^x}\frac{dx^x}{d\tau}\frac{dx^x}{d\tau}+\frac{dg_{yy}}{dx^x}\frac{dx^y}{d\tau}\frac{dx^y}{d\tau}+\frac{dg_{zz}}{dx^x}\frac{dx^z}{d\tau}\frac{dx^z}{d\tau}\right) && \text{write out sum}\\
0&=g_{xx}\frac{d^2x}{d\tau^2}-\frac{1}{2}\left(\cancelto{0}{\frac{dg_{tt}}{dx}}\left(\frac{dt}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{xx}}{dx}}\left(\frac{dx}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{yy}}{dx}}\left(\frac{dy}{d\tau}\right)^2+\cancelto{0}{\frac{dg_{zz}}{dx}}\left(\frac{dz}{d\tau}\right)^2\right) && \text{metric is constant}\\
0&=-\frac{d^2x}{d\tau^2} && \text{simplify}
\end{aligned}$$

The process works similarly for $\mu=y$ and $\mu=z$. The result is that the second derivative of each component with respect to proper time is zero. In other words, there is no acceleration, which is exactly what we expect for Special Relativity.

While the geodesic equation looks complicated, in practice it isn't so bad if you are careful about writing out all of the terms and noticing when you can leave terms out because they are zero.

3.2: Constants of Motion

By applying the geodesic equation to the Schwarzschild metric, we can see what an inertial worldline looks like. Recall that, for the Schwarzschild metric,

$$g_{\mu\nu} = \begin{cases} 1-\frac{2M}{r} & \mu=\nu=t \\ -\frac{1}{1-\frac{2M}{r}} & \mu=\nu=r \\ r^2 & \mu=\nu=\theta \\ r^2\sin^2\theta & \mu=\nu=\phi \end{cases}$$

and the geodesic equation is

$$0 = \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\frac{dg_{\alpha\beta}}{dx^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}.$$

Let's apply the geodesic equation for $\mu = t$.

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) - \frac{1}{2}\left(\frac{dg_{tt}}{dt}\left(\frac{dt}{d\tau}\right)^2 + \frac{dg_{rr}}{dt}\left(\frac{dr}{d\tau}\right)^2 + \frac{dg_{\theta\theta}}{dt}\left(\frac{d\theta}{d\tau}\right)^2 + \frac{dg_{\phi\phi}}{dt}\left(\frac{d\phi}{d\tau}\right)^2\right) && \text{write out sum} \\ 0 &= \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) - \frac{1}{2}\left(\cancelto{0}{\frac{dg_{tt}}{dt}}\left(\frac{dt}{d\tau}\right)^2 + \cancelto{0}{\frac{dg_{rr}}{dt}}\left(\frac{dr}{d\tau}\right)^2 + \cancelto{0}{\frac{dg_{\theta\theta}}{dt}}\left(\frac{d\theta}{d\tau}\right)^2 + \cancelto{0}{\frac{dg_{\phi\phi}}{dt}}\left(\frac{d\phi}{d\tau}\right)^2\right) && \text{metric has no t-dependence} \\ 0 &= \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) && \text{simplify} \\ g_{tt}\frac{dt}{d\tau} &= \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = \text{constant} && \text{zero derivative means constant} \end{aligned}$$

What we find from here is that this particular quantity is a constant of motion. In particular, it is the *energy per unit mass*.

$$\frac{E}{m} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \tag{3.2.1}$$

Exercise 3.2.1

Any time we derive a new result in General Relativity, we should be able to show that the result reduces to something from Special Relativity in the appropriate limits. Do this for Equation 3.2.1.

Answer

We should recover the energy per unit mass from Special Relativity in the limits $M \to 0$ or $r \to \infty$. In either case, we get $\frac{E}{m} = \frac{dt}{d\tau}$, which was how we defined energy per unit mass in Special Relativity.

Exercise 3.2.2

According to Equation 3.2.1, what happens to $\frac{dt}{d\tau}$ as a particle approaches the event horizon from outside. What does that physically mean?

Answer

As $r \to 2M$ from the outside, the quantity $1-\frac{2M}{r}$ gets smaller and smaller, which means that $\frac{dt}{d\tau}$ must get larger and larger in order for $\frac{E}{m}$ to remain constant. What this means physically is that for every tick on the wristwatch of the infalling particle, a much *larger* amount of time passes on the wristwatch of someone who is very far away (recall that t in Schwarzschild coordinates is *faraway time*). As $r \to 2M$, $\frac{dt}{d\tau} \to \infty$, which means that an infinite amount of time passes on the wristwatch of the faraway observer for every clock tick on the wristwatch of the infalling particle.

By applying the geodesic equation to $\mu = \phi$, we get another constant of motion: the **angular momentum per unit mass**.

$$\frac{L}{m} = r^2\sin^2\theta\frac{d\phi}{d\tau} \tag{3.2.2}$$

Exercise 3.2.3

Consider an inertial particle orbiting in the equatorial plane (i.e. $\theta = 90^\circ$) of a black hole at a constant r-coordinate. Describe the motion according to Equation 3.2.2. What if the motion is elliptical?

Answer

If r is a constant, then $\frac{d\phi}{d\tau}$ is a constant as well. In other words, the particle moves in a plane at a constant angular speed around the black hole. If the motion is elliptical, then r varies. In order for $\frac{L}{m}$ to remain constant, the angular speed must be larger when r is small and smaller when r is large. This is exactly what happens in elliptical orbits according to Newtonian mechanics.

Exercise 3.2.4

Argue that applying the geodesic equation to $\mu = \theta$ wouldn't yield any information about orbits not already contained in Equation 3.2.2.

Answer

Given that the Schwarschild metric is spherically symmetric, orbits in the θ direction can't be any different qualitatively than orbits in the ϕ direction. In fact, any orbital plane can be the equatorial plane just by rotating the sphere.

Box 3.2.1

Apply the geodesic equation to the Schwarzschild metric with $\mu = \phi$ in order to derive Equation 3.2.2.

3.3: Effective Potential

In the previous section, we applied the geodesic equation to three of the four Schwarzschild coordinates. Applying it to the r-coordinate yields a result that is complicated enough to warrant its own section. The result is

$$\left(\frac{dr}{d\tau}\right)^2=\left(\frac{E}{m}\right)^2-\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right). \tag{3.3.1}$$

Now let's look at this term by term. On the left, there is something that looks like a velocity squared. Next is a term that is constant and represents the total energy. Finally, there is a term that depends only on position and other constants. This is very similar to a familiar equation from Newtonian Mechanics:

$$KE=\frac{1}{2}mv^2=E_{\text{total}}-PE \tag{3.3.2}$$

If you make a graph of potential energy, then you can use the graph to come to conclusions about the motion. For example, let's graph $PE=\frac{1}{2}kx^2$. The total energy is a constant, which is a horizontal line on the graph. The kinetic energy is then given by the difference of the two curves, as shown in Fig. 3.3.1.

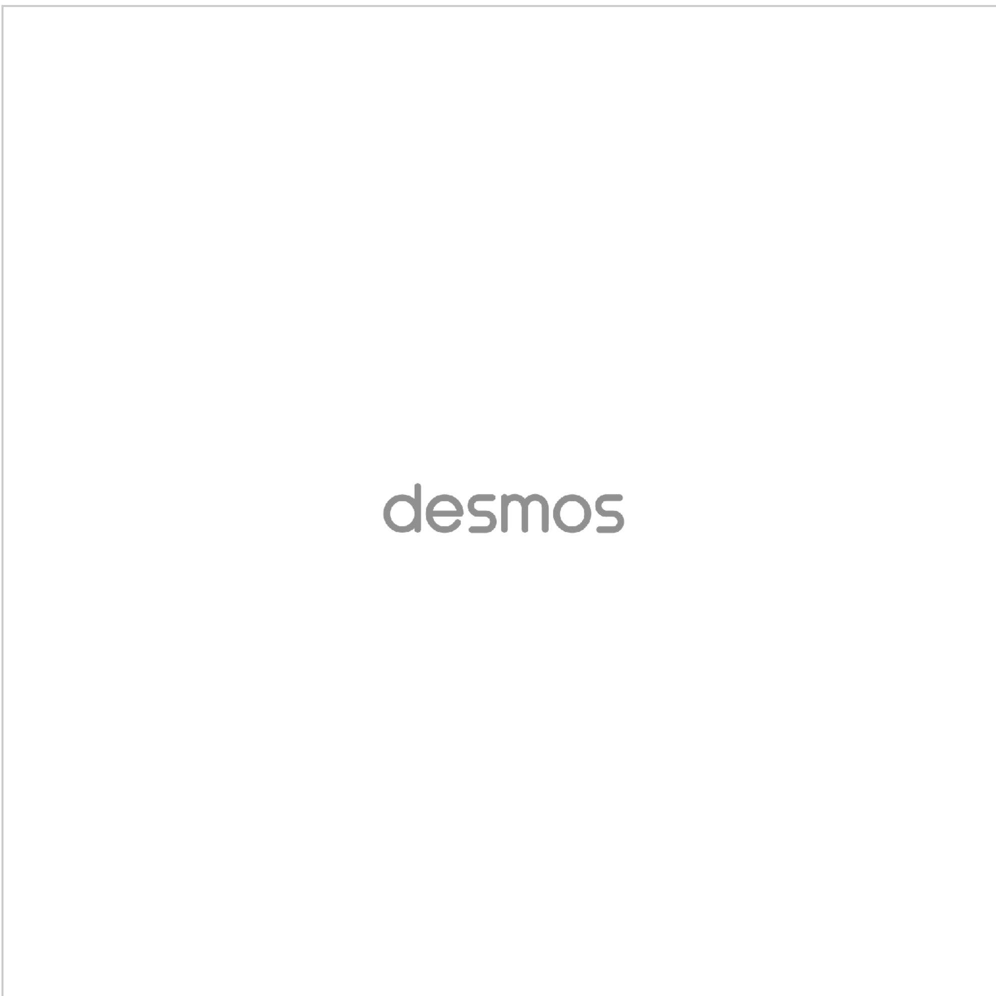


Figure 3.3.1: A graph of potential energy $PE=\frac{1}{2}kx^2$. The kinetic energy is the difference between the total energy (horizontal line) and the potential energy. (Copyright; author via source)

Since kinetic energy must always be greater than or equal to zero, we see that the object is forced to oscillate between the points of intersection (called **turning points**) of the two curves. Furthermore, the smaller the total energy the smaller the range of oscillation.

The point, though, is that we can obtain qualitative information about the velocity by plotting the other two parts. We can repeat the process with Equation 3.3.1 by plotting $\frac{E}{m}$ (as a horizontal line) and

$$\frac{PE_{\text{eff}}(r)}{m}=\sqrt{\left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right)}, \tag{3.3.3}$$

where $PE_{\text{eff}}(r)$ is called the **effective potential energy**.

Definition: Effective Potential Energy

An effective potential energy is not technically a potential energy but can still be used to obtain qualitative information about motion.

Fig. 3.3.2 plots this effective potential energy for $\frac{L}{m}=3.75M$ with dimensionless axes.

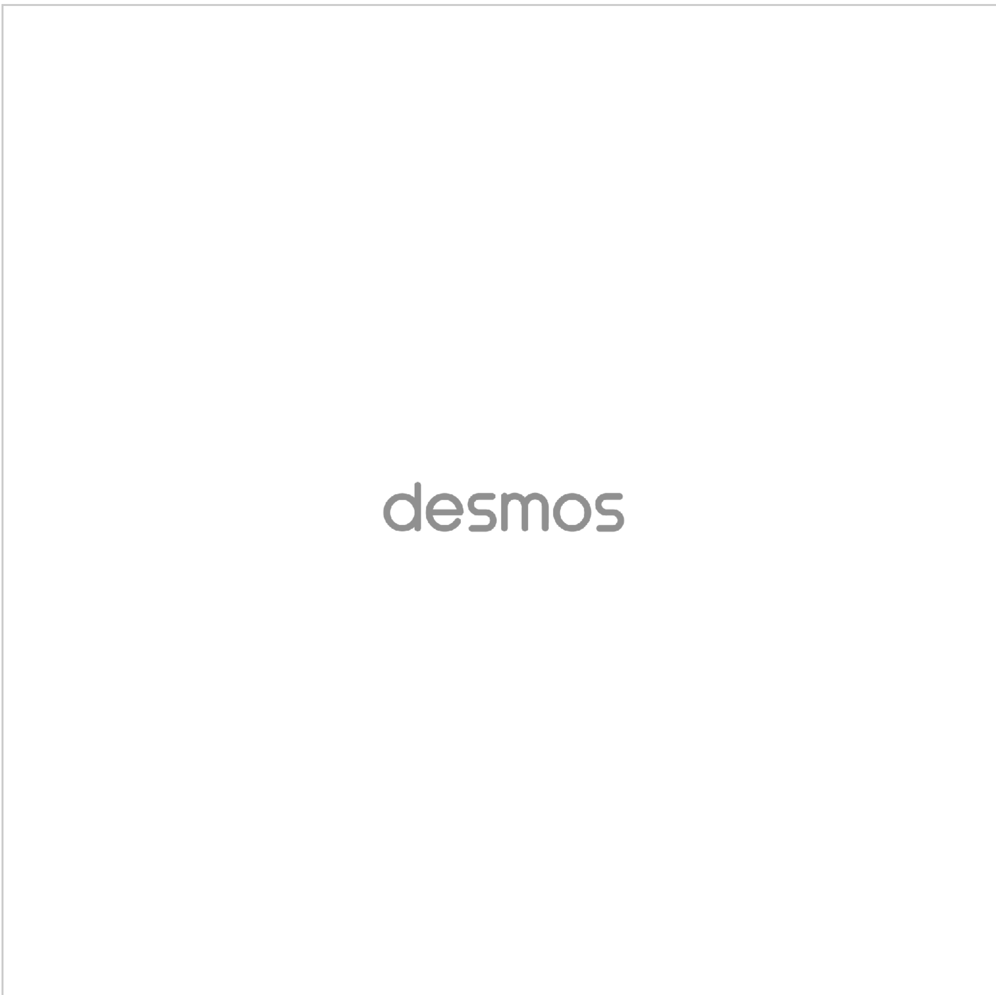


Figure 3.3.2: Schwarzschild effective potential energy for $\frac{L}{m}=3.75M$. The difference between the two curves tells us about the radial component of the velocity, $\frac{dr}{d\tau}$. (Copyright; author via source)

Exercise 3.3.1

Describe the motion depicted in Fig. 3.3.2. How would the motion be different if $\frac{E}{m}=0.98$?

Answer

There are three turning points (i.e. points of intersection) on the graph, so the motion depends on which region the particle is in. If $6.5 \leq \frac{r}{M} \leq 15.5$, then the particle is in a trapped orbit. We know this because the particle has angular momentum, which mean it is orbiting around, but there is a minimum and a maximum r-value to that orbit (given by the turning points). This could, for example, be an elliptical orbit, though we can't say for sure that it is *exactly* an ellipse. The other possibility is if $\frac{r}{M} \leq 3.5$, in which case there is a maximum orbital r-value but no minimum. That particle may swing around a bit, but it is doomed to fall to the event horizon.

If the total energy were $\frac{E}{m} = 0.98$, then there would only be one turning point instead of three (it would intersect way off to the right). In that case, the orbit would have maximim r-value but no minimum, which again means that it is doomed to fall to the event horizon.

? Exercise 3.3.2

Fig. 3.3.2 shows the effective potential energy for $\frac{L}{m} = 0$. Describe the resulting motion. Is that what you would expect?

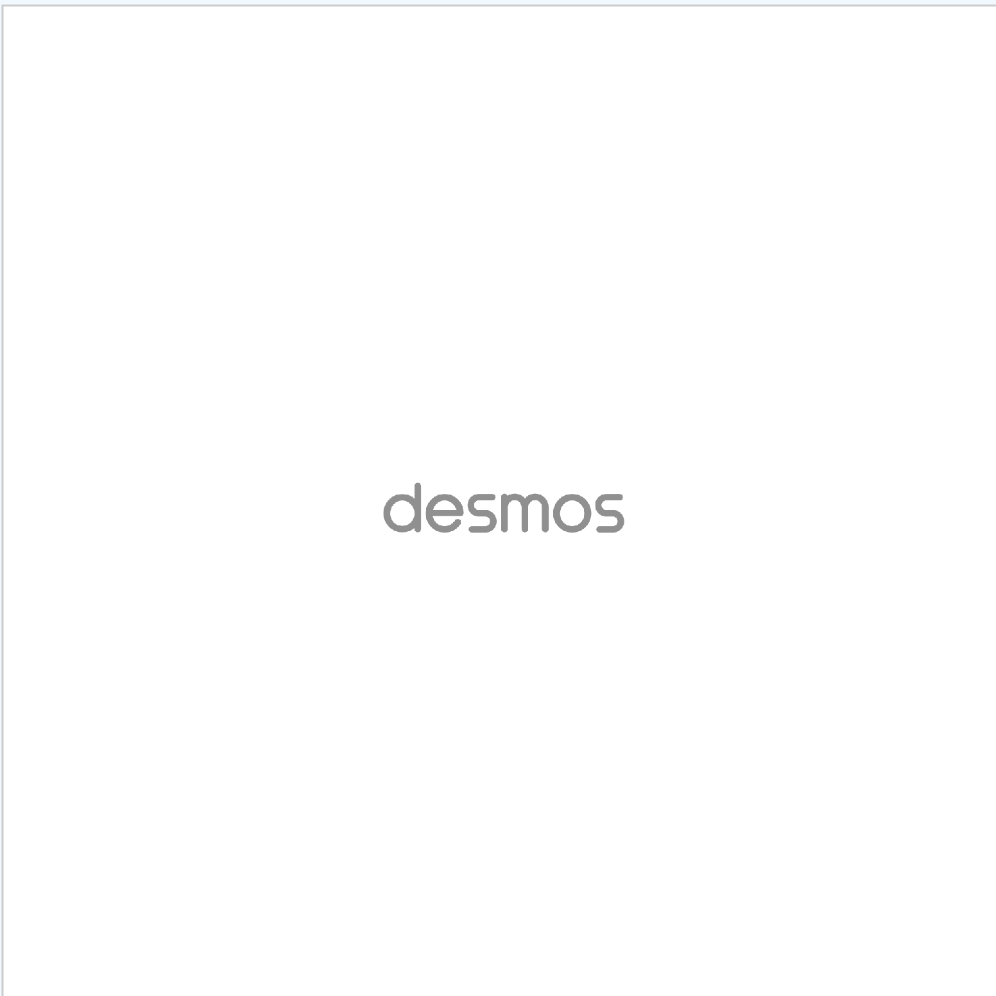


Figure 3.3.3: Schwarzschild effective potential energy for $\frac{L}{m} = 0$. (Copyright; author via source)

Answer

Zero angular momentum corresponds to a particle moving radially. The fact that there is only one turning point makes sense, since it shows that a particle initially moving radially away will reach a maximum r-value then fall back toward the event horizon. Note that the particle never passes the event horizon, which is consistent with what we learned from looking at light cones.

Box 3.3.1

Circular orbits are possible wherever the effective potential energy has zero slope. Take the derivative of the effective potential energy with respect to r, then set it equal to zero. Use it to show that

$$r^2 - \frac{L^2}{Mm^2}r + 3\frac{L^2}{m^2} = 0.$$

Box 3.3.2

Solve the quadratic equation from Box 3.3.1 to show that

$$r = \frac{L^2}{2m^2M}\left[1 \pm \sqrt{1 - \frac{12M^2m^2}{L^2}}\right].$$

Box 3.3.3

Using the result from the previous box, what is the minimum value of $\frac{L}{m}$ for which you get a real number for r? Express your answer in terms of M.

Box 3.3.4

Use the minimum value of $\frac{L}{m}$ from the previous box to determine the r-value of the orbit (in terms of M). This is called the **Innermost Stable Circular Orbit** (ISCO).

3.4: Local Inertial Reference Frames

The constants of motion E and L from a previous section are expressed in global coordinates, which means that they don't correspond to anything directly measurable. To turn them into something measurable, we have to use a global-to-local coordinate transformation. We did this back in Section 2.6. As a reminder the results were

$$\begin{aligned} dt_{\text{LIRF}} &= \sqrt{1-\frac{2M}{r}}\,dt \\ dx_{\text{LIRF}} &= r\,d\phi \\ dy_{\text{LIRF}} &= \frac{1}{\sqrt{1-\frac{2M}{r}}}dr \end{aligned}.$$

After applying the transformation, we get that the energy in the LIRF is

$$\frac{E_{\text{LIRF}}}{m} = \gamma = \frac{1}{\sqrt{1-\frac{2M}{r}}}\frac{E}{m}, \tag{3.4.1}$$

where $\gamma = \frac{1}{\sqrt{1-v_{\text{LIRF}}^2}}$ (recall that Special Relativity rules in the LIRF, so we can use the Special Relativity definition of energy). What this allows us to do is take a known value of E and use it to determine the speed at any given r-coordinate.

Exercise 3.4.1

Is E_{LIRF} a constant of motion? Is it an invariant?

Answer

Given that E_{LIRF} depends on speed, you may think that it is not a constant of motion. Recall, however, that an LIRF exists only for an infinitesimal amount of time, which means that the speed is effectively constant and therefore that E_{LIRF} is a constant of motion. It is not an invariant because it has different values in different reference frames.

Exercise 3.4.2

According to Equation 3.4.1, does speed increase, decrease, or remain the same as an object approaches the event horizon?

Answer

Let's start by solving Equation 3.4.1for the speed.

$$\begin{aligned} \frac{1}{\sqrt{1-v_{\text{LIRF}}^2}} &= \frac{1}{\sqrt{1-\frac{2M}{r}}}\frac{E}{m} \\ \sqrt{1-v_{\text{LIRF}}^2} &= \sqrt{1-\frac{2M}{r}}\frac{m}{E} && \text{take reciprocal of both sides} \\ 1-v_{\text{LIRF}}^2 &= \left(1-\frac{2M}{r}\right)\left(\frac{m}{E}\right)^2 && \text{square both sides} \\ v_{\text{LIRF}}^2 &= 1-\left(1-\frac{2M}{r}\right)\left(\frac{m}{E}\right)^2 && \text{isolate } v_{\text{LIRF}}^2 \\ v_{\text{LIRF}} &= \sqrt{1-\left(1-\frac{2M}{r}\right)\left(\frac{m}{E}\right)^2} && \text{final answer} \end{aligned}$$

As $r \to 2M$, $v_{\text{LIRF}} \to 1$. In other words, the object approaches the speed of light! This might seem like a contradiction since in global coordinates we saw objects seemed to slow down at the event horizon. But there is no reason that someone hovering just outside the event horizon of a black hole needs to make the same observations as someone who is very far away. As weird as it seems, there is no contradiction. Welcome to General Relativity!

? Exercise 3.4.3

While nobody actually measures E, anyone in an LIRF can determine it. Explain how.

Answer

Equation 3.4.1provides a prescription for how to do it. Anyone in a LIRF can take local measurements of speed. Given that they know the r-value of their reference frame, they can simply plug in to determine E. Even though observers at different r-coordinates will have different measurements of the energy in their LIRF, they will all agree on the global energy E.

? Exercise 3.4.4

Suppose a small stone of mass m is released from rest very far from a black hole of mass M. What is the global energy E of the stone? Determine v_{LIRF} for the stone as a function of r.

Answer

The global energy is given by $E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} m$. Very far from the black hole, $dt = d\tau$ and $\frac{2M}{r} \to 0$, which means that $E = m$. We can plug this into Equation 3.4.1to determine v_{LIRF} . In fact, we already derived v_{LIRF} as a function of r in a previous exercise, so all we have to do is substitute $E = m$. As a reminder, the result was

$$v_{\text{LIRF}} = \sqrt{1 - \left(1 - \frac{2M}{r}\right)\left(\frac{m}{E}\right)^2}.$$

Substituting $E = m$, we get

$$v_{\text{LIRF}} = \sqrt{\frac{2M}{r}}. \tag{3.4.2}$$

We see that the stone accelerates as it approaches the event horizon.

Box 3.4.1

Use the definitions of the global-to-local coordinate transformations along with $\frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$ and $\frac{E_{\text{LIRF}}}{m} = \frac{dt_{\text{LIRF}}}{d\tau}$ to derive Equation 3.4.1.

Box 3.4.2

In one of the exercises, we showed how to determine v_{LIRF} as a function of r for a small stone released from rest infinitely far away from a non-spinning black hole of mass M. Let's repeat the process for a stone that is released from rest from a position that is *not* infinitely far away. That is, the stone is released from some radius $r = r_0$, where r_0 is a finite number.

a) Write an expression for the global map energy per unit mass $\frac{E}{m}$ in terms of r_0.
b) Use your previous answer as well as the fact that $\frac{E}{m}$ is a constant of motion to determine an expression for v_{LIRF} in terms of r. That is, write a function that could be used to determine the LIRF velocity at *any* r for a stone that is released from rest at r_0. Your answer should only contain the variables r, r_0, and M. (You can check your answer by taking the limit $r_0 \to \infty$.)

Box 3.4.3

Now let's repeat the process from the previous Box, except this time for a stone that is *thrown inward* at a speed v_0 from infinitely far away.

a) Write an expression for the global map energy per unit mass $\frac{E}{m}$ in terms of v_0.
b) Use your previous answer as well as the fact that $\frac{E}{m}$ is a constant of motion to determine an expression for v_{LIRF} in

terms of r. Your answer should only contain the variables r, v_0, and M. (You can check your answer by taking the limit $v_0 \rightarrow 0$.)

Box 3.4.4

We can use the LIRF transformations to determine the speed of circular orbits in General Relativity. It can be shown (though we won't do so here) that

$$v_{\text{LIRF}} = \left[1 - \left(\frac{mr}{L}\right)^2\right]^{-1/2}. \tag{3.4.3}$$

We also saw in a previous section that, for a circular orbit,

$$\left(\frac{L}{m}\right)^2 = \frac{Mr^2}{r-3M}.$$

Substitute this into Equation 3.4.3 to show that

$$v_{\text{LIRF}} = \sqrt{\frac{\frac{M}{r}}{1-\frac{2M}{r}}} \tag{3.4.4}$$

then use it to determine v_{LIRF} at the minimum stable orbit ($r = 6M$) and the minimum unstable circular orbit ($r = 3M$).

3.5: Inside the Black Hole

The previous section looked at local inertial reference frames in the vicinity of a black hole, which we could use to determine speed as a function of the r-coordinate. We found that as an object approaches the event horizon, it's speed approaches the speed of light. Things break down, however, if $r<2M$ because the LIRF energy becomes imaginary and the object's speed exceeds the speed of light. This actually happens because it is not possible to construct an LIRF inside of a black hole.

To see what happens inside of a black hole, we need to use a metric other than the Schwarzschild metric since the Schwarzschild metric has problems at $r=2M$. There is actually another spherically symmetric solution to the Einstein Field Equations called the **Global Rain** metric. The word "rain" is a reference to the fact that the T-coordinate is measured by steadily infalling clocks. The Global Rain spacetime interval is given by the following.

$$\begin{aligned} d\tau^2 &= \left(1-\frac{2M}{r}\right)dT^2 - 2\sqrt{\frac{2M}{r}}dT\,dr - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \\ &= \begin{pmatrix} dT & dr & d\theta & d\phi \end{pmatrix} \begin{pmatrix} 1-\frac{2M}{r} & 2\sqrt{\frac{2M}{r}} & 0 & 0 \\ 2\sqrt{\frac{2M}{r}} & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} dT \\ dr \\ d\theta \\ d\phi \end{pmatrix} \end{aligned} \tag{3.5.1}$$

The r-, θ, and ϕ-coordinates are the same as before, but now we have T instead of t. Notice also that we have an off-diagonal term in the metric. While that may seem weird, it is perfectly allowable. Just don't think too much about what it means.

Exercise 3.5.1

Argue that the global rain metric is spherically symmetric and reduces to flat spacetime in the appropriate limits.

Answer

The metric is spherically symmetric because the metric components depend only on r and not θ or ϕ. In the limits $M\to 0$ and $r\to\infty$, we get

$$d\tau^2 = dT^2 - dr^2 - r^2 d\theta^2 - r^2\sin^2 d\phi^2,$$

which is a flat metric in spherical coordinates.

We can see what the light cones look like by setting $d\tau=0$ in Equation 3.5.1. The result is shown in Fig. 3.5.1.

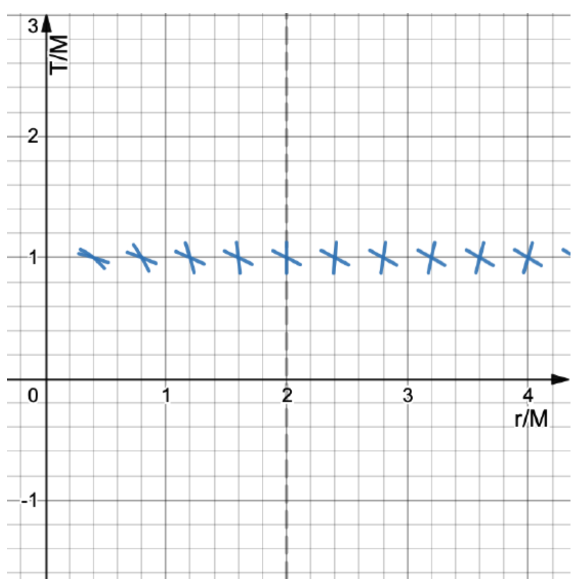


Figure 3.5.1: Light cones in global rain coordinates. (Copyright; author via source)

As shown in Fig. 3.5.1, the light cones are similar to those of flat spacetime far away from the black hole. The closer you get to the center, more the light cone tilts to the left.

Exercise 3.5.2

According to Fig. 3.5.1, is the event horizon a barrier to motion in global rain coordinates?

Answer

In Schwarzschild coordinates, we saw that light cones squeezed infinitely the closer you got to the event horizon, which indicated that for particles just outside the event horizon, there was no future inside. Here, however, we see no such squeezing. Particles just outside the event horizon do have a possible future within the event horizon.

What this indicates is that the problems that we saw at $r = 2M$ are simply due to the particular coordinate system we were using, similar to how the instruction "walk north" stops making sense once you get to the north pole. It's not that the north pole is an actual barrier, it's that the instruction stops making sense given how we define the word "north."

Exercise 3.5.3

At any particular point inside a black hole, you can draw one ingoing worldline of light and one "outgoing" worldline of light. Why did I put "outgoing" in quotation marks?

Answer

As you can see inside the black hole, there are no worldlines of light that move to larger values of r. This means that even light is not fast enough to escape. The rules of geometry inside of a black hole are so messed up that every direction is inward.

Box 3.5.1

Show that $d\tau^2 = dT^2 - \left(dr + \sqrt{\frac{2M}{r}}dT\right)^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$ is equivalent to Equation 3.5.1.

Box 3.5.2

Show that the slopes of the worldlines of light in global rain coordinates are given by $\frac{dT}{dr} = \frac{1}{-\sqrt{\frac{2M}{r}} \pm 1}$.

CHAPTER OVERVIEW

4: Light Orbits

Thumbnail: Einstein cross: four images of the same astronomical object, produced by a gravitational lens. Image used wtih permission (Public Domain; NASA and ESA).

4.1: Impact Parameter

In this chapter we will investigate the paths that photons take in Schwarzschild spacetime. That may seem like a silly question at first since it is often said that light travels in straight lines, but remember that General Relativity messes with the rules of geometry, so photons can travel in what appears to us as a curved path.

In the previous chapter, we learned that the motion of particles in Schwarzschild spacetime can be described using the energy per unit mass,

$$\frac{E}{m}=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau},$$

and the angular momentum per unit mass,

$$\frac{L}{m}=r^2\sin^2\theta\frac{d\phi}{d\tau}.$$

These don't make sense applied to a photon because $m=0$ and $d\tau=0$ for photons, which makes both sides undefined. If we take the *ratio* of these two quantities, however, then the problematic parts cancel out. Let's define this ratio as b.

$$b=\frac{L/m}{E/m} \qquad \text{definition}$$

$$b=\frac{L}{E} \qquad \text{cancel out m}$$

$$b=\frac{r^2\sin^2\theta\frac{d\phi}{d\tau}}{\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}} \qquad \text{substitute}$$

$$b=\frac{r^2\sin^2\theta}{1-\frac{2M}{r}}\frac{d\phi}{dt} \qquad \text{cancel } d\tau \text{ for final answer}$$

In principle, if you have a value for b, then you can determine $\phi(t)$. But that raises the question of how you determine b if neither E/m nor L/m is well-defined for a photon. It turns out that b is what is commonly referred to in physics at the **impact parameter** of the photon (see Box 4.1.1).

Definition: Impact Parameter

The impact parameter of a photon is basically how much it would miss the Schwarzschild center (i.e. the singularity) by if it were to move undeflected. More formally, it is the perpendicular distance between the line of motion and the radial line that is parallel to it when the photon is very far away. See Fig. 4.1.1.

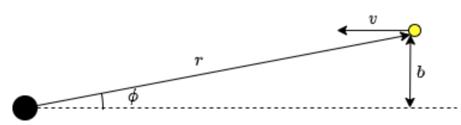


Figure 4.1.1: The impact parameter b is the perpendicular distance between the line of motion and the radial line that is parallel to it when the photon is very far away. (Copyright; author via source)

Box 4.1.1

You may recall from introductory mechanics that the magnitude of the angular momentum of a particle can be written as $L=rp\sin\phi$ where p is the linear momentum. If you look at the triangle in Fig. 4.1.1, you can see that $r\sin\phi=b$. Use these, along with the energy-momentum relationship $m^2=E^2-p^2$ to show that $\frac{L}{E}=b$ for a photon. (Note that this particular energy-momentum relationship only holds in flat spacetime, but that's okay since the stated definition of b is for a photon that is very far away.)

4.2: Effective Potential

The previous section discussed how ϕ evolves with t for a photon, but in order to get a full picture of the motion we also need $r(t)$. You can use the Schwarzschild metric along with the definition of the impact parameter to show that (see Box 4.2.1)

$$\frac{1}{b^2}=\left[\frac{1}{b\left(1-\frac{2M}{r}\right)}\frac{dr}{dt}\right]^2+\frac{1-\frac{2M}{r}}{r^2}. \tag{4.2.1}$$

This establishes a relationship, albeit a complicated one, between r and $\frac{dr}{dt}$. In *principle*, we can use it along with the angular equation from the previous section to completely determine the motion of a photon in Schwarzschild spacetime. In *practice*, however, this is basically impossible to do by hand, which means we must either use a computer or use the qualitative *effective potential energy* technique that we discussed earlier. Fortunately, Equation 4.2.1 is already in a form where the left hand side is a constant, and the right hand side is a sum of one term that involves speed and one term that involves position.

$$\begin{gathered}
\frac{1}{b^2}\Rightarrow \text{total energy} \\
\left[\frac{1}{b\left(1-\frac{2M}{r}\right)}\frac{dr}{dt}\right]^2\Rightarrow \text{radial "kinetic energy"} \\
\frac{1-\frac{2M}{r}}{r^2}\Rightarrow \text{effective potential energy}
\end{gathered}$$

Figure $4.2.1$ plots the effective potential for a photon in Schwarzschild spacetime.

Figure 4.2.1: Effective potential for a photon in Schwarzschild spacetime. (Copyright; author via source)

Exercise 4.2.1

Describe the various photon paths according to Figure 4.2.1.

Answer

The photon paths depend on the impact parameter. Figure 4.2.2shows three possible classes of paths.

Figure 4.2.2: Photon paths in Schwarzschild spacetime can be broken into three different classes. (Copyright; author via source)

The upper-most horizontal line is for photons with a small impact parameter. These photons come in from large r-values and go straight to $r=0$; they are called **plunge orbits**. The lower-most horizontal line is for photons with a large impact parameter, and they divide into two groups. One group (the part of the line that is above and to the right of the red curve) comes in from large r-values, swings around the central mass at $r=0$, then escapes back to large r-values; these orbits are called **bounce orbits**. The second group (above and to the left of the red curve) originates near the black hole and then plunges in; these orbits are called **trapped orbits**. The third class of paths is the middle horizontal line, which touches the very top of the red curve. If a photon comes into existence at that exact intersection point, it will follow a circular orbit. This ciarcular orbit is *unstable*, though, so it won't last for very long.

Note that this graph can't show the "swing around" part of orbits, but recall that ϕ also changes with time according to the previous section.

Box 4.2.1

Start with the Schwarzschild metric for a photon in the equatorial plane ($\theta=90^{\circ}$):

$$0=\left(1-\frac{2M}{r}\right)dt^2-\frac{1}{1-\frac{2M}{r}}dr^2-r^2d\phi^2.$$

Divide both sides by $\left(1-\frac{2M}{r}\right)dt^2$, then use the definition of the impact parameter $b=\frac{r^2}{1-\frac{2M}{r}}\frac{d\phi}{dt}$ and rearrange to prove Equation 4.2.1.

Box 4.2.2

There exists an unstable circular orbit at the maximum of the effective potential energy function $PE_{\text{eff}}(r)=\frac{1-\frac{2M}{r}}{r^2}$. Use calculus to show that the r-coordinate of this circular orbit is $r=3M$.

Box 4.2.3

In Box 4.2.2, you showed that an unstable circular orbit occurs at $r=3M$. Determine the value of the effective potential energy at that r-coordinate, then use that to show that the impact parameter that leads to a circular orbit is $b=\sqrt{27}M$. This is called the **critical impact parameter** because it separates photon orbits that plunge into the black hole from photon orbits that swing around and escape.

4.3: Lensing

When a photon passes a massive object, the path of the photon is deflected. If the impact parameter is small, the photon may be deflected and collide with the massive object (or fall into the singularity), or the photon may "swing around" the massive object (possibly multiple times) and escape. A quantitative calculation of the path is beyond the scope of this course. In the last section, we discussed a method to get qualitative information about the path. In this section, we will look at how to get quantitative information about the light path under very specific conditions. Specifically, in this section we are going to assume that:

1. the source of the light is very far from the **gravitational lens**
2. the gravitational lens is very far from the observer
3. the lensing object is between the source and the observer
4. the light travels in a straight line from the source to the lens, then suddenly turns and travels in another straight line to the observer

Definition: Gravitational Lens

A gravitational lens is a distribution of mass that causes light to bend around it.

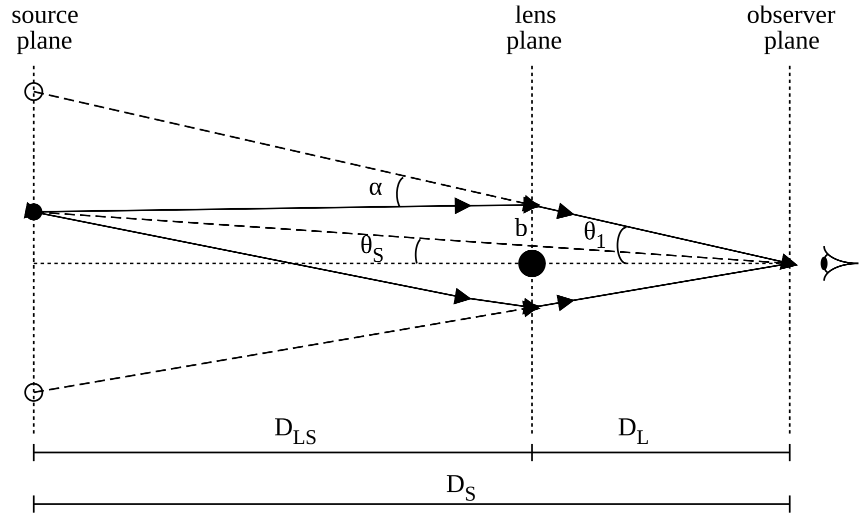


Figure 4.3.1: Light from a source (solid circle) travels in a straight line to the lensing plane, then turns and enters the observer's eye. Dashed lines show where the observer will think the light is coming from (i.e. the image location). Note that this is only a 2D depiction. There will also be paths that come out of or go into the page and then bend back to the observer, which will cause the image to be smeared out over a ring instead of two points. (Krishnavedala, CC0, via Wikimedia Commons)

Figure 4.3.1 illustrates the lensing process and defines key distances and angles. They are:

$$\begin{aligned} D_L &= \text{distance from observer to lens} \\ D_S &= \text{distance from observer to source} \\ D_{LS} &= \text{distance from lens to source} \\ \theta_S &= \text{angle between line of sight to lens and line of sight to source} \\ \theta_1 &= \text{angle between line of sight to lens and line of sight to image} \end{aligned}$$

As illustrated in the figure, there are two possible light paths from the source that will reach the observer's eye. The dashed lines show where the observer will think the light is coming from, which is where the images will appear. But this only shows the paths of light in one plane. There are also light beams that come out of the page that will bend back and enter the observer's eye. And there are light beams that go into the page that will will bend back and enter the observer's eye. The result is that the image of the source appears to be smeared out over a ring, called an **Einstein ring**. Figure 4.3.2 shows a cartoon depiction of an Einstein ring, and Figure 4.3.3 shows an actual image from the Hubble telescope with an Einstein ring.

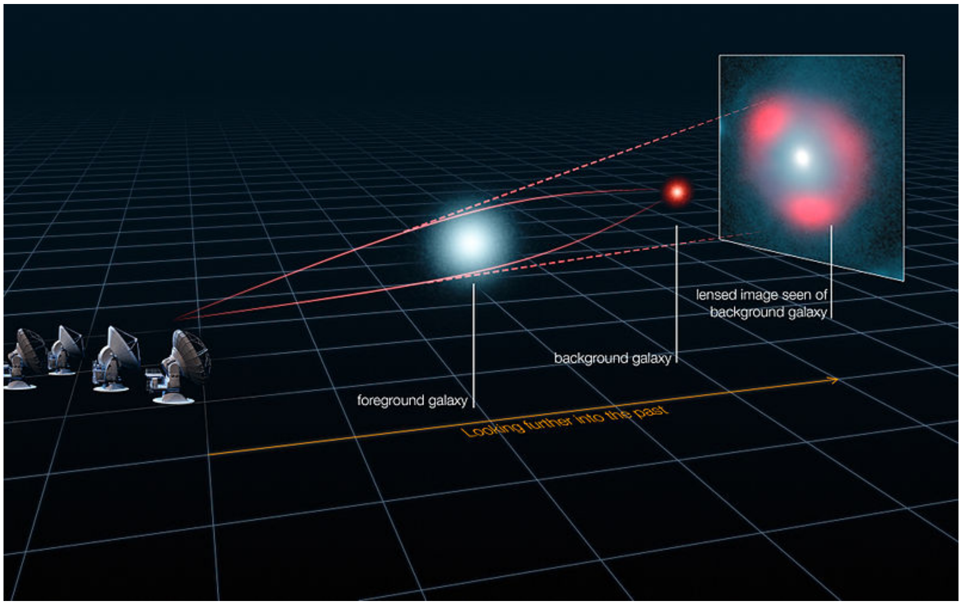


Figure 4.3.2: A cartoon depiction of an Einstein ring. (ALMA (ESO/NRAO/NAOJ), L. Calçada (ESO), Y. Hezaveh et al., CC BY 4.0, via Wikimedia Commons)

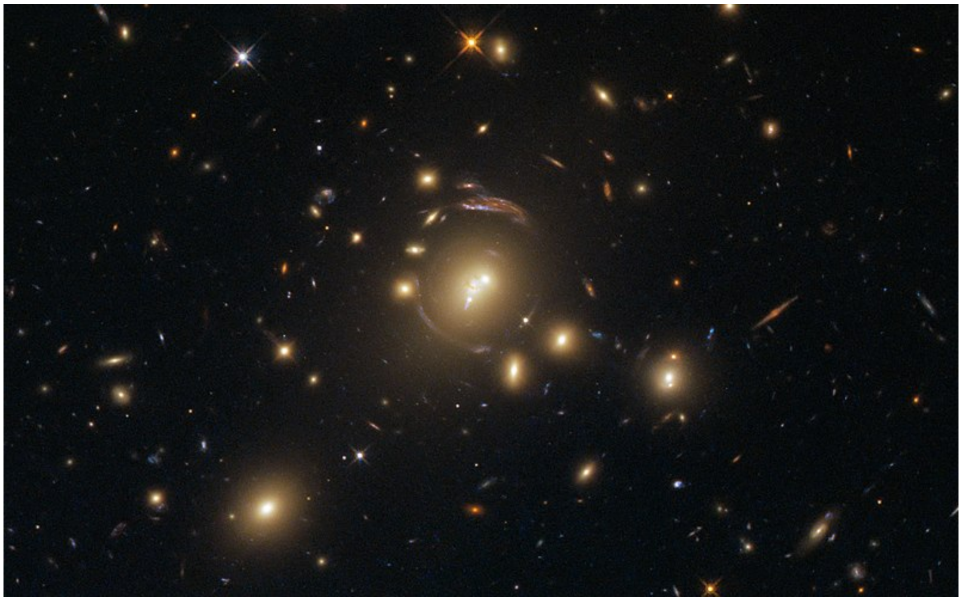


Figure 4.3.3: An actual Einstein ring. (ESA/Hubble, CC BY 4.0, via Wikimedia Commons)

The angular size θ_E of the Einstein ring is given by

$$\theta_E = \sqrt{\frac{4MD_{LS}}{D_L D_S}}. \qquad \text{(radians)} \tag{4.3.1}$$

The angles of the two images in the plane are

$$\theta_1 = \frac{\theta_S}{2} \pm \frac{1}{2}\sqrt{\theta_S^2 + 4\theta_E^2}. \tag{4.3.2}$$

? Exercise 4.3.1

Given that the Einstein ring is a circle, why is θ_E not just 360 degrees or 2π radians? Why is the size measured as an angle instead of a length?

Answer

Figure 4.3.4 shows how an object on the sky fills out a portion of a circle for which the observer is at the center. The angular diameter refers to the portion of the large "celestial sphere" on which the object falls. The Einstein ring angle θ_E is half of the angular diameter.

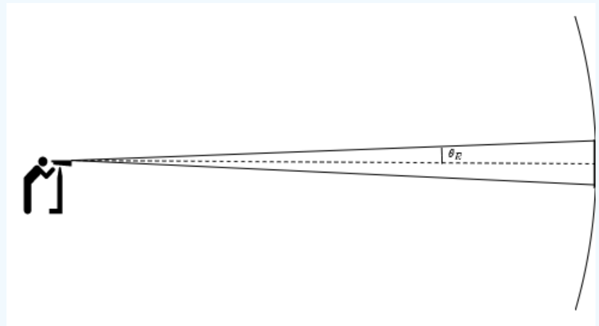


Figure 4.3.4: The angular size of the ring refers to the angle that the ring makes on the celestial sphere where the observer is the center. The Einstein ring angle if half the angular diameter. (Copyright; author via source)

The reason the size is given as an angle instead of a length is because the actual diameter of the ring depends on how far away it is, which we don't observe directly. The directly observable quantity is the angle.

? Exercise 4.3.2

Which, if any, of the angles θ_1 and/or θ_E depends on whether the lens, source, and observer are colinear? Under what conditions is the observation angle equal to the Einstein ring angle?

Answer

The quantity that captures whether the observer, lens, and source are colinear is the source angle θ_S. The Einstein ring angle does not depend on θ_S, but the observation angle θ_1 does.

If $\theta_S = 0$, then the observation angle and the Einstein ring angle are the same.

The Einstein ring angle tells you the size of the ring, but not where to look to find it. The observation angle θ_1 tells you where to look.

? Exercise 4.3.3

One of the applications of gravitational lensing is using the size of an Einstein ring to deduce the amount of mass in the lensing object. This requires knowing the distances to the lensing object and the source of the light, which is its own challenge. For now, let's assume we know that the distance from earth to the lensing object is 2 billion light years, and from earth to the light source is 6 billion light years. The angular size of the Einstein ring θ_E is 1 arcsecond (there are 60 arcminutes in a degree and 60 arcseconds in an arcminute). Calculate the mass of the lensing object in units of the mass of the sun.

Answer

The given distances are $D_L = 2$ billion light years and $D_S = 6$ billion light years, which means that $D_{LS} = 4$ billion light years. We can solve Equation 4.3.1for M and substitute the known values.

$$M = \frac{\theta_E^2 D_L D_S}{4 D_{LS}}$$

We also need to convert the angle to radians.

$$1 \text{ arcsecond} \times \frac{1 \text{ arcminute}}{60 \text{ arcsecond}} \times \frac{1 \text{ degree}}{60 \text{ arcminute}} \times \frac{\pi \text{ rad}}{180 \text{ degree}} = 4.85 \times 10^{-6} \text{ rad}$$

When we plug in, we have to be careful with units. The radian units effectively get dropped (since a radian is defined as a ratio of two lengths), but the mass will be in terms of light years.

$$M=\frac{\left(4.85\times 10^{-6}\text{ rad}\right)^2(2\text{ billion light years})(6\text{ billion light years})}{4(4\text{ billion light years})}=1.76\times 10^{-11}\text{ billion light years}$$

Previously we learned that the mass of the sun is 1477 m. We also know that the conversion between distance and time is $1\text{ s}=3\times 10^{8}\text{ m}$. We can use these to convert the mass of the lensing object to solar masses, $M_{\odot}$.

$$1.76\times 10^{-11}\text{ billion light years}=1.76\times 10^{-2}\text{ light years}\times\frac{3.154\times 10^{7}\text{ s}}{1\text{ year}}\times\frac{3\times 10^{8}\text{ m}}{1\text{ s}}\times\frac{1M_{\odot}}{1477\text{ m}}=1.13\times 10^{10}M_{\odot}$$

The mass of the lensing object is equivalent to about 10 billion suns, which is like a small galaxy.

Box 4.3.1

The "Cosmic Horseshoe" (Fig. 4.3.5) is a name given to an Einstein ring resulting from a gravitational lens in the Leo constellation. The lensing galaxy is 5.2 billion light-years from earth while the source galaxy is 10.3 billion light-years from earth.

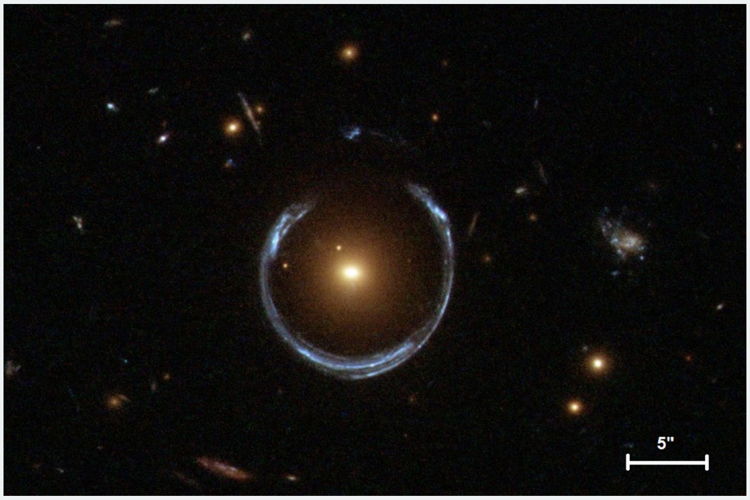
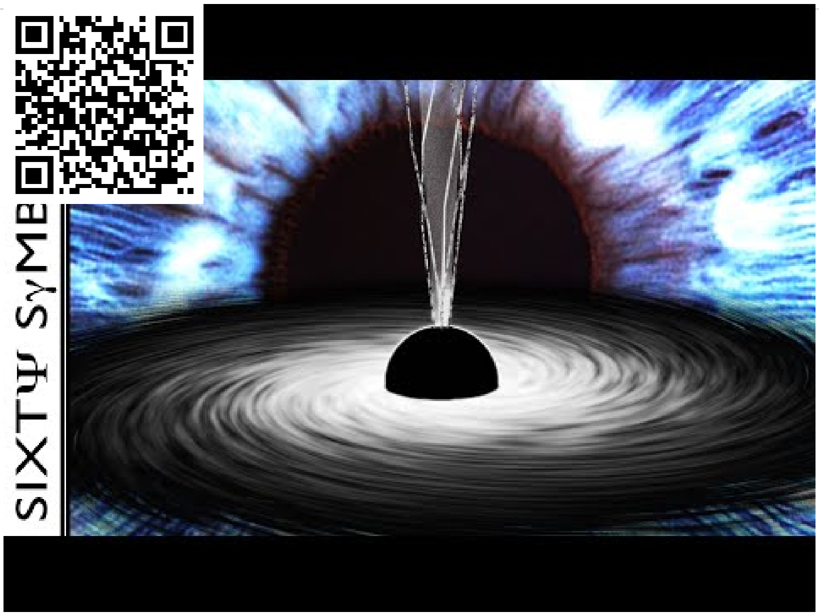


Figure 4.3.5: The Cosmic Horseshoe. (Image credit: https://home.strw.leidenuniv.nl/~jarle/Teaching/Practicum2015/assets/2015-04-19-einstein-ring.pdf)

a. Use the image in Figure $4.3.5$ as well as the gravitational lens equations to approximate the mass of the lensing galaxy.
b. Given the position of the ring relative to the lensing galaxy, what can you say about the position of the source galaxy relative to the lensing galaxy? In other words, is earth colinear with the two galaxies?

4.4: Video Resources



CHAPTER OVERVIEW

5: Spinning Black Holes

Thumbnail: This artist's concept illustrates a supermassive black hole with millions to billions times the mass of our sun. Supermassive black holes are enormously dense objects buried at the hearts of galaxies. (Public Domain; NASA/JPL-Caltech).

5.1: The Kerr Metric

The Schwarzschild metric assumes that the object at the center is completely stationary. Almost all spherical objects in space, however, spin. This is even true for black holes, which form when a star collapses. Since the progenitor star spins, by conservation of angular momentum the resulting black hole must spin as well. This spinning destroys the symmetry, which in turn means that we need a different metric to describe spinning, spherical objects. As with the Schwarzschild metric, though, we still need to make assumptions. They are:

1. the universe is empty except for a spherical mass
2. the mass itself is spherically symmetric, but it spins in the ϕ direction
3. the properties of the central mass do not depend on t.

In addition, we know that the resulting metric should reduce to the Schwarzschild metric when the spin is zero. If you put these assumptions and conditions into the Einstein Field Equations, you get

$$\begin{aligned} d\tau^2 &= \left(1-\frac{2Mr}{r^2+a^2\cos^2\theta}\right)dt^2+\frac{4Mra\sin^2\theta}{r^2+a^2\cos^2\theta}dt\,d\phi-\left(\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2}\right)dr^2 \\ &\quad-\left(r^2+a^2\cos^2\theta\right)d\theta^2-\left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{5.1.1}$$

where

$$a=\frac{J}{M} \tag{5.1.2}$$

is the angular momentum J per unit mass of the spinning object. The metric in Equation 5.1.1 is called the **Kerr metric**.

Definition: Kerr Metric

The Kerr metric is a metric for a spinning, spherical mass.

Note

Previously we used L to denote the angular momentum of a particle in an orbit. Here we use J to call out the fact that this is a different type of angular momentum. That is, it's the angular momentum of the spherical mass in the center.

Exercise 5.1.1

Verify that the Kerr metric reduces to the Schwarzschild metric when the central mass is not spinning.

Answer

If the central mass is not spinning, then $a=0$.

$$\begin{aligned} d\tau^2 &= \left(1-\frac{2Mr}{r^2}\right)dt^2-\left(\frac{r^2}{r^2-2Mr}\right)dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 && \text{set } a=0 \text{ in Eq. 5.1.1} \\ d\tau^2 &= \left(1-\frac{2M}{r}\right)dt^2-\frac{r^2}{r^2\left(1-\frac{2M}{r}\right)}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 && \text{simplify} \\ d\tau^2 &= \left(1-\frac{2M}{r}\right)dt^2-\frac{1}{1-\frac{2M}{r}}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 && \text{simplify for final answer} \end{aligned}$$

Exercise 5.1.2

Estimate the value of a for Earth. Express your answer in terms of M. (Recall that $J=I\omega$, where $I=\frac{2}{5}MR^2$ for a uniform sphere.)

Answer

Substituting $I = \frac{2}{5}MR^2$, we get $a = \frac{J}{M} = \frac{2}{5}R^2\omega$. The angular speed of earth is 2π radians per day, which we can convert to radians per second. We can then use the speed of light as a conversion factor to partially cancel out the units from the R^2.

$$\omega = 2\pi\frac{\text{rad}}{\text{day}} \times \frac{1\text{ day}}{24\text{ h}} \times \frac{1\text{ h}}{3600\text{ s}} \times \frac{1\text{ s}}{3\times10^8\text{ m}} = 2.42\times10^{-13}\text{ m}^{-1}$$

Now we can substitute $R = 6.37\times10^6\text{ m}$ to find a.

$$a = \frac{2}{5}\left(6.37\times10^6\text{ m}\right)^2\left(2.42\times10^{-13}\text{ m}^{-1}\right) = 3.93\text{ m}$$

In an earlier exercise, we calculated the mass of earth in meters at 0.0044 m, which means that

$$a = 3.93\text{ m}\times\frac{M}{0.0044\text{ m}} = 893M.$$

Exercise 5.1.3

Just like the Schwarzschild metric, the Kerr metric has problematic r-values for which some metric components are either zero or undefined. What are those r-values?

Answer

The coefficient of the dt^2 term is zero when $r^2 + a^2\cos^2\theta = 2Mr$. Let's solve this for r.

$$\begin{aligned}
r^2 + a^2\cos^2\theta &= 2Mr \\
r^2 - 2Mr + a^2\cos^2\theta &= 0 && \text{set up as quadratic} \\
r &= \frac{2M \pm \sqrt{4M^2 - 4a^2\cos^2\theta}}{2} && \text{use quadratic equation} \\
r &= M \pm \sqrt{M^2 - a^2\cos^2\theta} && \text{simplify}
\end{aligned}$$

Here we can see that r is only real-valued if $a^2\cos^2\theta \le M^2$, which in turn implies that there is a maximum spin for the black hole (since imaginary metric components are a no-no).

Another place to check is $r^2 - 2Mr + a^2 = 0$. That is similar to the previous case except that it is missing the $\cos^2\theta$, so the result is $r = M \pm \sqrt{M^2 - a^2}$. Here again we see that there is a limit on the spin.

We will talk more about these special r-values later. The big takeaway is that there are *two* horizons instead of one when the central mass spins.

5.2: Constants of Motion

When studying the Schwarzschild metric, we used the geodesic equation to determine constants of motion, which in turn gave us hints about how particles move. Let's repeat the process for the Kerr metric. As a reminder, the geodesic equation is

$$0=\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^{\nu}}{d\tau}\right)-\frac{1}{2}\frac{dg_{\alpha\beta}}{dx^{\mu}}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau}.$$

Also, for the sake of simplicity, for the remainder of this section will we look only at motion in the equatorial plane ($\theta=90^{\circ}$).

You can show (see Box 5.2.1) that setting $\mu=t$ leads to

$$\frac{E_{\text{Kerr}}}{m}=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}+\frac{2Ma}{r}\frac{d\phi}{d\tau} \tag{5.2.1}$$

and that setting $\mu=\phi$ (see Box 5.2.2) leads to

$$\frac{L_{\text{Kerr}}}{m}=\left(r^{2}+a^{2}+\frac{2Ma^{2}}{r}\right)\frac{d\phi}{d\tau}-\frac{2Ma}{r}\frac{dt}{d\tau}. \tag{5.2.2}$$

Using these to determine the exact motion of particles is mathematically challenging. We can, however, use them to make some broad claims. The most interesting claim that I am going to make is that particles released from rest can end up moving *around* the central mass; this is called *frame dragging*. You will demonstrate the plausibility of this claim in the following exercises.

Exercise 5.2.1

A rock is released from rest very far away from a spinning black hole. Argue that $\frac{E_{\text{Kerr}}}{m}=1$ and that $\frac{L_{\text{Kerr}}}{m}=0$.

Answer

Very far away, $\lim_{r\to\infty}1-\frac{2M}{r}=1$, $\lim_{r\to\infty}\frac{2Ma}{r}=0$, and $\lim_{r\to\infty}\frac{dt}{d\tau}=1$. If the rock is released from rest, then $\frac{d\phi}{dt}=0$ at that moment. We then end up with

$$\frac{E_{\text{Kerr}}}{m}=(1-0)(1)+0=1$$

and

$$\frac{L_{\text{Kerr}}}{m}=\left(r^{2}+a^{2}+0\right)(0)-(0)(1)=0.$$

Exercise 5.2.2

Eqs. 5.2.1 and 5.2.2 can be combined to yield

$$\frac{d\phi}{d\tau}=\frac{\frac{2Ma}{r}\frac{E_{\text{Kerr}}}{m}-\left(1-\frac{2M}{r}\right)\frac{L_{\text{Kerr}}}{m}}{\frac{4M^{2}a^{a}}{r}+\left(1-\frac{2M}{r}\right)\left(r^{2}+a^{2}+\frac{2Ma^{2}}{r}\right)}. \tag{5.2.3}$$

Argue that a rock released from rest from very far away from a spinning black hole will inevitably end up moving in the ϕ direction.

Answer

By substituting $\frac{E_{\text{Kerr}}}{m}=1$ and $\frac{L_{\text{Kerr}}}{m}=0$, we get

$$\frac{d\phi}{d\tau}=\frac{2Ma/r}{\frac{4M^{2}a^{2}}{r^{2}}+\left(1-\frac{2M}{r}\right)\left(r^{2}+a^{2}+\frac{2Ma^{2}}{r}\right)}.$$

As long as $r > 2M$, then the denominator will be positive and $\frac{d\phi}{d\tau}$ will have the same sign as a. In other words, the rock will always end up moving around the black hole in the same direction as the spin of the black hole.

Box 5.2.1

Set $\mu = 0$ in the geodesic equation and use it to show that $\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}+\frac{2Ma}{r}\frac{d\phi}{d\tau}$ is constant. (If you are wondering how we make the leap to calling it energy specifically, just set $a = 0$ to see that it reduces to the energy per unit mass for the Schwarzschild metric.)

Box 5.2.2

Set $\mu = \phi$ in the geodesic equation and use it to show that $\left(r^2+a^2+\frac{2Ma^2}{r}\right)\frac{d\phi}{d\tau}-\frac{2Ma}{r}\frac{dt}{d\tau}$ is a constant. (As before, simply set $a = 0$ to see how we justify calling this angular momentum per unit mass.)

Box 5.2.3

Equation 5.2.1 can be written as $\frac{E_{\text{Kerr}}}{m} = g_{tt}\frac{dt}{d\tau}+g_{t\phi}\frac{d\phi}{d\tau}$, and Equation 5.2.2 can be written as $\frac{L_{\text{Kerr}}}{m} = g_{t\phi}\frac{dt}{d\tau}+g_{\phi\phi}\frac{d\phi}{d\tau}$. Solve the latter for $\frac{dt}{d\tau}$, then substitute into the former. Solve for $\frac{d\phi}{d\tau}$ using algebra, then substitute the appropriate expressions for g_{tt} and $g_{t\phi}$ to prove Equation 5.2.3.

5.3: The Ergoregion

When studying the Schwarzschild metric, we saw that $r = 2M$ was interesting for two reasons. First, the coefficient of dt^2 (g_{tt}) in the metric became zero. We can call this an **infinite redshift surface** because the time between two consecutive crests of a light wave coming from just outside the event horizon is infinitely big for a faraway observer (shifted to a lower frequency). Second, we saw that the signs of g_{tt} and g_{rr} switched at $r = 2M$, which we interpreted as meaning that $r = 2M$ is the "point of no return" since the inevitable end was $r = 0$. We called this the **event horizon**. We could also call it the **static limit surface** because it is only possible to stay stationary outside. For a Schwarzschild black hole, the infinite redshift surface, the static limit surface, and the event horizon happen to have the same r-value, but that isn't necessarily the case for other black holes. For a spinning black hole, the infinite redshift surface and the static limit surface are still the same, but they are different from the event horizon. The region in between the infinite redshift surface and the event horizon is called the **ergoregion**.

Definition: Infinite Redshift Surface

The infinite redshift surface of a black hole is the set of all r-values for which $g_{tt} = 0$. A light wave emitted outward from this surface will be shifted to a lower frequency, eventually reaching a frequency of zero at an r-value of infinity.

Definition: Static Limit Surface

The static limit surface of a black hole is the set of all r-values below which a test particle cannot stay stationary.

Definition: Ergoregion

The ergoregion is the region between the infinite redshift surface and the event horizon.

By setting $g_{tt} = 0$, we can show that the infinite redshift surface r_{IR} is given by (see Box 5.3.1)

$$r_{\mathrm{IR}} = M + \sqrt{M^2 - a^2 \cos^2 \theta}. \tag{5.3.1}$$

Note

Setting $g_{tt} = 0$ in the Kerr metric and solving for r actually yields *two* solutions, but one of those solutions falls within the event horizon. Since light from just outside that surface can't escape to the outside world, we can ignore that solution.

It turns out that the infinite redshift surface is also the static limit surface (see Box 5.3.2), but *not* because particles within the ergoregion must fall to smaller values of r. Instead, particles within the ergoregion must move in the direction of the black hole's spin.

Finally, by setting $\frac{1}{g_{rr}} = 0$ in the Kerr metric, we can show that the event horizon is given by (see Box 5.3.3)

$$r_{\mathrm{EH}} = M + \sqrt{M^2 - a^2}. \tag{5.3.2}$$

Note

Once again, you actually get two solutions when you set $g_{rr} = 0$, but we ignore the smaller one. If the point of defining an event horizon is to identify the point of no return, then the smaller one is rendered irrelevant.

Fig. 5.3.1 shows a polar plot of the infinite redshift surface and the event horizon.

Figure 5.3.1: A polar plot of the infinite redshift surface (green) and the event horizon (purple) for $a = 0.95M$. This shows a cross-section, with the rotation axis in the center and vertical. Click "edit graph on desmos" to see how changing the spin parameter changes each. (Copyright; author via source)

? Exercise 5.3.1

Does a faster spin cause the ergoregion to become bigger or smaller?

Answer

If you click on "edit graph on desmos" in Fig. 5.3.1, you can choose the value of a using a slider. You should find that the ergoregion shrinks the closer a is to zero. This makes sense since $a = 0$ is the Schwarzschild limit, where there is no ergoregion.

? Exercise 5.3.2

What is the maximum spin for which there is an event horizon?

Answer

According to Equation 5.3.2, the r-value of the event horizon is only a real number if $M^2 - a^2 \geq 0$, which means that there is only an event horizon if $a \leq M$.

Exercise 5.3.3

How is the ergoregion different from the region in the interior of the event horizon?

Answer

It is impossible to stay stationary in both the ergoregion and the interior of the event horizon. The difference is that a particle is able to maintain a constant r-value inside the ergoregion while it is forced to move to $r = 0$ inside the event horizon.

Box 5.3.1

Set $g_{tt} = 0$ in the Kerr metric to prove Equation 5.3.1.

Box 5.3.2

Argue that a particle can only stay stationary ($dr = d\theta = d\phi = 0$) outside of a Kerr black hole if $g_{tt} > 0$. (Remember that $g_{tt} = 0$ defines the infinite redshift surface, so this proves that the static limit surface is the same as the infinite redshift surface.)

Box 5.3.3

Set $\frac{1}{g_{rr}} = 0$ in the Kerr metric to prove Equation 5.3.2.

CHAPTER OVERVIEW

6: Gravitational Waves

Thumbnail: Two-dimensional representation of gravitational waves generated by two neutron stars orbiting each other. (Public Domain; NASA).

6.1: What are Gravitational Waves?

In 2016, scientists published the first paper announcing the detection of gravitational waves. This was a very big deal, not only because it verified a very old prediction of General Relativity, but also because it was an experimental feat.

To understand what a gravitational wave is, let's first try to understand how charged particles interact through electromagnetism. We know that oppositely charged particles attract one another with a force that depends on the distance between them. Suppose we have a positively charged particle named Paula and a negatively charged particle named Nigel. What happens to the strength of the attractive force between Paula and Nigel if you suddenly move Paula? Does the force exerted on Nigel change immediately, or is there a delay? We now know that the answer is that there is a delay. When you suddenly move Paula, it creates an electromagnetic wave that propagates outward, and only when the wave reaches Nigel does Nigel register that anything changed. This is the principle behind all wireless communication; you wiggle the electrons in an antenna, which creates an electromagnetic wave that propagates outward. When that electromagnetic wave strikes another antenna, it causes the electrons in that antenna to wiggle as well (albeit a much weaker wiggle because the intensity of the wave decreases with distance).

Now think of the earth revolving around the sun. If you suddenly moved the sun, would the earth feel the effect immediately, or would there be a delay? If we were studying Newtonian mechanics, we would say that it happens immediately. But General Relativity predicts that there would be a delay because the movement of the sun creates a wave in spacetime that propagates outward and eventually strikes the earth. Much like with an antenna, though, the intensity of the wave decreases with distance. And since space is largely empty with vast distances between objects, the effects of these gravitational waves are very small.

The other obstacle in detecting gravitational waves is that there are not that many good sources. In order to have any hope of detecting a gravitational wave, you need a very heavy object with a large acceleration. Fortunately, black holes are very heavy, and sometimes two black holes get near enough to each other that they start to orbit one another. The vast majority of detected gravitational waves have come from such sources, and they will be the primary focus of this chapter.

6.2: Gravitational Wave Metric

Gravitational waves were predicted by General Relativity long before they were actually detected. How? Well, a lot of the theoretical predictions of General Relativity came about just from playing around with the Einstein Field Equations. You can propose a matter distribution and see what metric is consistent with that, or you can propose a metric and see what that implies. The metric corresponding to a gravitational plane wave traveling in the z-direction, for example, could be written as

$$d\tau^2 = dt^2 - (1+h)dx^2 - (1-h)dy^2 - dz^2, \tag{6.2.1}$$

where $h \ll 1$. This is very similar to the metric from Special Relativity, except that g_{xx} and g_{yy} are very slightly different from 1 (recall that they are exactly equal to 1 in Special Relativity). If you put Equation 6.2.1 into the Einstein Field Equations, you find that h is an oscillating function (the proof is beyond the scope of this course). Fig. 6.2.1 shows what the coordinate grid for such a metric could look like. If you start with a ring of particles in the plane of the page, then a gravitational wave traveling perpendicular to the page will cause the ring to stretch in one direction while squeezing in the other direction. As the wave continues to move, the part that was stretched becomes squeezed while the part that was squeezed becomes stretched.

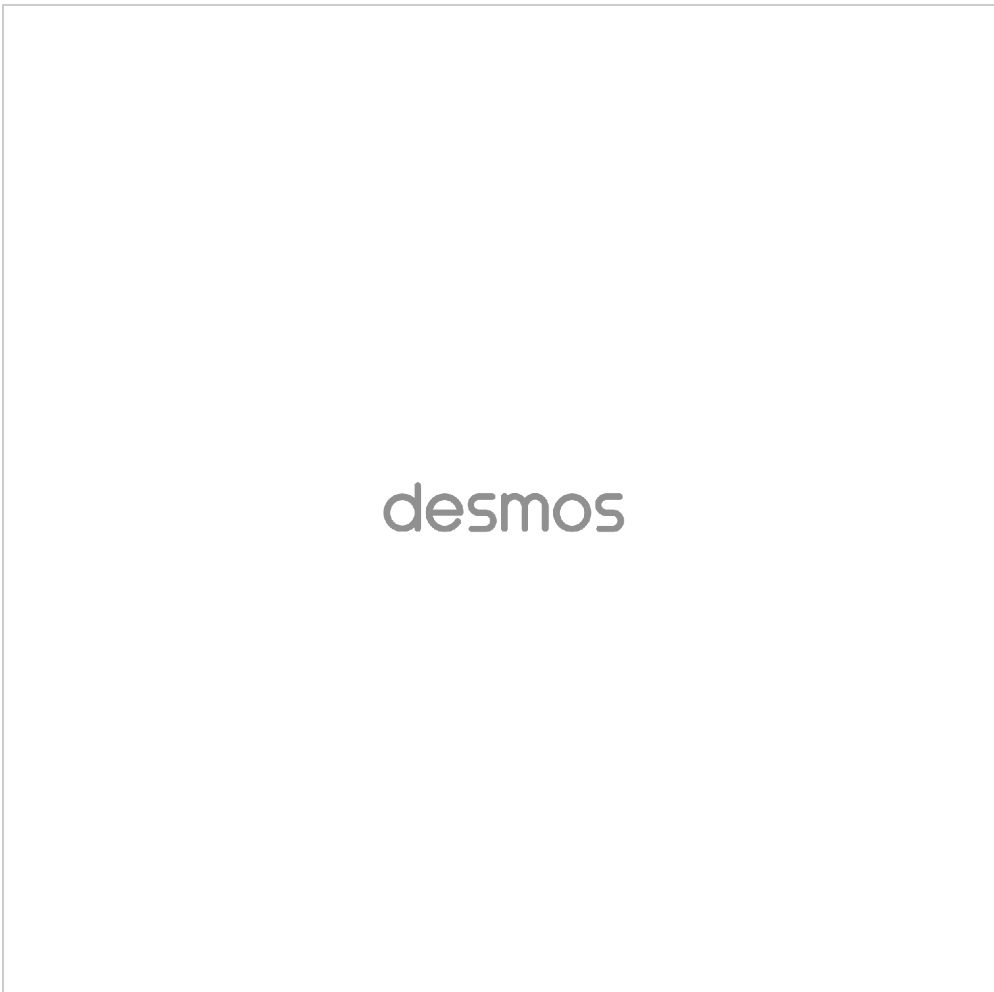


Figure 6.2.1: This animation shows what happens to a ring of particles when a gravitational wave passes by. The wave is traveling perpendicular to the page. (Image credit)

Notice that the coordinate locations of the particles in Fig. 6.2.1 do not change. Nevertheless, the distances between the particles change with time. This is the result of the fact that the coordinates in Equation 6.2.1 are *global coordinates*, where coordinate separations don't necessarily correspond to anything measurable. To get measurable quantities, like distance, we need to superimpose a *local inertial reference frame* (LIRF) on top of the global coordinate system. Then we use the fact that, for any LIRF,

$$d\tau^2 = dt_{\text{LIRF}}^2 - dx_{\text{LIRF}}^2 - dy_{\text{LIRF}}^2 - dz_{\text{LIRF}}^2.$$

By laying the axes of the LIRF along the axes of the global coordinate system, you can relate the two. The result is

$$\Delta t_{\text{LIRF}} = \Delta t \tag{6.2.2}$$

$$\Delta x_{\text{LIRF}} = \sqrt{1+h}\,\Delta x \qquad \approx \left(1+\frac{h}{2}\right)\Delta x \qquad (h \ll 1) \tag{6.2.3}$$

$$\Delta y_{\text{LIRF}} = \sqrt{1-h}\,\Delta y \qquad \approx \left(1-\frac{h}{2}\right)\Delta y \qquad (h \ll 1) \tag{6.2.4}$$

$$\Delta z_{\text{LIRF}} = \Delta z \tag{6.2.5}$$

where we have used the binomial approximation $(1+x)^n \approx 1+nx$ if $x \ll 1$.

Note

Large intervals (Δ's) come from adding up (i.e. integrating) many infinitesimally small intervals (d's). If all of the small intervals are the same size (i.e. the integrand is constant with respect to the integration variable), then you can freely switch between d's and Δ's as we did in going from the gravitational wave metric to the LIRF coordinate transformations.

According to these equations, measured distances between events on the x- or y-axes are either slightly bigger or slightly smaller than their respective global coordinate separations, with h representing the fractional change, or **gravitational wave strain**.

Definition: Gravitational Wave Strain

Gravitational wave strain is the amount that distances are stretched or compressed by a passing gravitational wave, relative to the original length. It is a dimensionless number.

Exercise 6.2.1

The Laser Interferometer Gravitational Wave Observatory (LIGO) works by firing a laser beam at a mirror that is 4 km away. Suppose a gravitational wave with $h = 10^{-21}$ passes by during that time. What is the maximum change in laser-to-mirror distance caused by the gravitational wave? Does the answer depend on the orientation of the gravitational wave?

Answer

In a local inertial reference frame, the separation of two points along, say, the x-axis is $\Delta x_{\text{LIRF}} = \left(1+\frac{h}{2}\right)\Delta x = \Delta x + \frac{h}{2}\Delta x$. In this case, $\Delta x = 4$ km and $\frac{h}{2}\Delta x = \left(\frac{10^{-21}}{2}\right)(4000 \text{ m}) = 2\times 10^{-18}$ m is the *change* in length. That is approximately 1/1000th of the width of the nucleus of an atom! (These are actually realistic numbers.)

The answer does depend on orientation of the wave. Our calculation assumes that the wave travels exactly in the z-direction (perpendicular to earth) and that one of the "stretchy" axes of the wave is parallel to the x-axis. In general, then, the actual change in length will be smaller than what we calculated.

6.3: LIGO

The first discovery of a gravitational wave occurred at the Laser Interferometer Gravitational Wave Observatory (LIGO) in September of 2015. LIGO actually consists of *two* detectors, one in Hanford, Washington, and the other in Livingston, Louisiana (see Figure $6.3.1$). One of the reasons for having two detectors is because any real gravitational wave would necessarily have to travel though both detectors; if only one detector sees what it thinks is a gravitational wave, then it could just be noise. Another benefit of having two detectors is that it helps in determining where the gravitational wave came from; whichever detector registers the wave first must be closer to the source of the wave.

Figure $6.3.1$: Images of LIGO facilities. (Image credit: LIGO collaboration website)

The detectors themselves are interferometers with arms that each have a length of 4 km, as shown in Figure $6.3.2$. The purpose of an interferometer is to split a single beam of light into two parts and then recombine them such that they interfere with one another. If the two parts travel exactly the same distance or if the path difference is an integer multiple of the wavelength, then the light beams interfere constructively. If the path difference is an odd multiple of a half-wavelength, then the light beams interfere completely destructively.

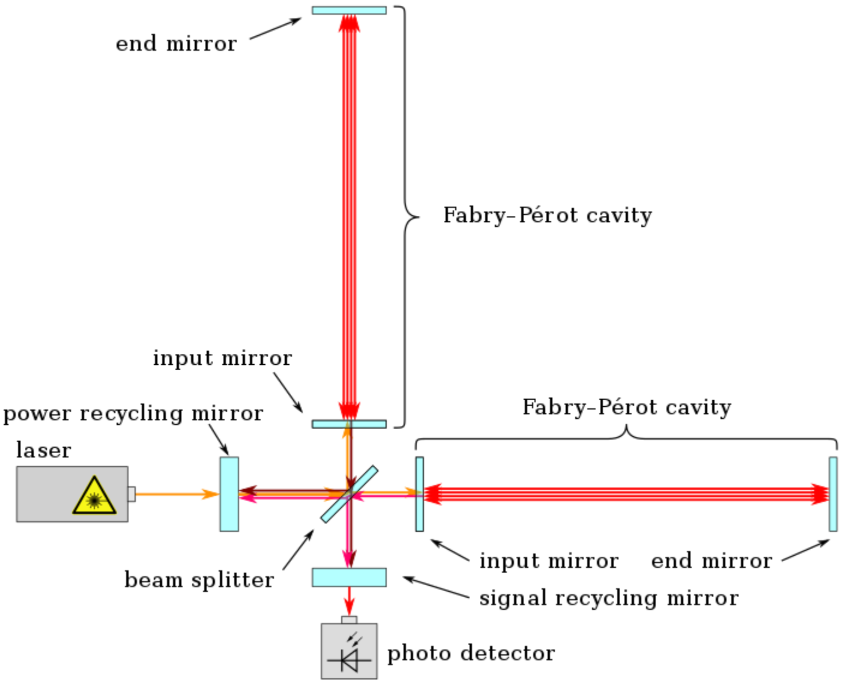


Figure $6.3.2$: A simplified depiction of the LIGO interferometer. (Image credit: Wikimedia commons)

In most interferometers, the mirrors are fixed to a surface such that their positions can be adjusted. By observing how the level of interference changes as you adjust the positions of the mirrors, you can determine the wavelength of the light (for example). In LIGO, the mirrors are instead suspended by systems that aim to decouple them from ground disturbances as much as possible, which makes them similar to free-floating objects. In this way, changes in the path lengths of the two arms can be caused by gravitational waves. The tricky part, however, is determining what motions of the mirrors are due to gravitational waves versus other sources. To do that, you can run the detector in the absence of gravitational waves (which is most of the time) and see what the signal at the detector looks like. This is called **strain noise** (Figure $6.3.3$). Then you look for deviations from the noise. It also helps that gravitational waves have a very distinct signal, which we will talk about later.

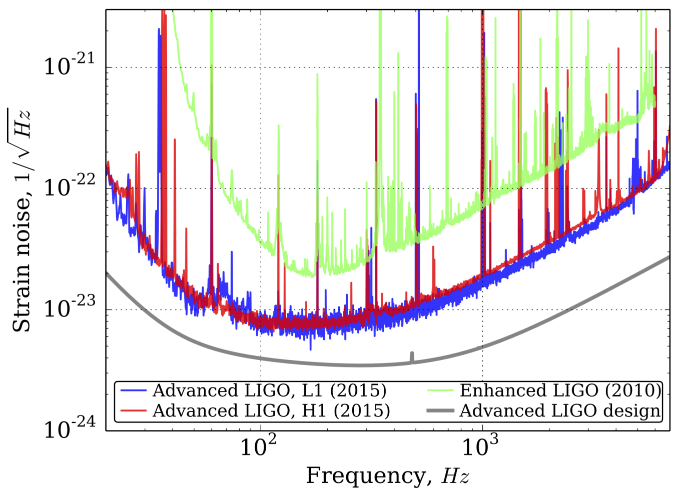


Figure $6.3.3$: This graph shows how much noise there is over a range of frequencies for three different generations of LIGO. Any gravitational wave signal with amplitude below the curve can't be detected. (Image credit: ligo.org)

Note: Spectral Densities

You may be wondering why the vertical axis in Figure $6.3.3$ has such unusual units. In experiment, a detector's sensitivity often varies with the frequency of the incoming signal. This is often described using a **power spectral density** (PSD) graph, which indicates how much energy is received by the detector per unit time over a range of frequencies. (If that sounds confusing, try imagining that the range of frequencies is discrete instead of continuous. The PSD shows how much energy is received at each of those frequencies.) The integral over a particular range of frequencies (i.e. the area under the curve) then tells you the total power received by the detector. Related to the PSD is something called an **amplitude spectral density** (ASD) graph. The power contained in a wave is proportional to the square of the amplitude of the wave, so you can get an ASD graph by taking the square root of the PSD's vertical axis. That is why you see $\sqrt{\mathrm{Hz}}$ in the denominator of the unit for the vertical axis in Figure $6.3.3$.

Figure $6.3.4$ breaks down the noise by source. Unsurprisingly, motion of the ground itself is a significant problem, but only at very low frequencies. Also note that a significant source of noise is **radiation pressure**, which is pressure exerted on the mirrors from photons. A typical task in Modern Physics courses is to learn how to calculate the momentum of a photon, and here we see why that actually matters.

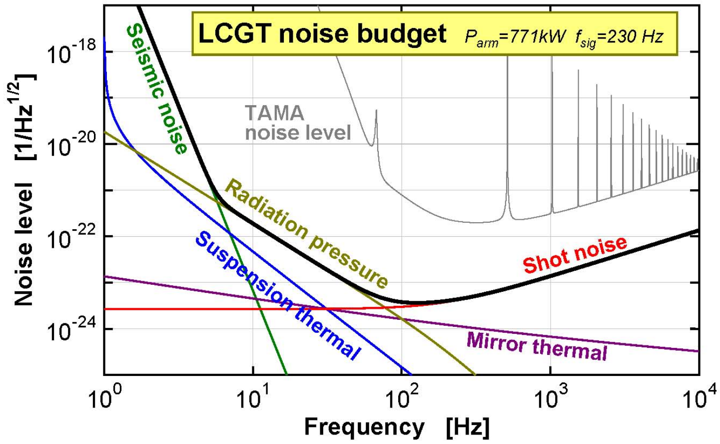


Figure 6.3.4: LIGO noise budget. (Image credit: LIGO collaboration web site)

You can also see from Fig. $6.3.4$ that the peak sensitivity occurs on the order of 100 Hz. It is not a coincidence that this happens to be right around the frequency of gravitational waves produced by binary black hole mergers (we will talk more about that in the next section). That is good experimental design!

Exercise 6.3.1

One concern that some people have when they learn how a gravitational wave interferometer works is that the wavelength of the laser light will be stretched along with the gravitational wave, which would defeat the purpose of using an interferometer in the first place. It seems especially bad when you consider the fact that the laser actually bounces back and forth between the mirrors 300 times before being received by the detector (this effectively increases the path length difference). This is not actually a problem, though, because while the interferometer arms are there for the entire stretch-compress cycle, a laser pulse travels fast enough that it is only present for a tiny portion of the stretch-compress cycle. In other words, the laser pulse doesn't exist for long enough for its wavelength to even be stretched or compressed by the gravitational wave. Do the math to argue that this is the case.

Answer

The light takes an 8 km round trip (4 km out and another 4 km back) 300 times. The time it takes to make this trip is

$$\Delta t=\frac{(8000 \text{ m})(300)}{3\times 10^8 \text{ m/s}}=0.008 \text{ s}.$$

Meanwhile, the period of a gravitational wave is approximately $T=\frac{1}{f}\approx\frac{1}{100 \text{ Hz}}=0.01 \text{ s}$. The time the light spends in the interferometer, then, is less than one tenth of the time of a full stretch-compress cycle. So while the wavelength of the light will be stretched by the gravitational wave by a little bit, it isn't stretched by so much as to defeat the purpose of using an interferometer.

6.4: Gravitational Wave Sources

In general, gravitational waves are produced by moving masses. This can include the collapse of stars or objects in orbit. One caveat, though, is that the motion can't be spherically symmetric. A perfectly spherical star, then, would not produce gravitational waves when it collapses or explodes, but an asymmetric star would. While perfectly spherical stars don't actually exist, the degree to which a typical star deviates from perfect spherical symmetry is small enough that wouldn't expect very strong gravitational waves to come from it. For this reason, the primary source that we can expect to detect gravitational waves from is objects in orbit. While many objects in space orbit around something, only the most massive objects have any hope of producing gravitational waves large enough for us to detect them. Furthermore, the orbit has to be fast enough to produce gravitational waves with a frequency on the order of 100 Hz. This vastly reduces the possible orbital scenarios that we can detect. So far, for example, we have only been able to detect binary systems (that is, two objects orbiting around each other) that consist of black holes and/or neutron stars.

Let's analyze a simple binary system consisting of objects with mass M_1 and M_2 separated by a distance r. As long as r is much greater than the Schwarszchild radius and the orbital speed is non-relativistic, the total energy can be approximated using Newtonian mechanics. The result (see Box 6.4.1), in natural units, is

$$E=-\frac{M_1 M_2}{2r}. \tag{6.4.1}$$

The gravitational waves, meanwhile, carry energy away, which makes the total energy of the binary system *more negative*.

Exercise 6.4.1

Use Equation 6.4.1 to argue that, as the binary system loses energy, the distance between the orbiting objects decreases.

Answer

Fig. 6.4.1 shows a plot of the total energy as a function of r.

Figure 6.4.1: Total energy of a binary system as a function of r. (Copyright; author via source)

You can see from the graph that energy loss corresponds to smaller values of r.

It can be shown (though we will not do so here), that the rate of energy loss is

$$\frac{dE}{dt} = -\frac{32}{5r^5}\left(M_1 M_2\right)^2\left(M_1 + M_2\right). \tag{6.4.2}$$

Equation 6.4.2 can then be used to determine the rate at which the distance between them decreases. The result (see Box 6.4.2) is

$$\frac{dr}{dt} = -\frac{64}{5r^3} M_1 M_2\left(M_1 + M_2\right). \tag{6.4.3}$$

This result shows two important things. First, $\frac{dr}{dt}$ is negative, which confirms that the distance between the orbiting objects decreases with time. Second, the rate at which the distance decreases gets larger as the orbiting objects get closer together.

Finally, the orbital speed increases as the distance between them decreases (see Box 6.4.3), which increases the frequency of the emitted gravitational waves. The result is a **chirp**. That is, there is a very rapid increase in the frequency of the gravitational wave in the moments just before the orbiting objects merge. Fig. 6.4.2 shows what this looks like for an actual gravitational wave event at three different detectors (Virgo is another detector that came online after the original detection by Hanford and Livingston).

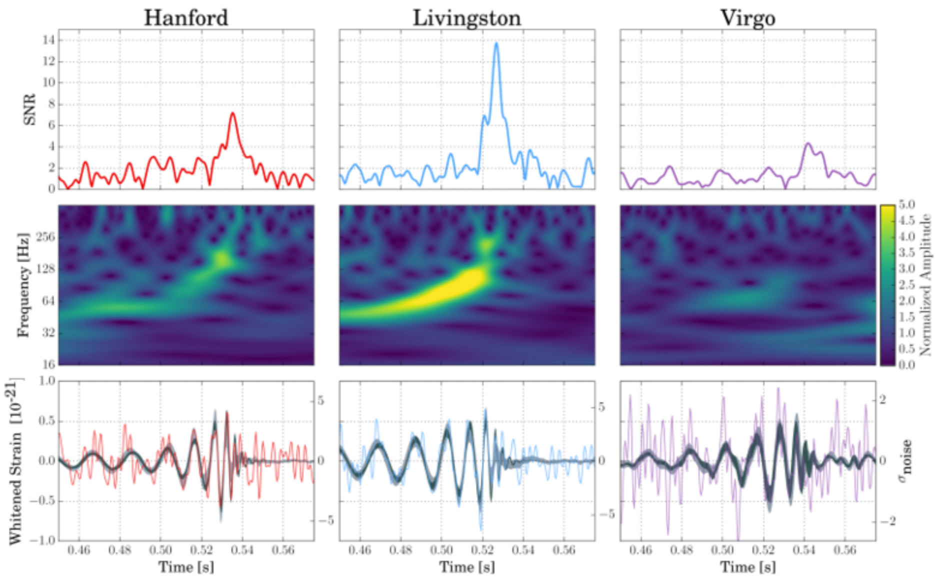


Figure 6.4.2: The top row shows the signal-to-noise ratio for a gravitational wave event at three different detectors. The middle row shows a spectrogram, which depicts both frequency and amplitude as a function of time. The bottom row shows the gravitational wave strain as a function of time. Both the middle row and the bottom row show that the frequency rapidly increases just before the merger. (Image credit: Wikimedia commons)

The point of this is that we have a theoretical expectation for what a gravitational wave signal should look like, and the properties of that signal depend on the masses of the orbiting objects, the distance between them, and the distance from us to the binary system. Gravitational wave detectors continuously look for signals with this characteristic shape. When they do find one, they perform a parameter fit, which gives us information about the binary system. Then by looking at the time delay between the signal at the different detectors, we can determine where on the sky on the signal came from.

Box 6.4.1

The total energy in Equation 6.4.1 is a combination of potential energy and kinetic energy. The potential energy, in natural units, is

$$PE = -\frac{M_1 M_2}{r}$$

while the kinetic energy is

$$KE = \frac{1}{2}\mu v^2,$$

where $\mu = \frac{M_1 M_2}{M_1 + M_2}$ is called the **reduced mass** (reduced mass is used when two objects revolve around a point between them rather than one object orbiting around another stationary object). Meanwhile, the gravitational force of attraction is given by

$$F = \frac{M_1 M_2}{r^2}.$$

Finally, the centripretal force is

$$F = \mu \frac{v^2}{r}.$$

Use these to derive Equation 6.4.1.

Box 6.4.2

Take the derivative of Equation 6.4.1 with respect to time, then equate it to Equation 6.4.2 in order to prove Equation 6.4.3. (Hint: the r in Equation 6.4.1 is a function of time, so you must use the chain rule when taking the derivative.)

Box 6.4.3

Use information from Box 6.4.1 to argue that orbital speed increases as the distance between the orbiting objects decreases.

CHAPTER OVERVIEW

7: Cosmology

Thumbnail: A star-forming region in the Large Magellanic Cloud (CC BY-4.0; ESA/Hubble via Wikipedia)

7.1: Modeling the Universe

In General Relativity, metrics come about by first proposing a matter/energy distribution and then solving the Einstein Field Equations. If we want to know the metric to apply to a star or black hole, for example, we can think of the universe as containing a single point mass with spherical symmetry. By imposing this constraint, we solve the Einstein Field Equations and discover the Schwarzschild metric.

Suppose, however, that we want to determine a metric for the universe as a whole? What does the distribution of matter look like? What constraints can we impose? To answer that question, take a look at Figures $7.1.1$ and $7.1.2$. The former is an image of a portion of sky that is only 2.4 arc-minutes on each side yet contains approximately 10,000 galaxies. The latter is a temperature map of the early universe, where temperature roughly approximates matter distribution. The colors show variations relative to the mean of approximately 10^{-5}.

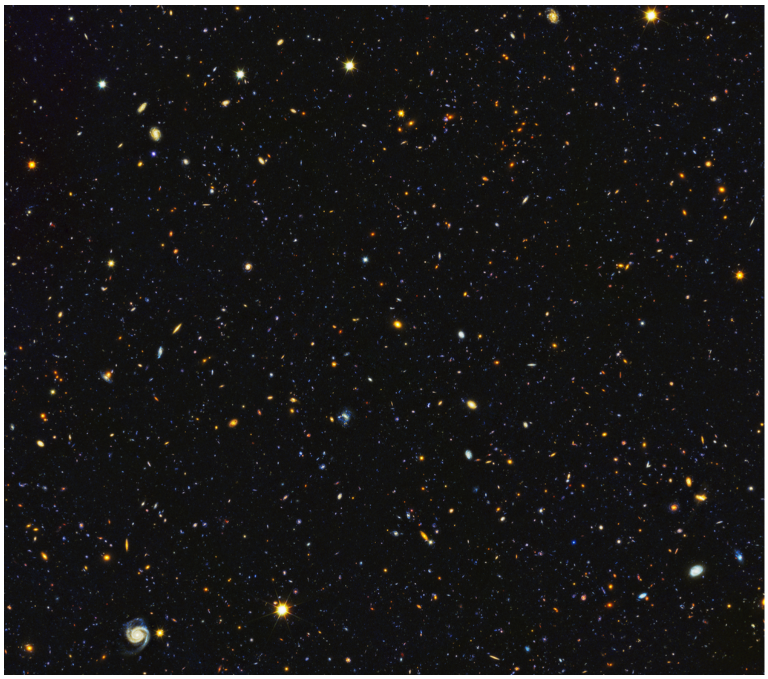


Figure $7.1.1$: This image is called the Hubble Ultra-Deep Field. It contains roughly 10,000 galaxies and represents a portion of sky that is only 2.4 arc-minutes on each side. (Image credit:

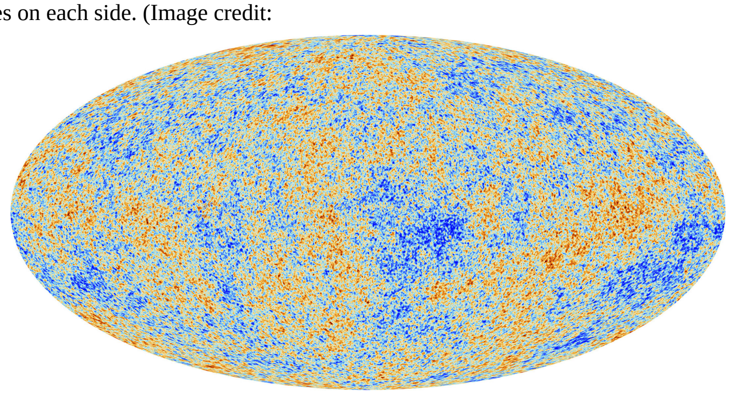


Figure $7.1.2$: An image of the Cosmic Microwave Background (CMB) from the Planck space telescope. Color indicates temperature variations from the mean, which approximately follows matter distribution. The scale of the variations relative to the mean is approximately 10^{-5}. (Image credit: ESA)

Exercise 7.1.1

Based on Figs. 7.1.1 and 7.1.2, how would you describe the matter distribution of the universe?

Answer

Both images show that matter is distributed roughly uniformly, as there is no obvious concentration in one location. Fig. 7.1.2 does show some "hotspots" and "coldspots" but the scale of those differences is incredibly tiny.

Figure $7.1.2$ shows the concentration of matter in the early universe across the entire **celestial sphere** (that is, a sphere centered on earth). Because the variations from one location to another are so small, no matter what direction you look in you will see approximately the same concentration of matter. We say that the universe is **isotropic** from our perspective.

Definition: Isotropic

A system of particles is isotropic from a particular point of view if its properties are the same no matter what direction you look in.

The problem with isotropy is that it is very observer-centric. That is, it describes what things look like from one single point of view. If we want to describe the universe as a whole, then we also have to address the question of what things look like from the point of view of someone in the Andromeda galaxy, or any other galaxy for that matter. If everything does look the same from the point of view of someone in any other galaxy, then we say that the distribution of matter is **homogeneous**.

Definition: Homogeneous

In a **homogeneous** system of particles, the properties of the system appear to be the same from all points of view.

Exercise $7.1.2$

Can you think of an example of something that is isotropic? Can you think of an example of something that is homogenous? Can you think of an example of something that is isotropic but not homogeneous? Homogeneous but not isotropic?

Answer

One example of something that is isotropic is a Gobstopper, as shown in Figure $7.1.3$ It is a hard candy that consists of many concentric spherical layers. It is isotropic as viewed from the center.

Figure $7.1.3$: A Gobstopper is a hard candy that consists of spherical layers. (Image credit: Wikipedia)

An example of something that is homogeneous is the surface of a pool cue ball. If you were to miniaturize yourself and stand on the surface, the surface would look the same to you no matter where you stood.

The Gobstopper is isotropic but not homogeneous because it does not look the same if you hopped to a location anywhere other than the center.

The pool cue ball is homogeneous but not isotropic because if you were standing on the surface and looked up versus looked down, you would see very different things.

Exercise 7.1.3

Can a system of particles be homogeneous if it has edges?

Answer

If a system of particles has edges, then you would see something different if you stood on one of the edges versus standing somewhere in the middle. Therefore anything with edges technically is not homogeneous. (Note that we are defining homogeneity different from how homogeneous mixtures are defined in chemistry, where mixtures can be homogeneous even if they are in a container.)

To be clear, while we can safely say that the universe appears to be approximately isotropic from our point of view, we can't necessarily say that it is homogeneous because we can't observe the universe from other locations. If the universe *isn't* homogeneous, though, then we can't make any claims about the matter distribution of the universe as a whole since the properties of the universe that we observe would just be due to the specialness of our own local patch. Therefore, we will *assume* that our local patch is not special in any way because we have no other option (or at least no option other than giving up). We call this the **cosmological principle**.

Definition: Cosmological Principle

The cosmological principle is the assertion that the properties of the universe are the same for all observers when viewed on a large enough scale. This implies that the universe is isotropic and homogeneous.

7.2: The Friedmann-Lemaitre-Robertson-Walker Metric

Solving the Einstein Field Equations given the conditions that the universe is homogeneous and isotropic yields what we call the Friedmann-Lemaitre-Robertson-Walker (FLRW) Metric, named after four scientists who each made individual contributions. It can be written as

$$d\tau^2 = dt^2 - a^2(t)\left(dr^2 + S^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{7.2.1}$$

where

$$S(r) = \begin{cases} R\sin\left(\frac{r}{R}\right) & \text{spatial geometry: sphere} \\ r & \text{spatial geometry: flat} \\ R\sinh\left(\frac{r}{R}\right) & \text{spatial geometry: saddle-like} \end{cases} \tag{7.2.2}$$

and where R represents the **radius of curvature** of the universe, and $a(t)$ is called the **scale factor**. Note that the function $S(r)$ depends on the overall curvature of the universe, as depicted in Figure $7.2.1$. The overall curvature, as we will see later, depends on the density of all of the stuff in the universe. I would also like to point out that **sphere**, **flat**, and **saddle-like** refer to the rules of geometry and do *not* mean that all of the stuff in the universe is arranged in such a way as to look like it all falls on the surface of a sphere, a plane, or a saddle.

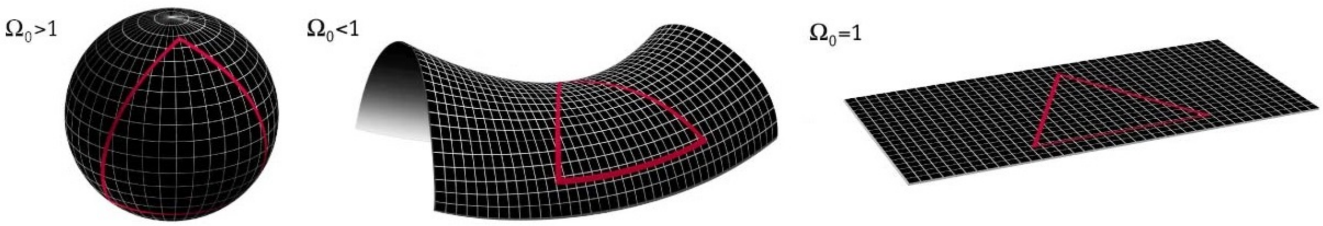


Figure $7.2.1$: Three possible geometries of the universe. In a **spherical** universe (top), the three angles of a triangle add up to more than 180 degrees. In a **saddle-like** universe (middle), the three angles of a triangle add up to less than 180 degrees. In a **flat** universe, the three angles of a triangle add up to 180 degrees. (Image credit: Wikipedia)

In practice, $S(r)$ is not that important to us. It only affects how we calculate the lengths of arcs on the sky. Astronomy, however, is done by looking at light that travels radially toward us (we are always free to choose ourselves as the origin of the coordinate system).

❓ Exercise 7.2.1

Suppose we had a dial that could control the value of R. In principle, we should be able to dial up or dial down the value of R and eventually produce a spatially flat geometry. What R value achieves that?

Answer

Since R represents the spatial curvature of the universe, it would make sense that $R \to \infty$ should yield a flat universe. While you may think that $\lim_{R\to\infty} R\sin\left(\frac{r}{R}\right) = 0$, it actually turns out that $\lim_{R\to\infty} R\sin\left(\frac{r}{R}\right) = r$. The same is true for $R\sinh\left(\frac{r}{R}\right)$. In both cases, the limit $R \to \infty$ reduces to the expression for $S(r)$ for a flat geometry.

❓ Exercise 7.2.2

If you were to draw a triangle on a piece of paper and measure the three angles, you would find that they add up to 180 degrees. What does that tell you about the overall geometry of the universe?

Answer

It implies that the universe has a flat geometry. There is one other possibility, though, which is that the radius of curvature is so large that we are unable to detect any deviation from flat geometry by drawing a triangle on a small patch of spacetime. It is similar to how you can't really tell that the earth is a sphere by looking only at a small portion.

We will talk later about how to determine $a(t)$ and R based on observations. For now, let's focus on using the metric itself. Suppose for example, that a star emits a beam of light and we want to determine how far away that star is based on its coordinates. We are always free to choose the origin of our coordinate system such that we are at the origin, which also means that we and the star have the same ϕ coordinate. Let's call the star's r-coordinate as it emits the light beam as r_{emit}. To find the proper distance between the us and the star, we take measurements of r-coordinates at the same t-coordinate. We are then left with

$$\begin{aligned} d\sigma^2 = -d\tau^2 &= -\cancelto{0}{dt^2} + a^2(t)\left(dr^2 + S^2(r)\left(\cancelto{0}{d\theta^2} + \sin^2\theta\, \cancelto{0}{d\phi^2}\right)\right) \\ d\sigma &= a(t)dr \\ \Delta\sigma &= \int_0^{r_{\text{emit}}} a(t)dr \end{aligned}$$

The result is that the ruler distance to the star at the moment it emitted the beam of light is

$$\Delta\sigma = a(t)r_{\text{emit}}. \tag{7.2.3}$$

Here we see why $a(t)$ is called the **scale factor**: Given two objects that have a spatial coordinate separation, the ruler distance between them is scaled by $a(t)$.

One way to understand the scale factor is to imagine a grid of points, as shown in Fig. 7.2.2. As time goes on the grid stretches. Not only does that increase the distance between the two indicated points, but it increases the distance between *every possible* pair of points. In that sense, $a(t)$ represents the scale of the entire universe.

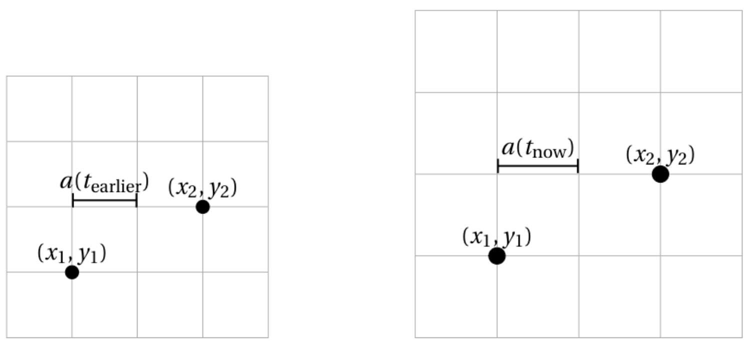


Figure 7.2.2: Two points are at fixed locations on a grid. As time passes, the grid stretches, which increases the distance between all possible pairs of points. (Copyright; author via source)

One interesting consequence of the fact that distances between two points change with time is that it is no longer true that light takes 1 year to reach us from an object that is 1 light-year away. To figure out the relationship between distance and time, let's look at the metric again. Suppose a distant light source emits light at $(r,t) = (r_{\text{emit}}, t_{\text{emit}})$ and we receive that light at $(r,t) = (0, t_0)$. Because $d\tau = 0$ for light, the FLRW metric reduces to

$$\begin{aligned} 0 &= dt^2 - a^2(t)dr^2 \\ dr^2 &= \frac{dt^2}{a^2(t)} \\ dr &= \pm\frac{dt}{a(t)} \\ \int_{r_{\text{emit}}}^{0} dr &= \pm\int_{t_{\text{emit}}}^{t_0} \frac{dt}{a(t)} \\ -r_{\text{emit}} &= \pm\int_{t_{\text{emit}}}^{t_0} \frac{dt}{a(t)} \end{aligned}.$$

Therefore

$$r_{\text{emit}} = \pm \int_{t_0}^{t_{\text{emit}}} \frac{dt}{a(t)}, \tag{7.2.4}$$

where we use whatever sign gives us a positive r-coordinate.

? Exercise 7.2.3

Observations indicate that the scale factor was zero approximately 14 billion years ago and that it has been monotonically increasing since then. It is also conventional to set $a(t_0) = 1$ (where the subscript of "0" represents "today"). Determine an expression for $a(t)$ assuming that it has increased linearly.

Answer

We want a linear function (i.e. $a(t) = mt + b$)with $a(0) = 0$ and $a(t_0) = 1$. Substituting $t = 0$, we get $a(0) = b$, which implies that $b = 0$. Substituting $t = t_0$, we get $a(t_0) = mt_0$, which implies that $m = \frac{1}{t_0}$. Therefore $a(t) = \frac{1}{t_0}t$.

? Exercise 7.2.4

A star emitted a beam of light 3 billion years ago and just now reaches us. What was the r-coordinate of that star at the moment it emitted the light? (Use the $a(t)$ that you determined in the previous exercise.) What was the distance between us and the star when the light was emitted?

Answer

Eq.7.2.4tells us how to determine the r-coordinate at which a beam of light was emitted given $a(t)$ and the amount of time it took to reach us. We will set $t_{\text{emit}} = 11$ billion years and $t_0 = 14$ billion years.

$$\begin{aligned} r_{\text{emit}} &= \pm \int_{t_0}^{t_{\text{emit}}} \frac{dt}{\frac{1}{t_0}t} \\ &= \pm \int_{t_0}^{t_{\text{emit}}} \frac{t_0\, dt}{t} && \text{simplify} \\ &= \pm t_0 [\ln t]_{t_0}^{t_{\text{emit}}} && \text{evaluate integral} \\ &= \pm t_0 \ln\left(\frac{t_{\text{emit}}}{t_0}\right) && \text{substitute bounds} \\ &= \pm (14 \text{ billion years}) \ln\left(\frac{11 \text{ billion years}}{14 \text{ billion years}}\right) && \text{substitute numbers} \\ &= 3.38 \text{ billion light-years} && \text{calculate} \end{aligned}$$

At first this answer may seem confusing since, in an expanding universe, light emitted from 3.38 billion light-years away should take *longer*, not shorter, than 3.38 billion years to reach us. Remember, though, that being at an r-coordinate of 3.38 billion light-years doesn't mean that the star was 3.38 billion light-years away from us. For that we use Equation 7.2.3

$$\Delta\sigma = a(t_{\text{emit}})\, r_{\text{emit}} = \left(\frac{11}{14}\right)(3.38 \text{ billion light-years}) = 2.65 \text{ billion light-years}$$

This answer makes sense. The time it takes for light to reach us from a star that used to be 2.65 billion light-years away from us is slightly longer than 2.65 billion years because the universe expanded in the meantime.

7.3: Redshift

In the previous section we saw that the proper distance between two objects can increase or decrease with time depending on the function $a(t)$, even if the objects are "stationary". What if those two "objects" are the crests of a light wave? As the light wave travels, then distance between those two crests can change. This causes the wavelength, and therefore the frequency, of the wave to change. We refer to this as **cosmological redshift**.

Definition: Cosmological Redshift

Cosmological redshift is the increase in wavelength that light undergoes as a result of traveling through an expanding universe.

Note

Redshift has two other causes in addition to the expansion of the universe. The first is due to **peculiar velocity**, which is the local motion of a star. If we were to look at a galaxy edge-on, for example, stars on one side would be moving toward us while stars on the other side would be moving away from us. Light from the stars moving toward us would be blueshifted while light from stars moving away from us would be redshifted. The other source of redshift is called **gravitational redshift**, which results when light loses energy by climbing out a gravitational potential energy well (as we saw with the infinite redshift surface of a black hole).

We will see later that $a(t)$ increases with time, which increases the wavelength and "reddens" it, as shown in Figure $7.3.1$.

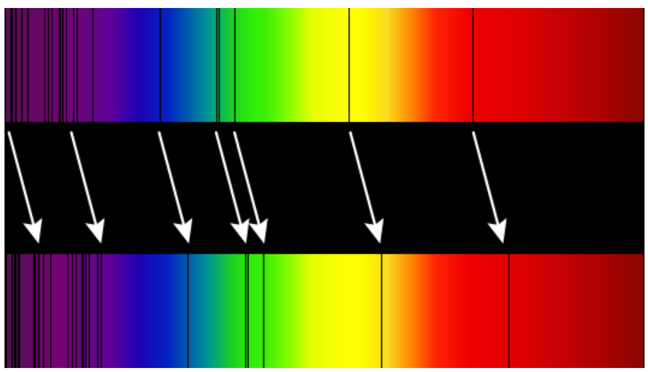


Figure $7.3.1$: Absorption lines of the sun as compared with absorption lines from a distant supercluster of galaxies. (Image credit: Wikipedia)

Redshift z is the fraction by which the wavelength λ increases. We write this as

$$\lambda_0 = (1+z)\lambda_{\text{emit}} \tag{7.3.1}$$

where λ_{emit} is the emitted wavelength, λ_0 is the received wavelength (once again using the subscript "0" to represent "today"), and z is the redshift. The factor $1+z$ is often called the **stretch factor**. Due to the fact that proper distances scale with the scale factor, it can be shown that

$$1+z = \frac{\lambda_0}{\lambda_{\text{emit}}} = \frac{a(t_0)}{a(t_{\text{emit}})}. \tag{7.3.2}$$

Note that, according to Equation 7.3.1, in order to determine redshift based on the measured wavelength of the received light, you must know the wavelength of the light when it was emitted. For monochromatic light, that is very difficult to determine. Fortunately, stars emit light with a predictable pattern of discrete wavelengths. Light from stars that are far away has the same pattern but shifted toward the red end of the spectrum, as shown in Figure $7.3.1$. So redshift is actually very easy for astronomers to determine.

Note

If you spend any amount of time with astronomers, you will find that they often use redshift as a proxy for time or distance. That is, they will use redshift to refer to how old or how far away something is (or was). Redshift is related to how old and how far away something is, but distances and times require knowing $a(t)$, while redshift does not.

Box 7.3.1

Some of the most distant observed objects emitted light so long ago that the light is redshifted by $z=6$.

a. If the light was emitted in the visible range of the spectrum, in what range of the spectrum do we receive it?
b. What was the scale factor at the time of emission?
c. At what time was the light emitted? Use $a(t)=\frac{1}{t_0}t$.
d. How far away was the light source when it emitted the light?
e. How far away is the light source now?

7.4: Evidence of Expansion

In 1929, Edwin Hubble combined Vesto Slipher's redshift and velocity measurements of galaxies with Henrietta Swan Leavitt's distance-finding procedure and discovered that there is a roughly linear relationship between a galaxy's distance from us and its recessional velocity (i.e. its velocity away from us), as shown in Figure 7.4.1.

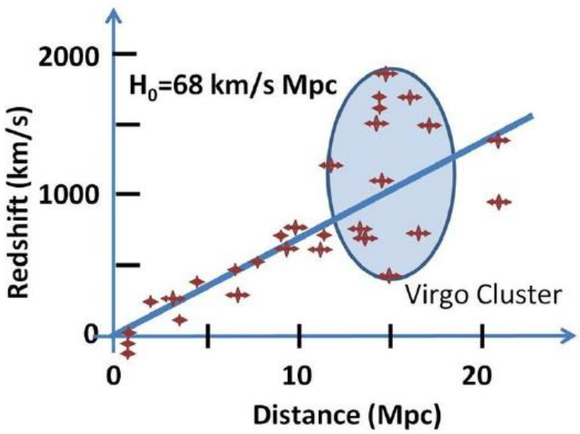


Figure 7.4.1: Hubble diagram depicting galaxy recessional velocity as a function of distance. (Image credit: Wikipedia)

This can be written as

$$v = H_0 d \tag{7.4.1}$$

where v is the recessional velocity, d is the distance, and H_0 is a constant of proportionality that we call the **Hubble constant**.

Since recessional velocity is measured in km/s and distance is measured in Mpc (megaparsecs), the slope has units of $\frac{\text{km/s}}{\text{Mpc}}$. There is currently disagreement about the value of H_0. Of the various methods of determining the Hubble constant, though, all of them put its value at approximately

$$H_0 = 70\,\frac{\text{km/s}}{\text{Mpc}}. \tag{7.4.2}$$

One potential issue with Equation 7.4.1 is that it makes it seem like we are the center of the universe, which seems to contradict the cosmological principle. One way to resolve this apparent contradiction is with the Raisin Bread Model of the expanding universe. As a loaf of raisin bread bakes, it expands and the raisins all move farther away from every other raisin. The raisins themselves do not expand, and they don't even change their relative locations within the "space" that they exist in. Similarly, if the universe as a whole expands, then clusters of galaxies will all move farther away from every other cluster of galaxies, thus making every point of view look like it is the center of the expansion.

? Exercise 7.4.1

Taken to its extreme, Equation 7.4.1 implies that there is no upper limit to recessional velocity. That is, the speed of light is not an upper limit. How is that possible?

Answer

In an expanding universe, objects can move farther apart from one another while not changing their coordinate positions. So in a sense, they are not technically *moving*. Rather, the space between them expands, and the expansion of space is not limited by the speed of light.

7.5: The Friedmann Equation

Up until now, we haven't talked about how to determine the scale factor function $a(t)$. In 1922, Alexander Friedmann combined the FLRW metric with the Einstein Field equations and discovered a way to determine $a(t)$. We now call it the **Friedmann Equation**

$$H^2(t) = \frac{8\pi\rho_{\text{tot}}(t)}{3} - \frac{K}{a^2(t)} \tag{7.5.1}$$

where the **Hubble Parameter** $H(t)$ is defined as

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \tag{7.5.2}$$

and where ρ_{tot} is the mass-energy density of the universe. The constant K determines whether the geometry of the universe is spherical ($K > 0$), flat ($K = 0$), or saddle-like ($K < 0$).

> **Note**
>
> The "dot" over the $a(t)$ in Equation 7.5.2 is a shorthand notation for a derivative with respect to time.

By setting $K = 0$, we can determine an expression for the **critical density** $\rho_{\text{crit}}(t)$. A density greater than the critical density will produce a universe with a spherical geometry, and anything less will produce a universe with a saddle-like geometry.

> **Definition: Critical Density**
>
> The critical density is the density of the stuff in the universe that would produce a universe with flat geometry.

Setting $K = 0$ in Equation 7.5.1 in solving for $\rho(t)$ yields

$$\rho_{\text{crit}}(t) = \frac{3H^2(t)}{8\pi}. \tag{7.5.3}$$

As you can see, the critical density changes with time. The critical density *today* is written as

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi}. \tag{7.5.4}$$

> **Exercise 7.5.1**
>
> Use the conversion 1 pc = 3.1×10^{16} m and 1 kg = 7.42×10^{-28} m to determine the value of the critical density *today* in $\frac{\text{kg}}{\text{m}^3}$. How many hydrogen atoms per cubic meter is that equivalent to?
>
> **Answer**
>
> Let's start by converting the Hubble constant.
>
> $$H_0 = 70\frac{\text{km/s}}{\text{Mpc}} \times \frac{1000\text{ m}}{1\text{ km}} \times \frac{1\text{ Mpc}}{10^6\text{ pc}} \times \frac{1\text{ pc}}{3.1\times10^{16}\text{ m}} = 2.26\times10^{-18}\frac{1}{\text{s}}$$
>
> Then we square it.
>
> $$H_0^2 = 5.1\times10^{-36}\frac{1}{\text{s}^2}$$
>
> Now we need to figure out how to get this in units of $\frac{\text{kg}}{\text{m}^3}$. We can start by using the speed of light to convert seconds to meters.
>
> $$H_0^2 = 5.1\times10^{-36}\frac{1}{\text{s}^2}\times\left(\frac{1\text{ s}}{3\times10^8\text{ m}}\right)^2 = 5.67\times10^{-53}\frac{1}{\text{m}^2}$$

Now we can write $\frac{1}{\mathrm{m}^2}$ as $\frac{\mathrm{m}}{\mathrm{m}^3}$ and convert the top to kilograms.

$$H_0^2 = 5.67 \times 10^{-53} \frac{\mathrm{m}}{\mathrm{m}^3} \times \frac{1\ \mathrm{kg}}{7.42 \times 10^{-28}\ \mathrm{m}} = 7.6 \times 10^{-26} \frac{\mathrm{kg}}{\mathrm{m}^3}$$

Therefore the critical density is

$$\rho_{\mathrm{crit},0} = \frac{3}{8\pi}\left(7.6 \times 10^{-26} \frac{\mathrm{kg}}{\mathrm{m}^3}\right) = 9.1 \times 10^{-27} \frac{\mathrm{kg}}{\mathrm{m}^3}.$$

The mass of a hydrogen atom is about 1.67×10^{-27} kg, so the critical density is equivalent to a little over five hydrogen atoms per cubic meter of space. That is a very tiny number, especially considering the fact that the density of everything around us is *much* bigger than that. On the other hand, space is *really* big. As we will find out later, it appears as the universe as a whole has a density extremely close to the critical density. That goes to show you just how much empty space there is in the universe.

7.6: Contents of the Universe

If the universe is expanding, then the density will change with time. Furthermore, different components of the universe change their densities in different ways as the universe expands. These components can be broken into three general categories.

- **Matter**: Matter consists of particles whose mass is much greater than their kinetic energy. Stars, gas, neutrinos, and dark matter are all considered matter. As the universe expands, the density of matter goes down as the cube of the scale factor (since density of matter is inversely proportional to volume).

$$\rho_m(t) = \frac{\rho_{m,0}}{a^3(t)} \tag{7.6.1}$$

- **Radiation**: Radiation consists of particles whose mass is much less than their total energy. Today this category consists mostly of photons, but long ago neutrinos were a significant component. The density of radiation is inversely proportional to the fourth power of the scale factor. This is because, in addition to the effect of a scaling volume, the energy of radiation is also redshifted as the universe expands.

$$\rho_r(t) = \frac{\rho_{r,0}}{a^4(t)} \tag{7.6.2}$$

- **Dark Energy**: Dark energy is something that hasn't been observed, and we don't know what it is. We introduce it as a component of the universe simply because it is required in order for the Friedmann equation to match current observations. Unlike matter and radiation, the density of dark energy is constant. As the universe expands, then, the total amount of dark energy increases.

$$\rho_{\text{dark energy}} \equiv \rho_\Lambda \sim 7\times 10^{-30}\,\frac{\text{g}}{\text{cm}^3} \tag{7.6.3}$$

The sum total of all of these represents the total density. Fig. 7.6.1 shows how each component, along with the total, evolves with the scale factor. Each component has a time in which it dominates over the others. At first the universe was radiation-dominated, then matter-dominated, and we are now in the transition point to the dark-energy-dominated regime.

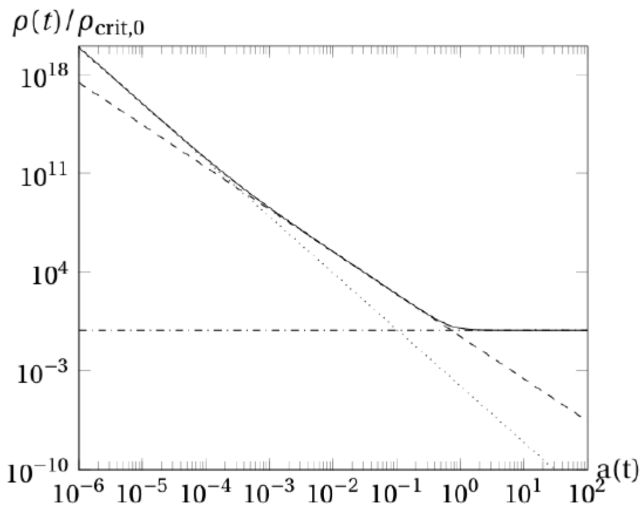


Figure 7.6.1: Evolution of density of different components of the universe as a function of scale factor. The dashed line represents matter, the dotted line represents radiation, the dash-dot line represents dark energy, and the solid line is the total. (Copyright; author via source)

In cosmology it is also convenient to define

$$\Omega_0 = \frac{\rho(t_0)}{\rho_{\text{crit},0}}. \tag{7.6.4}$$

If the current density is equal to the critical density, then this fraction is equal to 1. Meanwhile, $\Omega_0 > 1$ for an overdense universe and $\Omega_0 < 1$ for an underdense universe. Combining Eqs. 7.6.4, 7.6.1, 7.6.2, and 7.6.3 allows us to rewrite the Friedmann Equation (see Box 7.6.1) as

$$\dot{a}^2(t) = H_0^2\left(\frac{\Omega_{m,0}}{a(t)} + \frac{\Omega_{r,0}}{a^2(t)} + \Omega_\Lambda a^2(t)\right) - K \tag{7.6.5}$$

This is a differential equation for $a(t)$, meaning that if we have values for $\Omega_{m,0}$, $\Omega_{r,0}$, Ω_Λ, and K, then we can determine $a(t)$. Current observational experiments have constrained the values to

$$\Omega_{\mathrm{m},0} = 0.27 \pm 0.03 \tag{7.6.6}$$

$$\Omega_{\mathrm{r},0} = 8.4 \times 10^{-5} \tag{7.6.7}$$

$$\Omega_\Lambda = 0.73 \pm 0.03. \tag{7.6.8}$$

Notice that the sum is indistinguishable from 1, which implies that the universe either has a flat geometry ($K = 0$) or is very close to having a flat geometry. These numbers also make Equation 7.6.5 impossible to solve by hand, which means that we have to solve it either numerically or by some other qualitative means. Fortunately, the equation is written in exactly the form needed to define an effective potential since it contains a kinetic-energy-like term ($\dot{a}^2(t)$) and a constant ($-K$). What remains is the effective potential.

$$V_{\text{eff}}(a) = -H_0^2\left(\frac{\Omega_{m,0}}{a(t)} + \frac{\Omega_{r,0}}{a^2(t)} + \Omega_\Lambda a^2(t)\right) \tag{7.6.9}$$

Fig. 7.6.2 depicts a graph of the effective potential given current best estimates of $\Omega_{m,0} = 0.27$, $\Omega_{r,0} \approx 0$, and $\Omega_\Lambda = 0.73$.

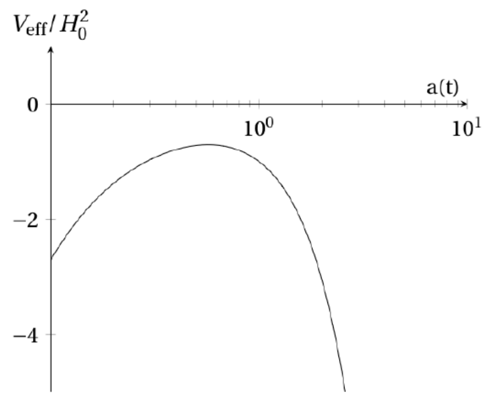


Figure 7.6.2: A graph of effective potential for a universe in which $\Omega_{m,0} = 0.27$, $\Omega_{r,0} \approx 0$, and $\Omega_\Lambda = 0.73$. (Copyright; author via source)

Exercise 7.6.1

The "total energy" in the effective potential plot shown in Fig. 7.6.2 is a horizontal line at $-K$. How are the different possible universes affected by the overall geometry of the universe? Assume that $a(0) = 0$.

Answer

As usual with effective potential plots, I think it is helpful to imagine placing a marble on the graph and giving it a kick. If $K = 0$ or $K < 0$, the horizontal line depicting the "total energy" does not intersect the effective potential plot. This means that there are no turning points, so the marble will go over the hump and roll down the other side. In terms of the universe, this corresponds to the scale factor $a(t)$ increasing without limit: an eternally expanding universe. If K is a sufficiently large positive number, however, then it will intersect the effective potential curve. Using our marble analogy, the marble would move to the right (corresponding to an increasing scale factor) but then would come to a stop at the turning point and roll back to the left. This represents a universe that initially expands but then eventually collapses on itself. This is referred to as the Big Crunch. We don't have to worry about that, however, since experimental observations indicate that $K \approx 0$.

The lesson from the previous exercise is that it appears as though the universe will expand forever. As objects move farther and farther away from us, they eventually reach a point at which they are moving away from us faster than the speed of light. At that point, the light they emit will not be able to reach us, effectively removing them from the observable portion of the universe. As time goes on, the amount of stuff within the observable portion of the universe will get smaller and smaller. While the idea of a completely empty night sky may be sad, don't worry; our sun will be long dead by then.

Box 7.6.1

The total density of the universe can be written as the sum of the densities of matter, radiation, and dark energy. Use this along with Eqs. 7.6.4, 7.6.1, 7.6.2, and 7.6.3 to derive Equation 7.6.5.

Box 7.6.2

While Equation 7.6.5 can't be solved by hand using the current observed values of the densities of each component, it can be solved in the case that the universe consists of only one of the three components. Here is the process, for example, if the universe is flat ($K = 0$), consists only of radiation ($\Omega_{r,0} = 1$), and starts with initial condition $a(0) = 0$.

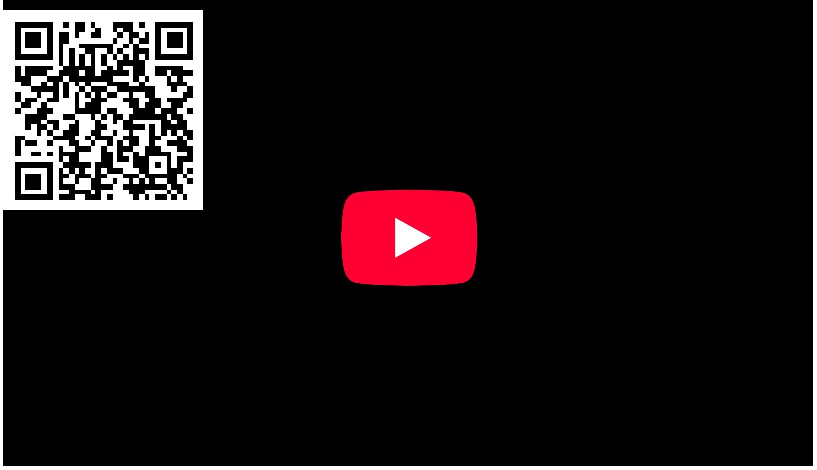
$$\begin{aligned}
\dot{a}^2(t) &= H_0^2\left(\frac{\cancelto{0}{\Omega_{m,0}}}{a(t)} + \frac{\cancelto{1}{\Omega_{r,0}}}{a^2(t)} + \cancelto{0}{\Omega_\Lambda} a^2(t)\right) - \cancelto{0}{K} && \text{substitute values into Friedmann Equation} \\
\frac{da}{dt} &= \frac{H_0}{a} && \text{take square root of both sides} \\
a\,da &= H_0\,dt && \text{put all a's on one side and t's on the other} \\
\int_0^{a(t)} a\,da &= \int_0^t H_0\,dt && \text{integrate both sides} \\
\frac{1}{2}a^2(t) - 0 &= H_0 t - 0 && \text{evaluate integral} \\
a(t) &= \sqrt{2H_0 t} && \text{solve for a(t)}
\end{aligned}$$

Repeat this process to solve for $a(t)$ for a flat universe that consists only of matter and for which $a(0) = 0$.

Box 7.6.3

Repeat the process from the previous box to solve for $a(t)$ for a flat universe that consists only of dark energy and for which $a(0) = a_0$. Why can we not use $a(0) = 0$ in this case? (Explain both mathematically and physically.)

7.7: Video Resources



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