MC : PHYSICS 213 - MODERN PHYSICS

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CHAPTER OVERVIEW

1: Relativity

The theory of relativity led to a profound change in the way we perceive space and time. The “common sense” rules that we use to relate space and time measurements in the Newtonian worldview differ seriously from the correct rules at speeds near the speed of light. Unlike Newtonian mechanics, which describes the motion of particles, or Maxwell's equations, which specify how the electromagnetic field behaves, special relativity is not restricted to a particular type of phenomenon. Instead, its rules on space and time affect all fundamental physical theories.

Thumbnail: The light cone consists of all the world lines followed by light from the event A at the vertex of the cone. (CC BY 4.0; OpenStax)

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1.1: Prelude to Relativity

The special theory of relativity was proposed in 1905 by **Albert Einstein** (1879–1955). It describes how time, space, and physical phenomena appear in different frames of reference that are moving at constant velocity with respect to each other. This differs from Einstein's later work on general relativity, which deals with any frame of reference, including accelerated frames.

Figure 1.1.1: Special relativity explains how time passes slightly differently on Earth and within the rapidly moving global positioning satellite (GPS). GPS units in vehicles could not find their correct location on Earth without taking this correction into account. (credit: USAF)

The theory of relativity led to a profound change in the way we perceive space and time. The "common sense" rules that we use to relate space and time measurements in the Newtonian worldview differ seriously from the correct rules at speeds near the speed of light. For example, the special theory of relativity tells us that measurements of length and time intervals are not the same in reference frames moving relative to one another. A particle might be observed to have a lifetime of $1.0 \times 10^{-8} s$ in one reference frame, but a lifetime of $2.0 \times 10^{-8} s$ in another; and an object might be measured to be 2.0 m long in one frame and 3.0 m long in another frame. These effects are usually significant only at speeds comparable to the speed of light, but even at the much lower speeds of the global positioning satellite, which requires extremely accurate time measurements to function, the different lengths of the same distance in different frames of reference are significant enough that they need to be taken into account.

Unlike **Newtonian mechanics**, which describes the motion of particles, or Maxwell's equations, which specify how the electromagnetic field behaves, special relativity is not restricted to a particular type of phenomenon. Instead, its rules on space and time affect all fundamental physical theories.

The modifications of Newtonian mechanics in special relativity do not invalidate classical Newtonian mechanics or require its replacement. Instead, the equations of relativistic mechanics differ meaningfully from those of classical Newtonian mechanics only for objects moving at relativistic speeds (i.e., speeds less than, but comparable to, the speed of light). In the macroscopic world that you encounter in your daily life, the relativistic equations reduce to classical equations, and the predictions of classical Newtonian mechanics agree closely enough with experimental results to disregard relativistic corrections.

1.2: Invariance of Physical Laws

Learning Objectives

By the end of this section, you will be able to:

- Describe the theoretical and experimental issues that Einstein's theory of special relativity addressed.
- State the two postulates of the special theory of relativity.

Suppose you calculate the hypotenuse of a right triangle given the base angles and adjacent sides. Whether you calculate the hypotenuse from one of the sides and the cosine of the base angle, or from the Pythagorean theorem, the results should agree. Predictions based on different principles of physics must also agree, whether we consider them principles of mechanics or principles of electromagnetism.

Albert Einstein pondered a disagreement between predictions based on electromagnetism and on assumptions made in classical mechanics. Specifically, suppose an observer measures the velocity of a light pulse in the observer's own **rest frame**; that is, in the frame of reference in which the observer is at rest. According to the assumptions long considered obvious in classical mechanics, if an observer measures a velocity $\vec{v}$ in one frame of reference, and that frame of reference is moving with velocity $\vec{u}$ past a second reference frame, an observer in the second frame measures the original velocity as

$$\vec{v}' = \vec{v} + \vec{u}.$$

This sum of velocities is often referred to as *Galilean relativity*. If this principle is correct, the pulse of light that the observer measures as traveling with speed **c** travels at speed **c** + **u** measured in the frame of the second observer. If we reasonably assume that the laws of electrodynamics are the same in both frames of reference, then the predicted speed of light (in vacuum) in both frames should be

$$c = 1/\sqrt{\epsilon_0 \mu_0}.$$

Each observer should measure the same speed of the light pulse with respect to that observer's own rest frame. To reconcile difficulties of this kind, Einstein constructed his special theory of relativity, which introduced radical new ideas about time and space that have since been confirmed experimentally.

Inertial Frames

All velocities are measured relative to some frame of reference. For example, a car's motion is measured relative to its starting position on the road it travels on; a projectile's motion is measured relative to the surface from which it is launched; and a planet's orbital motion is measured relative to the star it orbits. The frames of reference in which mechanics takes the simplest form are those that are not accelerating. Newton's first law, the law of inertia, holds exactly in such a frame.

Definition: Inertial Reference Frame

An **inertial frame of reference** is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted upon by an outside force.

For example, to a passenger inside a plane flying at constant speed and constant altitude, physics seems to work exactly the same as when the passenger is standing on the surface of Earth. When the plane is taking off, however, matters are somewhat more complicated. In this case, the passenger at rest inside the plane concludes that a net force **F** on an object is not equal to the product of mass and acceleration, **ma**. Instead, **F** is equal to **ma** plus a fictitious force. This situation is not as simple as in an inertial frame. Special relativity handles accelerating frames as a constant and velocities as relative to the observer. General relativity treats both velocity and acceleration as relative to the observer, thus making the use of curved space-time.

Einstein's First Postulate

Not only are the principles of classical mechanics simplest in inertial frames, but they are the same in all inertial frames. Einstein based the first postulate of his theory on the idea that this is true for all the laws of physics, not merely those in mechanics.

> **FIRST POSTULATE OF SPECIAL RELATIVITY**
>
> The laws of physics are the same in all *inertial frames of reference*.

This postulate denies the existence of a special or preferred inertial frame. The laws of nature do not give us a way to endow any one inertial frame with special properties. For example, we cannot identify any inertial frame as being in a state of "absolute rest." We can only determine the relative motion of one frame with respect to another.

There is, however, more to this postulate than meets the eye. The laws of physics include only those that satisfy this postulate. We will see that the definitions of energy and momentum must be altered to fit this postulate. Another outcome of this postulate is the famous equation $E = mc^2$, which relates energy to mass.

Einstein's Second Postulate

The second postulate upon which Einstein based his theory of special relativity deals with the **speed of light**. Late in the nineteenth century, the major tenets of classical physics were well established. Two of the most important were the laws of electromagnetism and Newton's laws. Investigations such as Young's double-slit experiment in the early 1800s had convincingly demonstrated that light is a wave. Maxwell's equations of electromagnetism implied that electromagnetic waves travel at $c = 3.00 \times 10^8\, m/s$ in a vacuum, but they do not specify the frame of reference in which light has this speed. Many types of waves were known, and all travelled in some medium. Scientists therefore assumed that some medium carried the light, even in a vacuum, and that light travels at a speed **c** relative to that medium (often called "the aether").

Starting in the mid-1880s, the American physicist A.A. Michelson, later aided by E.W. Morley, made a series of direct measurements of the speed of light. They intended to deduce from their data the speed v at which Earth was moving through the mysterious medium for light waves. The speed of light measured on Earth should have been $c+v$ when Earth's motion was opposite to the medium's flow at speed u past the Earth, and $c-v$ when Earth was moving in the same direction as the medium (Figure $1.2.1$). The results of their measurements were startling.

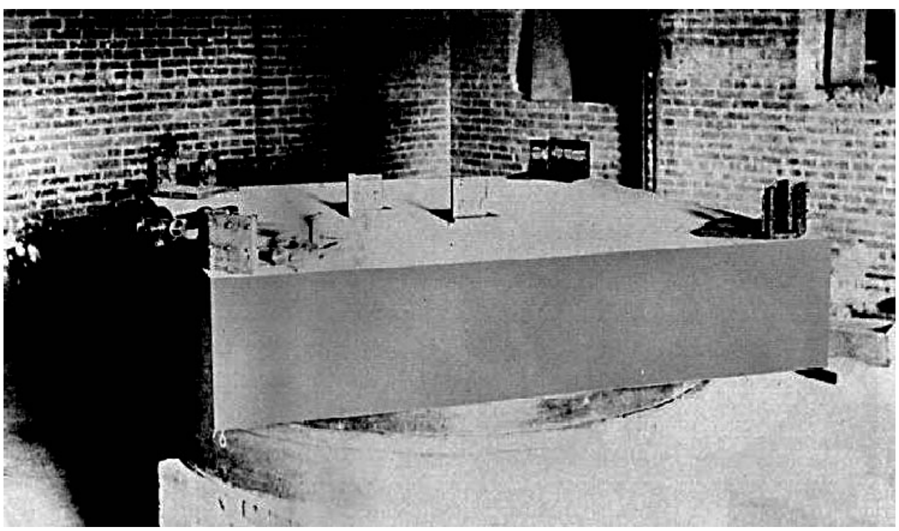

Figure $1.2.1$: Michelson and Morley's interferometric setup, mounted on a stone slab that floats in an annular trough of mercury.

The eventual conclusion derived from this result is that light, unlike mechanical waves such as sound, does not need a medium to carry it. Furthermore, the Michelson-Morley results implied that the speed of light **c** is independent of the motion of the source relative to the observer. That is, everyone observes light to move at speed **c** regardless of how they move relative to the light source or to one another. For several years, many scientists tried unsuccessfully to explain these results within the framework of Newton's laws.

> **MICHELSON-MORLEY EXPERIMENT**
>
> The Michelson-Morley experiment demonstrated that the speed of light in a vacuum is independent of the motion of Earth about the Sun.

In addition, there was a contradiction between the principles of electromagnetism and the assumption made in Newton's laws about relative velocity. Classically, the velocity of an object in one frame of reference and the velocity of that object in a second frame of reference relative to the first should combine like simple vectors to give the velocity seen in the second frame. If that were correct, then two observers moving at different speeds would see light traveling at different speeds. Imagine what a light wave would look like to a person traveling along with it (in vacuum) at a speed c. If such a motion were possible, then the wave would be stationary relative to the observer. It would have electric and magnetic fields whose strengths varied with position but were constant in time. This is not allowed by Maxwell's equations. So either Maxwell's equations are different in different inertial frames, or an object with mass cannot travel at speed c. Einstein concluded that the latter is true: An object with mass cannot travel at speed c. Maxwell's equations are correct, but Newton's addition of velocities is not correct for light.

Not until 1905, when Einstein published his first paper on special relativity, was the currently accepted conclusion reached. Based mostly on his analysis that the laws of electricity and magnetism would not allow another speed for light, and only slightly aware of the Michelson-Morley experiment, Einstein detailed his second postulate of special relativity.

SECOND POSTULATE OF SPECIAL RELATIVITY

Light travels in a vacuum with the same speed c in any direction in all inertial frames.

In other words, the speed of light has the same definite speed for any observer, regardless of the relative motion of the source. This deceptively simple and counterintuitive postulate, along with the first postulate, leave all else open for change. Among the changes are the loss of agreement on the time between events, the variation of distance with speed, and the realization that matter and energy can be converted into one another. We describe these concepts in the following sections.

Exercise 1.2.1

Explain how special relativity differs from general relativity.

Answer

Special relativity applies only to objects moving at constant velocity, whereas general relativity applies to objects that undergo acceleration

1.3: Relativity of Simultaneity

Learning Objectives

By the end of this section, you will be able to:

- Show from Einstein's postulates that two events measured as simultaneous in one inertial frame are not necessarily simultaneous in all inertial frames.
- Describe how simultaneity is a relative concept for observers in different inertial frames in relative motion.

Do time intervals depend on who observes them? Intuitively, it seems that the time for a process, such as the elapsed time for a foot race (Figure $1.3.1$), should be the same for all observers. In everyday experiences, disagreements over elapsed time have to do with the accuracy of measuring time. No one would be likely to argue that the actual time interval was different for the moving runner and for the stationary clock displayed. Carefully considering just how time is measured, however, shows that elapsed time does depends on the relative motion of an observer with respect to the process being measured.

Figure $1.3.1$: Elapsed time for a foot race is the same for all observers, but at relativistic speeds, elapsed time depends on the motion of the observer relative to the location where the process being timed occurs. (credit: "Jason Edward Scott Bain"/Flickr)

Consider how we measure elapsed time. If we use a stopwatch, for example, how do we know when to start and stop the watch? One method is to use the arrival of light from the event. For example, if you're in a moving car and observe the light arriving from a traffic signal change from green to red, you know it's time to step on the brake pedal. The timing is more accurate if some sort of electronic detection is used, avoiding human reaction times and other complications.

Now suppose two observers use this method to measure the time interval between two flashes of light from flash lamps that are a distance apart (Figure $1.3.2$). An observer **A** is seated midway on a rail car with two flash lamps at opposite sides equidistant from her. A pulse of light is emitted from each flash lamp and moves toward observer **A**, shown in frame (a) of the figure. The rail car is moving rapidly in the direction indicated by the velocity vector in the diagram. An observer **B** standing on the platform is facing the rail car as it passes and observes both flashes of light reach him simultaneously, as shown in frame (c). He measures the distances from where he saw the pulses originate, finds them equal, and concludes that the pulses were emitted simultaneously.

However, because of Observer **A**'s motion, the pulse from the right of the railcar, from the direction the car is moving, reaches her before the pulse from the left, as shown in frame (b). She also measures the distances from within her frame of reference, finds them equal, and concludes that the pulses were not emitted simultaneously.

The two observers reach conflicting conclusions about whether the two events at well-separated locations were simultaneous. Both frames of reference are valid, and both conclusions are valid. Whether two events at separate locations are simultaneous depends on the motion of the observer relative to the locations of the events.

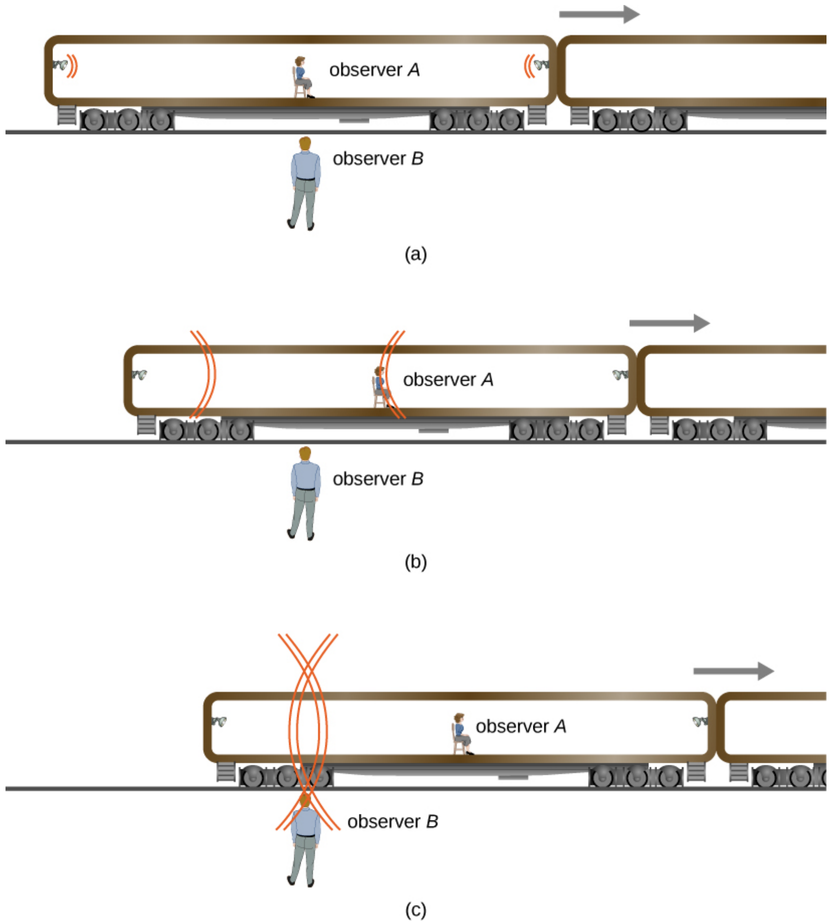

Figure 1.3.2: (a) Two pulses of light are emitted simultaneously relative to observer B. (c) The pulses reach observer B’s position simultaneously. (b) Because of A’s motion, she sees the pulse from the right first and concludes the bulbs did not flash simultaneously. Both conclusions are correct.

Here, the relative velocity between observers affects whether two events a distance apart are observed to be simultaneous. **Simultaneity is not absolute**. We might have guessed (incorrectly) that if light is emitted simultaneously, then two observers halfway between the sources would see the flashes simultaneously. But careful analysis shows this cannot be the case if the speed of light is the same in all inertial frames.

This type of **thought experiment** (in German, “Gedankenexperiment”) shows that seemingly obvious conclusions must be changed to agree with the postulates of relativity. The validity of thought experiments can only be determined by actual observation, and careful experiments have repeatedly confirmed Einstein’s theory of relativity.

1.4: Time Dilation

Learning Objectives

By the end of this section, you will be able to:

- Explain how time intervals can be measured differently in different reference frames.
- Describe how to distinguish a proper time interval from a dilated time interval.
- Describe the significance of the muon experiment.
- Explain why the twin paradox is not a contradiction.
- Calculate time dilation given the speed of an object in a given frame.

The analysis of simultaneity shows that Einstein's postulates imply an important effect: Time intervals have different values when measured in different inertial frames. Suppose, for example, an astronaut measures the time it takes for a pulse of light to travel a distance perpendicular to the direction of his ship's motion (relative to an earthbound observer), bounce off a mirror, and return (Figure $1.4.1a$). How does the elapsed time that the astronaut measures in the spacecraft compare with the elapsed time that an earthbound observer measures by observing what is happening in the spacecraft?

Examining this question leads to a profound result. The elapsed time for a process depends on which observer is measuring it. In this case, the time measured by the astronaut (within the spaceship where the astronaut is at rest) is smaller than the time measured by the earthbound observer (to whom the astronaut is moving). The time elapsed for the same process is different for the observers, because the distance the light pulse travels in the astronaut's frame is smaller than in the earthbound frame, as seen in Figure $1.4.1b$. Light travels at the same speed in each frame, so it takes more time to travel the greater distance in the earthbound frame.

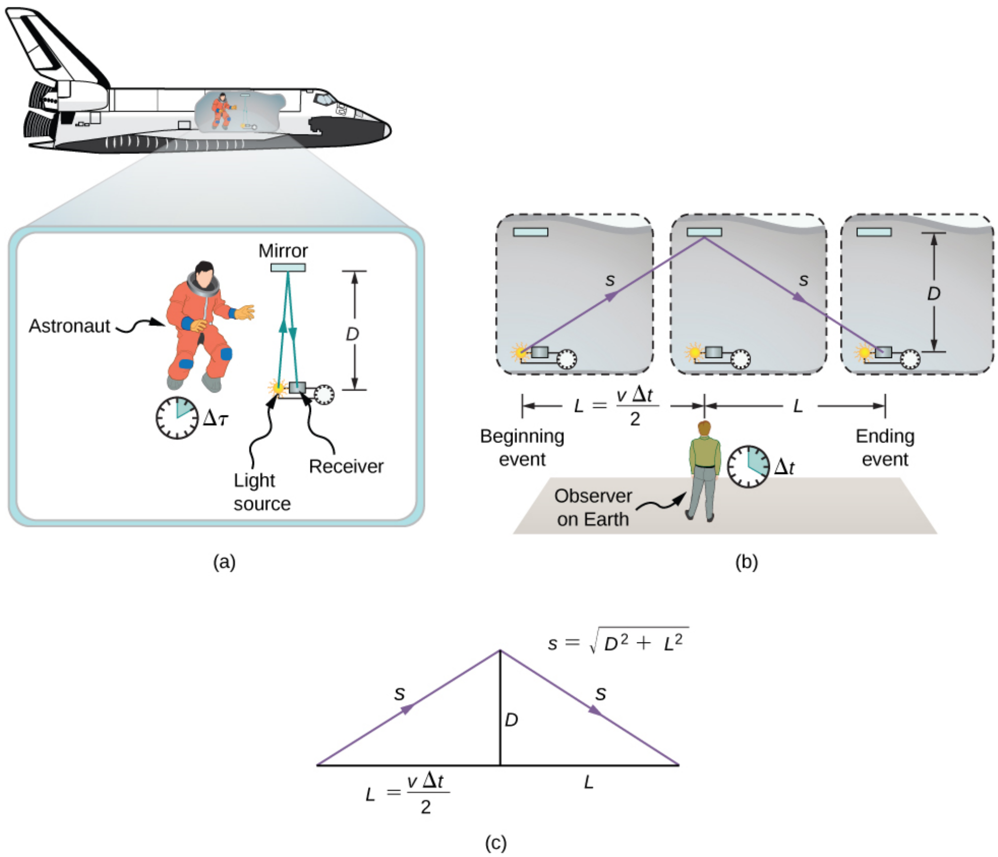

Figure 1.4.1: (a) An astronaut measures the time $\Delta\tau$ for light to travel distance 2D in the astronaut's frame. (b) A NASA scientist on Earth sees the light follow the longer path 2s and take a longer time Δt. (c) These triangles are used to find the relationship between the two distances D and s.

> **Definition: Time Dilation**
>
> **Time dilation** is the lengthening of the time interval between two events for an observer in an inertial frame that is moving with respect to the rest frame of the events (in which the events occur at the same location).

To quantitatively compare the time measurements in the two inertial frames, we can relate the distances in Figure $1.4.1b$ to each other, then express each distance in terms of the time of travel (respectively either Δt or $\Delta\tau$) of the pulse in the corresponding reference frame. The resulting equation can then be solved for Δt in terms of $\Delta\tau$.

The lengths D and L in Figure $1.4.1c$ are the sides of a right triangle with hypotenuse s. From the Pythagorean theorem,

$$s^2 = D^2 + L^2.$$

The lengths $2s$ and $2L$ are, respectively, the distances that the pulse of light and the spacecraft travel in time Δt in the earthbound observer's frame. The length D is the distance that the light pulse travels in time $\Delta\tau$ in the astronaut's frame. This gives us three equations:

$$2s = c\Delta t$$

$$2L = v\Delta t;$$

$$2D = c\Delta\tau.$$

Note that we used Einstein's second postulate by taking the speed of light to be **c** in both inertial frames. We substitute these results into the previous expression from the Pythagorean theorem:

$$\begin{aligned} s^2 &= D^2 + L^2 \\ \left(c\frac{\Delta t}{2}\right)^2 &= \left(c\frac{\Delta\tau}{2}\right)^2 + \left(v\frac{\Delta t}{2}\right)^2 \end{aligned}$$

Then we rearrange to obtain

$$(c\Delta t)^2 - (v\Delta t)^2 = (c\Delta\tau)^2.$$

Finally, solving for Δt in terms of $\Delta\tau$ gives us

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-(v/c)^2}}.$$

This is equivalent to

$$\Delta t = \gamma\Delta\tau, \tag{1.4.1}$$

where γ is the relativistic factor (often called the **Lorentz factor**) given by

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

and v and c are the speeds of the moving observer and light, respectively.

Note the asymmetry between the two measurements. Only one of them is a measurement of the time interval between two events—the emission and arrival of the light pulse—at the same position. It is a measurement of the time interval in the rest frame of a single clock. The measurement in the earthbound frame involves comparing the time interval between two events that occur at different locations. The time interval between events that occur at a single location has a separate name to distinguish it from the time measured by the earthbound observer, and we use the separate symbol $\Delta\tau$ to refer to it throughout this chapter.

> **Definition: Proper Time**
>
> The **proper time** interval $\Delta\tau$ between two events is the time interval measured by an observer for whom both events occur at the same location.

The equation relating δt and $\Delta\tau$ is truly remarkable. First, as stated earlier, elapsed time is not the same for different observers moving relative to one another, even though both are in inertial frames. A proper time interval $\Delta\tau$ for an observer who, like the astronaut, is moving with the apparatus, is smaller than the time interval for other observers. It is the smallest possible measured time between two events. The earthbound observer sees time intervals within the moving system as dilated (i.e., lengthened) relative to how the observer moving relative to Earth sees them within the moving system. Alternatively, according to the earthbound observer, less time passes between events within the moving frame. Note that the shortest elapsed time between events is in the inertial frame in which the observer sees the events (e.g., the emission and arrival of the light signal) occur at the same point.

This time effect is real and is not caused by inaccurate clocks or improper measurements. Time-interval measurements of the same event differ for observers in relative motion. The dilation of time is an intrinsic property of time itself. All clocks moving relative to an observer, including biological clocks, such as a person's heartbeat, or aging, are observed to run more slowly compared with a clock that is stationary relative to the observer.

Note that if the relative velocity is much less than the speed of light (v << c), then v^2/c^2 is extremely small, and the elapsed times Δt and $\Delta\tau$ are nearly equal. At low velocities, physics based on modern relativity approaches classical physics—everyday experiences involve very small relativistic effects. However, for speeds near the speed of light, v^2/c^2 is close to one, so $\sqrt{1-v^2/c^2}$ is very small and Δt becomes significantly larger than $\Delta\tau$.

Half-Life of a Muon

There is considerable experimental evidence that the equation $\Delta t = \gamma\Delta\tau$ is correct. One example is found in cosmic ray particles that continuously rain down on Earth from deep space. Some collisions of these particles with nuclei in the upper atmosphere result in short-lived particles called **muons**. The half-life (amount of time for half of a material to decay) of a muon is 1.52 μs when it is at rest relative to the observer who measures the half-life. This is the proper time interval $\Delta\tau$. This short time allows very few muons to reach Earth's surface and be detected if Newtonian assumptions about time and space were correct. However, muons produced by cosmic ray particles have a range of velocities, with some moving near the speed of light. It has been found that the muon's half-life as measured by an earthbound observer (Δt) varies with velocity exactly as predicted by the equation $\Delta t = \gamma\Delta\tau$. The faster the muon moves, the longer it lives. We on Earth see the muon last much longer than its half-life predicts within its own rest frame. As viewed from our frame, the muon decays more slowly than it does when at rest relative to us. A far larger fraction of muons reach the ground as a result.

Before we present the first example of solving a problem in relativity, we state a strategy you can use as a guideline for these calculations.

PROBLEM-SOLVING STRATEGY: RELATIVITY

1. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). Look in particular for information on relative velocity **v**.
2. Identify exactly what needs to be determined in the problem (identify the unknowns).
3. Make certain you understand the conceptual aspects of the problem before making any calculations (express the answer as an equation). Decide, for example, which observer sees time dilated or length contracted before working with the equations or using them to carry out the calculation. If you have thought about who sees what, who is moving with the event being observed, who sees proper time, and so on, you will find it much easier to determine if your calculation is reasonable.
4. Determine the primary type of calculation to be done to find the unknowns identified above (do the calculation). You will find the section summary helpful in determining whether a length contraction, relativistic kinetic energy, or some other concept is involved.

Note **that you should not round off during the calculation**. As noted in the text, you must often perform your calculations to many digits to see the desired effect. You may round off at the very end of the problem solution, but do not use a rounded number in a subsequent calculation. Also, check the answer to see if it is reasonable: Does it make sense? This may be more difficult for relativity, which has few everyday examples to provide experience with what is reasonable. But you can look for velocities greater than **c** or relativistic effects that are in the wrong direction (such as a time contraction where a dilation was expected).

Example 1.4.1A: Time Dilation in a High-Speed Vehicle

The Hypersonic Technology Vehicle 2 (HTV-2) is an experimental rocket vehicle capable of traveling at 21,000 km/h (5830 m/s). If an electronic clock in the HTV-2 measures a time interval of exactly 1-s duration, what would observers on Earth measure the time interval to be?

Strategy

Apply the time dilation formula to relate the proper time interval of the signal in HTV-2 to the time interval measured on the ground.

Solution

1. Identify the knowns: $\Delta\tau = 1\ s; v = 5830 m/s$.
2. Identify the unknown: Δt.
3. Express the answer as an equation:

$$\Delta t = \gamma \Delta\tau = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

4. Do the calculation. Use the expression for γ to determine Δt from $\Delta\tau$:

$$\begin{aligned} \Delta t &= \frac{1\ s}{\sqrt{1 - \left(\frac{5830\ m/s}{3.00 \times 10^8 m/s}\right)^2}} \\ &= 1.000000000189\ s \\ &= 1\ s + 1.89 \times 10^{-10} s. \end{aligned}$$

Significance

The very high speed of the HTV-2 is still only 10^{-5} times the speed of light. Relativistic effects for the HTV-2 are negligible for almost all purposes, but are not zero.

What Speeds are Relativistic?

How fast must a vehicle travel for 1 second of time measured on a passenger's watch in the vehicle to differ by 1% for an observer measuring it from the ground outside?

Strategy

Use the time dilation formula to find **v/c** for the given ratio of times.

Solution

1. Identify the known:

$$\frac{\Delta\tau}{\Delta t} = \frac{1}{1.01}.$$

2. Identify the unknown: **v/c**.
3. Express the answer as an equation:

$$\begin{aligned}\Delta t &= \gamma\Delta\tau \\ &= \frac{1}{\sqrt{1-v^2/c^2}}\Delta\tau \\ \frac{\Delta\tau}{\Delta t} &= \sqrt{1-v^2/c^2} \\ \left(\frac{\Delta\tau}{\Delta t}\right)^2 &= 1-\frac{v^2}{c^2} \\ \frac{v}{c} &= \sqrt{1-(\Delta\tau/\Delta t)^2}.\end{aligned}$$

4. Do the calculation:

$$\frac{v}{c} = \sqrt{1-(1/1.01)^2} = 0.14.$$

Significance

The result shows that an object must travel at very roughly 10% of the speed of light for its motion to produce significant relativistic time dilation effects.

Calculating Δt for a Relativistic Event

Suppose a cosmic ray colliding with a nucleus in Earth's upper atmosphere produces a muon that has a velocity $v = 0.950c$. The muon then travels at constant velocity and lives 2.20 μs as measured in the muon's frame of reference. (You can imagine this as the muon's internal clock.) How long does the muon live as measured by an earthbound observer (Figure $1.4.2$)?

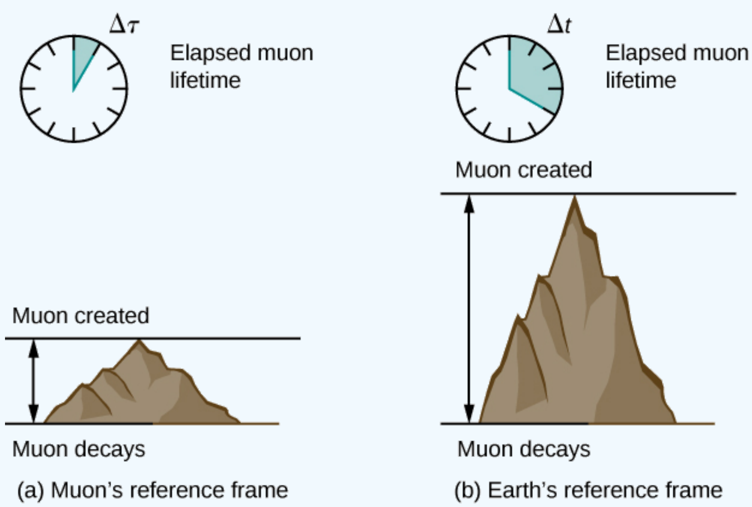

Figure $1.4.2$: A muon in Earth's atmosphere lives longer as measured by an earthbound observer than as measured by the muon's internal clock.

As we will discuss later, in the muon's reference frame, it travels a shorter distance than measured in Earth's reference frame.

Strategy

A clock moving with the muon measures the proper time of its decay process, so the time we are given is $\Delta\tau = 2.20\mu s$. The earthbound observer measures Δt as given by the equation $\Delta t = \gamma\Delta\tau$. Because the velocity is given, we can calculate the time in Earth's frame of reference.

Solution

1. Identify the knowns: $v = 0.950c$; $\delta\tau = 2.20\mu s$.
2. Identify the unknown: Δt.
3. Express the answer as an equation. Use:

$$\Delta t = \gamma\Delta\tau.$$

with

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

4. Do the calculation. Use the expression for γ to determine Δt from $\Delta\tau$:

$$\begin{aligned}\Delta t &= \gamma\Delta\tau. \\ &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\delta\tau \\ &= \frac{2.20\mu s}{\sqrt{1-(0.950)^2}} \\ &= 7.05\ \mu s.\end{aligned}$$

Remember to keep extra significant figures until the final answer.

Significance

One implication of this example is that because $\gamma = 3.20$ at 95.0% of the speed of light ($v = 0.950c$), the relativistic effects are significant. The two time intervals differ by a factor of 3.20, when classically they would be the same. Something moving at 0.950**c** is said to be highly relativistic.

✔ Example 1.4.1B: Relativistic Television

A non-flat screen, older-style television display (Figure $1.4.3$) works by accelerating electrons over a short distance to relativistic speed, and then using electromagnetic fields to control where the electron beam strikes a fluorescent layer at the front of the tube. Suppose the electrons travel at $6.00 \times 10^7 m/s$ through a distance of 0.200m0.200m from the start of the beam to the screen.

a. What is the time of travel of an electron in the rest frame of the television set?
b. What is the electron's time of travel in its own rest frame?

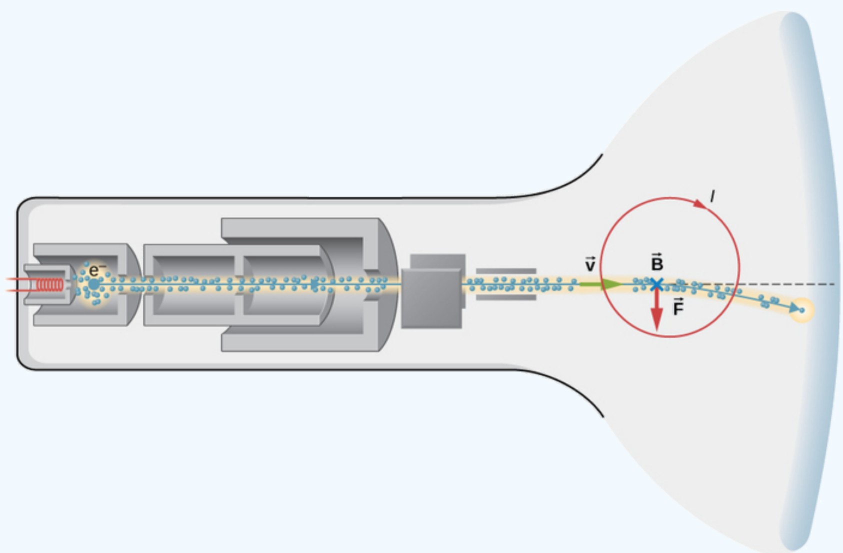

Figure $1.4.3$: The electron beam in a cathode ray tube television display.

Strategy for (a)

(a) Calculate the time from $vt = d$. Even though the speed is relativistic, the calculation is entirely in one frame of reference, and relativity is therefore not involved.

Solution

1. Identify the knowns:

$$v = 6.00 \times 10^7 m/s\, d = 0.200\, m.$$

2. Identify the unknown: the time of travel Δt.
3. Express the answer as an equation:

$$\Delta t = \frac{d}{v}.$$

4. Do the calculation:

$$\begin{aligned} t &= \frac{0.200\, m}{6.00 \times 10^7\ m/s} \\ &= 3.33 \times 10^{-9}\ s. \end{aligned}$$

Significance

The time of travel is extremely short, as expected. Because the calculation is entirely within a single frame of reference, relativity is not involved, even though the electron speed is close to **c**.

Strategy for (b)

(b) In the frame of reference of the electron, the vacuum tube is moving and the electron is stationary. The electron-emitting cathode leaves the electron and the front of the vacuum tube strikes the electron with the electron at the same location. Therefore we use the time dilation formula to relate the proper time in the electron rest frame to the time in the television frame.

Solution

1. Identify the knowns (from part a):

$$\Delta t = 3.33 \times 10^{-9}\ s;\ v = 6.00 \times 10^7\ m/s;\ d = 0.200\, m.$$

2. Identify the unknown: τ.
3. Express the answer as an equation:

$$\Delta t = \gamma \Delta \tau = \frac{\Delta \tau}{\sqrt{1 - v^2/c^2}}.$$

4. Do the calculation:

$$\begin{aligned} \Delta \tau &= (3.33 \times 10^{-9} s) \sqrt{1 - \left(\frac{6.00 \times 10^7 m/s}{3.00 \times 10^8 m/s} \right)^2} \\ &= 3.26 \times 10^{-9} s. \end{aligned}$$

Significance

The time of travel is shorter in the electron frame of reference. Because the problem requires finding the time interval measured in different reference frames for the same process, relativity is involved. If we had tried to calculate the time in the electron rest frame by simply dividing the 0.200 m by the speed, the result would be slightly incorrect because of the relativistic speed of the electron.

Exercise 1.4.1

What is γ if $v = 0.650c$?

Answer

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.650c)}{c^2}}} = 1.32$$

The Twin Paradox

An intriguing consequence of time dilation is that a space traveler moving at a high velocity relative to Earth would age less than the astronaut's earthbound twin. This is often known as the **twin paradox**. Imagine the astronaut moving at such a velocity that $\gamma = 30.0$, as in Figure $1.4.4$. A trip that takes 2.00 years in her frame would take 60.0 years in the earthbound twin's frame. Suppose the astronaut travels 1.00 year to another star system, briefly explores the area, and then travels 1.00 year back. An astronaut who was 40 years old at the start of the trip would be would be 42 when the spaceship returns. Everything on Earth, however, would have aged 60.0 years. The earthbound twin, if still alive, would be 100 years old.

The situation would seem different to the astronaut in Figure $1.4.4$. Because motion is relative, the spaceship would seem to be stationary and Earth would appear to move. (This is the sensation you have when flying in a jet.) Looking out the window of the spaceship, the astronaut would see time slow down on Earth by a factor of $\gamma = 30.0$. Seen from the spaceship, the earthbound sibling will have aged only 2/30, or 0.07, of a year, whereas the astronaut would have aged 2.00 years.

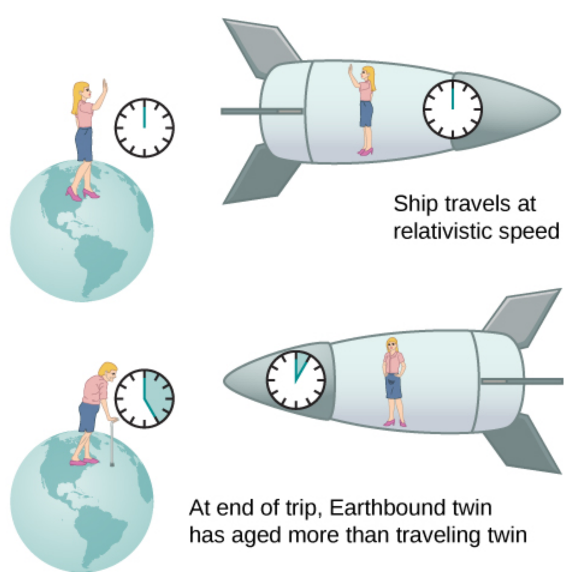

Figure $1.4.4$: The twin paradox consists of the conflicting conclusions about which twin ages more as a result of a long space journey at relativistic speed.

The paradox here is that the two twins cannot both be correct. As with all paradoxes, conflicting conclusions come from a false premise. In fact, the astronaut's motion is significantly different from that of the earthbound twin. The astronaut accelerates to a high velocity and then decelerates to view the star system. To return to Earth, she again accelerates and decelerates. The spacecraft is not in a single inertial frame to which the time dilation formula can be directly applied. That is, the astronaut twin changes inertial references. The earthbound twin does not experience these accelerations and remains in the same inertial frame. Thus, the situation is not symmetric, and it is incorrect to claim that the astronaut observes the same effects as her twin. The lack of symmetry between the twins will be still more evident when we analyze the journey later in this chapter in terms of the path the astronaut follows through four-dimensional space-time.

In 1971, American physicists Joseph Hafele and Richard Keating verified time dilation at low relative velocities by flying extremely accurate atomic clocks around the world on commercial aircraft. They measured elapsed time to an accuracy of a few nanoseconds and compared it with the time measured by clocks left behind. Hafele and Keating's results were within experimental uncertainties of the predictions of relativity. Both special and general relativity had to be taken into account, because gravity and accelerations were involved as well as relative motion.

Exercise 1.4.2A

a. A particle travels at $1.90 \times 10^8\ m/s$ and lives $2.1 \times 10^8\ s$ when at rest relative to an observer. How long does the particle live as viewed in the laboratory?

Answer

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.10 \times 10^{-8} s}{\sqrt{1 - \frac{(1.90 \times 10^8\ m/s)^2}{(3.00 \times 10^8\ m/s)^2}}} = 2.71 \times 10^{-8}\ s.$$

Exercise 1.4.2B

Spacecraft A and B pass in opposite directions at a relative speed of $4.00 \times 10^7\ m/s$. An internal clock in spacecraft A causes it to emit a radio signal for 1.00 s. The computer in spacecraft B corrects for the beginning and end of the signal having traveled different distances, to calculate the time interval during which ship A was emitting the signal. What is the time interval that the computer in spacecraft B calculates?

Answer

Only the relative speed of the two spacecraft matters because there is no absolute motion through space. The signal is emitted from a fixed location in the frame of reference of **A**, so the proper time interval of its emission is $\tau = 1.00\ s$. The duration of the signal measured from frame of reference **B** is then

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1.00\ s}{\sqrt{1 - \frac{(4.00 \times 10^7\ m/s)^2}{(3.00 \times 10^8\ m/s)^2}}} = 1.01\ s.$$

1.5: Length Contraction

Learning Objectives

By the end of this section, you will be able to:

- Explain how simultaneity and length contraction are related.
- Describe the relation between length contraction and time dilation and use it to derive the length-contraction equation.

The length of the train car in Figure $1.5.1$ is the same for all the passengers. All of them would agree on the simultaneous location of the two ends of the car and obtain the same result for the distance between them. But simultaneous events in one inertial frame need not be simultaneous in another. If the train could travel at relativistic speeds, an observer on the ground would see the simultaneous locations of the two endpoints of the car at a different distance apart than observers inside the car. Measured distances need not be the same for different observers when relativistic speeds are involved.

Figure 1.5.1: People might describe distances differently, but at relativistic speeds, the distances really are different. (credit: “russavia”/Flickr).

Proper Length

Two observers passing each other always see the same value of their relative speed. Even though time dilation implies that the train passenger and the observer standing alongside the tracks measure different times for the train to pass, they still agree that relative speed, which is distance divided by elapsed time, is the same. If an observer on the ground and one on the train measure a different time for the length of the train to pass the ground observer, agreeing on their relative speed means they must also see different distances traveled.

The muon discussed previously illustrates this concept (Figure $1.5.2$). To an observer on Earth, the muon travels at 0.950**c** for 7.05 μs from the time it is produced until it decays. Therefore, it travels a distance relative to Earth of:

$$\begin{aligned} L_0 &= v\Delta t \\ &= (0.950)(3.00\times 10^8\, m/s)(7.05\times 10^{-6}\, s) \\ &= 2.01\, km. \end{aligned}$$

In the muon frame, the lifetime of the muon is 2.20 μs. In this frame of reference, the Earth, air, and ground have only enough time to travel:

$$\begin{aligned} L &= v\Delta r \\ &= (0.950)(3.00\times 10^8\, m/s)(2.20\times 10^{-6}\, s) \\ &= 0.627\, km. \end{aligned}$$

The distance between the same two events (production and decay of a muon) depends on who measures it and how they are moving relative to it.

> ✎ Definition: Proper Length
>
> **Proper length** L_0 is the distance between two points measured by an observer who is at rest relative to both of the points.

The earthbound observer measures the proper length L_0 because the points at which the muon is produced and decays are stationary relative to Earth. To the muon, Earth, air, and clouds are moving, so the distance **L** it sees is not the proper length.

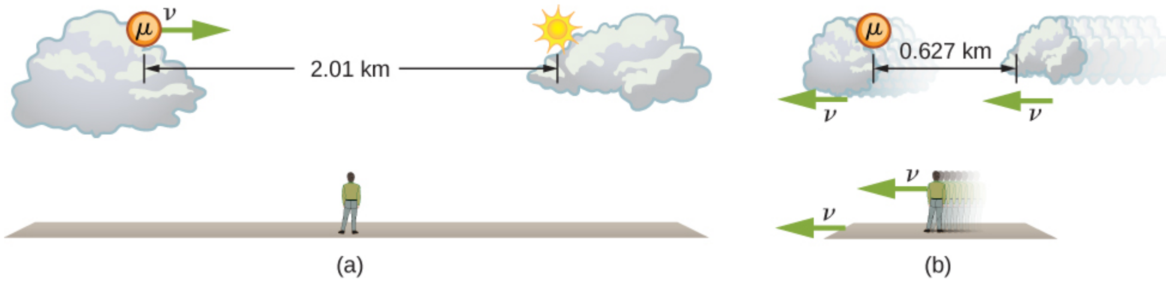

Figure 1.5.2: (a) The earthbound observer sees the muon travel 2.01 km. (b) The same path has length 0.627 km seen from the muon's frame of reference. The Earth, air, and clouds are moving relative to the muon in its frame, and have smaller lengths along the direction of travel.

Length Contraction

To relate distances measured by different observers, note that the velocity relative to the earthbound observer in our muon example is given by

$$v = \frac{L_0}{\Delta t}.$$

The time relative to the earthbound observer is Δt, because the object being timed is moving relative to this observer. The velocity relative to the moving observer is given by

$$v = \frac{L}{\Delta \tau}.$$

The moving observer travels with the muon and therefore observes the proper time $\Delta\tau$. The two velocities are identical; thus,

$$\frac{L_0}{\Delta t} = \frac{L}{\Delta \tau}. \tag{1.5.1}$$

We know that $\Delta t = \gamma \Delta \tau$ and substituting this into Equation 1.5.1 gives

$$L = \frac{L_0}{\gamma}.$$

Substituting for γ gives an equation relating the distances measured by different observers.

> ✎ Definition: Lenght Contraction
>
> **Length contraction** is the decrease in the measured length of an object from its proper length when measured in a reference frame that is moving with respect to the object:
>
> $$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \tag{1.5.2}$$
>
> where L_0 is the length of the object in its rest frame, and L is the length in the frame moving with velocity v.

If we measure the length of anything moving relative to our frame, we find its length **L** to be smaller than the proper length L_0 that would be measured if the object were stationary. For example, in the muon's rest frame, the distance Earth moves between where the muon was produced and where it decayed is shorter than the distance traveled as seen from the Earth's frame. Those points are fixed relative to Earth but are moving relative to the muon. Clouds and other objects are also contracted along the direction of motion as seen from muon's rest frame.

Thus, two observers measure different distances along their direction of relative motion, depending on which one is measuring distances between objects at rest.

But what about distances measured in a direction perpendicular to the relative motion? Imagine two observers moving along their **x**-axes and passing each other while holding meter sticks vertically in the **y**-direction. Figure $1.5.3$ shows two meter sticks M and M' that are at rest in the reference frames of two boys S and S', respectively. A small paintbrush is attached to the top (the 100-cm mark) of stick M'. Suppose that S' is moving to the right at a very high speed **v** relative to S, and the sticks are oriented so that they are perpendicular, or transverse, to their relative velocity vector. The sticks are held so that as they pass each other, their lower ends (the 0-cm marks) coincide. Assume that when S looks at his stick M afterwards, he finds a line painted on it, just below the top of the stick. Because the brush is attached to the top of the other boy's stick M', S can only conclude that stick M' is less than 1.0 m long.

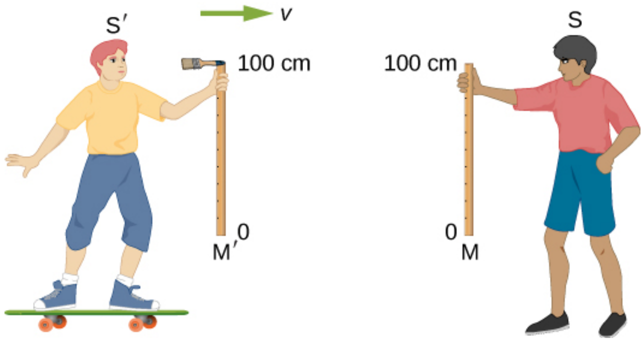

Figure $1.5.3$: Meter sticks M and M'M' are stationary in the reference frames of observers S and S', respectively. As the sticks pass, a small brush attached to the 100-cm mark of M' paints a line on M.

Now when the boys approach each other, S', like S, sees a meter stick moving toward him with speed **v**. Because their situations are symmetric, each boy must make the same measurement of the stick in the other frame. So, if S measures stick M' to be less than 1.0 m long, S' must measure stick M to be also less than 1.0 m long, and S' must see his paintbrush pass over the top of stick M and not paint a line on it. In other words, after the same event, one boy sees a painted line on a stick, while the other does not see such a line on that same stick!

Einstein's first postulate requires that the laws of physics (as, for example, applied to painting) predict that S and S', who are both in inertial frames, make the same observations; that is, S and S' must either both see a line painted on stick M, or both not see that line. We are therefore forced to conclude our original assumption that S saw a line painted below the top of his stick was wrong! Instead, S finds the line painted right at the 100-cm mark on M. Then both boys will agree that a line is painted on M, and they will also agree that both sticks are exactly 1 m long. We conclude then that measurements of a transverse *length must be the same* **in different inertial frames**.

Example $1.5.4$: Calculating Length Contraction

Suppose an astronaut, such as the twin in the twin paradox discussion, travels so fast that $\gamma = 30.00$. (a) The astronaut travels from Earth to the nearest star system, Alpha Centauri, 4.300 light years (ly) away as measured by an earthbound observer. How far apart are Earth and Alpha Centauri as measured by the astronaut? (b) In terms of **c**, what is the astronaut's velocity relative to Earth? You may neglect the motion of Earth relative to the sun (Figure $1.5.4$).

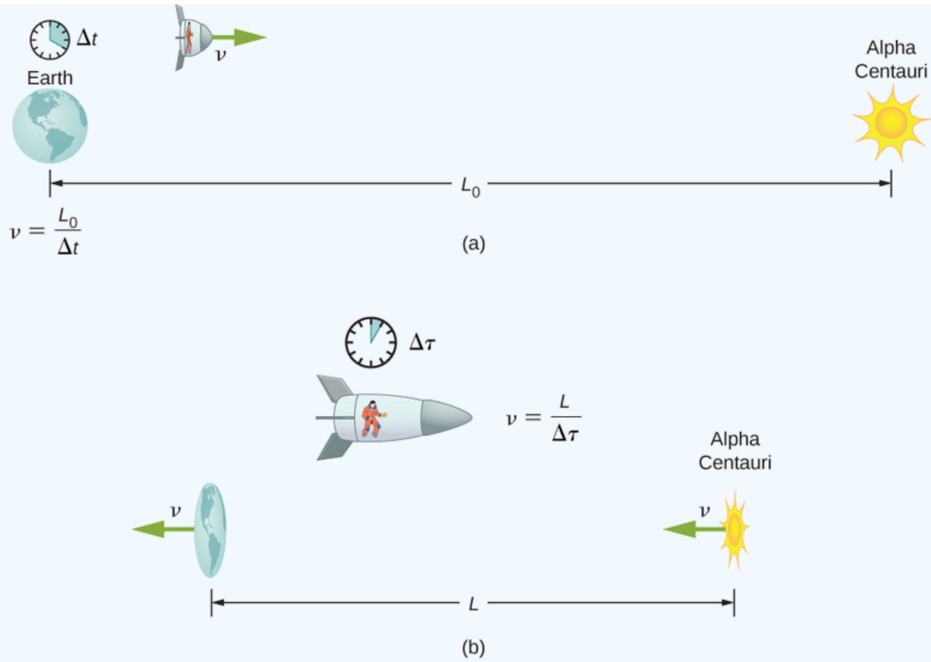

Figure 1.5.4: (a) The earthbound observer measures the proper distance between Earth and Alpha Centauri. (b) The astronaut observes a length contraction because Earth and Alpha Centauri move relative to her ship. She can travel this shorter distance in a smaller time (her proper time) without exceeding the speed of light.

Strategy

First, note that a light year (ly) is a convenient unit of distance on an astronomical scale—it is the distance light travels in a year. For part (a), the 4.300-ly distance between Alpha Centauri and Earth is the proper distance L_0, because it is measured by an earthbound observer to whom both stars are (approximately) stationary. To the astronaut, Earth and Alpha Centauri are moving past at the same velocity, so the distance between them is the contracted length **L**. In part (b), we are given γ, so we can find v by rearranging the definition of γ to express v in terms of c.

Solution for (a)

For part (a):

1. Identify the knowns: $L_0 = 4.300\,ly; \gamma = 30.00$.
2. Identify the unknown: **L**.
3. Express the answer as an equation: $L = \dfrac{L_0}{\gamma}$.
4. Do the calculation:

$$\begin{aligned} L &= \frac{L_0}{\gamma} \\ &= \frac{4.300\,ly}{30.00} \\ &= 0.1433\,ly. \end{aligned}$$

Solution for (b)

For part (b):

1. Identify the known: $\gamma = 30.00$.
2. Identify the unknown: **v** in terms of **c**.
3. Express the answer as an equation. Start with:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Then solve for the unknown **v/c** by first squaring both sides and then rearranging:

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}.$$

4. Do the calculation:

$$\begin{aligned}\frac{v}{c} &= \sqrt{1 - \frac{1}{\gamma^2}} \\ &= \sqrt{1 - \frac{1}{(30.00)^2}} \\ &= 0.99944\end{aligned}$$

or

$$v = 0.9994\,c.$$

Significance: Remember not to round off calculations until the final answer, or you could get erroneous results. This is especially true for special relativity calculations, where the differences might only be revealed after several decimal places. The relativistic effect is large here ($\gamma = 30.00$), and we see that **v** is approaching (not equaling) the speed of light. Because the distance as measured by the astronaut is so much smaller, the astronaut can travel it in much less time in her frame.

People traveling at extremely high velocities could cover very large distances (thousands or even millions of light years) and age only a few years on the way. However, like emigrants in past centuries who left their home, these people would leave the Earth they know forever. Even if they returned, thousands to millions of years would have passed on Earth, obliterating most of what now exists. There is also a more serious practical obstacle to traveling at such velocities; immensely greater energies would be needed to achieve such high velocities than classical physics predicts can be attained. This will be discussed later in the chapter.

Why don't we notice length contraction in everyday life? The distance to the grocery store does not seem to depend on whether we are moving or not. Examining Equation 1.5.2, we see that at low velocities ($v \ll c$), the lengths are nearly equal, which is the classical expectation. However, length contraction is real, if not commonly experienced. For example, a charged particle such as an electron traveling at relativistic velocity has electric field lines that are compressed along the direction of motion as seen by a stationary observer (Figure 1.5.5). As the electron passes a detector, such as a coil of wire, its field interacts much more briefly, an effect observed at particle accelerators such as the 3-km-long Stanford Linear Accelerator (SLAC). In fact, to an electron traveling down the beam pipe at SLAC, the accelerator and Earth are all moving by and are length contracted. The relativistic effect is so great that the accelerator is only 0.5 m long to the electron. It is actually easier to get the electron beam down the pipe, because the beam does not have to be as precisely aimed to get down a short pipe as it would to get down a pipe 3 km long. This, again, is an experimental verification of the special theory of relativity.

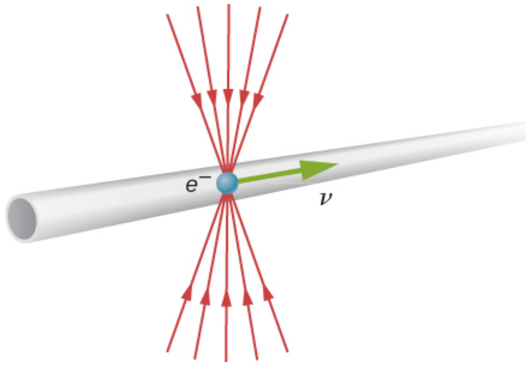

Figure 1.5.5: The electric field lines of a high-velocity charged particle are compressed along the direction of motion by length contraction, producing an observably different signal as the particle goes through a coil.

? Exercise 1.5.1

A particle is traveling through Earth's atmosphere at a speed of $0.750c$. To an earthbound observer, the distance it travels is 2.50 km. How far does the particle travel as viewed from the particle's reference frame?

Answer

$$L = L_0\sqrt{1-\frac{v^2}{c^2}} = (2.50\,km)\sqrt{1-\frac{(0.750c)^2}{c^2}} = 1.65\,km$$

1.6: The Lorentz Transformation

Learning Objectives

- Describe the Galilean transformation of classical mechanics, relating the position, time, velocities, and accelerations measured in different inertial frames
- Derive the corresponding Lorentz transformation equations, which, in contrast to the Galilean transformation, are consistent with special relativity
- Explain the Lorentz transformation and many of the features of relativity in terms of four-dimensional space-time

We have used the postulates of relativity to examine, in particular examples, how observers in different frames of reference measure different values for lengths and the time intervals. We can gain further insight into how the postulates of relativity change the Newtonian view of time and space by examining the transformation equations that give the space and time coordinates of events in one inertial reference frame in terms of those in another. We first examine how position and time coordinates transform between inertial frames according to the view in Newtonian physics. Then we examine how this has to be changed to agree with the postulates of relativity. Finally, we examine the resulting Lorentz transformation equations and some of their consequences in terms of four-dimensional space-time diagrams, to support the view that the consequences of special relativity result from the properties of time and space itself, rather than electromagnetism.

The Galilean Transformation Equations

An event is specified by its location and time (x, y, z, t) relative to one particular inertial frame of reference S. As an example, (x, y, z, t) could denote the position of a particle at time t, and we could be looking at these positions for many different times to follow the motion of the particle. Suppose a second frame of reference S' moves with velocity v with respect to the first. For simplicity, assume this relative velocity is along the x-axis. The relation between the time and coordinates in the two frames of reference is then

$$x = x' + vt \tag{1.6.1}$$

$$y = y' \tag{1.6.2}$$

$$x = z'. \tag{1.6.3}$$

Implicit in these equations is the assumption that time measurements made by observers in both S and S' are the same. That is,

$$t = t' \tag{1.6.4}$$

Equations 1.6.1-1.6.4 are known collectively as the **Galilean transformation**.

We can obtain the Galilean velocity and acceleration transformation equations by differentiating these equations with respect to time. We use u for the velocity of a particle throughout this chapter to distinguish it from v, the relative velocity of two reference frames. Note that, for the Galilean transformation, the increment of time used in differentiating to calculate the particle velocity is the same in both frames, $dt = dt'$. Differentiation yields

$$u_x = u'_x + v,\; u_y = u'_y,\; u_z = u'_z \tag{1.6.5}$$

and

$$a_x = a'_x,\; a_y = a'_y,\; a_z = a'_z. \tag{1.6.6}$$

We denote the velocity of the particle by u rather than v to avoid confusion with the velocity v of one frame of reference with respect to the other. Velocities in each frame differ by the velocity that one frame has as seen from the other frame. Observers in both frames of reference measure the same value of the acceleration. Because the mass is unchanged by the transformation, and distances between points are uncharged, observers in both frames see the same forces $F = ma$ acting between objects and the same form of Newton's second and third laws in all inertial frames. The laws of mechanics are consistent with the first postulate of relativity.

The Lorentz Transformation Equations

The Galilean transformation nevertheless violates Einstein's postulates, because the velocity equations state that a pulse of light moving with speed c along the **x**-axis would travel at speed $c-v$ in the other inertial frame. Specifically, the spherical pulse has radius $r=ct$ at time t in the unprimed frame, and also has radius $r'=ct'$ at time t' in the primed frame. Expressing these relations in Cartesian coordinates gives

$$x^2+y^2+z^2-c^2t^2 = 0 \tag{1.6.7}$$

$$x'^2+y'^2+z'^2-c^2t'^2 = 0. \tag{1.6.8}$$

The left-hand sides Equations 1.6.7 and 1.6.8 can be set equal because both are zero. Because $y=y'$ and $z=z'$, we obtain

$$x^2-c^2t^2=x'^2-c^2t'^2. \tag{1.6.9}$$

This cannot be satisfied for nonzero relative velocity v of the two frames if we assume the Galilean transformation results in $t=t'$ with $x=x'+vt'$.

To find the correct set of transformation equations, assume the two coordinate systems S and S' in Figure 1.6.1. First suppose that an event occurs at $(x',0,0,t')$ in S' and at $(x,0,0,t)$ in S, as depicted in Figure 1.6.1.

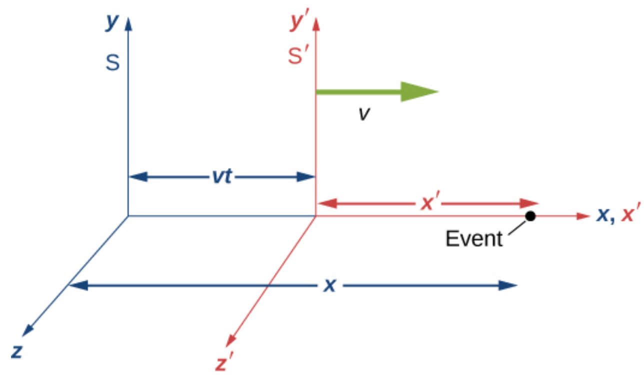

Figure 1.6.1: An event occurs at $(x,0,0,t)$ in S and at $(x',0,0,t')$ in S'. The Lorentz transformation equations relate events in the two systems.

Suppose that at the instant that the origins of the coordinate systems in **S** and S' coincide, a flash bulb emits a spherically spreading pulse of light starting from the origin. At time **t**, an observer in **S** finds the origin of S' to be at $x=vt$. With the help of a friend in **S**, the S' observer also measures the distance from the event to the origin of S' and finds it to be $x'\sqrt{1-v^2/c^2}$. This follows because we have already shown the postulates of relativity to imply length contraction. Thus the position of the event in **S** is

$$x=vt+x'\sqrt{1-v^2/c^2} \tag{1.6.10}$$

and

$$x'=\frac{x-vt}{\sqrt{1-v^2/c^2}}. \tag{1.6.11}$$

The postulates of relativity imply that the equation relating distance and time of the spherical wave front:

$$x^2+y^2+z^2-c^2t^2=0 \tag{1.6.12}$$

must apply both in terms of primed and unprimed coordinates, which was shown above to lead to Equation:

$$x^2-c^2t^2=x'^2-c^2t'^2. \tag{1.6.13}$$

We combine this with Equation 1.6.11 that relates x and x' to obtain the relation between t and t':

$$t'=\frac{t-vx/c^2}{\sqrt{1-v^2/c^2}}. \tag{1.6.14}$$

The equations relating the time and position of the events as seen in S are then

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}. \tag{1.6.15}$$

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}. \tag{1.6.16}$$

$$y = y' \tag{1.6.17}$$

$$z = z'. \tag{1.6.18}$$

This set of equations, relating the position and time in the two inertial frames, is known as the *Lorentz transformation*. They are named in honor of H.A. Lorentz (1853–1928), who first proposed them. Interestingly, he justified the transformation on what was eventually discovered to be a fallacious hypothesis. The correct theoretical basis is Einstein's special theory of relativity.

The reverse transformation expresses the variables in S in terms of those in S'. Simply interchanging the primed and unprimed variables and substituting gives:

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z.$$

Example 1.6.1: Using Lorentz Transformation for Time

Spacecraft S' is on its way to Alpha Centauri when Spacecraft **S** passes it at relative speed **c**/2. The captain of S' sends a radio signal that lasts 1.2 s according to that ship's clock. Use the Lorentz transformation to find the time interval of the signal measured by the communications officer of spaceship **S**.

Solution

1. Identify the known: $\Delta t' = t'_2 - t'_1 = 1.2 s$; $\Delta x' = x'_2 - x'_1 = 0$.
2. Identify the unknown: $\Delta t = t_2 - t_1$.
3. Express the answer as an equation. The time signal starts as (x', t'_1) and stops at (x', t'_1). Note that the x' coordinate of both events is the same because the clock is at rest in S'. Write the first Lorentz transformation equation in terms of $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, and similarly for the primed coordinates, as:

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Because the position of the clock in S' is fixed, $\Delta x' = 0$, and the time interval Δt becomes:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

4. Do the calculation.

With $\Delta t' = 1.2\ s$ this gives:

$$\Delta t = \frac{1.2\ s}{\sqrt{1 - \left(\frac{1}{2}\right)^2}}$$

$$= 1.6\ s.$$

Note that the Lorentz transformation reproduces the time dilation equation.

Example 1.6.2: Using the Lorentz Transformation for Length

A surveyor measures a street to be $L = 100\,m$ long in Earth frame S. Use the Lorentz transformation to obtain an expression for its length measured from a spaceship S', moving by at speed $0.20c$, assuming the x coordinates of the two frames coincide at time $t = 0$.

Solution

1. Identify the known: $L = 100\,m$; $v = 0.20c$; $\Delta\tau = 0$.
2. Identify the unknown: L'.
3. Express the answer as an equation. The surveyor in frame S has measured the two ends of the stick simultaneously, and found them at rest at x_2 and x_1 a distance $L = x_2 - x_1 = 100\,m$ apart. The spaceship crew measures the simultaneous location of the ends of the sticks in their frame. To relate the lengths recorded by observers in S' and S, respectively, write the second of the four Lorentz transformation equations as:

$$\begin{aligned} x'_2 - x'_1 &= \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x_2 - x_2}{\sqrt{1 - v^2/c^2}} \\ &= \frac{L}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

4. Do the calculation. Because $x_2 - x_1 = 100\,m$, the length of the moving stick is equal to:

$$\begin{aligned} L' &= (100\,m)\sqrt{1 - v^2/c^2} \\ &= (100\,m)\sqrt{1 - (0.20)^2} = 98.0\,m. \end{aligned}$$

Note that the Lorentz transformation gave the length contraction equation for the street.

Example 1.6.3: Lorentz Transformation and Simultaneity

The observer shown in Figure 1.6.2 standing by the railroad tracks sees the two bulbs flash simultaneously at both ends of the 26 m long passenger car when the middle of the car passes him at a speed of **c**/2. Find the separation in time between when the bulbs flashed as seen by the train passenger seated in the middle of the car.

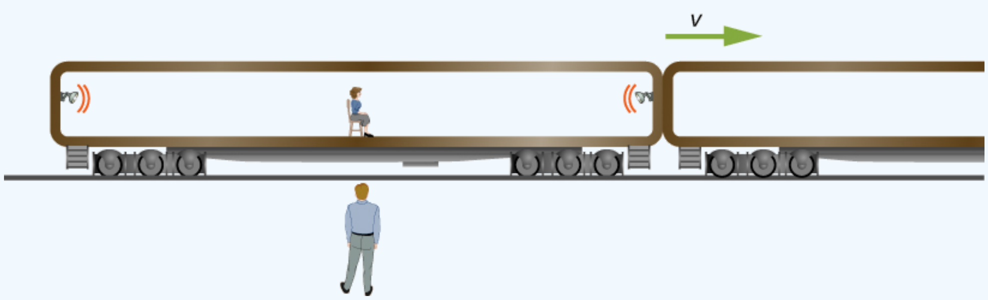

Figure 1.6.2: An person watching a train go by observes two bulbs flash simultaneously at opposite ends of a passenger car. There is another passenger inside of the car observing the same flashes but from a different perspective.

Solution

1. Identify the known: $\Delta t = 0$.

   Note that the spatial separation of the two events is between the two lamps, not the distance of the lamp to the passenger.

2. Identify the unknown: $\Delta t' = t'_2 - t'_1$. .

   Again, note that the time interval is between the flashes of the lamps, not between arrival times for reaching the passenger.

3. Express the answer as an equation:

$$\Delta t=\frac{\Delta t'+v\Delta x'/c^2}{\sqrt{1-v^2/c^2}}.$$

4. Do the calculation:

$$0=\frac{\Delta t'+\frac{c}{2}(26\,m)/c^2}{\sqrt{1-v^2/c^2}}$$

$$\Delta t'=-\frac{26\,m/s}{2c}=-\frac{26\,m/s}{2(3.00\times 10^8\,m/s)}$$

$$=-4.33\times 10^{-8}\,s.$$

Significance

The sign indicates that the event with the larger x'_2 namely, the flash from the right, is seen to occur first in the S'S′ frame, as found earlier for this example, so that $t_2<t_1$.

Space-Time

Relativistic phenomena can be analyzed in terms of events in a four-dimensional **space-time**. When phenomena such as the twin paradox, time dilation, length contraction, and the dependence of simultaneity on relative motion are viewed in this way, they are seen to be characteristic of the nature of space and time, rather than specific aspects of electromagnetism.

In three-dimensional space, positions are specified by three coordinates on a set of Cartesian axes, and the displacement of one point from another is given by:

$$(\Delta x,\,\Delta y,\,\Delta z)=(x_2-x_1,\,y_2-y-1,\,z_2-z_1). \tag{1.6.19}$$

The distance Δr between the points is

$$\Delta r^2=(\Delta x)^2+(\Delta y)^2+(\Delta z)^2. \tag{1.6.20}$$

The distance Δr is invariant under a rotation of axes. If a new set of Cartesian axes rotated around the origin relative to the original axes are used, each point in space will have new coordinates in terms of the new axes, but the distance $\Delta r'$ given by

$$\Delta r'^2=(\Delta x')^2+(\Delta y')^2+(\Delta z')^2. \tag{1.6.21}$$

That has the same value that Δr^2 had. Something similar happens with the Lorentz transformation in space-time.

Define the separation between two events, each given by a set of **x**, **y**, **z**¸ and **ct** along a four-dimensional Cartesian system of axes in space-time, as

$$(\Delta x,\,\Delta y,\,\Delta z,\,c\Delta t)=(x_2-x_1,\,y_2-y_1,\,z_2-z_1,\,c(t_2-t_1)). \tag{1.6.22}$$

Also define the space-time interval Δs between the two events as

$$\Delta s^2=(\Delta x)^2+(\Delta y)^2+(\Delta z)^2-(c\Delta t)^2. \tag{1.6.23}$$

If the two events have the same value of **ct** in the frame of reference considered, Δs would correspond to the distance Δr between points in space.

The path of a particle through space-time consists of the events (**x**, **y**, **z**¸ **ct**) specifying a location at each time of its motion. The path through space-time is called the **world line** of the particle. The world line of a particle that remains at rest at the same location is a straight line that is parallel to the time axis. If the particle moves at constant velocity parallel to the **x**-axis, its world line would be a sloped line $x=vt$, corresponding to a simple displacement vs. time graph. If the particle accelerates, its world line is curved. The increment of **s** along the world line of the particle is given in differential form as

$$ds^2=(dx)^2+(dy)^2+(dz)^2-c^2(dt)^2. \tag{1.6.24}$$

Just as the distance Δr is invariant under rotation of the space axes, the space-time interval:

$$\Delta s^2=(\Delta x)^2+(\Delta y)^2+(\Delta z)^2-(c\Delta t)^2. \tag{1.6.25}$$

is invariant under the Lorentz transformation. This follows from the postulates of relativity, and can be seen also by substitution of the previous Lorentz transformation equations into the expression for the space-time interval:

$$\begin{aligned} \Delta s^2 &= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2. \\ &= \left(\frac{\Delta x' + v\Delta t'}{\sqrt{1-v^2/c^2}}\right)^2 + (\Delta y')^2 + (\Delta z')^2 - \left(c\frac{\Delta t' + \dfrac{v\Delta x'}{c^2}}{\sqrt{1-v^2/c^2}}\right)^2 \\ &= (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - (c\Delta t')^2 \\ &= \Delta s'^2. \end{aligned}$$

In addition, the Lorentz transformation changes the coordinates of an event in time and space similarly to how a three-dimensional rotation changes old coordinates into new coordinates:

Lorentz transformation (x, t coordinates):	Axis–rotation around z - a axis (x, t coordinates):
$x' = (\gamma)x + (-\beta\gamma)ct$	$x' = (\cos\theta)x + (\sin\theta)y$
$ct' = (-\beta\gamma)x + (\gamma)ct$	$y' = (-\sin\theta)x + (\cos\theta)y$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$; $\beta = v/c$.

Lorentz transformations can be regarded as generalizations of spatial rotations to space-time. However, there are some differences between a three-dimensional axis rotation and a Lorentz transformation involving the time axis, because of differences in how the metric, or rule for measuring the displacements Δr and Δs, differ. Although Δr is invariant under spatial rotations and Δs is invariant also under Lorentz transformation, the Lorentz transformation involving the time axis does not preserve some features, such as the axes remaining perpendicular or the length scale along each axis remaining the same.

Note that the quantity Δs^2 can have either sign, depending on the coordinates of the space-time events involved. For pairs of events that give it a negative sign, it is useful to define $c^2\Delta\tau^2$ as $-\Delta s^2$. The significance of $c^2\Delta\tau$ as just defined follows by noting that in a frame of reference where the two events occur at the same location, we have $\Delta x = \Delta y = \Delta z = 0$ and therefore (from the equation for $\Delta s^2 = -c^2\Delta\tau^2$):

$$c^2\Delta\tau^2 = -\Delta s^2 = (c^2\Delta t)^2. \tag{1.6.26}$$

Therefore $c^2\Delta\tau$ is the time interval $c^2\Delta t$ in the frame of reference where both events occur at the same location. It is the same interval of proper time discussed earlier. It also follows from the relation between Δs and that $c^2\Delta\tau$ that because Δs is Lorentz invariant, the proper time is also Lorentz invariant. All observers in all inertial frames agree on the proper time intervals between the same two events.

Exercise 1.6.1

Show that if a time increment dt elapses for an observer who sees the particle moving with velocity v, it corresponds to a proper time particle increment for the particle of $d\tau = \gamma dt$.

Answer

Start with the definition of the proper time increment:

$$d\tau = \sqrt{-(ds)^2/c^2} = \sqrt{dt^2 - (dx^2 + dx^2 + dx^2)/c^2}\,. \tag{1.6.27}$$

where (dx, dy, dx, cdt) are measured in the inertial frame of an observer who does not necessarily see that particle at rest. This therefore becomes

$$d\tau = \sqrt{-(ds)^2/c^2} = \sqrt{dt^2 - [(dx)^2 + (dy)^2 + (dz)^2]/c^2} \tag{1.6.28}$$

$$dt\sqrt{1-\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]/c^2} \tag{1.6.29}$$

$$dt\sqrt{1-v^2/c^2} \tag{1.6.30}$$

$$dt=\gamma d\tau. \tag{1.6.31}$$

The Light Cone

We can deal with the difficulty of visualizing and sketching graphs in four dimensions by imagining the three spatial coordinates to be represented collectively by a horizontal axis, and the vertical axis to be the **ct**-axis. Starting with a particular event in space-time as the origin of the space-time graph shown, the world line of a particle that remains at rest at the initial location of the event at the origin then is the time axis. Any plane through the time axis parallel to the spatial axes contains all the events that are simultaneous with each other and with the intersection of the plane and the time axis, as seen in the rest frame of the event at the origin.

It is useful to picture a **light cone** on the graph, formed by the world lines of all light beams passing through the origin event **A**, as shown in Figure $1.6.3$. The light cone, according to the postulates of relativity, has sides at an angle of 45° if the time axis is measured in units of **ct**, and, according to the postulates of relativity, the light cone remains the same in all inertial frames. Because the event **A** is arbitrary, every point in the space-time diagram has a light cone associated with it.

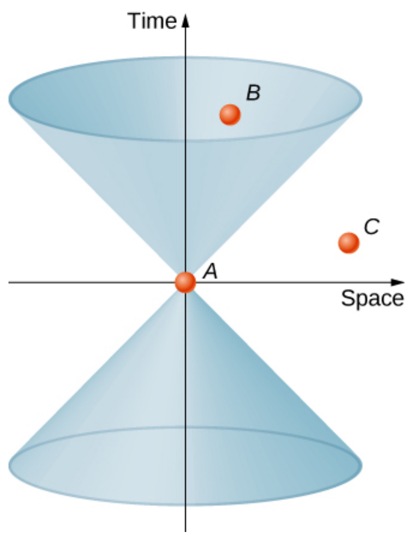

Figure $1.6.3$: The light cone consists of all the world lines followed by light from the event A at the vertex of the cone.

Consider now the world line of a particle through space-time. Any world line outside of the cone, such as one passing from **A** through **C**, would involve speeds greater than **c**, and would therefore not be possible. Events such as **C** that lie outside the light cone are said to have a space-like separation from event **A**. They are characterized by:

$$\Delta s^2_{AC}=(x_A-x_C)^2+(y_A-y_C)^2+(z_A-z_C)^2-(c\Delta t)^2>0. \tag{1.6.32}$$

An event like **B** that lies in the upper cone is reachable without exceeding the speed of light in vacuum, and is characterized by

$$\Delta s^2_{AB}=(x_A-x_B)^2+(y_A-y_B)^2+(z_A-z_B)^2-(c\Delta t)^2<0. \tag{1.6.33}$$

The event is said to have a time-like separation from **A**. Time-like events that fall into the upper half of the light cone occur at greater values of **t** than the time of the event **A** at the vertex and are in the future relative to **A**. Events that have time-like separation from A and fall in the lower half of the light cone are in the past, and can affect the event at the origin. The region outside the light cone is labeled as neither past nor future, but rather as “elsewhere.”

For any event that has a space-like separation from the event at the origin, it is possible to choose a time axis that will make the two events occur at the same time, so that the two events are simultaneous in some frame of reference. Therefore, which of the events with space-like separation comes before the other in time also depends on the frame of reference of the observer. Since space-like separations can be traversed only by exceeding the speed of light; this violation of which event can cause the other provides another argument for why particles cannot travel faster than the speed of light, as well as potential material for science fiction about time travel. Similarly for any event with time-like separation from the event at the origin, a frame of reference can be found that will make the events occur at the same location. Because the relations

$$\Delta s^2_{AC} = (x_A - x_C)^2 + (y_A - x_C)^2 + (z_A - z_C)^2 - (c\Delta t)^2 > 0. \tag{1.6.34}$$

and

$$\Delta s^2_{AB} = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - (c\Delta t)^2 < 0. \tag{1.6.35}$$

are Lorentz invariant, whether two events are time-like and can be made to occur at the same place or space-like and can be made to occur at the same time is the same for all observers. All observers in different inertial frames of reference agree on whether two events have a time-like or space-like separation.

The twin paradox seen in space-time

The **twin paradox** discussed earlier involves an astronaut twin traveling at near light speed to a distant star system, and returning to Earth. Because of time dilation, the space twin is predicted to age much less than the earthbound twin. This seems paradoxical because we might have expected at first glance for the relative motion to be symmetrical and naively thought it possible to also argue that the earthbound twin should age less.

To analyze this in terms of a space-time diagram, assume that the origin of the axes used is fixed in Earth. The world line of the earthbound twin is then along the time axis.

The world line of the astronaut twin, who travels to the distant star and then returns, must deviate from a straight line path in order to allow a return trip. As seen in Figure $1.6.4$, the circumstances of the two twins are not at all symmetrical. Their paths in space-time are of manifestly different length. Specifically, the world line of the earthbound twin has length $2c\Delta t$, which then gives the proper time that elapses for the earthbound twin as $2\Delta t$. The distance to the distant star system is $\Delta x = v\Delta t$. The proper time that elapses for the space twin is $2\Delta\tau$ where

$$c^2\Delta\tau^2 = -\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2. \tag{1.6.36}$$

This is considerably shorter than the proper time for the earthbound twin by the ratio

$$\frac{c\Delta\tau}{c\Delta t} = \sqrt{\frac{(c\Delta t)^2 - (\Delta x)^2}{(c\Delta t)^2}} = \sqrt{\frac{(c\Delta t)^2 - (v\Delta t)^2}{(c\Delta t)^2}} = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma}. \tag{1.6.37}$$

consistent with the time dilation formula. The twin paradox is therefore seen to be no paradox at all. The situation of the two twins is not symmetrical in the space-time diagram. The only surprise is perhaps that the seemingly longer path on the space-time diagram corresponds to the smaller proper time interval, because of how $\Delta\tau$ and Δs depend on Δx and Δt.

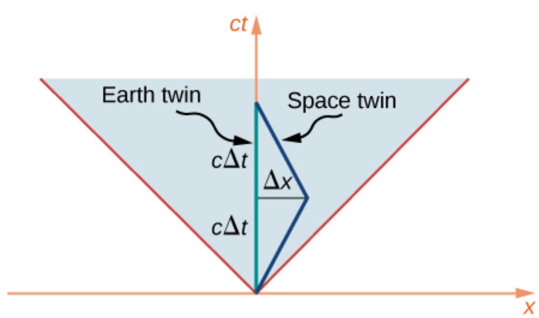

Figure $1.6.4$. The space twin and the earthbound twin, in the twin paradox example, follow world lines of different length through space-time.

Lorentz Transformations in Space-time

We have already noted how the Lorentz transformation leaves

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2. \tag{1.6.38}$$

unchanged and corresponds to a rotation of axes in the four-dimensional space-time. If the S and S' frames are in relative motion along their shared **x**-direction the space and time axes of S' are rotated by an angle αα as seen from S, in the way shown in shown in Figure 1.6.5, where:

$$tan\,\alpha = \frac{v}{c} = \beta. \tag{1.6.39}$$

This differs from a rotation in the usual three-dimension sense, insofar as the two space-time axes rotate toward each other symmetrically in a scissors-like way, as shown. The rotation of the time and space axes are both through the same angle. The mesh of dashed lines parallel to the two axes show how coordinates of an event would be read along the primed axes. This would be done by following a line parallel to the x' and one parallel to the t'-axis, as shown by the dashed lines. The length scale of both axes are changed by:

$$ct' = ct\sqrt{\frac{1+\beta^2}{1-\beta^2}};\; x' = x\sqrt{\frac{1+\beta^2}{1-\beta^2}}. \tag{1.6.40}$$

The line labeled "v = c" at 45° to the **x**-axis corresponds to the edge of the light cone, and is unaffected by the Lorentz transformation, in accordance with the second postulate of relativity. The "v = c" line, and the light cone it represents, are the same for both the **S** and S' frame of reference.

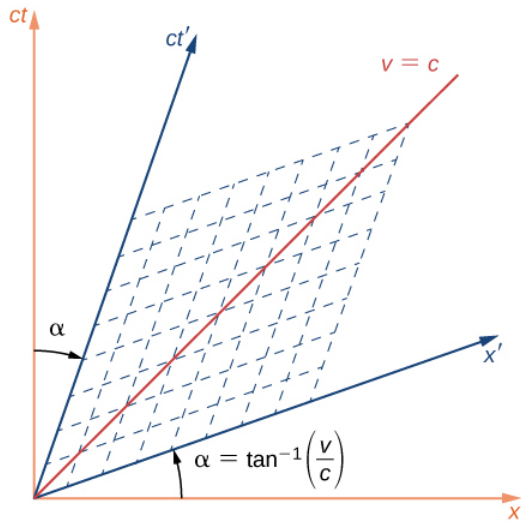

Figure 1.6.5: The Lorentz transformation results in new space and time axes rotated in a scissors-like way with respect to the original axes.

Simultaneity

Simultaneity of events at separated locations depends on the frame of reference used to describe them, as given by the scissors-like "rotation" to new time and space coordinates as described. If two events have the same t values in the unprimed frame of reference, they need not have the same values measured along the ct'-axis, and would then **not** be simultaneous in the primed frame.

As a specific example, consider the near-light-speed train in which flash lamps at the two ends of the car have flashed simultaneously in the frame of reference of an observer on the ground. The space-time graph is shown Figure 1.6.6. The flashes of the two lamps are represented by the dots labeled "Left flash lamp" and "Right flash lamp" that lie on the light cone in the past. The world line of both pulses travel along the edge of the light cone to arrive at the observer on the ground simultaneously. Their arrival is the event at the origin. They therefore had to be emitted simultaneously in the unprimed frame, as represented by the point labeled as t (both). But time is measured along the ct'-axis in the frame of reference of the observer seated in the middle of the train car. So in her frame of reference, the emission event of the bulbs labeled as t' (left) and t' (right) were not simultaneous.

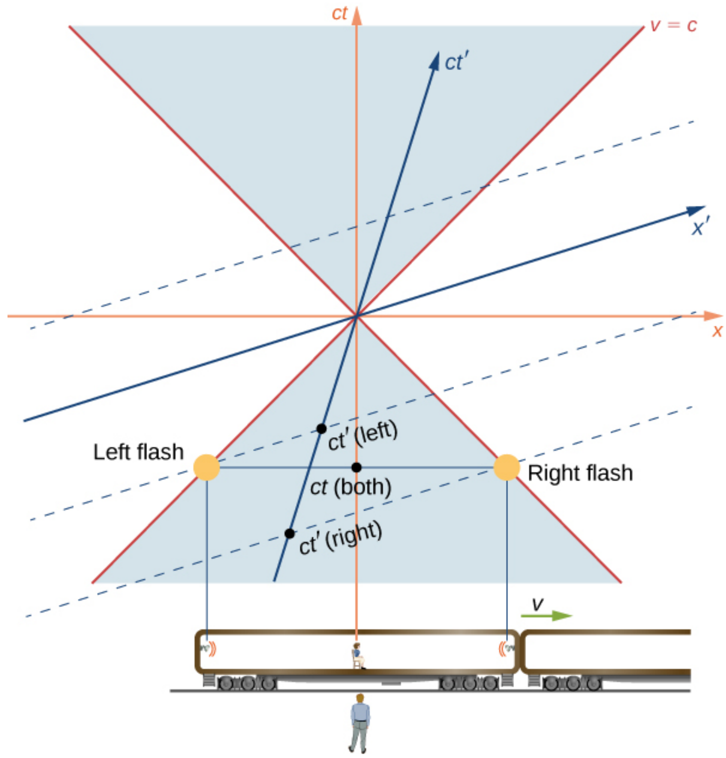

Figure 1.6.6: The train example revisited. The flashes occur at the same time t (both) along the time axis of the ground observer, but at different times, along the t't′ time axis of the passenger.

In terms of the space-time diagram, the two observers are merely using different time axes for the same events because they are in different inertial frames, and the conclusions of both observers are equally valid. As the analysis in terms of the space-time diagrams further suggests, the property of how simultaneity of events depends on the frame of reference results from the properties of space and time itself, rather than from anything specifically about electromagnetism.

Contributors and Attributions

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1.7: Relativistic Velocity Transformation

Learning Objectives

By the end of this section, you will be able to:

- Derive the equations consistent with special relativity for transforming velocities in one inertial frame of reference into another.
- Apply the velocity transformation equations to objects moving at relativistic speeds.
- Examine how the combined velocities predicted by the relativistic transformation equations compare with those expected classically.

Remaining in place in a kayak in a fast-moving river takes effort. The river current pulls the kayak along. Trying to paddle against the flow can move the kayak upstream relative to the water, but that only accounts for part of its velocity relative to the shore. The kayak's motion is an example of how velocities in Newtonian mechanics combine by vector addition. The kayak's velocity is the vector sum of its velocity relative to the water and the water's velocity relative to the riverbank. However, the relativistic addition of velocities is quite different.

Velocity Transformations

Imagine a car traveling at night along a straight road, as in Figure $1.7.1$. The driver sees the light leaving the headlights at speed c within the car's frame of reference. If the Galilean transformation applied to light, then the light from the car's headlights would approach the pedestrian at a speed $u=v+c$, contrary to Einstein's postulates.

Figure $1.7.1$: According to experimental results and the second postulate of relativity, light from the car's headlights moves away from the car at speed c and toward the observer on the sidewalk at speed c.

Both the distance traveled and the time of travel are different in the two frames of reference, and they must differ in a way that makes the speed of light the same in all inertial frames. The correct rules for transforming velocities from one frame to another can be obtained from the Lorentz transformation equations.

Relativistic Transformation of Velocity

Suppose an object **P** is moving at constant velocity $u=(u'_x, u'_y, u'_z)$ as measured in the S' frame. The S' frame is moving along its x'-axis at velocity v. In an increment of time dt', the particle is displaced by dx' along the x'-axis. Applying the Lorentz transformation equations gives the corresponding increments of time and displacement in the unprimed axes:

$$dt = \gamma(dt' + v dx'/c^2) \tag{1.7.1}$$

$$dx = \gamma(dx' + v dt') \tag{1.7.2}$$

$$dy = dy' \tag{1.7.3}$$

$$dz = dz'. \tag{1.7.4}$$

The velocity components of the particle seen in the unprimed coordinate system are then

$$\frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma(dt' + v\,dx'/c^2)} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \tag{1.7.5}$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + v\,dx'/c^2)} = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \tag{1.7.6}$$

$$\frac{dz}{dt} = \frac{dz'}{\gamma(dt' + v\,dx'/c^2)} = \frac{\frac{dz'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \tag{1.7.7}$$

We thus obtain the equations for the velocity components of the object as seen in frame S:

$$u_x = \left(\frac{u'_x + v}{1 + vu'_x/c^2}\right), \; u_y = \left(\frac{u'_y/\gamma}{1 + vu'_x/c^2}\right), \; u_z = \left(\frac{u'_z/\gamma}{1 + vu'_x/c^2}\right).$$

Compare this with how the Galilean transformation of classical mechanics says the velocities transform, by adding simply as vectors:

$$u_x = u'_x + u, \; u_y = u'_y, \; u_z = u'_z.$$

When the relative velocity of the frames is much smaller than the speed of light, that is, when $v \gg c$, the special relativity velocity addition law reduces to the Galilean velocity law. When the speed v of S' relative to S is comparable to the speed of light, the relativistic velocity addition law gives a much smaller result than the classical (Galilean) velocity addition does.

Example 1.7.1: Velocity Transformation Equations for Light

Suppose a spaceship heading directly toward Earth at half the speed of light sends a signal to us on a laser-produced beam of light (Figure 1.7.2). Given that the light leaves the ship at speed c as observed from the ship, calculate the speed at which it approaches Earth.

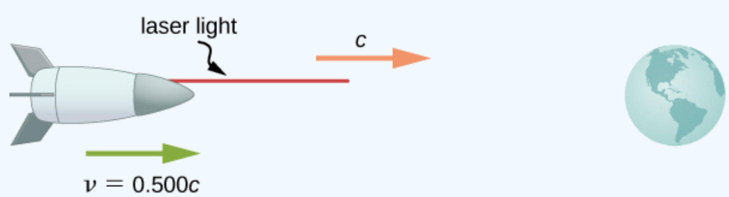


Figure 1.7.2: How fast does a light signal approach Earth if sent from a spaceship traveling at $0.500c$?

Strategy

Because the light and the spaceship are moving at relativistic speeds, we cannot use simple velocity addition. Instead, we determine the speed at which the light approaches Earth using relativistic velocity addition.

Solution

Identify the knowns: $v = 0.500c$; $u' = c$.

Identify the unknown: u.

Express the answer as an equation: $u = \dfrac{v+u'}{1+\dfrac{vu'}{c^2}}$.

Do the calculation:

$$\begin{aligned} u &= \frac{v+u'}{1+\frac{vu'}{c^2}} \\ &= \frac{0.500c+c}{1+\frac{(0.500c)(c)}{c^2}} \\ &= \frac{(0.500+1)c}{\left(\frac{c^2+0.500c^2}{c^2}\right)} = c. \end{aligned}$$

Significance

Relativistic velocity addition gives the correct result. Light leaves the ship at speed c and approaches Earth at speed c. The speed of light is independent of the relative motion of source and observer, whether the observer is on the ship or earthbound.

Velocities cannot add to greater than the speed of light, provided that v is less than c and u' does not exceed c. The following example illustrates that relativistic velocity addition is not as symmetric as classical velocity addition.

Example 1.7.2: Relativistic Package Delivery

Suppose the spaceship in the previous example approaches Earth at half the speed of light and shoots a canister at a speed of $0.750c$(Figure 1.7.3).

a. At what velocity does an earthbound observer see the canister if it is shot directly toward Earth?
b. If it is shot directly away from Earth?

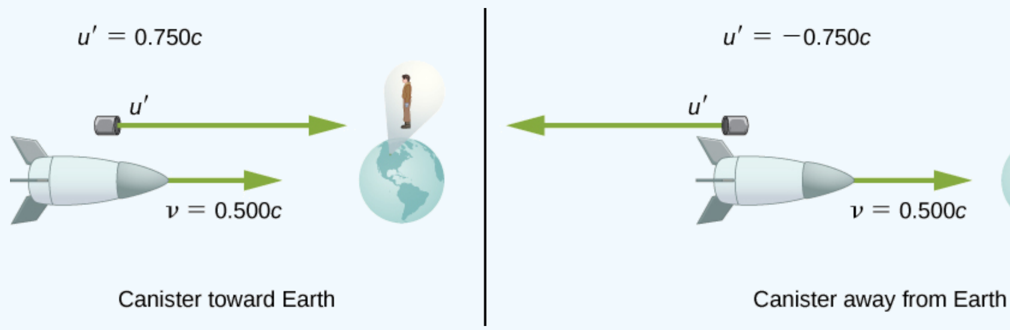


Figure 1.7.3: A canister is fired at 0.7500c toward Earth or away from Earth.

Strategy

Because the canister and the spaceship are moving at relativistic speeds, we must determine the speed of the canister by an earthbound observer using relativistic velocity addition instead of simple velocity addition.

Solution for (a)

1. Identify the knowns: $v = 0.500c$; $u' = 0.750c$.
2. Identify the unknown: u.
3. Express the answer as an equation: $u = \dfrac{v+u'}{1+\dfrac{vu'}{c^2}}$.
4. Do the calculation:

$$\begin{aligned} u &= \frac{v+u'}{1+\frac{vu'}{c^2}} \\ &= \frac{0.500c+0.750c}{1+\frac{(0.500c)(0.750c)}{c^2}} \\ &= 0.909c. \end{aligned}$$

Solution for (b)

1. Identify the knowns: $v = 0.500c$; $u' = -0.750c$.
2. Identify the unknown: u.
3. Express the answer as an equation: $u = \frac{v+u'}{1+\frac{vu'}{c^2}}$.
4. Do the calculation:

$$\begin{aligned} u &= \frac{v+u'}{1+\frac{vu'}{c^2}} \\ &= \frac{0.500c+(-0.750c)}{1+\frac{(0.500c)(-0.750c)}{c^2}} \\ &= -0.400c. \end{aligned}$$

Significance

The minus sign indicates a velocity away from Earth (in the opposite direction from v), which means the canister is heading toward Earth in part (a) and away in part (b), as expected. But relativistic velocities do not add as simply as they do classically. In part (a), the canister does approach Earth faster, but at less than the vector sum of the velocities, which would give $1.250c$. In part (b), the canister moves away from Earth at a velocity of −0.400c, which is faster than the −0.250c expected classically. The differences in velocities are not even symmetric: In part (a), an observer on Earth sees the canister and the ship moving apart at a speed of 0.409**c**, and at a speed of 0.900**c** in part (b).

Exercise 1.7.1

Distances along a direction perpendicular to the relative motion of the two frames are the same in both frames. Why then are velocities perpendicular to the **x**-direction different in the two frames?

Answer

Although displacements perpendicular to the relative motion are the same in both frames of reference, the time interval between events differ, and differences in dt and dt' lead to different velocities seen from the two frames.

1.8: Doppler Effect for Light

Learning Objectives

By the end of this section, you will be able to:

- Explain the origin of the shift in frequency and wavelength of the observed wavelength when observer and source moved toward or away from each other
- Derive an expression for the relativistic Doppler shift
- Apply the Doppler shift equations to real-world examples

As discussed in the chapter on sound, if a source of sound and a listener are moving farther apart, the listener encounters fewer cycles of a wave in each second, and therefore lower frequency, than if their separation remains constant. For the same reason, the listener detects a higher frequency if the source and listener are getting closer. The resulting Doppler shift in detected frequency occurs for any form of wave. For sound waves, however, the equations for the Doppler shift differ markedly depending on whether it is the source, the observer, or the air, which is moving. Light requires no medium, and the Doppler shift for light traveling in vacuum depends only on the relative speed of the observer and source.

The Relativistic Doppler Effect

Suppose an observer in S sees light from a source in S' moving away at velocity v (Figure 1.8.1). The wavelength of the light could be measured within S' — for example, by using a mirror to set up standing waves and measuring the distance between nodes. These distances are proper lengths with S' as their rest frame, and change by a factor $\sqrt{1-v^2/c^2}$ when measured in the observer's frame S, where the ruler measuring the wavelength in S' is seen as moving.

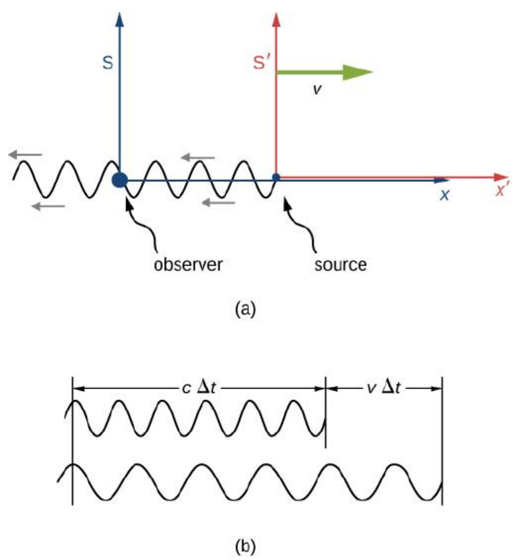


Figure 1.8.1: (a) When a light wave is emitted by a source fixed in the moving inertial frame S', the observer in S sees the wavelength measured in S'. to be shorter by a factor $\sqrt{1-v^2/c^2}$. (b) Because the observer sees the source moving away within S, the wave pattern reaching the observer in S is also stretched by the factor $(c\Delta t+v\Delta t)/(c\Delta t)=1+v/c$.

If the source were stationary in **S**, the observer would see a length cΔt of the wave pattern in time Δt. But because of the motion of S' relative to **S**, considered solely within **S**, the observer sees the wave pattern, and therefore the wavelength, stretched out by a factor of

$$\frac{c\Delta t_{period}+v\Delta t_{period}}{c\Delta t_{period}}=1+\frac{v}{c}$$

as illustrated in (b) of Figure 1.8.1. The overall increase from both effects gives

$$\begin{aligned}\lambda_{obs} &= \lambda_{src}\left(1+\frac{v}{c}\right)\sqrt{\frac{1}{1-\frac{v^2}{c^2}}} \\ &= \lambda_{src}\left(1+\frac{v}{c}\right)\sqrt{\frac{1}{\left(1+\frac{v}{c}\right)\left(1-\frac{v}{c}\right)}} \\ &= \lambda_{src}\sqrt{\frac{\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)}}\end{aligned}$$

where λ_{src} is the wavelength of the light seen by the source in S' and λ_{obs} is the wavelength that the observer detects within **S**.

Red Shifts and Blue Shifts

The observed wavelength λ_{obs} of electromagnetic radiation is longer (called a "red shift") than that emitted by the source when the source moves away from the observer. Similarly, the wavelength is shorter (called a "blue shift") when the source moves toward the observer. The amount of change is determined by

$$\lambda_{obs} = \lambda_s\sqrt{\frac{\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)}}$$

where λ_s is the wavelength in the frame of reference of the source, and v is the relative velocity of the two frames S and S'. The velocity v is positive for motion away from an observer and negative for motion toward an observer. In terms of source frequency and observed frequency, this equation can be written as

$$f_{obs} = f_s\sqrt{\frac{\left(1-\frac{v}{c}\right)}{\left(1+\frac{v}{c}\right)}} \tag{1.8.1}$$

Notice that the signs are different from those of the wavelength equation.

Example 1.8.1: Calculating a Doppler Shift

Suppose a galaxy is moving away from Earth at a speed 0.825**c**. It emits radio waves with a wavelength of

0.525 m. What wavelength would we detect on Earth?

Strategy

Because the galaxy is moving at a relativistic speed, we must determine the Doppler shift of the radio waves using the relativistic Doppler shift instead of the classical Doppler shift.

Solution

1. Identify the knowns: $u = 0.825c$; $\lambda_s = 0.525\,m$.
2. Identify the unknown: λ_{obs}.
3. Express the answer as an equation:

$$\lambda_{obs} = \lambda_s\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}.$$

4. Do the calculation:

$$\begin{aligned}\lambda_{obs} &= \lambda_s\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \\ &= (0.525\,m)\sqrt{\frac{1+\frac{0.825c}{c}}{1-\frac{0.825c}{c}}} \\ &= 1.70\,m.\end{aligned}$$

Significance

Because the galaxy is moving away from Earth, we expect the wavelengths of radiation it emits to be redshifted. The wavelength we calculated is 1.70 m, which is redshifted from the original wavelength of 0.525 m. You will see in Particle Physics and Cosmology that detecting redshifted radiation led to present-day understanding of the origin and evolution of the universe.

❓ Exercise 1.8.1

Suppose a space probe moves away from Earth at a speed 0.350**c**. It sends a radio-wave message back to Earth at a frequency of 1.50 GHz. At what frequency is the message received on Earth?

Solution

We can substitute the data directly into the equation for relativistic Doppler frequency (Equation 1.8.1):

$$\begin{aligned} f_{obs} &= f_s \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \\ &= (1.50\,GHz)\sqrt{\frac{1-\frac{0.350c}{c}}{1+\frac{0.350c}{c}}} \\ &= 1.04\,GHz. \end{aligned}$$

The relativistic Doppler effect has applications ranging from Doppler radar storm monitoring to providing information on the motion and distance of stars. We describe some of these applications in the exercises.

1.9: Relativistic Momentum

Learning Objectives

By the end of this section, you will be able to:

- Define relativistic momentum in terms of mass and velocity
- Show how relativistic momentum relates to classical momentum
- Show how conservation of relativistic momentum limits objects with mass to speeds less than c

Momentum is a central concept in physics. The broadest form of Newton's second law is stated in terms of momentum. Momentum is conserved whenever the net external force on a system is zero. This makes momentum conservation a fundamental tool for analyzing collisions (Figure 1.9.1). Much of what we know about subatomic structure comes from the analysis of collisions of accelerator-produced relativistic particles, and momentum conservation plays a crucial role in this analysis.

Figure 1.9.1: Momentum is an important concept for these football players from the University of California at Berkeley and the University of California at Davis. A player with the same velocity but greater mass collides with greater impact because his momentum is greater. For objects moving at relativistic speeds, the effect is even greater.

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Does the law of conservation of momentum survive this requirement at high velocities? It can be shown that the momentum calculated as merely $\vec{p} = m\frac{d\vec{x}}{dt}$, even if it is conserved in one frame of reference, may not be conserved in another after applying the Lorentz transformation to the velocities. The correct equation for momentum can be shown, instead, to be the classical expression in terms of the increment dτ of proper time of the particle, observed in the particle's rest frame:

$$\begin{aligned} \vec{p} &= m\frac{d\vec{x}}{d\tau} = m\frac{d\vec{x}}{dt}\frac{dt}{d\tau} \\ &= m\frac{d\vec{x}}{dt}\frac{1}{\sqrt{1-u^2/c^2}} \\ &= \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \\ &= \gamma m\vec{u}. \end{aligned}$$

Definition: Relativistic Momentum and Rest Mass

Relativistic momentum $\vec{p}$ is classical momentum multiplied by the relativistic factor γ:

$$\vec{p} = \gamma m \vec{u} \tag{1.9.1}$$

where m is the **rest mass** of the object, $\vec{u}$ is its velocity relative to an observer, and γ is the **relativistic factor**:

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{1.9.2}$$

Note that we use u for velocity here to distinguish it from relative velocity v between observers. The factor γ that occurs here has the same form as the previous relativistic factor γ except that it is now in terms of the velocity of the particle u instead of the relative velocity **v** of two frames of reference.

With **p** expressed in this way, total momentump_{tot} is conserved whenever the net external force is zero, just as in classical physics. Again we see that the relativistic quantity becomes virtually the same as the classical quantity at low velocities, where u/c is small and γ is very nearly equal to 1. Relativistic momentum has the same intuitive role as classical momentum. It is greatest for large masses moving at high velocities, but because of the factor γ, relativistic momentum approaches infinity as u approaches c (Figure 1.9.2). This is another indication that an object with mass cannot reach the speed of light. If it did, its momentum would become infinite—an unreasonable value.

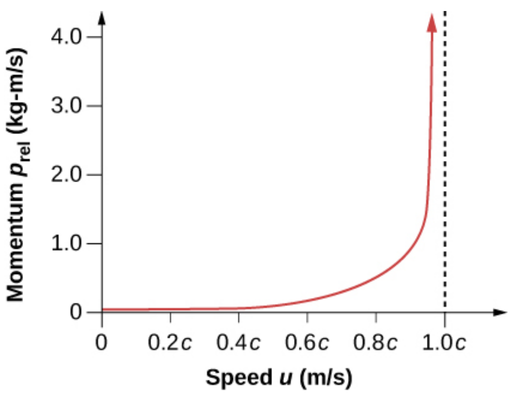


Figure 1.9.2: Relativistic momentum approaches infinity as the velocity of an object approaches the speed of light.

Mass vs. Rest mass

The relativistically correct definition of momentum (Equation 1.9.1) is sometimes taken to imply that mass varies with velocity: $m_{var} = \gamma m$, particularly in older textbooks. However, note that m is the mass of the object as measured by a person at rest relative to the object. Thus, m is defined to be the rest mass, which could be measured at rest, perhaps using gravity. When a mass is moving relative to an observer, the only way that its mass can be determined is through collisions or other means involving momentum. Because the mass of a moving object cannot be determined independently of momentum, the only meaningful mass is rest mass. Therefore, when we use the term "mass," assume it to be identical to "rest mass."

Relativistic momentum is defined in such a way that conservation of momentum holds in all inertial frames. Whenever the net external force on a system is zero, relativistic momentum is conserved, just as is the case for classical momentum. This has been verified in numerous experiments.

Exercise 1.9.1

What is the momentum of an electron traveling at a speed $0.985c$? The rest mass of the electron is $9.11 \times 10^{-31} kg$.

Answer

Substitute the data into Equation 1.9.1:

$$
\begin{aligned}
p &= \gamma m u \\
&= \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} \\
&= \frac{(9.11\times 10^{-31}\,kg)(0.985)(3.00\times 10^{8}\,m/s)}{\sqrt{1-\frac{(0.985c)^2}{c^2}}} \\
&= 1.56\times 10^{-21}\,kg-m/s.
\end{aligned}
$$

1.10: Relativistic Energy

Learning Objectives

By the end of this section, you will be able to:

- Explain how the work-energy theorem leads to an expression for the relativistic kinetic energy of an object
- Show how the relativistic energy relates to the classical kinetic energy, and sets a limit on the speed of any object with mass
- Describe how the total energy of a particle is related to its mass and velocity
- Explain how relativity relates to energy-mass equivalence, and some of the practical implications of energy-mass equivalence

The tokamak in Figure $1.10.1$ is a form of experimental fusion reactor, which can change mass to energy. Nuclear reactors are proof of the relationship between energy and matter.

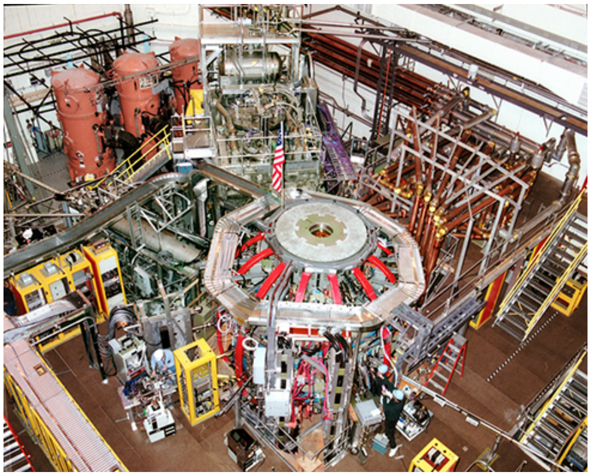


Figure 1.10.1: The National Spherical Torus Experiment (NSTX) is a fusion reactor in which hydrogen isotopes undergo fusion to produce helium. In this process, a relatively small mass of fuel is converted into a large amount of energy. (credit: Princeton Plasma Physics Laboratory)

Conservation of energy is one of the most important laws in physics. Not only does energy have many important forms, but each form can be converted to any other. We know that classically, the total amount of energy in a system remains constant. Relativistically, energy is still conserved, but energy-mass equivalence must now be taken into account, for example, in the reactions that occur within a nuclear reactor. Relativistic energy is intentionally defined so that it is conserved in all inertial frames, just as is the case for relativistic momentum. As a consequence, several fundamental quantities are related in ways not known in classical physics. All of these relationships have been verified by experimental results and have fundamental consequences. The altered definition of energy contains some of the most fundamental and spectacular new insights into nature in recent history.

Kinetic Energy and the Ultimate Speed Limit

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Einstein showed that the law of conservation of energy of a particle is valid relativistically, but for energy expressed in terms of velocity and mass in a way consistent with relativity. Consider first the relativistic expression for the kinetic energy. We again use u for velocity to distinguish it from relative velocity v between observers. Classically, kinetic energy is related to mass and speed by the familiar expression

$$K = \frac{1}{2}mu^2. \tag{1.10.1}$$

The corresponding relativistic expression for kinetic energy can be obtained from the work-energy theorem. This theorem states that the net work on a system goes into kinetic energy. Specifically, if a force, expressed as

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d(\gamma\vec{u})}{dt} \tag{1.10.2}$$

accelerates a particle from rest to its final velocity, the work done on the particle should be equal to its final kinetic energy. In mathematical form, for one-dimensional motion:

$$\begin{aligned} K &= \int F dx = \int m \frac{d}{dt}(\gamma u) dx \\ &= m \int \frac{d(\gamma u)}{dt} \frac{dx}{dt} dt \\ &= m \int u \frac{d}{dt}\left(\frac{u}{\sqrt{1-(u/c)^2}}\right) dt. \end{aligned}$$

Integrate this by parts to obtain

$$\begin{aligned} K &= \left.\frac{mu^2}{\sqrt{1-(u/c)^2}}\right|_0^u - m \int \frac{u}{\sqrt{1-(u/c)^2}} \frac{du}{dt} dt \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} - m \int \frac{u}{\sqrt{1-(u/c)^2}} du \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} - mc^2 \left(\sqrt{1-(u/c)^2}\right)\Big|_0^u \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} + \frac{mu^2}{\sqrt{1-(u/c)^2}} - mc^2 \\ &= mc^2\left[\frac{(u^2/c^2)+1-(u^2/c^2)}{\sqrt{1-(u/c)^2}}\right] - mc^2 \\ &= \frac{mc^2}{\sqrt{1-(u/c)^2}} - mc^2. \end{aligned}$$

Therefore, the **relativistic kinetic energy** of any particle of mass m is

$$K_{rel} = (\gamma - 1)mc^2. \tag{1.10.3}$$

When an object is motionless, its speed is $u = 0$ and

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 1 \tag{1.10.4}$$

so that $K_{rel} = 0$ at rest, as expected. However, the expression for relativistic kinetic energy (such as total energy and rest energy) does not look much like the classical $\frac{1}{2}mu^2$. To show that the expression for K_{rel} reduces to the classical expression for kinetic energy at low speeds, we use the binomial expansion to obtain an approximation for $(1+\varepsilon)^n$ valid for small ε:

$$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2!}\varepsilon^2 + \frac{n(n-1)(n-2)}{3!}\varepsilon^3 + \cdots \approx 1 + n\varepsilon \tag{1.10.5}$$

by neglecting the very small terms in $\varepsilon^2$and higher powers of ε. Choosing $\varepsilon = -u^2/c^2$ and $n = -\frac{1}{2}$ leads to the conclusion that γ at nonrelativistic speeds, where $\varepsilon = u/c$ is small, satisfies

$$\gamma = (1 - u^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\left(\frac{u^2}{c^2}\right). \tag{1.10.6}$$

A binomial expansion is a way of expressing an algebraic quantity as a sum of an infinite series of terms. In some cases, as in the limit of small speed here, most terms are very small. Thus, the expression derived here for γ is not exact, but it is a very accurate approximation. Therefore, at low speed:

$$\gamma - 1 \approx \frac{1}{2}\left(\frac{u^2}{c^2}\right). \tag{1.10.7}$$

Entering this into the expression for relativistic kinetic energy (Equation 1.10.3) gives

$$\begin{aligned} K_{rel} &\approx \left[\frac{1}{2}\left(\frac{u^2}{c^2}\right)\right] mc^2 \\ &\approx \frac{1}{2}mu^2 \\ &\approx K_{class}. \end{aligned}$$

That is, relativistic kinetic energy becomes the same as classical kinetic energy when $u \ll c$.

It is even more interesting to investigate what happens to kinetic energy when the speed of an object approaches the speed of light. We know that γ becomes infinite as u approaches c, so that K_{rel} also becomes infinite as the velocity approaches the speed of light (Figure 1.10.2). The increase in K_{rel} is far larger than in K_{class} as v approaches c. An infinite amount of work (and, hence, an infinite amount of energy input) is required to accelerate a mass to the speed of light.

> *No object with mass can attain the speed of light.*

The speed of light is the ultimate speed limit for any particle having mass. All of this is consistent with the fact that velocities less than **c** always add to less than c. Both the relativistic form for kinetic energy and the ultimate speed limit being c have been confirmed in detail in numerous experiments. No matter how much energy is put into accelerating a mass, its velocity can only approach—not reach—the speed of light.

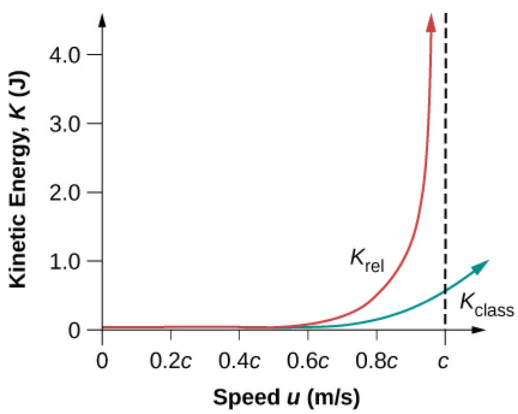


Figure 1.10.2:This graph of K_{rel} versus velocity shows how kinetic energy increases without bound as velocity approaches the speed of light. Also shown is K_{class}, the classical kinetic energy.

Example 1.10.1: Comparing Kinetic Energy

An electron has a velocity $v = 0.990c$.

a. Calculate the kinetic energy in MeV of the electron.
b. Compare this with the classical value for kinetic energy at this velocity. (The mass of an electron is $9.11 \times 10^{-31} kg$.)

Strategy

The expression for relativistic kinetic energy is always correct, but for (a), it must be used because the velocity is highly relativistic (close to c). First, we calculate the relativistic factor γ, and then use it to determine the relativistic kinetic energy. For (b), we calculate the classical kinetic energy (which would be close to the relativistic value if v were less than a few percent of c) and see that it is not the same.

Solution for (a)

1. Identify the knowns: $v = 0.990c$; $m = 9.11 \times 10^{-31} kg$
2. Identify the unknown: K_{rel}.
3. Express the answer as an equation: $K_{rel} = (\gamma - 1)mc^2$ with $\gamma = \dfrac{1}{\sqrt{1 - u^2/c^2}}$.

4. Do the calculation. First calculate γ. Keep extra digits because this is an intermediate calculation:

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-u^2/c^2}} \\ &= \frac{1}{\sqrt{1-\frac{(0.990c)^2}{c^2}}} \\ &= 7.0888.\end{aligned}$$

Now use this value to calculate the kinetic energy (Equatoin 1.10.3):

$$\begin{aligned}K_{rel} &= (\gamma-1)mc^2 \\ &= (7.0888-1)(9.11\times10^{-31}\,kg)(3.00\times10^8\,m/s^2) \\ &= 4.9922\times10^{-13}\,J\end{aligned}$$

5. Convert units:

$$\begin{aligned}K_{rel} &= (4.9922\times10^{-13}\,J)\left(\frac{1\,MeV}{1.60\times10^{-13}J}\right) \\ &= 3.12\,MeV.\end{aligned}$$

Solution for (b)

1. List the knowns: $v = 0.990c$; $m = 9.11\times10^{-31}kg$.
2. List the unknown: K_{rel}
3. Express the answer as an equation:
4. Do the calculation:

$$\begin{aligned}K_{class} &= \frac{1}{2}mu^2 \\ &= \frac{1}{2}(9.11\times10^{-31}kg)(0.990)^2(3.00\times10^8\,m/s)^2 \\ &= 4.0179\times10^{-14}J.\end{aligned}$$

5. Convert units:

$$\begin{aligned}K_{class} &= 4.0179\times10^{-14}J\left(\frac{1\,MeV}{1.60\times10^{-13}J}\right) \\ &= 0.251\,MeV.\end{aligned}$$

Significance

As might be expected, because the velocity is 99.0% of the speed of light, the classical kinetic energy differs significantly from the correct relativistic value. Note also that the classical value is much smaller than the relativistic value. In fact, $K_{rel}/K_{class} = 12.4$ in this case. This illustrates how difficult it is to get a mass moving close to the speed of light. Much more energy is needed than predicted classically. Ever-increasing amounts of energy are needed to get the velocity of a mass a little closer to that of light. An energy of 3 MeV is a very small amount for an electron, and it can be achieved with present-day particle accelerators. SLAC, for example, can accelerate electrons to over $50\times10^9 eV = 50,000\,MeV$.

Is there any point in getting **v** a little closer to **c** than 99.0% or 99.9%? The answer is yes. We learn a great deal by doing this. The energy that goes into a high-velocity mass can be converted into any other form, including into entirely new particles. In the Large Hadron Collider in Figure 1.10.1, charged particles are accelerated before entering the ring-like structure. There, two beams of particles are accelerated to their final speed of about 99.7% the speed of light in opposite directions, and made to collide, producing totally new species of particles. Most of what we know about the substructure of matter and the collection of exotic short-lived particles in nature has been learned this way. Patterns in the characteristics of these previously unknown particles hint at a basic substructure for all matter. These particles and some of their characteristics will be discussed in a later chapter on particle physics.

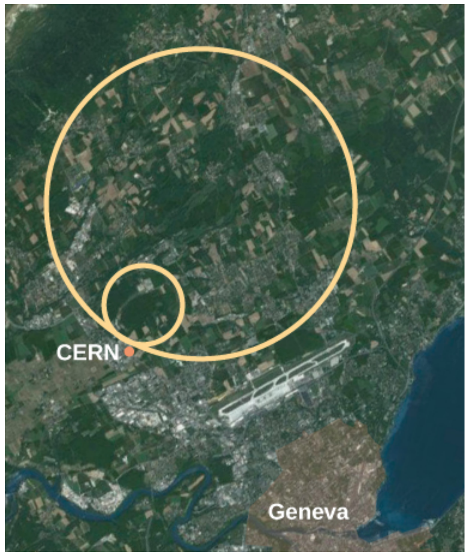


Figure 1.10.3: The European Organization for Nuclear Research (called CERN after its French name) operates the largest particle accelerator in the world, straddling the border between France and Switzerland.

Total Relativistic Energy

The expression for kinetic energy can be rearranged to:

$$\begin{aligned} E &= \frac{mc^2}{\sqrt{1-u^2/c^2}} \\ &= K + mc^2. \end{aligned}$$

Einstein argued in a separate article, also later published in 1905, that if the energy of a particle changes by ΔE, its mass changes by $\Delta m = \Delta E/C^2$. Abundant experimental evidence since then confirms that mc^2 corresponds to the energy that the particle of mass m has when at rest. For example, when a neutral pion of mass m at rest decays into two photons, the photons have zero mass but are observed to have total energy corresponding to mc^2 for the pion. Similarly, when a particle of mass m decays into two or more particles with smaller total mass, the observed kinetic energy imparted to the products of the decay corresponds to the decrease in mass. Thus, E is the total relativistic energy of the particle, and mc^2 is its rest energy.

Definition: Total Energy

Total energy (E) of a particle is

$$E = \gamma mc^2 \tag{1.10.8}$$

where m is mass, c is the speed of light, $\gamma = \dfrac{1}{\sqrt{1-\dfrac{u^2}{c^2}}}$, and u is the velocity of the mass relative to an observer.

Definition: Rest Enerey

Rest energy of an object is

$$E_0 = mc^2. \tag{1.10.9}$$

Equation 1.10.9 is the correct form of Einstein's most famous equation, which for the first time showed that energy is related to the mass of an object at rest. For example, if energy is stored in the object, its rest mass increases. This also implies that mass can be destroyed to release energy. The implications of these first two equations regarding relativistic energy are so broad that they were not completely recognized for some years after Einstein published them in 1905, nor was the experimental proof that they are correct widely recognized at first. Einstein, it should be noted, did understand and describe the meanings and implications of his theory.

Example 1.10.2: Calculating Rest Energy

Calculate the rest energy of a 1.00-g mass.

Strategy

One gram is a small mass—less than one-half the mass of a penny. We can multiply this mass, in SI units, by the speed of light squared to find the equivalent rest energy.

Solution

1. Identify the knowns: $m = 1.00 \times 10^{-3} kg$; $c = 3.00 \times 10^{8} m/s$.
2. Identify the unknown: E_0.
3. Express the answer as an equation: $E_0 = mc^2$.
4. Do the calculation:

$$E_0 = mc^2 = (1.00 \times 10^{-3} kg)(3.00 \times 10^{8} m/s)^2 = 9.00 \times 10^{13} kg \cdot m^2/s^2.$$

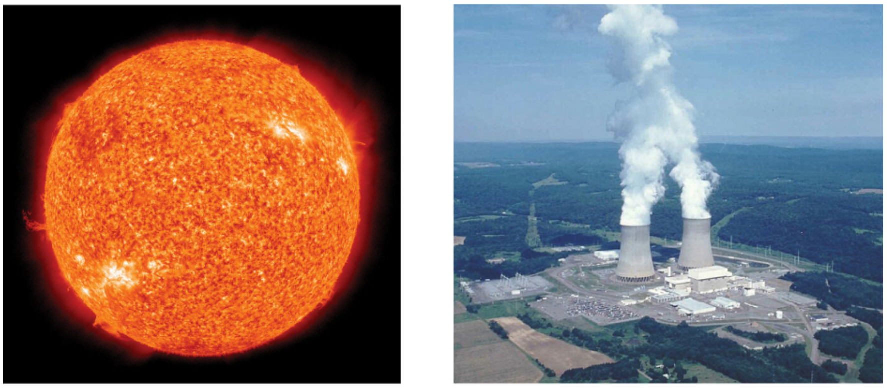
5. Convert units. Noting that $1\ kg \cdot m^2/s^2 = 1\ J$, we see the rest energy is:

$$E_0 = 9.00 \times 10^{13}\ J.$$

Significance

This is an enormous amount of energy for a 1.00-g mass. Rest energy is large because the speed of light **c** is a large number and c^2 is a very large number, so that mc^2 is huge for any macroscopic mass. The $9.00 \times 10^{13} J$ rest mass energy for 1.00 g is about twice the energy released by the Hiroshima atomic bomb and about 10,000 times the kinetic energy of a large aircraft carrier.

Today, the practical applications of *the conversion of mass into another form of energy*, such as in nuclear weapons and nuclear power plants, are well known. But examples also existed when Einstein first proposed the correct form of relativistic energy, and he did describe some of them. Nuclear radiation had been discovered in the previous decade, and it had been a mystery as to where its energy originated. The explanation was that, in some nuclear processes, a small amount of mass is destroyed and energy is released and carried by nuclear radiation. But the amount of mass destroyed is so small that it is difficult to detect that any is missing. Although Einstein proposed this as the source of energy in the radioactive salts then being studied, it was many years before there was broad recognition that mass could be and, in fact, commonly is, converted to energy (Figure 1.10.4).



(a) (b)

Figure 1.10.4: (a) The sun and (b) the Susquehanna Steam Electric Station both convert mass into energy—the sun via nuclear fusion, and the electric station via nuclear fission. (credit a: modification of work by NASA; credit b: modification of work by “ChNPP”/Wikimedia Commons)

Because of the relationship of rest energy to mass, we now consider mass to be a form of energy rather than something separate. There had not been even a hint of this prior to Einstein’s work. Energy-mass equivalence is now known to be the source of the sun’s energy, the energy of nuclear decay, and even one of the sources of energy keeping Earth’s interior hot.

Stored Energy and Potential Energy

What happens to energy stored in an object at rest, such as the energy put into a battery by charging it, or the energy stored in a toy gun's compressed spring? The energy input becomes part of the total energy of the object and thus increases its rest mass. All stored and potential energy becomes mass in a system. In seeming contradiction, the principle of conservation of mass (meaning total mass is constant) was one of the great laws verified by nineteenth-century science. Why was it not noticed to be incorrect? The following example helps answer this question.

Example 1.10.3: Calculating Rest Mass

A car battery is rated to be able to move 600 ampere-hours $(A \cdot h)$ of charge at 12.0 V.

a. Calculate the increase in rest mass of such a battery when it is taken from being fully depleted to being fully charged, assuming none of the chemical reactants enter or leave the battery.
b. What percent increase is this, given that the battery's mass is 20.0 kg?

Strategy

In part (a), we first must find the energy stored as chemical energy E_{batt} in the battery, which equals the electrical energy the battery can provide. Because $E_{batt} = qV$, we have to calculate the charge q in $600\ A \cdot h$, which is the product of the current I and the time t. We then multiply the result by 12.0 V. We can then calculate the battery's increase in mass using $E_{batt} = (\Delta m)c^2$. Part (b) is a simple ratio converted into a percentage.

Solution for (a)

1. Identify the knowns:

$$I \cdot t = 600\ A \cdot h;\ V = 12.0\ V;\ c = 3.00 \times 10^8\ m/s.$$

2. Identify the unknown: Δm.
3. Express the answer as an equation:

$$\begin{aligned} E_{batt} &= (\Delta m)c^2 \\ \Delta m &= \frac{E_{batt}}{c^2} \\ &= \frac{qV}{c^2} \\ &= \frac{(It)V}{c^2}. \end{aligned}$$

4. Do the calculation:

$$\Delta m = \frac{(600\ A \cdot h)(12.0\ V)}{(3.00 \times 10^8)^2}.$$

5. Write amperes A as coulombs per second (C/s), and convert hours into seconds:

$$\begin{aligned} \Delta m &= \frac{(600\ C/s \cdot h)\left(\frac{3600\ s}{1\ h}\right)(12.0\ J/C)}{(3.00 \times 10^8\ m/s)^2} \\ &= 2.88 \times 10^{-10}\ kg. \end{aligned}$$

where we have used the conversion $1\ kg \cdot m^2/s^2 = 1\ J.$.

Solution for (b)

For part (b):

1. Identify the knowns: $\delta m = 2.88 \times 10^{-10} kg$; $m = 20.0\ kg$.
2. Identify the unknown: % change.
3. Express the answer as an equation:

$$\% \, increase = \frac{\delta m}{m} \times 100\%. \tag{1.10.10}$$

4. Do the calculation:

$$\begin{aligned} \% \, increase &= \frac{\Delta m}{m} \times 100\% \\ &= \frac{2.88 \times 10^{-10} \, kg}{20.0 \, kg} \times 100\% \\ &= 1.44 \times 10^{-9} \% \end{aligned}$$

Significance

Both the actual increase in mass and the percent increase are very small, because energy is divided by c^2, a very large number. We would have to be able to measure the mass of the battery to a precision of a billionth of a percent, or 1 part in 10^{11}, to notice this increase. It is no wonder that the mass variation is not readily observed. In fact, this change in mass is so small that we may question how anyone could verify that it is real. The answer is found in nuclear processes in which the percentage of mass destroyed is large enough to be measured accurately. The mass of the fuel of a nuclear reactor, for example, is measurably smaller when its energy has been used. In that case, stored energy has been released (converted mostly into thermal energy to power electric generators) and the rest mass has decreased. A decrease in mass also occurs from using the energy stored in a battery, except that the stored energy is much greater in nuclear processes, making the change in mass measurable in practice as well as in theory.

Relativistic Energy and Momentum

We know classically that kinetic energy and momentum are related to each other, because:

$$K_{class} = \frac{p^2}{2m} = \frac{(mu)^2}{2m} = \frac{1}{2} mu^2. \tag{1.10.11}$$

Relativistically, we can obtain a relationship between energy and momentum by algebraically manipulating their defining equations. This yields:

$$E^2 = (pc)^2 + (mc^2)^2, \tag{1.10.12}$$

where E is the relativistic total energy,

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}} \tag{1.10.13}$$

and p is the relativistic momentum. This relationship between relativistic energy and relativistic momentum is more complicated than the classical version, but we can gain some interesting new insights by examining it. First, total energy is related to momentum and rest mass. At rest, momentum is zero, and the equation gives the total energy to be the rest energy mc^2 (so this equation is consistent with the discussion of rest energy above). However, as the mass is accelerated, its momentum p increases, thus increasing the total energy. At sufficiently high velocities, the rest energy term $(mc^2)^2$ becomes negligible compared with the momentum term $(pc)^2$; thus, $E = pc$ at extremely relativistic velocities.

If we consider momentum p to be distinct from mass, we can determine the implications of the equation

$$E^2 = (pc)^2 + (mc^2)^2, \tag{1.10.14}$$

for a particle that has no mass. If we take m to be zero in this equation, then $E = pc, or p = E/c$. Massless particles have this momentum. There are several massless particles found in nature, including photons (which are packets of electromagnetic radiation). Another implication is that a massless particle must travel at speed **c** and only at speed **c**. It is beyond the scope of this text to examine the relationship in the equation $E^2 = (pc)^2 + (mc^2)^2$ in detail, but you can see that the relationship has important implications in special relativity.

Exercise 1.10.1

What is the kinetic energy of an electron if its speed is $0.992c$?

Answer

$$\begin{aligned} K_{rel} &= (\gamma - 1)mc^2 = \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right) mc^2 \\ &= \left(\frac{1}{\sqrt{1 - \frac{(0.992c)^2}{c^2}}} - 1 \right) (9.11 \times 10^{-31}\, kg)(3.00 \times 10^8\, m/s)^2 \\ &= 5.67 \times 10^{-13}\, J \end{aligned}$$

Contributors and Attributions

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1.A: Relativity (Answers)

Check Your Understanding

5.1. Special relativity applies only to objects moving at constant velocity, whereas general relativity applies to objects that undergo acceleration.

5.2. $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.650c)^2}{c^2}}} = 1.32$

5.3. a. $\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.10\times10^{-8}s}{\sqrt{1-\frac{(1.90\times10^8 m/s)^2}{(3.00\times10^8 m/s)^2}}} = 2.71\times10^{-8}s$.

b. Only the relative speed of the two spacecraft matters because there is no absolute motion through space. The signal is emitted from a fixed location in the frame of reference of A, so the proper time interval of its emission is $\tau = 1.00s$. The duration of the signal measured from frame of reference B is then

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1.00s}{\sqrt{1-\frac{(4.00\times10^7 m/s)^2}{(3.00\times10^8 m/s)^2}}} = 1.01s\ .$$

5.4. $L = L_0\sqrt{1-\frac{v^2}{c^2}} = (2.50km)\sqrt{1-\frac{(0.750c)^2}{c^2}} = 1.65km$

5.5. Start with the definition of the proper time increment:

$$d\tau = \sqrt{-(ds)^2/c^2} = \sqrt{dt^2-(dx^2+dx^2+dx^2)/c^2}\ .$$

where (dx, dy, dx, cdt) are measured in the inertial frame of an observer who does not necessarily see that particle at rest. This therefore becomes

$$\begin{aligned} d_\tau &= \sqrt{-(ds)^2/c^2} = \sqrt{dt^2-[(dx)^2+(dy)^2+(dz)^2]/c^2} \\ &= dt\sqrt{1-[(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2]/c^2} \\ &= dt\sqrt{1-v^2/c^2} \end{aligned}$$

$$dt = \gamma d\tau.$$

5.6. Although displacements perpendicular to the relative motion are the same in both frames of reference, the time interval between events differ, and differences in **dt** and dt' lead to different velocities seen from the two frames.

5.7. We can substitute the data directly into the equation for relativistic Doppler frequency:

$$f_{obs} = f_s\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = (1.50GHz)\sqrt{\frac{1-\frac{0.350c}{c}}{1+\frac{0.350c}{c}}} = 1.04GHz.$$

5.8. Substitute the data into the given equation:

$$p = \gamma mu = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{(9.11\times10^{-31}kg)(0.985)(3.00\times10^8 m/s)}{\sqrt{1-\frac{(0.985c)^2}{c^2}}} = 1.56\times10^{-21}kg-m/s\ .$$

5.9.

$$K_{rel} = (\gamma-1)mc^2 = (\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}mc^2 = (\frac{1}{\sqrt{1-\frac{(0.992c)^2}{c^2}}-1}(9.11\times10^{-31}kg)(3.00\times10^8 m/s)^2 = 5.67\times10^{-13}J$$

Conceptual Questions

1. the second postulate, involving the speed of light; classical physics already included the idea that the laws of mechanics, at least, were the same in all inertial frames, but the velocity of a light pulse was different in different frames moving with respect to each other

3. yes, provided the plane is flying at constant velocity relative to the Earth; in that case, an object with no force acting on it within the plane has no change in velocity relative to the plane and no change in velocity relative to the Earth; both the plane and the ground are inertial frames for describing the motion of the object

5. The observer moving with the process sees its interval of proper time, which is the shortest seen by any observer.

7. The length of an object is greatest to an observer who is moving with the object, and therefore measures its proper length.

9. a. No, not within the astronaut's own frame of reference.

b. He sees Earth clocks to be in their rest frame moving by him, and therefore sees them slowed.

c. No, not within the astronaut's own frame of reference.

d. Yes, he measures the distance between the two stars to be shorter.

e. The two observers agree on their relative speed.

11. There is no measured change in wavelength or frequency in this case. The relativistic Doppler effect depends only on the relative velocity of the source and the observer, not any speed relative to a medium for the light waves.

13. It shows that the stars are getting more distant from Earth, that the universe is expanding, and doing so at an accelerating rate, with greater velocity for more distant stars.]

15. Yes. This can happen if the external force is balanced by other externally applied forces, so that the net external force is zero.

17. Because it loses thermal energy, which is the kinetic energy of the random motion of its constituent particles, its mass decreases by an extremely small amount, as described by energy-mass equivalence.

19. Yes, in principle there would be a similar effect on mass for any decrease in energy, but the change would be so small for the energy changes in a chemical reaction that it would be undetectable in practice.

21. Not according to special relativity. Nothing with mass can attain the speed of light.

Problems

23. a. 1.0328;

b. 1.15

25. $5.96 \times 10^{-8} s$

27. 0.800c

29. 0.140c

31. 48.6 m

33. Using the values given in Example 5.3:

a. 1.39 km;

b. 0.433 km;

c. 0.433 km

35. a. 10.0c;

b. The resulting speed of the canister is greater than c, an impossibility.

c. It is unreasonable to assume that the canister will move toward the earth at 1.20**c.**

37. The angle α approaches 45°, and the $t' -$ and $x' - axes$ rotate toward the edge of the light cone.

39. 15 m/s east

41. 32 m/s

43. a. The second ball approaches with velocity $-v$ and comes to rest while the other ball continues with velocity $-v$;

b. This conserves momentum.

45. a. $t_1' = 0; x_1' = 0$;

$t_2' = \tau; x_2' = 0$;

b. $t_1' = 0; x_1' = 0$;

$t_2' = \frac{\tau}{\sqrt{1 - v^2/c^2}}; x_2' = \frac{-v\tau}{\sqrt{1 - v^2/c^2}}$

47. 0.615c

49. 0.696c

51. (Proof)

53. $4.09 \times 10^{-19} kg \cdot m/s$

55. a. $3.000000015 \times 10^{13} kg \cdot m/s$;

b. 1.000000005

57. $2.988 \times 10^8 m/s$

59. 0.512 MeV according to the number of significant figures stated. The exact value is closer to 0.511 MeV.

61. $2.3 \times 10^{-30} kg$; to two digits because the difference in rest mass energies is found to two digits

63. a. $1.11 \times 10^{27} kg$;

b. $5.56 \times 10 - 5$

65. a. $7.1 \times 10^{-3} kg$;

b. $7.1 \times 10^{-3} = 7.1 \times 10^{-3}$;

c. $\frac{\Delta m}{m}$ is greater for hydrogen

67. a. 208;

b. 0.999988c; six digits used to show difference from c

69. a. $6.92 \times 10^5 J$;

b. 1.54

71. a. 0.914c;

b. The rest mass energy of an electron is 0.511 MeV, so the kinetic energy is approximately 150% of the rest mass energy. The electron should be traveling close to the speed of light.

Additional Problems

73. a. 0.866c;

b. 0.995c

75. a. 4.303 y to four digits to show any effect;

b. 0.1434 y;

c. $1/\sqrt{(1 - v^2/c^2)} = 29.88$..

77. a. 4.00;

b. $v = 0.867c$

79. a. A sends a radio pulse at each heartbeat to B, who knows their relative velocity and uses the time dilation formula to calculate the proper time interval between heartbeats from the observed signal.

b. $(66 beats/min)\sqrt{1-v^2/c^2} = 57.1$ beats/min

81. a. first photon: $(0,0,0)$ at $t = t'$; second photon:

$$t' = \frac{-vx/c^2}{\sqrt{1-v^2/c^2}} = \frac{-(c/2)(1.00m)/c^2}{\sqrt{0.75}} = \frac{0.577m}{c} = 1.93 \times 10^{-9} s$$

$$x' = \frac{x}{\sqrt{1-v^2/c^2}} = \frac{1.00m}{\sqrt{0.75}}$$ =1.15m\)

b. simultaneous in A, not simultaneous in B

83. $t' = \frac{t - vx/c^2}{\sqrt{1-v^2/c^2}} = \frac{(4.5 \times 10^{-4} \text{ s}) - (0.6c)\left(\frac{150 \times 10^3 \text{ m}}{c^2}\right)}{\sqrt{1-(0.6)^2}}$

$= 1.88 \times 10^{-4}\ s$

$$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}} = \frac{150 \times 10^3 \text{ m} - (0.60)\left(3.00 \times 10^8 \text{ m/s}\right)\left(4.5 \times 10^{-4} \text{ s}\right)}{\sqrt{1-(0.6)^2}}$$

$= 8.6 \times 10^4\ m = 86\ km$

$y = y' = 15\ km$

$z = z' = 1\ km$

85. $\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1-v^2/c^2}}$

$$0 = \frac{\Delta t' + v(500m)/c^2}{\sqrt{1-v^2/c^2}};$$

since $v \ll c$, we can ignore the term v^2/c^2 and find

$$\Delta t' = -\frac{(50m/s)(500m)}{(3.00 \times 10^8 m/s)^2} = -2.78 \times 10^{-13} s$$

The breakdown of Newtonian simultaneity is negligibly small, but not exactly zero, at realistic train speeds of 50 m/s.

87. $\Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1-v^2/c^2}}$

$$0 = \frac{(0.30s) - \frac{(v)(2.0 \times 10^9 m)}{(3.00 \times 10^8 m/s)^2}}{\sqrt{1-v^2/c^2}}$$

$$v = \frac{(0.30s)}{(2.0 \times 10^9 m)}(3.00 \times 10^8 m/s)^2$$

$$v = 1.35 \times 10^7 m/s$$

89. Note that all answers to this problem are reported to five significant figures, to distinguish the results.

a. 0.99947c;

b. $1.2064 \times 10^{11} y$;

c. $1.2058 \times 10^{11} y$

91. a. –0.400c;

b. –0.909c

93. a. 1.65 km/s;

b. Yes, if the speed of light were this small, speeds that we can achieve in everyday life would be larger than 1% of the speed of light and we could observe relativistic effects much more often.

95. 775 MHz

97. a. $1.12 \times 10^{-8} m/s$;

b. The small speed tells us that the mass of a protein is substantially smaller than that of even a tiny amount of macroscopic matter.

99. a. $F = \frac{dp}{dt} = \frac{d}{dt}\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right) = \frac{du}{dt}\left(\frac{m}{\sqrt{1-u^2/c^2}}\right) - \frac{1}{2}\frac{mu^2}{(1-u^2/c^2)^{3/2}} 2\frac{du}{dt} = \frac{m}{(1-u^2/c^2)^{3/2}}\frac{du}{dt}$;

b. $F = \frac{m}{(1-u^2/c^2)^{3/2}}\frac{du}{dt} = \frac{1kg}{(1-(\frac{1}{2})^2)^{3/2}}(1m/s^2) = 1.53N$

101. 90.0 MeV

103. a. $\gamma^2 - 1$;

b. yes

105. 1.07×10^3

107. a. $6.56 \times 10^{-8} kg$;

b. $m = (200L)(1m^3/1000L)(750kg/m^3) = 150kg$ therefore, $\frac{\Delta m}{m} = 4.37 \times 10^{-10}$

109. a. 0.314c;

b. 0.99995c (Five digits used to show difference from c)

111. a. 1.00 kg;

b. This much mass would be measurable, but probably not observable just by looking because it is 0.01% of the total mass.

113. a. $6.06 \times 10^{11} kg/s$;

b. $4.67 \times 10^{10} y$;

c. $4.27 \times 10^9 kg$;

d. 0.32

1.E: Relativity (Exercises)

Conceptual Questions

5.1 Invariance of Physical Laws

1. Which of Einstein’s postulates of special relativity includes a concept that does not fit with the ideas of classical physics? Explain.

2. Is Earth an inertial frame of reference? Is the sun? Justify your response.

3. When you are flying in a commercial jet, it may appear to you that the airplane is stationary and Earth is moving beneath you. Is this point of view valid? Discuss briefly.

5.3 Time Dilation

4. (a) Does motion affect the rate of a clock as measured by an observer moving with it?

(b) Does motion affect how an observer moving relative to a clock measures its rate?

5. To whom does the elapsed time for a process seem to be longer, an observer moving relative to the process or an observer moving with the process? Which observer measures the interval of proper time?

6. (a) How could you travel far into the future of Earth without aging significantly?

(b) Could this method also allow you to travel into the past?

5.4 Length Contraction

7. To whom does an object seem greater in length, an observer moving with the object or an observer moving relative to the object? Which observer measures the object’s proper length?

8. Relativistic effects such as time dilation and length contraction are present for cars and airplanes. Why do these effects seem strange to us?

9. Suppose an astronaut is moving relative to Earth at a significant fraction of the speed of light.

(a) Does he observe the rate of his clocks to have slowed?

(b) What change in the rate of earthbound clocks does he see?

(c) Does his ship seem to him to shorten?

(d) What about the distance between two stars that lie in the direction of his motion? (e) Do he and an earthbound observer agree on his velocity relative to Earth?

5.7 Doppler Effect for Light

10. Explain the meaning of the terms “red shift” and “blue shift” as they relate to the relativistic Doppler effect.

11. What happens to the relativistic Doppler effect when relative velocity is zero? Is this the expected result?

12. Is the relativistic Doppler effect consistent with the classical Doppler effect in the respect that λ_{obs} is larger for motion away?

13. All galaxies farther away than about 50×10^6 **ly** exhibit a red shift in their emitted light that is proportional to distance, with those farther and farther away having progressively greater red shifts. What does this imply, assuming that the only source of red shift is relative motion?

5.8 Relativistic Momentum

14. How does modern relativity modify the law of conservation of momentum?

15. Is it possible for an external force to be acting on a system and relativistic momentum to be conserved? Explain.

5.9 Relativistic Energy

16. How are the classical laws of conservation of energy and conservation of mass modified by modern relativity?

17. What happens to the mass of water in a pot when it cools, assuming no molecules escape or are added? Is this observable in practice? Explain.

18. Consider a thought experiment. You place an expanded balloon of air on weighing scales outside in the early morning. The balloon stays on the scales and you are able to measure changes in its mass. Does the mass of the balloon change as the day progresses? Discuss the difficulties in carrying out this experiment.

19. The mass of the fuel in a nuclear reactor decreases by an observable amount as it puts out energy. Is the same true for the coal and oxygen combined in a conventional power plant? If so, is this observable in practice for the coal and oxygen? Explain.

20. We know that the velocity of an object with mass has an upper limit of c. Is there an upper limit on its momentum? Its energy? Explain.

21. Given the fact that light travels at c , can it have mass? Explain.

22. If you use an Earth-based telescope to project a laser beam onto the moon, you can move the spot across the moon's surface at a velocity greater than the speed of light. Does this violate modern relativity? (Note that light is being sent from the Earth to the moon, not across the surface of the moon.)

Problems

5.3 Time Dilation

23. (a) What is γ if $v = 0.250c$?

(b) If $v = 0.500c$?

24. (a) What is γ if $v = 0.100c$?

(b) If $v = 0.900c$?

25. Particles called π-mesons are produced by accelerator beams. If these particles travel at $2.70 \times 10^8 m/s$ and live $2.60 \times 10^{-8} s$ when at rest relative to an observer, how long do they live as viewed in the laboratory?

26. Suppose a particle called a kaon is created by cosmic radiation striking the atmosphere. It moves by you at $0.980c$, and it lives $1.24 \times 10^{-8} s$ when at rest relative to an observer. How long does it live as you observe it?

27. A neutral π-meson is a particle that can be created by accelerator beams. If one such particle lives $1.40 \times 10^{-16} s$ as measured in the laboratory, and $0.840 \times 10^{-16} s$ when at rest relative to an observer, what is its velocity relative to the laboratory?

28. A neutron lives 900 s when at rest relative to an observer. How fast is the neutron moving relative to an observer who measures its life span to be 2065 s?

29. If relativistic effects are to be less than 1%, then γ must be less than 1.01. At what relative velocity is $\gamma = 1.01$?

30. If relativistic effects are to be less than 3%, then γ must be less than 1.03. At what relative velocity is $\gamma = 1.03$?

5.4 Length Contraction

31. A spaceship, 200 m long as seen on board, moves by the Earth at 0.970c. What is its length as measured by an earthbound observer?

32. How fast would a 6.0 m-long sports car have to be going past you in order for it to appear only 5.5 m long?

33. (a) How far does the muon in Example 5.1 travel according to the earthbound observer?

(b) How far does it travel as viewed by an observer moving with it? Base your calculation on its velocity relative to the Earth and the time it lives (proper time).

(c) Verify that these two distances are related through length contraction $\gamma = 3.20$.

34. (a) How long would the muon in Example 5.1 have lived as observed on Earth if its velocity was $0.0500c$?

(b) How far would it have traveled as observed on Earth?

(c) What distance is this in the muon's frame?

35. Unreasonable Results A spaceship is heading directly toward Earth at a velocity of 0.800c. The astronaut on board claims that he can send a canister toward the Earth at 1.20c relative to Earth.

(a) Calculate the velocity the canister must have relative to the spaceship.

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

5.5 The Lorentz Transformation

36. Describe the following physical occurrences as events, that is, in the form **(x, y, z, t)**:

(a) A postman rings a doorbell of a house precisely at noon.

(b) At the same time as the doorbell is rung, a slice of bread pops out of a toaster that is located 10 m from the door in the east direction from the door.

(c) Ten seconds later, an airplane arrives at the airport, which is 10 km from the door in the east direction and 2 km to the south.

37. Describe what happens to the angle $\alpha = tan(v/c)$, and therefore to the transformed axes in Figure 5.17, as the relative velocity **v** of the S and S' frames of reference approaches **c**.

38. Describe the shape of the world line on a space-time diagram of

(a) an object that remains at rest at a specific position along the **x**-axis;

(b) an object that moves at constant velocity **u** in the **x**-direction;

(c) an object that begins at rest and accelerates at a constant rate of in the positive **x**-direction.

39. A man standing still at a train station watches two boys throwing a baseball in a moving train. Suppose the train is moving east with a constant speed of 20 m/s and one of the boys throws the ball with a speed of 5 m/s with respect to himself toward the other boy, who is 5 m west from him. What is the velocity of the ball as observed by the man on the station?

40. When observed from the sun at a particular instant, Earth and Mars appear to move in opposite directions with speeds 108,000 km/h and 86,871 km/h, respectively. What is the speed of Mars at this instant when observed from Earth?

41. A man is running on a straight road perpendicular to a train track and away from the track at a speed of 12 m/s. The train is moving with a speed of 30 m/s with respect to the track. What is the speed of the man with respect to a passenger sitting at rest in the train?

42. A man is running on a straight road that makes **30°** with the train track. The man is running in the direction on the road that is away from the track at a speed of 12 m/s. The train is moving with a speed of 30 m/s with respect to the track. What is the speed of the man with respect to a passenger sitting at rest in the train?

43. In a frame at rest with respect to the billiard table, a billiard ball of mass m moving with speed v strikes another billiard ball of mass m at rest. The first ball comes to rest after the collision while the second ball takes off with speed v in the original direction of the motion of the first ball. This shows that momentum is conserved in this frame.

(a) Now, describe the same collision from the perspective of a frame that is moving with speed v in the direction of the motion of the first ball.

(b) Is the momentum conserved in this frame?

44. In a frame at rest with respect to the billiard table, two billiard balls of same mass **m** are moving toward each other with the same speed **v**. After the collision, the two balls come to rest.

(a) Show that momentum is conserved in this frame.

(b) Now, describe the same collision from the perspective of a frame that is moving with speed **v** in the direction of the motion of the first ball.

(c) Is the momentum conserved in this frame?

45. In a frame S, two events are observed: event 1: a pion is created at rest at the origin and event 2: the pion disintegrates after time τ. Another observer in a frame S' is moving in the positive direction along the positive x-axis with a constant speed v and observes the same two events in his frame. The origins of the two frames coincide at $t = t' = 0$.

(a) Find the positions and timings of these two events in the frame S' (a) according to the Galilean transformation, and

(b) according to the Lorentz transformation.

5.6 Relativistic Velocity Transformation

46. If two spaceships are heading directly toward each other at 0.800**c**, at what speed must a canister be shot from the first ship to approach the other at 0.999**c** as seen by the second ship?

47. Two planets are on a collision course, heading directly toward each other at 0.250**c**. A spaceship sent from one planet approaches the second at 0.750**c** as seen by the second planet. What is the velocity of the ship relative to the first planet?

48. When a missile is shot from one spaceship toward another, it leaves the first at 0.950**c** and approaches the other at 0.750**c**. What is the relative velocity of the two ships?

49. What is the relative velocity of two spaceships if one fires a missile at the other at 0.750**c** and the other observes it to approach at 0.950**c**?

50. Prove that for any relative velocity **v** between two observers, a beam of light sent from one to the other will approach at speed **c** (provided that **v** is less than **c**, of course).

51. Show that for any relative velocity **v** between two observers, a beam of light projected by one directly away from the other will move away at the speed of light (provided that **v** is less than **c**, of course).

5.7 Doppler Effect for Light

52. A highway patrol officer uses a device that measures the speed of vehicles by bouncing radar off them and measuring the Doppler shift. The outgoing radar has a frequency of 100 GHz and the returning echo has a frequency 15.0 kHz higher. What is the velocity of the vehicle? Note that there are two Doppler shifts in echoes. Be certain not to round off until the end of the problem, because the effect is small.

5.8 Relativistic Momentum

53. Find the momentum of a helium nucleus having a mass of $6.68 \times 10^{-27} kg$ that is moving at 0.200**c**.

54. What is the momentum of an electron traveling at 0.980**c**?

55. (a) Find the momentum of a $1.00 \times 10^9 - kg$ asteroid heading towards Earth at 30.0 km/s.

(b) Find the ratio of this momentum to the classical momentum. (Hint: Use the approximation that $\gamma = 1 + (1/2)v^2/c^2$ at low velocities.)

56. (a) What is the momentum of a 2000-kg satellite orbiting at 4.00 km/s? (b) Find the ratio of this momentum to the classical momentum. (Hint: Use the approximation that $\gamma = 1 + (1/2)v^2/c^2$ at low velocities.)

57. What is the velocity of an electron that has a momentum of $3.04 \times 10^{-21} kg \cdot m/s$? Note that you must calculate the velocity to at least four digits to see the difference from **c**.

58. Find the velocity of a proton that has a momentum of $4.48 \times 10^{-19} kg \cdot m/s$.

5.9 Relativistic Energy

59. What is the rest energy of an electron, given its mass is $9.11 \times 10^{-31} kg$? Give your answer in joules and MeV.

60. Find the rest energy in joules and MeV of a proton, given its mass is $1.67 \times 10^{-27} kg$.

61. If the rest energies of a proton and a neutron (the two constituents of nuclei) are 938.3 and 939.6 MeV, respectively, what is the difference in their mass in kilograms?

62. The Big Bang that began the universe is estimated to have released $10^{68} J$ of energy. How many stars could half this energy create, assuming the average star's mass is $4.00 \times 10^{30} kg$?

63. A supernova explosion of a $2.00 \times 10^{31} kg$ star produces $1.00 \times 10^{44} J$ of energy.

(a) How many kilograms of mass are converted to energy in the explosion?

(b) What is the ratio $\Delta m/m$ of mass destroyed to the original mass of the star?

64. (a) Using data from Potential Energy of a System, calculate the mass converted to energy by the fission of 1.00 kg of uranium.

(b) What is the ratio of mass destroyed to the original mass, $\Delta m/m$?

65. (a) Using data from Potential Energy of a System, calculate the amount of mass converted to energy by the fusion of 1.00 kg of hydrogen.

(b) What is the ratio of mass destroyed to the original mass, $\Delta m/m$?

(c) How does this compare with $\Delta m/m$ for the fission of 1.00 kg of uranium?

66. There is approximately $10^{34} J$ of energy available from fusion of hydrogen in the world's oceans.

(a) If $10^{33} J$ of this energy were utilized, what would be the decrease in mass of the oceans?

(b) How great a volume of water does this correspond to?

(c) Comment on whether this is a significant fraction of the total mass of the oceans.

67. A muon has a rest mass energy of 105.7 MeV, and it decays into an electron and a massless particle.

(a) If all the lost mass is converted into the electron's kinetic energy, find γ for the electron.

(b) What is the electron's velocity?

68. A π-meson is a particle that decays into a muon and a massless particle. The π-meson has a rest mass energy of 139.6 MeV, and the muon has a rest mass energy of 105.7 MeV. Suppose the π-meson is at rest and all of the missing mass goes into the muon's kinetic energy. How fast will the muon move?

69. (a) Calculate the relativistic kinetic energy of a 1000-kg car moving at 30.0 m/s if the speed of light were only 45.0 m/s.

(b) Find the ratio of the relativistic kinetic energy to classical.

70. Alpha decay is nuclear decay in which a helium nucleus is emitted. If the helium nucleus has a mass of $6.80 \times 10^{-27} kg$ and is given 5.00 MeV of kinetic energy, what is its velocity?

71. (a) Beta decay is nuclear decay in which an electron is emitted. If the electron is given 0.750 MeV of kinetic energy, what is its velocity?

(b) Comment on how the high velocity is consistent with the kinetic energy as it compares to the rest mass energy of the electron.

Additional Problems

72. (a) At what relative velocity is $\gamma = 1.50$?

(b) At what relative velocity is $\gamma = 100$?γ=100?

73. (a) At what relative velocity is $\gamma = 2.00$?

(b) At what relative velocity is $\gamma = 10.0$?

74. Unreasonable Results (a) Find the value of γ required for the following situation. An earthbound observer measures 23.9 h to have passed while signals from a high-velocity space probe indicate that 24.0 h have passed on board.

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

75. (a) How long does it take the astronaut in Example 5.5 to travel 4.30 ly at $0.99944c$ (as measured by the earthbound observer)?

(b) How long does it take according to the astronaut?

(c) Verify that these two times are related through time dilation with $\gamma = 30.00$ as given.

76. (a) How fast would an athlete need to be running for a 100-m race to look 100 yd long?

(b) Is the answer consistent with the fact that relativistic effects are difficult to observe in ordinary circumstances? Explain.

77. (a) Find the value of γ for the following situation. An astronaut measures the length of his spaceship to be 100 m, while an earthbound observer measures it to be 25.0 m.

(b) What is the speed of the spaceship relative to Earth?

78. A clock in a spaceship runs one-tenth the rate at which an identical clock on Earth runs. What is the speed of the spaceship?

79. An astronaut has a heartbeat rate of 66 beats per minute as measured during his physical exam on Earth. The heartbeat rate of the astronaut is measured when he is in a spaceship traveling at 0.5c with respect to Earth by an observer (A) in the ship and by an observer (B) on Earth.

(a) Describe an experimental method by which observer B on Earth will be able to determine the heartbeat rate of the astronaut when the astronaut is in the spaceship.

(b) What will be the heartbeat rate(s) of the astronaut reported by observers A and B?

80. A spaceship (A) is moving at speed **c/2** with respect to another spaceship (B). Observers in A and B set their clocks so that the event at **(x, y, z, t)** of turning on a laser in spaceship B has coordinates (0, 0, 0, 0) in A and also (0, 0, 0, 0) in B. An observer at the origin of B turns on the laser at $t = 0$ and turns it off at $t = \tau$ in his time. What is the time duration between on and off as seen by an observer in A?

81. Same two observers as in the preceding exercise, but now we look at two events occurring in spaceship A. A photon arrives at the origin of A at its time $t = 0$ and another photon arrives at $(x = 1.00m, 0, 0)$ at $t = 0$ in the frame of ship A.

(a) Find the coordinates and times of the two events as seen by an observer in frame B.

(b) In which frame are the two events simultaneous and in which frame are they are not simultaneous?

82. Same two observers as in the preceding exercises. A rod of length 1 m is laid out on the **x**-axis in the frame of B from origin to $(x = 1.00m, 0, 0)$. What is the length of the rod observed by an observer in the frame of spaceship A?

83. An observer at origin of inertial frame S sees a flashbulb go off at $x = 150km, y = 15.0km$, and $z = 1.00km$ at time $t = 4.5 \times 10^{-4} s$. At what time and position in the S' system did the flash occur, if S' is moving along shared **x**-direction with S at a velocity $v = 0.6c$?

84. An observer sees two events $1.5 \times 10^{-8} s$ apart at a separation of 800 m. How fast must a second observer be moving relative to the first to see the two events occur simultaneously?

85. An observer standing by the railroad tracks sees two bolts of lightning strike the ends of a 500-m-long train simultaneously at the instant the middle of the train passes him at 50 m/s. Use the Lorentz transformation to find the time between the lightning strikes as measured by a passenger seated in the middle of the train.

86. Two astronomical events are observed from Earth to occur at a time of 1 s apart and a distance separation of $1.5 \times 10^9 m$ from each other.

(a) Determine whether separation of the two events is space like or time like.

(b) State what this implies about whether it is consistent with special relativity for one event to have caused the other?

87. Two astronomical events are observed from Earth to occur at a time of 0.30 s apart and a distance separation of $2.0 \times 10^9 m$ from each other. How fast must a spacecraft travel from the site of one event toward the other to make the events occur at the same time when measured in the frame of reference of the spacecraft?

88. A spacecraft starts from being at rest at the origin and accelerates at a constant rate **g**, as seen from Earth, taken to be an inertial frame, until it reaches a speed of **c/2**.

(a) Show that the increment of proper time is related to the elapsed time in Earth's frame by:

$$d\tau = \sqrt{1 - v2/c^2}\, dt$$

.

(b) Find an expression for the elapsed time to reach speed **c/2** as seen in Earth's frame.

(c) Use the relationship in (a) to obtain a similar expression for the elapsed proper time to reach c/2 as seen in the spacecraft, and determine the ratio of the time seen from Earth with that on the spacecraft to reach the final speed.

89. (a) All but the closest galaxies are receding from our own Milky Way Galaxy. If a galaxy $12.0 \times 10^9 ly$ away is receding from us at 0.900c, at what velocity relative to us must we send an exploratory probe to approach the other galaxy at 0.990c as measured from that galaxy?

(b) How long will it take the probe to reach the other galaxy as measured from Earth? You may assume that the velocity of the other galaxy remains constant.

(c) How long will it then take for a radio signal to be beamed back? (All of this is possible in principle, but not practical.)

90. Suppose a spaceship heading straight toward the Earth at 0.750**c** can shoot a canister at 0.500**c** relative to the ship.

(a) What is the velocity of the canister relative to Earth, if it is shot directly at Earth?

(b) If it is shot directly away from Earth?

91. Repeat the preceding problem with the ship heading directly away from Earth.

92. If a spaceship is approaching the Earth at 0.100**c** and a message capsule is sent toward it at 0.100**c** relative to Earth, what is the speed of the capsule relative to the ship?

93. (a) Suppose the speed of light were only 3000 m/s. A jet fighter moving toward a target on the ground at 800 m/s shoots bullets, each having a muzzle velocity of 1000 m/s. What are the bullets' velocity relative to the target?

(b) If the speed of light was this small, would you observe relativistic effects in everyday life? Discuss.

94. If a galaxy moving away from the Earth has a speed of 1000 km/s and emits 656 nm light characteristic of hydrogen (the most common element in the universe).

(a) What wavelength would we observe on Earth?

(b) What type of electromagnetic radiation is this? (c) Why is the speed of Earth in its orbit negligible here?

95. A space probe speeding towards the nearest star moves at $0.250c$and sends radio information at a broadcast frequency of 1.00 GHz. What frequency is received on Earth?

96. Near the center of our galaxy, hydrogen gas is moving directly away from us in its orbit about a black hole. We receive 1900 nm electromagnetic radiation and know that it was 1875 nm when emitted by the hydrogen gas. What is the speed of the gas?

97. (a) Calculate the speed of a $1.00 - \mu g$ particle of dust that has the same momentum as a proton moving at 0.999**c**.

(b) What does the small speed tell us about the mass of a proton compared to even a tiny amount of macroscopic matter?

98. (a) Calculate γ for a proton that has a momentum of $1.00 kg \cdot m/s$.

(b) What is its speed? Such protons form a rare component of cosmic radiation with uncertain origins.

99. Show that the relativistic form of Newton's second law is

(a) $F = m\dfrac{du}{dt}\dfrac{1}{(1 - u^2/c^2)^{3/2}}$;

(b) Find the force needed to accelerate a mass of 1 kg by 1 m/s^2 when it is traveling at a velocity of **c/2**.

100. A positron is an antimatter version of the electron, having exactly the same mass. When a positron and an electron meet, they annihilate, converting all of their mass into energy.

(a) Find the energy released, assuming negligible kinetic energy before the annihilation.

(b) If this energy is given to a proton in the form of kinetic energy, what is its velocity?

(c) If this energy is given to another electron in the form of kinetic energy, what is its velocity?

101. What is the kinetic energy in MeV of a π-meson that lives $1.40 \times 10^{-16} s$ as measured in the laboratory, and $0.840 \times 10^{-16} s$ when at rest relative to an observer, given that its rest energy is 135 MeV?

102. Find the kinetic energy in MeV of a neutron with a measured life span of 2065 s, given its rest energy is 939.6 MeV, and rest life span is 900s.

103. (a) Show that $(pc)^2/(mc^2)^2 = \gamma^2 - 1$. This means that at large velocities $pc >> mc^2$.

(b) Is $E \approx pc$ when $\gamma = 30.0$, as for the astronaut discussed in the twin paradox?

104. One cosmic ray neutron has a velocity of $0.250c$relative to the Earth.

(a) What is the neutron's total energy in MeV?

(b) Find its momentum.

(c) Is $E \approx pc$ in this situation? Discuss in terms of the equation given in part (a) of the previous problem.

105. What is γ for a proton having a mass energy of 938.3 MeV accelerated through an effective potential of 1.0 TV (teravolt)?

106. (a) What is the effective accelerating potential for electrons at the Stanford Linear Accelerator, if $\gamma = 1.00 \times 10^5$ for them?

(b) What is their total energy (nearly the same as kinetic in this case) in GeV?

107. (a) Using data from Potential Energy of a System, find the mass destroyed when the energy in a barrel of crude oil is released.

(b) Given these barrels contain 200 liters and assuming the density of crude oil is $750 kg/m^3$, what is the ratio of mass destroyed to original mass, $\Delta m/m$?

108. (a) Calculate the energy released by the destruction of 1.00 kg of mass.

(b) How many kilograms could be lifted to a 10.0 km height by this amount of energy?

109. A Van de Graaff accelerator utilizes a 50.0 MV potential difference to accelerate charged particles such as protons.

(a) What is the velocity of a proton accelerated by such a potential?

(b) An electron?

110. Suppose you use an average of $500 kW \cdot h$ of electric energy per month in your home.

(a) How long would 1.00 g of mass converted to electric energy with an efficiency of 38.0% last you?

(b) How many homes could be supplied at the $500 kW \cdot h$ per month rate for one year by the energy from the described mass conversion?

111. (a) A nuclear power plant converts energy from nuclear fission into electricity with an efficiency of 35.0%. How much mass is destroyed in one year to produce a continuous 1000 MW of electric power?

(b) Do you think it would be possible to observe this mass loss if the total mass of the fuel is $10^4 kg$?

112. Nuclear-powered rockets were researched for some years before safety concerns became paramount.

(a) What fraction of a rocket's mass would have to be destroyed to get it into a low Earth orbit, neglecting the decrease in gravity? (Assume an orbital altitude of 250 km, and calculate both the kinetic energy (classical) and the gravitational potential energy needed.)

(b) If the ship has a mass of $1.00 \times 10^5 kg$ (100 tons), what total yield nuclear explosion in tons of TNT is needed?

113. The sun produces energy at a rate of 3.85×10^{26} W by the fusion of hydrogen. About **0.7%** of each kilogram of hydrogen goes into the energy generated by the Sun.

(a) How many kilograms of hydrogen undergo fusion each second?

(b) If the sun is 90.0% hydrogen and half of this can undergo fusion before the sun changes character, how long could it produce energy at its current rate?

(c) How many kilograms of mass is the sun losing per second?

(d) What fraction of its mass will it have lost in the time found in part (b)?

114. Show that $E^2 - p^2c^2$ for a particle is invariant under Lorentz transformations.

1.S: Relativity (Summary)

Key Terms

classical (Galilean) velocity addition	method of adding velocities when $v << c$; velocities add like regular numbers in one-dimensional motion: $u = v + u'$, where **v** is the velocity between two observers, **u** is the velocity of an object relative to one observer, and u' is the velocity relative to the other observer
event	occurrence in space and time specified by its position and time coordinates (x, y, z, t) measured relative to a frame of reference
first postulate of special relativity	laws of physics are the same in all inertial frames of reference
Galilean relativity	if an observer measures a velocity in one frame of reference, and that frame of reference is moving with a velocity past a second reference frame, an observer in the second frame measures the original velocity as the vector sum of these velocities
Galilean transformation	relation between position and time coordinates of the same events as seen in different reference frames, according to classical mechanics
inertial frame of reference	reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted on by an outside force
length contraction	decrease in observed length of an object from its proper length L_0 to length **L** when its length is observed in a reference frame where it is traveling at speed **v**
Lorentz transformation	relation between position and time coordinates of the same events as seen in different reference frames, according to the special theory of relativity
Michelson-Morley experiment	investigation performed in 1887 that showed that the speed of light in a vacuum is the same in all frames of reference from which it is viewed
proper length	L_0; the distance between two points measured by an observer who is at rest relative to both of the points; for example, earthbound observers measure proper length when measuring the distance between two points that are stationary relative to Earth
proper time	$\Delta\tau$ is the time interval measured by an observer who sees the beginning and end of the process that the time interval measures occur at the same location
relativistic kinetic energy	kinetic energy of an object moving at relativistic speeds
relativistic momentum	$\vec{p}$, the momentum of an object moving at relativistic velocity; $\vec{p} = \gamma m\vec{u}$
lativistic velocity addition	method of adding velocities of an object moving at a relativistic speeds
rest energy	energy stored in an object at rest: $E_0 = mc^2$
rest frame	frame of reference in which the observer is at rest

rest mass	mass of an object as measured by an observer at rest relative to the object
second postulate of special relativity	light travels in a vacuum with the same speed **c** in any direction in all inertial frames
special theory of relativity	theory that Albert Einstein proposed in 1905 that assumes all the laws of physics have the same form in every inertial frame of reference, and that the speed of light is the same within all inertial frames
speed of light	ultimate speed limit for any particle having mass
time dilation	lengthening of the time interval between two events when seen in a moving inertial frame rather than the rest frame of the events (in which the events occur at the same location)
total energy	sum of all energies for a particle, including rest energy and kinetic energy, given for a particle of mass m and speed **u** by $E=\gamma mc^2$, where $\gamma=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$
world line	path through space-time

Key Equations

Time dilation	$\Delta t=\frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}}=\gamma\tau$
Lorentz factor	$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$
Length contraction	$L=L_0\sqrt{1-\frac{v^2}{c^2}}=\frac{L_0}{\gamma}$
Galilean transformation	$x=x'+vt, y=y', z=z', t=t'$
Lorentz transformation	$t=\frac{t'+vx'/c^2}{\sqrt{1-v^2/c^2}}$ $x=\frac{x'+vt'}{\sqrt{1-v^2/c^2}}$ $y=y'$ $z=z'$
Inverse Lorentz transformation	$t'=\frac{t-vx/c^2}{\sqrt{1-v^2/c^2}}$ $x'=\frac{x-vt}{\sqrt{1-v^2/c^2}}$ $y'=y$ $z'=z$
Space-time invariants	$(\Delta s)^2=(\Delta x)^2+(\Delta y)^2+(\Delta z)^2-c^2(\Delta t)^2$ $(\Delta\tau)^2=-(\Delta s)^2/c^2=(\Delta t)^2-\frac{[(\Delta x)^2+(\Delta y)^2+(\Delta z)^2]}{c^2}$
Relativistic velocity addition	$u_x=(\frac{u'_x+v}{1+vu'_x/c^2}), u_y=(\frac{u'_y/\gamma}{1+vu'_x/c^2}), u_z=(\frac{u'_z/\gamma}{1+vu'_x/c^2})$

Relativistic Doppler effect for wavelength	$\lambda_{obs} = \lambda_s \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$
Relativistic Doppler effect for frequency	$f_{obs} = f_s \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$
Relativistic momentum	$\vec{p} = \gamma m\vec{u} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}$
Relativistic total energy	$E = \gamma mc^2$,where $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$
Relativistic kinetic energy	$K_{rel} = (\gamma - 1)mc^2$, where $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

Summary

5.1 Invariance of Physical Laws

- Relativity is the study of how observers in different reference frames measure the same event.
- Modern relativity is divided into two parts. Special relativity deals with observers in uniform (unaccelerated) motion, whereas general relativity includes accelerated relative motion and gravity. Modern relativity is consistent with all empirical evidence thus far and, in the limit of low velocity and weak gravitation, gives close agreement with the predictions of classical (Galilean) relativity.
- An inertial frame of reference is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted upon by an outside force.
- Modern relativity is based on Einstein's two postulates. The first postulate of special relativity is that the laws of physics are the same in all inertial frames of reference. The second postulate of special relativity is that the speed of light c is the same in all inertial frames of reference, independent of the relative motion of the observer and the light source.
- The Michelson-Morley experiment demonstrated that the speed of light in a vacuum is independent of the motion of Earth about the sun.

5.2 Relativity of Simultaneity

- Two events are defined to be simultaneous if an observer measures them as occurring at the same time (such as by receiving light from the events).
- Two events at locations a distance apart that are simultaneous for an observer at rest in one frame of reference are not necessarily simultaneous for an observer at rest in a different frame of reference.

5.3 Time Dilation

- Two events are defined to be simultaneous if an observer measures them as occurring at the same time. They are not necessarily simultaneous to all observers—simultaneity is not absolute.
- Time dilation is the lengthening of the time interval between two events when seen in a moving inertial frame rather than the rest frame of the events (in which the events occur at the same location).
- Observers moving at a relative velocity **v** do not measure the same elapsed time between two events. Proper time $\Delta\tau$ is the time measured in the reference frame where the start and end of the time interval occur at the same location. The time interval Δt measured by an observer who sees the frame of events moving at speed **v** is related to the proper time interval $\Delta\tau$ of the events by the equation:

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\Delta\tau,$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

- The premise of the twin paradox is faulty because the traveling twin is accelerating. The journey is not symmetrical for the two twins.
- Time dilation is usually negligible at low relative velocities, but it does occur, and it has been verified by experiment.
- The proper time is the shortest measure of any time interval. Any observer who is moving relative to the system being observed measures a time interval longer than the proper time.

5.4 Length Contraction

- All observers agree upon relative speed.
- Distance depends on an observer's motion. Proper length L_0 is the distance between two points measured by an observer who is at rest relative to both of the points.
- Length contraction is the decrease in observed length of an object from its proper length L_0 to length L when its length is observed in a reference frame where it is traveling at speed **v**.
- The proper length is the longest measurement of any length interval. Any observer who is moving relative to the system being observed measures a length shorter than the proper length.

5.5 The Lorentz Transformation

- The Galilean transformation equations describe how, in classical nonrelativistic mechanics, the position, velocity, and accelerations measured in one frame appear in another. Lengths remain unchanged and a single universal time scale is assumed to apply to all inertial frames.
- Newton's laws of mechanics obey the principle of having the same form in all inertial frames under a Galilean transformation, given by

$$x = x' + vt, y = y', z = z', t = t' \quad .$$

The concept that times and distances are the same in all inertial frames in the Galilean transformation, however, is inconsistent with the postulates of special relativity.

- The relativistically correct Lorentz transformation equations are

Lorentz transformation	Inverse Lorentz transformation
$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$ $x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$ √ $y = y'$ $z = z'$	$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$ $x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$ $y' = y$ $z' = z$

We can obtain these equations by requiring an expanding spherical light signal to have the same shape and speed of growth, c, in both reference frames.

- Relativistic phenomena can be explained in terms of the geometrical properties of four-dimensional space-time, in which Lorentz transformations correspond to rotations of axes.
- The Lorentz transformation corresponds to a space-time axis rotation, similar in some ways to a rotation of space axes, but in which the invariant spatial separation is given by Δs rather than distances Δr, and that the Lorentz transformation involving the time axis does not preserve perpendicularity of axes or the scales along the axes.
- The analysis of relativistic phenomena in terms of space-time diagrams supports the conclusion that these phenomena result from properties of space and time itself, rather than from the laws of electromagnetism.

5.6 Relativistic Velocity Transformation

- With classical velocity addition, velocities add like regular numbers in one-dimensional motion: $u = v + u'$, where **v** is the velocity between two observers, **u** is the velocity of an object relative to one observer, and u'u′ is the velocity relative to the other observer.
- Velocities cannot add to be greater than the speed of light.
- Relativistic velocity addition describes the velocities of an object moving at a relativistic velocity.

5.7 Doppler Effect for Light

- An observer of electromagnetic radiation sees relativistic Doppler effects if the source of the radiation is moving relative to the observer. The wavelength of the radiation is longer (called a red shift) than that emitted by the source when the source moves away from the observer and shorter (called a blue shift) when the source moves toward the observer. The shifted wavelength is described by the equation:

$$\lambda_{obs} = \lambda_s \sqrt{1 + \frac{v}{c}} 1 - \frac{v}{c}\ .$$

where λ_{obs} is the observed wavelength, λ_s is the source wavelength, and **v** is the relative velocity of the source to the observer.

5.8 Relativistic Momentum

- The law of conservation of momentum is valid for relativistic momentum whenever the net external force is zero. The relativistic momentum is $p = \gamma m u$, where m is the rest mass of the object, **u** is its velocity relative to an observer, and the relativistic factor is $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$.
- At low velocities, relativistic momentum is equivalent to classical momentum.
- Relativistic momentum approaches infinity as **u** approaches **c**. This implies that an object with mass cannot reach the speed of light.

5.9 Relativistic Energy

- The relativistic work-energy theorem is $W_{net} = E - E_0 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$.
- Relativistically, $W_{net} = K_{rel}$ where K_{rel} is the relativistic kinetic energy.
- An object of mass m at velocity u has kinetic energy $K_{rel} = (\gamma - 1)mc^2$, where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$.
- At low velocities, relativistic kinetic energy reduces to classical kinetic energy.
- No object with mass can attain the speed of light, because an infinite amount of work and an infinite amount of energy input is required to accelerate a mass to the speed of light.
- Relativistic energy is conserved as long as we define it to include the possibility of mass changing to energy.
- The total energy of a particle with mass m traveling at speed u is defined as $E = \gamma mc^2$, where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$ and **u** denotes the velocity of the particle.
- The rest energy of an object of mass m is $E_0 = mc^2$, meaning that mass is a form of energy. If energy is stored in an object, its mass increases. Mass can be destroyed to release energy.
- We do not ordinarily notice the increase or decrease in mass of an object because the change in mass is so small for a large increase in energy. The equation $E^2 = (pc)^2 + (mc^2)^2$ relates the relativistic total energy **E** and the relativistic momentum p. At extremely high velocities, the rest energy mc^2 becomes negligible, and $E = pc$.

CHAPTER OVERVIEW

2: Waves

In this chapter, we will study the physics of wave motion. We concentrate on mechanical waves, which are disturbances that move through a medium such as air or water. Like simple harmonic motion studied in the preceding chapter, the energy transferred through the medium is proportional to the amplitude squared. The concepts presented in this chapter will be the foundation for many interesting topics, from the transmission of information to the concepts of quantum mechanics.

Thumbnail: Surfer at Mavericks, one of the world's premier big wave surfing locations. (Surfer: Andrew Davis). (CC SA-BY 2.0; Shalom Jacobovitz).

Contributors and Attributions

2.1: Prelude to Wave

In this chapter, we study the physics of wave motion. We concentrate on mechanical waves, which are disturbances that move through a medium such as air or water. Like simple harmonic motion studied in the preceding chapter, the energy transferred through the medium is proportional to the amplitude squared.

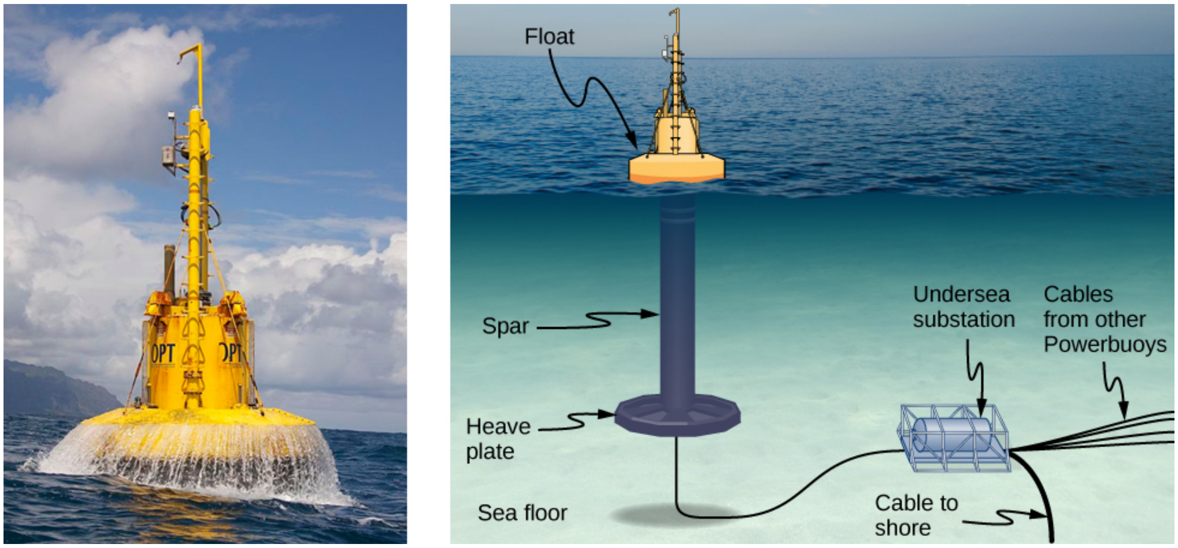


Figure 2.1.1: From the world of renewable energy sources comes the electric power-generating buoy. Although there are many versions, this one converts the up-and-down motion, as well as side-to-side motion, of the buoy into rotational motion in order to turn an electric generator, which stores the energy in batteries.

Surface water waves in the ocean are transverse waves in which the energy of the wave travels horizontally while the water oscillates up and down due to some restoring force. In Figure 2.1.1, a buoy is used to convert the awesome power of ocean waves into electricity. The up-and-down motion of the buoy generated as the waves pass is converted into rotational motion that turns a rotor in an electric generator. The generator charges batteries, which are in turn used to provide a consistent energy source for the end user. This model was successfully tested by the US Navy in a project to provide power to coastal security networks and was able to provide an average power of 350 W. The buoy survived the difficult ocean environment, including operation off the New Jersey coast through Hurricane Irene in 2011.

The concepts presented in this chapter will be the foundation for many interesting topics, from the transmission of information to the concepts of quantum mechanics.

2.2: Simple Harmonic Motion

Learning Objectives

- Define the terms period and frequency
- List the characteristics of simple harmonic motion
- Explain the concept of phase shift
- Write the equations of motion for the system of a mass and spring undergoing simple harmonic motion
- Describe the motion of a mass oscillating on a vertical spring

When you pluck a guitar string, the resulting sound has a steady tone and lasts a long time (Figure $2.2.1$). The string vibrates around an equilibrium position, and one oscillation is completed when the string starts from the initial position, travels to one of the extreme positions, then to the other extreme position, and returns to its initial position. We define **periodic motion** to be any motion that repeats itself at regular time intervals, such as exhibited by the guitar string or by a child swinging on a swing. In this section, we study the basic characteristics of oscillations and their mathematical description.

Figure $2.2.1$: When a guitar string is plucked, the string oscillates up and down in periodic motion. The vibrating string causes the surrounding air molecules to oscillate, producing sound waves. (credit: Yutaka Tsutano)

Period and Frequency in Oscillations

In the absence of friction, the time to complete one oscillation remains constant and is called the **period (T)**. Its units are usually seconds, but may be any convenient unit of time. The word ‘period’ refers to the time for some event whether repetitive or not, but in this chapter, we shall deal primarily in periodic motion, which is by definition repetitive.

A concept closely related to period is the frequency of an event. **Frequency (f)** is defined to be the number of events per unit time. For periodic motion, frequency is the number of oscillations per unit time. The relationship between frequency and period is

$$f = \frac{1}{T}. \tag{2.2.1}$$

The SI unit for frequency is the **hertz** (Hz) and is defined as one **cycle per second**:

$$1\ Hz = 1\ cycle/sec\ or\ 1\ Hz = \frac{1}{s} = 1\ s^{-1}. \tag{2.2.2}$$

A cycle is one complete **oscillation**

Example 2.2.1: Determining the Frequency of Medical Ultrasound

Ultrasound machines are used by medical professionals to make images for examining internal organs of the body. An ultrasound machine emits high-frequency sound waves, which reflect off the organs, and a computer receives the waves, using them to create a picture. We can use the formulas presented in this module to determine the frequency, based on what we know about oscillations. Consider a medical imaging device that produces ultrasound by oscillating with a period of 0.400 μs. What is the frequency of this oscillation?

Strategy

The period (T) is given and we are asked to find frequency (f).

Solution

Substitute 0.400 μs for T in f = $\frac{1}{T}$:

$$f = \frac{1}{T} = \frac{1}{0.400 \times 10^{-6}\ s}.$$

Solve to find

$$f = 2.50 \times 10^{6}\ Hz.$$

Significance

This frequency of sound is much higher than the highest frequency that humans can hear (the range of human hearing is 20 Hz to 20,000 Hz); therefore, it is called ultrasound. Appropriate oscillations at this frequency generate ultrasound used for noninvasive medical diagnoses, such as observations of a fetus in the womb.

Characteristics of Simple Harmonic Motion

A very common type of periodic motion is called **simple harmonic motion (SHM)**. A system that oscillates with SHM is called a **simple harmonic oscillator**.

Simple Harmonic Motion

In simple harmonic motion, the acceleration of the system, and therefore the net force, is proportional to the displacement and acts in the opposite direction of the displacement.

A good example of SHM is an object with mass m attached to a spring on a frictionless surface, as shown in Figure $2.2.2$. The object oscillates around the equilibrium position, and the net force on the object is equal to the force provided by the spring. This force obeys Hooke's law $F_s = -kx$, as discussed in a previous chapter.

If the net force can be described by Hooke's law and there is no **damping** (slowing down due to friction or other nonconservative forces), then a simple harmonic oscillator oscillates with equal displacement on either side of the equilibrium position, as shown for an object on a spring in Figure $2.2.2$. The maximum displacement from equilibrium is called the **amplitude (A)**. The units for amplitude and displacement are the same but depend on the type of oscillation. For the object on the spring, the units of amplitude and displacement are meters.

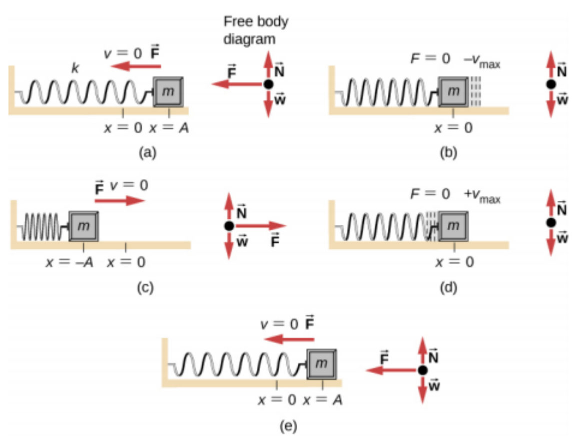


Figure $2.2.2$: - An object attached to a spring sliding on a frictionless surface is an uncomplicated simple harmonic oscillator. In the above set of figures, a mass is attached to a spring and placed on a frictionless table. The other end of the spring is attached to the wall. The position of the mass, when the spring is neither stretched nor compressed, is marked as x = 0 and is the equilibrium position. (a) The mass is displaced to a position x = A and released from rest. (b) The mass accelerates as it moves in the negative x-direction, reaching a maximum negative velocity at x = 0 . (c) The mass continues to move in the negative x-direction, slowing until it comes to a stop at x = −A . (d) The mass now begins to accelerate in the positive x direction, reaching a positive maximum velocity at x = 0 . (e) The mass then continues to move in the positive direction until it stops at x = A . The mass continues in SHM that has an amplitude A and a period T. The object's maximum speed occurs as it passes through equilibrium. The stiffer the spring is, the smaller the period T. The greater the mass of the object is, the greater the period T.

What is so significant about SHM? For one thing, the period T and frequency f of a simple harmonic oscillator are independent of amplitude. The string of a guitar, for example, oscillates with the same frequency whether plucked gently or hard.

Two important factors do affect the period of a simple harmonic oscillator. The period is related to how stiff the system is. A very stiff object has a large **force constant (k)**, which causes the system to have a smaller period. For example, you can adjust a diving board's stiffness—the stiffer it is, the faster it vibrates, and the shorter its period. Period also depends on the mass of the oscillating system. The more massive the system is, the longer the period. For example, a heavy person on a diving board bounces up and down more slowly than a light one. In fact, the mass m and the force constant k are the only factors that affect the period and frequency of SHM. To derive an equation for the period and the frequency, we must first define and analyze the equations of motion. Note that the force constant is sometimes referred to as the **spring constant**.

Equations of SHM

Consider a block attached to a spring on a frictionless table (Figure $2.2.3$). The **equilibrium** position (the position where the spring is neither stretched nor compressed) is marked as x = 0 . At the equilibrium position, the net force is zero.

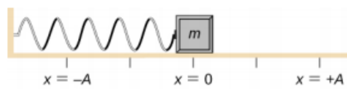


Figure $2.2.3$: A block is attached to a spring and placed on a frictionless table. The equilibrium position, where the spring is neither extended nor compressed, is marked as x = 0.

Work is done on the block to pull it out to a position of x = + A, and it is then released from rest. The maximum x-position (A) is called the amplitude of the motion. The block begins to oscillate in SHM between x = + A and x = −A, where A is the amplitude of the motion and T is the period of the oscillation. The period is the time for one oscillation. Figure $2.2.4$ shows the motion of the block as it completes one and a half oscillations after release.

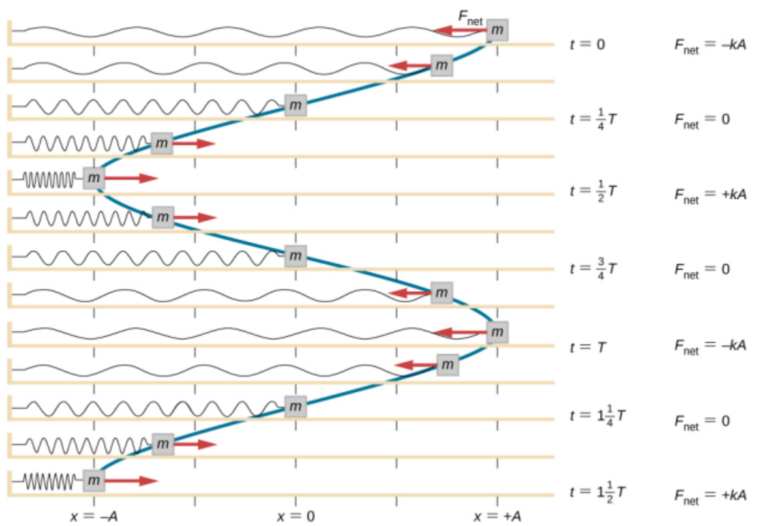


Figure $2.2.4$: A block is attached to one end of a spring and placed on a frictionless table. The other end of the spring is anchored to the wall. The equilibrium position, where the net force equals zero, is marked as x = 0 m. Work is done on the block, pulling it out to x = + A, and the block is released from rest. The block oscillates between x = + A and x = −A. The force is also shown as a vector.

Figure $2.2.4$ shows a plot of the position of the block versus time. When the position is plotted versus time, it is clear that the data can be modeled by a cosine function with an amplitude A and a period T. The cosine function cosθ repeats every multiple of 2π, whereas the motion of the block repeats every period T. However, the function cos$\left(\frac{2\pi}{T}t\right)$ repeats every integer multiple of the period. The maximum of the cosine function is one, so it is necessary to multiply the cosine function by the amplitude A.

$$x(t) = A\cos\left(\frac{2\pi}{T}t\right) = A\cos(\omega t). \tag{2.2.3}$$

Recall from the chapter on rotation that the angular frequency equals $\omega = \frac{d\theta}{dt}$. In this case, the period is constant, so the angular frequency is defined as 2π divided by the period, $\omega = \frac{2\pi}{T}$.

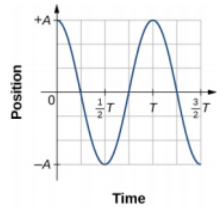


Figure 2.2.5: A graph of the position of the block shown in Figure 2.2.4 as a function of time. The position can be modeled as a periodic function, such as a cosine or sine function.

The equation for the position as a function of time $x(t) = A\cos(\omega t)$ is good for modeling data, where the position of the block at the initial time t = 0.00 s is at the amplitude A and the initial velocity is zero. Often when taking experimental data, the position of the mass at the initial time t = 0.00 s is not equal to the amplitude and the initial velocity is not zero. Consider 10 seconds of data collected by a student in lab, shown in Figure 2.2.6.

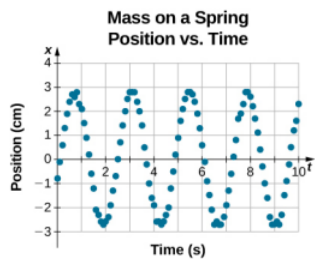


Figure 2.2.6: Data collected by a student in lab indicate the position of a block attached to a spring, measured with a sonic range finder. The data are collected starting at time t = 0.00s, but the initial position is near position x ≈ − 0.80 cm ≠ 3.00 cm, so the initial position does not equal the amplitude x_0 = + A. The velocity is the time derivative of the position, which is the slope at a point on the graph of position versus time. The velocity is not v = 0.00 m/s at time t = 0.00 s, as evident by the slope of the graph of position versus time, which is not zero at the initial time.

The data in Figure 2.2.6 can still be modeled with a periodic function, like a cosine function, but the function is shifted to the right. This shift is known as a **phase shift** and is usually represented by the Greek letter phi (ϕ). The equation of the position as a function of time for a block on a spring becomes

$$x(t) = A\cos(\omega t + \phi). \tag{2.2.4}$$

This is the generalized equation for SHM where t is the time measured in seconds, ω is the angular frequency with units of inverse seconds, A is the amplitude measured in meters or centimeters, and ϕ is the phase shift measured in radians (Figure 2.2.7). It should be noted that because sine and cosine functions differ only by a phase shift, this motion could be modeled using either the cosine or sine function.

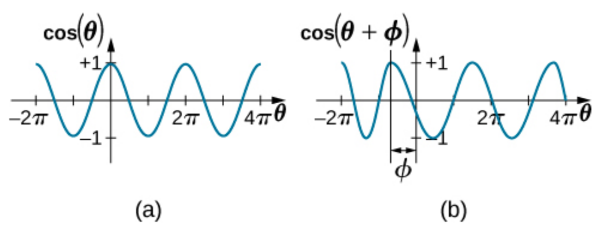


Figure 2.2.7: (a) A cosine function. (b) A cosine function shifted to the right by an angle ϕ. The angle ϕ is known as the phase shift of the function.

The velocity of the mass on a spring, oscillating in SHM, can be found by taking the derivative of the position equation:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}(A\cos(\omega t + \phi)) = -A\omega\sin(\omega t + \varphi) = -v_{max}\sin(\omega t + \phi). \tag{2.2.5}$$

Because the sine function oscillates between –1 and +1, the maximum velocity is the amplitude times the angular frequency, v_{max} = Aω. The maximum velocity occurs at the equilibrium position (x = 0) when the mass is moving toward x = + A. The maximum velocity in the negative direction is attained at the equilibrium position (x = 0) when the mass is moving toward x = −A and is equal to $-v_{max}$.

The acceleration of the mass on the spring can be found by taking the time derivative of the velocity:

$$a(t)=\frac{dv}{dt}=\frac{d}{dt}(-A\omega\sin(\omega t+\phi))=-A\omega^2\cos(\omega t+\varphi)=-a_{max}\cos(\omega t+\phi). \tag{2.2.6}$$

The maximum acceleration is $a_{max} = A\omega^2$. The maximum acceleration occurs at the position (x = −A), and the acceleration at the position (x = −A) and is equal to $-a_{max}$.

Summary of Equations of Motion for SHM

In summary, the oscillatory motion of a block on a spring can be modeled with the following equations of motion:

$$x(t) = A\cos(\omega t+\phi) \tag{2.2.7}$$

$$v(t) = -v_{max}\sin(\omega t+\phi) \tag{2.2.8}$$

$$a(t) = -a_{max}\cos(\omega t+\phi) \tag{2.2.9}$$

with

$$x_{max} = A \tag{2.2.10}$$

$$v_{max} = A\omega \tag{2.2.11}$$

$$a_{max} = A\omega^2. \tag{2.2.12}$$

Here, A is the amplitude of the motion, T is the period, ϕ is the phase shift, and $\omega=\frac{2\pi}{T}$ = 2πf is the angular frequency of the motion of the block.

Example 15.2: Determining the Equations of Motion for a Block and a Spring

A 2.00-kg block is placed on a frictionless surface. A spring with a force constant of k = 32.00 N/m is attached to the block, and the opposite end of the spring is attached to the wall. The spring can be compressed or extended. The equilibrium position is marked as x = 0.00 m. Work is done on the block, pulling it out to x = + 0.02 m. The block is released from rest and oscillates between x = + 0.02 m and x = −0.02 m. The period of the motion is 1.57 s. Determine the equations of motion.

Strategy

We first find the angular frequency. The phase shift is zero, ϕ = 0.00 rad, because the block is released from rest at x = A = + 0.02 m. Once the angular frequency is found, we can determine the maximum velocity and maximum acceleration.

Solution

The angular frequency can be found and used to find the maximum velocity and maximum acceleration:

$$\begin{aligned} \omega &= \frac{2\pi}{1.57\ s}=4.00\ s^{-1}; \\ v_{max} &= A\omega=(0.02\ m)(4.00\ s^{-1})=0.08\ m/s; \\ a_{max} &= A\omega^2=(0.02;m)(4.00\ s^{-1})^2=0.32\ m/s^2. \end{aligned}$$

All that is left is to fill in the equations of motion:

$$\begin{aligned} x(t) &= a\cos(\omega t+\phi)=(0.02\ m)\cos(4.00\ s^{-1}t); \\ v(t) &= -v_{max}\sin(\omega t+\phi)=(-0.8\ m/s)\sin(4.00\ s^{-1}t); \\ a(t) &= -a_{max}\cos(\omega t+\phi)=(-0.32\ m/s^2)\cos(4.00\ s^{-1}t). \end{aligned}$$

Significance

The position, velocity, and acceleration can be found for any time. It is important to remember that when using these equations, your calculator must be in radians mode.

The Period and Frequency of a Mass on a Spring

One interesting characteristic of the SHM of an object attached to a spring is that the angular frequency, and therefore the period and frequency of the motion, depend on only the mass and the force constant, and not on other factors such as the amplitude of the motion. We can use the equations of motion and Newton's second law ($\vec{F}_{net} = m\vec{a}$) to find equations for the angular frequency, frequency, and period.

Consider the block on a spring on a frictionless surface. There are three forces on the mass: the weight, the normal force, and the force due to the spring. The only two forces that act perpendicular to the surface are the weight and the normal force, which have equal magnitudes and opposite directions, and thus sum to zero. The only force that acts parallel to the surface is the force due to the spring, so the net force must be equal to the force of the spring:

$$\begin{aligned} F_x &= -kx; \\ ma &= -kx; \\ m\frac{d^2x}{dt^2} &= -kx; \\ \frac{d^2x}{dt^2} &= -\frac{k}{m}x. \end{aligned}$$

Substituting the equations of motion for x and a gives us

$$-A\omega^2 \cos(\omega t + \phi) = -\frac{k}{m} A \cos(\omega t + \phi). \tag{2.2.13}$$

Cancelling out like terms and solving for the angular frequency yields

$$\omega = \sqrt{\frac{k}{m}}. \tag{2.2.14}$$

The angular frequency depends only on the force constant and the mass, and not the amplitude. The angular frequency is defined as $\omega = \frac{2\pi}{T}$, which yields an equation for the period of the motion:

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{2.2.15}$$

The period also depends only on the mass and the force constant. The greater the mass, the longer the period. The stiffer the spring, the shorter the period. The frequency is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{2.2.16}$$

Vertical Motion and a Horizontal Spring

When a spring is hung vertically and a block is attached and set in motion, the block oscillates in SHM. In this case, there is no normal force, and the net effect of the force of gravity is to change the equilibrium position. Consider Figure $2.2.8$. Two forces act on the block: the weight and the force of the spring. The weight is constant and the force of the spring changes as the length of the spring changes.

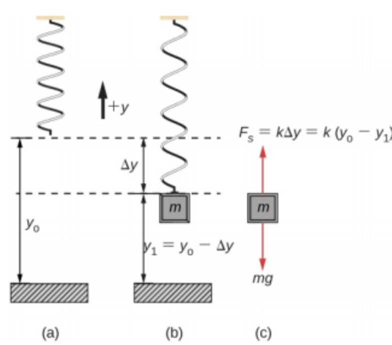


Figure $2.2.8$: A spring is hung from the ceiling. When a block is attached, the block is at the equilibrium position where the weight of the block is equal to the force of the spring. (a) The spring is hung from the ceiling and the equilibrium position is marked as yo . (b) A mass is attached to the spring and a new equilibrium position is reached ($y_1 = y_0 - \Delta y$) when the force provided by the spring equals the weight of the mass. (c) The free-body diagram of the mass shows the two forces acting on the mass: the weight and the force of the spring.

When the block reaches the equilibrium position, as seen in Figure 2.2.8, the force of the spring equals the weight of the block, $F_{net} = F_s - mg = 0$, where

$$-k(-\Delta y) = mg. \tag{2.2.17}$$

From the figure, the change in the position is $\Delta y = y_0 - y_1$ and since $-k(-\Delta y) = mg$, we have

$$k(y_0 - y_1) - mg = 0. \tag{2.2.18}$$

If the block is displaced and released, it will oscillate around the new equilibrium position. As shown in Figure 2.2.9, if the position of the block is recorded as a function of time, the recording is a periodic function. If the block is displaced to a position y, the net force becomes $F_{net} = k(y_0 - y) - mg$. But we found that at the equilibrium position, $mg = k\Delta y = ky_0 - ky_1$. Substituting for the weight in the equation yields

$$F_{net} = ky_0 - ky - (ky_0 - ky_1) = k(y_1 - y). \tag{2.2.19}$$

Recall that y_1 is just the equilibrium position and any position can be set to be the point y = 0.00 m. So let's set y_1 to y = 0.00 m. The net force then becomes

$$\begin{aligned} F_{net} &= -ky; \\ m\frac{d^2y}{dt^2} &= -ky. \end{aligned}$$

This is just what we found previously for a horizontally sliding mass on a spring. The constant force of gravity only served to shift the equilibrium location of the mass. Therefore, the solution should be the same form as for a block on a horizontal spring, $y(t) = A\cos(\omega t + \phi)$. The equations for the velocity and the acceleration also have the same form as for the horizontal case. Note that the inclusion of the phase shift means that the motion can actually be modeled using either a cosine or a sine function, since these two functions only differ by a phase shift.

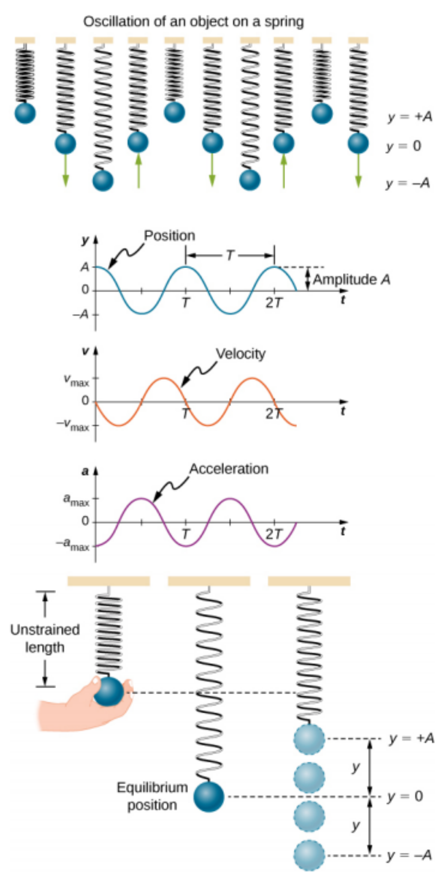


Figure 2.2.9: Graphs of y(t), v(t), and a(t) versus t for the motion of an object on a vertical spring. The net force on the object can be described by Hooke's law, so the object undergoes SHM. Note that the initial position has the vertical displacement at its maximum value A; v is initially zero and then negative as the object moves down; the initial acceleration is negative, back toward the equilibrium position and becomes zero at that point.

2.3: Energy in Simple Harmonic Motion

Learning Objectives

- Describe the energy conservation of the system of a mass and a spring
- Explain the concepts of stable and unstable equilibrium points

To produce a deformation in an object, we must do work. That is, whether you pluck a guitar string or compress a car's shock absorber, a force must be exerted through a distance. If the only result is deformation, and no work goes into thermal, sound, or kinetic energy, then all the work is initially stored in the deformed object as some form of potential energy.

Consider the example of a block attached to a spring on a frictionless table, oscillating in SHM. The force of the spring is a conservative force (which you studied in the chapter on potential energy and conservation of energy), and we can define a potential energy for it. This potential energy is the energy stored in the spring when the spring is extended or compressed. In this case, the block oscillates in one dimension with the force of the spring acting parallel to the motion:

$$W=\int_{x_i}^{x_f} F_x dx \int_{x_i}^{x_f} -kx dx=\left[-\frac{1}{2} k x^2\right]_{x_i}^{x_f}=-\left[\frac{1}{2} k x_f^2-\frac{1}{2} k x_i^2\right]=-\left[U_f-U_i\right]=-\Delta U. \tag{2.3.1}$$

When considering the energy stored in a spring, the equilibrium position, marked as $x_i = 0.00$ m, is the position at which the energy stored in the spring is equal to zero. When the spring is stretched or compressed a distance x, the potential energy stored in the spring is

$$U=\frac{1}{2} k x^2. \tag{2.3.2}$$

Energy and the Simple Harmonic Oscillator

To study the energy of a simple harmonic oscillator, we need to consider all the forms of energy. Consider the example of a block attached to a spring, placed on a frictionless surface, oscillating in SHM. The potential energy stored in the deformation of the spring is

$$U=\frac{1}{2} k x^2. \tag{2.3.3}$$

In a simple harmonic oscillator, the energy oscillates between kinetic energy of the mass $K = \frac{1}{2}mv^2$ and potential energy $U = \frac{1}{2}kx^2$ stored in the spring. In the SHM of the mass and spring system, there are no dissipative forces, so the total energy is the sum of the potential energy and kinetic energy. In this section, we consider the conservation of energy of the system. The concepts examined are valid for all simple harmonic oscillators, including those where the gravitational force plays a role.

Consider Figure $2.3.1$, which shows an oscillating block attached to a spring. In the case of undamped SHM, the energy oscillates back and forth between kinetic and potential, going completely from one form of energy to the other as the system oscillates. So for the simple example of an object on a frictionless surface attached to a spring, the motion starts with all of the energy stored in the spring as **elastic potential energy**. As the object starts to move, the elastic potential energy is converted into kinetic energy, becoming entirely kinetic energy at the equilibrium position. The energy is then converted back into elastic potential energy by the spring as it is stretched or compressed. The velocity becomes zero when the kinetic energy is completely converted, and this cycle then repeats. Understanding the conservation of energy in these cycles will provide extra insight here and in later applications of SHM, such as alternating circuits.

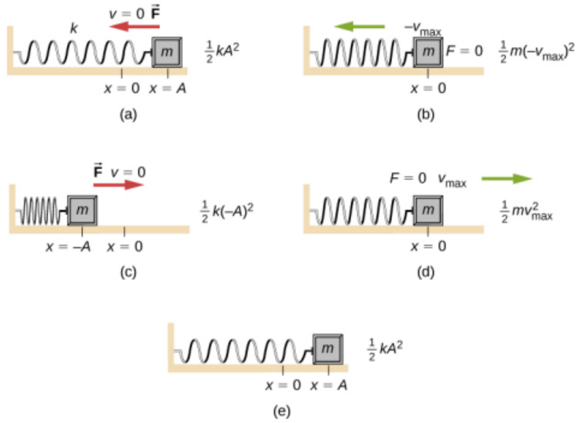


Figure 2.3.1: The transformation of energy in SHM for an object attached to a spring on a frictionless surface. (a) When the mass is at the position x = + A, all the energy is stored as potential energy in the spring $U = \frac{1}{2}kA^2$. The kinetic energy is equal to zero because the velocity of the mass is zero. (b) As the mass moves toward x = −A, the mass crosses the position x = 0. At this point, the spring is neither extended nor compressed, so the potential energy stored in the spring is zero. At x = 0, the total energy is all kinetic energy where $K = \frac{1}{2}m(-v_{max})^2$. (c) The mass continues to move until it reaches x = −A where the mass stops and starts moving toward x = + A. At the position x = −A, the total energy is stored as potential energy in the compressed $U = \frac{1}{2}k(-A)^2$ and the kinetic energy is zero. (d) As the mass passes through the position x = 0, the kinetic energy is $K = \frac{1}{2}mv_{max}^2$ and the potential energy stored in the spring is zero. (e) The mass returns to the position x = + A, where K = 0 and $U = \frac{1}{2}kA^2$.

Consider Figure 2.3.1 , which shows the energy at specific points on the periodic motion. While staying constant, the energy oscillates between the kinetic energy of the block and the potential energy stored in the spring:

$$E_{Total} = U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2. \tag{2.3.4}$$

The motion of the block on a spring in SHM is defined by the position $x(t) = A\cos\omega t + \phi)$ with a velocity of $v(t) = -A\omega\sin(\omega t + \phi)$. Using these equations, the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ and $\omega = \sqrt{\frac{k}{m}}$, we can find the total energy of the system:

$$\begin{aligned} E_{Total} &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}mA^2\left(\frac{k}{m}\right)\sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}kA^2\sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2(\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)) \\ &= \frac{1}{2}kA^2. \end{aligned}$$

The total energy of the system of a block and a spring is equal to the sum of the potential energy stored in the spring plus the kinetic energy of the block and is proportional to the square of the amplitude $E_{Total} = \left(\frac{1}{2}\right)kA^2$. The total energy of the system is constant.

A closer look at the energy of the system shows that the kinetic energy oscillates like a sine-squared function, while the potential energy oscillates like a cosine-squared function. However, the total energy for the system is constant and is proportional to the amplitude squared. Figure 2.3.2 shows a plot of the potential, kinetic, and total energies of the block and spring system as a function of time. Also plotted are the position and velocity as a function of time. Before time t = 0.0 s, the block is attached to the spring and placed at the equilibrium position. Work is done on the block by applying an external force, pulling it out to a position of x = + A. The system now has potential energy stored in the spring. At time t = 0.00 s, the position of the block is equal to the amplitude, the potential energy stored in the spring is equal to $U = \frac{1}{2}kA^2$, and the force on the block is maximum and points in the negative x-direction ($F_S = -kA$). The velocity and kinetic energy of the block are zero at time t = 0.00 s. At time t = 0.00 s, the block is released from rest.

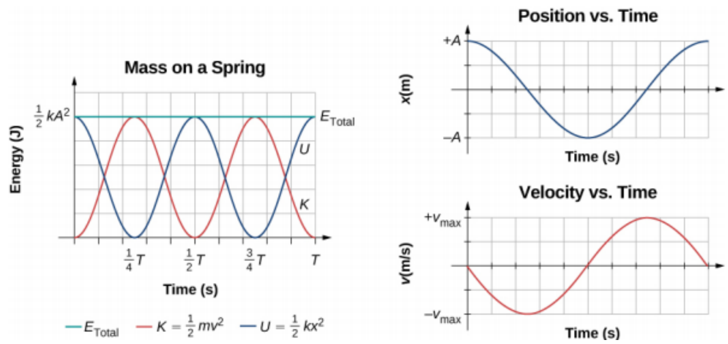


Figure 2.3.2: Graph of the kinetic energy, potential energy, and total energy of a block oscillating on a spring in SHM. Also shown are the graphs of position versus time and velocity versus time. The total energy remains constant, but the energy oscillates between kinetic energy and potential energy. When the kinetic energy is maximum, the potential energy is zero. This occurs when the velocity is maximum and the mass is at the equilibrium position. The potential energy is maximum when the speed is zero. The total energy is the sum of the kinetic energy plus the potential energy and it is constant.

Oscillations About an Equilibrium Position

We have just considered the energy of SHM as a function of time. Another interesting view of the simple harmonic oscillator is to consider the energy as a function of position. Figure 2.3.3 shows a graph of the energy versus position of a system undergoing SHM.

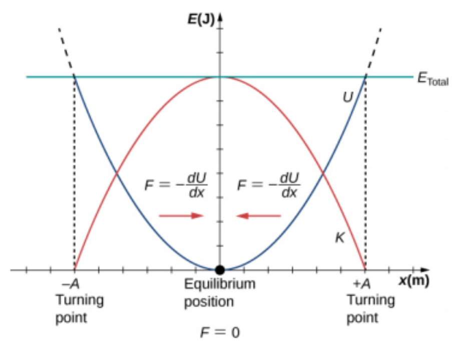


Figure 2.3.3: A graph of the kinetic energy (red), potential energy (blue), and total energy (green) of a simple harmonic oscillator. The force is equal to $F = -\frac{dU}{dx}$. The equilibrium position is shown as a black dot and is the point where the force is equal to zero. The force is positive when $x < 0$, negative when $x > 0$, and equal to zero when $x = 0$.

The potential energy curve in Figure 2.3.3 resembles a bowl. When a marble is placed in a bowl, it settles to the equilibrium position at the lowest point of the bowl ($x = 0$). This happens because a restoring force points toward the equilibrium point. This equilibrium point is sometimes referred to as a fixed point. When the marble is disturbed to a different position ($x = +A$), the marble oscillates around the equilibrium position. Looking back at the graph of potential energy, the force can be found by looking at the slope of the potential energy graph ($F = -\frac{dU}{dx}$). Since the force on either side of the fixed point points back toward the equilibrium point, the equilibrium point is called a **stable equilibrium point**. The points $x = A$ and $x = -A$ are called the turning points. (See Potential Energy and Conservation of Energy.) Stability is an important concept. If an equilibrium point is stable, a slight disturbance of an object that is initially at the stable equilibrium point will cause the object to oscillate around that point. The stable equilibrium point occurs because the force on either side is directed toward it. For an unstable equilibrium point, if the object is disturbed slightly, it does not return to the equilibrium point.

Consider the marble in the bowl example. If the bowl is right-side up, the marble, if disturbed slightly, will oscillate around the stable equilibrium point. If the bowl is turned upside down, the marble can be balanced on the top, at the equilibrium point where the net force is zero. However, if the marble is disturbed slightly, it will not return to the equilibrium point, but will instead roll off the bowl. The reason is that the force on either side of the equilibrium point is directed away from that point. This point is an unstable equilibrium point.

Figure 2.3.4 shows three conditions. The first is a stable equilibrium point (a), the second is an unstable equilibrium point (b), and the last is also an unstable equilibrium point (c), because the force on only one side points toward the equilibrium point.

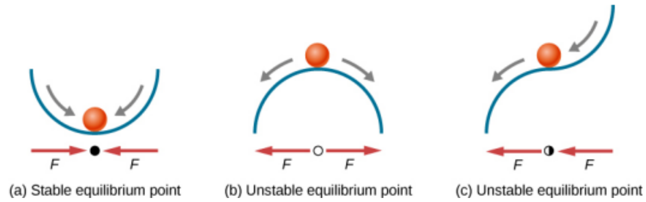


Figure 2.3.4: Examples of equilibrium points. (a) Stable equilibrium point; (b) unstable equilibrium point; (c) unstable equilibrium point (sometimes referred to as a half-stable equilibrium point).

The process of determining whether an equilibrium point is stable or unstable can be formalized. Consider the potential energy curves shown in Figure 2.3.5. The force can be found by analyzing the slope of the graph. The force is F = $-\frac{dU}{dx}$. In (a), the fixed point is at x = 0.00 m. When x < 0.00 m, the force is positive. When x > 0.00 m, the force is negative. This is a stable point. In (b), the fixed point is at x = 0.00 m. When x < 0.00 m, the force is negative. When x > 0.00 m, the force is also negative. This is an unstable point.

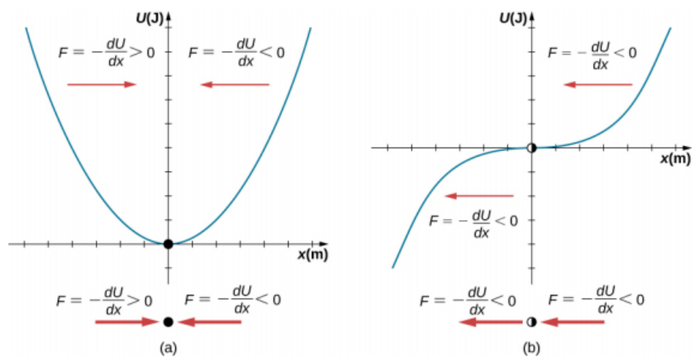


Figure 2.3.5: Two examples of a potential energy function. The force at a position is equal to the negative of the slope of the graph at that position. (a) A potential energy function with a stable equilibrium point. (b) A potential energy function with an unstable equilibrium point. This point is sometimes called half-stable because the force on one side points toward the fixed point.

A practical application of the concept of stable equilibrium points is the force between two neutral atoms in a molecule. If two molecules are in close proximity, separated by a few atomic diameters, they can experience an attractive force. If the molecules move close enough so that the electron shells of the other electrons overlap, the force between the molecules becomes repulsive. The attractive force between the two atoms may cause the atoms to form a molecule. The force between the two molecules is not a linear force and cannot be modeled simply as two masses separated by a spring, but the atoms of the molecule can oscillate around an equilibrium point when displaced a small amount from the equilibrium position. The atoms oscillate due the attractive force and repulsive force between the two atoms.

Consider one example of the interaction between two atoms known as the van Der Waals interaction. It is beyond the scope of this chapter to discuss in depth the interactions of the two atoms, but the oscillations of the atoms can be examined by considering one example of a model of the potential energy of the system. One suggestion to model the potential energy of this molecule is with the Lennard-Jones 6-12 potential:

$$U(x) = 4\epsilon\left[\left(\frac{\sigma}{x}\right)^{12} - \left(\frac{\sigma}{x}\right)^{6}\right]. \tag{2.3.5}$$

A graph of this function is shown in Figure 2.3.6. The two parameters ϵ and σ are found experimentally.

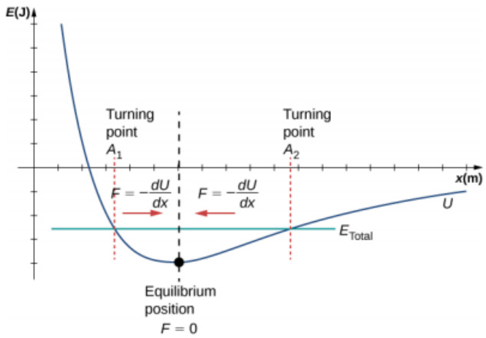


Figure $2.3.6$: The Lennard-Jones potential energy function for a system of two neutral atoms. If the energy is below some maximum energy, the system oscillates near the equilibrium position between the two turning points.

From the graph, you can see that there is a potential energy well, which has some similarities to the potential energy well of the potential energy function of the simple harmonic oscillator discussed in Figure $2.3.3$. The Lennard-Jones potential has a stable equilibrium point where the potential energy is minimum and the force on either side of the equilibrium point points toward equilibrium point. Note that unlike the simple harmonic oscillator, the potential well of the Lennard-Jones potential is not symmetric. This is due to the fact that the force between the atoms is not a Hooke's law force and is not linear. The atoms can still oscillate around the equilibrium position x_{min} because when $x < x_{min}$, the force is positive; when $x > x_{min}$, the force is negative. Notice that as x approaches zero, the slope is quite steep and negative, which means that the force is large and positive. This suggests that it takes a large force to try to push the atoms close together. As x becomes increasingly large, the slope becomes less steep and the force is smaller and negative. This suggests that if given a large enough energy, the atoms can be separated.

If you are interested in this interaction, find the force between the molecules by taking the derivative of the potential energy function. You will see immediately that the force does not resemble a Hooke's law force ($F = -kx$), but if you are familiar with the binomial theorem:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots, \tag{2.3.6}$$

the force can be approximated by a Hooke's law force.

Velocity and Energy Conservation

Getting back to the system of a block and a spring in Figure $2.3.1$, once the block is released from rest, it begins to move in the negative direction toward the equilibrium position. The potential energy decreases and the magnitude of the velocity and the kinetic energy increase. At time $t = \frac{T}{4}$, the block reaches the equilibrium position $x = 0.00$ m, where the force on the block and the potential energy are zero. At the equilibrium position, the block reaches a negative velocity with a magnitude equal to the maximum velocity $v = -A\omega$. The kinetic energy is maximum and equal to $K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega\omega^2 = \frac{1}{2}kA^2$. At this point, the force on the block is zero, but momentum carries the block, and it continues in the negative direction toward $x = -A$. As the block continues to move, the force on it acts in the positive direction and the magnitude of the velocity and kinetic energy decrease. The potential energy increases as the spring compresses. At time $t = \frac{T}{2}$, the block reaches $x = -A$. Here the velocity and kinetic energy are equal to zero. The force on the block is $F = +kA$ and the potential energy stored in the spring is $U = \frac{1}{2}kA^2$. During the oscillations, the total energy is constant and equal to the sum of the potential energy and the kinetic energy of the system,

$$E_{Total} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2. \tag{2.3.7}$$

The equation for the energy associated with SHM can be solved to find the magnitude of the velocity at any position:

$$|v| = \sqrt{\frac{k}{m}(A^2 - x^2)}. \tag{2.3.8}$$

The energy in a simple harmonic oscillator is proportional to the square of the amplitude. When considering many forms of oscillations, you will find the energy proportional to the amplitude squared.

❓ Exercise 15.1

Why would it hurt more if you snapped your hand with a ruler than with a loose spring, even if the displacement of each system is equal?

❓ Exercise 15.2

Identify one way you could decrease the maximum velocity of a simple harmonic oscillator.

2.4: Comparing Simple Harmonic Motion and Circular Motion

Learning Objectives

- Describe how the sine and cosine functions relate to the concepts of circular motion
- Describe the connection between simple harmonic motion and circular motion

An easy way to model Simple Harmonic Motion (SHM) is by considering uniform circular motion. Figure $2.4.1$ shows one way of using this method. A peg (a cylinder of wood) is attached to a vertical disk, rotating with a constant angular frequency.

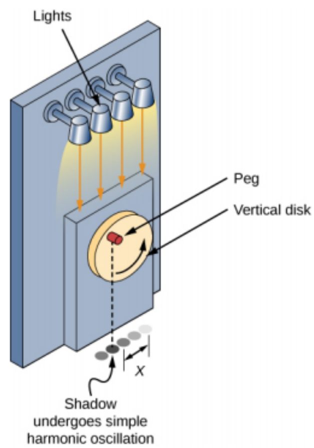
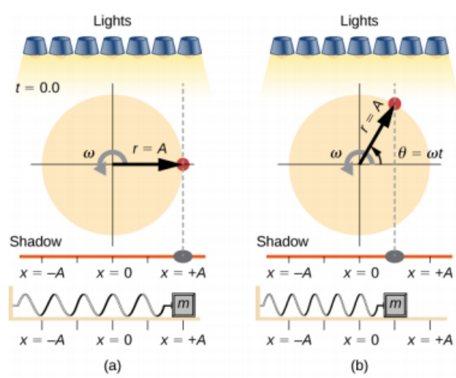


Figure $2.4.1$: SHM can be modeled as rotational motion by looking at the shadow of a peg on a wheel rotating at a constant angular frequency.

Figure $2.4.2$ shows a side view of the disk and peg. If a lamp is placed above the disk and peg, the peg produces a shadow. Let the disk have a radius of r = A and define the position of the shadow that coincides with the center line of the disk to be x = 0.00 m. As the disk rotates at a constant rate, the shadow oscillates between x = + A and x = −A. Now imagine a block on a spring beneath the floor as shown in Figure $2.4.2$.



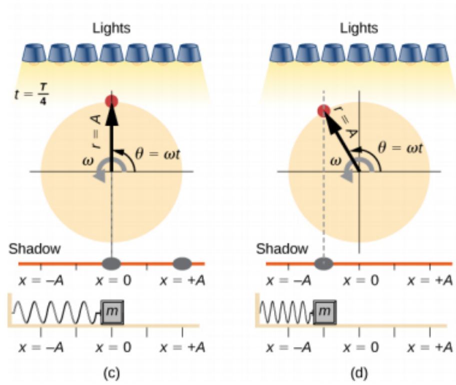


Figure 2.4.2: Light shines down on the disk so that the peg makes a shadow. If the disk rotates at just the right angular frequency, the shadow follows the motion of the block on a spring. If there is no energy dissipated due to nonconservative forces, the block and the shadow will oscillate back and forth in unison. In this figure, four snapshots are taken at four different times. (a) The wheel starts at θ = 0° and the shadow of the peg is at x = + A, representing the mass at position x = + A. (b) As the disk rotates through an angle $\theta = \omega$t, the shadow of the peg is between x = + A and x = 0. (c) The disk continues to rotate until θ = 90°, at which the shadow follows the mass to x = 0. (d) The disk continues to rotate, the shadow follows the position of the mass.

If the disk turns at the proper angular frequency, the shadow follows along with the block. The position of the shadow can be modeled with the equation

$$x(t) = A\cos(\omega t). \tag{2.4.1}$$

Recall that the block attached to the spring does not move at a constant velocity. How often does the wheel have to turn to have the peg's shadow always on the block? The disk must turn at a constant angular frequency equal to 2π times the frequency of oscillation (ω = 2πf).

Figure 2.4.3 shows the basic relationship between uniform circular motion and SHM. The peg lies at the tip of the radius, a distance A from the center of the disk. The x-axis is defined by a line drawn parallel to the ground, cutting the disk in half. The y-axis (not shown) is defined by a line perpendicular to the ground, cutting the disk into a left half and a right half. The center of the disk is the point (x = 0, y = 0). The projection of the position of the peg onto the fixed x-axis gives the position of the shadow, which undergoes SHM analogous to the system of the block and spring. At the time shown in the figure, the projection has position x and moves to the left with velocity v. The tangential velocity of the peg around the circle equals $\bar{v}_{max}$ of the block on the spring. The x-component of the velocity is equal to the velocity of the block on the spring.

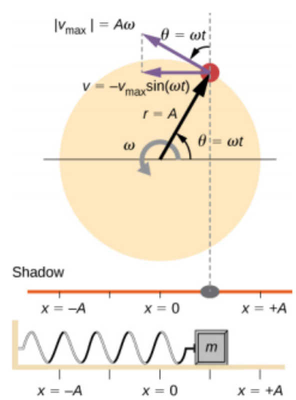


Figure 2.4.3: - A peg moving on a circular path with a constant angular velocity ω is undergoing uniform circular motion. Its projection on the x-axis undergoes SHM. Also shown is the velocity of the peg around the circle, v_{max}, and its projection, which is v. Note that these velocities form a similar triangle to the displacement triangle.

We can use Figure 2.4.3 to analyze the velocity of the shadow as the disk rotates. The peg moves in a circle with a speed of v_{max} = Aω. The shadow moves with a velocity equal to the component of the peg's velocity that is parallel to the surface where the shadow is being produced:

$$v = -v_{max}\sin(\omega t). \tag{2.4.2}$$

It follows that the acceleration is

$$a = -a_{max}\cos(\omega t). \tag{2.4.3}$$

? Exercise 15.3

Identify an object that undergoes uniform circular motion. Describe how you could trace the SHM of this object.

2.5: Traveling Waves

Learning Objectives

- Describe the basic characteristics of wave motion
- Define the terms wavelength, amplitude, period, frequency, and wave speed
- Explain the difference between longitudinal and transverse waves, and give examples of each type
- List the different types of waves

We saw in Oscillations that oscillatory motion is an important type of behavior that can be used to model a wide range of physical phenomena. Oscillatory motion is also important because oscillations can generate waves, which are of fundamental importance in physics. Many of the terms and equations we studied in the chapter on oscillations apply equally well to wave motion (Figure 2.5.1).

Figure 2.5.1: An ocean wave is probably the first picture that comes to mind when you hear the word "wave." Although this breaking wave, and ocean waves in general, have apparent similarities to the basic wave characteristics we will discuss, the mechanisms driving ocean waves are highly complex and beyond the scope of this chapter. It may seem natural, and even advantageous, to apply the concepts in this chapter to ocean waves, but ocean waves are nonlinear, and the simple models presented in this chapter do not fully explain them. (credit: Steve Jurvetson)

Types of Waves

A **wave** is a disturbance that propagates, or moves from the place it was created. There are three basic types of waves: mechanical waves, electromagnetic waves, and matter waves.

Basic **mechanical waves** are governed by Newton's laws and require a medium. A medium is the substance a mechanical waves propagates through, and the medium produces an elastic restoring force when it is deformed. Mechanical waves transfer energy and momentum, without transferring mass. Some examples of mechanical waves are water waves, sound waves, and seismic waves. The medium for water waves is water; for sound waves, the medium is usually air. (Sound waves can travel in other media as well; we will look at that in more detail in Sound.) For surface water waves, the disturbance occurs on the surface of the water, perhaps created by a rock thrown into a pond or by a swimmer splashing the surface repeatedly. For sound waves, the disturbance is a change in air pressure, perhaps created by the oscillating cone inside a speaker or a vibrating tuning fork. In both cases, the disturbance is the oscillation of the molecules of the fluid. In mechanical waves, energy and momentum transfer with the motion of the wave, whereas the mass oscillates around an equilibrium point. (We discuss this in Energy and Power of a Wave.) Earthquakes generate seismic waves from several types of disturbances, including the disturbance of Earth's surface and pressure disturbances under the surface. Seismic waves travel through the solids and liquids that form Earth. In this chapter, we focus on mechanical waves.

Electromagnetic waves are associated with oscillations in electric and magnetic fields and do not require a medium. Examples include gamma rays, X-rays, ultraviolet waves, visible light, infrared waves, microwaves, and radio waves. Electromagnetic waves can travel through a vacuum at the speed of light, $v = c = 2.99792458 \times 10^8$ m/s. For example, light from distant stars travels through the vacuum of space and reaches Earth. Electromagnetic waves have some characteristics that are similar to mechanical waves; they are covered in more detail in Electromagnetic Waves.

Matter waves are a central part of the branch of physics known as quantum mechanics. These waves are associated with protons, electrons, neutrons, and other fundamental particles found in nature. The theory that all types of matter have wavelike properties was first proposed by Louis de Broglie in 1924. Matter waves are discussed in Photons and Matter Waves.

Mechanical Waves

Mechanical waves exhibit characteristics common to all waves, such as amplitude, wavelength, period, frequency, and energy. All wave characteristics can be described by a small set of underlying principles.

The simplest mechanical waves repeat themselves for several cycles and are associated with simple harmonic motion. These simple harmonic waves can be modeled using some combination of sine and cosine functions. For example, consider the simplified surface water wave that moves across the surface of water as illustrated in Figure $2.5.2$. Unlike complex ocean waves, in surface water waves, the medium, in this case water, moves vertically, oscillating up and down, whereas the disturbance of the wave moves horizontally through the medium. In Figure $2.5.2$, the waves causes a seagull to move up and down in simple harmonic motion as the wave crests and troughs (peaks and valleys) pass under the bird. The crest is the highest point of the wave, and the trough is the lowest part of the wave. The time for one complete oscillation of the up-and-down motion is the wave's period T. The wave's frequency is the number of waves that pass through a point per unit time and is equal to f = $\frac{1}{T}$. The period can be expressed using any convenient unit of time but is usually measured in seconds; frequency is usually measured in hertz (Hz), where 1 Hz = 1 s^{-1}.

The length of the wave is called the **wavelength** and is represented by the Greek letter lambda (λ), which is measured in any convenient unit of length, such as a centimeter or meter. The wavelength can be measured between any two similar points along the medium that have the same height and the same slope. In Figure $2.5.2$, the wavelength is shown measured between two crests. As stated above, the period of the wave is equal to the time for one oscillation, but it is also equal to the time for one wavelength to pass through a point along the wave's path.

The amplitude of the wave (A) is a measure of the maximum displacement of the medium from its equilibrium position. In the figure, the equilibrium position is indicated by the dotted line, which is the height of the water if there were no waves moving through it. In this case, the wave is symmetrical, the crest of the wave is a distance +A above the equilibrium position, and the trough is a distance −A below the equilibrium position. The units for the amplitude can be centimeters or meters, or any convenient unit of distance.

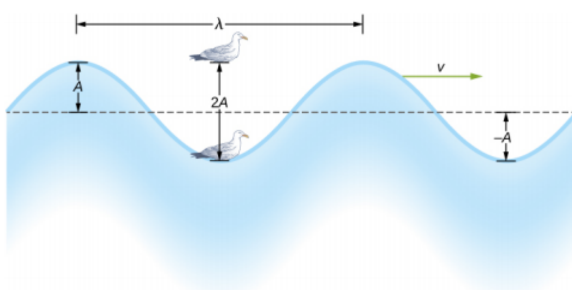


Figure $2.5.2$: An idealized surface water wave passes under a seagull that bobs up and down in simple harmonic motion. The wave has a wavelength λ, which is the distance between adjacent identical parts of the wave. The amplitude A of the wave is the maximum displacement of the wave from the equilibrium position, which is indicated by the dotted line. In this example, the medium moves up and down, whereas the disturbance of the surface propagates parallel to the surface at a speed v.

The water wave in the figure moves through the medium with a propagation velocity $\vec{v}$. The magnitude of the **wave velocity** is the distance the wave travels in a given time, which is one wavelength in the time of one period, and the **wave speed** is the magnitude of wave velocity. In equation form, this is

$$v=\frac{\lambda}{T}=\lambda f. \tag{2.5.1}$$

This fundamental relationship holds for all types of waves. For water waves, v is the speed of a surface wave; for sound, v is the speed of sound; and for visible light, v is the speed of light.

Transverse and Longitudinal Waves

We have seen that a simple mechanical wave consists of a periodic disturbance that propagates from one place to another through a medium. In Figure $2.5.3$(a), the wave propagates in the horizontal direction, whereas the medium is disturbed in the vertical direction. Such a wave is called a **transverse wave**. In a transverse wave, the wave may propagate in any direction, but the disturbance of the medium is perpendicular to the direction of propagation. In contrast, in a **longitudinal wave** or compressional wave, the disturbance is parallel to the direction of propagation. Figure 2.5.3(b) shows an example of a longitudinal wave. The size of the disturbance is its amplitude A and is completely independent of the speed of propagation v.

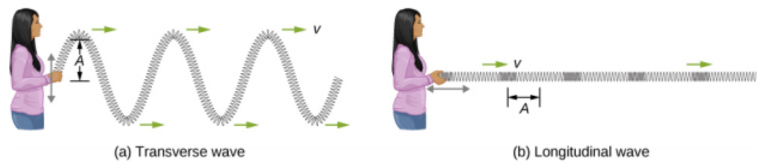


Figure 2.5.3: (a) In a transverse wave, the medium oscillates perpendicular to the wave velocity. Here, the spring moves vertically up and down, while the wave propagates horizontally to the right. (b) In a longitudinal wave, the medium oscillates parallel to the propagation of the wave. In this case, the spring oscillates back and forth, while the wave propagates to the right.

A simple graphical representation of a section of the spring shown in Figure 2.5.3(b) is shown in Figure 2.5.4. Figure 2.5.4(a) shows the equilibrium position of the spring before any waves move down it. A point on the spring is marked with a blue dot. Figure 2.5.4(b) through (g) show snapshots of the spring taken one-quarter of a period apart, sometime after the end of` the spring is oscillated back and forth in the x-direction at a constant frequency. The disturbance of the wave is seen as the compressions and the expansions of the spring. Note that the blue dot oscillates around its equilibrium position a distance A, as the longitudinal wave moves in the positive x-direction with a constant speed. The distance A is the amplitude of the wave. The y-position of the dot does not change as the wave moves through the spring. The wavelength of the wave is measured in part (d). The wavelength depends on the speed of the wave and the frequency of the driving force.

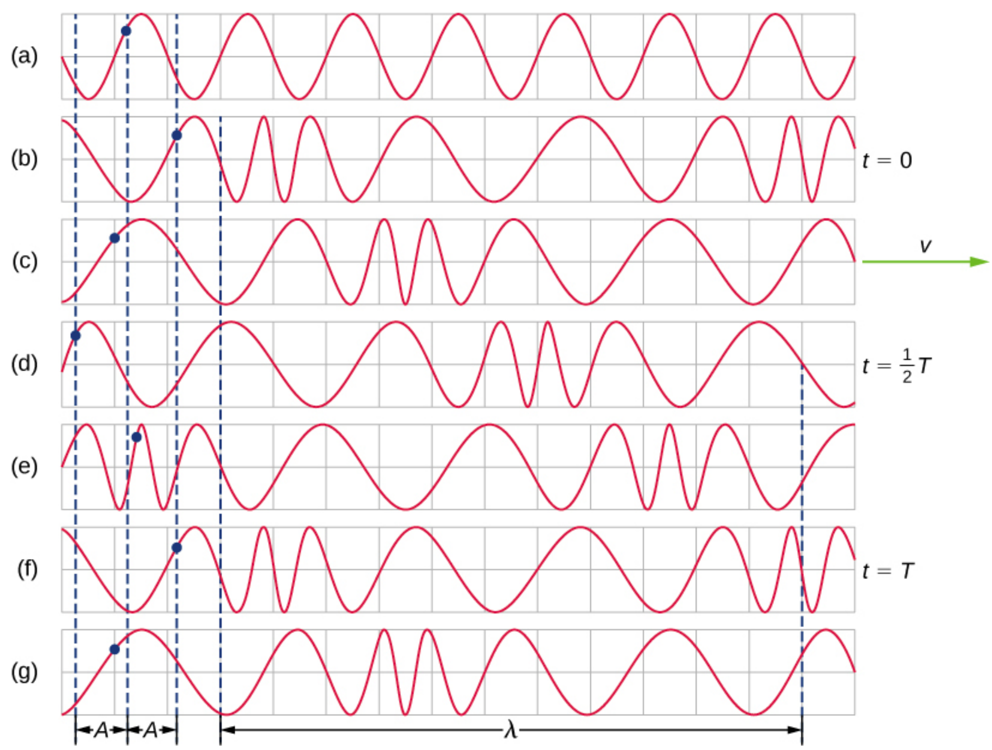


Figure 2.5.4: (a) This is a simple, graphical representation of a section of the stretched spring shown in Figure 2.5.3(b), representing the spring's equilibrium position before any waves are induced on the spring. A point on the spring is marked by a blue dot. (b–g) Longitudinal waves are created by oscillating the end of the spring (not shown) back and forth along the x-axis. The longitudinal wave, with a wavelength λ, moves along the spring in the +x-direction with a wave speed v. For convenience, the wavelength is measured in (d). Note that the point on the spring that was marked with the blue dot moves back and forth a distance A from the equilibrium position, oscillating around the equilibrium position of the point.

Waves may be transverse, longitudinal, or a combination of the two. Examples of transverse waves are the waves on stringed instruments or surface waves on water, such as ripples moving on a pond. Sound waves in air and water are longitudinal. With sound waves, the disturbances are periodic variations in pressure that are transmitted in fluids. Fluids do not have appreciable shear strength, and for this reason, the sound waves in them are longitudinal waves. Sound in solids can have both longitudinal and transverse components, such as those in a seismic wave. Earthquakes generate seismic waves under Earth's surface with both longitudinal and transverse components (called compressional or P-waves and shear or S-waves, respectively). The components of seismic waves have important individual characteristics—they propagate at different speeds, for example. Earthquakes also have surface waves that are similar to surface waves on water. Ocean waves also have both transverse and longitudinal components.

Example 16.1: Wave on a String

A student takes a 30.00-m-long string and attaches one end to the wall in the physics lab. The student then holds the free end of the rope, keeping the tension constant in the rope. The student then begins to send waves down the string by moving the end of the string up and down with a frequency of 2.00 Hz. The maximum displacement of the end of the string is 20.00 cm. The first wave hits the lab wall 6.00 s after it was created. (a) What is the speed of the wave? (b) What is the period of the wave? (c) What is the wavelength of the wave?

Strategy

a. The speed of the wave can be derived by dividing the distance traveled by the time.
b. The period of the wave is the inverse of the frequency of the driving force.
c. The wavelength can be found from the speed and the period v = $\frac{\lambda}{T}$.

Solution

a. The first wave traveled 30.00 m in 6.00 s:

$$v = \frac{30.00\ m}{6.00\ s} = 5.00\ m/s. \tag{2.5.2}$$

b. . The period is equal to the inverse of the frequency:

$$T = \frac{1}{f} = \frac{1}{2.00\ s^{-1}} = 0.50\ s. \tag{2.5.3}$$

c. The wavelength is equal to the velocity times the period:

$$\lambda = vT = (5.00\ m/s)(0.50\ s) = 2.50\ m. \tag{2.5.4}$$

Significance

The frequency of the wave produced by an oscillating driving force is equal to the frequency of the driving force.

Exercise 16.1

g When a guitar string is plucked, the guitar string oscillates as a result of waves moving through the string. The vibrations of the string cause the air molecules to oscillate, forming sound waves. The frequency of the sound waves is equal to the frequency of the vibrating string. Is the wavelength of the sound wave always equal to the wavelength of the waves on the string?

Example 16.2: Characteristics of a Wave

A transverse mechanical wave propagates in the positive x-direction through a spring (as shown in Figure $2.5.3$(a)) with a constant wave speed, and the medium oscillates between +A and −A around an equilibrium position. The graph in Figure $2.5.5$ shows the height of the spring (y) versus the position (x), where the xaxis points in the direction of propagation. The figure shows the height of the spring versus the x-position at t = 0.00 s as a dotted line and the wave at t = 3.00 s as a solid line. (a) Determine the wavelength and amplitude of the wave. (b) Find the propagation velocity of the wave. (c) Calculate the period and frequency of the wave.

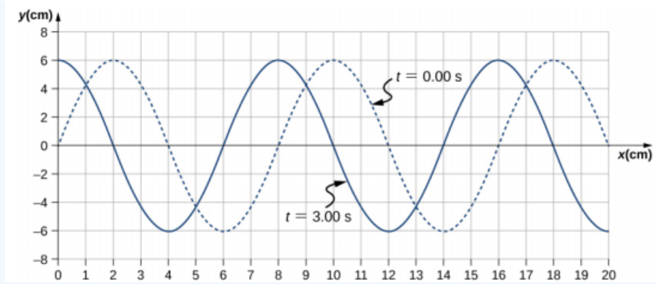


Figure 2.5.5: A transverse wave shown at two instants of time.

Strategy

a. The amplitude and wavelength can be determined from the graph.
b. Since the velocity is constant, the velocity of the wave can be found by dividing the distance traveled by the wave by the time it took the wave to travel the distance.
c. The period can be found from v = $\frac{\lambda}{T}$ and the frequency from f = $\frac{1}{T}$.

Solution

a. Read the wavelength from the graph, looking at the purple arrow in Figure 2.5.6. Read the amplitude by looking at the green arrow. The wavelength is λ = 8.00 cm and the amplitude is A = 6.00 cm.

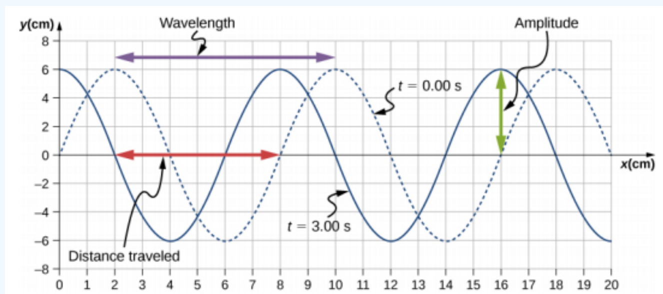


Figure 2.5.6: Characteristics of the wave marked on a graph of its displacement.

b. The distance the wave traveled from time t = 0.00 s to time t = 3.00 s can be seen in the graph. Consider the red arrow, which shows the distance the crest has moved in 3 s. The distance is 8.00 cm − 2.00 cm = 6.00 cm. The velocity is

$$v = \frac{\Delta x}{\Delta t} = \frac{8.00\ cm - 2.00\ cm}{3.00\ s - 0.00\ s} = 2.00\ cm/s. \tag{2.5.5}$$

c. The period is T = $\frac{\lambda}{v}$ = $\frac{8.00\ cm}{2.00\ cm/s}$ = 4.00\; s and the frequency is f = $\frac{1}{T}$ = $\frac{1}{4.00\ s}$ = 0.25 Hz.

Significance

Note that the wavelength can be found using any two successive identical points that repeat, having the same height and slope. You should choose two points that are most convenient. The displacement can also be found using any convenient point.

? Exercise 16.2

The propagation velocity of a transverse or longitudinal mechanical wave may be constant as the wave disturbance moves through the medium. Consider a transverse mechanical wave: Is the velocity of the medium also constant?

2.6: Mathematics of Waves

Learning Objectives

- Model a wave, moving with a constant wave velocity, with a mathematical expression
- Calculate the velocity and acceleration of the medium
- Show how the velocity of the medium differs from the wave velocity (propagation velocity)

In the previous section, we described periodic waves by their characteristics of wavelength, period, amplitude, and wave speed of the wave. Waves can also be described by the motion of the particles of the medium through which the waves move. The position of particles of the medium can be mathematically modeled as **wave functions**, which can be used to find the position, velocity, and acceleration of the particles of the medium of the wave at any time.

Pulses

A **pulse** can be described as wave consisting of a single disturbance that moves through the medium with a constant amplitude. The pulse moves as a pattern that maintains its shape as it propagates with a constant wave speed. Because the wave speed is constant, the distance the pulse moves in a time Δt is equal to Δx = vΔt (Figure $2.6.1$).

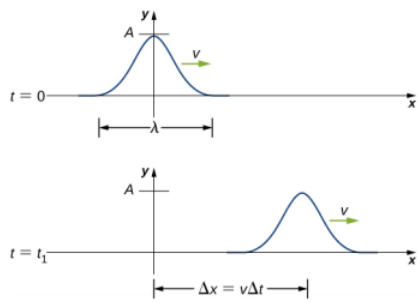


Figure $2.6.1$: The pulse at time t = 0 is centered on x = 0 with amplitude *A*. The pulse moves as a pattern with a constant shape, with a constant maximum value *A*. The velocity is constant and the pulse moves a distance Δx = vΔt in a time Δt. The distance traveled is measured with any convenient point on the pulse. In this figure, the crest is used.

Modeling a One-Dimensional Sinusoidal Wave Using a Wave Function

Consider a string kept at a constant tension F_T where one end is fixed and the free end is oscillated between $y = +A$ and $y = -A$ by a mechanical device at a constant frequency. Figure $2.6.2$ shows snapshots of the wave at an interval of an eighth of a period, beginning after one period ($t = T$).

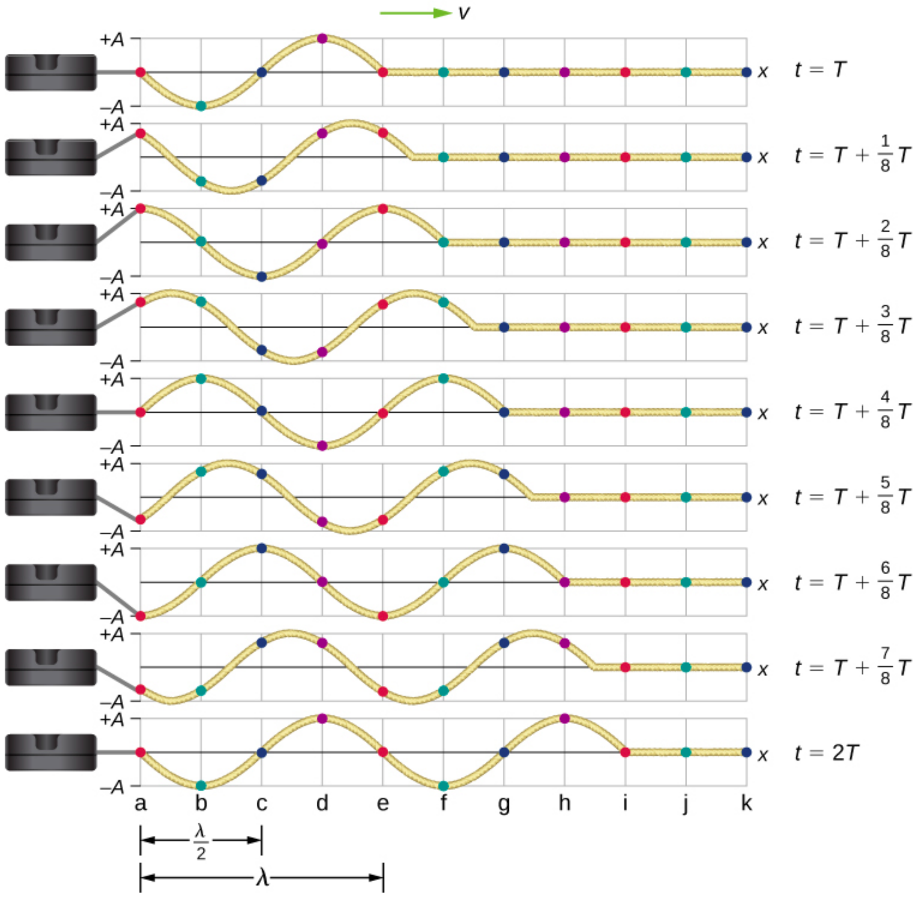


Figure $2.6.2$: Snapshots of a transverse wave moving through a string under tension, beginning at time $t = T$ and taken at intervals of $\frac{1}{8}T$. Colored dots are used to highlight points on the string. Points that are a wavelength apart in the *x*-direction are highlighted with the same color dots.

Notice that each select point on the string (marked by colored dots) oscillates up and down in simple harmonic motion, between $y = +A$ and $y = -A$, with a period T. The wave on the string is sinusoidal and is translating in the positive x-direction as time progresses.

At this point, it is useful to recall from your study of algebra that if $f(x)$ is some function, then $f(x-d)$ is the same function translated in the positive x-direction by a distance d. The function $f(x+d)$ is the same function translated in the negative x-direction by a distance d. We want to define a wave function that will give the y-position of each segment of the string for every position x along the string for every time t.

Looking at the first snapshot in Figure $2.6.2$, the y-position of the string between $x = 0$ and $x = \lambda$ can be modeled as a sine function. This wave propagates down the string one wavelength in one period, as seen in the last snapshot. The wave therefore moves with a constant wave speed of $v = \lambda/T$.

Recall that a sine function is a function of the angle θ, oscillating between +1 and −1, and repeating every 2π radians (Figure $2.6.3$). However, the y-position of the medium, or the wave function, oscillates between $+A$ and $-A$, and repeats every wavelength λ.

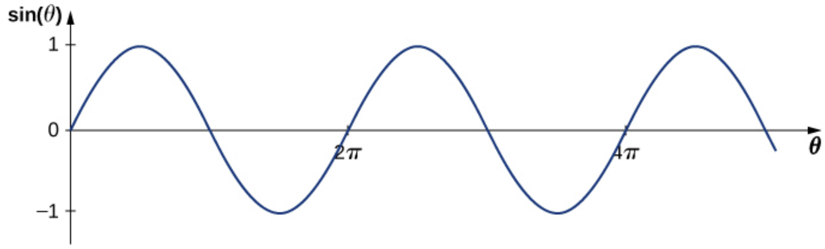


Figure 2.6.3: A sine function oscillates between +1 and −1 every 2π radians.

To construct our model of the wave using a periodic function, consider the ratio of the angle and the position,

$$\frac{\theta}{x} = \frac{2\pi}{\lambda},$$
$$\theta = \frac{2\pi}{\lambda}x.$$

Using $\theta = \frac{2\pi}{\lambda}x$ and multiplying the sine function by the amplitude *A*, we can now model the *y*-position of the string as a function of the position *x*:

$$y(x) = A\sin\left(\frac{2\pi}{\lambda}x\right).$$

The wave on the string travels in the positive *x*-direction with a constant velocity *v*, and moves a distance *vt* in a time *t*. The wave function can now be defined by

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right).$$

It is often convenient to rewrite this wave function in a more compact form. Multiplying through by the ratio $\frac{2\pi}{\lambda}$ leads to the equation

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}vt\right).$$

The value $\frac{2\pi}{\lambda}$ is defined as the **wave number**. The symbol for the wave number is *k* and has units of inverse meters, m^{-1}:

$$k \equiv \frac{2\pi}{\lambda} \tag{2.6.1}$$

Recall from Oscillations that the angular frequency is defined as $\omega \equiv \frac{2\pi}{T}$. The second term of the wave function becomes

$$\frac{2\pi}{\lambda}vt = \frac{2\pi}{\lambda}\left(\frac{\lambda}{T}\right)t = \frac{2\pi}{T}t = \omega t.$$

The wave function for a simple harmonic wave on a string reduces to

$$y(x,t) = A\sin(kx \mp \omega t)$$

where *A* is the amplitude, $k = \frac{2\pi}{\lambda}$ is the wave number, $\omega = \frac{2\pi}{T}$ is the angular frequency, the minus sign is for waves moving in the positive *x*-direction, and the plus sign is for waves moving in the negative *x*-direction. The velocity of the wave is equal to

$$v = \frac{\lambda}{T} = \frac{\lambda}{T}\left(\frac{2\pi}{2\pi}\right) = \frac{\omega}{k}. \tag{2.6.2}$$

Think back to our discussion of a mass on a spring, when the position of the mass was modeled as

$$x(t) = A\cos(\omega t + \phi).$$

The angle ϕ is a phase shift, added to allow for the fact that the mass may have initial conditions other than $x = +A$ and $v = 0$. For similar reasons, the initial phase is added to the wave function. The wave function modeling a sinusoidal wave, allowing for an initial phase shift ϕ, is

$$y(x,t) = A\sin(kx \mp \omega t + \phi) \tag{2.6.3}$$

The value

$$(kx \mp \omega t + \phi) \tag{2.6.4}$$

is known as the phase of the wave, where ϕ is the initial phase of the wave function. Whether the temporal term ωt is negative or positive depends on the direction of the wave. First consider the minus sign for a wave with an initial phase equal to zero (ϕ = 0). The phase of the wave would be $(kx = \omega t)$. Consider following a point on a wave, such as a crest. A crest will occur when $\sin(kx - \omega t = 1.00$, that is, when $kx - \omega t = n\pi + \frac{\pi}{2}$, for any integral value of n. For instance, one particular crest occurs at $kx - \omega t = \frac{\pi}{2}$. As the wave moves, time increases and x must also increase to keep the phase equal to $\frac{\pi}{2}$. Therefore, the minus sign is for a wave moving in the positive x-direction. Using the plus sign, $kx + \omega t = \frac{\pi}{2}$. As time increases, x must decrease to keep the phase equal to $\frac{\pi}{2}$. The plus sign is used for waves moving in the negative x-direction. In summary, $y(x,t) = A\sin(kx - \omega t + \phi)$ models a wave moving in the positive x-direction and $y(x,t) = A\sin(kx + \omega t + \phi)$ models a wave moving in the negative x-direction.

Equation 2.6.3 is known as a simple harmonic wave function. A wave function is any function such that $f(x,t) = f(x - vt)$. Later in this chapter, we will see that it is a solution to the linear wave equation. Note that $y(x,t) = A\cos(kx + \omega t + \phi')$ works equally well because it corresponds to a different phase shift $\phi' = \phi - \frac{\pi}{2}$.

Finding the characteristics of a sinusoidal wave

1. To find the amplitude, wavelength, period, and frequency of a sinusoidal wave, write down the wave function in the form $y(x,t) = A\sin(kx - \omega t + \phi)$.
2. The amplitude can be read straight from the equation and is equal to A.
3. The period of the wave can be derived from the angular frequency $\left(T = \frac{2\pi}{\omega}\right)$.
4. The frequency can be found using $f = \frac{1}{T}$
5. The wavelength can be found using the wave number $\left(\lambda = \frac{2\pi}{k}\right)$.

Example 2.6.1: Characteristics of a traveling wave on a string

A transverse wave on a taut string is modeled with the wave function

$$\begin{aligned} y(x,t) &= A\sin(kx - wt) \\ &= (0.2\ \text{m})\sin\left(6.28\ \text{m}^{-1}x - 1.57\ \text{s}^{-1}t\right) \end{aligned}$$

Find the amplitude, wavelength, period, and speed of the wave.

Strategy

All these characteristics of the wave can be found from the constants included in the equation or from simple combinations of these constants.

Solution

1. The amplitude, wave number, and angular frequency can be read directly from the wave equation:

$$y(x,t) = A\sin(kx - wt) = 0.2\ \text{m}\sin\left(6.28\ \text{m}^{-1}x - 1.57\ \text{s}^{-1}t\right)$$
$$\left(A = 0.2\ \text{m}; k = 6.28\ \text{m}^{-1}; \omega = 1.57\ \text{s}^{-1}\right)$$

2. The wave number can be used to find the wavelength:

$$\begin{aligned} k &= \frac{2\pi}{\lambda} \\ \lambda &= \frac{2\pi}{k} = \frac{2\pi}{6.28\ \text{m}^{-1}} = 1.0\ \text{m} \end{aligned} \tag{2.6.5}$$

3. The period of the wave can be found using the angular frequency:

$$\begin{aligned} \omega &= \frac{2\pi}{T} \\ T &= \frac{2\pi}{\omega} = \frac{2\pi}{1.57\ \text{s}^{-1}} = 4\ \text{s} \end{aligned} \tag{2.6.6}$$

4. The speed of the wave can be found using the wave number and the angular frequency. The direction of the wave can be determined by considering the sign of $kx \mp \omega t$: A negative sign suggests that the wave is moving in the positive *x*-direction:

$$|v| = \frac{\omega}{k} = \frac{1.57 \text{ s}^{-1}}{6.28 \text{ m}^{-1}} = 0.25 \text{ m/s}$$

Significance

All of the characteristics of the wave are contained in the wave function. Note that the wave speed is the speed of the wave in the direction parallel to the motion of the wave. Plotting the height of the medium *y* versus the position *x* for two times t = 0.00 s and t = 0.80 s can provide a graphical visualization of the wave (Figure $2.6.4$).

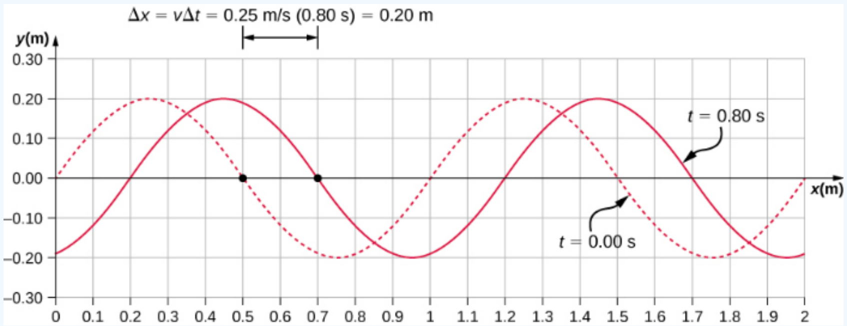


Figure $2.6.4$: A graph of height of the wave *y* as a function of position *x* for snapshots of the wave at two times. The dotted line represents the wave at time t = 0.00 s and the solid line represents the wave at t = 0.80 s. Since the wave velocity is constant, the distance the wave travels is the wave velocity times the time interval. The black dots indicate the points used to measure the displacement of the wave. The medium moves up and down, whereas the wave moves to the right.

There is a second velocity to the motion. In this example, the wave is transverse, moving horizontally as the medium oscillates up and down perpendicular to the direction of motion. The graph in Figure $2.6.5$ shows the motion of the medium at point x = 0.60 m as a function of time. Notice that the medium of the wave oscillates up and down between y = +0.20 m and y = −0.20 m every period of 4.0 seconds.

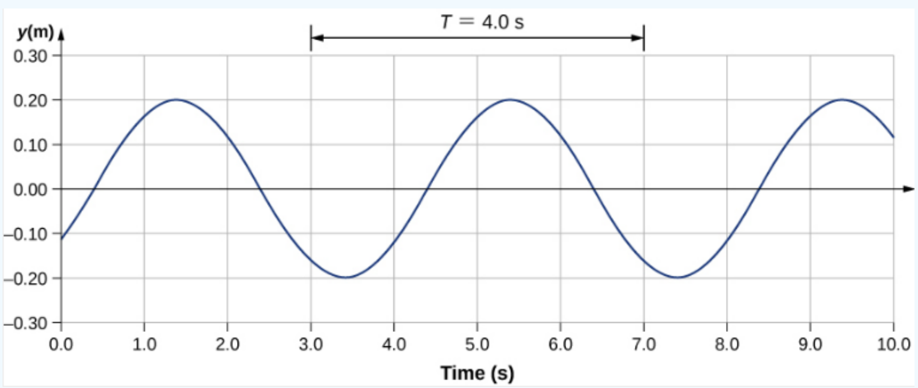


Figure $2.6.5$: A graph of height of the wave *y* as a function of time *t* for the position x = 0.6 m. The medium oscillates between y = +0.20 m and y = −0.20 m every period. The period represented picks two convenient points in the oscillations to measure the period. The period can be measured between any two adjacent points with the same amplitude and the same velocity, $(\partial y/\partial t)$. The velocity can be found by looking at the slope tangent to the point on a *y*-versus-*t* plot. Notice that at times t = 3.00 s and t = 7.00 s, the heights and the velocities are the same and the period of the oscillation is 4.00 s.

? Exercise $2.6.1$

The wave function above is derived using a sine function. Can a cosine function be used instead?

Velocity and Acceleration of the Medium

As seen in Example $2.6.2$, the wave speed is constant and represents the speed of the wave as it propagates through the medium, not the speed of the particles that make up the medium. The particles of the medium oscillate around an equilibrium position as the wave propagates through the medium. In the case of the transverse wave propagating in the x-direction, the particles oscillate up and down in the y-direction, perpendicular to the motion of the wave. The velocity of the particles of the medium is not constant, which means there is an acceleration. The velocity of the medium, which is perpendicular to the wave velocity in a transverse wave, can be found by taking the partial derivative of the position equation with respect to time. The partial derivative is found by taking the derivative of the function, treating all variables as constants, except for the variable in question. In the case of the partial derivative with respect to time t, the position x is treated as a constant. Although this may sound strange if you haven't seen it before, the object of this exercise is to find the transverse velocity at a point, so in this sense, the x-position is not changing. We have

$$\begin{aligned} y(x,t) &= A\sin(kx-\omega t+\phi) \\ v_y(x,t) &= \frac{\partial y(x,t)}{\partial t} = \frac{\partial}{\partial t}[A\sin(kx-\omega t+\phi)] \\ &= -A\omega\cos(kx-\omega t+\phi) \\ &= -v_{y\ max}\cos(kx-\omega t+\phi). \end{aligned}$$

The magnitude of the maximum velocity of the medium is $|v_{y\ max}| = A\omega$. This may look familiar from the Oscillations and a mass on a spring.

We can find the acceleration of the medium by taking the partial derivative of the velocity equation with respect to time,

$$\begin{aligned} a_y(x,t) &= \frac{\partial v_y(x,t)}{\partial t} = \frac{\partial}{\partial t}[-A\omega\cos(kx-\omega t+\phi)] \\ &= -A\omega^2\sin(kx-\omega t+\phi) \\ &= -a_{y\ max}\sin(kx-\omega t+\phi). \end{aligned}$$

The magnitude of the maximum acceleration is $|a_{y\ max}| = A\omega^2$. The particles of the medium, or the mass elements, oscillate in simple harmonic motion for a mechanical wave.

The Linear Wave Equation

We have just determined the velocity of the medium at a position x by taking the partial derivative, with respect to time, of the position y. For a transverse wave, this velocity is perpendicular to the direction of propagation of the wave. We found the acceleration by taking the partial derivative, with respect to time, of the velocity, which is the second time derivative of the position:

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2}{\partial t^2}[A\sin(kx-\omega t+\phi)] = -A\omega^2\sin(kx-\omega t+\phi). \tag{2.6.7}$$

Now consider the partial derivatives with respect to the other variable, the position x, holding the time constant. The first derivative is the slope of the wave at a point x at a time t,

$$slope = \frac{\partial y(x,t)}{\partial x} = \frac{\partial}{\partial x}[A\sin(kx-\omega t+\phi)] = Ak\cos(kx-\omega t+\phi). \tag{2.6.8}$$

The second partial derivative expresses how the slope of the wave changes with respect to position—in other words, the curvature of the wave, where

$$curvature = \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{\partial^2}{\partial x^2}[A\sin(kx-\omega t+\phi)] = -Ak^2\sin(kx-\omega t+\phi). \tag{2.6.9}$$

The ratio of the acceleration and the curvature leads to a very important relationship in physics known as the **linear wave equation**. Taking the ratio and using the equation v = $\frac{\omega}{k}$ yields the linear wave equation (also known simply as the wave equation or the equation of a vibrating string),

$$\begin{aligned} \frac{\frac{\partial^2 y(x,t)}{\partial t^2}}{\frac{\partial^2 y(x,t)}{\partial x^2}} &= \frac{-A\omega^2\sin(kx-\omega t+\phi)}{-Ak^2\sin(kx-\omega t+\phi)} \\ &= \frac{\omega^2}{k^2} = v^2, \\ \frac{\partial^2 y(x,t)}{\partial x^2} &= \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}. \end{aligned} \tag{2.6.10}$$

Equation 2.6.10 is the linear wave equation, which is one of the most important equations in physics and engineering. We derived it here for a transverse wave, but it is equally important when investigating longitudinal waves. This relationship was also derived using a sinusoidal wave, but it successfully describes any wave or pulse that has the form y(x, t) = f(x ∓ vt). These waves result due to a linear restoring force of the medium—thus, the name linear wave equation. Any wave function that satisfies this equation is a linear wave function.

An interesting aspect of the linear wave equation is that if two wave functions are individually solutions to the linear wave equation, then the sum of the two linear wave functions is also a solution to the wave equation. Consider two transverse waves that propagate along the x-axis, occupying the same medium. Assume that the individual waves can be modeled with the wave functions y_1(x, t) = f(x ∓ vt) and y_2(x, t) = g(x ∓ vt), which are solutions to the linear wave equations and are therefore linear wave functions. The sum of the wave functions is the wave function

$$y_1(x,t)+y_2(x,t)=f(x\mp vt)+g(x\mp vt). \tag{2.6.11}$$

Consider the linear wave equation:

$$\begin{aligned}\frac{\partial^2(f+g)}{\partial x^2}&=\frac{1}{v^2}\frac{\partial^2(f+g)}{\partial t^2}\\ \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 g}{\partial x^2}&=\frac{1}{v^2}\left(\frac{\partial^2 f}{\partial t^2}+\frac{\partial^2 g}{\partial t^2}\right).\end{aligned}$$

This has shown that if two linear wave functions are added algebraically, the resulting wave function is also linear. This wave function models the displacement of the medium of the resulting wave at each position along the x-axis. If two linear waves occupy the same medium, they are said to interfere. If these waves can be modeled with a linear wave function, these wave functions add to form the wave equation of the wave resulting from the interference of the individual waves. The displacement of the medium at every point of the resulting wave is the algebraic sum of the displacements due to the individual waves.

Taking this analysis a step further, if wave functions y1 (x, t) = f(x ∓ vt) and y2 (x, t) = g(x ∓ vt) are solutions to the linear wave equation, then Ay_1(x, t) + By_2(x, y), where A and B are constants, is also a solution to the linear wave equation. This property is known as the principle of superposition. Interference and superposition are covered in more detail in Interference of Waves.

✓ Example 2.6.2: Interference of Waves on a String

Consider a very long string held taut by two students, one on each end. Student A oscillates the end of the string producing a wave modeled with the wave function y_1(x, t) = A sin(kx − ωt) and student B oscillates the string producing at twice the frequency, moving in the opposite direction. Both waves move at the same speed v = $\frac{\omega}{k}$. The two waves interfere to form a resulting wave whose wave function is y_R(x, t) = y_1(x, t) + y_2(x, t). Find the velocity of the resulting wave using the linear wave equation $\frac{\partial^2 y(x,t)}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$.

Strategy

First, write the wave function for the wave created by the second student. Note that the angular frequency of the second wave is twice the frequency of the first wave (2ω), and since the velocity of the two waves are the same, the wave number of the second wave is twice that of the first wave (2k). Next, write the wave equation for the resulting wave function, which is the sum of the two individual wave functions. Then find the second partial derivative with respect to position and the second partial derivative with respect to time. Use the linear wave equation to find the velocity of the resulting wave.

Solution

1. Write the wave function of the second wave: y_2(x, t) = A sin(2kx + 2ωt).
2. Write the resulting wave function:

$$y_R(x,t)=y_1(x,t)+y(x,t)=A\sin(kx-\omega t)+A\sin(2kx+2\omega t). \tag{2.6.12}$$

3. Find the partial derivatives:

$$\begin{aligned}
\frac{\partial y_R(x,t)}{\partial x} &= -Ak\cos(kx-\omega t)+2Ak\cos(2kx+2\omega t), \\
\frac{\partial^2 y_R(x,t)}{\partial x^2} &= -Ak^2\sin(kx-\omega t)-4Ak^2\sin(2kx+2\omega t), \\
\frac{\partial y_R(x,t)}{\partial t} &= -A\omega\cos(kx-\omega t)+2A\omega\cos(2kx+2\omega t), \\
\frac{\partial^2 y_R(x,t)}{\partial t^2} &= -A\omega^2\sin(kx-\omega t)-4A\omega^2\sin(2kx+2\omega t).
\end{aligned}$$

4. Use the wave equation to find the velocity of the resulting wave:

$$\begin{aligned}
\frac{\partial^2 y(x,t)}{\partial x^2} &= \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}, \\
-Ak^2\sin(kx-\omega t)+4Ak^2\sin(2kx+2\omega t) &= \frac{1}{v^2}\left(-A\omega^2\sin(kx-\omega t)-4A\omega^2\sin(2kx+2\omega t)\right), \\
k^2\left(-A\sin(kx-\omega t)+4A\sin(2kx+2\omega t)\right) &= \frac{\omega^2}{v^2}\left(-A\sin(kx-\omega t)-4A\sin(2kx+2\omega t)\right), \\
k^2 &= \frac{\omega^2}{v^2}, \\
|v| &= \frac{\omega}{k}.
\end{aligned}$$

Significance

The speed of the resulting wave is equal to the speed of the original waves $\left(v=\frac{\omega}{k}\right)$. We will show in the next section that the speed of a simple harmonic wave on a string depends on the tension in the string and the mass per length of the string. For this reason, it is not surprising that the component waves as well as the resultant wave all travel at the same speed.

? Exercise 16.4

The wave equation $\frac{\partial^2 y(x,t)}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$ works for any wave of the form y(x, t) = f(x ∓ vt). In the previous section, we stated that a cosine function could also be used to model a simple harmonic mechanical wave. Check if the wave

$$y(x,t)=(0.50\ m)\cos(0.20\pi\ m^{-1}x-4.00\pi s^{-1}t+\frac{\pi}{10}) \tag{2.6.13}$$

is a solution to the wave equation.

Any disturbance that complies with the wave equation can propagate as a wave moving along the x-axis with a wave speed v. It works equally well for waves on a string, sound waves, and electromagnetic waves. This equation is extremely useful. For example, it can be used to show that electromagnetic waves move at the speed of light.

2.7: Wave Speed on a Stretched String

Learning Objectives

- Determine the factors that affect the speed of a wave on a string
- Write a mathematical expression for the speed of a wave on a string and generalize these concepts for other media

The speed of a wave depends on the characteristics of the medium. For example, in the case of a guitar, the strings vibrate to produce the sound. The speed of the waves on the strings, and the wavelength, determine the frequency of the sound produced. The strings on a guitar have different thickness but may be made of similar material. They have different **linear densities**, where the linear density is defined as the mass per length,

$$\mu = \frac{\text{mass of string}}{\text{length of string}} = \frac{m}{l}. \tag{2.7.1}$$

In this chapter, we consider only string with a constant linear density. If the linear density is constant, then the mass (Δm) of a small length of string (Δx) is $\Delta m = \mu \Delta x$. For example, if the string has a length of 2.00 m and a mass of 0.06 kg, then the linear density is $\mu = \frac{0.06\ kg}{2.00\ m}$ = 0.03 kg/m. If a 1.00-mm section is cut from the string, the mass of the 1.00-mm length is

$$\Delta m = \mu \Delta x = (0.03\ kg/m)(0.001\ m) = 3.00 \times 10^{-5}\ kg.$$

The guitar also has a method to change the tension of the strings. The tension of the strings is adjusted by turning spindles, called the tuning pegs, around which the strings are wrapped. For the guitar, the linear density of the string and the tension in the string determine the speed of the waves in the string and the frequency of the sound produced is proportional to the wave speed.

Wave Speed on a String under Tension

To see how the speed of a wave on a string depends on the tension and the linear density, consider a pulse sent down a taut string (Figure $2.7.1$). When the taut string is at rest at the equilibrium position, the tension in the string F_T is constant. Consider a small element of the string with a mass equal to $\Delta m = \mu \Delta x$. The mass element is at rest and in equilibrium and the force of tension of either side of the mass element is equal and opposite.

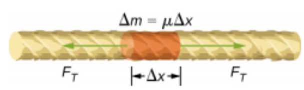


Figure $2.7.1$: Mass element of a string kept taut with a tension F_T. The mass element is in static equilibrium, and the force of tension acting on either side of the mass element is equal in magnitude and opposite in direction.

If you pluck a string under tension, a transverse wave moves in the positive x-direction, as shown in Figure $2.7.2$. The mass element is small but is enlarged in the figure to make it visible. The small mass element oscillates perpendicular to the wave motion as a result of the restoring force provided by the string and does not move in the x-direction. The tension F_T in the string, which acts in the positive and negative x-direction, is approximately constant and is independent of position and time.

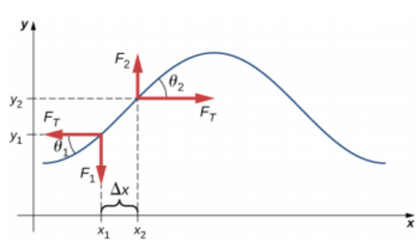


Figure $2.7.2$: A string under tension is plucked, causing a pulse to move along the string in the positive x-direction.

Assume that the inclination of the displaced string with respect to the horizontal axis is small. The net force on the element of the string, acting parallel to the string, is the sum of the tension in the string and the restoring force. The x-components of the force of tension cancel, so the net force is equal to the sum of the y-components of the force. The magnitude of the x-component of the force is equal to the horizontal force of tension of the string F_T as shown in Figure $2.7.2$. To obtain the y-components of the force, note that tan $\theta_1 = -\frac{F_1}{F_T}$ and $\tan \theta_2 = \frac{F_2}{F_T}$. The $\tan \theta$ is equal to the slope of a function at a point, which is equal to the partial derivative of y with respect to x at that point. Therefore, $\frac{F_1}{F_T}$ is equal to the negative slope of the string at x_1 and $\frac{F_2}{F_T}$ is equal to the slope of the string at x_2:

$$\frac{F_1}{F_T} = -\left(\frac{\partial y}{\partial x}\right)_{x_1} \; and \; \frac{F_2}{F_T} = \left(\frac{\partial y}{\partial x}\right)_{x_2}. \tag{2.7.2}$$

The net force is on the small mass element can be written as

$$F_{net} = F_1 + F_2 = F_T\left[\left(\frac{\partial y}{\partial x}\right)_{x_2} - \left(\frac{\partial y}{\partial x}\right)_{x_1}\right]. \tag{2.7.3}$$

Using Newton's second law, the net force is equal to the mass times the acceleration. The linear density of the string μ is the mass per length of the string, and the mass of the portion of the string is $\mu\Delta$x,

$$F_T\left[\left(\frac{\partial y}{\partial x}\right)_{x_2} - \left(\frac{\partial y}{\partial x}\right)_{x_1}\right] = \Delta ma = \mu\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right). \tag{2.7.4}$$

Dividing by $F_T\Delta x$ and taking the limit as Δx approaches zero,

$$\lim_{\Delta x \to 0} \frac{\left[\left(\frac{\partial y}{\partial x}\right)_{x_2} - \left(\frac{\partial y}{\partial x}\right)_{x_1}\right]}{\Delta x} = \frac{\mu}{F_T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F_T}\frac{\partial^2 y}{\partial t^2}.$$

Recall that the linear wave equation is

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}. \tag{2.7.5}$$

Therefore,

$$\frac{1}{v^2} = \frac{\mu}{F_T}. \tag{2.7.6}$$

Solving for v, we see that the speed of the wave on a string depends on the tension and the linear density

Speed of a Wave on a String Under Tension

The speed of a pulse or wave on a string under tension can be found with the equation

$$|v| = \sqrt{\frac{F_T}{\mu}} \tag{2.7.7}$$

where F_T is the tension in the string and μ is the mass per length of the string.

Example 16.5: The Wave Speed of a Guitar Spring

On a six-string guitar, the high E string has a linear density of $\mu_{High\ E}$ = 3.09 x 10^{-4} kg/m and the low E string has a linear density of $\mu_{Low\ E}$ = 5.78 x 10^{-3} kg/m. (a) If the high E string is plucked, producing a wave in the string, what is the speed of the wave if the tension of the string is 56.40 N? (b) The linear density of the low E string is approximately 20 times greater than that of the high E string. For waves to travel through the low E string at the same wave speed as the high E, would the tension need to be larger or smaller than the high E string? What would be the approximate tension? (c) Calculate the tension of the low E string needed for the same wave speed.

Strategy

a. The speed of the wave can be found from the linear density and the tension $v = \sqrt{\frac{F_T}{\mu}}$.

b. From the equation v = $\sqrt{\frac{F_T}{\mu}}$, if the linear density is increased by a factor of almost 20, the tension would need to be increased by a factor of 20.
c. Knowing the velocity and the linear density, the velocity equation can be solved for the force of tension $F_T = \mu v^2$.

Solution

a. Use the velocity equation to find the speed:

$$v=\sqrt{\frac{F_T}{\mu}}=\sqrt{\frac{56.40\ N}{3.09\times 10^{-4}\ kg/m}}=427.23\ m/s. \tag{2.7.8}$$

b. The tension would need to be increased by a factor of approximately 20. The tension would be slightly less than 1128 N.
c. Use the velocity equation to find the actual tension:

$$F_T=\mu v^2=(5.78\times 10^{-3}\ kg/m)(427.23\ m/s)^2=1055.00\ N. \tag{2.7.9}$$

d. This solution is within 7% of the approximation.

Significance

The standard notes of the six string (high E, B, G, D, A, low E) are tuned to vibrate at the fundamental frequencies (329.63 Hz, 246.94 Hz, 196.00 Hz, 146.83 Hz, 110.00 Hz, and 82.41 Hz) when plucked. The frequencies depend on the speed of the waves on the string and the wavelength of the waves. The six strings have different linear densities and are “tuned” by changing the tensions in the strings. We will see in Interference of Waves that the wavelength depends on the length of the strings and the boundary conditions. To play notes other than the fundamental notes, the lengths of the strings are changed by pressing down on the strings.

? Exercise 16.5

The wave speed of a wave on a string depends on the tension and the linear mass density. If the tension is doubled, what happens to the speed of the waves on the string?

Speed of Compression Waves in a Fluid

The speed of a wave on a string depends on the square root of the tension divided by the mass per length, the linear density. In general, the speed of a wave through a medium depends on the elastic property of the medium and the inertial property of the medium.

$$|v|=\sqrt{\frac{elastic\ property}{inertial\ property}} \tag{2.7.10}$$

The elastic property describes the tendency of the particles of the medium to return to their initial position when perturbed. The inertial property describes the tendency of the particle to resist changes in velocity.

The speed of a longitudinal wave through a liquid or gas depends on the density of the fluid and the bulk modulus of the fluid,

$$v=\sqrt{\frac{\beta}{\rho}}. \tag{2.7.11}$$

Here the bulk modulus is defined as B = $-\frac{\Delta P}{\frac{\Delta V}{V_0}}$, where ΔP is the change in the pressure and the denominator is the ratio of the change in volume to the initial volume, and $\rho \equiv \frac{m}{V}$ is the mass per unit volume. For example, sound is a mechanical wave that travels through a fluid or a solid. The speed of sound in air with an atmospheric pressure of 1.013 x 10^5 Pa and a temperature of 20°C is $v_s \approx$ 343.00 m/s. Because the density depends on temperature, the speed of sound in air depends on the temperature of the air. This will be discussed in detail in Sound.

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2.8: Energy and Power of a Wave

> **Learning Objectives**
>
> - Explain how energy travels with a pulse or wave
> - Describe, using a mathematical expression, how the energy in a wave depends on the amplitude of the wave

All waves carry energy, and sometimes this can be directly observed. Earthquakes can shake whole cities to the ground, performing the work of thousands of wrecking balls (Figure $2.8.1$). Loud sounds can pulverize nerve cells in the inner ear, causing permanent hearing loss. Ultrasound is used for deep-heat treatment of muscle strains. A laser beam can burn away a malignancy. Water waves chew up beaches.

Figure $2.8.1$: The destructive effect of an earthquake is observable evidence of the energy carried in these waves. The Richter scale rating of earthquakes is a logarithmic scale related to both their amplitude and the energy they carry.

In this section, we examine the quantitative expression of energy in waves. This will be of fundamental importance in later discussions of waves, from sound to light to quantum mechanics.

Energy in Waves

The amount of energy in a wave is related to its amplitude and its frequency. Large-amplitude earthquakes produce large ground displacements. Loud sounds have high-pressure amplitudes and come from larger-amplitude source vibrations than soft sounds. Large ocean breakers churn up the shore more than small ones. Consider the example of the seagull and the water wave earlier in the chapter (Figure 16.2.2). Work is done on the seagull by the wave as the seagull is moved up, changing its potential energy. The larger the amplitude, the higher the seagull is lifted by the wave and the larger the change in potential energy.

The energy of the wave depends on both the amplitude and the frequency. If the energy of each wavelength is considered to be a discrete packet of energy, a high-frequency wave will deliver more of these packets per unit time than a low-frequency wave. We will see that the average rate of energy transfer in mechanical waves is proportional to both the square of the amplitude and the square of the frequency. If two mechanical waves have equal amplitudes, but one wave has a frequency equal to twice the frequency of the other, the higher-frequency wave will have a rate of energy transfer a factor of four times as great as the rate of energy transfer of the lower-frequency wave. It should be noted that although the rate of energy transport is proportional to both the square of the amplitude and square of the frequency in mechanical waves, the rate of energy transfer in electromagnetic waves is proportional to the square of the amplitude, but independent of the frequency.

Power in Waves

Consider a sinusoidal wave on a string that is produced by a string vibrator, as shown in Figure $2.8.2$. The string vibrator is a device that vibrates a rod up and down. A string of uniform linear mass density is attached to the rod, and the rod oscillates the string, producing a sinusoidal wave. The rod does work on the string, producing energy that propagates along the string. Consider a mass element of the string with a mass Δm, as seen in Figure $2.8.2$. As the energy propagates along the string, each mass element of the string is driven up and down at the same frequency as the wave. Each mass element of the string can be modeled as a simple harmonic oscillator. Since the string has a constant linear density $\mu = \frac{\Delta m}{\Delta x}$, each mass element of the string has the mass Δm = $\mu\Delta$x.

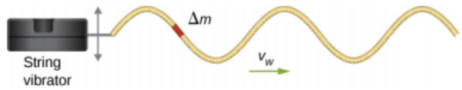


Figure 2.8.2: A string vibrator is a device that vibrates a rod. A string is attached to the rod, and the rod does work on the string, driving the string up and down. This produces a sinusoidal wave in the string, which moves with a wave velocity v. The wave speed depends on the tension in the string and the linear mass density of the string. A section of the string with mass Δm oscillates at the same frequency as the wave.

The total mechanical energy of the wave is the sum of its kinetic energy and potential energy. The kinetic energy $K = \frac{1}{2}mv^2$ of each mass element of the string of length Δx is $\Delta K = \frac{1}{2}(\Delta m){v_y}^2$, as the mass element oscillates perpendicular to the direction of the motion of the wave. Using the constant linear mass density, the kinetic energy of each mass element of the string with length Δx is

$$\Delta K = \frac{1}{2}(\mu \Delta x)v_y^2.$$

A differential equation can be formed by letting the length of the mass element of the string approach zero,

$$dK = \lim_{\Delta x \to 0} \frac{1}{2}(\mu \Delta x)v_y^2 = \frac{1}{2}(\mu\ dx)v_y^2.$$

Since the wave is a sinusoidal wave with an angular frequency ω, the position of each mass element may be modeled as $y(x, t) = A \sin(kx - \omega t)$. Each mass element of the string oscillates with a velocity $v_y = \frac{\partial y(x,t)}{\partial t} = -A\omega \cos(kx - \omega t)$. The kinetic energy of each mass element of the string becomes

$$\begin{aligned} dK &= \frac{1}{2}(\mu\ dx)[-A\omega \cos(kx-\omega t)]^2 \\ &= \frac{1}{2}(\mu\ dx)[A^2\omega^2 \cos^2(kx-\omega t)]. \end{aligned}$$

The wave can be very long, consisting of many wavelengths. To standardize the energy, consider the kinetic energy associated with a wavelength of the wave. This kinetic energy can be integrated over the wavelength to find the energy associated with each wavelength of the wave:

$$\begin{aligned} dK &= \frac{1}{2}(\mu\ dx)[A^2\omega^2 \cos^2(kx-\omega t)] \\ \int_0^{K_\lambda} dK &= \int_0^\lambda \frac{1}{2}\mu A^2\omega^2 \cos^2(kx-\omega t)dx = \frac{1}{2}\mu A^2\omega^2 \int_0^\lambda \cos^2(kx)dx, \\ K_{lambda} &= \frac{1}{2}\mu A^2\omega^2 \left[\frac{1}{2}x + \frac{1}{4k}\sin(2kx)\right]_0^\lambda \\ &= \frac{1}{2}\mu A^2\omega^2 \left[\frac{1}{2}\lambda + \frac{1}{4k}\sin(2k\lambda) - \frac{1}{4k}\sin(0)\right] \\ &= \frac{1}{4}\mu A^2\omega^2\lambda. \end{aligned}$$

There is also potential energy associated with the wave. Much like the mass oscillating on a spring, there is a conservative restoring force that, when the mass element is displaced from the equilibrium position, drives the mass element back to the equilibrium position. The potential energy of the mass element can be found by considering the linear restoring force of the string, In Oscillations, we saw that the potential energy stored in a spring with a linear restoring force is equal to $U = \frac{1}{2}k_s x^2$, where the equilibrium position is defined as $x = 0.00$ m. When a mass attached to the spring oscillates in simple harmonic motion, the angular frequency is equal to $\omega = \frac{k_s}{m}$. As each mass element oscillates in simple harmonic motion, the spring constant is equal to $k_s = \Delta m\ \omega^2$. The potential energy of the mass element is equal to

$$\Delta U = \frac{1}{2}k_s x^2 = \frac{1}{2}\Delta m \omega^2 x^2.$$

Note that k_s is the spring constant and not the wave number $k = \frac{2\pi}{\lambda}$. This equation can be used to find the energy over a wavelength. Integrating over the wavelength, we can compute the potential energy over a wavelength:

$$\begin{aligned} dU &= \frac{1}{2}k_s x^2 = \frac{1}{2}\mu\omega^2 x^2 dx, \\ U_\lambda &= \frac{1}{2}\mu\omega^2 A^2 \int_0^\lambda \sin^2(kx)dx = \frac{1}{4}\mu A^2\omega^2\lambda. \end{aligned}$$

The potential energy associated with a wavelength of the wave is equal to the kinetic energy associated with a wavelength. The total energy associated with a wavelength is the sum of the potential energy and the kinetic energy:

$$\begin{aligned} E_\lambda &= U_\lambda + K_\lambda \\ &= \frac{1}{4}\mu A^2\omega^2\lambda + \frac{1}{4}\mu A^2\omega^2\lambda \\ &= \frac{1}{2}\mu A^2\omega^2\lambda. \end{aligned}$$

The time-averaged power of a sinusoidal mechanical wave, which is the average rate of energy transfer associated with a wave as it passes a point, can be found by taking the total energy associated with the wave divided by the time it takes to transfer the energy. If the velocity of the sinusoidal wave is constant, the time for one wavelength to pass by a point is equal to the period of the wave, which is also constant. For a sinusoidal mechanical wave, the time-averaged power is therefore the energy associated with a wavelength divided by the period of the wave. The wavelength of the wave divided by the period is equal to the velocity of the wave,

$$P_{ave} = \frac{E_\lambda}{T} = \frac{1}{2}\mu A^2\omega^2\frac{\lambda}{T} = \frac{1}{2}\mu A^2\omega^2 v. \tag{2.8.1}$$

Note that this equation for the time-averaged power of a sinusoidal mechanical wave shows that the power is proportional to the square of the amplitude of the wave and to the square of the angular frequency of the wave. Recall that the angular frequency is equal to $\omega = 2\pi f$, so the power of a mechanical wave is equal to the square of the amplitude and the square of the frequency of the wave.

✓ Example 16.6: Power Supplied by a String Vibrator

Consider a two-meter-long string with a mass of 70.00 g attached to a string vibrator as illustrated in Figure 2.8.2. The tension in the string is 90.0 N. When the string vibrator is turned on, it oscillates with a frequency of 60 Hz and produces a sinusoidal wave on the string with an amplitude of 4.00 cm and a constant wave speed. What is the time-averaged power supplied to the wave by the string vibrator?

Strategy

The power supplied to the wave should equal the time-averaged power of the wave on the string. We know the mass of the string (m_s) , the length of the string (L_s) , and the tension (F_T) in the string. The speed of the wave on the string can be derived from the linear mass density and the tension. The string oscillates with the same frequency as the string vibrator, from which we can find the angular frequency.

Solution

1. Begin with the equation of the time-averaged power of a sinusoidal wave on a string:

$$P = \frac{1}{2}\mu A^2\omega^2 v. \tag{2.8.2}$$

   The amplitude is given, so we need to calculate the linear mass density of the string, the angular frequency of the wave on the string, and the speed of the wave on the string.
2. We need to calculate the linear density to find the wave speed:

$$\mu = \frac{m_s}{L_s} = \frac{0.070\ kg}{2.00\ m} = 0.035\ kg/m. \tag{2.8.3}$$

3. The wave speed can be found using the linear mass density and the tension of the string:

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{90.00\ N}{0.035\ kg/m}} = 50.71\ m/s. \tag{2.8.4}$$

4. The angular frequency can be found from the frequency:

$$\omega = 2\pi f = 2\pi(60\ s^{-1}) = 376.80\ s^{-1}. \tag{2.8.5}$$

5. Calculate the time-averaged power:

$$P = \frac{1}{2}\mu A^2\omega^2 v = \frac{1}{2}(0.035\ kg/m)(0.040\ m)^2(376.80\ s^{-1})^2(50.71\ m/s) = 201.5\ W. \tag{2.8.6}$$

Significance

The time-averaged power of a sinusoidal wave is proportional to the square of the amplitude of the wave and the square of the angular frequency of the wave. This is true for most mechanical waves. If either the angular frequency or the amplitude of the wave were doubled, the power would increase by a factor of four. The timeaveraged power of the wave on a string is also proportional to the speed of the sinusoidal wave on the string. If the speed were doubled, by increasing the tension by a factor of four, the power would also be doubled.

? Exercise 16.6

Is the time-averaged power of a sinusoidal wave on a string proportional to the linear density of the string?

The equations for the energy of the wave and the time-averaged power were derived for a sinusoidal wave on a string. In general, the energy of a mechanical wave and the power are proportional to the amplitude squared and to the angular frequency squared (and therefore the frequency squared).

Another important characteristic of waves is the intensity of the waves. Waves can also be concentrated or spread out. Waves from an earthquake, for example, spread out over a larger area as they move away from a source, so they do less damage the farther they get from the source. Changing the area the waves cover has important effects. All these pertinent factors are included in the definition of **intensity** (I) as power per unit area:

$$I = \frac{P}{A}, \tag{2.8.7}$$

where P is the power carried by the wave through area A. The definition of intensity is valid for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter (W/m^2). Many waves are spherical waves that move out from a source as a sphere. For example, a sound speaker mounted on a post above the ground may produce sound waves that move away from the source as a spherical wave. Sound waves are discussed in more detail in the next chapter, but in general, the farther you are from the speaker, the less intense the sound you hear. As a spherical wave moves out from a source, the surface area of the wave increases as the radius increases ($A = 4\pi r^2$). The intensity for a spherical wave is therefore

$$I = \frac{P}{4\pi r^2}. \tag{2.8.8}$$

If there are no dissipative forces, the energy will remain constant as the spherical wave moves away from the source, but the intensity will decrease as the surface area increases.

In the case of the two-dimensional circular wave, the wave moves out, increasing the circumference of the wave as the radius of the circle increases. If you toss a pebble in a pond, the surface ripple moves out as a circular wave. As the ripple moves away from the source, the amplitude decreases. The energy of the wave spreads around a larger circumference and the amplitude decreases proportional to $\frac{1}{r}$, and not $\frac{1}{r^2}$, as in the case of a spherical wave.

2.9: Interference of Waves

Learning Objectives

- Explain how mechanical waves are reflected and transmitted at the boundaries of a medium
- Define the terms interference and superposition
- Find the resultant wave of two identical sinusoidal waves that differ only by a phase shift

Up to now, we have been studying mechanical waves that propagate continuously through a medium, but we have not discussed what happens when waves encounter the boundary of the medium or what happens when a wave encounters another wave propagating through the same medium. Waves do interact with boundaries of the medium, and all or part of the wave can be reflected. For example, when you stand some distance from a rigid cliff face and yell, you can hear the sound waves reflect off the rigid surface as an echo. Waves can also interact with other waves propagating in the same medium. If you throw two rocks into a pond some distance from one another, the circular ripples that result from the two stones seem to pass through one another as they propagate out from where the stones entered the water. This phenomenon is known as interference. In this section, we examine what happens to waves encountering a boundary of a medium or another wave propagating in the same medium. We will see that their behavior is quite different from the behavior of particles and rigid bodies. Later, when we study modern physics, we will see that only at the scale of atoms do we see similarities in the properties of waves and particles.

Reflection and Transmission

When a wave propagates through a medium, it reflects when it encounters the boundary of the medium. The wave before hitting the boundary is known as the incident wave. The wave after encountering the boundary is known as the reflected wave. How the wave is reflected at the boundary of the medium depends on the boundary conditions; waves will react differently if the boundary of the medium is fixed in place or free to move (Figure $2.9.1$). A **fixed boundary condition** exists when the medium at a boundary is fixed in place so it cannot move. A **free boundary condition** exists when the medium at the boundary is free to move.

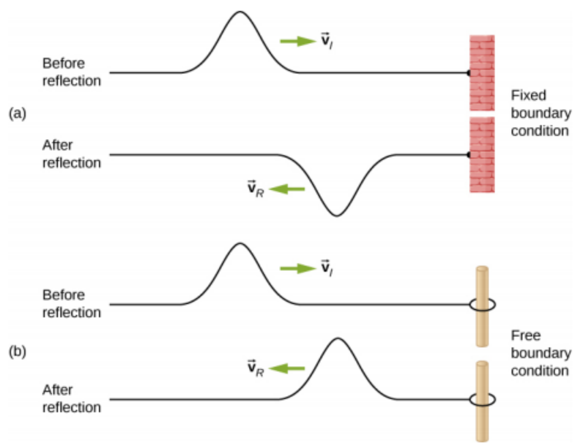


Figure $2.9.1$: (a) One end of a string is fixed so that it cannot move. A wave propagating on the string, encountering this fixed boundary condition, is reflected 180°(π rad) out of phase with respect to the incident wave. (b) One end of a string is tied to a solid ring of negligible mass on a frictionless lab pole, where the ring is free to move. A wave propagating on the string, encountering this free boundary condition, is reflected in phase 0°(0 rad) with respect to the wave.

Figure $2.9.1a$ shows a fixed boundary condition. Here, one end of the string is fixed to a wall so the end of the string is fixed in place and the medium (the string) at the boundary cannot move. When the wave is reflected, the amplitude of the reflected way is exactly the same as the amplitude of the incident wave, but the reflected wave is reflected $180^o \pi$ rad) out of phase with respect to the incident wave. The phase change can be explained using Newton's third law: Recall that Newton's third law states that when object A exerts a force on object B, then object B exerts an equal and opposite force on object A. As the incident wave encounters the wall, the string exerts an upward force on the wall and the wall reacts by exerting an equal and opposite force on the string. The reflection at a fixed boundary is inverted. Note that the figure shows a crest of the incident wave reflected as a trough. If the incident wave were a trough, the reflected wave would be a crest.

Figure $2.9.1b$ shows a free boundary condition. Here, one end of the string is tied to a solid ring of negligible mass on a frictionless pole, so the end of the string is free to move up and down. As the incident wave encounters the boundary of the medium, it is also reflected. In the case of a free boundary condition, the reflected wave is in phase with respect to the incident wave. In this case, the wave encounters the free boundary applying an upward force on the ring, accelerating the ring up. The ring travels up to the maximum height equal to the amplitude of the wave and then accelerates down towards the equilibrium position due to the tension in the string. The figure shows the crest of an incident wave being reflected in phase with respect to the incident wave as a crest. If the incident wave were a trough, the reflected wave would also be a trough. The amplitude of the reflected wave would be equal to the amplitude of the incident wave.

In some situations, the boundary of the medium is neither fixed nor free. Consider Figure $2.9.2a$, where a low-linear mass density string is attached to a string of a higher linear mass density. In this case, the reflected wave is out of phase with respect to the incident wave. There is also a transmitted wave that is in phase with respect to the incident wave. Both the transmitted and reflected waves have amplitudes less than the amplitude of the incident wave. If the tension is the same in both strings, the wave speed is higher in the string with the lower linear mass density.

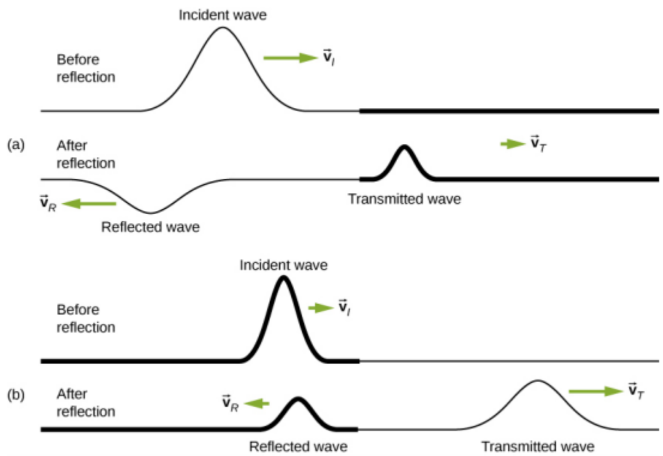


Figure $2.9.2$: Waves traveling along two types of strings: a thick string with a high linear density and a thin string with a low linear density. Both strings are under the same tension, so a wave moves faster on the low-density string than on the high-density string. (a) A wave moving from a low-speed to a high-speed medium results in a reflected wave that is 180°(π rad) out of phase with respect to the incident pulse (or wave) and a transmitted wave that is in phase with the incident wave. (b) When a wave moves from a low-speed medium to a high-speed medium, both the reflected and transmitted wave are in phase with respect to the incident wave.

$2.9.2b$ shows a high-linear mass density string is attached to a string of a lower linear density. In this case, the reflected wave is in phase with respect to the incident wave. There is also a transmitted wave that is in phase with respect to the incident wave. Both the incident and the reflected waves have amplitudes less than the amplitude of the incident wave. Here you may notice that if the tension is the same in both strings, the wave speed is higher in the string with the lower linear mass density.

Superposition and Interference

Most waves do not look very simple. Complex waves are more interesting, even beautiful, but they look formidable. Most interesting mechanical waves consist of a combination of two or more traveling waves propagating in the same medium. The principle of superposition can be used to analyze the combination of waves.

Consider two simple pulses of the same amplitude moving toward one another in the same medium, as shown in Figure $2.9.3$. Eventually, the waves overlap, producing a wave that has twice the amplitude, and then continue on unaffected by the encounter. The pulses are said to interfere, and this phenomenon is known as **interference**.

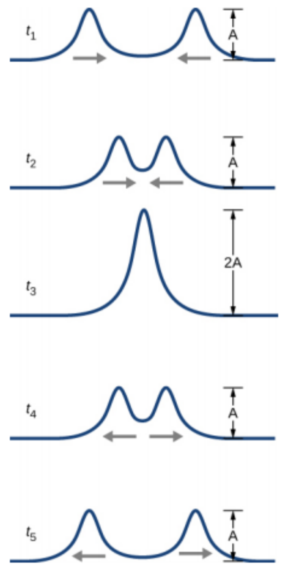


Figure $2.9.3$: Two pulses moving toward one another experience interference. The term interference refers to what happens when two waves overlap.

To analyze the interference of two or more waves, we use the principle of superposition. For mechanical waves, the principle of **superposition** states that if two or more traveling waves combine at the same point, the resulting position of the mass element of the medium, at that point, is the algebraic sum of the position due to the individual waves. This property is exhibited by many waves observed, such as waves on a string, sound waves, and surface water waves. Electromagnetic waves also obey the superposition principle, but the electric and magnetic fields of the combined wave are added instead of the displacement of the medium. Waves that obey the superposition principle are linear waves; waves that do not obey the superposition principle are said to be nonlinear waves. In this chapter, we deal with linear waves, in particular, sinusoidal waves.

The superposition principle can be understood by considering the linear wave equation. In Mathematics of a Wave, we defined a linear wave as a wave whose mathematical representation obeys the linear wave equation. For a transverse wave on a string with an elastic restoring force, the linear wave equation is

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}. \tag{2.9.1}$$

Any wave function $y(x, t) = y(x \mp vt)$, where the argument of the function is linear $(x \mp vt)$ is a solution to the linear wave equation and is a linear wave function. If wave functions $y_1(x, t)$ and $y_2(x, t)$ are solutions to the linear wave equation, the sum of the two functions $y_1(x, t) + y_2(x, t)$ is also a solution to the linear wave equation. Mechanical waves that obey superposition are normally restricted to waves with amplitudes that are small with respect to their wavelengths. If the amplitude is too large, the medium is distorted past the region where the restoring force of the medium is linear.

Waves can interfere constructively or destructively. Figure $2.9.4$ shows two identical sinusoidal waves that arrive at the same point exactly in phase. Figure $2.9.4a$ and $2.9.4b$ show the two individual waves, Figure $2.9.4c$ shows the resultant wave that results from the algebraic sum of the two linear waves. The crests of the two waves are precisely aligned, as are the troughs. This superposition produces **constructive interference**. Because the disturbances add, constructive interference produces a wave that has twice the amplitude of the individual waves, but has the same wavelength.

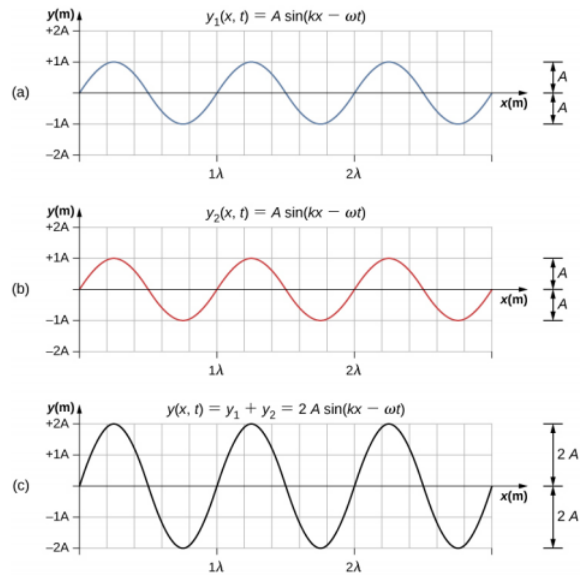


Figure $2.9.4$: Constructive interference of two identical waves produces a wave with twice the amplitude, but the same wavelength.

Figure $2.9.5$ shows two identical waves that arrive exactly 180° out of phase, producing **destructive interference**. Figure $2.9.5a$ and $2.9.5b$ show the individual waves, and Figure $2.9.5c$ shows the superposition of the two waves. Because the troughs of one wave add the crest of the other wave, the resulting amplitude is zero for destructive interference—the waves completely cancel.

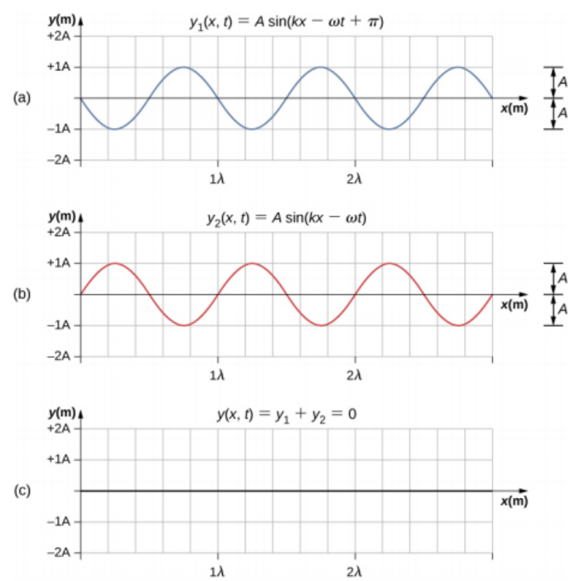


Figure $2.9.5$: Destructive interference of two identical waves, one with a phase shift of 180°(π rad), produces zero amplitude, or complete cancellation.

When linear waves interfere, the resultant wave is just the algebraic sum of the individual waves as stated in the principle of superposition. Figure $2.9.6$ shows two waves (red and blue) and the resultant wave (black). The resultant wave is the algebraic sum of the two individual waves.

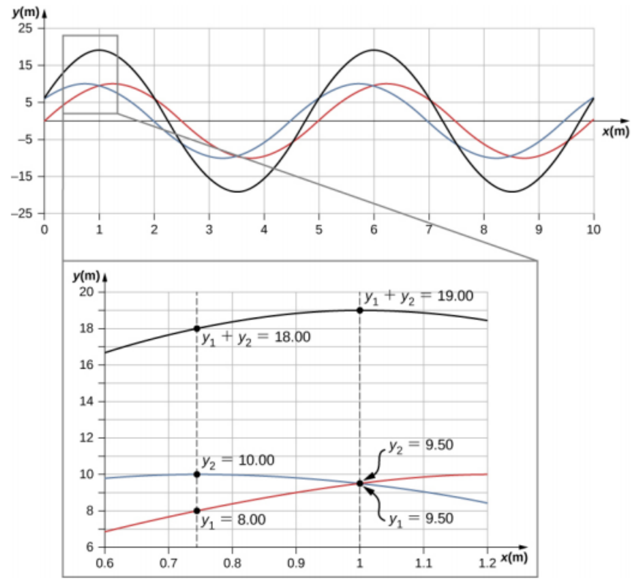


Figure 2.9.6: When two linear waves in the same medium interfere, the height of resulting wave is the sum of the heights of the individual waves, taken point by point. This plot shows two waves (red and blue) added together, along with the resulting wave (black). These graphs represent the height of the wave at each point. The waves may be any linear wave, including ripples on a pond, disturbances on a string, sound, or electromagnetic waves.

The superposition of most waves produces a combination of constructive and destructive interference, and can vary from place to place and time to time. Sound from a stereo, for example, can be loud in one spot and quiet in another. Varying loudness means the sound waves add partially constructively and partially destructively at different locations. A stereo has at least two speakers creating sound waves, and waves can reflect from walls. All these waves interfere, and the resulting wave is the superposition of the waves.

We have shown several examples of the superposition of waves that are similar. Figure 2.9.7 illustrates an example of the superposition of two dissimilar waves. Here again, the disturbances add, producing a resultant wave.

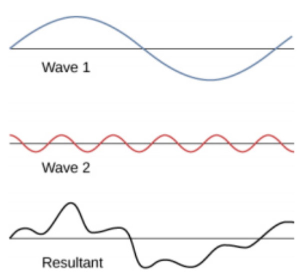


Figure 2.9.7: Superposition of nonidentical waves exhibits both constructive and destructive interference.

At times, when two or more mechanical waves interfere, the pattern produced by the resulting wave can be rich in complexity, some without any readily discernable patterns. For example, plotting the sound wave of your favorite music can look quite complex and is the superposition of the individual sound waves from many instruments; it is the complexity that makes the music interesting and worth listening to. At other times, waves can interfere and produce interesting phenomena, which are complex in their appearance and yet beautiful in simplicity of the physical principle of superposition, which formed the resulting wave. One example is the phenomenon known as standing waves, produced by two identical waves moving in different directions. We will look more closely at this phenomenon in the next section.

Simulation

Try this simulation to make waves with a dripping faucet, audio speaker, or laser! Add a second source or a pair of slits to create an interference pattern. You can observe one source or two sources. Using two sources, you can observe the interference patterns that result from varying the frequencies and the amplitudes of the sources.

Superposition of Sinusoidal Waves that Differ by a Phase Shift

Many examples in physics consist of two sinusoidal waves that are identical in amplitude, wave number, and angular frequency, but differ by a phase shift:

$$\begin{aligned} y_1(x,t) &= A\sin(kx-\omega t+\phi), \\ y_2(x,t) &= A\sin(kx-\omega t). \end{aligned}$$

When these two waves exist in the same medium, the resultant wave resulting from the superposition of the two individual waves is the sum of the two individual waves:

$$y_R(x,t) = y_1(x,t)+y_2(x,t) = A\sin(kx-\omega t+\phi)+A\sin(kx-\omega t). \tag{2.9.2}$$

The resultant wave can be better understood by using the trigonometric identity:

$$\sin u+\sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right), \tag{2.9.3}$$

where u = kx - ωt + ϕ and v = kx - ωt. The resulting wave becomes

$$\begin{aligned} y_R(x,t) &= y_1(x,t)+y_2(x,t) = A\sin(kx-\omega t+\phi)+A\sin(kx-\omega t) \\ &= 2A\sin\left(\frac{(kx-\omega t+\phi)+(kx-\omega t)}{2}\right)\cos\left(\frac{(kx-\omega t+\phi)-(kx-\omega t)}{2}\right) \\ &= 2A\sin\left(kx-\omega t+\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right). \end{aligned}$$

This equation is usually written as

$$y_R(x,t) = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx-\omega t+\frac{\phi}{2}\right). \tag{2.9.4}$$

The resultant wave has the same wave number and angular frequency, an amplitude of A_R = [2A cos $\left(\frac{\phi}{2}\right)$], and a phase shift equal to half the original phase shift. Examples of waves that differ only in a phase shift are shown in Figure 2.9.7.

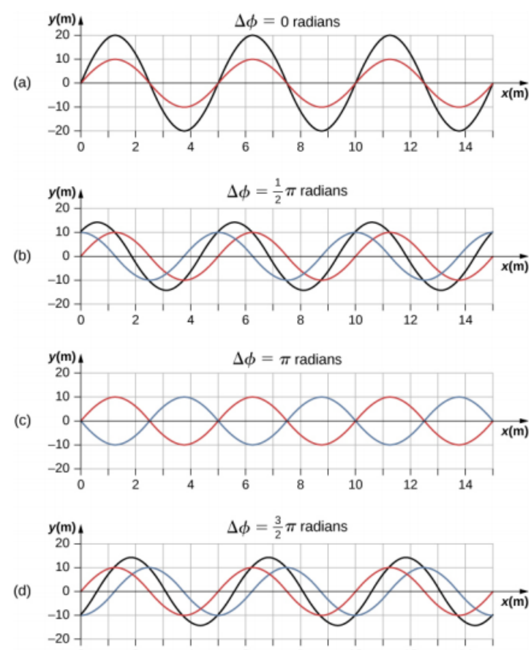


Figure 2.9.8: Superposition of two waves with identical amplitudes, wavelengths, and frequency, but that differ in a phase shift. The red wave is defined by the wave function y_1(x, t) = A sin(kx − ωt) and the blue wave is defined by the wave function y_2(x, t) = A sin(kx − ωt + ϕ). The black line shows the result of adding the two waves. The phase difference between the two waves are (a) 0.00 rad, (b) $\frac{\pi}{2}$ rad, (c) π rad, and (d) $\frac{3\pi}{2}$ rad.

The red and blue waves each have the same amplitude, wave number, and angular frequency, and differ only in a phase shift. They therefore have the same period, wavelength, and frequency. The green wave is the result of the superposition of the two waves. When the two waves have a phase difference of zero, the waves are in phase, and the resultant wave has the same wave number and angular frequency, and an amplitude equal to twice the individual amplitudes (part (a)). This is constructive interference. If the phase difference is 180°, the waves interfere in destructive interference (part (c)). The resultant wave has an amplitude of zero. Any other phase difference results in a wave with the same wave number and angular frequency as the two incident waves but with a phase shift of $\frac{\phi}{2}$ and an amplitude equal to $2A\cos\left(\frac{\phi}{2}\right)$. Examples are shown in parts (b) and (d).

2.10: Standing Waves and Resonance

Learning Objectives

- Describe standing waves and explain how they are produced
- Describe the modes of a standing wave on a string
- Provide examples of standing waves beyond the waves on a string

Throughout this chapter, we have been studying traveling waves, or waves that transport energy from one place to another. Under certain conditions, waves can bounce back and forth through a particular region, effectively becoming stationary. These are called **standing waves**.

Another related effect is known as resonance. In Oscillations, we defined resonance as a phenomenon in which a small-amplitude driving force could produce large-amplitude motion. Think of a child on a swing, which can be modeled as a physical pendulum. Relatively small-amplitude pushes by a parent can produce large-amplitude swings. Sometimes this resonance is good—for example, when producing music with a stringed instrument. At other times, the effects can be devastating, such as the collapse of a building during an earthquake. In the case of standing waves, the relatively large amplitude standing waves are produced by the superposition of smaller amplitude component waves.

Standing Waves

Sometimes waves do not seem to move; rather, they just vibrate in place. You can see unmoving waves on the surface of a glass of milk in a refrigerator, for example. Vibrations from the refrigerator motor create waves on the milk that oscillate up and down but do not seem to move across the surface. Figure $2.10.1$ shows an experiment you can try at home. Take a bowl of milk and place it on a common box fan. Vibrations from the fan will produce circular standing waves in the milk. The waves are visible in the photo due to the reflection from a lamp. These waves are formed by the superposition of two or more traveling waves, such as illustrated in Figure $2.10.2$ for two identical waves moving in opposite directions. The waves move through each other with their disturbances adding as they go by. If the two waves have the same amplitude and wavelength, then they alternate between constructive and destructive interference. The resultant looks like a wave standing in place and, thus, is called a standing wave.

Figure $2.10.1$: Standing waves are formed on the surface of a bowl of milk sitting on a box fan. The vibrations from the fan causes the surface of the milk to oscillate. The waves are visible due to the reflection of light from a lamp.

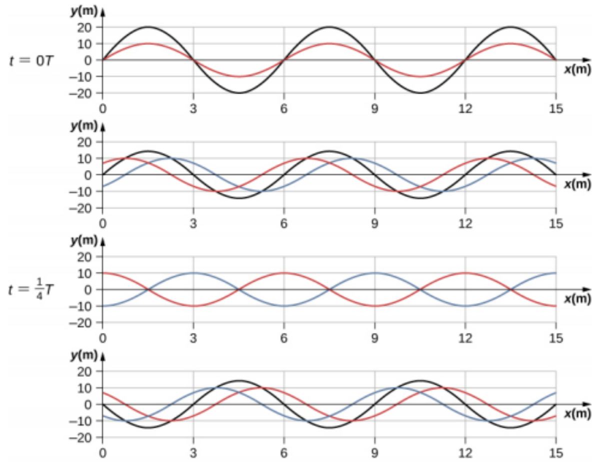


Figure 2.10.2: Time snapshots of two sine waves. The red wave is moving in the −x-direction and the blue wave is moving in the +x-direction. The resulting wave is shown in black. Consider the resultant wave at the points x = 0 m, 3 m, 6 m, 9 m, 12 m, 15 m and notice that the resultant wave always equals zero at these points, no matter what the time is. These points are known as fixed points (nodes). In between each two nodes is an antinode, a place where the medium oscillates with an amplitude equal to the sum of the amplitudes of the individual waves.

Consider two identical waves that move in opposite directions. The first wave has a wave function of $y_1(x, t) = A \sin(kx - \omega t)$ and the second wave has a wave function $y_2(x, t) = A \sin(kx + \omega t)$. The waves interfere and form a resultant wave

$$\begin{aligned} y(x,t) &= y_1(x,t) + y_2(x,t), \\ &= A\sin(kx-\omega t) + A\sin(kx+\omega t). \end{aligned}$$

This can be simplified using the trigonometric identity

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \tag{2.10.1}$$

where $\alpha = kx$ and $\beta = \omega t$, giving us

$$y(x,t) = A[\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t) + \sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t)], \tag{2.10.2}$$

which simplifies to

$$y(x,t) = 2A\sin(kx)\cos(\omega t). \tag{2.10.3}$$

Notice that the resultant wave is a sine wave that is a function only of position, multiplied by a cosine function that is a function only of time. Graphs of y(x,t) as a function of x for various times are shown in Figure 2.10.6. The red wave moves in the negative x-direction, the blue wave moves in the positive x-direction, and the black wave is the sum of the two waves. As the red and blue waves move through each other, they move in and out of constructive interference and destructive interference.

Initially, at time t = 0, the two waves are in phase, and the result is a wave that is twice the amplitude of the individual waves. The waves are also in phase at the time $t = \frac{T}{2}$. In fact, the waves are in phase at any integer multiple of half of a period:

$$t = n\frac{T}{2} \text{ where } n = 0, 1, 2, 3.... \text{ (in phase).}$$

At other times, the two waves are 180°(π radians) out of phase, and the resulting wave is equal to zero. This happens at

$$t = \frac{1}{4}T, \frac{3}{4}T, \frac{5}{4}T, ..., \frac{n}{4}T \text{ where } n = 1, 3, 5.... \text{ (out of phase).}$$

Notice that some x-positions of the resultant wave are always zero no matter what the phase relationship is. These positions are called **nodes**. Where do the nodes occur? Consider the solution to the sum of the two waves

$$y(x,t) = 2A\sin(kx)\cos(\omega t). \tag{2.10.4}$$

Finding the positions where the sine function equals zero provides the positions of the nodes.

$$\begin{aligned}\sin(kx) &= 0\\ kx &= 0, \pi, 2\pi, 3\pi, \ldots\\ \frac{2\pi}{\lambda}x &= 0, \pi, 2\pi, 3\pi, \ldots\\ x &= 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots = n\frac{\lambda}{2} \quad n = 0, 1, 2, 3, \ldots\end{aligned}$$

There are also positions where y oscillates between y = ±A. These are the **antinodes**. We can find them by considering which values of x result in sin(kx) = ±1.

$$\begin{aligned}\sin(kx) &= \pm 1\\ kx &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\\ \frac{2\pi}{\lambda}x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\\ x &= \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots = n\frac{\lambda}{4} \quad n = 1, 3, 5, \ldots\end{aligned}$$

What results is a standing wave as shown in Figure $2.10.3$, which shows snapshots of the resulting wave of two identical waves moving in opposite directions. The resulting wave appears to be a sine wave with nodes at integer multiples of half wavelengths. The antinodes oscillate between y = ±2A due to the cosine term, cos(ωt) , which oscillates between ±1.

The resultant wave appears to be standing still, with no apparent movement in the x-direction, although it is composed of one wave function moving in the positive, whereas the second wave is moving in the negative x-direction. Figure $2.10.3$ shows various snapshots of the resulting wave. The nodes are marked with red dots while the antinodes are marked with blue dots.

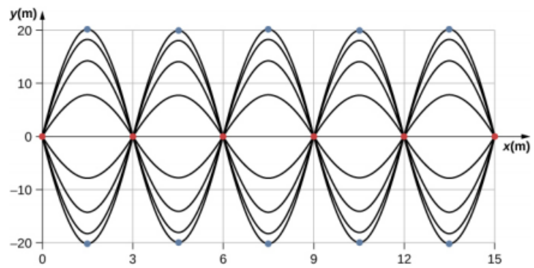


Figure $2.10.3$: When two identical waves are moving in opposite directions, the resultant wave is a standing wave. Nodes appear at integer multiples of half wavelengths. Antinodes appear at odd multiples of quarter wavelengths, where they oscillate between y = ±A. The nodes are marked with red dots and the antinodes are marked with blue dots.

A common example of standing waves are the waves produced by stringed musical instruments. When the string is plucked, pulses travel along the string in opposite directions. The ends of the strings are fixed in place, so nodes appear at the ends of the strings—the boundary conditions of the system, regulating the resonant frequencies in the strings. The resonance produced on a string instrument can be modeled in a physics lab using the apparatus shown in Figure $2.10.4$.

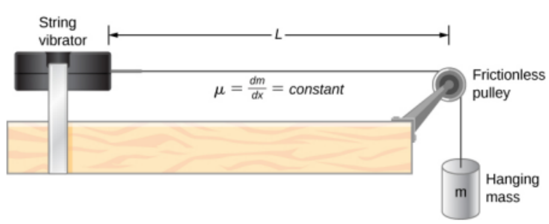


Figure $2.10.4$: A lab setup for creating standing waves on a string. The string has a node on each end and a constant linear density. The length between the fixed boundary conditions is L. The hanging mass provides the tension in the string, and the speed of the waves on the string is proportional to the square root of the tension divided by the linear mass density.

The lab setup shows a string attached to a string vibrator, which oscillates the string with an adjustable frequency f. The other end of the string passes over a frictionless pulley and is tied to a hanging mass. The magnitude of the tension in the string is equal to the weight of the hanging mass. The string has a constant linear density (mass per length) μ and the speed at which a wave travels down the string equals $v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{mg}{\mu}}$ Equation 16.7. The symmetrical boundary conditions (a node at each end) dictate the possible frequencies that can excite standing waves. Starting from a frequency of zero and slowly increasing the frequency, the first mode n = 1 appears as shown in Figure 2.10.5. The first mode, also called the fundamental mode or the first harmonic, shows half of a wavelength has formed, so the wavelength is equal to twice the length between the nodes λ_1 = 2L. The **fundamental frequency**, or first harmonic frequency, that drives this mode is

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L}, \tag{2.10.5}$$

where the speed of the wave is v = $\sqrt{\frac{F_T}{\mu}}$. Keeping the tension constant and increasing the frequency leads to the second harmonic or the n = 2 mode. This mode is a full wavelength λ_2 = L and the frequency is twice the fundamental frequency:

$$f_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2f_1. \tag{2.10.6}$$

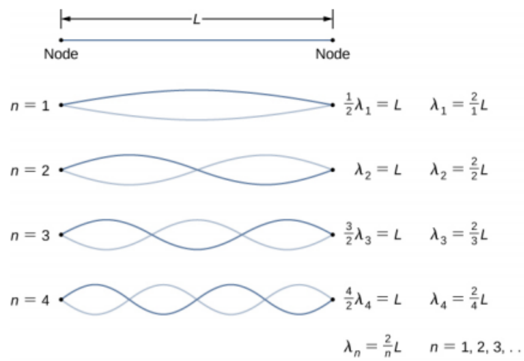


Figure 2.10.5: Standing waves created on a string of length L. A node occurs at each end of the string. The nodes are boundary conditions that limit the possible frequencies that excite standing waves. (Note that the amplitudes of the oscillations have been kept constant for visualization. The standing wave patterns possible on the string are known as the normal modes. Conducting this experiment in the lab would result in a decrease in amplitude as the frequency increases.)

The next two modes, or the third and fourth harmonics, have wavelengths of $\lambda_3 = \frac{2}{3}$ L and $\lambda_4 = \frac{2}{4}$ L, driven by frequencies of f_3 = $\frac{3v}{2L}$ = 3f_1 and f_4 = $\frac{4v}{2L}$ = 4f_1. All frequencies above the frequency f1 are known as the **overtones**. The equations for the wavelength and the frequency can be summarized as:

$$\lambda_n = \frac{2}{n}L \quad n = 1, 2, 3, 4, 5 \ldots \tag{2.10.7}$$

$$f_n = n\frac{v}{2L} = nf_1 \quad n = 1, 2, 3, 4, 5 \ldots \tag{2.10.8}$$

The standing wave patterns that are possible for a string, the first four of which are shown in Figure 2.10.5, are known as the **normal modes**, with frequencies known as the normal frequencies. In summary, the first frequency to produce a normal mode is called the fundamental frequency (or first harmonic). Any frequencies above the fundamental frequency are overtones. The second frequency of the n = 2 normal mode of the string is the first overtone (or second harmonic). The frequency of the n = 3 normal mode is the second overtone (or third harmonic) and so on.

The solutions shown as Equation 2.10.7 and Equation 2.10.8 are for a string with the boundary condition of a node on each end. When the boundary condition on either side is the same, the system is said to have symmetric boundary conditions. Equation 2.10.7 and Equation 2.10.8 are good for any symmetric boundary conditions, that is, nodes at both ends or antinodes at both ends.

✔ Example 2.10.1: Standing Waves on a String

Consider a string of L = 2.00 m. attached to an adjustable-frequency string vibrator as shown in Figure 2.10.6. The waves produced by the vibrator travel down the string and are reflected by the fixed boundary condition at the pulley. The string, which has a linear mass density of μ = 0.006 kg/m, is passed over a frictionless pulley of a negligible mass, and the tension is provided by a 2.00-kg hanging mass. (a) What is the velocity of the waves on the string? (b) Draw a sketch of the first three normal modes of the standing waves that can be produced on the string and label each with the wavelength. (c) List the frequencies that the string vibrator must be tuned to in order to produce the first three normal modes of the standing waves.

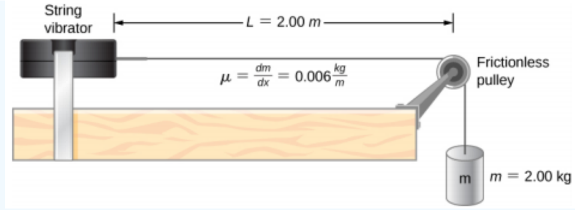


Figure 2.10.6: A string attached to an adjustable-frequency string vibrator.

Strategy

a. The velocity of the wave can be found using v = $\sqrt{\frac{F_T}{\mu}}$. The tension is provided by the weight of the hanging mass.
b. The standing waves will depend on the boundary conditions. There must be a node at each end. The first mode will be one half of a wave. The second can be found by adding a half wavelength. That is the shortest length that will result in a node at the boundaries. For example, adding one quarter of a wavelength will result in an antinode at the boundary and is not a mode which would satisfy the boundary conditions. This is shown in Figure 2.10.7.
c. Since the wave speed velocity is the wavelength times the frequency, the frequency is wave speed divided by the wavelength.

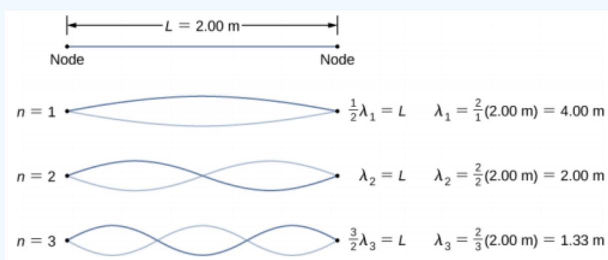
Figure 2.10.7 - (a) The figure represents the second mode of the string that satisfies the boundary conditions of a node at each end of the string. (b)This figure could not possibly be a normal mode on the string because it does not satisfy the boundary conditions. There is a node on one end, but an antinode on the other.

Solution

a. Begin with the velocity of a wave on a string. The tension is equal to the weight of the hanging mass. The linear mass density and mass of the hanging mass are given:

$$v=\sqrt{\frac{F_T}{\mu}}=\sqrt{\frac{mg}{\mu}}=\sqrt{\frac{(2\ kg)(9.8\ m/s)}{0.006\ kg/m}}=57.15\ m/s. \tag{2.10.9}$$

b. The first normal mode that has a node on each end is a half wavelength. The next two modes are found by adding a half of a wavelength.



c. The frequencies of the first three modes are found by using f = $\frac{v_w}{\lambda}$.

$$f_1=\frac{v_w}{\lambda_1}=\frac{57.15\ m/s}{4.00\ m}=14.29\ Hz$$
$$f_2=\frac{v_w}{\lambda_2}=\frac{57.15\ m/s}{2.00\ m}=28.58\ Hz$$
$$f_3=\frac{v_w}{\lambda_3}=\frac{57.15\ m/s}{1.333\ m}=42.87\ Hz$$

Significance

The three standing modes in this example were produced by maintaining the tension in the string and adjusting the driving frequency. Keeping the tension in the string constant results in a constant velocity. The same modes could have been produced by keeping the frequency constant and adjusting the speed of the wave in the string (by changing the hanging mass.)

Simulation

Visit this simulation to play with a 1-D or 2-D system of coupled mass-spring oscillators. Vary the number of masses, set the initial conditions, and watch the system evolve. See the spectrum of normal modes for arbitrary motion. See longitudinal or transverse modes in the 1-D system.

Exercise 2.10.1

The equations for the wavelengths and the frequencies of the modes of a wave produced on a string:

$$\lambda_n = \frac{2}{n}L \quad n = 1, 2, 3, 4, 5 \ldots and$$
$$f_n = n\frac{v}{2L} = nf_1 \quad n = 1, 2, 3, 4, 5 \ldots$$

were derived by considering a wave on a string where there were symmetric boundary conditions of a node at each end. These modes resulted from two sinusoidal waves with identical characteristics except they were moving in opposite directions, confined to a region L with nodes required at both ends. Will the same equations work if there were symmetric boundary conditions with antinodes at each end? What would the normal modes look like for a medium that was free to oscillate on each end? Don't worry for now if you cannot imagine such a medium, just consider two sinusoidal wave functions in a region of length L, with antinodes on each end.

The free boundary conditions shown in the last Check Your Understanding may seem hard to visualize. How can there be a system that is free to oscillate on each end? In Figure 2.10.8 are shown two possible configuration of a metallic rods (shown in red) attached to two supports (shown in blue). In part (a), the rod is supported at the ends, and there are fixed boundary conditions at both ends. Given the proper frequency, the rod can be driven into resonance with a wavelength equal to length of the rod, with nodes at each end. In part (b), the rod is supported at positions one quarter of the length from each end of the rod, and there are free boundary conditions at both ends. Given the proper frequency, this rod can also be driven into resonance with a wavelength equal to the length of the rod, but there are antinodes at each end. If you are having trouble visualizing the wavelength in this figure, remember that the wavelength may be measured between any two nearest identical points and consider Figure 2.10.9.

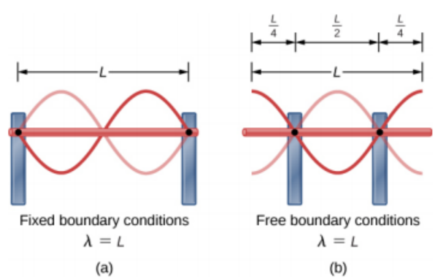


Figure 2.10.8: (a) A metallic rod of length L (red) supported by two supports (blue) on each end. When driven at the proper frequency, the rod can resonate with a wavelength equal to the length of the rod with a node on each end. (b) The same metallic rod of length L (red) supported by two supports (blue) at a position a quarter of the length of the rod from each end. When driven at the proper frequency, the rod can resonate with a wavelength equal to the length of the rod with an antinode on each end.

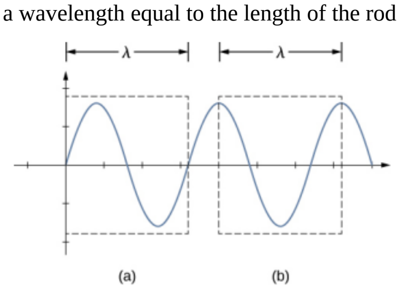


Figure 2.10.9: A wavelength may be measure between the nearest two repeating points. On the wave on a string, this means the same height and slope. (a) The wavelength is measured between the two nearest points where the height is zero and the slope is maximum and positive. (b) The wavelength is measured between two identical points where the height is maximum and the slope is zero.

Note that the study of standing waves can become quite complex. In Figure 16.32(a), the n = 2 mode of the standing wave is shown, and it results in a wavelength equal to L. In this configuration, the n = 1 mode would also have been possible with a standing wave equal to 2L. Is it possible to get the n = 1 mode for the configuration shown in part (b)? The answer is no. In this configuration, there are additional conditions set beyond the boundary conditions. Since the rod is mounted at a point one quarter of the length from each side, a node must exist there, and this limits the possible modes of standing waves that can be created. We leave it as an exercise for the reader to consider if other modes of standing waves are possible. It should be noted that when a system is driven at a frequency that does not cause the system to resonate, vibrations may still occur, but the amplitude of the vibrations will be much smaller than the amplitude at resonance.

A field of mechanical engineering uses the sound produced by the vibrating parts of complex mechanical systems to troubleshoot problems with the systems. Suppose a part in an automobile is resonating at the frequency of the car's engine, causing unwanted vibrations in the automobile. This may cause the engine to fail prematurely. The engineers use microphones to record the sound produced by the engine, then use a technique called Fourier analysis to find frequencies of sound produced with large amplitudes and then look at the parts list of the automobile to find a part that would resonate at that frequency. The solution may be as simple as changing the composition of the material used or changing the length of the part in question.

There are other numerous examples of resonance in standing waves in the physical world. The air in a tube, such as found in a musical instrument like a flute, can be forced into resonance and produce a pleasant sound, as we discuss in Sound.

At other times, resonance can cause serious problems. A closer look at earthquakes provides evidence for conditions appropriate for resonance, standing waves, and constructive and destructive interference. A building may vibrate for several seconds with a driving frequency matching that of the natural frequency of vibration of the building—producing a resonance resulting in one building collapsing while neighboring buildings do not. Often, buildings of a certain height are devastated while other taller buildings remain intact. The building height matches the condition for setting up a standing wave for that particular height. The span of the roof is also important. Often it is seen that gymnasiums, supermarkets, and churches suffer damage when individual homes suffer far less damage. The roofs with large surface areas supported only at the edges resonate at the frequencies of the earthquakes, causing them to collapse. As the earthquake waves travel along the surface of Earth and reflect off denser rocks, constructive interference occurs at certain points. Often areas closer to the epicenter are not damaged, while areas farther away are damaged.

2.E: Waves (Exercises)

Conceptual Questions

16.1 Traveling Waves

1. Give one example of a transverse wave and one example of a longitudinal wave, being careful to note the relative directions of the disturbance and wave propagation in each.
2. A sinusoidal transverse wave has a wavelength of 2.80 m. It takes 0.10 s for a portion of the string at a position x to move from a maximum position of y = 0.03 m to the equilibrium position y = 0. What are the period, frequency, and wave speed of the wave?
3. What is the difference between propagation speed and the frequency of a mechanical wave? Does one or both affect wavelength? If so, how?
4. Consider a stretched spring, such as a slinky. The stretched spring can support longitudinal waves and transverse waves. How can you produce transverse waves on the spring? How can you produce longitudinal waves on the spring?
5. Consider a wave produced on a stretched spring by holding one end and shaking it up and down. Does the wavelength depend on the distance you move your hand up and down?
6. A sinusoidal, transverse wave is produced on a stretched spring, having a period T. Each section of the spring moves perpendicular to the direction of propagation of the wave, in simple harmonic motion with an amplitude A. Does each section oscillate with the same period as the wave or a different period? If the amplitude of the transverse wave were doubled but the period stays the same, would your answer be the same?
7. An electromagnetic wave, such as light, does not require a medium. Can you think of an example that would support this claim?

16.2 Mathematics of Waves

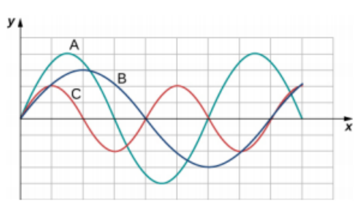
8. If you were to shake the end of a taut spring up and down 10 times a second, what would be the frequency and the period of the sinusoidal wave produced on the spring?
9. If you shake the end of a stretched spring up and down with a frequency f, you can produce a sinusoidal, transverse wave propagating down the spring. Does the wave number depend on the frequency you are shaking the spring?
10. Does the vertical speed of a segment of a horizontal taut string through which a sinusoidal, transverse wave is propagating depend on the wave speed of the transverse wave?
11. In this section, we have considered waves that move at a constant wave speed. Does the medium accelerate?
12. If you drop a pebble in a pond you may notice that several concentric ripples are produced, not just a single ripple. Why do you think that is?

16.3 Wave Speed on a Stretched String

13. If the tension in a string were increased by a factor of four, by what factor would the wave speed of a wave on the string increase?
14. Does a sound wave move faster in seawater or fresh water, if both the sea water and fresh water are at the same temperature and the sound wave moves near the surface?

$$\rho_w \approx 1000\ kg/m^3, \rho_s \approx 1030\ kg/m^3, B_w = 2.15 \times 10^9\ Pa, B_s = 2.34 \times 10^9\ Pa \tag{2.E.1}$$

15. Guitars have strings of different linear mass density. If the lowest density string and the highest density string are under the same tension, which string would support waves with the higher wave speed?
16. Shown below are three waves that were sent down a string at different times. The tension in the string remains constant. (a) Rank the waves from the smallest wavelength to the largest wavelength. (b) Rank the waves from the lowest frequency to the highest frequency.



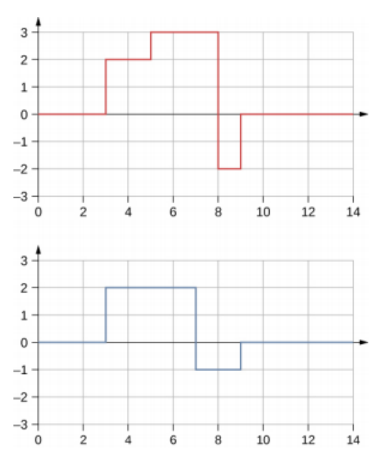
17. Electrical power lines connected by two utility poles are sometimes heard to hum when driven into oscillation by the wind. The speed of the waves on the power lines depend on the tension. What provides the tension in the power lines?
18. Two strings, one with a low mass density and one with a high linear density are spliced together. The higher density end is tied to a lab post and a student holds the free end of the low-mass density string. The student gives the string a flip and sends a pulse down the strings. If the tension is the same in both strings, does the pulse travel at the same wave velocity in both strings? If not, where does it travel faster, in the low density string or the high density string?

16.4 Energy and Power of a Wave

19. Consider a string with under tension with a constant linear mass density. A sinusoidal wave with an angular frequency and amplitude produced by some external driving force. If the frequency of the driving force is decreased to half of the original frequency, how is the time-averaged power of the wave affected? If the amplitude of the driving force is decreased by half, how is the time-averaged power affected? Explain your answer.
20. Circular water waves decrease in amplitude as they move away from where a rock is dropped. Explain why.
21. In a transverse wave on a string, the motion of the string is perpendicular to the motion of the wave. If this is so, how is possible to move energy along the length of the string?
22. The energy from the sun warms the portion of the earth facing the sun during the daylight hours. Why are the North and South Poles cold while the equator is quite warm?
23. The intensity of a spherical waves decreases as the wave moves away from the source. If the intensity of the wave at the source is I_0, how far from the source will the intensity decrease by a factor of nine?

16.5 Interference of Waves

24. An incident sinusoidal wave is sent along a string that is fixed to the wall with a wave speed of v. The wave reflects off the end of the string. Describe the reflected wave.
25. A string of a length of 2.00 m with a linear mass density of μ = 0.006 kg/m is attached to the end of a 2.00-m-long string with a linear mass density of μ = 0.012 kg/m. The free end of the higher-density string is fixed to the wall, and a student holds the free end of the low-density string, keeping the tension constant in both strings. The student sends a pulse down the string. Describe what happens at the interface between the two strings.
26. A long, tight spring is held by two students, one student holding each end. Each student gives the end a flip sending one wavelength of a sinusoidal wave down the spring in opposite directions. When the waves meet in the middle, what does the wave look like?
27. Many of the topics discussed in this chapter are useful beyond the topics of mechanical waves. It is hard to conceive of a mechanical wave with sharp corners, but you could encounter such a wave form in your digital electronics class, as shown below. This could be a signal from a device known as an analog to digital converter, in which a continuous voltage signal is converted into a discrete signal or a digital recording of sound. What is the result of the superposition of the two signals?



28. A string of a constant linear mass density is held taut by two students, each holding one end. The tension in the string is constant. The students each send waves down the string by wiggling the string. (a) Is it possible for the waves to have different wave speeds? (b) Is it possible for the waves to have different frequencies? (c) Is it possible for the waves to have different wavelengths?

16.6 Standing Waves and Resonance

29. A truck manufacturer finds that a strut in the engine is failing prematurely. A sound engineer determines that the strut resonates at the frequency of the engine and suspects that this could be the problem. What are two possible characteristics of the strut can be modified to correct the problem?
30. Why do roofs of gymnasiums and churches seem to fail more than family homes when an earthquake occurs?
31. Wine glasses can be set into resonance by moistening your finger and rubbing it around the rim of the glass. Why?
32. Air conditioning units are sometimes placed on the roof of homes in the city. Occasionally, the air conditioners cause an undesirable hum throughout the upper floors of the homes. Why does this happen? What can be done to reduce the hum?
33. Consider a standing wave modeled as $y(x, t) = 4.00 \text{ cm} \sin (3 \text{ m}^{-1} x) \cos (4 \text{ s}^{-1} t)$. Is there a node or an antinode at x = 0.00 m? What about a standing wave modeled as $y(x, t) = 4.00 \text{ cm} \sin (3 \text{ m}^{-1} x + \frac{\pi}{2}) \cos (4 \text{ s}^{-1} t)$? Is there a node or an antinode at the x = 0.00 m position?

Problems

16.1 Traveling Waves

34. Storms in the South Pacific can create waves that travel all the way to the California coast, 12,000 km away. How long does it take them to travel this distance if they travel at 15.0 m/s?
35. Waves on a swimming pool propagate at 0.75 m/s. You splash the water at one end of the pool and observe the wave go to the opposite end, reflect, and return in 30.00 s. How far away is the other end of the pool?
36. Wind gusts create ripples on the ocean that have a wavelength of 5.00 cm and propagate at 2.00 m/s. What is their frequency?
37. How many times a minute does a boat bob up and down on ocean waves that have a wavelength of 40.0 m and a propagation speed of 5.00 m/s?
38. Scouts at a camp shake the rope bridge they have just crossed and observe the wave crests to be 8.00 m apart. If they shake the bridge twice per second, what is the propagation speed of the waves?
39. What is the wavelength of the waves you create in a swimming pool if you splash your hand at a rate of 2.00 Hz and the waves propagate at a wave speed of 0.800 m/s?
40. What is the wavelength of an earthquake that shakes you with a frequency of 10.0 Hz and gets to another city 84.0 km away in 12.0 s?
41. Radio waves transmitted through empty space at the speed of light ($v = c = 3.00 \times 10^8$ m/s) by the Voyager spacecraft have a wavelength of 0.120 m. What is their frequency?
42. Your ear is capable of differentiating sounds that arrive at each ear just 0.34 ms apart, which is useful in determining where low frequency sound is originating from. (a) Suppose a low-frequency sound source is placed to the right of a person, whose ears are approximately 18 cm apart, and the speed of sound generated is 340 m/s. How long is the interval between when the sound arrives at the right ear and the sound arrives at the left ear? (b) Assume the same person was scuba diving and a low-frequency sound source was to the right of the scuba diver. How long is the interval between when the sound arrives at the right ear and the sound arrives at the left ear, if the speed of sound in water is 1500 m/s? (c) What is significant about the time interval of the two situations?
43. (a) Seismographs measure the arrival times of earthquakes with a precision of 0.100 s. To get the distance to the epicenter of the quake, geologists compare the arrival times of S- and P-waves, which travel at different speeds. If S- and P-waves travel at 4.00 and 7.20 km/s, respectively, in the region considered, how precisely can the distance to the source of the earthquake be determined? (b) Seismic waves from underground detonations of nuclear bombs can be used to locate the test site and detect violations of test bans. Discuss whether your answer to (a) implies a serious limit to such detection. (Note also that the uncertainty is greater if there is an uncertainty in the propagation speeds of the S- and P-waves.)
44. A Girl Scout is taking a 10.00-km hike to earn a merit badge. While on the hike, she sees a cliff some distance away. She wishes to estimate the time required to walk to the cliff. She knows that the speed of sound is approximately 343 meters per second. She yells and finds that the echo returns after approximately 2.00 seconds. If she can hike 1.00 km in 10 minutes, how long would it take her to reach the cliff?
45. A quality assurance engineer at a frying pan company is asked to qualify a new line of nonstick-coated frying pans. The coating needs to be 1.00 mm thick. One method to test the thickness is for the engineer to pick a percentage of the pans manufactured, strip off the coating, and measure the thickness using a micrometer. This method is a destructive testing method. Instead, the engineer decides that every frying pan will be tested using a nondestructive method. An ultrasonic transducer is used that produces sound waves with a frequency of f = 25 kHz. The sound waves are sent through the

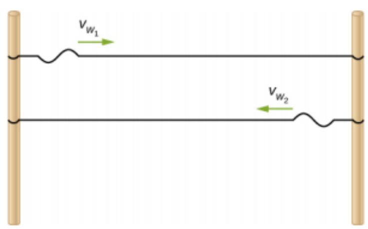
coating and are reflected by the interface between the coating and the metal pan, and the time is recorded. The wavelength of the ultrasonic waves in the coating is 0.076 m. What should be the time recorded if the coating is the correct thickness (1.00 mm)?

16.2 Mathematics of Waves

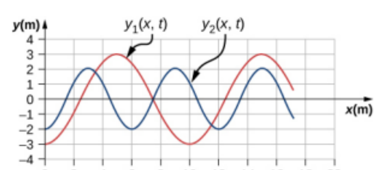
46. A pulse can be described as a single wave disturbance that moves through a medium. Consider a pulse that is defined at time t = 0.00 s by the equation $y(x) = \frac{6.00\ m^3}{x^2+2.00\ m^2}$ centered around x = 0.00 m. The pulse moves with a velocity of v = 3.00 m/s in the positive x-direction. (a) What is the amplitude of the pulse? (b) What is the equation of the pulse as a function of position and time? (c) Where is the pulse centered at time t = 5.00 s ?
47. A transverse wave on a string is modeled with the wave function $y(x, t) = (0.20\ cm)\sin(2.00\ m^{-1}\ x - 3.00\ s^{-1}\ t + \frac{\pi}{16})$. What is the height of the string with respect to the equilibrium position at a position x = 4.00 m and a time t = 10.00 s?
48. Consider the wave function $y(x, t) = (3.00\ cm)\sin(0.4\ m^{-1}\ x + 2.00\ s^{-1}\ t + \frac{\pi}{10})$. What are the period, wavelength, speed, and initial phase shift of the wave modeled by the wave function?
49. A pulse is defined as $y(x, t) = e^{-2.77\left(\frac{2.00(x-2.00\ m/s(t))}{5.00\ m}\right)^2}$. Use a spreadsheet, or other computer program, to plot the pulse as the height of medium y as a function of position x. Plot the pulse at times t = 0.00 s and t = 3.00 s on the same graph. Where is the pulse centered at time t = 3.00 s ? Use your spreadsheet to check your answer.
50. A wave is modeled at time t = 0.00 s with a wave function that depends on position. The equation is $y(x) = (0.30\ m)\sin(6.28\ m^{-1}\ x)$. The wave travels a distance of 4.00 meters in 0.50 s in the positive x-direction. Write an equation for the wave as a function of position and time.
51. A wave is modeled with the function $y(x, t) = (0.25\ m)\cos(0.30\ m^{-1}\ x - 0.90\ s^{-1}\ t + \frac{\pi}{3})$. Find the (a) amplitude, (b) wave number, (c) angular frequency, (d) wave speed, (e) initial phase shift, (f) wavelength, and (g) period of the wave.
52. A surface ocean wave has an amplitude of 0.60 m and the distance from trough to trough is 8.00 m. It moves at a constant wave speed of 1.50 m/s propagating in the positive x-direction. At t = 0, the water displacement at x = 0 is zero, and vy is positive. (a) Assuming the wave can be modeled as a sine wave, write a wave function to model the wave. (b) Use a spreadsheet to plot the wave function at times t = 0.00 s and t = 2.00 s on the same graph. Verify that the wave moves 3.00 m in those 2.00 s.
53. A wave is modeled by the wave function $y(x, t) = (0.30\ m)\sin\left[\frac{2\pi}{4.50\ m}(x - 18.00\ m/s\ t)\right]$. What are the amplitude, wavelength, wave speed, period, and frequency of the wave?
54. A transverse wave on a string is described with the wave function $y(x, t) = (0.50\ cm)\sin(1.57\ m^{-1}\ x - 6.28\ s^{-1}\ t)$. (a) What is the wave velocity of the wave? (b) What is the magnitude of the maximum velocity of the string perpendicular to the direction of the motion?
55. A swimmer in the ocean observes one day that the ocean surface waves are periodic and resemble a sine wave. The swimmer estimates that the vertical distance between the crest and the trough of each wave is approximately 0.45 m, and the distance between each crest is approximately 1.8 m. The swimmer counts that 12 waves pass every two minutes. Determine the simple harmonic wave function that would describes these waves.
56. Consider a wave described by the wave function $y(x, t) = 0.3\ m\sin(2.00\ m^{-1}\ x - 628.00\ s^{-1}\ t)$. (a) How many crests pass by an observer at a fixed location in 2.00 minutes? (b) How far has the wave traveled in that time?
57. Consider two waves defined by the wave functions $y_1(x, t) = 0.50\ m\sin\left(\frac{2\pi}{3.00\ m}x + \frac{2\pi}{4.00\ s}t\right)$ and $y_2(x, t) = 0.50\ m\sin\left(\frac{2\pi}{6.00\ m}x - \frac{2\pi}{4.00\ s}t\right)$. What are the similarities and differences between the two waves?
58. Consider two waves defined by the wave functions $y_1(x, t) = 0.20\ m\sin\left(\frac{2\pi}{6.00\ m}x - \frac{2\pi}{4.00\ s}t\right)$ and $y_2(x, t) = 0.20\ m\cos\left(\frac{2\pi}{6.00\ m}x - \frac{2\pi}{4.00\ s}t\right)$. What are the similarities and differences between the two waves?
59. The speed of a transverse wave on a string is 300.00 m/s, its wavelength is 0.50 m, and the amplitude is 20.00 cm. How much time is required for a particle on the string to move through a distance of 5.00 km?

16.3 Wave Speed on a Stretched String

60. Transverse waves are sent along a 5.00-m-long string with a speed of 30.00 m/s. The string is under a tension of 10.00 N. What is the mass of the string?
61. A copper wire has a density of $\rho = 8920\ kg/m^3$, a radius of 1.20 mm, and a length L. The wire is held under a tension of 10.00 N. Transverse waves are sent down the wire. (a) What is the linear mass density of the wire? (b) What is the speed of the waves through the wire?
62. A piano wire has a linear mass density of $\mu = 4.95 \times 10^{-3}$ kg/m. Under what tension must the string be kept to produce waves with a wave speed of 500.00 m/s?
63. A string with a linear mass density of $\mu = 0.0060$ kg/m is tied to the ceiling. A 20-kg mass is tied to the free end of the string. The string is plucked, sending a pulse down the string. Estimate the speed of the pulse as it moves down the string.
64. A cord has a linear mass density of $\mu = 0.0075$ kg/m and a length of three meters. The cord is plucked and it takes 0.20 s for the pulse to reach the end of the string. What is the tension of the string?
65. A string is 3.00 m long with a mass of 5.00 g. The string is held taut with a tension of 500.00 N applied to the string. A pulse is sent down the string. How long does it take the pulse to travel the 3.00 m of the string?
66. A sound wave travels through a column of nitrogen at STP. Assuming a density of $\rho = 1.25\ kg/m^3$ and a bulk modulus of $\beta = 1.42 \times 10^5$ Pa, what is the approximate speed of the sound wave?
67. What is the approximate speed of sound traveling through air at a temperature of T = 28°C ?
68. Transverse waves travel through a string where the tension equals 7.00 N with a speed of 20.00 m/s. What tension would be required for a wave speed of 25.00 m/s?
69. Two strings are attached between two poles separated by a distance of 2.00 m as shown below, both under the same tension of 600.00 N. String 1 has a linear density of $\mu_1 = 0.0025$ kg/m and string 2 has a linear mass density of $\mu_2 = 0.0035$ kg/m. Transverse wave pulses are generated simultaneously at opposite ends of the strings. How much time passes before the pulses pass one another?



70. Two strings are attached between two poles separated by a distance of 2.00 meters as shown in the preceding figure, both strings have a linear density of $\mu_1 = 0.0025$ kg/m, the tension in string 1 is 600.00 N and the tension in string 2 is 700.00 N. Transverse wave pulses are generated simultaneously at opposite ends of the strings. How much time passes before the pulses pass one another?
71. The note E_4 is played on a piano and has a frequency of f = 393.88. If the linear mass density of this string of the piano is $\mu = 0.012$ kg/m and the string is under a tension of 1000.00 N, what is the speed of the wave on the string and the wavelength of the wave?
72. Two transverse waves travel through a taut string. The speed of each wave is v = 30.00 m/s. A plot of the vertical position as a function of the horizontal position is shown below for the time t = 0.00 s. (a) What is the wavelength of each wave? (b) What is the frequency of each wave? (c) What is the maximum vertical speed of each string?



73. A sinusoidal wave travels down a taut, horizontal string with a linear mass density of μ = 0.060 kg/m . The maximum vertical speed of the wave is vy max = 0.30 cm/s. The wave is modeled with the wave equation $y(x, t) = A \sin(6.00\ m^{-1}\ x - 24.00\ s^{-1}\ t)$. (a) What is the amplitude of the wave? (b) What is the tension in the string?
74. The speed of a transverse wave on a string is v = 60.00 m/s and the tension in the string is $F_T = 100.00$ N . What must the tension be to increase the speed of the wave to v = 120.00 m/s?

16.4 Energy and Power of a Wave

75. A string of length 5 m and a mass of 90 g is held under a tension of 100 N. A wave travels down the string that is modeled as $y(x, t) = 0.01\ m \sin(15.7\ m^{-1}\ x - 1170.12\ s^{-1})$. What is the power over one wavelength?
76. Ultrasound of intensity $1.50 \times 10^2\ W/m^2$ is produced by the rectangular head of a medical imaging device measuring 3.00 cm by 5.00 cm. What is its power output?
77. The low-frequency speaker of a stereo set has a surface area of $A = 0.05\ m^2$ and produces 1 W of acoustical power. (a) What is the intensity at the speaker? (b) If the speaker projects sound uniformly in all directions, at what distance from the speaker is the intensity $0.1\ W/m^2$?
78. To increase the intensity of a wave by a factor of 50, by what factor should the amplitude be increased?
79. A device called an insolation meter is used to measure the intensity of sunlight. It has an area of $100\ cm^2$ and registers 6.50 W. What is the intensity in W/m^2 ?
80. Energy from the Sun arrives at the top of Earth's atmosphere with an intensity of $1400\ W/m^2$. How long does it take for 1.80×10^9 J to arrive on an area of $1.00\ m^2$?
81. Suppose you have a device that extracts energy from ocean breakers in direct proportion to their intensity. If the device produces 10.0 kW of power on a day when the breakers are 1.20 m high, how much will it produce when they are 0.600 m high?
82. A photovoltaic array of (solar cells) is 10.0% efficient in gathering solar energy and converting it to electricity. If the average intensity of sunlight on one day is $70.00\ W/m^2$, what area should your array have to gather energy at the rate of 100 W? (b) What is the maximum cost of the array if it must pay for itself in two years of operation averaging 10.0 hours per day? Assume that it earns money at the rate of 9.00 cents per kilowatt-hour.
83. A microphone receiving a pure sound tone feeds an oscilloscope, producing a wave on its screen. If the sound intensity is originally $2.00 \times 10^{-5}\ W/m^2$, but is turned up until the amplitude increases by 30.0%, what is the new intensity?
84. A string with a mass of 0.30 kg has a length of 4.00 m. If the tension in the string is 50.00 N, and a sinusoidal wave with an amplitude of 2.00 cm is induced on the string, what must the frequency be for an average power of 100.00 W?
85. The power versus time for a point on a string ($\mu = 0.05$ kg/m) in which a sinusoidal traveling wave is induced is shown in the preceding figure. The wave is modeled with the wave equation $y(x, t) = A \sin(20.93\ m^{-1}\ x - \omega t)$. What is the frequency and amplitude of the wave?
86. A string is under tension F_{T1}. Energy is transmitted by a wave on the string at rate P_1 by a wave of frequency f_1. What is the ratio of the new energy transmission rate P_2 to P_1 if the tension is doubled?
87. A 250-Hz tuning fork is struck and the intensity at the source is I1 at a distance of one meter from the source. (a) What is the intensity at a distance of 4.00 m from the source? (b) How far from the tuning fork is the intensity a tenth of the intensity at the source? 88. A sound speaker is rated at a voltage of P = 120.00 V and a current of I = 10.00 A. Electrical power consumption is P = IV. To test the speaker, a signal of a sine wave is applied to the speaker. Assuming that the sound wave moves as a spherical wave and that all of the energy applied to the speaker is converted to sound energy, how far from the speaker is the intensity equal to $3.82\ W/m^2$?
88. The energy of a ripple on a pond is proportional to the amplitude squared. If the amplitude of the ripple is 0.1 cm at a distance from the source of 6.00 meters, what was the amplitude at a distance of 2.00 meters from the source?

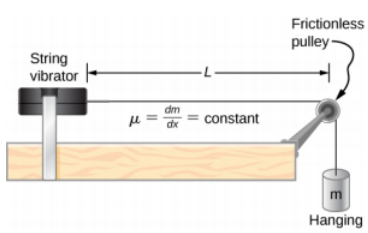
16.5 Interference of Waves

90. Consider two sinusoidal waves traveling along a string, modeled as $y_1(x, t) = 0.3\ m \sin(4\ m^{-1}\ x + 3\ s^{-1}\ t)$ and $y_2(x, t) = 0.6\ m \sin(8\ m^{-1}\ x - 6\ s^{-1}\ t)$. What is the height of the resultant wave formed by the interference of the two waves at the position x = 0.5 m at time t = 0.2 s?
91. Consider two sinusoidal sine waves traveling along a string, modeled as $y_1(x, t) = 0.3\ m \sin(4\ m^{-1}\ x + 3\ s^{-1}\ t + \frac{\pi}{3})$ and $y_2(x, t) = 0.6\ m \sin(8\ m^{-1}\ x - 6\ s^{-1}\ t)$. What is the height of the resultant wave formed by the interference of the two waves at the position x = 1.0 m at time t = 3.0 s?
92. Consider two sinusoidal sine waves traveling along a string, modeled as $y_1(x, t) = 0.3\ m \sin(4\ m^{-1}\ x - 3\ s^{-1}\ t)$ and $y_2(x, t) = 0.3\ m \sin(4\ m^{-1}\ x + 3\ s^{-1}\ t)$. What is the wave function of the resulting wave? [Hint: Use the trig identity $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$]
93. Two sinusoidal waves are moving through a medium in the same direction, both having amplitudes of 3.00 cm, a wavelength of 5.20 m, and a period of 6.52 s, but one has a phase shift of an angle ϕ. What is the phase shift if the resultant wave has an amplitude of 5.00 cm? [Hint: Use the trig identity $\sin u + \sin v = 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$]

94. Two sinusoidal waves are moving through a medium in the positive x-direction, both having amplitudes of 6.00 cm, a wavelength of 4.3 m, and a period of 6.00 s, but one has a phase shift of an angle ϕ = 0.50 rad. What is the height of the resultant wave at a time t = 3.15 s and a position x = 0.45 m?
95. Two sinusoidal waves are moving through a medium in the positive x-direction, both having amplitudes of 7.00 cm, a wave number of k = 3.00 m−1 , an angular frequency of ω = 2.50 s^{-1}, and a period of 6.00 s, but one has a phase shift of an angle $\phi = \frac{\pi}{12}$ rad. What is the height of the resultant wave at a time t = 2.00 s and a position x = 0.53 m?
96. Consider two waves $y_1(x, t)$ and $y_2(x, t)$ that are identical except for a phase shift propagating in the same medium. (a) What is the phase shift, in radians, if the amplitude of the resulting wave is 1.75 times the amplitude of the individual waves? (b) What is the phase shift in degrees? (c) What is the phase shift as a percentage of the individual wavelength?
97. Two sinusoidal waves, which are identical except for a phase shift, travel along in the same direction. The wave equation of the resultant wave is $y_R(x, t) = 0.70\ \text{m} \sin(3.00\ \text{m}^{-1}\ x - 6.28\ \text{s}^{-1}\ t + \frac{\pi}{16}\ \text{rad})$. What are the angular frequency, wave number, amplitude, and phase shift of the individual waves?
98. Two sinusoidal waves, which are identical except for a phase shift, travel along in the same direction. The wave equation of the resultant wave is $y_R(x, t) = 0.35\ \text{cm} \sin(6.28\ \text{m}^{-1}\ x - 1.57\ \text{s}^{-1}\ t + \frac{\pi}{4})$. What are the period, wavelength, amplitude, and phase shift of the individual waves?
99. Consider two wave functions, $y_1(x, t) = 4.00\ \text{m} \sin(\pi\ \text{m}^{-1}\ x - \pi\ \text{s}^{-1}\ t)$ and $y_2(x, t) = 4.00\ \text{m} \sin(\pi\ \text{m}^{-1}\ x - \pi\ \text{s}^{-1}\ t + \frac{\pi}{3})$. (a) Using a spreadsheet, plot the two wave functions and the wave that results from the superposition of the two wave functions as a function of position (0.00 ≤ x ≤ 6.00 m) for the time t = 0.00 s. (b) What are the wavelength and amplitude of the two original waves? (c) What are the wavelength and amplitude of the resulting wave?
100. Consider two wave functions, $y_2(x, t) = 2.00\ \text{m} \sin(\frac{\pi}{2}\ \text{m}^{-1}\ x - \frac{\pi}{3}\ \text{s}^{-1}\ t)$ and $y_2(x, t) = 2.00\ \text{m} \sin(\frac{\pi}{2}\ \text{m}^{-1}\ x - \frac{\pi}{3}\ \text{s}^{-1}\ t + \frac{\pi}{6})$. (a) Verify that $y_R = 2A \cos\left(\frac{\phi}{2}\right) \sin(kx - \omega t + \frac{\phi}{2})$ is the solution for the wave that results from a superposition of the two waves. Make a column for x, y_1, y_2, $y_1 + y_2$, and $y_R = 2A \cos\left(\frac{\phi}{2}\right) \sin(kx - \omega t + \frac{\phi}{2})$. Plot four waves as a function of position where the range of x is from 0 to 12 m.
101. Consider two wave functions that differ only by a phase shift, $y_1(x, t) = A \cos(kx - \omega t)$ and $y_2(x, t) = A \cos(kx - \omega t + \phi)$. Use the trigonometric identities $\cos u + \cos v = 2 \cos\left(\frac{u-v}{2}\right) \cos\left(\frac{u+v}{2}\right)$ and $\cos(-\theta) = \cos(\theta)$ to find a wave equation for the wave resulting from the superposition of the two waves. Does the resulting wave function come as a surprise to you?

16.6 Standing Waves and Resonance

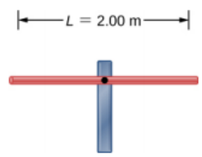
102. A wave traveling on a Slinky® that is stretched to 4 m takes 2.4 s to travel the length of the Slinky and back again. (a) What is the speed of the wave? (b) Using the same Slinky stretched to the same length, a standing wave is created which consists of three antinodes and four nodes. At what frequency must the Slinky be oscillating?
103. A 2-m long string is stretched between two supports with a tension that produces a wave speed equal to v_w = 50.00 m/s. What are the wavelength and frequency of the first three modes that resonate on the string?
104. Consider the experimental setup shown below. The length of the string between the string vibrator and the pulley is L = 1.00 m. The linear density of the string is μ = 0.006 kg/m. The string vibrator can oscillate at any frequency. The hanging mass is 2.00 kg. (a)What are the wavelength and frequency of n = 6 mode? (b) The string oscillates the air around the string. What is the wavelength of the sound if the speed of the sound is v_s = 343.00 m/s?



105. A cable with a linear density of μ = 0.2 kg/m is hung from telephone poles. The tension in the cable is 500.00 N. The distance between poles is 20 meters. The wind blows across the line, causing the cable resonate. A standing waves pattern

is produced that has 4.5 wavelengths between the two poles. The speed of sound at the current temperature T = 20 °C is 343.00 m/s. What are the frequency and wavelength of the hum?

106. Consider a rod of length L, mounted in the center to a support. A node must exist where the rod is mounted on a support, as shown below. Draw the first two normal modes of the rod as it is driven into resonance. Label the wavelength and the frequency required to drive the rod into resonance.



107. Consider two wave functions $y(x, t) = 0.30 \text{ cm} \sin(3 \text{ m}^{-1} x - 4 \text{ s}^{-1} t)$ and $y(x, t) = 0.30 \text{ cm} \sin(3 \text{ m}^{-1} x + 4 \text{ s}^{-1} t)$. Write a wave function for the resulting standing wave.

108. A 2.40-m wire has a mass of 7.50 g and is under a tension of 160 N. The wire is held rigidly at both ends and set into oscillation. (a) What is the speed of waves on the wire? The string is driven into resonance by a frequency that produces a standing wave with a wavelength equal to 1.20 m. (b) What is the frequency used to drive the string into resonance?

109. A string with a linear mass density of 0.0062 kg/m and a length of 3.00 m is set into the n = 100 mode of resonance. The tension in the string is 20.00 N. What is the wavelength and frequency of the wave?

110. A string with a linear mass density of 0.0075 kg/m and a length of 6.00 m is set into the n = 4 mode of resonance by driving with a frequency of 100.00 Hz. What is the tension in the string?

111. Two sinusoidal waves with identical wavelengths and amplitudes travel in opposite directions along a string producing a standing wave. The linear mass density of the string is $\mu = 0.075$ kg/m and the tension in the string is $F_T = 5.00$ N. The time interval between instances of total destructive interference is $\Delta t = 0.13$ s. What is the wavelength of the waves?

112. A string, fixed on both ends, is 5.00 m long and has a mass of 0.15 kg. The tension if the string is 90 N. The string is vibrating to produce a standing wave at the fundamental frequency of the string. (a) What is the speed of the waves on the string? (b) What is the wavelength of the standing wave produced? (c) What is the period of the standing wave?

113. A string is fixed at both end. The mass of the string is 0.0090 kg and the length is 3.00 m. The string is under a tension of 200.00 N. The string is driven by a variable frequency source to produce standing waves on the string. Find the wavelengths and frequency of the first four modes of standing waves.

114. The frequencies of two successive modes of standing waves on a string are 258.36 Hz and 301.42 Hz. What is the next frequency above 100.00 Hz that would produce a standing wave?

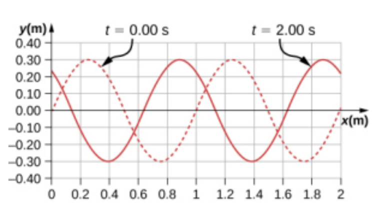
115. A string is fixed at both ends to supports 3.50 m apart and has a linear mass density of $\mu = 0.005$ kg/m. The string is under a tension of 90.00 N. A standing wave is produced on the string with six nodes and five antinodes. What are the wave speed, wavelength, frequency, and period of the standing wave?

116. Sine waves are sent down a 1.5-m-long string fixed at both ends. The waves reflect back in the opposite direction. The amplitude of the wave is 4.00 cm. The propagation velocity of the waves is 175 m/s. The n = 6 resonance mode of the string is produced. Write an equation for the resulting standing wave.

Additional Problems

117. Ultrasound equipment used in the medical profession uses sound waves of a frequency above the range of human hearing. If the frequency of the sound produced by the ultrasound machine is f = 30 kHz, what is the wavelength of the ultrasound in bone, if the speed of sound in bone is v = 3000 m/s?

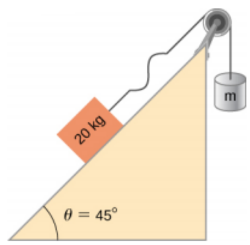
118. Shown below is the plot of a wave function that models a wave at time t = 0.00 s and t = 2.00 s. The dotted line is the wave function at time t = 0.00 s and the solid line is the function at time t = 2.00 s. Estimate the amplitude, wavelength, velocity, and period of the wave.



119. The speed of light in air is approximately v = 3.00 x 10^8 m/s and the speed of light in glass is v = 2.00 x 10^8 m/s. A red laser with a wavelength of λ = 633.00 nm shines light incident of the glass, and some of the red light is transmitted to the glass. The frequency of the light is the same for the air and the glass. (a) What is the frequency of the light? (b) What is the wavelength of the light in the glass?
120. A radio station broadcasts radio waves at a frequency of 101.7 MHz. The radio waves move through the air at approximately the speed of light in a vacuum. What is the wavelength of the radio waves?
121. A sunbather stands waist deep in the ocean and observes that six crests of periodic surface waves pass each minute. The crests are 16.00 meters apart. What is the wavelength, frequency, period, and speed of the waves?
122. A tuning fork vibrates producing sound at a frequency of 512 Hz. The speed of sound of sound in air is v = 343.00 m/s if the air is at a temperature of 20.00 °C. What is the wavelength of the sound?
123. A motorboat is traveling across a lake at a speed of v_b = 15.00 m/s. The boat bounces up and down every 0.50 s as it travels in the same direction as a wave. It bounces up and down every 0.30 s as it travels in a direction opposite the direction of the waves. What is the speed and wavelength of the wave?
124. Use the linear wave equation to show that the wave speed of a wave modeled with the wave function y(x, t) = 0.20 m sin(3.00 m^{-1} x + 6.00 s^{-1} t) is v = 2.00 m/s. What are the wavelength and the speed of the wave?
125. Given the wave functions $y_1(x, t) = A \sin(kx - \omega t)$ and $y_2(x, t) = A \sin(kx - \omega t + \phi)$ with $\phi \neq \frac{\pi}{2}$, show that $y_1(x, t) + y_2(x, t)$ is a solution to the linear wave equation with a wave velocity of $v = \frac{\omega}{k}$.
126. A transverse wave on a string is modeled with the wave function y(x, t) = 0.10 m sin(0.15 m^{-1} x + 1.50 s^{-1} t + 0.20). (a) Find the wave velocity. (b) Find the position in the y-direction, the velocity perpendicular to the motion of the wave, and the acceleration perpendicular to the motion of the wave, of a small segment of the string centered at x = 0.40 m at time t = 5.00 s.
127. A sinusoidal wave travels down a taut, horizontal string with a linear mass density of μ = 0.060 kg/m. The magnitude of maximum vertical acceleration of the wave is ay max = 0.90 cm/s^2 and the amplitude of the wave is 0.40 m. The string is under a tension of F_T = 600.00 N . The wave moves in the negative x-direction. Write an equation to model the wave.
128. A transverse wave on a string (μ = 0.0030 kg/m) is described with the equation y(x, t) = 0.30 m sin $\left(\frac{2\pi}{4.00\ m}(x - 16.00\ m/s\ t)\right)$. What is the tension under which the string is held taut?
129. A transverse wave on a horizontal string (μ = 0.0060 kg/m) is described with the equation y(x, t) = 0.30 m sin $\left(\frac{2\pi}{4.00\ m(x - v_w t)}\right)$. The string is under a tension of 300.00 N. What are the wave speed, wave number, and angular frequency of the wave?
130. A student holds an inexpensive sonic range finder and uses the range finder to find the distance to the wall. The sonic range finder emits a sound wave. The sound wave reflects off the wall and returns to the range finder. The round trip takes 0.012 s. The range finder was calibrated for use at room temperature T = 20 °C, but the temperature in the room is actually T = 23 °C. Assuming that the timing mechanism is perfect, what percentage of error can the student expect due to the calibration?
131. A wave on a string is driven by a string vibrator, which oscillates at a frequency of 100.00 Hz and an amplitude of 1.00 cm. The string vibrator operates at a voltage of 12.00 V and a current of 0.20 A. The power consumed by the string vibrator is P = IV. Assume that the string vibrator is 90% efficient at converting electrical energy into the energy associated with the vibrations of the string. The string is 3.00 m long, and is under a tension of 60.00 N. What is the linear mass density of the string?
132. A traveling wave on a string is modeled by the wave equation y(x, t) = 3.00 cm sin(8.00 m^{-1} x + 100.00 s^{-1} t). The string is under a tension of 50.00 N and has a linear mass density of μ = 0.008 kg/m. What is the average power transferred by the wave on the string?
133. A transverse wave on a string has a wavelength of 5.0 m, a period of 0.02 s, and an amplitude of 1.5 cm. The average power transferred by the wave is 5.00 W. What is the tension in the string?
134. (a) What is the intensity of a laser beam used to burn away cancerous tissue that, when 90.0% absorbed, puts 500 J of energy into a circular spot 2.00 mm in diameter in 4.00 s? (b) Discuss how this intensity compares to the average intensity of sunlight (about) and the implications that would have if the laser beam entered your eye. Note how your answer depends on the time duration of the exposure.
135. Consider two periodic wave functions, $y_1(x, t) = A \sin(kx - \omega t)$ and $y_2(x, t) = A \sin(kx - \omega t + \phi)$. (a) For what values of ϕ will the wave that results from a superposition of the wave functions have an amplitude of 2A? (b) For what values of

ϕ will the wave that results from a superposition of the wave functions have an amplitude of zero?

136. Consider two periodic wave functions, $y_1(x, t) = A \sin(kx - \omega t)$ and $y_2(x, t) = A \cos(kx - \omega t + \phi)$. (a) For what values of ϕ will the wave that results from a superposition of the wave functions have an amplitude of 2A? (b) For what values of ϕ will the wave that results from a superposition of the wave functions have an amplitude of zero?
137. A trough with dimensions 10.00 meters by 0.10 meters by 0.10 meters is partially filled with water. Small-amplitude surface water waves are produced from both ends of the trough by paddles oscillating in simple harmonic motion. The height of the water waves are modeled with two sinusoidal wave equations, $y_1(x, t) = 0.3\ m \sin(4\ m^{-1}\ x - 3\ s^{-1}\ t)$ and $y_2(x, t) = 0.3\ m \cos(4\ m^{-1}\ x + 3\ s^{-1}\ t - \frac{\pi}{2})$. What is the wave function of the resulting wave after the waves reach one another and before they reach the end of the trough (i.e., assume that there are only two waves in the trough and ignore reflections)? Use a spreadsheet to check your results. [Hint: Use the trig identities $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$ and $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$]
138. A seismograph records the S- and P-waves from an earthquake 20.00 s apart. If they traveled the same path at constant wave speeds of $v_S = 4.00$ km/s and $v_P = 7.50$ km/s, how far away is the epicenter of the earthquake?
139. Consider what is shown below. A 20.00-kg mass rests on a frictionless ramp inclined at 45°. A string with a linear mass density of $\mu = 0.025$ kg/m is attached to the 20.00-kg mass. The string passes over a frictionless pulley of negligible mass and is attached to a hanging mass (m). The system is in static equilibrium. A wave is induced on the string and travels up the ramp. (a) What is the mass of the hanging mass (m)? (b) At what wave speed does the wave travel up the string?

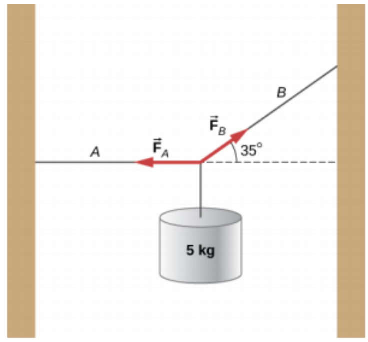


140. Consider the superposition of three wave functions $y(x, t) = 3.00\ cm \sin(2\ m^{-1}\ x - 3\ s^{-1}\ t)$, $y(x, t) = 3.00\ cm \sin(6\ m^{-1}\ x + 3\ s^{-1}\ t)$, and $y(x, t) = 3.00\ cm \sin(2\ m^{-1}\ x - 4\ s^{-1}\ t)$. What is the height of the resulting wave at position x = 3.00 m at time t = 10.0 s?
141. A string has a mass of 150 g and a length of 3.4 m. One end of the string is fixed to a lab stand and the other is attached to a spring with a spring constant of $k_s = 100$ N/m. The free end of the spring is attached to another lab pole. The tension in the string is maintained by the spring. The lab poles are separated by a distance that stretches the spring 2.00 cm. The string is plucked and a pulse travels along the string. What is the propagation speed of the pulse?
142. A standing wave is produced on a string under a tension of 70.0 N by two sinusoidal transverse waves that are identical, but moving in opposite directions. The string is fixed at x = 0.00 m and x = 10.00 m. Nodes appear at x = 0.00 m, 2.00 m, 4.00 m, 6.00 m, 8.00 m, and 10.00 m. The amplitude of the standing wave is 3.00 cm. It takes 0.10 s for the antinodes to make one complete oscillation. (a) What are the wave functions of the two sine waves that produce the standing wave? (b) What are the maximum velocity and acceleration of the string, perpendicular to the direction of motion of the transverse waves, at the antinodes?
143. A string with a length of 4 m is held under a constant tension. The string has a linear mass density of $\mu = 0.006$ kg/m. Two resonant frequencies of the string are 400 Hz and 480 Hz. There are no resonant frequencies between the two frequencies. (a) What are the wavelengths of the two resonant modes? (b) What is the tension in the string?

Challenge Problems

144. A copper wire has a radius of 200 μm and a length of 5.0 m. The wire is placed under a tension of 3000 N and the wire stretches by a small amount. The wire is plucked and a pulse travels down the wire. What is the propagation speed of the pulse? (Assume the temperature does not change: ($\rho = 8.96\ g/cm^3$, $Y = 1.1 \times 10^{11}$ N/m).
145. A pulse moving along the x axis can be modeled as the wave function $y(x, t) = 4.00\ m\ e^{-\left(\frac{x + (2.00\ m/s)t}{1.00\ m}\right)^2}$. (a) What are the direction and propagation speed of the pulse? (b) How far has the wave moved in 3.00 s? (c) Plot the pulse using a spreadsheet at time t = 0.00 s and t = 3.00 s to verify your answer in part (b).

146. A string with a linear mass density of μ = 0.0085 kg/m is fixed at both ends. A 5.0-kg mass is hung from the string, as shown below. If a pulse is sent along section A, what is the wave speed in section A and the wave speed in section B?



147. Consider two wave functions $y_1(x, t) = A \sin(kx - \omega t)$ and $y_2(x, t) = A \sin(kx + \omega t + \phi)$. What is the wave function resulting from the interference of the two wave? [Hint: $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$ and $\phi = \frac{\phi}{2} + \frac{\phi}{2}$).
148. The wave function that models a standing wave is given as yR (x, t) = 6.00 cm sin(3.00 m^{-1} x + 1.20 rad) cos(6.00 s^{-1} t + 1.20 rad). What are two wave functions that interfere to form this wave function? Plot the two wave functions and the sum of the sum of the two wave functions at t = 1.00 s to verify your answer.
149. Consider two wave functions $y_1(x, t) = A \sin(kx - \omega t)$ and $y_2(x, t) = A \sin(kx + \omega t + \phi)$. The resultant wave form when you add the two functions is y_R = 2A sin(kx + $\frac{\phi}{2}$) cos(\omega t + \frac{\phi}{2}\)). Consider the case where A = 0.03 m^{-1}, k = 1.26 m^{-1}, $\omega = \pi$ s^{-1}, and $\phi = \frac{\pi}{10}$. (a) Where are the first three nodes of the standing wave function starting at zero and moving in the positive x direction? (b) Using a spreadsheet, plot the two wave functions and the resulting function at time t = 1.00 s to verify your answer.

2.S: Waves (Summary)

Key Terms

antinode	location of maximum amplitude in standing waves
constructive interference	when two waves arrive at the same point exactly in phase; that is, the crests of the two waves are prec
destructive interference	when two identical waves arrive at the same point exactly out of phase; that is, precisely al
fixed boundary condition	when the medium at a boundary is fixed in place so it cannot move
free boundary condition	exists when the medium at the boundary is free to move
fundamental frequency	lowest frequency that will produce a standing wave
intensity (I)	power per unit area
interference	overlap of two or more waves at the same point and time
linear wave equation	equation describing waves that result from a linear restoring force of the medium; any function that is a solution to t the positive x-direction or the negative x-direction with a constant wave spe
longitudinal wave	wave in which the disturbance is parallel to the direction of propagatio
mechanical wave	wave that is governed by Newton's laws and requires a medium
node	point where the string does not move; more generally, nodes are where the wave disturbance is
normal mode	possible standing wave pattern for a standing wave on a string
overtone	frequency that produces standing waves and is higher than the fundamental fre
pulse	single disturbance that moves through a medium, transferring energy but not
standing wave	wave that can bounce back and forth through a particular region, effectively becomi
superposition	phenomenon that occurs when two or more waves arrive at the same poi
transverse wave	wave in which the disturbance is perpendicular to the direction of propaga
wave	disturbance that moves from its source and carries energy
wave function	mathematical model of the position of particles of the medium
wave number	$\frac{2\pi}{\lambda}$
wave speed	magnitude of the wave velocity
wave velocity	velocity at which the disturbance moves; also called the propagation veloc
wavelength	distance between adjacent identical parts of a wave

Key Equations

Wave speed	$v = \frac{\lambda}{T} = \lambda f$
Linear mass density	$\mu = \frac{mass\ of\ the\ string}{length\ of\ the\ string}$
Speed of a wave or pulse on a string under tension	$\|v\| = \sqrt{\frac{F_T}{\mu}}$
Speed of a compression wave in a fluid	$v = \sqrt{\frac{B}{\rho}}$
Resultant wave from superposition of two sinusoidal waves that are identical except for a phase shift	\[y_{R} (x,t) = \Bigg[2A \cos \left(\dfrac{\phi}{2}\right) \Bigg] \sin \left(kx - \omega t + \d
Wave number	$k = \equiv \frac{2\pi}{\lambda}$
Wave speed	$v = \frac{\omega}{k}$
Periodic wave	$y(x,t) = A\sin(kx \mp \omega + \phi)$
Phase of a wave	$kx \mp \omega t + \phi$
Linear wave equation	$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v_w^2}\frac{\partial^2 y(x,t)}{\partial t^2}$
Power in a wave for one wavelength	$P_{ave} = \frac{E_\lambda}{T} = \frac{1}{2}\mu A^2\omega^2\frac{\lambda}{T} = \frac{1}{2}\mu A^2\omega^2 v$
Intensity	$I = \frac{P}{A}$
Intensity for a spherical wave	$I = \frac{P}{2\pi r^2}$
Equation of a standing wave	\[y (x,t) = [2A \sin (kx)] \cos (\omega t)$$
Wavelength for symmetric boundary conditions	$\lambda_n = \frac{2}{n}L, \qquad n = 1,2,3,4,5\ldots$

Frequency for symmetric boundary conditions	$f_n = n\frac{v}{2L} = nf_1, \qquad n = 1,2,3,4,5\dots$

Summary

16.1 Traveling Waves

- A wave is a disturbance that moves from the point of origin with a wave velocity v.
- A wave has a wavelength λ, which is the distance between adjacent identical parts of the wave. Wave velocity and wavelength are related to the wave's frequency and period by v = $\frac{\lambda}{T}$ = λf.
- Mechanical waves are disturbances that move through a medium and are governed by Newton's laws.
- Electromagnetic waves are disturbances in the electric and magnetic fields, and do not require a medium.
- Matter waves are a central part of quantum mechanics and are associated with protons, electrons, neutrons, and other fundamental particles found in nature.
- A transverse wave has a disturbance perpendicular to the wave's direction of propagation, whereas a longitudinal wave has a disturbance parallel to its direction of propagation.

16.2 Mathematics of Waves

- A wave is an oscillation (of a physical quantity) that travels through a medium, accompanied by a transfer of energy. Energy transfers from one point to another in the direction of the wave motion. The particles of the medium oscillate up and down, back and forth, or both up and down and back and forth, around an equilibrium position.
- A snapshot of a sinusoidal wave at time t = 0.00 s can be modeled as a function of position. Two examples of such functions are y(x) = A sin (kx + ϕ) and y(x) = A cos (kx + ϕ).
- Given a function of a wave that is a snapshot of the wave, and is only a function of the position x, the motion of the pulse or wave moving at a constant velocity can be modeled with the function, replacing x with x $\mp$ vt. The minus sign is for motion in the positive direction and the plus sign for the negative direction.
- The wave function is given by y(x, t) = A sin (kx − ωt + ϕ) where k = $\frac{2\pi}{\lambda}$ is defined as the wave number, $\omega = \frac{2\pi}{T}$ is the angular frequency, and ϕ is the phase shift.
- The wave moves with a constant velocity v_w, where the particles of the medium oscillate about an equilibrium position. The constant velocity of a wave can be found by v = $\frac{\lambda}{T}$ = $\frac{\omega}{k}$.

16.3 Wave Speed on a Stretched String

- The speed of a wave on a string depends on the linear density of the string and the tension in the string. The linear density is mass per unit length of the string.
- In general, the speed of a wave depends on the square root of the ratio of the elastic property to the inertial property of the medium.
- The speed of a wave through a fluid is equal to the square root of the ratio of the bulk modulus of the fluid to the density of the fluid.
- The speed of sound through air at T = 20 °C is approximately v_s = 343.00 m/s.

16.4 Energy and Power of a Wave

- The energy and power of a wave are proportional to the square of the amplitude of the wave and the square of the angular frequency of the wave.
- The time-averaged power of a sinusoidal wave on a string is found by P_{ave} = $\frac{1}{2}\mu A^2\omega^2 v$, where μ is the linear mass density of the string, A is the amplitude of the wave, ω is the angular frequency of the wave, and v is the speed of the wave.
- Intensity is defined as the power divided by the area. In a spherical wave, the area is A = 4πr^2 and the intensity is I = $\frac{P}{4\pi r^2}$. As the wave moves out from a source, the energy is conserved, but the intensity decreases as the area increases.

16.5 Interference of Waves

- Superposition is the combination of two waves at the same location.
- Constructive interference occurs from the superposition of two identical waves that are in phase.
- Destructive interference occurs from the superposition of two identical waves that are 180° (π radians) out of phase.
- The wave that results from the superposition of two sine waves that differ only by a phase shift is a wave with an amplitude that depends on the value of the phase difference.

16.6 Standing Waves and Resonance

- A standing wave is the superposition of two waves which produces a wave that varies in amplitude but does not propagate.
- Nodes are points of no motion in standing waves.
- An antinode is the location of maximum amplitude of a standing wave.
- Normal modes of a wave on a string are the possible standing wave patterns. The lowest frequency that will produce a standing wave is known as the fundamental frequency. The higher frequencies which produce standing waves are called overtones.

Contributors and Attributions

CHAPTER OVERVIEW

3: Photons and Matter Waves

In this chapter, you will learn about the energy quantum, a concept that was introduced in 1900 by the German physicist Max Planck to explain blackbody radiation. We discuss how Albert Einstein extended Planck's concept to a quantum of light (a "photon") to explain the photoelectric effect. We also show how American physicist Arthur H. Compton used the photon concept in 1923 to explain wavelength shifts observed in X-rays. After a discussion of Bohr's model of hydrogen, we describe how matter waves were postulated in 1924 by Louis-Victor de Broglie to justify Bohr's model and we examine the experiments conducted in 1923–1927 by Clinton Davisson and Lester Germer that confirmed the existence of de Broglie's matter waves.

Thumbnails: An experimental setup to study the photoelectric effect. The anode and cathode are enclosed in an evacuated glass tube. The voltmeter measures the electric potential difference between the electrodes, and the ammeter measures the photocurrent. The incident radiation is monochromatic.

Contributors and Attributions

3.1: Prelude to Photons and Matter Waves

Two of the most revolutionary concepts of the twentieth century were the description of light as a collection of particles, and the treatment of particles as waves. These wave properties of matter have led to the discovery of technologies such as electron microscopy, which allows us to examine submicroscopic objects such as grains of pollen, as shown above.

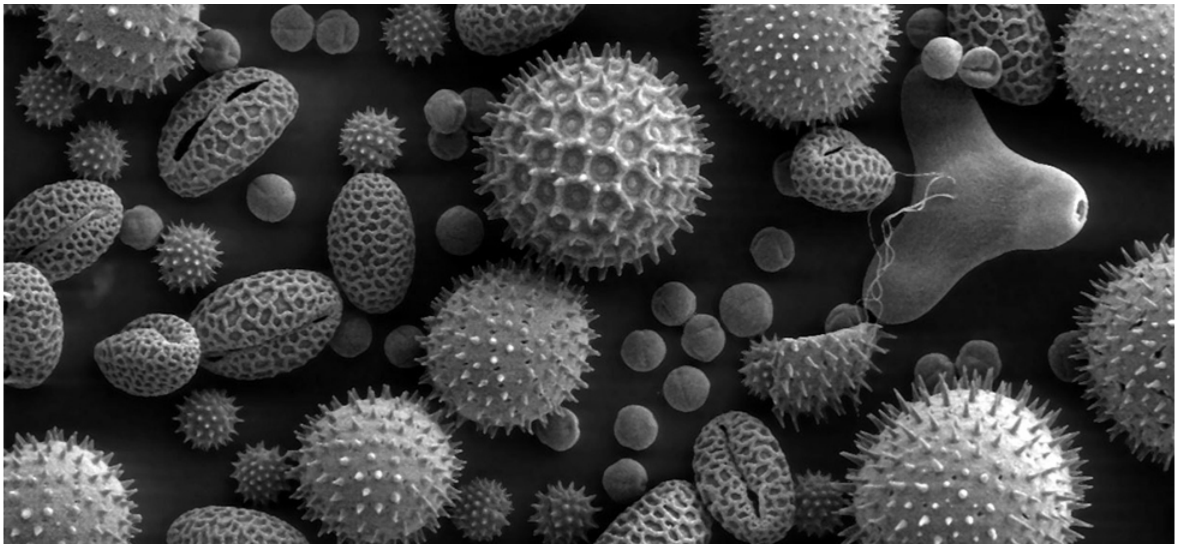


Figure 3.1.1: In this image of pollen taken with an electron microscope, the bean-shaped grains are about 50 μm long. Electron microscopes can have a much higher resolving power than a conventional light microscope because electron wavelengths can be 100,000 times shorter than the wavelengths of visible-light photons. (credit: modification of work by Dartmouth College Electron Microscope Facility).

In this chapter, you will learn about the energy quantum, a concept that was introduced in 1900 by the German physicist Max Planck to explain blackbody radiation. We discuss how Albert Einstein extended Planck's concept to a quantum of light (a "photon") to explain the photoelectric effect. We also show how American physicist Arthur H. Compton used the photon concept in 1923 to explain wavelength shifts observed in X-rays. After a discussion of Bohr's model of hydrogen, we describe how matter waves were postulated in 1924 by Louis-Victor de Broglie to justify Bohr's model and we examine the experiments conducted in 1923–1927 by Clinton Davisson and Lester Germer that confirmed the existence of de Broglie's matter waves.

Contributors and Attributions

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3.2: Blackbody Radiation

Learning Objectives

By the end of this section you will be able to:

- Apply Wien's and Stefan's laws to analyze radiation emitted by a blackbody
- Explain Planck's hypothesis of energy quanta

All bodies emit electromagnetic radiation over a range of wavelengths. In an earlier chapter, we learned that a cooler body radiates less energy than a warmer body. We also know by observation that when a body is heated and its temperature rises, the perceived wavelength of its emitted radiation changes from infrared to red, and then from red to orange, and so forth. As its temperature rises, the body glows with the colors corresponding to ever-smaller wavelengths of the electromagnetic spectrum. This is the underlying principle of the incandescent light bulb: A hot metal filament glows red, and when heating continues, its glow eventually covers the entire visible portion of the electromagnetic spectrum. The temperature ($\mathbf{T}$) of the object that emits radiation, or the **emitter**, determines the wavelength at which the radiated energy is at its maximum. For example, the Sun, whose surface temperature is in the range between 5000 K and 6000 K, radiates most strongly in a range of wavelengths about 560 nm in the visible part of the electromagnetic spectrum. Your body, when at its normal temperature of about 300 K, radiates most strongly in the infrared part of the spectrum.

Radiation that is incident on an object is partially absorbed and partially reflected. At thermodynamic equilibrium, the rate at which an object absorbs radiation is the same as the rate at which it emits it. Therefore, a good absorber of radiation (any object that absorbs radiation) is also a good emitter. A perfect **absorber** absorbs all electromagnetic radiation incident on it; such an object is called a **blackbody**.

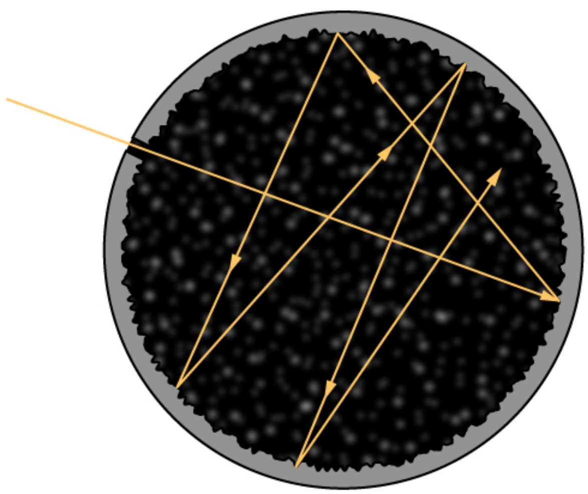

Figure $3.2.1$: A blackbody is physically realized by a small hole in the wall of a cavity radiator.

Although the blackbody is an idealization, because no physical object absorbs 100% of incident radiation, we can construct a close realization of a blackbody in the form of a small hole in the wall of a sealed enclosure known as a cavity radiator, as shown in Figure $3.2.1$. The inside walls of a cavity radiator are rough and blackened so that any radiation that enters through a tiny hole in the cavity wall becomes trapped inside the cavity. At thermodynamic equilibrium (at temperature $\mathbf{T}$), the cavity walls absorb exactly as much radiation as they emit. Furthermore, inside the cavity, the radiation entering the hole is balanced by the radiation leaving it. The emission spectrum of a blackbody can be obtained by analyzing the light radiating from the hole. Electromagnetic waves emitted by a blackbody are called **blackbody radiation**.

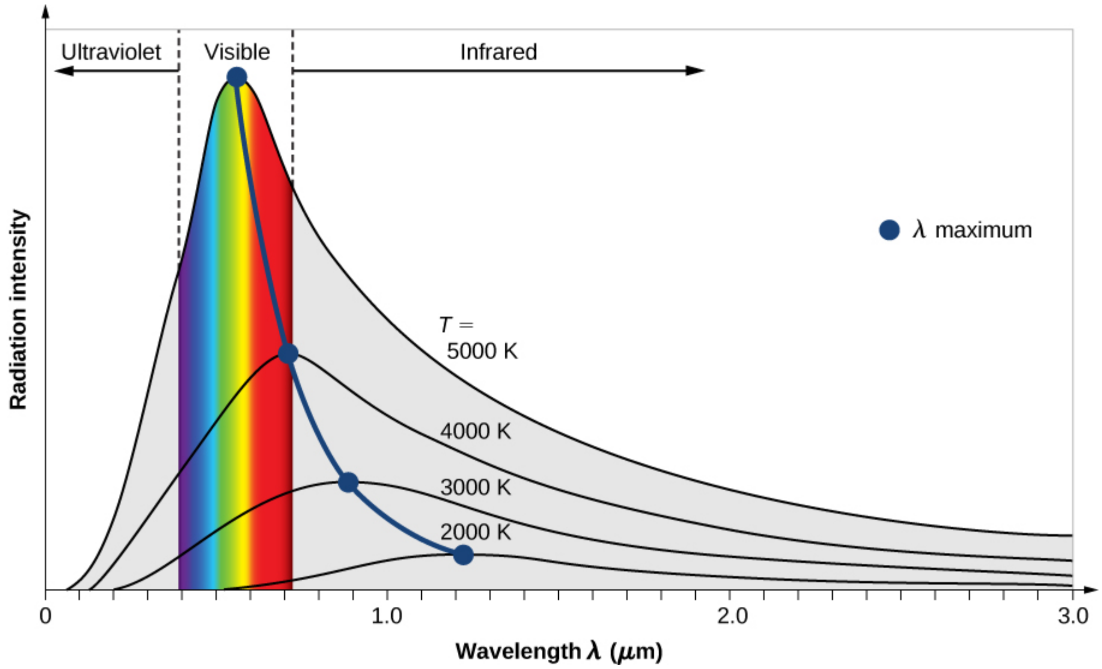

Figure $3.2.2$: The intensity of blackbody radiation versus the wavelength of the emitted radiation. Each curve corresponds to a different blackbody temperature, starting with a low temperature (the lowest curve) to a high temperature (the highest curve).

The intensity $I(\lambda, T)$ of blackbody radiation depends on the wavelength λ of the emitted radiation and on the temperature **T** of the blackbody (Figure $3.2.2$). The function $I(\lambda, T)$ is the **power** intensity that is radiated per unit wavelength; in other words, it is the power radiated per unit area of the hole in a cavity radiator per unit wavelength. According to this definition, $I(\lambda, T)d\lambda$ is the power per unit area that is emitted in the wavelength interval from λ to $\lambda + d\lambda$. The intensity distribution among wavelengths of radiation emitted by cavities was studied experimentally at the end of the nineteenth century. Generally, radiation emitted by materials only approximately follows the blackbody radiation curve (Figure $3.2.3$); however, spectra of common stars do follow the blackbody radiation curve very closely.

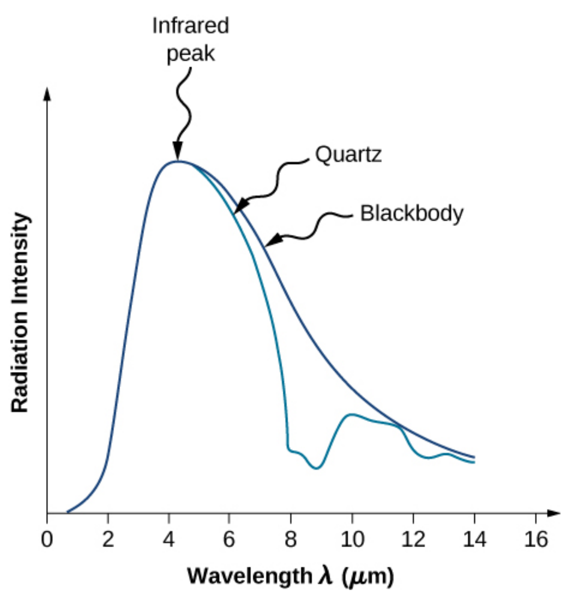

Figure $3.2.3$: The spectrum of radiation emitted from a quartz surface (blue curve) and the blackbody radiation curve (black curve) at 600 K.

Two important laws summarize the experimental findings of blackbody radiation: ***Wien's displacement law*** and **Stefan's law**. Wien's displacement law is illustrated in Figure $3.2.2$ by the curve connecting the maxima on the intensity curves. In these curves, we see that the hotter the body, the shorter the wavelength corresponding to the emission peak in the radiation curve. Quantitatively, Wien's law reads

$$\lambda_{max} T = 2.898 \times 10^{-3} m \cdot K \tag{3.2.1}$$

where λ_{max} is the position of the maximum in the radiation curve. In other words, λ_{max} is the wavelength at which a blackbody radiates most strongly at a given temperature **T**. Note that in Equation 3.2.1, the temperature is in kelvins. Wien's displacement law allows us to estimate the temperatures of distant stars by measuring the wavelength of radiation they emit.

✔ Example 3.2.1: Temperatures of Distant Stars

On a clear evening during the winter months, if you happen to be in the Northern Hemisphere and look up at the sky, you can see the constellation Orion (The Hunter). One star in this constellation, Rigel, flickers in a blue color and another star, Betelgeuse, has a reddish color, as shown in Figure 3.2.4. Which of these two stars is cooler, Betelgeuse or Rigel?

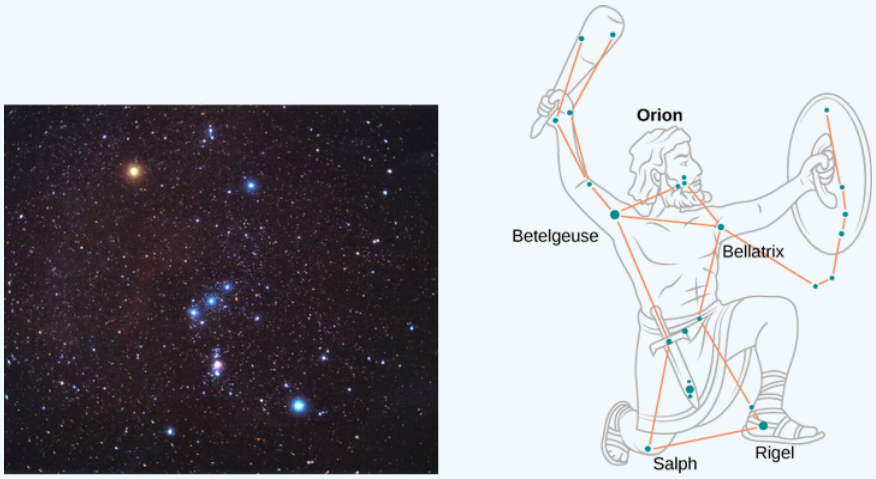

Figure 3.2.4: In the Orion constellation, the red star Betelgeuse, which usually takes on a yellowish tint, appears as the figure's right shoulder (in the upper left). The giant blue star on the bottom right is Rigel, which appears as the hunter's left foot. (credit left: modification of work by NASA c/o Matthew Spinelli)

Strategy

We treat each star as a blackbody. Then according to Wien's law, its temperature is inversely proportional to the wavelength of its peak intensity. The wavelength $\lambda_{max}^{(blue)}$ of blue light is shorter than the wavelength $\lambda_{max}^{(red)}$ of red light. Even if we do not know the precise wavelengths, we can still set up a proportion.

Solution

Writing Wien's law for the blue star and for the red star, we have

$$\begin{aligned} \lambda_{max}^{(red)} T_{(red)} &= 2.898 \times 10^{-3} m \cdot K \\ &= \lambda_{max}^{(blue)} T_{(blue)} \end{aligned}$$

When simplified, this gives

$$T_{(red)} = \frac{\lambda_{max}^{(blue)}}{\lambda_{max}^{(red)}} T_{(blue)} < T_{(blue)}$$

Therefore, Betelgeuse is cooler than Rigel.

Significance

Note that Wien's displacement law tells us that the higher the temperature of an emitting body, the shorter the wavelength of the radiation it emits. The qualitative analysis presented in this example is generally valid for any emitting body, whether it is a big object such as a star or a small object such as the glowing filament in an incandescent lightbulb.

? Exercise 3.2.1

The flame of a peach-scented candle has a yellowish color and the flame of a Bunsen's burner in a chemistry lab has a bluish color. Which flame has a higher temperature?

Answer

Bunsen's burner

The second experimental relation is **Stefan's law,** which concerns the total power of blackbody radiation emitted across the entire spectrum of wavelengths at a given temperature. In 3.2.2, this total power is represented by the area under the blackbody radiation curve for a given **T**. As the temperature of a blackbody increases, the total emitted power also increases. Quantitatively, Stefan's law expresses this relation as

$$P(T) = \sigma A T^4$$

where A is the surface area of a blackbody, T is its temperature (in kelvins), and σ is the **Stefan–Boltzmann constant**, $\sigma = 5.670 \times 10^{-8} W/(m^2 \cdot K^4)$. Stefan's law enables us to estimate how much energy a star is radiating by remotely measuring its temperature.

✓ Example 3.2.2: Power Radiated by Stars

A star such as our Sun will eventually evolve to a "red giant" star and then to a "white dwarf" star. A typical white dwarf is approximately the size of Earth, and its surface temperature is about $2.5 \times 10^4 K$. A typical red giant has a surface temperature of $3.0 \times 10^3 K$ and a radius ~100,000 times larger than that of a white dwarf. What is the average radiated power per unit area and the total power radiated by each of these types of stars? How do they compare?

Strategy

If we treat the star as a blackbody, then according to Stefan's law, the total power that the star radiates is proportional to the fourth power of its temperature. To find the power radiated per unit area of the surface, we do not need to make any assumptions about the shape of the star because **P**/**A** depends only on temperature. However, to compute the total power, we need to make an assumption that the energy radiates through a spherical surface enclosing the star, so that the surface area is $A = 4\pi R^2$, where **R** is its radius.

Solution

A simple proportion based on Stefan's law gives

$$\frac{P_{dwarf}/A_{dwarf}}{P_{giant}/A_{giant}} = \frac{\sigma T^4_{dwarf}}{\sigma T^4_{giant}} = \left(\frac{T_{dwarf}}{T_{giant}}\right)^4 = 4820 \tag{3.2.2}$$

The power emitted per unit area by a white dwarf is about 5000 times that the power emitted by a red giant. Denoting this ratio by $a = 4.8 \times 10^3$, Equation 3.2.2 gives

$$\frac{P_{dwarf}}{P_{giant}} = \alpha \frac{A_{dwarf}}{A_{giant}} = \alpha \frac{4\pi R^2_{dwarf}}{4\pi R^2_{giant}} = \alpha \left(\frac{R_{dwarf}}{R_{giant}}\right)^2 = 4.8 \times 10^{-7}$$

We see that the total power emitted by a white dwarf is a tiny fraction of the total power emitted by a red giant. Despite its relatively lower temperature, the overall power radiated by a red giant far exceeds that of the white dwarf because the red giant has a much larger surface area. To estimate the absolute value of the emitted power per unit area, we again use Stefan's law. For the white dwarf, we obtain

$$\frac{P_{dwarf}}{A_{dwarf}} = \sigma T^4_{dwarf} = 5.670 \times 10^{-8} \frac{W}{m^2 \cdot K^4} (2.5 \times 10^4\ K)^4 = 2.2 \times 10^{10} \frac{W}{m^2} \tag{3.2.3}$$

The analogous result for the red giant is obtained by scaling the result for a white dwarf:

$$\frac{P_{giant}}{A_{giant}} = \frac{2.2 \times 10^{10}}{4.82 \times 10^3} \frac{W}{m^2} = 4.56 \times 10^6 \frac{W}{m^2} \cong 4.6 \times 10^{-6} \frac{W}{m^2}$$

Significance

To estimate the total power emitted by a white dwarf, in principle, we could use Equation 3.2.3. However, to find its surface area, we need to know the average radius, which is not given in this example. Therefore, the solution stops here. The same is also true for the red giant star.

Exercise $3.2.2A$

An iron poker is being heated. As its temperature rises, the poker begins to glow—first dull red, then bright red, then orange, and then yellow. Use either the blackbody radiation curve or Wien's law to explain these changes in the color of the glow.

Answer

The wavelength of the radiation maximum decreases with increasing temperature.

Exercise $3.2.2B$

Suppose that two stars, α and β, radiate exactly the same total power. If the radius of star α is three times that of star β, what is the ratio of the surface temperatures of these stars? Which one is hotter?

Answer

$T_\alpha / T_\beta = 1/\sqrt{3} \cong 0.58$, so the star β is hotter.

The term "blackbody" was coined by Gustav R. Kirchhoff in 1862. The blackbody radiation curve was known experimentally, but its shape eluded physical explanation until the year 1900. The physical model of a blackbody at temperature **T** is that of the electromagnetic waves enclosed in a cavity (Figure $3.2.1$) and at thermodynamic equilibrium with the cavity walls. The waves can exchange energy with the walls. The objective here is to find the energy density distribution among various modes of vibration at various wavelengths (or frequencies). In other words, we want to know how much energy is carried by a single wavelength or a band of wavelengths. Once we know the energy distribution, we can use standard statistical methods (similar to those studied in a previous chapter) to obtain the blackbody radiation curve, Stefan's law, and Wien's displacement law. When the physical model is correct, the theoretical predictions should be the same as the experimental curves.

In a classical approach to the blackbody radiation problem, in which radiation is treated as waves (as you have studied in previous chapters), the modes of electromagnetic waves trapped in the cavity are in equilibrium and continually exchange their energies with the cavity walls. There is no physical reason why a wave should do otherwise: Any amount of energy can be exchanged, either by being transferred from the wave to the material in the wall or by being received by the wave from the material in the wall. This classical picture is the basis of the model developed by Lord Rayleigh and, independently, by Sir James Jeans. The result of this classical model for blackbody radiation curves is known as the **Rayleigh–Jeans law**. However, as shown in Figure $3.2.5$, the **Rayleigh–Jeans law** fails to correctly reproduce experimental results. In the limit of short wavelengths, the Rayleigh–Jeans law predicts infinite radiation intensity, which is inconsistent with the experimental results in which radiation intensity has finite values in the ultraviolet region of the spectrum. This divergence between the results of classical theory and experiments, which came to be called the **ultraviolet catastrophe**, shows how classical physics fails to explain the mechanism of blackbody radiation.

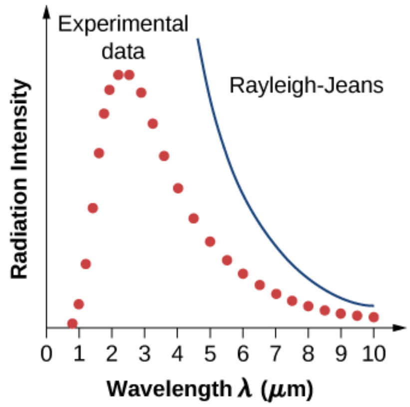

Figure 3.2.5: The ultraviolet catastrophe: The Rayleigh–Jeans law does not explain the observed blackbody emission spectrum.

The blackbody radiation problem was solved in 1900 by Max **Planck**. Planck used the same idea as the Rayleigh–Jeans model in the sense that he treated the electromagnetic waves between the walls inside the cavity classically, and assumed that the radiation is in equilibrium with the cavity walls. The innovative idea that Planck introduced in his model is the assumption that the cavity radiation originates from atomic oscillations inside the cavity walls, and that these oscillations can have only **discrete** values of energy. Therefore, the radiation trapped inside the cavity walls can exchange energy with the walls only in discrete amounts. Planck's hypothesis of discrete energy values, which he called **quanta**, assumes that the oscillators inside the cavity walls have **quantized energies**. This was a brand new idea that went beyond the classical physics of the nineteenth century because, as you learned in a previous chapter, in the classical picture, the energy of an oscillator can take on any continuous value. Planck assumed that the energy of an oscillator (E_n) can have only discrete, or quantized, values:

$$E_n = nhf, \; where \, n = 1, 2, 3, ... \tag{3.2.4}$$

In Equation 3.2.4, f is the frequency of Planck's oscillator. The natural number n that enumerates these discrete energies is called a quantum number. The physical constant h is called **Planck's constant**:

$$h = 6.626 \times 10^{-34} J \cdot s = 4.136 \times 10^{-15} eV \cdot s \tag{3.2.5}$$

Each discrete energy value corresponds to a **quantum state of a Planck oscillator**. Quantum states are enumerated by quantum numbers. For example, when Planck's oscillator is in its first $n1$ quantum state, its energy is $E_1 = hf$; when it is in the $n = 2$ quantum state, its energy is $E_2 = 2hf$; when it is in the $n = 3$ quantum state, $E_3 = 3hf$; and so on.

Note that Equation 3.2.4 shows that there are infinitely many quantum states, which can be represented as a sequence {**hf**, 2**hf**, 3**hf**,…, (**n** – 1)**hf**, **nhf**, (**n** + 1)**hf**,…}. Each two consecutive quantum states in this sequence are separated by an energy jump, $\delta E = hf$. An oscillator in the wall can receive energy from the radiation in the cavity (absorption), or it can give away energy to the radiation in the cavity (emission). The absorption process sends the oscillator to a higher quantum state, and the emission process sends the oscillator to a lower quantum state. Whichever way this exchange of energy goes, the smallest amount of energy that can be exchanged is **hf**. There is no upper limit to how much energy can be exchanged, but whatever is exchanged must be an integer multiple of **hf**. If the energy packet does not have this exact amount, it is neither absorbed nor emitted at the wall of the blackbody.

PLANCK'S QUANTUM HYPOTHESIS

Planck's hypothesis of energy quanta states that the amount of energy emitted by the oscillator is carried by the quantum of radiation, ΔE:

$$\Delta E = hf \tag{3.2.6}$$

Recall that the frequency of electromagnetic radiation is related to its wavelength and to the speed of light by the fundamental relation $f\lambda = c$. This means that we can express Equation 3.2.5 equivalently in terms of wavelength λ. When included in the computation of the energy density of a blackbody, Planck's hypothesis gives the following theoretical expression for the power intensity of emitted radiation per unit wavelength:

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \tag{3.2.7}$$

where **c** is the speed of light in vacuum and kBkB is Boltzmann's constant, $k_B = 1.380 \times 10^{-23} J/K$. The theoretical formula expressed in Equation 3.2.7 is called **Planck's blackbody radiation law**. This law is in agreement with the experimental blackbody radiation curve (Figure $3.2.2$). In addition, Wien's displacement law and Stefan's law can both be derived from Equation 3.2.7. To derive Wien's displacement law, we use differential calculus to find the maximum of the radiation intensity curve $I(\lambda, T)$. To derive Stefan's law and find the value of the Stefan–Boltzmann constant, we use integral calculus and integrate $I(\lambda, T)$ to find the total power radiated by a blackbody at one temperature in the entire spectrum of wavelengths from $\lambda = 0$ to $\lambda = \infty$. This derivation is left as an exercise later in this chapter.

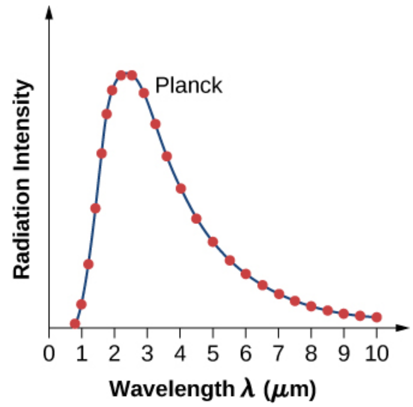

Figure $3.2.6$: Planck's theoretical result (continuous curve) and the experimental blackbody radiation curve (dots).

✔ Example 3.2.3: Planck's Quantum Oscillator

A quantum oscillator in the cavity wall in Figure $3.2.1$ is vibrating at a frequency of $5.0 \times 10^{14} Hz$. Calculate the spacing between its energy levels.

Strategy

Energy states of a quantum oscillator are given by Equation 3.2.4. The energy spacing ΔE is obtained by finding the energy difference between two adjacent quantum states for quantum numbers **n** + 1 and **n**.

Solution

We can substitute the given frequency and Planck's constant directly into the equation:

$$\begin{aligned} \Delta E &= E_{n+1} - En = (n+1)hf - nhf \\ &= hf \\ &= (6.626 \times 10^{-34}\, J \cdot s)(5.0 \times 10^{14}\, Hz) \\ &= 3.3 \times 10^{-19}\, J \end{aligned}$$

Significance

Note that we do not specify what kind of material was used to build the cavity. Here, a quantum oscillator is a theoretical model of an atom or molecule of material in the wall.

? Exercise 3.2.3

A molecule is vibrating at a frequency of $5.0 \times 10^{14}\, Hz$. What is the smallest spacing between its vibrational energy levels?

Answer

$3.3 \times 10^{-19} J$

Example 3.2.4: Quantum Theory Applied to a Classical Oscillator

A 1.0-kg mass oscillates at the end of a spring with a spring constant of 1000 N/m. The amplitude of these oscillations is 0.10 m. Use the concept of quantization to find the energy spacing for this classical oscillator. Is the energy quantization significant for macroscopic systems, such as this oscillator?

Strategy

We use Equation 3.2.6 as though the system were a quantum oscillator, but with the frequency **f** of the mass vibrating on a spring. To evaluate whether or not quantization has a significant effect, we compare the quantum energy spacing with the macroscopic total energy of this classical oscillator.

Solution

For the spring constant, $k = 1.0 \times 10^3 N/m$, the frequency **f** of the mass, $m = 1.0\, kg$, is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{1.0 \times 10^3 N/m}{1.0\, kg}} \simeq 5.0\, Hz$$

The energy quantum that corresponds to this frequency is

$$\Delta E = hf = (6.626 \times 10^{-34} J \cdot s)(5.0\, Hz) = 3.3 \times 10^{-33} J$$

When vibrations have amplitude $A = 0.10\, m$, the energy of oscillations is

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(1000\, N/m)(0.1\, m)^2 = 5.0\, J$$

Significance

Thus, for a classical oscillator, we have $\Delta E/E \approx 10^{-34}$. We see that the separation of the energy levels is immeasurably small. Therefore, for all practical purposes, the energy of a classical oscillator takes on continuous values. This is why classical principles may be applied to macroscopic systems encountered in everyday life without loss of accuracy.

Exercise 3.2.4

Would the result in Example 3.2.4 be different if the mass were not 1.0 kg but a tiny mass of 1.0 µg, and the amplitude of vibrations were 0.10 µm?

Answer

No, because then $\Delta E/E \approx 10^{-21}$

When Planck first published his result, the hypothesis of energy quanta was not taken seriously by the physics community because it did not follow from any established physics theory at that time. It was perceived, even by Planck himself, as a useful mathematical trick that led to a good theoretical "fit" to the experimental curve. This perception was changed in 1905 when Einstein published his explanation of the photoelectric effect, in which he gave Planck's energy quantum a new meaning: that of a particle of light.

3.3: Photoelectric Effect

Learning Objectives

By the end of this section you will be able to:

- Describe physical characteristics of the photoelectric effect
- Explain why the photoelectric effect cannot be explained by classical physics
- Describe how Einstein's idea of a particle of radiation explains the photoelectric effect

When a metal surface is exposed to a monochromatic electromagnetic wave of sufficiently short wavelength (or equivalently, above a threshold frequency), the incident radiation is absorbed and the exposed surface emits electrons. This phenomenon is known as the **photoelectric effect**. Electrons that are emitted in this process are called **photoelectrons**.

The experimental setup to study the photoelectric effect is shown schematically in Figure $3.3.1$. The target material serves as the anode, which becomes the emitter of photoelectrons when it is illuminated by monochromatic radiation. We call this electrode the **photoelectrode**. Photoelectrons are collected at the cathode, which is kept at a lower potential with respect to the anode. The potential difference between the electrodes can be increased or decreased, or its polarity can be reversed. The electrodes are enclosed in an evacuated glass tube so that photoelectrons do not lose their kinetic energy on collisions with air molecules in the space between electrodes.

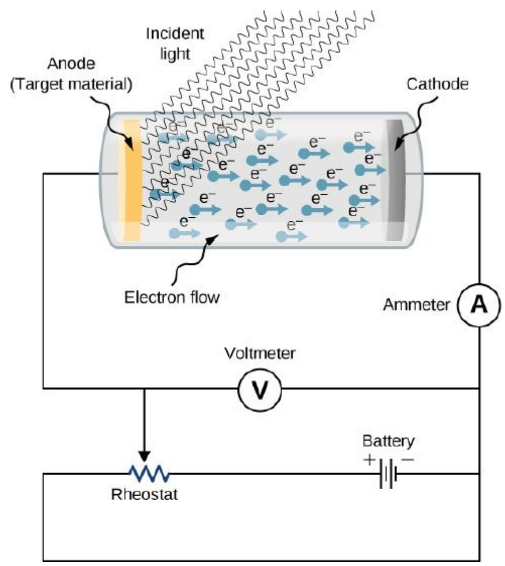

Figure $3.3.1$: An experimental setup to study the photoelectric effect. The anode and cathode are enclosed in an evacuated glass tube. The voltmeter measures the electric potential difference between the electrodes, and the ammeter measures the photocurrent. The incident radiation is monochromatic.

When the target material is not exposed to radiation, no current is registered in this circuit because the circuit is broken (note, there is a gap between the electrodes). But when the target material is connected to the negative terminal of a battery and exposed to radiation, a current is registered in this circuit; this current is called the **photocurrent**. Suppose that we now reverse the potential difference between the electrodes so that the target material now connects with the positive terminal of a battery, and then we slowly increase the voltage. The photocurrent gradually dies out and eventually stops flowing completely at some value of this reversed voltage. The potential difference at which the photocurrent stops flowing is called the **stopping potential**.

Characteristics of the Photoelectric Effect

The photoelectric effect has three important characteristics that cannot be explained by classical physics: (1) the absence of a lag time, (2) the independence of the kinetic energy of photoelectrons on the intensity of incident radiation, and (3) the presence of a cut-off frequency. Let's examine each of these characteristics.

The absence of lag time

When radiation strikes the target material in the electrode, electrons are emitted almost instantaneously, even at very low intensities of incident radiation. This absence of lag time contradicts our understanding based on classical physics. Classical physics predicts that for low-energy radiation, it would take significant time before irradiated electrons could gain sufficient energy to leave the electrode surface; however, such an energy buildup is not observed.

The intensity of incident radiation and the kinetic energy of photoelectrons

Typical experimental curves are shown in Figure $3.3.2$, in which the photocurrent is plotted versus the applied potential difference between the electrodes. For the positive potential difference, the current steadily grows until it reaches a plateau. Furthering the potential increase beyond this point does not increase the photocurrent at all. A higher intensity of radiation produces a higher value of photocurrent. For the negative potential difference, as the absolute value of the potential difference increases, the value of the photocurrent decreases and becomes zero at the stopping potential. For any intensity of incident radiation, whether the intensity is high or low, the value of the stopping potential always stays at one value.

To understand why this result is unusual from the point of view of classical physics, we first have to analyze the energy of photoelectrons. A photoelectron that leaves the surface has kinetic energy K. It gained this energy from the incident electromagnetic wave. In the space between the electrodes, a photoelectron moves in the electric potential and its energy changes by the amount $q\Delta V$, where ΔV is the potential difference and $q=-e$. Because no forces are present but electric force, by applying the work-energy theorem, we obtain the energy balance $\Delta K - e\Delta V = 0$ for the photoelectron, where ΔK is the change in the photoelectron's kinetic energy. When the stopping potential $-\Delta V_s$ is applied, the photoelectron loses its initial kinetic energy K_i and comes to rest. Thus, its energy balance becomes $(0-K_i) - e(-\Delta V_s) = 0$, so that $K_i = e\Delta V_s$. In the presence of the stopping potential, the largest kinetic energy K_{max} that a photoelectron can have is its initial kinetic energy, which it has at the surface of the photoelectrode. Therefore, the largest kinetic energy of photoelectrons can be directly measured by measuring the stopping potential:

$$K_{max} = e\Delta V_s. \tag{3.3.1}$$

At this point we can see where the classical theory is at odds with the experimental results. In classical theory, the photoelectron absorbs electromagnetic energy in a continuous way; this means that when the incident radiation has a high intensity, the kinetic energy in Equation 3.3.1 is expected to be high. Similarly, when the radiation has a low intensity, the kinetic energy is expected to be low. But the experiment shows that the maximum kinetic energy of photoelectrons is independent of the light intensity.

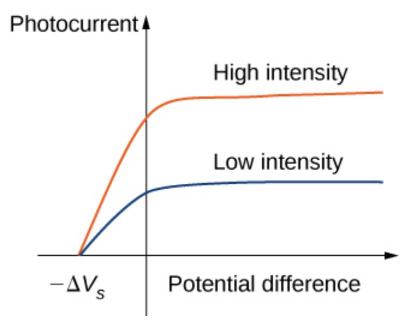

Figure $3.3.2$: The detected photocurrent plotted versus the applied potential difference shows that for any intensity of incident radiation, whether the intensity is high (upper curve) or low (lower curve), the value of the stopping potential is always the same.

The presence of a cut-off frequency

For any metal surface, there is a minimum frequency of incident radiation below which photocurrent does not occur. The value of this **cut-off frequency** for the photoelectric effect is a physical property of the metal: Different materials have different values of cut-off frequency. Experimental data show a typical linear trend (Figure $3.3.3$). The kinetic energy of photoelectrons at the surface grows linearly with the increasing frequency of incident radiation. Measurements for all metal surfaces give linear plots with one slope. None of these observed phenomena is in accord with the classical understanding of nature. According to the classical description, the kinetic energy of photoelectrons should not depend on the frequency of incident radiation at all, and there should be no cut-off frequency. Instead, in the classical picture, electrons receive energy from the incident electromagnetic wave in a continuous way, and the amount of energy they receive depends only on the intensity of the incident light and nothing else. So in the classical understanding, as long as the light is shining, the photoelectric effect is expected to continue.

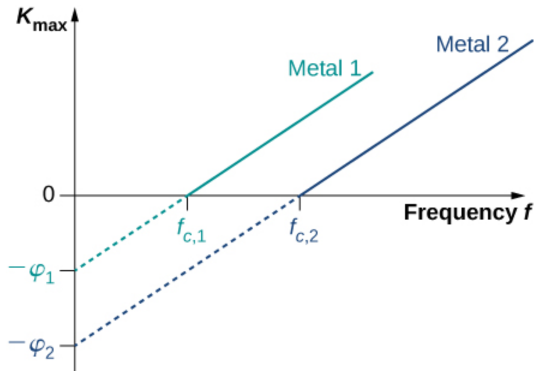

Figure $3.3.3$: Kinetic energy of photoelectrons at the surface versus the frequency of incident radiation. The photoelectric effect can only occur above the cut-off frequency f_c. Measurements for all metal surfaces give linear plots with one slope. Each metal surface has its own cut-off frequency.

The Work Function

The photoelectric effect was explained in 1905 by A. **Einstein**. Einstein reasoned that if Planck's hypothesis about energy quanta was correct for describing the energy exchange between electromagnetic radiation and cavity walls, it should also work to describe energy absorption from electromagnetic radiation by the surface of a photoelectrode. He postulated that an electromagnetic wave carries its energy in discrete packets. Einstein's postulate goes beyond Planck's hypothesis because it states that the light itself consists of energy quanta. In other words, it states that electromagnetic waves are quantized.

In Einstein's approach, a beam of monochromatic light of frequency f is made of photons. A **photon** is a particle of light. Each photon moves at the speed of light and carries an energy quantum E_f. A photon's energy depends only on its frequency f. Explicitly, the energy of a photon is

$$E_f = hf \tag{3.3.2}$$

where h is Planck's constant. In the photoelectric effect, photons arrive at the metal surface and each photon gives away **all** of its energy to only **one** electron on the metal surface. This transfer of energy from photon to electron is of the "all or nothing" type, and there are no fractional transfers in which a photon would lose only part of its energy and survive. The essence of a **quantum phenomenon** is either a photon transfers its entire energy and ceases to exist or there is no transfer at all. This is in contrast with the classical picture, where fractional energy transfers are permitted. Having this quantum understanding, the energy balance for an electron on the surface that receives the energy E_f from a photon is

$$E_f = K_{max} + \phi$$

where K_max is the kinetic energy, given by Equation 3.3.1, that an electron has at the very instant it gets detached from the surface. In this energy balance equation, ϕ is the energy needed to detach a photoelectron from the surface. This energy ϕ is called the work function of the metal. Each metal has its characteristic work function, as illustrated in Table $3.3.1$. To obtain the kinetic energy of photoelectrons at the surface, we simply invert the energy balance equation and use Equation 3.3.2 to express the energy of the absorbed photon. This gives us the expression for the kinetic energy of photoelectrons, which explicitly depends on the frequency of incident radiation:

$$K_{max} = hf - \phi \tag{3.3.3}$$

Equation 3.3.3 has a simple mathematical form but its physics is profound. We can now elaborate on the physical meaning behind this equation.

Table $3.3.1$: Typical Values of the Work Function for Some Common Metals

Metal	**ϕ (eV)**
Na	2.46
Al	4.08
Pb	4.14

Metal	ϕ (eV)
Zn	4.31
Fe	4.50
Cu	4.70
Ag	4.73
Pt	6.35

In Einstein's interpretation, interactions take place between individual electrons and individual photons. The absence of a lag time means that these one-on-one interactions occur instantaneously. This interaction time cannot be increased by lowering the light intensity. The light intensity corresponds to the number of photons arriving at the metal surface per unit time. Even at very low light intensities, the photoelectric effect still occurs because the interaction is between one electron and one photon. As long as there is at least one photon with enough energy to transfer it to a bound electron, a photoelectron will appear on the surface of the photoelectrode.

The existence of the cut-off frequency f_c for the photoelectric effect follows from Equation 3.3.3 because the kinetic energy K_{max} of the photoelectron can take only positive values. This means that there must be some threshold frequency for which the kinetic energy is zero, $0 = hf_c - \phi$. In this way, we obtain the explicit formula for cut-off frequency:

$$f_c = \frac{\phi}{h}. \tag{3.3.4}$$

Cut-off frequency depends only on the work function of the metal and is in direct proportion to it. When the work function is large (when electrons are bound fast to the metal surface), the energy of the threshold photon must be large to produce a photoelectron, and then the corresponding threshold frequency is large. Photons with frequencies larger than the threshold frequency f_c always produce photoelectrons because they have $K_{max} > 0$. Photons with frequencies smaller than f_c do not have enough energy to produce photoelectrons. Therefore, when incident radiation has a frequency below the cut-off frequency, the photoelectric effect is not observed. Because frequency f and wavelength λ of electromagnetic waves are related by the fundamental relation $\lambda f = c$ (where cc is the speed of light in vacuum), the cut-off frequency has its corresponding **cut-off wavelength** λ_c:

$$\lambda_c = \frac{c}{f_c} = \frac{c}{\phi/h} = \frac{hc}{\phi}. \tag{3.3.5}$$

In this equation, $hc = 1240\,eV \cdot nm$. Our observations can be restated in the following equivalent way: When the incident radiation has wavelengths longer than the cut-off wavelength, the photoelectric effect does not occur.

3.3.1: Photoelectric Effect for Silver

Radiation with wavelength 300 nm is incident on a silver surface. Will photoelectrons be observed?

Strategy

Photoelectrons can be ejected from the metal surface only when the incident radiation has a shorter wavelength than the cut-off wavelength. The work function of silver is $\phi = 4.73\,eV$ (Table 3.3.1). To make the estimate, we use Equation 3.3.5.

Solution

The threshold wavelength for observing the photoelectric effect in silver is

$$\begin{aligned}\lambda_c &= \frac{hc}{\phi} \\ &= \frac{1240\,eV \cdot nm}{4.73\,eV} = 262\,nm.\end{aligned}$$

The incident radiation has wavelength 300 nm, which is longer than the cut-off wavelength; therefore, photoelectrons are not observed.

Significance

If the photoelectrode were made of sodium instead of silver, the cut-off wavelength would be 504 nm and photoelectrons would be observed.

Equation 3.3.3 in Einstein's model tells us that the maximum kinetic energy of photoelectrons is a linear function of the frequency of incident radiation, which is illustrated in Figure $3.3.3$. For any metal, the slope of this plot has a value of Planck's constant. The intercept with the K_{max}-axis gives us a value of the work function that is characteristic for the metal. On the other hand, K_{max} can be directly measured in the experiment by measuring the value of the stopping potential δV_s (see Equation 3.3.1) at which the photocurrent stops. These direct measurements allow us to determine experimentally the value of Planck's constant, as well as work functions of materials.

Einstein's model also gives a straightforward explanation for the photocurrent values shown in Figure $3.3.3$. For example, doubling the intensity of radiation translates to doubling the number of photons that strike the surface per unit time. The larger the number of photons, the larger is the number of photoelectrons, which leads to a larger photocurrent in the circuit. This is how radiation intensity affects the photocurrent. The photocurrent must reach a plateau at some value of potential difference because, in unit time, the number of photoelectrons is equal to the number of incident photons and the number of incident photons does not depend on the applied potential difference at all, but only on the intensity of incident radiation. The stopping potential does not change with the radiation intensity because the kinetic energy of photoelectrons (see Equation 3.3.3) does not depend on the radiation intensity.

✓ Example 3.3.2: Work Function and Cut-Off Frequency

When a 180-nm light is used in an experiment with an unknown metal, the measured photocurrent drops to zero at potential – 0.80 V. Determine the work function of the metal and its cut-off frequency for the photoelectric effect.

Strategy

To find the cut-off frequency f_c, we use Equation 3.3.4, but first we must find the work function ϕ. To find ϕ, we use Equation 3.3.1 and Equation 3.3.3. Photocurrent drops to zero at the stopping value of potential, so we identify $\Delta V_s = 0.8V$.

Solution

We use Equation 3.3.1 to find the kinetic energy of the photoelectrons:

$$K_{max} = e\Delta V_s = e(0.80V) = 0.80\,eV.$$

Now we solve Equation for ϕ:

$$\phi = hf - K_{max} = \frac{hc}{\lambda} - K_{max} = \frac{1240\,eV\cdot m}{180\,nm} - 0.80\,eV = 6.09eV.$$

Finally, we use Equation to find the cut-off frequency:

$$f_c = \frac{\phi}{h}\frac{6.09\,eV}{4.136\times 10^{-15} eV\cdot s} = 1.47\times 10^{-15} Hz.$$

Significance

In calculations like the one shown in this example, it is convenient to use Planck's constant in the units of $eV\cdot s$ and express all energies in eV instead of joules.

✓ Example 3.3.3: The Photon Energy and Kinetic Energy of Photoelectrons

A 430-nm violet light is incident on a calcium photoelectrode with a work function of 2.71 eV. Find the energy of the incident photons and the maximum kinetic energy of ejected electrons.

Strategy

The energy of the incident photon is $E_f = hf = hc/\lambda$,, where we use $f\lambda = c$. To obtain the maximum energy of the ejected electrons, we use Equation 3.3.5.

Solution

$$E_f = \frac{hc}{\lambda} = \frac{1240\,eV\cdot nm}{430\,nm} = 2.88\,eV,\; K_{max} = E_f - \phi = 2.88\,eV - 2.71\,eV = 0.17\,eV$$

Significance

In this experimental setup, photoelectrons stop flowing at the stopping potential of 0.17 V.

Exercise 3.3.1

A yellow 589-nm light is incident on a surface whose work function is 1.20 eV. What is the stopping potential? What is the cut-off wavelength?

Answer

$-0.91\,V$ 1040 nm

Exercise 3.3.2

Cut-off frequency for the photoelectric effect in some materials is $8.0\times10^{13}Hz.$ When the incident light has a frequency of $1.2\times10^{14}Hz$, the stopping potential is measured as – 0.16 V. Estimate a value of Planck's constant from these data (in units J·sJ·s and eV·seV·s) and determine the percentage error of your estimation.

Answer

$h=6.40\times10^{-34}J\cdot s=4.0\times10^{-15}eV\cdot s\ \ ;-3.5\%$

3.4: The Compton Effect

Learning Objectives

By the end of this section, you will be able to:

- Describe Compton's experiment
- Explain the Compton wavelength shift
- Describe how experiments with X-rays confirm the particle nature of radiation

Two of Einstein's influential ideas introduced in 1905 were the theory of special relativity and the concept of a light quantum, which we now call a photon. Beyond 1905, Einstein went further to suggest that freely propagating electromagnetic waves consisted of photons that are particles of light in the same sense that electrons or other massive particles are particles of matter. A beam of monochromatic light of wavelength λ (or equivalently, of frequency f) can be seen either as a classical wave or as a collection of photons that travel in a vacuum with one speed, c (the speed of light), and all carrying the same energy, $E_f = hf$. This idea proved useful for explaining the interactions of light with particles of matter.

Momentum of a Photon

Unlike a particle of matter that is characterized by its rest mass m_0, a photon is massless. In a vacuum, unlike a particle of matter that may vary its speed but cannot reach the speed of light, a photon travels at only one speed, which is exactly the speed of light. From the point of view of Newtonian classical mechanics, these two characteristics imply that a photon should not exist at all. For example, how can we find the linear momentum or kinetic energy of a body whose mass is zero? This apparent paradox vanishes if we describe a photon as a relativistic particle. According to the theory of special relativity, any particle in nature obeys the relativistic energy equation

$$E^2 = p^2c^2 + m_0^2c^4. \tag{3.4.1}$$

This relation can also be applied to a photon. In Equation 3.4.1, E is the total energy of a particle, p is its linear momentum, and m_0 is its rest mass. For a photon, we simply set $m_0 = 0$ in Equation 3.4.1, which leads to the expression for the momentum p_f of a photon

$$p_f = \frac{E_f}{c}. \tag{3.4.2}$$

Here the photon's energy E_f is the same as that of a light quantum of frequency f, which we introduced to explain the photoelectric effect:

$$E_f = hf = \frac{hc}{\lambda}. \tag{3.4.3}$$

The wave relation that connects frequency f with wavelength λ and speed c also holds for photons:

$$\lambda f = c \tag{3.4.4}$$

Therefore, a photon can be equivalently characterized by either its energy and wavelength, or its frequency and momentum. Equations 3.4.3 and 3.4.4 can be combined into the explicit relation between a photon's momentum and its wavelength:

$$p_f = \frac{h}{\lambda}. \tag{3.4.5}$$

Notice that this equation gives us only the magnitude of the photon's momentum and contains no information about the direction in which the photon is moving. To include the direction, it is customary to write the photon's momentum as a vector:

$$\vec{p}_f = \hbar\vec{l}. \tag{3.4.6}$$

In Equation 3.4.6, $\hbar = h/2\pi$ is the **reduced Planck's constant** (pronounced "h-bar"), which is just Planck's constant divided by the factor 2π. Vector $\vec{l}$ is called the "wave vector" or propagation vector (the direction in which a photon is moving). The propagation vector shows the direction of the photon's linear momentum vector. The magnitude of the wave vector is

$$k=|\vec{k}|=2\pi/\lambda$$

and is called the **wave number**. Notice that this equation does not introduce any new physics. We can verify that the magnitude of the vector in Equation 3.4.6 is the same as that given by Equation 3.4.2.

The Compton Effect

The **Compton effect** is the term used for an unusual result observed when X-rays are scattered on some materials. By classical theory, when an electromagnetic wave is scattered off atoms, the wavelength of the scattered radiation is expected to be the same as the wavelength of the incident radiation. Contrary to this prediction of classical physics, observations show that when X-rays are scattered off some materials, such as graphite, the scattered X-rays have different wavelengths from the wavelength of the incident X-rays. This classically unexplainable phenomenon was studied experimentally by Arthur H. Compton and his collaborators, and Compton gave its explanation in 1923.

To explain the shift in wavelengths measured in the experiment, Compton used Einstein's idea of light as a particle. The Compton effect has a very important place in the history of physics because it shows that electromagnetic radiation cannot be explained as a purely wave phenomenon. The explanation of the Compton effect gave a convincing argument to the physics community that electromagnetic waves can indeed behave like a stream of photons, which placed the concept of a photon on firm ground.

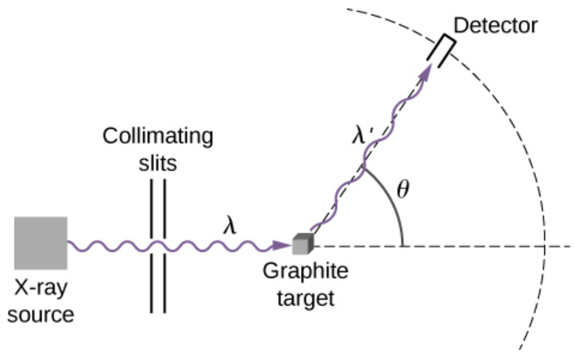

Figure 3.4.1: Experimental setup for studying Compton scattering.

The schematics of Compton's experimental setup are shown in Figure 3.4.1. The idea of the experiment is straightforward: Monochromatic X-rays with wavelength λ are incident on a sample of graphite (the "target"), where they interact with atoms inside the sample; they later emerge as scattered X-rays with wavelength λ'. A detector placed behind the target can measure the intensity of radiation scattered in any direction θ with respect to the direction of the incident X-ray beam. This **scattering angle**, θ, is the angle between the direction of the scattered beam and the direction of the incident beam. In this experiment, we know the intensity and the wavelength λ of the incoming (incident) beam; and for a given scattering angle θ, we measure the intensity and the wavelength λ' of the outgoing (scattered) beam. Typical results of these measurements are shown in Figure 3.4.2, where the x-axis is the wavelength of the scattered X-rays and the y-axis is the intensity of the scattered X-rays, measured for different scattering angles (indicated on the graphs). For all scattering angles (except for $\theta=0^\circ$), we measure two intensity peaks. One peak is located at the wavelength λ, which is the wavelength of the incident beam. The other peak is located at some other wavelength, λ'. The two peaks are separated by $\Delta\lambda$, which depends on the scattering angle θ of the outgoing beam (in the direction of observation). The separation $\Delta\lambda$ is called the **Compton shift**.

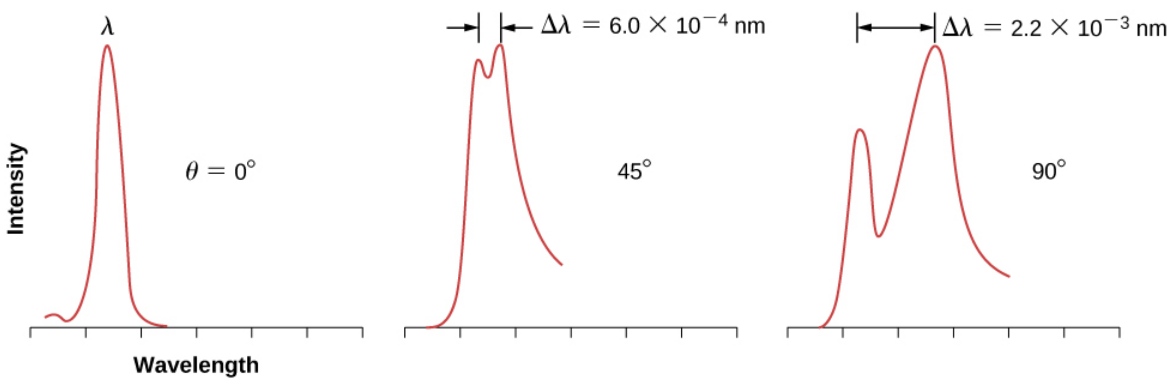

Figure 3.4.2. The experimental data in this figure are plotted in arbitrary units so that the height of the profile reflects the intensity of the scattered beam above background noise.

Compton Shift

As given by Compton, the explanation of the Compton shift is that in the target material, graphite, valence electrons are loosely bound in the atoms and behave like free electrons. Compton assumed that the incident X-ray radiation is a stream of photons. An incoming photon in this stream collides with a valence electron in the graphite target. In the course of this collision, the incoming photon transfers some part of its energy and momentum to the target electron and leaves the scene as a scattered photon. This model explains in qualitative terms why the scattered radiation has a longer wavelength than the incident radiation. Put simply, a photon that has lost some of its energy emerges as a photon with a lower frequency, or equivalently, with a longer wavelength. To show that his model was correct, Compton used it to derive the expression for the Compton shift. In his derivation, he assumed that both photon and electron are relativistic particles and that the collision obeys two commonsense principles:

1. the conservation of linear momentum and
2. the conservation of total relativistic energy.

In the following derivation of the Compton shift, E_f and $\vec{p}_f$ denote the energy and momentum, respectively, of an incident photon with frequency f. The photon collides with a relativistic electron at rest, which means that immediately before the collision, the electron's energy is entirely its rest mass energy, m_0c^2. Immediately after the collision, the electron has energy E and momentum $\vec{p}$, both of which satisfy Equation 3.4.3. Immediately after the collision, the outgoing photon has energy $\vec{\tilde{E}}_f$, momentum $\vec{\tilde{p}}_f$, and frequency f'. The direction of the incident photon is horizontal from left to right, and the direction of the outgoing photon is at the angle θ, as illustrated in Figure 3.4.1. The scattering angle θ is the angle between the momentum vectors $\vec{p}_f$ and $\vec{\tilde{p}}_f$, and we can write their scalar product:

$$\vec{p} \cdot \vec{\tilde{p}}_f = p_f \vec{p}_f \cos\theta. \tag{3.4.7}$$

Following Compton's argument, we assume that the colliding photon and electron form an isolated system. This assumption is valid for weakly bound electrons that, to a good approximation, can be treated as free particles. Our first equation is the conservation of energy for the photon-electron system:

$$E_f + m_0c^2 = \tilde{E}_f + E. \tag{3.4.8}$$

The left side of this equation is the energy of the system at the instant immediately before the collision, and the right side of the equation is the energy of the system at the instant immediately after the collision. Our second equation is the conservation of linear momentum for the photon–electron system where the electron is at rest at the instant immediately before the collision:

$$\vec{p}_f = \vec{\tilde{p}}_f + \vec{p}. \tag{3.4.9}$$

The left side of this equation is the momentum of the system right before the collision, and the right side of the equation is the momentum of the system right after collision. The entire physics of Compton scattering is contained in these three preceding equations—the remaining part is algebra. At this point, we could jump to the concluding formula for the Compton shift, but it is beneficial to highlight the main algebraic steps that lead to Compton's formula, which we give here as follows.

We start with rearranging the terms in Equation 3.4.8 and squaring it:

$$[(E_f - \tilde{E}_f) + m_0c^2]^2 = E^2.$$

In the next step, we substitute Equation 3.4.3 for E^2, simplify, and divide both sides by c^2 to obtain

$$(E_f/c - \tilde{E}_f/c)^2 + 2m_0c(E_f/c - \tilde{E}_f/c) = p^2.$$

Now we can use Equation 3.4.5 to express this form of the energy equation in terms of momenta. The result is

$$(p_f - \tilde{p}_f)^2 + 2m_0c(p_f - \tilde{p}_f) = p^2. \tag{3.4.10}$$

To eliminate p^2, we turn to the momentum equation Equation 3.4.9, rearrange its terms, and square it to obtain

$$\begin{aligned}(\vec{p}_f - \vec{\tilde{p}}_f)^2 &= p^2 \\ &= p_f^2 + \tilde{p}_f^2 - 2p_f\tilde{p}_f \cos\theta.\end{aligned}$$

The product of the momentum vectors is given by Equation 3.4.7. When we substitute this result for p^2 in Equation 3.4.10, we obtain the energy equation that contains the scattering angle θ:

$$(p_f - \tilde{p}_f)^2 + 2m_0c(p_f - \tilde{p}_f) = p_f^2 + \tilde{p}_f^2 - 2p_f\tilde{p}_f \cos\theta.$$

With further algebra, this result can be simplified to

$$\frac{1}{\tilde{p}_f} - \frac{1}{p_f} = \frac{1}{m_0c}(1 - \cos\theta). \tag{3.4.11}$$

Now recall Equation 3.4.5 and write: $1/\tilde{p}_f = \lambda'/h$ and $1/p_f = \lambda/h$. When these relations are substituted into Equation 3.4.11, we obtain the relation for the Compton shift:

$$\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\theta). \tag{3.4.12}$$

The factor h/m_0c is called the **Compton wavelength** of the electron:

$$\lambda_c = \frac{h}{m_0c} = 0.00243\,nm = 2.43\,pm. \tag{3.4.13}$$

Denoting the shift as $\Delta\lambda = \lambda' - \lambda$, the concluding result can be rewritten as

$$\Delta\lambda = \lambda_c(1 - \cos\theta). \tag{3.4.14}$$

This formula for the Compton shift describes outstandingly well the experimental results shown in Figure $3.4.2$. Scattering data measured for molybdenum, graphite, calcite, and many other target materials are in accord with this theoretical result. The nonshifted peak shown in Figure $3.4.1$ is due to photon collisions with tightly bound inner electrons in the target material. Photons that collide with the inner electrons of the target atoms in fact collide with the entire atom. In this extreme case, the rest mass in Equation 3.4.13 must be changed to the rest mass of the atom. This type of shift is four orders of magnitude smaller than the shift caused by collisions with electrons and is so small that it can be neglected.

Compton scattering is an example of **inelastic scattering**, in which the scattered radiation has a longer wavelength than the wavelength of the incident radiation. In today's usage, the term "Compton scattering" is used for the inelastic scattering of photons by free, charged particles. In Compton scattering, treating photons as particles with momenta that can be transferred to charged particles provides the theoretical background to explain the wavelength shifts measured in experiments; this is the evidence that radiation consists of photons.

Example 3.4.1: Compton Scattering

An incident 71-pm X-ray is incident on a calcite target. Find the wavelength of the X-ray scattered at a 30°30° angle. What is the largest shift that can be expected in this experiment?

Strategy

To find the wavelength of the scattered X-ray, first we must find the Compton shift for the given scattering angle, $\theta = 30^\circ$. We use Equation 3.4.14. Then we add this shift to the incident wavelength to obtain the scattered wavelength. The largest

Compton shift occurs at the angle θ when $1-\cos\theta$ has the largest value, which is for the angle $\theta = 180^\circ$.

Solution

The shift at $\theta = 30^\circ$ is

$$\begin{aligned}\Delta\lambda &= \lambda_c(1-\cos 30^\circ)\\ &= 0.134\lambda_c\\ &= (0.134)(2.43)\,pm\\ &= 0.32\end{aligned}$$

This gives the scattered wavelength:

$$\begin{aligned}\lambda' &= \lambda + \Delta\lambda\\ &= (71+0.325)\,pm\\ &= 71.325\,pm.\end{aligned}$$

The largest shift is

$$\begin{aligned}(\Delta\lambda)_{max} &= \lambda_c(1-\cos 180^\circ)\\ &= 2(2.43\,pm)\\ &= 4.86\,pm.\end{aligned}$$

Significance

The largest shift in wavelength is detected for the backscattered radiation; however, most of the photons from the incident beam pass through the target and only a small fraction of photons gets backscattered (typically, less than 5%). Therefore, these measurements require highly sensitive detectors.

Exercise 3.4.1

An incident 71-pm X-ray is incident on a calcite target. Find the wavelength of the X-ray scattered at a 60° angle. What is the smallest shift that can be expected in this experiment?

Answer

$(\Delta\lambda)_{min} = 0\,m$ at a 0° angle; $71.0\,pm + 0.5\lambda_c = 72.215\,pm$

3.5: Bohr's Model of the Hydrogen Atom

Learning Objectives

By the end of this section, you will be able to:

- Explain the difference between the absorption spectrum and the emission spectrum of radiation emitted by atoms
- Describe the Rutherford gold foil experiment and the discovery of the atomic nucleus
- Explain the atomic structure of hydrogen
- Describe the postulates of the early quantum theory for the hydrogen atom
- Summarize how Bohr's quantum model of the hydrogen atom explains the radiation spectrum of atomic hydrogen

Historically, Bohr's model of the hydrogen atom is the very first model of atomic structure that correctly explained the radiation spectra of atomic hydrogen. The model has a special place in the history of physics because it introduced an early quantum theory, which brought about new developments in scientific thought and later culminated in the development of quantum mechanics. To understand the specifics of Bohr's model, we must first review the nineteenth-century discoveries that prompted its formulation.

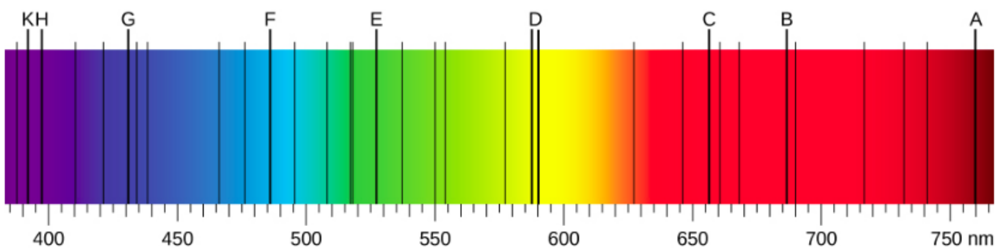

Figure 3.5.1: In the solar emission spectrum in the visible range from 380 nm to 710 nm, Fraunhofer lines are observed as vertical black lines at specific spectral positions in the continuous spectrum. Highly sensitive modern instruments observe thousands of such lines.

When we use a prism to analyze white light coming from the sun, several dark lines in the solar spectrum are observed (Figure 3.5.1). Solar absorption lines are called **Fraunhofer lines** after Joseph von **Fraunhofer**, who accurately measured their wavelengths. During 1854–1861, Gustav Kirchhoff and Robert Bunsen discovered that for the various chemical elements, the line **emission spectrum** of an element exactly matches its line **absorption spectrum**. The difference between the absorption spectrum and the emission spectrum is explained in Figure 3.5.2.

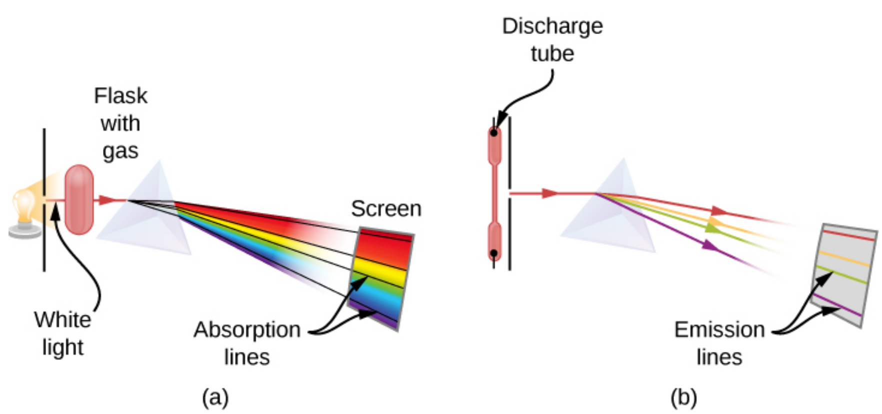

Figure 3.5.2: Observation of line spectra: (a) setup to observe absorption lines; (b) setup to observe emission lines. (a) White light passes through a cold gas that is contained in a glass flask. A prism is used to separate wavelengths of the passed light. In the spectrum of the passed light, some wavelengths are missing, which are seen as black absorption lines in the continuous spectrum on the viewing screen. (b) A gas is contained in a glass discharge tube that has electrodes at its ends. At a high potential difference between the electrodes, the gas glows and the light emitted from the gas passes through the prism that separates its wavelengths. In the spectrum of the emitted light, only specific wavelengths are present, which are seen as colorful emission lines on the screen.

An absorption spectrum is observed when light passes through a gas. This spectrum appears as black lines that occur only at certain wavelengths on the background of the continuous spectrum of white light (Figure 3.5.2). The missing wavelengths tell us which wavelengths of the radiation are absorbed by the gas. The emission spectrum is observed when light is emitted by a gas. This spectrum is seen as colorful lines on the black background (Figures $3.5.3$ and $3.5.4$). Positions of the emission lines tell us which wavelengths of the radiation are emitted by the gas. Each chemical element has its own characteristic emission spectrum. For each element, the positions of its emission lines are exactly the same as the positions of its absorption lines. This means that atoms of a specific element absorb radiation only at specific wavelengths and radiation that does not have these wavelengths is not absorbed by the element at all. This also means that the radiation emitted by atoms of each element has exactly the same wavelengths as the radiation they absorb.

Figure $3.5.3$: The emission spectrum of atomic hydrogen: The spectral positions of emission lines are characteristic for hydrogen atoms. (credit: "Merikanto"/Wikimedia Commons)

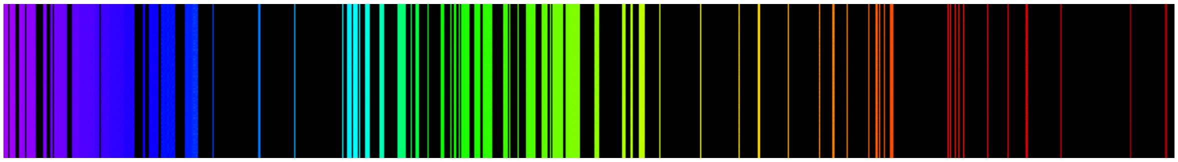

Figure $3.5.4$: The emission spectrum of atomic iron: The spectral positions of emission lines are characteristic for iron atoms.

Emission spectra of the elements have complex structures; they become even more complex for elements with higher atomic numbers. The simplest spectrum, shown in Figure $3.5.4$, belongs to the hydrogen atom. Only four lines are visible to the human eye. As you read from right to left in Figure $3.5.4$, these lines are: red (656 nm), called the $H-\alpha$ line; aqua (486 nm), blue (434 nm), and violet (410 nm). The lines with wavelengths shorter than 400 nm appear in the ultraviolet part of the spectrum (Figure $3.5.4$, far left) and are invisible to the human eye. There are infinitely many invisible spectral lines in the series for hydrogen.

An empirical formula to describe the positions (wavelengths) λ of the hydrogen emission lines in this series was discovered in 1885 by Johann **Balmer**. It is known as the **Balmer** formula:

$$\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right). \tag{3.5.1}$$

The constant $R_H = 1.09737 \times 10^7 m^{-1}$ is called the **Rydberg constant for hydrogen**. In Equation 3.5.1, the positive integer **n** takes on values $n = 3, 4, 5, 6$ for the four visible lines in this series. The series of emission lines given by the Balmer formula is called the Balmer series for hydrogen. Other emission lines of hydrogen that were discovered in the twentieth century are described by the **Rydberg formula**, which summarizes all of the experimental data:

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{3.5.2}$$

where $n_i = n_f > n_i$ (in integer steps).

When $n_f = 1$, the series of spectral lines is called the **Lyman series**. When $n_f = 2$, the series is called the Balmer series, and in this case, the Rydberg formula coincides with the Balmer formula (Equation 3.5.1). When $n_f = 3$, the series is called the **Paschen series**. When $n_f = 4$, the series is called the **Brackett series**. When $n_f = 5$, the series is called the **Pfund series**. When $n_f = 6$, we have the **Humphreys series**. As you may guess, there are infinitely many such spectral bands in the spectrum of hydrogen because n_f can be any positive integer number.

The Rydberg formula for hydrogen gives the exact positions of the spectral lines as they are observed in a laboratory; however, at the beginning of the twentieth century, nobody could explain why it worked so well. The Rydberg formula remained unexplained until the first successful model of the hydrogen atom was proposed in 1913.

Example 3.5.1: Limits of the Balmer Series

Calculate the longest and the shortest wavelengths in the Balmer series.

Strategy

We can use either the Balmer formula (Equation 3.5.1) or the Rydberg formula (Equation 3.5.2). The longest wavelength is obtained when $1/n_i$ is largest, which is when $n_i = n_f + 1 = 3$, because $n_f = 2$ for the Balmer series. The smallest wavelength is obtained when $1/n_i$ is smallest, which is $1/n_i \to 0$ when $n_i \to \infty$.

Solution

The long-wave limit:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{2^2} - \frac{1}{3^2}\right) = (1.09737 \times 10^7)\frac{1}{m}\left(\frac{1}{4} - \frac{1}{9}\right) \Rightarrow \lambda = 656.3\, nm$$

The short-wave limit:

$$\frac{1}{\lambda} = R_H \left(\frac{1}{2^2} - 0\right) = (1.09737 \times 10^7)\frac{1}{m}\left(\frac{1}{4}\right) \Rightarrow \lambda = 364.6\, nm.$$

Significance

Note that there are infinitely many spectral lines lying between these two limits.

? Exercise 3.5.1

What are the limits of the Lyman series? Can you see these spectral lines?

Answer

121.5 nm and 91.1 nm; no, these spectral bands are in the ultraviolet

The key to unlocking the mystery of atomic spectra is in understanding atomic structure. Scientists have long known that matter is made of atoms. According to nineteenth-century science, atoms are the smallest indivisible quantities of matter. This scientific belief was shattered by a series of groundbreaking experiments that proved the existence of subatomic particles, such as electrons, protons, and neutrons.

The electron was discovered and identified as the smallest quantity of electric charge by J.J. **Thomson** in 1897 in his cathode ray experiments, also known as β-ray experiments: A **β**-ray is a beam of electrons. In 1904, Thomson proposed the first model of atomic structure, known as the "plum pudding" model, in which an atom consisted of an unknown positively charged matter with negative electrons embedded in it like plums in a pudding. Around 1900, E. **Rutherford**, and independently, Paul Ulrich Villard, classified all radiation known at that time as α-**rays**, β-**rays**, and γ-**rays** (a γ-ray is a beam of highly energetic photons). In 1907, Rutherford and Thomas Royds used spectroscopy methods to show that positively charged particles of α-radiation (called α-particles) are in fact doubly ionized atoms of helium. In 1909, Rutherford, Ernest Marsden, and Hans Geiger used α-particles in their famous scattering experiment that disproved Thomson's model (see Linear Momentum and Collisions).

In the **Rutherford gold foil experiment** (also known as the Geiger–Marsden experiment), α-particles were incident on a thin gold foil and were scattered by gold atoms inside the foil (see Types of Collisions). The outgoing particles were detected by a 360° scintillation screen surrounding the gold target (for a detailed description of the experimental setup, see Linear Momentum and Collisions). When a scattered particle struck the screen, a tiny flash of light (scintillation) was observed at that location. By counting the scintillations seen at various angles with respect to the direction of the incident beam, the scientists could determine what fraction of the incident particles were scattered and what fraction were not deflected at all. If the plum pudding model were correct, there would be no back-scattered α-particles. However, the results of the Rutherford experiment showed that, although a sizable fraction of α-particles emerged from the foil not scattered at all as though the foil were not in their way, a significant fraction of α-particles were back-scattered toward the source. This kind of result was possible only when most of the mass and the entire positive charge of the gold atom were concentrated in a tiny space inside the atom.

In 1911, Rutherford proposed a **nuclear model of the atom**. In Rutherford's model, an atom contained a positively charged nucleus of negligible size, almost like a point, but included almost the entire mass of the atom. The atom also contained negative electrons that were located within the atom but relatively far away from the nucleus. Ten years later, Rutherford coined the name **proton** for the nucleus of hydrogen and the name **neutron** for a hypothetical electrically neutral particle that would mediate the binding of positive protons in the nucleus (the neutron was discovered in 1932 by James **Chadwick**). Rutherford is credited with the discovery of the atomic nucleus; however, the Rutherford model of atomic structure does not explain the Rydberg formula for the hydrogen emission lines.

Bohr's model of the hydrogen atom, proposed by Niels **Bohr** in 1913, was the first quantum model that correctly explained the hydrogen emission spectrum. Bohr's model combines the classical mechanics of planetary motion with the quantum concept of photons. Once Rutherford had established the existence of the atomic nucleus, Bohr's intuition that the negative electron in the hydrogen atom must revolve around the positive nucleus became a logical consequence of the inverse-square-distance law of electrostatic attraction. Recall that Coulomb's law describing the attraction between two opposite charges has a similar form to Newton's universal law of gravitation in the sense that the gravitational force and the electrostatic force are both decreasing as $1/r^2$, where **r** is the separation distance between the bodies. In the same way as Earth revolves around the sun, the negative electron in the hydrogen atom can revolve around the positive nucleus. However, an accelerating charge radiates its energy. Classically, if the electron moved around the nucleus in a planetary fashion, it would be undergoing centripetal acceleration, and thus would be radiating energy that would cause it to spiral down into the nucleus. Such a planetary hydrogen atom would not be stable, which is contrary to what we know about ordinary hydrogen atoms that do not disintegrate. Moreover, the classical motion of the electron is not able to explain the discrete emission spectrum of hydrogen.

To circumvent these two difficulties, Bohr proposed the following **three postulates of Bohr's model**:

1. The negative electron moves around the positive nucleus (proton) in a circular orbit. All electron orbits are centered at the nucleus. Not all classically possible orbits are available to an electron bound to the nucleus.
2. The allowed electron orbits satisfy the **first quantization condition**: In the **n**th orbit, the angular momentum L_n of the electron can take only discrete values:

   $$L_n = n\hbar, \; where \, n = 1, 2, 3, \ldots$$

   This postulate says that the electron's angular momentum is quantized. Denoted by r_n and v_n, respectively, the radius of the **n**th orbit and the electron's speed in it, the first quantization condition can be expressed explicitly as

   $$m_e v_n r_n = n\hbar. \tag{3.5.3}$$

3. An electron is allowed to make transitions from one orbit where its energy is E_n to another orbit where its energy is E_m. When an atom absorbs a photon, the electron makes a transition to a higher-energy orbit. When an atom emits a photon, the electron transits to a lower-energy orbit. Electron transitions with the simultaneous photon absorption or photon emission take place **instantaneously**. The allowed electron transitions satisfy the **second quantization condition**:

   $$hf = |E_n - E_m|$$

   where hf is the energy of either an emitted or an absorbed photon with frequency f. The second quantization condition states that an electron's change in energy in the hydrogen atom is quantized.

These three postulates of the early quantum theory of the hydrogen atom allow us to derive not only the Rydberg formula, but also the value of the Rydberg constant and other important properties of the hydrogen atom such as its energy levels, its ionization energy, and the sizes of electron orbits. Note that in Bohr's model, along with two nonclassical quantization postulates, we also have the classical description of the electron as a particle that is subjected to the Coulomb force, and its motion must obey Newton's laws of motion. The hydrogen atom, as an isolated system, must obey the laws of conservation of energy and momentum in the way we know from classical physics. Having this theoretical framework in mind, we are ready to proceed with our analysis.

Electron Orbits

To obtain the size r_n of the electron's **n**th orbit and the electron's speed v_n in it, we turn to Newtonian mechanics. As a charged particle, the electron experiences an electrostatic pull toward the positively charged nucleus in the center of its circular orbit. This electrostatic pull is the centripetal force that causes the electron to move in a circle around the nucleus. Therefore, the magnitude of centripetal force is identified with the magnitude of the electrostatic force:

$$\frac{m_e v_n^2}{r_n} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n^2}. \tag{3.5.4}$$

Here, e denotes the value of the elementary charge. The negative electron and positive proton have the same value of charge,

$$|q| = e.$$

When Equation 3.5.4 is combined with the first quantization condition given by Equation 3.5.3, we can solve for the speed, v_n, and for the radius, r_n:

$$v_n = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar}\frac{1}{n} \tag{3.5.5}$$

$$r_n = 4\pi\epsilon_0\frac{\hbar^2}{m_e e^2}n^2. \tag{3.5.6}$$

Note that these results tell us that the electron's speed as well as the radius of its orbit depend only on the index **n** that enumerates the orbit because all other quantities in the preceding equations are fundamental constants. We see from Equation 3.5.6 that the size of the orbit grows as the square of **n**. This means that the second orbit is four times as large as the first orbit, and the third orbit is nine times as large as the first orbit, and so on. We also see from Equation 3.5.5 that the electron's speed in the orbit decreases as the orbit size increases. The electron's speed is largest in the first Bohr orbit, for $n = 1$, which is the orbit closest to the nucleus. The radius of the first Bohr orbit is called the **Bohr radius of hydrogen**, denoted as a_0. Its value is obtained by setting $n = 1$ in Equation 3.5.6:

$$a_0 = 4\pi\epsilon_0\frac{\hbar^2}{m_e e^2} = 5.29\times 10^{-11} m = 0.529\ \text{Å}.$$

We can substitute a_0 in Equation 3.5.6 to express the radius of the **n**th orbit in terms of a_0:

$$r_n = a_0 n^2. \tag{3.5.7}$$

This result means that the electron orbits in hydrogen atom are **quantized** because the orbital radius takes on only specific values of a_0, $4a_0$, $9a_0$, $16a_0$... given by Equation 3.5.7, and no other values are allowed.

Electron Energies

The total energy E_n of an electron in the **n**th orbit is the sum of its kinetic energy K_n and its electrostatic potential energy U_n. Utilizing Equation 3.5.5, we find that

$$K_n = \frac{1}{2}m_e v_n^2 = \frac{1}{32\pi^2\epsilon^2}\frac{m_e e^4}{\hbar^2}\frac{1}{n^2}. \tag{3.5.8}$$

Recall that the electrostatic potential energy of interaction between two charges q_1 and q_2 that are separated by a distance r_{12} is $(1/4\pi\epsilon_0)q_1q_2/r_{12}$. Here, $q_1 = +e$ is the charge of the nucleus in the hydrogen atom (the charge of the proton), $q_2 = -e$ is the charge of the electron and $r_{12} = r_n$ is the radius of the **n**-th orbit. Now we use Equation 3.5.6 to find the potential energy of the electron:

$$U_n = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n} = -\frac{1}{16\pi^2\epsilon_0^2}\frac{m_e e^4}{\hbar^2}\frac{1}{n^2}. \tag{3.5.9}$$

The total energy of the electron is the sum of Equation 3.5.8 and Equation 3.5.9:

$$E_n = K_n + U_n = -\frac{1}{32\pi^2\epsilon_0^2}\frac{m_e e^4}{\hbar^2}\frac{1}{n^2}. \tag{3.5.10}$$

Note that the energy depends only on the index **n** because the remaining symbols in Equation 3.5.10 are physical constants. The value of the constant factor in Equation 3.5.10 is

$$E_0 = \frac{1}{32\pi^2\epsilon_0^2}\frac{m_e e^4}{\hbar^2} = \frac{1}{8\epsilon_0^2}\frac{m_e e^4}{h^2} = 2.17\times 10^{-18} J = 13.6\, eV. \tag{3.5.11}$$

It is convenient to express the electron's energy in the **n**th orbit in terms of this energy, as

$$E_n = -E_0\frac{1}{n^2}. \tag{3.5.12}$$

Now we can see that the electron energies in the hydrogen atom are **quantized** because they can have only discrete values of $-E_0$, $-E_0/4$, $-E_0/9$, $-E_0/16, \ldots$ given by Equation 3.5.12, and no other energy values are allowed. This set of allowed electron energies is called the **energy spectrum of hydrogen** (Figure $3.5.5$). The index **n** that enumerates energy levels in Bohr's model is called the energy **quantum number**. We identify the energy of the electron inside the hydrogen atom with the energy of the hydrogen atom. Note that the smallest value of energy is obtained for $n = 1$, so the hydrogen atom cannot have energy smaller than that. This smallest value of the electron energy in the hydrogen atom is called the **ground state energy of the hydrogen atom** and its value is

$$E_1 = -E_0 = -13.6\,eV. \tag{3.5.13}$$

The hydrogen atom may have other energies that are higher than the ground state. These higher energy states are known as **excited energy states of a hydrogen atom.**

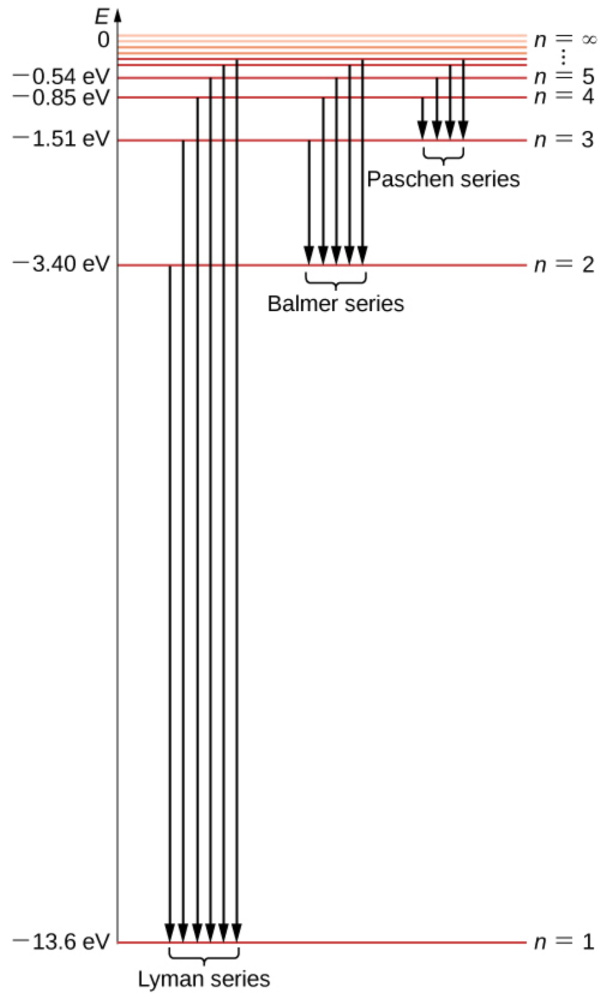

Figure $3.5.5$: The energy spectrum of the hydrogen atom. Energy levels (horizontal lines) represent the bound states of an electron in the atom. There is only one ground state, n=1,n=1, and infinite quantized excited states. The states are enumerated by the quantum number n = 1,2,3,4,....Vertical lines illustrate the allowed electron transitions between the states. Downward arrows illustrate transitions with an emission of a photon with a wavelength in the indicated spectral band.

There is only one ground state, but there are infinitely many excited states because there are infinitely many values of **n** in Equation 3.5.12. We say that the electron is in the "first exited state" when its energy is E_n (when $n = 2$), the second excited state when its energy is E_3 (when $n = 3$) and, in general, in the **n**th exited state when its energy is $E_n + 1$. There is no highest-of-all excited state; however, there is a limit to the sequence of excited states. If we keep increasing n in Equation 3.5.12, we find that the limit is $-lim_{n\to\infty} E_0/n^2 = 0$. In this limit, the electron is no longer bound to the nucleus but becomes a free electron. An electron remains bound in the hydrogen atom as long as its energy is negative. An electron that orbits the nucleus in the first Bohr orbit, closest to the nucleus, is in the ground state, where its energy has the smallest value. In the ground state, the electron is most strongly bound to the nucleus and its energy is given by Equation 3.5.13. If we want to remove this electron from the atom, we must supply it with enough energy, E_∞, to at least balance out its ground state energy E_1:

$$E_\infty + E_1 = 0 \Rightarrow E_\infty = -E_1 = -(-E_0) = E_0 = 13.6\,eV. \tag{3.5.14}$$

The energy that is needed to remove the electron from the atom is called the **ionization energy**. The ionization energy E_∞ that is needed to remove the electron from the first Bohr orbit is called the **ionization limit of the hydrogen atom**. The ionization limit in Equation 3.5.14 that we obtain in Bohr's model agrees with experimental value.

Spectral Emission Lines of Hydrogen

To obtain the wavelengths of the emitted radiation when an electron makes a transition from the **n**-th orbit to the **m**-th orbit, we use the second of Bohr's quantization conditions and Equation 3.5.12 for energies. The emission of energy from the atom can occur only when an electron makes a transition from an excited state to a lower-energy state. In the course of such a transition, the emitted photon carries away the difference of energies between the states involved in the transition. The transition cannot go in the other direction because the energy of a photon cannot be negative, which means that for emission we must have $E_n > E_m$ and $n > m$. Therefore, the third of Bohr's postulates gives

$$\begin{aligned} hf &= |E_n - E_m| \\ &= E_n - E_m \\ &= -E_0\frac{1}{n^2} + E_m\frac{1}{m^2} \\ &= E_0\left(\frac{1}{m^2} - \frac{1}{n^2}\right). \end{aligned} \tag{3.5.15}$$

Now we express the photon's energy in terms of its wavelength, $hf = hc/\lambda$, and divide both sides of Equation 3.5.15 by hc. The result is

$$\frac{1}{\lambda} = \frac{E_0}{hc}\left(\frac{1}{m^2} - \frac{1}{n^2}\right). \tag{3.5.16}$$

The value of the constant in this equation is

$$\frac{E_0}{hc} = \frac{13.6\,eV}{(4.136\times 10^{-15} eV\cdot s)(2.997\times 10^8 m/s)} = 1.097\times 10^7\frac{1}{m}. \tag{3.5.17}$$

This value is exactly the Rydberg constant R_H in the Rydberg heuristic formula Equation 3.5.2. In fact, Equation 3.5.16 is identical to the Rydberg formula, because for a given **m**, we have $n = m+1,\ m+2, \ldots$. In this way, the Bohr quantum model of the hydrogen atom allows us to derive the experimental Rydberg constant from first principles and to express it in terms of fundamental constants. Transitions between the allowed electron orbits are illustrated in Figure $3.5.5$.

We can repeat the same steps that led to Equation 3.5.16 to obtain the wavelength of the absorbed radiation; this again gives Equation 3.5.16 but this time for the positions of absorption lines in the absorption spectrum of hydrogen. The only difference is that for absorption, the quantum number m is the index of the orbit occupied by the electron before the transition (lower-energy orbit) and the quantum number n is the index of the orbit to which the electron makes the transition (higher-energy orbit). The difference between the electron energies in these two orbits is the energy of the absorbed photon.

✓ Example $3.5.2$: Size and Ionization Energy of the Hydrogen Atom in an Excited State

If a hydrogen atom in the ground state absorbs a 93.7-nm photon, corresponding to a transition line in the Lyman series, how does this affect the atom's energy and size? How much energy is needed to ionize the atom when it is in this excited state? Give your answers in absolute units, and relative to the ground state.

Strategy

Before the absorption, the atom is in its ground state. This means that the electron transition takes place from the orbit $m = 1$ to some higher **n**th orbit. First, we must determine nn for the absorbed wavelength $\lambda = 93.7\,nm$. Then, we can use Equation 3.5.12 to find the energy E_n of the excited state and its ionization energy $E_{\infty,n}$, and use Equation 3.5.7 to find the radius r_n of the atom in the excited state. To estimate **n**, we use Equation 3.5.16.

Solution

Substitute $m = 1$ and λ = 93.7 nm in Equation 3.5.16 and solve for n. You should not expect to obtain a perfect integer answer because of rounding errors, but your answer will be close to an integer, and you can estimate **n** by taking the integral part of your answer:

$$\begin{aligned} \frac{1}{\lambda} &= R_H \left(\frac{1}{1^1} - \frac{1}{n^2} \right) \\ \Rightarrow n &= \frac{1}{\sqrt{1 - \frac{1}{\lambda R_H}}} \\ &= \frac{1}{\sqrt{1 - \frac{1}{(93.7 \times 10^{-9} m)(1.097 \times 10^7 m^{-1})}}} \\ &= 6.07 \\ \Rightarrow n &= 6. \end{aligned}$$

The radius of the $n = 6$ orbit is

$$r_n = a_0 n^2 = a_0 6^2 = 36 a_0 = 36(0.529 \times 10^{-10}\, m) = 19.04 \times 10^{-10}\, m \cong 19.0\, \text{Å}.$$

Thus, after absorbing the 93.7-nm photon, the size of the hydrogen atom in the excited $n = 6$ state is 36 times larger than before the absorption, when the atom was in the ground state. The energy of the fifth excited state ($n = 6$) is:

$$E_n = -\frac{E_0}{n^2} = -\frac{E_0}{6^2} = -\frac{E_0}{36} = -\frac{13.6\, eV}{36} \cong -0.378\, eV.$$

After absorbing the 93.7-nm photon, the energy of the hydrogen atom is larger than it was before the absorption. Ionization of the atom when it is in the fifth excited state ($n = 6$) requites 36 times less energy than is needed when the atom is in the ground state:

$$E_{\infty,6} = -E_6 = -(-0.378\, eV) = 0.378\, eV.$$

Significance

We can analyze any spectral line in the spectrum of hydrogen in the same way. Thus, the experimental measurements of spectral lines provide us with information about the atomic structure of the hydrogen atom.

? Exercise 3.5.2

When an electron in a hydrogen atom is in the first excited state, what prediction does the Bohr model give about its orbital speed and kinetic energy? What is the magnitude of its orbital angular momentum?

Answer

$v_2 = 1.1 \times 10^6 m/s \cong 0.0036\, c$;

$L_2 = 2\hbar K_2 = 3.4\, eV$

Bohr's model of the hydrogen atom also correctly predicts the spectra of some hydrogen-like ions. **Hydrogen-like ions** are atoms of elements with an atomic number **Z** larger than one ($Z = 1$ for hydrogen) but with all electrons removed except one. For example, an electrically neutral helium atom has an atomic number $Z = 2$. This means it has two electrons orbiting the nucleus with a charge of $q = +Ze$. When one of the orbiting electrons is removed from the helium atom (we say, when the helium atom is singly ionized), what remains is a hydrogen-like atomic structure where the remaining electron orbits the nucleus with a charge of $q = +Ze$. This type of situation is described by the Bohr model. Assuming that the charge of the nucleus is not $+e$ but $+Ze$, we can repeat all steps, beginning with Equation 3.5.4, to obtain the results for a hydrogen-like ion:

$$r_n = \frac{a_0}{Z} n^2 \tag{3.5.18}$$

where a_0 is the Bohr orbit of hydrogen, and

$$E_n = -Z^2 E_0 \frac{1}{n^2} \tag{3.5.19}$$

where E_0 is the ionization limit of a hydrogen atom. These equations are good approximations as long as the atomic number **Z** is not too large.

The Bohr model is important because it was the first model to postulate the quantization of electron orbits in atoms. Thus, it represents an early quantum theory that gave a start to developing modern quantum theory. It introduced the concept of a quantum number to describe atomic states. The limitation of the early quantum theory is that it cannot describe atoms in which the number of electrons orbiting the nucleus is larger than one. The Bohr model of hydrogen is a semi-classical model because it combines the classical concept of electron orbits with the new concept of quantization. The remarkable success of this model prompted many physicists to seek an explanation for why such a model should work at all, and to seek an understanding of the physics behind the postulates of early quantum theory. This search brought about the onset of an entirely new concept of “matter waves.”

3.6: De Broglie’s Matter Waves

Learning Objectives

By the end of this section, you will be able to:

- Describe de Broglie’s hypothesis of matter waves
- Explain how the de Broglie’s hypothesis gives the rationale for the quantization of angular momentum in Bohr’s quantum theory of the hydrogen atom
- Describe the Davisson–Germer experiment
- Interpret de Broglie’s idea of matter waves and how they account for electron diffraction phenomena

Compton’s formula established that an electromagnetic wave can behave like a particle of light when interacting with matter. In 1924, Louis **de Broglie** proposed a new speculative hypothesis that electrons and other particles of matter can behave like waves. Today, this idea is known as **de Broglie’s hypothesis of matter waves**. In 1926, De Broglie’s hypothesis, together with Bohr’s early quantum theory, led to the development of a new theory of **wave quantum mechanics** to describe the physics of atoms and subatomic particles. Quantum mechanics has paved the way for new engineering inventions and technologies, such as the laser and magnetic resonance imaging (MRI). These new technologies drive discoveries in other sciences such as biology and chemistry.

According to de Broglie’s hypothesis, massless photons as well as massive particles must satisfy one common set of relations that connect the energy E with the frequency f, and the linear momentum p with the wavelength λ. We have discussed these relations for photons in the context of Compton’s effect. We are recalling them now in a more general context. Any particle that has energy and momentum is a **de Broglie wave** of frequency f and wavelength λ:

$$E = hf \tag{3.6.1}$$

$$\lambda = \frac{h}{p} \tag{3.6.2}$$

Here, E and p are, respectively, the relativistic energy and the momentum of a particle. De Broglie’s relations are usually expressed in terms of the wave vector $\vec{k}$, $k = 2\pi/\lambda$, and the wave frequency $\omega = 2\pi f$, as we usually do for waves:

$$E = \hbar\omega$$
$$\vec{p} = \hbar\vec{k}$$

Wave theory tells us that a wave carries its energy with the **group velocity**. For matter waves, this group velocity is the velocity u of the particle. Identifying the energy **E** and momentum **p** of a particle with its **relativistic energy** mc^2 and its **relativistic momentum** mu, respectively, it follows from de Broglie relations that matter waves satisfy the following relation:

$$\lambda f = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p} = \frac{mc^2}{mu} = \frac{c^2}{u} = \frac{c}{\beta} \tag{3.6.3}$$

where $\beta = u/c$. When a particle is massless we have $u = c$ and Equation 3.6.3 becomes $\lambda f = c$.

Example 3.6.1: How Long are de Broglie Matter Waves?

Calculate the de Broglie wavelength of:

1. a 0.65-kg basketball thrown at a speed of 10 m/s,
2. a nonrelativistic electron with a kinetic energy of 1.0 eV, and
3. a relativistic electron with a kinetic energy of 108 keV.

Strategy

We use Equation 3.6.3 to find the de Broglie wavelength. When the problem involves a nonrelativistic object moving with a nonrelativistic speed **u**, such as in (a) when $\beta = u/c \ll 1$, we use nonrelativistic momentum **p**. When the nonrelativistic approximation cannot be used, such as in (c), we must use the relativistic momentum $p = mu = m_0\gamma u = E_0\gamma\beta/c$, where the rest mass energy of a particle is $E_0 = mc^2$ and γ is the Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$. The total energy E of a particle is given by Equation 3.6.1 and the kinetic energy is $K = E - E_0 = (\gamma - 1)E_0$. When the kinetic energy is known, we can invert Equation 6.4.2 to find the momentum

$$p = \sqrt{\left(E^2 - E_0^2\right)/c^2} = \sqrt{K\left(K + 2E_0\right)}/c$$

and substitute into Equation 3.6.3 to obtain

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K\left(K+2E_0\right)}} \tag{3.6.4}$$

Depending on the problem at hand, in this equation we can use the following values for **hc**:

$$hc = \left(6.626 \times 10^{-34}\ \mathrm{J \cdot s}\right)\left(2.998 \times 10^8\ \mathrm{m/s}\right) = 1.986 \times 10^{-25}\ \mathrm{J \cdot m} = 1.241\ \mathrm{eV} \cdot \mu\mathrm{m}$$

Solution

a. For the basketball, the kinetic energy is

$$K = mu^2/2 = (0.65\ \mathrm{kg})(10\ \mathrm{m/s})^2/2 = 32.5\ \mathrm{J}$$

and the rest mass energy is

$$E_0 = mc^2 = (0.65\ \mathrm{kg})\left(2.998 \times 10^8\ \mathrm{m/s}\right)^2 = 5.84 \times 10^{16}\ \mathrm{J}$$

We see that $K/\left(K + E_0\right) \ll 1$ and use $p = mu = (0.65\ \mathrm{kg})(10\ \mathrm{m/s}) = 6.5\ \mathrm{J \cdot s/m}$:

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\ \mathrm{J \cdot s}}{6.5\ \mathrm{J \cdot s/m}} = 1.02 \times 10^{-34}\ \mathrm{m}$$

b. For the nonrelativistic electron,

$$E_0 = mc^2 = \left(9.109 \times 10^{-31}\mathrm{kg}\right)\left(2.998 \times 10^8 \mathrm{m/s}\right)^2 = 511\mathrm{keV}$$

and when $K = 1.0\ eV$, we have $K/(K + E_0) = (1/512) \times 10^{-3} \ll 1$, so we can use the nonrelativistic formula. However, it is simpler here to use Equation 3.6.4:

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K\left(K+2E_0\right)}} = \frac{1.241\ \mathrm{eV} \cdot \mu\mathrm{m}}{\sqrt{(1.0\ \mathrm{eV})[1.0\ \mathrm{eV} + 2(511\ \mathrm{keV})]}} = 1.23\ \mathrm{nm}$$

If we use nonrelativistic momentum, we obtain the same result because 1 eV is much smaller than the rest mass of the electron.

c. For a fast electron with $K = 108\ keV$, relativistic effects cannot be neglected because its total energy is $E = K = E_0 = 108\ keV + 511\ keV = 619\ keV$ and $K/E = 108/619$ is not negligible:

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K\left(K+2E_0\right)}} = \frac{1.241\ \mathrm{eV} \cdot \mu\mathrm{m}}{\sqrt{108\ \mathrm{keV}[108\ \mathrm{keV} + 2(511\ \mathrm{keV})]}} = 3.55\ \mathrm{pm}$$

.

Significance

We see from these estimates that De Broglie’s wavelengths of macroscopic objects such as a ball are immeasurably small. Therefore, even if they exist, they are not detectable and do not affect the motion of macroscopic objects.

Exercise 3.6.1

What is de Broglie’s wavelength of a nonrelativistic proton with a kinetic energy of 1.0 eV?

Answer

1.7 pm

Using the concept of the electron matter wave, de Broglie provided a rationale for the quantization of the electron's angular momentum in the hydrogen atom, which was postulated in Bohr's quantum theory. The physical explanation for the first Bohr quantization condition comes naturally when we assume that an electron in a hydrogen atom behaves not like a particle but like a wave. To see it clearly, imagine a stretched guitar string that is clamped at both ends and vibrates in one of its normal modes. If the length of the string is **l** (Figure 3.6.1), the wavelengths of these vibrations cannot be arbitrary but must be such that an integer **k** number of half-wavelengths $\lambda/2$ fit exactly on the distance **l** between the ends. This is the condition $l = k\lambda/2$ for a standing wave on a string. Now suppose that instead of having the string clamped at the walls, we bend its length into a circle and fasten its ends to each other. This produces a circular string that vibrates in normal modes, satisfying the same standing-wave condition, but the number of half-wavelengths must now be an even number k, $k = 2n$, and the length **l** is now connected to the radius r_n of the circle. This means that the radii are not arbitrary but must satisfy the following standing-wave condition:

$$2\pi r_n = 2n\frac{\lambda}{2}. \tag{3.6.5}$$

If an electron in the **n**th Bohr orbit moves as a wave, by Equation 3.6.5 its wavelength must be equal to $\lambda = 2\pi r_n/n$. Assuming that Equation 3.6.4 is valid, the electron wave of this wavelength corresponds to the electron's linear momentum, $p = h/\lambda = nh/(2\pi r_n) = n\hbar/r_n$. In a circular orbit, therefore, the electron's angular momentum must be

$$L_n = r_n p = r_n\frac{n\hbar}{r_n} = n\hbar. \tag{3.6.6}$$

This equation is the first of Bohr's quantization conditions, given by Equation 6.5.6. Providing a physical explanation for Bohr's quantization condition is a convincing theoretical argument for the existence of matter waves.

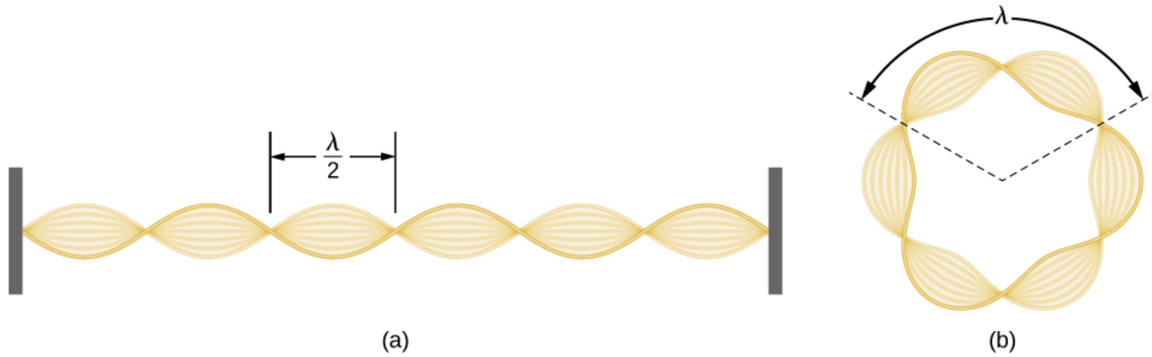

Figure 3.6.1: Standing-wave pattern: (a) a stretched string clamped at the walls; (b) an electron wave trapped in the third Bohr orbit in the hydrogen atom.

✓ Example 3.6.2: The Electron Wave in the Ground State of Hydrogen

Find the de Broglie wavelength of an electron in the ground state of hydrogen.

Strategy

We combine the first quantization condition in Equation 3.6.6 with Equation 6.5.6 and use Equation 6.5.9 for the first Bohr radius with $n = 1$.

Solution

When $n = 1$ and $r_n = a_0 = 0.529\ \text{Å}$, the Bohr quantization condition gives $a_0 p = 1 \cdot \hbar \Rightarrow p = \hbar/a_0$. The electron wavelength is:

$$\lambda = h/p = h/\hbar/a_0 = 2\pi a_0 = 2\pi(0.529\ \text{Å}) = 3.324\ \text{Å}.$$

Significance

We obtain the same result when we use Equation 3.6.4 directly.

Exercise 3.6.2

Find the de Broglie wavelength of an electron in the third excited state of hydrogen.

Answer

$\lambda = 2\pi n a_0 = 2(3.324\ \text{Å}) = 6.648\ \text{Å}$

Experimental confirmation of matter waves came in 1927 when C. Davisson and L. Germer performed a series of electron-scattering experiments that clearly showed that electrons do behave like waves. Davisson and Germer did not set up their experiment to confirm de Broglie's hypothesis: The confirmation came as a byproduct of their routine experimental studies of metal surfaces under electron bombardment.

In the particular experiment that provided the very first evidence of electron waves (known today as the **Davisson–Germer experiment**), they studied a surface of nickel. Their nickel sample was specially prepared in a high-temperature oven to change its usual polycrystalline structure to a form in which large single-crystal domains occupy the volume. Figure 3.6.2 shows the experimental setup. Thermal electrons are released from a heated element (usually made of tungsten) in the electron gun and accelerated through a potential difference ΔV, becoming a well-collimated beam of electrons produced by an electron gun. The kinetic energy K of the electrons is adjusted by selecting a value of the potential difference in the electron gun. This produces a beam of electrons with a set value of linear momentum, in accordance with the conservation of energy:

$$e\Delta V = K = \frac{p^2}{2m} \Rightarrow p = \sqrt{2me\Delta V} \tag{3.6.7}$$

The electron beam is incident on the nickel sample in the direction normal to its surface. At the surface, it scatters in various directions. The intensity of the beam scattered in a selected direction φφ is measured by a highly sensitive detector. The detector's angular position with respect to the direction of the incident beam can be varied from φ=0° to φ=90°. The entire setup is enclosed in a vacuum chamber to prevent electron collisions with air molecules, as such thermal collisions would change the electrons' kinetic energy and are not desirable.

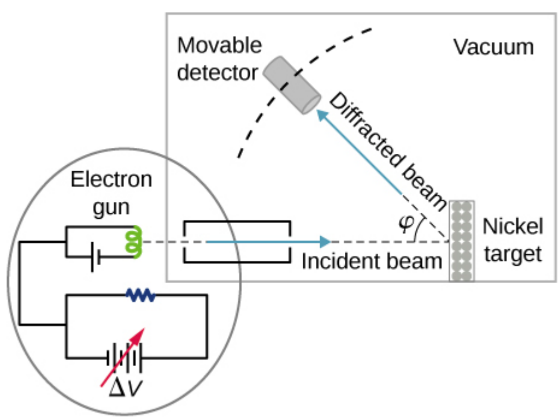

Figure 3.6.2: Schematics of the experimental setup of the Davisson–Germer diffraction experiment. A well-collimated beam of electrons is scattered off the nickel target. The kinetic energy of electrons in the incident beam is selected by adjusting a variable potential, ΔV, in the electron gun. Intensity of the scattered electron beam is measured for a range of scattering angles φ, whereas the distance between the detector and the target does not change.

When the nickel target has a polycrystalline form with many randomly oriented microscopic crystals, the incident electrons scatter off its surface in various random directions. As a result, the intensity of the scattered electron beam is much the same in any direction, resembling a diffuse reflection of light from a porous surface. However, when the nickel target has a regular crystalline structure, the intensity of the scattered electron beam shows a clear maximum at a specific angle and the results show a clear diffraction pattern (see Figure 3.6.3). Similar diffraction patterns formed by X-rays scattered by various crystalline solids were studied in 1912 by father-and-son physicists William H. Bragg and William L. Bragg. The Bragg law in X-ray crystallography provides a connection between the wavelength λ of the radiation incident on a crystalline lattice, the lattice spacing, and the position of the interference maximum in the diffracted radiation (see Diffraction).

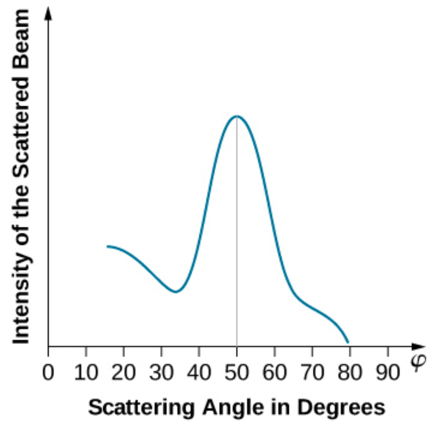

Figure 3.6.3: The experimental results of electron diffraction on a nickel target for the accelerating potential in the electron gun of about ΔV = 54 V:The intensity maximum is registered at the scattering angle of about φ=50°

The lattice spacing of the Davisson–Germer target, determined with X-ray crystallography, was measured to be $a = 2.15$ Å. Unlike X-ray crystallography in which X-rays penetrate the sample, in the original Davisson–Germer experiment, only the surface atoms interact with the incident electron beam. For the surface diffraction, the maximum intensity of the reflected electron beam is observed for scattering angles that satisfy the condition nλ = a sin φ (see Figure 3.6.4). The first-order maximum (for n=1) is measured at a scattering angle of φ≈50° at ΔV≈54 V, which gives the wavelength of the incident radiation as λ=(2.15 Å) sin 50° = 1.64 Å. On the other hand, a 54-V potential accelerates the incident electrons to kinetic energies of K = 54 eV. Their momentum, calculated from Equation 3.6.7, is $p = 2.478 \times 10^{-5}\ eV \cdot s/m$. When we substitute this result in Equation 3.6.4, the de Broglie wavelength is obtained as

$$\lambda = \frac{h}{p} = \frac{4.136 \times 10^{-15} \mathrm{eV \cdot s}}{2.478 \times 10^{-5} \mathrm{eV \cdot s/m}} = 1.67 \text{Å}. \tag{3.6.8}$$

The same result is obtained when we use K = 54eV in Equation 3.6.7. The proximity of this theoretical result to the Davisson–Germer experimental value of λ = 1.64 Å is a convincing argument for the existence of de Broglie matter waves.

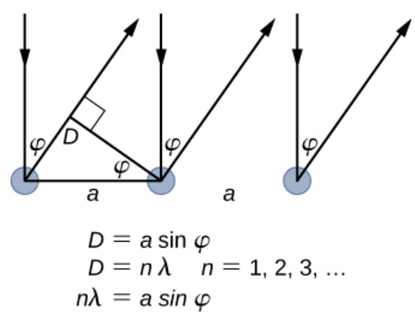

Figure 3.6.4: In the surface diffraction of a monochromatic electromagnetic wave on a crystalline lattice structure, the in-phase incident beams are reflected from atoms on the surface. A ray reflected from the left atom travels an additional distance D = a sin θ to the detector, where a is the lattice spacing. The reflected beams remain in-phase when D is an integer multiple of their wavelength λ.. The intensity of the reflected waves has pronounced maxima for angles φ satisfying nλ = a sin φ.

Diffraction lines measured with low-energy electrons, such as those used in the Davisson–Germer experiment, are quite broad (Figure 3.6.3) because the incident electrons are scattered only from the surface. The resolution of diffraction images greatly improves when a higher-energy electron beam passes through a thin metal foil. This occurs because the diffraction image is created by scattering off many crystalline planes inside the volume, and the maxima produced in scattering at Bragg angles are sharp (Figure 3.6.5).

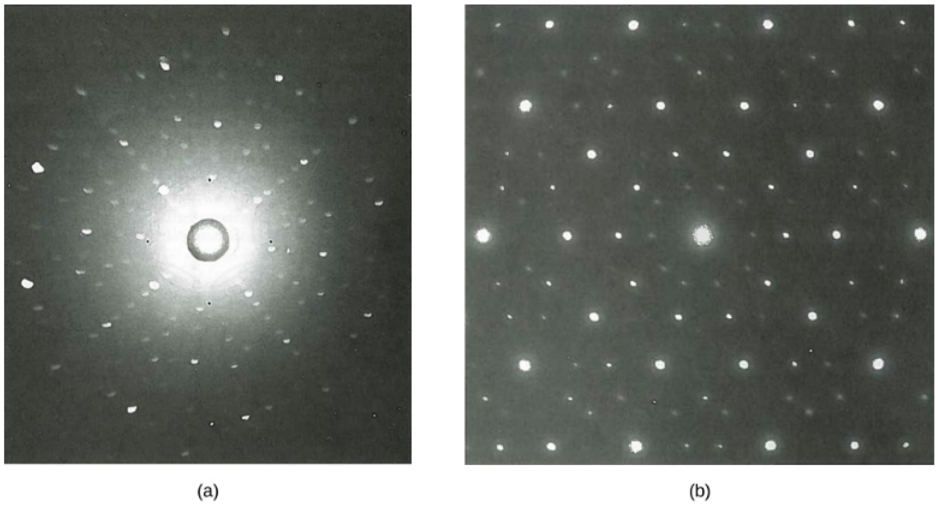

Figure 3.6.5: Diffraction patterns obtained in scattering on a crystalline solid: (a) with X-rays, and (b) with electrons. The observed pattern reflects the symmetry of the crystalline structure of the sample.

Since the work of Davisson and Germer, de Broglie's hypothesis has been extensively tested with various experimental techniques, and the existence of de Broglie waves has been confirmed for numerous elementary particles. Neutrons have been used in scattering experiments to determine crystalline structures of solids from interference patterns formed by neutron matter waves. The neutron has zero charge and its mass is comparable with the mass of a positively charged proton. Both neutrons and protons can be seen as matter waves. Therefore, the property of being a matter wave is not specific to electrically charged particles but is true of all particles in motion. Matter waves of molecules as large as carbon C_{60} have been measured. All physical objects, small or large, have an associated matter wave as long as they remain in motion. The universal character of de Broglie matter waves is firmly established.

Example $3.6.3A$: Neutron Scattering

Suppose that a neutron beam is used in a diffraction experiment on a typical crystalline solid. Estimate the kinetic energy of a neutron (in eV) in the neutron beam and compare it with kinetic energy of an ideal gas in equilibrium at room temperature.

Strategy

We assume that a typical crystal spacing **a** is of the order of 1.0 Å. To observe a diffraction pattern on such a lattice, the neutron wavelength λ must be on the same order of magnitude as the lattice spacing. We use Equation 3.6.7 to find the momentum **p** and kinetic energy **K**. To compare this energy with the energy E_T of ideal gas in equilibrium at room temperature $T = 300\,K$, we use the relation $K = 3/2k_BT$, where $k_B = 8.62 \times 10^{-5}eV/K$ is the Boltzmann constant.

Solution

We evaluate **pc** to compare it with the neutron's rest mass energy $E_0 = 940\,MeV$:

$$p = \frac{h}{\lambda} \Rightarrow pc = \frac{hc}{\lambda} = \frac{1.241 \times 10^{-6} eV \cdot m}{10^{-10} m} = 12.41\,keV.$$

We see that $p^2c^2 << E_0^2$ and we can use the nonrelativistic kinetic energy:

$$K = \frac{p^2}{2m_n} = \frac{h^2}{2\lambda^2 m_n} = \frac{(6.63 \times 10^{-34} J \cdot s)^2}{(2 \times 10^{-20} m^2)(1.66 \times 10^{-27} kg)} = 1.32 \times 10^{-20} J = 82.7\,meV.$$

Kinetic energy of ideal gas in equilibrium at 300 K is:

$$K_T = \frac{3}{2}k_BT = \frac{3}{2}(8.62 \times 10^{-5} eV/K)(300\,K) = 38.8\,MeV.$$

We see that these energies are of the same order of magnitude.

Significance

Neutrons with energies in this range, which is typical for an ideal gas at room temperature, are called "thermal neutrons."

Example $3.6.3B$: Wavelength of a Relativistic Proton

In a supercollider at CERN, protons can be accelerated to velocities of 0.75 **c**. What are their de Broglie wavelengths at this speed? What are their kinetic energies?

Strategy

The rest mass energy of a proton is $E_0 = m_0c^2 = (1.672 \times 10^{-27}kg)(2.998 \times 10^8 m/s)^2 = 938\,MeV$. When the proton's velocity is known, we have β = 0.75 and $\beta\gamma = 0.75/\sqrt{1-0.75^2} = 1.714$. We obtain the wavelength λλ and kinetic energy **K** from relativistic relations.

Solution

$$\lambda = \frac{h}{p} = \frac{hc}{\beta\gamma E_0} = \frac{1.241\,eV\cdot\mu m}{1.714(938\,MeV)} = 0.77\,fm$$

$$K = E_0(\gamma - 1) = 938\,MeV(1/\sqrt{1-0.75^2} - 1) = 480.1\,MeV$$

Significance

Notice that because a proton is 1835 times more massive than an electron, if this experiment were performed with electrons, a simple rescaling of these results would give us the electron's wavelength of (1835)0.77 fm = 1.4 pm and its kinetic energy of 480.1 MeV /1835 = 261.6 keV.

Exercise $3.6.3$

Find the de Broglie wavelength and kinetic energy of a free electron that travels at a speed of 0.75**c**.

Answer

$\lambda = 1.417\,pm;\; K = 261.56\,keV$

3.7: Wave-Particle Duality

Learning Objectives

By the end of this section, you will be able to:

- Identify phenomena in which electromagnetic waves behave like a beam of photons and particles behave like waves
- Describe the physics principles behind electron microscopy
- Summarize the evolution of scientific thought that led to the development of quantum mechanics

The energy of radiation detected by a radio-signal receiving antenna comes as the energy of an electromagnetic wave. The same energy of radiation detected by a photocurrent in the photoelectric effect comes as the energy of individual photon particles. Therefore, the question arises about the nature of electromagnetic radiation: Is a photon a wave or is it a particle? Similar questions may be asked about other known forms of energy. For example, an electron that forms part of an electric current in a circuit behaves like a particle moving in unison with other electrons inside the conductor. The same electron behaves as a wave when it passes through a solid crystalline structure and forms a diffraction image. Is an electron a wave or is it a particle? The same question can be extended to all particles of matter—elementary particles, as well as compound molecules—asking about their true physical nature. At our present state of knowledge, such questions about the true nature of things do not have conclusive answers. All we can say is that **wave-particle duality** exists in nature: Under some experimental conditions, a particle appears to act as a particle, and under different experimental conditions, a particle appears to act a wave. Conversely, under some physical circumstances electromagnetic radiation acts as a wave, and under other physical circumstances, radiation acts as a beam of photons.

This dualistic interpretation is not a new physics concept brought about by specific discoveries in the twentieth century. It was already present in a debate between Isaac Newton and Christiaan Huygens about the nature of light, beginning in the year 1670. According to Newton, a beam of light is a collection of corpuscles of light. According to Huygens, light is a wave. The corpuscular hypothesis failed in 1803, when Thomas Young announced his **double-slit interference** experiment with light (see Figure 3.7.1), which firmly established light as a wave. In James Clerk Maxwell's theory of electromagnetism (completed by the year 1873), light is an electromagnetic wave. Maxwell's classical view of radiation as an electromagnetic wave is still valid today; however, it is unable to explain blackbody radiation and the photoelectric effect, where light acts as a beam of photons.

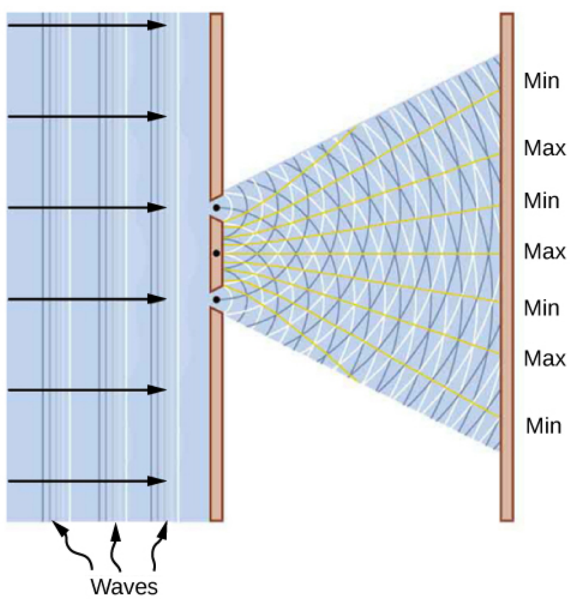

Figure 3.7.1: Young's double-slit experiment explains the interference of light by making an analogy with the interference of water waves. Two waves are generated at the positions of two slits in an opaque screen. The waves have the same wavelengths. They travel from their origins at the slits to the viewing screen placed to the right of the slits. The waves meet on the viewing screen. At the positions marked "Max" on the screen, the meeting waves are in-phase and the combined wave amplitude is enhanced. At positions marked "Min," the combined wave amplitude is zero. For light, this mechanism creates a bright-and-dark fringe pattern on the viewing screen.

A similar dichotomy existed in the interpretation of electricity. From Benjamin Franklin's observations of electricity in 1751 until J.J. Thomson's discovery of the electron in 1897, electric current was seen as a flow in a continuous electric medium. Within this theory of electric fluid, the present theory of electric circuits was developed, and electromagnetism and electromagnetic induction were discovered. Thomson's experiment showed that the unit of negative electric charge (an electron) can travel in a vacuum without any medium to carry the charge around, as in electric circuits. This discovery changed the way in which electricity is understood today and gave the electron its particle status. In Bohr's early quantum theory of the hydrogen atom, both the electron and the proton are particles of matter. Likewise, in the Compton scattering of X-rays on electrons, the electron is a particle. On the other hand, in electron-scattering experiments on crystalline structures, the electron behaves as a wave.

A skeptic may raise a question that perhaps an electron might always be nothing more than a particle, and that the diffraction images obtained in electron-scattering experiments might be explained within some macroscopic model of a crystal and a macroscopic model of electrons coming at it like a rain of ping-pong balls. As a matter of fact, to investigate this question, we do not need a complex model of a crystal but just a couple of simple slits in a screen that is opaque to electrons. In other words, to gather convincing evidence about the nature of an electron, we need to repeat the Young double-slit experiment with electrons. If the electron is a wave, we should observe the formation of interference patterns typical for waves, such as those described in Figure $3.7.1$, even when electrons come through the slits one by one. However, if the electron is a not a wave but a particle, the interference fringes will not be formed.

The very first double-slit experiment with a beam of electrons, performed by Claus Jönsson in Germany in 1961, demonstrated that a beam of electrons indeed forms an interference pattern, which means that electrons collectively behave as a wave. The first double-slit experiments with **single** electrons passing through the slits one-by-one were performed by Giulio **Pozzi** in 1974 in Italy and by Akira **Tonomura** in 1989 in Japan. They show that interference fringes are formed gradually, even when electrons pass through the slits individually. This demonstrates conclusively that electron-diffraction images are formed because of the wave nature of electrons. The results seen in double-slit experiments with electrons are illustrated by the images of the interference pattern in Figure $3.7.2$.

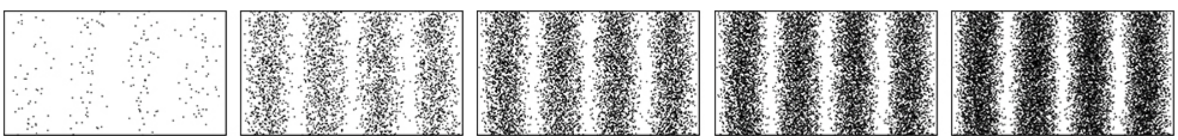

Figure $3.7.2$: Computer-simulated interference fringes seen in the Young double-slit experiment with electrons. One pattern is gradually formed on the screen, regardless of whether the electrons come through the slits as a beam or individually one-by-one.

Example $3.7.1$: Double-Slit Experiment with Electrons

In one experimental setup for studying interference patterns of electron waves, two slits are created in a gold-coated silicon membrane. Each slit is 62-nm wide and 4-μm long, and the separation between the slits is 272 nm. The electron beam is created in an electron gun by heating a tungsten element and by accelerating the electrons across a 600-V potential. The beam is subsequently collimated using electromagnetic lenses, and the collimated beam of electrons is sent through the slits. Find the angular position of the first-order bright fringe on the viewing screen.

Strategy

Recall that the angular position θ of the **n**th order bright fringe that is formed in Young's two-slit interference pattern (discussed in a previous chapter) is related to the separation, **d**, between the slits and to the wavelength, λ, of the incident light by the equation dsin θ = nλ, where n = 0, ± 1, ± 2,.... The separation is given and is equal to d = 272 nm. For the first-order fringe, we take n = 1. The only thing we now need is the wavelength of the incident electron wave.

Since the electron has been accelerated from rest across a potential difference of ΔV = 600 V, its kinetic energy is K = e ΔV = 600 eV. The rest-mass energy of the electron is E_0 = 511 keV.

We compute its de Broglie wavelength as that of a nonrelativistic electron because its kinetic energy K is much smaller than its rest energy E_0, K $\ll E_0$.

Solution

The electron's wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_e K}} = \frac{h}{\sqrt{2E_0/c^2 K}} = \frac{hc}{\sqrt{2E_0 K}} = \frac{1.241 \times 10^{-6}\, eV \cdot m}{\sqrt{2(511\, keV)(600\, eV)}} = 0.050\, nm.$$

This λ is used to obtain the position of the first bright fringe:

$$\sin\theta = \frac{1 \cdot \lambda}{d} = \frac{0.050\, nm}{272\, nm} = 0.000184 \Rightarrow \theta = 0.010^\circ.$$

Significance

Notice that this is also the angular resolution between two consecutive bright fringes up to about n = 1000. For example, between the zero-order fringe and the first-order fringe, between the first-order fringe and the second-order

? Exercise 3.7.1

For the situation described in Example $3.7.1$, find the angular position of the fifth-order bright fringe on the viewing screen.

Answer

0.052^o

The wave-particle dual nature of matter particles and of radiation is a declaration of our inability to describe physical reality within one unified classical theory because separately neither a classical particle approach nor a classical wave approach can fully explain the observed phenomena. This limitation of the classical approach was realized by the year 1928, and a foundation for a new statistical theory, called quantum mechanics, was put in place by Bohr, Edwin **Schrödinger**, Werner **Heisenberg**, and Paul **Dirac**. Quantum mechanics takes de Broglie's idea of matter waves to be the fundamental property of all particles and gives it a statistical interpretation. According to this interpretation, a wave that is associated with a particle carries information about the probable positions of the particle and about its other properties. A single particle is seen as a moving **wave packet** such as the one shown in Figure $3.7.3$. We can intuitively sense from this example that if a particle is a **wave packet**, we will not be able to measure its exact position in the same sense as we cannot pinpoint a location of a wave packet in a vibrating guitar string. The uncertainty, Δx, in measuring the particle's position is connected to the uncertainty, Δp, in the simultaneous measuring of its linear momentum by Heisenberg's uncertainty principle:

$$\Delta x \Delta p \geq \frac{1}{2}\hbar. \tag{3.7.1}$$

Heisenberg's principle expresses the law of nature that, at the quantum level, our perception is limited. For example, if we know the exact position of a body (which means that Δx = 0 in Equation 3.7.1) at the same time we cannot know its momentum, because then the uncertainty in its momentum becomes infinite (because Δp ≥ 0.5 ℏ/Δx in Equation 3.7.1). The Heisenberg uncertainty principle sets the limit on the precision of **simultaneous** measurements of position and momentum of a particle; it shows that the best precision we can obtain is when we have an equals sign (=) in Equation 3.7.1, and we cannot do better than that, even with the best instruments of the future. Heisenberg's principle is a consequence of the wave nature of particles.

Figure $3.7.3$: In this graphic, a particle is shown as a wave packet and its position does not have an exact value.

We routinely use many electronic devices that exploit wave-particle duality without even realizing the sophistication of the physics underlying their operation. One example of a technology based on the particle properties of photons and electrons is a charge-coupled device, which is used for light detection in any instrumentation where high-quality digital data are required, such as in digital cameras or in medical sensors. An example in which the wave properties of electrons is exploited is an electron microscope.

In 1931, physicist Ernst **Ruska** - building on the idea that magnetic fields can direct an electron beam just as lenses can direct a beam of light in an optical microscope—developed the first prototype of the electron microscope. This development originated the field of **electron microscopy**. In the **transmission electron microscope (TEM)**, shown in Figure $3.7.4$, electrons are produced by a hot tungsten element and accelerated by a potential difference in an electron gun, which gives them up to 400 keV in kinetic energy. After leaving the electron gun, the electron beam is focused by electromagnetic lenses (a system of condensing lenses) and transmitted through a specimen sample to be viewed. The image of the sample is reconstructed from the transmitted electron beam. The magnified image may be viewed either directly on a fluorescent screen or indirectly by sending it, for example, to a digital camera or a computer monitor. The entire setup consisting of the electron gun, the lenses, the specimen, and the fluorescent screen are enclosed in a vacuum chamber to prevent the energy loss from the beam. Resolution of the TEM is limited only by spherical aberration (discussed in a previous chapter). Modern high-resolution models of a TEM can have resolving power greater than 0.5 Å and magnifications higher than 50 million times. For comparison, the best resolving power obtained with light microscopy is currently about 97 nm. A limitation of the TEM is that the samples must be about 100-nm thick and biological samples require a special preparation involving chemical “fixing” to stabilize them for ultrathin slicing.

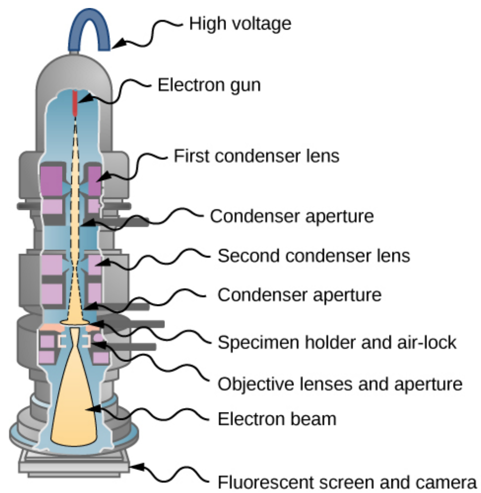

Figure $3.7.4$: TEM: An electron beam produced by an electron gun is collimated by condenser lenses and passes through a specimen. The transmitted electrons are projected on a screen and the image is sent to a camera. (credit: modification of work by Dr. Graham Beards).

Such limitations do not appear in the **scanning electron microscope (SEM)**, which was invented by Manfred von Ardenne in 1937. In an SEM, a typical energy of the electron beam is up to 40 keV and the beam is not transmitted through a sample but is scattered off its surface. Surface topography of the sample is reconstructed by analyzing back-scattered electrons, transmitted electrons, and the emitted radiation produced by electrons interacting with atoms in the sample. The resolving power of an SEM is better than 1 nm, and the magnification can be more than 250 times better than that obtained with a light microscope. The samples scanned by an SEM can be as large as several centimeters but they must be specially prepared, depending on electrical properties of the sample.

High magnifications of the TEM and SEM allow us to see individual molecules. High resolving powers of the TEM and SEM allow us to see fine details, such as those shown in the SEM micrograph of pollen at the beginning of this chapter (Figure 6.1.1).

✓ Example 3.7.2: Resolving Power of an Electron Microscope

If a 1.0-pm electron beam of a TEM passes through a 2.0-μm circular opening, what is the angle between the two just-resolvable point sources for this microscope?

Solution

We can directly use a formula for the resolving power, $\Delta\theta$, of a microscope when the wavelength of the incident radiation is $\lambda = 1.0\,pm$ and the diameter of the aperture is $D = 2.0\,\mu m$:

$$\Delta\theta = 1.22\frac{\lambda}{D} = 1.22\frac{1.0\,pm}{2.0\,\mu m} = 6.10\times 10^{-7} rad = 3.50\times 10.5^{o}.$$

Significance

Note that if we used a conventional microscope with a 400-nm light, the resolving power would be only 14°, which means that all of the fine details in the image would be blurred.

Exercise $3.7.2$

Suppose that the diameter of the aperture in Example $3.7.2$ is halved. How does it affect the resolving power?

Answer

doubles it

3.A: Photons and Matter Waves (Answer)

Check Your Understanding

6.1. Bunsen's burner

6.2. The wavelength of the radiation maximum decreases with increasing temperature.

6.3. $T_\alpha / T_\beta = 1/\sqrt{3} \cong 0.58$, so the star β is hotter.

6.4. $3.3 \times 10^{-19} J$

6.5. No, because then $\Delta E/E \approx 10^{-21}$

6.6. $-0.91V; 1040nm$

6.7. $h = 6.40 \times 10^{-34} J \cdot s = 4.0 \times 10^{-15} eV \cdot s$; -3.5

6.8. $(\Delta\lambda)_{min} = 0m$ at a 0° angle; $71.0pm + 0.5\lambda_c = 72.215pm$

6.9. 121.5 nm and 91.1 nm; no, these spectral bands are in the ultraviolet

6.10. $v_2 = 1.1 \times 10^6 m/s \cong 0.0036c$; $L_2 = 2\hbar K_2 = 3.4eV$

6.11. 1.7 pm

6.12. $\lambda = 2\pi n a_0 = 2(3.324\text{Å}) = 6.648\text{Å}$

6.13. $\lambda = 1.417pm$; $K = 261.56keV$

6.14. 0.052°

6.15. doubles it

Conceptual Questions

1. yellow

3. goes from red to violet through the rainbow of colors

5. would not differ

7. human eye does not see IR radiation

9. No

11. from the slope

13. Answers may vary

15. the particle character

17. Answers may vary

19. no; yes

21. no

23. right angle

25. no

27. They are at ground state.

29. Answers may vary

31. increase

33. for larger n

35. Yes, the excess of 13.6 eV will become kinetic energy of a free electron.

37. no

39. X-rays, best resolving power

41. proton

43. negligibly small de Broglie’s wavelengths

45. to avoid collisions with air molecules

47. Answers may vary

49. Answers may vary

51. yes

53. yes

Problems

55. a. 0.81 eV;

b. 2.1×10^{23};

c. 2 min 20 sec

57. a. 7245 K;

b. 3.62 μm

59. about 3 K

61. 4.835×10^{18}Hz; 0.620 Å

63. 263 nm; no

65. 369 eV

67. 4.09 eV

69. 5.60 eV

71. a. 1.89 eV;

b. 459 THz;

c. 1.21 V

73. 264 nm; UV

75. $1.95 \times 10^{6} m/s$

77. $1.66 \times 10^{-32} kg \cdot m/s$

79. 5620 eV

81. $6.63 \times 10^{-23} kg \cdot m/s$; 124 keV

83. 82.9 fm; 15 MeV

85. (Proof)

87. $\Delta\lambda_{30}/\Delta\lambda_{45} = 45.74$

89. 121.5 nm

91. a. 0.661 eV;

b. –10.2 eV;

c. 1.511 eV

93. 3038 THz

95. 97.33 nm

97. a. h/π;

b. 3.4 eV;

c. – 6.8 eV;

d. – 3.4 eV

99. $n = 4$

101. 365 nm; UV

103. no

105. 7

107. 145.5 pm

109. 20 fm; 9 fm

111. a. 2.103 eV;

b. 0.846 nm

113. 80.9 pm

115. $2.21 \times 10^{-20} m/s$

117. 9.929×10^{32}

119. $\gamma = 1060; 0.00124 fm$

121. 24.11 V

123. a. $P = 2I/c = 8.67 \times 10^{-6} N/m^2$;

b. $a = PA/m = 8.67 \times 10^{-4} m/s^2$;

c. $74.91 m/s$

125. $x = 4.965$

Additional Problems

127. $7.124 \times 10^{16} W/m^3$

129. 1.034 eV

131. 5.93×10^{18}

133. 387.8 nm

135. a. 4.02×10^{15};

b. 0.533 mW

137. a. 4.02×10^{15};

b. 0.533 mW;

c. 0.644 mA;

d. 2.57 ns

139. a. 0.132 pm;

b. 9.39 MeV;

c. 0.047 MeV

141. a. 2 kJ;

b. $1.33 \times 10^{-5} kg \cdot m/s$;

c. $1.33 \times 10^{-5} N$;

d. yes

143. a. 0.003 nm;

b. 105.56°

145. $n=3$

147. a. $a_0/2$;

b. $-54.4eV/n^2$;

c. $a_0/3, -122.4eV/n^2$

149. a. 36;

b. 18.2 nm;

c. UV

151. 396 nm; 5.23 neV

153. 7.3 keV

155. 728 m/s; $1.5\mu V$

157. $\lambda = hc/\sqrt{K(2E_0+K)} = 3.705nm, K = 100keV$

159. $\Delta\lambda_c^{(electron)}/\Delta\lambda^{(proton)}c_{=}m_p/m_e = 1836$

161. (Proof)

163. $5.1\times10^{17}Hz.$

3.E: Photons and Matter Waves (Exercise)

Conceptual Questions

6.1 Blackbody Radiation

1. Which surface has a higher temperature – the surface of a yellow star or that of a red star?

2. Describe what you would see when looking at a body whose temperature is increased from 1000 K to 1,000,000 K.

3. Explain the color changes in a hot body as its temperature is increased.

4. Speculate as to why UV light causes sunburn, whereas visible light does not.

5. Two cavity radiators are constructed with walls made of different metals. At the same temperature, how would their radiation spectra differ?

6. Discuss why some bodies appear black, other bodies appear red, and still other bodies appear white.

7. If everything radiates electromagnetic energy, why can we not see objects at room temperature in a dark room?

8. How much does the power radiated by a blackbody increase when its temperature (in K) is tripled?

6.2 Photoelectric Effect

9. For the same monochromatic light source, would the photoelectric effect occur for all metals?

10. In the interpretation of the photoelectric effect, how is it known that an electron does not absorb more than one photon?

11. Explain how you can determine the work function from a plot of the stopping potential versus the frequency of the incident radiation in a

photoelectric effect experiment. Can you determine the value of Planck's constant from this plot?

12. Suppose that in the photoelectric-effect experiment we make a plot of the detected current versus the applied potential difference. What information do we obtain from such a plot? Can we determine from it the value of Planck's constant? Can we determine the work function of the metal?

13. Speculate how increasing the temperature of a photoelectrode affects the outcomes of the photoelectric effect experiment.

14. Which aspects of the photoelectric effect cannot be explained by classical physics?

15. Is the photoelectric effect a consequence of the wave character of radiation or is it a consequence of the particle character of radiation? Explain briefly.

16. The metals sodium, iron, and molybdenum have work functions 2.5 eV, 3.9 eV, and 4.2 eV, respectively. Which of these metals will emit photoelectrons when illuminated with 400 nm light?

6.3 The Compton Effect

17. Discuss any similarities and differences between the photoelectric and the Compton effects.

18. Which has a greater momentum: an UV photon or an IR photon?

19. Does changing the intensity of a monochromatic light beam affect the momentum of the individual photons in the beam? Does such a change affect the net momentum of the beam?

20. Can the Compton effect occur with visible light? If so, will it be detectable?

21. Is it possible in the Compton experiment to observe scattered X-rays that have a shorter wavelength than the incident X-ray radiation?

22. Show that the Compton wavelength has the dimension of length.

23. At what scattering angle is the wavelength shift in the Compton effect equal to the Compton wavelength?

6.4 Bohr’s Model of the Hydrogen Atom

24. Explain why the patterns of bright emission spectral lines have an identical spectral position to the pattern of dark absorption spectral lines for a given gaseous element.

25. Do the various spectral lines of the hydrogen atom overlap?

26. The Balmer series for hydrogen was discovered before either the Lyman or the Paschen series. Why?

27. When the absorption spectrum of hydrogen at room temperature is analyzed, absorption lines for the Lyman series are found, but none are found for the Balmer series. What does this tell us about the energy state of most hydrogen atoms at room temperature?

28. Hydrogen accounts for about 75% by mass of the matter at the surfaces of most stars. However, the absorption lines of hydrogen are strongest (of highest intensity) in the spectra of stars with a surface temperature of about 9000 K. They are weaker in the sun spectrum and are essentially nonexistent in very hot (temperatures above 25,000 K) or rather cool (temperatures below 3500 K) stars. Speculate as to why surface temperature affects the hydrogen absorption lines that we observe.

29. Discuss the similarities and differences between Thomson’s model of the hydrogen atom and Bohr’s model of the hydrogen atom.

30. Discuss the way in which Thomson’s model is nonphysical. Support your argument with experimental evidence.

31. If, in a hydrogen atom, an electron moves to an orbit with a larger radius, does the energy of the hydrogen atom increase or decrease?

32. How is the energy conserved when an atom makes a transition from a higher to a lower energy state?

33. Suppose an electron in a hydrogen atom makes a transition from the **(n+1)**th orbit to the nth orbit. Is the wavelength of the emitted photon longer for larger values of **n**, or for smaller values of **n**?

34. Discuss why the allowed energies of the hydrogen atom are negative.

35. Can a hydrogen atom absorb a photon whose energy is greater than 13.6 eV?

36. Why can you see through glass but not through wood?

37. Do gravitational forces have a significant effect on atomic energy levels?

38. Show that Planck’s constant has the dimensions of angular momentum.

6.5 De Broglie’s Matter Waves

39. Which type of radiation is most suitable for the observation of diffraction patterns on crystalline solids; radio waves, visible light, or X-rays? Explain.

40. Speculate as to how the diffraction patterns of a typical crystal would be affected if $\gamma - rays$ were used instead of X-rays.

41. If an electron and a proton are traveling at the same speed, which one has the shorter de Broglie wavelength?

42. If a particle is accelerating, how does this affect its de Broglie wavelength?

43. Why is the wave-like nature of matter not observed every day for macroscopic objects?

44. What is the wavelength of a neutron at rest? Explain.

45. Why does the setup of Davisson–Germer experiment need to be enclosed in a vacuum chamber? Discuss what result you expect when the chamber is not evacuated.

6.6 Wave-Particle Duality

46. Give an example of an experiment in which light behaves as waves. Give an example of an experiment in which light behaves as a stream of photons.

47. Discuss: How does the interference of water waves differ from the interference of electrons? How are they analogous?

48. Give at least one argument in support of the matter-wave hypothesis.

49. Give at least one argument in support of the particle-nature of radiation.

50. Explain the importance of the Young double-slit experiment.

51. Does the Heisenberg uncertainty principle allow a particle to be at rest in a designated region in space?

52. Can the de Broglie wavelength of a particle be known exactly?

53. Do the photons of red light produce better resolution in a microscope than blue light photons? Explain.

54. Discuss the main difference between an SEM and a TEM.

Problems

6.1 Blackbody Radiation

55. A 200-W heater emits a 1.5-µm radiation.

(a) What value of the energy quantum does it emit?

(b) Assuming that the specific heat of a 4.0-kg body is $0.83 kcal/kg \cdot K$, how many of these photons must be absorbed by the body to increase its temperature by 2 K?

(c) How long does the heating process in (b) take, assuming that all radiation emitted by the heater gets absorbed by the body?

56. A 900-W microwave generator in an oven generates energy quanta of frequency 2560 MHz.

(a) How many energy quanta does it emit per second?

(b) How many energy quanta must be absorbed by a pasta dish placed in the radiation cavity to increase its temperature by 45.0 K? Assume that the dish has a mass of 0.5 kg and that its specific heat is $0.9 kcal/kg \cdot K$.

(c) Assume that all energy quanta emitted by the generator are absorbed by the pasta dish. How long must we wait until the dish in (b) is ready?

57. (a) For what temperature is the peak of blackbody radiation spectrum at 400 nm?

(b) If the temperature of a blackbody is 800 K, at what wavelength does it radiate the most energy?

58. The tungsten elements of incandescent light bulbs operate at 3200 K. At what wavelength does the filament radiate maximum energy?

59. Interstellar space is filled with radiation of wavelength $970 \mu m$. This radiation is considered to be a remnant of the "big bang." What is the corresponding blackbody temperature of this radiation?

60. The radiant energy from the sun reaches its maximum at a wavelength of about 500.0 nm. What is the approximate temperature of the sun's surface?

6.2 Photoelectric Effect

61. A photon has energy 20 keV. What are its frequency and wavelength?

62. The wavelengths of visible light range from approximately 400 to 750 nm. What is the corresponding range of photon energies for visible light?

63. What is the longest wavelength of radiation that can eject a photoelectron from silver? Is it in the visible range?

64. What is the longest wavelength of radiation that can eject a photoelectron from potassium, given the work function of potassium 2.24 eV? Is it in the visible range?

65. Estimate the binding energy of electrons in magnesium, given that the wavelength of 337 nm is the longest wavelength that a photon may have to eject a photoelectron from magnesium photoelectrode.

66. The work function for potassium is 2.26 eV. What is the cutoff frequency when this metal is used as photoelectrode? What is the stopping potential when for the emitted electrons when this photoelectrode is exposed to radiation of frequency 1200 THz?

67. Estimate the work function of aluminum, given that the wavelength of 304 nm is the longest wavelength that a photon may have to eject a photoelectron from aluminum photoelectrode.

68. What is the maximum kinetic energy of photoelectrons ejected from sodium by the incident radiation of wavelength 450 nm?

69. A 120-nm UV radiation illuminates a silver-plated electrode. What is the maximum kinetic energy of the ejected photoelectrons?

70. A 400-nm violet light ejects photoelectrons with a maximum kinetic energy of 0.860 eV from sodium photoelectrode. What is the work function of sodium?

71. A 600-nm light falls on a photoelectric surface and electrons with the maximum kinetic energy of 0.17 eV are emitted. Determine

(a) the work function and

(b) the cutoff frequency of the surface.

(c) What is the stopping potential when the surface is illuminated with light of wavelength 400 nm?

72. The cutoff wavelength for the emission of photoelectrons from a particular surface is 500 nm. Find the maximum kinetic energy of the ejected photoelectrons when the surface is illuminated with light of wavelength 600 nm.

73. Find the wavelength of radiation that can eject 2.00-eV electrons from calcium electrode. The work function for calcium is 2.71 eV. In what range is this radiation?

74. Find the wavelength of radiation that can eject 0.10-eV electrons from potassium electrode. The work function for potassium is 2.24 eV. In what range is this radiation?

75. Find the maximum velocity of photoelectrons ejected by an 80-nm radiation, if the work function of photoelectrode is 4.73 eV.

6.3 The Compton Effect

76. What is the momentum of a 589-nm yellow photon?

77. What is the momentum of a 4-cm microwave photon?

78. In a beam of white light (wavelengths from 400 to 750 nm), what range of momentum can the photons have?

79. What is the energy of a photon whose momentum is $3.0 \times 10^{-24} kg \cdot m/s$?

80. What is the wavelength of

(a) a 12-keV X-ray photon;

(b) a 2.0-MeV γ-ray photon?

81. Find the momentum and energy of a 1.0-Å photon.

82. Find the wavelength and energy of a photon with momentum $5.00 \times 10^{-29} kg \cdot m/s$.

83. A γ-ray photon has a momentum of $8.00 \times 10^{-21} kg \cdot m/s$. Find its wavelength and energy.

84. (a) Calculate the momentum of a $2.5 - \mu m$ photon.

(b) Find the velocity of an electron with the same momentum.

(c) What is the kinetic energy of the electron, and how does it compare to that of the photon?

85. Show that $p = h/\lambda$ and $E_f = hf$ are consistent with the relativistic formula $E^2 = p^2c^2 + m_0^2c^2$.

86. Show that the energy E in eV of a photon is given by $E = 1.241 \times 10^{-6} eV \cdot m/\lambda$, where λ is its wavelength in meters.

87. For collisions with free electrons, compare the Compton shift of a photon scattered as an angle of 30° to that of a photon scattered at 45°.

88. X-rays of wavelength 12.5 pm are scattered from a block of carbon. What are the wavelengths of photons scattered at

(a) 30°;

(b) 90°; and,

(c) 180°?

6.4 Bohr's Model of the Hydrogen Atom

89. Calculate the wavelength of the first line in the Lyman series and show that this line lies in the ultraviolet part of the spectrum.

90. Calculate the wavelength of the fifth line in the Lyman series and show that this line lies in the ultraviolet part of the spectrum.

91. Calculate the energy changes corresponding to the transitions of the hydrogen atom:

(a) from $n=3$ to $n=4$;

(b) from $n=2$ to $n=1$; and

(c) from $n=3$ to $n=\infty$.

92. Determine the wavelength of the third Balmer line (transition from $n=5$ to $n=2$).

93. What is the frequency of the photon absorbed when the hydrogen atom makes the transition from the ground state to the $n=4$ state?

94. When a hydrogen atom is in its ground state, what are the shortest and longest wavelengths of the photons it can absorb without being ionized?

95. When a hydrogen atom is in its third excided state, what are the shortest and longest wavelengths of the photons it can emit?

96. What is the longest wavelength that light can have if it is to be capable of ionizing the hydrogen atom in its ground state?

97. For an electron in a hydrogen atom in the $n=2$ state, compute:

(a) the angular momentum;

(b) the kinetic energy;

(c) the potential energy; and

(d) the total energy.

98. Find the ionization energy of a hydrogen atom in the fourth energy state.

99. It has been measured that it required 0.850 eV to remove an electron from the hydrogen atom. In what state was the atom before the ionization happened?

100. What is the radius of a hydrogen atom when the electron is in the first excited state?

101. Find the shortest wavelength in the Balmer series. In what part of the spectrum does this line lie?

102. Show that the entire Paschen series lies in the infrared part of the spectrum.

103. Do the Balmer series and the Lyman series overlap? Why? Why not? (Hint: calculate the shortest Balmer line and the longest Lyman line.)

104. (a) Which line in the Balmer series is the first one in the UV part of the spectrum?

(b) How many Balmer lines lie in the visible part of the spectrum?

(c) How many Balmer lines lie in the UV?

105. A $4.653-\mu m$ emission line of atomic hydrogen corresponds to transition between the states $n_f=5$ and n_i. Find n_i.

6.5 De Broglie's Matter Waves

106. At what velocity will an electron have a wavelength of 1.00 m?

107. What is the de Broglie wavelength of an electron travelling at a speed of $5.0\times10^{6} m/s$?

108. What is the de Broglie wavelength of an electron that is accelerated from rest through a potential difference of 20 kV?

109. What is the de Broglie wavelength of a proton whose kinetic energy is 2.0 MeV? 10.0 MeV?

110. What is the de Broglie wavelength of a 10-kg football player running at a speed of 8.0 m/s?

111. (a) What is the energy of an electron whose de Broglie wavelength is that of a photon of yellow light with wavelength 590 nm?

(b) What is the de Broglie wavelength of an electron whose energy is that of the photon of yellow light?

112. The de Broglie wavelength of a neutron is 0.01 nm. What is the speed and energy of this neutron?

113. What is the wavelength of an electron that is moving at a 3% of the speed of light?

114. At what velocity does a proton have a 6.0-fm wavelength (about the size of a nucleus)? Give your answer in units of **c**.

115. What is the velocity of a 0.400-kg billiard ball if its wavelength is 7.50 fm?

116. Find the wavelength of a proton that is moving at 1.00% of the speed of light (when $\beta = 0.01$).

6.6 Wave-Particle Duality

117. An AM radio transmitter radiates 500 kW at a frequency of 760 kHz. How many photons per second does the emitter emit?

118. Find the Lorentz factor γ and de Broglie's wavelength for a 50-GeV electron in a particle accelerator.

119. Find the Lorentz factor γ and de Broglie's wavelength for a 1.0-TeV proton in a particle accelerator.

120. What is the kinetic energy of a 0.01-nm electron in a TEM?

121. If electron is to be diffracted significantly by a crystal, its wavelength must be about equal to the spacing, **d**, of crystalline planes. Assuming $d = 0.250 nm$, estimate the potential difference through which an electron must be accelerated from rest if it is to be diffracted by these planes.

122. X-rays form ionizing radiation that is dangerous to living tissue and undetectable to the human eye. Suppose that a student researcher working in an X-ray diffraction laboratory is accidentally exposed to a fatal dose of radiation. Calculate the temperature increase of the researcher under the following conditions: the energy of X-ray photons is 200 keV and the researcher absorbs 4×10^{13} photons per each kilogram of body weight during the exposure. Assume that the specific heat of the student's body is $0.83 kcal/kg \cdot K$.

123. Solar wind (radiation) that is incident on the top of Earth's atmosphere has an average intensity of $1.3 kW/m^2$. Suppose that you are building a solar sail that is to propel a small toy spaceship with a mass of 0.1 kg in the space between the International Space Station and the moon. The sail is made from a very light material, which perfectly reflects the incident radiation. To assess whether such a project is feasible, answer the following questions, assuming that radiation photons are incident only in normal direction to the sail reflecting surface.

(a) What is the radiation pressure (force per m^2) of the radiation falling on the mirror-like sail?

(b) Given the radiation pressure computed in (a), what will be the acceleration of the spaceship when the sail has of an area of $10.0 m^2$?

(c) Given the acceleration estimate in (b), how fast will the spaceship be moving after 24 hours when it starts from rest?

124. Treat the human body as a blackbody and determine the percentage increase in the total power of its radiation when its temperature increases from 98.6° F to 103° F.

125. Show that Wien's displacement law results from Planck's radiation law. (Hint: substitute $x = hc/\lambda kT$ and write Planck's law in the form $I(x,T) = Ax^5/(e^x - 1)$, where $A = 2\pi (kT)^5/(h^4c^3)$. Now, for fixed **T**, find the position of the maximum in **I(x,T)** by solving for **x** in the equation $dI(x,T)/dx = 0$.)

126. Show that Stefan's law results from Planck's radiation law. **Hint:** To compute the total power of blackbody radiation emitted across the entire spectrum of wavelengths at a given temperature, integrate Planck's law over the entire spectrum

$P(T) = \int_0^\infty I(\lambda, T) d\lambda$. Use the substitution $x = hc/\lambda kT$ and the tabulated value of the integral $\int_0^\infty dx x^3/(e^x - 1) = \pi^4/15$.

Additional Problems

127. Determine the power intensity of radiation per unit wavelength emitted at a wavelength of 500.0 nm by a blackbody at a temperature of 10,000 K.

128. The HCl molecule oscillates at a frequency of 87.0 THz. What is the difference (in eV) between its adjacent energy levels?

129. A quantum mechanical oscillator vibrates at a frequency of 250.0 THz. What is the minimum energy of radiation it can emit?

130. In about 5 billion years, the sun will evolve to a red giant. Assume that its surface temperature will decrease to about half its present value of 6000 K, while its present radius of $7.0 \times 10^8 m$ will increase to $1.5 \times 10^{11} m$ (which is the current Earth-sun distance). Calculate the ratio of the total power emitted by the sun in its red giant stage to its present power.

131. A sodium lamp emits 2.0 W of radiant energy, most of which has a wavelength of about 589 nm. Estimate the number of photons emitted per second by the lamp.

132. Photoelectrons are ejected from a photoelectrode and are detected at a distance of 2.50 cm away from the photoelectrode. The work function of the photoelectrode is 2.71 eV and the incident radiation has a wavelength of 420 nm. How long does it take a photoelectron to travel to the detector?

133. If the work function of a metal is 3.2 eV, what is the maximum wavelength that a photon can have to eject a photoelectron from this metal surface?

134. The work function of a photoelectric surface is 2.00 eV. What is the maximum speed of the photoelectrons emitted from this surface when a 450-nm light falls on it?

135. A 400-nm laser beam is projected onto a calcium electrode. The power of the laser beam is 2.00 mW and the work function of calcium is 2.31 eV.

(a) How many photoelectrons per second are ejected?

(b) What net power is carried away by photoelectrons?

136. (a) Calculate the number of photoelectrons per second that are ejected from a $1.00 - mm^2$ area of sodium metal by a 500-nm radiation with intensity $1.30 kW/m^2$ (the intensity of sunlight above Earth's atmosphere).

(b) Given the work function of the metal as 2.28 eV, what power is carried away by these photoelectrons?

137. A laser with a power output of 2.00 mW at a 400-nm wavelength is used to project a beam of light onto a calcium photoelectrode. (a) How many photoelectrons leave the calcium surface per second? (b) What power is carried away by ejected photoelectrons, given that the work function of calcium is 2.31 eV? (c) Calculate the photocurrent. (d) If the photoelectrode suddenly becomes electrically insulated and the setup of two electrodes in the circuit suddenly starts to act like a 2.00-pF capacitor, how long will current flow before the capacitor voltage stops it?

138. The work function for barium is 2.48 eV. Find the maximum kinetic energy of the ejected photoelectrons when the barium surface is illuminated with:

(a) radiation emitted by a 100-kW radio station broadcasting at 800 kHz;

(b) a 633-nm laser light emitted from a powerful He-Ne laser; and

(c) a 434-nm blue light emitted by a small hydrogen gas discharge tube.

139. (a) Calculate the wavelength of a photon that has the same momentum as a proton moving with 1% of the speed of light in a vacuum.

(b) What is the energy of this photon in MeV?

(c) What is the kinetic energy of the proton in MeV?

140. (a) Find the momentum of a 100-keV X-ray photon.

(b) Find the velocity of a neutron with the same momentum.

(c) What is the neutron's kinetic energy in eV?

141. The momentum of light, as it is for particles, is exactly reversed when a photon is reflected straight back from a mirror, assuming negligible recoil of the mirror. The change in momentum is twice the photon's incident momentum, as it is for the particles. Suppose that a beam of light has an intensity $1.0kW/m^2$ and falls on a $-2.0-m^2$ area of a mirror and reflects from it.

(a) Calculate the energy reflected in 1.00 s.

(b) What is the momentum imparted to the mirror?

(c) Use Newton's second law to find the force on the mirror.

(d) Does the assumption of no-recoil for the mirror seem reasonable?

142. A photon of energy 5.0 keV collides with a stationary electron and is scattered at an angle of 60°. What is the energy acquired by the electron in the collision?

143. A 0.75-nm photon is scattered by a stationary electron. The speed of the electron's recoil is $1.5\times10^6 m/s$.

(a) Find the wavelength shift of the photon.

(b) Find the scattering angle of the photon.

144. Find the maximum change in X-ray wavelength that can occur due to Compton scattering. Does this change depend on the wavelength of the incident beam?

145. A photon of wavelength 700 nm is incident on a hydrogen atom. When this photon is absorbed, the atom becomes ionized. What is the lowest possible orbit that the electron could have occupied before being ionized?

146. What is the maximum kinetic energy of an electron such that a collision between the electron and a stationary hydrogen atom in its ground state is definitely elastic?

147. Singly ionized atomic helium He^{+1} is a hydrogen-like ion.

(a) What is its ground-state radius?

(b) Calculate the energies of its four lowest energy states.

(c) Repeat the calculations for the Li^{2+} ion.

148. A triply ionized atom of beryllium Be^{3+} is a hydrogen-like ion. When Be^{3+} is in one of its excited states, its radius in this nth state is exactly the same as the radius of the first Bohr orbit of hydrogen. Find **n** and compute the ionization energy for this state of Be^{3+}.

149. In extreme-temperature environments, such as those existing in a solar corona, atoms may be ionized by undergoing collisions with other atoms. One example of such ionization in the solar corona is the presence of C^{5+} ions, detected in the Fraunhofer spectrum.

(a) By what factor do the energies of the C^{5+} ion scale compare to the energy spectrum of a hydrogen atom?

(b) What is the wavelength of the first line in the Paschen series of C^{5+}?

(c) In what part of the spectrum are these lines located?

150. (a) Calculate the ionization energy for He^+.

(b) What is the minimum frequency of a photon capable of ionizing He^+?

151. Experiments are performed with ultra cold neutrons having velocities as small as 1.00 m/s. Find the wavelength of such an ultracold neutron and its kinetic energy.

152. Find the velocity and kinetic energy of a 6.0-fm neutron. (Rest mass energy of neutron is $E_0=940MeV$.)

153. The spacing between crystalline planes in the NaCl crystal is 0.281 nm, as determined by X-ray diffraction with X-rays of wavelength 0.170 nm. What is the energy of neutrons in the neutron beam that produces diffraction peaks at the same locations as the peaks obtained with the X-rays?

154. What is the wavelength of an electron accelerated from rest in a 30.0-kV potential difference?

155. Calculate the velocity of a $1.0-\mu m$ electron and a potential difference used to accelerate it from rest to this velocity.

156. In a supercollider at CERN, protons are accelerated to velocities of 0.25**c**. What are their wavelengths at this speed? What are their kinetic energies? If a beam of protons were to gain its kinetic energy in only one pass through a potential difference, how high would this potential difference have to be? (Rest mass energy of a proton is $E_0 = 938 MeV$).

157. Find the de Broglie wavelength of an electron accelerated from rest in an X-ray tube in the potential difference of 100 keV. (Rest mass energy of an electron is $E_0 = 511 keV$.)

158. The cutoff wavelength for the emission of photoelectrons from a particular surface is 500 nm. Find the maximum kinetic energy of the ejected photoelectrons when the surface is illuminated with light of wavelength 450 nm.

159. Compare the wavelength shift of a photon scattered by a free electron to that of a photon scattered at the same angle by a free proton.

160. The spectrometer used to measure the wavelengths of the scattered X-rays in the Compton experiment is accurate to $5.0 \times 10^{-4} nm$. What is the minimum scattering angle for which the X-rays interacting with the free electrons can be distinguished from those interacting with the atoms?

161. Consider a hydrogen-like ion where an electron is orbiting a nucleus that has charge $q = +Ze$. Derive the formulas for the energy E_n of the electron in nth orbit and the orbital radius r_n.

162. Assume that a hydrogen atom exists in the $n = 2$ excited state for $10^{-8} s$ before decaying to the ground state. How many times does the electron orbit the proton nucleus during this time? How long does it take Earth to orbit the sun this many times?

163. An atom can be formed when a negative muon is captured by a proton. The muon has the same charge as the electron and a mass 207 times that of the electron. Calculate the frequency of the photon emitted when this atom makes the transition from $n = 2$ to the $n = 1$ state. Assume that the muon is orbiting a stationary proton.

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3.S: Photons and Matter Waves (Summary)

Key Terms

absorber	any object that absorbs radiation
absorption spectrum	wavelengths of absorbed radiation by atoms and molecules
Balmer formula	describes the emission spectrum of a hydrogen atom in the visible-light range
Balmer series	spectral lines corresponding to electron transitions to/from the $n=2$ state of the hydrogen atom, described by the Balmer formula
blackbody	perfect absorber/emitter
blackbody radiation	radiation emitted by a blackbody
Bohr radius of hydrogen	radius of the first Bohr's orbit
Bohr's model of the hydrogen atom	first quantum model to explain emission spectra of hydrogen
Brackett series	spectral lines corresponding to electron transitions to/from the $n=4$ state
Compton effect	the change in wavelength when an X-ray is scattered by its interaction with some materials
Compton shift	difference between the wavelengths of the incident X-ray and the scattered X-ray
Compton wavelength	physical constant with the value $\lambda_c = 2.43 pm$
cut-off frequency	frequency of incident light below which the photoelectric effect does not occur
cut-off wavelength	wavelength of incident light that corresponds to cut-off frequency
Davisson–Germer experiment	historically first electron-diffraction experiment that revealed electron waves
de Broglie wave	matter wave associated with any object that has mass and momentum
de Broglie's hypothesis of matter waves	particles of matter can behave like waves
double-slit interference experiment	Young's double-slit experiment, which shows the interference of waves
electron microscopy	microscopy that uses electron waves to "see" fine details of nano-size objects
emission spectrum	wavelengths of emitted radiation by atoms and molecules
emitter	any object that emits radiation
energy of a photon	quantum of radiant energy, depends only on a photon's frequency
energy spectrum of hydrogen	set of allowed discrete energies of an electron in a hydrogen atom
excited energy states of the H atom	energy state other than the ground state
Fraunhofer lines	dark absorption lines in the continuum solar emission spectrum
ground state energy of the hydrogen atom	energy of an electron in the first Bohr orbit of the hydrogen atom

group velocity	velocity of a wave, energy travels with the group velocity
Heisenberg uncertainty principle	sets the limits on precision in simultaneous measurements of momentum and position of a particle
Humphreys series	spectral lines corresponding to electron transitions to/from the $n=6$ state
hydrogen-like atom	ionized atom with one electron remaining and nucleus with charge $+Ze$
inelastic scattering	scattering effect where kinetic energy is not conserved but the total energy is conserved
ionization energy	energy needed to remove an electron from an atom
ionization limit of the hydrogen atom	ionization energy needed to remove an electron from the first Bohr orbit
Lyman series	spectral lines corresponding to electron transitions to/from the ground state
nuclear model of the atom	heavy positively charged nucleus at the center is surrounded by electrons, proposed by Rutherford
Paschen series	spectral lines corresponding to electron transitions to/from the $n=3$ state
Pfund series	spectral lines corresponding to electron transitions to/from the $n=5$ state
photocurrent	in a circuit, current that flows when a photoelectrode is illuminated
photoelectric effect	emission of electrons from a metal surface exposed to electromagnetic radiation of the proper frequency
photoelectrode	in a circuit, an electrode that emits photoelectrons
photoelectron	electron emitted from a metal surface in the presence of incident radiation
photon	particle of light
Planck's hypothesis of energy quanta	energy exchanges between the radiation and the walls take place only in the form of discrete energy quanta
postulates of Bohr's model	three assumptions that set a frame for Bohr's model
power intensity	energy that passes through a unit surface per unit time
propagation vector	vector with magnitude $2\pi/\lambda$ that has the direction of the photon's linear momentum
quantized energies	discrete energies; not continuous
quantum number	index that enumerates energy levels
quantum phenomenon	in interaction with matter, photon transfers either all its energy or nothing
quantum state of a Planck's oscillator	any mode of vibration of Planck's oscillator, enumerated by quantum number
reduced Planck's constant	Planck's constant divided by 2π
Rutherford's gold foil experiment	first experiment to demonstrate the existence of the atomic nucleus

Rydberg constant for hydrogen	physical constant in the Balmer formula
Rydberg formula	experimentally found positions of spectral lines of hydrogen atom
scattering angle	angle between the direction of the scattered beam and the direction of the incident beam
Stefan–Boltzmann constant	physical constant in Stefan’s law
stopping potential	in a circuit, potential difference that stops photocurrent
wave number	magnitude of the propagation vector
wave quantum mechanics	theory that explains the physics of atoms and subatomic particles
wave-particle duality	particles can behave as waves and radiation can behave as particles
work function	energy needed to detach photoelectron from the metal surface
α-particle	doubly ionized helium atom
α-ray	beam of α-particles (alpha-particles)
β-ray	beam of electrons
γ-ray	beam of highly energetic photons

Key Equations

Wien’s displacement law	$\lambda_{max}T = 2.898 \times 10^{-3} m \cdot K$
Stefan’s law	$P(T) = \sigma A T^4$
Planck’s constant	$h = 6.626 \times 10-^{34} J \cdot s = 4.136 \times 10^{-15} eV \cdot s$
Energy quantum of radiation	$\Delta E = hf$
Planck’s blackbody radiation law	$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B^T} - 1}$
Maximum kinetic energy of a photoelectron	$K_{max} = e\Delta V_s$
Energy of a photon	$E_f = hf$
Energy balance for photoelectron	$K_{max} = hf - \phi$
Cut-off frequency	$f_c = \frac{\phi}{h}$
Relativistic invariant energy equation	$E^2 = p^2c^2 + m_0^2c^4$
Energy-momentum relation for photon	$p_f = \frac{E_f}{c}$
Energy of a photon	$E_f = hf = \frac{hc}{\lambda}$
Magnitude of photon’s momentum	$p_f = \frac{h}{\lambda}$
Photon’s linear momentum vector	$\vec{p_f} = \hbar\vec{k}$
The Compton wavelength of an electron	$\lambda_c = \frac{h}{m_0 c} = 0.00243 nm$
The Compton shift	$\Delta\lambda = \lambda_c(1 - cos\theta)$
The Balmer formula	$\frac{1}{\lambda} = R_H(\frac{1}{2^2} - \frac{1}{n^2})$

The Rydberg formula	$\frac{1}{\lambda} = R_H(\frac{1}{n_f^2} - \frac{1}{n_i^2}), n_i = n_f + 1, n_f + 2, \ldots$
Bohr's first quantization condition	$L_n = n\hbar, n = 1, 2, \ldots$
Bohr's second quantization condition	$h_f = \lvert E_n - E_m \rvert$
Bohr's radius of hydrogen	$a_0 = 4\pi\varepsilon 0 \frac{\hbar^2}{m_e e^2} = 0.529\text{Å}$
Bohr's radius of the *n*th orbit	$r_n = a_0 n^2$
Ground-state energy value, ionization limit	$E_0 = \frac{1}{8\varepsilon_0^2} \frac{m_e e^4}{h^2} = 13.6eV$
Electron's energy in the *n*th orbit	$E_n = -E_0 \frac{1}{n^2}$
Ground state energy of hydrogen	$E_1 = -E_0 = -13.6eV$
The *n*th orbit of hydrogen-like ion	$r_n = \frac{a_0}{Z} n^2$
The *n*th energy of hydrogen-like ion	$E_n = -Z^2 E_0 \frac{1}{n^2}$
Energy of a matter wave	$E = hf$
The de Broglie wavelength	$\lambda = \frac{h}{p}$
The frequency-wavelength relation for matter waves	$\lambda f = \frac{c}{\beta}$
Heisenberg's uncertainty principle	$\Delta x \Delta p \geq \frac{1}{2}\hbar$

Summary

6.1 Blackbody Radiation

- All bodies radiate energy. The amount of radiation a body emits depends on its temperature. The experimental Wien's displacement law states that the hotter the body, the shorter the wavelength corresponding to the emission peak in the radiation curve. The experimental Stefan's law states that the total power of radiation emitted across the entire spectrum of wavelengths at a given temperature is proportional to the fourth power of the Kelvin temperature of the radiating body.
- Absorption and emission of radiation are studied within the model of a blackbody. In the classical approach, the exchange of energy between radiation and cavity walls is continuous. The classical approach does not explain the blackbody radiation curve.
- To explain the blackbody radiation curve, Planck assumed that the exchange of energy between radiation and cavity walls takes place only in discrete quanta of energy. Planck's hypothesis of energy quanta led to the theoretical Planck's radiation law, which agrees with the experimental blackbody radiation curve; it also explains Wien's and Stefan's laws.

6.2 Photoelectric Effect

- The photoelectric effect occurs when photoelectrons are ejected from a metal surface in response to monochromatic radiation incident on the surface. It has three characteristics: (1) it is instantaneous, (2) it occurs only when the radiation is above a cut-off frequency, and (3) kinetic energies of photoelectrons at the surface do not depend of the intensity of radiation. The photoelectric effect cannot be explained by classical theory.
- We can explain the photoelectric effect by assuming that radiation consists of photons (particles of light). Each photon carries a quantum of energy. The energy of a photon depends only on its frequency, which is the frequency of the radiation. At the surface, the entire energy of a photon is transferred to one photoelectron.
- The maximum kinetic energy of a photoelectron at the metal surface is the difference between the energy of the incident photon and the work function of the metal. The work function is the binding energy of electrons to the metal surface. Each metal has its own characteristic work function.

6.3 The Compton Effect

- In the Compton effect, X-rays scattered off some materials have different wavelengths than the wavelength of the incident X-rays. This phenomenon does not have a classical explanation.
- The Compton effect is explained by assuming that radiation consists of photons that collide with weakly bound electrons in the target material. Both electron and photon are treated as relativistic particles. Conservation laws of the total energy and of momentum are obeyed in collisions.
- Treating the photon as a particle with momentum that can be transferred to an electron leads to a theoretical Compton shift that agrees with the wavelength shift measured in the experiment. This provides evidence that radiation consists of photons.
- Compton scattering is an inelastic scattering, in which scattered radiation has a longer wavelength than that of incident radiation.

6.4 Bohr's Model of the Hydrogen Atom

- Positions of absorption and emission lines in the spectrum of atomic hydrogen are given by the experimental Rydberg formula. Classical physics cannot explain the spectrum of atomic hydrogen.
- The Bohr model of hydrogen was the first model of atomic structure to correctly explain the radiation spectra of atomic hydrogen. It was preceded by the Rutherford nuclear model of the atom. In Rutherford's model, an atom consists of a positively charged point-like nucleus that contains almost the entire mass of the atom and of negative electrons that are located far away from the nucleus.
- Bohr's model of the hydrogen atom is based on three postulates: (1) an electron moves around the nucleus in a circular orbit, (2) an electron's angular momentum in the orbit is quantized, and (3) the change in an electron's energy as it makes a quantum jump from one orbit to another is always accompanied by the emission or absorption of a photon. Bohr's model is semi-classical because it combines the classical concept of electron orbit (postulate 1) with the new concept of quantization (postulates 2 and 3).
- Bohr's model of the hydrogen atom explains the emission and absorption spectra of atomic hydrogen and hydrogen-like ions with low atomic numbers. It was the first model to introduce the concept of a quantum number to describe atomic states and to postulate quantization of electron orbits in the atom. Bohr's model is an important step in the development of quantum mechanics, which deals with many-electron atoms.

6.5 De Broglie's Matter Waves

- De Broglie's hypothesis of matter waves postulates that any particle of matter that has linear momentum is also a wave. The wavelength of a matter wave associated with a particle is inversely proportional to the magnitude of the particle's linear momentum. The speed of the matter wave is the speed of the particle.
- De Broglie's concept of the electron matter wave provides a rationale for the quantization of the electron's angular momentum in Bohr's model of the hydrogen atom.
- In the Davisson–Germer experiment, electrons are scattered off a crystalline nickel surface. Diffraction patterns of electron matter waves are observed. They are the evidence for the existence of matter waves. Matter waves are observed in diffraction experiments with various particles.

6.6 Wave-Particle Duality

- Wave-particle duality exists in nature: Under some experimental conditions, a particle acts as a particle; under other experimental conditions, a particle acts as a wave. Conversely, under some physical circumstances, electromagnetic radiation acts as a wave, and under other physical circumstances, radiation acts as a beam of photons.
- Modern-era double-slit experiments with electrons demonstrated conclusively that electron-diffraction images are formed because of the wave nature of electrons.
- The wave-particle dual nature of particles and of radiation has no classical explanation.
- Quantum theory takes the wave property to be the fundamental property of all particles. A particle is seen as a moving wave packet. The wave nature of particles imposes a limitation on the simultaneous measurement of the particle's position and momentum. Heisenberg's uncertainty principle sets the limits on precision in such simultaneous measurements.
- Wave-particle duality is exploited in many devices, such as charge-couple devices (used in digital cameras) or in the electron microscopy of the scanning electron microscope (SEM) and the transmission electron microscope (TEM).

CHAPTER OVERVIEW

4: Quantum Mechanics

Quantum mechanics is a powerful framework for understanding the motions and interactions of particles at small scales, such as atoms and molecules. The ideas behind quantum mechanics often appear quite strange. In many ways, our everyday experience with the macroscopic physical world does not prepare us for the microscopic world of quantum mechanics. The purpose of this chapter is to introduce you to this exciting world.

Thumbnail: Schrödinger took the absurd implications of this thought experiment (a cat simultaneously dead and alive) as an argument against the Copenhagen interpretation. However, this interpretation remains the most commonly taught view of quantum mechanics.

Contributors and Attributions

4.1: Prelude to Quantum Mechanics

Quantum mechanics is a powerful framework for understanding the motions and interactions of particles at small scales, such as atoms and molecules. The ideas behind quantum mechanics often appear quite strange. In many ways, our everyday experience with the macroscopic physical world does not prepare us for the microscopic world of quantum mechanics. The purpose of this chapter is to introduce you to this exciting world.

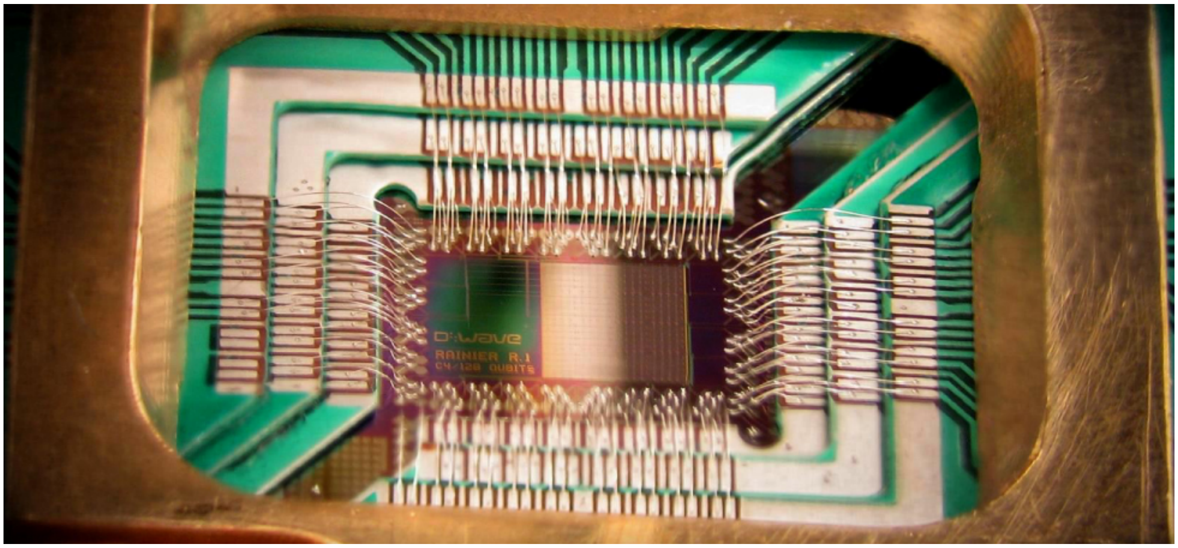


Figure 4.1.1: A D-wave qubit processor: The brain of a quantum computer that encodes information in quantum bits to perform complex calculations. (credit: modification of work by D-Wave Systems, Inc.)

Pictured above is a quantum-computer processor. This device is the “brain” of a quantum computer that operates at near-absolute zero temperatures. Unlike a digital computer, which encodes information in binary digits (definite states of either zero or one), a quantum computer encodes information in quantum bits or qubits (mixed states of zero and one). Quantum computers are discussed in the first section of this chapter.

4.2: Wave functions

Learning Objectives

By the end of this section, you will be able to:

- Describe the statistical interpretation of the wavefunction
- Use the wavefunction to determine probabilities
- Calculate expectation values of position, momentum, and kinetic energy

In the preceding chapter, we saw that particles act in some cases like particles and in other cases like waves. But what does it mean for a particle to "act like a wave"? What precisely is "waving"? What rules govern how this wave changes and propagates? How is the wavefunction used to make predictions? For example, if the amplitude of an electron wave is given by a function of position and time, $\Psi(x,t)$, defined for all **x**, **where** exactly is the electron? The purpose of this chapter is to answer these questions.

Using the Wavefunction

A clue to the physical meaning of the wavefunction $\Psi(x,t)$ is provided by the two-slit interference of monochromatic light (Figure $4.2.1$) that behave as electromagnetic waves. The wavefunction of a light wave is given by **E**(**x**,**t**), and its energy density is given by $|E|^2$, where **E** is the electric field strength. The energy of an individual photon depends only on the frequency of light, $\epsilon_{photon} = hf$, so $|E|^2$ is proportional to the number of photons. When light waves from S_1 interfere with light waves from S_2 at the viewing screen (a distance **D** away), an interference pattern is produced ($4.2.1a$). Bright fringes correspond to points of constructive interference of the light waves, and dark fringes correspond to points of destructive interference of the light waves ($4.2.1b$).

Suppose the screen is initially unexposed to light. If the screen is exposed to very weak light, the interference pattern appears gradually (Figure $4.2.1c$, left to right). Individual photon hits on the screen appear as dots. The dot density is expected to be large at locations where the interference pattern will be, ultimately, the most intense. In other words, the probability (per unit area) that a single photon will strike a particular spot on the screen is proportional to the square of the total electric field, $|E|^2$ at that point. Under the right conditions, the same interference pattern develops for matter particles, such as electrons.

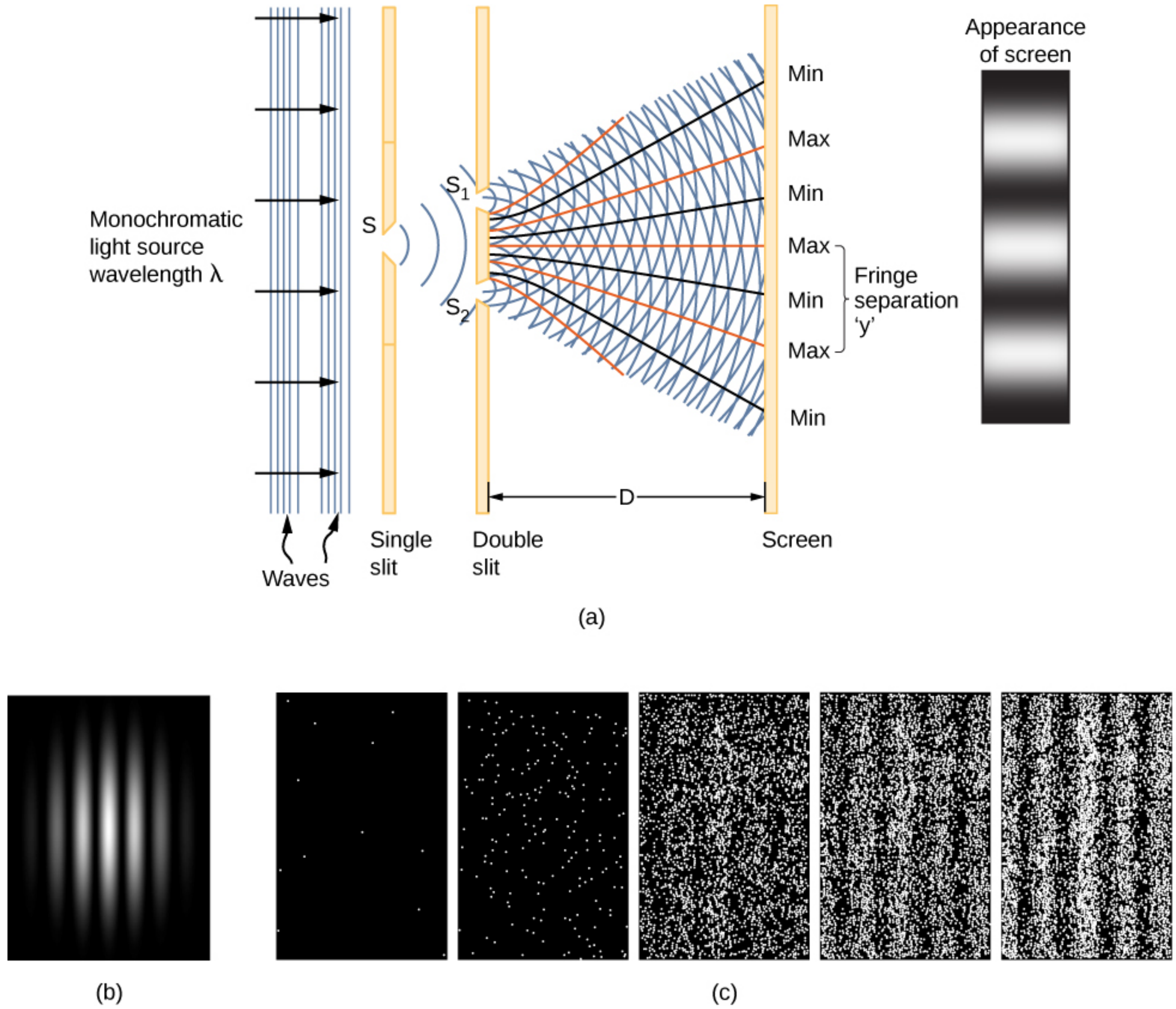


Figure 4.2.1: Two-slit interference of monochromatic light. (a) Schematic of two-slit interference; (b) light interference pattern; (c) interference pattern built up gradually under low-intensity light (left to right).

> **Note**
>
> Visit this interactive simulation to learn more about quantum wave interference.

The square of the matter wave $|\Psi|^2$ in one dimension has a similar interpretation as the square of the electric field $|E|^2$. It gives the probability that a particle will be found at a particular position and time per unit length, also called the **probability density**. The probability (P) a particle is found in a narrow interval (**x**, **x + dx**) at time **t** is therefore

$$P(x, x+dx) = |\Psi(x,t)|^2 dx. \tag{4.2.1}$$

(Later, we define the magnitude squared for the general case of a function with "imaginary parts.") This probabilistic interpretation of the wavefunction is called the **Born interpretation**. Examples of wavefunctions and their squares for a particular time t are given in Figure 4.2.2.

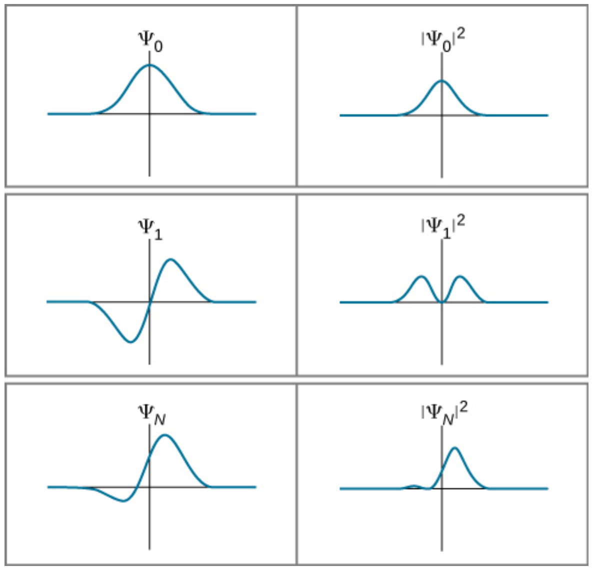


Figure 4.2.2: Several examples of wavefunctions and the corresponding square of their wavefunctions.

If the wavefunction varies slowly over the interval Δx, the probability a particle is found in the interval is approximately

$$P(x, x+\Delta x) \approx |\Psi(x,t)|^2 \delta x. \tag{4.2.2}$$

Notice that squaring the wavefunction ensures that the probability is positive. (This is analogous to squaring the electric field strength—which may be positive or negative—to obtain a positive value of intensity.) However, if the wavefunction does not vary slowly, we must integrate:

$$P(x, x+\Delta x) = \int_x^{x+\Delta x} |\Psi(x,t)|^2 dx. \tag{4.2.3}$$

This probability is just the area under the function $|\Psi(x,t)|^2$ between x and $x+\Delta x$. The probability of finding the particle "somewhere" (the normalization condition) is

$$P(-\infty, +\infty) = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1. \tag{4.2.4}$$

For a particle in two dimensions, the integration is over an area and requires a double integral; for a particle in three dimensions, the integration is over a volume and requires a triple integral. For now, we stick to the simple one-dimensional case.

Example $4.2.1A$: Where Is the Ball? (Part I)

A ball is constrained to move along a line inside a tube of length L. The ball is equally likely to be found anywhere in the tube at some time t. What is the probability of finding the ball in the left half of the tube at that time? (The answer is 50%, of course, but how do we get this answer by using the probabilistic interpretation of the quantum mechanical wavefunction?)

Strategy

The first step is to write down the wavefunction. The ball is equally like to be found anywhere in the box, so one way to describe the ball with a **constant** wavefunction (Figure $4.2.3$). The normalization condition can be used to find the value of the function and a simple integration over half of the box yields the final answer.

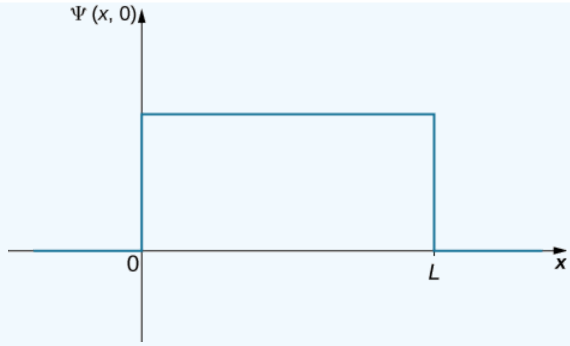


Figure 4.2.3: Wavefunction for a ball in a tube of length L.

Solution

The wavefunction of the ball can be written as $\Psi(x,t) = C(0<x<L)$, where C is a constant, and $\Psi(x,t) = 0$ otherwise. We can determine the constant **C** by applying the normalization condition (we set $t=0$ to simplify the notation):

$$P(\infty, +\infty) = \int_{-\infty}^{\infty} |C|^2 dx = 1.$$

This integral can be broken into three parts: (1) negative infinity to zero, (2) zero to **L**, and (3) **L** to infinity. The particle is constrained to be in the tube, so $C=0$ outside the tube and the first and last integrations are zero. The above equation can therefore be written

$$P(x=0, L) = \int_{0}^{L} |C|^2 dx = 1.$$

The value **C** does not depend on **x** and can be taken out of the integral, so we obtain

$$|C|^2 \int_0^L dx = 1.$$

Integration gives

$$C = \sqrt{\frac{1}{L}}.$$

To determine the probability of finding the ball in the first half of the box ($0<x<L$), we have

$$\begin{aligned} P(x=0, L/2) &= \int_0^{L/2} \left|\sqrt{\frac{1}{L}}\right|^2 dx \\ &= \left(\frac{1}{L}\right)\frac{L}{2} \\ &= 0.50. \end{aligned} \tag{4.2.5}$$

Significance

The probability of finding the ball in the first half of the tube is 50%, as expected. Two observations are noteworthy. First, this result corresponds to the area under the constant function from $x=0$ to $L/2$ (the area of a square left of **L**/2). Second, this calculation requires an integration of the **square** of the wavefunction. A common mistake in performing such calculations is to forget to square the wavefunction before integration.

Example 4.2.1B: Where Is the Ball? (Part II)

A ball is again constrained to move along a line inside a tube of length **L**. This time, the ball is found preferentially in the middle of the tube. One way to represent its wavefunction is with a simple cosine function (Figure 4.2.4). What is the probability of finding the ball in the last one-quarter of the tube?

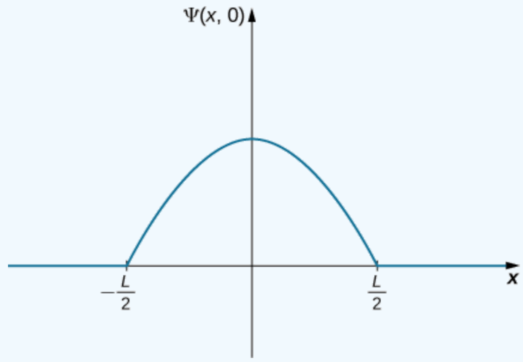


Figure 4.2.4: Wavefunction for a ball in a tube of length L, where the ball is preferentially in the middle of the tube.

Strategy

We use the same strategy as before. In this case, the wavefunction has two unknown constants: One is associated with the wavelength of the wave and the other is the amplitude of the wave. We determine the amplitude by using the boundary conditions of the problem, and we evaluate the wavelength by using the normalization condition. Integration of the square of the wavefunction over the last quarter of the tube yields the final answer. The calculation is simplified by centering our coordinate system on the peak of the wavefunction.

Solution

The wavefunction of the ball can be written

$$\Psi(x,t) = A\cos(kx)(-L/2 < x < L/2),$$

where A is the amplitude of the wavefunction and $k = 2\pi/\lambda$ is its **wavenumber**. Beyond this interval, the amplitude of the wavefunction is zero because the ball is confined to the tube. Requiring the wavefunction to terminate at the right end of the tube gives

$$\Psi\left(x = \frac{L}{2}, 0\right) = 0.$$

Evaluating the wavefunction at $x = L/2$ gives

$$A\cos(kL/2) = 0.$$

This equation is satisfied if the argument of the cosine is an integral multiple of $\pi/2$, $3\pi/2$, $5\pi/2$, and so on. In this case, we have

$$\frac{kL}{2} = \frac{\pi}{2},$$

or

$$k = \frac{\pi}{L}.$$

Applying the normalization condition gives $A = \sqrt{2/L}$, so the wavefunction of the ball is

$$\Psi(x,0) = \sqrt{\frac{2}{L}}\, cos(\pi x/L),\ -L/2 < x < L/2.$$

To determine the probability of finding the ball in the last quarter of the tube, we square the function and integrate:

$$P(x = L/4, L/2) = \int_{L/4}^{L/2} \left| \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) \right|^2 dx = 0.091.$$

Significance

The probability of finding the ball in the last quarter of the tube is 9.1%. The ball has a definite wavelength ($\lambda = 2L$). If the tube is of macroscopic length ($L = 1\, m$), the momentum of the ball is

$$p = \frac{h}{\lambda} = \frac{h}{2L} \approx 10^{-36} m/s.$$

This momentum is much too small to be measured by any human instrument.

An Interpretation of the Wavefunction

We are now in position to begin to answer the questions posed at the beginning of this section. First, for a traveling particle described by $\Psi(x,t) = A\, \sin(kx - \omega t)$, what is "waving?" Based on the above discussion, the answer is a mathematical function that can, among other things, be used to determine where the particle is likely to be when a position measurement is performed. Second, how is the wavefunction used to make predictions? If it is necessary to find the probability that a particle will be found in a certain interval, square the wavefunction and integrate over the interval of interest. Soon, you will learn soon that the wavefunction can be used to make many other kinds of predictions, as well.

Third, if a matter wave is given by the wavefunction $\Psi(x,t)$, **where** exactly is the particle? Two answers exist: (1) when the observer **is not** looking (or the particle is not being otherwise detected), the particle is everywhere ($x = -\infty, +\infty$); and (2) when the observer **is** looking (the particle is being detected), the particle "jumps into" a particular position state ($x, x + dx$) with a probability given by

$$P(x, x+dx) = |\Psi(x,t)|^2 dx$$

via a process called **state reduction** or **wavefunction collapse**. This answer is called the **Copenhagen interpretation** of the wavefunction, or of quantum mechanics.

To illustrate this interpretation, consider the simple case of a particle that can occupy a small container either at x_1 or x_2 (Figure 4.2.5). In classical physics, we assume the particle is located either at x_1 or x_2 when the observer is not looking. However, in quantum mechanics, the particle may exist in a state of indefinite position—that is, it may be located at x_1 **and** x_2 when the observer is not looking. The assumption that a particle can only have one value of position (when the observer is not looking) is abandoned. Similar comments can be made of other measurable quantities, such as momentum and energy.

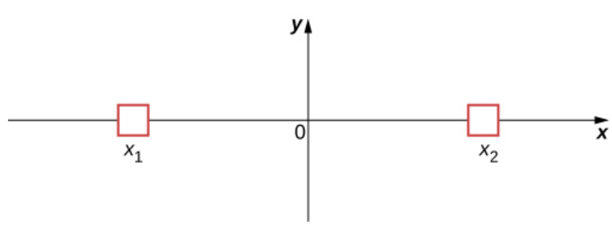


Figure 4.2.5: A two-state system of position of a particle.

The bizarre consequences of the Copenhagen interpretation of quantum mechanics are illustrated by a creative thought experiment first articulated by Erwin Schrödinger (**National Geographic**, 2013) (4.2.6):

"A cat is placed in a steel box along with a Geiger counter, a vial of poison, a hammer, and a radioactive substance. When the radioactive substance decays, the Geiger detects it and triggers the hammer to release the poison, which subsequently kills the cat. The radioactive decay is a random [probabilistic] process, and there is no way to predict when it will happen. Physicists say the atom exists in a state known as a superposition—both decayed and not decayed at the same time. Until the box is opened, an observer doesn't know whether the cat is alive or dead—because the cat's fate is intrinsically tied to whether or not the atom has decayed and the cat would [according to the Copenhagen interpretation] be "living and dead ... in equal parts" until it is observed."

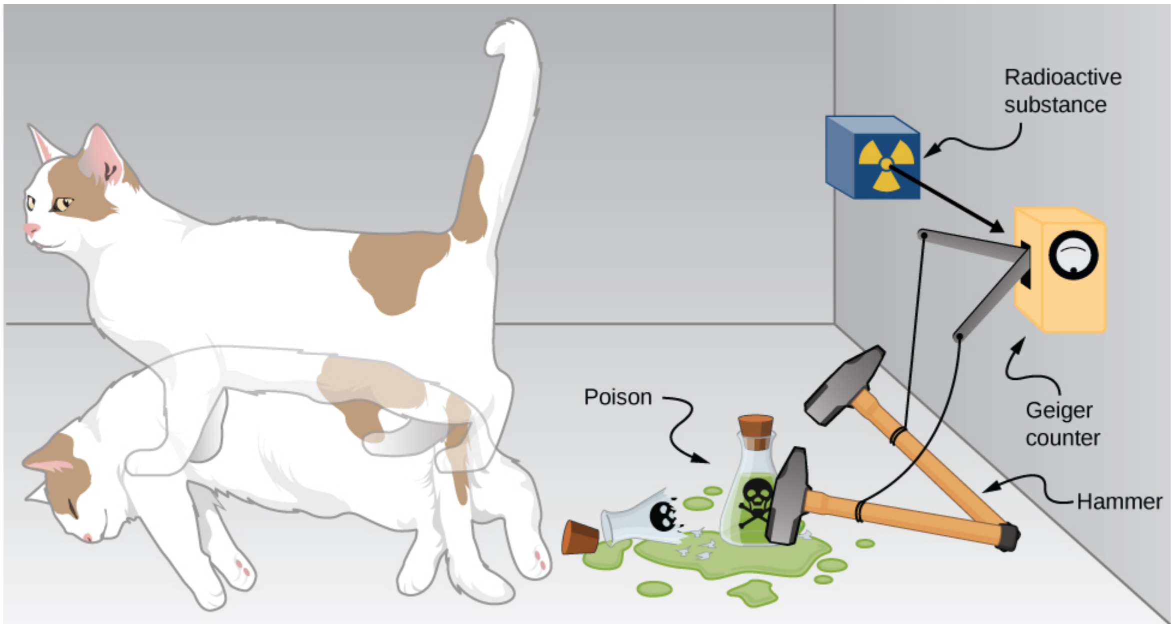


Figure 4.2.6: Schrödinger’s cat.

Schrödinger took the absurd implications of this thought experiment (a cat simultaneously dead and alive) as an argument against the Copenhagen interpretation. However, this interpretation remains the most commonly taught view of quantum mechanics.

Quantum Computing: qubits

Two-state systems (left and right, atom decays and does not decay, and so on) are often used to illustrate the principles of quantum mechanics. These systems find many applications in nature, including electron spin and mixed states of particles, atoms, and even molecules. Two-state systems are also finding application in the quantum computer, as mentioned in the introduction of this chapter. Unlike a digital computer, which encodes information in binary digits (zeroes and ones), a quantum computer stores and manipulates data in the form of quantum bits, or qubits. In general, a **qubit** is not in a state of zero or one, but rather in a mixed state of zero **and** one. If a large number of qubits are placed in the same quantum state, the measurement of an individual qubit would produce a zero with a probability **p**, and a one with a probability $q = 1 - p$. Some scientists believe that quantum computers are the future of the computer industry.

Complex Conjugates

Later in this section, you will see how to use the wavefunction to describe particles that are “free” or bound by forces to other particles. The specific form of the wavefunction depends on the details of the physical system. A peculiarity of quantum theory is that these functions are usually **complex functions**. A complex function is one that contains one or more imaginary numbers ($i = \sqrt{-1}$). Experimental measurements produce real (nonimaginary) numbers only, so the above procedure to use the wavefunction must be slightly modified. In general, the probability that a particle is found in the narrow interval $(x, x + dx)$ at time t is given by

$$P(x, x + dx) = |\Psi(x, t)|^2 dx = \Psi^*(x, t)\, \Psi(x, t)\, dx, \tag{4.2.6}$$

where $\Psi^*(x, t)$ is the complex conjugate of the wavefunction. The complex conjugate of a function is obtaining by replacing every occurrence of $i = \sqrt{-1}$ in that function with $-i$. This procedure eliminates complex numbers in all predictions because the product $\Psi^*(x, t)\, \Psi(x, t)$ is always a real number.

Exercise 4.2.1

If $a = 3 + 4i$, what is the product $a^* a$?

Answer

$$(3 + 4i)(3 - 4i) = 9 - 16i^2 = 25$$

Consider the motion of a free particle that moves along the **x**-direction. As the name suggests, a free particle experiences no forces and so moves with a constant velocity. As we will see in a later section of this chapter, a formal quantum mechanical treatment of a free particle indicates that its wavefunction has real **and** complex parts. In particular, the wavefunction is given by

$$\Psi(x,t) = A\,\cos(kx-\omega t) + iA\,\sin(kx-\omega t), \tag{4.2.7}$$

where A is the **amplitude**, k is the **wave number**, and ω is the **angular frequency**. Euler's formula

$$\underbrace{e^{i\phi} = \cos(\phi) + i\,\sin(\phi)}_{\text{Euler's formula}}$$

can be used to rewrite Equation 4.2.7 in the form

$$\Psi(x,t) = Ae^{i(kx-\omega t)} = Ae^{i\phi},$$

where ϕ is the phase angle. If the wavefunction varies slowly over the interval Δx, the probability of finding the particle in that interval is

$$P(x, x+\Delta x) \approx \Psi^*(x,t)\,\Psi(x,t)\,\Delta x = (A^* e^{-i\phi})(Ae^{i\phi})\,\Delta x = (A^*A)\Delta x.$$

If A has real and complex parts ($a+ib$, where a and b are real constants), then

$$A^*A = (a-ib)(a+ib) = a^2+b^2.$$

Notice that the complex numbers have vanished. Thus,

$$P(x, x+\Delta x) \approx |A|^2\delta x$$

is a real quantity. The interpretation of $\Psi^*(x,t)\,\Psi(x,t)$ as a probability density ensures that the predictions of quantum mechanics can be checked in the "real world."

Exercise 4.2.2

Suppose that a particle with energy **E** is moving along the **x**-axis and is confined in the region between 0 and **L**. One possible wavefunction is

$$\psi(x,t) = \begin{cases} Ae^{-iEt/\hbar}\sin\dfrac{\pi x}{L} & 0 \le x \le L \\ 0 & x < 0 \text{ and } x > L \end{cases}$$

Determine the normalization constant.

Answer

$A = \sqrt{2/L}$

Expectation Values

In classical mechanics, the solution to an equation of motion is a function of a measurable quantity, such as $x(t)$, where x is the position and t is the time. Note that the particle has one value of position for any time t. In quantum mechanics, however, the solution to an equation of motion is a wavefunction, $\Psi(x,t)$. The particle has many values of position for any time t, and only the probability density of finding the particle, $|\Psi(x,t)|^2$, can be known. The average value of position for a large number of particles with the same wavefunction is expected to be

$$\langle x \rangle = \int_{-\infty}^{\infty} xP(x,t)\,dx = \int_{-\infty}^{\infty} x\Psi^*(x,t)\,\Psi(x,t)\,dx. \tag{4.2.8}$$

This is called the expectation value of the position. It is usually written

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,x\Psi(x,t)\,dx. \tag{4.2.9}$$

where the x is sandwiched between the wavefunctions. The reason for this will become apparent soon. Formally, x is called the **position operator**.

At this point, it is important to stress that a wavefunction can be written in terms of other quantities as well, such as velocity (v), momentum (p), and kinetic energy (K). The expectation value of momentum, for example, can be written

$$\langle p\rangle = \int_{-\infty}^{\infty} \Psi^*(p,t)\, p\Psi(p,t)\, dp, \tag{4.2.10}$$

where dp is used instead of dx to indicate an infinitesimal interval in momentum. In some cases, we know the wavefunction in position, $\Psi(x,t)$, but seek the expectation of momentum. The procedure for doing this is

$$\langle p\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\left(-i\hbar\frac{d}{dx}\right)\Psi(x,t)\, dx, \tag{4.2.11}$$

where the quantity in parentheses, sandwiched between the wavefunctions, is called the **momentum operator** in the **x**-direction. [The momentum operator in Equation 4.2.11 is said to be the position-space representation of the momentum operator.] The momentum operator must act (operate) on the wavefunction to the right, and then the result must be multiplied by the complex conjugate of the wavefunction on the left, before integration. The momentum operator in the **x**-direction is sometimes denoted

$$\langle p\rangle = -i\hbar\frac{d}{dx}, \tag{4.2.12}$$

Momentum operators for the **y**- and **z**-directions are defined similarly. This operator and many others are derived in a more advanced course in modern physics. In some cases, this derivation is relatively simple. For example, the kinetic energy operator is just

$$(K)_{op} = \frac{1}{2}m(v_x)^2_{op} \tag{4.2.13}$$

$$= \frac{(p_x)^2_{op}}{2m} \tag{4.2.14}$$

$$= \frac{\left(-i\hbar\frac{d}{dx}\right)^2}{2m} \tag{4.2.15}$$

$$= \frac{-\hbar^2}{2m}\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right). \tag{4.2.16}$$

Thus, if we seek an expectation value of kinetic energy of a particle in one dimension, two successive ordinary derivatives of the wavefunction are required before integration.

Symmetry can simplify calculations

Expectation-value calculations are often simplified by exploiting the symmetry of wavefunctions. Symmetric wavefunctions can be even or odd. An **even function** is a function that satisfies

$$\psi(x) = \psi(-x). \tag{4.2.17}$$

In contrast, an odd function is a function that satisfies

$$\psi(x) = -\psi(-x). \tag{4.2.18}$$

An example of even and odd functions is shown in Figure 4.2.7. An even function is symmetric about the **y**-axis. This function is produced by reflecting $\psi(x)$ for $x>0$ about the vertical **y**-axis. By comparison, an odd function is generated by reflecting the function about the **y**-axis and then about the **x**-axis. (An odd function is also referred to as an **anti-symmetric function**.)

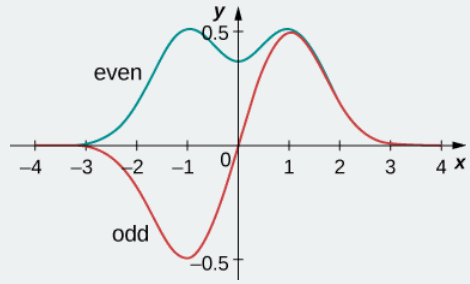


Figure 4.2.7: Examples of even and odd wavefunctions.

In general, an even function times an even function produces an even function. A simple example of an even function is the product $x^2 e^{-x^2}$ (even times even is even). Similarly, an odd function times an odd function produces an even function, such as **x** sin **x** (odd times odd is even). However, an odd function times an even function produces an odd function, such as $x^2 e^{-x^2}$ (odd times even is odd). The integral over all space of an odd function is zero, because the total area of the function above the **x**-axis cancels the (negative) area below it. As the next example shows, this property of odd functions is very useful.

✓ Example 4.2.2A: Expectation Value (Part I)

The normalized wavefunction of a particle is

$$\psi(x) = e^{-|x|/x_0}/\sqrt{x_0}.$$

Find the expectation value of position.

Strategy

Substitute the wavefunction into Equation 4.2.9 and evaluate. The position operator introduces a multiplicative factor only, so the position operator need not be "sandwiched."

Solution

First multiply, then integrate:

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty} dx\, x|\psi(x)|^2 \\ &= \int_{-\infty}^{\infty} dx\, x\left|\frac{e^{-|x|/x_0}}{\sqrt{x_0}}\right|^2 \\ &= \frac{1}{x_0}\int_{-\infty}^{\infty} dx\, x e^{-2|x|/x_0} \\ &= 0.\end{aligned}$$

Significance

The function in the integrand ($xe^{-2|x|/x_0}$) is odd since it is the product of an odd function (**x**) and an even function ($e^{-2|x|/x_0}$). The integral vanishes because the total area of the function about the **x**-axis cancels the (negative) area below it. The result ($\langle x\rangle = 0$) is not surprising since the probability density function is symmetric about $x = 0$.

✓ Example 4.2.2B: Expectation Value (Part II)

The time-dependent wavefunction of a particle confined to a region between 0 and **L** is

$$\psi(x,t) = A\, e^{-i\omega t}\sin(\pi x/L)$$

where ω is angular frequency and E is the energy of the particle. (**Note:** The function varies as a sine because of the limits (0 to **L**). When $x = 0$, the sine factor is zero and the wavefunction is zero, consistent with the boundary conditions.) Calculate the expectation values of position, momentum, and kinetic energy.

Strategy

We must first normalize the wavefunction to find **A**. Then we use the operators to calculate the expectation values.

Solution

Computation of the normalization constant:

$$\begin{aligned} 1 &= \int_0^L dx\, \psi^*(x)\psi(x) \\ &= \int_0^L dx\, \left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)\left(Ae^{-i\omega t}\sin\frac{\pi x}{L}\right) \\ &= A^2\int_0^L dx\, \sin^2\frac{\pi x}{L} \\ &= A^2\frac{L}{2} \\ \Rightarrow A &= \sqrt{\frac{2}{L}}. \end{aligned}$$

The expectation value of position is

$$\begin{aligned} \langle x\rangle &= \int_0^L dx\, \psi^*(x)x\psi(x) \\ &= \int_0^L dx\, \left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)x\left(Ae^{-i\omega t}\sin\frac{\pi x}{L}\right) \\ &= A^2\int_0^L dx\, x\,\sin^2\frac{\pi x}{L} \\ &= A^2\frac{L^2}{4} \\ \Rightarrow A &= \frac{L}{2}. \end{aligned}$$

The expectation value of momentum in the **x**-direction also requires an integral. To set this integral up, the associated operator must— by rule—act to the right on the wavefunction $\psi(x)$:

$$\begin{aligned} -i\hbar\frac{d}{dx}\psi(x) &= -i\hbar\frac{d}{dx}Ae^{-i\omega t}\sin\frac{\pi x}{L} \\ &= -i\frac{Ah}{2L}e^{-i\omega t}\cos\frac{\pi x}{L}. \end{aligned}$$

Therefore, the expectation value of momentum is

$$\begin{aligned} \langle p\rangle &= \int_0^L dx\, \left(Ae^{+i\omega t}sin\frac{\pi x}{L}\right)\left(-i\frac{Ah}{2L}e^{-i\omega t}cos\frac{\pi x}{L}\right) \\ &= -i\frac{A^2h}{4L}\int_0^L dx\, \sin\frac{2\pi x}{L} \\ &= 0. \end{aligned}$$

The function in the integral is a sine function with a wavelength equal to the width of the well, **L**—an odd function about $x = L/2$. As a result, the integral vanishes.

The expectation value of kinetic energy in the **x**-direction requires the associated operator to act on the wavefunction:

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) &= -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}Ae^{-i\omega t}\sin\frac{\pi x}{L} \\ &= -\frac{\hbar^2}{2m}Ae^{-i\omega t}\frac{d^2}{dx^2}\sin\frac{\pi x}{L} \\ &= \frac{Ah^2}{8mL^2}e^{-i\omega t}\sin\frac{\pi x}{L}. \end{aligned}$$

Thus, the expectation value of the kinetic energy is

$$\begin{aligned} \langle K\rangle &= \int_0^L dx\left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)\left(\frac{Ah^2}{8mL^2}e^{-i\omega t}\sin\frac{\pi x}{L}\right) \\ &= \frac{A^2h^2}{8mL^2}\int_0^L dx\,\sin^2\frac{\pi x}{L} \\ &= \frac{A^2h^2}{8mL^2}\frac{L}{2} \\ &= \frac{h^2}{8mL^2}. \end{aligned}$$

Significance

The average position of a large number of particles in this state is $L/2$. The average momentum of these particles is zero because a given particle is equally likely to be moving right or left. However, the particle is not at rest because its average kinetic energy is not zero. Finally, the probability density is

$$|\psi|^2 = (2/L)\sin^2(\pi x/L).$$

This probability density is largest at location $L/2$ and is zero at $x=0$ and at $x=L$. Note that these conclusions do not depend explicitly on time.

? Exercise 4.2.3

For the particle in the above example, find the probability of locating it between positions 0 and $L/4$.

Answer

$(1/2-1/\pi)/2 = 9\%$

Quantum mechanics makes many surprising predictions. However, in 1920, Niels Bohr (founder of the Niels Bohr Institute in Copenhagen, from which we get the term "Copenhagen interpretation") asserted that the predictions of quantum mechanics and classical mechanics must agree for all macroscopic systems, such as orbiting planets, bouncing balls, rocking chairs, and springs. This **correspondence principle** is now generally accepted. It suggests the rules of classical mechanics are an approximation of the rules of quantum mechanics for systems with very large energies. Quantum mechanics describes both the microscopic and macroscopic world, but classical mechanics describes only the latter.

4.3: The Heisenberg Uncertainty Principle

Learning Objectives

By the end of this section, you will be able to:

- Describe the physical meaning of the position-momentum uncertainty relation
- Explain the origins of the uncertainty principle in quantum theory
- Describe the physical meaning of the energy-time uncertainty relation

Heisenberg's uncertainty principle is a key principle in quantum mechanics. Very roughly, it states that if we know **everything** about where a particle is located (the uncertainty of position is small), we know **nothing** about its momentum (the uncertainty of momentum is large), and vice versa. Versions of the uncertainty principle also exist for other quantities as well, such as energy and time. We discuss the momentum-position and energy-time uncertainty principles separately.

Momentum and Position

To illustrate the momentum-position uncertainty principle, consider a free particle that moves along the **x**-direction. The particle moves with a constant velocity u and momentum $p = mu$. According to de Broglie's relations, $p = \hbar k$ and $E = \hbar\omega$. As discussed in the previous section, the wavefunction for a free particle is given by

$$\begin{aligned}\psi_k(x,t) &= A[\cos(\omega t - kx) - i\,\sin(\omega t - kx)] \\ &= A\,e^{-i(\omega t - kx)} \\ &= A\,e^{-i(\omega t - kx)} \\ &= A\,e^{-i\omega t}e^{ikx}\end{aligned}$$

and the probability density $|\psi_k(x,t)|^2 = A^2$ is **uniform** and independent of time. The particle is equally likely to be found anywhere along the **x**-axis but has definite values of wavelength and wave number, and therefore momentum. The uncertainty of position is infinite (we are completely uncertain about position) and the uncertainty of the momentum is zero (we are completely certain about momentum). This account of a free particle is consistent with Heisenberg's uncertainty principle.

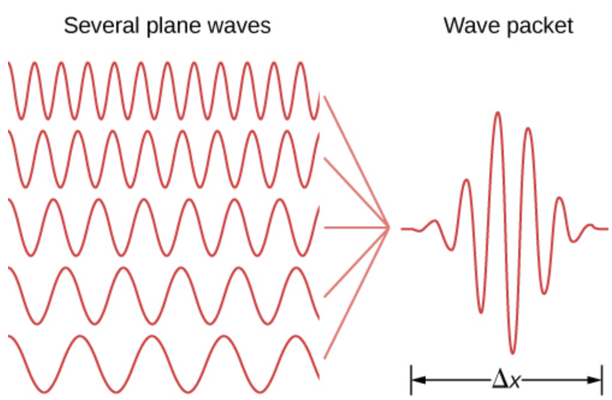


Figure 4.3.1: Adding together several plane waves of different wavelengths can produce a wave that is relatively localized.

Similar statements can be made of localized particles. In quantum theory, a localized particle is modeled by a linear superposition of free-particle (or plane-wave) states called a **wave packet**. An example of a wave packet is shown in Figure 4.3.1. A wave packet contains many wavelengths and therefore by de Broglie's relations many momenta—possible in quantum mechanics! This particle also has many values of position, although the particle is confined mostly to the interval Δx. The particle can be better localized (Δx can be decreased) if more plane-wave states of different wavelengths or momenta are added together in the right way (Δp is increased). According to Heisenberg, these uncertainties obey the following relation.

Definition: The Heisenberg's Uncertainty Principle

The product of the uncertainty in position of a particle and the uncertainty in its momentum can never be less than one-half of the reduced Planck constant:

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{4.3.1}$$

This relation expresses Heisenberg's uncertainty principle. It places limits on what we can know about a particle from simultaneous measurements of position and momentum. If Δx is large, Δp is small, and vice versa. Equation 4.3.1 can be derived in a more advanced course in modern physics. Reflecting on this relation in his work **The Physical Principles of the Quantum Theory**, Heisenberg wrote "Any use of the words 'position' and 'velocity' with accuracy exceeding that given by [the relation] is just as meaningless as the use of words whose sense is not defined."

Note that the uncertainty principle has nothing to do with the precision of an experimental apparatus. Even for perfect measuring devices, these uncertainties would remain because they originate in the wave-like nature of matter. The precise value of the product $\Delta x \Delta p$ depends on the specific form of the wavefunction. Interestingly, the Gaussian function (or bell-curve distribution) gives the minimum value of the uncertainty product:

$$\Delta x \Delta p = \frac{\hbar}{2}$$

Example 4.3.1: The Uncertainty Principle Large and Small

Determine the minimum uncertainties in the positions of the following objects if their speeds are known with a precision of $1.0 \times 10^{-3} m/s$:

a. an electron and
b. a bowling ball of mass 6.0 kg.

Strategy

Given the uncertainty in speed $\Delta u = 1.0 \times 10^{-3} m/s$, we have to first determine the uncertainty in momentum $\Delta p = m\Delta u$ and then invert Equation 4.3.1 to find the uncertainty in position

$$\Delta x = \frac{\hbar}{2\Delta p}.$$

Solution

a. For the electron:

$$\begin{aligned} \Delta p &= m\Delta u \\ &= (9.1 \times 10^{-31} kg)(1.0 \times 10^{-3} m/s) \\ &= 9.1 \times 10^{-34} kg \cdot m/s, \end{aligned}$$

$$\begin{aligned} \Delta x &= \frac{\hbar}{2\Delta p} \\ &= 5.8\, cm. \end{aligned}$$

b. For the bowling ball:

$$\begin{aligned} \Delta p &= m\Delta u \\ &= (6.0\, kg)(1.0 \times 10^{-3} m/s) \\ &= 6.0 \times 10^{-3} kg \cdot m/s, \end{aligned}$$

$$\begin{aligned} \Delta x &= \frac{\hbar}{2\Delta p} \\ &= 8.8 \times 10^{-33} m. \end{aligned}$$

Significance

Unlike the position uncertainty for the electron, the position uncertainty for the bowling ball is immeasurably small. Planck's constant is very small, so the limitations imposed by the uncertainty principle are not noticeable in macroscopic systems such as a bowling ball.

Example 4.3.2: Uncertainty and the Hydrogen Atom

Estimate the ground-state energy of a hydrogen atom using Heisenberg's uncertainty principle. (**Hint**: According to early experiments, the size of a hydrogen atom is approximately 0.1 nm.)

Strategy

An electron bound to a hydrogen atom can be modeled by a particle bound to a one-dimensional box of length $L = 0.1\,nm$. The ground-state wavefunction of this system is a half wave. This is the largest wavelength that can "fit" in the box, so the wavefunction corresponds to the lowest energy state. Note that this function is very similar in shape to a Gaussian (bell curve) function. We can take the average energy of a particle described by this function (**E**) as a good estimate of the ground state energy (E_0). This average energy of a particle is related to its average of the momentum squared, which is related to its momentum uncertainty.

Solution

To solve this problem, we must be specific about what is meant by "uncertainty of position" and "uncertainty of momentum." We identify the uncertainty of position (Δx) with the standard deviation of position (σ_x), and the uncertainty of momentum (Δp) with the standard deviation of momentum (σ_p). For the Gaussian function, the uncertainty product is

$$\sigma_x \sigma_p = \frac{\hbar}{2},$$

where

$$\sigma_x^2 = x^2 - \bar{x}^2$$

and

$$\sigma_p^2 = p^2 - \bar{p}^2.$$

The particle is equally likely to be moving left as moving right, so $\bar{p}^2 = 0$. Also, the uncertainty of position is comparable to the size of the box, so $\sigma_x = L$. The estimated ground state energy is therefore

$$\begin{aligned} E_0 &= E_{Gaussian} \\ &= \frac{\bar{p}^2}{m} \\ &= \frac{\sigma_p^2}{2m} \\ &= \frac{1}{2m}\left(\frac{\hbar}{2\sigma_x}\right)^2 \\ &= \frac{1}{2m}\left(\frac{\hbar}{2L}\right)^2 \\ &= \frac{\hbar^2}{8mL^2}. \end{aligned}$$

$$\begin{aligned} E_0 &= \frac{(\hbar c)^2}{8(mc^2)L^2} \\ &= \frac{(197.3\,eV \cdot nm)^2}{8(0.511 \cdot 10^6 eV)(0.1\,nm)^2} \\ &= 0.952\,eV \approx 1\,eV. \end{aligned}$$

Multiplying numerator and denominator by c^2 gives

Significance

Based on early estimates of the size of a hydrogen atom and the uncertainty principle, the ground-state energy of a hydrogen atom is in the eV range. The ionization energy of an electron in the ground-state energy is approximately 10 eV, so this prediction is roughly confirmed. (**Note:** The product ℏcℏc is often a useful value in performing calculations in quantum mechanics.)

Energy and Time

Another kind of uncertainty principle concerns uncertainties in simultaneous measurements of the energy of a quantum state and its lifetime,

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{4.3.2}$$

where ΔE is the uncertainty in the energy measurement and Δt is the uncertainty in the lifetime measurement. The **energy-time uncertainty principle** does not result from a relation of the type expressed by Equation 4.3.1 for technical reasons beyond this discussion. Nevertheless, the general meaning of the energy-time principle is that a quantum state that exists for only a short time cannot have a definite energy. The reason is that the frequency of a state is inversely proportional to time and the frequency connects with the energy of the state, so to measure the energy with good precision, the state must be observed for many cycles.

To illustrate, consider the excited states of an atom. The finite lifetimes of these states can be deduced from the shapes of spectral lines observed in atomic emission spectra. Each time an excited state decays, the emitted energy is slightly different and, therefore, the emission line is characterized by a **distribution** of spectral frequencies (or wavelengths) of the emitted photons. As a result, all spectral lines are characterized by spectral widths. The average energy of the emitted photon corresponds to the theoretical energy of the excited state and gives the spectral location of the peak of the emission line. Short-lived states have broad spectral widths and long-lived states have narrow spectral widths.

Example 4.3.3: Atomic Transitions

An atom typically exists in an excited state for about $\Delta t = 10^{-8} s$. Estimate the uncertainty Δf in the frequency of emitted photons when an atom makes a transition from an excited state with the simultaneous emission of a photon with an average frequency of $f = 7.1 \times 10^{14} Hz$. Is the emitted radiation monochromatic?

Strategy

We invert Equation 4.3.2 to obtain the energy uncertainty $\Delta E \approx \hbar/2\Delta t$ and combine it with the photon energy $E = hf$ to obtain Δf. To estimate whether or not the emission is monochromatic, we evaluate $\Delta f/f$.

Solution

The spread in photon energies is $\Delta E = h\Delta f$. Therefore,

$$\Delta E \approx \frac{\hbar}{2\Delta t} \Rightarrow h\Delta t \approx \frac{\hbar}{2\Delta t} \Rightarrow \Delta f \approx \frac{1}{4\pi\Delta t} = \frac{1}{4\pi(10^{-8} s)} = 8.0 \times 10^6 \; Hz,$$

$$\frac{\Delta f}{f} = \frac{8.0 \times 10^6 \; Hz}{7.1 \times 10^{14} \; Hz} = 1.1 \times 10^{-8}.$$

Significance

Because the emitted photons have their frequencies within 1.1×10^{-6} percent of the average frequency, the emitted radiation can be considered monochromatic.

Exercise 4.3.1

A sodium atom makes a transition from the first excited state to the ground state, emitting a 589.0-nm photon with energy 2.105 eV. If the lifetime of this excited state is $1.6 \times 10^{-8} s$, what is the uncertainty in energy of this excited state? What is the width of the corresponding spectral line?

Answer

$4.1 \times 10^{-8} eV$; $1.1 \times 10^{-5} nm$

4.4: The Schrödinger Equation

Learning Objectives

By the end of this section, you will be able to:

- Describe the role Schrödinger's equation plays in quantum mechanics
- Explain the difference between time-dependent and -independent Schrödinger's equations
- Interpret the solutions of Schrödinger's equation

In the preceding two sections, we described how to use a quantum mechanical wavefunction and discussed Heisenberg's uncertainty principle. In this section, we present a complete and formal theory of quantum mechanics that can be used to make predictions. In developing this theory, it is helpful to review the wave theory of light. For a light wave, the electric field $E(x,t)$ obeys the relation

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}, \tag{4.4.1}$$

where c is the speed of light and the symbol ∂ represents a **partial derivative**. (Recall from Oscillations that a partial derivative is closely related to an ordinary derivative, but involves functions of more than one variable. When taking the partial derivative of a function by a certain variable, all other variables are held constant.) A light wave consists of a very large number of photons, so the quantity $|E(x,t)|^2$ can interpreted as a probability density of finding a single photon at a particular point in space (for example, on a viewing screen).

There are many solutions to this equation. One solution of particular importance is

$$E(x,t) = A\,\sin(kx - \omega t), \tag{4.4.2}$$

where A is the amplitude of the electric field, k is the wave number, and ω is the angular frequency. Combing this equation with Equation 4.4.1 gives

$$k^2 = \frac{\omega^2}{c^2}, \tag{4.4.3}$$

According to de Broglie's equations, we have $p = \hbar k$ and $E = \hbar\omega$. Substituting these equations into Equation 4.4.3 gives

$$p = \frac{E}{c},$$

or

$$E = pc. \tag{4.4.4}$$

Therefore, according to Einstein's general energy-momentum equation (Equation 5.10.26), Equation 4.4.4 describes a particle with a zero **rest mass**. This is consistent with our knowledge of a photon.

This process can be reversed. We can begin with the energy-momentum equation of a particle and then ask what wave equation corresponds to it. The energy-momentum equation of a nonrelativistic particle in one dimension is

$$E = \frac{p^2}{2m} + U(x,t),$$

where **p** is the momentum, **m** is the mass, and **U** is the potential energy of the particle. The wave equation that goes with it turns out to be a key equation in quantum mechanics, called **Schrödinger's time-dependent equation**.

THE TIME-DEPENDENT SCHRÖDINGER EQUATION

The equation describing the energy and momentum of a wavefunction is known as the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + U(x,t)\,\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}. \tag{4.4.5}$$

As described in Potential Energy and Conservation of Energy, the force on the particle described by this equation is given by

$$F=-\frac{\partial U(x,t)}{\partial x}. \tag{4.4.6}$$

This equation plays a role in quantum mechanics similar to Newton's second law in classical mechanics. Once the potential energy of a particle is specified—or, equivalently, once the force on the particle is specified—we can solve this differential equation for the wavefunction. The solution to Newton's second law equation (also a differential equation) in one dimension is a function **x**(**t**) that specifies where an object is at any time **t**. The solution to Schrödinger's time-dependent equation provides a tool—the wavefunction—that can be used to determine where the particle is **likely** to be. This equation can be also written in two or three dimensions. Solving Schrödinger's time-dependent equation often requires the aid of a computer.

Consider the special case of a free particle. A free particle experiences no force ($F=0$).Based on Equation 4.4.6, this requires only that

$$U(x,t)=U_0=constant. \tag{4.4.7}$$

For simplicity, we set $U_0=0$. Schrödinger's equation then reduces to

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}=i\hbar\frac{\partial\Psi(x,t)}{\partial t}. \tag{4.4.8}$$

A valid solution to this equation is

$$\Psi(x,t)=Ae^{i(kx-\omega t)}. \tag{4.4.9}$$

Not surprisingly, this solution contains an **imaginary number** ($i=\sqrt{-1}$) because the differential equation itself contains an imaginary number. As stressed before, however, quantum-mechanical predictions depend only on $|\Psi(x,t)|^2$, which yields completely real values. Notice that the real plane-wave solutions, $\Psi(x,t)=A\,sin(kx-\omega t)$ and $\Psi(x,t)=A\,cos(kx-\omega t)$, do not obey Schrödinger's equation. The temptation to think that a wavefunction can be seen, touched, and felt in nature is eliminated by the appearance of an imaginary number. In Schrödinger's theory of quantum mechanics, the wavefunction is merely a tool for calculating things.

If the potential energy function (**U**) does not depend on time, it is possible to show that

$$\Psi(x,t)=\psi(x)\,e^{-i\omega t} \tag{4.4.10}$$

satisfies Schrödinger's time-dependent equation, where $\psi(x)$ is a **time**-independent function and e−iωte−iωt is a **space**-independent function. In other words, the wavefunction is **separable** into two parts: a space-only part and a time-only part. The factor $e^{-i\omega t}$ is sometimes referred to as a **time-modulation factor** since it modifies the space-only function. According to de Broglie, the energy of a matter wave is given by $E=\hbar\omega$, where **E** is its total energy. Thus, the above equation can also be written as

$$\Psi(x,t)=\psi(x)\,e^{-iEt/\hbar}. \tag{4.4.11}$$

Any linear combination of such states (mixed state of energy or momentum) is also valid solution to this equation. Such states can, for example, describe a localized particle (see Figure 7.3.1)

Exercise 4.4.1

A particle with mass **m** is moving along the **x**-axis in a potential given by the potential energy function $U(x)=0.5m\,\omega^2x^2$. Compute the product $\Psi(x,t)^*U(x)\,\Psi(x,t)$. Express your answer in terms of the time-independent wavefunction, $\psi(x)$.

Answer:

$0.5\,m\omega^2x^2\,\psi(x)^*\psi(x)$

Combining Equation 4.4.11 and Equation 4.4.5, Schrödinger's time-dependent equation reduces to the **Schrödinger's time-independent equation**.

THE TIME-INDEPENDENT SCHRÖDINGER EQUATION

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+U(x)\,\psi(x)=E\,\psi(x), \tag{4.4.12}$$

where E is the total energy of the particle (a real number).

Notice that we use "big psi" (Ψ) for the time-dependent wavefunction and "little psi" (ψ) for the time-independent wavefunction. The wave-function solution to this equation must be multiplied by the time-modulation factor to obtain the time-dependent wavefunction.

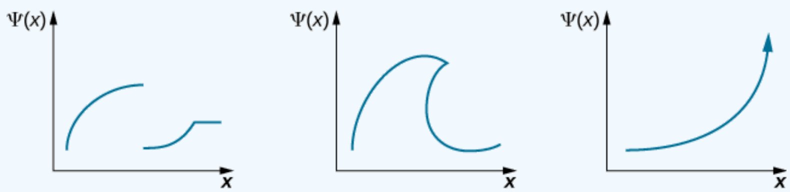
In the next sections, we solve Schrödinger's time-independent equation for three cases: a quantum particle in a box, a simple harmonic oscillator, and a quantum barrier. These cases provide important lessons that can be used to solve more complicated systems. The time-independent wavefunction $\psi(x)$ solutions must satisfy three conditions:

- $\psi(x)$ must be a continuous function.
- The first derivative of $\psi(x)$ with respect to space, $d\psi(x)/dx$, must be continuous, unless $V(x)=\infty$.
- $\psi(x)$ must not diverge ("blow up") at $x=\pm\infty$.

The first condition avoids sudden jumps or gaps in the wavefunction. The second condition requires the wavefunction to be smooth at all points, except in special cases. (In a more advanced course on quantum mechanics, for example, potential spikes of infinite depth and height are used to model solids). The third condition requires the wavefunction be normalizable. This third condition follows from Born's interpretation of quantum mechanics. It ensures that $|\psi(x)|^2$ is a finite number so we can use it to calculate probabilities.

Exercise $4.4.2$

Which of the following wavefunctions is a valid wave-function solution for Schrödinger's equation?



Answer:

None. The first function has a discontinuity; the second curve is not even a function - it is double-valued; and the third function diverges so is not normalizable.

4.5: The Quantum Particle in a Box

Learning Objectives

By the end of this section, you will be able to:

- Describe how to set up a boundary-value problem for the stationary Schrödinger equation
- Explain why the energy of a quantum particle in a box is quantized
- Describe the physical meaning of stationary solutions to Schrödinger's equation and the connection of these solutions with time-dependent quantum states
- Explain the physical meaning of Bohr's correspondence principle

In this section, we apply Schrödinger's equation to a particle bound to a one-dimensional box. This special case provides lessons for understanding quantum mechanics in more complex systems. The energy of the particle is quantized as a consequence of a standing wave condition inside the box.

Consider a particle of mass m that is allowed to move only along the **x**-direction and its motion is confined to the region between hard and rigid walls located at $x=0$ and at $x=L$ (Figure $4.5.1$). Between the walls, the particle moves freely. This physical situation is called the **infinite square well**, described by the potential energy function

$$U(x)=\begin{cases}0 & 0\leq x\leq L\\ \infty & x<0\ and\ x>L\end{cases}\tag{4.5.1}$$

Combining this equation with Schrödinger's time-independent wave equation gives

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=E\psi(x),\ for\, 0\leq x\leq L\tag{4.5.2}$$

where E is the **total energy of the particle**. What types of solutions do we expect? The energy of the particle is a positive number, so if the value of the wavefunction is positive (right side of the equation), the curvature of the wavefunction is negative, or concave down (left side of the equation). Similarly, if the value of the wavefunction is negative (right side of the equation), the curvature of the wavefunction is positive or concave up (left side of equation). This condition is met by an oscillating wavefunction, such as a sine or cosine wave. Since these waves are confined to the box, we envision standing waves with fixed endpoints at $x=0$ and $x=L$.

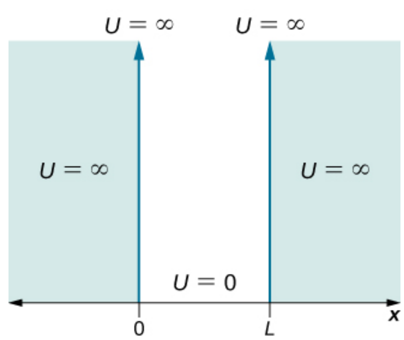


Figure $4.5.1$: The potential energy function that confines the particle in a one-dimensional box

Solutions $\psi(x)$ to this equation have a probabilistic interpretation. In particular, the square $|\psi(x)|^2$ represents the probability density of finding the particle at a particular location **x**. This function must be integrated to determine the probability of finding the particle in some interval of space. We are therefore looking for a normalizable solution that satisfies the following normalization condition:

$$\int_0^L dx|\psi(x)|^2=1.\tag{4.5.3}$$

The walls are rigid and impenetrable, which means that the particle is never found beyond the wall. Mathematically, this means that the solution must vanish at the walls:

$$\psi(0) = \psi(L) = 0. \tag{4.5.4}$$

We expect oscillating solutions, so the most general solution to this equation is

$$\psi_k(x) = A_k \cos kx + B_k \sin kx \tag{4.5.5}$$

where k is the wave number, and A_k and B_k are constants. Applying the boundary condition expressed by Equation 4.5.3 gives

$$\psi_k(0) = A_k \cos(k \cdot 0) + B_k \sin(k \cdot 0) = A_k = 0. \tag{4.5.6}$$

Because we have $A_k = 0$, the solution must be

$$\psi_k(x) = B_k \sin kx. \tag{4.5.7}$$

If B_k is zero, then $\psi_k(x) = 0$ for all values of x and the normalization condition (Equation 4.5.3) cannot be satisfied. Assuming $B_k \neq 0$, Equation 4.5.4 for $x = L$ then gives

$$0 = B_k \sin(kL) \Rightarrow \sin(kL) = 0 \Rightarrow kL = n\pi,\ n = 1, 2, 3, \ldots \tag{4.5.8}$$

We discard the $n = 0$ solution because $\psi(x)$ for this quantum number would be zero everywhere—an un-normalizable and therefore unphysical solution. Substituting Equation 4.5.7 into Equation 4.5.2 gives

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}(B_k \sin(kx)) = E(B_k \sin(kx)). \tag{4.5.9}$$

Computing these derivatives leads to

$$E = E_k = \frac{\hbar^2 k^2}{2m}. \tag{4.5.10}$$

According to de Broglie, $p = \hbar k$, so this expression implies that the total energy is equal to the kinetic energy, consistent with our assumption that the "particle moves freely." Combining the results of Equation 4.5.8 and 4.5.10 gives

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2},\ n = 1, 2, 3, \ldots \tag{4.5.11}$$

Strange!

Equation 4.5.11 argues that a particle bound to a one-dimensional box can only have certain discrete (quantized) values of energy. Further, the particle **cannot** have a zero kinetic energy—it is impossible for a particle bound to a box to be "at rest."

To evaluate the allowed wavefunctions that correspond to these energies, we must find the normalization constant B_n. We impose the normalization condition Equation 4.5.3 on the wavefunction

$$\psi_n(x) = B_n \sin \frac{n\pi x}{L} \tag{4.5.12}$$

We start with the normalization condition (Equation 4.5.3)

$$1 = \int_0^L dx |\psi_n(x)|^2 \tag{4.5.13}$$

$$= \int_0^L dx\, B_n^2 \sin^2 \frac{n\pi}{L} x \tag{4.5.14}$$

$$= B^2 n \int_0^2 dx\, \sin^2 \frac{n\pi}{L} x \tag{4.5.15}$$

$$= B_n^2 \frac{L}{2} \tag{4.5.16}$$

$$\Rightarrow B_n = \sqrt{\frac{2}{L}}. \tag{4.5.17}$$

Hence, the wavefunctions that correspond to the energy values given in Equation 4.5.11 are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \; n = 1, 2, 3, \ldots \tag{4.5.18}$$

For the lowest energy state or ground state energy, we have

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2}, \; \psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right). \tag{4.5.19}$$

All other energy states can be expressed as

$$E_n = n^2 E_1, \; \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right). \tag{4.5.20}$$

The index n is called the **energy quantum number** or **principal quantum number**. The state for $n = 2$ is the first excited state, the state for $n = 3$ is the second excited state, and so on. The first three quantum states (for n = 1, 2, and 3) of a particle in a box are shown in Figure $4.5.2$. The wavefunctions in Equation 4.5.20 are sometimes referred to as the "states of definite energy." Particles in these states are said to occupy **energy levels**, which are represented by the horizontal lines in Figure $4.5.2$. Energy levels are analogous to rungs of a ladder that the particle can "climb" as it gains or loses energy.

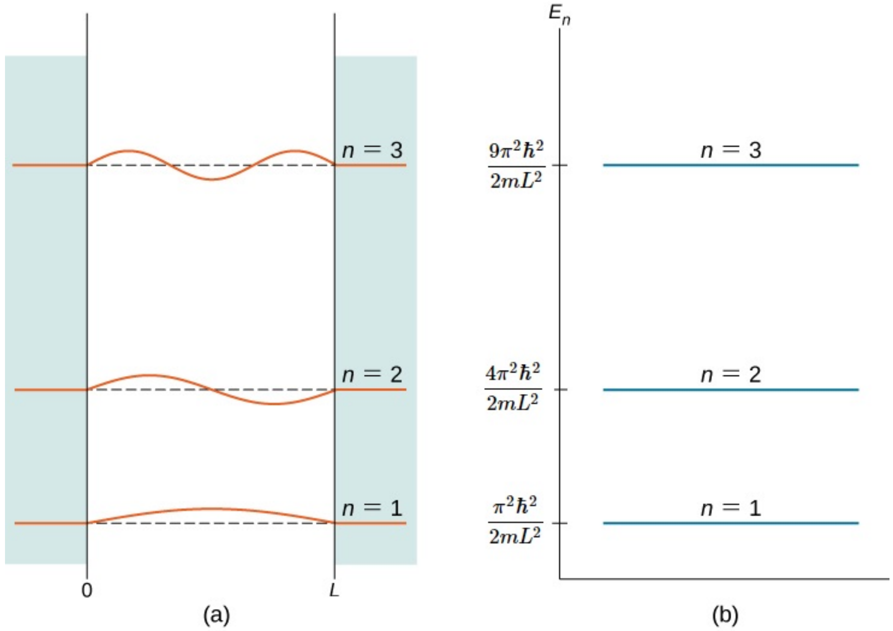


Figure $4.5.2$: The first three quantum states of a quantum particle in a box for principal quantum numbers n = 1,2,and 3: (a) standing wave solutions and (b) allowed energy states.

Stationary States

The wavefunctions in Equation 4.5.20 are also called **stationary states** and **standing wave states**. These functions are "stationary," because their probability density functions, $|\Psi(x,t)|^2$, do not vary in time, and "standing waves" because their real and imaginary parts oscillate up and down like a standing wave—like a rope waving between two children on a playground. Stationary states are states of definite energy (Equation 4.5.20), but linear combinations of these states, such as $\psi(x) = a\psi_1 + b\psi_2$ (also solutions to Schrödinger's equation) are states of mixed energy.

Energy quantization is a consequence of the boundary conditions. If the particle is not confined to a box but wanders freely, the allowed energies are continuous. However, in this case, only certain energies $(E_1, 4E_1, 9E_1, \ldots)$ are allowed. The energy difference between adjacent energy levels is given by

$$\Delta E_{n+1,n} = E_{n+1} - E_n = (n+1)^2 E_1 - n^2 E_1 = (2n+1)E_1.$$

Conservation of energy demands that if the energy of the system changes, the energy difference is carried in some other form of energy. For the special case of a charged particle confined to a small volume (for example, in an atom), energy changes are often carried away by photons. The frequencies of the emitted photons give us information about the energy differences (spacings) of the system and the volume of containment—the size of the "box" (Equation 4.5.19).

✓ Example 4.5.1: A Simple Model of the Nucleus

Suppose a proton is confined to a box of width $L = 1.00 \times 10^{-14} m$ (a typical nuclear radius). What are the energies of the ground and the first excited states? If the proton makes a transition from the first excited state to the ground state, what are the energy and the frequency of the emitted photon?

Strategy

If we assume that the proton confined in the nucleus can be modeled as a quantum particle in a box, all we need to do is to use Equation 4.5.11 to find its energies E_1 and E_2. The mass of a proton is $m = 1.76 \times 10^{-27} kg$. The emitted photon carries away the energy difference $\Delta E = E_2 - E_1$. We can use the relation $E_f = hf$ to find its frequency **f**.

Solution

The ground state:

$$\begin{aligned} E_1 &= \frac{\pi^2 \hbar^2}{2mL^2} \\ &= \frac{\pi^2 (1.05 \times 10^{-34} J \cdot s)}{2(1.67 \times 10^{-27} kg)(1.00 \times 10^{-14} m)^2} \\ &= 3.28 \times 10^{-13} J \\ &= 2.05\, MeV \end{aligned}$$

The first excited state:

$$E_2 = 2^2 E_1 = 4(2.05\, MeV) = 8.20\, MeV.$$

The energy of the emitted photon is

$$E_f = \Delta E = E_2 - E_1 = 8.20\, MeV - 2.05\, MeV = 6.15\, MeV.$$

The frequency of the emitted photon is

$$f = \frac{E_f}{h} = \frac{6.15\, MeV}{4.14 \times 10^{-21} MeV \cdot s} = 1.49 \times 10^{21} Hz.$$

Significance

This is the typical frequency of a gamma ray emitted by a nucleus. The energy of this photon is about 10 million times greater than that of a visible light photon.

The expectation value of the position for a particle in a box is given by

$$\langle x \rangle = \int_0^L dx\, \psi_n^*(x) x \psi_n(x) = \int_0^L dx\, x |\psi_n^*(x)|^2 = \int_0^L dx\, x \frac{2}{L} sin^2 \frac{n\pi x}{L} = \frac{L}{2}. \tag{4.5.21}$$

We can also find the expectation value of the momentum or average momentum of a large number of particles in a given state:

$$\langle p\rangle = \int_0^L dx\psi_n^*(x)\left[-i\hbar\frac{d}{dx}\psi_n(x)\right] \tag{4.5.22}$$

$$= -i\hbar\int_0^L dx\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\left[\frac{d}{dx}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right] \tag{4.5.23}$$

$$= -i\frac{2\hbar}{L}\int_0^L dx\sin\frac{n\pi x}{L}\left[\frac{n\pi}{L}\cos\frac{n\pi x}{L}\right] \tag{4.5.24}$$

$$= -i\frac{2n\pi\hbar}{L^2}\int_0^L dx\frac{1}{2}\sin\frac{2n\pi x}{L} \tag{4.5.25}$$

$$= -i\frac{n\pi\hbar}{L^2}\frac{L}{2n\pi}\int_0^{2\pi n} d\varphi\sin\varphi \tag{4.5.26}$$

$$= -i\frac{\hbar}{2L}\cdot 0 \tag{4.5.27}$$

$$= 0. \tag{4.5.28}$$

Thus, for a particle in a state of definite energy, the average position is in the middle of the box and the average momentum of the particle is zero—as it would also be for a classical particle. Note that while the minimum energy of a classical particle can be zero (the particle can be at rest in the middle of the box), the minimum energy of a quantum particle is nonzero and given by Equation 4.5.19. The average particle energy in the **nth** quantum state—its expectation value of energy—is

$$E_n = \langle E\rangle = n^2\frac{\pi^2\hbar^2}{2m}. \tag{4.5.29}$$

The result is not surprising because the standing wave state is a state of definite energy. Any energy measurement of this system must return a value equal to one of these allowed energies.

Our analysis of the quantum particle in a box would not be complete without discussing Bohr's correspondence principle. This principle states that for large quantum numbers, the laws of quantum physics must give identical results as the laws of classical physics. To illustrate how this principle works for a quantum particle in a box, we plot the probability density distribution

$$|\psi_n(x)|^2 = \frac{2}{L}sin^2(n\pi x/L) \tag{4.5.30}$$

for finding the particle around location x between the walls when the particle is in quantum state ψ_n. Figure 4.5.3 shows these probability distributions for the ground state, for the first excited state, and for a highly excited state that corresponds to a large quantum number. We see from these plots that when a quantum particle is in the ground state, it is most likely to be found around the middle of the box, where the probability distribution has the largest value. This is not so when the particle is in the first excited state because now the probability distribution has the zero value in the middle of the box, so there is no chance of finding the particle there. When a quantum particle is in the first excited state, the probability distribution has two maxima, and the best chance of finding the particle is at positions close to the locations of these maxima. This quantum picture is unlike the classical picture.

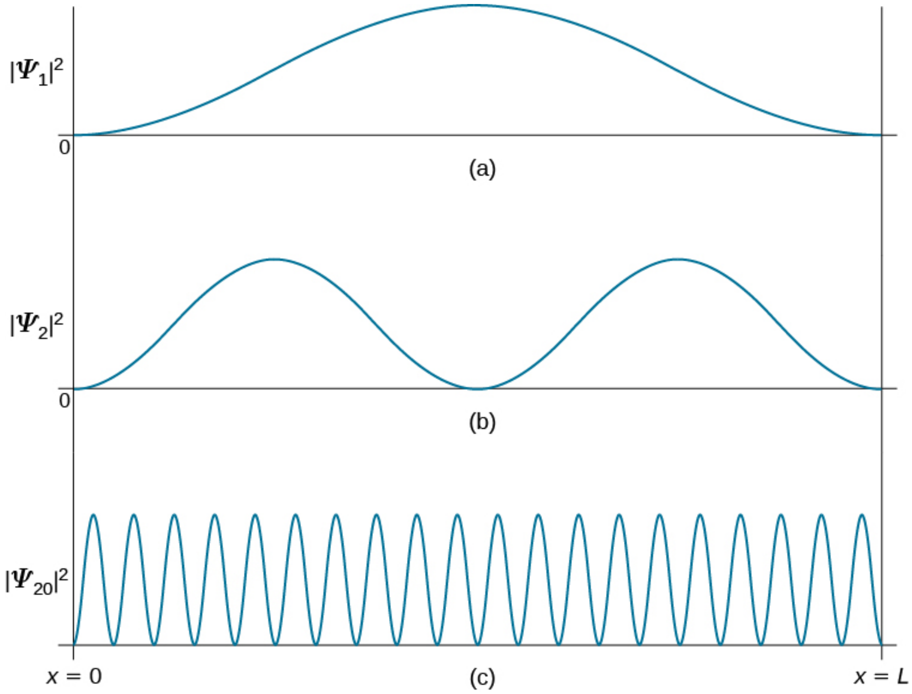


Figure 4.5.3: The probability density distribution $|\psi_n(x)|^2$ for a quantum particle in a box for: (a) the ground state, $n = 1$; (b) the first excited state, $n = 2$; and, (c) the nineteenth excited state, $n = 20$.

The probability density of finding a classical particle between x and $x + \Delta x$ depends on how much time Δt the particle spends in this region. Assuming that its speed **u** is constant, this time is $\Delta t = \Delta x / u$, which is also constant for any location between the walls. Therefore, the probability density of finding the classical particle at x is uniform throughout the box, and there is no preferable location for finding a classical particle. This classical picture is matched in the limit of large quantum numbers. For example, when a quantum particle is in a highly excited state, shown in Figure 4.5.3, the probability density is characterized by rapid fluctuations and then the probability of finding the quantum particle in the interval Δx does not depend on where this interval is located between the walls.

Example 4.5.2: A Classical Particle in a Box

A small 0.40-kg cart is moving back and forth along an air track between two bumpers located 2.0 m apart. We assume no friction; collisions with the bumpers are perfectly elastic so that between the bumpers, the car maintains a constant speed of 0.50 m/s. Treating the cart as a quantum particle, estimate the value of the principal quantum number that corresponds to its classical energy.

Strategy

We find the kinetic energy **K** of the cart and its ground state energy E_1 as though it were a quantum particle. The energy of the cart is completely kinetic, so $K = n^2 E_1$ (Equation 4.5.20). Solving for **n** gives $n = (K/E_1)^{1/2}$.

Solution

The kinetic energy of the cart is

$$K = \frac{1}{2} m u^2 = \frac{1}{2}(0.40\,kg)(0.50\,m/s)^2 = 0.050\,J.$$

The ground state of the cart, treated as a quantum particle, is

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (1.05 \times 10^{-34} J \cdot s)^2}{2(0.40\,kg)(2.0\,m)^2} = 1.700 \times 10^{-68} J.$$

Therefore,

$$n = (K/E_1)^{1/2} = (0.050/1.700 \times 10^{-68})^{1/2} = 1.2 \times 10^{33}.$$

Significance

We see from this example that the energy of a classical system is characterized by a very large quantum number. Bohr's correspondence principle concerns this kind of situation. We can apply the formalism of quantum mechanics to any kind of system, quantum or classical, and the results are correct in each case. In the limit of high quantum numbers, there is no advantage in using quantum formalism because we can obtain the same results with the less complicated formalism of classical mechanics. However, we cannot apply classical formalism to a quantum system in a low-number energy state.

Exercise 4.5.1

(a) Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. What is the probability of finding a quantum particle in its ground state somewhere between $x = 0$ and $x = L/4$? (b) Repeat question (a) for a classical particle.

Solution

a. 9.1%; b. 25%

Having found the stationary states $\psi_n(x)$ and the energies E_n by solving the time-independent Schrödinger equation (Equation 4.5.2), we use Equation 7.4.12 to write wavefunctions $\Psi_n(x,t)$ that are solutions of the time-dependent Schrödinger's equation given by Equation 7.4.7. For a particle in a box this gives

$$\Psi_n(x,t) = e^{-i\omega_n t}\psi_n(x) = \sqrt{\frac{2}{L}} e^{-iE_n t/\hbar} sin\frac{n\pi x}{L}, \; n = 1, 2, 3, \ldots \tag{4.5.31}$$

where the energies are given by Equation 4.5.11.

The quantum particle in a box model has practical applications in a relatively newly emerged field of optoelectronics, which deals with devices that convert electrical signals into optical signals. This model also deals with nanoscale physical phenomena, such as a nanoparticle trapped in a low electric potential bounded by high-potential barriers.

4.6: The Quantum Harmonic Oscillator

Learning Objectives

By the end of this section, you will be able to:

- Describe the model of the quantum harmonic oscillator
- Identify differences between the classical and quantum models of the harmonic oscillator
- Explain physical situations where the classical and the quantum models coincide

Oscillations are found throughout nature, in such things as electromagnetic waves, vibrating molecules, and the gentle back-and-forth sway of a tree branch. In previous chapters, we used Newtonian mechanics to study macroscopic oscillations, such as a block on a spring and a simple pendulum. In this chapter, we begin to study oscillating systems using quantum mechanics. We begin with a review of the classic harmonic oscillator.

The Classic Harmonic Oscillator

A simple harmonic oscillator is a particle or system that undergoes harmonic motion about an equilibrium position, such as an object with mass vibrating on a spring. In this section, we consider oscillations in one-dimension only. Suppose a mass moves back-and-forth along the x-direction about the equilibrium position, $x=0$. In classical mechanics, the particle moves in response to a linear restoring force given by $F_x = -kx$, where x is the displacement of the particle from its equilibrium position. The motion takes place between two turning points, $x \pm A$, where **A** denotes the amplitude of the motion. The position of the object varies periodically in time with angular frequency $\omega = \sqrt{k/m}$, which depends on the mass **m** of the oscillator and on the force constant k of the net force, and can be written as

$$x(t) = A\,\cos(\omega t + \phi). \tag{4.6.1}$$

The total energy E of an oscillator is the sum of its kinetic energy $K = mu^2/2$ and the elastic potential energy of the force $U(x) = kx^2/2$,

$$E = \frac{1}{2}mu^2 + \frac{1}{2}kx^2. \tag{4.6.2}$$

At turning points $x = \pm A$, the speed of the oscillator is zero; therefore, at these points, the energy of oscillation is solely in the form of potential energy $E = kA^2/2$. The plot of the potential energy $U(x)$ of the oscillator versus its position x is a parabola (Figure $4.6.1$). The potential-energy function is a quadratic function of x, measured with respect to the equilibrium position. On the same graph, we also plot the total energy E of the oscillator, as a horizontal line that intercepts the parabola at $x = \pm A$. Then the kinetic energy K is represented as the vertical distance between the line of total energy and the potential energy parabola.

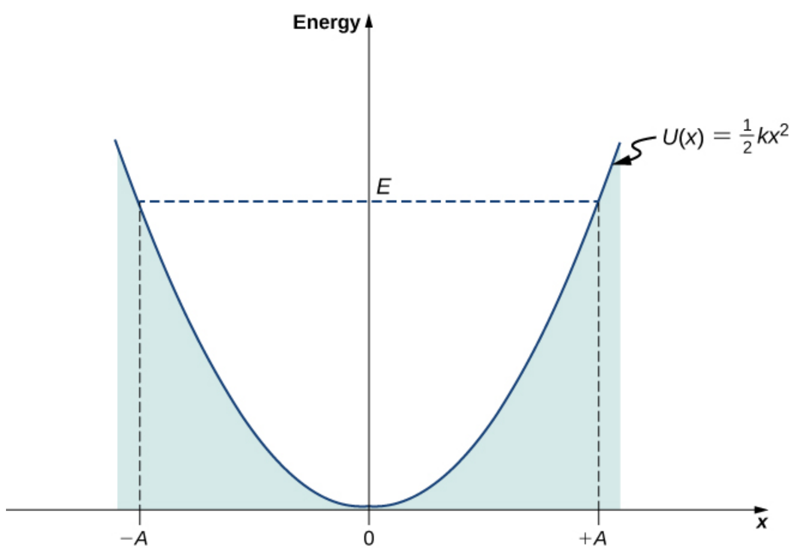


Figure $4.6.1$: The potential energy well of a classical harmonic oscillator: The motion is confined between turning points at $x=-A$ and at $x=+A$. The energy of oscillations is $E = kA^2/2$.

In this plot, the motion of a classical oscillator is confined to the region where its kinetic energy is nonnegative, which is what the energy relation Equation 4.6.2 says. Physically, it means that a classical oscillator can never be found beyond its turning points, and its energy depends only on how far the turning points are from its equilibrium position. The energy of a classical oscillator changes in a continuous way. The lowest energy that a classical oscillator may have is zero, which corresponds to a situation where an object is at rest at its equilibrium position. The zero-energy state of a classical oscillator simply means no oscillations and no motion at all (a classical particle sitting at the bottom of the potential well in Figure $4.6.1$). When an object oscillates, no matter how big or small its energy may be, it spends the longest time near the turning points, because this is where it slows down and reverses its direction of motion. Therefore, the probability of finding a classical oscillator between the turning points is highest near the turning points and lowest at the equilibrium position. (Note that this is not a statement of preference of the object to go to lower energy. It is a statement about how quickly the object moves through various regions.)

The Quantum Harmonic Oscillator

One problem with this classical formulation is that it is not general. We cannot use it, for example, to describe vibrations of diatomic molecules, where quantum effects are important. A first step toward a quantum formulation is to use the classical expression $k = m\omega^2$ to limit mention of a "spring" constant between the atoms. In this way the potential energy function can be written in a more general form,

$$U(x) = \frac{1}{2}m\omega^2 x^2. \tag{4.6.3}$$

Combining this expression with the time-independent Schrödinger equation gives

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x). \tag{4.6.4}$$

To solve Equation 4.6.4, that is, to find the allowed energies E and their corresponding wavefunctions $\psi(x)$ - we require the wavefunctions to be symmetric about $x = 0$ (the bottom of the potential well) and to be normalizable. These conditions ensure that the probability density $|\psi(x)|^2$ must be finite when integrated over the entire range of **x** from $-\infty$ to $+\infty$. How to solve Equation 4.6.4 is the subject of a more advanced course in quantum mechanics; here, we simply cite the results. The allowed energies are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{4.6.5}$$

$$= \frac{2n+1}{2}\hbar\omega \tag{4.6.6}$$

with $n = 0, 1, 2, 3, \ldots$

The wavefunctions that correspond to these energies (the stationary states or states of definite energy) are

$$\psi_n(x) = N_n e^{-\beta^2x^2/2} H_n(\beta x),\ n = 0, 1, 2, 3, \ldots \tag{4.6.7}$$

where $\beta = \sqrt{m\omega/\hbar}$, N_n is the normalization constant, and $H_n(y)$ is a polynomial of degree n called a **Hermite polynomial**. The first four Hermite polynomials are

- $H_0(y) = 1$
- $H_1(y) = 2y$
- $H_2(y) = 4y^2 - 2$
- $H_3(y) = 8y^3 - 12y$.

A few sample wavefunctions are given in Figure $4.6.2$. As the value of the principal number increases, the solutions alternate between even functions and odd functions about $x = 0$.

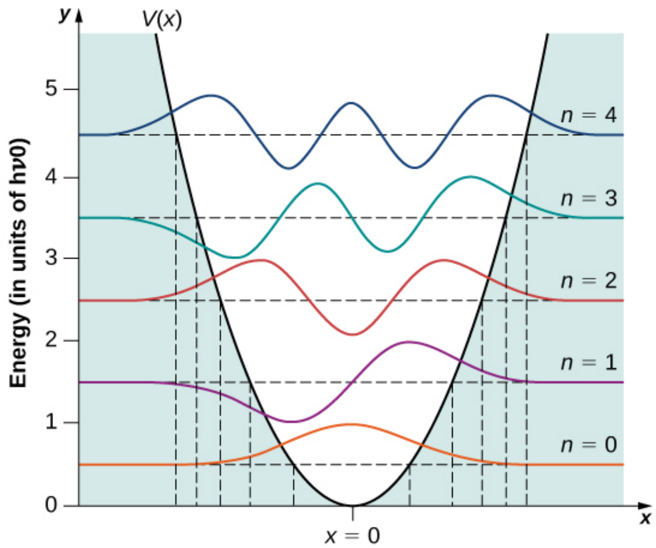


Figure 4.6.2: The first five wavefunctions of the quantum harmonic oscillator. The classical limits of the oscillator's motion are indicated by vertical lines, corresponding to the classical turning points at $x = \pm A$ of a classical particle with the same energy as the energy of a quantum oscillator in the state indicated in the figure.

Example 4.6.1: Classical Region of Harmonic Oscillations

Find the amplitude A of oscillations for a classical oscillator with energy equal to the energy of a quantum oscillator in the quantum state n.

Strategy

To determine the amplitude A, we set the classical energy $E = kx^2/2 = m\omega^2\ A^2/2$ equal to E_n given by Equation 4.6.6.

Solution

We obtain

$$\begin{aligned} E_n &= m\omega^2 A_n^2/2 \\ A_n &= \sqrt{\frac{2}{m\omega^2} E_n} \\ &= \sqrt{\frac{2}{m\omega^2} \frac{2n+1}{2} \hbar\omega} \\ &= \sqrt{(2n+1)\frac{\hbar}{m\omega}}. \end{aligned}$$

Significance

As the quantum number **n** increases, the energy of the oscillator and therefore the amplitude of oscillation increases (for a fixed natural angular frequency. For large **n**, the amplitude is approximately proportional to the square root of the quantum number.

Several interesting features appear in this solution. Unlike a classical oscillator, the measured energies of a quantum oscillator can have only energy values given by Equation 4.6.6. Moreover, unlike the case for a quantum particle in a box, the allowable energy levels are evenly spaced,

$$\Delta E = E_{n+1} - E_n \tag{4.6.8}$$

$$= \frac{2(n+1)+1}{2}\hbar\omega - \frac{2n+1}{2}\hbar\omega \tag{4.6.9}$$

$$= \hbar\omega = hf. \tag{4.6.10}$$

When a particle bound to such a system makes a transition from a higher-energy state to a lower-energy state, the smallest-energy quantum carried by the emitted photon is necessarily hf. Similarly, when the particle makes a transition from a lower-energy state to a higher-energy state, the smallest-energy quantum that can be absorbed by the particle is hf. A quantum oscillator can absorb or emit energy only in multiples of this smallest-energy quantum. This is consistent with Planck's hypothesis for the energy exchanges between radiation and the cavity walls in the blackbody radiation problem.

Example 4.6.2: Vibrational Energies of the Hydrogen Chloride Molecule

The HCl diatomic molecule consists of one chlorine atom and one hydrogen atom. Because the chlorine atom is 35 times more massive than the hydrogen atom, the vibrations of the HCl molecule can be quite well approximated by assuming that the Cl atom is motionless and the H atom performs harmonic oscillations due to an elastic molecular force modeled by Hooke's law. The infrared vibrational spectrum measured for hydrogen chloride has the lowest-frequency line centered at $f = 8.88 \times 10^{13} Hz$. What is the spacing between the vibrational energies of this molecule? What is the force constant **k** of the atomic bond in the HCl molecule?

Strategy

The lowest-frequency line corresponds to the emission of lowest-frequency photons. These photons are emitted when the molecule makes a transition between two adjacent vibrational energy levels. Assuming that energy levels are equally spaced, we use Equation 4.6.10 to estimate the spacing. The molecule is well approximated by treating the Cl atom as being infinitely heavy and the H atom as the mass m that performs the oscillations. Treating this molecular system as a classical oscillator, the force constant is found from the classical relation $k = m\omega^2$.

Solution

The energy spacing is

$$\begin{aligned}\Delta E &= hf \\ &= (4.14 \times 10^{-15} eV \cdot s)(8.88 \times 10^{13} Hz) \\ &= 0.368\, eV.\end{aligned}$$

The force constant is

$$\begin{aligned}k &= m\omega^2 \\ &= m\,(2\pi f)^2 \\ &= (1.67 \times 10^{-27} kg)(2\pi \times 8.88 \times 10^{13}\, Hz)^2 \\ &= 520\, N/m.\end{aligned}$$

Significance

The force between atoms in an HCl molecule is surprisingly strong. The typical energy released in energy transitions between vibrational levels is in the infrared range. As we will see later, transitions in between vibrational energy levels of a diatomic molecule often accompany transitions between rotational energy levels.

Exercise 4.6.1

The vibrational frequency of the hydrogen iodide HI diatomic molecule is $6.69 \times 10^{13}\, Hz$.

a. What is the force constant of the molecular bond between the hydrogen and the iodine atoms?
b. What is the energy of the emitted photon when this molecule makes a transition between adjacent vibrational energy levels?

Answer a

295 N/m

Answer b

0.277 eV

The quantum oscillator differs from the classic oscillator in three ways:

- First, the ground state of a quantum oscillator is $E_0 = \hbar\omega/2$, not zero. In the classical view, the lowest energy is zero. The nonexistence of a zero-energy state is common for all quantum-mechanical systems because of omnipresent fluctuations that are a consequence of the Heisenberg uncertainty principle. If a quantum particle sat motionless at the bottom of the potential well, its momentum as well as its position would have to be simultaneously exact, which would violate the Heisenberg uncertainty principle. Therefore, the lowest-energy state must be characterized by uncertainties in momentum and in position, so the ground state of a quantum particle must lie above the bottom of the potential well.
- Second, a particle in a quantum harmonic oscillator potential can be found with nonzero probability outside the interval $-A \leq x \leq +A$. In a classic formulation of the problem, the particle would not have any energy to be in this region. The probability of finding a ground-state quantum particle in the classically forbidden region is about 16%.
- Third, the probability density distributions $|\psi_n(x)|^2$ for a quantum oscillator in the ground low-energy state, $\psi_0(x)$, is largest at the middle of the well $(x = 0)$. For the particle to be found with greatest probability at the center of the well, we expect that the particle spends the most time there as it oscillates. This is opposite to the behavior of a classical oscillator, in which the particle spends most of its time moving with relative small speeds near the turning points.

Exercise 4.6.2

Find the expectation value of the position for a particle in the ground state of a harmonic oscillator using symmetry.

Answer b

$$\langle x \rangle = 0$$

Quantum probability density distributions change in character for excited states, becoming more like the classical distribution when the quantum number gets higher. We observe this change already for the first excited state of a quantum oscillator because the distribution $|\psi_1(x)|^2$ peaks up around the turning points and vanishes at the equilibrium position, as seen in Figure 4.6.2. In accordance with Bohr's correspondence principle, in the limit of high quantum numbers, the quantum description of a harmonic oscillator converges to the classical description, which is illustrated in Figure 4.6.3. The classical probability density distribution corresponding to the quantum energy of the $n = 12$ state is a reasonably good approximation of the quantum probability distribution for a quantum oscillator in this excited state. This agreement becomes increasingly better for highly excited states.

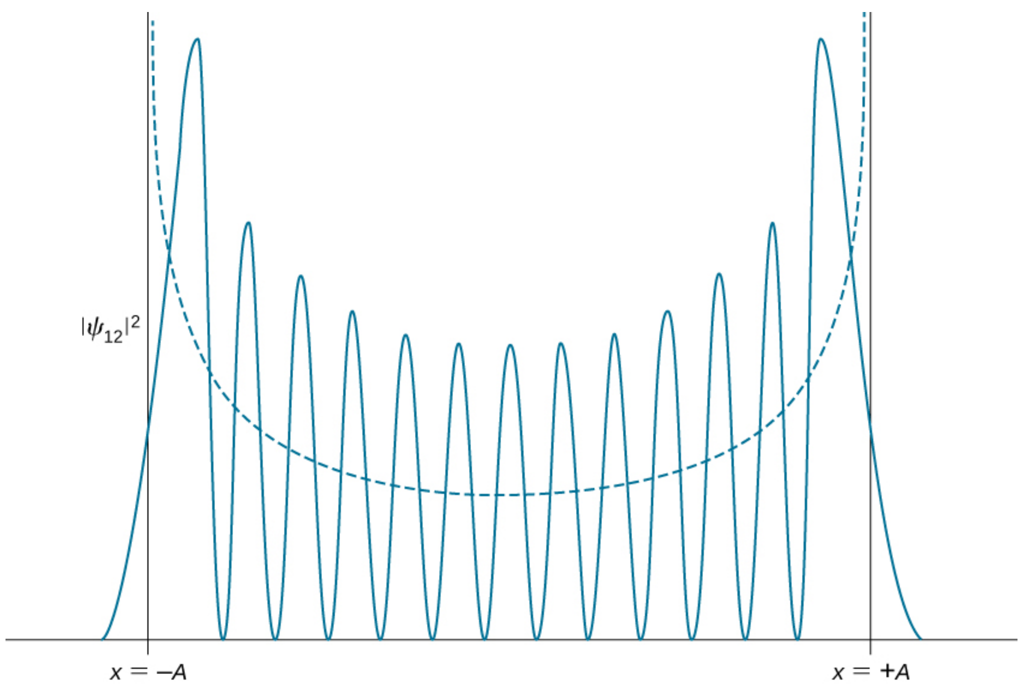


Figure 4.6.1: The probability density distribution for finding the quantum harmonic oscillator in its $n = 12$ quantum state. The dashed curve shows the probability density distribution of a classical oscillator with the same energy.

4.7: Quantum Tunneling of Particles through Potential Barriers

Learning Objectives

By the end of this section, you will be able to:

- Describe how a quantum particle may tunnel across a potential barrier
- Identify important physical parameters that affect the tunneling probability
- Identify the physical phenomena where quantum tunneling is observed
- Explain how quantum tunneling is utilized in modern technologies

Quantum tunneling is a phenomenon in which particles penetrate a potential energy barrier with a height greater than the total energy of the particles. The phenomenon is interesting and important because it violates the principles of classical mechanics. Quantum tunneling is important in models of the Sun and has a wide range of applications, such as the scanning tunneling microscope and the tunnel diode.

Tunneling and Potential Energy

To illustrate **quantum tunneling**, consider a ball rolling along a surface with a kinetic energy of 100 J. As the ball rolls, it encounters a hill. The potential energy of the ball placed atop the hill is 10 J. Therefore, the ball (with 100 J of kinetic energy) easily rolls over the hill and continues on. In classical mechanics, the probability that the ball passes over the hill is exactly 1—it makes it over every time. If, however, the height of the hill is increased—a ball placed atop the hill has a potential energy of 200 J—the ball proceeds only part of the way up the hill, stops, and returns in the direction it came. The total energy of the ball is converted entirely into potential energy before it can reach the top of the hill. We do not expect, even after repeated attempts, for the 100-J ball to ever be found beyond the hill. Therefore, the probability that the ball passes over the hill is exactly 0, and probability it is turned back or “reflected” by the hill is exactly 1. The ball **never** makes it over the hill. The existence of the ball beyond the hill is an impossibility or “energetically forbidden.”

However, according to quantum mechanics, the ball has a wave function and this function is defined over all space. The wave function may be highly localized, but there is always a chance that as the ball encounters the hill, the ball will suddenly be found beyond it. Indeed, this probability is appreciable if the “wave packet” of the ball is wider than the barrier.

View this interactive simulation for a simulation of tunneling.

In the language of quantum mechanics, the hill is characterized by a **potential barrier**. A finite-height square barrier is described by the following potential-energy function:

$$U(x)=\begin{cases}0, & \text{when } x<0\\ U_0, & \text{when } 0\le x\le L\\ 0, & \text{when } x>L\end{cases} \tag{4.7.1}$$

The potential barrier is illustrated in Figure 4.7.1. When the height U_0 of the barrier is infinite, the wave packet representing an incident quantum particle is unable to penetrate it, and the quantum particle bounces back from the barrier boundary, just like a classical particle. When the width L of the barrier is infinite and its height is finite, a part of the wave packet representing an incident quantum particle can filter through the barrier boundary and eventually perish after traveling some distance inside the barrier.

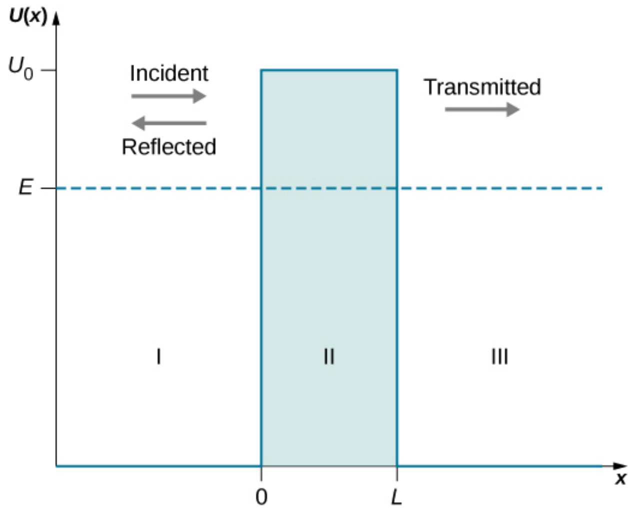


Figure 4.7.1: A potential energy barrier of height U_0 creates three physical regions with three different wave behaviors. In region I where $x < 0$, an incident wave packet (incident particle) moves in a potential-free zone and coexists with a reflected wave packet (reflected particle). In region II, a part of the incident wave that has not been reflected at $x = 0$ moves as a transmitted wave in a constant potential $U(x) = +U_0$ and tunnels through to region III at $x = L$. In region III for $x > L$, a wave packet (transmitted particle) that has tunneled through the potential barrier moves as a free particle in potential-free zone. The energy E of the incident particle is indicated by the horizontal line.

When both the width L and the height U_0 are finite, a part of the quantum wave packet incident on one side of the barrier can penetrate the barrier boundary and continue its motion inside the barrier, where it is gradually attenuated on its way to the other side. A part of the incident quantum wave packet eventually emerges on the other side of the barrier in the form of the transmitted wave packet that tunneled through the barrier. How much of the incident wave can tunnel through a barrier depends on the barrier width L and its height U_0, and on the energy E of the quantum particle incident on the barrier. This is the physics of tunneling.

Barrier penetration by quantum wave functions was first analyzed theoretically by Friedrich **Hund** in 1927, shortly after Schrödinger published the equation that bears his name. A year later, George **Gamow** used the formalism of quantum mechanics to explain the radioactive α-decay of atomic nuclei as a quantum-tunneling phenomenon. The invention of the tunnel diode in 1957 made it clear that quantum tunneling is important to the semiconductor industry. In modern nanotechnologies, individual atoms are manipulated using a knowledge of quantum tunneling.

Tunneling and the Wavfunction

Suppose a uniform and time-independent beam of electrons or other quantum particles with energy E traveling along the **x**-axis (in the positive direction to the right) encounters a potential barrier described by Equation 4.7.1. The question is: What is the probability that an individual particle in the beam will tunnel through the potential barrier? The answer can be found by solving the boundary-value problem for the time-independent Schrödinger equation for a particle in the beam. The general form of this equation is given by Equation 4.7.2, which we reproduce here:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+U(x)\psi(x)=E\psi(x), \tag{4.7.2}$$

where $-\infty < x < +\infty$.

The potential function $U(x)$ in Equation 4.7.2 is defined by Equation 4.7.1. We assume that the given energy E of the incoming particle is smaller than the height U_0 of the potential barrier, $E < U_0$, because this is the interesting physical case. Knowing the energy E of the incoming particle, our task is to solve Equation 4.7.2 for a function $\psi(x)$ that is continuous and has continuous first derivatives for all **x**. In other words, we are looking for a "smooth-looking" solution (because this is how wave functions look) that can be given a probabilistic interpretation so that $|\psi(x)|^2 = \psi^*(x)\psi(x)$ is the probability density.

We divide the real axis into three regions with the boundaries defined by the potential function in Equation 4.7.1 (illustrated in Figure 4.7.1) and transcribe Equation 4.7.2 for each region. Denoting by $\psi_I(x)$ the solution in region I for $x < 0$, by $\psi_{II}(x)$ the solution in region II for $0 \le x \le L$, and by $\psi_{III}(x)$ the solution in region III for $x > L$, the stationary Schrödinger equation has the following forms in these three regions:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I(x)}{dx^2} = E\psi_I(x), \tag{4.7.3}$$

in region I: $-\infty < x < 0$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}(x)}{dx^2} + U_0\psi_{II}(x) = E\psi_{II}(x) \tag{4.7.4}$$

in region II: $0 < x < L$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{III}(x)}{dx^2} = E\psi_{III}(x) \tag{4.7.5}$$

in region III: $L < x < +\infty$,

The continuity condition at region boundaries requires that:

$$\psi_I(0) = \psi_{II}(0) \tag{4.7.6}$$

at the boundary between regions I and II

and

$$\psi_{II}(L) = \psi_{III}(L) \tag{4.7.7}$$

at the boundary between regions II and III.

The “smoothness” condition requires the first derivative of the solution be continuous at region boundaries:

$$\left.\frac{d\psi_I(x)}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}(x)}{dx}\right|_{x=0} \tag{4.7.8}$$

at the boundary between regions I and II

and

$$\left.\frac{d\psi_{II}(x)}{dx}\right|_{x=L} = \left.\frac{d\psi_{III}(x)}{dx}\right|_{x=L} \tag{4.7.9}$$

at the boundary between regions II and III.

In what follows, we find the functions $\psi_I(x)$, $\psi_{II}(x)$, and $\psi_{III}(x)$.

We can easily verify (by substituting into the original equation and differentiating) that in regions I and III, the solutions must be in the following general forms:

$$\psi_I(x) = Ae^{+ikx} + Be^{-ikx} \tag{4.7.10}$$

$$\psi_{III}(x) = Fe^{+ikx} + Ge^{-ikx} \tag{4.7.11}$$

where $k = \sqrt{2mE}/\hbar$ is a **wave number** and the complex exponent denotes oscillations,

$$e^{\pm ikx} = \cos kx \pm i \sin kx.$$

The constants A, B, F, and G in Equations 4.7.10 and 4.7.11 may be complex. These solutions are illustrated in Figure 4.7.2. In region I, there are two waves—one is incident (moving to the right) and one is reflected (moving to the left)—so none of the constants A and B in Equation 4.7.10 may vanish. In region III, there is only one wave (moving to the right), which is the transmitted wave, so the constant G must be zero in Equation 4.7.11, $G = 0$. We can write explicitly that the incident wave is $\psi_{in}(x) = Ae^{+ikx}$ and that the reflected wave is $\psi_{ref}(x) = Be^{-ikx}$, and that the transmitted wave is $\psi_{tra}(x) = Fe^{+ikx}$. The amplitude of the incident wave is

$$\begin{aligned}|\psi_{in}(x)|^2 &= \psi^*_{in}(x)\psi_{in}(x)\\ &= (Ae^{+ikx})^*Ae^{+ikx}\\ &= A^*e^{-ikx}Ae^{+ikx}\\ &= A^*A = |A|^2.\end{aligned}$$

Similarly, the amplitude of the reflected wave is $|\psi_{ref}(x)|^2 = |B|^2$ and the amplitude of the transmitted wave is $|\psi_{tra}(x)|^2 = |F|^2$. We know from the theory of waves that the square of the wave amplitude is directly proportional to the wave intensity. If we want to know how much of the incident wave tunnels through the barrier, we need to compute the square of the amplitude of the transmitted wave. The **transmission probability** or **tunneling probability** is the ratio of the transmitted intensity ($|F|^2$) to the incident intensity ($|A|^2$), written as

$$T(L,E) = \frac{|\psi_{tra}(x)|^2}{|\psi_{in}(x)|^2} \tag{4.7.12}$$

$$= \frac{|F|^2}{|A|^2} \tag{4.7.13}$$

$$= \left|\frac{F}{A}\right|^2 \tag{4.7.14}$$

where L is the width of the barrier and E is the total energy of the particle. This is the probability an individual particle in the incident beam will tunnel through the potential barrier. Intuitively, we understand that this probability must depend on the barrier height U_0.

In region II, the terms in equation Equation 4.7.4 can be rearranged to

$$\frac{d^2\psi_{II}(x)}{dx^2} = \beta^2\psi_{II}(x) \tag{4.7.15}$$

where β^2 is positive because $U_0 > E$ and the parameter β is a real number,

$$\beta^2 = \frac{2m}{\hbar^2}(U_0 - E). \tag{4.7.16}$$

The general solution to Equation 4.7.15 is not oscillatory (unlike in the other regions) and is in the form of exponentials that describe a gradual attenuation of $\psi_{II}(x)$,

$$\psi_{II}(x) = Ce^{-\beta x} + De^{+\beta x}. \tag{4.7.17}$$

The two types of solutions in the three regions are illustrated in Figure 4.7.2.

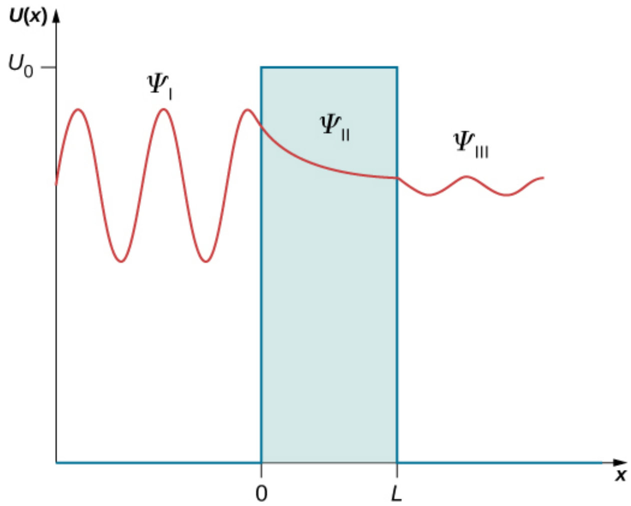


Figure 4.7.2: Three types of solutions to the stationary Schrödinger equation for the quantum-tunneling problem: Oscillatory behavior in regions I and III where a quantum particle moves freely, and exponential-decay behavior in region II (the barrier region) where the particle moves in the potential U_0.

Now we use the boundary conditions to find equations for the unknown constants. Equations 4.7.10 and 4.7.17 are substituted into Equation 4.7.6 to give

$$A + B = C + D.$$

Equations 4.7.17 and 4.7.11 are substituted into Equation 4.7.7 to give

$$Ce^{-\beta L} + De^{+\beta L} = Fe^{+ikL}.$$

Similarly, we substitute Equations 4.7.10 and 4.7.17 into Equation 4.7.8, differentiate, and obtain

$$-ik(A - B) = \beta(D - C).$$

Similarly, the boundary condition Equation 4.7.9 reads explicitly

$$\beta(De^{+\beta L} - Ce^{-\beta L}) = +ikFe^{+ikL}.$$

We now have four equations for five unknown constants. However, because the quantity we are after is the transmission coefficient (T), defined in Equation 4.7.14 by the fraction F/A, the number of equations is exactly right because when we divide each of the above equations by A, we end up having only four unknown fractions: B/A, C/A, D/A, and F/A, three of which can be eliminated to find F/A. The actual algebra that leads to expression for F/A is pretty lengthy, but it can be done either by hand or with a help of computer software. The end result is

$$\frac{F}{A} = \frac{e^{-ikL}}{\cosh(\beta L) + i(\gamma/2)\sinh(\beta L)}. \tag{4.7.18}$$

In deriving Equation 4.7.18, to avoid the clutter, we use the substitutions $\gamma \equiv \beta/k - k/\beta$, and the definition of hyperbolic functions:

$$\cosh y = \frac{e^y + e^{-y}}{2}$$

and

$$\sinh y = \frac{e^y - e^{-y}}{2}.$$

We substitute Equation 4.7.18 into Equation 4.7.14 and obtain the exact expression for the transmission coefficient for the barrier,

$$T(L,E) = \left(\frac{F}{A}\right)^* \frac{F}{A} = \frac{e^{+ikL}}{\cosh(\beta L) - i(\gamma/2)\sinh(\beta L)} \cdot \frac{e^{-ikL}}{\cosh(\beta L) + i(\gamma/2)\sinh(\beta L)}.$$

or

$$T(L, E) = \frac{1}{\cosh^2(\beta L) + (\gamma/2)^2 \sinh^2(\beta L)}. \tag{4.7.19}$$

where

$$\left(\frac{\gamma}{2}\right)^2 = \frac{1}{4}\left(\frac{1 - E/U_0}{E/U_0} + \frac{E/U_0}{1 - E/U_0} - 2\right).$$

For a wide and high barrier that transmits poorly, Equation 4.7.19 can be approximated by

$$T(L, E) \approx 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right) e^{-2\beta L}. \tag{4.7.20}$$

Whether it is the exact expression (Equation 4.7.19) or the approximate expression (Equation 4.7.20), we see that the tunneling effect very strongly depends on the width L of the potential barrier. In the laboratory, we can adjust both the potential height U_0 and the width L to design nano-devices with desirable transmission coefficients.

Example 4.7.1: Transmission Coefficient

Two copper nanowires are insulated by a copper oxide nano-layer that provides a 10.0-eV potential barrier. Estimate the tunneling probability between the nanowires by 7.00-eV electrons through a 5.00-nm thick oxide layer. What if the thickness of the layer were reduced to just 1.00 nm? What if the energy of electrons were increased to 9.00 eV?

Strategy

Treating the insulating oxide layer as a finite-height potential barrier, we use Equation 4.7.20. We identify $U_0 = 10.0\,eV$, $E_1 = 7.00\,eV$, $E_2 = 9.00\,eV$, $L_1 = 5.00\,nm$, and $L_2 = 1.00\,nm$. We use Equation 4.7.16 to compute the exponent. Also, we need the rest mass of the electron $m = 511\,keV/c^2$ and Planck's constant $\hbar = 0.1973\,keV\cdot nm/c$. It is typical for this type of estimate to deal with very small quantities that are often not suitable for handheld calculators. To make correct estimates of orders, we make the conversion $e^y = 10^{y/ln\,10}$.

Solution

Constants:

$$\frac{2m}{\hbar^2} = \frac{2(511\,keV/c^2)}{(0.1973\,keV\cdot nm/c^2)^2} = 26,254\frac{1}{keV\cdot(nm)^2},$$

$$\beta = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)} = \sqrt{26,254\frac{(10.0\,eV - E)}{keV\cdot(nm)^2}} = \sqrt{26.254(10.00 - E)/eV}\frac{1}{nm}.$$

For a lower-energy electron with $E_1 = 7.00\,eV$:

$$\beta_1 = \sqrt{26.254(10.00\,eV - E_1)/eV}\frac{1}{nm} = \sqrt{26.254(10.00 - 7.00)}\frac{1}{nm} = \frac{8.875}{nm},$$

$$T(L, E) = 16\frac{E_1}{U_0}\left(1 - \frac{E_1}{U_0}\right) e^{-2\beta_1 L} = 16\frac{7}{10}\left(1 - \frac{7}{10}\right) e^{-17.75\,L/nm} = 3.36\,e^{-17.75\,L/nm}$$

For a higher-energy electron with $E_2 = 9.00\,eV$:

$$\beta_2 = \sqrt{26.254(10.00\,eV - E_2)/eV}\frac{1}{nm} = \sqrt{26.254(10.00 - 9.00)}\frac{1}{nm} = \frac{5.124}{nm},$$

$$T(L, E_2) = 16\frac{E_2}{U_0}\left(1 - \frac{E_2}{U_0}\right) e^{-2\beta_2 L} = 16\frac{9}{10}\left(1 - \frac{9}{10}\right) e^{-10.25\,L/nm} = 1.44\,e^{-10.25\,L/nm}$$

For a broad barrier with $L_1 = 5.00\,nm$:

$$T(L_1, E_1) = 3.36e^{-17.75\,L_1/nm} = 3.36e^{-17.75\cdot 5.00nm/nm} = 3.36e^{-88} = 3.36(6.2\times 10^{-39}) = 2.1\% \times 10^{-36}$$

$$T(L_1, E_2) = 1.44e^{-10.25\,L_1/nm} = 1.44e^{-10.25\cdot 5.00\,nm/nm} = 1.44^{-51.2} = 1.44(5.81\times 10^{-12}) = 8.36\% \times 10^{-25}$$

For a narrower barrier with $L_2 = 1.00\,nm$:

$$T(L_2, E_1) = 3.36e^{-17.75\,L_2/nm} = 3.36e^{-17.75\cdot 1.00\,nm/nm} = 3.36e^{-17.75} = 3.36(5.1\times 10^{-7}) = 1.7\%\times 10^{-4},$$

$$T(L_2, E_2) = 1.44e^{-10.25\,L_2/nm} = 1.44e^{-10.25\cdot 1.00\,nm/nm} = 1.44e^{-10.25} = 1.44(3.53\times 10^{-5}) = 5.09\%\times 10^{-7}.$$

Significance

We see from these estimates that the probability of tunneling is affected more by the width of the potential barrier than by the energy of an incident particle. In today's technologies, we can manipulate individual atoms on metal surfaces to create potential barriers that are fractions of a nanometer, giving rise to measurable tunneling currents. One of many applications of this technology is the scanning tunneling microscope (STM), which we discuss later in this section.

? Exercise 4.7.1

A proton with kinetic energy 1.00 eV is incident on a square potential barrier with height 10.00 eV. If the proton is to have the same transmission probability as an electron of the same energy, what must the width of the barrier be relative to the barrier width encountered by an electron?

Answer

$$L_{proton}/L_{electron} = \sqrt{m_e/m_p} = 2.3\%$$

Radioactive Decay

In 1928, Gamow identified quantum tunneling as the mechanism responsible for the **radioactive decay** of atomic nuclei. He observed that some isotopes of thorium, uranium, and bismuth disintegrate by emitting α-particles (which are doubly ionized helium atoms or, simply speaking, helium nuclei). In the process of emitting an α-particle, the original nucleus is transformed into a new nucleus that has two fewer neutrons and two fewer protons than the original nucleus. The α-particles emitted by one isotope have approximately the same kinetic energies. When we look at variations of these energies among isotopes of various elements, the lowest kinetic energy is about 4 MeV and the highest is about 9 MeV, so these energies are of the same order of magnitude. This is about where the similarities between various isotopes end.

When we inspect half-lives (a half-life is the time in which a radioactive sample loses half of its nuclei due to decay), different isotopes differ widely. For example, the half-life of polonium-214 is 160 µs and the half-life of uranium is 4.5 billion years. Gamow explained this variation by considering a 'spherical-box' model of the nucleus, where α-particles can bounce back and forth between the walls as free particles. The confinement is provided by a strong nuclear potential at a spherical wall of the box. The thickness of this wall, however, is not infinite but finite, so in principle, a nuclear particle has a chance to escape this nuclear confinement. On the inside wall of the confining barrier is a high nuclear potential that keeps the α-particle in a small confinement. But when an α-particle gets out to the other side of this wall, it is subject to electrostatic Coulomb repulsion and moves away from the nucleus. This idea is illustrated in Figure 4.7.3. The width L of the potential barrier that separates an α-particle from the outside world depends on the particle's kinetic energy E. This width is the distance between the point marked by the nuclear radius R and the point R_0 where an α-particle emerges on the other side of the barrier, $L = R_0 - R$. At the distance R_0, its kinetic energy must at least match the electrostatic energy of repulsion, $E = (4\pi\epsilon_0)^{-1} Ze^2/R_0$ (where +Ze is the charge of the nucleus). In this way we can estimate the width of the nuclear barrier,

$$L = \frac{e^2}{4\pi\epsilon_0}\frac{Z}{E} - R.$$

We see from this estimate that the higher the energy of α-particle, the narrower the width of the barrier that it is to tunnel through. We also know that the width of the potential barrier is the most important parameter in tunneling probability. Thus, highly energetic α-particles have a good chance to escape the nucleus, and, for such nuclei, the nuclear disintegration half-life is short. Notice that this process is highly nonlinear, meaning a small increase in the α-particle energy has a disproportionately large enhancing effect on the tunneling probability and, consequently, on shortening the half-life. This explains why the half-life of polonium that emits 8-MeV α-particles is only hundreds of milliseconds and the half-life of uranium that emits 4-MeV α-particles is billions of years.

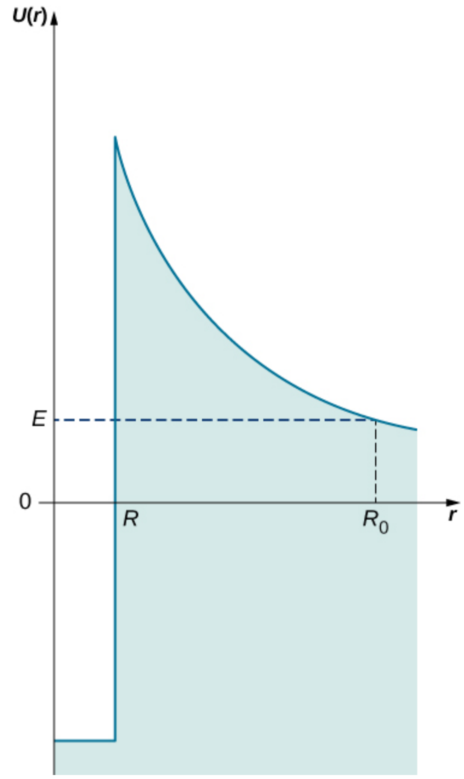


Figure 4.7.3: The potential energy barrier for an α-particle bound in the nucleus: To escape from the nucleus, an α-particle with energy E must tunnel across the barrier from distance R to distance R_0 away from the center.

Field Emission

Field emission is a process of emitting electrons from conducting surfaces due to a strong external electric field that is applied in the direction normal to the surface (Figure 4.7.4). As we know from our study of electric fields in earlier chapters, an applied external electric field causes the electrons in a conductor to move to its surface and stay there as long as the present external field is not excessively strong. In this situation, we have a constant electric potential throughout the inside of the conductor, including its surface. In the language of potential energy, we say that an electron inside the conductor has a constant potential energy $U(x) - -U_0$ (here, the **x** means inside the conductor).

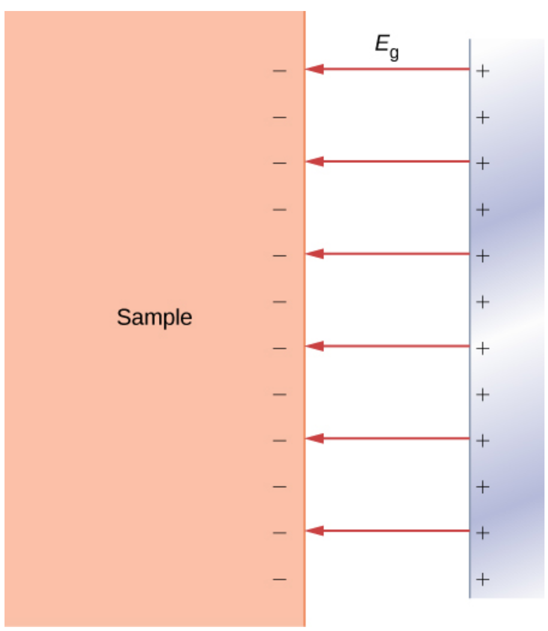


Figure 4.7.4: A normal-direction external electric field at the surface of a conductor: In a strong field, the electrons on a conducting surface may get detached from it and accelerate against the external electric field away from the surface.

In the situation represented in Figure 4.7.4, where the external electric field is uniform and has magnitude E_g, if an electron happens to be outside the conductor at a distance **x** away from its surface, its potential energy would have to be $U(x) = -eE_g x$ (here, **x** denotes distance to the surface). Taking the origin at the surface, so that $x = 0$ is the location of the surface, we can represent the potential energy of conduction electrons in a metal as the potential energy barrier shown in Figure 4.7.5. In the absence of the external field, the potential energy becomes a step barrier defined by $U(x \leq 0) = -U_0$ and by $U(x > 0) = 0$.

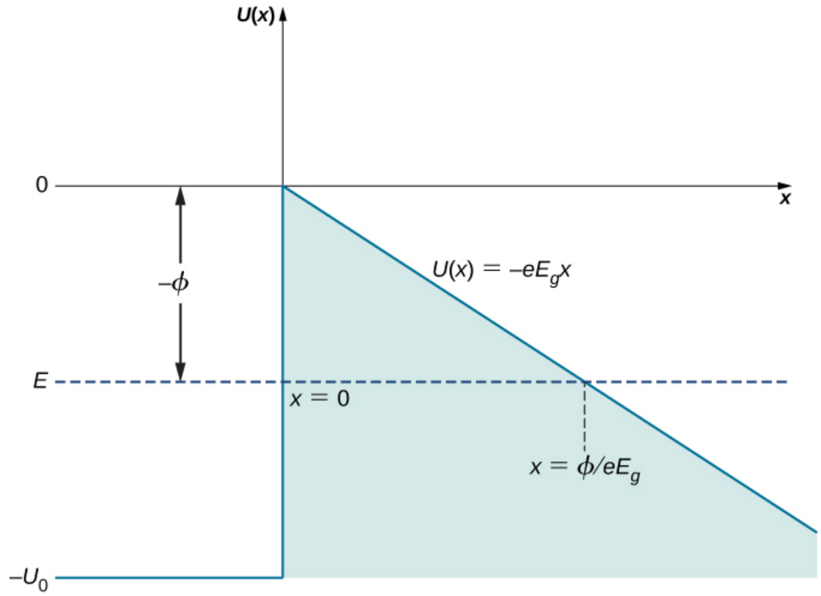


Figure 4.7.5: The potential energy barrier at the surface of a metallic conductor in the presence of an external uniform electric field E_g normal to the surface: It becomes a step-function barrier when the external field is removed. The work function of the metal is indicated by ϕ.

When an external electric field is strong, conduction electrons at the surface may get detached from it and accelerate along electric field lines in a direction antiparallel to the external field, away from the surface. In short, conduction electrons may escape from the surface. The field emission can be understood as the quantum tunneling of conduction electrons through the potential barrier at the conductor's surface. The physical principle at work here is very similar to the mechanism of α-emission from a radioactive nucleus.

Suppose a conduction electron has a kinetic energy **E** (the average kinetic energy of an electron in a metal is the work function ϕ for the metal and can be measured, as discussed for the photoelectric effect in Photons and Matter Waves), and an external electric field can be locally approximated by a uniform electric field of strength E_g. The width **L** of the potential barrier that the electron must cross is the distance from the conductor's surface to the point outside the surface where its kinetic energy matches the value of its potential energy in the external field. In Figure 4.7.5, this distance is measured along the dashed horizontal line $U(x) = E$ from $x = 0$ to the intercept with $U(x) = -eE_g x$, so the barrier width is

$$L = \frac{e^{-1}E}{E_g} = \frac{e^{-1}\phi}{E_g}$$

We see that **L** is inversely proportional to the strength E_g of an external field. When we increase the strength of the external field, the potential barrier outside the conductor becomes steeper and its width decreases for an electron with a given kinetic energy. In turn, the probability that an electron will tunnel across the barrier (conductor surface) becomes exponentially larger. The electrons that emerge on the other side of this barrier form a current (tunneling-electron current) that can be detected above the surface. The tunneling-electron current is proportional to the tunneling probability. The tunneling probability depends nonlinearly on the barrier width **L**, and **L** can be changed by adjusting E_g. Therefore, the tunneling-electron current can be tuned by adjusting the strength of an external electric field at the surface. When the strength of an external electric field is constant, the tunneling-electron current has different values at different elevations **L** above the surface.

scanning tunneling microscope

The quantum tunneling phenomenon at metallic surfaces, which we have just described, is the physical principle behind the operation of the scanning tunneling microscope (STM), invented in 1981 by Gerd Binnig and Heinrich Rohrer. The STM device consists of a scanning tip (a needle, usually made of tungsten, platinum-iridium, or gold); a piezoelectric device that controls the tip's elevation in a typical range of 0.4 to 0.7 nm above the surface to be scanned; some device that controls the motion of the tip along the surface; and a computer to display images. While the sample is kept at a suitable voltage bias, the scanning tip moves along the surface (Figure 4.7.6) and the tunneling-electron current between the tip and the surface is registered at each position.

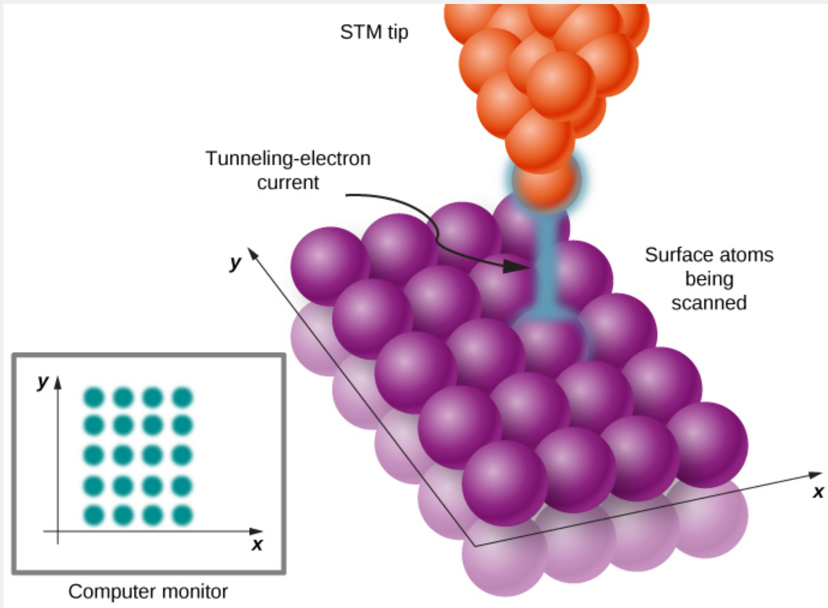


Figure 4.7.6: In STM, a surface at a constant potential is being scanned by a narrow tip moving along the surface. When the STM tip moves close to surface atoms, electrons can tunnel from the surface to the tip. This tunneling-electron current is continually monitored while the tip is in motion. The amount of current at location (x,y) gives information about the elevation of the tip above the surface at this location. In this way, a detailed topographical map of the surface is created and displayed on a computer monitor.

The amount of the current depends on the probability of electron tunneling from the surface to the tip, which, in turn, depends on the elevation of the tip above the surface. Hence, at each tip position, the distance from the tip to the surface is measured by measuring how many electrons tunnel out from the surface to the tip. This method can give an unprecedented resolution of about 0.001 nm, which is about 1% of the average diameter of an atom. In this way, we can see individual atoms on the surface, as in the image of a carbon nanotube in Figure 4.7.7.

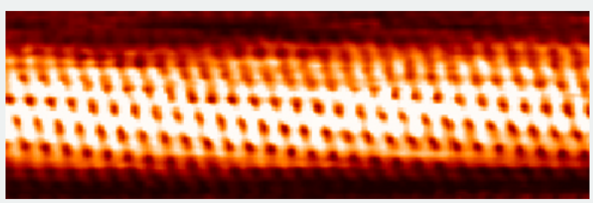


Figure 4.7.7: An STM image of a carbon nanotube: Atomic-scale resolution allows us to see individual atoms on the surface. STM images are in gray scale, and coloring is added to bring up details to the human eye.

Resonant Quantum Tunneling

Quantum tunneling has numerous applications in semiconductor devices such as electronic circuit components or integrated circuits that are designed at nanoscales; hence, the term '**nanotechnology**.' For example, a diode (an electric-circuit element that causes an electron current in one direction to be different from the current in the opposite direction, when the polarity of the bias voltage is reversed) can be realized by a tunneling junction between two different types of semiconducting materials. In such a **tunnel diode**, electrons tunnel through a single potential barrier at a contact between two different semiconductors. At the junction, tunneling-electron current changes nonlinearly with the applied potential difference across the junction and may rapidly decrease as the bias voltage is increased. This is unlike the Ohm's law behavior that we are familiar with in household circuits. This kind of rapid behavior (caused by quantum tunneling) is desirable in high-speed electronic devices.

Another kind of electronic nano-device utilizes **resonant tunneling** of electrons through potential barriers that occur in **quantum dots**. A quantum dot is a small region of a semiconductor nanocrystal that is grown, for example, in a silicon or aluminum arsenide crystal. Figure $4.7.8a$ shows a quantum dot of gallium arsenide embedded in an aluminum arsenide wafer. The quantum-dot region acts as a potential well of a finite height (Figure $4.7.8b$) that has two finite-height potential barriers at dot boundaries. Similarly, as for a quantum particle in a box (that is, an infinite potential well), lower-lying energies of a quantum particle trapped in a finite-height potential well are quantized. The difference between the box and the well potentials is that a quantum particle in a box has an infinite number of quantized energies and is trapped in the box indefinitely, whereas a quantum particle trapped in a potential well has a finite number of quantized energy levels and can tunnel through potential barriers at well boundaries to the outside of the well. Thus, a quantum dot of gallium arsenide sitting in aluminum arsenide is a potential well where low-lying energies of an electron are quantized, indicated as E_{dot} in part (b) in the figure. When the energy $E_{electron}$ of an electron in the outside region of the dot does not match its energy E_{dot} that it would have in the dot, the electron does not tunnel through the region of the dot and there is no current through such a circuit element, even if it were kept at an electric voltage difference (bias). However, when this voltage bias is changed in such a way that one of the barriers is lowered, so that E_{dot} and $E_{electron}$ become aligned, as seen in part (c) of the figure, an electron current flows through the dot. When the voltage bias is now increased, this alignment is lost and the current stops flowing. When the voltage bias is increased further, the electron tunneling becomes improbable until the bias voltage reaches a value for which the outside electron energy matches the next electron energy level in the dot. The word ‘resonance’ in the device name means that the tunneling-electron current occurs only when a selected energy level is matched by tuning an applied voltage bias, such as in the operation mechanism of the resonant-tunneling diode just described. Resonant-tunneling diodes are used as super-fast nano-switches.

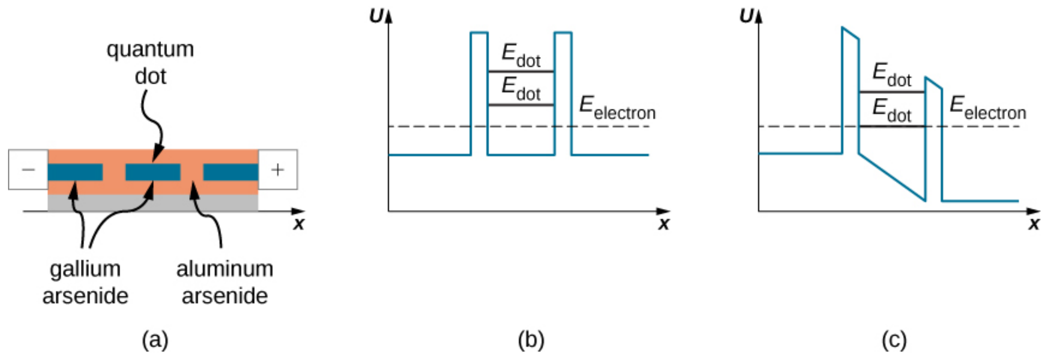


Figure $4.7.8$: Resonant-tunneling diode: (a) A quantum dot of gallium arsenide embedded in aluminum arsenide. (b) Potential well consisting of two potential barriers of a quantum dot with no voltage bias. Electron energies $E_{electron}$ **in aluminum arsenide are not aligned with their energy levels** E_{dot} **in the quantum dot, so electrons do not tunnel through the dot. (c) Potential well of the dot with a voltage bias across the device. A suitably tuned voltage difference distorts the well so that electron-energy levels in the dot are aligned with their energies in aluminum arsenide, causing the electrons to tunnel through the dot.**

4.A: Quantum Mechanics (Answers)

Check Your Understanding

7.1. $(3+4i)(3-4i)=9-16i^2=25$

7.2. $A=\sqrt{2/L}$

7.3. $(1/2-1/\pi)/2=9$

7.4. $4.1\times 10^{-8}eV; 1.1\times 10^{-5}nm$

7.5. $0.5m\omega^2x^2\psi(x)*\psi(x)$

7.6. None. The first function has a discontinuity; the second function is double-valued; and the third function diverges so is not normalizable.

7.7. a. 9.1%;

b. 25%

7.8. a. 295 N/m;

b. 0.277 eV

7.9. $\langle x\rangle=0$

7.10. $L_{proton}/L_{electron}=\sqrt{m_e/m_p}=2.3$

Conceptual Questions

1. $1/\sqrt{L}$, where $L=length$; **1/L**, where $L=length$

3. The wave function does not correspond directly to any measured quantity. It is a tool for predicting the values of physical quantities.

5. The average value of the physical quantity for a large number of particles with the same wave function.

7. Yes, if its position is completely unknown. Yes, if its momentum is completely unknown.

9. No. According to the uncertainty principle, if the uncertainty on the particle's position is small, the uncertainty on its momentum is large. Similarly, if the uncertainty on the particle's position is large, the uncertainty on its momentum is small.

11. No, it means that predictions about the particle (expressed in terms of probabilities) are time-independent.

13. No, because the probability of the particle existing in a narrow (infinitesimally small) interval at the discontinuity is undefined.

15. No. For an infinite square well, the spacing between energy levels increases with the quantum number **n**. The **smallest** energy measured corresponds to the transition from **n** = 2 to 1, which is three times the ground state energy. The largest energy measured corresponds to a transition from $n=\infty$ to 1, which is infinity. (Note: Even particles with extremely large energies remain bound to an infinite square well—they can never "escape")

17. No. This energy corresponds to $n=0.25$, but **n** must be an integer.

19. Because the smallest allowed value of the quantum number n for a simple harmonic oscillator is 0. No, because quantum mechanics and classical mechanics agree only in the limit of large nn.

21. Yes, within the constraints of the uncertainty principle. If the oscillating particle is localized, the momentum and therefore energy of the oscillator are distributed.

23. doubling the barrier width

25. No, the restoring force on the particle at the walls of an infinite square well is infinity.

27. $\left|\psi(x)\right|^2 \, sin^2 \omega t$

29. (a) and (e), can be normalized

31. a. $A = \sqrt{2\alpha/\pi}$;

b. $probability = 29.3$;

c. $\langle x \rangle = 0 \langle x \rangle = 0$;

d. $\langle p \rangle = 0$;

e. $\langle K \rangle = \alpha^2 \hbar^2/2m$

33. a. $\Delta p \geq 2.11 \times 10^{-34} N \cdot s$;

b. $\Delta v \geq 6.31 \times 10^{-8} m$;

c. $\Delta v/\sqrt{k_B T/m_\alpha} = 5.94 \times 10^{-11}$

35. $\Delta \tau \geq 1.6 \times 10^{-25} s$

37. a. $\Delta f \geq 1.59 MHz$;

b. $\Delta \omega/\omega_0 = 3.135 \times 10^{-9}$

39. Carrying out the derivatives yields $k^2 = \dfrac{\omega^2}{c^2}$.

41. Carrying out the derivatives (as above) for the sine function gives a cosine on the right side the equation, so the equality fails. The same occurs for the cosine solution.

43. $E = \hbar^2 k^2/2m$

45. $\hbar^2 k^2 \hbar$; The particle has definite momentum and therefore definite momentum squared.

47. 9.4 eV, 64%

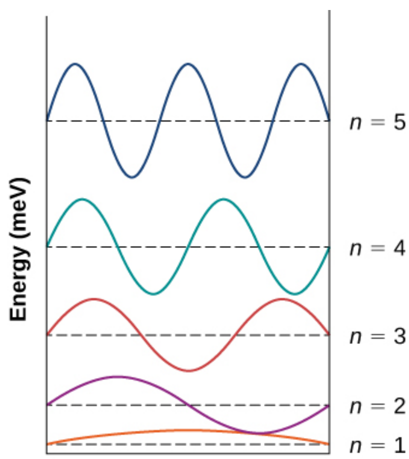
49. 0.38 nm

51. 1.82 MeV

53. 24.7 nm

55. 6.03Å

57. a.



The wave functions for the n=1 through n=5 states of the electron in an infinite square well are shown. Each function is displaced vertically by its energy, measured in meV. The n=1 state is the first half wave of the sine function. The n=2 function is the first full wave of the sine function. The n=3 function is the first one and a half waves of the sine function. The n=4 function is the first two waves of the sine function. The n=5 function is the first two and a half waves of the sine function. ;

b. $\lambda_{5\to 3} = 12.9nm, \lambda_{3\to 1} = 25.8nm, \lambda_{4\to 3} = 29.4nm$

59. proof

61. $6.662 \times 10^{14} Hz$

63. $n \approx 2.037 \times 10^{30}$

65. $\langle x \rangle = 0.5m\omega^2 \langle x^2 \rangle = \hbar\omega/4$; $\langle K \rangle = \langle E \rangle - \langle U \rangle = \hbar\omega/4$

67. proof

69. A complex function of the form, $Ae^{i\phi}$, satisfies Schrödinger's time-independent equation. The operators for kinetic and total energy are linear, so any linear combination of such wave functions is also a valid solution to Schrödinger's equation. Therefore, we conclude that Equation 7.68 satisfies Equation 7.61, and Equation 7.69 satisfies Equation 7.63.

71. a. 4.21%;

b. 0.84%;

c. 0.06%

73. a. 0.13%;

b. close to 0%

75. 0.38 nm

Additional Problems

77. proof

79. a. 4.0 %;

b. 1.4 %;

c. 4.0%;

d. 1.4%

81. a. $t = mL^2/h = 2.15 \times 10^{26} years$;

b. $E_1 = 1.46 \times 10^{-66} J, K = 0.4J$

83. proof

85. 1.2 N/m

87. 0

Challenge Problems

89. 19.2µm;11.5µm19.2µm;11.5µm

91. 3.92%

93. proof

4.E: Quantum Mechanics (Exercises)

Conceptual Questions

7.1 Wave Functions

1. What is the physical unit of a wave function, $\Psi(x,t)$? What is the physical unit of the square of this wave function?

2. Can the magnitude of a wave function $(\Psi*(x,t)\Psi(x,t))$ be a negative number? Explain.

3. What kind of physical quantity does a wave function of an electron represent?

4. What is the physical meaning of a wave function of a particle?

5. What is the meaning of the expression “expectation value?” Explain.

7.2 The Heisenberg Uncertainty Principle

6. If the formalism of quantum mechanics is ‘more exact’ than that of classical mechanics, why don’t we use quantum mechanics to describe the motion of a leaping frog? Explain.

7. Can the de Broglie wavelength of a particle be known precisely? Can the position of a particle be known precisely?

8. Can we measure the energy of a free localized particle with complete precision?

9. Can we measure both the position and momentum of a particle with complete precision?

7.3 The Schrödinger Equation

10. What is the difference between a wave function $\psi(x,y,z)$ and a wave function $\Psi(x,y,z,t)$ for the same particle?

11. If a quantum particle is in a stationary state, does it mean that it does not move?

12. Explain the difference between time-dependent and -independent Schrödinger’s equations.

13. Suppose a wave function is discontinuous at some point. Can this function represent a quantum state of some physical particle? Why? Why not?

7.4 The Quantum Particle in a Box

14. Using the quantum particle in a box model, describe how the possible energies of the particle are related to the size of the box.

15. Is it possible that when we measure the energy of a quantum particle in a box, the measurement may return a smaller value than the ground state energy? What is the highest value of the energy that we can measure for this particle?

16. For a quantum particle in a box, the first excited state (Ψ_2) has zero value at the midpoint position in the box, so that the probability density of finding a particle at this point is exactly zero. Explain what is wrong with the following reasoning: “If the probability of finding a quantum particle at the midpoint is zero, the particle is never at this point, right? How does it come then that the particle can cross this point on its way from the left side to the right side of the box?

7.5 The Quantum Harmonic Oscillator

17. Is it possible to measure energy of $0.75\hbar\omega$ for a quantum harmonic oscillator? Why? Why not? Explain.

18. Explain the connection between Planck’s hypothesis of energy quanta and the energies of the quantum harmonic oscillator.

19. If a classical harmonic oscillator can be at rest, why can the quantum harmonic oscillator never be at rest? Does this violate Bohr’s

correspondence principle?

20. Use an example of a quantum particle in a box or a quantum oscillator to explain the physical meaning of Bohr’s correspondence principle.

21. Can we simultaneously measure position and energy of a quantum oscillator? Why? Why not?

7.6 The Quantum Tunneling of Particles through Potential Barriers

22. When an electron and a proton of the same kinetic energy encounter a potential barrier of the same height and width, which one of them will

tunnel through the barrier more easily? Why?

23. What decreases the tunneling probability most: doubling the barrier width or halving the kinetic energy of the incident particle?

24. Explain the difference between a box-potential and a potential of a quantum dot.

25. Can a quantum particle 'escape' from an infinite potential well like that in a box? Why? Why not?

26. A tunnel diode and a resonant-tunneling diode both utilize the same physics principle of quantum tunneling. In what important way are they different?

Problems

7.1 Wave Functions

27. Compute $|\Psi(x,t)|^2$ for the function $\Psi(x,t)=\psi(x) sin\omega t$, where ω is a real constant.

28. Given the complex-valued function $f(x,y)=(x-iy)/(x+iy)$, calculate $|f(x,y)|^2$.

29. Which one of the following functions, and why, qualifies to be a wave function of a particle that can move along the entire real axis?

(a) $\psi(x)=Ae^{-x^2}$;

(b) $\psi(x)=Ae^{-x}$;

(c) $\psi(x)=Atanx$;

(d) $\psi(x)=A(sinx)/x$;

(e) $\psi(x)=Ae^{-|x|}$.

30. A particle with mass **m** moving along the **x**-axis and its quantum state is represented by the following wave function: $\Psi(x,t)=\begin{cases}0 & x<0\\ Axe^{-\alpha x}e^{-iEt/\hbar} & ,x\geq 0\end{cases}$, where $\alpha=2.0\times 10^{10}m^{-1}$.

(a) Find the normalization constant.

(b) Find the probability that the particle can be found on the interval $0\leq x\leq L$.

(c) Find the expectation value of position.

(d) Find the expectation value of kinetic energy.

31. A wave function of a particle with mass m is given by $\psi(x)=\begin{cases}Acos\alpha x & -\frac{\pi}{2\alpha}\leq x\leq +\frac{\pi}{2\alpha}\\ 0 & otherwise\end{cases}$, where $\alpha=1.00\times 10^{10}/m$.

(a) Find the normalization constant.

(b) Find the probability that the particle can be found on the interval $0\leq x\leq 0.5\times 10^{-10}m$.

(c) Find the particle's average position.

(d) Find its average momentum.

(e) Find its average kinetic energy $-0.5\times 10^{-10}m\leq x\leq +0.5\times 10^{-10}m$.

7.2 The Heisenberg Uncertainty Principle

32. A velocity measurement of an α-particle has been performed with a precision of 0.02 mm/s. What is the minimum uncertainty in its position?

33. A gas of helium atoms at 273 K is in a cubical container with 25.0 cm on a side.

(a) What is the minimum uncertainty in momentum components of helium atoms?

(b) What is the minimum uncertainty in velocity components?

(c) Find the ratio of the uncertainties in

(b) to the mean speed of an atom in each direction.

34. If the uncertainty in the y-component of a proton's position is 2.0 pm, find the minimum uncertainty in the simultaneous measurement of the proton's y-component of velocity. What is the minimum uncertainty in the simultaneous measurement of the proton's xx-component of velocity?

35. Some unstable elementary particle has a rest energy of 80.41 GeV and an uncertainty in rest energy of 2.06 GeV. Estimate the lifetime of this particle.

36. An atom in a metastable state has a lifetime of 5.2 ms. Find the minimum uncertainty in the measurement of energy of the excited state.

37. Measurements indicate that an atom remains in an excited state for an average time of 50.0 ns before making a transition to the ground state with the simultaneous emission of a 2.1-eV photon.

(a) Estimate the uncertainty in the frequency of the photon.

(b) What fraction of the photon's average frequency is this?

38. Suppose an electron is confined to a region of length 0.1 nm (of the order of the size of a hydrogen atom).

(a) What is the minimum uncertainty of its momentum?

(b) What would the uncertainty in momentum be if the confined length region doubled to 0.2 nm?

7.3 The Schrödinger Equation

39. Combine Equation $7.4.1$ and Equation $7.4.2$ to show $k^2 = \frac{\omega^2}{c^2}$.

40. Show that $\Psi(x,t) = Ae^{i(kx-\omega t)}$ is a valid solution to Schrödinger's time-dependent equation.

41. Show that $\Psi(x,t) = Asin(kx-\omega t)$ and $\Psi(x,t) = Acos(kx-\omega t)$ do not obey Schrödinger's time-dependent equation.

42. Show that when $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are solutions to the time-dependent Schrödinger equation and A,B are numbers, then a function $\Psi(x,t)$ that is a superposition of these functions is also a solution: $\Psi(x,t) = A\Psi_1(x,t) + B\Psi_1(x,t)$.

43. A particle with mass m is described by the following wave function: $\psi(x) = Acoskx + Bsinkx$, where A, B, and k are constants. Assuming that the particle is free, show that this function is the solution of the stationary Schrödinger equation for this particle and find the energy that the particle has in this state.

44. Find the expectation value of the kinetic energy for the particle in the state, $\Psi(x,t) = Ae^{i(kx-\omega t)}$. What conclusion can you draw from your solution?

45. Find the expectation value of the square of the momentum squared for the particle in the state, $\Psi(x,t) = Ae^{i(kx-\omega t)}$. What conclusion can you draw from your solution?

46. A free proton has a wave function given by $\Psi(x,t) = Ae^{i(5.02\times10^{11}x-8.00\times10^{15}t)}$. The coefficient of **x** is inverse meters (m^{-1}) and the coefficient on **t** is inverse seconds (s^{-1}). Find its momentum and energy.

7.4 The Quantum Particle in a Box

47. Assume that an electron in an atom can be treated as if it were confined to a box of width $2.0 Å$. What is the ground state energy of the electron? Compare your result to the ground state kinetic energy of the hydrogen atom in the Bohr's model of the hydrogen atom.

48. Assume that a proton in a nucleus can be treated as if it were confined to a one-dimensional box of width 10.0 fm.

(a) What are the energies of the proton when it is in the states corresponding to $n = 1, n = 2,$ and $n = 3$?

(b) What are the energies of the photons emitted when the proton makes the transitions from the first and second excited states to the ground state?

49. An electron confined to a box has the ground state energy of 2.5 eV. What is the width of the box?

50. What is the ground state energy (in eV) of a proton confined to a one-dimensional box the size of the uranium nucleus that has a radius of approximately 15.0 fm?

51. What is the ground state energy (in eV) of an αα-particle confined to a one-dimensional box the size of the uranium nucleus that has a radius of approximately 15.0 fm?

52. To excite an electron in a one-dimensional box from its first excited state to its third excited state requires 20.0 eV. What is the width of the box?

53. An electron confined to a box of width 0.15 nm by infinite potential energy barriers emits a photon when it makes a transition from the first excited state to the ground state. Find the wavelength of the emitted photon.

54. If the energy of the first excited state of the electron in the box is 25.0 eV, what is the width of the box?

55. Suppose an electron confined to a box emits photons. The longest wavelength that is registered is 500.0 nm. What is the width of the box?

56. Hydrogen H_2 molecules are kept at 300.0 K in a cubical container with a side length of 20.0 cm. Assume that you can treat the molecules as though they were moving in a one-dimensional box.

(a) Find the ground state energy of the hydrogen molecule in the container.

(b) Assume that the molecule has a thermal energy given by $k_BT/2$ and find the corresponding quantum number n of the quantum state that would correspond to this thermal energy.

57. An electron is confined to a box of width 0.25 nm.

(a) Draw an energy-level diagram representing the first five states of the electron.

(b) Calculate the wavelengths of the emitted photons when the electron makes transitions between the fourth and the second excited states, between the second excited state and the ground state, and between the third and the second excited states.

58. An electron in a box is in the ground state with energy 2.0 eV.

(a) Find the width of the box.

(b) How much energy is needed to excite the electron to its first excited state?

(c) If the electron makes a transition from an excited state to the ground state with the simultaneous emission of 30.0-eV photon, find the quantum number of the excited state?

7.5 The Quantum Harmonic Oscillator

59. Show that the two lowest energy states of the simple harmonic oscillator, $\psi_0(x)$ and $\psi_1(x)$ from

$$\psi_n(x) = N_n e^{-\beta^2 x^2/2} H_n(\beta x)$$

with $n = 0, 1, 2, 3, \ldots$ satisfy the relatant time-independent Schrödinger equation

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x).$$

60. If the ground state energy of a simple harmonic oscillator is 1.25 eV, what is the frequency of its motion?

61. When a quantum harmonic oscillator makes a transition from the $(n+1)$ state to the n state and emits a 450-nm photon, what is its frequency?

62. Vibrations of the hydrogen molecule H_2 can be modeled as a simple harmonic oscillator with the spring constant $k = 1.13 \times 10^3 N/m$ and mass $m = 1.67 \times 10^{-27} kg$.

(a) What is the vibrational frequency of this molecule?

(b) What are the energy and the wavelength of the emitted photon when the molecule makes transition between its third and second excited states?

63. A particle with mass 0.030 kg oscillates back-and-forth on a spring with frequency 4.0 Hz. At the equilibrium position, it has a speed of 0.60 m/s. If the particle is in a state of definite energy, find its energy quantum number.

64. Find the expectation value $\langle x^2 \rangle$ of the square of the position for a quantum harmonic oscillator in the ground state. Note: $\int_{-\infty}^{+\infty} dx x^2 e^{-ax^2} = \sqrt{\pi}(2a^{3/2})^{-1}$.

65. Determine the expectation value of the potential energy for a quantum harmonic oscillator in the ground state. Use this to calculate the expectation value of the kinetic energy.

66. Verify that $\psi_1(x)$ given by Equation 7.57 is a solution of Schrödinger's equation for the quantum harmonic oscillator.

67. Estimate the ground state energy of the quantum harmonic oscillator by Heisenberg's uncertainty principle. Start by assuming that the product of the uncertainties Δx and Δp is at its minimum. Write Δp in terms of Δx and assume that for the ground state $x \approx \Delta x$ and $p \approx \Delta p$, then write the ground state energy in terms of **x**. Finally, find the value of x that minimizes the energy and find the minimum of the energy.

68. A mass of 0.250 kg oscillates on a spring with the force constant 110 N/m. Calculate the ground energy level and the separation between the adjacent energy levels. Express the results in joules and in electron-volts. Are quantum effects important?

7.6 The Quantum Tunneling of Particles through Potential Barriers

69. Show that the wave function in

(a) Equation 7.68 satisfies Equation 7.61, and

(b) Equation 7.69 satisfies Equation 7.63.

70. A 6.0-eV electron impacts on a barrier with height 11.0 eV. Find the probability of the electron to tunnel through the barrier if the barrier width is

(a) 0.80 nm and

(b) 0.40 nm.

71. A 5.0-eV electron impacts on a barrier of with 0.60 nm. Find the probability of the electron to tunnel through the barrier if the barrier height is

(a) 7.0 eV;

(b) 9.0 eV; and

(c) 13.0 eV.

72. A 12.0-eV electron encounters a barrier of height 15.0 eV. If the probability of the electron tunneling through the barrier is 2.5 %, find its width.

73. A quantum particle with initial kinetic energy 32.0 eV encounters a square barrier with height 41.0 eV and width 0.25 nm. Find probability that the particle tunnels through this barrier if the particle is

(a) an electron and,

(b) a proton.

74. A simple model of a radioactive nuclear decay assumes that α-particles are trapped inside a well of nuclear potential that walls are the barriers of a finite width 2.0 fm and height 30.0 MeV. Find the tunneling probability across the potential barrier of the wall for αα-particles having kinetic energy

(a) 29.0 MeV and

(b) 20.0 MeV. The mass of the α-particle is $m = 6.64 \times 10^{-27} kg$.

75. A muon, a quantum particle with a mass approximately 200 times that of an electron, is incident on a potential barrier of height 10.0 eV. The kinetic energy of the impacting muon is 5.5 eV and only about 0.10% of the squared amplitude of its incoming wave function filters through the barrier. What is the barrier's width?

76. A grain of sand with mass 1.0 mg and kinetic energy 1.0 J is incident on a potential energy barrier with height 1.000001 J and width 2500 nm. How many grains of sand have to fall on this barrier before, on the average, one passes through?

Additional Problems

77. Show that if the uncertainty in the position of a particle is on the order of its de Broglie's wavelength, then the uncertainty in its momentum is on the order of the value of its momentum.

78. The mass of a ρ-meson is measured to be $770 MeV/c^2$ with an uncertainty of $100 MeV/c^2$. Estimate the lifetime of this meson.

79. A particle of mass **m** is confined to a box of width **L**. If the particle is in the first excited state, what are the probabilities of finding the particle in a region of width 0.020 L around the given point **x**:

(a) $x = 0.25L$;

(b) $x = 0.40L$;

(c) $x = 0.75L$; and

(d) $x = 0.90L$.

80. A particle in a box [0;**L**] is in the third excited state. What are its most probable positions?

81. A 0.20-kg billiard ball bounces back and forth without losing its energy between the cushions of a 1.5 m long table

(a) If the ball is in its ground state, how many years does it need to get from one cushion to the other? You may compare this time interval to the age of the universe.

(b) How much energy is required to make the ball go from its ground state to its first excited state? Compare it with the kinetic energy of the ball moving at 2.0 m/s.

82. Find the expectation value of the position squared when the particle in the box is in its third excited state and the length of the box is **L**.

83. Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. Show that the function $\psi(x) = A sin kx$ is the solution to the stationary Schrödinger equation for the particle in a box only if $k = \sqrt{2mE}/\hbar$. Explain why this is an acceptable wave function only if **k** is an integer multiple of π/L.

84. Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. Explain why the function $\psi(x) = A cos kx$ is not a solution to the stationary Schrödinger equation for the particle in a box.

85. Atoms in a crystal lattice vibrate in simple harmonic motion. Assuming a lattice atom has a mass of $9.4 \times 10^{-26} kg$, what is the force constant of the lattice if a lattice atom makes a transition from the ground state to first excited state when it absorbs a $525-\mu m$ photon?

86. A diatomic molecule behaves like a quantum harmonic oscillator with the force constant 12.0 N/m and mass $5.60 \times 10^{-26} kg$.

(a) What is the wavelength of the emitted photon when the molecule makes the transition from the third excited state to the second excited state?

(b) Find the ground state energy of vibrations for this diatomic molecule.

87. An electron with kinetic energy 2.0 MeV encounters a potential energy barrier of height 16.0 MeV and width 2.00 nm. What is the probability that the electron emerges on the other side of the barrier?

88. A beam of mono-energetic protons with energy 2.0 MeV falls on a potential energy barrier of height 20.0 MeV and of width 1.5 fm. What percentage of the beam is transmitted through the barrier?

Challenge Problems

89. An electron in a long, organic molecule used in a dye laser behaves approximately like a quantum particle in a box with width 4.18 nm. Find the emitted photon when the electron makes a transition from the first excited state to the ground state and from the second excited state to the first excited state.

90. In STM, an elevation of the tip above the surface being scanned can be determined with a great precision, because the tunneling-electron current between surface atoms and the atoms of the tip is extremely sensitive to the variation of the separation gap between them from point to point along the surface. Assuming that the tunneling-electron current is in direct proportion to the tunneling probability and that the tunneling probability is to a good approximation expressed by the exponential function $e^{-2\beta L}$ with $\beta = 10.0/nm$, determine the ratio of the tunneling current when the tip is 0.500 nm above the surface to the current when the tip is 0.515 nm above the surface.

91. If STM is to detect surface features with local heights of about 0.00200 nm, what percent change in tunneling-electron current must the STM electronics be able to detect? Assume that the tunneling-electron current has characteristics given in the preceding problem.

92. Use Heisenberg's uncertainty principle to estimate the ground state energy of a particle oscillating on an spring with angular frequency, $\omega = \sqrt{k/m}$, where **k** is the spring constant and m is the mass.

93. Suppose an infinite square well extends from $-L/2$ to $+L/2$. Solve the time-independent Schrödinger's equation to find the allowed energies and stationary states of a particle with mass m that is confined to this well. Then show that these solutions can be obtained by making the coordinate transformation $x' = x - L/2$ for the solutions obtained for the well extending between 0 and **L**.

94. A particle of mass m confined to a box of width L is in its first excited state $\psi_2(x)$.

(a) Find its average position (which is the expectation value of the position).

(b) Where is the particle most likely to be found?

4.S: Quantum Mechanics (Summary)

Key Terms

anti-symmetric function	odd function
Born interpretation	states that the square of a wave function is the probability density
complex function	function containing both real and imaginary parts
Copenhagen interpretation	states that when an observer *is not* looking or when a measurement is not being made, the particle has many values of measurable quantities, such as position
correspondence principle	in the limit of large energies, the predictions of quantum mechanics agree with the predictions of classical mechanics
energy levels	states of definite energy, often represented by horizontal lines in an energy "ladder" diagram
energy quantum number	index that labels the allowed energy states
energy-time uncertainty principle	energy-time relation for uncertainties in the simultaneous measurements of the energy of a quantum state and of its lifetime
even function	in one dimension, a function symmetric with the origin of the coordinate system
expectation value	average value of the physical quantity assuming a large number of particles with the same wave function
field emission	electron emission from conductor surfaces when a strong external electric field is applied in normal direction to conductor's surface
ground state energy	lowest energy state in the energy spectrum
Heisenberg's uncertainty principle	places limits on what can be known from a simultaneous measurements of position and momentum; states that if the uncertainty on position is small then the uncertainty on momentum is large, and vice versa
infinite square well	potential function that is zero in a fixed range and infinitely beyond this range
momentum operator	operator that corresponds to the momentum of a particle
nanotechnology	technology that is based on manipulation of nanostructures such as molecules or individual atoms to produce nano-devices such as integrated circuits
normalization condition	requires that the probability density integrated over the entire physical space results in the number one
odd function	in one dimension, a function antisymmetric with the origin of the coordinate system
position operator	operator that corresponds to the position of a particle
potential barrier	potential function that rises and falls with increasing values of position
principal quantum number	energy quantum number
probability density	square of the particle's wave function

quantum dot	small region of a semiconductor nanocrystal embedded in another semiconductor nanocrystal, acting as a potential well for electrons
quantum tunneling	phenomenon where particles penetrate through a potential energy barrier with a height greater than the total energy of the particles
resonant tunneling	tunneling of electrons through a finite-height potential well that occurs only when electron energies match an energy level in the well, occurs in quantum dots
resonant-tunneling diode	quantum dot with an applied voltage bias across it
scanning tunneling microscope (STM)	device that utilizes quantum-tunneling phenomenon at metallic surfaces to obtain images of nanoscale structures
Schrӧdinger's time-dependent equation	equation in space and time that allows us to determine wave functions of a quantum particle
Schrӧdinger's time-independent equation	equation in space that allows us to determine wave functions of a quantum particle; this wave function must be multiplied by a time-modulation factor to obtain the time-dependent wave function
standing wave state	stationary state for which the real and imaginary parts of Ψ(x,t)Ψ(x,t) oscillate up and down like a standing wave (often modeled with sine and cosine functions)
state reduction	hypothetical process in which an observed or detected particle "jumps into" a definite state, often described in terms of the collapse of the particle's wave function
stationary state	state for which the probability density function, $\lvert\Psi(x,t)\rvert^2$, does not vary in time
time-modulation factor	factor $e^{-i\omega t}$ that multiplies the time-independent wave function when the potential energy of the particle is time independent
transmission probability	also called tunneling probability, the probability that a particle will tunnel through a potential barrier
tunnel diode	electron tunneling-junction between two different semiconductors
tunneling probability	also called transmission probability, the probability that a particle will tunnel through a potential barrier
wave function	function that represents the quantum state of a particle (quantum system)
wave function collapse	equivalent to state reduction
wave packet	superposition of many plane matter waves that can be used to represent a localized particle

Key Equations

Normalization condition in one dimension	$P(x=-\infty,+\infty)=\int_{-\infty}^{\infty} \mid \Psi(x,t) \mid^2 dx = 1$
Probability of finding a particle in a narrow interval of position in one dimension $(x, x+dx)$	$P(x,x+dx)=\Psi^*(x,t)\Psi(x,t)dx$
Expectation value of position in one dimension	$\langle x\rangle=\int_{-\infty}^{\infty}\Psi^*(x,t)x\Psi(x,t)dx$
Heisenberg's position-momentum uncertainty principle	$\Delta x\Delta p \geq \frac{\hbar}{2}$

Heisenberg’s energy-time uncertainty principle	$\Delta E \Delta t \geq \frac{\hbar}{2}$
Schrödinger’s time-dependent equation	$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + U(x,t)\Psi(x,t) = i\hbar\frac{\partial \Psi(x,t)}{\partial t}$
General form of the wave function for a time-independent potential in one dimension	$\Psi(x,t) = \psi(x)e^{-i\omega}$
Schrödinger’s time-independent equation	$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$
Schrödinger’s equation (free particle)	$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$
Allowed energies (particle in box of length L)	$E_n = n^2\frac{\pi^2\hbar^2}{2mL^2}, n = 1, 2, 3, \ldots$
Stationary states (particle in a box of length L)	$\psi_n(x) = \sqrt{\frac{2}{L}} sin\frac{n\pi x}{L}, n = 1, 2, 3, \ldots$
Potential-energy function of a harmonic oscillator	$U(x) = \frac{1}{2}m\omega^2 x^2$
Schrödinger equation (harmonic oscillator)	$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x)$
The energy spectrum	$E_n = (n + \frac{1}{2})\hbar\omega, n = 0, 1, 2, 3, \ldots$
The energy wave functions	$\psi_n(x) = N_n e^{-\beta^2x^2/2}H_n(\beta x), n = 0, 1, 2, 3, \ldots$
Potential barrier	$U(x) = \begin{cases} 0, & \text{when } x < 0 \\ U_0, & \text{when } 0 \leq x \leq L \\ 0, & \text{when } x > L \end{cases}$
Definition of the transmission coefficient	$T(L,E) = \frac{\lvert\psi_{tra}(x)\rvert^2}{\lvert\psi_{in}(x)\rvert^2}$
A parameter in the transmission coefficient	$\beta^2 = \frac{2m}{\hbar^2}(U_0 - E)$
Transmission coefficient, exact	$T(L,E) = \frac{1}{cosh^2\beta L + (\gamma/2)^2 sinh^2\beta L}$
Transmission coefficient, approximate	$T(L,E) = 16\frac{E}{U_0}(1 - \frac{E}{U_0})e^{-2\beta L}$

Summary

7.1: Wavefunctions

- In quantum mechanics, the state of a physical system is represented by a wave function.
- In Born’s interpretation, the square of the particle’s wave function represents the probability density of finding the particle around a specific location in space.
- Wave functions must first be normalized before using them to make predictions.
- The expectation value is the average value of a quantity that requires a wave function and an integration.

7.2: The Heisenberg Uncertainty Principle

- The Heisenberg uncertainty principle states that it is impossible to simultaneously measure the *x*-components of position and of momentum of a particle with an arbitrarily high precision. The product of experimental uncertainties is always larger than or equal to $\hbar/2$.
- The limitations of this principle have nothing to do with the quality of the experimental apparatus but originate in the wave-like nature of matter.

- The energy-time uncertainty principle expresses the experimental observation that a quantum state that exists only for a short time cannot have a definite energy.

7.3: The Schrödinger Equation

- The Schrödinger equation is the fundamental equation of wave quantum mechanics. It allows us to make predictions about wave functions.
- When a particle moves in a time-independent potential, a solution of the time-dependent Schrödinger equation is a product of a time-independent wave function and a time-modulation factor.
- The Schrödinger equation can be applied to many physical situations.

7.4: The Quantum Particle in a Box

- Energy states of a quantum particle in a box are found by solving the time-independent Schrödinger equation.
- To solve the time-independent Schrödinger equation for a particle in a box and find the stationary states and allowed energies, we require that the wave function terminate at the box wall.
- Energy states of a particle in a box are quantized and indexed by principal quantum number.
- The quantum picture differs significantly from the classical picture when a particle is in a low-energy state of a low quantum number.
- In the limit of high quantum numbers, when the quantum particle is in a highly excited state, the quantum description of a particle in a box coincides with the classical description, in the spirit of Bohr's correspondence principle.

7.5: The Quantum Harmonic Oscillator

- The quantum harmonic oscillator is a model built in analogy with the model of a classical harmonic oscillator. It models the behavior of many physical systems, such as molecular vibrations or wave packets in quantum optics.
- The allowed energies of a quantum oscillator are discrete and evenly spaced. The energy spacing is equal to Planck's energy quantum.
- The ground state energy is larger than zero. This means that, unlike a classical oscillator, a quantum oscillator is never at rest, even at the bottom of a potential well, and undergoes quantum fluctuations.
- The stationary states (states of definite energy) have nonzero values also in regions beyond classical turning points. When in the ground state, a quantum oscillator is most likely to be found around the position of the minimum of the potential well, which is the least-likely position for a classical oscillator.
- For high quantum numbers, the motion of a quantum oscillator becomes more similar to the motion of a classical oscillator, in accordance with Bohr's correspondence principle.

7.6 The Quantum Tunneling of Particles through Potential Barriers

- A quantum particle that is incident on a potential barrier of a finite width and height may cross the barrier and appear on its other side. This phenomenon is called 'quantum tunneling.' It does not have a classical analog.
- To find the probability of quantum tunneling, we assume the energy of an incident particle and solve the stationary Schrödinger equation to find wave functions inside and outside the barrier. The tunneling probability is a ratio of squared amplitudes of the wave past the barrier to the incident wave.
- The tunneling probability depends on the energy of the incident particle relative to the height of the barrier and on the width of the barrier. It is strongly affected by the width of the barrier in a nonlinear, exponential way so that a small change in the barrier width causes a disproportionately large change in the transmission probability.
- Quantum-tunneling phenomena govern radioactive nuclear decays. They are utilized in many modern technologies such as STM and nano-electronics. STM allows us to see individual atoms on metal surfaces. Electron-tunneling devices have revolutionized electronics and allow us to build fast electronic devices of miniature sizes.

CHAPTER OVERVIEW

5: The Schrödinger Equation

Contributors and Attributions

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5.1: Schrödinger’s Equation

5.2: Solving the 1D Infinite Square Well

Imagine a (non-relativistic) particle trapped in a one-dimensional well of length L. Inside the well there is no potential energy, and the particle is trapped inside the well by "walls" of infinite potential energy.

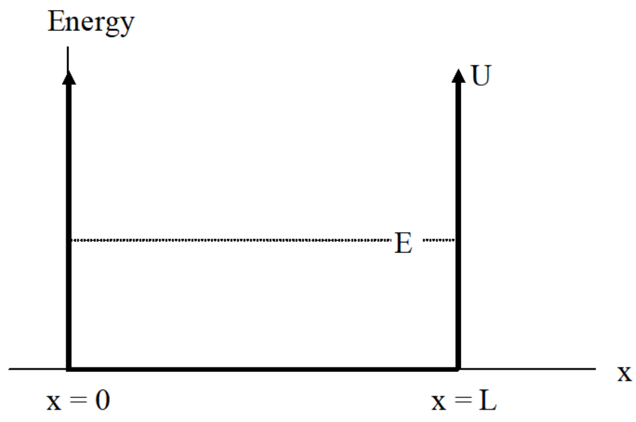

Figure $5.2.1$

Since this potential is a piece-wise function, Schrödinger's equation must be solved in the three regions separately. In the region $x > L$ (and $x < 0$), the equation is:

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}\psi(x)+U(x)\psi x=E\psi(x) \tag{5.2.1}$$

$$-\frac{h^2}{2m}\frac{d^2}{dx^2}\psi(x)+(\infty)\psi(x)=E\psi(x) \tag{5.2.2}$$

$$(\infty)\psi(x)=E\psi(x) \tag{5.2.3}$$

This has solutions of $E=\infty$, which is impossible (no particle can have infinite energy) or $\psi=0$. Since $\psi=0$, the particle can never be found outside of the well.

In the region $0 < x < L$, the equation is:

$$\begin{aligned}&-\tfrac{h^2}{2m}\tfrac{d^2}{dx^2}\psi(x)+U(x)\psi(x)=E\psi(x)\\&-\tfrac{h^2}{2m}\tfrac{d^2}{dx^2}\psi(x)+(0)\psi(x)=E\psi(x)\\&-\tfrac{h^2}{2m}\tfrac{d^2}{dx^2}\psi(x)=E\psi(x)\\&\tfrac{d^2}{dx^2}\psi(x)=-\tfrac{2mE}{h^2}\psi(x)\end{aligned} \tag{5.2.4}$$

The general solution to this equation is

$$\psi(x)=A\sin(kx)+B\cos(kx) \text{ with } \quad k=\sqrt{\tfrac{2mE}{h^2}}$$

The solutions in the different regions of space must be continuous across the boundaries at $x=0$ and $x=L$ (the derivatives of the solutions must also be continuous if the potential energy is continuous at these points).

At $x=0$, equate the two solutions:

$$\begin{aligned}&\psi\left(x=0^{-}\right)=\psi\left(x=0^{+}\right)\\&0=A\sin(k0)+B\cos(k0)\\&0=\mathrm{B}\end{aligned} \tag{5.2.5}$$

therefore, the cosine portion of the solution has amplitude zero.

At $x = L$, equate the two solutions:

$$\begin{aligned} &\psi\left(x=L^{-}\right)=\psi\left(x=L^{+}\right) \\ &A \sin (k L)=0 \\ &k L=n \pi \\ &k=\frac{n \pi}{L} \end{aligned} \tag{5.2.6}$$

This leads to allowed wavefunctions:

$$\psi(x)=A \sin \left(\frac{n \pi x}{L}\right) \tag{5.2.7}$$

and allowed energies:

$$\begin{aligned} &k=\sqrt{\frac{2 m E}{h^{2}}} \\ &\left(\frac{n \pi}{L}\right)^{2}=\frac{2 m E}{h^{2}} \\ &\left(\frac{n \pi}{L}\right)^{2}=\left(\frac{2 \pi}{h}\right)^{2} 2 m E \\ &E=\frac{n^{2} h^{2}}{8 m L^{2}} \\ &E=n^{2} \frac{(h c)^{2}}{8 m c^{2} L^{2}} \end{aligned} \tag{5.2.8}$$

These solutions are represented below. On the left is the standard textbook representation, where the wavefunction is superimposed on the graph of energy. Remember, however, that what is "waving" above and below the energy level isn't a measure of energy (i.e., the energy is a constant value at each level), but rather related to the probability of finding the particle at each location in space. The representation on the right tries to represent the regions where it is more likely to find the particle (the darker regions) and the regions where it is less likely to find the particle.

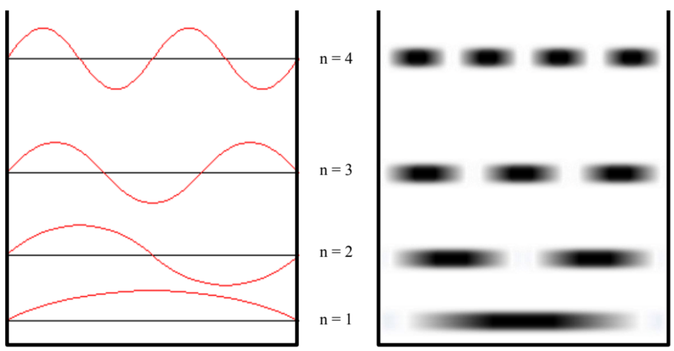

Figure 5.2.1: To the left the wavefunction, to the right a representation of the probability of finding the particle at a specific position for the various quantum states.

Additionally, since the probability of finding the particle somewhere in the well is equal to 1, and the probability of finding the particle is equal to the wavefunction squared,

using an integral table yields the result

This result has a number of extremely important features.

- The particle can only have certain, discrete values for energy. In classical physics, a particle trapped in a region of space can have any, continuous value for energy. The restriction of a bound particle to specific, quantized values of allowed energy is a hallmark of quantum mechanics.

- The particle cannot be at rest. Notice that the lowest possible energy for the particle is in the $n=1$ state, which has non-zero energy. This is termed the zero-point energy, and can be understood as a consequence of the Heisenberg uncertainty principle. This is in complete contrast to classical physics.
- The particle has regions of high and low probability of being found in the well. The probability of finding the particle is equal to the square of the wavefunction, which is not a constant value. Unlike classical physics, where the particle is equally likely to be anywhere in the well, in quantum mechanics there exist positions where the particle will never be found, and regions where the probability of finding the particle is greatly enhanced. for example, in the $n=2$ state, the particle will never be found in the center of the well, while it is very likely to find the particle at the positions $x=L/4$ and $x=3L/4$.

The 1D Infinite Well

An electron is trapped in a one-dimensional infinite potential well of length $4.0\times10^{-10}\ m$. Find the three longest wavelength photons emitted by the electron as it changes energy levels in the well.

The allowed energy states of a particle of mass m trapped in an infinite potential well of length L are

$$E=n^2\frac{(hc)^2}{8mc^2L^2} \tag{5.2.9}$$

Therefore, the electron has allowed energy levels given by

$$\begin{aligned} E &= n^2\frac{(hc)^2}{8mc^2L^2} \\ E &= n^2\frac{(1240)^2}{8(511000\text{eV})(0.4\text{nm})^2} \\ E &= n^2(2.35\text{eV}) \end{aligned} \tag{5.2.10}$$

As the electron changes energy levels, the energy released by the electron is in the form of a photon.

$$\begin{aligned} E_{photon} &= E_{initial\ electron}-E_{final\ electron} \\ E_{photon} &= (n_i^2-n_f^2)2.35\text{eV} \end{aligned} \tag{5.2.11}$$

Thus, the emitted wavelengths are

$$\begin{aligned} \lambda &= \frac{hc}{E_{photon}} \\ \lambda &= \frac{1240\text{eV nm}}{(n_i^2-n_f^2)2.35\text{eV nm}} \\ \lambda &= \frac{528\text{ nm}}{(n_i^2-n_f^2)} \end{aligned} \tag{5.2.12}$$

The longest wavelength photons involve the smallest value of $n_i^2-n_f^2$. These are:

$$\begin{array}{ccc} n_i & n_f & \lambda \\ \hline \\ 2 & 1 & 176\text{nm} \\ 3 & 2 & 106\text{ nm} \\ 4 & 3 & 75.6\text{ nm} \end{array} \tag{5.2.13}$$

5.3: The Pauli Exclusion Principle

Imagine five neutrons and four protons trapped in an atomic nucleus. Model the nucleus as an infinite potential well of length 10 fm. Find the total kinetic energy of the nucleons. Ignore the mass difference between protons and neutrons.

The allowed energy levels in this infinite square well are:

$$\begin{aligned} E &= n^2 \frac{(hc)^2}{8mc^2L^2} \\ E &= n^2 \frac{(1240 \text{ MeV fm})^2}{8(938 \text{ MeV})(10 \text{ fm})^2} \\ E &= n^2 (2.35 \text{ MeV}) \end{aligned} \tag{5.3.1}$$

Although all of the particles would love to occupy the lowest energy state, the Pauli Exclusion Principle states that no two identical fermions can occupy the exact same quantum state. Thus, only two neutrons (and two protons) can occupy the lowest energy state, one with spin "up" and one with spin "down". In this way, the well is filled from the bottom up as indicted below:

n	E (2.05 MeV)	# neutrons	# protons
1	1	2	2
2	4	2	2
3	9	1	

(5.3.2)

Therefore, the total kinetic energy of the nucleons is:

$$KE = [4(1) + 4(4) + 1(9)]2.05 \text{ MeV} = 59.5 \text{ MeV} \tag{5.3.3}$$

5.4: Expectation Values, Observables, and Uncertainty

An electron is trapped in a one-dimensional infinite potential well of length L. Find the expectation values of the electron's position and momentum in the ground state of this well. Show that the uncertainties in these values do not violate the uncertainty principle.

Imagine an electron was trapped in the well described above and we repeatedly measured its location in the well. Due to the wave nature of the electron, we would get different values for these positions but, after many measurements, we could average these values to determine the expectation value of the electron's position.

You may be tempted to just refer to this as the average value of the electron's position, but if you take the wave-like nature of the electron seriously, and you should, the electron does not have a position until it is measured, so it is senseless to refer to the average value of something that doesn't even exist! What you are averaging is your measurements of the electron's position, not its pre-existing position. To avoid this metaphysical conundrum, we will call the value that we most likely expect to measure the expectation value of the variable.

The expectation value of the position (given by the symbol <x>) can be determined by a simple weighted average of the product of the probability of finding the electron at a certain position and the position, or

$$<x>=\int_0^L x\,\mathrm{Prob}(x)dx \tag{5.4.1}$$

$$<x>=\int_0^L (\Psi(x))x(\Psi(x))dx \tag{5.4.2}$$

What may strike you as somewhat strange is why I placed the factor of x between the two factors of the wavefunction. Mathematically, it doesn't matter where I place the x, but it turns out that for other variables the placement of the variable of interest must be "between" the two wavefunctions. Before we explore why this is the case, let's finish the calculation.

$$<x>=\int_0^L \left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)x\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)dx \tag{5.4.3}$$

$$<x>=\frac{2}{L}\int_0^L x\sin^2\left(\frac{\pi x}{L}\right)dx$$

This integral begs for a u-substitution of:

$$u=\frac{\pi x}{L} \tag{5.4.4}$$

$$<x>=\frac{2}{L}\int_0^\pi \left(\frac{2}{L}u\right)\sin^2(u)\left(\frac{L}{\pi}du\right) \tag{5.4.5}$$

$$<x>=\frac{2L}{\pi^2}\int_0^\pi u\sin^2(u)du$$

$$<x>=\frac{2L}{\pi^2}\frac{\pi^2}{4}$$

$$<x>=\frac{L}{2}$$

I'll agree that this seems like a stupid amount of work just to determine that the expectation value of a particle's position in an infinite well is in the center of the well, but it's always nice when learning a new mathematical technique to apply it to a situation in which you know the answer.

Now let's move on to the expectation value of the electron's momentum. You should be tempted to write:

$$<p>=\int_0^L (\Psi(x))p(\Psi(x))dx \tag{5.4.6}$$

The only problem with this, of course, is that we have to express the momentum of the electron in terms of its position in order to do the integral. How can we do that? Well, the momentum of the electron is related (by DeBroglie) to its wavelength, and the wavelength is dependent on how "curvy" the wavefunction is at any point, and the "curviness" of the wavefunction is related to the spatial derivative of the wavefunction. Thus,

$$p \propto \frac{d}{dx} \tag{5.4.7}$$

Not to get overly philosophical here, but in quantum mechanics all that exists is the wavefunction. Everything that is observable in nature must somehow be extracted from the wavefunction. This means that quantities like momentum can only be determined by manipulating the wavefunction is some way, in this case by taking a spatial derivative. Thus, quantities like momentum (or kinetic energy) are represented not by the "formulas" you are familiar with from classical mechanics but by mathematical operators, basically actions that must be taken on the wavefunction in order to squeeze from it the information you are interested in. This is why the placement of the variable in the formula for expectation values is so important. The operator for momentum acts on one "copy" of the wavefunction, and then the result is multiplied by the other "copy" and then integrated over all of space. We are almost ready to do this, but first we need to complete the description of the momentum operator.

If you recall from above, I showed that Schrödinger's equation is consistent with the idea of energy conservation:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x)+U(x)\Psi(x)=E\Psi(x) \tag{5.4.8}$$

$$\frac{p^2}{2m}+U(x)=E \tag{5.4.9}$$

Carefully comparing these two relationships leads to the conclusion that the operator representing momentum may be,

$$p=i\hbar ddx \tag{5.4.10}$$

where

$$i=\sqrt{-1} \tag{5.4.11}$$

Thus, if you want to determine the momentum of a wavefunction, you must take a spatial derivative and then multiply the result by –ih. Should you be concerned that this implies that momentum is not "real"? The short answer is no.

Let's determine the expectation value of the momentum of the electron:

$$\begin{aligned}
&<p>=\int_0^L\left(\sqrt{\frac{2}{L}}\sin(\frac{\pi x}{L})\right)\left(-i\hbar\frac{d}{dx}\left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right)dx\right. \\
&<p>=-i\hbar\frac{2}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\left(\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right)\right)dx \\
&<p>=-i\hbar 2\pi\frac{2}{L}\int_0^L\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)dx \\
&<p>=-ih\int_0^\pi\sin(u)\cos(u)\left(\frac{L}{\pi}du\right) \\
&<p>=-ih\frac{L}{\pi}\int_0^\pi\sin(u)\cos(u)du \\
&<p>=-ih\frac{L}{\pi}\int_0^\pi\frac{1}{2}\sin(2u)du \\
&<p>=-ih\frac{L}{\pi}(0) \\
&<p>=0
\end{aligned} \tag{5.4.12}$$

So the expectation value of the momentum of a particle in an infinite square well is zero? Of course it is! The allowed energy levels in a well can be thought of as the standing waves that "fit" in the well. The whole idea of a standing wave is that there is no net flow of energy (or momentum) in either direction. That's why we call it a standing wave!

Now what about the uncertainties in these values? Obviously, every time we measure the position of the electron it won't be in the center of the well (just equally likely on the right and the left) and every time we measure the momentum of the particle it won't be at rest (just equally likely "moving" to the right or the left). The uncertainty in these values gives you an idea of the spread in possible measurements you should expect if you made a large number of measurements. This idea of the spread in a collection of data is simply the idea of the standard deviation.
Mathematically, the standard deviation of a set of position data is determined by

$$\sigma_x = \sqrt{<x^2> - <x>^2} \tag{5.4.13}$$

i.e., the difference between the expectation value of the square of x and the expectation value of x squared. Thus, to find the uncertainty in position, we need the expectation value of x2:

$$\begin{aligned} <x^2> &= \int_0^L \left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right) x^2 \left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right) dx \\ <x^2> &= \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{\pi x}{L}\right) dx \\ <x^2> &= \frac{2}{L}\int_0^\pi \left(\frac{L}{\pi}u\right)^2 \sin^2(u)\left(\frac{L}{\pi}du\right) \\ <x^2> &= \frac{2L^2}{\pi^3}\int_0^\pi u^2\sin^2(u)du \\ <x^2> &= \frac{2L^2}{\pi^3}\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right) \\ <x^2> &= 0.283L^2 \end{aligned} \tag{5.4.14}$$

So the uncertainty in position is:

$$\begin{aligned} \sigma_x &= \sqrt{0.283L^2 - (0.5L)^2} \\ \sigma_x &= \sqrt{0.283L^2 - 0.25L^2} \\ \sigma_x &= 0.182L \end{aligned} \tag{5.4.15}$$

The uncertainty in momentum is:

$$\sigma_p = \sqrt{<p^2> - <p>^2} \tag{5.4.16}$$

and the expectation value of p2 is:

$$
\begin{aligned}
&< p^2 >= \int_0^L \left(\sqrt{\frac{2}{L}}\right) \sin bigg(\frac{\pi x}{L})\bigg)\left(-\hbar^2 \frac{d^2}{dx^2}\left(\sqrt{\frac{2}{L}}\right) \sin bigg(\frac{\pi x}{L})\right)\right) \\
&< p^2 >= -\hbar^2 \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right)\left(-\frac{\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right)\right) dx \\
&< p^2 >= \frac{2\pi^2\hbar^2}{L^3} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \\
&< p^2 >= \frac{2\pi\hbar^2}{L^2} \int_0^\pi \sin^2(u)\left(\frac{L}{\pi} du\right) \\
&< p^2 >=< p^2 >= \frac{2\pi\hbar^2}{L^2} \int_0^\pi \sin^2(u) du \\
&< p^> = \frac{2\pi\hbar^2}{L^2}\left(\frac{\pi}{2}\right) \\
&< p^2 >= \frac{\pi^2\hbar^2}{L^2} \\
&< p^2 >= \frac{h^2}{4L^2}
\end{aligned} \tag{5.4.17}
$$

Does this result look familiar? If not, compare it to the ground state energy …

So the uncertainty in momentum is:

$$
\begin{aligned}
\sigma_p &= \sqrt{\left(\frac{h^2}{4L^2}\right) - (0)^2} \\
\sigma_p &= \frac{h}{2L}
\end{aligned} \tag{5.4.18}
$$

Note that the uncertainty in the momentum is actually equal to the absolute value of the momentum. (The electron has a wavelength of 2L, so the above expression is actually just DeBroglie's relationship for the momentum of the electron.) This can be interpreted as the electron having a momentum magnitude of h/2L but having an unknown direction for this momentum. Thus the momentum is:

$$
p = 0 \pm \frac{h}{2L} \tag{5.4.19}
$$

Finally, we can verify that the uncertainty in position and momentum are consistent with the uncertainty principle:

$$
\begin{aligned}
&\sigma_x \sigma_p \geq \hbar/2 (0.182\text{ L})\left(\frac{h}{2L}\right) \geq \frac{h}{4\pi} \\
&0.091h \geq 0.080h
\end{aligned} \tag{5.4.20}
$$

Heisenberg can sleep easy since the ground state of the infinite well displays no violation of his principle!

5.5: The 2D Infinite Square Well

Twelve electrons are trapped in a two-dimensional infinite potential well of x-length 0.40 nm and y-width 0.20 nm. Find the total kinetic energy of the system.

Since the x- and y-directions in space are independent, Schrödinger's equation can be separated into an x-equation and a y-equation. The solutions to these equations are identical to the one-dimensional infinite square well. Thus, the allowed energy states of a particle of mass m trapped in a two-dimensional infinite potential well can be written as:

$$\begin{aligned} E &= n_x^2 \frac{(hc)^2}{8mc^2 L_x^2} + n_y^2 \frac{(hc)^2}{8mc^2 L_y^2} \\ E &= \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)\frac{(hc)^2}{8mc^2} \end{aligned} \tag{5.5.1}$$

with wavefunction:

$$\Psi(x,y) = A \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L_y}\right) \tag{5.5.2}$$

Therefore, the allowed energy levels are given by

$$\begin{aligned} E &= \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)\frac{(hc)^2}{8mc^2} \\ E &= \left(\frac{n_x^2}{4^2} + \frac{2^2}{L_y^2}\right)\frac{(1240 \text{ eV nm})^2}{8(5111000 \text{ eV})^2(0.1 \text{ nm})^2} \\ E &= \left(\frac{n_x^2}{16} + \frac{n_y^2}{4}\right) 37.6 \text{ eV} \end{aligned} \tag{5.5.3}$$

Rather than deal with fractions, multiply and divide by 16:

$$E = (n_x^2 + 4n_y^2) 2.35 \text{ eV} \tag{5.5.4}$$

To help calculate the total kinetic energy of the system, list the first few lowest allowed energy states:

(5.5.5)

Level	n_x	n_y	E (2.35 eV)	# electrons
1	1	1	5	2
2	2	1	8	2
3	3	1	13	2
4	1	2	17	2
5	2 4	2 1	20	4
6	3	2	25	0

The states (nx, ny) = (2, 2) and (4, 1) are termed degenerate because two completely different wavefunctions have the same energy. The state (2, 2) looks like this:

while the state (4, 1) looks like this:

Since these are different wavefunctions, two electrons (spin up and spin down) can occupy each state. Thus, four electrons have the same energy. Thus, the twelve electrons will have total kinetic energy of

$$KE = [2(5) + 2(8) + 2(13) + 2(17) + 4(20)]2.35 \text{ eV} = 390 \text{ eV} \tag{5.5.6}$$

5.6: Solving the 1D Semi-Infinite Square Well

Imagine a particle trapped in a one-dimensional well of length L. Inside the well there is no potential energy. However, the "right-hand wall" of the well (and the region beyond this wall) has a finite potential energy. This means that it is possible for the particle to escape the well if it had enough energy.

Again, since this potential is a piece-wise function, Schrödinger's equation must be solved in the three regions separately.
In the region x < 0, we have already seen that since the potential is infinite there is no chance of finding the particle in this region. Thus, Ψ = 0 in this region.
In the region 0 < x < L, the equation and solution should look familiar:

$$\begin{aligned} -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+U(x)\Psi(x) &= E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+(0)\Psi(x) &= E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d}{dx^2}\Psi(x) &= E\Psi(x) \\ \frac{d^2}{dx^2}\Psi(x) &= -\frac{2mE}{\hbar^2}\Psi(x) \end{aligned} \tag{5.6.1}$$

The general solution to this equation is

$$\Psi(x) = A\sin(kx)+B\cos(kx)\text{with}k=\sqrt{\frac{2mE}{\hbar^2}} \tag{5.6.2}$$

In order for this solution to be continuous with the solution for x < 0, the coefficient B must equal zero. Thus,

$$\Psi(x) = A\sin(kx) \tag{5.6.3}$$

In the region x > L, the equation is:

$$\begin{aligned} -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+U(x)\Psi(x)=E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)=(E-U)\Psi(x) \\ \frac{d^2}{dx^2}\Psi(x)=\frac{2m(U-E)}{\hbar^2}\Psi(x) \end{aligned} \tag{5.6.4}$$

The general solution is:

$$\Psi(x) = Ce^{aX}+De^{-aX} \text{ with } \alpha=\sqrt{\frac{2m(U-E)}{\hbar^2}} \tag{5.6.5}$$

Since this region contains the point x = +∞, C must equal zero or the wavefunction will diverge. Therefore,

$$\Psi(x) = De^{-aX} \tag{5.6.6}$$

The wave function, and the derivative of the wave function, must be continuous across the boundary at x = L. Forcing continuity leads to:

$$\begin{aligned} \Psi(x=L^-)=\Psi(x=L^+) \\ A\sin(kL)=De^{-aL} \end{aligned} \tag{5.6.7}$$

and forcing the continuity of the derivative leads to:

$$\begin{aligned} \Psi(x=L^-)=\Psi(x=L^+) \\ kA\cos(kL)=-\alpha De^{-aL} \end{aligned} \tag{5.6.8}$$

Substituting the first equation into the second equation yields:

\[\begin}align} & kA\cos(kL) = - \alpha(A \sin(kL)) \\ \nonumber & \tan(kL) = - \dfrac{k}{\alpha} \\ \nonumber & \tan \bigg(\sqrt{\dfrac{2mE}{\hbar^2}L \bigg) = - \dfrac{

This last result is a transcendental equation for the allowed energy levels. If the potential energy and width of the well are known, the allowed energy levels can be determined by using a solver or graphing the function.

The 1D Semi-Infinite Well

Determine the allowed energy levels for a proton trapped in a semi-infinite square well of width 5.0 fm and depth 60 MeV.

Applying the previous result:

\tan \bigg \sqrt{\frac{2mc^2L^2E}{(\hbar c)^2}} \bigg) = - \sqrt{\frac{E}{U-E}}

\tan \bigg \sqrt{\frac{2(938 \text{ MeV} (5.0 \text{ fm})^2 E}{(194.7 \text{ MeV fm})^2}} \bigg) = - \sqrt{\frac{E}{60-E}}

\tan \bigg \sqrt{1.204 E} \bigg) = - \sqrt{\frac{E}{60-E}}

with E in MeV.
The solutions to this equation, which represent the allowed energy levels for the proton, are 6.53, 25.75, and 55.08 MeV.

5.7: Barrier Penetration and Tunneling

Estimate the tunneling probability for an 10 MeV proton incident on a potential barrier of height 20 MeV and width 5 fm.

Consider the square barrier shown above. In the regions x < 0 and x > L the wavefunction has the oscillatory behavior we've seen before, and can be modeled by linear combinations of sines and cosines. These regions are referred to as allowed regions because the kinetic energy of the particle (KE = E – U) is a real, positive value.

Now consider the region 0 < x < L. In this region, the wavefunction decreases exponentially, and takes the form

$$\Psi(x) = Ae^{-\alpha X} \tag{5.7.1}$$

This is referred to as a forbidden region since the kinetic energy is negative, which is forbidden in classical physics. However, the probability of finding the particle in this region is not zero but rather is given by:

$$P(x) = A^2 e^{-2aX} \tag{5.7.2}$$

Thus, the particle can penetrate into the forbidden region. If the particle penetrates through the entire forbidden region, it can "appear" in the allowed region x > L. This is referred to as quantum tunneling and illustrates one of the most fundamental distinctions between the classical and quantum worlds.

A typical measure of the extent of an exponential function is the distance over which it drops to 1/e of its original value. This occurs when $x = \frac{1}{2a}$. This distance, called the penetration depth, δ, is given by

$$\delta = \frac{1}{2\alpha} \tag{5.7.3}$$

$$\delta = \frac{\hbar x}{\sqrt{8mc^2(U-E)}} \tag{5.7.4}$$

where

- U is the depth of the potential and
- E is the energy state of the wavefunction.

The penetration depth defines the approximate distance that a wavefunction extends into a forbidden region of a potential. Using this definition, the tunneling probability (T), the probability that a particle can tunnel through a classically impermeable barrier, is given by

$$T \approx e^{-x/\delta} \tag{5.7.5}$$

For this example, the probability that the proton can pass through the barrier is

$$\delta = \frac{\hbar c}{\sqrt{8mc^2(U-E)}} \tag{5.7.6}$$

$$\delta = \frac{197.3 \text{ MeVfm}}{\sqrt{8(938 \text{ MeV}}}(20 \text{ MeV -10 MeV}) \tag{5.7.7}$$

$$\delta = 0.720 \text{ fm} \tag{5.7.8}$$

Thus, there is about a one-in-a-thousand chance that the proton will tunnel through the barrier.

Tunneling In and Out

In a crude approximation of a collision between a proton and a heavy nucleus, imagine an 10 MeV proton incident on a symmetric potential well of barrier height 20 MeV, barrier width 5 fm, well depth -50 MeV, and well width 15 fm. Estimate the probability that the proton tunnels into the well. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

In this approximation of nuclear fusion, an incoming proton can tunnel into a pre-existing nuclear well. Once in the well, the proton will remain for a certain amount of time until it tunnels back out of the well. Although the potential outside of the well is due to electric repulsion, which has the 1/r dependence shown below,

we will approximate it by a rectangular barrier:

The tunneling probability into the well was calculated above and found to be

$$T \approx 0.97x10^{-3} \tag{5.7.9}$$

All that remains is to determine how long this proton will remain in the well until tunneling back out.
First, notice that the probability of tunneling out of the well is exactly equal to the probability of tunneling in, since all of the parameters of the barrier are exactly the same.

Remember, T is now the probability of escape per collision with a well wall, so the inverse of T must be the number of collisions needed, on average, to escape. If we can determine the number of seconds between collisions, the product of this number and the inverse of T should be the lifetime ( ) of the state:

The time per collision is just the time needed for the proton to traverse the well. This is simply the width of the well (L) divided by the speed of the proton:

$$\tau = \left(\frac{L}{v}\right)\left(\frac{1}{T}\right) \tag{5.7.10}$$

The speed of the proton can be determined by relativity,

$$KE = (\gamma - 1)mc^2 \tag{5.7.11}$$

$$60 \text{ MeV} = (\gamma - 1)(938.3 \text{ MeV} \tag{5.7.12}$$

$$\gamma = 1.064 \tag{5.7.13}$$

$$v = 0.34c \tag{5.7.14}$$

$$v = 1.0x10^8 \text{ m/s} \tag{5.7.15}$$

Therefore the lifetime of the state is:

$$\tau = \left(\frac{15x10^{-15} \text{ m}}{1.0x10^8 \text{ m/s}}\right)\left(\frac{1}{0.97x10^{-3}}\right. \tag{5.7.16}$$

$$\tau = 1.5x10^{-19} \text{ s} \tag{5.7.17}$$

5.8: The Time-Dependent Schrödinger Equation

In this chapter, we investigated solutions of the time-independent Schrödinger equation. These solutions are offered referred to as stationary states because their spatial shape does not change with time, leading to probabilities that are constant in time. However, they are not "stationary" in the sense of having no time dependence. Imagine a guitar string vibrating in its fundamental mode. The "shape" of the string is constant; it just vibrates back and forth through space. To explore the rate at which the quantum wavefunction "vibrates" we need to solve the time-dependent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\delta^2}{\delta x^2}\Psi(x,t)+U(x)\Psi(x,t)=i\hbar\frac{\delta}{\delta t}\Psi(x,t) \tag{5.8.1}$$

Note that the wavefunction is now a function of both space and time, and the derivatives in the equation are partial derivatives. Also note that the potential energy function, U, is constant with respect to time.

Let's try to solve this partial differential equation by separation of variables. To do this, we'll assume the solution takes the form:

$$\Psi(x,t)=\psi(x)T(t) \tag{5.8.2}$$

Substituting this into the differential equation yields:

$$-\frac{\hbar^2}{2m}T(t)\frac{d^2}{dx^2}\psi(x)+U(x)\psi(x)T(t)=i\hbar\psi(x)\frac{d}{dt}T(t) \tag{5.8.3}$$

Dividing both sides by the wavefunction gives:

$$-\frac{\hbar^2}{2m}\frac{\frac{d^2\psi(x)}{dx^2}}{\psi(x)}+U(x)=i\hbar\frac{\frac{dT(t)}{dt}}{T(t)} \tag{5.8.4}$$

Since the left-hand side is only a function of x and the right-hand side is only a function of t, they can only be equal if both sides equal a constant value. If we call that constant E,

$$-\frac{\hbar^2}{2m}\frac{\frac{d^2\psi(x)}{dx^2}}{\psi(x)}+U(x)=E=i\hbar\frac{\frac{dT(t)}{dt}}{T(t)} \tag{5.8.5}$$

the left-side becomes the time-independent Schrödinger equation and the right-hand side becomes:

$$i\hbar\frac{dT(t)}{dt}=ET(t) \tag{5.8.6}$$

$$\frac{dT(t)}{dt}=-i\omega T(t)\text{ with }\omega=\frac{E}{\hbar} \tag{5.8.7}$$

This equation has the solution

$$T(t)=Ae^{-i\omega t} \tag{5.8.8}$$

Although you may not be familiar with imaginary arguments in the exponential function, mathematicians will tell you it is a really cool way to compactly write the sine and cosine functions:

$$T(t)=Ae^{-i\omega t}=A\left[\cos(\omega t)-i\sin(\omega t)\right] \tag{5.8.9}$$

So what does this mean?

Sines and cosines are familiar functions used to describe oscillations, so this means that the wavefunction simply oscillates at an angular frequency ( ) proportional to the energy. Unlike a guitar string, however, this oscillation is not through physical space, but rather through some much more abstract space in which the square of the amplitude of oscillation is related to probability. Moreover, this space cannot be adequately represented without using imaginary numbers.

You may be concerned that the imaginary nature of the wavefunction will somehow creep into the probability of measuring some physical quantity. You shouldn't be. Although we've stated numerous times that probabilities depend on the square of the wavefunction, actually they depend on the product of the wavefunction and its complex conjugate . Thus, the effect of the temporal part of the wavefunction on all probabilities is given by:

$$\begin{aligned} \mathrm{P}rob(t) &= T\backslash*(t)T(t) \\ \mathrm{P}rob(t) &= (Ae^{+\omega t})(Ae^{-i\omega t}) \\ \mathrm{P}rob(t) &= A^2e^0 \\ \mathrm{P}rob(t) &= A^2 \end{aligned} \tag{5.8.10}$$

In fact, if we set A = 1, the temporal part of the wavefunction will have no effect on the probabilities calculated earlier in this chapter. Thus, as long as the potential energy function is constant in time, Schrödinger's equation is separable and all of our work studying the time-independent equation is valid, as long as we remember that these solutions are actually oscillating in time according to the description given above.

5.9: The Schrödinger Equation Activities

A 1.0 kg ball is thrown directly upward at 10 m/s and the zero-point of gravitational potential energy is located at the position at which the ball leaves the thrower's hand. All heights are measured relative to the zero-point. Below is a graph of gravitational potential energy (U = mgh) vs. height. Answer the following questions.

a. Draw a line on the graph representing the total energy of the ball. Using the graph, determine the maximum height reached by the ball.

b. Using the graph, determine the kinetic energy of the ball when it is at one-half of its maximum height.

c. What would happen to the kinetic energy of the ball if it was at 1 m above its maximum height. What would this imply about the velocity of the ball at this location?

d. Does the ball spend more time, less time or the same amount time in the position interval between 3 m and 4 m or in the position interval between 4 m and 5 m? Why?

e. If a determination of the ball's position is made at a random time, is it more likely, less likely or equally likely to find the ball in the position interval between 3 m and 4 m or in the position interval between 4 m and 5 m? Why?
A 2.0 kg cart oscillates on a horizontal, frictionless surface attached to a k = 50 N/m spring. The cart passes through the spring equilibrium length (s = 0 m) at a speed of 10 m/s. Below is a graph of elastic potential energy (U = ½ ks2) vs. spring deformation (s). Answer the following questions.

a. Draw a line on the graph representing the total energy of the cart. Using the graph, determine the maximum elongation of the spring.

b. Using the graph, determine the kinetic energy of the cart when the spring is at one-half of its maximum elongation.

c. What would happen to the kinetic energy of the cart if it were 1m beyond the maximum elongation of the spring? What would this imply about the velocity of the cart at this location?

d. Does the cart spend more time, less time or the same amount time in the position interval between 0 m and 1 m or in the position interval between 1 m and 2 m? Why?

e. If a determination of the cart's position is made at a random time, is it more likely, less likely or equally likely to find the cart in the position interval between 0 m and 1 m or in the position interval between 1 m and 2 m? Why?

Below is a graph of potential energy (U) vs. position for a region of space occupied by a single macroscopic particle. There are no forces acting on the particle in this region of space other than the force that gives rise to the potential energy. The total energy of the particle is also indicated on the graph. The letters A through F mark six positions within this region of space.

a. Rank the potential energy of the particle at these positions.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

b. Rank the kinetic energy of the particle at these positions.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

c. If a determination of the particle's position is made at a random time, rank the probability of finding the particle in the immediate vicinity of these positions.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your rankings:

Below is a graph of potential energy (U) vs. position for a region of space occupied by a single macroscopic particle. There are no forces acting on the particle in this region of space other than the force that gives rise to the potential energy. The total energy of the particle is also indicated on the graph. The letters A through F mark six positions within this region of space.

a. Rank the potential energy of the particle at these positions. If the particle is never at a position leave it out of the ranking.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

b. Rank the kinetic energy of the particle at these positions. If the particle is never at a position leave it out of the ranking.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

c. If a determination of the particle's position is made at a random time, rank the probability of finding the particle in the immediate vicinity of these positions.

Largest 1. _____ 2. _____ 3. _____ 4. _____ 5. _____ 6. _____ Smallest

_____ The ranking cannot be determined based on the information provided.

Explain the reason for your rankings:

For each of the potential energy functions below, carefully sketch the wavefunction corresponding to the energy level indicated.

a.

b.

c.

d.

For each of the potential energy functions below, carefully sketch the wavefunction corresponding to the energy level indicated.

a.

b.

c.

d.

For each of the potential energy functions below, carefully sketch the continuation of the wavefunction as it passes the barrier or well.

a.

b.

c.

d.

A hypothetical atom has the four electron energy levels shown below.

a. How many spectral lines can be emitted by transitions between these four energy levels?

b. The transition from level ____ to level ____ emits the longest wavelength photon.

c. The transition from level ____ to level ____ involves the absorption of the largest energy photon.

d. If the energy of the n = 3 level was somehow reduced, how many of the six spectral lines would change energy?

e. If the energy of the n = 3 level was somehow reduced, which electron transition(s) would change to larger energy?

f. In which level(s) is the electron most likely to be detected in the immediate vicinity of x = 0?

g. In which level(s) does the electron spend the most time outside of the atom?

g. In which level(s) is the electron most likely to be detected within the right half of the atom?

A proton is trapped in an infinite potential well of length 1.0 x 10-15 m. Find the three longest wavelength photons emitted by the proton as it changes energy levels in the well.

Mathematical Analysis

An electron is trapped in an infinite potential well of length 1.0 x 10-10 m. Find the three longest wavelength photons emitted by the electron as it changes energy levels in the well.

Mathematical Analysis

Imagine eight neutrons trapped in an atomic nucleus. Model the nucleus as an infinite potential well of length 5.0 x 10-15 m.
a. Find the total kinetic energy of the eight neutrons.
b. If four of the neutrons changed into protons, calculate the new total kinetic energy. Ignore the mass difference between protons and neutrons.

Mathematical Analysis

Imagine eight neutrons and five protons trapped in an atomic nucleus. Model the nucleus as an infinite potential well of length 5.0 x 10-15 m.
a. Find the total kinetic energy of the nucleons.
b. If neutrons and protons can freely change into each other, what will happen?
c. Calculate the minimum total kinetic energy of the nucleons. Ignore the mass difference between protons and neutrons.

Mathematical Analysis

A photon is trapped in an infinite potential well of length L. Find the allowed energies for the photon. (Hint: You cannot use the Schrödinger equation to solve this problem.)

Mathematical Analysis

You may occasionally feel trapped in a classroom. If so, you may find yourself unable to be completely stationary. Assuming a 10 m wide classroom, and a 65 kg student, estimate your minimum energy and velocity.

Mathematical Analysis

An electron is trapped in an infinite potential well of length 1.0 x 10-10 m.
a. According to classical physics (i.e., common sense), what is the probability that the electron will be found in the middle fifth

(from 0.4 x 10-10 m to 0.6 x 10-10 m) of the well?
b. In its ground state, what is the probability that the electron will be found in the middle fifth of the well?
c. In its n=2 state, what is the probability that the electron will be found in the middle fifth of the well?
d. Based on the shape of the wavefunction, explain why (b) is greater than (a), and (a) is greater than (c)? (If they aren't, you did the problem incorrectly!)

Mathematical Analysis

A neutron is trapped in an infinite potential well of length 4.0 x 10-15 m.
a. According to classical physics (i.e., common sense), what is the probability that the neutron will be found in the left quarter (from 0.0 x 10-15 m to 1.0 x 10-15 m) of the well?
b. In its ground state, what is the probability that the neutron will be found in the left quarter of the well?
c. In its n=2 state, what is the probability that the neutron will be found in the left quarter of the well?
d. Based on the shape of the wavefunctions, do your answers for a, b, and c have the correct relative size?

Mathematical Analysis

A particle is trapped in a one-dimensional infinite potential well of length L.
a. Find the expectation values of the particle's position and momentum in the first excited state of this well. Compare these results to the results for the ground state.
b. Find the uncertainty in the particle's position and momentum in the first excited state of this well. Compare these results to the results for the ground state.
c. Show that the uncertainties in these values do not violate the uncertainty principle.

Mathematical Analysis

A particle is trapped in a one-dimensional infinite potential well of length L.
a. Find the expectation value of the particle's position as a function of energy level, n.
b. Find the uncertainty in the particle's position as a function of energy level, n.
c. In the limit of very large n, what is the uncertainty in the particle's position? (In classical physics, this value would be 0.289L.)

Mathematical Analysis

A particle is trapped in a one-dimensional infinite potential well of length L.
a. Find the expectation value of the particle's kinetic energy in the ground state of this well. (Hint: Kinetic energy can be expressed as p2/2m. This should allow you to construct an operator for kinetic energy.)
b. Find the uncertainty in the particle's kinetic energy in the ground state of this well.
c. Carefully explain what your answer for (b) implies about the time the particle can remain in the ground state.

Mathematical Analysis

Classically, a particle trapped in a one-dimensional infinite potential well of length L would have an equal probability of being detected anywhere in the well. Thus, its "classical" wavefunction would be:

where C is a constant.
a. Find the value of C by setting the total probability of finding the particle in the well equal to 1.
b. Find the expectation value of the particle's position.
c. Find the uncertainty in the particle's position.

Mathematical Analysis

A particle trapped in a one-dimensional parabolic potential well centered on x = 0 has a ground-state wavefunction given by:

Note that this type of well extends from x = -∞ to x = +∞.
a. Find A.
b. Find the expectation values of the particle's position and momentum in the ground state of this well.
c. Find the uncertainty in the particle's position and momentum in the ground state of this well.

d. Show that the uncertainties in these values do not violate the uncertainty principle.

Mathematical Analysis

A particle of mass m is trapped in an infinite potential well of x-length L and y-width 3L. For each of the pairs of quantum numbers below, state the locations (other than the boundaries) where the probability of detecting the particle is zero.
a. (nx, ny) = (1, 1)
b. (nx, ny) = (3, 1)
c. (nx, ny) = (2, 3)

Mathematical Analysis

A particle of mass m is trapped in an infinite potential well of x-length L, y-width L, and z-height 2L. For each of the triplets of quantum numbers below, state the locations (other than the boundaries) where the probability of detecting the particle is zero.
a. (nx, ny, nz) = (1, 1, 1)
b. (nx, ny, nz) = (2, 2, 1)
c. (nx, ny, nz) = (1, 2, 2)

Mathematical Analysis

A particle of mass m is trapped in an infinite potential well of x-length L and y-width L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny E ( )
1 1 1

A particle of mass m is trapped in an infinite potential well of x-length 3L and y-width 2L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny E ( )
1 1 1

Eight neutrons and five protons are trapped in an infinite potential well of x-length 4.0 x 10-15 m and y-width 4.0 x 10-15 m. Ignore the mass difference between protons and neutrons.
a. Determine the 5 lowest energy levels and list them below.
b. Find the total kinetic energy of the 13 particles.
c. Would the total energy decrease if a neutron turned into a proton? If so, by how much?

Mathematical Analysis

Level nx ny E ( ) # neutrons # protons
1 1 1 2 2

Eight protons and eleven neutrons are trapped in an infinite potential well of x-length 4.0 x 10-15 m and y-width 2.0 x 10-15 m. Ignore the mass difference between protons and neutrons.

a. Determine the 5 lowest energy levels and list them below.

b. Find the total kinetic energy of the 19 particles.

c. Would the total energy decrease if a neutron turned into a proton? If so, by how much?

Mathematical Analysis

Level nx ny E ( ) # neutrons # protons
1 1 1 2 2

A particle of mass m is trapped in an infinite potential well of x-length L, y-width L, and z-height L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny nz E ( )
1 1 1 1

A particle of mass m is trapped in an infinite potential well of x-length L, y-width L, and z-height 2L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny nz E ( )
1 1 1 1

A particle of mass m is trapped in an infinite potential well of x-length L, y-width 2L, and z-height 2L. Determine the 5 lowest energy levels and list them below.

Mathematical Analysis

Level nx ny nz E ( )
1 1 1 1

Find the allowed energy levels for a proton trapped in a semi-infinite potential well of width 3.0 fm and depth 40 MeV. Compare these values to those obtained assuming the well is infinitely deep.

Mathematical Analysis

Find the allowed energy levels for an electron trapped in a semi-infinite potential well of width 1.0 nm and depth 5.0 eV. Compare these values to those obtained assuming the well is infinitely deep.

Mathematical Analysis

Find the allowed energy levels for a neutron trapped in a semi-infinite potential well of width 7.0 fm and depth 50 MeV. Compare these values to those obtained assuming the well is infinitely deep.

Mathematical Analysis

A proton is incident on a rectangular potential barrier of height 50 MeV and width 5 fm. What is the approximate probability that the proton will tunnel through the barrier for each of the incident kinetic energies below?
a. 20 MeV
b. 30 MeV
c. 40 MeV
d. 45 MeV

Mathematical Analysis

a. Estimate the tunneling probability for an 18 MeV proton incident on a symmetric potential well with barrier height 20 MeV, barrier width 3 fm, well depth -50 MeV, and well width 15 fm.
b. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

Mathematical Analysis

a. Estimate the tunneling probability for a 5 MeV alpha particle incident on a symmetric potential well with barrier height 40 MeV, barrier width 8 fm, well depth -50 MeV, and well width 15 fm.
b. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

Mathematical Analysis

a. Estimate the tunneling probability for a 1.0 MeV proton incident on a symmetric potential well with barrier height 1.5 MeV, barrier width 1 fm, well depth -10 MeV, and well width 3.0 fm.
b. If the proton successfully tunnels into the well, estimate the lifetime of the resulting state.

Mathematical Analysis

In a sparkplug, a potential difference of about 20,000 V is needed for a spark to jump the 1.5 mm gap. Modeling this as a rectangular potential barrier of height 20 keV and width 1.5 mm:
a. Estimate the probability of an electron tunneling across the sparkplug gap when the potential difference across the gap is only 10,000 V. Based on this answer, should auto mechanics worry about quantum mechanics?
b. At what potential difference would tunneling provide a one-in-a-billion chance of a premature spark?

Mathematical Analysis

In a transistor a rule of thumb is “60 mV per decade”, meaning that a voltage change of 60 mV should cause a tenfold increase (or decrease) in current. In a tunneling transistor we can image a rectangular potential barrier of width 10 nm with a height that can be adjusted by the applied voltage. What height barrier is needed so that a change of 60 mV (60meV of energy) causes a tenfold change in current?

Mathematical Analysis

In a scanning tunneling microscope (STM), a slender metal tip is positioned very close to a sample under study. Although no electric contact is made between tip and sample, electrons from the tip can tunnel across the empty space to the sample, resulting in an electric current. Since this current is exponentially dependent on the separation between the tip and the sample, incredibly precise measurements of surface features are possible.

Approximating the potential barrier between tip and sample to be a rectangular barrier with height equal to a typical metallic work function (4.0 eV), find the change in separation between tip and sample that will result in a 10% change in tunneling current. This change in separation is approximately the resolution of the STM.

Mathematical Analysis

In an ammonia molecule (NH3), the nitrogen atom is equally likely to be “above” or “below” the plane formed by the three hydrogen atoms. In fact, the nitrogen atom tunnels back and forth between these two equivalent orientations at an incredibly stable frequency. The stable frequency of ammonia inversion was used as the standard for the first generation of atomic clocks.

Approximating the potential barrier between the above and below orientations as a rectangular barrier with height U = 0.26 eV, and width L = 0.038 nm, find the frequency with which the nitrogen oscillates between these two states. The energy of the nitrogen atom in either well is E = 0.25 eV.

Mathematical Analysis

5.A: Solving the Finite Well (Project)

Imagine a particle trapped in a one-dimensional well of length 2L. Inside the well there is no potential energy while the region outside the well has a finite potential energy. This potential energy function is referred to as the finite square well.

I. General Solution

Since the potential energy is a piece-wise function, Schrödinger's equation must be solved in the three regions separately.

A. Region I: x < -L
1. Apply and solve Schrödinger's equation in region I. Your solution should have two arbitrary constants. Make use of the definition:

2. Clearly explain why one of your two constants must equal zero, and then record your simplified solution below.

B. Region II: -L < x < L
1. Apply and solve Schrödinger's equation in region II. Your solution should have two arbitrary constants. Make use of the definition:

C. Region III: x > L
1. Apply and solve Schrödinger's equation in region III. Your solution should have two arbitrary constants. Again make use of the definition:

2. Clearly explain why one of your two constants must equal zero, and then record your simplified solution below.

D. Sketching Solutions
1. Sketch the wavefunctions of the four lowest energy states on the diagram below.

Your sketched solutions should fall into one of two categories. Symmetric solutions (n = 1, 3, …) are those that are symmetric about the y-axis. Antisymmetric solutions (n = 2, 4, …) are, you guessed it, antisymmetric when reflected about the y-axis. We will solve for these two types of solutions separately.

E. Symmetric Solutions
1. Clearly explain why one of the two constants in region II must equal zero for symmetric solutions.

2. What is the relationship between the constant in region I and the constant in region III for symmetric solutions?

3. Record your simplified solutions in the three regions below.
Region I:

Region II:

Region III:

The wave function, and the derivative of the wave function, must be continuous across the boundary at x = L. This will lead to two simultaneous equations that can be solved for the energy levels of symmetric solutions.
4. Force the wavefunction to be continuous across the x = L boundary and simplify the resulting expression.

5. Force the derivative of the wavefunction to be continuous across the x = L boundary and simplify the resulting expression.

6. Solve the two equations above for an expression involving and k. Simplify this expression into a transcendental equation for the energy levels of symmetric solutions.

F. Antisymmetric Solutions
1. Clearly explain why one of the two constants in region II must equal zero for antisymmetric solutions.

2. What is the relationship between the constant in region I and the constant in region III for antisymmetric solutions?

3. Record your simplified solutions in the three regions below.
Region I:

Region II:

Region III:

4. Force the wavefunction to be continuous across the x = L boundary and simplify the resulting expression.

5. Force the derivative of the wavefunction to be continuous across the x = L boundary and simplify the resulting expression.

6. Solve the two equations above for an expression involving and k. Simplify this expression into a transcendental equation for the energy levels of antisymmetric solutions.

II. Specific Solution

Find all the allowed energy levels for an electron trapped in a finite square well of total width 2.00 nm and depth 1.00 eV.

5.A: Solving the Hydrogen Atom (Project)

Enough with pretending atoms are three-dimensional, infinite square wells! It's time to tackle an atom for real. (Before we get too excited, the atom under analysis is hydrogen. All other atoms are impossible to solve analytically.)

I. Schrödinger's Equation in Spherical Coordinates

The time-independent Schrödinger's equation in Cartesian coordinates is:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi(x,y,z)}{\partial x^2}+\frac{\partial^2\Psi(x,y,z)}{\partial y^2}+\frac{\partial^2\Psi(x,y,z)}{\partial z^2}\right)+U(x,y,z)\Psi(x,y,z)=E\Psi(x,y,z) \tag{5.A.1}$$

Of course, this is not really the best coordinate system to use to address the hydrogen atom. A better coordinate system is spherical coordinates. In spherical coordinates, the Schrödinger's equation for the hydrogen atom looks like this:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r,\theta,\phi)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi(r,\theta,\phi)}{\partial\theta}\right)+\frac{R(r)}{r^2\sin^2\theta}\frac{\partial^2\Psi(r,\theta,\phi)}{\partial\phi^2}\right]+U(r,\theta,\phi)\Psi(r,\theta,\phi) \\ =E\Psi(r,\theta,\phi) \tag{5.A.2}$$

Granted, this doesn't make the equation look very nice, but it makes the solution of the equation possible. If you want to know more about why the spatial derivative part looks like this, go talk to a math professor.

II. Solving Schrödinger's Equation for Hydrogen

For hydrogen, the potential energy function is simply:

$$U(r,\theta,\phi)=-\frac{ke^2}{r} \tag{5.A.3}$$

Since the potential energy only depends on r, perhaps we can separate the r-dependence in the equation from the angular dependence. Let's assume:

$$\Psi(r,\theta,\phi)=R(r)Y(\theta,\phi) \tag{5.A.4}$$

Substituting these in gives:

$$-\frac{\hbar^2}{2m}\left[\frac{Y(\theta,\phi)}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{R(r)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{R(r)}{r^2\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right]-\frac{ke^2}{r}R(r)Y(\theta,\phi) \\ =ER(r)Y(\theta,\phi) \tag{5.A.5}$$

Next, divide through by the wavefunction, $R(r)Y(\theta,\phi)$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{1}{r^2Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{1}{r^2Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right]-\frac{ke^2}{r}=E \tag{5.A.6}$$

Move the potential energy to the right side and multiply through by $\frac{2mr^2}{\hbar^2}$

$$-\left[\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{1}{Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{1}{Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\right]=\frac{2mr^2}{\hbar^2}\left(E+\frac{ke^2}{r}\right) \tag{5.A.7}$$

Move the remaining r-dependence to the right-hand side:

$$-\frac{1}{Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)-\frac{1}{Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}=\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{2mr^2}{\hbar^2}\left(E+\frac{ke^2}{r}\right) \tag{5.A.8}$$

The left-side of the equation is a function only of θ and ϕ, and the right-side is a function only of r. For the two sides to be equal, they must both equal the same constant. Making what may seem like an odd choice for this constant yields two differential equations. The radial equation:

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{2mr^2}{\hbar^2}\left(E+\frac{ke^2}{r}\right)=\lambda(\lambda+1) \tag{5.A.9}$$

and the angular equation:

$$-\frac{1}{Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)-\frac{1}{Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}=\lambda(\lambda+1) \tag{5.A.10}$$

II.A: The Angular Equation

Re-arranging the angular equation a bit leads to:

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)+\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}=-\lambda(\lambda+1)Y(\theta,\phi)\sin^2\theta \tag{5.A.11}$$

You may not recognize this equation, but Laplace solved it in 1782 and named the solutions spherical harmonics. Just as sines and cosines can be used to model vibrations in rectangular coordinates (on guitar strings, in 3D infinite wells, etc.) spherical harmonics model vibrations in spherical coordinates. In some sense, you can imagine them as the fundamentally distinct ways in which a spherical surface can vibrate.

Spherical harmonics are labeled by a pair of integers, m and l, and typically written as $Y_\lambda^m(\theta,\phi$

- m is termed the magnetic quantum number and represents the number of complete waves that wrap around the sphere in the azimuthal (ϕ) direction. Thus, if m = 0, there is no change as you move around the sphere. If m = 1, one wavelength wraps around the sphere so that half of the sphere has positive "displacement" and the other half negative. m can also be negative. In this case, the displacements are flipped, and the "wave" travels the other way around the sphere.
- l is termed the orbital quantum number and controls the polar (θ) direction. The value of (l – |m|) represents the number of nodes in the polar direction. l cannot be negative.
- In ye olde fashioned chemistry notation, l = 0, 1, 2, 3, … are referred to as s, p, d, f, …
- |m| is always less than or equal to l.

The diagrams below may help you better visualize spherical harmonics. The nodal lines are clearly marked.

1. Draw the nodal lines for the spherical harmonics listed below. Label the sectors + or -.

The first few spherical harmonics are listed below.

$$Y_0^0(\theta,\phi)=\frac{1}{2}\sqrt{\frac{1}{\pi}} \tag{5.A.12}$$

$$Y_1^{-1}(\theta,\phi)=\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi} \tag{5.A.13}$$

$$Y_1^0(\theta,\phi)=\frac{1}{2}\sqrt{frac3\pi}\cos\theta \tag{5.A.14}$$

$$Y_1^1(\theta,\phi)=\frac{-1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi} \tag{5.A.15}$$

$$Y_2^{-2}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\phi} \tag{5.A.16}$$

$$Y_2^{-1}(\theta,\phi)=\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\phi} \tag{5.A.17}$$

$$Y_2^0(\theta,\phi)=\frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta-1) \tag{5.A.18}$$

$$Y_2^1(\theta,\phi)=\frac{-1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\phi} \tag{5.A.19}$$

$$Y_2^2(\theta,\phi)=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi} \tag{5.A.20}$$

Note lack of node in θ direction
Note node at $\theta=\pi/2$

2. At what values of θ does $Y_2^0(\theta,\phi)$ have nodes?

3. $Y_3^1(\theta,\phi)=\frac{-1}{8}\sqrt{\frac{21}{\pi}}\sin\theta(5\cos^2\theta-1)e^{i\theta}$. Where are the nodal lines?

These functions are all normalized so that the integral of $Y(\theta, phi)^2$ over the surface of the sphere is 1:

$$\int_0^{2\pi}\int_0^{\pi}(Y_i^m(\theta,\phi))^2\sin\theta d\theta d\phi=1 \tag{5.A.21}$$

4. $Y_3^{-3}C\sin^3(\theta)e^{-3i\phi}$. Find C by normalizing this spherical harmonic.

5. Can you see a trend in the spherical harmonics when l = |m|? Determine $Y_5^5(\theta,\phi)$, including the constant.

B. The Radial Equation

Re-arranging the radial equation a bit leads to:
\[\frac{d}{dr}\bigg(r^2\frac{dR(r)}{dr}\bigg)+\frac{2mr^2}{\hbar^2}\bigg(E+\frac{ke^2}{r}\bigg)R(r)-\lambda(\lambda+1)R(r)=0\)

If we define a0, termed the Bohr radius, as

$$a_0=\frac{\hbar^2}{mke^2}=5.29\times10^{-11}\text{ m} \tag{5.A.22}$$

then

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\left[\frac{2mE}{\hbar^2}r^2+\frac{2}{a_0}r-\lambda(\lambda+1)\right]R(r)=0 \tag{5.A.23}$$

Again, you may not recognize this equation, but the 19th century mathematician Leguerre did and the solutions involve *Laguerre polynomials*. These solutions depend on both l and a new integer, n.

The radial wavefunction depends on a pair of integers, n and l, and is typically written as $R_{n\lambda}(r)$.
• All solutions are the product of a polynomial of degree (n -1) and the term \(e^{-r/na_0\).
• n is termed the principal quantum number and determines the spatial extent of the wavefunction and its energy. Since all solutions drop off as , large n states extend farther from the nucleus.
• The energy of the electron is dependent on n, and given by the formula:

$$E=-\frac{1}{n^2}\frac{ke^2}{2a_0}=-\frac{13.6\text{ eV}}{n^2} \tag{5.A.24}$$

• n is greater than zero, and l is always less than n.

The first few radial wavefunctions are listed below. (Note Z = 1 for hydrogen)

6. Which of the radial wavefunctions listed above are non-zero at the origin, r = 0? In ye olde fashioned chemistry notation, what are these wavefunctions called?

7. Which of the radial wavefunctions listed above have nodes, i.e., radii where the probability of finding the electron is zero? (Do not include r = 0 or r = ∞ as nodes.)

8. Find the nodes for each of the radial wavefunctions listed above, in terms of a0.

9. Can you spot a relationship between the number of nodes and the values of n and l? What is it?

These functions are all normalized so that the total probability of finding the electron somewhere is 1. In spherical coordinates, this means:

$$\int_0^{\infty}(R_{n\lambda}(r))^2r^2dr=1 \tag{5.A.25}$$

10. $R_{43}(r)=C\left(\frac{r}{a_0}\right)^3e^{-r/4a_0}$. Find C by normalizing this radial wavefunction.

The graphic below shows both the radial wavefunction and the radial probability distribution . You should be able to use this to check your node calculations.

Combining the radial wavefunctions with the spherical harmonics generates the complete wavefunction for the electron. If you want to see this in three dimensions, go stand in front of the chemistry labs. If you want to see two dimensional pictures, just look below.

Remember:
• The number of polar nodes is l –|m|
• The number of radial nodes is n – (l + 1)

III. Angular Momentum

The energy levels for hydrogen are the same as Bohr derived using a much simpler model. (In fact, you reproduced this derivation in the Emission Spectrum laboratory.) However, the angular momentum properties are quite different.

The vector expression for angular momentum is:

$$L = r \times p \tag{5.A.26}$$

Thus, a particle circling around the z-axis counter-clockwise would have angular momentum in the +z-direction.

Of course, thinking of the electron as having a well-defined trajectory like this will get you into trouble with Heisenberg.
A better picture would be the one below, at left. In this state, the electron is basically circling around the z-axis, but with quite a bit of "play" in its plane of rotation. We interpret this as the electron's angular momentum precessing around the z-axis, like a wobbly top. The z-component of its angular momentum is constant, but the total angular momentum vector traces a cone around the z-axis, providing a bit of angular momentum in both the x- and y-directions. Note that Lx and Ly are not well-defined like Lz. They each average to zero but at all times there is some angular momentum in the xy-plane.

Now consider the state at right. You should be able to see that the angular momentum vector lies in the xy-plane, with Lz = 0. (By the argument above you may believe that it occasionally "wanders" off the xy-plane creating a temporary z-component, which averages to zero, but since m = 0 there is no "wave" encircling the z-axis and hence Lz is exactly zero.)

Finally, the state at left has no angular momentum at all. The probability "cloud" is completely independent of both and so there is no "motion" in any angular direction.

Rather than examining the wavefunction for every possible situation conceptually, I'm now just going to tell you the formulas for the total angular momentum and the z-component of angular momentum:

$$|L| = \sqrt{\lambda(\lambda+1)}\hbar \tag{5.A.27}$$

$$L_s = m\hbar \tag{5.A.28}$$

When no external electric or magnetic fields are present, the allowed energy levels do not depend on the angular momentum state. For each value of l, there are (2l + 1) degenerate energy states.

In the l = 2 states illustrated at right, the magnitude of the angular momentum is always

$$|L| = \sqrt{2(2+1)}\hbar = 2.44\hbar \tag{5.A.29}$$

This represents the length of the vector precessing around the z-axis. The projection of this vector on the z-axis can take on one of five different values determined by m. With no external field to break the ±z-symmetry all five of these states have the same energy.

11. For each of the five states above, determine the angular momentum in the xy-plane.

All this talk about precessing around the z-axis may have you convinced that at any particular moment we can determine the exact direction of and therefore calculate not only the amount of angular momentum in the xy-plane, but the exact components in both x and y. However, if this was true (i.e., if we knew Lx, Ly, and Lz at the same time) it would mean we know both the position and linear momentum of the electron at the same time. Again, Heisenberg wouldn't like this! We need to think of the electron "cloud" as somehow precessing around the z-axis, with no definite direction at any particular time.

12. Consider the n = 4 state. What is the total degeneracy of this state? (Do not include electron spin.) List the angular momentum and z-component of angular momentum for each sub-state.

It may confuse you for me to say that all of the states above (4s, 4p, 4d, and 4f in chemistry notation) have the same energy, but for hydrogen they do. However, you've learned in chemistry that these states have different energies and are thus filled in a specific order. That's only the case for multi-electron atoms. Basically, if you incorporate the interactions of all the electrons the degeneracy between these states splits.

For example, s-state electrons have a radial wavefunction that is non-zero at the location of the nucleus while the other states don't. Thus, when the s-state electrons are closer to the nucleus than the other states, they effectively "screen" the nuclear charge so that the other states "see" less positive charge. This makes the other states less tightly bound. If there are Z protons in the nucleus, s-state electrons interact with all Z protons while p-states only "see" approximately Z - 2 protons, d-states "see" Z – 8, etc. This effect, electron screening, and other electron-electron interactions split the degeneracy present for a single-electron atom.

IV. Emission Spectrum

IV.A: Selection Rules

In the Bohr model of hydrogen, the emission spectrum is simple. Every time the electron drops from a higher to a lower energy level a photon is emitted with energy equal to the difference between these levels. However, the real situation is more complicated. One reason is that the photon has angular momentum.

Since the photon has angular momentum, and angular momentum is conserved, the atom can only undergo photon emission if the total angular momentum of the atom changes. This requirement creates a selection rule:

$$\Delta \lambda=\pm 1 \tag{5.A.30}$$

The angular momentum carried away by the photon can either come from the z-component or the xy-plane component:

$$\Delta m=0, \pm 1 \tag{5.A.31}$$

Thus, s-states cannot decay to s-states, p-states cannot decay to p-states, etc. Some allowed transitions are illustrated at right.

13. Consider the state (4,1,1). List all decays from this state that are "allowed" by the selection rules above.

Decays that do not obey the selection rules above are not strictly forbidden, they just occur at greatly reduced rates.

IV.B: Zeeman Effect

Another complication to the simple emission spectrum described by Bohr occurs if the atom is in the presence of an external magnetic field, which for convenience we will imagine as oriented in the z-direction.

If m is not equal to 0, we can imagine the electron as orbiting around the z-axis. This orbiting electron creates a magnetic field that interacts with the external magnetic field. If these two fields are in opposite directions the electron "wants" to flip over and align with the external field. This can be quantified as an additional source of potential energy given by:

$$U=m \mu_B B \tag{5.A.32}$$

where μ_B is the Bohr Magneton, defined as

$$\mu_B=\frac{e \nu}{2 m} \tag{5.A.33}$$

$$\mu_B=5.79 \times 10^{-5} \text{ eV} / \text{T} \tag{5.A.34}$$

Imagine the 2p state illustrated at right in the presence of a magnetic field in the +z-direction. If the electron is orbiting counterclockwise (m = 1) it creates a magnetic field oriented in the –z-direction, since it is negatively charged. Since these fields have opposite orientation, this electron state has additional potential energy. The opposite is true for the m = -1 state. The m = 0 is unaffected by the magnetic field. Thus, the threefold degeneracy in the 2p state is broken in the presence of a magnetic field.

Due to this breaking of the energy degeneracy of 2p, rather than a single emission line corresponding to the 2p to 1s transition there are three closely spaced emission lines. This splitting of spectral lines due to the presence of an external magnetic field is termed the *Zeeman effect.*

14. Consider the 2p state in the presence of a 2.0 T magnetic field. If this state decays to the 1s state, find the shift in wavelength of the m = ±1 states relevant to the m = 0 state. What is this shift as a percentage of the original wavelength?

C. Intrinsic Spin

One last complication, I promise.

The angular momentum we have been discussing up to this point is termed orbital angular momentum and is due, not surprisingly given its name, to the orbital motion of the electron. In addition, the electron has an intrinsic angular momentum, commonly referred to as spin. This angular momentum is not really due to the electron "spinning" in space but rather represents a fundamental characteristic of the electron, like its charge or its mass.

The electron's spin vector adds to its orbital angular momentum vector to form its total angular momentum vector, . All of these vectors may precess around the z-axis.

The spin vector obeys similar mathematics to the angular momentum vector, although the spin quantum number, s, is always equal to ½.

If you are following along you should now expect me to discuss how the z-portion of the spin can interact with an external magnetic field to further split each sub-state into two different energy levels, a spin-up and a spin-down sub-state. In fact, there is a splitting involving a potential energy term given by:

$$U = \pm\mu_B B \tag{5.A.35}$$

However, this splitting is not due to an external magnetic field but rather the magnetic field generated by the electron's own orbital motion. If Lz and Sz are in the same direction (either both "up" or both "down") the potential energy is positive while if they are in opposite directions the potential energy is negative. This interaction between the spin and angular momentum is termed spin-orbit coupling. The difference between these levels is given by:

$$\Delta E = E_{parallel} - E_{antiparallel} = \frac{mc^2\alpha^4}{n^5} \tag{5.A.36}$$

where , the fine structure constant, is very close to 1/137.

15. Consider the (2,1,1) state with no external magnetic field. If this state decays to the 1s state, find the shift in wavelength of the LS-parallel state with the LS-antiparallel state. What is this shift as a percentage of the original wavelength?

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