5.6: Solving the 1D Semi-Infinite Square Well

Imagine a particle trapped in a one-dimensional well of length L. Inside the well there is no potential energy. However, the "right-hand wall" of the well (and the region beyond this wall) has a finite potential energy. This means that it is possible for the particle to escape the well if it had enough energy.

Again, since this potential is a piece-wise function, Schrödinger's equation must be solved in the three regions separately.
In the region x < 0, we have already seen that since the potential is infinite there is no chance of finding the particle in this region. Thus, Ψ = 0 in this region.
In the region 0 < x < L, the equation and solution should look familiar:

$$\begin{aligned} -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+U(x)\Psi(x) &= E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+(0)\Psi(x) &= E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d}{dx^2}\Psi(x) &= E\Psi(x) \\ \frac{d^2}{dx^2}\Psi(x) &= -\frac{2mE}{\hbar^2}\Psi(x) \end{aligned} \tag{5.6.1}$$

The general solution to this equation is

$$\Psi(x) = A\sin(kx)+B\cos(kx)\text{with}k=\sqrt{\frac{2mE}{\hbar^2}} \tag{5.6.2}$$

In order for this solution to be continuous with the solution for x < 0, the coefficient B must equal zero. Thus,

$$\Psi(x) = A\sin(kx) \tag{5.6.3}$$

In the region x > L, the equation is:

$$\begin{aligned} -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)+U(x)\Psi(x)=E\Psi(x) \\ -\frac{\hbar}{2m}\frac{d^2}{dx^2}\Psi(x)=(E-U)\Psi(x) \\ \frac{d^2}{dx^2}\Psi(x)=\frac{2m(U-E)}{\hbar^2}\Psi(x) \end{aligned} \tag{5.6.4}$$

The general solution is:

$$\Psi(x) = Ce^{aX}+De^{-aX}\text{ with }\alpha=\sqrt{\frac{2m(U-E)}{\hbar^2}} \tag{5.6.5}$$

Since this region contains the point x = +∞, C must equal zero or the wavefunction will diverge. Therefore,

$$\Psi(x) = De^{-aX} \tag{5.6.6}$$

The wave function, and the derivative of the wave function, must be continuous across the boundary at x = L. Forcing continuity leads to:

$$\begin{aligned} \Psi(x=L^-)=\Psi(x=L^+) \\ A\sin(kL)=De^{-aL} \end{aligned} \tag{5.6.7}$$

and forcing the continuity of the derivative leads to:

$$\begin{aligned} \Psi(x=L^-)=\Psi(x=L^+) \\ kA\cos(kL)=-\alpha De^{-aL} \end{aligned} \tag{5.6.8}$$

Substituting the first equation into the second equation yields:

\[\begin}align} & kA\cos(kL) = - \alpha(A \sin(kL)) \\ \nonumber & \tan(kL) = - \dfrac{k}{\alpha} \\ \nonumber & \tan \bigg(\sqrt{\dfrac{2mE}{\hbar^2}L \bigg) = - \dfrac{

This last result is a transcendental equation for the allowed energy levels. If the potential energy and width of the well are known, the allowed energy levels can be determined by using a solver or graphing the function.

The 1D Semi-Infinite Well

Determine the allowed energy levels for a proton trapped in a semi-infinite square well of width 5.0 fm and depth 60 MeV.

Applying the previous result:

\tan \bigg \sqrt{\frac{2mc^2L^2E}{(\hbar c)^2}} \bigg) = - \sqrt{\frac{E}{U-E}}

\tan \bigg \sqrt{\frac{2(938 \text{ MeV} (5.0 \text{ fm})^2 E}{(194.7 \text{ MeV fm})^2}} \bigg) = - \sqrt{\frac{E}{60-E}}

\tan \bigg \sqrt{1.204 E} \bigg) = - \sqrt{\frac{E}{60-E}}

with E in MeV.
The solutions to this equation, which represent the allowed energy levels for the proton, are 6.53, 25.75, and 55.08 MeV.
