11: Photons: quantization of a single electromagnetic field mode

Objectives

In this chapter we introduce the concept of the photon, as the quantum of a mode of the electromagnetic field. We show that the photon number has a well defined value in an energy eigenstate of the mode. But these number states are very far from our everyday concept of a monochromatic field. Light in a highly coherent monochromatic laser field is in a so called coherent state, which we introduce shortly.

Prerequisites

lements of electrodynamics, Maxwell equations. Energy density and energy of the field. Chapter 2. The classical and the quantum harmonic oscillator (See Chapter 2).

Hamilton function of a single mode

We shall consider two perfectly reflecting parallel mirrors one at $z=0$, the other at $z=L$, and let their surface be F. We also consider a **single transversal electromagnetic standing wave** where the linearly polarized electric field oscillates in the x direction with the classical expression:

$$\mathbf{E}=\hat{\mathbf{x}} q(t) A \sin k z \quad (11.1)$$

where $q(t)$ is a time dependent function, to be determined. In order to satisfy the boundary condition that the parallel component of $\mathbf{E}$ must vanish on the surfaces of the mirrors, we obtain that k cannot be arbitrary, it must be of the form

$$k=n \pi / L \quad (11.2)$$

where $n=1,2, \ldots$. Different nn-s correspond to different k-s. Choosing one specific value of k means a specific standing wave, which is called a **mode** of the field. Let us assume that $q(t)$ is dimensionless, then A has the dimension of the electric field. According to the law of electrodynamics, a time dependent electric field is always accompanied with a magnetic field, as it follows from the Maxwell equation (Faraday's law of induction):

$$\dot{\mathbf{B}}=-\nabla \times \mathbf{E}=-\left|\begin{array}{ccc}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_{x} & \partial_{y} & \partial_{z} \\ q(t) A \sin k z & 0 & 0\end{array}\right|=-\hat{\mathbf{y}} q(t) A k \cos k z \quad (11.3)$$

We have then

$$\mathbf{B}=-\hat{\mathbf{y}} A s(t) k \cos k z, \quad \text{where} \quad \dot{s}(t)=q(t) \quad (11.4)$$

where we have assumed, that the magnetic field has no static component. Then from the other Maxwell equation:

$$\dot{\mathbf{E}}=c^{2} \nabla \times \mathbf{B}=-c^{2} s(t) A k\left|\begin{array}{ccc}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_{x} & \partial_{y} & \partial_{z} \\ 0 & \cos k z & 0\end{array}\right| \quad (11.5)$$

we get:

$$\begin{aligned} \hat{\mathbf{x}} \dot{q}(t) A \sin k z &= -\hat{\mathbf{x}} c^{2} s(t) A k^{2} \sin k z \\ \dot{q}(t) &= -c^{2} k^{2} s(t)=-\omega^{2} s(t) \quad (11.6) \\ \ddot{q}(t)+\omega^{2} q(t) &= 0 \end{aligned}$$

where the circular frequency of the mode

$$\omega=c k \quad (11.7)$$

has been introduced. The solution for $q(t)$ is

$$q(t)=q_{0} \cos \left(\omega t+\varphi_{0}\right) \quad (11.8)$$

which is well known for the standing wave.

We know that this is the solution for the position of the equation of motion of a classical linear harmonic oscillator of circular frequency ω.

Now a crucial point comes to the play. We shall consider this solution as one corresponding to a fictitious harmonic oscillator of mass M and circular frequency ω, that can be obtained from a classical Hamiltonian:

$$\mathcal{H}=\frac{p^2}{2M}+\frac{M\omega^2 q^2}{2} \quad (11.9)$$

where M is of dimension $\text{energy} \times \text{time}^2$, and p is defined as the canonical momentum by

$$p:=M\dot{q}=-M\omega^2 s(t) \quad (11.10)$$

p is of the dimension of action, $\text{energy} \times \text{time}$.

Problem 11.1

Consider the Lagrange function defined as $\mathcal{L}=\frac{1}{2}M\dot{q}^2-\frac{1}{2}M\omega^2 q^2$, and show that the canonical momentum is $p=M\dot{q}$, derive the Hamilton function and the canonical equations of motion.

Let us calculate now the energy W carried by the electromagnetic field in the cavity. This will be the volume integral of the energy density of the electromagnetic field over the cavity formed by the mirrors:

$$\begin{aligned} W &= F\int_0^L\left(\frac{1}{2}\varepsilon_0\mathbf{E}^2+\frac{1}{2\mu_0}\mathbf{B}^2\right)dz= \\ &= F\frac{1}{2}A^2(\varepsilon_0 q^2\underbrace{\int_0^L \sin^2 kz dz}_{=L/2}+s^2(t)\frac{1}{\mu_0}\frac{\omega^2}{c^2}\underbrace{\int_0^L \cos^2 kz dz}_{=L/2})= \\ &= \frac{FL}{2}A^2\frac{\varepsilon_0}{2}\left(q^2+\omega^2 s^2\right)=\frac{VA^2}{2}\frac{\varepsilon_0}{2}\left(q^2+\frac{p^2}{M^2\omega^2}\right)=\frac{VA^2}{2}\varepsilon_0\frac{1}{M\omega^2}\left(\frac{M\omega^2 q^2}{2}+\frac{p^2}{2M}\right)\end{aligned} \quad (11.11)$$

where $V=FL$ is the volume of this cavity. If we compare this result with the form of the Hamilton function, we see that we can identify the total energy W with the Hamilton function $\mathcal{H}$, if we make the following identification.

$$M=\frac{\varepsilon_0 VA^2}{2\omega^2} \quad (11.12)$$

We see that the dynamics of a given mode is the same as that of a harmonic oscillator. In the method we applied here it is easily seen that the potential energy of the oscillator is that of the electric field, while the kinetic energy of the oscillator corresponds to the energy of the magnetic field. This analogy was realized in the end of the 19th century by Lord Rayleigh and J. Jeans, and this line of thought was also kept by M. Planck when he derived his law of the blackbody radiation. When doing so, however, he prescribed that the many modes in the cavity exchange their energy in quanta, whose energy is $h\nu=\hbar\omega$. In order to come to this point, we consider the quantum mechanics of the oscillator associated with the mode by the method above.

For sake of simplicity we have chosen here a standing wave mode, but similar considerations would hold for a **running** transverse wave, where the electric field strength is of the form:

$$\mathbf{E}=\hat{\mathbf{x}}A\left(\alpha(t)e^{ikz}+\alpha^*(t)e^{-ikz}\right) \quad (11.13)$$

Problem 11.2

Calculate the magnetic field from the Maxwell equations and derive the equation determining the time dependence of the complex function $\alpha(t)$.

Quantization of the mode

Given the Hamilton function of the field mode as an oscillator, the quantization is performed by replacing the canonical variables q and p by linear and selfadjoint operators Q and P, and stipulating the canonical commutation relations:

$$[Q,P]=i\hbar \quad (11.14)$$

where on the right hand side we have Planck's constant $\hbar = h/2\pi$, as it is required by the experimental consequences of this prescription. This method of describing the electromagnetic field mode as a quantized harmonic oscillator was the idea of P. Dirac. The mathematical step of introducing operators instead of the classical quantities results in the adequate and experimentally verified description of the quantum properties of the electromagnetic field. Note that according to the previous section Q corresponded to the electric part E, while P to the magnetic part B of the field. So the noncommutativity of these operators means that they cannot have both a sharp value in quantum electrodynamics. The energy of a single mode will be an operator, as well, and it is the Hamilton operator of the mode:

$$H = \frac{1}{2}\left(\frac{P^2}{M} + M\omega^2 Q^2\right) \quad (11.15)$$

As we know, the eigenstates of the Hamiltonian play an exceptional role in quantum theory, and we are going to determine these states.

According to their present definition Q and P do not represent coordinate and momentum in the strict sense, therefore the eigenstates of the H operators cannot be considered as functions of the q coordinate $u_n(q)$ in the ordinary sense. Therefore we turn here to a more abstract point, and we shall use the notation introduced by Dirac. A state of this quantum system will be denoted in general by the symbol $|\psi\rangle$ and the eigenstates of H by $|u_n\rangle$, called a ket. The eigenvalue equation of H can then be written as:

$$H\,|u_n\rangle = \varepsilon_n\,|u_n\rangle \ (11.16)$$

where ε_n is a number, while $|u_n\rangle$ is the eigenket of H. In order to proceed further, we introduce the following new dimensionless operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{M\omega}{\hbar}}Q + i\frac{P}{\sqrt{M\hbar\omega}}\right) = \sqrt{\frac{M\omega}{2\hbar}}\left(Q + i\frac{P}{M\omega}\right) \\ a^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{M\omega}{\hbar}}Q - i\frac{P}{\sqrt{M\hbar\omega}}\right) = \sqrt{\frac{M\omega}{2\hbar}}\left(Q - i\frac{P}{M\omega}\right) \end{aligned} \quad (11.17)$$

These are not self-adjoint, they are the adjoints of each other. It is not difficult to calculate their commutator

$$\left[a, a^\dagger\right] = \frac{1}{2\hbar}([Q, -iP] + [iP, Q]) = 1 \ (11.18)$$

and to express the Hamiltonian in the following form:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \quad (11.19)$$

Problem 11.3

Prove (11.18) and (11.19).

The eigenvalues and the eigenfunctions of H can be found from the eigenvalues and the eigenkets of the operator

$$N := a^\dagger a \ (11.20)$$

It is easily seen that the eigenkets of $H = \hbar\omega(N + 1/2)$ will be identical to that of N:

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad (11.21)$$

and the eigenvalues are in the relation

(11.22)

The following theorems can be proven:

- The eigenvalues nn of $N := a^\dagger a$ are the nonnegative integers:

$$n = 0, 1, 2, \ldots (11.23)$$

- To each of the energy eigenvalues εnεn there corresponds one single eigenket $|u_n\rangle$, denoted often simply by $|n\rangle$. This means that ε_n is nondegenerate

$$H|n\rangle = \hbar\omega(n + 1/2)|n\rangle \quad (11.24)$$

The ground state energy of the mode is $\hbar\omega/2$ corresponding to $n = 0$. The other eigenstates have higher energies which differ by integer multiples of $\hbar\omega$ from each other. This is the quantization of the energy in the mode, found heuristically by Einstein in 1905, in one of his groundbreaking works about the photoelectric effect. **If the mode is in the state $|n\rangle$, we say that there are nn photons in the mode** with energy $n\hbar\omega + \hbar\omega/2$, where the ground state energy $\hbar\omega/2$ is often called as the zero point energy. We see that the value of $\hbar$ is fixed by those measurements on black body radiation and the photoeffect, which state the discreteness of the energy of the field, and the energy quanta are proportional to the (circular) frequency of the mode.

The set of states $|n\rangle$, i.e. $|0\rangle, |1\rangle, |2\rangle \ldots$are called **photon number states**, or simply **number states of the field mode.**

- The effect of the operator $a^\dagger$ on the eigenkets is the following

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad (11.25)$$

which means that $a^\dagger$ raises the state $|n\rangle$ to $|n+1\rangle$, which has an energy by $\hbar\omega$ more. We say that $a^\dagger$ **creates a photon in the mode, $a^\dagger$ is called therefore a raising or a creation operator**. The numerical factor $\sqrt{n+1}$ is introduced for convenience, it is actually a normalization constant.

- The effect of the operator aa on the eigenkets is the following

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad (11.26)$$

which means that aa lowers the state $|n\rangle$ to $|n-1\rangle$, which has an energy by $\hbar\omega$ less. We say that aa **annihilates a photon in the mode, a is called therefore a lowering or an annihilation operator**. Once again the numerical factor $\sqrt{n}$ is introduced for convenience, it is actually a normalization constant.

- The state $|n=0\rangle := |0\rangle$ is the lowest energy state, or ground state called often the **vacuum**. According to (11.26)

$$a|0\rangle = 0 \quad (11.27)$$

where the 0 on the right hand side is not a state. This just means that there are no states with energy below the **vacuum**.

Problem 11.4

Using (11.26) and (11.25) show the validity of the eigenvalue equation (11.21) of $N = a^\dagger a$.

Problem 11.5

Prove that a number state can be created from the vacuum according to the formula

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \quad (11.28)$$

Other states of the field

The photon number states are very important in quantum electrodynamics, but they are only an exceptional set. There are several other quantum states of a single mode. It can be proven, however that any state characterizing the field can be expanded in terms of the number states, as a generally infinite linear combination of them:

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \quad (11.29)$$

where we require the equality $\sum_{n=0}^{\infty} |c_n|^2 = 1$. It turns out that the actual generation of number states in a cavity mode is a very difficult experimental task.

The time dependence of the state expanded in terms of the number state follows from the general rules of quantum mechanics, and it is given by:

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-i(n+1/2)\omega t}|n\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty} c_n e^{-in\omega t}|n\rangle \quad (11.30)$$

where we put into the exponentials the energy eigenvalues belonging to the given stationary state $|n\rangle$.

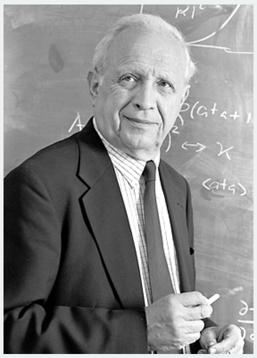
Coherent states

We note here that the states that are most close to our everyday concept of a monochromatic field mode, are not the number states. The light in a highly coherent monochromatic laser field, say that of a red He-Ne laser at $632.8\mathrm{nm}$ is in a so called coherent state of the mode. The coherent states are labelled by a complex number, denoted usually by α, the state itself is then $|\alpha\rangle$, which has the following expansion in terms of the number states:

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (11.31)$$

We stress that α is an arbitrary complex number. The coherent states are not eigenstates of the mode Hamiltonian and of N, therefore the number of photons has not a well defined value in them. Coherent states of an oscillator have already been introduced by Schrödinger in the very early days of quantum mechanics. The use of the coherent states in photon optics was initiated by R. Glauber (Nobel prize 2005).

Further Reading



Under this link you can find the Nobel Lecture of Roy Glauber with the title: One Hundred Years of Light Quanta.
http://www.nobelprize.org/nobel_prizes/physics/laureates/2005/glauber-lecture.html

Problem 11.6

Prove that the coefficients in this expansion obey the condition $\sum_{n=0}^{\infty} |c_n|^2 = 1$.

The intensity of the field turns out to be proportional to $|\alpha|^2$, which is the expectation value of the photon number in the coherent state: .

The operators of field strengths

Now we come back to the electric and magnetic fields in the mode. In the expression (11.1) we replace the classical amplitude qq by its quantum counterpart, by the Q operator. Using the expression of Q by a and $a^\dagger$, and the identification (11.12): ($\frac{\varepsilon_0 V A^2}{2\omega^2} = M$) we get:

$$\mathbf{E} = \hat{\mathbf{x}} Q A \sin kz = \hat{\mathbf{x}} \frac{a + a^\dagger}{\sqrt{2}} \sqrt{\frac{\hbar}{M\omega}} A \sin kz = \hat{\mathbf{x}} \left(a + a^\dagger\right) \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin kz \quad (11.32)$$

In this way the electric filed strength has become an operator. Similarly the magnetic field is obtained by using the quantum variant of $s = -p/\left(M\omega^2\right)$

(11.33)

Interestingly the perpendicular components of the $mathbfE$ and the $\mathbf{B}$ fields do not commute with each other, except at the points in space where the fields are zero:

$$\begin{aligned} \mathbf{B} = -\hat{\mathbf{y}} A S k \cos kz = \hat{\mathbf{y}} A \frac{P}{M\omega^2} \frac{\omega}{c} \cos kz = \hat{\mathbf{y}} A \sqrt{\frac{M\hbar\omega}{2}} \frac{1}{M\omega^2} \frac{\omega}{c} i \left(a^\dagger - a\right) \cos kz = \\ = \hat{\mathbf{y}} A \sqrt{\frac{\hbar}{2M\omega}} \frac{1}{c} i \left(a^\dagger - a\right) \cos kz = \hat{\mathbf{y}} \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \frac{1}{c} i \left(a^\dagger - a\right) \cos kz \end{aligned} \quad (11.34)$$

Concluding remarks

We have considered in this section the simplest way to introduce the correct notion of the photon, which can be considered as the amount of the excitation of a field mode. The photon number has a well defined value in an energy eigenstate of the mode. There are however many more other interesting states of a mode, which could be created and detected in experiments. One of the most important among them is the coherent state, which we introduced shortly. Number states, coherent states, as well as other, more complicated ones have also practical applications in high precision measurements performed by light waves and in other technical fields, too. Another important point here is that a cavity or free space can sustain, of course, a great number of modes, instead of one. These modes may have identical or different frequencies, and their number in a frequency interval is an important parameter of the given physical situation. The extension of the theory to multimode fields and the discussion of other possible states is the realm of quantum optics, which could be a subject of another full semester course.