1: Wave-corpuscular duality of photons and massive particles

Objectives

In this chapter we review the main experimental results showing the dual behaviour of light. In the last section the concept of probability amplitude playing a keyrole in quantummechanics is introduced

Prerequisites

Elements of classical mechanics, interference of light. Elements of calculus and probability theory. The concept of probability distribution.

The double slit experiment with light

The nature of light, whether it consists of corpuscles or is it a wave of an unidentified medium, called ether, intrigued scientists for a long time before the beginning of the 20th century, and there was a long standing dispute between the two views. A seemingly final solution came along with the numerous interference experiments in the beginning of the 19th century performed by Thomas Young, Augustine Fresnel and others, which had shown the wave nature of light. Here we present the double slit experiment of Young. It is interesting that about a 100 years later in 1909 Geoffrey Taylor repeated the experiment with a very weak beam that resulted in the same fringes after a sufficiently long time.

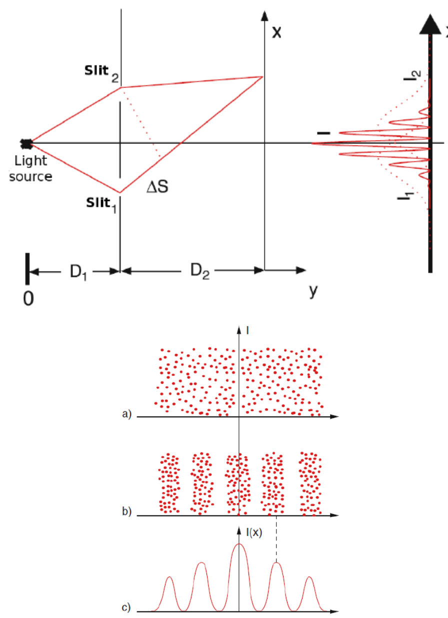


Figure 1.1: The double slit experiment of Young. Secondary light waves emerging from the two slits show interference fringes on the screen along the *x* direction. This was considered for a long time as the proof of the wave nature of light.

The photon hypothesis

It turned out however, that nature is more complicated than simple man's imagination. The first step towards a deeper understanding of what light is was done by Max Planck in 1900. In order to explain correctly the spectral distribution of radiation emitted from a hot body like the sun, or a heated piece of iron, he had to assume that light waves emerging from the body took up energy from it **in quanta**. The smallest amount of quantum is determined by the frequency ν of the emitted light and a universal constant h so that a single quantum has

$$\varepsilon = h\nu \quad (1.1)$$

energy, where the requirement to be in agreement with the experimental observations determined the value of

$$h = 6.63 \times 10^{-34}\,\mathrm{Js} \quad (1.2)$$

called the Planck constant. This is an extremely small value therefore even for visible light the energy of a single quantum is only of the order of 10^{-19}J, that is why we do not observe this granular property of light.

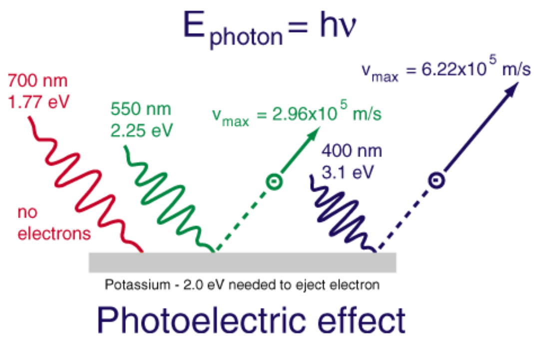


Figure 1.2: Schematic drawing demonstrating the photoelectric effect. http://hyperphysics.phy-astr.gsu.edu/hbase/mod1.html

The next fundamental step was when A. Einstein extended Planck's ideas to the explanation of the photoelectric effect discovered by Hallwachs and Stoletov and studied in detail by P. Lenard and Herz. If light falls on the surface of a metal, it becomes positively charged. The reason for this is that electrons i.e. negatively charged particles called photoelectrons are emitted from the surface of the metal. On the details of this phenomenon we refer here to courses of experimental physics. The observations were explained by Einstein, by the assumption that the electrons in the metal absorb the energy of the light field in quanta. Following the idea of Planck, Einstein assumed that the energy of a single light quantum is

$$\varepsilon = h\nu \quad (1.3)$$

where ν is the frequency of the light, h is Planck's constant. A frequently used notation is

$$\hbar = \frac{h}{2\pi} \quad (1.4)$$

and with the angular frequency $\omega = 2\pi\nu$ we can write Einstein's relation as

$$\varepsilon = \hbar\omega \quad (1.5)$$

which is used more often is theoretical physics. The energy quantum of light obtained the name *photon*. The kinetic energy of the emitted electrons is given by the following equation:

$$E_{kin} = h\nu - W_a \quad (1.6)$$

where W_a – called "work function" – is the energy needed to extract the electron from the metal, which is characteristic for the metal. If $h\nu$ is smaller than W_a, there is no electron emission. For alkaline metals Na, Cs even visible light shows the effect, while for most metals the threshold frequency determined by $h\nu_t = W_a$ falls into the ultraviolet region.

The assumptions of Einstein were proven exactly later by experiments of Millikan in the photoelectric effect. The value of E_{kin} can be determined by the countervoltage U_0 given to the electrode registering the electrons. The value of q_0U_0 – where q_0 is the electron's charge – that stops the electrons gives W_a. Eq. (1.6) is Einstein's photoelectric equation, which can be proven plotting U_0 versus the light frequency ν.

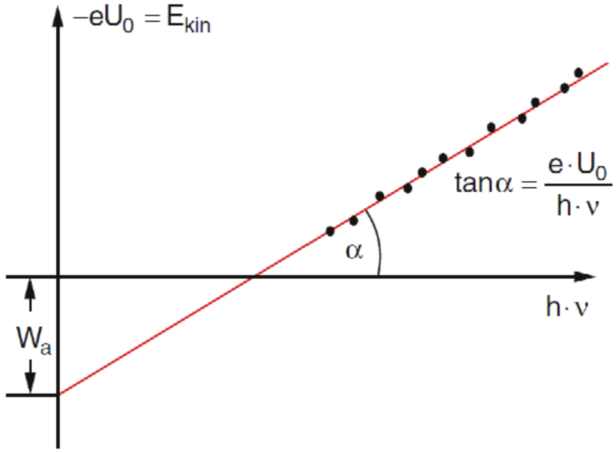
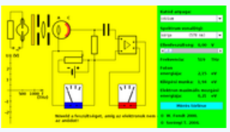


Figure 1.3: Plotting countervoltage U_0 versus the light frequency ν.

Animation



The photoelectric effect can be investigated with this simulation. A virtual experiment can be performed to determine the Planck-constant and the “work function” W_a.

http://www.walter-fendt.de/ph14e/photoeffect.htm

Photon momentum

According to Einstein the photons must posses not only energy but momentum, as well: A light quantum with frequency ν, whose wavelength in vacuum is $\lambda = c/\nu$, has momentum

$$\mathbf{p} = \frac{h\nu}{c}\hat{\mathbf{k}} = \frac{h}{\lambda}\hat{\mathbf{k}} = \hbar\mathbf{k} \quad (1.7)$$

where $\hat{\mathbf{k}}$ is a unit vector pointing in the direction of the propagation of light field which is assumed to be a monochromatic plane wave, $\mathbf{k} = k\hat{\mathbf{k}}$ is the vector called wave-number with absolute value $k = 2\pi/\lambda$. This value of $\mathbf{p}$ is in accordance with the fundamental relation of relativity, valid for any particle of rest mass m_0:

$$E^2/c^2 - p^2 = (m_0 c)^2 \quad (1.8)$$

assuming that the photon’s rest mass is m_0=0.

The Compton effect

Among the several evidences of the photon hypothesis a very important one is the Compton effect, which is seen in figure 1.4. In this experiment, performed by Arthur Compton, an X ray of wavelength λ_0 falls on a sample and ejects electrons from it. There appear also a secondary X ray, whose wavelength λ_s and direction, given by the angle φ is different from the original one. These can be measured by the experiment shown in the figure 1.4.

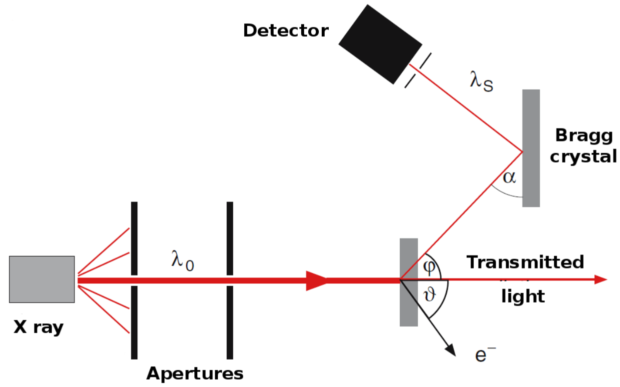


Figure 1.4: Experimental setup for the Compton scattering experiment.

We can consider the effect as the scattering of the photon on the electron where both of them are classical point-like objects. The conservation of energy and momentum requires:

$$\begin{aligned} h\nu_0 + E_0 &= h\nu_s + E_e \\ \frac{h\nu_0}{c} &= \frac{h\nu_s}{c}\cos\varphi + p_e\cos\vartheta \\ 0 &= \frac{h\nu_s}{c}\sin\varphi - p_e\sin\vartheta \end{aligned} \quad (1.9)$$

Here $E_0 = m_e c^2$ is the rest energy of the electron, E_e and p_e are the energy and momentum of the electron after the collision

$$E_e = \frac{m_e c^2}{\sqrt{1-v^2/c^2}}, \quad p_e = \frac{m_e v}{\sqrt{1-v^2/c^2}} \quad (1.10)$$

according to the relativistic formulas.

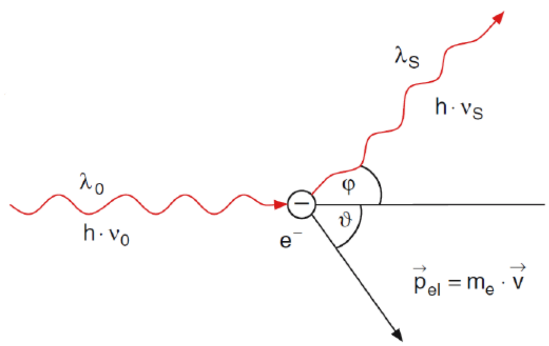


Figure 1.5: The X ray photon approaching from the left is scattered on the electron.

Both of them are treated as classical point-like objects From the three equations above, and using $\nu = c/\lambda$ we get with some algebra the result

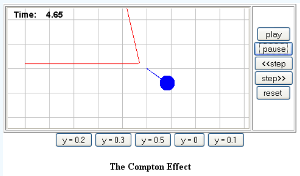
$$\lambda_s - \lambda_0 = \frac{h}{m_e c}(1-\cos\varphi) \quad (1.11)$$

(Strictly speaking the result is valid for free electrons in rest. In the experiments the electrons are set free from atoms, where they have some negative binding energy, but this can be neglected compared to other energies occurring here.) The expression above connects the change of the wavelength of the X ray photon during the scattering and the scattering angle φ. Compton's experiments approved the theoretical result. The length

$$\lambda_C = \frac{h}{m_e c} = 2.43 \times 10^{-12}\ \mathrm{m} \quad (1.12)$$

is called the Compton wavelength of the electron. Note that sometimes the value $\tilde{\lambda}_C = \frac{\hbar}{m_e c} = \frac{\lambda_C}{2\pi}$ is called the Compton wavelength.

Animation



Study the Compton effect with the animation on the page and answer the questions appearing there.

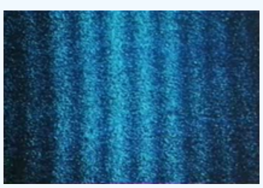
http://physics.bu.edu/~duffy/semeste...5_compton.html

According to these experiments the electromagnetic field can manifest itself of being of discrete nature, and for a given frequency and direction of propagation, it interacts as a particle with atomic matter with the corresponding energy and momentum.

Probability waves for massive particles

After the photon hypothesis had proven to be true, many physicists tried to reconcile the two views, how it is possible that light behaves as a kind of a wave, whereas it consists of quanta or particles that have energy and momentum. While these attempts were not successful, another view emerged. L. de Broglie a French physicist set up the hypothesis that perhaps the behaviour of the electrons in atoms can be explained, if one assumes that electrons – that were known to be massive particles and possessing well defined negative charge – might also behave like waves. Later on, this assumption was proven experimentally by G. P. Thomson, and independently by Davisson. The property that an electron just like photons can behave either as a particle or as a wave is called duality. This very unexpected property was resolved by Max Born in 1926 by purely theoretical arguments. But before explaining Born's idea, let us turn to an experiment revealing explicitly the duality property. This is the famous double slit experiment – one of the most interesting experiments in physics – first performed with electrons by Clauss Jönsson in 1961 and popularized later by Akira Tonomura.

Animation



The experiment seen in Fig. 1.6 was first performed by Clauss Jönsson in 1961. A famous variant of the experiment was done by Akira Tonomura, he made the result visibly available, the video of which is shown here.

http://titan.physx.u-szeged.hu/~mmqu...eslitTono.mpeg

Looking at the experiment in more detail, it turns out that the picture is formed by individual pointlike impacts of particles, as it is seen in the sequence of figure 1.6. A result definitely different from the case of water waves.

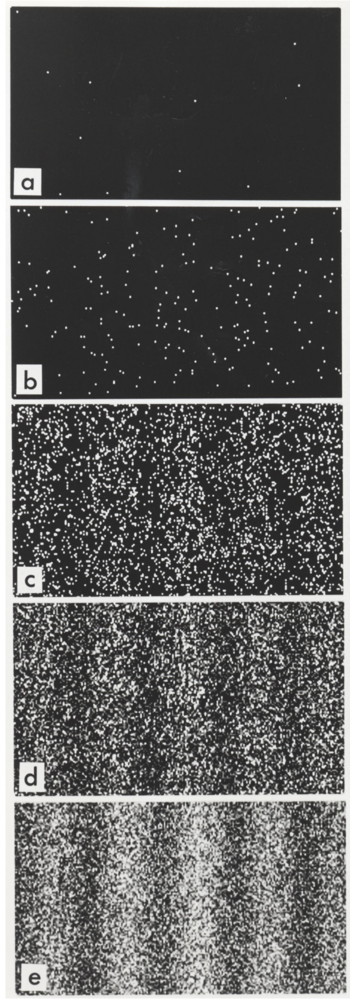


Figure 1.6: Result of the double slit experiment with electrons. Sequence of figures a, b, c, d, e shows the electrons reaching the screen with increasing exposition time. The interference pattern is formed by individual pointlike impacts of the particles. To understand the wavelike behaviour of quantum particles it is necessary to introduce a probabilistic interpretation.http://commons.wikimedia.org/wiki/Fi...Tanamura_2.jp

This extraordinarily surprising phenomenon can be explained with the fundamental concept of quantum mechanics: called the **probability amplitude**. We shall consider only one of the coordinates, the *x* coordinate which is perpendicular to the fringes seen on the screen where the particles are detected. We do not know exactly what was the state of the particle before arriving at the screen. This state will be denoted by ψ. But we know that after falling on the screen we realize that it is at a place say x_1. This event will be characterized by a complex number to be denoted by $\langle x_1 \mid \psi\rangle$. So to each *x* we attribute a complex number $\langle x \mid \psi\rangle$. Considering all the possible *x* values we get a function of *x*, which is a complex valued function of the real variable – the position of the particle – *x*:

$$\langle x \mid \psi\rangle=\psi(x) \quad (1.13)$$

This function will be used to characterize the state of the particle, and this function is called the **coordinate wave function**. The important law of quantum mechanics is that the probability of finding the particle around *x* in a small interval *dx* is $|\psi(x)|^2 dx$. In the language of probability theory, $|\psi(x)|^2$ is the probability density function of the coordinate of the particle which is a random variable. So we have

$$|\langle x \mid \psi\rangle|^2=|\psi(x)|^2=\psi^*(x) \psi(x)=: \rho(x) \geq 0 \quad (1.14)$$

In quantum mechanics we use the notation $\rho(x)$ for the probability distribution. The probability of finding it at a point *x* in the interval $x_1<x<x_2$ is given by:

$$P\left(x_1<x<x_2\right)=\int_{x_1}^{x_2}|\psi(x)|^2 dx=\int_{x_1}^{x_2} \rho(x) dx \quad (1.15)$$

which is valid for arbitrary $x_1<x_2$. The probability of finding the particle somewhere on the real axis must be 1, therefore:

$\int_{-\infty}^{\infty} \rho(x) dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$ (1.16)

Accordingly the wave function describing a particle must be such that the integral of its absolute value square between −∞ and ∞ should give 1.

The interference pattern seen in Fig 1.6 can be explained by the rules of quantum mechanics, as we shall do it in the following. The particle which was assumed to be in a state ψψ can arrive to the first slit on the screen, and this state, i.e. particle in the first slit will be denoted by ψ_1. The corresponding amplitude is then $\langle \varphi_1 \mid \psi \rangle$. Continuing its path it arrives somewhere to the second screen, where it is registered at the point *x*. The amplitude corresponding to this second part of its way is $\langle x \mid \varphi_1 \rangle$. The total amplitude belonging to the process that the particle arrives at xx through slit 1 is the product of the two amplitudes $\langle x \mid \varphi_1 \rangle \langle \varphi_1 \mid \psi \rangle$. Similarly, the amplitude of the other possible path of the particle, that it went through slit 2 before arriving at *x* is $\langle x \mid \varphi_2 \rangle \langle \varphi_2 \mid \psi \rangle$, where ψ_2 is the state of the particle, when it is at the second slit. The total amplitude is the sum of the amplitudes of the two possible paths.

$$\begin{aligned} \langle x \mid \psi \rangle &= \langle x \mid \varphi_1 \rangle \langle \varphi_1 \mid \psi \rangle + \langle x \mid \varphi_2 \rangle \langle \varphi_2 \mid \psi \rangle, \text{ or equivalently} \\ \psi(x) &= \varphi_1(x) \langle \varphi_1 \mid \psi \rangle + \varphi_2(x) \langle \varphi_2 \mid \psi \rangle = c_1 \varphi_1(x) + c_2 \varphi_2(x) \end{aligned} \tag{1.17}$$

where $c_1 = \langle \varphi_1 \mid \psi \rangle$, $c_2 = \langle \varphi_2 \mid \psi \rangle$. These latter amplitudes have been set to be two (complex) numbers instead of functions, because we have assumed here that the slits are small, and we need not distinguish between the different positions within the slits. What small means, will become more definite a little later. The sum of the amplitudes above is the superposition of the two possibilities. The probability of finding the particle at position xx is the absolute value square of the sum of the two possible amplitudes:

$$\begin{aligned} |\psi(x)|^2 &= |c_1 \varphi_1(x)|^2 + |c_2 \varphi_2(x)|^2 + c_1^* c_2 \varphi_1^*(x) \varphi_2(x) + c_1 c_2^* \varphi_1(x) \varphi_2^*(x) \\ &= |c_1|^2 |\varphi_1(x)|^2 + |c_2|^2 |\varphi_2(x)|^2 + 2 \operatorname{Re}(c_1^* c_2 \varphi_1^*(x) \varphi_2(x)) \end{aligned} \tag{1.18}$$

The interference appears because of the last term. There will be places where the particles arrive with a small probability, these will be the dark fringes, whereas there will be places where the detection probabilities are large. Quantum mechanics gives the explicit form of the two functions $\varphi_1(x)$ and $\varphi_2(x)$, but we will not deal with this here.

We cannot tell which of the slits the particle went through, the interference pattern is the witness that it went simultaneously through both. It can be shown, that if we try to detect somehow which path the particle has taken, then a successful detection will destroy the interference pattern, so we cannot see simultaneously the interference pattern and distinguish between the two ways the particle went along. The same experiment has been performed with particles larger than electrons, like neutrons, atoms, small molecules. More recently the effect was demonstrated with the molecular object called fullerene, consisting of 60 Carbon atoms.

We did not emphasize so far, but it is important that the amplitude, i.e. the wave function depends also on time. If we wish to make use of time dependent wave functions we will denote it by a capital Greek letter: $\Psi(x,t)$. How does such a function look in any concrete case? This depends on the physical situation in question, and one of the important problems of quantum mechanics is to find the form of this function. In general we have to prescribe the property of square integrability with respect to *x*, because for any fixed value of the time variable *t* $|\Psi(x,t)|^2$ is the probability density of the coordinate of the particle. This interpretation of the wave function was first given by Max Born. Accordingly:

$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$ (1.19)

at any time instant. We say then that $\Psi(x,t)$ is normalized to unity. This requirement will be (i) refined later and (ii) will necessitate certain prescriptions on how the wave function should depend on the time variable.

If we wish to give the position of the particle in three dimensional space, instead of the line, then the wave function shall depend on all three coordinates of the points in space: $\Psi(x,y,z,t) = \Psi(\mathbf{r},t)$. Here the random variable will be the position vector of the particle. The probability density is then the function $|\Psi(\mathbf{r},t)|^2$, for which the normalization is prescribed in three dimensions.

Max Born (1882-1970)

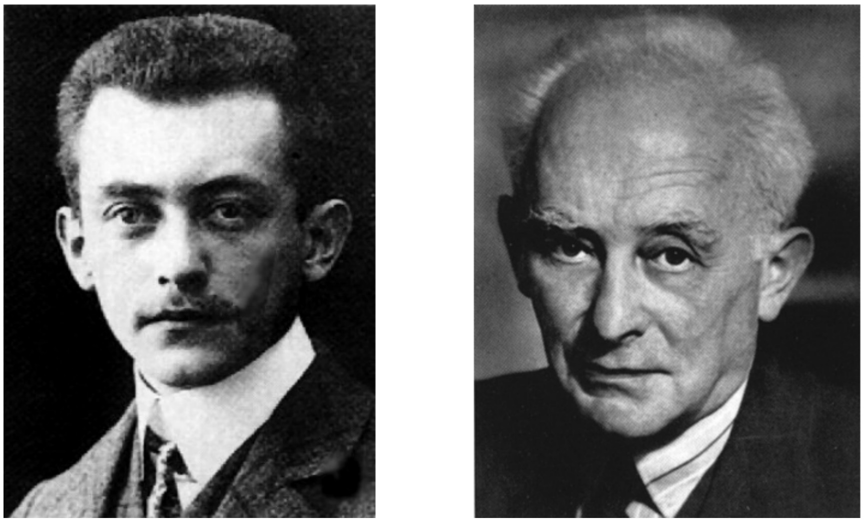


Figure 1.7:

Coming back to one dimension, a frequently used wave function (real in this specific case) for a particle localized more or less around the origin is $\psi(x) = \mathcal{N} e^{-x^2/4\sigma_0^2}$ at a given instant, where σ_0 is a constant independent of x, while $\mathcal{N}$ is a normalization factor. The corresponding probability density is $|\psi(x)|^2 =: \rho(x) = \mathcal{N}^2 e^{-x^2/2\sigma_0^2}$. Its integral along the real axis must give 1, this is the condition that determines the factor $\mathcal{N}$. As it is known from probability theory, this is the probability density of the so called standard normal, or Gaussian distribution function, if $\mathcal{N} = \left(2\pi\sigma^2\right)^{-1/4}$.

Problem 1.1

Show that the normalization factor in $|\psi(x)|^2 =: \rho(x) = \mathcal{N}^2 e^{-x^2/2\sigma_0^2}$ is $\mathcal{N} = \left(2\pi\sigma^2\right)^{-1/4}$.

According to what we said before, the function can depend on time as well, so that instead of σ_0 we can have a time dependent $\sigma(t)$ in it, and then of course (\mathcal{N}\) will be also time dependent. Besides the Gaussian wave function above there are many other possibilities depending on the physical problem in question.

Later on, we shall consider in detail the problem of the dynamics of the state, which means we shall consider how the state, i.e. the wave function changes with time given a force or a potential acting on the particle.

Problem 1.2

Investigating the photoeffect with a photocell where the cathode is covered by Cesium the electrons are stopped if the voltage is larger than 0.33V for a light wave of wavelength 546nm, while the corresponding value for 365nm is 1.46V. Derive the value of Planck's constant from these results. What is the work function of Cesium? How much is the stopping voltage using light of 436nm? What is the threshold wavelength producing electron emission?

Problem 1.3

Estimate the cost of one green photon in the case of a traditional 100W light bulb working with 5% efficiency, and switched on for 1 second.

Problem 1.4

Derive the result

$$\lambda_s - \lambda_0 = \frac{h}{m_e c}(1 - \cos\varphi) \quad (1.20)$$

for Compton scattering using the equations expressing conservation of energy and momentum:

$$\begin{aligned} h\nu_0 + E_0 &= h\nu_s + E_e \\ \frac{h\nu_0}{c} &= \frac{h\nu_s}{c}\cos\varphi + p_e\cos\vartheta \\ 0 &= \frac{h\nu_s}{c}\sin\varphi - p_e\sin\vartheta \end{aligned} \tag{1.21}$$