1.8: Electromagnetic Energy

The total energy stored in the electromagnetic field per unit of volume at a point **r** is equal to the sum of the electric and the magnetic energy densities. We postulate that the results for the energy densities derived in electrostatics and magnetostatics are also valid for the fast-oscillating fields in optics; hence we assume that the total electromagnetic energy density is given by:

$$U_{em}(r,t)=\frac{E}{2}\varepsilon(r,t)\cdot\varepsilon(r,t)+\frac{\mu_0}{2}H(r,t)\cdot H(r,t).$$

It is to be noticed that we assume in this section that the permittivity is real, i.e. there is no absorption and the permittivity does not include the conductivity.

Time dependent electromagnetic fields propagate energy. The flow of electromagnetic energy at a certain position r and time t is given by the Poynting vector, which is defined by

$$S(r,t)=\varepsilon(r,t)\times H(r,t).$$

More precisely, the flow of electromagnetic energy through a small surface dS with normal **n** at point **r** is given by

$$S(r,t)\cdot\hat{n}dS.$$

If this scalar product is positive, the energy flow is in the direction of **n**, otherwise it is in the direction of **−n**. Hence the direction of the vector **S**(**r**, t) is the direction of the flow of energy at point r and the length |**S**(**r**, t)| is the amount of the flow of energy, per unit of time and per unit of area perpendicular to the direction of **S**. This quantity has unit J/(sm^2). That the Poynting vector gives the flow of energy can be seen in a dielectric for which dispersion may be neglected by the following derivation. We consider the change with time of the total electromagnetic energy in a volume V :

$$\frac{d}{dt}\iiint\limits_V U_{em}(r,t)\,dV=\iiint\limits_V E\frac{\partial\varepsilon(r,t)}{\partial t}\cdot\varepsilon(r,t)+\mu_0\frac{\partial H(r,t)}{\partial t}\cdot H(r,t)\,dV$$

By substituting (1.3.12), (1.3.13) and using

$$-A\cdot\nabla\times B+B\cdot\nabla\times A=\nabla\cdot(A\times B),$$

which holds for any two vector fields, we find

$$\iiint\limits_V E\frac{\partial\varepsilon(r,t)}{\partial t}\cdot\varepsilon(r,t)+\mu_0\frac{\partial H(r,t)}{\partial t}\cdot H(r,t)\,dV=\iiint\limits_V \varepsilon(r,t)\cdot\nabla\times H(r,t)-H(r,t)\cdot\nabla\times\varepsilon(r,t)\,dV-\iiint\limits_V\sigma$$
$$\varepsilon(r,t)\cdot\varepsilon(r,t)\,dV-\iiint\limits_V\varepsilon(r,t)\cdot J_{ext}(r,t)\,dV=-\iiint\limits_V\nabla\cdot(\varepsilon\times H)\,dV-\iiint\limits_V\sigma\varepsilon(r,t)\cdot\varepsilon(r,t)\,dV-\iiint\limits_V\varepsilon(r,t)$$
$$\cdot J_{ext}(r,t)\,dV=-\iint\limits_S(\varepsilon\times H)\cdot\hat{n}\,dS-\iiint\limits_V\sigma\varepsilon(r,t)\cdot\varepsilon(r,t)\,dV-\iiint\limits_V\varepsilon(r,t)\cdot J_{ext}(r,t)\,dV,$$

where S is the surface bounding volume V and **n** is the unit normal on S pointing out of V. Hence,

$$\frac{d}{dt}\iiint\limits_V U_{em}(r,t)\,dV+\iiint\limits_V\sigma\varepsilon(r,t)\cdot\varepsilon(r,t)\,dV+\iiint\limits_V\varepsilon(r,t)\cdot J(r,t)\,dV=-\iint\limits_S S(r,t)\cdot\hat{n}\,dS.$$

This equation says that the rate of change with time of the electromagnetic energy in a volume V plus the work done by the field on the conduction and external currents inside V is equal to the influx of electromagnetic energy through the boundary of V.

Remark. The energy flux **S** and the energy density U_{em} depend quadratically on the field. For Uem the quadratic dependence on the electric and magnetic fields is clear. To see that the Poynting vector is also quadratic in the electromagnetic field, one should realise that the electric and magnetic fields are inseparable: they together form the electromagnetic field. Stated differently: if the amplitude of the electric field is doubled, then also that of the magnetic field is doubled and hence the Poynting vector is increased by the factor 4. Therefore, when computing the Poynting vector or the electromagnetic energy density of a time-harmonic electromagnetic field, the real-valued vector fields should be used, i.e. the complex fields should **NOT** be used.An exception is the calculation of the long-time average of the Poynting vector or the energy density. As we will show in the next section, the **time averages** of the energy flux and energy density of time-harmonic fields can actually be expressed quite conveniently in terms of the complex field amplitudes.

If we subsitute the real fields (1.6.12), (1.6.13) of the plane wave in the Poynting vector and the electromagnetic energy density we get:

$$S(z,t) = \varepsilon(z,t) \times H(z,t) = (\frac{E}{\mu_0})^{1/2} |A|^2 cos^2(kz - \omega t + \phi)\hat{z},$$

$$U_{em}(z,t) = E|A|^2 cos^2(kz - \omega t + \phi).$$

We see that the energy flow of a plane wave is in the direction of the wave vector, which is also the direction of the phase velocity. Furthermore, it changes with time at frequency 2ω.