1.4: Maxwell Equations in Matter

Atoms are neutral and consist of a positively charged kernel surrounded by a negatively charged electron cloud. In an electric field, the centres of mass of the positive and negative charges get displaced with respect to each other. Therefore, an atom in an electric field behaves like an electric dipole. In polar molecules, the centres of mass of the positive and negative charges are permanently separated, even without an electric field. But without an electric field, they are randomly orientated and therefore have no net effect, while in the presence of an electric field they line up parallel to the field. Whatever the precise mechanism, an electric field induces a certain net dipole moment density per unit volume **P**(**r**) [C/m^2] i in matter which is proportional to the local electric field **ε** (**r**):

$$P(r,t) = E_0 \chi_e \varepsilon(r,t),$$

where χ_e is a dimensionless quantity, the electric susceptibility of the material. We stress that **ε** is the total local field at the position of the dipole, i.e. it contains the contribution of all other dipoles, which are also excited and radiate an electromagnetic field themselves. Only in the case of diluted gasses, the influence of the other dipoles in matter can be neglected and the local electric field is simply given by the field emitted by the external source.

A dipole moment density that changes with time corresponds to a current density $\mathbf{J}_p$ [Ampere/m^2=C/(m^2 s)] and a charge density ρ_p [C/m^3] given by

$$J_p(r,t) = \frac{\partial P(r,t)}{\partial t} = E_0 \chi \frac{\partial \varepsilon(r,t)}{\partial t},$$

$$\rho_p(r,t) = -\bigtriangledown \cdot P(r,t) = -\bigtriangledown \cdot (E_0 \chi \varepsilon),$$

All materials conduct electrons to a certain extent, although the conductivity σ [Ampere/(Volt m)=C/(Volt s] differs greatly between dielectrics, semi-conductors and metals (the conductivity of copper is 10^7 times that of a good conductor such as sea water and 10^{19} times that of glass). The current density $\mathbf{J}_c$ and the charge density corresponding to the conduction electrons satisfy:

$$J_c = \rho \varepsilon,$$

$$\frac{\partial \rho_c}{\partial t} = -\bigtriangledown \cdot J_c = -\bigtriangledown \cdot (\rho \varepsilon),$$

where (1.4.4) is Ohm's Law. The total current density on the right-hand side of Maxwell's Law (1.2.4) is the sum of $\mathbf{J}_p$, $\mathbf{J}_c$ and an external current density $\mathbf{J}_{ext}$, which we assume to be known. Similarly, the total charge density at the right of (1.2.5) is the sum of ρ_p, ρ_c and a given external charge density ρ_{ext}. The latter is linked to the external current density by the law of conservation of charge (1.2.2). Hence, (1.2.4) and (1.2.5) become

$$\bigtriangledown \times \frac{B}{\mu_0} = E_0 E_0 \frac{\partial \varepsilon}{\partial t} + J_p + J_c + J_{ext} = E_0(1+\chi)\frac{\partial \varepsilon}{\partial t} + \sigma \varepsilon + J_{ext}$$

$$\bigtriangledown \cdot E_0 \varepsilon = \rho_p + \rho_c + \rho_{ext} = -\bigtriangledown \cdot (E_0 \chi \varepsilon) + \rho_c + \rho_{ext}.$$

We define the permittivity E by

$$E = E_0(1+\chi_e).$$

Then (1.4.6) and (1.4.7) can be written as

$$\bigtriangledown \times \frac{B}{\mu_0} = E \frac{\partial \varepsilon}{\partial t} + \sigma \varepsilon + J_{ext}$$

$$\bigtriangledown \cdot (E\varepsilon) = \rho_c + \rho_{ext}.$$

It is verified in Problem 1 that in a conductor any accumulation of charge is extremely quickly reduced to zero. Therefore we may assume that

$$\rho_c = 0.$$

If the material is magnetic, the magnetic permeability is different from vacuum and is written as $\mu = \mu_0(1 + \chi_m)$, where χ_m is the magnetic susceptibility. In the Maxwell equations, one should then replace μ_0 by μ. However, at optical frequencies magnetic effects are negligible (except in ferromagnetic materials, which are rare). **We will therefore always assume that the magnetic permeability is that of vacuum**: $\mu = \mu_0$.

It is customary to define the magnetic field by **H** = **B**/μ_0 [Ampere/m=C/(ms)]. By using the magnetic field **H** instead of the magnetic induction **B**, Maxwell's equations become more symmetric:

$$\bigtriangledown \times \varepsilon = -\mu_0 \frac{\partial H}{\partial t}, Faraday's\ Law$$

$$\bigtriangledown \times H = E\frac{\partial \varepsilon}{\partial t} + \sigma\varepsilon + J_{ext}, Maxwell's\ Law$$

$$\bigtriangledown \cdot E\varepsilon = \rho_{ext}, Gauss's\ Law$$

$$\bigtriangledown \cdot H = 0. no\ magnetic\ charge.$$

This is the form in which we will be using the Maxwell equations in matter in this book. It is seen that the Maxwell equations in matter are identical to those in vacuum, with E substituted for E_0.
