1.9: Time-Averaged Energy

Optical frequencies are in the range of 5×10^{14} Hz and the fastest detectors working at optical frequencies have integration times larger than 10^{-10} s. Hence there is no detector which can measure the time fluctuations of the electromagnetic fields at optical frequencies and any detector always measures an average value, taken over an interval of time that is very large compared to the period $2\pi/\omega$ of the light wave, typically at least a factor 10^5 longer. We therefore compute averages over such time intervals of the Poynting vector and of the electromagnetic energy. Because the Poynting vector and energy density depend nonlinearly (quadratically) on the field amplitudes, we can not perform the computations using the complex amplitudes and take the real part afterwards, but have instead to start from the real quantities. Nevertheless, it turns out that the final result can be conveniently expressed in terms of the complex field amplitudes.

Consider two time-harmonic functions:

$$A(t) = Re[Ae^{-i\omega t}] = |A| cos(\phi_A - \omega t)$$

$$B(t) = Re[Be^{-i\omega t}] = |B| cos(\phi_B - \omega t)$$

with $A = |A| \exp(i\phi A)$ and $B = |B| \exp(i\phi B)$ the complex amplitudes. For a general function of time $f(t)$ we define the time average over an interval T at a certain time t, by

$$\frac{1}{T} \int_{t-T/2}^{t+T/2} f(t')\, dt'.$$

where T is much larger (say a factor of 10^5) than the period of the light. It is obvious that for time-harmonic fields the average does not depend on the precise time t at which it is computed. and we therefore take t = 0 and write

$$\langle f(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, dt.$$

With A(t) = Re [$Ae^{-i\omega t}$]=1/2[$Ae^{-i\omega t}+A*e^{i\omega t}$],where A∗ is the complex conjugate of A; and with a similar expression for B(t), it follows that

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A(t)B(t)\, dt = \lim_{T\to\infty} \frac{1}{4T} \int_{-T/2}^{T/2} [AB* + A*B + ABe^{-2i\omega t} + A*B*e^{2i\omega t}]\, dt =$$

$$\lim_{T\to\infty} \frac{1}{4T} [AB* + A*B + AB\frac{e^{i\omega t} - e^{-i\omega t}}{2iT\omega} + A*B*\frac{e^{i\omega t} - e^{-i\omega t}}{2iT\omega}] = \frac{1}{2} Re[AB*],$$

This important result will be used over and over again. In words:

The average of the product of two time-harmonic quantities over a long time interval compared with the period, is half the real part of the product of the complex amplitude of one quantity and the complex conjugate of the other.

If we apply this to Poynting's vector of a general time-harmonic electromagnetic field:**ε**(**r**, t) = Re [**E**(**r**)$e^{-i\omega t}$], **H**(**r**, t) = Re [**H**(**r**)$e^{-i\omega t}$],then we find that the time-averaged energy flow denoted by **S**(**r**) is given by

$$S(r) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} S(r,t)\, dt = \frac{1}{2} Re[E \times H*].$$

Similarly, the time-averaged electromagnetic energy density is:

$$\langle U_{en}(r) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} U_{en}(r,t)\, dt = \frac{1}{2} E\varepsilon(r) \cdot \varepsilon(r)* + \frac{\mu_0}{2} H(r) \cdot H(r)* = \frac{1}{2} E|\varepsilon(r)|^2 + \frac{\mu_0}{2} |H(r)|^2.$$

For the special case of plane wave (1.6.12), (1.6.13) in a medium without absorption, we get:

$$S=\frac{1}{2}(\frac{E}{\mu_0})^{1/2}Re[AA*]\hat{z}=\frac{1}{2}(\frac{E}{\mu_0})^{1/2}|A|^2\hat{z}.$$

The length of vector (1.9.8) is the time-averaged flow of energy per unit of area in the direction of the plane wave and is commonly called the **intensity** of the wave. For the time-averaged electromagnetic energy density of the plane wave, we get:

$$\langle U_{en}\rangle=\frac{1}{2}E|A|^2+\frac{1}{2\mu_0}\mu_0E|A|^2=E|A|^2.$$

For a plane wave both the time-averaged energy flux and the time-averaged energy density are proportional to the modulus squared of the complex electric field.