6.4: Rayleigh-Sommerfeld Diffraction Integral

Another method to propagate a wave field is by using the **Rayleigh-Sommerfeld** integral. A very good approximation of this integral states that each point in the plane $z=0$ emits spherical waves, and to find the field in a point (x, y, z), we have to add the contributions from all these point sources together. This corresponds to the Huygens-Fresnel principle postulated earlier in Section 5.6. Because a more rigorous derivation starting from the Helmholtz equation would be complicated and lengthy, we will just give the final result:

$$\begin{aligned} U(x, y, z) &= \frac{1}{i\lambda} \iint U\left(x', y', 0\right) \frac{z e^{ik\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\left(x-x'\right)^2+\left(y-y'\right)^2+z^2} \mathrm{d}x' \mathrm{d}y' \\ &= \frac{1}{i\lambda} \iint U\left(x', y', 0\right) \frac{z e^{ikr}}{r} \mathrm{d}x' \mathrm{d}y' \end{aligned}$$

where we defined

$$r=\sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+z^2}\,.$$

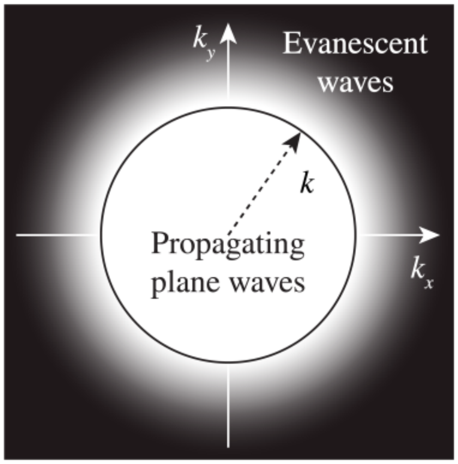

Figure 6.4.1: The spatial frequencies k_x, k_y of the plane waves in the angular spectrum of a time-harmonic field which propagates in the z-direction. There are two types of waves: the propagating waves with spatial frequencies inside the cricle: $\sqrt{k_x^2+k_y^2}<k=2\pi/\lambda$ and which have phase depending on the propagation distance z but constant amplitude, and the evanescent waves for which $\sqrt{k_x^2+k_y^2}>k$ and of which the amplitude decreases exponentially during propagation.

Remarks.

1. The formula (6.4.1) is not completely rigorous: a term that is a factor $1/(kr)$ smaller (and in practive is therefore is very much smaller) has been omitted.
2. In (6.4.1) there is an additional factor z/r compared to the expressions for a time-harmonic spherical wave as given in (1.53) and at the right-hand side of (5.44). This factor means that the spherical waves in the Rayleigh-Sommerfeld diffraction integral have amplitudes that depend on the angle of radiation (although their wave front is spherical), the amplitude being largest in the forward direction.
3. **Equivalence of the two propagation methods**. The angular spectrum method amounts to a multiplication by $\exp(izk_z)$ in Fourier space, while the Rayleigh-Sommerfeld integral is a convolution. It is one of the properties of the Fourier transform that a multiplication in Fourier space corresponds to a convolution in real space and vice versa. Indeed a mathematical result called **Weyl's identity** implies that the rigorous version of (6.4.1) and the plane wave expansion (i.e. angular spectrum method) give identical results.