4.3: Creating and Manipulating Polarisation States

We have seen how Maxwell's equations allow the existence of plane waves with many different states of polarisation. But how can we create these states, and how do these states manifest themselves?

Natural light often does not have a definite polarisation. Instead, the polarisation fluctuates rapidly with time. In order to turn such randomly polarised light into linearly polarised light in a certain direction, we must extinguish the light polarised in the perpendicular direction, so that the remaining light is linearly polarised along the required direction. One could do this by using light reflected under the Brewster angle (which extinguishes p-polarised light), or one could let light pass through a dichroic crystal, which is a material which absorbs light polarised perpendicular to its so-called optic axis. A third method is sending the light through a wire grid polariser, which consists of a metallic grating with sub-wavelength slits. Such a grating only transmits the electric field component that is perpendicular to the slits.

So suppose that with one of these methods we have obtained linearly polarised light. Then the question rises how the state of linear polarisation can be changed into circularly or elliptically polarised light. Or how the state of linear polarisation can be rotated over a certain angle. We have seen that the polarisation state depends on the ratio of the amplitudes and on the phase difference $\varphi_y - \varphi_x$ of the orthogonal components $\mathcal{E}_y$ and $\mathcal{E}_x$ of the electric field. Thus, to change linearly polarised light to some other state of polarisation, a certain phase shift (say $\Delta\varphi_x$) must be introduced to one component (say $\mathcal{E}_x$), and another phase shift $\Delta\varphi_y$ to the orthogonal component $\mathcal{E}_y$. We can achieve this with a **birefringent crystal**, such as calcite. What is special about such a crystal is that it has two refractive indices: light polarised in a certain direction experiences a refractive index of n_o, while light polarised perpendicular to it feels another refractive index n_e (the subscripts o and e stand for "ordinary" and "extraordinary"), but for our purpose we do not need to understand this terminology. The direction for which the refractive index is smallest (which can be either n_o or n_e) is called the **fast axis** because its phase velocity is largest, and the other direction is the **slow axis**. Because there are two different refractive indices, one can see double images through a birefringent crystal. The difference between the two refractive indices $\Delta n = n_e - n_o$ is called the **birefringence**.

Suppose $n_e > n_o$ and that the fast axis, which corresponds to n_o is aligned with $\mathcal{E}_x$, while the slow axis (which then has refractive index n_e) is aligned with $\mathcal{E}_y$. If the wave travels a distance d through the crystal, $\mathcal{E}_y$ will accumulate a phase $\Delta\varphi_y = \frac{2\pi n_e}{\lambda}d$, and $\mathcal{E}_x$ will accumulate a phase $\Delta\varphi_x = \frac{2\pi n_o}{\lambda}d$. Thus, after propagation through the crystal the phase difference $\varphi_y - \varphi_x$ has increased by

$$\Delta\varphi_y - \Delta\varphi_x = \frac{2\pi}{\lambda}d\left(n_e - n_o\right).$$

Jones Matrices

By letting light pass through crystals of different thicknesses d, we can create different phase differences between the orthogonal field components, and this way we can create different states of polarisation. To be specific, let $\mathbf{J}$, as given by (4.1.4), be the Jones vector of the plane wave before the crystal. Then we have, for the Jones vector after the passage through the crystal:

$$\tilde{\mathbf{J}} = \mathcal{M}\mathbf{J},$$

where

$$\mathcal{M} = \begin{pmatrix} e^{\frac{2\pi i}{\lambda}dn_o} & 0 \\ 0 & e^{\frac{2\pi i}{\lambda}dn_e} \end{pmatrix} = e^{\frac{2\pi i}{\lambda}dn_o}\begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{\lambda}d(n_e - n_o)} \end{pmatrix}.$$

A matrix such as $\mathcal{M}$, which transfers one state of polarisation of a plane wave in another, is called a **Jones matrix**. Depending on the phase difference which a wave accumulates by traveling through the crystal, these devices are called **quarter-wave plates** (phase difference $\pi/2$), **half-wave plates** (phase difference π), or **full-wave plates** (phase difference 2π). The applications of these wave plates will be discussed in later sections.

Consider as example the Jones matrix which described the change of linear polarised light into circular polarisation. Assume that we have diagonally (linearly) polarised light, so that

$$J = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We want to change it to circularly polarised light, for which

$$J=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix},$$

where one can check that indeed $\varphi_y-\varphi_x=\pi/2$. This can be done by passing the light through a crystal such that $\mathcal{E}_y$ accumulates a phase difference of $\pi/2$ with respect to $\mathcal{E}_x$. The transformation by which this is accomplished can be written as

$$\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}.$$

The matrix on the left is the Jones matrix describing the operation of a quarter-wave plate.

Another important Jones matrix is the **rotation matrix**. In the preceding discussion it was assumed that the fast and slow axes were aligned with the x - and y-direction (i.e. they were parallel to $\mathcal{E}_x$ and $\mathcal{E}_y$). Suppose now that the slow and fast axes of the wave plate no longer coincide with $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$, but rather with some other $\widehat{\mathbf{x}}'$ and $\widehat{\mathbf{y}}'$ as in Figure 4.3.1. In that case we apply a basis transformation: the electric field vector which is expressed in the $\widehat{\mathbf{x}},\widehat{\mathbf{y}}$ basis should first be expressed in the $\widehat{\mathbf{x}}',\widehat{\mathbf{y}}'$ basis before applying the Jones matrix of the wave plate to it. After applying the Jones matrix, the electric field has to be transformed back from the $\widehat{\mathbf{x}}',\widehat{\mathbf{y}}'$ basis to the $\widehat{\mathbf{x}},\widehat{\mathbf{y}}$ basis.

Let $\mathbf{E}$ be given in terms of its components on the $\hat{\mathbf{x}},\hat{\mathbf{y}}$ basis:

$$\mathbf{E}=E_x\widehat{\mathbf{x}}+E_y\widehat{\mathbf{y}}.$$

To find the components $E_{x'},E_{y'}$ on the $\widehat{\mathbf{x}}',\widehat{\mathbf{y}}'$ basis:

$$\mathbf{E}=E_{x'}\widehat{\mathbf{x}}'+E_{y'}\widehat{\mathbf{y}}',$$

we first write the unit vectors $\widehat{\mathbf{x}}'$ and $\widehat{\mathbf{y}}'$ in terms of the basis $\hat{\mathbf{x}},\hat{\mathbf{y}}$ (see Figure $4.3.1$)

$$\begin{aligned}\widehat{\mathbf{x}}'&=\cos\theta\widehat{\mathbf{x}}+\sin\theta\widehat{\mathbf{y}},\\ \widehat{\mathbf{y}}'&=-\sin\theta\widehat{\mathbf{x}}+\cos\theta\widehat{\mathbf{y}}.\end{aligned}$$

By substituting ($4.3.9$) and ($4.3.10$) into ($4.3.8$) we find

$$\begin{aligned}\mathbf{E}&=E_{x'}\widehat{\mathbf{x}}'+E_{y'}\widehat{\mathbf{y}}'\\ &=E_{x'}(\cos\theta\widehat{\mathbf{x}}+\sin\theta\widehat{\mathbf{y}})+E_{y'}(-\sin\theta\widehat{\mathbf{x}}+\cos\theta\widehat{\mathbf{y}}),\\ &=(\cos\theta E_{x'}-\sin\theta E_{y'})\,\widehat{\mathbf{x}}+(\sin\theta E_x+\cos\theta E_y)\,\widehat{\mathbf{y}}.\end{aligned}$$

Comparing with ($4.3.7$) implies

$$\begin{pmatrix}E_x\\ E_y\end{pmatrix}=\begin{pmatrix}E_{x'}\cos\theta-E_{y'}\sin\theta\\ E_{x'}\sin\theta+E_{y'}\cos\theta\end{pmatrix}=\mathcal{R}_\theta\begin{pmatrix}E_{x'}\\ E_{y'}\end{pmatrix},$$

where $\mathcal{R}_\theta$ is the rotation matrix over an angle θ in the anti-clockwise direction:

$$\mathcal{R}_\theta\equiv\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

That $\mathcal{R}(\theta)$ indeed is a rotation over angle θ in the anti-clockwise direction is easy to see by considering what happens when $\mathcal{R}_\theta$ is applied to the vector $(1,0)^T$. Since $\mathcal{R}_\theta^{-1}=\mathcal{R}_{-\theta}$ we get:

$$\begin{pmatrix}E_{x'}\\ E_{y'}\end{pmatrix}=\mathcal{R}_{-\theta}\begin{pmatrix}E_x\\ E_y\end{pmatrix}.$$

This relationship expresses the components $E_{x'},E_{y'}$ of the Jones vector on the $\hat{\mathbf{x}}',\hat{\mathbf{y}}'$ basis, which is aligned with the fast and slow axes of the crystal, in terms of the components E_x and E_y on the original basis $\widehat{\mathbf{x}},\widehat{\mathbf{y}}$. If the matrix $\mathcal{M}$ describes the Jones matrix as defined in ($4.3.3$), then the matrix M_θ for the same wave plate but with x' as slow and y' as fast axis, is, with respect to the $\widehat{\mathbf{x}},\widehat{\mathbf{y}}$ basis, given by:

$$\mathcal{M}_\theta=\mathcal{R}_\theta\mathcal{M}\mathcal{R}_{-\theta}.$$

For more information on basis transformations, see Appendix F.

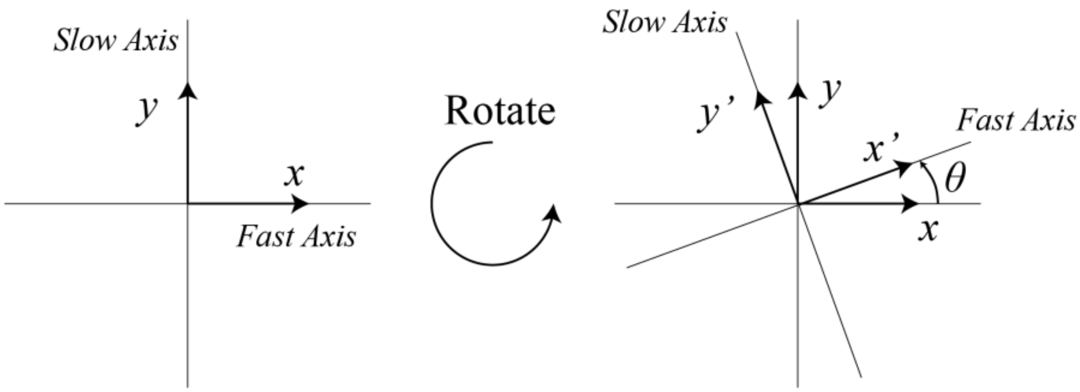

Figure $4.3.1$: If the wave plate is rotated, the fast and slow axis no longer correspond to y and x. Instead, we have to introduce a new coordinate system y', x'.

4.2.2 Linear Polarisers

A polariser that only transmits horizontally polarised light is described by the Jones matrix:

$$\mathcal{M}_{LP} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, horizontally polarised light is completely transmitted, while vertically polarised light is not transmitted at all. More generally, for light that is polarised at an angle α, we get

$$\mathcal{M}_{\alpha} = \mathcal{M}_{LP} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ 0 \end{pmatrix}.$$

The amplitude of the transmitted field is reduced by the factor $\cos\alpha$, which implies that the intensity of the transmitted light is reduced by the factor $\cos^2\alpha$. This relation is known as **Malus' law**.

4.2.3 Degree of Polarisation

Natural light such as sun light is unpolarised. The instantaneous polarisation of unpolarised light fluctuates rapidly in a random manner. A linear polariser produces linear polarised light from unpolarised light.

Light that is a mixture of polarised and unpolarised light is called partially polarised. The **degree of polarisation** is defined as the fraction of the total intensity that is polarised:

$$\text{degree of polarisation } = \frac{I_{pol}}{I_{pol} + I_{\text{unpol}}}.$$

It follows from ($4.3.17$) that the intensity transmitted by a linear polariser when unpolarised light is passed incident, is the average value of $\cos^2\alpha$ namely $\frac{1}{2}$, times the incident intensity.

4.2.4 Quarter-Wave Plates

A quarter-wave plate introduces a phase shift of $\pi/2$, so its Jones matrix is

$$\mathcal{M}_{QWP} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

because $\exp(i\pi/2) = i$. To describe the actual transmission through the quarter-wave plate, the matrix should be multiplied by some global phase factor, but because we only care about the phase difference between the field components, this global phase factor can be omitted without problem. The quarter-wave plate is typically used to convert linearly polarised light to elliptically polarised light and vice-versa. If the incident light is linearly polarised at angle α, the state of polarisation after the quater wave plate is

$$\begin{pmatrix} \cos\alpha \\ i\sin\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}.$$

In particular, if incident light is linear polarised under 45°, or equivalently, if the quarter wave plate is rotated over this angle, it will transform linearly polarised light into circularly polarised light (and vice versa).

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A demonstration is shown.

4.2.5 Half-Wave Plates

A half-wave plate introduces a phase shift of π, so its Jones matrix is

$$\mathcal{M}_{HWP} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

because $\exp(i\pi) = -1$. An important application of the half-wave plate is to **change the orientation of linearly polarised light**. After all, what this matrix does is mirroring the polarisation state in the x-axis. Thus, if we choose our mirroring axis correctly (i.e. if we choose the orientation of the wave plate correctly), we can change the direction in which the light is linearly polarised arbitrarily. A demonstration is shown in. To give an example: the polarisation of a wave that is parallel to the x-direction, can be rotated over angle α by rotating the crystal such that the slow axis makes angle $\alpha/2$ with the x-axis. Upon propagation through the crystal, the fast axis gets an additional phase of π, due to which the electric vector makes angle α with the x-axis (see Figure $4.3.2$).

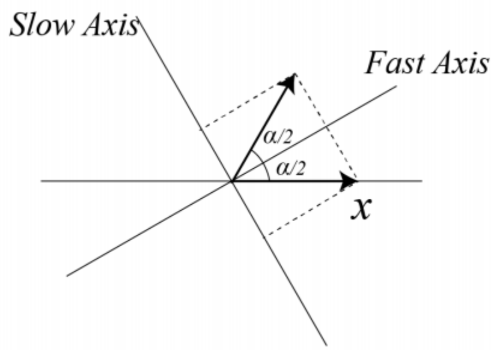

Figure $4.3.2$: Rotation of horizontally polarised light over an angle α using a half-wave plate.

4.2.6 Full-Wave Plates

A full-wave plate introduces a phase difference of 2π, which is the same as introducing no phase difference between the two field components. So what can possibly be an application for a full-wave plate? We need to recall from Eq. (($4.3.1$)) that the phase difference is 2π only for a particular wavelength. If we send through linearly (say vertically) polarised light of other wavelengths, these will become elliptically polarised, while the light with the correct wavelength λ_0 will stay vertically polarised. If we then let all the light pass through a horizontal polariser, the light with wavelength λ_0 will be completely extinguished, while the light of other wavelengths will be able to pass through at least partially. Therefore, **full-wave plates can be used to filter out specific wavelengths of light**.