5.3: Interference of Monochromatic Fields of the Same Frequency

Let us first recall the basic concepts of interference. What causes interference is the fact that light is a wave, which means that it not only has an **amplitude** but also a **phase**. Suppose for example we evaluate a time-harmonic field in two points

$$\mathcal{U}_1(t)=\cos(\omega t), \quad \mathcal{U}_2(t)=\cos(\omega t+\varphi).$$

Here φ denotes the phase difference between the fields at the two points. If $\varphi=0$, or φ is a multiple of 2π, the fields are **in phase**, and when they are added they interfere **constructively**

$$\mathcal{U}_1(t)+\mathcal{U}_2(t)=\cos(\omega t)+\cos(\omega t+2m\pi)=2\cos(\omega t).$$

However, when $\varphi=\pi$, or more generally $\varphi=\pi+2m\pi$, for some integer m, then the waves are **out of phase**, and when they are superimposed, they interfere **destructively**.

$$\begin{aligned}\mathcal{U}_1(t)+\mathcal{U}_2(t) &=\cos(\omega t)+\cos(\omega t+\pi+2m\pi)\\ &=\cos(\omega t)-\cos(\omega t)\\ &=0.\end{aligned}$$

We can sum the two fields for arbitrary φ more conveniently using complex notation:

$$\mathcal{U}_1(t)=\operatorname{Re}\left[e^{-i\omega t}\right], \quad \mathcal{U}_2(t)=\operatorname{Re}\left[e^{-i\omega t}e^{-i\varphi}\right].$$

Adding gives

$$\begin{aligned}\mathcal{U}_1(t)+\mathcal{U}_2(t) &=\operatorname{Re}\left[e^{-i\omega t}\left(1+e^{-i\varphi}\right)\right]\\ &=\operatorname{Re}\left[e^{-i\omega t}e^{-i\varphi/2}\left(e^{i\varphi/2}+e^{-i\varphi/2}\right)\right]\\ &=\operatorname{Re}\left[e^{-i\omega t}e^{-i\varphi/2}2\cos(\varphi/2)\right]\\ &=2\cos(\varphi/2)\cos(\omega t+\varphi/2)\end{aligned}$$

For $\varphi=2m\pi$ and $\varphi=\pi+2m\pi$ we retrieve the results obtained before. It is important to note that what we see or detect physically (say, the 'brightness' of light) does not correspond to the quantities $\mathcal{U}_1,\mathcal{U}_2$. After all, $\mathcal{U}_1$ and $\mathcal{U}_2$ can attain negative values, while there is no such thing as 'negative brightness'. What $\mathcal{U}_1$ and $\mathcal{U}_2$ describe are the **fields**, which may be positive or negative. The 'brightness' or the **irradiance** or **intensity** is given by taking an average over a long time of $\mathcal{U}(t)^2$ (see (1.7.8), we shall omit the factor $\sqrt{\epsilon/\mu_0}$). As explained in Chapter 2 , we see and measure only the long time-average of $\mathcal{U}(t)^2$, because at optical frequencies $\mathcal{U}(t)^2$

fluctuates very rapidly. We recall the definition of the time average over an interval of length T at a specific time t given in (1.8.4) in Chapter 2:

$$\langle f(t)\rangle=\frac{1}{T}\int_t^{t+T} f\left(t'\right)\mathrm{d}t',$$

where T is a time interval that is the response time of a typical detector, which is 10^{-9} s for a very fast detector, but this is still extremely long compared to the period of visible light which is of the order of 10^{-14} s. For a time-harmonic function, the long-time average is equal to the average over one period of the field and hence **it is independent of the time t at which it is taken**. Indeed for (5.3.5) we get

$$\begin{aligned} I &=\left\langle\left(\mathcal{U}_1(t)+\mathcal{U}_2(t)\right)^2\right\rangle\\ &=4\cos^2(\varphi/2)\left\langle\cos^2(\omega t+\varphi/2)\right\rangle\\ &=1+\cos(\varphi)\end{aligned}$$

where $T\omega>>1$. It is important to note that one can use complex notation to obtain the factor $1+\cos(\varphi)$ more easily. Let us write

$$\mathcal{U}_1(t)=\operatorname{Re}\left\{U_1e^{-i\omega t}\right\}, \quad \mathcal{U}_2(t)=\operatorname{Re}\left\{U_2e^{-i\omega t}\right\},$$

where

$$U_1 = 1, \quad U_2 = e^{-i\varphi}.$$

Then we find

$$\begin{aligned} |U_1 + U_2|^2 &= \left|1 + e^{-i\varphi}\right|^2 \\ &= \left(1 + e^{i\varphi}\right)\left(1 + e^{-i\varphi}\right) \\ &= 1 + 1 + e^{-i\varphi} + e^{i\varphi} \\ &= 2 + 2\cos(\varphi), \end{aligned}$$

hence

$$I = \frac{1}{2}|U_1 + U_2|^2.$$

To see why this works, recall Eq. (1.8.5) and choose $A = B = U_1 + U_2$.

Remark. To shorten the formulae, we will omit in this chapter the factor $1/2$ in front of the time-averaged intensity.

Hence we define $I_1 = |U_1|^2$ and $I_2 = |U_2|^2$, and we then find for the time-averaged intensity of the sum of U_1 and U_2 :

$$\begin{aligned} I &= |U_1 + U_2|^2 = (U_1 + U_2)^* (U_1 + U_2) \\ &= |U_1|^2 + |U_2|^2 + U_1^* U_2 + U_1 U_2^* \\ &= I_1 + I_2 + 2\,\mathrm{Re}[U_1^* U_2]. \end{aligned}$$

Here, $2\,\mathrm{Re}[U_1^* U_2]$ is known as the interference term. In the famous double-slit experiment (which we will discuss in a later section), we can interpret the terms as follows: let us say U_1 is the field that comes from slit 1 , and U_2 comes from slit 2 . If only slit 1 is open, we measure on the screen intensity I_1, and if only slit 2 is open, we measure I_2. If both slits are open, we would not measure $I_1 + I_2$, but we would observe fringes due to the interference term $2\,\mathrm{Re}[U_1^* U_2]$.

More generally, the intensity of a sum of multiple time-harmonic fields U_j all having the same frequency is given by the **coherent sum**

$$I = \left|\sum_j U_j\right|^2$$

However, we will see in the next section that sometimes the fields are unable to interfere. In that case all the interference terms of the coherent sum vanish, and the intensity is given by the **incoherent sum**

$$I = \sum_j |U_j|^2.$$