5.5: Temporal Coherence and the Michelson Interferometer

To investigate the coherence of a field, the most general approach is to make the field in two different points $\mathbf{r}_1$ and $\mathbf{r}_2$ interfere for some time delay τ and observe the fringe contrast. This means that one lets the fields $U(\mathbf{r}_1, t)$ and $U(\mathbf{r}_2, t-\tau)$ interfere. It is however customary to first look at the field in one point and let it interfere with itself but delayed in time, i.e. interfering $U(\mathbf{r}, t)$ with $U(\mathbf{r}, t-\tau)$. This special case is called **temporal coherence**. The other special case is **spatial coherence** in which the coherence of fields at two points is considered without time delay, by interfering $U(\mathbf{r}_1, t)$ and $U(\mathbf{r}_2, t)$. Spatial coherence will be treated later.

Because, when studying temporal coherence, the point $\mathbf{r}$ is always the same, we omit it from the formula. Furthermore, for easier understanding of the phenomena, we assume for the time being that the field considered is emitted by a single atom (i.e. a point source).

Temporal coherence is closely related to the spectral content of the light: if the light consists of fewer frequencies (think of monochromatic light), then it is more temporally coherent. To study the interference of $U(t)$ with $U(t-\tau)$, a Michelson interferometer, shown in Figure $5.5.1$, is a suitable setup. The light that goes through one arm takes time t to reach the detector, while the light that goes through the other (longer) arm takes time $t+\tau$ which means that it was radiated earlier. Therefore, the detector observes the time-averaged intensity $\left\langle |U(t)+U(t-\tau)|^2\right\rangle$. As remarked before, this averaged intensity does not depend on the time the average is taken, it only depends on the time difference τ between the two beams. We have

$$\begin{aligned}I(\tau) &= \left\langle |U(t)+U(t-\tau)|^2\right\rangle \\ &= \left\langle |U(t)|^2\right\rangle + \left\langle |U(t-\tau)|^2\right\rangle + 2\,\mathrm{Re}\langle U(t)U(t-\tau)^*\rangle \\ &= 2\left\langle |U(t)|^2\right\rangle + 2\,\mathrm{Re}\langle U(t)U(t-\tau)^*\rangle.\end{aligned}$$

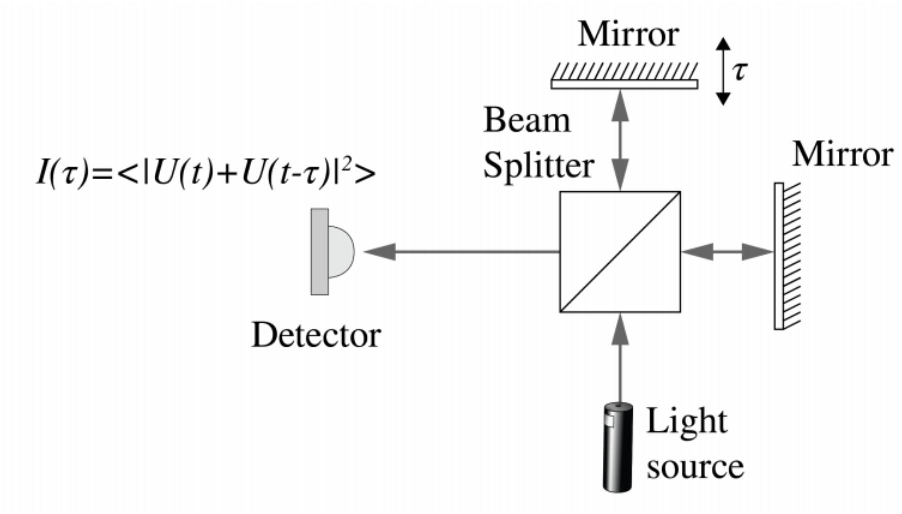

Figure $5.5.1$: A Michelson interferometer to study the temporal coherence of a field. A beam is split in two by a beam splitter, and the two beams propagate over different distances which corresponds to a time difference τ and then interfere at the detector.

So far we have considered a field that originates from a single atom. The total field emitted by an extended source is the sum of fields $U_i(t)$ corresponding to all atoms i. In studying time coherence we assume that these fields are propagating more or less parallel and that the light has a fixed polarisation, so that the fields can be added algebraically. The total complex field produced by a large number N of atoms is

$$U(t) = U_1(t) + \ldots + U_N(t).$$

During the integration time of the detector the fields U_i experience thousands of random phase jumps and therefore they do not interfere: **the point sources of the extended source are mutually fully incoherent**. The detected total intensity ($5.5.1$) is thus the sum of the intensities of the individual atoms:

$$\begin{aligned}I(\tau)&=\left\langle\left|U(t)+U(t-\tau)\right|^2\right\rangle\\&=\left\langle\sum_i\left|U_i(t)+U_i(t-\tau)\right]\left|^2\right\rangle\\&=\sum_i 2\left\{\left\langle\left|U_i(t)\right|^2\right\rangle+\operatorname{Re}\langle U_i(t)U_i(t-\tau)^*\rangle\right\},\end{aligned}$$

The expression $\left\langle\left|U(t)+U(t-\tau)\right|^2\right\rangle$ does not depend on position, so it cannot describe interference fringes in space. To better observe what happens when τ is varied, we introduce interference fringes in space by tilting one beam so that the observed interference pattern is given by

$$\begin{aligned}I(x,\tau)&=\left\langle\left|U(t)e^{ik_xx}+U(t-\tau)\right|^2\right\rangle\\&=2\left\langle\left|U(t)\right|^2\right\rangle+2\operatorname{Re}\left\langle U(t)U(t-\tau)^*e^{ik_xx}\right\rangle.\end{aligned}$$

If τ is changed, the maxima of the interference pattern translate as function of x, which is easy to observe. How interference fringes for tilted collimated beams are observed in a Michelson interferometer is demonstrated in. It is possible to obtain different fringe patterns using diverging beams instead of collimated beams, as is demonstrated in.

The **self coherence function** $\Gamma(\tau)$ is defined by

$$\Gamma(\tau)=\langle U(t)U(t-\tau)^*\rangle \qquad \text{self-coherence.}$$

The intensity of $U(t)$ is

$$I_0=\left\langle\left|U(t)\right|^2\right\rangle=\Gamma(0).$$

The **complex degree of self-coherence** is defined by:

$$\gamma(\tau)=\frac{\Gamma(\tau)}{\Gamma(0)}. \qquad \text{complex degree of self-coherence}$$

This is a complex number with modulus between 0 and 1 :

$$0\leq|\gamma(\tau)|\leq 1,$$

The observed intensity can then be written:

$$I(x,\tau)=2I_0\left\{1+\operatorname{Re}\left[\gamma(\tau)e^{ik_xx}\right]\right\},$$

Recall that we vary τ by varying the length of one of the arms in the Michelson interferometer.

We consider some special cases. Suppose $U(t)$ is a monochromatic wave

$$U(t)=e^{-i\omega t}.$$

In that case we get for the self-coherence

$$\Gamma(\tau)=\left\langle e^{-i\omega t}e^{i\omega(t-\tau)}\right\rangle=e^{-i\omega\tau},$$

and

$$\gamma(\tau)=e^{-i\omega\tau}.$$

Hence the interference pattern is given by

$$I(x,\tau)=2\left[1+\cos(\omega\tau-k_xx)\right].$$

So for monochromatic light we expect to see a cosine interference pattern, which shifts as we change the arm length of the interferometer (i.e. change τ). No matter how large τ, a clear interference pattern should be observed.

Next we consider what happens when the light is a superposition of two frequencies:

$$U(t)=\frac{e^{-i(\bar{\omega}+\Delta\omega/2)t}+e^{-i(\bar{\omega}-\Delta\omega/2)t}}{2},$$

where $\Delta\omega \ll \bar{\omega}$. Then:

$$\begin{aligned}\Gamma(\tau) &= \frac{1}{4}\left\langle\left(e^{-i(\bar{\omega}+\Delta\omega/2)t}+e^{-i(\bar{\omega}-\Delta\omega/2)t}\right)\left(e^{i(\bar{\omega}+\Delta\omega/2)(t-\tau)}+e^{i(\bar{\omega}-\Delta\omega/2)(t-\tau)}\right)\right\rangle \\ &\approx \frac{e^{-i(\bar{\omega}+\Delta\omega/2)\tau}+e^{-i(\bar{\omega}-\Delta\omega/2)\tau}}{4} \\ &= \cos(\Delta\omega\tau/2)\frac{e^{-i-\bar{\omega}\tau}}{2}\end{aligned}$$

where in the second line the time average of terms that oscillate with time is set to zero because the averaging is done over a time interval of duration T satisfying $T\Delta\bar{\omega} \gg 1$. Hence, the complex degree of self-coherence is:

$$\gamma(\tau)=\cos(\Delta\omega\tau/2)e^{-i\bar{\omega}\tau}$$

and ($5.5.8$) becomes

$$I(x,\tau)=\left\{1+\operatorname{Re}\left[\gamma(\tau)e^{ik_x x}\right]\right\}=\left[1+\cos(\Delta\omega\tau/2)\cos(\bar{\omega}\tau-k_x x)\right].$$

The interference term is the product of the function $\cos(\bar{\omega}\tau-k_x x)$), which is a rapidly oscillating function of τ, and a slowly varying envelope $\cos(\Delta\omega\tau/2)$. It is interesting to note that the envelope, and hence $\gamma(\tau)$, vanishes for some periodically spaced τ, which means that for certain τ the degree of self-coherence vanishes and no interference fringes form. Note that if $\Delta\omega$ is increased, the intervals between the zeroes of $\gamma(\tau)$ decrease.

If more frequencies are added, the envelope function is not a cosine function but on average decreases with τ. The typical value of τ below which interferences are observed is roughly equal to half the first zero of the envelope function. This value is called the coherence time $\Delta\tau_c$. We conclude with some further interpretations of the degree of self-coherence $\gamma(\tau)$.

- In stochastic signal analysis $\Gamma(\tau)=\langle U(t)U(t-\tau)^*\rangle$ is called the autocorrelation of $U(t)$. Informally, one can interpret the autocorrelation function as the ability to predict the field U at time t given the field at time $t-\tau$.
- The Wiener-Khinchine theorem says that the **Fourier transform of the self coherence function is the spectral power density of $U(t)$** :

$$\hat{\Gamma}(\omega)=|\hat{U}(\omega)|^2.$$

This result can be proved for stationary fields using Parseval's indentity. Using the uncertainty principle, we can see that the larger the spread of the frequencies of $U(t)$ (i.e. the larger the bandwidth), the more sharply peaked $\Gamma(\tau)$ is. Thus, the light gets temporally less coherent when it consists of a broader range of frequencies.