4.4: How to verify whether a Jones Matrix is a Linear Polariser or a Wave Plate?

If the direction of either the slow or fast axis is given, it is easy to write down the Jones matrix of a birefringent plate. Similarly, for a linear polariser it is simple to write down the Jones matrix if one knows the direction in which the polariser absorbs or transmits all the light. But suppose that you are given an arbitrary complex $(2,2)$-matrix:

$$\mathcal{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

How can one determine whether this matrix corresponds to a linear polariser or to a wave plate? Note that the elements of a Jones matrix are in general complex.

1. **Linear Polariser** The matrix corresponds to a linear polariser if there is a real vector which remains invariant under $\mathcal{M}$ and the real vector that orthogonal to this vector is mapped to zero. In other words, there must be an orthogonal basis of **real** eigenvectors and one of the eigenvalues must be 1 and the other 0 . Hence, to check that a given matrix corresponds to a linear polariser, one should verify that one eigenvalue is 1 and the other is 0 and furthermore that the eigenvectors are **real** vectors. It is important to check that the eigenvectors are indeed real because if they are not, they do not correspond to particular polarisation directions.
2. **Wave plate** To show that a matrix corresponds to a wave plate, there should exist two real orthogonal eigenvectors with, in general, complex eigenvalues of modulus 1. In fact, one of the eigenvectors corresponds to the fast axis with refractive index n_1, say, and the other to the slow axis with refractive index n_2, say. The eigenvalues are then

   $$e^{ikn_1d}, \quad e^{ikn_2d},$$

   where d is the thickness of the plate and k is the wave number. Hence to verify that a $(2,2)$-matrix corresponds to a wave plate, one has to compute the eigenvalues and check that these have modulus 1 and that the corresponding eigenvectors are real vectors and orthogonal.