6.4: Normalization and Orthogonality

Although we aren't yet going to learn rules for doing general inner products between state vectors, there are two cases where the inner product of two state vectors produces a simple answer. The first is not intrinsic to the mathematical representation, but rather something we will insist for state vectors that properly represent real physical states. For a complete state vector $|\psi\rangle$ to be a proper quantum mechanical state, it must satisfy the condition

$$\langle\psi \mid \psi\rangle=1$$

We say that this means that the state vector is normalized. It is possible to have non-normalized state vectors. For instance, in the equation

$$|+x\rangle=\tfrac{1}{\sqrt{2}}|+z\rangle+\tfrac{1}{\sqrt{2}}|-z\rangle$$

the two parts of the sum on the right side are themselves ket vectors. However, because they are valid state vectors multiplied by a constant, they are not normalized themselves. We will show later that this definition of $|+x\rangle$ is, however, normalized.

The second rule is that state vectors that represent different possible states corresponding to different possible measurements of a given observable must be orthogonal. Mathematically, this is expressed as:

$$\langle\phi_1 \mid \phi_2\rangle=0$$

if $|\phi_1\rangle$ and $|\phi_2\rangle$ are two different states corresponding to definite states for a given observable. For example, the states $|+z\rangle$ and $|-z\rangle$ correspond to two states of the same observable, specifically, the z component of angular momentum. The first corresponds to that component being measured along $+z$, the second to it being measured along $-z$. The orthogonality condition is then:

$$\langle+z \mid -z\rangle=0$$

As an example of doing these calculations with a more complicated state, consider the state $|+x\rangle$. If this state is properly normalized, then we should have $\langle+x \mid +x\rangle=1$. Do we? Well, first, we have to construct the bra vector that goes along with the ket vector:

$$\begin{aligned}
|+x\rangle &= \frac{1}{\sqrt{2}}|+z\rangle+\frac{1}{\sqrt{2}}|-z\rangle \\
\langle+x| &= \frac{1}{\sqrt{2}}|+z\rangle+\frac{1}{\sqrt{2}}|-z\rangle
\end{aligned}$$

Notice that in the case of a compound ket vector, to get the bra vector we just turn all ket vectors on the right side into bra vectors, and replace all the numbers with their complex conjugates (which is trivial here, since all the numbers are real). Now we have what we need to figure out the inner product. Just substitute in our expressions for $|+x\rangle$ and $\langle+x|$:

$$\begin{aligned}
\langle+x \mid +x\rangle &= \left(\frac{1}{\sqrt{2}}\left\langle+z\right|+\frac{1}{\sqrt{2}}\langle-z|\right)\left(\frac{1}{\sqrt{2}}|+z\rangle+\frac{1}{\sqrt{2}}|-z\rangle\right) \\
&= \frac{1}{2}\langle+z \mid +z\rangle+\frac{1}{2}\langle+z \mid -z\rangle+\frac{1}{2}\langle-z \mid +z\rangle+\frac{1}{2}\langle-z \mid -z\rangle
\end{aligned}$$

That looks very complicated, but now we can use the orthogonality condition we know is true for the z states, as we've defined them as good states corresponding to the z component of z angular momentum. We know that $\langle+z \mid +z\rangle=1$, $\langle-z \mid -z\rangle=1$, $\langle-z \mid +z\rangle=0$, and $\langle+z \mid -z\rangle=0$ from normalization and orthogonality. Substitute these in:

$$\begin{aligned}
\langle+x \mid +x\rangle &= \frac{1}{2}(1)+\frac{1}{2}(0)+\frac{1}{2}(0)+\frac{1}{2}(1) \\
\langle+x \mid +x\rangle &= 1
\end{aligned}$$

So the state is properly normalized! I leave it as an exercise for the alert reader to show that $|+x\rangle$ and $|-x\rangle$ are orthogonal.