10.2: Expectation Values

You can also use the matrix representation of operators to figure out expectation values. Suppose that you have an electron in the state:

$$|\psi\rangle=\sqrt{\frac{1}{3}}|+z\rangle+\sqrt{\frac{2}{3}}|-z\rangle \tag{10.5}$$

What are the expectation values of for spin along the x-axis??

First, we construct the column vector representation of this state $|\psi\rangle$:

$$|\psi\rangle=\begin{bmatrix}\sqrt{1/3}\\ \sqrt{2/3}\end{bmatrix} \tag{10.6}$$

The corresponding bra vector is represented by a row vector:

$$\langle\psi|=\begin{bmatrix}\sqrt{1/3} & \sqrt{2/3}\end{bmatrix} \tag{10.7}$$

To figure out the expectation value of x-spin, we sandwich the $\hat{S}_x$ operator in between the bra and ket vectors for this state:

$$\begin{aligned}\langle s_x\rangle &= \left\langle\psi\left|\hat{S}_x\right|\psi\right\rangle\\ &= \begin{bmatrix}\sqrt{1/3} & \sqrt{2/3}\end{bmatrix}\left(\frac{\hbar}{2}\right)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{1/3}\\ \sqrt{2/3}\end{bmatrix}\\ &= \left(\frac{\hbar}{2}\right)\begin{bmatrix}\sqrt{1/3} & \sqrt{2/3}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{1/3}\\ \sqrt{2/3}\end{bmatrix}\end{aligned} \tag{10.8}$$

(all we did between the last two lines was pull the scalar constant $\hbar/2$ out front). We've got a row vector times a matrix times a column vector. That may look intimidating, but we know how to do the matrix times the column vector, so let's do that first. That will leave us with a row vector times a column vector; we know how to work that out as well, leaving us with just a scalar. A scalar is what we need for an expectation value.

$$\begin{aligned}\langle s_x\rangle &= \left(\frac{\hbar}{2}\right)[\sqrt{1/3}\sqrt{2/3}]\begin{bmatrix}\sqrt{2/3}\\ \sqrt{1/3}\end{bmatrix}\\ &= \left(\frac{\hbar}{2}\right)\left(\sqrt{\frac{1}{3}}\right)\left(\sqrt{\frac{2}{3}}\right)+\left(\sqrt{\frac{2}{3}}\right)\left(\sqrt{\frac{1}{3}}\right)\\ &= \left(\frac{\hbar}{2}\right)\frac{2\sqrt{2}}{3}\\ &= \frac{\sqrt{2}}{3}\hbar\end{aligned} \tag{10.9}$$

That's a plausible expectation value. It's neither $\hbar/2$ nor $-\hbar/2$, which means that this is not a definite state for x spin. That's good, because the state is clearly not the same as $|+x\rangle$ when you write out that state in terms of $|+z\rangle$ and $|-z\rangle$. It's between those two. However, from just looking at the state, while you can fairly quickly see that $|-z\rangle$ has more amplitude than $|+z\rangle$, and thus a measurement of z spin will yield $-\hbar/2$ more often than $+\hbar/2$, it's not obvious at all just looking at the state which value of x spin would be more common, and thus whether the x expectation value should be positive or negative. In this case, you have to perform the calculation. The matrix formulation of the spin operators makes the calculations faster and easier than they would be when you explicit writing out everything in terms of the z basis states.

We could also quickly figure out what the amplitude for measuring positive x spin is with this formalism. Remember that for a particle in state $|\psi\rangle$, the amplitude for finding positive x spin is $\langle+x\mid\psi\rangle$. Putting together the $\langle+x|$ bra vector with the column vector for $|\psi\rangle$ above, we get:

$$\begin{aligned}
\langle +x \mid \psi\rangle &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{1/3} \\ \sqrt{2/3} \end{bmatrix} \\
&= \left(\sqrt{\frac{1}{2}}\right)\left(\sqrt{\frac{1}{3}}\right) + \left(\sqrt{\frac{1}{2}}\right)\left(\sqrt{\frac{2}{3}}\right) \\
&= \sqrt{\frac{1}{6}} + \sqrt{\frac{1}{3}} \\
&= 0.9856
\end{aligned} \tag{10.10}$$

That's a high positive amplitude, corresponding to a probability of 0.97 that positive x spin will be measured for this state. Again, without performing the calculations, this is not at all obvious. However, this high probability for positive x spin is consistent with the fact that the x spin expectation value $\langle s_x\rangle$ is positive and only a little bit less than $\hbar/2$.