6.2: Positive and Completely Positive Maps

We considered the evolution of the density operator under a family of Kraus operators in Eq. (4.30):

$$\rho \to \rho' = \mathscr{E}(\rho) = \sum_k A_k \rho A_k^\dagger, \tag{6.10}$$

where $\sum_k A_k^\dagger A_k = \mathbb{I}$ (that is, $\mathscr{E}$ is trace-preserving). When $\mathscr{E}$ transforms any positive operator into another positive operator, we call it a positive map. We may be tempted to conclude that all positive maps correspond to physically allowed transformations. After all, it maps density operators to density operators. Unfortunately, Nature (or Mathematics?) is not that tidy.

Consider the transpose of the density operator $\rho \to \rho^T$, which acts according to

$$\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j| \to \rho^T = \sum_{ij} \rho_{ji} |i\rangle\langle j|. \tag{6.11}$$

You can verify immediately that the trace is preserved in this operation (check this!), and ρ^T is again a positive operator since the eigenvalues are identical to those of ρ. For example, consider the qubit state $(|0\rangle + i|1\rangle)/\sqrt{2}$. The density operator and its transpose are

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad \text{and} \quad \rho^T = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \tag{6.12}$$

The transpose therefore corresponds to the state $(|0\rangle - i|1\rangle)/\sqrt{2}$. Now consider that the qubit is part of an entangled state $(|00\rangle + |11\rangle)/\sqrt{2}$. The density operator is given by

$$\rho = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|), \tag{6.13}$$

and the partial transpose on the first qubit is

$$\rho^T = \frac{1}{2}(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|). \tag{6.14}$$

The eigenvalues of ρ are all positive, but ρ^T has a negative eigenvalue! So ρ^T cannot be a density operator. Consequently, it is not correct to say that positive maps correspond to physical processes. We need to put another restriction on maps.

From the example of the partial transpose, we can deduce that maps must not only be positive for the system S that they act on, but also positive on larger systems that include S as a subsystem. When this is the case, we call the map completely positive. There is a very important theorem in mathematics, called Kraus' Representation Theorem, which states that maps of the form in Eq. (6.10) with the restriction that $\sum_k A_k^\dagger A_k = \mathbb{I}$ is a completely positive map, and moreover, that any completely positive map can be expressed in this form.

Exercises

1. a. Show that $\sum_k A_k^\dagger A_k = \mathbb{I}$,
   b. prove that any non-Hermitian square matrix can be written as $A + iB$, with A and B Hermitian,
   c. prove that $L_0 = -\frac{1}{2}\sum_{k\neq 0} L_k^\dagger L_k$.
2. Consider a two-level system $(|0\rangle, |1\rangle)$ that has a dephasing process, modelled by the Lindblad operators $L_1 = \gamma|0\rangle\langle 1|$ and $L_2 = \gamma|1\rangle\langle 0|$.
   a. write down the Lindblad equation (choose $H = 0$ for simplicity).
   b. Calculate the evolution of the pure states $|0\rangle$ and $|+\rangle$ at $t = 0$. Hint: write the density matrix in the Pauli matrix basis $\{\mathbb{I}, X, Y, Z\}$. What can you say about the equilibrium state of the system?
   c. Calculate and plot the entropy $S(\rho)$ of the state $\rho(t)$ as a function of γ and t.
3. Calculate the eigenvalues of ρ^T in Eq. (6.14).