7.4: Composite Systems with Angular Momentum

Now consider two systems, 1 and 2, with total angular momentum $\mathbf{J}_1$ and J_2, respectively. The total angular momentum is again additive, and given by

$$\mathbf{J}=\mathbf{J}_1+\mathbf{J}_2 \equiv \mathbf{J}_1 \otimes \mathbb{I}+\mathbb{I} \otimes \mathbf{J}_2. \tag{7.53}$$

Completely analogous to the addition of spin and orbital angular momentum, we can construct the commuting operators J^2, J_z, $\mathbf{J}_1^2$, and J_2^2, but not J_{1z} and J_{2z}. Again, we construct two natural bases for the total angular momentum of the composite system, namely

$$\{|j, m\rangle\} \quad \text{or} \quad \{|j_1, m_1\rangle \otimes |j_2, m_2\rangle\} \equiv \{|j_1, j_2, m_1, m_2\rangle\} \tag{7.54}$$

We want to know how the two bases relate to each other, because sometimes we wish to talk about the angular momentum of the composite system, and at other times we are interested in the angular momentum of the subsystems. Since the second basis (as well as the first) in Eq. (7.54) forms a complete orthonormal basis, we can write

$$|j, m\rangle=\sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle \langle j_1, j_2, m_1, m_2 \mid j, m\rangle \tag{7.55}$$

The amplitudes $\langle j_1, j_2, m_1, m_2 \mid j, m\rangle$ are called Clebsch-Gordan coefficients, and we will now present a general procedure for calculating them.

Let's first consider a simple example of two spin$-\frac{1}{2}$ systems, such as two electrons. The spin basis for each electron is given by $\left|\frac{1}{2}, \frac{1}{2}\right\rangle=|\uparrow\rangle$ and $\left|\frac{1}{2},-\frac{1}{2}\right\rangle=|\downarrow\rangle$. The spin basis for the two electrons is therefore

$$|j_1, j_2, m_1, m_2\rangle \in \{|\uparrow, \uparrow\rangle, |\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle, |\downarrow, \downarrow\rangle\}. \tag{7.56}$$

The total spin is given by $j=\frac{1}{2}+\frac{1}{2}=1$ and $j=\frac{1}{2}-\frac{1}{2}=0$, so the four basis states for total angular momentum are

$$|j, m\rangle \in \{|1,1\rangle, |1,0\rangle, |1,-1\rangle, |0,0\rangle\} \tag{7.57}$$

The latter state is the eigenstate for $j=0$. The maximum total angular momentum state $|1,1\rangle$ can occur only when the two electron spins a parallel, and we therefore have

$$|1,1\rangle=|\uparrow, \uparrow\rangle=\left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes\left|\frac{1}{2}, \frac{1}{2}\right\rangle \tag{7.58}$$

To find the expansion of the other total angular momentum eigenstates in terms of spin eigenstates we employ the following trick: use that $J_{\pm}=J_{1\pm}+J_{2\pm}$. We can then apply $J_{\pm}$ to the state $|1,1\rangle$, and $J_{1\pm}+J_{2\pm}$ to the state $\left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes\left|\frac{1}{2}, \frac{1}{2}\right\rangle$. This yields

$$J_{-}|1,1\rangle=\hbar \sqrt{j(j+1)-m(m-1)}\,|1,0\rangle=\hbar \sqrt{2}|1,0\rangle \tag{7.59}$$

Similarly, we calculate

$$J_{1-}\left|\frac{1}{2}, \frac{1}{2}\right\rangle=\hbar \sqrt{\frac{1}{2}\left(\frac{3}{2}\right)-\frac{1}{2}\left(-\frac{1}{2}\right)}\left|\frac{1}{2},-\frac{1}{2}\right\rangle=\hbar\left|\frac{1}{2},-\frac{1}{2}\right\rangle, \tag{7.60}$$

and a similar result for J_{2-}. Therefore, we find that

$$\hbar \sqrt{2}|1,0\rangle=\hbar|\uparrow, \downarrow\rangle+\hbar|\downarrow, \uparrow\rangle \quad \Longrightarrow \quad |1,0\rangle=\frac{|\uparrow, \downarrow\rangle+|\downarrow, \uparrow\rangle}{\sqrt{2}}. \tag{7.61}$$

Applying J_{-} again yields

$$|1,-1\rangle=|\downarrow, \downarrow\rangle \tag{7.62}$$

This agrees with the construction of adding parallel spins. The three total angular momentum states

$$\begin{aligned}|1,1\rangle &= |\uparrow,\uparrow\rangle, \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle), \\ |1,-1\rangle &= |\downarrow,\downarrow\rangle\end{aligned} \tag{7.63}$$

form a so-called triplet of states with $j = 1$. We now have to find the final state corresponding to $j = 0, m = 0$. The easiest way to find it at this point is to require orthonormality of the four basis states, and this gives us the singlet state

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle). \tag{7.64}$$

The singlet state has zero total angular momentum, and it is therefore invariant under rotations.

In general, this procedure of finding the Clebsch-Gordan coefficients results in multiplets of constant j. In the case of two spins, we have a tensor product of two two-dimensional spaces, which are decomposed in two subspaces of dimension 3 (the triplet) and 1 (the singlet), respectively. We write this symbolically as

$$2 \otimes 2 = 3 \oplus 1. \tag{7.65}$$

If we had combined a spin 1 particle with a spin $\frac{1}{2}$ particle, the largest multiplet would have been due to $j = 1 + \frac{1}{2} = \frac{3}{2}$, which is a 4-dimensional subspace, and the smallest subspace is due to $j = 1 - \frac{1}{2} = \frac{1}{2}$, which is a two-dimensional subspace:

$$3 \otimes 2 = 4 \oplus 2. \tag{7.66}$$

In general, the total angular momentum of two systems with angular momentum k and l is decomposed into multiplets according to the following rule $(k \geq l)$:

$$(2k+1) \otimes (2l+1) = [2(k+l)+1] \oplus [2(k+l)-1] \oplus \ldots \oplus [2(k-l)+1], \tag{7.67}$$

or in terms of the dimensions of the subspaces $(n \geq m)$:

$$n \otimes m = (n+m-1) \oplus (n+m-3) \oplus \ldots \oplus (n-m+1). \tag{7.68}$$

? Exercises

1. Angular momentum algebra.
   a. Prove the algebra given in Eq. (7.2). Also show that $\left[L^2, L_i\right] = 0$, and verify the commutation relations in Eq. (7.7).
   b. Show that $L^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$. Use the fact that $\left[L_-, L^2\right] = 0$.
2. Pauli matrices.
   a. Check that the matrix representation of the spin$-\frac{1}{2}$ operators obey the commutation relations.
   b. Calculate the matrix representation of the Pauli matrices for $s = 1$.
   c. Prove that $\exp[-i\theta \cdot \sigma]$ is a 2×2 unitary matrix.
3. Isospin $\mathbf{I}$ describes certain particle families called multiplets, and the components of the isospin obey the commutation relations $[I_i, I_j] = i\epsilon_{ijk}I_k$.
   a. What is the relation between spin and isospin?
   b. Organize the nucleons (proton and neutron), the pions $\left(\pi^+, \pi^0, \text{ and } \pi^-\right)$, and the delta baryons $\left(\Delta^{++}, \Delta^+, \Delta^0, \text{ and } \Delta^-\right)$ into multiplets. You will have to determine their isospin quantum number.
   c. Give all possible decay channels of the delta baryons into pions and nucleons (use charge and baryon number conservation).
   d. Calculate the relative decay ratios of Δ^+ and Δ^0 into the different channels.
4. A simple atom has orbital and spin angular momentum, and the Hamiltonian for the atom contains a spin-orbit coupling term $H_{\text{so}} = g\hbar\mathbf{L}\cdot\mathbf{S}$, where $g\hbar$ is the coupling strength.
   a. Are orbital and spin angular momentum good quantum numbers for this system? What about total angular momentum?
   b. Use first-order perturbation theory to calculate the energy shift due to the spin-orbit coupling term.
   c. Calculate the transition matrix elements of H_{so} in the basis $\{|l,m;s,m_s\rangle\}$.
5. Multiplets.

a. A spin $\frac{3}{2}$ particle and a spin 2 particle form a composite system. How many multiplets are there, and what is the dimension of the largest multiplet?
b. How many multiplets do two systems with equal angular momentum have?