10.3: Total Angular Momentum

In 3D space, if you have three components of a vector $\vec{v}$, then the magnitude of that vector squared is $v^2 = v_x^2 + v_y^2 + v_z^2$. Angular momentum is a vector, and so this rule would apply to angular momentum as well. However, in quantum mechanics, we see that angular momentum behaves very differently from how it does in classical physics. In particular, if an object has a definite z component of angular momentum, then it has an indefinite x component of angular momentum. Does that mean that total angular momentum must also be indefinite? In order to answer this question, we must ask it in a proper quantum manner.

In quantum mechanics, we associate each observable quantity with an operator. We can then use that operator on one of its eigenstates (i.e. a state where the observable has a definite value) to pull out the value of the observable as the eigenvalue a in the equation $\hat{A}|\phi\rangle = a|\phi\rangle$. If the system is not in an eigenstate, we can figure out the "expectation value" $\langle a\rangle$ (i.e. the weighted average of all values that could be observed) using the operator in the equation

$$\langle a\rangle = \langle\psi|\hat{A}|\psi\rangle \tag{10.11}$$

What then is the operator that corresponds to total angular momentum? By analogy to classical physics, we can guess that the operator for total angular momentum squared is:

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \tag{10.12}$$

This begs the question as to what $\hat{S}_x^2$ means; we know how to square numbers and variables, but how do you square an operator? For $\hat{S}^2$, we will just treat that as an operator itself, the "total spin squared" operator. We'll not treat the superscript 2 as squaring, rather we'll just consider it part of the name. As for the others, what really matters about an operator is what it does to a vector representing a quantum state that it's supposed to operate on. In order to make this definition of the spin angular momentum squared operator to work, we need to interpret them as follows:

$$\hat{S}_x^2|\psi\rangle = \hat{S}_x\hat{S}_x|\psi\rangle \tag{10.13}$$

In other words, first apply the $\hat{S}_x$ operator to the state $|\psi\rangle$, and then apply the $\hat{S}_x$ operator again to the vector that resulted from the first application. As an example, let's consider the $\hat{S}_x^2$ operator on the state $|+z\rangle$:

$$\begin{aligned}\hat{S}_x^2|+z\rangle &= \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\\ &= \frac{\hbar^2}{4}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\end{aligned} \tag{10.14}$$

All we've done in the first step is pulled the scalar constants out front. To perform this matrix multiplication, first we must multiply the rightmost matrix by the vector, and then we can multiply the first matrix by the result.[1]

$$\begin{aligned}\hat{S}_x^2|+z\rangle &= \frac{\hbar^2}{4}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{\hbar^2}{4}\begin{bmatrix}1\\ 0\end{bmatrix}\\ &= \frac{\hbar^2}{4}|+z\rangle\end{aligned} \tag{10.15}$$

Interestingly, it seems that $|+z\rangle$ is in fact an eigenstate of $\hat{S}_x^2$, even though it's not an eigenstate of $\hat{S}_x$!

Armed with these techniques, it is possible to show that any properly normalized spin-1/2 state $|\psi\rangle$ is an eigenstate of $\hat{S}_2$ with eigenvalue $\frac{3}{4}\hbar^2$. Although it may be surprising that $|+z\rangle$ is an eigenstate of $\hat{S}_x^2$, in retrospect it should not be surprising that all states are eigenstates of the total angular momentum operator. We've been saying all along that the total angular momentum of an electron is $\frac{\sqrt{3}}{2}\hbar$; what can be in an indefinite state is the components of that angular momentum along various axes.

[1]If you know how to multiply 2×2 matrices, you can do the matrix multiplication first if you wish. As we will see, the commutative property does not apply to matrix multiplication, but the associative property does.