11.3.2: Uncertainty in Quantum Mechanics

In order to bring this into quantum mechanics, we already know how to calculate the average $\langle a\rangle$, which we call the "expectation value". If the state of the system is $|\psi\rangle$ and the operator corresponding to the observable a is $\hat{A}$, then

$$\langle a\rangle=\langle\psi|\hat{A}|\psi\rangle \tag{11.20}$$

Similarly, now that we recognize that we can interpret $\hat{A}^2$ as just applying the operator $\hat{A}$ twice, we can calculate $\left\langle a^2\right\rangle$:

$$\left\langle a^2\right\rangle=\left\langle\psi\left|\hat{A}^2\right|\psi\right\rangle \tag{11.21}$$

For example, let's consider the state $|\psi\rangle=|+z\rangle$ and the observable spin-z. We expect the uncertainty here to be zero, because we know exactly what we'll get if we measure spin-z. Let's see if it works out that way:

$$\begin{aligned}\langle s_z\rangle &=\left\langle\psi\left|\hat{S}_z\right|\psi\right\rangle\\ &=\frac{\hbar}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ &=\frac{\hbar}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ &=\frac{\hbar}{2}\end{aligned} \tag{11.22}$$

As expected, the expectation value for spin-z is $+\hbar/2$. For the other part:

$$\begin{aligned}\left\langle s_z^2\right\rangle &=\left\langle+z\left|\hat{S}_z\hat{S}_z\right|-z\right\rangle\\ &=\frac{\hbar^2}{4}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ &=\frac{\hbar^2}{4}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ &=\frac{\hbar^2}{4}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ &=\frac{\hbar^2}{4}\end{aligned} \tag{11.23}$$

If we take the difference $\left\langle {s_z}^2\right\rangle-\langle s_z\rangle^2$, we get $\hbar^2/4-\hbar^2/4=0$, as expected.

What if we want to know the uncertainty on S_x for this state?

$$\begin{aligned}\langle s_x\rangle &=\left\langle+z\left|\hat{S}_x\right|+z\right\rangle\\ &=\frac{\hbar}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\ &=\frac{\hbar}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ &=0\end{aligned} \tag{11.24}$$

If the system is in the state $|+z\rangle$, we know that we have a 50% chance each for finding spin-x to be $+\hbar/2$ or $-\hbar/2$. Thus, it's no surprise that the average value of spin-x is zero, even though zero isn't a value we might measure. To figure out the variance:

$$
\begin{aligned}
\left\langle s_x^2\right\rangle &= \left\langle +z \left| \hat{S}_x \hat{S}_x \right| +z \right\rangle \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{\hbar^2}{4}
\end{aligned} \tag{11.25}
$$

Thus, in this case, the formal uncertainty Δs_x on the x-spin is $\hbar/2$.