8.3: Observables Based on Creation and Annihilation Operators

So far, we have considered only the basis states of many particles for a single observable A. What about other observables, in particular those that do not commute with A? We can make a similar construction. Suppose an observable B has eigenvalues b_j. We can construct creation and annihilation operators $\hat{b}_j^\dagger$ and $\hat{b}_j$ that act according to

$$\begin{aligned}\hat{b}_j^\dagger \left|m_1, m_2, \ldots, m_j, \ldots\right\rangle &= \sqrt{m_j+1}\left|m_1, m_2, \ldots, m_j+1, \ldots\right\rangle, \\ \hat{b}_j \left|m_1, m_2, \ldots, m_j, \ldots\right\rangle &= \sqrt{m_j}\left|m_1, m_2, \ldots, m_j-1, \ldots\right\rangle.\end{aligned} \tag{8.36}$$

where m_j is the number of particles with value b_j. Typically, the basis states of two observables are related via a single unitary transformation $\left|b_j\right\rangle = U\left|a_j\right\rangle$ for all j. How does this relate the creation and annihilation operators?

To answer this, let's look at the single particle states. We can write the single-particle eigenstates $\left|a_j\right\rangle$ and $\left|b_j\right\rangle$ as

$$\left|a_j\right\rangle = \hat{a}_j^\dagger|\varnothing\rangle \quad \text{and} \quad \left|b_j\right\rangle = \hat{b}_j^\dagger|\varnothing\rangle. \tag{8.37}$$

We assume that U does not change the vacuum[6], so $U|\varnothing\rangle = |\varnothing\rangle$. This means that we can relate the two eigenstates via

$$\left|b_j\right\rangle = U\left|a_j\right\rangle = U\hat{a}_j^\dagger|\varnothing\rangle = U\hat{a}_j^\dagger\left(U^\dagger U\right)|\varnothing\rangle = U\hat{a}_j^\dagger U^\dagger|\varnothing\rangle = \hat{b}_j^\dagger|\varnothing\rangle, \tag{8.38}$$

where we have strategically inserted the identity $\mathbb{I} = U^\dagger U$. This leads to the operator transformation

$$\hat{b}_j^\dagger = U\hat{a}_j^\dagger U^\dagger. \tag{8.39}$$

The Hermitian adjoint is easily calculated as $\hat{b}_j = U\hat{a}U^\dagger$. It is left as an exercise for you to prove that

$$\hat{b}_j^\dagger = \sum_k u_{jk}\hat{a}_k^\dagger \quad \text{and} \quad \hat{b}_j = \sum_k u_{kj}^*\hat{a}_k, \tag{8.40}$$

where $u_{jk} = \left\langle a_k \mid b_j\right\rangle$.

How do we construct operators using the creation and annihilation operators? Suppose that a one-particle observable H has eigenvalues E_j and eigenstates $|j\rangle$. This can be, for example the Hamiltonian of the system, which ensures that the physical values of the particles (the eigenvalues) are additive. The operator for many identical particles then becomes

$$H = \sum_j E_j\hat{n}_j = \sum_j E_j\hat{a}_j^\dagger\hat{a}_j, \tag{8.41}$$

which transforms according to Eq. (8.40). More generally, the operator may not be written in the eigenbasis $\left|n_1, n_2, \ldots\right\rangle$, in which case it has the form

$$H = \sum_{ij} H_{ij}\hat{b}_i^\dagger\hat{b}_j, \tag{8.42}$$

where H_{ij} are matrix elements. The creation and annihilation operators $\hat{a}_j^\dagger$ and $\hat{a}_j$ diagonalise H, and are sometimes called normal modes. The reason for this is that the creation and annihilation operators for bosons obey the same mathematical rules as the raising and lowering operators for the harmonic oscillator. The index j then denotes different oscillators. A system of coupled oscillators can be decomposed into normal modes, which are themselves isolated harmonic oscillators.

[6] This is a natural assumption when we are confined to the single particle Hilbert space, but there are general unitary transformations for which this does not hold, such as the transformation to an accelerated frame.