13.1: Where we are so far

We have focused primarily on electron spin so far because it's a simple quantum system (there are only two basis states!), and yet it still shows much of the peculiar nature of reality on the quantum level. In particular, we've seen the following things in the theory of quantum mechanics:

- A "system" (e.g. the angular momentum vector of an electron) may be an an indefinite state, also sometimes called a "mixture" of states, where an observable doesn't have a set value. Rather, the state of the system is such that if the observable were measured, there is a probability of different values being observed. The mathematical theory represents this by allowing states to be sums of coefficients times orthogonal basis states. For example, with angular momentum of a spin-1/2 particle such as an electron, the basis states are $|+z\rangle$ and $|-z\rangle$.
- Observables may take on quantized values. For example, every time you measure the z component of angular momentum of an electron, you get either $+\hbar/2$ or $-\hbar/2$. This is in sharp contrast to what you'd see in classical physics.
- What propagates in quantum mechanics is amplitudes. For example, if an electron is in state $|\psi\rangle$, the amplitude to measure it to have z angular momentum $+\hbar/2$ is $\langle +z \mid \psi\rangle$. The probability, which is what we can really find in experiments, is the absolute square of the amplitude; in this example, that would be $|\langle +z \mid \psi\rangle|^2$.
- Different observables may be orthogonal (the second use of this term). If they are, then a system can not be in a definite state for those two observables at the same time. The projections of angular momentum along different axes are orthogonal; position and momentum along the same direction are orthogonal.
- Observables in quantum mechanics are paired with operators. A quantum mechanical operator operates on a quantum state (represented by a ket vector), and the result of that operation is another (non-normalized) quantum states (i.e. another ket vector). For example, if we call the z component of angular momentum spin-z or just s_z, the operator that goes with it is $\hat{S}_z$, the spin-z operator. Operators are quite abstract, and form a mathematical part of the theory that is useful, but is difficult to interpret and associate directly with something that you could observe.
- A state that is a definite state for a given observable is an eigenstate of that operator. (We would also say that the ket vector that represents that state is an eigenvector of the operator; if we're representing operators as matrices, then the column vector that represents the state is an eigenvector of the operator.) An operator working on one of its eigenstate returns a constant times the same state. That constant is called the eigenvalue associated with the eigenstate. If this operator corresponds to an observable, that eigenvalue must be a real number, and corresponds to the physical measurement you'd make of that observable. For example:

$$\hat{S}_z|+z\rangle = \frac{\hbar}{2}|+z\rangle \tag{13.1}$$

  This equation is the eigenvalue equation, in this case specifically for the z-spin operator and the $|+z\rangle$ state. The state $|+z\rangle$ is a state of definite z-spin, so it is an eigenstate of the z-spin operator $\hat{S}_z$. The eigenvalue equation for this state and this operator includes the constant $\hbar/2$, which is the actual value of the z component of spin angular momentum that an electron in state $|+z\rangle$ has.

- You can find the expectation value for a system in state $|\psi\rangle$ for a given observable by sandwiching the observable's operator between $\langle\psi|$ and $|\psi\rangle$. For example, the expectation value for z-spin for a given electron is $\left\langle \psi \middle| \hat{S}_z \middle| \psi \right\rangle$. The expectation value is the average value you'd get if you measured the observable for that state. That is, if you took a large number of systems in that state and measured the observable for all of those systems, you'd get different results, with probabilities for each result predicted by the mathematics of quantum mechanics. The average of all those results would be the expectation values.

Although the eigenvalue equation is fairly abstract, it's a very important part of the mathematical theory of quantum mechanics. The only direct connection it has to what we might observe in the lab is that it extracts (in the form of the eigenvalue) the measured quantity for the observable that you'd get for a given eigenstate (i.e. definite state) of that observable's operator. However, the operator itself doesn't represent any particular physical operation you might perform in the lab.

From a broader point of view, the eigenvalue equation is the equation you can use to figure out what states are possible definite states for a given operator, and therefore what values you might measure for the observable associated with that operator.