10.1: Spin Operators

We've been talking about three different spin observables for a spin-1/2 particle: the component of angular momentum along, respectively, the x, y, and z axes. In quantum mechanics, there is an operator that corresponds to each observable. The operators for the three components of spin are $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. If we use the column vector representation of the various spin eigenstates above, then we can use the following representation for the spin operators:

$$\hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{10.2}$$

It is also conventional to define the three "Pauli spin matrices" σ_x, σ_y, and σ_z, which are:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{10.3}$$

Clearly, then, the spin operators can be built from the corresponding Pauli matrices just by multiplying each one by $\hbar/2$.

You can verify that this is a good representation of the spin operators by making sure that all all of the various observations about spin states are reproduced by using these operators and these vectors to predict them from the theory. For example, $|+y\rangle$ is an eigenstate for the y component of spin, so the column vector representation of $|+y\rangle$ needs to be an eigenvector of $\hat{S}_y$. Is it? Let's try it:

$$\begin{aligned} \hat{S}_y|+y\rangle &= \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} \\ &= \frac{\hbar}{2}\begin{bmatrix} (0)(1/\sqrt{2})+(-i)(i/\sqrt{2}) \\ (i)(1/\sqrt{2})+(0)(i/\sqrt{2}) \end{bmatrix} \\ &= \frac{\hbar}{2}\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} \\ &= \frac{\hbar}{2}|+y\rangle \end{aligned} \tag{10.4}$$

In at least this case, the matrix and column vector representations of $\hat{S}_y$ and $|+y\rangle$ are working.