# 12.4: Fermions and Bosons

In quantum mechanics, there are two kinds of particles. **Fermions** are particles that are antisymmetric under the exchange operator; that is, if $|\xi\rangle$ is a two-particle state for two indistinguishable fermions, $\hat{P}_{12}|\xi\rangle = -|\xi\rangle$. **Bosons** are particles that are symmetric under the exchange operator; that is, if $|\xi\rangle$ is a two-particle state for two indistinguishable bosons, $\hat{P}_{12}|\xi\rangle = |\xi\rangle$. This is summarized below:

$$\hat{P}_{12}|\xi\rangle = \begin{cases} |\xi\rangle \text{ for a two-boson state} \\ -|\xi\rangle \text{ for a two-fermion state} \end{cases} \tag{12.12}$$

Which particles are which? Particles that have half-integral spin— which includes the spin-1/2 electrons we've been talking about all this time— are fermions. Other fermions include protons, neutrons, quarks, and neutrinos. Particles with integral spin are bosons. Bosons include photons, pions, and the force carriers for the weak and strong nuclear forces.

How do you create a two-fermion state with a total $z$ component of angular momentum equal to zero? The most obvious first thing to guess is just to assign each particle angular momentum in a different direction, so that they cancel:

$$|\xi\rangle = |+z_1\rangle\,|-z_2\rangle \tag{12.13}$$

However, this state doesn't work! Why not? Consider the operation of the exchange operator on it:

$$\hat{P}_{12}\,|+z_1\rangle\,|-z_2\rangle = |+z_2\rangle\,|-z_1\rangle \tag{12.14}$$

We started with particle one having positive $z$-spin and particle 2 having negative $z$-spin. After the exchange, it's the other way around. However, this isn't the same state, nor is it a constant times the original state. On other words, this state is not an eigenstate of the exchange operator. Therefore, it's not a valid quantum state if particle 1 and particle 2 are indistinguishable particles (e.g. if they're two electrons).

A valid two-fermion spin state with total angular momentum zero would be:

$$|\xi\rangle = \frac{1}{\sqrt{2}}|+z_1\rangle\,|-z_2\rangle - \frac{1}{\sqrt{2}}|+z_2\rangle\,|-z_1\rangle \tag{12.15}$$

To verify that this works, let's try the exchange operator on this state:

$$\begin{aligned} \hat{P}_{12}|\xi\rangle &= \hat{P}_{12}\left(\frac{1}{\sqrt{2}}|+z_1\rangle\,|-z_2\rangle - \frac{1}{\sqrt{2}}|+z_2\rangle\,|-z_1\rangle\right) \\ &= \frac{1}{\sqrt{2}}\hat{P}_{12}\,|+z_1\rangle\,|-z_2\rangle - \frac{2}{\sqrt{2}}\hat{P}_{12}\,|+z_2\rangle\,|-z_1\rangle \\ &= \frac{1}{\sqrt{2}}|+z_2\rangle\,|-z_1\rangle - \frac{1}{\sqrt{2}}|+z_1\rangle\,|-z_2\rangle \\ &= -|\xi\rangle \end{aligned} \tag{12.16}$$

Sure enough, this state is an eigenstate of the exchange operator. What's more, the eigenvalue is −1, which is required for fermions. (If you're wondering about why we mess about with all of the $1/\sqrt{2}$ coefficients, we do that so that $|\xi\rangle$ is properly normalized. You can verify that this is the case, and indeed doing so would be good practice in doing algebra with bra and ket vector representations of multiple particle states.)