9.3: An Atom in a Cavity

Consider a two-level atom with bare energy eigenstates $|g\rangle$ and $|e\rangle$ and energy splitting $\hbar\omega_0$. The free Hamiltonian H_0 of the atom is given by

$$H_0 = \frac{1}{2}\hbar\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.34}$$

The atom interacts with an electromagnetic wave, and the interaction is approximately the coupling between the dipole moment $\hat{\mathbf{d}}$ associated with the $|g\rangle \leftrightarrow |e\rangle$ transition and the electric field **E**, leading to the interaction Hamiltonian

$$H_{\text{int}} = -\hat{\mathbf{d}} \cdot \mathbf{E}, \tag{9.35}$$

where $\hat{\mathbf{d}} = -e\hat{\mathbf{r}}$, and **E** is the classical, complex-valued electric field at the position of the atom. For an atom at the position **r** = 0, the electric field can be written as

$$\mathbf{E} = \mathscr{E}_0 \boldsymbol{\epsilon} e^{i\omega t} + \mathscr{E}_0 \boldsymbol{\epsilon}^* e^{-i\omega t}, \tag{9.36}$$

where $\mathscr{E}_0$ is the real amplitude of the electric field ϵ is the polarsation vector of the wave. The off-diagonal matrix elements of H_{int} are given by

$$\langle e | H_{\text{int}} | g\rangle = e\mathscr{E}_0 \langle e|\hat{\mathbf{r}}|g\rangle \cdot \epsilon e^{i\omega t} + \text{ c.c.}, \tag{9.37}$$

and c.c. denotes the complex conjugate. Since $\hat{\mathbf{r}}$ has odd parity, the diagonal matrix elements $\langle g | H_{\text{int}} | g\rangle$ and $\langle e | H_{\text{int}} | e\rangle$ vanish. When we define $\mathbf{r}_{eg} \equiv \langle e|\hat{\mathbf{r}}|g\rangle$, the total Hamiltonian becomes

$$H = \begin{pmatrix} \frac{1}{2}\hbar\omega_0 & e\mathscr{E}_0 \mathbf{r}_{eg}^* \cdot \left(\epsilon e^{i\omega t} + \epsilon^* e^{-i\omega t}\right) \\ e\mathscr{E}_0 \mathbf{r}_{eg} \cdot \left(\epsilon e^{i\omega t} + \epsilon^* e^{-i\omega t}\right) & -\frac{1}{2}\hbar\omega_0 \end{pmatrix}. \tag{9.38}$$

Using the **Rotating Wave Approximation** (see exercise 9.1), this Hamiltonian can be written as

$$H = \frac{\hbar}{2} \begin{pmatrix} v & \Omega^* \\ \Omega & -v \end{pmatrix}, \tag{9.39}$$

where we made the substitution

$$v = \omega - \omega_0 \quad \text{and} \quad \Omega = \frac{2e\mathscr{E}_0}{\hbar} \mathbf{r}_{eg} \cdot \epsilon. \tag{9.40}$$

We can use the standard matrix techniques in quantum mechanics to solve for the eigenvalues, the eigenstates, and the time evolution of the atom.

Next, we consider the situation where atom is placed inside a cavity of volume V, and the electric field in the cavity has angular frequency ω with wave vector k propagating in the z-direction. Assume that the length of the cavity is a multiple of $\lambda/2$, such that ω is a resonant cavity mode. The field is very weak, so that the classical description of **E** is no longer sufficient. In particular, the field is made of photons, i.e., identical bosons. Consequently, we need to express **E** in terms of bosonic creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, which create photons of frequency ω. Since the intensity of the field is proportional to both $\mathscr{E}_0^2$ and $\hat{a}^\dagger\hat{a}$, we expect the electric field to be proportional to the creation and annihilation operators themselves. Hermiticity requires that it is proportional to the sum of the creation and annihilation operators. Furthermore, the electric field is a transverse standing wave cavity mode and must vanish at the mirrors due to the boundary conditions imposed by Maxwell's equations. The spatial amplitude variation therefore includes a factor sin kz. For linear polarisation the operator form of **E** then becomes

$$\hat{\mathbf{E}}(z,t) = \epsilon \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) \sin kz, \tag{9.41}$$

where we assumed that ϵ is real[7]. The creation and annihilation operators are thus the amplitude operators of the field. Note that by the analogy with the harmonic oscillator, the electric field operator acts as a position operator of a particle in a harmonic potential well characterised by ω.

We again consider the dipole approximation of the atom in the field, and the Hamiltonian is written as

$$H_{\text{int}} = -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}} = e\hat{\mathbf{r}} \cdot \epsilon \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \sin kz \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right). \tag{9.42}$$

The operator $\hat{\mathbf{r}}$ can be written as

$$\hat{\mathbf{r}} = \mathbf{r}_{eg} |e\rangle \langle g| + \mathbf{r}_{eg}^* |g\rangle \langle e|, \tag{9.43}$$

and for notational simplicity, we define the coupling constant g as

$$g = e\mathbf{r}_{eg} \cdot \epsilon \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \sin kz. \tag{9.44}$$

We again calculate the matrix elements of H_{int} as before, but this time we write the operator in terms of $|g\rangle\langle e|$ and $|e\rangle\langle g|$:

$$H_{\text{int}} = g|e\rangle\langle g| \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right) + g^* |g\rangle\langle e| \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right). \tag{9.45}$$

It is convenient to express $|g\rangle\langle e|$ and $|e\rangle\langle g|$ in terms of the two-level raising and lowering operators σ_+ and σ_-:

$$\sigma_+ = |e\rangle\langle g| \quad \text{and} \quad \sigma_- = |g\rangle\langle e|, \tag{9.46}$$

with the commutation relation

$$[\sigma_+, \sigma_-] = 2\sigma_3 \quad \text{with} \quad \sigma_3 = |g\rangle\langle g| - |e\rangle\langle e| \tag{9.47}$$

The interaction Hamiltonian becomes

$$H_{\text{int}} = g\sigma_+ \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right) + g^* \sigma_- \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right). \tag{9.48}$$

Including the free Hamiltonian for the field and the two-level atom, this becomes in the Rotating Wave Approximation

$$H_{\text{JC}} = \frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\omega\hat{a}^\dagger\hat{a} + g\sigma_+\hat{a} + g^*\sigma_-\hat{a}^\dagger, \tag{9.49}$$

This is the Jaynes-Cummings Hamiltonian for a two-level atom with energy splitting $\hbar\omega_0$ interacting with a cavity mode of frequency ω. To achieve the strongest coupling g, the volume of the cavity should be small, and the atom should sit at an anti-node of the field.

Quantities are conserved when they commute with the Hamiltonian. We can identify two observables that satisfy this requirement, namely the number of electrons

$$\hat{P}_e = |g\rangle\langle g| + |e\rangle\langle e| = \mathbb{I}, \tag{9.50}$$

and the total number of excitations

$$\hat{N}_e = \hat{a}^\dagger\hat{a} + |e\rangle\langle e| \tag{9.51}$$

This means that the Hamiltonian will not couple states with different total excitations.

In a real system, the cavity will not be perfect, and the excited state of the atom will suffer from spontaneous emission into modes other than the cavity mode. This can be modelled by a Lindblad equation for the joint state ρ of the atom and the cavity mode. The Lindblad operator for a leaky cavity is proportional to the annihilation operator $\hat{a}$, with a constant of proportionality $\sqrt{\kappa}$ that denotes the leakage rate. The spontaneous emission of the atom is modelled by the Lindblad operator $\sqrt{\gamma}\sigma_-$. The Lindblad equation then becomes

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H_{\text{JC}}, \rho] + \gamma\sigma_-\rho\sigma_+ - \frac{\gamma}{2}\{\sigma_+\sigma_-, \rho\} + \kappa\hat{a}\rho\hat{a}^\dagger - \frac{\kappa}{2}\left\{\hat{a}^\dagger\hat{a}, \rho\right\}. \tag{9.52}$$

The research field of cavity quantum electrodynamics (or cavity QED) is devoted in a large part to solving this equation.

---

[7]This is true for linear polarisation. For elliptical polarisation ϵ will be complex. The subsequent derivation will be slightly modified (with more terms in H_{int}), but no extra technical or conceptual difficulties arise.