# 13.3: Free Particles and the de Broglie Wavelength

A particle is called a “free particle” if its potential is constant. That is, there are no potential energy wells or barriers anywhere. It’s simplest to choose that constant potential energy to be zero, as that reduces the Schrödinger equation to:

$$\hat{K}\psi(x) = E\psi(x) \tag{13.4}$$

(in the one-dimensional case). Solutions to this equation are called “plane-wave” solutions. They are states with definite momentum $p = E^2/2m$ (which is exactly what you’d expect if you compare momentum and kinetic energy in classical physics). However, their position is completely undetermined; there is equal probability for any $x$, which is what you’d expect for a state of definite momentum given the Heisenberg Uncertainty Principle. The functional form of $\psi(x)$ is just a standard wave:

$$\psi(x) = A\cos(2\pi x/\lambda) \pm iA\sin(2\pi x/\lambda) \tag{13.5}$$

where $A$ is a constant (a complex number) that normalizes the wave. The ± depends on whether the wave is moving to the right (i.e. momentum is in the $+x$ direction) or to the left (i.e. momentum is in the $-x$ direction). The normalization condition will only put a constraint on the absolute square of $A$, meaning that there will be many complex numbers that satisfy it. As such, there isn’t one single solution to this equation; however, all of the solutions do give the same predictions for measurable things such as the probability of finding the electron at a given spot. The value $\lambda$ that shows up in these equations is the wavelength; that is it’s the range of $x$ over which it takes the sine or the cosine to go through one complete cycle. Note that although $\psi(x)$ varies with space, the probability of finding $x$ at any given position, $|\langle x \mid \psi\rangle|^2$, does not! See Section 13.7 for more details about this.

In this case, the energy levels are not quantized. $E$ can be anything in the equation above. A different energy $E$ does correspond to a different wavelength in the plane wave represented by $\psi$. In these solutions, the energy $E$ of the particle is related to the wavelength $\lambda$ of the wave function by:

$$E = \frac{h^2}{2m\lambda^2} \tag{13.6}$$

It’s more traditional to express this wavelength, called the de Broglie wavelength, in terms of the momentum of the particle:

$$\lambda = \frac{h}{p} \tag{13.7}$$

You can get this equation directly from the previous equation by using the relationship $E = p^2/2m$, that results from the combination of kinetic energy $E = \frac{1}{2}mv^2$ and momentum $p = mv$. The constant $h$ here is a version of Plank’s Constant, related to $\hbar$ by $h = 2\pi\hbar$.

For example, what is the de Broglie wavelength of an electron moving at 1×10$^6$ m/s (a “typical” speed for an atomic electron)? We would plug the right numbers into this equation:

$$\begin{aligned} \lambda &= \frac{h}{p} = \frac{h}{mv} \\ &= \frac{6.626 \times 10^{-34}\ \text{kg m}^2\ \text{s}^{-1}}{\left(9.109 \times 10^{-31}\ \text{kg}\right)\left(1 \times 10^6\ \text{m s}^{-1}\right)} \\ &= 7 \times 10^{-10}\ \text{m} = 0.7\ \text{nm} \end{aligned} \tag{13.8}$$

For comparison, this is about 1/1000 the wavelength of visible red light.

Many of the physical effects peculiar to quantum mechanics show up as wave interference between different components of a wave function $\psi(x)$. All waves, including those that derive from classical physics (such as waves on a string, sound waves, or electromagnetic (i.e. light) waves), show interference effects. The fact that the wave function, this abstract mathematical object which is used to figure out things about the state of a particle, also shows interference effects is what we mean when we say that sometimes particles behave like waves. In general, the longer the wavelength of a wave (i.e. the larger $\lambda$ is), the easier it is to see interference effects. The de Broglie wavelength indicates that wavelength is inversely proportional to momentum. For a non-relativistic particle (which is implied here, as the Schr¨odinger equation assumes non-relativistic particles), $p = mv$. Thus, for particles moving at a given velocity, the larger $m$ is, the smaller $\lambda$ is. This is why it is so difficult to observe quantum interference effects for larger objects; the effective wavelength, and thus the typical separations that you’d need to see those effects, becomes tiny.