1.3: Hermitian and Unitary Operators

Next, we will consider two special types of operators, namely Hermitian and unitary operators. An operator A is Hermitian if and only if $A^{\dagger}=A$.

Lemma

An operator is Hermitian if and only if it has real eigenvalues: $A^{\dagger}=A \Leftrightarrow a_j \in \mathbb{R}$.

Proof

The eigenvalue equation of A implies that

$$A\left|a_j\right\rangle=a_j\left|a_j\right\rangle \Rightarrow\left\langle a_j\right| A^{\dagger}=a_j^*\left\langle a_j\right|, \tag{1.27}$$

which means that $\left\langle a_j|A| a_j\right\rangle=a_j$ and $\left\langle a_j\left|A^{\dagger}\right| a_j\right\rangle=a_j^*$. It is now straightforward to show that $A=A^{\dagger}$ implies $a_j=a_j^*$, or $a_j \in \mathbb{R}$. Conversely, $a_j \in \mathbb{R}$ implies $a_j=a_j^*$, and

$$\left\langle a_j|A| a_j\right\rangle=\left\langle a_j\left|A^{\dagger}\right| a_j\right\rangle \tag{1.28}$$

Let $|\psi\rangle=\sum_k c_k\left|a_k\right\rangle$. Then

$$\begin{aligned}\langle\psi|A| \psi\rangle & =\sum_j\left|c_j\right|^2\left\langle a_j|A| a_j\right\rangle=\sum_j\left|c_j\right|^2\left\langle a_j\left|A^{\dagger}\right| a_j\right\rangle=\sum_j\left|c_j\right|^2\left\langle a_j\left|A^{\dagger}\right| a_j\right\rangle \\ & =\left\langle\psi\left|A^{\dagger}\right| \psi\right\rangle\end{aligned} \tag{1.29}$$

for all $|\psi\rangle$, and therefore $A=A^{\dagger}$.

Next, we construct the exponent of an operator A according to $U=\exp (i c A)$. We have included the complex number c for completeness. At first sight, you may wonder what it means to take the exponent of an operator. Recall, however, that the exponent has a power expansion:

$$U=\exp (i c A)=\sum_{n=0}^{\infty} \frac{(i c)^n}{n!} A^n \tag{1.30}$$

The n^{th} power of an operator is straightforward: just multiply A n times with itself. The expression in Eq. (1.30) is then well defined, and the exponent is taken as an abbreviation of the power expansion. In general, we can construct any function of operators, as long as we can define the function in terms of a power expansion:

$$f(A)=\sum_{n=0}^{\infty} f_n A^n \tag{1.31}$$

This can also be extended to functions of multiple operators, but now we have to be very careful in the case where these operators do not commute. Familiar rules for combining normal functions no longer apply (see exercise 4b).

We can construct the adjoint of the operator U according to

$$U^{\dagger}=\left(\sum_{n=0}^{\infty} \frac{(i c)^n}{n!} A^n\right)^{\dagger}=\sum_{n=0}^{\infty} \frac{\left(-i c^*\right)^n}{n!} A^{\dagger n}=\exp \left(-i c^* A^{\dagger}\right) \tag{1.32}$$

In the special case where $A=A^{\dagger}$ and c is real, we calculate

$$U U^{\dagger}=\exp (i c A) \exp \left(-i c^* A^{\dagger}\right)=\exp (i c A) \exp (-i c A)=\exp [i c(A-A)]=\mathbb{I}, \tag{1.33}$$

since A commutes with itself. Similarly, $U^{\dagger} U=\mathbb{I}$. Therefore, $U^{\dagger}=U^{-1}$, and an operator with this property is called unitary. Each unitary operator can be generated by a Hermitian (self-adjoint) operator A and a real number c. A is called the generator of U. We often write $U=U_A(c)$. Unitary operators are basis transformations.