5.2: Entanglement

Consider the following experiment: Alice and Bob each blindly draw a marble from a vase that contains one black and one white marble. Let’s call the state of the write marble $|0\rangle$ and the state of the black marble $|1\rangle$. If we describe this classical experiment quantum mechanically (we can always do this, because classical physics is contained in quantum physics), then there are two possible states, $|1,0\rangle$ and $|0,1\rangle$. Since blind drawing is a statistical procedure, the state of the marbles held by Alice and Bob is the mixed state

$$\rho=\frac{1}{2}|0,1\rangle\left\langle 0,1\right|+\frac{1}{2}\left|1,0\right\rangle\langle 1,0| \tag{5.5}$$

From Alice’s perspective, the state of her marble is obtained by tracing over Bob’s marble:

$$\rho_A=\operatorname{Tr}_B(\rho)=\frac{1}{2}|0\rangle\left\langle 0\right|+\frac{1}{2}\left|1\right\rangle\langle 1| \tag{5.6}$$

This is what we expect: Alice has a 50:50 probability of finding “white” or “black” when she looks at her marble (i.e., when she measures the colour of the marble).

Next, consider what the state of Bob’s marble is when Alice finds a white marble. Just from the setup we know that Bob’s marble must be black, because there was only one white and one black marble in the vase. Let’s see if we can reproduce this in our quantum mechanical description. Finding a white marble can be described mathematically by a projection operator $|0\rangle\langle 0|$ (see Eq. (2.24)). We need to include this operator in the trace over Alice’s marble’s Hilbert space:

$$\rho_B=\frac{\operatorname{Tr}_A(|0\rangle_A\langle 0|\rho)}{\operatorname{Tr}(|0\rangle_A\langle 0|\rho)}=|1\rangle\langle 1|, \tag{5.7}$$

which we set out to prove: if Alice finds that when she sees that her marble is white, she describes the state of Bob’s marble as black. Based on the setup of this experiment, Alice knows instantaneously what the state of Bob’s marble is as soon as she looks at her own marble. There is nothing spooky about this; it just shows that the marbles held by Alice and Bob are correlated.

Next, consider a second experiment: By some procedure, the details of which are not important right now, Alice and Bob each hold a two-level system (a qubit) in the pure state

$$|\Psi\rangle_{AB}=\frac{|0,1\rangle+|1,0\rangle}{\sqrt{2}} \tag{5.8}$$

Since $|1,0\rangle$ and $|0,1\rangle$ are valid quantum states, by virtue of the first postulate of quantum mechanics $|\Psi\rangle_{AB}$ is also a valid quantum mechanical state. It is not difficult to see that these systems are also correlated in the states $|0\rangle$ and $|1\rangle$: When Alice finds the value “0”, Bob must find the value “1”, and vice versa. We can write this state as a density operator

$$\begin{aligned}\rho &=\frac{1}{2}(|0,1\rangle+|1,0\rangle)(\langle 0,1|+\langle 1,0|)\\ &=\frac{1}{2}(|0,1\rangle\langle 0,1|+|0,1\rangle\langle 1,0|+|1,0\rangle\langle 0,1|+|1,0\rangle\langle 1,0|).\end{aligned} . \tag{5.9}$$

Notice the two extra terms with respect to Eq. (5.5). If Alice now traces out Bob’s system, she finds that the state of her marble is

$$\rho_A=\operatorname{Tr}_B(\rho)=\frac{1}{2}|0\rangle\left\langle 0\right|+\frac{1}{2}\left|1\right\rangle\langle 1|. \tag{5.10}$$

In other words, even though the total system was in a pure state, the subsystem held by Alice (and Bob, check this) is mixed! We can try to put the two states back together:

$$\begin{aligned}\rho_A\otimes\rho_B &=\left(\frac{1}{2}|0\rangle\left\langle 0\right|+\frac{1}{2}\left|1\right\rangle\langle 1|\right)\otimes\left(\frac{1}{2}|0\rangle\left\langle 0\right|+\frac{1}{2}\left|1\right\rangle\langle 1|\right)\\ &=\frac{1}{4}(|0,0\rangle\langle 0,0|+|0,1\rangle\langle 0,1|+|1,0\rangle\langle 1,0|+|1,1\rangle\langle 1,1|)\end{aligned} , \tag{5.11}$$

but this is not the state we started out with! It is also a mixed state, instead of the pure state we started with. Since mixed states mean incomplete knowledge, there must be some information in the combined system that does not reside in the subsystems alone!

This is called entanglement.

Entanglement arises because states like $(|0,1\rangle+|1,0\rangle)/\sqrt{2}$ cannot be written as a tensor product of two pure states $|\psi\rangle\otimes|\phi\rangle$. These latter states are called separable. In general a state is separable if and only if it can be written as

$$\rho=\sum_j p_j\rho_j^{(A)}\otimes\rho_j^{(B)} \tag{5.12}$$

Classical correlations such as the black and white marbles above fall into the category of separable states.

So far, we have considered the quantum states in the basis $\{|0\rangle,|1\rangle\}$. However, we can also describe the same system in the rotated basis $\{|\pm\rangle\}$ according to

$$|0\rangle=\frac{|+\rangle+|-\rangle}{\sqrt{2}} \quad \text{and} \quad |1\rangle=\frac{|+\rangle-|-\rangle}{\sqrt{2}} \tag{5.13}$$

The entangled state $|\Psi\rangle_{AB}$ can then be written as

$$\frac{|0,1\rangle+|1,0\rangle}{\sqrt{2}}=\frac{|+,+\rangle-|-,-\rangle}{\sqrt{2}}, \tag{5.14}$$

which means that we have again perfect correlations between the two systems with respect to the states $|+\rangle$ and $|-\rangle$. Let's do the same for the state ρ in Eq. (5.5) for classically correlated marbles. After a bit of algebra, we find that

$$\begin{aligned}\rho=\ &\frac{1}{4}(|++\rangle\langle++|+|+-\rangle\langle+-|+|-+\rangle\langle-+|+|--\rangle\langle--|\\ &-|++\rangle\langle--|-|--\rangle\langle++|-|+-\rangle\langle-+|-|-+\rangle\langle+-|).\end{aligned} \tag{5.15}$$

Now there are no correlations in the conjugate basis $\{|\pm\rangle\}$, which you can check by calculating the conditional probabilities of Bob's state given Alice's measurement outcomes. This is another key difference between classically correlated states and entangled states. A good interpretation of entanglement is that entangled systems exhibit correlations that are stronger than classical correlations. We will shortly see how these stronger correlations can be used in information processing.

We have seen that operators, just like states, can be combined into tensor products:

$$A\otimes B|\phi\rangle\otimes|\psi\rangle=A|\phi\rangle\otimes B|\psi\rangle. \tag{5.16}$$

And just like states, some operators cannot be written as $A\otimes B$:

$$C=\sum_k A_k\otimes B_k \tag{5.17}$$

This is the most general expression of an operator in the Hilbert space $\mathscr{H}_1\otimes\mathscr{H}_2$. In Dirac notation this becomes

$$C=\sum_{jklm}\phi_{jklm}|\phi_j\rangle\langle\phi_k|\otimes|\phi_l\rangle\langle\phi_m|=\sum_{jklm}\phi_{jklm}|\phi_j,\phi_l\rangle\langle\phi_k,\phi_m|. \tag{5.18}$$

As an example, the Bell operator is diagonal on the Bell basis:

$$|\Phi^\pm\rangle=\frac{|0,0\rangle\pm|1,1\rangle}{\sqrt{2}} \text{ and } |\Psi^\pm\rangle=\frac{|0,1\rangle\pm|1,0\rangle}{\sqrt{2}}. \tag{5.19}$$

The eigenvalues of the Bell operator are not important, as long as they are not degenerate (why?). A measurement of the Bell operator projects onto an eigenstate of the operator, which is an entangled state. Consequently, we cannot implement such composite measurements by measuring each subsystem individually, because those individual measurements would project onto pure states of the subsystems. And we have seen that the subsystems of pure entangled states are mixed states.

A particularly useful technique when dealing with two systems is the so-called Schmidt decomposition. In general, we can write any pure state over two systems as a superposition of basis states:

$$|\Psi\rangle=\sum_{j=1}^{d_A}\sum_{k=1}^{d_B}c_{jk}|\phi_j\rangle_A|\psi_k\rangle_B, \tag{5.20}$$

where d_A and d_B are the dimensions of the Hilbert spaces of system A and B, respectively, and we order the systems such that $d_B \geq d_A$. It turns out that we can always simplify this description and write $|\Psi\rangle$ as a single sum over basis states. We state it as a theorem:

Theorem 5.2.1

Let $|\Psi\rangle$ be a pure state of two systems, A and B with Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ of dimension d_A and $d_B \geq d_A$, respectively. There exist orthonormal basis vectors $|a_j\rangle_A$ for system A and $|b_j\rangle_B$ for system B such that

$$|\Psi\rangle = \sum_j \lambda_j |a_j\rangle_A |b_j\rangle_B, \tag{5.21}$$

with real, positive Schmidt coefficients λ_j, and $\sum_j \lambda_j^2 = 1$. This decomposition is unique, and the sum runs at most to d_A, the dimension of the smallest Hilbert space. Traditionally, we order the Schmidt coefficients in descending order: $\lambda_1 \geq \lambda_2 \geq \ldots$ The total number of non-zero λ_i is the Schmidt number.

Proof

The proof can be found in many graduate texts on quantum mechanics and quantum information theory.

Given the Schmidt decomposition for a bi-partite system, we can immediately write down the reduced density matrices for the sub-systems:

$$\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \sum_j \lambda_j^2 |a_j\rangle_A \langle a_j|, \tag{5.22}$$

and

$$\rho_B = \mathrm{Tr}_A(|\Psi\rangle\langle\Psi|) = \sum_j \lambda_j^2 |b_j\rangle_B \langle b_j|. \tag{5.23}$$

The basis states $|a_j\rangle_A$ and $|b_j\rangle_B$ may have completely different physical meanings; here we care only that the states of the decomposition can be labelled with a single index, as opposed to two indices.

Conversely, when we have a single system in a mixed state

$$\rho = \sum_j p_j |a_j\rangle \langle a_j|, \tag{5.24}$$

we can always construct a pure state $|\Psi\rangle$ that obeys $\left(\lambda_j = \sqrt{p_j}\right)$

$$|\Psi\rangle = \sum_j \lambda_j |a_j, b_j\rangle, \tag{5.25}$$

By virtue of the Schmidt decomposition. The state $|\Psi\rangle$ is called the purification of ρ. Since many theorems are easier to prove for pure states than for mixed states, purifications can make our work load significantly lighter.

When there is more than one non-zero λ_j in Eq. (5.25), the state $|\Psi\rangle$ is clearly entangled: there is no alternative choice of λ_j due to the uniqueness of the Schmidt decomposition that would result in $\lambda_1' = 1$ and all others zero. Moreover, the more uniform the values of λ_j, the more the state is entangled. One possible measure for the amount of entanglement in $|\Psi\rangle$ is the Shannon entropy H.

$$H = -\sum_j \lambda_j^2 \log_2 \lambda_j^2 \tag{5.26}$$

This is identical to the von Neumann entropy S of the reduced density matrix ρ of $|\Psi\rangle$ given in Eq. (5.24):

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho) \tag{5.27}$$

Both entropies are measured in classical bits.

How do we find the Schmidt decomposition? Consider the state $|\Psi\rangle$ from Eq. (5.20). The (not necessarily square) matrix C with elements c_{jk} needs to be transformed into a single array of numbers λ_j. This is achieved by applying the singular-value decomposition:

$$c_{jk} = \sum_i u_{ji} d_{ii} v_{ik}, \tag{5.28}$$

where u_{ji} and v_{ik} are elements of unitary matrices U and V, respectively, and d_{ii} is a diagonal matrix with singular values λ_i. The vectors in the Schmidt decomposition become

$$|a_i\rangle = \sum_j u_{ji} |\phi_j\rangle \quad \text{and} \quad |b_i\rangle = \sum_k v_{ik} |\psi_k\rangle . \tag{5.29}$$

This is probably a good time to remind ourselves about the singular-value decomposition. All we need to do is find U and V, the rest is just matrix multiplication. To find U, we diagonalize $CC^\dagger$ and find its eigenvectors. These form the columns of U. Similarly, we diagonalize $C^\dagger C$ and arrange the eigenvectors in columns to find V. If C is an $n \times m$ matrix, U should be $n \times n$ and V should be $m \times m$.