5.3: Quantum teleportation

Probably the most extraordinary use of the quantum correlations present in entanglement is quantum teleportation. Alice and Bob share two entangled qubits, labelled 2 (held by Alice) and 3 (held by Bob), in the state $\left(|0,0\rangle_{23}+|1,1\rangle_{23}\right)/\sqrt{2}$. In addition, Alice holds a qubit in the

$$state|\psi\rangle_1=\alpha|0\rangle_1+\beta|1\rangle_1. \tag{5.30}$$

The object of quantum teleportation is to transfer the state of qubit 1 to qubit 3, without either Alice or Bob gaining any information about α or β. To make things extra hard, the three qubits must not change places (so Alice cannot take qubit 1 and bring it to Bob).

Classically, this is an impossible task: we cannot extract enough information about α and β with a single measurement to reproduce $|\psi\rangle$ faithfully, otherwise we could violate the no-cloning theorem. However, in quantum mechanics it can be done (without violating no-cloning). Write the total state as

$$|\chi\rangle=|\psi\rangle_1|\Phi^+\rangle_{23}=\frac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|011\rangle+\beta|100\rangle+\beta|111\rangle) \tag{5.31}$$

Alice now performs a Bell measurement on her two qubits (1 and 2), which project them onto one of the Bell states $|\Phi^\pm\rangle_{12}$ or $|\Psi^\pm\rangle_{12}$. We write $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ in the Bell basis:

$$\begin{aligned}
|00\rangle &= \frac{|\Phi^+\rangle+|\Phi^-\rangle}{\sqrt{2}} \\
|01\rangle &= \frac{|\Psi^+\rangle+|\Psi^-\rangle}{\sqrt{2}} \\
|10\rangle &= \frac{|\Psi^+\rangle-|\Psi^-\rangle}{\sqrt{2}} \\
|11\rangle &= \frac{|\Phi^+\rangle-|\Phi^-\rangle}{\sqrt{2}}.
\end{aligned} \tag{5.32}$$

We can use these substitutions to write the state $|\chi\rangle$ before the measurement as

$$\begin{aligned}
|\chi\rangle = &\frac{1}{2}\left[\alpha|\Phi^+\rangle_{12}|0\rangle_3+\alpha|\Phi^-\rangle_{12}|0\rangle_3+\alpha|\Psi^+\rangle_{12}|1\rangle_3+\alpha|\Psi^-\rangle_{12}|1\rangle_3\right. \\
&\left.+\beta|\Psi^+\rangle_{12}|0\rangle_3-\beta|\Psi^-\rangle_{12}|0\rangle_3+\beta|\Phi^+\rangle_{12}|1\rangle_3-\beta|\Phi^-\rangle_{12}|1\rangle_3\right] \\
= &\frac{1}{2}\left[|\Phi^+\rangle(\alpha|0\rangle+\beta|1\rangle)+|\Phi^-\rangle(\alpha|0\rangle-\beta|1\rangle)\right. \\
&\left.+|\Psi^+\rangle(\beta|0\rangle+\alpha|1\rangle)+|\Psi^-\rangle(\beta|0\rangle-\alpha|1\rangle)\right]
\end{aligned} \tag{5.33}$$

Alice finds one of four possible outcomes:

$$\begin{aligned}
\Phi^+: &\quad \mathrm{Tr}_{12}(|\Phi^+\rangle\langle\Phi^+||\chi\rangle\langle\chi|)\to|\psi\rangle_3=\alpha|0\rangle+\beta|1\rangle \\
\Phi^-: &\quad \mathrm{Tr}_{12}(|\Phi^-\rangle\langle\Phi^-||\chi\rangle\langle\chi|)\to|\psi\rangle_3=\alpha|0\rangle-\beta|1\rangle \\
\Psi^+: &\quad \mathrm{Tr}_{12}(|\Psi^+\rangle\langle\Psi^+||\chi\rangle\langle\chi|)\to|\psi\rangle_3=\alpha|1\rangle+\beta|0\rangle \\
\Psi^-: &\quad \mathrm{Tr}_{12}(|\Psi^-\rangle\langle\Psi^-||\chi\rangle\langle\chi|)\to|\psi\rangle_3=\alpha|1\rangle-\beta|0\rangle
\end{aligned} \tag{5.34}$$

From these outcomes, it is clear that the state held by Bob is different for the different measurement outcomes of Alice's Bell measurement. Let this sink in for a moment: After setting up the entangled state between Alice and Bob, who may be literally light years apart, Bob has done absolutely nothing to his qubit, yet its state is different depending on Alice's measurement outcome. This suggests that there is some instantaneous communication taking place, possibly violating causality!

In order to turn the state of Bob's qubit into the original state, Alice needs to send the measurement outcome to Bob. This will take two classical bits, because there are four outcomes. The correction operators that Bob need to apply are as follows:

$$\Phi^+:\mathbb{I},\quad \Phi^-:Z,\quad \Psi^+:X,\quad \Psi^-:ZX \tag{5.35}$$

So in each case Bob needs to do something different to his qubit. To appreciate how remarkable this protocol is, here are some of its relevant properties:

1. No matter is transported, only the state of the system;
2. neither Alice nor Bob learns anything about α or β;
3. any attempt to use quantum teleportation for signaling faster than light is futile!

? Exercises

1. Derive Eq. (5.15), and show that $\rho_B = \mathbb{I}/2$ when Alice's qubit is projected onto $|+\rangle$.
2. Quantum teleportation. Write the Bell states as

$$|\psi_{nm}\rangle = (|0, 0\oplus n\rangle + (-1)^m |1, 1\oplus n\rangle)/\sqrt{2},$$

   where ⊕ denotes addition modulo 2 and $n, m = 0, 1$.

   a. Write $|\psi_{nm}\rangle$ in terms of $|\psi_{00}\rangle$ and the Pauli operators X and Z acting on the second qubit.
   b. Using the shared Bell state $|\psi_{nm}\rangle$ between Alice and Bob, and the two-bit measurement outcome (j, k) for Alice's Bell measurement, determine the correction operator for Bob.
   c. We now generalize to N-dimensional systems. We define the N^2 entangled states

$$|\psi_{nm}\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi i j n/N} |j, j\oplus m\rangle\ ,$$

      where $n \oplus m = n + m \mod N$. Prove that this is an orthonormal basis.
   d. Give the teleportation protocol for the N-dimensional Hilbert space.
   e. What is Bob's state before he learns Alice's measurement outcome?
3. Imperfect measurements.
   a. A two-qubit system (held by Alice and Bob) is in the anti-symmetric Bell state $|\Psi^-\rangle$. Calculate the state of Bob's qubit if Alice measures her qubit in the state $|0\rangle$. Hint: write the measurement procedure as a partial trace over Alice's qubit.
   b. Now Alice's measurement is imperfect, and when her apparatus indicates "0", there was actually a small probability p that the qubit was projected onto the state $|1\rangle$. What is Bob's state?
   c. If Alice's (imperfect) apparatus has only two measurement outcomes, what will Bob's state be if she finds outcome "1"?