7.1: Orbital Angular Momentum

From classical physics we know that the orbital angular momentum of a particle is given by the cross product of its position and momentum

$$\mathbf{L}=\mathbf{r}\times\mathbf{p} \quad \text{or} \quad L_i=\epsilon_{ijk}r_jp_k, \tag{7.1}$$

where we used Einstein's summation convention for the indices. In quantum mechanics, we can find the operator for orbital angular momentum by promoting the position and momentum observables to operators. The resulting orbital angular momentum operator turns out to be rather complicated, due to a combination of the cross product and the fact that position and momentum do not commute. As a result, the components of orbital momentum do not commute with each other. When we use $[r_j,p_k]=i\hbar\delta_{jk}$, the commutation relation for the components of $\mathbf{L}$ becomes

$$[L_i,L_j]=i\hbar\epsilon_{ijk}L_k. \tag{7.2}$$

A set of relations like this is called an algebra, and the algebra here is called closed since we can take the commutator of any two elements L_i and L_j, and express it in terms of another element L_k. Another (simpler) closed algebra is $[x,p_x]=i\hbar\mathbb{I}$ and $[x,\mathbb{I}]=[p_x,\mathbb{I}]=0$.

Since the components of angular momentum do not commute, we cannot find simultaneous eigenstates for L_x, L_y, and L_z. We will choose one of them, traditionally denoted by L_z, and construct its eigenstates. It turns out that there is another operator, a function of all L_is, that commutes with any component L_j, namely $\mathbf{L}^2=L_x^2+L_y^2+L_z^2$. This operator is unique, in that there is no other operator that differs from L^2 in a nontrivial way and still commutes with all L_is. We can now construct simultaneous eigenvectors for L_z and $\mathbf{L}^2$.

Since we are looking for simultaneous eigenvectors for the square of the angular momentum and the z-component, we expect that the eigenvectors will be determined by two quantum numbers, l, and m. First, and without any prior knowledge, we can formally write down the eigenvalue equation for L_z as

$$L_z|l,m\rangle=m\hbar|l,m\rangle, \tag{7.3}$$

where m is some real number, and we included $\hbar$ to fit the dimensions of angular momentum. We will now proceed with the derivation of the eigenvalue equation for $\mathbf{L}^2$, and determine the possible values for l and m.

From the definition of L^2, we have $\mathbf{L}^2-L_z^2=L_x^2+L_y^2$, and

$$\left\langle l,m\left|\mathbf{L}^2-L_z^2\right|l,m\right\rangle=\left\langle l,m\left|L_x^2+L_y^2\right|l,m\right\rangle\geq 0 \tag{7.4}$$

The spectrum of L_z is therefore bounded by

$$l\leq m\leq l \tag{7.5}$$

for some value of l. We derive the eigenvalues of $\mathbf{L}^2$ given these restrictions. First, we define the ladder operators

$$L_\pm=L_x\pm iL_y \quad \text{with} \quad L_-=L_+^\dagger. \tag{7.6}$$

The commutation relations with L_z and $\mathbf{L}^2$ are

$$[L_z,L_\pm]=\pm\hbar L_\pm, \quad [L_+,L_-]=2\hbar L_z, \quad \left[L_\pm,\mathbf{L}^2\right]=0. \tag{7.7}$$

Next, we calculate $L_z\left(L_+|l,m\rangle\right)$:

$$\begin{aligned} L_z\left(L_+|l,m\rangle\right) &=\left(L_+L_z+[L_z,L_+]\right)|l,m\rangle=m\hbar L_+|l,m\rangle+\hbar L_+|l,m\rangle \\ &=(m+1)\hbar L_+|l,m\rangle. \end{aligned} \tag{7.8}$$

Therefore $L_+|l,m\rangle\propto|l,m+1\rangle$. By similar reasoning we find that $L_-|l,m\rangle\propto|l,m-1\rangle$. Since we already determined that $-l\leq m\leq l$, we must also require that

$$L_+|l,l\rangle=0 \quad \text{and} \quad L_-|l,-l\rangle=0. \tag{7.9}$$

Counting the states between $-l$ and $+l$ in steps of one, we find that there are $2l+1$ different eigenstates for L_z. Since $2l+1$ is a positive integer, l must be a half-integer $\left(l=0,\frac{1}{2},1,\frac{3}{2},2,\ldots\right)$. Later we will restrict this further to $l=0,1,2,\ldots$

The next step towards finding the eigenvalues of $\mathbf{L}^2$ is to calculate the following identity:

$$L_-L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 + i[L_x, L_y] = \mathbf{L}^2 - L_z^2 - \hbar L_z \tag{7.10}$$

We can then evaluate

$$L_-L_+|l,l\rangle = 0 \Rightarrow \left(\mathbf{L}^2 - L_z^2 - \hbar L_z\right)|l,l\rangle = \mathbf{L}^2|l,l\rangle - \left(l^2 + l\right)\hbar^2|l,l\rangle = 0 \tag{7.11}$$

It is left as an exercise (see exercise 1b) to show that

$$\mathbf{L}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle. \tag{7.12}$$

We now have derived the eigenvalues for L_z and $\mathbf{L}^2$.

One aspect of our algebraic treatment of angular momentum we still have to determine is the matrix elements of the ladder operators. We again use the relation between $L_\pm$, and L_z and $\mathbf{L}^2$:

$$\langle l,m\,|L_-L_+|\,l,m\rangle = \sum_{j=-l}^{l} \langle l,m\,|L_-|\,l,j\rangle\,\langle l,j\,|L_+|\,l,m\rangle\,. \tag{7.13}$$

Both sides can be rewritten as

$$\left\langle l,m\left|\mathbf{L}^2 - L_z^2 - \hbar L_z\right|l,m\right\rangle = \langle l,m\,|L_-|\,l,m+1\rangle\,\langle l,m+1\,|L_+|\,l,m\rangle\,, \tag{7.14}$$

where on the right-hand-side we used that only the $m+1$-term survives. This leads to

$$[l(l+1) - m(m+1)]\hbar^2 = |\langle l,m+1\,|L_+|\,l,m\rangle|^2. \tag{7.15}$$

The ladder operators then act as

$$L_+|l,m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\,|l,m+1\rangle, \tag{7.16}$$

and

$$L_-|l,m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\,|l,m-1\rangle. \tag{7.17}$$

We have seen that the angular momentum L is quantized, and that this gives rise to a discrete state space parameterized by the quantum numbers l and m. However, we still have to restrict the values of l further, as mentioned above. We cannot do this using only the algebraic approach (i.e., using the commutation relations for L_i), and we have to consider the spatial properties of angular momentum. To this end, we write L_i as

$$L_i = -i\hbar\epsilon_{ijk}\left(x_j\frac{\partial}{\partial x_k}\right), \tag{7.18}$$

which follows directly from the promotion of **r** and **p** in Eq. (7.1) to quantum mechanical operators. In spherical coordinates,

$$r = \sqrt{x^2+y^2+z^2}, \quad \phi = \arctan\left(\frac{y}{x}\right), \quad \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right), \tag{7.19}$$

the angular momentum operators can be written as

$$\begin{aligned}
L_x &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\\
L_y &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\\
L_z &= -i\hbar\frac{\partial}{\partial\phi},\\
\mathbf{L}^2 &= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].
\end{aligned} \tag{7.20}$$

The eigenvalue equation for L_z then becomes

$$L_z\psi(r,\theta,\phi) = -i\hbar\frac{\partial}{\partial\phi}\psi(r,\theta,\phi) = m\hbar\psi(r,\theta,\phi) \tag{7.21}$$

We can solve this differential equation to find that

$$\psi(r,\theta,\phi) = \zeta(r,\theta)e^{im\phi}. \tag{7.22}$$

A spatial rotation over 2π must return the wave function to its original value, because $\psi(r,\theta,\phi)$ must have a unique value at each point in space. This leads to $\psi(r,\theta,\phi+2\pi) = \psi(r,\theta,\phi)$ and

$$e^{im(\phi+2\pi)} = e^{im\phi}, \quad \text{or} \quad e^{2\pi im} = 1 \tag{7.23}$$

This means that m is an integer, which in turn means that l must be an integer also.