9.4: Outlook: Quantum Field Theory

We have surreptitiously introduced the basic elements of non-relativistic quantum field theory. Consider again the Heisenberg model, where we described a lattice of spins with nearestneighbour interactions. If we take the limit of the lattice constant $a \to 0$ we end up with a continuum of creation and annihilation operators for each point in space. This is a field.

Traditionally we construct a quantum field theory from harmonic oscillators at each point in space. To this end, we characterise a classical harmonic oscillator with mass M in terms of its displacement q and velocity $\dot{q}$. The equations of motion for the classical harmonic oscillator are the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{9.53}$$

where L is the Lagrangian

$$L = \frac{1}{2}M\dot{q}^2 - \frac{1}{2}Kq^2, \tag{9.54}$$

and K can be thought of as a spring constant. Substituting this L into Eq. (9.53) yields the familiar differential equations for the harmonic oscillator $\ddot{q} + \Omega^2 q = 0$, with $\Omega^2 = K/M$.

Next, we arrange N particles in a one-dimensional lattice of length L and lattice constant a, where $L = Na$. Each particle's displacement is coupled to the displacement of it's nearest neighbours by a spring with constant K. The equations of motion of this set of coupled particles is given by

$$\ddot{q}_n = \Omega^2\left[(q_{n+1} - q_n) + (q_{n-1} - q_n)\right]. \tag{9.55}$$

We take the limit of $a \to 0$ and $N \to \infty$ while keeping $L = Na$ fixed. Our variable $q_n(t)$ then becomes a field $u(x,t)$, and it takes only a few lines of algebra to show that

$$\ddot{u}(x,t) = a^2\Omega^2 u''(x,t) = v^2 u''(x,t) \quad \text{with} \quad \lim_{a\to 0} a\Omega = v. \tag{9.56}$$

This is a wave equation, and v is the velocity of the wave. We can generalise this immediately to three dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{9.57}$$

In quantum field theory, we consider only these waves as the field excitations, and ignore the underlying particle structure we used to arrive at this result. We have already done something similar when we considered the spin waves in the Heisenberg model. Note also that the finite speed of propagation of waves means that we can make the theory Lorentz invariant when $v = c$, the speed of light, and Eq. (9.57) becomes $\partial_\mu\partial^\mu u = 0$.

The wave equation is typically derived from a Lagrangian L, or in the case of a field theory, the Lagrangian density $\mathscr{L}$. A massless scalar field is described by the Lorentz-invariant Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right] = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi), \tag{9.58}$$

where for technical reasons we redefined $\phi = u/\sqrt{a}$. The dispersion relation for such a field is $c^2k^2 = \omega^2$, with k the wave number and ω the frequency of the wave. Similarly, a massive field is described by

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \tag{9.59}$$

The Euler-Lagrange equation for this Lagrangian density is the so-called Klein-Gordon equation

$$\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi = 0. \tag{9.60}$$

The mass term leads to a new dispersion relation

$$c^2\hbar^2k^2 - \hbar^2\omega^2 + m^2c^4 = 0, \tag{9.61}$$

and the group velocity for wave packets is

$$v_g = \frac{d\omega}{dk} = \frac{c}{\sqrt{1+\mu^2}} \quad \text{with} \quad \mu = \frac{mc}{\hbar k}. \tag{9.62}$$

For relativistic particles the momentum $\hbar k$ is much larger than the rest mass mc, and therefore v_g approaches c.

We can solve the Klein-Gordon equation formally by writing

$$\phi(\mathbf{r}, t) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} + a^*(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}\right), \tag{9.63}$$

where $a_{\mathbf{k}}$ is the complex amplitude of a wave with wave vector $\mathbf{k}$. The field is essentially a superposition of (non-interacting) eigenmodes labelled by $\mathbf{k}$, and we call this a free field. We can introduce interactions between the waves in the field by adding higher-order terms to $\mathscr{L}$ with coupling constants $v, \lambda, \ldots$

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{v}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 - \cdots \tag{9.64}$$

The solution to the Klein-Gordon equation is now no longer the correct solution to the new equations of motion, but when the interaction is reasonably weak we can use the solutions $\phi(\mathbf{r}, t)$ of the free field as a starting point in a perturbation expansion.

So far, everything in this section has been a classical treatment. In order to extend the theory to quantum mechanics we have to quantise the field. We achieve this by promoting the amplitudes in ϕ (and therefore ϕ itself) to operators that obey commutation of anti-commutation relations)

$$\hat{\phi}(\mathbf{r}, t) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} \left(\hat{a}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}} t} + \hat{a}^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_{\mathbf{k}} t}\right). \tag{9.65}$$

These are the creation and annihilation operators for excitations of the field. For the Klein-Gordon equation they obey the bosonic commutation relations

$$[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = \left[\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = 0 \quad \text{and} \quad \left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = \delta^3(\mathbf{k} - \mathbf{k}'). \tag{9.66}$$

The state of the field can then be written as a superposition of Fock states. The field $\hat{\phi}$ has now become an observable.

In quantum field theory, the excitations of the field are interpreted as particles. All fundamental particles like quarks, electrons, photons, and the Higgs boson are excitations of a corresponding field. So the excitations of the Higgs field are Higgs bosons, and the excitations of the electromagnetic field are photons. Spin$-\frac{1}{2}$ particles obey the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\hat{\psi} = 0, \tag{9.67}$$

where the γ^μ are 4×4 matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \tag{9.68}$$

and ψ is a four-dimensional vector field called a spinor field. The solution to the free Dirac field can be written as

$$\hat{\psi}(\mathbf{r}, t) = \sum_s \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} \left[\hat{b}_s(\mathbf{k}) u_s(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_{\mathbf{k}} t} + \hat{d}_s^\dagger(\mathbf{k}) v_s(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_{\mathbf{k}} t}\right], \tag{9.69}$$

where $s = \pm\frac{1}{2}\hbar$ is the spin value, and $u_s(\mathbf{k})$ and $v_s(\mathbf{k})$ are two spinors carrying the spin component of the field

$$u_s(\mathbf{k}) = \mathscr{N} \begin{pmatrix} \chi_s \\ \frac{\hbar c \sigma \cdot \mathbf{k}}{\hbar\omega + mc^2} \chi_s \end{pmatrix} \quad \text{and} \quad v_s(\mathbf{k}) = \mathscr{N} \begin{pmatrix} -\frac{\hbar c \cdot \sigma \cdot \mathbf{k}}{\hbar\omega + mc^2} \chi_s \\ \chi_s \end{pmatrix}, \tag{9.70}$$

where $\mathscr{N}$ is a normalisation constant, $\boldsymbol{\sigma}$ is a vector of Pauli matrices, and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.71}$$

The creation and annihilation operators $\hat{b}_s(\mathbf{k})$ and $\hat{d}_s^\dagger(\mathbf{k})$ obey anti-commutation relations:

$$\begin{aligned}
&\left\{\hat{b}_s(\mathbf{k}), \hat{b}_r\left(\mathbf{k}^{\prime}\right)\right\}=\left\{\hat{b}_s^{\dagger}(\mathbf{k}), \hat{b}_r^{\dagger}\left(\mathbf{k}^{\prime}\right)\right\}=0 \quad \text{and} \quad \left\{\hat{b}_s(\mathbf{k}), \hat{b}_r^{\dagger}\left(\mathbf{k}^{\prime}\right)\right\}=\delta_{rs} \delta^3\left(\mathbf{k}-\mathbf{k}^{\prime}\right), \\
&\left\{\hat{d}_s(\mathbf{k}), \hat{d}_r\left(\mathbf{k}^{\prime}\right)\right\}=\left\{\hat{d}_s^{\dagger}(\mathbf{k}), \hat{d}_r^{\dagger}\left(\mathbf{k}^{\prime}\right)\right\}=0 \quad \text{and} \quad \left\{\hat{d}_s(\mathbf{k}), \hat{d}_r^{\dagger}\left(\mathbf{k}^{\prime}\right)\right\}=\delta_{rs} \delta^3\left(\mathbf{k}-\mathbf{k}^{\prime}\right), \\
&\left\{\hat{b}_s(\mathbf{k}), \hat{d}_r\left(\mathbf{k}^{\prime}\right)\right\}=\left\{\hat{b}_s^{\dagger}(\mathbf{k}), \hat{d}_r^{\dagger}\left(\mathbf{k}^{\prime}\right)\right\}=0 \quad \text{and} \quad \left\{\hat{b}_s(\mathbf{k}), \hat{d}_r^{\dagger}\left(\mathbf{k}^{\prime}\right)\right\}=\left\{\hat{d}_s(\mathbf{k}), \hat{b}_r^{\dagger}\left(\mathbf{k}^{\prime}\right)\right\}=0.
\end{aligned} \tag{9.72}$$

This means that the excitations of the Dirac field are fermions with spin $\frac{1}{2}$, such as the electron. You see immediately that $\hat{\psi}$ is not Hermitian due to the appearance of $\hat{d}^\dagger$. This means that $\hat{\psi}$ is not an observable and we cannot think of the Dirac field as a quantised version of a classical observable field. There is no classical analog to the Dirac field. This is a consequence of the fact that the anti-particle of the Dirac excitations are not the same as the particle itself. E.g., the positron is different from the electron. Anti-particles are a quintessentially quantum mechanical phenomenon.

There is of course a lot more to quantum field theory than this. For example, the techniques for doing the perturbation expansion of interacting fields leads to Feynman diagrams, and renormalisation theory must be employed to deal with the infinities that crop up in the perturbation theory. Furthermore, one has to choose the right Lagrangian density, and principles such as gauge invariance and CPT invariance are imposed to constrain the possible choices. This leads ultimately to the extraordinary successful Standard Model of particle physics. It the most fundamental theory of Nature that we have, and it is tested to unprecedented accuracy.

? Exercises

1. The Hamiltonian for a two-level atom in the presence of an electromagnetic wave, as given in Eq. (9.38) depends on the time t. This makes it difficult to solve the Schrödinger equation, so in this exercise we will get rid of the time dependence by applying the Rotating Wave Approximation.
   a. If $|\psi'(t)\rangle = U(t)|\psi(t)\rangle$, find the Schrödinger equation for $|\psi'(t)\rangle$. What is the new Hamiltonian?
   b. Choose $U(t)$ such that
   $$U(t)=\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix},$$
   and H is given by Eq. (9.38). Show that the new Hamiltonian is given by Eq. (9.39).
2. A two-level atom is placed in a perfect cavity with an electromagnetic field of frequency ω.
   a. Show that the Jaynes-Cummings Hamiltonian can be written as a direct sum of 2×2 matrices H_n, and specify H_n.
   b. Diagonalize H_n to find the energy values of the system, and calculate the eigenstates.
   c. At $t=0$, the system is in the state $|\psi(0)\rangle = |e,n\rangle$. Calculate the state $|\psi(t)\rangle$ when the light is on resonance with the atomic transition $(\omega=\omega_0)$.
   d. Calculate the amount of entanglement between the atom and the cavity field. Use the relative entropy as the entanglement measure.
   e. How long does the atom need to reside in the cavity in order to achieve maximum entanglement?