7.3: Total Angular Momentum

In general, a particle may have both spin and orbital angular momentum. Since $\mathbf{L}$ and $\mathbf{S}$ have the same dimensions, we can ask what is the total angular momentum $\mathbf{J}$ of the particle. We write this as

$$\mathbf{J}=\mathbf{L}+\mathbf{S}\equiv\mathbf{L}\otimes\mathbb{I}+\mathbb{I}\otimes\mathbf{S}, \tag{7.46}$$

which emphasizes that orbital and spin angular momentum are described in distinct Hilbert spaces.

Since $[L_i, S_j]=0$, we have

$$\begin{aligned}[J_i, J_j] &= [L_i+S_i, L_j+S_j]=[L_i, L_j]+[S_i, S_j] \\ &= i\hbar\epsilon_{ijk}L_k+i\hbar\epsilon_{ijk}S_k=i\hbar\epsilon_{ijk}\left(L_k+S_k\right) \\ &= i\hbar\epsilon_{ijk}J_k\end{aligned} \tag{7.47}$$

In other words, $\mathbf{J}$ obeys the same algebra as $\mathbf{L}$ and $\mathbf{S}$, and we can immediately carry over the structure of the eigenvalues and eigenvectors from $\mathbf{L}$ and $\mathbf{S}$.

In addition, $\mathbf{L}$ and $\mathbf{S}$ must be added as vectors. However, only one of the components of the total angular momentum can be sharp (i.e., having a definite value). Recall that l and s are magnitudes of the orbital and spin angular momentum, respectively. We can determine the extremal values of $\mathbf{J}$, denoted by $\pm j$, by adding and subtracting the spin from the orbital angular momentum, as shown in Figure 3:

$$|l-s|\leq j\leq l+s. \tag{7.48}$$

For example, when $l=1$ and $s=\frac{1}{2}$, the possible values of j are $j=\frac{1}{2}$ and $j=\frac{3}{2}$.

The commuting operators for $\mathbf{J}$ are, first of all, $\mathbf{J}^2$ and J_z as we expect from the algebra, but also the operators $\mathbf{L}^2$ and $\mathbf{S}^2$. You may think that S_z and L_z also commute with these operators, but that it not the case:

$$\left[\mathbf{J}^2, L_z\right]=\left[(\mathbf{L}+\mathbf{S})^2, L_z\right]=\left[\mathbf{L}^2+2\mathbf{L}\cdot\mathbf{S}+\mathbf{S}^2, L_z\right]=2\left[\mathbf{L}, L_z\right]\cdot\mathbf{S}\neq 0 \tag{7.49}$$

We can construct a full basis for total angular momentum in terms of $\mathbf{J}^2$ and J_z, as before:

$$\mathbf{J}^2\left|j, m_j\right\rangle=\hbar^2 j(j+1)\left|j, m_j\right\rangle \text{ and } J_z\left|j, m_j\right\rangle=m_j\hbar\left|j, m_j\right\rangle. \tag{7.50}$$

Alternatively, we can construct spin and orbital angular momentum eigenstates directly as a tensor product of the eigenstates

$$\mathbf{L}^2|l, m\rangle\left|s, m_s\right\rangle=\hbar^2 l(l+1)|l, m\rangle\left|s, m_s\right\rangle \quad \text{and} \quad L_z|l, m\rangle\left|s, m_s\right\rangle=m\hbar|l, m\rangle\left|s, m_s\right\rangle, \tag{7.51}$$

and

$$\mathbf{S}^2|l, m\rangle\left|s, m_s\right\rangle=\hbar^2 s(s+1)|l, m\rangle\left|s, m_s\right\rangle \quad \text{and} \quad S_z|l, m\rangle\left|s, m_s\right\rangle=m_s\hbar|l, m\rangle\left|s, m_s\right\rangle. \tag{7.52}$$

Since the L_z and S_z do not commute with $\mathbf{J}^2$, the states $\left|j, m_j\right\rangle$ are not the same as the states $|l, m\rangle\left|s, m_s\right\rangle$.