8.2: Creation and Annihilation Operators

The second, and particularly powerful way to implement the description of identical particles is via creation and annihilation operators. To see how this description arises, consider some single-particle Hermitian operator A with eigenvalues a_j. On physical grounds, and regardless of distinguishability, we require that n_j particles in the eigenstate $|a_j\rangle$ of A must have a total physical value $n_j \times a_j$ for the observable A. We can repeat this for all eigenvalues a_j, and obtain a potentially infinite set of basis vectors

$$|n_1, n_2, n_3, \ldots\rangle,$$

for all integer values of n_j, including zero. You should convince yourself that this exhausts all the possible ways any number of particles can be distributed over the eigenvalues a_j. The spectrum of A can be bounded or unbounded, and discrete or continuous. It may even be degenerate. For simplicity we consider here an unbounded, non-degenerate discrete spectrum.

A special state is given by

$$|\varnothing\rangle = |0, 0, 0, \ldots\rangle \tag{8.2.1}$$

which indicates the state of no particles, or the vacuum. The numbers n_j are called the occupation number, and any physical state can be written as a superposition of these states:

$$|\Psi\rangle = \sum_{n_1, n_2, n_3, \ldots = 0}^{\infty} c_{n_1, n_2, n_3, \ldots} |n_1, n_2, n_3, \ldots\rangle. \tag{8.2.2}$$

The basis states $|n_1, n_2, n_3, \ldots\rangle$ span a linear vector space called a **Fock space** $\mathscr{F}$. It is the direct sum of the Hilbert spaces for zero particles $\mathscr{H}_0$, one particle $\mathscr{H}_1$, two particles, etc.:

$$\mathscr{F} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3 \oplus \cdots \tag{8.2.3}$$

Since $|\Psi\rangle$ is now a superposition over different particle numbers, we require operators that change the particle number. These are the **creation** and **annihilation** operators, $\hat{a}^\dagger$ and $\hat{a}$ respectively. Up to a proportionality constant that we will determine later, the action of these operators is defined by

$$\begin{aligned} \hat{a}_j^\dagger |n_1, n_2, \ldots, n_j, \ldots\rangle &\propto |n_1, n_2, \ldots, n_j + 1, \ldots\rangle \\ \hat{a}_j |n_1, n_2, \ldots, n_j, \ldots\rangle &\propto |n_1, n_2, \ldots, n_j - 1, \ldots\rangle \end{aligned}$$

So the operator $\hat{a}_j^\dagger$ creates a particle in a state with eigenvalue a_j, and the operator $\hat{a}_j$ removes a particle in a state with eigenvalue a_j. These operators are each others' Hermitian adjoint, since removing a particle is the time reversal of adding a particle. Clearly, when an annihilation operator attempts to remove particles that are not there, the result must be zero:

$$\hat{a}_j |n_1, n_2, \ldots, n_j = 0, \ldots\rangle = 0 \tag{8.2.4}$$

The vacuum is then defined as the state that gives zero when acted on by any annihilation operator: $\hat{a}_j|\varnothing\rangle = 0$ for any j. Notice how we have so far sidestepped the problem of particle swapping; we exclusively used aspects of the total particle number.

What are the basic properties of these creation and annihilation operators? In particular, we are interested in their commutation relations. We will now derive these properties from what we have determined so far. First, note that we can create two particles with eigenvalues a_i and a_j in the system in any order, and the only difference this can make is in the normalization of the state:

$$\hat{a}_i^\dagger \hat{a}_j^\dagger |\Psi\rangle = \lambda \hat{a}_j^\dagger \hat{a}_i^\dagger |\Psi\rangle, \tag{8.2.5}$$

where λ is some complex number. Since state $|\Psi\rangle$ is certainly not zero, we require that

$$\hat{a}_k^\dagger \hat{a}_l^\dagger - \lambda \hat{a}_l^\dagger \hat{a}_k^\dagger = 0. \tag{8.2.6}$$

Since k and l are just dummy variables, we equally have

$$\hat{a}_l^\dagger \hat{a}_k^\dagger - \lambda \hat{a}_k^\dagger \hat{a}_l^\dagger = 0. \tag{8.2.7}$$

We now substitute Eq. (8.15) into Eq. (8.14) to eliminate $\hat{a}_l^\dagger \hat{a}_k^\dagger$. This leads to

$$\left(1 - \lambda^2\right) \hat{a}_k^\dagger \hat{a}_l^\dagger = 0, \tag{8.2.8}$$

and therefore

$$\lambda = \pm 1. \tag{8.2.9}$$

The relation between different creation operators can thus take two forms. They can obey a commutation relation when $\lambda = +1$:

$$\hat{a}_l^\dagger \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_l^\dagger = \left[\hat{a}_l^\dagger, \hat{a}_k^\dagger\right] = 0, \tag{8.2.10}$$

or they can obey an anti-commutation relation when $\lambda = -1$:

$$\hat{a}_l^\dagger \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_l^\dagger = \left\{\hat{a}_l^\dagger, \hat{a}_k^\dagger\right\} = 0 \tag{8.2.11}$$

While creating the particles in different temporal order is not the same as swapping two particles, it should not come as a surprise that there are two possible situations (the commutation relation and the anti-commutation relation). We encountered two possibilities in our previous approach as well, where we found that many-particle states are either symmetric or anti-symmetric. In fact, creation operators that obey the commutation relation produce symmetric states, while creation operators that obey the anti-commutation relation produce anti-symmetric states. We also see that the creation operators described by the anti-commutation relations naturally obey Pauli's exclusion principle. Suppose that we wish to create two identical particles in the same eigenstate $|a_j\rangle$. The anti-commutation relations say that $\left\{\hat{a}_j^\dagger, \hat{a}_j^\dagger\right\} = 0$, so

$$\hat{a}_j^{\dagger 2} = 0. \tag{8.2.12}$$

Any higher powers of $\hat{a}_j^\dagger$ will also be zero, and we can create at most one particle in the state $|a_j\rangle$.

Taking the adjoint of the commutation relations for the creation operators gives us the corresponding relations for the annihilation operators

$$\hat{a}_l \hat{a}_k - \hat{a}_k \hat{a}_l = [\hat{a}_l, \hat{a}_k] = 0, \tag{8.2.13}$$

or

$$\hat{a}_l \hat{a}_k + \hat{a}_k \hat{a}_l = \{\hat{a}_l, \hat{a}_k\} = 0. \tag{8.2.14}$$

The remaining question is now what the (anti-) commutation relations are for products of creation and annihilation operators.

We proceed along similar lines as before. Consider the operators $\hat{a}_j$ and $\hat{a}_k^\dagger$ with $j \neq k$, and apply them in different orders to a state $|\Psi\rangle$.

$$\hat{a}_i \hat{a}_j^\dagger |\Psi\rangle = \mu \hat{a}_j^\dagger \hat{a}_i |\Psi\rangle. \tag{8.2.15}$$

The same argumentation as before leads to $\mu = \pm 1$. For different j and k we therefore find

$$\left[\hat{a}_j, \hat{a}_k^\dagger\right] = 0 \quad \text{or} \quad \left\{\hat{a}_j, \hat{a}_k^\dagger\right\} = 0 \tag{8.2.16}$$

Now let's consider the case $j = k$. For the special case where $|\Psi\rangle = |\varnothing\rangle$, we find

$$\left(\hat{a}_j \hat{a}_k^\dagger - \mu \hat{a}_j^\dagger \hat{a}_k\right) |\varnothing\rangle = \hat{a}_j \hat{a}_k^\dagger |\varnothing\rangle = \delta_{jk} |\varnothing\rangle, \tag{8.2.17}$$

based on the property that $\hat{a}_j |\varnothing\rangle = 0$. When $l = k$,

$$\left(\hat{a}_k \hat{a}_k^\dagger - \mu \hat{a}_k^\dagger \hat{a}_k\right) |\varnothing\rangle = |\varnothing\rangle, \tag{8.2.18}$$

we find for the two possible values of μ

$$\hat{a}_k \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_k = 1 \quad \text{or} \quad \hat{a}_k \hat{a}_k^\dagger + \hat{a}_k^\dagger \hat{a}_k = 1 \tag{8.2.19}$$

which is equivalent to

$$\left[\hat{a}_k, \hat{a}_k^\dagger\right] = 1 \quad \text{or} \quad \left\{\hat{a}_k, \hat{a}_k^\dagger\right\} = 1. \tag{8.2.20}$$

To summarise, we have two sets of algebras for the creation and annihilation operators. The algebra in terms of the commutation relations is given by

$$[\hat{a}_k, \hat{a}_l] = \left[\hat{a}_k^\dagger, \hat{a}_l^\dagger\right] = 0 \quad \text{and} \quad \left[\hat{a}_k, \hat{a}_l^\dagger\right] = \delta_{kl}. \tag{8.2.21}$$

This algebra describes particles that obey Bose-Einstein statistics, or bosons. The algebra in terms of anti-commutation relations is given by

$$\{\hat{a}_k, \hat{a}_l\} = \left\{\hat{a}_k^\dagger, \hat{a}_l^\dagger\right\} = 0 \quad \text{and} \quad \left\{\hat{a}_k, \hat{a}_l^\dagger\right\} = \delta_{kl}. \tag{8.2.22}$$

This algebra describes particles that obey Fermi-Dirac statistics, or fermions.

Finally, we have to determine the constant of proportionality for the creation and annihilation operators. We have already required that $\hat{a}_j \hat{a}_k^\dagger |\varnothing\rangle = \delta_{jk} |\varnothing\rangle$. To determine the rest, we consider a new observable that gives us the total number of particles in the system. We denote this observable by $\hat{n}$, and we see that it must be additive over all particle numbers for the different eigenvalues of A:

$$\hat{n} = \sum_j \hat{n}_j, \tag{8.2.23}$$

where $\hat{n}_j$ is the number of particles in the eigenstate $|a_j\rangle$. The total number of particles does not change if we consider a different observable (although the distribution typically will), so this relation is also true when we count the particles in the states $|b_j\rangle$. Pretty much the only way we can achieve this is to choose

$$\hat{n} = \sum_j \hat{n}_j = \sum_j \hat{a}_j^\dagger \hat{a}_j = \sum_j \hat{b}_j^\dagger \hat{b}_j. \tag{8.2.24}$$

For the case of n_j particles in state $|a_j\rangle$ this gives

$$\hat{a}_j^\dagger \hat{a}_j |n_j\rangle = n_j |n_j\rangle, \tag{8.2.25}$$

where we ignored the particles in other states $|a_k\rangle$ with $k \neq j$ for brevity. For the Bose-Einstein case this leads to the relations

$$\hat{a}_j |n_j\rangle = \sqrt{n_j} |n_j - 1\rangle \quad \text{and} \quad \hat{a}_j^\dagger |n_j\rangle = \sqrt{n_j + 1} |n_j + 1\rangle. \tag{8.2.26}$$

For Fermi-Dirac statistics, the action of the creation and annihilation operators on number states becomes

$$\begin{aligned} \hat{a}_j |0\rangle_j &= 0 & \text{and} \quad \hat{a}_j^\dagger |0\rangle_j &= e^{-i\alpha} |1\rangle_j, \\ \hat{a}_j |1\rangle_j &= e^{i\alpha} |0\rangle_j & \text{and} \quad \hat{a}_j^\dagger |1\rangle_j &= 0. \end{aligned} \tag{8.2.27}$$

The phase factor $e^{i\alpha}$ can be chosen ± 1.