12.2: Notating Multiple Particle States

Before we go further, we need to refine our notation so that we can keep track of two different particles. We can construct a two-particle state by putting together two states for each individual particle with:

$$|\psi_1\rangle \otimes |\phi_2\rangle \tag{12.3}$$

The ⊗ operator indicates that we're putting these two states together to form a composite state. It's sometimes called a "direct product", but it's not really all that much like multiplication. Really, it just means that we're making some composed state that combines particle 1 in state $|\psi\rangle$ and particle 2 in state $|\phi\rangle$. The subscript indicates which particle we're talking about; the rest of the stuff inside the ket indicates the state of that particular particle.

For simplicity, we will often omit the ⊗ symbol in the "direct product", and just write the two states next to each other, e.g.

$$|\psi_1\rangle |\phi_2\rangle \tag{12.4}$$

Again, this does not mean that we're multiplying two ket vectors, which is something we can't do. Instead, it means that we're composing the states. If these were spin states, we would not represent this with two column vectors. Instead, we'd represent it with a single four-row column vector; the first two rows have the column vector representation of whatever state the first particle is in, and the second two rows have the column vector representation of whatever state the second particle is in.

If an operator operates on this state, it only affects the state for the particle it is an operator for. That is, if "spin-z for particle 2" is the observable we're talking about, then the operator $\hat{S}_{z2}$ only operates on (in this example) the state $|\phi_2\rangle$. Indeed, you can treat $|\phi_1\rangle$ as if it were a constant:

$$\hat{S}_{z2} |\psi_1\rangle |\phi_2\rangle = |\psi_1\rangle \hat{S}_{z2} |\phi_2\rangle \tag{12.5}$$

As an example, suppose that particle 1 is in the state $|+z\rangle$ and particle 2 is in the state $|-z\rangle$. If we apply the $\hat{S}_{z2}$ operator to this state, we get:

$$\begin{aligned} \hat{S}_{z2} |+z_1\rangle |-z_2\rangle &= |+z_1\rangle \hat{S}_{z2} |-z_2\rangle \\ &= |+z_1\rangle \left(\frac{-\hbar}{2}\right) |-z_2\rangle \\ &= \left(-\frac{\hbar}{2}\right) |+z_1\rangle |-z_2\rangle \end{aligned} \tag{12.6}$$

Here, we have taken advantage of the fact that $|-z_2\rangle$ is an eigenstate of $\hat{S}_{z2}$, and replaced the action of the operator with a simple multiplication by the eigenvalue.

There will be some operators (e.g. the forthcoming exchange operator) that don't operate on just one of the two particles, but on both at the same time.

Similarly, with inner products, bra versions of a state only "stick" to ket versions of a state on the straight side of the bra-ket notation if they are states for the same particle. Thus, suppose we had a composite state:

$$|\xi\rangle = |\psi_1\rangle |\phi_2\rangle \tag{12.7}$$

The corresponding bra vector is:

$$\langle\xi| = \langle\psi_1| \langle\phi_2| \tag{12.8}$$

Normalization of this state is then expressed as:

$$\begin{aligned} \langle\xi \mid \xi\rangle &= (\langle\psi_1 | \langle\phi_2|)(|\psi_1\rangle |\phi_2\rangle) \\ &= \langle\psi_1 \mid \psi_1\rangle \langle\phi_2 \mid \phi_2\rangle \\ &= 1 \end{aligned} \tag{12.9}$$

We've rearranged states here a bit. We moved the $|\psi_1\rangle$ from after the $\langle\phi_2|$ to before it. This should make you a little nervous; we've seen that with matrices and other things that aren't simple numbers, multiplication is not necessarily commutative. However, again, in this case, when it comes to inner products, a state for a different particle can be treated as a constant with respect to inner products for the first particle. As such, it's entirely legitimate to move $|\psi_1\rangle$ into, out of, and through inner products on particle 2 (at least in the case of the simple composed states we're talking about here).