11.2: Non-Commuting Operators

In the previous section, we saw that if a particle can be in a definite state for two observables, then the two operators associated with those observables will commute. The converse is therefore also true; if two operators do not commute, then it is not possible for a quantum state to have a definite value of the corresponding two observables at the same time.

We've already seen examples of this. A particle can't have a definite x spin and a definite y spin at the same time. If our theory is to be useful, then we would hope that $\hat{S}_x$ and $\hat{S}_y$ would not commute when they operate on a general normalized state $|\psi\rangle$. Let's try it first in one order:

$$\begin{aligned}\hat{S}_x\hat{S}_y|\psi\rangle &= \frac{\hbar^2}{4}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\\ &= \frac{\hbar^2}{4}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}-i\psi_2\\ i\psi_1\end{bmatrix}\\ &= \frac{\hbar^2}{4}\begin{bmatrix}i\psi_1\\ -i\psi_2\end{bmatrix}\\ &= i\frac{\hbar^2}{4}\begin{bmatrix}\psi_1\\ -\psi_2\end{bmatrix}\end{aligned} \tag{11.4}$$

Now let's try it in the other order:

$$\begin{aligned}\hat{S}_y\hat{S}_x|\psi\rangle &= \frac{\hbar^2}{4}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\\ &= \frac{\hbar^2}{4}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\psi_2\\ \psi_1\end{bmatrix}\\ &= \frac{\hbar^2}{4}\begin{bmatrix}-i\psi_1\\ i\psi_2\end{bmatrix}\\ &= -i\frac{\hbar^2}{4}\begin{bmatrix}\psi_1\\ -\psi_2\end{bmatrix}\end{aligned} \tag{11.5}$$

Clearly the two are not the same; one is the negative of the other. Therefore, $\hat{S}_x$ and $\hat{S}_y$ do not commute when operating on a general state ψ, as expected.

It is interesting to note the effect of $\hat{S}_z$ on this same general state:

$$\begin{aligned}\hat{S}_z|\psi\rangle &= \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\\ &= \frac{\hbar}{2}\begin{bmatrix}\psi_1\\ -\psi_2\end{bmatrix}\end{aligned} \tag{11.6}$$

Notice that except for the constant out front, the vector produced by $\hat{S}_z$ on this state is the same as the vector produced by $\hat{S}_x\hat{S}_y$ and $\hat{S}_y\hat{S}_x$. In fact, we can put the two together:

$$\begin{aligned}\left(\hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x\right)|\psi\rangle &= i\frac{\hbar^2}{2}|\psi\rangle\\ \left[\hat{S}_x, \hat{S}_y\right]|\psi\rangle &= i\hbar\hat{S}_z|\psi\rangle\end{aligned} \tag{11.7}$$

The term in brackets, $\left[\hat{S}_x, \hat{S}_y\right]$ is called the commutator of $\hat{S}_x$ and $\hat{S}_y$. It's defined by the term in parentheses above it: $\left(\hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x\right)$. It works out for the commutators of all three spin angular momentum operators that:

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z \tag{11.8}$$

$$\begin{aligned} \left[\hat{S}_y, \hat{S}_z\right] &= i\hbar\hat{S}_x \\ \left[\hat{S}_z, \hat{S}_x\right] &= i\hbar\hat{S}_y \end{aligned} \tag{11.9}$$