11.1: Eigenstates and Commuting Operators

You are probably used to the idea that multiplication is commutative. That is, if you have a product ab where a and b are scalars, you can write the multiplication in either order (ab or ba), and the product is exactly the same. This is not necessarily the case for matrix multiplication! If A and B are matrices, then $AB \neq BA$ in general. Sometimes it will be true, but not always. Because we can use matrices to represent operators in quantum mechanics, this means that operators don't commute in general. That is, for example, $\hat{S}_x \hat{S}_y |\psi\rangle \neq \hat{S}_y \hat{S}_x |\psi\rangle$.

Sometimes, however, operators do commute. Suppose that you have two observables A and B with corresponding operators $\hat{A}$ and $\hat{B}$. Suppose also that you have a state $|\phi\rangle$ that is a definite state for both A and B. That means that in our mathematical formalism, $|\phi\rangle$ must be an eigenvector for both $\hat{A}$ and $\hat{B}$:

$$\begin{aligned} \hat{A}|\phi\rangle &= a|\phi\rangle \\ \hat{B}|\phi\rangle &= b|\phi\rangle \end{aligned} \tag{11.1}$$

Here, a and b are the eigenvalues for $\hat{A}$ and $\hat{B}$ respectively. In other words, $|\phi\rangle$ has a definite value of observable A, and a is that value; likewise, it has a definite value of observable B, and b is that value.

Let us now consider the application of both of these operators to this state $|\phi\rangle$:

$$\begin{aligned} \hat{A}\hat{B}|\phi\rangle &= \hat{A}b|\phi\rangle \\ &= b\hat{A}|\phi\rangle \\ &= ba|\phi\rangle \\ &= ab|\phi\rangle \end{aligned} \tag{11.2}$$

where in the last step we've used the fact that a and b are real numbers, so the product of the two of them does in fact commute. Let's now try this in the other order:

$$\begin{aligned} \hat{B}\hat{A}|\phi\rangle &= \hat{B}a|\phi\rangle \\ &= a\hat{B}|\phi\rangle \\ &= ab|\phi\rangle \end{aligned} \tag{11.3}$$

Here, we can see that in fact the operators $\hat{A}$ and $\hat{B}$ do commute if they are operating on a state that is an eigenstate for both operators.

Remember that in the case of spin, we argued that $|+z\rangle$ and $|-z\rangle$ form a complete basis set of vectors; that is, any spin state $|\psi\rangle$ can be written as a sum of scalar constants times those two vectors. In general, a complete set of eigenvectors for a given operator do form a basis set that can be used to construct any vector that is part of the overall scheme that that operator is part of. (For instance, the projection of spin along all three axes are part of the same scheme, as they are all the same kinds of states— that is, spin angular momentum states.) Therefore, we can write any state $|\psi\rangle$ as a sum of constants times the eigenvectors for that operator. If $\hat{A}$ and $\hat{B}$ are two operators that share the same eigenvectors, then $\hat{A}\hat{B}|\psi\rangle = \hat{B}\hat{A}|\psi\rangle$. That is, the operation of these two operators on any state commutes. For that reason, we generally just say that the operators commute.