9.2 Spin Waves in Solids

Consider a system of spins with a nearest-neighbour interaction. For a uniform interaction in all directions, this is described by the Hamiltonian

$$H = \pm J \sum_{(i,j)} \mathbf{S}_i \cdot \mathbf{S}_j = \pm J \sum_{(i,j)} S_{z,i} S_{z,j} + \frac{1}{2}\left(S_{+,i} S_{-,j} + S_{-,i} S_{+,j}\right). \tag{9.17}$$

where $J > 0$ is the coupling strength between the spins, $\sum_{(i,j)}$ is the sum over all neighboring pairs, and $S_\pm = S_x \pm i S_y$. The physics described by this Hamiltonian is known as the Heisenberg model. The sign of the coupling (here made explicit) determines whether the spins want to lign up in parallel $(-J)$ or antiparallel $(+J)$. The former situation describes **ferromagnets**, while the later describes **anti-ferromagnets**. The spin operators for different sites $(i \neq j)$ commute with each other, while the spin operators at the same site $(i = j)$ obey the spin algebra of Eq. (7.24).

Both systems have a well-defined ground state. For the ferromagnet this is the tensor product of the ground state of each individual spin. We are interested in the behaviour of the excitations with respect to this ground state. Due to the large degeneracy in the system (all the spins are of the same species with the same coupling J) the excitations act as identical quasi-particles. Consequently we can describe them using creation and annihilation operators. It turns out that they behave like bosons. You can think of an excitation as a higher spin value at some site that propagates to its neighbors due to the interaction. This is called a spin wave.

Suppose that the spins are aligned in the positive z-direction (so we consider $-J$), and S_z has the maximum eigenvalue s. When the spin is lowered by $\hbar$, this creates an excitation in the system, because the spin is no longer lined up. So the z-component of the spin at site j is given by

$$S_{z,j} = s - \hat{a}_j^\dagger \hat{a}_j, \tag{9.18}$$

where $\hat{a}_j^\dagger \hat{a}_j$ is the operator for the number of excitations at site j. Since $S_\pm$ raise and lower the eigenvalue of S_z, we expect that $S_+ \propto$ and $S_- \propto \hat{a}^\dagger$. When we insist on the commutation relation $[S_+, S_-] = 2S_z$, they become

$$S_{+,j} = \left(2s - \hat{a}_j^\dagger \hat{a}_j\right)^{\frac{1}{2}} \hat{a}_j \quad \text{and} \quad S_{-,j} = \left(2s - \hat{a}_j^\dagger \hat{a}_j\right)^{\frac{1}{2}} \hat{a}_j^\dagger. \tag{9.19}$$

This is known as the Holstein-Primakoff transformation.

For small numbers, the operators $S_\pm$ can be approximated as

$$S_{+,j} \simeq \sqrt{2s}\,\hat{a}_j \quad \text{and} \quad S_{-,j} \simeq \sqrt{2s}\,\hat{a}_j^\dagger. \tag{9.20}$$

This allows us to write the Heisenberg Hamiltonian of Eq. (9.17) with $-J$ to lowest order as

$$H = -J \sum_{(i,j)} \left[s^2 + s\left(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger - \hat{a}_i^\dagger \hat{a}_i - \hat{a}_j^\dagger \hat{a}_j\right)\right]. \tag{9.21}$$

For a simple cubic lattice of side L, lattice constant a and total number of spins $N = (L/a)^3$ we expect the spin waves to have wave vectors

$$\mathbf{k} = \frac{2\pi}{L}(m, n, o) \quad \text{with} \quad m, n, o \in \mathbb{N}, \tag{9.22}$$

and $1 \leq m, n, o \leq L$. The spin sites must now be indicated by a vector $\mathbf{r}$ instead of a single number j, and the Fourier transformation of $\hat{a}_\mathbf{r}^\dagger$ and $\hat{a}_\mathbf{r}$ is given by

$$\hat{a}_\mathbf{r} = \frac{1}{\sqrt{N}} \sum_\mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{a}_\mathbf{k} \quad \text{and} \quad \hat{a}_\mathbf{r}^\dagger = \frac{1}{\sqrt{N}} \sum_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{a}_\mathbf{k}^\dagger, \tag{9.23}$$

which transforms the Heisenberg Hamiltonian to

$$
\begin{aligned}
H &= -3Js^2N - \frac{Js}{N}\sum_{\mathbf{r},\mathbf{dk},\mathbf{k}'} e^{i\mathbf{r}(\mathbf{k}-\mathbf{k}')}\left(e^{i\mathbf{d}\cdot\mathbf{q}}-1\right)\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}'} \\
&= -3Js^2N - \frac{Js}{N}\sum_{\mathbf{k}}\epsilon(\mathbf{k})\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}},
\end{aligned}
\tag{9.24}
$$

where $\mathbf{r}$ is the position of a lattice site, $\mathbf{d}$ is the vector from a site to its nearest neighbours, which takes care of the sum over nearest neighbours. This is a diagonal Hamiltonian with eigenenergies

$$\epsilon(\mathbf{k}) = 2J_s\left(3-\cos k_x a - \cos k_y a - \cos k_z a\right). \tag{9.25}$$

This is the dispersion relation for the spin waves, and to lowest order $\left(\cos x \simeq 1-\frac{1}{2}x^2\right)$ it is quadratic:

$$\epsilon(\mathbf{k}) = Jsa^2k^2. \tag{9.26}$$

Spin waves are important when we want to manipulate magnetic properties with high frequency, such as in microwave devices. They carry energy, and are therefore a mechanism for dissipation.

For the case of anti-ferromagnets $(+J)$, the ground state is harder to find. Consider an antiferromagnet that is again a simple cubic lattice with alternating spin $\pm s$ and lattice constant a. We can think of this lattice as two sub-lattices with constant spin, and redefine the spins on the $-s$ sub-lattice according to

$$S_x \rightarrow -S_x, \quad S_y \rightarrow S_y, \quad \text{and} \quad S_z \rightarrow -S_z. \tag{9.27}$$

These operators still obey the commutation relations of spin (which $\mathbf{S}\rightarrow-\mathbf{S}$ would not), and the Heisenberg Hamiltonian becomes

$$H = -J\sum_{(i,j)} S_{z,i}S_{z,j} + \frac{1}{2}\left(S_{+,i}S_{+,j} + S_{-,i}S_{-,j}\right). \tag{9.28}$$

When we apply the Holstein-Primakoff transformation to this Hamiltonian, to first order we obtain

$$H = -J\sum_{(i,j)}\left[s^2 + s\left(\hat{a}^{\dagger}_i\hat{a}_i + \hat{b}^{\dagger}_j\hat{b}_j + \hat{a}_i\hat{b}_j + \hat{a}^{\dagger}_i\hat{b}^{\dagger}_j\right)\right], \tag{9.29}$$

where $\hat{a}^{\dagger}_i$ and $\hat{a}_i$ are the creation and annihilation operators for the spin +s sub-lattice, and $\hat{b}^{\dagger}_j$ and $\hat{b}_j$ are the creation and annihilation operators for the original spin −s sub-lattice. After the Fourier transform of the creation and annihilation operators we get

$$H = -3Js^2N + 3Js\sum_{\mathbf{k}}\left[\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{b}^{\dagger}_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} + f(\mathbf{k})\left(\hat{a}_{\mathbf{k}}\hat{b}_{-\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}\hat{b}^{\dagger}_{-\mathbf{k}}\right)\right], \tag{9.30}$$

where $f(\mathbf{k}) = \frac{1}{3}(\cos k_x a + \cos k_y a + \cos k_z a)$.

To find the ground state we must diagonalise H so that it is a sum over number operators. This will involve mixing creation and annihilation operators. This is a unitary transformation that can be written as

$$\hat{a}_{\mathbf{k}} = u_{\mathbf{k}}\hat{c}_{\mathbf{k}} - v_{\mathbf{k}}\hat{d}^{\dagger}_{-\mathbf{k}} \quad \text{and} \quad \hat{b}_{-\mathbf{k}} = u_{\mathbf{k}}\hat{d}_{-\mathbf{k}} - v_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}} \tag{9.31}$$

This leads to the Hamiltonian

$$H = -3Js(s+1)N + \sum_{\mathbf{k}}\epsilon(\mathbf{k})\left(\hat{c}^{\dagger}_{\mathbf{k}}\hat{c}_{\mathbf{k}} + \hat{d}^{\dagger}_{-\mathbf{k}}\hat{d}_{-\mathbf{k}} + 1\right), \tag{9.32}$$

with the spin wave energy

$$\epsilon(\mathbf{k}) = 3J_s\left(1-f(\mathbf{k})^2\right)^{\frac{1}{2}} \tag{9.33}$$

For small $\mathbf{k}$ the dispersion relation of the spin wave is linear in the wave vector, $\epsilon(\mathbf{k}) \simeq \sqrt{3}J_{sak}$, which means that the spin waves behave markedly different in ferromagnets and anti-ferromagnets.