6.1: The Lindblad Equation

Next, we will derive the Lindblad equation, which is the direct extension of the Heisenberg equation for the density operator, i.e., the mixed state of a system. We have seen in Eq. (4.30) that formally, we can write the evolution of a density operator as a mathematical map $\mathscr{E}$, such that the density operator ρ transforms into

$$\rho \rightarrow \rho' = \mathscr{E}(\rho) \equiv \sum_k A_k \rho A_k^\dagger, \tag{6.1}$$

where the A_k are the Kraus operators. Requiring that ρ' is again a density operator $(\mathrm{Tr}(\rho') = 1)$ leads to the restriction $\sum_k A_k^\dagger A_k = \mathbb{I}$.

We want to describe an infinitesimal evolution of ρ, in order to give the continuum evolution later on. We therefore have that

$$\rho' = \rho + \delta\rho = \sum_k A_k \rho A_k^\dagger \tag{6.2}$$

Since $\delta\rho$ is very small, one of the Kraus operators must be close to the identity. Without loss of generality we choose this to be A_0, and then we can write

$$A_0 = \mathbb{I} + (L_0 - iK)\,\delta t \quad \text{and} \quad A_k = L_k \sqrt{\delta t}, \tag{6.3}$$

where we introduced the Hermitian operators L_0 and K, and the remaining L_k are not necessarily Hermitian. We could have written $A_0 = \mathbb{I} + L_0 \delta t$ and keep L_0 general (non-Hermitian as well), but it will be useful later on to explicitly decompose it into Hermitian parts. We can now write

$$\begin{aligned} A_0 \rho A_0^\dagger &= \rho + \left[(L_0 - iK)\,\rho + \rho\,(L_0 + iK)\right]\delta t + O\left(\delta t^2\right) \\ A_k \rho A_k^\dagger &= L_k \rho L_k^\dagger \delta t \end{aligned}. \tag{6.4}$$

We can substitute this into Eq. (6.2), to obtain up to first order in δt

$$\delta\rho = \left[(L_0\rho + \rho L_0) - i(K\rho - \rho K) + \sum_{k\neq 0} L_k \rho L_k^\dagger\right]\delta t. \tag{6.5}$$

We now give the continuum evolution by dividing by δt and taking the limit $\delta t \rightarrow dt$:

$$\frac{d\rho}{dt} = -i[K,\rho] + \{L_0,\rho\} + \sum_{k\neq 0} L_k \rho L_k^\dagger, \tag{6.6}$$

where $\{A, B\} = AB + BA$ is the anti-commutator of A and B. We are almost there, but we must determine what the different terms mean. Suppose we consider the free evolution of the system.

Eq. (6.6) must then reduce to the Heisenberg equation for the density operator ρ in Eq. (4.7), and we see that all L_k including L_0 are zero, and K is proportional to the Hamiltonian $K = H/\hbar$. Again from the general property that $\mathrm{Tr}(\rho) = 1$ we have

$$\mathrm{Tr}\left(\frac{d\rho}{dt}\right) = 0 \rightarrow L_0 = -\frac{1}{2}\sum_{k\neq 0} L_k^\dagger L_k. \tag{6.7}$$

This finally leads to the Lindblad equation

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho] + \frac{1}{2}\sum_k \left(2L_k\rho L_k^\dagger - \left\{L_k^\dagger L_k, \rho\right\}\right). \tag{6.8}$$

The operators L_k are chosen such that they model the relevant physical processes. This may sound vague, but in practice it will be quite clear. For example, modelling a transition $|1\rangle \rightarrow |0\rangle$ without keeping track of where the energy is going or coming from will require a single Lindblad operator

$$L = \gamma|0\rangle\langle 1|, \tag{6.9}$$

where γ is a real parameter indicating the strength of the transition. This can model both decay and excitations.