8.1: Symmetric and Anti-symmetric States

The indistinguishability of identical particles means that we have to adjust our quantum mechanical description of these objects. There are two ways of doing this, namely via a modification of the allowed states and via a restructuring of the observables[5]. In this section we consider the restricted state space, and in the next we will be considering the new observables.

First of all, since the total number of particles is an observable quantity (for example by measuring the total charge in the box), we can give the particles an artificial labelling. The wave functions of the two particles are then given by $|\psi(\mathbf{r}_1)\rangle_1$ for particle 1 at position $\boldsymbol{r}_1$, and $|\phi(\mathbf{r}_2)\rangle_2$ for particle 2 at position $\mathbf{r}_2$. Since we can swap the positions of the particle without observable consequences, we find that there are two states that denote the same physical situation:

$$|\psi(\mathbf{r}_1), \phi(\mathbf{r}_2)\rangle_{12} \quad \text{and} \quad |\psi(\mathbf{r}_2), \phi(\mathbf{r}_1)\rangle_{12}. \tag{8.1}$$

However, we wish that each physically distinct situation has exactly one quantum state. Since there is no preference for either state, we can denote the physical situation of identical particles at position $\mathbf{r}_1$ and $\mathbf{r}_2$ by the quantum state that is an equal weight over these two possibilities:

$$|\Psi(\mathbf{r}_1, \mathbf{r}_2)\rangle_{12} = \frac{|\psi(\mathbf{r}_1), \phi(\mathbf{r}_2)\rangle_{12} + e^{i\varphi}|\psi(\mathbf{r}_2), \phi(\mathbf{r}_1)\rangle_{12}}{\sqrt{2}}. \tag{8.2}$$

You can verify that swapping the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ of the indistinguishable particles incurs only a global (unobservable) phase. The question is now how we should choose ϕ.

Suppose that the two identical particles in the box are electrons. We know From Pauli's exclusion principle that the two electrons cannot be in the same state. Therefore, when $\phi = \psi$, the state in Eq. (8.2) should naturally disappear:

$$|\Psi(\mathbf{r}_1, \mathbf{r}_2)\rangle_{12} = \frac{|\psi(\mathbf{r}_1), \psi(\mathbf{r}_2)\rangle_{12} + e^{i\varphi}|\psi(\mathbf{r}_2), \psi(\mathbf{r}_1)\rangle_{12}}{\sqrt{2}} = 0, \tag{8.3}$$

which means that for particles obeying Pauli's exclusion principle we must choose $e^{i\varphi} = -1$. The quantum state of the two particles is anti-symmetric.

What about particles that do not obey Pauli's exclusion principle? These must be restricted to states that are orthogonal to the anti-symmetric states. In other words, they must be in states that are symmetric under the exchange of two particles. For the two identical particles in a box, we therefore choose the value $e^{i\varphi} = +1$, which makes the state orthogonal to the anti-symmetric state. The two possibilities for combining two identical particles are therefore

$$\begin{aligned} |\Psi_\mathrm{S}(\mathbf{r}_1, \mathbf{r}_2)\rangle &= \frac{|\psi(\mathbf{r}_1), \phi(\mathbf{r}_2)\rangle + |\psi(\mathbf{r}_2), \phi(\mathbf{r}_1)\rangle}{\sqrt{2}} \\ |\Psi_\mathrm{A}(\mathbf{r}_1, \mathbf{r}_2)\rangle &= \frac{|\psi(\mathbf{r}_1), \phi(\mathbf{r}_2)\rangle - |\psi(\mathbf{r}_2), \phi(\mathbf{r}_1)\rangle}{\sqrt{2}} \end{aligned} \tag{8.4}$$

These states include both the internal degrees of freedom, such as spin, and the external degrees of freedom. So two electrons can still be in the state $|\uparrow\uparrow\rangle$, as long as their spatial wave function is anti-symmetric. The particles that are in a symmetric overall quantum state are bosons, while the particles in an overall anti-symmetric state are fermions.

We can extend this to N particles in a fairly straightforward manner. For bosons, we sum over all possible permutations of $\mathbf{r}_1$ to $\mathbf{r}_N$:

$$|\Psi_\mathrm{S}(\mathbf{r}_1, \ldots, \mathbf{r}_N)\rangle = \frac{1}{\sqrt{N!}} \sum_{\mathrm{perm}(\mathbf{r}_1, \ldots, \mathbf{r}_N)} |\psi_1(\mathbf{r}_1), \ldots, \psi_N(\mathbf{r}_N)\rangle. \tag{8.5}$$

For fermions, the odd permutations pick up a relative minus sign:

$$|\Psi_\mathrm{A}(\mathbf{r}_1, \ldots, \mathbf{r}_N)\rangle = \frac{1}{\sqrt{N!}} \sum_{\text{even}} |\psi_1(\mathbf{r}_1), \ldots, \psi_N(\mathbf{r}_N)\rangle - \frac{1}{\sqrt{N!}} \sum_{\text{odd}} |\psi_1(\mathbf{r}_1), \ldots, \psi_N(\mathbf{r}_N)\rangle \tag{8.6}$$

This can be written compactly as the so-called **Slater determinant**

$$\Psi_{\mathrm{A}}\left(\mathbf{r}_{1},\ldots,\mathbf{r}_{N}\right)=\frac{1}{\sqrt{N!}}\left|\begin{array}{cccc}\psi_{1}\left(\mathbf{r}_{1}\right) & \psi_{1}\left(\mathbf{r}_{2}\right) & \ldots & \psi_{1}\left(\mathbf{r}_{N}\right)\\ \psi_{2}\left(\mathbf{r}_{1}\right) & \psi_{2}\left(\mathbf{r}_{2}\right) & \ldots & \psi_{2}\left(\mathbf{r}_{N}\right)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_{N}\left(\mathbf{r}_{1}\right) & \psi_{N}\left(\mathbf{r}_{2}\right) & \ldots & \psi_{N}\left(\mathbf{r}_{N}\right)\end{array}\right|, \tag{8.7}$$

where we removed the kets for notational convenience. The N particles in the state $|\Psi_{\mathrm{A}}\left(\mathbf{r}_{1},\ldots,\mathbf{r}_{N}\right)\rangle$ automatically obey the Pauli exclusion principle.

---

[5]This is sometimes called second quantisation. This is a misnomer, since quantisation occurs only once, when observables are promoted to operators.
