4.3: Imperfect Measurements

Postulates 3 and 5 determine what happens when a quantum system is subjected to a measurement. In particular, these postulates concern ideal measurements. In practice, however, we often have to deal with imperfect measurements that include noise. As a simple example, consider a photodetector: not every photon that hits the detector will result in a detector "click", which tells us that there indeed was a photon. How can we describe situations like these?

First, let's recap ideal measurements. We can ask the question what will be the outcome of a single measurement of the observable A. We know from Postulate 3 that the outcome m must be an eigenvalue a_m of A. If the spectral decomposition of A is given by

$$A=\sum_m a_m|m\rangle\langle m|, \tag{4.16}$$

then the probability of finding measurement outcome m is given by the Born rule

$$p(m)=|\langle m \mid \psi\rangle|^2=\operatorname{Tr}(P_m|\psi\rangle\langle\psi|)\equiv\langle P_m\rangle\,, \tag{4.17}$$

where we introduced the operator $P_m=|m\rangle\langle m|$. One key interpretation of $p(m)$ is as the expectation value of the operator P_m associated with measurement outcome m.

When a measurement does not destroy the system, the state of the system must be updated after the measurement to reflect the fact that another measurement of A immediately following the first will yield outcome m with certainty. The update rule is given by Eq. (2.24)

$$|\psi\rangle\rightarrow\frac{P_m|\psi\rangle}{\sqrt{\left\langle\psi\left|P_m^\dagger P_m\right|\psi\right\rangle}} \tag{4.18}$$

where the square root in the denominator is included to ensure proper normalization of the state after the measurement. However, this form is not so easily generalized to measurements yielding incomplete information, so instead we will write (using $P_m^2=P_m=P_m^\dagger$)

$$|\psi\rangle\langle\psi|\rightarrow\frac{P_m|\psi\rangle\langle\psi|P_m^\dagger}{\operatorname{Tr}\left(P_m|\psi\rangle\langle\psi|P_m^\dagger\right)} \tag{4.19}$$

From a physical perspective this notation is preferable, since the unobservable global phase of $\langle m \mid \psi\rangle$ no longer enters the description. Up to this point, both state preparation and measurement were assumed to be ideal. How must this formalism of ideal measurements be modified in order to take into account measurements that yield only partial information about the system? First, we must find the probability for measurement outcome m, and second, we have to formulate the update rule for the state after the measurement.

When we talk about imperfect measurements, we must have some way of knowing (or suspecting) how the measurement apparatus fails. For example, we may suspect that a photon hitting a photodetector has only a finite probability of triggering a detector "click". We therefore describe our measurement device with a general probability distribution $\{q_j(m)\}$, where $q_j(m)$ is the probability that the measurement outcome m in the detector is triggered by a system in the state $|\psi_j\rangle$. The probabilities $q_j(m)$ can be found by modelling the physical aspects of the measurement apparatus. The accuracy of this model can then be determined by experiment. The probability of measurement outcome m for the ideal case is given in Eq. (4.17) by $\langle P_m\rangle$. When the detector is not ideal, the probability of finding outcome m is given by the weighted average over all possible expectation values $\langle P_j\rangle$ that can lead to m:

$$\begin{aligned}
p(m) &=\sum_j q_j(m)\left\langle P_j\right\rangle=\sum_j q_j(m)\operatorname{Tr}(P_j|\psi\rangle\langle\psi|)\\
&=\operatorname{Tr}\left[\sum_j q_j(m)P_j|\psi\rangle\langle\psi|\right]\equiv\operatorname{Tr}(E_m|\psi\rangle\langle\psi|)\,,\\
&=\langle E_m\rangle
\end{aligned} \tag{4.20}$$

where we defined the operator $\boldsymbol{E}_m$ associated with outcome m as

$$E_m = \sum_j q_j(m) P_j \tag{4.21}$$

Each possible measurement outcome m has an associated operator E_m, the expectation value of which is the probability of obtaining m in the measurement. The set of $\boldsymbol{E}_m$ is called a Positive Operator-Valued Measure (POVM). While the total number of ideal measurement outcomes, modelled by P_m, must be identical to the dimension of the Hilbert space (ignoring the technical complication of degeneracy), this is no longer the case for the POVM described by E_m; there can be more or fewer measurement outcomes, depending on the physical details of the measurement apparatus. For example, the measurement of an electron spin in a Stern-Gerlach apparatus can have outcomes "up", "down", or "failed measurement". The first two are determined by the position of the fluorescence spot on the screen, and the last may be the situation where the electron fails to produce a spot on the screen. Here, the number of possible measurement outcomes is greater than the number of eigenstates of the spin operator. Similarly, photodetectors in Geiger mode have only two possible outcomes, namely a "click" or "no click" depending on whether the detector registered photons or not. However, the number of eigenstates of the intensity operator (the photon number states) is infinite.

Similar to the density operator, the POVM elements E_m have three key properties. First, the $p(m)$ are probabilities and therefore real, so for all states $|\psi\rangle$ we have

$$\langle E_m\rangle^* = \langle E_m\rangle \quad \Longleftrightarrow \langle\psi|E_m|\psi\rangle = \left\langle\psi\left|E_m^\dagger\right|\psi\right\rangle, \tag{4.22}$$

and E_m is therefore Hermitian. Second, since $\sum_m p(m) = 1$ we have

$$\sum_m p(m) = \sum_m \langle\psi|E_m|\psi\rangle = \left\langle\psi\left|\sum_m E_m\right|\psi\right\rangle = 1, \tag{4.23}$$

for all $|\psi\rangle$, and therefore

$$\sum_m E_m = \mathbb{I}, \tag{4.24}$$

where $\mathbb{I}$ is the identity operator. Finally, since $p(m) = \langle\psi|E_m|\psi\rangle \geq 0$ for all $|\psi\rangle$, the POVM element $\boldsymbol{E}_m$ is a positive operator. Note the close analogy of the properties of the POVM and the density operator. In particular, just as in the case of density operators, POVMs are defined by these three properties.

The second question about generalized measurements is how the measurement outcomes should be used to update the quantum state of the system. The rule for ideal von Neumann measurements is given in Eq. (4.19), which can be generalized immediately to density operators using the techniques presented above. This yields

$$\rho \rightarrow \frac{P_m \rho P_m^\dagger}{\operatorname{Tr}\left(P_m \rho P_m^\dagger\right)} \tag{4.25}$$

What if we have instead a measurement apparatus described by a POVM? Consistency with the Born rule in Eq. (4.20) requires that we again replace P_m, associated with measurement outcome m, with a probability distribution over all P_j:

$$\rho \rightarrow \sum_j q_j(m) \frac{P_j \rho P_j^\dagger}{\operatorname{Tr}\left[\sum_k q_k(m) P_k \rho P_k^\dagger\right]} = \sum_j q_j(m) \frac{P_j \rho P_j^\dagger}{\operatorname{Tr}(E_m \rho)}, \tag{4.26}$$

where we adjusted the normalization factor to maintain $\operatorname{Tr}(\rho) = 1$. We also used that the POVM element $\boldsymbol{E}_m$ in Eq. (4.21) can be written as

$$E_m = \sum_j q_j(m) P_j^\dagger P_j \tag{4.27}$$

If we rescale the P_j by a factor $\sqrt{q_j(m)}$, we obtain the standard form of the POVM

$$E_m = \sum_j A_{jm}^\dagger A_{jm} \tag{4.28}$$

where $A_{jm} = \sqrt{q_j(m)} P_j$ are the so-called the Kraus operators. Consequently, the update rule can be written as

$$\rho \to \frac{\sum_j A_{jm}\rho A_{jm}^\dagger}{\mathrm{Tr}(E_m\rho)} \tag{4.29}$$

which generally yields a mixed state (described by a density operator) after an incomplete measurement.

Finally, we can consider the case of some nonunitary evolution without a measurement (e.g., when the system interacts with its environment). We can model this purely formally by removing the index of the measurement outcome m from the description (since there are no measurement outcomes). The most general evolution then takes the form

$$\rho \to \rho' = \sum_k A_k \rho A_k^\dagger, \tag{4.30}$$

and the question is what form the Kraus operators A_k take. We will return to this in section 6.

? Exercises

1. The density matrix.
   a. Show that $\frac{1}{2}|0\rangle + \frac{1}{2}|+\rangle$ is not a properly normalized state.
   b. Show that $\mathrm{Tr}(\rho) = 1$ with ρ given by Eq. (4.3), and then prove that any density operator has unit trace and is Hermitian.
   c. Show that density operators are convex, i.e., that $\rho = w_1\rho_1 + w_2\rho_2$ with $w_1 + w_2 = 1(w_1, w_2 \geq 0)$, and ρ_1, ρ_2 again density operators.
   d. Calculate the expectation value of A using the two representations of ρ in terms of p_i and the spectral decomposition. What is the difference in the physical interpretation of p_j and λ_j?
2. Using the identity $\langle x|A|\psi\rangle = A\psi(x)$, and the resolution of the identity $\int dx|x\rangle\langle x| = \mathbb{I}$, calculate the expectation value for an operator A, given a mixed state of wave functions.
3. Calculate P^2 with P given by

$$P = \begin{pmatrix} a \\ b \end{pmatrix}(a^*, b^*) = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \quad \text{and} \quad |a|^2 + |b|^2 = 1 \tag{4.31}$$

4. Calculate the eigenvalues of the density matrix in Eq. (4.15), and show that $\gamma_1 \leq 2\gamma_2$. Hint: it is difficult to derive the inequality directly, so you should try to demonstrate that $\gamma_1 \geq 2\gamma_2$ leads to a contradiction.
5. A system with energy eigenstates $|E_n\rangle$ is in thermal equilibrium with a heat bath at temperature T. The probability for the system to be in state E_n is proportional to $e^{-E_n/kT}$.
   a. Write the Hamiltonian of the system in terms of $\boldsymbol{E}_n$ and $|E_n\rangle$.
   b. Give the normalized density operator ρ for the system as a function of the Hamiltonian.
   c. We identify the normalization of ρ with the partition function $\mathscr{Z}$. Calculate the average energy directly and via

$$\langle E\rangle = -\frac{\partial \ln \mathscr{Z}}{\partial \beta} \tag{4.32}$$

   of the system, where $\beta = 1/(kT)$.
   d. What is the entropy $S = k\ln\mathscr{Z}$ if the system is a harmonic oscillator? Comment on the limit $T \to 0$.
6. Lossy photodetectors.
   a. The state of a beam of light can be written in the photon number basis $|n\rangle$ as $|\psi\rangle = \sum_n c_n|n\rangle$. What are the possible measurement outcomes for a perfect photon detector? Calculate the probabilities of the measurement outcome using projection operators.
   b. Suppose that the detector can only tell the difference between the presence and absence of photons (a so-called “bucket detector”). How do we calculate the probability of finding the measurement outcomes?
   c. Real bucket detectors have a finite efficiency η, which means that each photon has a probability η of triggering the detector. Calculate the probabilities of the measurement outcomes.
   d. What other possible imperfections do realistic bucket detectors have?
7. An electron with spin state $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ and $|\alpha|^2 + |\beta|^2 = 1$ is sent through a SternGerlach apparatus to measure the spin in the z direction (i.e., the $|\uparrow\rangle, |\downarrow\rangle$ basis).

a. What are the possible measurement outcomes? If the position of the electron is measured by an induction loop rather than a screen, what is the state of the electron immediately after the measurement?
b. Suppose that with probability p the induction loops fail to give a current signifying the presence of an electron. What are the three possible measurement outcomes? Give the POVM.
c. Calculate the probabilities of the measurement outcomes, and the state of the electron imediately after the measurement (for all possible outcomes).