11.2: Faster Than Light?

A Dialog - a reason to examine the Lorentz tranformation

	No object travels faster than light.
So YOU say, but watch ME: I travel in a rocket that you observe to move at $4/5$ *light speed. Out the front of my rocket I fire a bullet that I observe to fly forward at 4/5 light speed. Then you measure this bullet to streak forward at* $4/5+4/5=8/5=1.6$ *light speed, which is greater than the speed of light. There!*	No!
Velocities do not add	
Why not? Is it not true that $4/5+4/5=1.6$ *?*	As a mathematical abstraction: always true. As a description of the world: only sometimes true! Example 1: Add 4/5 liter of alcohol to $4/5$ liter of water. The result? Less than $8/5=1.6$ liter of liquid! Why? Molecules of water interpenetrate molecules of alcohol to yield a combined volume less that the sum of the separate volumes. Example 2: Add the speed you measure for the bullet $(4/5)$ to the speed I measure for your rocket $(4/5)$. The result? The speed I measure for the bullet is $40/41=0.9756$. This remains less than the speed of light.
Why? And where did you get that number 40/41 for the bullet speed you measure?	I got the number from the Lorentz transformation, the subject of this Special Topic. The Lorentz transformation embodies a central feature of relativity: Space and time separations typically do not have the same values as observed in different frames.
Space and time separations between what?	Between events.
What events are we talking about here?	Event 1: You fire the bullet out the front of your rocket. Event 2: The bullet strikes a target ahead of you.
Events define velocities	
What do these events have to do with speed? We are arguing about speed!	Let the bullet hit the target four meters in front of you, as measured in your rocket. Then the space separation between event 1 and event 2 is 4 meters. Suppose the time of flight is 5 meters as measured by your clocks, the time separation between the two events. Then your bullet speed measurement is (4 meters of distance) $/(5$ meters of time$)=4/5$, as you said.
And what do YOU measure for the space and time separations in your laboratory frame?	For that we need the Lorentz coordinate transformation equations.
Interval: Only a start in reckoning spacetime separations in different frames	
Phooey! I know how to reckon spacetime separations in different frames. We have been doing it for several chapters!' From measurements in one frame we figure the spacetime interval, which has the same value in all frames. End of story.	No, not the end of the story, but at least its beginning. True, the invariant interval has the same value as derived from measurements in every frame. That allows you to predict the time between firing and impact as measured by the passenger riding on the bullet - and measured directly by the bullet passenger alone.

Predict how?	You know your space separation $x'=4$ meters (primes for rocket measurements), and your time separation, $t'=5$ meters. You know the space separation for the bullet rider, $x''=0$ (double primes for bullet measurements), since she is present at both the firing and the impact. From this you can use invariance of the interval to determine the wristwatch time between these events for the bullet rider: $$\left(t''\right)^2-\left(x''\right)^2=\left(t'\right)^2-\left(x'\right)^2$$ or $$\left(t''\right)^2-(0)^2=(5 \text{ meters })^2-(4 \text{ meters })^2=(3 \text{ meters })^2$$ so that $t''=3$ meters. This is the proper time, agreed on by all observers but measured directly only on the wristwatch of the bullet rider.
Need more to compare velocities in different frames	
Fine. Can't we use the same procedure to determine the space and time separations between these events in your laboratory frame, and thus the bullet speed for you?	Unfortunately not. We do reckon the same value for the interval. Use unprimed symbols for laboratory measurements. Then $t^2-x^2=(3 \text{ meters })^2$. That, however, is not sufficient to determine x or t separately. Therefore we cannot yet find their ratio x/t, which determines the bullet's speed in our frame.
Compare velocities using Lorentz transformation	
So how can we reckon these x and t separations in your laboratory frame, thereby allowing us to predict the bullet speed you measure?	Use the Lorentz transformation. This transformation reports that our laboratory space separation between firing and impact is $x=40/3$ meters and the time separation is slightly greater: $t=41/3$ meters. Then bullet speed in my laboratory frame is predicted to be $v=x/t=40/41=0.9756$. The results of our analysis in three reference frames are laid out in Table L-1.
Lorentz transformation previewed	

Is the Lorentz transformation generally useful, beyond the specific task of reckoning speeds as measured in different frames?	Oh yes! Generally, we insert into the Lorentz transformation the coordinates x', t' of an event determined in the rocket frame. The Lorentz transformation then grinds and whirs, finally spitting out the coordinates x, t of the same event measured in the laboratory frame. Following are the Lorentz transformation equations. Here v_{rel} is the relative velocity between rocket and laboratory frames. For our convenience we lay the positive x-axis along the direction of motion of the rocket as observed in the laboratory frame and choose a common reference event for the zero of time and space for both frames. $$x = \frac{x' + v_{\text{rel}} t'}{\left(1 - v_{\text{rel}}^2\right)^{1/2}}$$ $$t = \frac{v_{\text{rel}} x' + t'}{\left(1 - v_{\text{rel}}^2\right)^{1/2}}$$ $$y = y' \quad \text{and} \quad z = z'$$ Check for yourself that for the impact event of bullet with target (rocket coordinates: $x' = 4$ meters, $t' = 5$ meters; rocket speed in laboratory frame: $v_{\text{rel}} = 4/5$) one obtains laboratory coordinates $x = 40/3$ meters and $t = 41/3$ meters. Hence $v = x/t$ $= 40/41 = 0.9756$.

Table 11.2.1: How Fast the Bullet

	Bullet fired (coordinates of this event)	Bullet hits (coordinates of this event)	Speed of bullet (computed from frame coordinates)
Rocket frame (moves at $v_{\text{rel}} = 4/5$ as measured in laboratory)	$x' = 0$ $t' = 0$ n	$x' = 4$ meters $t' = 5$ meters n	as measured in rocket frame: $v' = 4/5 = 0.8$ n
Bullet frame (moves at $v' = 4/5$ as measured in rocket)	$x'' = 0$ $t'' = 0$ n	$x'' = 0$ $t'' = 3$ meters $=$ n (from invariance of the interval)	as measured in bullet frame: $v'' = 0$ n
Laboratory frame	$x = 0$ $t = 0$ n	$x = 40/3$ meters $t = 41/3$ meters n (from Lorentz transformation)	as measured in laboratory frame: $v = 40/41 = 0.9756$ n

Lorentz transformation: Useful but not fundamental

You say the Lorentz transformation is general. If it is so important, then why is this a special topic rather than a regular chapter?	The Lorentz transformation is powerful; it brings the technical ability to transform coordinates from frame to frame. It helps us predict how to add velocities, as outlined here. It describes the Doppler shift for light (see the exercises for this chapter). On the other hand, the Lorentz transformation is not fundamental; it does not expose deep new features of spacetime. But no matter! Physics has to get on with the world's work. One uses the method of describing separation best suited to the job at hand. On some occasions the useful fact to give about a luxury yacht is the 50-meter distance between bow and stern, a distance independent of the direction in which the yacht is headed. On another occasion it may be much more important to know that the bow is 30 meters east of the stern and 40 meters north of it as observed by its captain, who uses North-Star north.
Two foundations of Lorentz transformation	
What does the Lorentz transformation rest on? On what foundations is it based?	On two foundations: (1) The equations must be linear. That is, space and time coordinates enter the equations to the first power, not squared or cubed. This results from the requirement that you may choose any event as the zero of space and time. (2) The spacetime interval between two events must have the same value when computed from laboratory coordinate separations as when reckoned from rocket coordinate separations.
All right, I'll reserve judgment on the validity of what you claim, but show me the derivation itself.	Read on!