1.2: Surveying Spacetime

disagree on separations in space and time; agree on spacetime interval

The Parable of the Surveyors illustrates the naive state of physics before the discovery of **special relativity** by Einstein of Bern, Lorentz of Leiden, and Poincaré of Paris. Naive in what way? Three central points compare physics at the turn of the twentieth century with surveying before the student arrived to help Daytimers and Nighttimers.

The second: A sacred unit

First, surveyors in the mythical kingdom measured northward separations in a sacred unit, the mile, different from the unit used in measuring eastward separations. Similarly, people studying physics measured time in a sacred unit, called the second,[1] different from the unit used to measure space. No one suspected the powerful results of using the same unit for both, or of squaring and combining space and time separations when both were measured in meters. Time in meters is just the time it takes a light flash to go that number of meters. The conversion factor between seconds and meters is the speed of light, $c=299,792,458$ meters/second. The velocity of light c (in meters/second) multiplied by time t (in seconds) yields ct (in meters).

Speed of light converts seconds to meters

The speed of light is the only natural constant that has the necessary units to convert a time to a length.[2] Historically the value of the speed of light was regarded as a sacred number. It was not recognized as a mere conversion factor, like the factor of conversion between miles and meters — a factor that arose out of historical accident in humankind's choice of units for space and time, with no deeper physical significance.

Time between events: Different for different frames

Second, in the parable northward readings as recorded by two surveyors did not differ much because the two directions of north were inclined to one another by only the small angle of 1.15 degrees. At first our mythical student thought that small differences between Daytime and Nighttime northward measurements were due to surveying error alone. Analogously, we used to think of the separation in time between two electric sparks as the same, regardless of the motion of the observer. Only with the publication of Einstein's relativity paper in 1905 did we learn that the separation in time between two sparks really has different values for observers in different states of motion — in different **frames**.[3]

Think of John standing quietly in the front doorway of his laboratory building. Suddenly a rocket carrying Mary flashes through the front door past John, zooms down the middle of the long corridor, and shoots out the back door. An antenna projects from the side of Mary's rocket. As the rocket passes John, a spark jumps across the 1-millimeter gap between the antenna and a pen in John's shirt pocket. The rocket continues down the corridor. A second spark jumps 1 millimeter between the antenna and the fire extinguisher mounted on the wall 2 meters farther down the corridor. Still later other metal objects nearer the rear receive additional sparks from the passing rocket before it finally exits through the rear door.

One observer uses laboratory frame

John and Mary each measure the lapse of time between "pen spark" and "fire extinguisher spark." They use accurate and fast electronic clocks. John measures this time lapse as 33.6900 thousand-millionths of a second $(0.0000000336900$ second $=33.6900\times10^{-9}$ second$)$[4]. This equals 33.6900 **nanoseconds** in the terminology of high-speed electronic circuitry. (One nanosecond $=10^{-9}$ second.) Mary measures a slightly different value for the time lapse between the two sparks, 33.0228 nanoseconds[5]. For John the fire-extinguisher spark is separated in space by 2.0000meters from the pen spark.

Another observer uses rocket frame

For Mary in the rocket the pen spark and fire-extinguisher spark occur at the same place, namely at the end of her antenna. Thus for her their space separation equals zero.

Later, laboratory and rocket observers compare their space and time measurements between the various sparks (Table $1.2.1$). Space locations and time lapses in both frames are measured from the pen spark.

Table 1.2.1: Space and Time Locations of the Same Sparks as Seen by Two Observers

	Distance and time between sparks as measured by observer who is	
	standing in laboratory (John)	moving by in rocket (Mary)

	Distance (meters)	Time (nanoseconds)	Distance (meters)	Time (nanoseconds)
Reference spark (pen spark)	0	0	0	0
Spark A (fire-extinguisher spark)	2.0000	33.6900	0	33.0228
Spark B	3.0000	505350	0	49.5343
Spark C	5.0000	84.2250	0	82.5572
Spark D	8.0000	134.7600	0	132.0915

Discovery: Invariance of spacetime interval

The third point of comparison between the Parable of the Surveyors and the state of physics before special relativity is this: The mythical student's discovery of the concept of distance is matched by the Einstein-Poincare discovery in 1905 of the **invariant spacetime interval** (formal name **Lorentz interval**, but we often say just **interval**), a central theme of this book[6]. Let each time measurement in seconds be converted to meters by multiplying it by the "conversion factor c," the speed of light:

$$\begin{aligned} c &= 299,792,458 \text{ meters/second} \\ &= 2.99792458 \times 10^8 \text{ meters/second} \\ &= 0.299792458 \times 10^9 \text{ meters/second} \\ &= 0.299792458 \text{ meters/nanosecond} \end{aligned}$$

Then the square of the spacetime interval is calculated from the laboratory observer's measurements by subtracting the square of the space separation from the square of the time separation. Note the minus sign in equation (1.2.1).

$$(\text{ interval })^2 = \overbrace{\left[c \times \begin{pmatrix} \text{time} \\ \text{separation} \\ \text{(seconds)} \end{pmatrix} \right]^2}^{\textbf{Laboratory}} - \overbrace{\begin{bmatrix} \text{space} \\ \text{separation} \\ \text{(meters)} \end{bmatrix}^2}^{\textbf{Laboratory}} \tag{1.2.1}$$

The rocket calculation gives exactly the same value of the interval as the laboratory calculation,

$$(\text{ interval })^2 = \overbrace{\left[c \times \begin{pmatrix} \text{time} \\ \text{separation} \\ \text{(seconds)} \end{pmatrix} \right]^2}^{\textbf{Rocket}} - \overbrace{\begin{bmatrix} \text{space} \\ \text{separation} \\ \text{(meters)} \end{bmatrix}^2}^{\textbf{Rocket}}$$

even though the respective space and time separations are not the same. Two observers find different space and time separations, respectively, between pen spark and fire extinguisher spark, but when they calculate the spacetime interval between these sparks their results agree (Table 1.2.2).

Table 1.2.2: "Invariant Spacetime Interval" from Reference Spark to Spark A (Data from Table 1.2.1)

Laboratory measurements		Rocket measurements	
Time lapse	33.6900×10^{-9} seconds = 33.6900 nanoseconds	Time lapse	33.0228×10^{-9} seconds = 33.0228 nanoseconds
Multiply by c = 0.299792458 meters per nanosecond to convert to meters:	10.1000 meters	Multiply by c = 0.299792458 meters per nanosecond to convert to meters:	9.9000 meters
Square the value	102.010 (meters)^2	Square the value	98.010 (meters)^2

Laboratory measurements		Rocket measurements	
Spatial separation 2.000 meters	$\underline{-4.000\ (\text{meters})^2}$	Spatial separation zero	$\underline{-\quad 0\quad}$
Square the value and subtract	$=98.010\ (\text{meters})^2$	Square the value and subtract	$=98.010\ (\text{meters})^2$
Result of subtraction expressed as a number squared	$=(9.900\text{ meters })^2$	Result of subtraction expressed as a number squared	$=(9.900\text{ meters })^2$
This is the square of what measurement?	**9.900 meters**	This is the square of what measurement?	**9.900 meters**

Note

Note that both laboratory and rocket measurements have same spacetime internal from the refrence event.

The student surveyor found that invariance of distance was most simply written with both northward and eastward separations expressed in the same unit, the meter Likewise, invariance of the spacetime interval is most simply written with space and time separations expressed in the same unit. Time is converted to meters: t (meters) $= c \times t$ (seconds). Then the interval appears in simplified form:

$$(\text{ interval })^2=\left[\begin{array}{c}\text{time}\\ \text{separation}\\ (\text{meters})\end{array}\right]^2-\left[\begin{array}{c}\text{space}\\ \text{separation}\\ (\text{meters})\end{array}\right]^2$$

Space and time are part of spacetime

The **invariance of the spacetime interval** — its independence of the state of motion of the observer — forces us to recognize that time cannot be separated from space[7]. Space and time are part of a single entity, **spacetime**. Space has three dimensions: northward, eastward, and upward. Time has one dimension: onward! The interval combines all four dimensions in a single expression. The geometry of spacetime is truly four-dimensional.

To recognize the unity of spacetime we follow the procedure that makes a landscape take on depth — we look at it from several angles. That is why we compare space and time separations between events A and B as recorded by two different observers in relative motion.

Question and Answer

Why the minus sign in the equation for the interval? Pythagoras tells us to ADD the squares of northward and eastward separations to get the square of the distance. Who tells us to SUBTRACT the square of the space separation between events from the square of their time separation in order to get the square of the spacetime interval?

Answer

Shocked? Then you're well on the way to understanding the new world of very fast motion! This world goes beyond the three-dimensional textbook geometry of Euclid, in which distance is reckoned from a sum of squares. In this book we use another kind of geometry, called **Lorentz geometry**, more real, more powerful than Euclid for the world of the very fast. In Lorentz geometry the squared space separation is combined with the squared time separation in a new way—by *subtraction*. The result is the square of a new unity called the *spacetime interval* between events. The numerical value of this interval is *invariant*, the same for all observers, no matter how fast they are moving past one another. Proof? Every minute of every day an experiment somewhere in the world demonstrates it. In Chapter 3 we derive the invariance of the spacetime interval—with its minus sign —from experiments. They show the finding that no experiment conducted in a closed room will reveal whether that room is "at rest" or "in motion" (Einstein's Principle of Relativity). We won't wait until then to cash in on the idea of interval. We can begin to enjoy the payoff right now

Example 1.2.1

Another, even faster rocket follows the first, entering the ftont door, zipping down the long corridor, and exiting through the back doorway. Each time the rocket clock ticks it emits a spark. As before, the first spark jumps the 1 millimeter from the passing rocket antenna to the pen in the pocket of John, the laboratory observer. The second flash jumps when the rocket antenna reaches a doorknob 4.00000000 meters farther along the hall as measured by the laboratory observer, who records the time between these two sparks as 16.6782048 nanoseconds.

a. What is the time between sparks, measured in meters by John, the laboratory observer?

b. What is the value of the spacetime interval between the two events, calculated from John's laboratory measurements?

c. Predict: What is the value of the interval calculated from measurements in the new rocket frame?

d. What is the distance between sparks as measured in this rocket frame?

e. What is the time (in meters) between sparks as measured in this rocket frame? Compare with the time between the same sparks as measured by John in the laboratory frame.

f. What is the speed of this rocket as measured by John in the laboratory?

Solution

a. Time in meters equals time in nanoseconds multiplied by the conversion factor, the speed of light in meters per nanosecond. For John, the laboratory observer,

$16.6782048 \text{nanoseconds } \times 0.299792458 \text{meters/nanosecond} = 5.00000000 \text{meters}$

b. The square of the interval between two flashes is reckoned by subtracting the square of the space separation from the square of the time separation. Using laboratory figures:

$$\begin{aligned}
(\text{ interval })^2 &= (\text{ laboratory time })^2-(\text{ laboratory distance })^2 \\
&= (5 \text{ meters })^2-(4 \text{ meters })^2 \\
&= 25(\text{ meters })^2-16(\text{ meters })^2 \\
&= 9(\text{ meters })^2=(3 \text{ meters })^2
\end{aligned}$$

Therefore the interval between the two sparks has the value 3 meters (to nine significant figures).

c. We strongly assert in this chapter that the **spacetime interval is invariant** - it has the same value for whoever calculates it. Accordingly, the interval between the two sparks calculated from rocket observations has the same value as the interval (3 meters) calculated from laboratory measurements.

d. From the rocket rider's viewpoint, both sparks jump from the same place, namely the end of her antenna, and so distance between the sparks equals zero for the rocket rider.

e. We know the value of the spacetime interval between two sparks as computed in the rocket frame (c). And we know that the interval is computed by subtracting the square of the space separation from the square of the time separation in the rocket frame. Finally we know that the space separation in the rocket frame equals zero (d). Therefore the rocket time lapse between the two sparks equals the interval between them:

$$\begin{aligned}
(\text{ interval })^2 &= (\text{ rocket time })^2-(\text{ rocket distance })^2 \\
(3 \text{ meters })^2 &= (\text{ rocket time })^2-(\text{ zero })^2
\end{aligned}$$

from which 3 meters equals the rocket time between sparks. Compare this with 5 meters of light-travel time between sparks as measured in the laboratory frame.

f. Measured in the laboratory frame, the rocket moves 4 meters of distance (statement of the problem) in 5 meters of light-travel time (a). Therefore its speed in the laboratory is $4/5$ light speed. Why? Well, light moves 4 meters of distance in 4 meters of time. The rocket takes longer to cover this distance: 5 meters of time. Suppose that instead of 5 meters of time, the rocket had taken 8 meters of time, twice as long as light, to cover the 4 meters of distance. In that case it would be moving at $4/8$ - or half - the speed of light. In the present case the rocket travels the 4 meters of distance in 5 meters of time, so it moves at $4/5$ light speed. Therefore its speed equals

$$(4/5) \times 2.99792458 \times 10^8 \text{ meters/second } = 2.3983397 \times 10^8 \text{ meters/second}$$

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1 The second: A sacred unit

2 Speed of light converts seconds to meters

3 Time between events: Different for different frames

4 John observer uses laboratory frame

5 Mary observer uses rocket frame

6 Discovery: Invariance of spacetime interval

7 Space and time are part of spacetime
