1.E: Spacetime (Exercises)

Introduction to the Exercises

Important areas of current research can be analyzed very simply using the theory of relativity. This analysis depends heavily on a physical intuition, which develops with experience. Wide experience is not easy to obtain in the laboratory - simple experiments in relativity are difficult and expensive because the speed of light is so great. As alternatives to experiments, the exercises and problems in this text evoke a wide range of physical consequences of the properties of spacetime. These properties of spacetime recur here over and over again in different contexts:

- paradoxes
- puzzles
- derivations
- technical applications
- experimental results
- estimates
- precise calculations
- philosophical difficulties

The text presents all formal tools necessary to solve these exercises and problems, but intuition - a practiced way of seeing - is best developed without hurry. For this reason we suggest continuing to do more and more of these exercises in relativity after you have moved on to material outside this book. The mathematical manipulations in the exercises and problems are very brief: only a few answers take more than five lines to write down. On the other hand, the exercises require some "rumination time."

In some chapters, exercises are divided into two categories, Practice and Problems. The Practice exercises help you to get used to ideas in the text. The Problems apply these ideas to physical systems, thought experiments, and paradoxes.

WHEELER'S FIRST MORAL PRINCIPLE: *Never make a calculation until you know the answer.* Make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every paradox and puzzle. Courage: No one else needs to know what the guess is. Therefore make it quickly, by instinct. A right guess reinforces this instinct. A wrong guess brings the refreshment of surprise. In either case life as a spacetime expert, however long, is more fun!

Chapter 1 Exercises

Practice

1: comparing speeds

Compare the speeds of an automobile, a jet plane, an Earth satellite, Earth in its orbit around Sun, and a pulse of light. Do this by comparing the relative distance each travels in a fixed time. Arbitrarily choose the fixed time to give convenient distances. A car driving at the USA speed limit of 65 miles/hour (105 kilometers/hour) covers 1 meter of distance in about 35 milliseconds $=35 \times 10^{-3}$ second.

a. How far does a commercial jetliner go in 35 milliseconds? (speed: 650 miles/hour $=1046$ kilometers/hour)

b. How far does an Earth satellite go in 35 milliseconds? (speed: 17,000 miles/hour $\approx 27,350$ kilometers/hour)

c. How far does Earth travel in its orbit around Sun in 35 milliseconds? (speed: 30 kilometers/second)

d. How far does a light pulse go in a vacuum in 35 milliseconds? (speed: 3×10^{8} meters/second). This distance is roughly how many times the distance from Boston to San Francisco (5000 kilometers)?

2: images from Neptune

At 9:00 P.M. Pacific Daylight Time on August 24, 1989, the planetary probe *Voyager II* passed by the planet Neptune. Images of the planet were coded and transmitted to Earth by microwave relay.

It took 4 hours and 6 minutes for this microwave signal to travel from Neptune to Earth. Microwaves (electromagnetic radiation, like light, but of frequency lower than that of visible light), when propagating through interplanetary space, move at the "standard" light speed of one meter of distance in one meter of light-travel time, or $299,792,458$meters/ second. In the following, neglect any relative motion among Earth, Neptune, and *Voyager II*.

a. Calculate the distance between Earth and Neptune at fly-by in units of minutes, seconds, years, meters, and kilometers.

b. Calculate the time the microwave signal takes to reach Earth. Use the same units as in part **a**.

3: units of spacetime

Light moves at a speed of 3.0×10^8 meters/second. One mile is approximately equal to 1600 meters. One furlong is approximately equal to 200 meters.

a. How many meters of time in one day?

b. How many seconds of distance in one mile?

c. How many hours of distance in one furlong?

d. How many weeks of distance in one light-year?

e. How many furlongs of time in one hour?

4: time stretching and the spacetime interval

A rocket clock emits two flashes of light and the rocket observer records the time lapse (in seconds) between these two flashes. The laboratory observer records the time separation (in seconds) and space separation (in light-seconds) between the same pair of flashes. The results for both laboratory and rocket observers are recorded in the first line of the table.

Now a clock in a different rocket, moving at a different speed with respect to the laboratory, emits a different pair of flashes. The set of laboratory and rocket space and time separations are recorded on the second line of the table. And so on. Complete table $1.E.1$.

Table $1.E.1$: Space and Time Separations

	Rocket time lapse (seconds)	Laboratory time lapse (seconds)	Laboratory distance (light-seconds)
Example	20	29	21
a	?	10.72	5.95
b	20	?	99
c	66.8	72.9	?
d	?	8.34	6.58
e	21	22	?

5: where and when?

Two firecrackers explode at the same place in the laboratory and are separated by a time of 3 years as measured on a laboratory clock.

a. What is the spatial distance between these two events in a rocket in which the events are separated in time by 5 years as measured on rocket clocks?

b. What is the relative speed of the rocket and laboratory frames?

6: mapmaking in space

The table shows distances between cities. The units are kilometers. Assume all cities lie on the same flat plane.

a. Use a ruler and a compass (the kind of compass that makes circles) to construct a map of these cities. Choose a convenient scale, such as one centimeter on the map corresponds to ten kilometers on Earth.

Discussion: How to start? With three arbitrary decisions!

(1) Choose any city to be at the center of the map.

(2) Choose any second city to be "due north" - that is, along any arbitrary direction you select.

(3) Even with these choices, there are two places you can locate the third city; choose either of these two places arbitrarily.

b. If you rotate the completed map in its own plane - for example, turning it while keeping it flat on the table - does the resulting map also satisfy the distance entries above?

c. Hold up your map between you and a light, with the marks on the side of the paper facing the light. Does the map you see from the back also satisfy the table entries?

Discussion: In this exercise you use a table consisting only of distances between pairs of cities to construct a map of these cities from the point of view of a surveyor using a given direction for north. In Exercise 5-3 you use a table consisting only of space- time intervals between pairs of events to draw a “spacetime map” of these events from the point of view of one free-float observer. Exercise 7 previews this kind of spacetime map.

Table $1.E.2$: Distances Between Cities

	Distance to city							
	A	B	C	D	E	F	G	H
from city								
A	0	20.0	28.3	28.3	28.3	20.0	28.3	44.7
B		0	20.0	20.0	44.7	40.0	44.7	40.0
C			0	40.0	40.0	44.7	56.6	60.0
D				0	56.6	44.7	40.0	20.0
E					0	20.0	40.0	72.1
F						0	20.0	56.6
G							0	44.7
H								0

7: spacetime map

The laboratory space and time measurements of events 1 through 5 are plotted in the figure. Compute the value of the spacetime interval

a. between event 1 and event 2 .

b. between event 1 and event 3 .

c. between event 1 and event 4 .

d. between event 1 and event 5 .

e. A rocket moves with constant velocity from event 1 to event 2 . That is, events 1 and 2 occur at the same place in this rocket frame. What time lapse is recorded on the rocket clock between these two events?

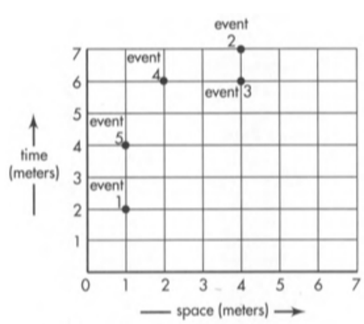

Figure $1.E.1$: Spacetime map of some events.

Problems

8: size of a computer

In one second some desktop computers can carry out one million instructions in sequence. One instruction might be, for instance, multiplying two numbers together. In technical jargon, such a computer operates at "one megaflop." Assume that carrying out one instruction requires transmission of data from the memory (where data is stored) to the processor (where the computation is carried out) and transmission of the result back to the memory for storage.

a. What is the maximum average distance between memory and processor in a "one-megaflop" computer? Is this maximum distance increased or decreased if the signal travels through conductors at one half the speed of light in a vacuum?

b. Computers are now becoming available that operate at "one gigaflop," that is, they carry out 10^9 sequential instructions per second. What is the maximum average distance between memory and processor in a "one-gigaflop" machine?

c. Estimate the overall maximum size of a "one-teraflop" machine, that is, a computer that can carry out 10^{12} sequential instructions per second.

d. Discussion question: In contrast with most current personal computers, a "parallel processing" computer contains several or many processors that work together on a computing task. One might think that a machine with 10,000 processors would complete a given computation task in $1/10,000$the time. However, many computational problems cannot be divided up in this way, and in any case some fraction of the computing capacity must be devoted to coordinating the team of processors. What limits on physical size does the speed of light impose on a parallel processing computer?

9: Trips to Andromeda by rocket

The Andromeda galaxy is approximately two million light-years distant from Earth as measured in the Earth-linked frame. Is it possible for you to travel from Earth to Andromeda in your lifetime? Sneak up on the answer to this question by considering a series of trips from Earth to Andromeda, each one faster than the one before. For simplicity, assume the Earth-Andromeda distance to be exactly two million light-years in the Earth frame, treat Earth and Andromeda as points, and neglect any relative motion between Earth and Andromeda.

a. *TRIP 1.* Your one-way trip takes a time 2.01 X 10^6 years (measured in the Earth-linked frame) to cover the distance of 2.00×10^6 light-years. How long does the trip last as measured in your rocket frame?

b. What is your rocket speed on Trip 1 as measured in the Earth-linked frame? Express this speed as a decimal fraction of the speed of light. Call this fraction, $v = v_{\text{conv}} / c$, where v_{conv} is speed in conventional units, such as meters/second.
Discussion: If your rocket moves at half the speed of light, it takes 4×10^6 years to cover the distance 2×10^6 lightyears. In this case

$$v = \frac{2 \times 10^6 \text{ light-years}}{4 \times 10^6 \text{ years}} = \frac{1}{2}$$

Therefore ...

c. *TRIP 2.* Your one-way Earth-Andromeda trip takes 2.001×10^6 years as measured in the Earth-linked frame. How long does the trip last as measured in your rocket frame? What is your rocket speed for Trip 2, expressed as a decimal fraction of the speed of light?

d. *TRIP 3.* Now set the rocket time for the one-way trip to 20 years, which is all the time you want to spend getting to Andromeda. In this case, what is your speed as a decimal fraction of the speed of light?
Discussion: Solutions to many exercises in this text are simplified by using the following approximation, which is the first two terms in the binomial expansion

$$(1+z)^n \approx 1 + nz \quad \text{if} \quad |z| << 1$$

Here n can be positive or negative, a fraction or an integer; z can be positive or negative, as long as its magnitude is very much smaller than unity. This approximation can be used twice in the solution to part **d**.

10: trip to Andromeda by Transporter

In the *Star Trek* series a so-called Transporter is used to "beam" people and their equipment from a starship to the surface of nearby planets and back. The Transporter mechanism is not explained, but it appears to work only locally. (If it could transport to remote locations, why bother with the starship at all?) Assume that one thousand years from now a Transporter exists that reduces people and things to data (elementary bits of information) and transmits the data by light or radio signal to remote locations. There a Receiver uses the data to reassemble travelers and their equipment out of local raw materials.

One of your descendants, named Samantha, is the first "transporternaut" to be beamed from Earth to the planet Zircon orbiting a star in the Andromeda Nebula, two million light-years from Earth. Neglect any relative motion between Earth and Zircon, and assume:
(1) transmission produces a Samantha identical to the original in every respect (except that she is 2 million light-years from home!),
and (2) the time required for disassembling Samantha on Earth and reassembling her on Zircon is negligible as measured in the common rest frame of Transporter and Receiver.

a. How much does Samantha age during her outward trip to Zircon?

b. Samantha collects samples and makes observations of the Zirconian civilization for one Earthyear, then beams back to Earth. How much has Samantha aged during her entire trip?

c. How much older is Earth and its civilization when Samantha returns?

d. Earth has been taken over by a tyrant, who wishes to invade Zircon. He sends one warrior and has him duplicated into attack battalions at the Receiver end. How long will the Earth tyrant have to wait to discover whether his ambition has been satisfied?

e. A second transporternaut is beamed to a much more remote galaxy that is moving away from Earth at 87 percent of the speed of light. This time, too, the traveler stays in the remote galaxy for one year *as measured by clocks moving with the galaxy* before returning to Earth by Transporter. How much has the transporternaut aged when she arrives back at Earth? (Careful!)

11: Time stretching with muons

At heights of 10 to 60 kilometers above Earth, cosmic rays continually strike nuclei of oxygen and nitrogen atoms and produce muons (muons: elementary particles of mass equal to 207 electron masses produced in some nuclear reactions). Some of the muons move vertically downward with a speed nearly that of light. Follow one of the muons on its way down. In a given sample of muons, half of them decay to other elementary particles in 1.5 microseconds $\left(1.5 \times 10^{-6}\right.$ seconds), measured with respect to a reference frame in which they are at rest. Half of the remainder decay in the next 1.5 microseconds, and so on. Analyze the results of this decay as observed in two different frames. Idealize the rather complicated actual experiment to the following roughly equivalent situation: All the muons are produced at the same height $(60$ kilometers); all have the same speed; all travel straight down; none are lost to collisions with air molecules on the way down.

a. Approximately how long a time will it take these muons to reach the surface of Earth, as measured in the Earth frame?

b. If the decay time were the same for Earth observers as for an observer traveling with the muons, approximately how many half-lives would have passed? Therefore what fraction of those created at a height of 60 kilometers would remain when they reached sea level on Earth? You may express your answer as a power of the fraction $1/2$.

c. An experiment determines that the fraction $1/8$ of the muons reaches sea level. Call the rest frame of the muons the rocket frame. In this rocket frame, how many half-lives have passed between creation of a given muon and its arrival as a survivor at sea level?

d. *In the rocket frame*, what is the space separation between birth of a survivor muon and its arrival at the surface of Earth? (Careful!)

e. From the rocket space and time separations, find the value of the spacetime interval between the birth event and the arrival event for a single surviving muon.

Reference: Nalini Easwar and Douglas A. MacIntire, *American Journal of Physics*, Volume 59, pages 589-592 (July 1991).

12: time stretching with π^+-mesons

Laboratory experiments on particle decay are much more conveniently done with π^+-mesons (pi-plus mesons) than with μ-mesons, as is seen in the table.

In a given sample of π^+-mesons half will decay to other elementary particles in 18 nanoseconds (18 X 10^{-9} seconds) measured in a reference frame in which the π^+-mesons are at rest. Half of the remainder will decay in the next 18 nanoseconds, and so on.

Exercise 1.12: time stretching with π^+-mesons

Particle	Time for half to decay (measured in rest frame)	"Characteristic distance" (speed of light multiplied by foregoing time)
muon (207 times electron mass)	1.5×10^{-6} second	450 meters
π^+-mesons (273 times electron mass)	18×10^{-9} seconds	5.4 meters

a. In a particle accelerator π^+-mesons are produced when a proton beam strikes an aluminum target inside the accelerator. Mesons leave this target with nearly the speed of light. If there were no time stretching and if no mesons were removed from the resulting beam by collisions, what would be the greatest distance from the target at which half of the mesons would remain undecayed?

b. The π^+-mesons of interest in a particular experiment have a speed 0.9978 that of light. By what factor is the predicted distance from the target for half-decay increased by time dilation over the previous prediction - that is, by what factor does this dilation effect allow one to increase the separation between the detecting equipment and target?