7.4: Momentum- "Space Part" of Momenergy

simply use proper time instead of Newton's so-called" "universal" time

Newton called momentum "quantity of motion." The expressions for momentum that spacetime physics gives us, the last three equations in $(7-2)$, seem at first sight to distinguish themselves by a trivial difference from the expressions for momentum given to us long ago by Newton's followers:

$$p_{x\,\text{Newton}}=m\frac{dx}{dt},\quad p_{y\,\text{Newton}}=m\frac{dy}{dt},\quad p_{z\,\text{Newton}}=m\frac{dz}{dt}$$

[valid for low velocity]

Newtonian versus relativistic expressions for momentum

That difference? Today, proper time $d\tau$ between nearby events on the worldline of the particle. Laboratory time, in older days, when the concepts of proper time and interval were unknown. The percentage difference between the two, trivial or even negligible under everyday circumstances, becomes enormous when the speed of the object approaches the speed of light.

We explore most simply the difference between relativistic and Newtonian predictions of momentum by analyzing a particle that travels with speed v in the x-direction only. Then the relation between displacement of this particle and its speed is $x=vt$. For small displacements, for example between two nearby spark events on the worldline, this becomes, in the mathematical limit of interest in calculus notation, $dx=vdt$.

The proper time between the two nearby sparks is always less than the laboratory time:

$$\begin{aligned} d\tau &= \left[(d\tau)^2\right]^{1/2}=\left[(dt)^2-(dx)^2\right]^{1/2}=\left[(dt)^2-(vdt)^2\right]^{1/2} \\ &=(dt)\left(1-v^2\right)^{1/2}=\frac{dt}{\gamma}\end{aligned} \tag{7.4.1}$$

where gamma, $\gamma=1/\left(1-v^2\right)^{1/2}$ is the time stretch factor (Section 5.8). This figure for the interval, or proper time, between the two nearby sparks we now substitute into equations $7.3.2)$ in order to learn how the relativistic expressions for energy and momentum depend on particle speed:

$$\begin{aligned} E &= m\frac{dt}{d\tau}=\frac{m}{\left(1-v^2\right)^{1/2}}=m\gamma \\ p_x &= m\frac{dx}{d\tau}=\frac{m(dx/dt)}{\left(1-v^2\right)^{1/2}}=\frac{mv_x}{\left(1-v^2\right)^{1/2}}=mv_x\gamma\end{aligned} \tag{7.4.2}$$

Low speed: Newton and Einstein agree on value of momentum

The momentum expression is the same as for Newtonian mechanics - mass m times velocity (dx/dt) - except for the factor $\left(1-v^2\right)^{1/2}$ in the denominator. That factor we can call 1 when the speed is small. For example, a commercial airliner moves through the air at approximately one millionth of the speed of light. Then the factor $(1-v^2)^{1/2}$ differs from unity by only five parts in 10^{12}. Even for an alpha particle (helium nucleus) ejected from a radioactive nucleus with approximately 5 percent of the speed of light, the correction to the Newtonian figure for momentum is only a little more than one part in a thousand. Thus for low speeds the momentum expressed in equation (7-5) reduces to the Newtonian version.

High speed: Relativity reveals much larger momentum

At a speed close to that of light, however, the particle acquires a momentum enormous compared with the Newtonian prediction. The unusually energetic cosmicray protons mentioned at the end of Section 5.8 crossed the Milky Way in 30 seconds of their own time, but a thousand centuries or 3×10^{12} seconds of Earth time. The ratio $dt/d\tau$ between Earth time and proper time is thus 10^{11}. That is also the ratio between the correct relativistic value of the protons' momentum and the Newtonian prediction.

Units: Both Newtonian and relativistic expressions for momentum contain speed, a ratio of distance to time. From the beginning we have measured distance and time in the same unit, for example meter. Therefore the ratio of distance to time is unit-free. In Section 2.8, we expressed speed as a dimensionless quantity, the fraction of light speed:

Unit of momentum: mass

$$\begin{aligned} v &= \frac{(\text{ meters of distance covered by particle })}{(\text{meters of time required to cover that distance})} \\ &= \frac{(\text{particle speed in meters / second })}{(\text{ speed of light in meters / second })} = \frac{v_{\text{conv}}}{c} \end{aligned} \tag{7.4.3}$$

In terms of speed v (called beta, β, by some authors), Newtonian and relativistic expressions for the magnitude of the momentum have the forms

$$p_{\text{Newton}} = mv \qquad [\text{valid for low speed}] \tag{7.4.4}$$

$$p = mv/\left(1-v^2\right)^{1/2} \qquad [\text{good at any speed}] \tag{7.4.5}$$

More Units: In order to convert momentum in units of mass to momentum in conventional units, such as kilogram meters/second, multiply expressions (7-6), $(7-7)$, and (7-8) by the speed of light c and use the subscript "conv" for "conventional"':

Conversion to conventional momentum units

$$\begin{aligned} p_{\text{conv Newton}} &= p_{\text{Newton}}\, c = mvc = m\left(v_{\text{conv}}/c\right)c = mv_{\text{conv}} \qquad [\text{low speed}]\ (7\text{-}9) \\ p_{\text{conv}} &= pc = \frac{mvc}{\left[1-v^2\right]^{1/2}} = \frac{m\left(v_{\text{conv}}/c\right)c}{\left[1-\left(v_{\text{conv}}/c\right)^2\right]^{1/2}} \\ &= \frac{mv_{\text{conv}}}{\left[1-\left(v_{\text{conv}}/c\right)^2\right]^{1/2}} \end{aligned} \tag{7.4.6}$$

Thus conversion from momentum in units of mass to momentum in conventional units is always accomplished by multiplying by the conversion factor c. This is true whether the expression for momentum being converted is Newtonian or relativistic. Table 7-1 at the end of the chapter summarizes these comparisons.

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