5.8: Stretch Factor

ratio of frame-clock time to wristwatch time

Different reference frames: different times between two events

A speeding beacon emits two flashes, F and S, in quick succession. These two flashes, as recorded in the rocket that carries the beacon, occur with a 6-meter separation in time but a zero separation in space. Zero space separation? Then 6 meters is the value of the interval, the proper time, the wristwatch time between F and S. As registered in the laboratory, in contrast, the second flash S occurs 10 meters of time later than the first flash F. The ratio between this frame time, 10 meters, and the proper time, 6 meters, between the two events we call the time stretch factor, or simply **stretch factor**. Some authors use the lowercase Greek letter gamma, γ, for the stretch factor, as we do occasionally. We will also use the Greek letter tau, τ, for proper time.[1]

Time lapse minimum for frame in which events occur at same place

The same two events register in the super-rocket frame that overtakes and passes the beacon - register with a separation in time of 20.88 meters. In this frame, the time stretch factor between the two events is $(20.88)/6 = 3.48$ In the beacon frame the stretch factor is unity: $6/6 = 1$. Why? Because in this beacon frame flashes F and S occur at the same place, so beacon-frame clocks record the proper time directly. This place proper time is less than the time between the two flashes as measured in either laboratory or super-rocket frame. The larger value of time observed in laboratory and super-rocket frames shows up in Figure $5.8.1$ (center and right). Among all conceivable frames, the separation in time between the two flashes evidently takes on its minimum value in the beacon frame itself, the value of the proper time τ.[2]

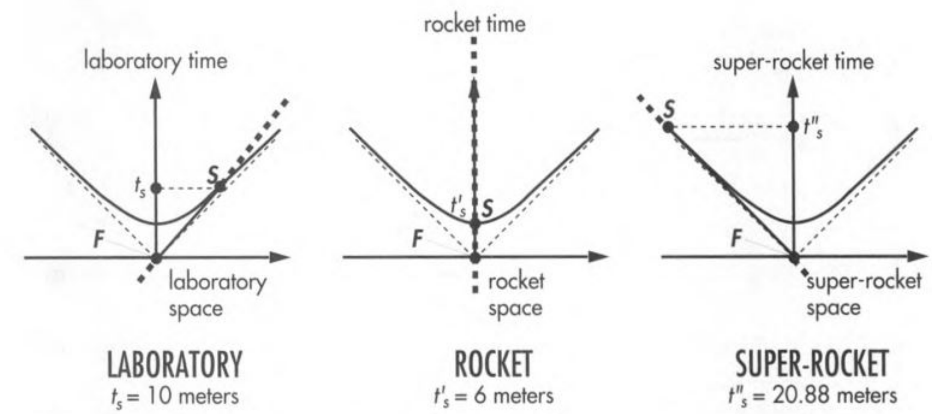


Figure $5.8.1$: **Spacetime maps of Figure 5.2.2, modified to show the worldline of the speeding beacon (heavy dashed line) and the segment of this line between emission F of the first flash and the secondflash S (solid section of worldline).** Emission F is taken as the zero of space and time. Time t_s of the second emission S is different as recorded in different frames. The shortest time is recorded in that frame in which the two events occur at the same place— in this case the rocket frame.

Question and Answer

Hold it! In Sections 5.6 and 5.7 you insisted that the time along a straight worldline is a MAXIMUM. Now you show us a straight worldline along which the time is - you say - a MINIMUM. Maximum or minimum? Please make up your mind!

Answer

The worldline taken by the beacon wristwatch from F to S is straight. It is straight ^ whether mapped in the beacon frame itself or in the rocket or super-rocket frame. The beacon racks up 6 meters of proper time regardless of the frame in which we reckon this time. When we turn from this wristwatch time to what different free-float frames show for the separation in map time (latticework time, frame time) between the two flashes, however, the record displays a minimal value for that separation in time only in the beacon frame itself.

In contrast. Figure $5.8.2$ (Figure 5.7.1 in simplified form) shows two different worldlines that join events 0 and B mapped in the same reference frame. In this case we compare two different proper times: a proper rime of 10 meters racked up by a wristwatch carried along the direct course from 0 to B, and a proper time of 6 meters recorded by the wristwatch carried along on the kinked worldline OQB. In every such comparison made in the context of flat spacetime, the direct worldline displays maximum proper time. Caution: Conditions can be different in curved spacetime (Chapter 9).

In summary, two points come to the fore in these comparisons of the time between two events. (1) Are we comparing map time (frame time, latticework time) between those two events, pure and simple, free of any talk about any wotldline that might connect those events? Then separation in time between those events is least as mapped in the free-float frame that shows them happening at the same place. (2) Or are we directing our arrention to a worldline that connects the two events? More specifically, to the time racked up by a wrisrwarch toted along that worldline? Then we have to ask, is that worldline straight? Then it registers maximal passage of proper time. Or does it have a kink? Then the proper time racked up is not maximal.

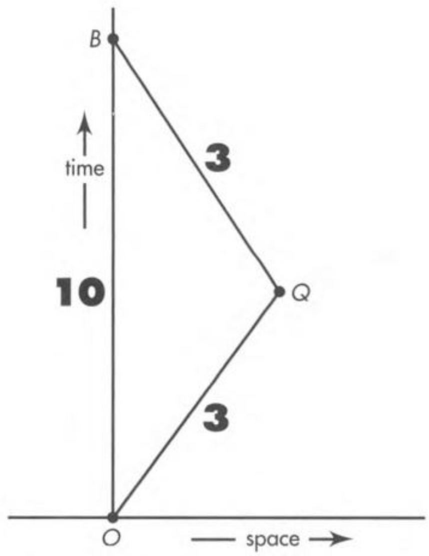


Figure 5.8.2: **Figure 5.7.1 stripped down to emphasize total proper time (wristwatch time), printed boldface along two different worldlines between the same two events O and B in a given reference frame.** Among all possible worldlines connecting events O and B, the straight worldline registers maximal lapse of proper time.

When we find ourselves in a free-float frame and see a beacon zooming past in a straight line with speed v, how much is the factor by which our frame-clock time is stretched relative to the beacon wristwatch time? Answer: The stretch factor is

$$(\text{stretch factor}) = \gamma = \frac{1}{\left(1-v^2\right)^{1/2}} \tag{5.8.1}$$

How can we derive this famous formula? If you do not cover up the following lines and derive this answer on your own, here is the reasoning: Start with measurements in the laboratory frame. We know that for this rocket

$$(\text{advance in proper time })^2 = (\text{ advance in lab time })^2 - (\text{ lab distance covered })^2$$

Stretch factor = frame time/proper time

However, we want to compare lapses in laboratory time and proper time; laboratory distance covered is not of interest. For the laboratory observer the proper clock moving along a straight worldline covers the distance between the two events in the time between the events. Therefore this distance and time are related by particle speed:[3]

$$(\text{lab distance covered}) = (\text{ speed}) \times (\text{advance in lab time})$$

Substitute this expression into the equation for proper time:

$$\begin{aligned} (\text{ proper time })^2 &= (\text{ lab time })^2 - (\text{ speed })^2 \times (\text{ lab time })^2 \\ &= (\text{ lab time })^2 \left[1 - (\text{ speed })^2\right] \end{aligned}$$

This leads to an expression for the square of the stretch factor:

$$\frac{(\text{ lab time })^2}{(\text{ proper time })^2} = (\text{ stretch factor })^2 = \frac{1}{1-(\text{ speed })^2} = \frac{1}{1-v^2}$$

where we use the symbol $v = v_{\text{conv}} / c$ for speed. The equation for the stretch factor becomes[4]

Stretch factor derived

$$(\text{ stretch factor }) = \gamma = \frac{1}{\left(1 - v^2\right)^{1/2}} \tag{5.8.1}$$

The stretch factor has the value unity when $v = 0$. For all other values of v the stretch factor is greater than unity. For very high relative speeds, speeds close to that of light $(v \rightarrow 1)$, the value of the stretch factor increases without limit.

The value of the stretch factor does not depend on the direction of motion of the rocket that moves from first event to second event: The speed is squared in equation 5.8.1, so any negative sign is lost.

Lorentz-contraction by same " stretch" factor

The stretch factor is the ratio of frame time to proper time between events, where speed $(= v)$ is the steady speed necessary for the proper clock to pass along a straight worldline from one event to the other in that frame.

> The stretch factor also describes the Lorentz contraction, the measured shortening of a moving object along its direction of motion when the observer determines the distance between the two ends at the same time. For example, suppose you travel at speed v between Earth and a star that lies distance L away as measured in the Earth frame. Your trip takes time $t = L/v$ in the Earth-linked frame. Proper time τ - your wristwatch time - is smaller than this by the stretch factor: $\tau = L/[v \times (\text{ stretch factor })] = (L/v)(1 - v^2)^{1/2}$. Now think of a very long rod that reaches from Earth to star and is at rest in the Earth frame. How long is that rod in your rocket frame? In your frame the rod is moving at speed v. One end of the rod, at the position of Earth, passes at speed v. A time τ later in your frame the other end of the rod arrives - along with the star - also moving at speed v according to your rocket measurements. From these data you calculate that the length of the rod in your rocket frame - call it L' - is equal to $L' = v\tau = v(L/v)\left(1 - v^2\right)^{1/2} = L \left(1 - v^2\right)^{1/2}$. This is a valid measure of length. By this method the rod is measured to be shorter.[5]

Stretch factor as a measure of speed

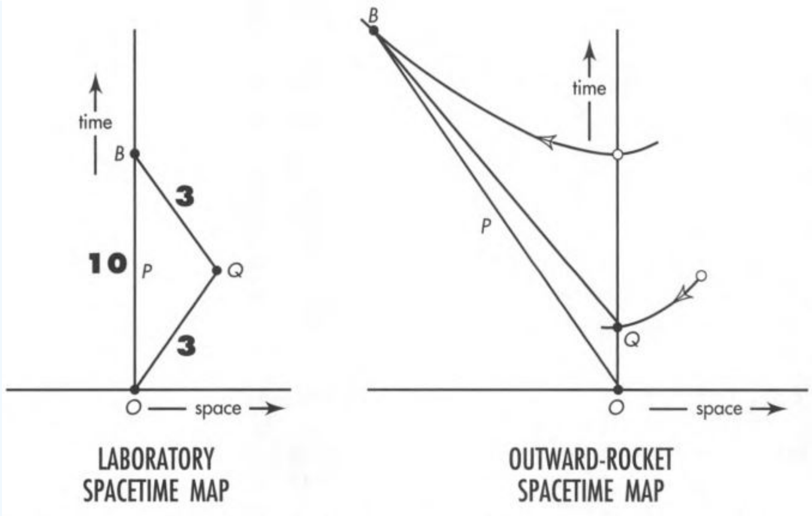
Finally, the stretch factor is often used as an alternative measure of particle speed: A particle moves with a speed such that the stretch factor is 10 . This statement assumes that the particle is moving with constant speed, so that the separation between any pair of events on the particle worldline has the same stretch factor as the separation between any other pair. This way of describing particle speed can be both convenient and powerful. We will see (Chapter 7) that the total energy of a particle is proportional to the stretch factor.[6]

Example $5.8.1$: Round Trip Observed in Different Frame

Return to the alternative worldlines between events O and B, shown in Figure 5.7.1 and the spacetime maps in this sample problem. Measure these worldlines from a rocket frame that moves outward with the particle from O to Q and keeps on going forever at the same constant velocity. Show that an observer in this outward-rocket frame predicts the same proper time— wristwatch time — for worldline OQB as that predicted in the laboratory frame. Similarly show that this outward-rocket-frame observer predicts the same proper time along the direct worldline OPB as does the laboratory observer. Finally, show that both observers predict the elapsed wristwatch time along OQB to be less than along OPB.

Solution

Here are laboratory and rocket spacetime maps for these round trips, simplified and drawn to reduced scale.



Laboratory and outward-rocket spacetime maps, each showing alternative worldlines (direct OPB and indirect OQB) between events O and B. Laboratory spacetime map: Figure 5.7.1, redrawn to a different scale. Proper times are shown on the laboratory spacetime map. Outward-rocket spacetime map: The rocket in which the outgoing particle is at rest. Portions of two invariant hyperbolas show how events Q and B transform. The direct worldline OPB has longer total proper time— greater aging— as computed using measurements from either frame.

Find x'_Q and t'_Q: First compute space and time locations of events Q and B in the outgoing rocket frame — right-hand map. (Event 0 is the reference event, x = 0 and t = 0 in all frames by convention.) We choose the rocket frame so that the worldline segment OQ lies vertical and the outbound rocket does not move in this frame. As a result, event Q occurs at rocket space origin: $x'_Q = 0$. (Primes refer to measurements in the outwardrocket frame.) The rocket time t'_Q for this event is just the wristwatch time between 0 and Q, because the wristwatch is at rest in this frame: $t'_Q = 3$ meters.

In summary, using a prime for rocket measurements:

$$x'_Q = 0$$
$$t'_Q = 3 \text{ meters}$$

Find x'_B and t'_B: In the laboratory frame, the particle moves to the right from event O to event Q, covering 4 meters of distance in 5 meters of time. Therefore its speed is the fraction $v = 4/5 = 0.8$ of light speed. As measured in the rocket frame, the laboratory frame moves to the left with speed $v = 0.8$, by symmetry. Use equation 5.8.1 with $v = 0.8$ to compute the value of the stretch factor:

$$\frac{1}{[1-v^2]^{1/2}} = \frac{1}{[1-(0.8)^2]^{1/2}} = \frac{1}{[1-0.64]^{1/2}} = \frac{1}{[0.36]^{1/2}} = \frac{1}{0.6} = \frac{10}{6} = \frac{5}{3}$$

This equals the ratio of rocket time period t'_B to proper time τ_B along the direct path OPB. Hence elapsed rocket time $t'_B = (5/3) \times 10$ meters $= 50/3$ meters of time. In this time, the laboratory moves to the left in the rocket frame by the distance $x'_B = -vt'_B = -(4/5)(50/3) = -200/15 = -40/3$meters. In summary for outgoing rocket:

$$x'_B = -\frac{40}{3} \text{ meters } = -13\frac{1}{3} \text{ meters}$$
$$t'_B = \frac{50}{3} \text{ meters } = 16\frac{2}{3} \text{ meters of time}$$

Events Q and B are plotted on the rocket spacetime map.

Compare Wristwatch Times: Now compute the total proper time-wristwatch time, aging-along alternative worldlines OPB and OQB using rocket measurements. Direct worldline OB has proper time τ_{OB} given by the regular expression for interval:

$$(\tau_{OB})^2 = (t'_{OB})^2 - (x'_{OB})^2 = \left(\frac{50}{3}\right)^2 - \left(-\frac{40}{3}\right)^2$$
$$= \frac{2500}{9} - \frac{1600}{9} = \frac{900}{9} = 100 \text{ (meters)}^2$$

whence $\tau_{OB} = 10$ meters computed from rocket measurements. This is the same value as computed in the laboratory frame (in which proper time equals laboratory time, since laboratory separation in space is zero).

Worldline OQB has two segments. On the first segment, OQ, proper time lapse is just equal to the rocket time span, 3 meters, since the space separation equals zero in the rocket frame. For the second segment of this worldline, QB, we need to compute elapsed time in this frame:

$$\begin{aligned} t'_{QB} &= t'_B - t'_Q = \frac{50}{3} - 3 = \frac{50}{3} - \frac{9}{3} = \frac{41}{3} \text{ meters} \\ x'_{QB} &= -\frac{40}{3} \text{ meters} \end{aligned}$$

Therefore,

$$\begin{aligned} (t_{\mathrm{QB}})^2 &= \left(t'_{QB}\right)^2 - \left(x'_{QB}\right)^2 = \left(\frac{41}{3}\right)^2 - \left(\frac{40}{3}\right)^2 \\ &= \frac{1681}{9} - \frac{1600}{9} = \frac{81}{9} = 9(\text{ meters })^2 \end{aligned}$$

whence $\tau_{QB} = 3$ meters. So the total increase in proper time-the total aging-along worldline OQB sums to $3 + 3 = 6$ meters as reckoned from outward-rocket measurements. This is the same as figured from laboratory measurements.

Exercise 5.8.2

How can these weird results be true? In our everyday lives why don't we bave to take account of clocks that record different elapsed times between events, and rods that we measure to be contracted as they speed by us?

Note

In answer, consider two events that occur at the same place in our frame. The proper clock moving in spacetime between these two events has speed zero for us. In this case the stretch factor has the value unity: the frame clock is the proper clock. The same is approximately true for events that are much closer together in space (measured in meters) than the time between them (also measured in meters). In these cases the proper clock moving between them has speed v - measured in meters/ meter - that is very much less than unity. That is, the proper clock moves very much slower than the speed of light. For such slow speeds, the stretch factor has a value that approaches unity; the proper clock records very nearly the same time lapse between two events as frame clocks. This is the situation for all motions on earth that we can follow by eye. For all such "ordinary-speed" motions, moving clocks and stationary clocks record essentially the same time lapses. This is the assumption of Newtonian mechanics: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external..."

A similar argument leads to the conclusion that Lorentz contraction is negligible for objects moving at everyday speeds. Newton's mechanics - with its unique measured time between events and its unique measured length for an object whether or not it moves - gives correct results for objects moving at everyday speeds. In contrast, for particle speeds approaching light speed (approaching one meter of distance traveled per meter of elapsed time in the laboratory frame), the denominator on the right of equation 5.8.1 approaches zero and the stretch factor increases without limit. Increased without limit, also, is the laboratory time between ticks of the zooming particle's wristwatch. This is the case for high-speed particles in accelerators and for cosmic rays, very high-energy particles (mostly protons) that continually pour into our atmosphere from space. Newton's mechanics gives results wildly in error when applied to these particles and their interactions; the laws of relativistic mechanics must be used.

More than one cosmic ray has been detected (indirectly by the resulting shower of particles in the atmosphere) moving so fast that it could cross our galaxy in 30 seconds as recorded on its own wristwatch. During this trip a thousand centuries pass as recorded by clocks on Earth! (See Exercise 7-7.)

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1 Different reference frames: different times between two events

2 Time lapse minimum for frame in which events occur at same place

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