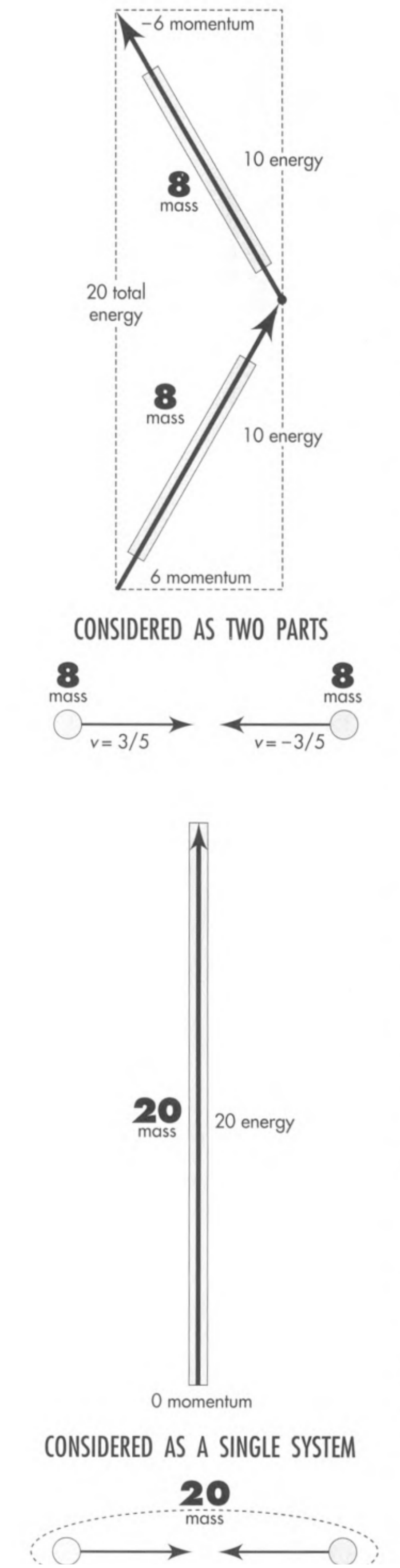
# 8.3: Mass of a System of Particles

## energies add. momenta add. masses do not add.

No one with any detective instincts will rest content with the vague thought that heat has mass. Where within our stuck-together wads of chewing gum or Rumford's barrel of water or Braginsky's quartz pellet is that mass located? In random motions of the atoms? Nonsense. Each atom has mass, yes. But does an atom acquire additional mass by virtue of any motion? Does motion have mass? No. Absolutely not. Then where, and in what form, does the extra mass reside? Answer: Not in any part, but in the system.

Heat resides not in the particles individually but in the system of particles. Heat arises not from motion of one particle but from relative motions of two or more particles. Heat is a system property.

The mass of a system is greater when system parts move relative to each other. Of this central point, no simpler example offers itself than a system composed of a single pair of masses. Our example? Two identical objects (Figure $8.3.1$). Each has mass 8 . Relative to the laboratory frame of reference each object has momentum 6 , but the two momenta are opposite in direction. The energy of each object is $E=\left(m^{2}+p^{2}\right)^{1/2}= \left(8^{2}+6^{2}\right)^{1/2}=10$.



−6 momentum
10 energy
8
mass
20 total
energy
8
mass
10 energy
6 momentum
CONSIDERED AS TWO PARTS
8
mass
8
mass
v = 3/5
v = −3/5
20
mass
20 energy
0 momentum
CONSIDERED AS A SINGLE SYSTEM
20
mass

Figure 8.3.1: Two noninteracting particles, magnitude - the length of the arrow of total momenergy, figured as we figure any each of mass 8 , are in relative motion. Taken spacetime interval - is system mass. Whether the system consists of a single object or together, they constitute a system of mass 20. of many objects, and whether these objects do or do not collide or otherwise interact with each other, this system mass never changes. That's why the concept of system mass makes sense!

The total momentum of the two-object system is $p_{\text{system}} = 6-6=0$ . The energy of the system is $E_{\text{system}} = 10+10=20$ . Therefore the mass of the system is $M_{\text{system}} = \left(E_{\text{system}}{}^2 - p_{\text{system}}{}^2\right)^{1/2} = \left\{(20)^2 - 0^2\right]^{1/2} = 20$ . Thus the mass of the system exceeds the sum of the masses of the two parts of the system. The mass of the system does not agree with the sum of the masses of its parts.

Energy is additive. Momentum is additive. But mass is *not* additive.

Ask where the *extra* $20-16=4$ units of mass are located? Silly question, any answer to which is also silly!

Ask where the 20 units of mass are located? Good question, with a good answer. The 20 units of mass belong to the system as a whole, not to any part individually.

Where is the life of a puppy located? Good question, with a good answer. Life is a property of the *system* of atoms we call a puppy, not a property of any part of the puppy.

Where is the *extra* ingredient added to atoms to yield a live puppy? Unacceptable question, any answer to which is also unacceptable. Life is not a property of any of the individual atoms of which the puppy is constituted. Nor is it a property of the space between the atoms. Nor is it an ingredient that has to be added to atoms. Life is a property of the puppy *system*.

Life is remarkable, but in one respect the two-object system that we are talking about is even more remarkable. Life requires organization, but the two-object system of Figure $8.3.1$ lacks organization. Neither mass interacts with the other. Yet the total energy of the two-object system, and its total momentum, regarded from first one frame of reference, then another, then another, take on values identical in every respect to the values they would have were we dealing throughout with a single object of mass 20 units. Totally unlinked, the two objects, viewed as a system, possess the dynamic attributes - energy, momentum, and mass - of a single object.

This wider idea of mass - the mass of an isolated *system* composed of disconnected objects: what right have we to give it the name "mass"? Nature, for whatever reason, demands conservation of total momenergy in every collision. Each collision, no matter how much it changes the momenergy of each participant, leaves unchanged the sum of their momenergies, regarded as a directed arrow in spacetime-a 4-vector. Encounter or no encounter, and however complex any encounter, system momenergy does not alter. Neither in spacetime direction nor in magnitude does it ever change. But the magnitude — the length of the arrow of total momenergy, figured as we figure any spacetime interval — is system mass. Whether the system consists of a single object or of many objects, and whether these objects do or do not collide or otherwise interact with each other, this system mass never changes. That's why the concept of system mass makes sense!

***Different free-float frames. Same system mass.***

An example? Again, two objects of mass 8 , again each moving toward a point midway between them at $v = (\text{ momentum })/(\text{ energy }) = (p=6)/(E=10) = 3/5$ the speed of light. Now, however, we analyze the two motions in a frame moving with the Same system mass. right-hand object (Figure 8-4). In this new frame the right-hand object is at rest: mass, $m=8$; momentum, $p=0$; energy, $E = \left[m^2+p^2\right]^{1/2} = 8$ . The left-hand object is approaching with a speed (addition of velocities: Section L. 7 of the Special Topic following Chapter 3; also Exercise 3-11)

$$v = \frac{3/5+3/5}{1+(3/5)(3/5)} = \frac{6/5}{34/25} = \frac{15}{17}$$

It has energy $E = m/\left[1-v^2\right]^{1/2} = 8/\left[1-(15/17)^2\right]^{1/2} = 17$ and momentum $p = vE = 15$. So much for the parts of the system! Now for the system itself. For the system the energy is $E_{\text{system}} = 8+17=25$ and the momentum is $p_{\text{system}} = 0+15=15$ .

Now for the test! Does the concept of system mass make sense? In other words, does system mass turn out to have the same value in the new frame as in the original frame? It does:

$$\begin{aligned} M_{\text{system}} &= \left(E_{\text{system}}^2 - p_{\text{system}}^2\right)^{1/2} = \left[(25)^2 - (15)^2\right]^{1/2} = [625 - 225]^{1/2} \\ &= [400]^{1/2} = 20 \end{aligned}$$

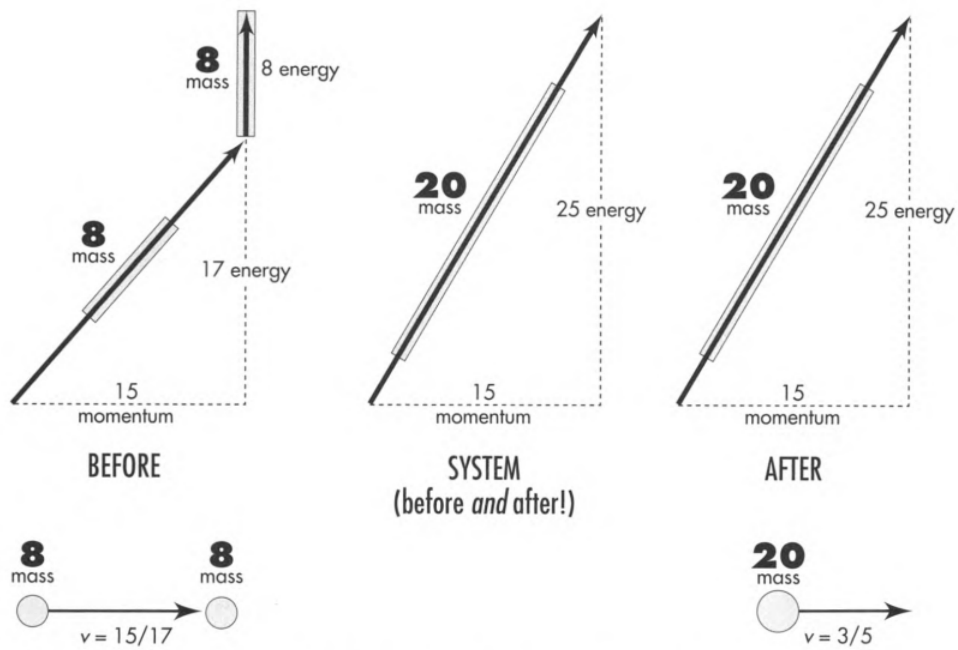



Figure 8.3.2: System of Figure 8.3.1 observed from a frame moving with the right-hand object. The right-band object is therefore initially at rest. Before: Arrows of momenergy for two objects before collision. Each object has a mass of eight units (shaded handles). The upper, vertical, arrow belongs to the particle originally at rest, the slanted arrow to the incoming particle. System: Addition of the two momenta (one of them zero!') gives the total momentum before collision. Similarly, addition of the two energies gives the total energy. Mass of the system - even before the two particles interact. - comes from the expression for the "hypotenuse" of a spacetimelike triangle. Result: 20 units of mass (shaded handle on center 4-vector)

$$(\text{ mass })^2 = (\text{ energy })^2 - (\text{ momentum })^2 = (25)^2 - (15)^2 = 625 - 225 = 400 = (20)^2$$

After: The two particles now collide and amalgamate to form one particle. Arrow of total momenergy after the amalgamation is identical to arrow of total momenergy before the collision. Mass of this two-object system exceeds the mass of one object plus the mass of the other, not only after the collision but also before. Mass is not an additive quantity.

### ✔ Example 8.3.1 MASS OF A SYSTEM OF MATERIAL PARTICLES \)

Compute $M_{\text{system}}$ for each of the following systems. The particles that make up these systems do not interact with one another. Express the system mass in terms of the unit mass $m$; do not use momenta or velocities in your answers. [Note: In the following diagrams, arrows represent (3-vector) momenta.]

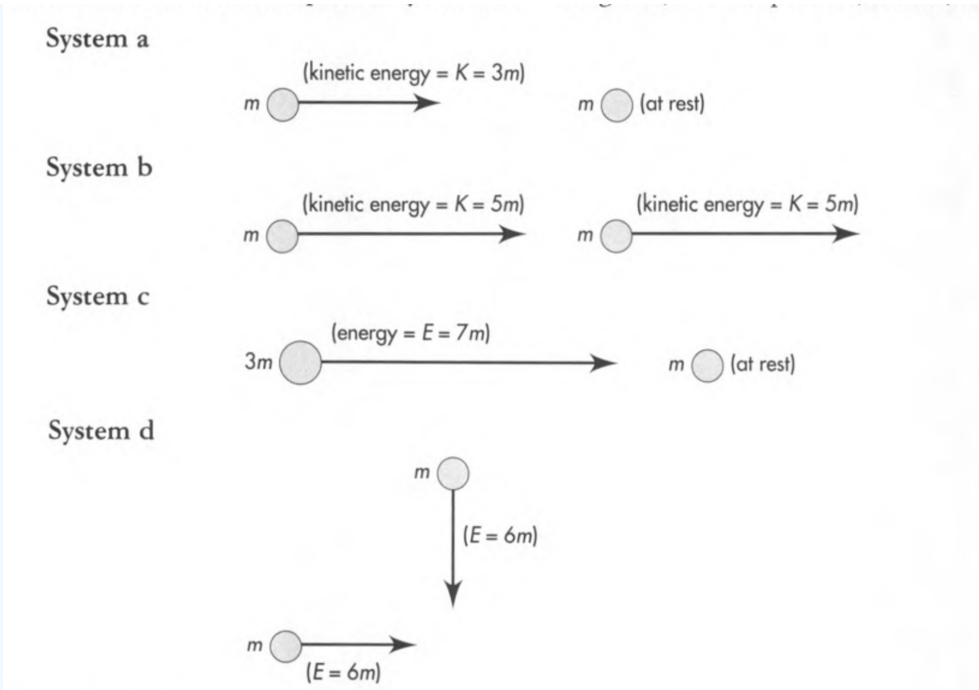



Figure 8.3.3:

### Solution

**System a:** System energy equals the rest energy of the two particles (the sum of their masses) plus the kinetic energy of the moving particle: $E_{\text{system}} = (m+m)+3m = 5m$. Squared momentum of the system equals that of the moving particle: $p_{\text{system}}^2 = p^2 = E^2 - m^2 = (4m)^2 - m^2 = 15m^2$. Mass of the system is reckoned from the difference between the squares of energy and momentum:

$$M_{\text{system}} = \left[E_{\text{system}}{}^2 - p_{\text{system}}{}^2\right]^{1/2} = \left[25m^2 - 15m^2\right]^{1/2} = [10]^{1/2} m = 3.162m$$

**System b:** System energy equals rest energy of the two particles plus kinetic energy of the two particles: $E_{\text{spseem}} = 2m + 10m = 12m$. Squared momentum of each particle is $p^2 = E^2 - m^2 = (6m)^2 - m^2 = 35m^2$ yielding $p = (35)^{1/2} m$. System momentum is twice this: $p_{\text{system}} = 2(35)^{1/2}$ m. The mass of the system is

$$\begin{aligned} M_{\text{spssem}} &= \left[E_{\text{sssem}}{}^2 - p_{\text{pssstem}}{}^2\right]^{1/2} = \left[(12m)^2 - \left\{2(35)^{1/2} m\right\}^2\right]^{1/2} \\ &= [144 - 140]^{1/2} m = [4]^{1/2} m = 2m \end{aligned}$$

In this one special case the mass of the system equals the sum of masses of the objects that make up the system. We could have seen this result immediately by observing the system from a reference frame that moves along with the particles. In this frame the particles are at rest and have zero total momentum; the total energy is identical to the sum of the individual rest energies (the individual masses). So in this case the mass of the system is equal to its energy, which is equal to the sum of masses. Moreover, system mass is an invariant. Thus $2m$ is the mass of the system as reckoned in all reference frames, including the one in which System b is pictured.

**System c**: Total energy $=$ system energy $= E_{\text{system}} = 7m + m = 8m$. System momentum equals the momentum of the moving particle: $p_{\text{pspstem}}{}^2 = E^2 - m^2 = (7m)^2 - (3m)^2 = 49m^2 - 9m^2 = 40m^2$. Hence the system mass is

$$M_{\text{system}} = \left[64m^2 - 40m^2\right]^{1/2} = [24]^{1/2} m = 4.899m$$

**System d**: This part of the problem serves as a reminder that momentum is a Euclidean 3 -vector. The squared momentum of each particle is $p^2 = E^2 - m^2 = 36m^2 \quad -m^2 = 35m^2$. Their total momentum is not the algebraic sum of the momenta, because they are vectors pointing in perpendicular directions. This perpendicular orientation allows us to equate the squared system momentum to the sum of the squares of the individual momenta: $p_{\text{system}}{}^2 = 35m^2 + 35m^2 = 70m^2$. System energy is the sum of the energies (energy is a scalar and adds like a scalar!): $E_{\text{system}} = 6m + 6m = 12m$. Hence system mass is

$$M_{\text{system}} = \left[144m^2 - 70m^2\right]^{1/2} = [74]^{1/2}m = 8.602m$$

Compare this result with that of System b, which also contained two particles, each of total energy $6m$.

Moreover, if the two objects collide and amalgamate, the system energy remains at the value 25 , the system momentum remains at the value 15 , and the system mass remains 20 , as illustrated in Figure $8.3.2$.

In summary, the mass of an isolated system has a value independent of the choice of frame of reference in which it is figured. System mass remains unchanged by encounters between the constituents of the system. And why? Because the system mass is the length (in the sense of spacetime interval) of the arrow of total momentum-energy. This momenergy total is unaffected by collisions among the parts or by any transformations, decays, or annihilations they may undergo. System mass does make sense!

### Question and Answer

*System!' System!' You keep talking about "system," even when the particles do not interact, as in the system of Figure $8.3.2$. It seems to me that you are totally arbitrary in the way you define a system. Who chooses which particles are in the system?*

#### Answer

We do! We can draw the dashed line around any collection of objects whatever, subject to this one restriction: no object in our system may interact with any external object or experience a force from outside the system. Our system must be $i$ solated. With that single limitation, the system we choose is arbitrary, has a conserved total energy, a conserved total momentum, and a system mass that is invariant-a mass that has the same value no matter in which free-float frame it is reckoned.

### Question and Answer

*I can't believe the story you tell. Those two mass-8 objects, you say, may fly past each other. Then your talk about the system mass is just talk, terminology. Or they may whang into each otber and amalgamate. Then your talk is all wrong, and for an obvious reason. As the objects collide they slow and come to rest relative to each other. At that instant and in that "'rest frame'" (the frame of Figure $8.3.2$), each bas zero momentum, and energy equal to its mass. So the total momentum of the system is zero, and its total energy is $8+8=16$ . That means a mass of 16 . Yet you claim 20 .*

#### Answer

Slow and come to rest? Yes. But that means force: "elastic," gravitational, electromagnetic, or nuclear force. That's the new and valuable point you make here. And those particles, pushing against that force, store up energy. This energy, too, has to be put into the bookkeeping. When amalgamating particles come to rest relative to one another, the energy of interaction "balances the books" - it so happens - and leads to a final mass of 20 , greater than the sum of masses of the original objects. For the figuring of system mass, however, we really don't have to get into this detail. It is enough for us to know that total momentum is conserved, $p_{\text{system}} = 0$ in Figure $8-3$, and total energy - in whatever way it is apportioned between the objects and the fields of force that act between them - is also conserved, $E_{\text{system}} = 20$. The length, in the sense of interval, of the 4-vector of momenergy for the system remains unchanged: $M_{\text{sysem}} = 20$.

Table $8.3.1$: CLEOPATRA'S VASE, HER BATH, AND INTERSTELLAR VACUUM: ILLUSTRATIVE FRACTIONAL CHANGES IN MASS OF SYSTEMS

| System before | System after | Fractional increase in system mass (to nearest power of 10) |
|---|---|---|
| One-kilogram vase | Vase sm ashed in to so many fragments that 100 $cm^2$ of glass-to-glass bonds are broken | $10^{-18}$ |
| Bath water at 15° C | Bath water at 40° C | $10^{-12}$ |
| Water $H_2O$ | Atomic hydrogen (H) and oxygen (O) | $10^{-9}$ |

| System before | System after | Fractional increase in system mass (to nearest power of 10) |
|---|---|---|
| Earth | All molecules of Earth lifted against the pull of their mutual gravity to infinite separation from one another | $10^{-9}$ |
| Hydrogen atom in lowest energy state | Electron withdrawn to infinite separation from nucleus | $10^{-8}$ |
| Deuteron | Deuteron separated into proton and neutron | $10^{-3}$ |
| Neutron star | Widely separated iron atoms at rest with respect to each other | $10^{-1}$ |
| A vacuum before it is zapped by converging photons | Electron-positron pair bound as a positronium atom | Infinite fractional increase |

***System energy increase? System mass can increase.***

What about a system that is not isolated? A system that has - and keeps - zero momentum, but receives an increment of energy? Then its mass rises by an amount exactly equal to that input of energy. The increase in mass is the same whether that energy goes into altering the relative motion of the parts of the system or increasing the energy of interaction between them or some combination of motion and interaction. Supply energy to a system by heating it or setting it into internal vibration or fracturing the bonds between its parts? Each is a guaranteed way to increase the mass of the system (Table 8-1)!