5.5: Length Along a Path

straight line has shortest length between two given points in space

Measure length of curved path with tape measure . . .

Distance is a central idea in all applications of Euclidean geometry. For instance, using a flexible tape measure it is easy to quantify the total distance along a winding path that starts at one point (point O in Figure $5.5.1$) and ends at another point (point B). Another way to measure distance along the curved path is to lay a series of short straight sticks end to end along the path. Provided the straight sticks are short enough to conform to the gently curving path, total distance along the path equals the sum of lengths of the sticks.[1]

. . . or with short straight sticks laid end to end along path

The length of a short stick laid between any two nearby points on the path - for instance, points 3 and 4 in Figure $5.5.1$ - can also be calculated using the northward separation and the eastward separation between the two ends of the stick as measured by a surveyor.[2]

$$(\text{ length })^2=(\text{ northward separation })^2+(\text{ eastward separation })^2$$

All surveyors agree on length of path

Distance is invariant for surveyors. Therefore the length of this stick is the same when calculated by any surveyor, even though the northward and eastward separations between two ends of the stick have different values, respectively, for different surveyors. The length of another stick laid elsewhere along the path is also agreed on by all surveyors despite their use of different northward directions. Therefore the sum of the lengths of all short sticks laid along the path has the same value for all surveyors. This sum equals the value of the total length of the path, on which all surveyors agree. And this total length is just the length measured using the flexible tape.[3]

It is possible to proceed from O to B along quite another path - for example along straight line OB in Figure $5.5.1$. The length of this alternative path is evidently different from that of the original curved path. This feature of Euclidean geometry is so well known as to occasion hardly any comment and certainly no surprise: In Euclidean geometry a curved path between two specified points is longer than a straight path between them. The existence of this difference of length between two paths violates no law. No one would claim that a tape measure fails to perform properly when laid along a curved path.

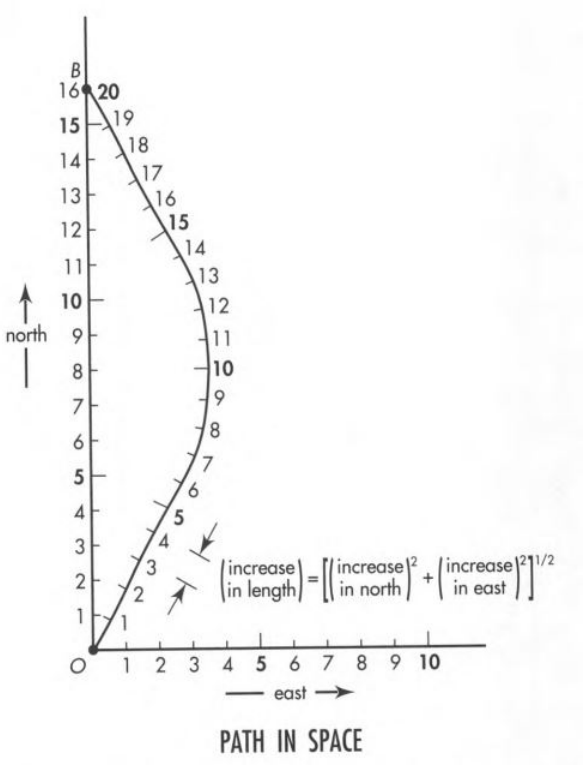

Figure 5.5.1: **Length along a winding path starting a t the town square.** Notice that the total length along the winding path from point O to point B is greater than the length along the straight northward axis from O to B.

Among all possible paths between two points in space, the straight-line path is unique. All surveyors agree that this path has the shortest length. When we speak of "the distance between two points," we ordinarily mean the length of this straight path.[4]

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1 Measure length of curved path with tape measure...

2 ...or with short straight sticks laid end to end along path

3 All surveyors agree on length of path

4 Straight path in space has shortest length
