11.6: Inverse Lorentz Transformation

from laboratory event coordinates, reckon rocket coordinates

Equations 11.5.5 and 11.5.6 provide laboratory coordinates of an event when one knows the rocket coordinates of the same event. But suppose that one already knows the laboratory coordinates of the event and wishes to predict the coordinates of the event measured by the rocket observer. What equations should be used for this purpose?

An algebraic manipulation of equations 11.5.5 and 11.5.6 provides the answer. The first two of these equations can be thought of as two equations in the two unknowns x' and t'. Solve for these unknowns in terms of the now-knowns x and t. To do this, multiply both sides of the second equation by v_{rel} and subtract corresponding sides of the resulting second equation from the first. Terms in x' cancel to yield

Long derivation of inverse Lorentz transformation

$$t - v_{\text{rel}} x = \gamma t' - v_{\text{rel}}^2 \gamma t' = \gamma \left(1 - v_{\text{rel}}^2\right) t' = \frac{\gamma}{\gamma^2} t' = \frac{t'}{\gamma}$$

Here we have used the definition $\gamma^2 = 1/\left(1 - v_{\text{rel}}^2\right)$. The equation for t' can then be written

$$t' = -v_{\text{rel}}\, \gamma x + \gamma t$$

A similar procedure leads to the equation for x'. Multiply the first of equations 11.5.5 and 11.5.6 by v_{rel} and subtract corresponding sides of the first equation from the second -try it! The y and z components are respectively equal in both frames, as before.

Inverse Lorentz transformation

Then the **inverse Lorentz transformation equations** become

$$\begin{aligned} t' &= -v_{\text{rel}}\, \gamma x + \gamma t \\ x' &= \gamma x - v_{\text{rel}}\, \gamma t \\ y' &= y \\ z' &= z \end{aligned}$$

Or, substituting again for gamma, $\gamma = 1/\left(1 - v_{\text{rel}}^2\right)^{1/2}$

$$\begin{aligned} t' &= \frac{-v_{\text{rel}} x + t}{\left(1 - v_{\text{rel}}^2\right)^{1/2}} \\ x' &= \frac{x - v_{\text{rel}} t}{\left(1 - v_{\text{rel}}^2\right)^{1/2}} \\ y' &= y \quad \text{and} \quad z' = z \end{aligned}$$

Equations $11.6.1$ and $11.6.2$ transform coordinates of an event known in the laboratory frame to coordinates in the rocket frame.

> A simple but powerful argument from symmetry leads to the same result. The symmetry argument is based on the relative velocity between laboratory and rocket frames. With respect to the laboratory, the rocket by convention moves with known speed in the positive x-direction. With respect to the rocket, the laboratory moves with the same speed but in the opposite direction, the negative x-direction. This convention about positive and negative directions - not a law of physics! - is the only difference between laboratory and rocket frames that can be observed from either frame. Lorentz transformation equations must reflect this single difference. In consequence, the "inverse" (laboratoryto-rocket) transformation can be obtained from the "direct" (rocket-to-laboratory) transformation by changing the sign of relative velocity, v_{rel}, in the equations and interchanging laboratory and rocket labels (primed and unprimed coordinates). Carrying out this operation on the Lorentz transformation equations 11.5.5 and 11.5.6 yields the inverse transformation equations $11.6.1$ and $11.6.2$).