

7.3: Momenergy Components and Magnitude

space part? momentum of the object time part: energy of the object magnitude: mass of the object

Break down momenergy for examination

Accidents of history have given us not one word, momenergy, but two words, momentum and energy, to describe mass in motion. Before Einstein, mass and motion were described not in the unified context of spacetime but in terms of space and time separately, as that division shows itself in some chosen free-float frame. Often we still think in those separated terms. But the single concept spacetime location of an event unites the earlier two ideas of its position in space and the time of its happening. In the

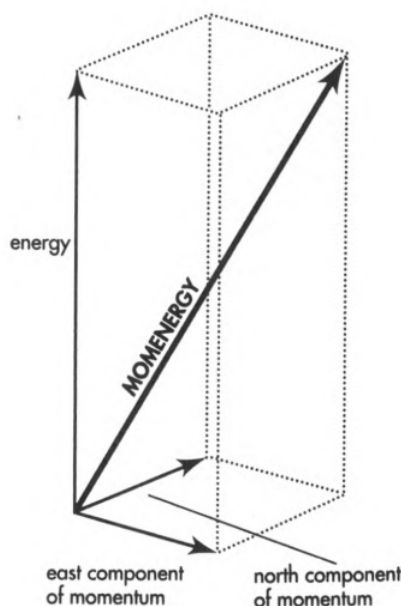


Figure 7.3.1: Momenergy arrow of a moving object translated into the language of momentum and energy, shown for the special case in which upward momentum (vertical momentum) equals zero. The momenergy arrow itself has an existence and direction (in that great baystack of worldlines and events that we call spacetime) independent of the choice, or even presence, of any free-float frame. In contrast, separate measures of momentum and energy do depend on choice of frame. They point parallel to, that is in the same direction as, the corresponding space and time directions of the chosen frame itself. See Figures 7 – 3 and 7 – 4 for a still more revealing representation of the proportion between momenergy and its components. same way we combine momentum and energy of a moving object into the single idea of momenergy arrow. Having assembled it, we now break momenergy down again, seeking new insight by examining its separate parts.

The unity of momenergy dissolves - in our thinking - into the separateness of momentum and energy when we choose a free-float frame, say the laboratory. In that laboratory frame the spacetime separation between two nearby events on the worldline of a particle resolves itself into four different separations: one in laboratory time and one in each of three perpendicular space directions, such as north, east, and upward. With each spacetime separation goes a separate part, a separate portion, a separate component of momenergy in the laboratory free-float frame (Figure 7.3.1).

The "space parts" of momenergy of a particle are its three components of momentum relative to a chosen frame. Their general form is not strange to us - mass times a velocity component. The "time part," however, is new to us, foreshadowing important insights into the nature of energy (Section 7.5). The four components are

$$\begin{pmatrix} \text{eastward} \\ \text{component} \\ \text{of} \\ \text{momenergy} \end{pmatrix} = \begin{pmatrix} \text{eastward} \\ \text{component} \\ \text{of} \\ \text{momentum} \end{pmatrix}$$

$$\begin{aligned}
 &= (\text{mass}) \times \frac{(\text{eastward displacement})}{(\text{proper time for that displacement})} \\
 \left(\begin{array}{c} \text{northward} \\ \text{component} \\ \text{of} \\ \text{momenergy} \end{array} \right) &= \left(\begin{array}{c} \text{northward} \\ \text{component} \\ \text{of} \\ \text{momentum} \end{array} \right) \\
 &= (\text{mass}) \times \frac{(\text{northward displacement})}{(\text{proper time for that displacement})} \\
 \left(\begin{array}{c} \text{upward} \\ \text{component} \\ \text{of} \\ \text{momenergy} \end{array} \right) &= \left(\begin{array}{c} \text{upward} \\ \text{component} \\ \text{of} \\ \text{momentum} \end{array} \right) \\
 &= (\text{mass}) \times \frac{(\text{upward displacement})}{(\text{proper time for that displacement})} \\
 \left(\begin{array}{c} \text{time} \\ \text{component} \\ \text{of} \\ \text{momenergy} \end{array} \right) &= (\text{energy}) = (\text{mass}) \times \frac{(\text{time displacement})}{(\text{proper time for that displacement})}
 \end{aligned}$$

Momenergy components of particle in a given frame

The calculus version of these equations is deliciously brief. Here, as in Section 6.2, tau (τ) stands for proper time:

$$\begin{aligned}
 E &= m \frac{dt}{d\tau} \\
 p_x &= m \frac{dx}{d\tau} \\
 p_y &= m \frac{dy}{d\tau} \\
 p_z &= m \frac{dz}{d\tau}
 \end{aligned} \tag{7.3.1}$$

The components of the momenergy 4-vector we now have before us in simple form, but how much is the absolutely-number-one measure of this physical quantity, its magnitude? This magnitude we reckon as we figure the magnitude of any Lorentz 4-vector: magnitude squared is the difference of squares of the time part and the space part:

$$\begin{aligned}
 &(\text{magnitude of momenergy arrow})^2 \\
 &= (\text{energy})^2 - (\text{east momentum})^2 - (\text{north momentum})^2 - (\text{up momentum})^2 \\
 &= E^2 - (p_x)^2 - (p_y)^2 - (p_z)^2 \\
 &= m^2 \frac{(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}{(d\tau)^2} = m^2 \frac{(d\tau)^2}{(d\tau)^2} = m^2
 \end{aligned}$$

Magnitude of momenergy 4-vector: mass!

In brief, the magnitude of the momenergy 4-vector, or its square,

$$(\text{magnitude of momenergy arrow})^2 = E^2 - p^2 = m^2 \tag{7.3.2}$$

is identical with the particle mass, or its square. Moreover, this mass is a quantity characteristic of the particle and totally independent of its state of motion.

It's worthwhile to translate this story into operational language. Begin with a particle that is at rest. Its 4-vector of energy and momentum points in the pure timelike direction, all energy, no momentum. Let an accelerator boost that particle. The particle acquires momentum. The space component of the 4-vector, originally zero, grows to a greater and greater value. In other words, the momenergy 4-vector tilts more and more from the "vertical," that is, from a purely timelike direction. However, its magnitude remains totally unchanged, at the fixed value m . In consequence, the time component of that 4-vector, that is, the energy of the particle, undergoes a systematic alteration

✓ Example 7.3.1

The energy and momentum components of a particle, measured in the laboratory, are

$$E = 6.25 \text{ kilograms}$$

$$p_x = 1.25 \text{ kilograms}$$

$$p_y = p_z = 2.50 \text{ kilograms}$$

What is the value of its mass?

Solution

We obtain a value for mass using equation (7-3) :

$$\begin{aligned} m^2 &= E^2 - (p_x)^2 - (p_y)^2 - (p_z)^2 \\ &= [(6.25)^2 - (1.25)^2 - (2.50)^2 - (2.50)^2] (\text{kilograms})^2 \\ &= [39.06 - 1.56 - 6.25 - 6.25] (\text{kilograms})^2 \\ &= [39.06 - 14.06] (\text{kilograms})^2 \\ &= 25.00 (\text{kilograms})^2 \end{aligned}$$

Hence

$$m = 5.0 \text{ kilograms} \tag{7.3.3}$$

1. 20 mass

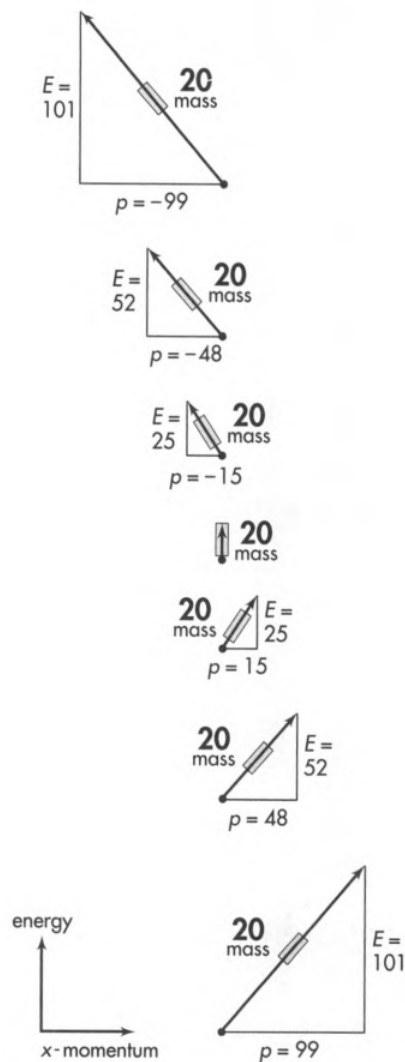


Figure 7.3.1: Different views of one and the same momenergy 4-vector of a particle in seven different free-float frames. The y- and z-components of momentum are assumed to equal zero, and frames are chosen to give integer values for energy and x-momentum components. The mass of the particle equals 20 units as reckoned in every free-float frame: $m^2 = E^2 - P^2$. This invariant value of the mass is shown by the thick "handle" on each vector. For a frame in which the particle is at rest (center diagram), the energy is equal to the mass and the handle covers the vector.

Does the momenergy 4-vector for this particle require for its existence any reference frame? No one would laugh more at such a misapprehension than the particle! The momenergy 4-vector has an existence in spacetime independent of any clocks and measuring rods. We, however, wish to assign to this 4-vector an energy and momentum. For that purpose we do require one or another free-float frame. Hence, the time component of that 4-vector, that is, the energy of the particle, undergoes a systematic alteration.

If the geometry of spacetime were Euclidean, this ever-growing tilt, this continuing rotation of the direction of the arrow of momenergy, would cause the vertical or time component to become ever shorter. However, spacetime is not Euclidean. It is Lorentzian, as appears in the minus sign in the equation for momenergy magnitude m : $m^2 = E^2 - p^2$. With momenergy magnitude, or particle mass m , being constant, and momentum p ever growing, Lorentz geometry itself tells us that the ever-growing tilt, the ever-larger momentum value, p , causes the time component of the momenergy - the energy E - not to shorten, as in a Euclidean spacetime, but to lengthen as the acceleration proceeds:

$$E = (m^2 + p^2)^{1/2} = \text{an increasing function of momentum, } p$$

This marvelously simple relation between energy and momentum, full of geometric as well as physical content, has by now been tried and verified in so many thousands of experiments of such varied kinds that it counts today as battle-tested.

Energy, momentum, and mass, expressed so far in the language of algebra, let themselves be displayed even more clearly in the language of pictures. Only one obstacle stands in the way. The paper is Euclidean and the vertical leg of a right triangle typically is shorter than the hypotenuse. In contrast, spacetime is Lorentzian, and the timelike dimension (the energy) is typically longer than the "hypotenuse" (the mass). We are indebted to our colleague William A. Shurcliff for a way to have our cake and eat it too, a device to employ Euclidean paper and yet display Lorentzian length. How? By laying over the hypotenuse of the Euclidean triangle a fat line or handle of length adjusted to the appropriate Lorentzian magnitude (Figure 7-3). The length of the handle represents the invariant value of the particle mass. This length remains the same, whatever the values of energy and momentum, values that differ as the particle is observed from one or another frame of reference in relative motion.

Figure 7.3.1 shows a few of the infinitely many different values of energy and momentum that one and the same particle can have as measured in different free-float frames. Each arrow, being depicted on a Euclidean sheet of paper, necessarily appears with an apparent length that increases with slope or particle speed. The handle on the arrow, by contrast, has the length appropriate to Lorentz geometry. This length represents particle mass, $m = 20$, a quantity independent of particle speed. The momenergy 4-vector of a material particle is always timelike. Why timelike? Because the momenergy 4-vector lies in the same spacetime direction as the worldline of the particle (Section 7.2). The events along the worldline have a timelike relationship: Time displacement between events is greater than the space displacement. One

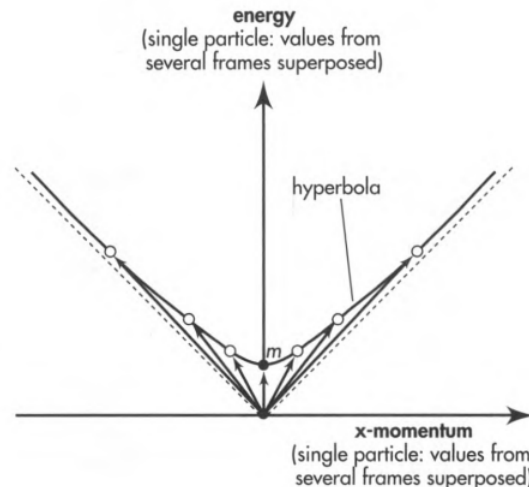


Figure 7.3.1: Momenergy 4-vector for the single particle of Figure 7 – 3 as observed in seven free-float frames, these plots then superposed on a composite momenergy diagram. Frames are chosen so that y - and z -components of momentum equal zero. Locus of the tips of the arrows traces out a hyperbola. The central short vertical arrow pointing to the dot labeled m represents momenergy as measured in the particle rest frame. In this frame momentum has value zero and energy- "rest energy"-equals the mass of the particle. For clarity, the handles have been omitted from the 4 vectors, which all have identical invariant magnitude $m = 20$. consequence is that the particle moves at less than the speed of light in every possible free-float frame.

E, p, m of particle in different frames related by hyperbola

The equation $E^2 - p^2 = m^2 = (\text{constant})$ is the formula for a hyperbola. Figure 7-4 E, p, m of particle in different frames related by hyperbola generates this hyperbola by superposing on the same figure spacetime vectors that For visual clarity the handles are omitted from these 4-vectors. However, each momenergy 4-vector has the same magnitude, equal to the particle mass, $m = 20$.

This page titled [7.3: Momenergy Components and Magnitude](#) is shared under a [CC BY 4.0](#) license and was authored, remixed, and/or curated by [Edwin F. Taylor & John Archibald Wheeler \(Self-Published \(via W. H. Freeman and Co.\)\)](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform.