7.5: Energy- "Time Part" of Momenergy

energy has two partis: rest energy (= mass) plus kinetic energy

What about the "time part" of the momentum-energy of a particle-the part we have called its energy? This is certainly a strange-looking beast! As measured in a particular free-float frame, say the laboratory, this time component as given in equation (7-5) is

Relativistic expression for energy

$$E=m\frac{dt}{d\tau}=\frac{m}{\left(1-v^2\right)^{1/2}}=m\gamma \tag{7.5.1}$$

Compare this with the Newtonian expression for kinetic energy, using K as the symbol for kinetic energy:

$$K_{\text{Newton}}=\frac{1}{2}mv^2 \quad [\text{valid for low speed}] \tag{7.5.2}$$

How does the relativistic expression for energy, equation $7.5.1$, compare with the Newtonian expression for kinetic energy $7.5.2$? To answer this question, first look at the behavior of these two expressions when particle speed equals zero. The Newtonian kinetic energy goes to zero. In contrast, at zero speed $1/\left(1-v^2\right)^{1/2}=1$ and the relativistic value for energy becomes equal to mass of the particle,

Rest energy of a particle equals its mass

$$E_{\text{rest}}=m \tag{7.5.3}$$

where E_{rest} is called rest energy of the particle. Rest energy of a particle is simply its mass. So the relativistic expression for energy does not go to zero at zero speed, while the Newtonian expression for kinetic energy does go to zero.

Is this an irreconcilable difference? The Newtonian formula does not contain an expression for rest energy, equal to the mass of the particle. But here is the distinction: The relativistic expression gives the value for total energy of the particle, while the Newtonian expression describes kinetic energy only (valid for low speed). However, in Newtonian mechanics any constant potential energy whatever can be added to the energy of a particle without changing the laws that describe its motion. One may think of the zero-speed limit of the relativistic expression for energy as providing this previously undetermined constant.

When we refer to energy of a particle we ordinarily mean total energy of the particle. As measured in a frame in which the particle is at rest, this total energy equals rest energy, the mass of the particle. As measured from frames in which the particle moves, total energy includes not only rest energy but also kinetic energy.

This leads us to define kinetic energy of a particle as energy above and beyond its rest energy:

Kinetic energy defined

$$(\text{ energy })=(\text{ rest energy })+(\text{ kinetic energy })$$

or

$$E=m+K \tag{7.5.4}$$

Example $7.5.1$ MOTION IN THE $X-DIRECTION$

An object of mass 3 kilograms moves 8 meters along the x-direction in 10 meters of time as measured in the laboratory. What is its energy and momentum? Its rest energy? Its kinetic energy? What value of kinetic energy would Newton predict for this object? Using relativistic expressions, verify that the velocity of this object equals its momentum divided by its energy.

Solution

From the statement of the problem:

$$\begin{aligned} m &= 3 \text{ kilograms} \\ t &= 10 \text{ meters} \\ x &= 8 \text{ meters} \\ y &= 0 \text{ meters} \\ z &= 0 \text{ meters} \end{aligned}$$

From this we obtain a value for the speed:

$$v=\frac{x}{t}=\frac{8 \text{ meters of distance}}{10 \text{ meters of time}}=0.8 \tag{7.5.5}$$

Use v to calculate the factor $1/\left(1-v^2\right)^{1/2}$ in equation (7-8):

$$\frac{1}{\left(1-v^2\right)^{1/2}}=\frac{1}{\left(1-(0.8)^2\right)^{1/2}}=\frac{1}{(1-0.64)^{1/2}}=\frac{1}{(0.36)^{1/2}}=\frac{1}{0.6}=\frac{5}{3} \tag{7.5.6}$$

From equation (7-11) the energy is

$$E=m/\left(1-v^2\right)^{1/2}=(3 \text{ kilograms })(5/3)=5 \text{ kilograms} \tag{7.5.7}$$

From equation (7-8) momentum has the magnitude

$$p=mv/\left(1-v^2\right)^{1/2}=(5/3)\times(3 \text{ kilograms })\times 0.8=4 \text{ kilograms} \tag{7.5.8}$$

Rest energy of the particle just equals its mass:

$$E_{\text{rest }}=m=3 \text{ kilograms} \tag{7.5.9}$$

From equation (7-15) kinetic energy K equals total energy minus rest energy:

$$K=E-m=5 \text{ kilograms }-3 \text{ kilograms }=2 \text{ kilograms} \tag{7.5.10}$$

The Newtonian prediction for kinetic energy is

$$K_{\text{Newton }}=\frac{1}{2}mv^2=\frac{1}{2}\times 3\times(0.8)^2=0.96 \text{ kilogram} \tag{7.5.11}$$

which is a lot smaller than the correct relativistic result. Even at the speed of light, the Newtonian prediction would be $K_{\text{Newton }}=1.5$ kilogram, whereas relativistic value would increase without limit.

Equation (7-16) says that velocity equals the ratio (magnitude of momentum) $/($ energy):

$$v=\frac{p}{E}=\frac{4 \text{ kilograms}}{5 \text{ kilograms}}=0.8 \tag{7.5.12}$$

This is the same value as reckoned directly from the given quantities.

From this comes the relativistic expression for kinetic energy K :

$$K=E-E_{\text{rest }}=E-m=\frac{m}{\left(1-v^2\right)^{1/2}}-m=m\left[\frac{1}{\left(1-v^2\right)^{1/2}}-1\right] \quad (7-15) \tag{7.5.13}$$

Box 7-2 elaborates the relation between this expression and the Newtonian expression (7.5.2). Notice that if we divide the respective sides of the momentum equation (7-8) by corresponding sides of the energy equation (7.5.1), the result gives particle speed:

$$v=\frac{p}{E} \tag{7.5.14}$$

We could have predicted this directly from the first figure in this chapter, Figure 7.1.1. Speed v is the tilt (slope) of the worldline from the vertical: (space displacement) /(time for this displacement). Momenergy points along the worldline, with space component p and time component E. Therefore momenergy slope p/E equals worldline slope v.

Conversion to conventional energy units

Still More Units: In order to convert energy in units of mass to energy in conventional units, such as joules, multiply the expressions above by the square of light speed, c^2, and use subscript "conv":

$$E_{\text{conv}} = Ec^2 = \frac{mc^2}{\left[1-(v_{\text{conv}}/c)^2\right]^{1/2}} \quad \text{[good at any speed]} \qquad (7\text{-}17)$$

$$K_{\text{conv}} = (E-E_{\text{rest}})c^2 = mc^2\left[\frac{E_{\text{conv rest}} = mc^2}{\left[1-(v_{\text{conv}}/c)^2\right]^{1/2}} - 1\right] \quad \text{[good at any speed]} \ (7\text{-}19)$$

$$\text{Conversion to conventional}$$

$$K_{\text{conv Newton}} = \frac{1}{2}mv^2c^2 = \frac{1}{2}m\left(\frac{v_{\text{conv}}}{c}\right)^2 c^2 = \frac{1}{2}mv_{\text{conv}}^2 \quad \text{[low speed only]} \ \ (7\text{-}20)$$

Thus conversion from energy in units of mass to energy in conventional units is always accomplished by multiplying by conversion factor c^2. This is true whether the expression for energy being converted is Newtonian or relativistic. Table $7-1$ at the end of the chapter summarizes these comparisons.

Equation (7-18) is the most famous equation in all physics. Historically, the factor c^2 captured the public imagination because it witnessed to the vast store of energy available in the conversion of even tiny amounts of mass to heat and radiation. The units of mc^2 are joules; the units of m are kilograms. However, we now recognize that joules and kilograms are units different only because of historical accident. The conversion factor c^2, like the factor of conversion from seconds to meters or miles to feet, can today be counted as a detail of convention rather than as a deep new principle.

Example 7.5.2 MOMENERGY COMPONENTS

For each of the following cases, write down the vector in the given frame in the form $[E, t_x t_j t_z]$. four components of the momentum-energy 4- Each particle has mass m.

a. A particle moves in the positive x-direction in the laboratory with kinetic energy equal to three times its rest energy.

b. The same particle is observed in a rocket in which its kinetic energy equals its mass.

c. Another particle moves in the y-direction in the laboratory frame with momentum equal to twice its mass.

d. Yet another particle moves in the negative x-direction in the laboratory with total energy equal to four times its mass.

e. Still another particle moves with equal x, y, and z momentum components in the laboratory and kinetic energy equal to four times its rest energy.

Solution

a. Total energy of the particle equals rest energy m plus kinetic energy $3m$. Therefore its total energy E equals $E = m + 3m = 4m$. The particle moves along the x-direction, so $p_y = p_z = 0$ and $p_x = p$, the total momentum. Substitute the value of E into the equation $m^2 = E^2 - p^2$ to obtain

$$p^2 = E^2 - m^2 = (4m)^2 - m^2 = 16m^2 - m^2 = 15m^2 \tag{7.5.15}$$

Hence $p_x = (15)^{1/2}\,\mathrm{m}$.

In summary, the components of the momenergy 4-vector are

$$[E, p_x, p_y, p_z] = \left[4m, (15)^{1/2}m, 0, 0\right] \tag{7.5.16}$$

Of course the magnitude of this momenergy 4-vector equals the mass of the particle m - true whatever its speed, its energy, or its momentum.

b. In this rocket frame, total energy - rest energy plus kinetic energy-has the value $E = 2m$. As before, $p^2 = E^2 - m^2 = (2m)^2 - m^2 = 4m^2 - m^2 = 3m^2$. Hence $p_x = 3^{1/2}m$ and components of the 4-vector are $[E, p_x, p_y, p_z] = \left[2m, 3^{1/2}m,, 0, 0\right].$

c. In this case $p_x = p_z = 0$ and $p_y = p = 2m$. Moreover, $E^2 = m^2 + p^2 = m^2 + (2m)^2 \quad = 5m^2$. So, finally, $[E, p_x, p_y, p_z] = \left[5^{1/2}m, 0, 2m, 0\right]$.

d. We are given directly that $E = 4m$, the same as in part a, except here the particle travels in the negative x-direction so has negative x-momentum. Hence:

$$[E, p_x, p_y, p_z] = \left[4m, -(15)^{1/2}m, 0, 0\right] \tag{7.5.17}$$

e. Total energy equals $E = 5m$. All momentum components have equal value, say

$$p_x = p_y = p_z = P \tag{7.5.18}$$

In this case we use the full equation that relates energy, momentum, and mass:

$$(p_x)^2 + (p_y)^2 + (p_z)^2 = 3P^2 = E^2 - m^2 = (5m)^2 - m^2 = 24m^2$$
$$\text{or } P^2 = 8m^2 \text{ and hence } [E, p_x, p_y, p_z] = \left[5m, 8^{1/2}m, 8^{1/2}m, 8^{1/2}m\right].$$

Box 7-2

Energy at relativistic speeds and energy at everyday speeds: How are expressions for these two cases related?

Energy in Terms of Momentum: In the limit of velocities low compared with the speed of light, the relativistically accurate expression for energy $E = \left(m^2 \; + p^2\right)^{1/2}$ reduces to $E = m + p^2/(2m) +$ corrections. To see why and how, and to estimate the corrections, it is convenient to work in dimensionless ratios. Thus we focus on the accurate expression in the form $E/m = \left[1 + (p/m)^2\right]^{1/2}$, or even simpler, $y = [1 + x]^{1/2}$, and on the approximation to this result, in the form

$$E/m = 1 + (1/2)(p/m)^2 + \text{ corrections, or } y = 1 + (1/2)x + \text{ corrections} \tag{7.5.19}$$

Example: $x = 0.21$. Then our approximation formula gives $y = (1.21)^{1/2} = 1 \; + 0.105 +$ a correction. The accurate result is $y = 1.100$, which is the square root of 1.21. In other words, the correction is negative and extremely small: correction $= -0.005$.

Energy in Terms of Velocity: In the limit of velocities low compared with the speed of light, the relativistically accurate expression for energy $E = m/\left(1 - \; v^2\right)^{1/2}$, reduces to $E = m + (1/2)mv^2 +$ corrections. It is convenient again to work in dimensionless ratios. Thus we focus on the accurate expression in the form $E/m = \left[1 - v^2\right]^{-1/2}$, or even simpler, $y = [1 - x]^{-1/2}$, and on the approximation to this result, in the form

$$E/m = 1 + (1/2)v^2 + \text{ corrections, or } y = 1 + (1/2)x + \text{ corrections} \tag{7.5.20}$$

Example: $x = 0.19$. Then our approximation formula gives $y = 1 + (1/2)$

$0.19 +$ a correction $= 1.095 +$ a correction. The accurate result is $y = [1 - \; 0.19]^{-1/2} = (0.81)^{-1/2} = (0.9)^{-1} = 1.1111\ldots$ In other words, the correction is positive and small: correction $= +0.01611$.

Another example: A jet plane. Take its speed to be exactly $v = 10^{-6}$. That speed, according to our approximation, brings with it a fractional augmentation of energy, a kinetic energy per unit mass, equal to $(1/2)v^2 = 5X\ 10^{-13}$ or 0.0000000000005 In contrast, the accurate expression $E/\mathrm{m} = \; \left[1 - v^2\right]^{-1/2}$ gives the result $E/m = 1.0000000000005000000000009375$ $000000000\ldots$.The 5 a little less than halfway down the length of this string of digits is no trifle, as anyone will testify who has seen the consequences of the crash of a jet plane into a skyscraper. However, the 9375 further down the line is approximately a million million times smaller and totally negligible in its practical consequences.

In brief, low speed gives rise to a kinetic energy which, relative to the mass, is given to good approximation by $(1/2)v^2$ or by $(1/2)(p/m)^2$. Moreover, the same one or other unit-free number la "fraction" because it is small compared to unity) automatically reveals to us the order of magnitude of the fractional correction we would have had to make in this fraction itself if we were to have insisted on a perfectly accurate figure for the kinetic energy.

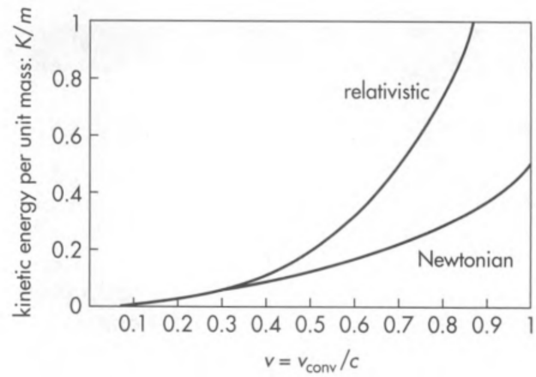

Figure 7.5.1: Kinetic energy as a function of speed, as predicted by relativity [equation (7-19), valid for all speeds] and by Newtonian mechanics [equation (7-20), valid for low speeds only]

Energy: Time part of momenergy 4-vector Mass: Magnitude of that 4-vector

Central to an understanding of the equation $E_{\text{rest}} = m$ or its equivalent $E_{\text{conv rest}} = mc^2$ is the subscript "rest." Energy is not the same as mass! Energy is only the time part of the momenergy 4 -vector. Mass is the magnitude of that 4 -vector. The energy of an object, expressed in conventional units, has the value mc^2 only when that object is observed from a frame in which it is at rest. Observed from all other free-float frames, the energy of the object is greater than its rest energy, as shown by equation $(7-17)$.

Figure $7.5.1$ compares relativistic and Newtonian predictions for kinetic energy per unit mass as a function of speed. At low speeds the values are indistinguishable (left side of the graph). When a particle moves with high speed, however, so that the factor $1/\left(1-v^2\right)^{1/2}$ has a value much greater than one, relativistic and Newtonian expressions do not yield at all the same value for kinetic energy (right side of the graph). Then one must choose which expression to use in analyzing collisions and other high-speed phenomena. We choose the relativistic expression because it leads to the same value of the total energy of an isolated system before and after any interaction between particles in the system - it leads to conservation of total energy of the system.

Relativity: All forms of energy automatically conserved

All this talk of reconciliation at low speeds obscures an immensely powerful feature of the relativistic expression for total energy of an isolated system of particles. Total energy is conserved in all interactions among particles in the system: elastic and inelastic collisions as well as creations, transformations, decays, and annihilations of particles. In contrast, total kinetic energy of a system calculated using the Newtonian formula for low-speed interactions is conserved only for elastic collisions. Elastic collisions are defined as collisions in which kinetic energy is conserved. In collisions that are not elastic, kinetic energy transforms into heat energy, chemical energy, potential energy, or other forms of energy. For Newtonian mechanics of low-speed particles, each of these forms of energy must be treated separately: Conservation of energy must be invoked as a separate principle, as something beyond Newtonian analysis of mechanical energy.

In relativity, all these energies are included automatically in the single time component of total momenergy of a system - total energy - which is always conserved for an isolated system. Chapter 8 discusses more fully the momenergy of a system of particles and the effects of interactions between particles on the energy and mass of the system.