5.E: Trekking through Spacetime (Exercises)

PRACTICE

5-1 more is less

The spacetime diagram shows two alternative worldlines from event A to event D. The table shows coordinates of numbered events in this frame. Time and space are measured in years.

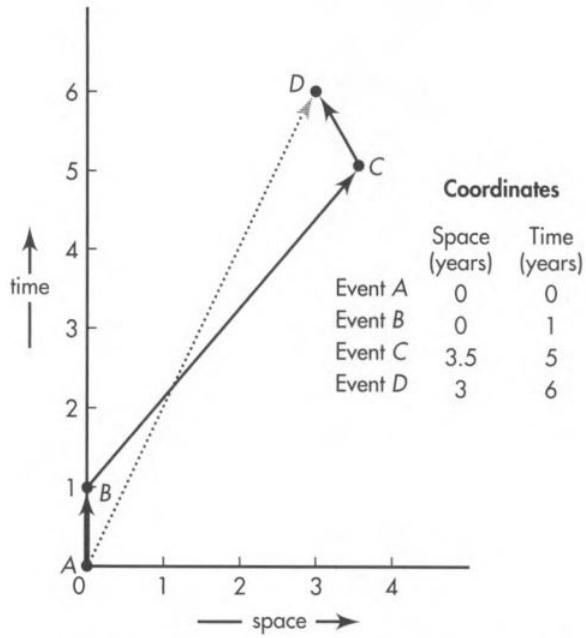

Figure $5.E.1$: Two alternative worldlines between initial event A and final event D.

a. One traveler moves along the solid segmented worldline from event A to events B, C, and D. Calculate the time increase on his wristwatch (proper clock)
   (1) between event A and event B.
   (2) between event B and event C.
   (3) between event C and event D.
   (4) Also calculate the total proper time along worldline A, B, C, D.
b. His twin sister moves along the straight dotted worldline from event A directly to event D. Calculate the time increase on her wristwatch between events A and D.
c. Which twin (solid-line or dotted-line traveler) is younger when they rejoin at event D?

5-2 transforming worldlines

The laboratory spacetime diagram in the figure shows two worldlines. One, the vertical line labeled B, is the worldline of an object that is at rest in this frame. The other, the segmented line that connects events 0, 1, 2, and 3, is the worldline of an object that moves at different speeds at different times in this frame. The proper time is written on each segment and invariant hyperbolas are drawn through events 1,2, and 3. The event table shows the space and time locations in this frame of the four events 0, 1, 2, and 3.

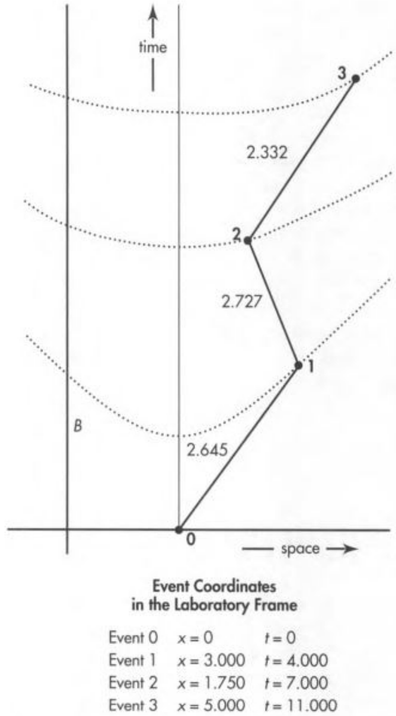

Figure $5.E.2$: Two worldlines as recorded in the laboratory frame. Numbers on the segmented worldline are proper times along each straight segment.

a. Trace the axes and hyperbolas onto a blank piece of paper. Sketch a qualitatively correct spacetime diagram for the same pair of worldlines observed in a frame in which the particle on the segmented worldline has zero velocity between event 1 and event 2.
b. What is the velocity, in this new frame, of the particle moving along worldline B?
c. On each straight portion of the segmented worldline for this new frame write the numerical value of the interval between the two connected events.

5-3 mapmaking in spacetime

Note: Recall Exercise 1-6, the corresponding mapmaking exercise in Chapter 1 .

Here is a table of timelike intervals between events, in meters. The events occur in the time sequence $ABCD$ in all frames and along a single line in space in all frames. (They do not occur along a single line on the spacetime map.)

INTERVAL	to event			
	A	B	C	D
from event				
A	0	1.0	3.161	5.196
B		0	2.0	4.0
C			0	2.0
D				0

a. Use a ruler and the hyperbola graph to construct a spacetime map of these events. Draw this map on thin paper so you can lay it over the hyperbola graph and see the hyperbolas.

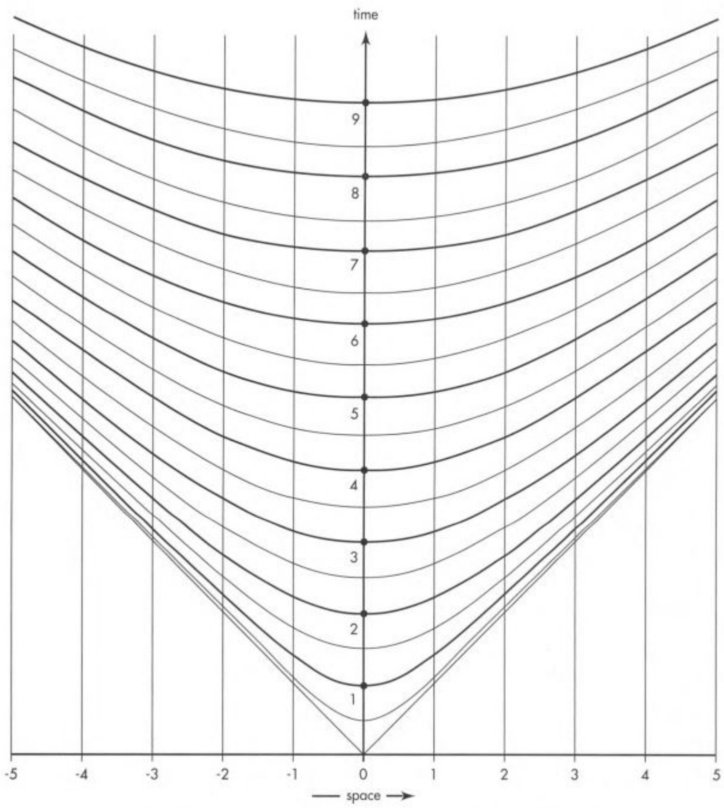

Figure $5.E.3$: Template of byperbolas ou can lay it over the hyperbola on thin paper so you can lay it over the hyperbola graph and see the hyperbolas.

Discussion: How to start? With three arbitrary decisions! (1) Choose event A to be at the origin of the spacetime map. (2) Choose event B to occur at the same place as event A. That is, event point B is located on the positive time axis with respect to event point A. After plotting B, use your ruler to draw this straight time axis through event points A and B. Keep this line parallel to the vertical lines on the hyperbola graph in all later constructions. (3) Even with these choices, there are two spacetime locations (x, t) at which you can locate the event point C; choose either of these two spacetime locations arbitrarily. Then go on to plot event D.

Analogy to surveying: In surveying (using Euclidean geometry) you locate all points a given distance from some stake by using that stake as origin and drawing a circle of radius equal to the desired distance. In a spacetime map (using Lorentz geometry) you locate all event points a given interval from some event by using that event point as origin and drawing a hyperbola with nearest point equal to the desired interval.

b. Now take a new piece of paper and draw a spacetime map for another reference frame. Choose event D to be at the origin of the spacetime map. This means that all other events occur before D. Hence turn the hyperbola plot upside down, so that the hyperbolas open downward. Choose event B to occur at the same place as D. Now find the locations of A and C using the same strategy as in part **a**.
c. Find an approximate value for the relative speed of the two frames for which you have made spacetime plots.
d. Hold one of your spacetime maps up to the light with the marks on the side of the paper facing the light. Does the map you see from the back also satisfy the table entries?

PROBLEMS

5-4 the pole and barn paradox

A worried student writes, "Relativity must be wrong. Consider a 20-meter pole carried so fast in the direction of its length that it appears to be only 10 meters long in the laboratory frame of reference. Let the runner who carries the pole enter a barn 10 meters long, as shown in the figure. At some instant the farmer can close the front door and the pole will be entirely enclosed in the barn. However, look at the same situation from the frame of reference of the runner. To him the barn appears to be contracted to half its length. How can a 20 -meter pole possibly fit into a 5-meter barn? Does not this unbelievable conclusion prove that relativity contains somewhere a fundamental logical inconsistency?"

Write a reply to the worried student explaining clearly and carefully how the pole and barn are treated by relativity without internal contradiction. Use the following outline or some other method.

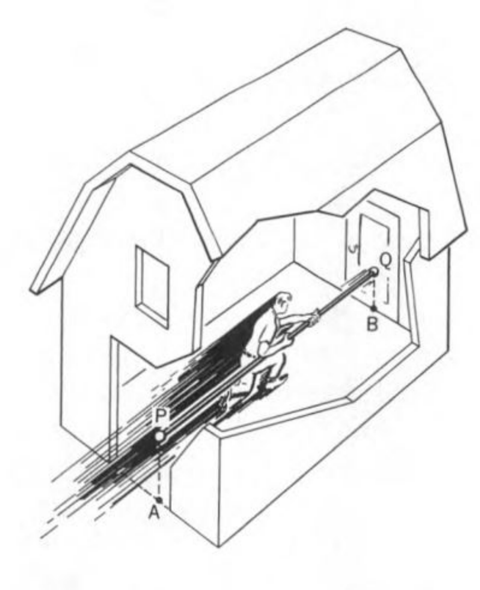

Figure $5.E.4$: Fast runner with "20-meter" pole enclosed in a "10-meter" ham. In the next instant he will burst through the hack door, which is made of paper.

a. Make two carefully labeled spacetime diagrams, one an xt diagram for the barn rest frame, the other an $x't'$ diagram for the runner rest frame. Referring to the figure, take the event "Q coincides with A" to be at the origin of both diagrams. In both plot the worldlines of A, B, P, and Q. Pay attention to the scale of both diagrams. Label both diagrams with the time (in meters) of the event "Q coincides with B" (derived from Lorentz transformation equations or otherwise). Do the same for the times of events "P coincides with A" and "P coincides with B."
b. **Discussion question:** Suppose the barn has no back door but rather a back wall of steel-reinforced concrete. What happens after the farmer closes the front door on the pole?
c. Replace the pole with a line of ten tennis balls the same length as the pole and moving together with the same velocity as the pole. The farmer's ten children line up inside the barn, and each catches and stops one tennis ball at the same time as the farmer closes the front door of the barn. Describe the stopping events as recorded by the observer riding on the last tennis ball. Plot them on your two diagrams.

5-5 radar speed trap

A highway patrolman aims a stationary radar transmitter backward along the highway toward oncoming traffic. A detector mounted next to the transmitter analyzes the radar wave reflected from an approaching car. An internal computer uses the shift in frequency of the reflected wave to reckon and display the car's speed. Analyze this shift in frequency as in parts **a-e** or with some other method. Treat the car as a simple mirror and assume that the radar signals move back and forth along one line on the highway. Radar is an electromagnetic wave that moves with the speed of light.

The figure shows the worldline of the car, worldlines of two adjacent maxima of the radar wave, and the wavelength λ of incident and reflected waves.

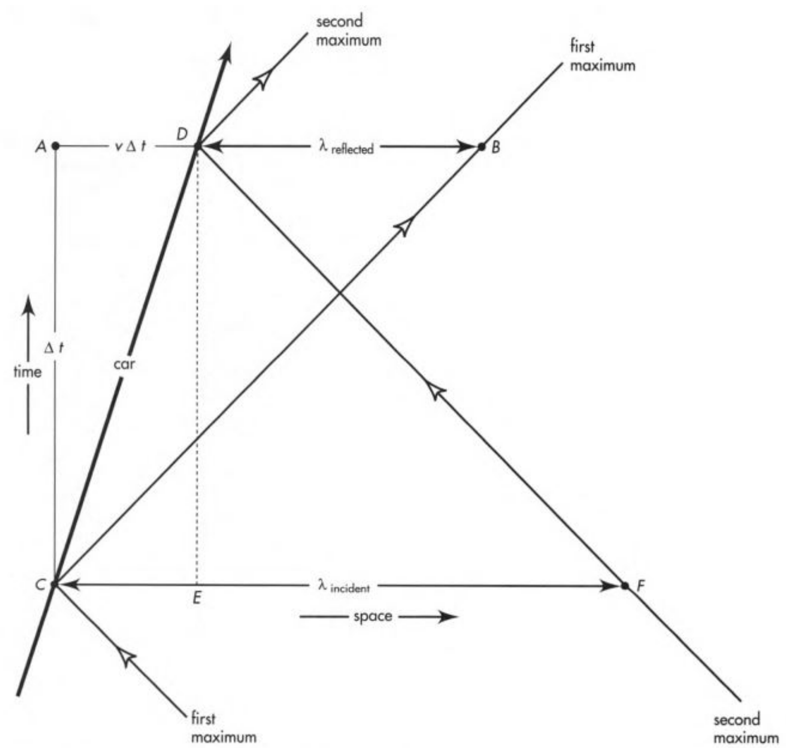

Figure 5.E.5: Worldlines of approaching car and two radar wave maxima that reflect from the car. The speed of the car is greatly exaggerated.

a. From the 45-degree right triangle ABC, show that

$$\Delta t = v\Delta t + \lambda_{\text{reflected}}$$

From the 45-degree right triangle DEF, show that

$$\Delta t = \lambda_{\text{incident}} - v\Delta t$$

Eliminate Δt from these two equations to find an expression for $\lambda_{\text{reflected}}$ in terms of $\lambda_{\text{incident}}$ and the automobile speed v.

b. The frequency f of radar (in cycles/second) is related to its wavelength λ in a vacuum by the formula $f = c/\lambda$, where c is the speed of light ($\approx$ the speed of radar waves in air). Derive an expression or frequency $f_{\text{reflected}}$ of the reflected radar signal in terms of frequency f_{incident} of the incident wave and the speed v of the oncoming automobile. Show that the result is

$$f_{\text{reflected}} = \left(\frac{1+v}{1-v}\right) f_{\text{incident}}$$

c. For an automobile moving at a speed $v = v_{\text{conv}}/c$ that is a small fraction of the speed of light, assume that the fractional change in frequency of reflected radar is small. Under this assumption, use the first two terms of the binomial expansion

$$(1-z)^n \approx 1 - nz \text{ for } |z| << 1$$

to show that the fractional change of frequency is given by the approximate expression

$$\frac{\Delta f}{f} \approx 2v$$

Substitute the speed of a car moving at 100 kilometers/hour ($= 27.8$ meters/second ≈ 60 miles/hour) and show that your assumption about the small fractional change is justified.

d. One radar gun used by the Massachusetts Highway Patrol operates at a frequency of 10.525×10^9 cycles/second. By how many cycles/second is the reflected beam shifted in frequency when reflected from a car approaching at 100 kilometers/ hour?
e. What discrimination between different frequency shifts must the unit have if it can distinguish the speed of a car moving at 100 kilometers/hour from the speed of one moving at 101 kilometers/ hour?

Reference: T. M. Kalotas and A. R. Lee, *American Journal of Physics*, Volume 58, pages 187-188 (February 1990).

5-6 a summer evening's fantasy

You are standing alone outdoors at dusk on the first day of summer. You see Sun setting due west and the planet Venus in the same direction. On the opposite horizon the full Moon is rising due east. An alien ship approaches from the east and lands beside you. The occupants inform you that they are from Proxima Centauri, which lies due east beyond the rising Moon. They say they have been traveling straight to Earth and that their reduced approach speed within the solar system was such that the time stretch factor gamma during the approach was $5/3$.

At the same instant that the aliens land, you see Sun explode. The aliens admit to you that earlier, on their way to Earth, they shot a laser light pulse at Sun, which caused this explosion. They warn that Sun's explosion emitted an immense pulse of particles moving at half the speed of light that will blow away Earth's atmosphere. In confirmation, shortly after the aliens land you notice that the planet Venus, lying in the direction of Sun, suddenly changes color.

You grab a passing human of the opposite sex and plead with the aliens to take you both away from Earth in order to establish the human gene pool elsewhere. They agree and set the dials to flee in an easterly direction away from Sun at top speed, with time stretch factor gamma of $25/7$. The takeoff is to be 7 minutes after the alien landing on Earth.

Do you make it?

Draw a detailed Earth spacetime diagram showing the events and worldlines of this story. Use the following information.

- Sun is 8 light-minutes from Earth.
- Venus is 2 light-minutes from Earth.
- Assume that Sun, Venus, Earth, and Moon all lie along a single direction in space and are relatively at rest during this short story. The incoming and outgoing paths of the alien ship lie along this same line in space.
- All takeoffs and landings involve instantaneous changes from initial to final speed.
- $5^2 - 3^2 = 4^2$ and $(25)^2 - (7)^2 = (24)^2$

a. Plot EVENTS labeled with the following NUMBERS.
   0. your location when the aliens land (at the origin)
   1. Sun explodes
   2. light from Sun explosion reaches you
   3. Venus's atmosphere blown away
   4. light from event 3 reaches you
   5. you and aliens depart Earth (you hope!)
   6. Earth atmosphere blown away
b. Plot WORLDLINES labeled with the following CAPITAL LETTERS.
   A. your worldline
   B. worldline of Earth
   C. aliens' worldline
   D. worldline of Sun
   E. worldline of Venus
   F. worldline of light from Sun's explosion
   G. worldline of the "speed-one-half" pulse of particles from Sun's explosion
   H. worldline of light emitted when Venus loses atmosphere
   J. terminal part of the worldline of the laser cannon pulse fired at Sun by the aliens
c. Write numerical values for the speed $v = v_{\text{conv}} / c$ on every segment of all worldlines.

5-7 the runner on the train paradox

A letter sent to the Massachusetts Institute of Technology by Hsien-Yen Tsao of Los Angeles poses the following paradox, which he asserts disproves the theory of relativity. The Chairman of the Physics Department sends the inquiry along to you, asking you to respond to Mr. Tsao. You determine to make the answer clear, concise, decisive, and polite-a personal test of your diplomacy and grasp of relativity.

The setting: A train travels at high speed. A runner on the train sprints toward the back of the train with the same speed (with respect to the train) as the train moves forward (with respect to Earth). Therefore the runner is not moving with respect to Earth.

The paradox: We know that, crudely speaking, clocks on the train run "slow" compared to the Earth clock. We also know that the runner's clock runs "slow" compared to the train clocks. Therefore the runner's clock should run "doubly slow" with respect to the Earth clock. But the runner is not moving with respect to Earth! 'Therefore the runner's clock must run at the same rate as the Earth clock. How can it possibly be that the runner's clock runs "doubly slow" with respect to the Earth clock and also runs at the same rate as the Earth clock?

5-8 the twin paradox put to rest - a worked example

Motto: The swinging line of simultaneity tells all!

Combine the Lorentz transformation with the spacetime diagram to clear up - once and for all! - the solution to the Twin Paradox. An astronaut travels from Earth to Canopus (Chapter 4) at speed $v_{\text{rel}} = 99/101$, arriving at Canopus $t' = 20$ years later according to her rocket clock, $t = 101$ years later according to Earth-linked clocks - which means that the stretch factor γ has the value 101/20.

The key idea is "lines of simultaneity" (boxed labels in the figure). A line of simultaneity connects events that occur "at the same time." But events simultaneous in the Earth ("laboratory") frame are typically not simultaneous in the rocket frame (Section 3.4). Horizontal is the line of simultaneity on the Earth ("laboratory") spacetime map that connects events occurring at the same time in the Earth frame. Totally different — not a horizontal line! —is a line of simultaneity on the Earth spacetime map that connects events simultaneous in the outgoing astronaut frame. To draw this line of outgoing-astronaut simultaneity, start with the inverse Lorentz transformation equation for time:

$$t' = -v_{\text{rel}}\gamma_x + \gamma t$$

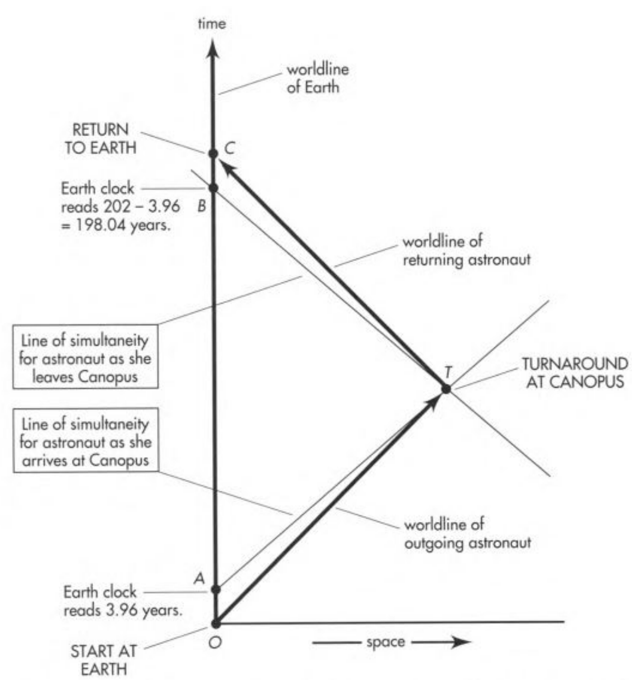

Figure $5.E.6$: Earth spacetime map of the trip to Canopus and hack. Atthe astronaut arrives at Canopus, her colleagues in her outgoing reference frame record along line AT events simultaneous with this arrival, including Earth-clock reading of 3-96 years at A. At Canopus the astronaut changes frames, thus changing the line of simultaneity, which swings to BT. As she leaves Canopus, her new colleagues take an Earth-clock reading of 198.04 years at B. At turnaround, the ticks on the Earth clock along worldline segment AB go from the outward-moving astronaut's future to the incoming astronaut’s past.

For the outgoing astronaut, $v_{\text{rel}} = 99/101$ and $\gamma = 101/20$. We want the line of simultaneity that passes through turnaround event T. So let $t' = 20$ years. Then:

$$20 = -(99/101)(101/20)x + (101/20)t$$

Multiply through by $20/101$:

$$400/101 = -(99/101)x + t$$

which yields

$$t = 0.980x + 3.96$$

This is the equation for a straight line passing through event points A and T in the spacetime diagram. It is the line of simultaneity for the outgoing astronaut, connecting all events simultaneous with the arrival of the rocket at Canopus (simultaneous in that frame). Among these events is event A, the Earth clock reading of 3.96 years, which occurs at Earth position $x = 0$. In brief, at the moment the rocket arrives at Canopus, the Earth clock reads 3.96 years as observed in the outgoing rocket frame.

To go back over the astronaut trip while looking at the spacetime map is (finally!) to solve the Twin Paradox. As the astronaut travels outward toward Canopus, many colleagues follow her at the same speed, with clocks synchronized in her frame. As they whiz past Earth, each records the reading on the Earth clock. Later analysis leads them to agree that the time between ticks of Earth's clock is longer than the time between ticks of their own outward-moving clocks. (They say, "The Earth clock runs slow.") At any event point on her outward worldline, the astronaut's line of simultaneity slopes upward to the right in the Earth spacetime diagram, as shown in the figure. Simultaneous with astronaut arrival at Canopus (event T, when all outward-moving clocks read 20 years), one of her colleagues reads a time 3.96 years on the Earth clock (event A).

Now the astronaut jumps from the outward-moving rocket to a returning rocket. She inherits a completely new set of colleagues, with a new set of synchronized clocks. The astronaut's new line of simultaneity slopes upward to the left in the Earth spacetime diagram. Simultaneous with her departure from Canopus (event T, when all inward-moving clocks read 20 years), one of her new colleagues reads a time $202 - 3.96 = 198.04$ years on the passing Earth clock (event B). Thereafter new colleague after new colleague streaks past Earth, recording the fact that Earth clock ticks are farther apart in time than the ticks on their own clocks. (They say, "The Earth clock runs slow.")

The analysis so far accounts for the short time segments OA and BC recorded by the Earth clock on its vertical worldline AC. What about the omitted time lapse AB ? This is recorded, sure enough, by the Earth clock plowing forward along worldline OC in its comfortable single free-float frame. However, the story of time AB is quite different for the turn-around astronaut. Before she reaches turnaround at T, events on line AB are in her future. All those Earth clock ticks are yet to be recorded by her outgoing colleagues. These events lie above her line of simultaneity AT as she arrives at Canopus at T. However, as she turns around, her line of simultaneity also slews forward, swinging from line AT to line BT. Suddenly the events on line AB - all those intermediate ticks of the Earth clock - are in the astronaut's past. These events lie below the line of simultaneity BT as she starts back at T. Her outward-moving colleague reads 3.96 years on the Earth clock as she reaches Canopus; an instant later on her clock, her new inward-moving colleague reads 198.04 on the Earth clock.

Shall we say that the Earth clock "jumps ahead" as the astronaut turns around? No! Utterly ridiculous! For what single observer does it jump ahead? Not for the Earth observer. Not for the outgoing set of clock readers. Not for the returning set of clock readers. For whom then? Nobody! At the same time as she reaches Canopus-old meaning of simultaneous! - the astronaut's outgoing colleague records 3.96 years for the Earth clock. At the same time as she leaves Canopus - new meaning of simultaneous! - her new ingoing colleague records 198.04 years on the Earth clock. The astronaut has nobody but herself to blame for her misperception of a "jump" in the Earth clock reading.

The "lost Earth time" AB in the figure makes consistent the story each observer tells about the clocks. Simple is the story told by the Earth observer: "My clock ticked along steadily at the 'proper' rate from astronaut departure to astronaut return. In contrast, ticks on the astronaut clock were far apart in time on both the outgoing and incoming legs of her trip. We agree that her total ticks are less than my total ticks: she is younger than I when we meet again." More complicated is the astronaut account of clock behavior: "Ticks on the Earth clock were far apart in time as I traveled to Canopus; ticks on the Earth clock were also far apart as I traveled home again. But as I turned around, a whole bunch of Earth clock ticks went from my future to my past. This accounts for the larger number of total ticks on the Earth clock than on my clock during the trip. We agree that I am younger when we meet again."

So saying, the astronaut renounces her profession and becomes a stand-up comedian.

Reference: E. Lowry, *American Journal of Physics*, Volume 31, page 59 (1963).