3.7: Invariance of the Interval Proved

laboratory and rocket observers agree on something important

Principle of Relativity leads to invariance of spacetime interval

The Principle of Relativity has a major consequence. It demands that the spacetime interval have the same value as measured by observers in every overlapping free-float frame; in brief, it demands "invariance of the interval."[1] Proof? Plan of attack: Determine the separation in space and the separation in time between two events, E and R, in the rocket frame. Then determine the quite different space and time separations between the same two events as measured in a free-float laboratory frame. Then look for - and find - what is invariant. It is the "interval." Now for the details (Figures $3.7.1$ and $3.7.2$).

Event E we take to be the reference event, the emission of a flash of light from the central laboratory and rocket reference clocks as they coincide at the zero of time (Section 2.6). The path of this flash is tracked by the recording clocks in the rocket lattice. Riding with the rocket, we examine that portion of the flash that flies straight "up" 3 meters to a mirror. There it reflects straight back down to the photodetector located at our rocket reference clock, where it is received and recorded. The act of reception constitutes the second event we consider. This event, R, is located at the rocket space origin, at the same location as the emission event E. Therefore, for the rocket observer, the space separation between event E and event R equals zero.

What is the time separation between events E and R in the rocket frame? The light travels 3 meters up to the mirror and 3 meters back down again, a total of 6 meters of distance. At the "standard" light speed of 1 meter of distance per meter of light-travel time, the flash takes a total of 6 meters of time to complete the round trip. In summary, for the rocket observer the event of reception, R, is separated from the event of emission, E, by zero meters in space and 6 meters in time.

What are the space and time separations of events E and R measured in the free-float laboratory frame? As measured in the laboratory, the rocket moves at high speed to the right (Figures $3.7.1$ and $3.7.2$). The rocket goes so fast that the simple up-down track of the light in the rocket frame appears in the laboratory to have the profile of a tent, with its right-hand corner - the place of reception of the light - 8 meters to the right of the starting point.

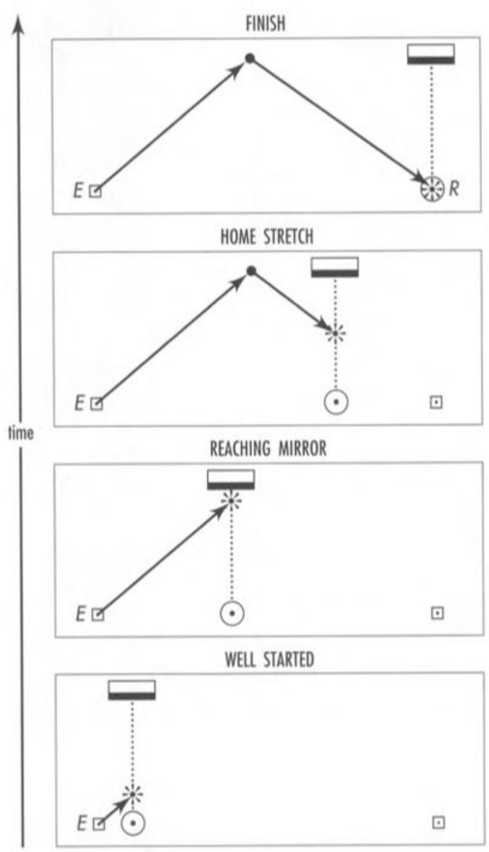

Figure $3.7.1$: **Plot of the flash path as recorded in the laboratory frame**. Time progresses from bottom to top: Well started: The flash (represented as an asterisk) bas been emitted (event E) from a moving rocket clock (shown as a circle) that coincided with a laboratory clock (shown as a square). **Reaching mirror** and **Home stretch**: The flash reaches a mirror and reflects from it. The mirror moves along in step with the rocket clock. **Finish**: The flash is received (event R) back at the same rocket clock, which has moved in the laboratory frame to coincide with a second laboratory clock. Figure $3.7.2$ shows the trajectory of the same flash in three different free-float frames.

When does the event of reception, R, take place as registered in the laboratory frame? Note that it occurs at the time 6 meters in the rocket frame. All we know about everyday events urges us to say, "Why, obviously it occurs at 6 meters of time in the laboratory frame too." But no. More binding than preconceived expectations are the demands of the Principle of Relativity. Among those demands none ranks higher than this: The speed of light has the standard value 1 meter of distance in 1 meter of light-travel time in every free-float frame.

Greater distance of travel for light flash: longer time!

Figure $3.7.3$ punches us in the eye with this point: The light flash travels *farther as* recorded in the laboratory frame than as recorded in the rocket frame. The perpendicular "altitude" of the mirror from the line along which the rocket reference clock moves has the same value in laboratory frame as in rocket frame no matter how fast the rocket - as shown in Section 3.6. Therefore on its slanted path toward and away from the mirror the flash must cover more distance in the laboratory frame than it does in the rocket frame. More distance covered means more time required at the "standard" light speed.[2] We conclude that the time between events E and R is greater in the laboratory frame than in the rocket frame - a staggering result that stood physics on its ear when first proposed. There is no way out.

In the laboratory frame the flash has to go "up" 3 meters, as before, and "down" again 3 meters. But in addition it has to go 8 meters to the right: 4 meters to the right while rising to hit the mirror, and 4 meters more to the right while falling again to the receptor. The Pythagorean Theorem, applied to the right triangles of Figure $3.7.3$, tells us that each slanted leg of the trip has length 5 meters:

$$(3 \text{ meters })^2+(4 \text{ meters })^2=(5 \text{ meters })^2$$

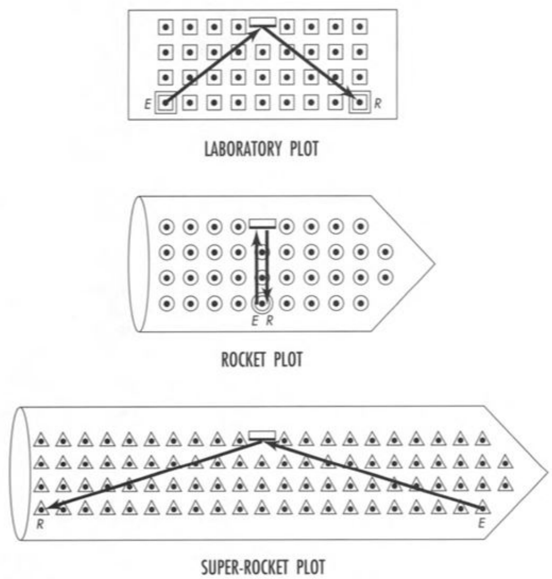

Figure $3.7.2$: **Plots of the path in space of a reflected flash of light as measured in three different frames, showing event E, emission of the flash, and event R, its reception after reflection.** Squares, circles, and triangles represent latticeworks of recording clocks in laboratory, rocket, and super-rocket frames, respectively. The super-rocket frame moves to the right with respect to the rocket, and with such relative speed that the event of reception, R, occurs to the left of the event of emission, E, as measured in the super-rocket frame. The reflecting mirror is fixed in the rocket, hence appears to move from left to right in the laboratory and from right to left in the super-rocket.

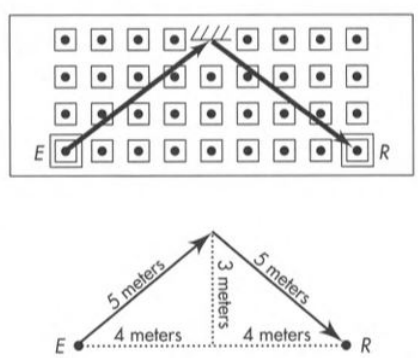

Figure $3.7.3$: **Laboratory plot of the path of the light flash**. The flash rises 3 meters while it moves to the right 4 meters. Then it falls 3 meters as it moves an additional 4 meters to the right. From the Pythagorean Theorem, the total length of the flash path equals 5 meters plus 5 meters or 10 meters. Therefore 10 meters of light-travel time is the separation in time between emission event E and reception event R as measured in the laboratory frame.

Thus the total length of the trip equals 10 meters, definitely longer than the length of the round trip, 6 meters, as observed in the rocket frame. Moreover, the light can cover that slanted and greater distance only at the standard rate of 1 meter of distance in 1 meter of light-travel time. Therefore there is no escape from saying that the time of reception as recorded in the laboratory frame equals 10 meters. Thus there is a great variance between what is recorded in the two frames (Figure $3.7.2$, Laboratory plot and Rocket plot): separation in time and in space between the emission E of a pulse of light and its reception R after reflection.

In spite of the difference in space separation between events E and R and the difference in time lapse between these events as measured in laboratory and rocket frames, there exists a measure of their separation that has the same value for both observers. This is the interval calculated from the difference of squares of time and space separations (Table $3.7.1$). For both observers the interval has the value 6 meters. The interval is an **invariant** between free-float frames.

Two central results are to be seen here, one of variance, the other of invariance. We discover first that typically there is not and cannot be an absolute time difference between two events. The difference in time depends on our choice of the free-float frame, which inertial frame we use to record events. There is no such thing as a simple concept of universal and absolute separation in time.

Between events: No absolute time, but invariant interval

Second, despite variance between the laboratory frame and the rocket frame in the values recorded for time and space separations individually, the difference between the Between events: No absolute time, but invariant interval squares of those separations is identical, that is, invariant with respect to choice of reference frame. The difference of squares obtained in this way defines the square of the interval. The invariant interval itself has the value 6 meters in this example.[3]

Table 3.7.1: Reckoning the Spacetime Interval From Rocket and Laboratory Measurements

	Rocket measurements	Laboratory measurements
Time from emission of the flash to its reception	6 meters	10 meters
Distance from the point of emission of the flash to its point of reception	0 meters	8 meters
Square of time	36 (meters)^2	100 (meters)^2
Square distance and subtract	- 0 (meters)^2	- 64 (meters)^2
Result of subtraction	= 36 (meters)^2	= 36 (meters)^2
This is the square of what measurement?	6 meters	6 meters

Note

Note the same spacetime interval for both rocket and laboratory measurements.

1 Principle of Relativity leads to invariance of spacetime interval

2 Greater distance of travel for light flash: longer time!

3 Between events: No absolute time, but invariant interval