11.5: Completing the Derivation

invariance of the interval completes the story

Equations (11.3.3) and (11.3.4) provide coefficients D and H called for in equation (11.4.1):

$$\begin{aligned} t &= Bx' + \gamma t' \\ x &= Gx' + v_{\text{rel}}\, \gamma t' \end{aligned} \tag{11.5.1}$$

About the two constants B and G we know nothing, for an elementary reason. All events so far considered occured at point $x' = 0$ in the rocket. Therefore the two coefficients B and G could have any finite values whatever without affecting the numerical results of the calculation. To determine B and G we turn our attention from an $x' = 0$ event to a more general event, one that occurs at a point with arbitrary rocket coordinates x' and t'.

Demanding invariance of interval...

Then we demand that the spacetime interval have the same numerical value in laboratory and rocket frames for any event whatever:

$$t^2 - x^2 = t'^2 - x'^2$$

Substitute expressions for t and x from equation $11.5.1$:

$$(Bx' + \gamma t')^2 - (Gx' + v_{\text{rel}}\, \gamma t')^2 = t'^2 - x'^2$$

On the left side, multiply out the squares. This leads to the rather cumbersome result

$$B^2 x'^2 + 2B\gamma x't' + \gamma^2 t'^2 - G^2 x'^2 - 2G v_{\text{rel}}\, \gamma x't' - v_{\text{rel}}^2\, \gamma^2 t'^2 = t'^2 - x'^2$$

Group together coefficients of t'^2, coefficients of x'^2, and coefficients of the cross-term $x't'$ to obtain

$$\gamma^2 \left(1 - v_{\text{rel}}^2\right) t'^2 + 2\gamma \left(B - v_{\text{rel}} G\right) x't' - \left(G^2 - B^2\right) x'^2 = t'^2 - x'^2 \tag{11.5.2}$$

. . . between any pair of events whatsoever...

Now, t' and x' can each take on any value whatsoever, since they represent the coordinates of an arbitrary event. Under these circumstances, it is impossible to satisfy equation $11.5.2$with a single choice of values of B and G unless they are chosen in a very special way. The quantities B and G must first be such as to make the coefficient of $x't'$ on the left side of equation $11.5.2$ vanish as it does on the right:

$$2\gamma \left(B - v_{\text{rel}}\, G\right) = 0$$

But γ can never equal zero. The value of $\gamma = 1/\left(1 - v_{\text{rel}}^2\right)^{1/2}$ equals unity when $v_{\text{rel}} = 0$ and is greater than this for any other values of v_{rel}. Hence the left side of this equation can be zero only if

$$\left(B - v_{\text{rel}}\, G\right) = 0 \quad \text{or} \quad B = v_{\text{rel}}\, G \tag{11.5.3}$$

Second, B and G must be such as to make the coefficient of x'^2 equal on the left and right of equation $11.5.2$; hence

$$G^2 - B^2 = 1 \tag{11.5.4}$$

... leads to completed form of Lorentz transformation.

Substitute B from equation $11.5.3$into equation $11.5.4$:

$$G^2 - \left(v_{\text{rel}}\, G\right)^2 = 1 \quad \text{or} \quad G^2 \left(1 - v_{\text{rel}}^2\right) = 1$$

Divide through by $\left(1 - v_{\text{rel}}^2\right)$ and take the square root of both sides:

$$G = \frac{1}{\left(1 - v_{\text{rel}}^2\right)^{1/2}}$$

But the right side is just the definition of the time stretch factor γ, so that

$$G = \gamma$$

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Substitute this into equation $11.5.3$ to find B :

$$B=v_{\text{rel}}\,\gamma$$

\]

These results plus equations 11.4.1 and $11.5.1$ yield the Lorentz transformation equations:

The Lorentz transformation

$$\begin{aligned}
&t=v_{\text{rel}}\,\gamma x'+\gamma t' \\
&x=\gamma x'+v_{\text{rel}}\,\gamma t' \\
&y=y' \\
&z=z'
\end{aligned}\tag{11.5.5}$$

or, substituting for the value of gamma, $\gamma=1/\left(1-v_{\text{rel}}^2\right)^{1/2}$:

$$\begin{aligned}
t&=\frac{v_{\text{rel}}x'+t'}{\left(1-v_{\text{rel}}^2\right)^{1/2}} \\
x&=\frac{x'+v_{\text{rel}}t'}{\left(1-v_{\text{rel}}^2\right)^{1/2}} \\
y&=y' \quad \text{and} \quad z=z'
\end{aligned}$$

In summary, the Lorentz transformation equations rest fundamentally on the required linearity of the transformation and on the invariance of the spacetime interval. Invariance of the interval was used twice in the derivation. First, we examined a pair of events both of which occur at the same fixed location in the rocket, so that rocket time between these events-proper time, wristwatch time-equals the space-time interval between them (Section 11.3). Second, we demanded that the interval also be invariant between every possible event and the reference event (the present section).