

7.E: Momenergy (Exercises)

PRACTICE

7-1 momenergy 4-vector

For each of the following cases, write down the four components of the momentum-energy (momenergy) 4-vector in the given frame in the form $[E, p_x, p_y, p_z]$. Assume that each particle has mass m . You may use square roots in your answer.

- A particle moves in the positive x -direction in the laboratory with total energy equal to five times its rest energy.
- Same particle as observed in a frame in which it is at rest.
- Another particle moves in the z -direction with momentum equal to three times its mass.
- Yet another particle moves in the negative y -direction with kinetic energy equal to four times its mass.
- Still another particle moves with total energy equal to ten times its mass and x -, y -, and z -components of momentum in the ratio 1 to 2 to 3.

7-2 system mass

Determine the mass of the system of particles shown in Figure 7-6. Is this system mass equal to the sum of the masses of the individual particles in the system? Does the mass of this system change as a result of the interaction? Does the momenergy 4-vector of the system change as a result of the interaction? (In Chapter 8 there is a lot more discussion about the mass of a system of particles.)

7-3 much ado about little

Two freight trains, each of mass 5×10^6 kilograms (5000 metric tons) travel in opposite directions on the same track with equal speeds of 42 meters/second (about 100 miles/hour). They collide head on and come to rest.

- Calculate in milligrams the kinetic energy for each train $(1/2)mv^2$ before the collision. (Newtonian expression OK for 100mph!) (1 milligram = 10^{-3} gram = 10^{-6} kilogram)
- After the collision, the mass of the trains plus the mass of the track plus the mass of the roadbed has increased by what number of milligrams? Neglect energy lost in the forms of sound and light.

7-4 fast protons

Each of the protons described in the table emits a flash of light every meter of its own (proper) time $d\tau$. Between successive flash emissions, each proton travels a distance given in the left column. Complete the table. Take the rest energy of the proton to be equal to $1\text{GeV} = 10^9\text{eV}$ and express momentum in the same units. Hints: Avoid calculating or using the speed v in relativistic particle problems; it is too close to unity to distinguish between protons of radically different energies. An accuracy of two significant figures is fine; don't give more. Recall: $E^2 - p^2 = m^2$ and $E = mdt/d\tau = m\gamma$ [note tau!].

| Lab distance Δx traveled between flashes (meters) | Momentum $\backslash (m \text{ d } x/d\tau)$ (GeV) | Energy (GeV) | Time stretch factor γ | Lab time between flashes (meters) |
|---|---|-----------------|---------------------------------|---|
| 0 | | | | |
| 0.1 | | | | |
| 1 | | | | |
| 10 | | | | |
| 10^3 | | | | |
| 10^6 | | | | |
| factor γ | | | | |

PROBLEMS

7-5 Lorentz transformation for momenergy components

The rocket observer measures energy and momentum components of a particle to have the values E' and p'_x , p'_y , and p'_z . What are the corresponding values of energy and momentum measured by the laboratory observer? The answer comes from the Lorentz transformation, equation (L-10) in the Special Topic following Chapter 3.

The moving particle emits a pair of sparks closely spaced in time as measured on its wristwatch. The rocket latticework of clocks records these emission events; so does the laboratory latticework of clocks. The rocket observer constructs components of particle momentum and energy, equation (7-2), from knowledge of particle mass m , the spacetime displacements dt' , dx' , dy' , and dz' derived from the event recordings, and the proper time $d\tau$ computed from these spacetime components. Laboratory momenergy components come from transforming the spacetime displacements. The Lorentz transformation, equation (L-10), for incremental displacements gives

$$\begin{aligned} dt &= \gamma dx' + \gamma dt' \\ dx &= \gamma dx' + v\gamma dt' \\ dy &= dy' \\ dz &= dz' \end{aligned}$$

a Multiply both sides of each equation by the invariant mass m and divide through by the invariant proper time $d\tau$. Recognizing the components of the momenergy 4-vector in equation (7-2), show that the transformation equations for momenergy are

$$\begin{aligned} E &= v\gamma p'_x + \gamma E' \\ p_x &= \gamma p'_x + v\gamma E' \\ p_y &= p'_y \\ p_z &= p'_z \end{aligned}$$

b Repeat the process for particle displacements dt , dx , dy , and dz recorded in the laboratory frame to derive the inverse transformations from laboratory to rocket.

$$\begin{aligned} E' &= -v\gamma p_x + \gamma E \\ p'_x &= \gamma p_x - v\gamma E \\ p'_y &= p_y \\ p'_z &= p_z \end{aligned}$$

7-6 fast electrons

The Two-Mile Stanford Linear Accelerator accelerates electrons to a final kinetic energy of 47 GeV (47×10^9 electron-volts; one electron-volt = 1.6×10^{-19} joule). The resulting high-energy electrons are used for experiments with elementary particles. Electromagnetic waves produced in large vacuum tubes ("klystron tubes") accelerate the electrons along a straight pipelike structure 10,000 feet long (approximately 3000 meters long). Take the rest energy of an electron to be $m \approx 0.5 \text{ MeV} = 0.5 \times 10^6$ electronvolts.

a Electrons increase their kinetic energy by approximately equal amounts for every meter traveled along the accelerator pipe as observed in the laboratory frame. What is this energy gain in MeV/meter? Suppose the Newtonian expression for kinetic energy were correct. In this case how far would the electron travel along the accelerator before its speed were equal to the speed of light?

b In reality, of course, even the 47-GeV electrons that emerge from the end of the accelerator have a speed v that is less than the speed of light. What is the value of the difference $(1 - v)$ between the speed of light and the speed of these electrons as measured in the laboratory frame? [Hint: For v very near the value unity, $1 - v^2 = (1 + v)(1 - v) \approx 2(1 - v)$.] Let a 47-GeV electron from this accelerator race a flash of light along an evacuated tube straight through Earth from one side to the other (Earth diameter 12,740 kilometers). How far ahead of the electron is the light flash at the end of this race? Express your answer in millimeters.

c How long is the "3000-meter" accelerator tube as recorded on the latticework of rocket clocks moving along with a 47-GeV electron emerging from the accelerator?

7-7 super cosmic rays

The Haverah Park extensive air shower array near Leeds, England, detects the energy of individual cosmic ray particles indirectly by the resulting shower of particles this cosmic ray creates in the atmosphere. Between 1968 and 1987 the Haverah Park array

detected more than 25,000 cosmic rays with energies greater than 4×10^{17} electron-volts, including 5 with an energy of approximately 10^{20} electron-volts. (rest energy of the proton $\approx 10^9$ electron-volts = 1.6×10^{-10} joule)

a Suppose a cosmic ray is a proton of energy 10^{20} electron-volts. How long would it take this proton to cross our galaxy as measured on the proton's wristwatch? The diameter of our galaxy is approximately 10^5 light-years. How many centuries would take as observed in our Earth-linked frame?

b The research workers at Haverah Park find no evidence of an upper limit to cosmic ray energies. A proton must have an energy of how many times its rest energy for the diameter of our galaxy to appear to it Lorentz-contracted to the diameter of the proton (about 1 femtometer, which is equal to 10^{-15} meters)? How many metric tons of mass would have to be converted to energy with 100-percent efficiency equals 1000 kilograms.

Reference: M. A. Lawrence, R. J. O. Reid, and A. A. Watson, Journal of Physics G: Nuclear and Particle Physics, Volume 17, pages 733 – 757 (1991).

7-8 rocket nucleus

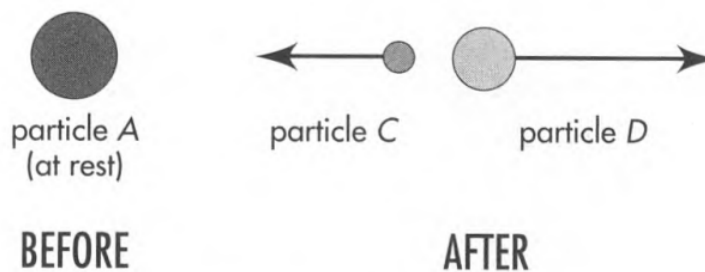


Figure 7.E. 1: Radioactive decay of a particle

A radioactive decay or "inverse collision" is observed in the laboratory frame, as shown in the figure.

Suppose that $m_A = 20$ units, $m_C = 2$ units, and $E_C = 5$ units.

a What is the total energy E_A of particle A ?

b From the conservation of energy, find the total energy E_D (rest plus kinetic) of particle D.

c Using the expression $E^2 - p^2 = m^2$ find the momentum p_C of particle C.

d From the conservation of momentum p_D of particle D

e What is the mass m_D of particle D ?

f Does $m_C + m_D$ after the collision equal m_A before the collision? Explain your answer.

9 Draw three momenergy diagrams for this reaction similar to those of Figure 7-6: BEFORE, SYSTEM, and AFTER. Plot positive and negative momentum along the positive and negative horizontal direction, respectively, and energy along the vertical direction. On the AFTER diagram draw the momenergy vectors for particles C and D head to tail so that they add up to the momenergy vector for the system. Place labeled mass handles on the arrows in all three diagrams, including the arrow for the system.

7-9 sticky collision

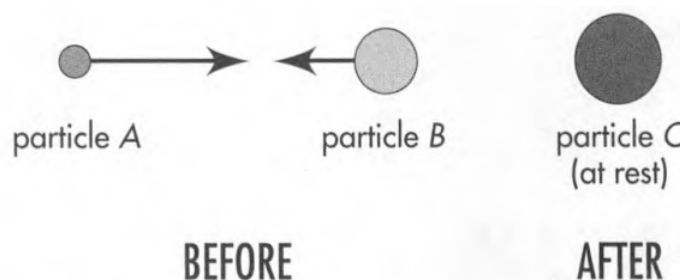


Figure 7.E. 2: Two particles collide to form a third at rest in the laboratory frame.

An inelastic collision is observed in the laboratory frame, as shown in the figure. Suppose that $m_A = 2$ units, $E_A = 6$ units, $m_C = 15$ units.

a From the conservation of energy, what is the energy E_B of particle B ?

b What is the momentum p_A of particle A ? Therefore what is the momentum p_B of particle B ?

c From $m^2 = E^2 - p^2$ find the mass m_B of par-

d Quick guess: Is the mass of particle C after the collision less than or greater than the sum of the masses of particles A and B before the collision? Validate your guess from the answer to part c.

7-10 colliding putty balls

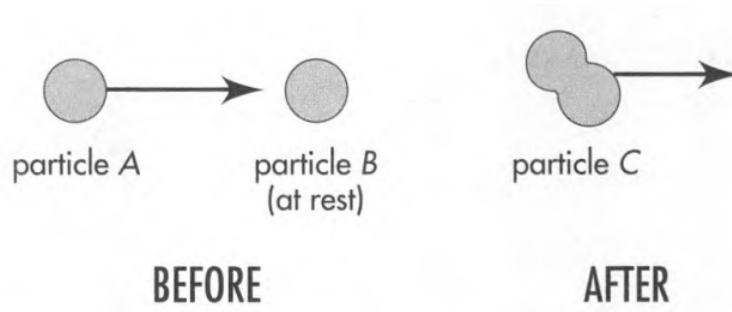


Figure 7.E. 3: Two putty balls stick together

A ball of putty of mass m and kinetic energy K streaks across the frozen ice of a pond and hits a second identical ball of putty initially at rest on the ice. The two stick together and skitter onward as one unit. Referring to the figure, find the mass of the combined particle using parts a-e or some other method.

a What is the total energy of the system before the collision? Keep the kinetic energy K explicitly, and don't forget the rest energies of both particles A and B . Therefore what is the total energy E_C of particle C after the collision?

b Using the equation $m^2 = E^2 - p^2 = (m + K)^2 - p^2$ find the momentum p_A of particle A before the collision. What is the total momentum of the system before the collision? Therefore what is the momentum p_C of particle C after the collision?

EXERCISE 7-10. Two putty balls stick together. c Again use the equation $m^2 = E^2 - p^2$ to find the mass m_C of particle C . Show that the result satisfies the equation

$$m_C^2 = (2m)^2 + 2mK = (2m)^2 \left(1 + \frac{K}{2m} \right) \quad (7.E.1)$$

d Examine the result of part c in two limiting cases. (1) The value of m_C in the Newtonian low-velocity limit in which kinetic energy is very much less than mass: $K/m \ll 1$. Is this what one expects from everyday living? (2) What is the value of m_C in the highly relativistic limit in which $K/m \gg 1$? What is the upper limit on the value of m_C ? Discussion: Submicroscopic particles moving at extreme relativistic speeds rarely stick together when they collide. Rather, their collision often leads to creation of additional particles. See Chapter 8 for examples.

e Discussion question: Are the results of part c changed if the resulting blob of putty rotates, whirling like a dumbbell about its center as it skitters along?

7-11 limits of Newtonian mechanics

a One electron-volt (eV) is equal to the increase of kinetic energy that a singly charged particle experiences when accelerated through a potential difference of one volt. One electron-volt is equal to 1.60×10^{-19} joules. Verify the rest energies of the electron and the proton (masses listed inside the back cover) in units of million electron-volts (MeV).

b The kinetic energy of a particle of a given velocity v is not correctly given by the expression $1/2mv^2$. The error

is one percent when the Newtonian kinetic energy has risen to a certain fraction of the rest energy. What fraction? Hint: Apply the first three terms of the binomial expansion

$$(1+z)^n = 1 + nz + \frac{1}{2}n(n-1)z^2 + \dots \quad (7.E.2)$$

to the relativistic expression for kinetic energy, an accurate enough approximation if $|z| \ll 1$. Let this point - where the error is one percent - be arbitrarily called the "limit of Newtonian mechanics." What is the speed of the particle at this limit? At what kinetic energy does a proton reach this limit (energy in MeV ? An electron?

c An electron in a modern color television tube is accelerated through a voltage as great as 25,000 volts and then directed by a magnetic field to a particular pixel of luminescent material on the inner face of the tube. Must the designer of color television tubes use special relativity in predicting the trajectories of these electrons?

7-12 derivation of the relativistic expression for momentum - a worked example

A very fast particle interacts with a very slow particle. If the collision is a glancing one, the slow particle may move as slowly after the collision as before. Reckon the momentum of the slow-moving particle using the Newtonian expression. Now demand that momentum be conserved in the collision. From this derive the relativistic expression for momentum of the fastmoving particle.

The top figure shows such a glancing collision. After the collision each particle has the same speed as before the collision, but each particle has changed its direction of motion.

Behind this figure is a story. Ten million years ago, and in another galaxy nearly ten million light-years distant, a supernova explosion launched a proton toward Earth. The energy of this proton far exceeded anything we can give to protons in our earthbound particle accelerators. Indeed, the speed of the proton so nearly approached that of light that the proton's wristwatch read a time lapse of only one second between launch and arrival at Earth.

We on Earth pay no attention to the proton's wristwatch. For our latticework of Earth-linked observers, ages have passed since the proton was launched. Today our remote outposts warn us that the streaking proton approaches Earth. Exactly one second on our clocks before the proton is due to arrive, we launch our own proton at the slow speed one meter/second almost perpendicular to the direction of the incoming proton (BEFORE part of the top figure). Our proton saunters the one meter to the impact point. The two protons meet. So perfect is our aim and timing that after the encounter our proton simply reverses direction and returns with the same speed we gave it originally (AFTER part of the top figure). The incoming proton also does not change speed, but it is deflected upward at the same angle at which it was originally slanting downward.

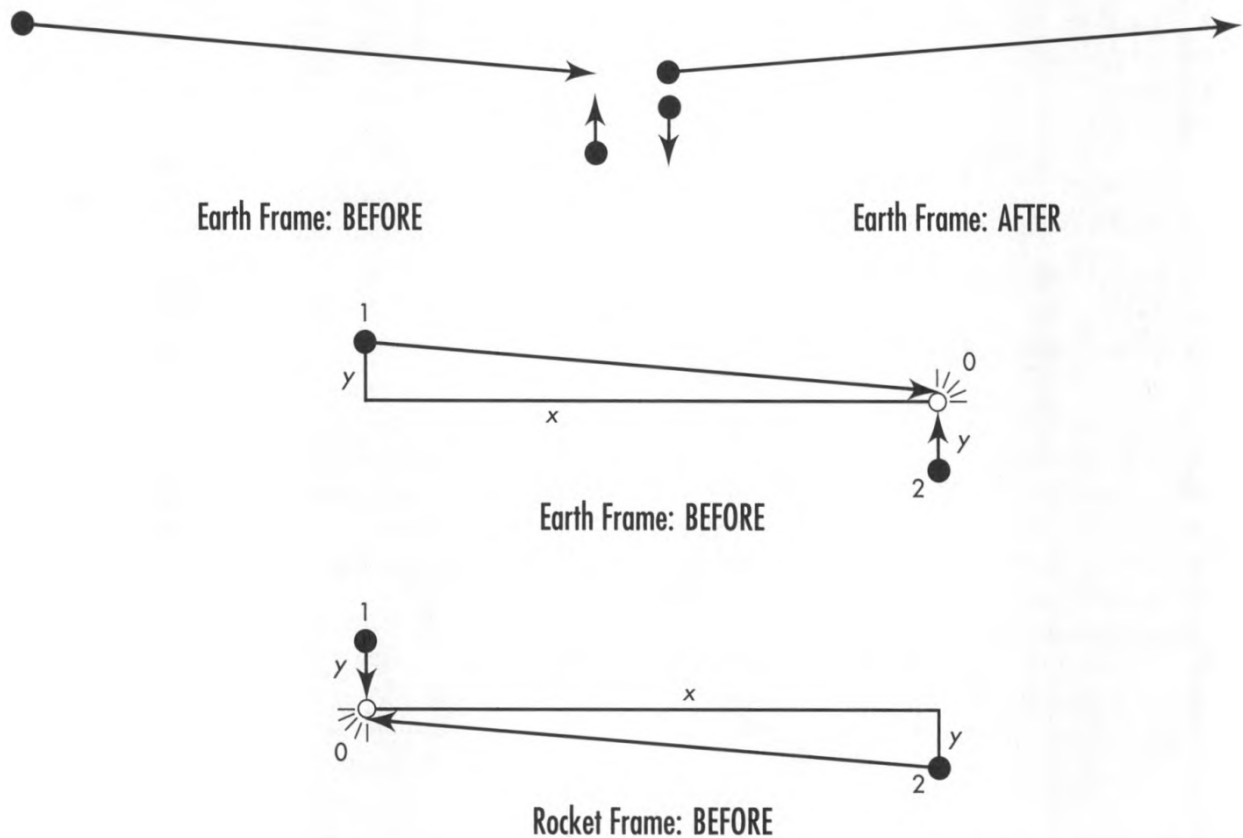


Figure 7.E. 4: A symmetric elastic collision between a fast proton and a slow proton in which each proton changes direction but not speed as a result of the encounter. Center: Events and separations as observed in Earth frame before the collision. Here $x = 10$ million light-years and $y = 1$ meter, so these figures are not to scale! Bottom; Events and separations as observed in the rocket frame before the collision.

How much does y -momentum of our slow-moving proton change during this encounter? Newton can tell us. At a particle speed of one meter/second, his expression for momentum, mv , is accurate. Our proton simply reverses its direction. Therefore the change in its momentum is just $2mv$, twice its original momentum in the y -direction.

What is the change in the y -momentum of the incoming proton, moving at extreme relativistic speed? We demand that the change in y -momentum of the fast proton be equal in magnitude and opposite in direction to the change in y -momentum of our slow proton. In brief, y -momentum is conserved. This demand, plus a symmetry argument, leads to the relativistic expression for momentum.

Key events in our story are numbered in the center figure. Event 1 is the launching of the proton from the supernova ten million years (in our frame) before the impact. Event 2 is the quiet launch of our local proton one second (in our frame) before the impact. Event 0 is the impact itself. The x -direction is chosen so that y -displacements of both protons have equal magnitude between launch and impact, namely one meter.

Now view the same events from a rocket moving along the x -axis at such a speed that events 1 and 0 are vertically above one another (bottom figure). For the rocket observer the transverse y -separations are the same as for the Earth observer (Section 3.6), so $y = 1$ meter in both frames. The order of events 1 and 2, however, is exactly reversed in time: For the rocket observer, we released our proton at high speed ten million years before impact and she releases hers one second before the collision. Otherwise the diagrams are symmetrical: To make the bottom figure look like the center one, exchange event numbers 1 and 2, then stand on your head!

Rocket observer and Earth observer do not agree on the time between events 1 and 0, but they agree on the proper time τ_{10} between them, namely one second. They also agree on the proper time τ_{20} between events 2 and 0. Moreover, because of the symmetry between the center and bottom figures, these two proper times have the same value: For the case we have chosen, the wristwatch (proper) time for each proton is one second between launch and impact.

$$\tau_{10} = \tau_{20} \quad (7.E.3)$$

We can use these quantities to construct expressions for the y -momenta of the two protons. Both are protons, so their masses m are the same and have the same invariant value for both observers. Because of the equality in magnitude of the y -displacements and the equality of τ_{20} and τ_{10} , we can write

$$m \frac{y}{\tau_{10}} = -m \frac{y}{\tau_{20}} \quad (7.E.4)$$

[both frames]

The final key idea in the derivation of the relativistic expression for momentum is that the slow-moving proton travels between events 2 and 0 in an Earth-measured time that is very close in value to the proper time between these events. The vertical separation y between events 2 and 0 is quite small: one meter. In the same units, the time between them has a large value in the Earth frame: one second, or 300 million meters of light-travel time. Therefore, for such a slow-moving proton, the proper time τ_{20} between events 2 and 0 is very close to the Earth time t_{20} between these events:

$$\tau_{20} \approx t_{20} \quad [\text{Earth frame only}] \quad (7.E.5)$$

Hence rewrite the both-frames equation for the Earth frame:

$$m \frac{y}{\tau_{10}} = -m \frac{y}{t_{20}} \quad [\text{Earth frame only}] \quad (7.E.6)$$

The right side of this equation gives the y -momentum of the slow proton before the collision, correctly calculated using the Newtonian formula. The change in momentum of the slow proton during the collision is twice this magnitude. Now look at the left side. We claim that the expression on the left side is the y -momentum of the very fast proton. The y -momentum of the fast proton also reverses in the collision, so the change is just twice the value of the left side. In brief, this equation embodies the conservation of the y component of total momentum in the collision. Conclusion: The left side of this equation yields the relativistic expression for y -momentum: mass times displacement divided by proper time for this displacement.

What would be wrong with using the Newtonian expression for momentum on the left side as well as on the right? That would mean using earth time t_{10} instead of proper time τ_{10} in the denominator of the left side. But t_{10} is the time it took the fast proton to reach Earth from the distant galaxy as recorded in the Earth frame - ten million years or 320 million million seconds! With this substitution, the equation would no longer be an equality; the left side would be 320 million million times smaller in value than the right side (smaller because t_{10} would appear in the denominator). Nothing shows more dramatically than this the radical difference between Newtonian and relativistic expressions for momentum - and the correctness of the relativistic expression that has proper time in the denominator.

This derivation of the relativistic expression for momentum deals only with its y -component. But the choice of y -direction is arbitrary. We could have interchanged y and x axes. Also the expression has been derived for particles moving with constant velocity before and after the collision. When velocity varies with time, the momentum is better expressed in terms of incremental changes in space and time. For a particle displacement dr between two events a proper time $d\tau$ apart, the expression for the magnitude of the momentum is

$$p = m \frac{dr}{d\tau} \quad (7.E.7)$$

One-sentence summary: In order to preserve conservation of momentum for relativistic collisions, simply replace Newton's "universal time" t in the expression for momentum with Einstein's invariant proper time τ .

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