

7.2: Momenergy Arrow

a spacetime arrow pointing along the worldline

What lies behind the name momentum-energy (momenergy)? What counts are its properties. We most easily uncover three central properties of momenergy by combining everyday observation with momenergy's essential feature: Total momenergy is conserved in any collision.

Momenergy of particle proportional to its mass

First, think of two pebbles of different sizes moving with the same velocity toward the windshield of a speeding car. One bounces off the windshield without anyone noticing; the other startles the occupants and leaves a scratch. Five times the mass? Five times the punch-delivering capacity! Five times the momenergy. Momenergy, in other words, is proportional to mass.

Momenergy a directed quantity

Second, momentum-energy of a particle depends on its direction of travel. A pebble coming from the front takes a bigger chip out of the windshield than a pebble of equal mass and identical speed glancing off the windshield from the side. Therefore momenergy is not measurable by a mere number. It is a directed quantity. Like an arrow of a certain length, it has magnitude and direction.

Momenergy a 4-vector

Our experience with the unity of spacetime leads us to expect that the momenergy arrow will have three parts, corresponding to three space dimensions, plus a fourth part corresponding to time. In what follows we find that momenergy is indeed a four-dimensional arrow in spacetime, **the momenergy 4-vector** (Box 7-1). Its three "space parts" represent the momentum of the object in the three chosen space directions. Its "time part" represents energy. The unity of momentum and energy springs from the unity of space and time.

Particle momenergy points along its worldline

In what direction does the momenergy 4-vector of a particle point? It points in the "same direction in spacetime" as the worldline of the particle itself (Figure 7-1). There is no other natural direction in which it can point! Spacetime itself has no structure that indicates or favors one direction rather than another. Only the motion of the particle itself gives a preferred direction in spacetime. The particle moves from one event to a nearby event along its worldline. In so doing, it undergoes a spacetime displacement, small changes in the three space positions along with an accompanying small advance in the time. The spacetime displacement has four parts: it is a 4-vector. The momenergy arrow points in the direction of another arrow, the arrow of the particle's spacetime 4-vector displacement. Momenergy runs parallel to worldline!

Momenergy independent of reference frame

Compare the worldline of an individual particle in spacetime with a single straw in a great barn filled with hay. This particular straw has a direction, an existence, and a meaning independent of any measuring method imagined by humans who stack the hay or by mice that live in it. Similarly, in the rich trelliswork of worldlines that course through spacetime, the arrowlike momenergy of the particle has an existence and definiteness independent of the choice - or even use - of any free-float frame of reference (Section 5.9).

Particle wristwatch logs time for momenergy

No frame of reference? Then no clock available to time motion from here to there! Or rather no clock except one that the particle itself carries, its own wristwatch that records proper time. Proper time for what? Proper time for spacetime displacement between two adjacent events on the worldline of the particle. Proper time provides the only natural way to clock the rate of motion of the particle; that is the third and final feature of momenergy.

In brief, the momenergy of a particle is a 4-vector: Its magnitude is proportional to its mass, it points in the direction of the particle's spacetime displacement, and it is reckoned using the proper time for that displacement. How are these properties combined to form momenergy? Simple! Use the recipe for Newtonian momentum: mass times displacement divided by time lapse for that displacement. Instead of Newtonian displacement in space, use Einstein's displacement in spacetime; instead of Newton's "universal time," use Einstein's proper time.

 Box 7-1

A vector is a mathematical object that has both magnitude and direction. The meanings of the terms magnitude and direction, however, differ between one geometry and another. Mathematics offers many geometries. The two geometries important to us in this book are Euclidean geometry and Lorentz geometry.

Euclidean geometry defines 3-vectors located in 3-dimensional space. Let a speeding particle emit two sparks. The particle's spatial displacement from first spark to second spark is a 3-vector. Each of the three components (northward, eastward, and upward) of this 3-vector displacement has a value larger or smaller, depending on the orientation of the coordinate system chosen. In contrast, the magnitude of the displacement - the distance traveled (computed as the square root of the sum of the squares of the three components of displacement) - has the same value in all coordinate systems.

Lorentz geometry defines 4-vectors located in 4-dimensional spacetime. Construct the 4-vector spacetime displacement from the three space components supplemented by the time component, the time between sparks emitted by the speeding particle. Each of these four components (including time) has a value larger or smaller, depending on the choice of free-float frame of reference from which it is measured. The square of the separation in time between the two sparks as so measured, diminished by the square of the separation in space in the chosen frame, yields the square of the spacetime interval between the two events. This interval has the same value in all free-float frames. It is also the proper time, the time between the two sparks read directly on the particle's wristwatch.

Newtonian mechanics combines (in various ways) time and mass of the particle with Euclidean 3-vector displacement of the particle to yield additional 3-vectors that describe particle motion: velocity, momentum, acceleration. Each 3-vector has magnitude and direction. The values of the three components of each 3-vector depend on the orientation of the chosen coordinate system. But for each 3-vector quantity, the 3-vector itself is the same, both in magnitude and direction in space, no matter what Euclidean coordinate system we choose. Every 3-vector exists even in the absence of any coordinate system at all! That is why the analysis of Newtonian mechanics can proceed in all its everyday applications independent of choice of coordinate system.

Relativistic mechanics combines (in various ways) proper time and mass of the particle with Lorentz 4-vector displacement of the particle to yield additional 4-vectors that describe particle motion. Central among these is the particle's momentum-energy 4-vector, or momenergy. Values of the four components of the momenergy 4-vector differ as measured in different freefloat frames in relative motion. But the momenergy 4-vector itself is the same, both in magnitude (mass!) and direction in spacetime, no matter what the frame. The momenergy 4-vector of a particle exists even in the absence of any reference frame at all! That is why the analysis of relativistic mechanics can proceed in all its power independent of choice of free-float frame of reference.

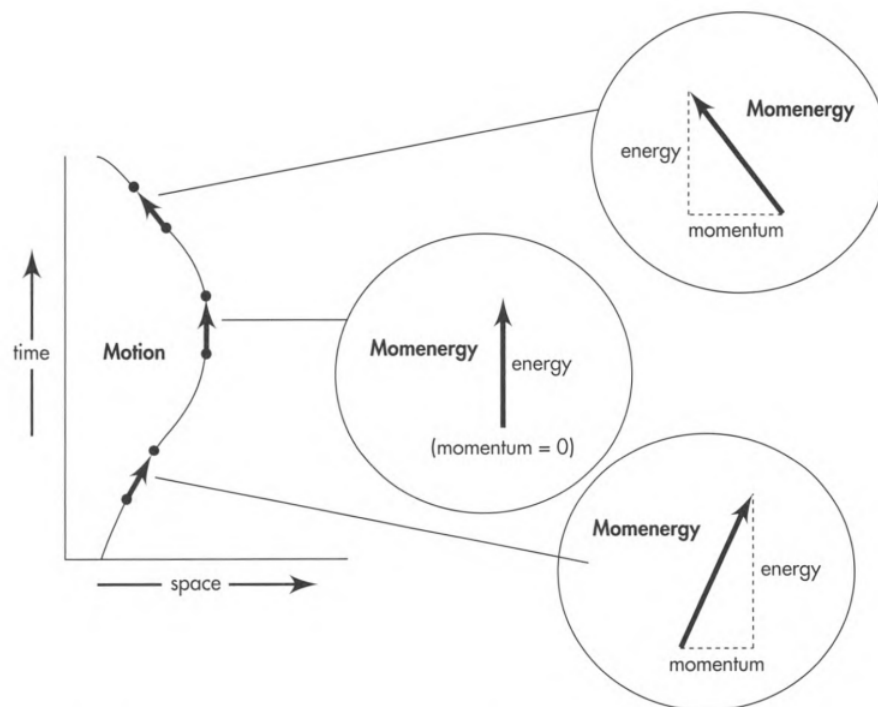


Figure 7.2.1: Motion and momenergy seen as identically directed arrows. The momenergy arrow is carried along the worldline with the particle. Under action of a force, the particle traces out a curved worldline. The momenergy arrow-its constant magnitude equal to the mass of the particle - continually alters its tilt to point in the same direction in spacetime as the worldline. (For the special case shown here, the particle moves in x and t , but not in y or z .)

Particle momenergy Magnitude: Mass Direction: Along spacetime displacement

The result expresses the momenergy 4-vector in terms of the spacetime displacement 4-vector:

$$(\text{momenergy}) = (\text{mass}) \times \frac{(\text{spacetime displacement})}{(\text{proper time for that displacement})} \quad (7.2.1)$$

In any given free-float frame, the momentum of the particle is the three "space parts" of the momenergy and the particle's energy is the "time part." This expression for momenergy is simple, and it works - works as employed in the law of conservation of momenergy: Total momenergy before reaction equals total momenergy after reaction. Investigators have observed and analyzed more than a million collisions, creations, transformations, decays, and annihilations of particles and radiation. They have failed to discover a single violation of the relativistic law of conservation of momenergy.

To arrive at a formula as important as (7-1) so painlessly may at first sight create doubts. These doubts have to be dismissed. Fact is, there is no room for any alternative - as we see by going step by step through the factors in this equation.

- **Statement 1 : m units of mass pursuing a given motion carry m times the momenergy of one unit of mass.** Reasoning: m identical objects racing along side by side carry m times the momentum and m times the energy - and therefore m times the momenergy - of an object of unit mass.
- **Statement 2: Momenergy points in the same direction in spacetime as worldline.** Reasoning: Where else can it point? Even the slightest difference in direction between momenergy and direction of motion along the worldline would bear witness to some crazy asymmetry in spacetime, for which no experiment in field-free space has ever given the slightest evidence.

Momenergy formula justified

- **Statement 3: The spacetime displacement between one event on the worldline and a nearby event on it specifies the direction of that worldline.** Reasoning: The very concept of direction implies that there exists a segment, AB , of the worldline short enough to be considered straight. And to fix the direction of this spacetime displacement AB , it suffices to know the location of any two events, A and B , on this short segment.
- **Statement 4: Worldline direction - and therefore momenergy - is independent of the magnitude of the spacetime displacement.** Reasoning: To pick an event B' on the worldline half as far from A as B along the short straight segment - thus

to cut in half the spacetime displacement - makes no change in the direction of the worldline, therefore no change in the direction of the momenergy, therefore no change in the momenergy itself.

- **Statement 5: The unit 4-vector (spacetime displacement)/(proper time for that displacement) defines and measures the direction of the worldline displacement and therefore the direction of the momenergy 4-vector.** Reasoning: What matters is not spacetime displacement individually, not proper time individually, but only their ratio. This ratio is the only directed quantity available to us to describe the rate of motion of the particle through spacetime.

Note

The spacetime displacement, AB , has a magnitude equal to the interval (or proper time or wristwatch time) the particle requires to pass from A to B . That is why the ratio in question is a unit 4-vector.

Proper time provides the only natural way, the only frame-independent means, to clock the particle. If instead we should incorrectly put frame time into the denominator - frame time measured by the array of clocks in a particular free-float frame - the value of this time would differ from one frame to another. Divided into the spacetime displacement, it would typically not yield a unit vector. The vector's magnitude would differ from one frame to another. Therefore we must use in the denominator the proper time to go from A to B , a proper time identical to the magnitude of the spacetime displacement AB in the numerator.

Statement 6: The momenergy 4-vector of the particle is

$$(\text{momenergy}) = (\text{mass}) \times \frac{(\text{spacetime displacement})}{(\text{proper time for that displacement})} \quad (7.2.2)$$

Reasoning: There is no other frame-independent way to construct a 4-vector that lies along the worldline and has magnitude equal to the mass.

Unit of momenergy: mass

Units: In this book, as in more and more present-day writing, space and time appear in the same unit: meter. Numerator and denominator on the right side of equation (7-1) have the unit of meter. Therefore their quotient is unit-free. As a result, the right side of the equation has the same unit as the first factor: mass. So the left side, the momenergy arrow, must also have the unit of mass. As the oneness of spacetime is emphasized by measuring space and time in the same unit, so the oneness of momenergy is clarified by measuring momentum and energy in the same unit: mass. Table 7-1 at the end of the chapter compares expressions for momentum and energy in units of mass with expressions in conventional units.

✓ Question and Answer

You say that the equation for momenergy is

$$(\text{momenergy}) = (\text{mass}) \times \frac{(\text{spacetime displacement})}{(\text{proper time for that displacement})} \quad (7.2.3)$$

I thought that "spacetime displacement" was the interval, which is the proper time. I know, however, that I am wrong, because if spacetime displacement and proper time were the same, then the numerator and denominator of the fraction would cancel, and momenergy would simply equal mass. Surely you would have told us of such simplicity. What have I missed?

Answer

It is easy to confuse a vector - or a 4-vector - with its magnitude.

In the expression for momenergy, the spacetime displacement is a 4-vector (Box 7-1). In the laboratory frame this displacement 4-vector has four components, $\{dt, dx, dy, dz\}$. In a free-float rocket moving in an arbitrary direction, the displacement 4-vector has four components, $\{dt', dx', dy', dz'\}$, typically different, respectively, from those in the laboratory frame.

A vector in space (a 3-vector) has not only a magnitude but also a direction independent of any coordinate system. ("Which way did they go?" "That-a-way!" - pointing.) Similarly, the spacetime displacement has a magnitude and direction in spacetime independent of any reference frame. This **spacetime direction** distinguishes the 4-vector displacement (the numerator above) from its magnitude, which is the proper time for that displacement (the denominator). This proper time (interval) can be

observed directly: it is the time lapse read off the wristwatch carried by the particle while it undergoes the spacetime displacement.

Unit 4-vector along worldline

In summary the fraction

$$\frac{(\text{spacetime displacement})}{(\text{proper time for that displacement})} \quad (7.2.4)$$

has a numerator that is a 4-vector. This 4-vector numerator has the same magnitude as the denominator. The resulting fraction is therefore a unit 4-vector pointing along the worldline of the particle. This unit 4-vector determines the direction of the particle's momenergy in spacetime. And the magnitude of the momenergy? It is the mass of the particle, the first term on the right of the expression at the top of this page. In brief, the momenergy of a particle is 4-vector of magnitude m pointing along its worldline in spacetime. This description is independent of reference frame.

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