8.E: Collide. Create. Annihilate. (Exercises)

You now have at your disposal the power of special relativity to provide physical insight and accurate predictions about an immense range of phenomena, from nucleus to galaxy. The following exercises give only a hint of this range. Even so, there are too many to carry out as a single assignment or even several assignments. For this reason — and to anchor your understanding of relativity— we recommend that you continue to enjoy these exercises as your study moves on to other subjects. The following table of contents is intended to help organize this ongoing attention.

Reminder: In these exercises the symbol v (in other texts sometimes called β) stands for speed as a fraction of the speed of light c. Let v_{conv} be the speed in conventional units; then $v \equiv v_{\text{conv}} / c$.

mass and energy

photons

creations, transformations, annihilation

Doppler shift

Complon scattering

Tests of relativity

free-for-all!

8-1 examples of conversion

a How much mass does a 100 -watt bulb dissipate (in heat and light) in one year?

b The total electrical energy generated on Earth during the year 1990 was probably between 1 and 2×10^{13} kilowatt-hours. To how much mass is this energy equivalent? In the actual production of this electrical energy is this much mass converted to energy? Less mass? More mass? Explain your answer.

c Eric Berman, pedaling a bicycle at full throttle, produces one-half horsepower of useful power (1 horsepower $= 746$ watts). The human body is about 25 percent efficient; that is, 75 percent of the food burned is converted to heat and only 25 percent is converted to useful work. How long a time will Eric have to ride to lose one kilogram by the conversion of mass to energy? How can reducing gymnasiums stay in business?

8-2 relativistic chemistry

One kilogram of hydrogen combines chemically with 8 kilograms of oxygen to form water; about 10^8 joules of energy is released.

a Ten metric tons (10^4 kilograms) of hydrogen combines with oxygen to produce water. Does the resulting water have a greater or less mass than the original hydrogen and oxygen? What is the magnitude of this difference in mass?

b A smaller amount of hydrogen and oxygen is weighed, then combined to form water, which is weighed again. A very good chemical balance is able to detect a fractional change in mass of 1 part in 10^8. By what factor is this sensitivity more than enough - or insufficient - to detect the fractional change in mass in this reaction?

PHOTONS

8-3 pressure of light

a Shine a one-watt flashlight beam on the palm of your hand. Can you feel it? Calculate the total force this beam exerts on your palm. Should you be able to feel it? A particle of what mass exerts the same force when you hold it at Earth's surface?

b From the solar constant (1.372 kilowatts/ square meter, Sample Problem 8-5) calculate the pressure of sunlight on an Earth satellite. Consider both reflecting and absorbing surfaces, and also "real" surfaces (partially absorbing). Why does the color of the light make no difference? c A spherical Earth satellite has radius $r=1$ meter and mass $m=1000$ kilograms. Assume that the satellite absorbs all the sunlight that falls on it. What is the acceleration of the satellite due to the force of sunlight, in units of g, the gravitational acceleration at Earth's surface? For a way to reduce this "disturbing" acceleration, see Figure 9-2.

d It may be that particles smaller than a certain size are swept out of the solar system by the pressure of sunlight. This certain size is determined by the equality of the outward force of sunlight and the inward gravitational attraction of Sun. Estimate this critical particle size, making any assumptions necessary for your estimate. List the assumptions with your answer. Does your estimated size depend on the particle's distance from Sun?

Reference: For pressure of light measurement in an elementary laboratory, see Robert Pollock, American Journal of Pbysics, Volume 31, pages $901-904$ (1963). Pollock's method of determining the pressure of light makes use of resonance to amplify a small effect to an easily measured magnitude. Dr. Pollock developed this experiment in collaboration with the same group of first-year students at Princeton University with whom the authors had the privilege to work out the presentation of relativity in the first edition of this book.

8-4 measurement of photon energy

A given radioactive source emits energetic photons (X-rays) or very energetic photons (gamma rays) with energies characteristic of the particular radioactive nucleus in question. Thus a precise energy measurement can often be used to determine the composition of even a tiny specimen. In the apparatus diagrammed in the figure on page 255 , only those events are detected in which a count on detector A (knocked-on electron) is accompanied by a count on detector B (scattered photon). What is the energy of the incoming photons that are detected in this way, in units of the rest energy of the electron?

8-5 Einstein's derivation: equivalence of energy and mass - a worked example

From the fact that light exerts pressure and carries energy, show that this energy is equivalent to mass and hence - by extension - show the equivalence of all energy to mass.

Commentary: The equivalence of energy and mass is such an important consequence that Einstein very early, after his relativistic derivation of this result, sought and found an alternative elementary physical line of reasoning that leads to the same conclusion. He envisaged a closed box of mass M initially at rest, as shown in the first figure. A directed burst of electro-

EXERCISE 8-4. Measurement of photon energy.

magnetic energy is emitted from the left wall. It travels down the length L of the box and is absorbed at the other end. The radiation carries an energy E. But it also carries momentum. This one sees from the following reasoning. The radiation exerts a pressure on the left wall during the emission. In consequence of this pressure the box receives a push to the left, and a momentum, p. But the momentum of the system as a whole was zero initially. Therefore the radiation carries a momentum p opposite to the momentum of the box. How can one use knowledge of the transport of energy and momentum by the radiation to deduce the mass equivalent of the radiation? Einstein got his answer from the argument that the center of mass of the system was not moving before the transport process and therefore cannot be in motion during the transport process. But the box obviously carries mass to the left. Therefore the radiation must carry mass to the right. So much for Einstein's reasoning in broad outline. Now for the details.

From relativity Einstein knew that the momentum p of a directed beam of radiation is equal to the energy E of that beam (Section 8.4; both p and E measured in units of mass). However, this was known before Einstein's relativity theory, both from Maxwell's theory of electromagnetic radiation and from direct observa-

EXERCISE 8-5,

first figure.

Transfer of mass by radiation. tion of the pressure exerted by light on a mirror suspended in a vacuum. This measurement had first successfully been carried out by E. F. Nichols and G. F. Hull between 1901 and 1903 . (By now the experiment has been so simplified and increased in sensitivity that it can be carried out in an elementary laboratory. See the reference for Exercise 8-3.)

Thus the radiation carries momentum and energy to the right while the box carries momentum and mass to the left. But the center of mass of the system, box plus radiation, cannot move. So the radiation must carry to the right not merely energy but mass. How much mass? To discover the answer is the object of these questions.

a What is the velocity of the box during the time of transit of the radiation?

b After the radiation is absorbed in the other end of the box, the system is once again at rest. How far has the box moved during the transit of the radiation?

c Now demand that the center of mass of the system be at the same location both before and after the flight of the radiation. From this argument, what is the mass equivalent of the energy that has been transported from one end of the box to the other?

Solution

a During the transit of the radiation the momentum of the box must be equal in magnitude and opposite in direction to the momentum p of the radiation. The box moves with a very low velocity v. Therefore the Newtonian formula Mv suffices to calculate its momentum:

$$Mv=-p=-E \tag{8.E.1}$$

From this relation we deduce the velocity of the box,

$$v=-E/M \tag{8.E.2}$$

b The transit time of the photon is very nearly $t=L$ meters of light-travel time. In this time the box moves a distance

$$\Delta x=vt=-EL/M \tag{8.E.3}$$

c If the radiation transported no mass from one end of the box to the other, and if the box were the sole object endowed with mass, then this displacement Δx would result in a net motion of the center of mass of the system to the left. But, Einstein reasoned, an isolated system with its center of mass originally at rest can never set itself into motion nor experience any shift in its center of mass. Therefore, he argued, there must be some countervailing displacement of a part of the mass of the system. This transport of mass to the right can be understood only as a new feature of the radiation itself. Consequently, during the time the box is moving to the left, the radiation must transport to the right some mass m, as yet of unknown magnitude, but such as to ensure that the center of mass of the system has not moved. The distance of transport is the full length L of the box diminished by the distance Δx through which the box has moved to the left in the meantime. But Δx is smaller than L in the ratio E/M. This ratio can be made as small as one pleases for any given transport of radiant energy E by making the mass M of the box sufficiently great. Therefore it is legitimate to take the distance moved by the radiation as equal to L itself. Thus, with arbitrarily high precision, the condition that the center of mass shall not move becomes

$$M\Delta x + mL = 0 \tag{8.E.4}$$

Calculate the mass m and find, using Δx from part **b**,

$$m = -\Delta x M/L = -(-EL/M)(M/L) \tag{8.E.5}$$

or, finally,

$$m = E \tag{8.E.6}$$

In conventional units, we have the famous equation

$$E_{\text{conv}} = mc^2 \tag{8.E.7}$$

We conclude that the process of emission, transport, and reabsorption of radiation of energy E is equivalent to the transport of a mass $m = E$ from one end of the box to the other end. The simplicity of this derivation and the importance of the result makes this analysis one of the most interesting in all of physics.

Discussion: The mass equivalence of radiant energy implies the mass equivalence of thermal energy and - by extension - of other forms of energy, ac- cording to the following reasoning. The energy that emerges from the left wall of the box may reside there originally as heat energy. This thermal energy excites a typical atom of the surface from its lowest energy state to a higher energy state. The atom returns from this higher state to a lower state and in the course of this change sends out the surplus energy in the form of radiation. This radiant energy traverses the box, is absorbed, and is ultimately converted back into thermal energy. Whatever the details of the mechanisms by which light is emitted and absorbed, the net effect is the transfer of heat energy from one end of the box to the other. To say that mass has to pass down the length of the box when radiation goes from one wall to the other therefore implies that mass moves when thermal energy changes location. The thermal energy in turn is derived from chemical energy or the energy of a nuclear transformation or from electrical energy. Moreover, thermal energy deposited at the far end of the tube can be converted back into one or another of these forms of energy. Therefore these forms of energy - and likewise all other forms of energy - are equivalent in their transport to the transport of mass in the amount $m = E$.

How can one possibly uphold the idea that a pulse of radiation transports mass? One already knows that a photon has zero mass, by virtue of the relation (Section 8.4)

$$(\text{ mass })^2 = (\text{ energy })^2 - (\text{ momentum })^2 = 0 \tag{8.E.8}$$

Moreover, what is true of the individual photon is true of the pulse of radiation made up of many such photons: The energy and momentum are equal in magnitude, so that the mass of the radiation necessarily vanishes. Is there not a fundamental inconsistency in saying in the same breath that the mass of the pulse is zero and that radiation of energy E transports the mass $m = E$ from one place to another?

The source of our difficulty is some confusion between two quite different concepts: (1) energy, the time component of the momentum-energy 4-vector, and (2) mass, the magnitude of this 4-vector. When the system divides itself into two parts (radiation going to the right and box recoiling to the left) the components of the 4-vectors of the radiation and of the recoiling box add up to identity with the components of the original 4-vector of the system before emission, as shown in the second figure. However, the magnitudes of the 4-vectors (magnitude = mass) are not additive. No one dealing with Euclidean geometry would expect the length of one side of a triangle to be equal to the sum of the lengths of the other two sides. Similarly in Lorentz geometry. The mass of the system (M) is not to be considered as equal to the sum of the mass of the radiation (zero) and the mass of the recoiling box (less than M). But components of 4vectors are additive; for example,

$$\left(\begin{array}{c}\text{energy of}\\ \text{system}\end{array}\right) = \left(\begin{array}{c}\text{energy of}\\ \text{radiation}\end{array}\right) + \left(\begin{array}{c}\text{energy of}\\ \text{recoiling box}\end{array}\right) \tag{8.E.9}$$

Thus we see that the energy of the recoiling box is $M - E$. Not only is the energy of the box reduced by the emission of radiation from the wall; also its mass is reduced (see shortened length of 4-vector in diagram). Thus the radiation takes away mass from the wall of the box even though this radiation has zero mass. The inequality

$$\left(\begin{array}{c}\text{mass of}\\ \text{system}\end{array}\right) \neq \left(\begin{array}{c}\text{mass of}\\ \text{radiation[zero]}\end{array}\right) + \left(\begin{array}{c}\text{mass of}\\ \text{recoiling box}\end{array}\right) \tag{8.E.10}$$

is as natural in spacetime geometry as is the inequality $5 \neq 3 + 4$ for a $3 - 4 - 5$ triangle in Euclidean geometry.

What about the gravitational attraction exerted by the system on a test object? Of course the redistribution of mass as the radiation moves from left to right makes some difference in the attraction. But let the test object be at a distance r so great that any such redistribution has a negligible effect on the attraction. In other words, all that counts for the pull on a unit test object is the total mass M as it appears in Newton's formula for gravitational force:

$$\left(\begin{array}{c}\text{force per}\\ \text{unit mass}\end{array}\right) = \frac{GM}{r^2} \tag{8.E.11}$$

experience a less-than-normal pull while the radiation is in transit down the box? Is not the mass of the radiation zero, and is not the mass of the recoiling box reduced below the original mass M of the system? So is not the total attracting mass less than normal during the process of transport? No! The mass of the system - one has to say again - is not equal to the sum of the masses of its several parts. It is instead equal to the magnitude of the total momentum-energy 4-vector of the system. And at no time does either the total momentum (in our case zero!) or the total energy of the system change-it is an isolated system. Therefore neither is there any change in the magnitude M of the total momentum-energy 4-vectors shown in the second figure. So, finally, there is never any change in the gravitational attraction.

There is one minor swindle in the way this problem has been presented: The box cannot in fact move as a rigid body. If it could, then information about the emission of the radiation from one end could be obtained from the motion of the other end before the arrival of the radiation itself-this information would be transmitted at a speed greater than that of light! Instead, the recoil from the emission of the radiation travels along the sides of the box as a vibrational wave, that is, with the speed of sound, so that this wave arrives at the other end long after the radiation does. In the meantime the absorption of the radiation at the second end causes a second vibrational wave which travels back along the sides of the box. The addition of the vibration of the box to the prob-

lem requires a more complicated analysis but does not change in any essential way the results of the exercise.

References: A. Einstein, Annalen der Pbysik, Volume 20, pages $627 - 633$ (1906). For a more careful treatment of the box, see A. P. French, Special Relativity (W. W. Norton, New York, 1968), pages $16 - 18$ and $27 - 28$.

8-6 gravitational red shift

Note: Exercises 8-6 and 8-7 assume an acquaintance with the following elementary facts of gravitation.

(1) A very small object - or a spherically symmetric object of any radius - with mass M attracts an object of mass m-also small or spherically symmetric - with a force

$$F = \frac{GMm}{r^2} \tag{8.E.12}$$

Here r is the distance between the centers of the two objects and G is the Newtonian constant of gravitation, $G = 6.67 \times 10^{-11}$ (meter) $^3/$ (kilogram-second ^{2}).

(2) The work required to move a test particle of unit mass from r to $r + dr$ against the gravitational pull of a fixed mass M is $GM\left(dr/r^2\right)$. Translated from conventional units of energy to units of mass this work is

$$dW_{\text{conv}} = \frac{GM}{c^2}\frac{dr}{r^2} = M^*\frac{dr}{r^2} \tag{8.E.13}$$

per unit of mass contained in the test particle.

(3) The symbol $M^{*}=GM/c^{2}$ in this formula has a simple meaning. It is the mass of the center of attraction translated from units of kilograms to units of meters. For example, the mass of Earth $\left(M_{\text{Earch}}=5.974\times10^{24}\right.$ kilograms) expressed in length units is $M_{\text{Earch}}^{*}=4.44\times10^{-3}$ meters, and the mass of Sun $\left(M_{\text{Sun}}=1.989\times10^{30}\ \text{kg}\right)$ is $M^{*}{}_{\text{Sun}}=1.48\times10^{3}$ meters

(4) Start the test particle at a distance r from the center of attraction of mass M and carry it to an infinite distance. The work required is $W=M^{*}/r$ in units of mass per unit of mass contained in the test particle.

So much for the minitutorial. Now to business.

a What fraction of your rest energy is converted to potential energy when you climb the Eiffel Tower (300 meters high) in Paris? Let g^{*} be the acceleration of gravity in meters / meter ^{2} at the surface of Earth:

$$g^{*}=\frac{GM_{\text{Earch}}}{c^{2}}\frac{1}{r_{\text{Earch}}^{2}}=\frac{M_{\text{Earth}}^{*}}{r_{\text{Earth}}^{2}}=\frac{g}{c^{2}} \tag{8.E.14}$$

b What fraction of one's rest energy is converted to potential energy when one climbs a very high ladder that reaches higher than the gravitational influence of Earth? Assume that Earth does not rotate and is alone in space. Does the fraction of the energy that is lost in either part a or part b depend on your original mass?

c Apply the result of part a to deduce the fractional energy change of a photon that rises vertically to a height z in a uniform gravitational field g^{*}. Photons have zero mass; one can say formally that they have only kinetic energy $E=K$. Thus photons have only one purse - the kinetic energy purse - from which to pay the potential energy tax as they rise in the gravitational field. Light of frequency f is composed of photons of energy $E=hf/c^{2}$ (see Exercise 8-31). Show that the fractional energy loss for photons rising in a gravitational field corresponds to the following fractional change in frequency:

$$\frac{\Delta f}{f}=-g^{*}z \quad [\text{uniform gravitational field}] \tag{8.E.15}$$

Note: We use f for frequency instead of the usual Greek nu, v, to avoid confusion with v for speed.

d Apply the result of part b to deduce the fractional energy loss of a photon escaping to infinity. (To apply b for this purpose is an approximation good to one percent when this fractional energy loss itself is less than two percent.) Specifically, let the photon start from a point on the surface of an astronomical object of mass M (kilograms) or M^{*} (meters) $=GM/c^{2}$ and radius r. From the fractional energy loss, show that the fractional change of frequency is given by the expression

$$\frac{\Delta f}{f}=-\frac{M^{*}}{r} \quad [\text{escape field of spherical object}] \tag{8.E.16}$$

This decrease in frequency is called the gravitational red shift because, for visible light, the shift is toward the lower-frequency (red) end of the visible spectrum.

e Calculate the fractional gravitational red shifts for light escaping from the surface of Earth and for light escaping from the surface of Sun.

Discussion: The results obtained in this exercise are approximately correct for light moving near Earth, Sun, and white dwarf (Exercise 8-7). Only general relativity correctly describes the motion of light very close to neutron star or black hole (Box 9-2).

8-7 densify of the companion of Sirius

Note: This exercise uses a result of Exercise 8-6.

Sirius (the Dog Star) is the brightest star in the heavens. Sirius and a small companion revolve about

EXERCISE 8-12 PHOTOPRODUCTION OF A PAIR BY TWO PHOTONS 259

one another. By analyzing this revolution using Newtonian mechanics, astronomers have determined that the mass of the companion of Sirius is roughly equal to the mass of our Sun (M is about 2×10^{30} kilograms; M^{*} is about 1.5×10^{3} meters). Light from the companion of Sirius is analyzed in a spectrometer. A spectral line from a certain element, identified from the pattern of lines, is shifted in frequency by a fraction 7×10^{-4} compared to the frequency of the same spectral line from the same element in the laboratory. (These figures are experimentally accurate to only one significant figure.) Assuming that this is a gravitational red shift (Exercise 8-6), estimate the average density of the companion of Sirius in grams/centimeter ^{3}. This type of star is called a white dwarf (Box 9-2).

CREATIONS,

TRANSFORMATIONS, ANNIHLATIONS

8-8 nuclear excitation

A nucleus of mass m initially at rest absorbs a gamma ray (photon) and is excited to a higher energy state such that its mass is now $1.01\ \mathrm{m}$.

a Find the energy of the incoming photon needed to carry out this excitation.

b Explain why the required energy of the incoming photon is greater than the change of mass of the nucleus.

AFTER

EXERCISE 8-8. Excitation of a nucleus by a gamma ray.

8-9 photon braking

A moving radioactive nucleus of known mass M emits a gamma ray (photon) in the forward direction and drops to its stable nonradioactive state of known mass m. Find the energy E_{A} of the incoming nucleus (BEFORE diagram in the figure) such that the resulting mass m nucleus is at rest (AFTER diagram). The unknown energy E_{C} of the outgoing gamma ray should not appear in your answer.

EXERCISE 8-9. Stopping a nucleus by emission of a gamma ray.

8-10 photon infegrify

Show that an isolated photon cannot split into two photons going in directions other than the original direction. (Hint: Apply the laws of conservation of momentum and energy and the fact that the third side of a triangle is shorter than the sum of the other two sides. What triangle?)

8-11 pair production by a lonely photon?

A gamma ray (high-energy photon, zero mass) can carry an energy greater than the rest energy of an electron-positron pair. (Remember that a positron has the same mass as the electron but opposite charge.) Nevertheless the process

(energetic gamma ray) $\longrightarrow$ (electron) $+$ (positron)

cannot occur in the absence of other matter or radiation.

a Prove that this process is incompatible with the laws of conservation of momentum and energy as employed in the laboratory frame of reference. Analyze the alleged creation in the frame in which electron and positron go off at equal but opposite angles $\pm\phi$ with the extended path of the incoming gamma ray.

b Repeat the demonstration - which then becomes much more impressive - in the center-of-momentum frame of the alleged pair, the frame of reference in which the total momentum of the two resulting particles is zero.

8-12 pholoproduction of a pair by two photons

Two gamma rays of different energies collide in a vacuum and disappear, bringing into being an electron - positron pair. For what ranges of energies of the two gamma rays, and for what range of angles between their initial directions of propagation, can this reaction occur? (Hint: Start with an analysis of the reaction at threshold; at threshold the electron and positron are relatively at rest.)

A positron $e^{+}$of mass m and kinetic energy K is annihilated on a target containing electrons e^{-}(same mass m) practically at rest in the laboratory frame:

$$e^{+}(\text{fast })+e^{-}(\text{at rest}) \longrightarrow \text{ radiation} \tag{8.E.17}$$

a By considering the collision in the center-of- momentum frame (the frame of reference in which with respect to the direction of the The

A positron $e^{+}$of mass m and kinetic energy K is annihilated on a target containing electrons e^{-}(same mass m) practically at rest in the laboratory frame:

$$e^{+}(\text{fast })+e^{-}(\text{at rest}) \longrightarrow \text{ radiation} \tag{8.E.18}$$

a By considering the collision in the center-of- momentum frame (the frame of reference in which with respect to the direction of the The

positron $e^{+}$of mass m and kinetic energy K is nnihilated on a target containing electrons e^{-}(same

$$\begin{array}{ll} \text{a matically at rest in the laboratory frame:} & \text{mated on a target containing electrons } e^{-}\text{(same} \\ \text{mass } m \text{) practictically at rest in the laboratory frame:} & e^{+}(\text{fast })+e^{-}(\text{at rest }) \longrightarrow \text{ radiation} \\ \text{By considering the collision in the center-of-} & \text{with respect to the direction of the incoming positron,} \\ \text{mentum frame (the frame of reference in which} & \text{as shown in the first figure.} \\ \text{me} & \end{array} \tag{8.E.19}$$

A positron $e^{+}$of mass m and kinetic energy K is annihilated on a target containing electrons e^{-}(same mass m) practically at rest in the laboratory frame:

$$e^{+}(\text{fast })+e^{-}(\text{at rest}) \longrightarrow \text{ radiation} \tag{8.E.20}$$

a By considering the collision in the center-of- momentum frame (the frame of reference in which the total momentum of the initial particles is equal to zero), show that it is necessary for at least two gamma rays (rather than one) to result from the annihilation. **b** Return to the laboratory frame, shown in the figure. The outgoing photons move on the line along which the positron approaches. Find an expression for the

A positron $e^{+}$of mass m and kinetic energy K is annihilated on a target containing electrons e^{-}(same mass m) practically at rest in the laboratory frame:

$$e^{+}(\text{fast })+e^{-}(\text{at rest}) \longrightarrow \text{ radiation} \tag{8.E.21}$$

a By considering the collision in the center-of- momentum frame (the frame of reference in which with respect to the direction of the The

A positron $e^{+}$of mass m and kinetic energy K is annihilated on a target containing electrons e^{-}(same mass m) practically at rest in the laboratory frame:

$$e^{+}(\text{fast })+e^{-}(\text{at rest}) \longrightarrow \text{ radiation} \tag{8.E.22}$$

a By considering the collision in the center-of- momentum frame (the frame of reference in which with respect to the direction of the The

A positron $e^{+}$of mass m and kinetic energy K is annihilated on a target containing electrons e^{-}(same mass m) practically at rest in the laboratory frame:

$$e^{+}(\text{fast })+e^{-}(\text{at rest}) \longrightarrow \text{ radiation} \tag{8.E.23}$$

a By considering the collision in the center-of- momentum frame (the frame of reference in which with respect to the direction of the The

positron $e^{+}$of mass m and kinetic energy K is nnihilated on a target containing electrons e^{-}(same

$$\begin{array}{ll} \text{a matically at rest in the laboratory frame:} & \text{mated on a target containing electrons } e^{-}\text{(same} \\ \text{mass } m \text{) practictically at rest in the laboratory frame:} & e^{+}(\text{fast })+e^{-}(\text{at rest }) \longrightarrow \text{ radiation} \\ \text{By considering the collision in the center-of-} & \text{with respect to the direction of the incoming positron,} \\ \text{mentum frame (the frame of reference in which} & \text{as shown in the first figure.} \\ \text{me} & \end{array} \tag{8.E.24}$$

EXERCISE 8-17 COLLIDERS $\mathbf{261}$

8-16 creation of proton- anfiprolon pair by an electron

What is the threshold kinetic energy K_{tb} of the incident electron for the following process?

$$\begin{array}{c} \text{electron (fast) } + \text{ proton (at rest) } \longrightarrow \\ \text{electron } + \text{ antiproton } + \text{ two protons} \end{array}$$

8-17 colliders

How much more violent is a collision of two protons that are moving toward one another from opposite directions than a collision of a moving proton with one at rest?

Discussion: When a moving particle strikes a stationary one, the energy available for the creation of new particles, for heating, and for other interactions - or, in brief, the available interaction energy- is less than the initial energy (the sum of the rest and kinetic energies of the initial two particles). Reason: The particles that are left over after the reaction have a net forward motion (law of conservation of momentum), the kinetic energy of which is available neither for giving these particles velocity relative to each other nor for producing more particles. For this reason much of the particle energy produced in accelerators is not available for studying interactions because it is carried away in the kinetic energy of the products of the collision.

However, in the center-of-momentum frame, the frame in which the total momentum of the system is equal to zero, no momentum need be carried away from the interaction. Therefore the energy available for interaction is equal to the total energy of the incoming particles.

Is there some way that the laboratory frame can be made also the center-of-momentum frame? One way is to build two particle accelerators and have the two beams collide head on. If the energy and masses of the particles in each beam are respectively the same, then the laboratory frame is the center-of-momentum frame and all the energy in each collision is available interaction energy. It is easier and cheaper to achieve the same efficiency by arranging to have particles moving in opposite directions in the same accelerator. A magnetic field keeps the particles in a circular path, "storing" them at their maximum energy for repeated tries at interaction. Such a facility is called a collider. The figure on page 262 gives some details of a particular collider.

a What is the total available interaction energy for each encounter in the laboratory frame of the Tevatron shown on page 262 ? b Now transform to a frame in which one of the incoming particles is at rest (transformation given in Exercise 7-5). This would be the situation if we tried to build an accelerator in which moving antiprotons hit a stationary target of, say, liquid hydrogen (made of protons and electrons). [Simplify: At 0.9 $\mathrm{TeV}=9 \times 10^{11} \mathrm{eV}$ what is the effective speed v of the proton? What is its momentum compared with its energy? What is the value of the time stretch factor $\gamma=E/m$?] If the target protons were at rest, what energy, in TeV, would the incoming antiproton need to have in order to yield the same interaction energy as that achieved in the Tevatron?

Wait a minute! You keep telling us that energy and momentum bave different values when measured with respect to different reference frames. Yet bere you assume the "interaction energy" is the same in the Tevatron laboratory frame as it is in the rest frame of a proton that moves with nearly the speed of light in the Tevatron frame. Is the energy of a system different in different frames, or is it the same?

There is an important distinction between the total energy of a system and the "available interaction energy," just as there is an important distinction between your money in the bank and "ready cash" in the bank that you can spend. If some of your money in the bank has been put in escrow for payment on a house you are buying, then you cannot spend that part of your bank money to buy a new car. Similarly, the total energy of the protonantiproton system is much smaller in the Tevatron laboratory frame than in the frame in which the proton is initially at rest, but all of the Tevatron laboratoryframe energy can be spent-used to create new particles, for example. In contrast, only a minute fraction of the energy in the frame in which the proton is initially at rest can be spent to create new particles, since total momentum must be conserved; most of the total energy is kept "in escrow" for this purpose. The number and kinds of new particles created must be the same for all observers! Therefore the "available interaction energy" must be the same for all observers. The central point here is that the Tevatron collider design makes all of the energy in the proton-antiproton system "available" for use in the laboratory.

EXERCISE 8-17 COLLIDERS

EXERCISE 8-17. Top: Aerial view of the Tevatron ring at Fermi National Accelerator Laboratory in Batavia, Illinois. The ring is 6.3 kilometers in circumference. Bottom: View along the tunnel of the Tevatron. Protons (positive charge) and antiprotons (antiparticle of the proton: same mass, negative charge) circulate in separate beams in opposite directions in the same vacuum cbamber in the lower ring of superconducting magnets sbown in the photo. The upper ring of regular magnets accelerates protons from $8 \mathrm{GeV}$ to $150 \mathrm{GeV}$. Some of these protons are injected into the lower set of magnets directly, rotating clockwise. Other protons strike a copper target and create antiprotons at a lower energy that are accumulated over approximately 15 bours in a separate ring (not shown) and then reaccelerated to $150 \mathrm{GeV}$ and inserted into the lower ring, circulating counterclockwise. (Opposite charge, opposite motion yields same magnetic force toward the center, bence counterrotation around the same circle.) Then particles in both beams in the lower ring of magnets are accelerated at the same time from $150 \mathrm{GeV}$ to a final energy of $0.9\, \mathrm{TeV}$ per particle. (1 teraelectron-volt $=10^{12}$ electronvolts, or approximately 1000 times the rest energy of the proton or antiproton.) After acceleration, the beams are switched magnetically so that they cross each other at multiple intersection points around the ring, allowing protons and antiprotons to collide in the laboratory center-of-momentum frame. Detectors at the points of intersection monitor products of the collithe points of intersection monitor products of the collisions. Protons and antiprotons that do not interact at one intersection are not wasted; they may interact at one intersection are not wasted; they may interact at another intersection point or on subsequent trips around the ring. The particles are allowed to coast around and around at full energy for as long as 24 bours as they interact. Question: Approximately bow many revolutions around the ring does a given proton many revolutions around the ring does a given proton or antiproton make in 24 bours? Photographs couror antiproton make in 24 bours? Photographs cour tesy of Fermi Laboratory.

DOPPLER SHIFT

8-18 Doppler shift along the x-direction

Note: Recall Exercise L-5 in the Special Topic on Lorentz Transformation, following Chapter 3 .

Apply the momenergy transformation equations (Exercise 7-5) to light moving in the positive x-direction for which $p_x=p=E$.

a Show that the relation between photon energy E' in the rocket frame and photon energy E in the laboratory frame is given by the equation

\begin{aligned} E &=\gamma(1+v) E^{\prime}=\frac{(1+v) E^{\prime}}{\left(1-v^{2}\right)^{1 / 2}} \\ &=\frac{(1+v) E^{\prime}}{[(1-v)(1+v)]^{1 / 2}}=\left[\frac{1+v}{1-v}\right]_{\begin{array}{c} 1 / 2 \end{array}

positive x-direction]

b Use the Einstein relation between photon en- ergy E and classical wave frequency f, namely $E_{conv}=$ ergy E and classical wave frequency f, namely $E_{\text{conv}}\ =bf$ or $E=bf/c^2$ and $E'=bf'/c^2$, to derive the transformation for frequency

$$f=\left[\frac{1+v}{1-v}\right]^{1/2} f' \tag{8.E.25}$$

[wave motion along positive x-direction]

This is the Doppler shift equation for light waves moving along the positive x-direction.

Note: We use f for frequency instead of the usual Greek nu, v, to avoid confusion with v for speed.

c Show that for a wave moving along the negative x-direction, the equation becomes

$$f=\left[\frac{1-v}{1+v}\right]^{1/2} f' \qquad \begin{array}{c}[\text{wave motion along} \\ \text{negative } x\text{-direction}\,]\end{array} \tag{8.E.26}$$

d Derive the corresponding equations that convert laboratory-measured frequency f to rocket-measured frequency f' for waves moving along both positive and negative x-directions.

8-19 Doppler equations

A photon moves in the xy laboratory plane in a direction that makes an angle ϕ with the x-axis, so that its components of momentum are $p_x=p\cos\phi$ and $p_y=p\sin\phi$ and $p_z=0$.

a Use the Lorentz transformation equations for the momentum-energy 4-vector (Exercise 7-5) and the relation $E^2-p^2=0$ for a photon to show that in the rocket frame, moving with speed v_{rel} along the laboratory x-direction, the photon has an energy E' given by the equation

$$E'=E\gamma\left(1-v_{\text{rel}}\cos\phi\right) \tag{8.E.27}$$

and moves in a direction that makes an angle ϕ' with the x'-axis given by the equation

$$\cos\phi'=\frac{\cos\phi-v_{\text{rel}}}{1-v_{\text{re}}\cos\phi} \tag{8.E.28}$$

b Derive the inverse equations for E and $\cos\phi$ as functions of E', $\cos\phi'$, and v_{rel} . Show that the results are

$$\begin{aligned} E &=E'\gamma\left(1+v_{\text{rel}}\cos\phi'\right) \\ \cos\phi &=\frac{\cos\phi'+v_{\text{rel}}}{1+v_{\text{rel}}\cos\phi'} \end{aligned}$$

c If the frequency of the light in the laboratory frame is f, what is the frequency f' of the light in the rocket frame? Use the Einstein relation between photon energy E and classical wave frequency f, namely $E_{\text{conv}}\ =hf$ or $E=bf/c^2$ and $E'=bf'/c^2$, to derive the transformations for frequency

$$\begin{aligned} f' &=f\gamma\left(1-v_{\text{rel}}\cos\phi\right) \\ f &=f'\gamma\left(1+v_{\text{rel}}\cos\phi'\right) \end{aligned}$$

This difference in frequency due to relative motion is called the Doppler shift.

Note: We use f for frequency instead of the usual Greek nu, v, to avoid confusion with v for speed

d For wave motion along the positive and negative x-direction, show that the results of this exercise reduce to the results of Exercise $8-18$.

e Discussion question: Do the Doppler equations enable one to determine the rest frame of the source that emits the photons?

8-20 the physicist and the - Prafiic light

A physicist is arrested for going through a red light. In court he pleads that he approached the intersection at such a speed that the red light looked green to him. The judge, a graduate of a physics class, changes the charge to speeding and fines the defendant one dollar for every kilometer/hour he exceeded the local speed limit of 30 kilometers/hour. What is the fine? Take the wavelength of green light to be 530 nanometers $= 530 \times 10^{-9}$ meter) and the wavelength of red light to be 650 nanometers. The relation between wavelength λ and frequency f for light is $f\lambda = c$. Notice that the light propagates in the negative x-direction $(\phi = \phi' = \pi)$.

8-21 speeding light bulb

A bulb that emits spectrally pure red light uniformly in all directions in its rest frame approaches the observer from a very great distance moving with nearly the speed of light along a straight-line path whose perpendicular distance from the observer is b. Both the color and the number of photons that reach the observer per second from the light bulb vary with time. Describe these changes qualitatively at several stages as the light bulb passes the observer. Consider both the Doppler shift and the headlight effect (Exercises 8-19 and L-9).

8-22 Doppler shift a the limb of Sun

Sun rotates once in about 25.4 days. The radius of Sun is about 7.0×10^{8} meters. Calculate the Doppler shift that we should observe for light of wavelength 500 nanometers $= 500 \times 10^{-9}$ meter) from the edge of Sun's disk (the limb) near the equator. Is this shift toward the red end or toward the blue end of the visible spectrum? Compare the magnitude of this Doppler shift with that of the gravitational red shift of light from Sun (Exercise 8-6).

8-23 the expanding universe

Note: Recall Exercise 3-10.

a Light from a distant galaxy is analyzed by a spectrometer. A spectral line of wavelength 730 nanometers $= 730 \times 10^{-9}$ meters is identified (from the pattern of other lines) to be one of the lines of hydrogen that, for hydrogen in the laboratory, has the wavelength 487 nanometers. If the shift in wavelength is a Doppler shift, how fast is the observed galaxy moving relative to Earth? Notice that the light propagates in a direction opposite to the direction of motion of the galaxy $(\phi = \phi' = \pi)$.

b There is independent evidence that the observed galaxy is 5×10^{9} light years away. Estimate the time when that galaxy parted company from our own galaxy - the Milky Way - using the simplifying assumption that the speed of recession was the same throughout the past (that is, not slowed down by the gravitational attractions between one galaxy and another). The astronomer Edwin Hubble discovered in 1929 that this time-whose reciprocal is called the Hubble constant, and which may itself therefore appropriately be called the Hubble timehas about the same value for all galaxies whose distances and speeds can be measured. Hence the concept of the expanding universe. c Will allowance for the past effect of gravitation in slowing the expansion increase or decrease the estimated time back to the start of this expansion?

Reference: E. Hubble, Proceedings of the U. S. National Academy of Sciences, Volume 15, pages 168-173 (1929).

8-24 fwin paradox using the Doppler shift

The Twin Paradox (Chapter 4 and Exercises 4-1 and 5-8) can be resolved elegantly using the Doppler shift as follows. Paul remains on Earth. His twin sister Penny travels at a high speed, v, to a distant star and returns to Earth at the same speed. Both Penny and Paul observe a distant variable star whose light gets alternately dimmer and then brighter with a frequency f in the Earth frame (f' in the rocket frame). This variable star is very much farther away than the length of Penny's path and is in a direction perpendicular to this path in the Earth frame. Both observers will count the same total number of pulsations of the variable star during Penny's round trip. Use this fact and the expression for the Doppler shift at the 90 -degree laboratory angle of observation (Exercise 8-19) to verify that at the end of the trip described in Chapter 4 , Penny will be only 20 years older while Paul will have aged 202 years.

Reference: E. Feenberg, American Journal of Pbysics, Volume 27 , page 190 (1959).

8-25 Doppler line broadening

The average kinetic energy of a molecule in a gas at temperature T degrees Kelvin is $(3/2)kT$. (The constant k is called the Boltzmann constant and has the value 1.38×10^{-23} joules/degree Kelvin). Molecules of gas move in random directions. Calculate the average speed from the low-velocity approximation of Newtonian mechanics. Estimate the fractional change in frequency due to the Doppler shift that will be observed in light emitted from a molecule in a gas at temperature T. Will this shift increase or decrease the observed frequency of the emitted light? This effect, called Doppler broadening of spectral lines, is one reason why a given spectral line from a gas excited in an electric discharge contains a range of frequencies around a central frequency.

8-26 $E_{\text{rost conv}} = mc^2$ from the Doppler shiff

Einstein's famous equation in conventional units, $E_{\text{rest conv}} = mc^2$, and the relativistic expression for energy can be derived from (1) the relativistic expression for momentum (derived separately, for example in Exercise 7-12), (2) the conservation laws, and (3) the Doppler shift (Exercise 8-18). In conventional units, a photon has energy $E_{\text{conv}} = bf$, where b is Planck's constant and f is the frequency of the corresponding classical wave. (We use f for frequency instead of the usual Greek nu, v, to avoid confusion with v for speed.) Divide by c^2 to convert to units of mass: $E = bf/c^2$. Expressed in units of mass, a photon has equal energy and momentum. Therefore the momentum of a photon is also given by the equation $p = hf/c^2$. Momentum does differ from energy, however, in that it is a 3 -vector. In one dimensional motion, the sign of the momentum (positive for motion to the right, negative for motion to the left) is important, as in the analysis below.

A - ricle of mass m_{before} emits two photons in opposite directions while remaining at rest in the laboratory frame. Conservation of momentum requires these two photons to have equal and opposite momenta and therefore to correspond to the same classical frequency f. In consequence, they also have the same energy.

a First result: Energy released $= \Delta m$. Now view this process from a rocket frame moving at speed $v = v_{\text{conv}}/c$ along the direction of flight of the two photons. The particle moves in this frame, but does not change velocity on emitting the photons. The photon emitted in the same direction as the rocket motion will be upshifted in energy (and in corresponding classical frequency) as compared with the energy observed in the laboratory; the other backward-moving photon will be downshifted. We can calculate this frequency shift using the Doppler formulas (Exercise 8-18). Use the expression $m\gamma v$ for momentum of a particle, equation (7-8), to state the conservation of momentum (notice the minus sign before the second photon term, representing the photon moving to the left):

$$\begin{aligned} m_{\text{before}} v\gamma = & m_{\text{after}} v\gamma + \frac{hf}{c^2}\left[\frac{1+v}{1-v}\right]^{1/} \\ & -\frac{bf}{c^2}\left[\frac{1-v}{1+v}\right]^{1/2} \end{aligned}$$

Simplify this expression to

$$m_{\text{before}} = m_{\text{after}} + 2bf/c^2 \tag{8.E.29}$$

or

$$m_{\text{before}} - m_{\text{after}} = \Delta m = 2bf/c^2 = \text{ energy released} \tag{8.E.30}$$

Conservation of momentum in both frames implies a change in particle mass equal to the total energy of the emitted photons. Multiply the mass- units result by c^2 to convert to conventional units and the equation in the well-known form

energy released (conventional units) $= (\Delta m)c^2$

b Second result: $E_{\text{rest}} = m$. Now add the condition that energy is conserved in the laboratory frame:

$$E_{\text{before}} = E_{\text{after}} + 2bf/c^2 \tag{8.E.31}$$

Compare equations (1) and (2). These two equations both describe a particle at rest. Show that they are consistent if $E_{\text{before}} = m_{\text{before}}$ and $E_{\text{after}} = m_{\text{after}}$ and that therefore in general

$$E_{\text{rest}} = m \tag{8.E.32}$$

or, in conventional units,

$$E_{\text{rest conv}} = mc^2 \tag{8.E.33}$$

c Third result: At any speed, $E=m\gamma$. Next add the condition that energy be conserved in the rocket frame. Place primes on expressions for rocketmeasured energy of the particle and use the Doppler equations to transform the classical frequency back to the laboratory value f. Show that the result is

$$E'_{\text{before}}=E'_{\text{after}}+\left(2bf/c^2\right)\gamma \tag{8.E.34}$$

The salient difference between equations (2) and (3) is that in the rocket frame the particle is in motion. Deduce that the general expression for energy of a particle includes the stretch factor gamma:

$$E=m\gamma \tag{8.E.35}$$

or, in conventional units,

$$E_{\text{conv}}=m\gamma c^2 \tag{8.E.36}$$

Reference: Fritz Rohrlich, American Journal of Physics, Volume 58, pages 348-349 (April 1990).

8-27 every fhing goes forward

"Everything goes forward" is a good rule of thumb for interactions between highly relativistic particles and stationary targets. In the laboratory frame, many particles and gamma rays resulting from collisions continue in essentially the same direction as the incoming particles.

The first figure (top) shows schematically the collision of two protons in the center-of-momentum frame, the frame in which the system has zero total momentum. A great many different particles are created in the collision, including a gamma ray (the fastest possible particle) that by chance moves perpendicular to the line of motion of the incoming particles: $\phi'=\pi/2$ radians.

The first figure (bottom) shows the same interaction in the laboratory frame, in which one proton is initially at rest. At what angle ϕ does the product gamma ray move in this frame?

a From the Doppler equations (Exercise 8-19), show that the outgoing angle ϕ for the gamma ray in the laboratory frame is given by the expression

$$\cos\phi=v_{\text{rel}} \tag{8.E.37}$$

b What is the speed v_{proton} of the rightwardmoving proton in the laboratory frame? We define the laboratory frame by riding at speed v_{rel} on the leftward-moving proton in the center-of-momentum frame. Therefore the rightward-moving proton also moves with speed v_{rel} in the center-of-momentum frame. Use the law of addition of velocities to find the speed of the rightward-moving proton in the laboratory frame (Section L.7 and Exercise 3.11).

$$v_{\text{procon}}=\frac{2v_{\text{rel}}}{1+v_{\text{rel}}^2} \tag{8.E.38}$$

c In order to solve equation (1) for ϕ, we need to know the value of v_{rel}. Equation (2) is a quadratic in v_{rel}. Show that the solution is

$$v_{\text{rel}}=\frac{1}{v_{\text{proton}}}\left[1-\frac{1}{\gamma_{\text{proton}}}\right] \tag{8.E.39}$$

Here γ_{proton} is the stretch factor γ using the proton velocity v_{proton}.

d We are interested in finding the angle ϕ when the incoming proton is highly relativistic. In this case $v_{\text{proton}}\approx 1$. From the approximation for small angles (ϕ expressed in radians)

$$\cos\phi\approx 1-\phi^2/2 \quad |\phi|<<1 \tag{8.E.40}$$

show that the angle ϕ is given approximately by the expression

$$\phi\approx\left[\frac{2}{\gamma_{\text{proton}}}\right]^{1/2} \tag{8.E.41}$$

e What is the value of ϕ in radians and in degrees for incident protons of energy $E=200\,\text{GeV}$? For incident protons of energy $2\times 10^4\,\text{GeV}$? ($1\,\text{GeV}=10^9$ electron-volts. Mass of the proton is approximately $1\,\text{GeV}$.)

BEFORE

CENTER-OF-MOMENTUM FRAME AFTER

BEFORE

AFTER LABORATORY FRAME

EXERCISE 8-27, first figure. In the center-of-momentum frame two incoming protons collide, creating many particles, among them a gamma ray that moves perpendicular to the original line of motion. In the laboratory frame, in which one proton is initially at rest, in what direction does the gamma ray move?

8-28 decay of $\pi^0-meson$

A π° meson (neutral pi-meson) moving in the x-direction with a kinetic energy in the laboratory frame equal to its mass m decays into two photons. In the

EXERCISE 8-28. Two photons resulting from the decay of a π° meson, as observed in rocket and laboratory frames. rocket frame in which the meson is at rest these photons are emitted in the positive and negative y'-directions, as shown in the figure. Find the energies of the two photons in the rocket frame (in units of the mass of the meson) and the energies and directions of propagation of the two photons in the laboratory frame.

COMPTON SCATTERING

8-29 Compton scaftering

Analyze Compton scattering of an incident photon that collides with and recoils from an electron that is initially at rest. Compton scattering in one dimension was discussed in Section 8.4. Here we analyze Compton scattering in two dimensions. The goal is to determine the reduced energy of the photon that has been scattered with a change of direction measured by the

EXERCISE 8-27, second figure. Forward spray of particles created in collisions near the middle of the picture. An incident particle, probably a charged π-meson, enters from the left with energy approximately 100 to 200 times its rest energy and strikes a nucleus of neon or bydrogen. Curving paths in the imposed magnetic field are probably knock-on electrons. These and the cascade of other particles move initially in the same direction as the incoming π-meson: "Everything goes forward." Photograph courtesy of Fermi Laboratory. angle.

8

BEFORE

AFTER

EXERCISE 8-29, first figure. Compton scattering of a photon from an electron initially at rest. The angle ϕ is called the scattering

angle ϕ. The angle ϕ is called the scattering angle.

$$\begin{array}{ll} E^2-p^2=m^2 & [\text{for an electron}] \\ E^2-p^2=0 & [\text{for a photon}] \end{array} \tag{8.E.42}$$

Discussion: The conservation of momentum is a vector conservation law. This means that the vector sum of the momenta after the collision equals the momentum of the photon before the collision. In other words, the vectors form a triangle, as shown in

$$p_D^2 = p_A^2 + p_C^2 - 2p_A p_C \cos\phi \tag{8.E.43}$$

a Now replace all momenta with energies (easy

EXERCISE 8-29, second figure. Conservation of vector momentum means that the momentum triangle is closed. Use the notation in the first figure. Do not use frequency or wavelength or Planck's constant or speed in your analysis - only the laws of conservation of momentum and energy plus equations:

Discussion: The conservacion the second figure. Apply the law of cosines to this figure: for photons, more awkward for the electron), com-

Detector at $\phi = 45^\circ$

9, third figure. Results of the Compton experiment EXERCISE 8-29, third figure. Results of the Compton experiment target. At each angle of the detector except $\phi = 0$ there are some photons scattered with loss of energy (electron recoils by itself) and photons scatiered with loss of energy (electron recoils by itself) and atom recoil as a unit). bine with the conservation of energy, and derive the Compton scattering formula:

$$E_{\text{scattered}} = \frac{E_{\text{incident}}}{1 + \frac{E_{\text{incident}}}{m}(1 - \cos\phi)} \tag{8.E.44}$$

Exercise 8-30 gives some examples of this result.

b Compton's original experiments showed that some photons were scattered without a measurable change of energy. These photons were scattered by electrons that did not leave the atom in which they were bound, so that the entire atom recoiled as a unit. Assume that the energy of the incoming photon is at most a few times the rest energy of the electron. In this case, show that the energy change is negligible for photons scattered by electrons tightly bound to an atom of average mass (say $10 \times 2000 \times$ mass of an electron). See the third figure.

Reference: A. H. Compton, Pbysical Review, Volume 22, pages $409 - 413$ (1923).

8-30 complon scattering examples

a A gamma ray photon of energy equal to twice the mass of the electron scatters from an electron initially at rest. Provide the following answers in units of $\mathbf{MeV}$. (Mass of the electron is $0.511\mathbf{MeV}$.) From the Compton scattering formula find the energy of the scattered photon for scattering angles 0,90, and 180 degrees. If you have access to a computer, calculate this energy at 10 - degree increments between zero and 180 degrees and plot the resulting curve of energy vs. angle.

$\mathbf{b}$ In a new set of experiments, the incident gamma ray has energy equal to five times the rest energy of the electron. Repeat the calculations of part a for this case.

8-31 energy of a photon and frequency of light

Planck found himself forced in 1900 to recognize that light of frequency f (vibrations/second) is composed of quanta (Planck's word) or photons (Einstein's later word), each endowed with an energy $E = hf/c^2$ (energy in units of mass) where b is a universal constant of proportionality called Planck's constant. How can Planck's formula possibly make sense when - as we now know - not only E b. 1 t also f depend upon the frame of reference in which the light is observed? (We use f for frequency instead of the usual Greek nu, v, to avoid confusion with v for speed.) a A photon moves along the positive x-axis. Results of Exercise 8-18 show the relation between the energy of this photon measured in the rocket frame and its energy measured in the laboratory frame. A classical electromagnetic wave moves along the positive x-axis. Results of Exercise L- 5 (at the end of the Special Topic following Chapter 3) show the relation between the frequency of this wave measured in the rocket frame and its energy measured in the laboratory frame. Compare these two results to show that if we associate photons with a light wave in one coordinate system, this association will hold in all coordinate systems.

$\mathbf{b}$ The theory of relativity does not tell us the value of Planck's constant b in the formula $E = \left(b/c^2\right) f$ that relates photon energy (in units of mass) to classical wave frequency. Experiment shows the constant h to have the value 6.63×10^{-34} joule-second. Show that if energy is measured in conventional units, the relation between energy and frequency has the form

$$E_{\text{conv}} = bf \quad [\text{energy in conventional units}] \tag{8.E.45}$$

c Show that the formula for Compton scattering (Exercise 8-29) becomes

$$f_{\text{scattered}} = \frac{f_{\text{incident}}}{1 + \frac{hf_{\text{incident}}}{mc^2}(1 - \cos\phi)} \tag{8.E.46}$$

In the 1920 s there was great resistance to the idea that when the electron is "shaken'" by the electric field of wave at one frequency it should scatter (reemit) this radiation at a lower frequency.

8-32 inverse Complon scaftering

In Compton's original experiment an X-ray photon scattered with reduced energy from an electron initially at rest. In contrast, a photon scattered from a moving electron can increase the energy of the photon. Such an interaction is called inverse Compton scattering. The figure (page 270) shows an example.

When a high-energy electron collides head on with a low-energy photon, what is the energy of the outgoing photon? Answer this question using parts $a - e$ or by some other method.

a Write down equations of conservation of energy and momentum, using subscripts A through D from the figure.

$\mathbf{b}$ Recall that the energy of a photon is equal to the magnitude of its momentum. Use this to simplify

AFTER

EXERCISE 8-32. Inverse Compton scattering. A low-energy photon is scattered by a bigh-energy electron.

the conservation equations, taking leftward momentum to be negative.

c We are not interested in the energy or the momentum of outgoing electron C. Therefore solve the energy equation for E_C and the momentum equation for p_C, square and subtract the two sides, and use $E_C^2 - p_C^2 = m$. What happens to E_A^2 and p_A^2 on the other side of the resulting equation? For now keep terms in the first power of p_A without substituting the awkward equivalent $p_A = \left(E_A^2 + m^2\right)^{1/2}$.

d Solve the resulting equation for the energy of the outgoing photon.

e Now consider an important special case in which the incoming electron is extremely energetic, with an energy of, say, thousands of times its rest energy as measured in the laboratory. Show that this case the incoming electron behaves in essential respects as a photon: $p_A \approx +E_A$. Simplify your equation of part d to show that under these circumstances the outgoing photon has the energy of the incoming electron no matter what the energy of the incoming photon.

TESTS OF RELATIVITY

Note: Exercises 8-33 through 8-39 form a connected tutorial on tests of relativity. Some of these exercises depend on each other and on earlier exercises, especially Exercise 8-6.

8-33 photon energy shift due To recoil of emitter

Note: This exercise uses the results of Exercise 8-25. A free particle of initial mass m_o and initially at rest emits a photon of energy E. The particle (now of mass m) recoils with velocity v, as shown in the figure.

BEFORE

$m_{\circ}$
(at rest)

AFIER

EXERCISE 8-33. Recoil of a particle that emits a photone

EXERCISE 8-33. Recoil of a particle that emits a photon.

a Write down the conservation laws in a form that makes no reference to velocity. Consider the case in which the fractional change in mass in the emission process is very small compared to unity. Show that for this special case the photon has an energy $E_{\circ}=m_{\circ}-m$. For the general case show that

$$E=E_{\circ}\left(1-\frac{E_{\circ}}{2 m_{\circ}}\right) \tag{8.E.47}$$

or

$$\frac{E-E_{\circ}}{E_{\circ}}=\frac{\Delta E}{E_{\circ}}=-\frac{E}{2 m_{\circ}} \tag{8.E.48}$$

b Show that this shift in energy for visible light $\left(E_{\circ \text{ conv }} \sim 3 \mathrm{eV}\right)$ emitted from atoms $\left(m c^{2} \sim 10 \times 10^{9} \text{ eV }\right)$ in a gas is very much less than the Doppler shift due to thermal motion (Exercise 8-25) even for temperatures as low as room temperature ($k T \sim 1 / 40$ eV).

8-34 recoilless processes

a A free atom of iron $^{57}\mathrm{Fe}$ - formed in a socalled "excited state" by the radioactive decay of cobalt $^{57}\mathrm{Co}$ - emits from its nucleus a gamma ray (high-energy photon) of energy $14.4 \mathrm{keV}$ and transforms to a "normal" $^{57}\mathrm{Fe}$ atom. By what fraction is the energy of the emitted ray shifted because of the recoil of the atom? The mass of the $^{57}\mathrm{Fe}$ atom is about equal to that of 57 protons.

b That not all emitted gamma rays experience this kind of frequency shift was the important discovery made in 1958 by R. L. Mössbauer at the age of 29. He showed that when radioactive nuclei embedded in a solid emit gamma rays, some significant fraction of these atoms fail to recoil as free atoms. Instead they behave as if locked rigidly to the rest of the solid. The recoil in these cases is communicated to the solid as a whole. The solid being heavier than one atom by many powers of 10 , these events are called recoilless processes. For gamma rays emitted in recoilless processes, the $m_{\circ}$ in Exercise $8-33$ is the mass of the entire chunk in which the iron atoms are embedded. When this chunk has a mass of one gram, by what fraction is the frequency of the emitted ray shifted in this "recoilless" process?

c The gamma rays emitted from excited $^{57}\mathrm{Fe}$ atoms do not have a precisely defined energy but are spread over a narrow energy range-or frequency range - or natural line width, shown as a bell-shaped curve in the figure. (The physical basis for this curve is explained by quantum physics.) The full width of this curve at half maximum is denoted by Δv. R. V. Pound and G. A. Rebka selected $^{57}\mathrm{Fe}$ for experiments

EXERCISE 8-34. Natural line width of photons emitted from $^{57}\mathrm{Fe}$.

with recoilless processes because the fractional ratio $\Delta f / f_{0}$ has the very small value 6×10^{-13} for the $14.4 \mathrm{keV}$ gamma ray from $^{57}\mathrm{Fe}$. How much is the natural line width, Δf, of ^{57} Fe expressed in cycles/second? Compare the fractional natural line width with the fractional shift due to recoil of a free iron atom. And compare it with the fractional shift of a gamma ray from a recoilless process.

Reference: For a more detailed account of Mössbauer's discovery for which the German scientist was awarded the Nobel prize in 1961 - see S. DeBenedetti, "The Mössbauer Effect," Scientific American, Volume 202, pages $72-80$ (April 1960). For the selection of $^{57}\mathrm{Fe}$, see R. V. Pound and G. A. Rebka, Jr., Physical Review Letters, Volume 3, pages 439-441 (1959).

Pound and Rebka's application of recoilless processes thus put into one's bands a resonance phenomenon sharp in frequency to the fantastic precision of 6 parts in 10^{13}. Exercise 8-35 deals with detection of this radiation. Exercise 8-36 uses motion (Doppler shift) as a means for producing controlled changes of a few parts in 10^{13} - or much larger changes - in the effective frequency of source or detector or both. To what uses can radiation of precisely defined frequency be put? There are many uses. For instance, the effect is the basis of important techniques in solid-state physics, molecular physics, and biophysics. One can detect the change in the natural frequency of radiation from $^{57}\mathrm{Fe}$ atoms caused by other atoms in the neighborhood - and by external magnetic fields - and in this way analyze the interaction between the iron atom its surroundings. Here we aim at detection of various effects predicted by relativity.

8-35 resonant scattering

The nucleus of normal $^{57}\mathrm{Fe}$ absorbs gamma rays at the resonant energy of $14.4 \mathrm{keV}$ much more strongly than it absorbs gamma rays of any nearby energy. The energy absorbed in this way is converted to internal energy of the nucleus and transmutes the $^{57}\mathrm{Fe}$ to the "excited state." After a time this excited nucleus drops back to the "normal state," emitting the excess energy in various forms in all directions. Therefore the number of gamma rays transmitted through a thin sheet containing $^{57}\mathrm{Fe}$ will be less at the $14.4 \mathrm{keV}$ resonance energy than at any nearby energy. This process is called resonant scattering.

a Show that when a gamma ray of the resonant energy $E_{\circ}$ is incident on a free iron atom initially at rest then the free nucleus cannot absorb the gamma ray at its resonant energy, because the process cannot satisfy both the law of conservation of momentum and the law of conservation of energy.

b Show that both conservation laws are satisfied when an iron atom embedded in a one-gram crystal absorbs such a gamma ray by a recoilless process, in which the entire crystal absorbs the momentum of the incident gamma ray. " "Satisfied"'? For momentum, yes; for energy, no. However, the fractional discrepancy in energy - equivalent to the fractional discrepancy in frequency - is less than 6 parts in 10^{13} and therefore small enough so that the iron nucleus is "unable to notice" the discrepancy and therefore absorbs the gamma ray.)

8-36 measurement of Doppler shift by resonant scaffering

In the experimental arrangement shown in the figure, a source containing excited $^{57}\mathrm{Fe}$ nuclei emits (among other radiations) gamma rays of energy $E_{\circ}$ by a recoilless process. An absorber containing $^{57}\mathrm{Fe}$ nuclei in the normal state absorbs some of these gamma rays by another recoilless process and reemits this energy in various forms in all directions. Thus the counting rate on a gamma ray counter placed as shown is less for an absorber containing normal $^{57}\mathrm{Fe}$ than for an equivalent absorber without normal $^{57}\mathrm{Fe}$. Now the source is moved toward the absorber with speed v.

a What must be the velocity of the source if the gamma rays are to arrive at the absorber shifted in frequency by 6 parts in 10^{13} ? Express your answer in centimeters/second.

EXERCISE 8-36. Resonant scattering of photons. b Will the counting rate of the counter increase or decrease under these circumstances?

c What will happen to this counting rate if the source is moved away from the absorber with the same speed?

d Make a rough plot of counting rate of the counter as a function of the source velocity toward the absorber (positive velocity) and away from the absorber (negative velocity).

e Discussion question: Does this method allow one to measure the "absolute velocity" of the source, in violation of the Principle of Relativity (Chapter 3)?

8-37 test of the gravitational red shift I

A 14.4-keV gamma ray emitted from $^{57}\mathrm{Fe}$ without recoil travels vertically upward in a uniform gravitational field. By what fraction will the energy of this photon be reduced in rising to a height z (Exercise 8-6)? An absorber located at this height must move with what speed and in what direction in order to absorb such gamma rays by recoilless processes? Calculate this velocity when the height is 22.5 meters. Plot the counting rate as a function of absorber velocity expected if (a) the gravitational red shift exists, and (b) there is no gravitational red shift. A frequency shift of $\Delta f / f_{\circ}=(2.56 \pm 0.03) \times 10^{-15}$ was determined in an experiment conducted by R. V. Pound and J. L. Snider. You will notice that this shift is very much smaller than the natural line width $\Delta f / f_{0}=6 \times 10^{-13}$ (see the figure for Exercise 8-34). Therefore the result depended on a careful exploration of the shape of this line and was derived statistically from a large number of photon counts.

References: Original experiment: R. V. Pound and G. A. Rebka, Jr., Pbysical Review Letters, Volume 4, pages $337-341$ (1960). Improved experiment: R. V. Pound and J. L. Snider, Pbysical Review, Volume 140, pages B788-B803 (1965)

8-38 test of the gravifational red shift II

On June 18,1976, a Scout D rocket was launched from Wallops island, Virginia, carrying an atomic hydrogen-maser clock as the payload. It achieved a maximum altitude of 10^7 meters. By means of microwave signals, its clock was compared with an identical clock at the surface of Earth. The experiment used continuous comparison of these two clocks as the payload rose and fell. Simplifying (and somewhat misrepresenting) the experiment, we report their result as a fractional frequency red shift at the top of the trajectory due to gravitational effects of $\Delta f / f = 0.945 \times 10^{-10} \pm 6.6 \times 10^{-15}$. Modify the analysis of Exercise 8-6 to make a prediction about this experiment and compare your prediction with the results of the Scout D rocket experiment.

References: Description of experiment and preliminary results: R. F. C. Vessot and M. W. Levine, General Relativity and Gravitation, Volume 10, Number 3, pages $181-204$ (1979). Final results: R. F. C. Vessot, M. W. Levine, and others, Pbysical Review Letters, Volume 45, pages 2081-2084 (1980). Popular explanation: Clifford M. Will, Was Einstein Right? (Basic Books, New York, 1986), pages 42-64.

8-39 test of the fwin paradox

For Penny to leave her twin brother Paul behind in the laboratory, go away at high speed, return, and find herself younger than stay-at-home Paul is so contrary to everyday experience that it is astonishing to find that the experiment has already been done and the prediction upheld! Chalmers Sherwin pointed out that the twins can be identical iron atoms just as well as living beings. Let one iron atom remain at rest. Let the other make one forth-and-back trip. Or many round trips. The percentage difference in aging of the twin atoms is the same after a million round trips as after one round trip - and it is easier to measure. How does one get the second atom to make many round trips? By embedding it in a hot piece of iron, so that it vibrates back and forth about a position of equilibrium (thermal agitation!). How does one measure the difference in aging? In the case of Penny and Paul the number of birthday firecrackers that each sets off during their separation are counted. In the experiment with iron atoms one compares not the number of flashes of firecrackers up to the time of meeting but the frequency of the photons emitted by recoilless processes, and thus - in effect - the number of ticks from two identical nuclear clocks in the course of one laboratory second. In other words, one compares the effective frequency of INTERNAL nuclear vibrations (not to be confused with the backand-forth vibration of the iron atom as a whole!) as observed in the laboratory for (a) an iron nucleus at rest and (b) an iron nucleus in a hot specimen.

It is difficult to obtain an iron nucleus at rest. Therefore the actual experiment compared the effective internal nuclear frequency for two crystals of iron with a difference of temperature ΔT. R. V. Pound and G. A. Rebka, Jr., measured that a sample warmed up by the amount $\Delta T = 1$ degree Kelvin underwent a fractional change in effective frequency of $\Delta f / f_0 = (-2.09 \pm 0.24) \times 10^{-15}$ (fewer vibrations; fewer clock ticks; fewer birthdays; more youthful!). (We use f for frequency instead of the usual Greek nu, v, to avoid confusion with v for speed.) To simplify thinking about the experiment, go back to the idea that one iron atom is at rest and the other is in thermal agitation at temperature T; predict the fractional lowering in number of internal vibrations in the hot sample per laboratory second; and compare with experiment.

Discussion: The figure compares the effective "ticks" of the two "internal nuclear clocks" in the laboratory time dt. Note that the speed of thermal agitation is about 10^{-5} the speed of light. What algebraic approximation suggests itself for the dis-

EXERCISE 8-39. Comparison of nuclear clock at rest with nuclear clock in thermal motion. crepancy factor $1-\left(1-v^2\right)^{1/2}$? How much is the deficit in number of "ticks" (for hot atom versus atom at rest) in the lapse of laboratory time dt ? Show that the cumulative deficit in number of "ticks" from the hot atom in one second is $f_{\circ}\left(v^2/2\right)_{\text{avg}}(1 \text{ second})$ where $\left(v^2\right)_{\text{avg}}$ means "the time average value of the square of the atomic speed'" (relative to the speed of light). Note that the mean kinetic energy of thermal agitation of a hot iron atom (mass $m_{\text{Fe}} = 57 m_{\text{proton}}$) is given by the classical kinetic theory of gases:

$$(1/2) m_{\text{Fe}}\left(v^2\right)_{\text{avg}} c^2 = (3/2) kT \tag{8.E.49}$$

Here k is Boltzmann's factor of conversion between two units of energy, degrees and joules (or degrees and ergs); $k = 1.38 \times 10^{-23}$ joule/degree Kelvin ($k = 1.38 \times 10^{-16}$ erg/degree Kelvin). How does the experimental result of Pound and Rebka compare with the result of your calculation?

References: Chalmers W. Sherwin, Pbysical Review, Volume 120 , pages $17-21(1960)$. R V. Pound and G. A. Rebka, Jr., Physical Review Letters, Volume 4, pages 274-275 (1960).

FREE-FOR-ALL!

8-40 momenfum without mass?

A small motor mounted on a board is powered by a battery mounted on top of it, as shown in the figure on page 274 . By means of a belt the motor drives a paddlewheel that stirs a puddle of water. The paddlewheel mechanism is mounted on the same board as the motor but a distance x away. The motor performs work at a rate dE/dt.

a How much mass is being transferred per second from the motor end of the board to the paddlewheel end of the board?

b Mass is being transferred over a distance x at a rate given by your answer to part a. What is the momentum associated with this transfer of mass? Since this momentum is small, Newtonian momentum concepts are adequate.

c Let the mounting board be initially at rest and supported by frictionless rollers on a horizontal table. The board will move! In which direction? What happens to this motion when the battery runs down? How far will the board have moved in this time?

d Show that an observer on the board sees the energy being transferred by the belt; an observer on the table sees the energy being transferred partly by the belt and partly by the board; an observer riding one way on the belt sees the energy being transferred partly by the belt moving in the other direction and partly by the board. Evidently it is not always possible

EXERCISE 8-40. Transfer of mass without net transfer of particles or radiation.

to make a statement satisfactory to all observers about the path by which energy travels from one place to another or about the speed at which this energy moves from one place to another!

8-41 The photon rocket and inferstellar fravel

The "perfect" rocket engine combines matter and antimatter in a controlled way to yield photons (highenergy gamma rays), all of which are directed out the rear of the rocket. Suppose we start with a spaceship of initial mass M_o, initially at rest. At burnout the remaining spaceship moves with speed v and has a mass equal to the fraction f of the original mass. For a given fraction f, we want to know the final rocket speed v or, better yet, the time stretch factor $\gamma = 1/\left(1-v^2\right)^{1/2}$. (Note: Here, f is not frequency.)

a What is the total energy of the system initially? Let E_{rad} stand for the total energy of radiation after burnout. Find an expression for the total energy of the system after burnout and set up the conservation of energy equation.

b Similarly, set up the conservation of momentum equation. What is the total momentum of the system initially? The momentum of the radiation at burnout? The momentum of the spaceship at burnout?

c Eliminate E_{rad} between the two conservation equations. Show that the result can be written

$$\gamma f + \gamma v f = 1 \tag{8.E.50}$$

d From the definition of γ, show that $\gamma v = \left(\gamma^2 - 1\right)^{1/2}$ and hence that the equation of part c can be written in the form

$$f^2 - 2\gamma f + 1 = 0 \tag{8.E.51}$$

e What is the value of the fraction $f =$ (final spaceship mass)/(initial spaceship mass) for a time stretch factor $\gamma = 10$? In your opinion, is it possible to construct a spaceship whose shell and payload is this small a fraction of takeoff mass?

f Substitute the result of part e into the conservation of energy equation in part a. Show that the total energy of emitted radiation is less than the mass of fuel consumed. Why?

9 Does your analysis apply to takeoff from Earth's surface? From Earth orbit? From somewhere else? What safety precautions apply to the backward blast of gamma rays?

h You are the astronaut assigned to this spaceship. Do you want to stop at your distant destination star or fly past at high speed? Do you want to return to Earth? Do you want to stop at Earth on your return or merely wave in passing? Must all fuel for the entire trip be on board at takeoff or can you refuel at your destination star? From your answers to these questions, plan your trip and find the resulting fractions of spaceship mass to initial mass for different stages of your trip.

i Discussion question: From your results for this exercise, what are your conclusions about the technical possibilities of human flight to the stars?

References: Adapted from A.P. French, Special Relativity (W.W. Norton, New York, 1968), pages $183-184$. See also J. R. Pierce, Proceedings of the IRE, Volume 47, pages 1053-1061 (1959).