1.4: Kinematics

Equations of Motion

Okay, enough of the definitions. Let's see how these things all fit together, and how they can be used. What we will be looking at are called the *equations of motion*, and this topic is often referred to as *kinematics*. It is important to note that we are not yet dealing with causes for these motions, but only the motions themselves.

We will mostly only deal with constant accelerations (unless otherwise specified), and since instantaneous acceleration is the derivative of velocity, it is not difficult to integrate it to get the instantaneous velocity as a function of time:

$$\left.\begin{array}{l} a=\dfrac{dv}{dt} \quad \Rightarrow \quad v\left(t\right)=\int a\;dt=at+const \\ const=v\left(t=0\right)\equiv v_o \end{array}\right\}\quad v\left(t\right)=at+v_o \tag{1.4.1}$$

The constant of integration is found by plugging $t=0$ into Equation 1.4.1, which results in the velocity of the object at the starting time, which is typically designated as v_o.

We can play exactly the same game to obtain the equation of motion for position as a function of time, since we know how it relates to the instantaneous velocity:

$$\left.\begin{array}{l} v=\dfrac{dx}{dt} \quad \Rightarrow \quad x\left(t\right)=\int vdt=\int\left(at+v_o\right)dt=\frac{1}{2}at^2+v_ot+const \\ const=x\left(t=0\right)\equiv x_o \end{array}\right\}\quad x\left(t\right)=\frac{1}{2}at^2+v_ot+x_o \tag{1.4.2}$$

Notice that if we have all the details of this last equation, we can obtain the velocity equation above simply by taking a derivative. We cannot go in the opposite direction without also obtaining the starting position.

Analyze This

A particle moves along the x-axis with an acceleration that varies linearly with time.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

*First, we note that this is **not** a case of constant acceleration, so equations 1.4.1 and 1.4.2 do not apply. But the calculus we employed to get to these equations does apply, so we just need some sort of mathematical expression for acceleration to repeat that process. We are given that the acceleration varies linearly with time, so translating this into a mathematical expression gives:*

$$a\left(t\right)=\lambda t+\beta$$

where λ and β are unknown constants. Note that the acceleration at time $t=0$ is just β, so it is more descriptive to just call it a_o from this point on.

Without these constants (and some others), we cannot compute values like speeds and positions at different times, but we can still do some calculus in terms of the unknowns. The velocity of the particle at a time t in terms of the acceleration is:

$$v\left(t\right)=\int a\left(t\right)dt=\int\left(\lambda t+a_o\right)dt=\frac{1}{2}\lambda t^2+a_ot+\gamma$$

where γ is the constant of integration. We note that at $t=0$ the velocity just equals γ, so hereafter we'll just call that constant v_o.

We can repeat this process for the position of the particle as a function of time (noting that the constant of integration this time is the position at time $t=0$):

$$x\left(t\right)=\int v\left(t\right)dt=\int\left(\frac{1}{2}\lambda t^2+a_ot+v_o\right)dt=\frac{1}{6}\lambda t^3+\frac{1}{2}a_ot^2+v_ot+x_o$$

Let's make an accounting of all the numbers we can encounter in a constant-acceleration situation:

- independent variable: t
- dependent variables: x , v

- constants of the motion: x_o , v_o , a (acceleration is constant by assumption)

With six numbers to work with, you can imagine there are many ways to set up a problem to solve for something unknown. Everything you need to solve any such problem is provided in the above equations. However, it is often easier to put those equations together to form a new equation, to cut down on the algebra needs for certain types of problems. The most common useful re-combining of these variables involves eliminating time from the two equations, since you may be given velocities and positions. The algebra is straightforward:

$$\left.\begin{array}{l} v_f=at+v_o \;\;\Rightarrow\;\; t=\dfrac{v_f-v_o}{a} \\ x_f-x_o=\frac{1}{2}at^2+v_ot \end{array}\right\} \;\; x_f-x_o=\frac{1}{2}a\left(\frac{v_f-v_o}{a}\right)^2+v_o\left(\frac{v_f-v_o}{a}\right) \;\;\Rightarrow\;\; 2a\left(x_f-x_o\right)={v_f}^2-{v_o}^2 \tag{1.4.3}$$

You can think of this equation as the “before/after” equation, because it deals only with starting and ending positions and velocities, and has eliminated time as an input variable.

While we are accumulating useful (though unnecessary) equations for motion with constant acceleration, we should also include the two equations that involve average velocity. The first is just a rewriting of the definition of average velocity, with the "final" position occurring at time t:

$$v_{ave}=\frac{x_f-x_o}{t}=\frac{x\left(t\right)-x_o}{t} \;\;\;\Rightarrow\;\;\; x\left(t\right)=v_{ave}t+x_o \tag{1.4.4}$$

The second equation is quite useful, though it applies *only* to motion involving constant acceleration:

$$v_{ave}=\frac{x_f-x_o}{t}=\frac{\frac{1}{2}at^2+v_ot}{t}=\frac{1}{2}at+v_o=\frac{1}{2}\left(v_f-v_o\right)+v_o \;\;\;\Rightarrow\;\;\; v_{ave}=\frac{v_o+v_f}{2} \tag{1.4.5}$$

For constant acceleration, the average velocity simply equals the arithmetic average of the starting and ending velocities. We will better see why it comes out this way when we start discussing graphing shortly.

Free-Fall

There is one type of straight-line motion that involves constant acceleration that we are all familiar with: free-fall.

We will look more closely at how to explain this in terms of forces in a future section, but assuming air resistance has a small effect (remember, we are devising a simplified model here), then it turns out (as shown by Galileo dropping stones from the Tower of Pisa, and more dramatically in the demonstration) that objects all accelerate at the same constant rate as they fall to Earth. This rate of acceleration is commonly given the symbol g, and it has the value:

$$\textit{acceleration due to gravity near the surface of the earth}=g=9.8\,\tfrac{m}{s^2}$$

Note the units of distance-per-time-squared are the units of acceleration. This acceleration is of course always directed downward, and depending on our choice of coordinate system, this can be either positive or negative. Once the coordinate system is selected, the sign for g stays the same no matter which way the object is moving. If the positive direction is chosen to be upward, and the object is moving upward, then its velocity is positive and the negative value of g leads to a slowing of the object’s motion. If it is moving down, then its velocity is negative, and the negative acceleration leads to the velocity becoming more negative (i.e. it is speeding up).

Analyze This

A ball is thrown vertically upward at the same instant that a second ball is dropped from rest directly above it.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Both balls are under the influence of the earth's gravity, and therefore both accelerate at a rate g downward. However, one starts with a velocity in the upward direction, while the other starts from rest. With both balls subject to the same velocity equation:

$$v\left(t\right)=-gt+v_o$$

their different values for v_o ensure that they will always have different velocities. In particular, when the ball thrown upward is eventually moving downward, then assuming there is no collision, its speed will always be less than the speed of the other ball. This means that if the two balls start sufficiently high above the Earth's surface, they are guaranteed to eventually collide.

Suppose the higher ball starts at a distance of y_o above the lower ball. Calling the initial speed of the lower ball v_o, and calling its starting height zero, we can write down the equations of motion of both balls:

$$\begin{array}{ll} upper: & y_{higher}\left(t\right)=-\frac{1}{2}gt^2+y_o \\ lower: & y_{lower}\left(t\right)=-\frac{1}{2}gt^2+v_ot \end{array}$$

Naturally, if we are interested in when the two balls collide, we simply set the two heights equal to each other. If we do this, this seemingly complicated situation reduces to something very simple, as the effects of the gravitational accelerations of the two balls cancel out, which means that they will collide when the lower ball traverses the separation with its initial velocity, ignoring the common acceleration:

$$y_{higher}=y_{lower}\quad\Rightarrow\quad v_ot_{collision}=y_o$$