5.4: Torque

Rotational Newton's Second Law

As we saw for linear motion, we can only go so far with energy conservation. If we want to analyze aspects of motion such as elapsed time and direction of motion, we need more than mechanical energy conservation to work with. In the linear case, we found that this meant that we had to use Newton's Second Law. We now seek the rotational equivalent of that law.

The rotational equivalent of the Newton's Second Law must relate the reaction of the system (rotational acceleration) to an external influence (rotational force), with the degree of this effect being determined by an internal property of the system (rotational mass). That is, we need a rotational substitute for all of the participants of this formula:

$$\overrightarrow{a}_{cm} = \frac{\overrightarrow{F}_{net}}{m} \tag{5.4.1}$$

We already found a rotational version of acceleration in our discussion of rotational kinematics – it is the angular acceleration. We even defined a direction for this vector using the right-hand rule. The center of mass qualification in the case above is unneeded for the rotational case, because the angular acceleration is the same about every point on a rigid object.

We have also determined an appropriate candidate for the "rotational mass" – the rotational inertia. This is certainly a reasonable choice, for a couple of reasons. First, from our direct experience we know that it is easier to swing an object (e.g. a baseball bat) when holding the heavier end than when holding the lighter end, so the degree to which an extended object "resists" angular acceleration is determined by the distribution of mass. Second, if the physics is to remain consistent, why would the quantity that plays the role of mass in kinetic energy be different from the quantity that plays the role of mass for the second law?

With those two quantities established, we can now get a glimpse into what the "rotational force" is by examining the units:

$$\left[\alpha\right] = \frac{\left[rotational\;force\right]}{\left[I\right]} \;\;\;\Rightarrow\;\;\; \left[rotational\;force\right] = \left[\frac{rad}{s^2}\right]\left[kg\cdot m^2\right] = \frac{kg\cdot m^2}{s^2} \tag{5.4.2}$$

This is weird... These are units of energy! We'll need to chalk this up to coincidence, since clearly the vector quantity of rotational force cannot be a measure of energy. One way to see the difference is to remember the presence of radians in the numerator, even though they are not physical units. We will soon see the source of this coincidence, and it shouldn't take long before the apparent ambiguity between this quantity and energy fades away.

Alert

While the physical units are the same as energy, we ***never*** *refer to the SI units of this quantity as "Joules." Using this term implies that we are talking about energy, which we are not. Generally we stick to "Newton-meters."*

We can't continue calling this vector "rotational force" forever, so we will henceforth refer to it by its proper name: *torque*. In keeping with our tradition of using Greek variables for rotational quantities, we will represent torque with $\overrightarrow{\tau}$, giving as our rotational Newton's second law:

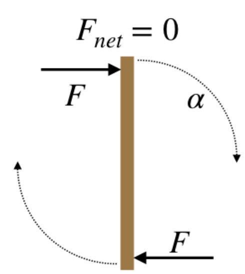
$$\overrightarrow{\alpha} = \frac{\overrightarrow{\tau}_{net}}{I} \tag{5.4.3}$$

Torque

In the cases of acceleration and inertia, we found a direct relationship between the linear and rotational quantities, so we would expect there to be a similar relationship between force and torque. Furthermore, since the linear/rotational bridge for acceleration and inertia both require a point of reference (the pivot), we would expect the same to be true for the bridge between force and torque.

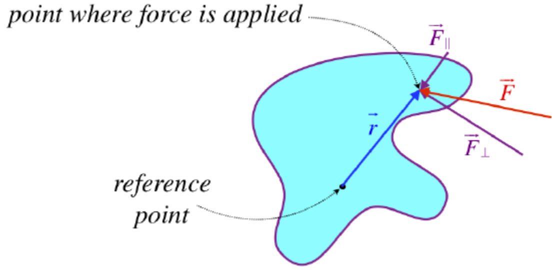
The first thing we notice is that an object can experience no net force and yet still experience a nonzero rotational acceleration:

Figure 5.4.1 – Zero Net Force Can Accelerate Rotationally



If the two forces shown in the figure above are moved so that act at the same point on the object, then it's clear that they also cancel rotationally. So apparently the place *where* the force acts is important to computing torque. If we choose a reference point (we will refer to this as a "pivot" in cases when it happens to be a fixed point, but in general it does not), then the application point of a force can be described by a position vector $\overrightarrow{r}$ that points from the reference point to the point where the force is applied. But there is still more that we have to worry about here. If two forces with the same magnitudes as those in the figure above were applied at the ends of the bar, but were pointing vertically, then no angular acceleration would result. Let's put all this together...

Figure 5.4.2 – Parts of a Force that Cause Angular Acceleration



The force vector can be decomposed into two perpendicular vectors – one that is parallel to the position vector, and one perpendicular to it. When it comes to causing the object to accelerate its rotation around the pivot, it's clear that the part of the force that is parallel to the position vector $\overrightarrow{F}_{\parallel}$ will have no effect, while the perpendicular part of the force $\overrightarrow{F}_{\perp}$ will.

If we were to perform experiments to test the effects of various force magnitudes, we would find that the angular acceleration is proportional to the magnitude of the force – push twice as hard in the same direction at the same point on the object, and its angular acceleration is twice as great around the same pivot. If we were to perform further experiments to test the effects of applying the force at different positions, we would find that the angular acceleration is proportional to the magnitude of the position vector – extend the position vector in the same direction to twice its original length and apply the same force in the same direction, and the angular acceleration is once again twice as great around the same pivot. Mathematically, we express the results of these experiments this way:

$$\left|\overrightarrow{\tau}\right| \sim \left|\overrightarrow{r}\right| \left|\overrightarrow{F}\right| \tag{5.4.4}$$

Notice that the units of this product work out correctly, so all we need to do is incorporate the "only the perpendicular part of $\overrightarrow{F}$ has an effect" into the math. If we call the angle between the position vector and the force vector θ, then the perpendicular component is $F\sin\theta$. Assuming there are no other constants involved (and there aren't any), we get, for the magnitude of the torque:

$$\left|\overrightarrow{\tau}\right| = \left|\overrightarrow{r}\right| \left|\overrightarrow{F}\right| \sin\theta \tag{5.4.5}$$

This looks familiar – we actually saw something just like it, way back in Equation 1.2.8. Torque *is* a vector that is derived from the product of two other vectors. Is it possible that it is simply a cross-product of these two vectors? The magnitude works, but what about direction? In Figure 5.4.2, the force will accelerate the rotation counterclockwise, which means that according to the right-hand-rule, the acceleration vector points out of the page. If we perform a cross-product of the position vector (up to the right) and the force vector (up to the left), the right-hand-rule results in a vector that also points out of the page. We therefore write:

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F} \tag{5.4.6}$$

Exercise

A rigid object is pivoted around the origin. The force vector given below acts on this object at the position also indicated below. Find the torque vector exerted on the object due to this force.

$$\overrightarrow{F} = 1.5N\;\hat{i} + 0.80N\;\hat{j} - 2.4N\;\hat{k}, \qquad position:\;(x,\;y,\;z) = (3.0m,\;0.0m,\;-2.0m)$$

Solution

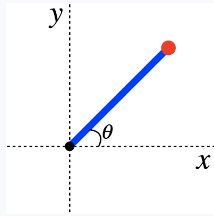
This is a straightforward calculation of a cross product:

$$\begin{aligned}\overrightarrow{\tau} &= \overrightarrow{r} \times \overrightarrow{F} \\ &= \left(3.0m\;\hat{i} + 0.0m\;\hat{j} - 2.0m\;\hat{k}\right) \times \left(1.5N\;\hat{i} + 0.80N\;\hat{j} - 2.4N\;\hat{k}\right) \\ &= \left[(3.0m)(0.80N) - (0.0m)(1.50N)\right]\hat{k} + \left[(0.0m)(-2.40N) - (-2.0m)(0.80N)\right]\hat{i} + \left[(-2.0m)(1.50N) - (3.0m)(-2.4N)\right]\hat{j} \\ &= 1.6Nm\;\hat{i} + 4.2Nm\;\hat{j} + 2.4Nm\;\hat{k}\end{aligned}$$

Analyze This

A small marble is attached to the end of a thin rigid rod with an equal mass, whose other end is held fixed at the origin. The rod starts at rest in the $x-y$ plane, and makes an angle θ up from the x-axis, as shown in the diagram. There is no gravity present, but the marble (not the rod) is subjected to a force from a potential energy field given by:

$$U(x,y) = \beta xy + U_o \;, \quad \beta = constant > 0$$



Analysis

We can use the potential energy function to determine the force at every point in space:

$$\vec F = -\frac{\partial U}{\partial x}\;\hat i - \frac{\partial U}{\partial y}\;\hat j = -\beta\left(y\;\hat i + x\;\hat j\right)$$

The torque exerted relative to the origin at the point (x,y) is the cross-product of the position vector there and the force vector there:

$$\vec\tau = \vec r\times\vec F = \left(x\;\hat i + y\;\hat j\right)\times\left[-\beta\left(y\;\hat i + x\;\hat j\right)\right] = \beta\left(y^2-x^2\right)\hat k$$

Let's call the length of the rod L. Then the coordinates of the marble in terms of L and θ are:

$$x = L\cos\theta \quad y = L\sin\theta$$

Plugging these in gives us the torque on the object in terms of θ.

$$\vec\tau = \beta L^2\left(\sin^2\theta - \cos^2\theta\right)\hat k = -\beta L^2\cos 2\theta\;\hat k$$

If we call the mass of the marble (and rod) m, we can also compute the moment of inertia of the object, and combine it with the torque to obtain its angular acceleration at any angle θ. When doing so, it is important to remember that the reference point for the moment of inertia must be the same as for the torque, which in this case is the origin. Here we have a rod pivoted about its end and a point mass a known distance from the pivot, so the moment of inertia is the sum of these contributions:

$$I = I_{rod} + I_{marble} = \frac{1}{3}mL^2 + mL^2 = \frac{4}{3}mL^2$$

So now, from Newton's 2nd Law for rotations:

$$\vec\alpha = \frac{\vec\tau}{I} = \frac{-\beta L^2\cos 2\theta\;\hat k}{\frac{4}{3}mL^2} = -\frac{3\beta\cos 2\theta}{4m}\;\hat k$$

The math is correct, and we have obtained a formula, but the exploration of our "analysis" can go much further. For example, we note that at $\theta = 45^o$, this acceleration is zero – there is no torque on the object. Does this make sense? Well, this angle occurs when $y = x > 0$, and plugging this back into the force vector that we found reveals that the force on the marble is not zero, but its direction in space is parallel to $-\hat i - \hat j$. When the marble is at $\theta = 45^o$, that direction is pointed directly at the origin. A force exerted on the marble that points directly through the rod is not going to cause the rod to accelerate rotationally, so this makes sense!

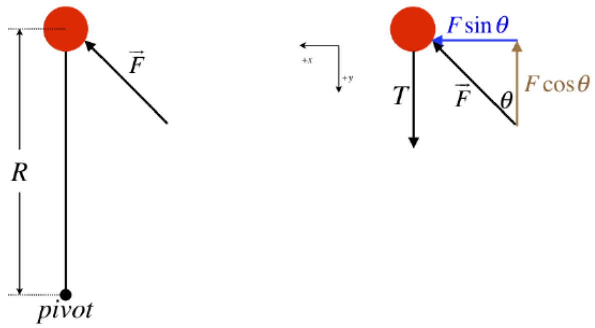
Another thing we note is that if the rod is turned slightly clockwise from $\theta = 45^o$, then there is a torque in the clockwise direction (the torque vector points in the $-\hat k$ direction, which is into the page, and from the RHR is clockwise). So a small clockwise nudge from $\theta = 45^o$ will cause the rotation to speed up. We similarly find that a small nudge from $\theta = 45^o$ in a counterclockwise direction results in a counterclockwise torque, speeding it up in that direction as well. Back in Section 3.7 we called a position like $\theta = 45^o$ a point of unstable equilibrium.

Finally, we note that since the sign of $\cos 2\theta$ changes between positive and negative as θ goes around a full circle, the torque alternates direction. It makes sense that the torque would not be in s single direction, because if the object makes a full turn in the direction of this torque and returns to where it started, it would have to be turning faster, as the single-direction torque keeps speeding it up. But this is impossible, because that would mean it gained KE, while the potential energy for a full 360^o turn comes back to its original value. Indeed we can therefore conclude that any potential energy function whatsoever that we care to use must result in torques that alternate between clockwise and counterclockwise!

Linking Rotational and Linear

Let’s do a sanity check on our definition of torque and its role in the rotational second law. We can do it very simply by choosing a single point mass tied to a string whose other end is held as a fixed pivot (we'll leave gravity out of this). We'll start with the linear version of Newton’s second law, and translate it into the rotational version.

Figure 5.4.3 – A Simple System Solved Two Ways



The forces in the x and y directions provide two equations through Newton's second law:

$$a_x = \frac{\sum F_x}{m} \quad \Rightarrow \quad a_{\parallel} = \frac{F\sin\theta}{m} \tag{5.4.7}$$

$$a_y = \frac{\sum F_y}{m} \quad \Rightarrow \quad a_{\perp} = \frac{T - F\cos\theta}{m} \tag{5.4.8}$$

Now we translate to rotational motion by first converting the parallel part of the acceleration into angular acceleration:

$$a_{\parallel} = R\alpha \tag{5.4.9}$$

Then convert mass into rotational inertia:

$$m = \frac{I}{R^2} \tag{5.4.10}$$

Plugging Equation 5.4.9 and Equation 5.4.10 into Equation 5.4.7 gives:

$$R\alpha = \frac{F\sin\theta}{\frac{I}{R^2}} \quad \Rightarrow \quad \alpha = \frac{FR\sin\theta}{I} = \frac{\tau}{I} \tag{5.4.11}$$

One important thing to note here is that while the torque and rotational inertia depend upon the pivot point (i.e. they are different values if we use a new reference point), the translation between the angular acceleration and linear acceleration exactly balances this difference. For example, if we replace the pivot defined above with a new one that is a distance $2R$ from the object, all of the math works out exactly the same. That is, the torque is twice as great and the rotational inertia is four times as great, resulting in a rotational acceleration that is half as large as before, but when it is multiplied by twice the radius to get the linear acceleration, the same result occurs, as it must.

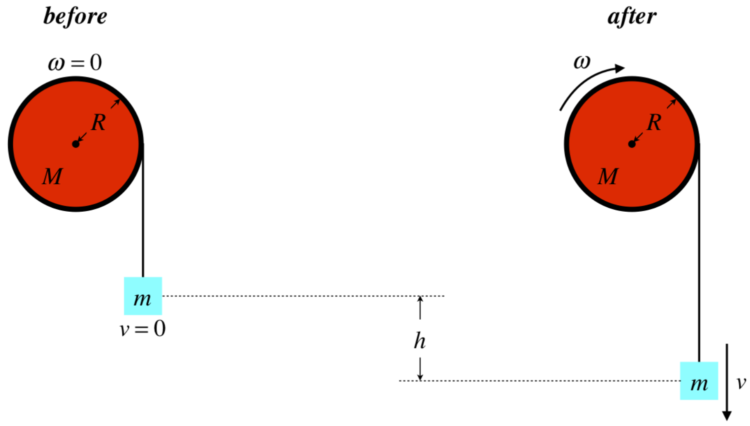
Solving Problems

Now we can do a whole set of problems involving torque causing rotational acceleration. There are many similarities with solving problems involving linear forces and accelerations, but here are some differences:

- Free-body diagrams now require that forces be placed appropriately on the objects, since torque depends upon force placement (no more using dots to represent the object).
- There usually is no need to resolve the torque vector into components. In fact, most problems can limit torque (and angular acceleration) to just "clockwise" and "counterclockwise" – the direction of the torque vector can be left until the end.
- One must either know or be able to calculate the rotational inertia of the object on which the torques acts.
- The perfect rolling condition is sometimes applied.

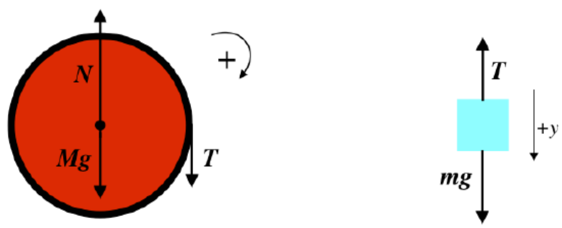
To get an idea of the process, we'll re-work the problem of the falling block unwinding a spool, this time using rotational second law instead of energy conservation:

Figure 5.4.4 – Falling Block Unwinds Spool (Redux)



Start with free-body diagrams:

Figure 5.4.5 – FBD's of Block and Spool



Next we need to right down the equations for Newton's second law for each object. The block is moving in a straight line, so we are already familiar with that one:

$$a_y = \frac{\sum F_y}{m} \quad \Rightarrow \quad a = \frac{mg - T}{m} = g - \frac{T}{m} \tag{5.4.12}$$

The spool is rotating, so we need to use the rotational version for it. Before we can sum the torques for the spool, we need to select a reference point, and its axle is a pretty obvious choice. The length of the position vector from this reference point to the where the gravity and normal forces act is zero, so those forces produce no torque around the axle (which makes sense – pushing on an axle should not cause something to spin around it). This leaves only the tension force. It acts tangent to the spool, so this force is perpendicular to the position vector connecting the pivot to the point where the force acts, which makes the magnitude of torque it produces equal to simply the product of the tension and the radius of the spool. The direction of this torque is positive, since it causes a clockwise acceleration and our FBD defines that as the positive direction. As this is the only torque, it is the net torque, and we have:

$$\alpha = \frac{\tau_{net}}{I} \quad \Rightarrow \quad \alpha = \frac{T \cdot R}{I} \tag{5.4.13}$$

Now we have to incorporate our constraints (our "other information"). We know that the spool is a uniform solid disk with mass M, giving us its rotational inertia. Also, we know that the rate at which the string exits the spool is related to the rotation rate of the spool according to the usual "no slipping" condition, so we have an equation relating the block's linear acceleration a to the spool's angular acceleration α:

$$I = \frac{1}{2}MR^2, \qquad \alpha = \frac{a}{R} \tag{5.4.14}$$

Putting these constraints into Equation 5.4.13 and combining this with Equation 5.4.12 gives:

$$\left.\begin{array}{l} \dfrac{a}{R} = \dfrac{T \cdot R}{\frac{1}{2}MR^2} \quad \Rightarrow \quad T = \dfrac{Ma}{2} \\ a = g - \dfrac{T}{m} \end{array}\right\} \quad \Rightarrow \quad a = \frac{2m}{2m+M}g \tag{5.4.15}$$

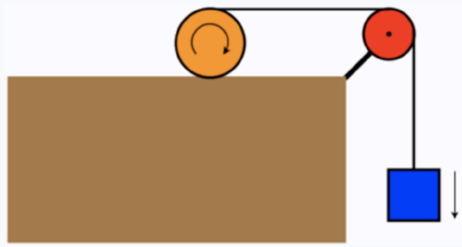
We see that the acceleration of the block is constant, so we can use a kinematics equation to determine the velocity after displacing a distance h from rest:

$$v_f^2 - v_o^2 = 2a\Delta y \quad \Rightarrow \quad v = \sqrt{\frac{4mgh}{2m+M}} \tag{5.4.16}$$

This agrees with our previous answer.

Analyze This (Again!)

Now that we have some new tools to work with, we can return to a physical system that we previously analyzed with energy conservation and re-analyze it using what we now know. As before, the spool rolls without slipping as the block descends, and here we will immediately assume that the mass of the block equals the mass of the spool.



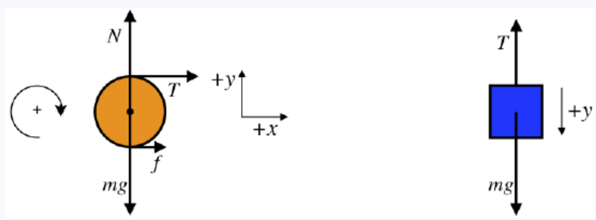
Analysis

We will only include in this analysis information we did not already obtain when we looked at this case previously. Since we are now interested in the effects of individual forces on linear and rotational motion, we can point out that rolling without slipping is only possible if static friction is acting between the horizontal surface and the spool. This introduces a limiting factor on the friction force, based on the coefficient of friction and the normal force between the surfaces. In particular, the maximum static friction force is given by:

$$f_{max} = \mu_s N = \mu_s mg$$

The final equality here comes from the fact that the surface is horizontal, so the normal force and gravitational forces must be exactly equal, with zero vertical acceleration.

Okay, now let's tackle the equations that come from Newton's second law. We of course start with force diagrams:



You might ask how we know that the friction force points in the direction indicated in the diagram. Technically, we don't yet know this, but we don't have to. If, in the course of our calculations, we find that the only way a solution can work out is if the value of f is negative, then the friction force must point the other way. We will see shortly that the direction on the diagram is in fact the only direction it can point.

Remembering from the previous time we analyzed this that the block at all times moves twice as fast as the spool (and therefore accelerates twice as much), there are three equations that come from Newton's second law for the cylinder (the horizontal and vertical linear net force equations, and the net torque equation), and there is one equation that comes out for the block:

$$\begin{array}{lll} & cylinder & block \\ x-direction: & a = \dfrac{T+f}{m} & \\ y-direction: & 0 = N - mg & 2a = \dfrac{mg-T}{m} \\ torques: & \alpha = \dfrac{TR-fR}{I} & \end{array}$$

Plugging in for the rotational inertia and the angular acceleration gives:

$$\frac{a}{R} = \frac{TR-fR}{\frac{1}{2}mR^2} \quad \Rightarrow \quad \frac{a}{2} = \frac{T-f}{m}$$

Adding this equation to the x-direction equation for the cylinder gives:

$$\frac{3}{2}a = \frac{T+f}{m} + \frac{T-f}{m} \quad \Rightarrow \quad T = \frac{3}{4}ma$$

Now combine this result with the y-direction equation for the block to get:

$$2a = \frac{mg - \frac{3}{4}ma}{m} \quad \Rightarrow \quad a = \frac{4}{11}g$$

In our previous analysis, if the masses of the spool and block were equal, we would find that after the block falls a distance h, the final velocity of the cylinder is:

$$v_c = \sqrt{\frac{4}{11}gh}$$

From the kinematic equation we have:

$$v_f^2 - v_o^2 = 2a\Delta x \quad \Rightarrow \quad v_c = \sqrt{2a\Delta x}$$

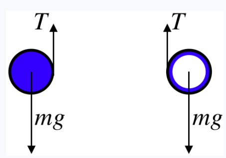
It appears at first blush that if we plug in the acceleration found above, that it disagrees with this answer by a factor of two within the radical. But there is one other thing to remember: The quantity Δx is the displacement of the accelerated object (in this case, the cylinder), but the drop of the block h is twice as great as the displacement of the cylinder, so putting in $\Delta x = \frac{1}{2}h$ confirms the answer.

Analyze This

Two ends of a massless rope are wound around two spools with equal masses and radii. One of the spools is a solid, uniform disk, while the other is a thin, hollow cylinder. The rope between them goes over a massless, frictionless pulley in a vertical plane. The spools are released from rest from the same height, and the rope does not slip over the pulley.

Analysis

There are a lot of potentially "moving parts" here (both spools and the pulley are free to turn), and keeping the motion constraints straight can be a bit daunting. At first blush, one might think that the solid spool will unwind faster than the hollow one, causing it to fall faster, just as it rolls faster down a slope. But making such generalizations is dangerous without careful analysis, so let's forge ahead with that, starting as we so often do, with free-body diagrams. The pulley is massless, which means the tension in the rope is the same on both sides, and since the pulley is held in place by the axle, a free-body diagram of it will not prove particularly useful. The two spools are another matter. Calling their common mass m, we get:



Well this is interesting – both spools have the same FBD's! With the same mass, and same starting conditions (starting at rest), Newton's 2nd law for linear motion ensures that they will have identical motions. Therefore they must remain side-by-side as they fall.

But wait, we know that both spools also experience the same torques about their centers, because the only force that exerts any torque about the centers is the tension, and this is the same for both spools, as is the radius of each spool. But equal torques will not result in equal angular accelerations, because they have different moments of inertia – the hollow spool (with the higher moment of inertia) will have to be unwinding slower than the solid spool at any given moment in time. How is this possible, if they have the same radius and are falling linearly at the same rate?

The answer is that the rope is moving! The solid spool is giving up rope faster than the hollow one, but the amount of rope between each of these and the pulley is the same. So some of the rope given up by the solid spool must be passing over the pulley to the other side – the pulley is rotating clockwise.

We have already gained a lot of insight into the physics of this situation, but the power of analysis is not to be underestimated - let's see if we can take it further...

First, let's apply Newton's 2nd law to the FBDs above:

$$mg - T = ma$$

Next we need to think about the rope constraints. We'll call the radius of the pulley R and the radii of the spools r. When the solid spool unwinds and angle $\Delta\theta_{solid}$, it gives up an amount of rope equal to $r\Delta\theta_{solid}$. Similarly, for the hollow spool, we know that it gives up a length of rope equal to $r\Delta\theta_{hollow}$. The sum of these quantities is the additional rope put into the system. The spools both drop equal distances at the same time, so on each side of the pulley, the rope gets longer by half this total:

$$\Delta y = \frac{1}{2}(r\Delta\theta_{solid} + r\Delta\theta_{hollow})$$

Two time derivatives of the y value is the linear acceleration of the spools (the same acceleration that appears in the first equation above). Two derivatives of the angles the spools rotate through are the angular accelerations of the spools, so we have:

$$a = \frac{1}{2}r\left(\alpha_{solid} + \alpha_{hollow}\right)$$

These angular accelerations come from torques exerted on the spools about their centers by the tension force (the gravity force acts through their centers, so it provides no torque about that axis). So from Newton's 2nd Law for rotational motion, and using what we know about their moments of inertia, we get:

$$\alpha = \frac{\tau}{I} \quad \Rightarrow \quad \begin{cases} \alpha_{solid} = \frac{Tr}{\frac{1}{2}mr^2} = \frac{2T}{mr} \\ \alpha_{hollow} = \frac{Tr}{mr^2} = \frac{T}{mr} \end{cases}$$

Plugging these values back in above gives:

$$a = \frac{1}{2}r\left(\frac{2T}{mr} + \frac{T}{mr}\right) = \frac{3T}{2m} \quad \Rightarrow \quad T = \frac{2}{3}ma$$

And using this in the first equation gives us the exact linear acceleration of the falling spools:

$$mg - \frac{2}{3}ma = ma \quad \Rightarrow \quad a = \frac{3}{5}g$$

Remarkable that we know the exact numerical answer without knowing the mass of a spool or the radii of the spools or the pulley.

Rotational Work

We have now discussed the rotational version of energy conservation and Newton's second law, so the link between these two topics – the work-energy theorem – should follow naturally. Rather than provide a derivation (which would really just resemble what we have done before for the linear case), we'll just write down the answer that makes sense from following our linear/rotational parallel.

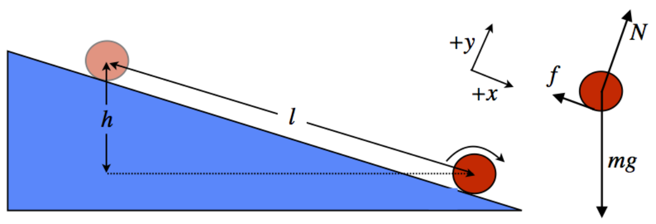
$$\begin{array}{llll} W_{A\rightarrow B}\left(linear\right) & = \int\limits_A^B \overrightarrow{F} \cdot \overrightarrow{dl} & = \Delta KE & = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \\ W_{A\rightarrow B}\left(rotational\right) & = \int\limits_A^B \overrightarrow{\tau} \cdot \overrightarrow{d\theta} & = \Delta KE & = \frac{1}{2}I\omega_B^2 - \frac{1}{2}I\omega_A^2 \end{array} \tag{5.4.17}$$

If we were so inclined, we could do the same unwinding-the-spool problem for a third time, this time with the rotational work-energy theorem. The approach looks slightly different, but when you actually sit down to do it, you see the same things come out of it as before. This time instead of relating the accelerations, we would relate the distance the mass drops to the angle the spool rotates.

Back when we discussed objects rolling down an inclined plane without slipping, we avoided talking about one potentially confusing point that we are now equipped to deal with. For a ball or cylinder to roll down, there has to be a friction force (otherwise it would merely slide). This friction force can only be static friction, because we are assuming there is no slipping, and we said that without any rubbing, the mechanical energy must be conserved. But this friction force acts *up the plane while the object moves down it*, which means that it does negative work on the object. This would seem to imply that mechanical energy should not be conserved, so how were we able to make the assumption that it is conserved?

The answer is, "Because the static friction force also does *positive* rotational work which adds energy to the object in rotational form, and this addition exactly balances the loss in linear form." This is not hard to prove. Start with a diagram and a FBD:

Figure 5.4.6 – Work Done on Cylinder by Static Friction as It Rolls Down Plane



Computing the work done by static friction for linear motion is very simple, since the friction force is constant and the motion is in a straight line:

$$W\left(x_1 \rightarrow x_2\right) = \int\limits_{x_1}^{x_2} \overrightarrow{F} \cdot \overrightarrow{dx} = -f \cdot l \tag{5.4.18}$$

As expected, this work takes energy out of the cylinder system. Next we compute the rotational work done on the cylinder. The torque is a constant equal to fR, and is acting in the same direction as the rotational displacement, so

$$W\left(\theta_1 \rightarrow \theta_2\right) = \int_{\theta_1}^{\theta_2} \overrightarrow{\tau} \cdot \overrightarrow{d\theta} = +(fR) \cdot \theta \tag{5.4.19}$$

Putting these together gives us the total work done on the cylinder by the static friction force. Note that since it rolls without slipping, the linear distance it travels is related to the angle through which it rotates by the usual relation:

$$W_{static\ friction} = W\left(x_1 \rightarrow x_2\right) + W\left(\theta_1 \rightarrow \theta_2\right) = -f \cdot l + fR\theta = f\left(-l + R\theta\right) = 0 \tag{5.4.20}$$

So we see that in fact the work done by static friction here only serves to convert linear kinetic energy into rotational kinetic energy, and our understanding of how thermal energy is generated remains intact.

Rotational Power

We spoke before about how sometimes we are interested in the rate at which work is done, calling this value "power." Well, as with everything else we studied in linear motion, there is of course a rotational version:

$$P = \frac{dW}{dt} = \overrightarrow{\tau} \cdot \overrightarrow{\omega} \tag{5.4.21}$$

You sometimes hear the silly "debate" about torque vs. horsepower for car & truck engines. This should make it clear what the difference is. Power delivered to the wheels is directly related to torque exerted on them, but it is dependent upon how fast they are turning. Engines that can still produce a lot of torque at high speeds are powerful. To get an idea of why it might be hard to maintain torque at high speeds, imagine pedaling a bike downhill – when you get going fast enough, it's difficult to push hard on (provide torque to) your pedals. So generally the effectiveness of an engine is defined by torque at low speeds and power at high speeds. If you want fast acceleration off the line or the ability to pull a stump out of the ground, you want torque. If you want to go fast or tow a heavy trailer up a hill at a steady speed, you want power.
