

6.3: Applications of Angular Momentum Conservation

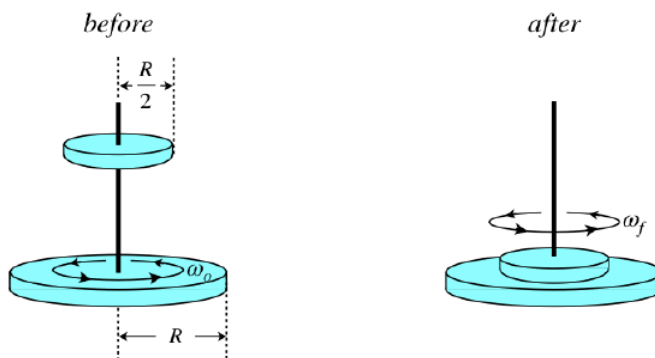
If we continue to follow the trail we blazed in linear motion, our next step is to consider what happens when we choose a system for which there are no external rotational impulses. For such a system, we can declare the angular momentum to be conserved before and after any event, however complicated the internal interactions might be.

In the linear case, we saw that the primary application of momentum conservation was related to collisions, because it was useful to ignore the complicated forces that come about between the colliding objects. What sorts of problems might angular momentum conservation be useful for solving? There are actually three basic varieties that commonly arise in classical mechanics, and we will look each one in turn.

Spinning Collisions

Two uniform solid disks with small holes in their centers, are threaded onto the same frictionless vertical cylindrical rod. One of the disks lies flat on a frictionless horizontal surface and is rotating at a speed ω_o around the rod, while the other disk is held at rest directly above it. Both disks are made from the same material, and have the same thickness, but the spinning disk has twice the radius of the stationary disk. The smaller disk is then dropped on top of the larger one, and after a short time the kinetic friction force between the two disks brings them both to the same rotational speed, which is a fraction of the larger disk's original speed. Find this fraction, and the fraction of the original kinetic energy still left the system afterward (it loses some from work done by kinetic friction).

Figure 6.3.1 – Rotating Disk Inelastic Collision



This is clearly the rotational version of a perfectly inelastic collision, as both of the objects end up moving together. We solve it in the same way that we solve the linear counterpart – by noting that the only torques involved are internal to the two-disk system, which means that the total angular momentum is the same before and after the collision.

$$\left. \begin{array}{l} \text{before: } L_{tot} = I_1 \omega_o + I_2 (0) \\ \text{after: } L_{tot} = (I_1 + I_2) \omega_f \end{array} \right\} \Rightarrow \omega_f = \frac{I_1}{I_1 + I_2} \omega_o \quad (6.3.1)$$

Because it has the same thickness and is made from the same material, the ratio of the two disks' masses equals the ratio of their areas. With one-half the radius, the smaller disk therefore has one-fourth the mass, and we get:

$$I_1 = \frac{1}{2} M R^2 \Rightarrow I_2 = \frac{1}{2} \left(\frac{1}{4} M \right) \left(\frac{1}{2} R \right)^2 = \frac{1}{16} I_1 \Rightarrow \omega_f = \frac{I_1}{I_1 + \frac{1}{16} I_1} \omega_o = \frac{16}{17} \omega_o \quad (6.3.2)$$

Now for the fraction of kinetic energy leftover:

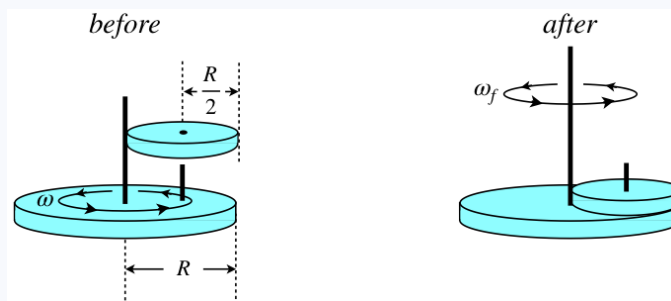
$$\left. \begin{array}{l} KE_o = \frac{1}{2} I_1 \omega_o^2 \\ KE_f = \frac{1}{2} (I_1 + I_2) \omega_f^2 \end{array} \right\} \Rightarrow \frac{KE_f}{KE_o} = \frac{\frac{1}{2} (I_1 + I_2) \omega_f^2}{\frac{1}{2} I_1 \omega_o^2} = \frac{\left(\frac{17}{16} I_1 \right) \left(\frac{16}{17} \omega_o \right)^2}{I_1 \omega_o^2} = \frac{16}{17} \quad (6.3.3)$$

We can actually achieve this last answer even more easily using $L_o = L_f$:

$$\frac{KE_f}{KE_o} = \frac{\frac{1}{2} I_f \omega_f^2}{\frac{1}{2} I_o \omega_o^2} = \frac{I_o}{I_f} \frac{(I_f \omega_f)^2}{(I_o \omega_o)^2} = \frac{I_o}{I_f} \frac{L_f^2}{L_o^2} = \frac{I_1}{I_1 + I_2} \quad (6.3.4)$$

Analyze This

A large uniform-density disk of radius R rotates in a horizontal plane around a frictionless axle with a rotational speed ω . This disk includes a vertical post located a distance half its radius from its axle, and onto this axle is placed (very suddenly) a second disk with half the radius of the larger disk. The second disk is made of the same uniform material as the larger disk, and has the same thickness.



Analysis

The angular momentum about the axle of the two-disk system is conserved, as there are no external torques introduced. The starting angular momentum is clear: The smaller disk starts with zero angular momentum, and the larger disk's angular momentum is:

$$L_{\text{before}} = L_{\text{large}} + L_{\text{small}} = I_{\text{large}}\omega + 0 = \frac{1}{2}MR^2\omega$$

Expressing the angular momentum after depends upon whether the two disks become a single "rigid body". A rigid body is an object for which all of its parts remain at their same relative positions. If there is no friction between the two disks, then as the large disk rotates, the smaller disk retains its same orientation relative to the Earth. This is because without friction, the large disk can only affect the motion of the smaller disk through its interaction at the axle, and this cannot produce any torque with which to start the smaller disk rotating. On the other hand, if there is friction between the two disks, then eventually they have no relative sliding, and the two are effectively behaving like a single rigid body. Let's address each case separately...

no friction

In this case, after the small disk is dropped onto the axle, it has only orbital angular momentum, as it does not spin on its axis. If we call its mass m , then this orbital angular momentum is found using its speed and the radius of its orbit, $r = \frac{R}{2}$:

$$L_{\text{small}} = mv_{\perp}r = m\omega_f\left(\frac{R}{2}\right)^2 = \frac{1}{4}mR^2\omega_f$$

With half the radius, the area of the smaller disk is one quarter the area of the larger disk, which means it has one quarter the mass of the larger disk. Putting $m = \frac{1}{4}M$ into the angular momentum of the smaller disk and adding it to the angular momentum of the larger disk gives the "after" angular momentum:

$$L_{\text{after}} = \frac{1}{2}MR^2\omega_f + \frac{1}{4}\left(\frac{1}{4}M\right)R^2\omega_f = \frac{9}{16}MR^2\omega_f$$

Invoking angular momentum conservation gives the final angular velocity in terms of the initial angular velocity:

$$L_{\text{after}} = L_{\text{before}} \Rightarrow \omega_f = \frac{8}{9}\omega$$

friction

If the two disks experience kinetic friction after they make contact, and we wait until they stop skidding across each other, then the "after" situation is a rigid body. We can solve this in two equivalent ways: 1. We can compute the moment of inertia of the two-disk system using the parallel axis theorem and the additivity property, then multiply this by the final angular speed to get the final angular momentum. 2. We can continue to treat the disks separately, and add-in the spinning portion of the smaller disk's angular momentum to the "after" tally. We will do method #2 here.

Note that once the disks are no longer sliding, the rotation rate of the smaller disk about its axis in the Earth frame is the same rate as the larger disk – it reaches the same orientation relative to the Earth that it started with after one full revolution. The spin portion of the angular momentum is therefore the moment of inertia of the small disk about its center times the final angular speed:

$$L_{spin} = I_{center} \omega_f = \frac{1}{2} m r^2 \omega_f = \frac{1}{2} \left(\frac{1}{4} M \right) \left(\frac{1}{2} R \right)^2 \omega_f = \frac{1}{32} M R^2 \omega_f$$

Adding this into the L_{after} and recomputing the final rotational speed gives:

$$L_{after} = \frac{19}{32} M R^2 \omega_f \Rightarrow \omega_f = \frac{16}{19} \omega$$

Changing Rotational Inertia

A child stands on the outer edge of a merry-go-round, which is spinning around a fixed axle on a horizontal frictionless surface. The merry-go-round is a solid, uniform disk with ten times the mass of the child, and is spinning at a rotational speed ω_o . The child then slowly walks to the center of the merry-go-round. What, if anything, happens to the rotational speed of the merry-go-round?

Figure 6.3.2 – System Changes Rotational Inertia While Rotating



Before we invoke angular momentum conservation and launch into the mathematics, it might help to think about this in a "less evolved" manner – let's think about the *internal* interactions in the child + merry-go-round system. When the child takes a step, toward the center, they are moving from a faster moving part of the merry-go-round to a slower part. This means that the merry-go-round will exert a static friction force on the feet of the child tangent to the circular motion, acting to slow them down. There is, of course, a Newton's third law pair friction force on the merry-go-round by the feet of the child in the opposite direction, which results in a torque that acts to speed up the merry-go-round's rotation. So we would expect the linear speed of the child to slow with every step, as the merry-go-round's rotational speed increases. The details of these changes are hard to work out using the details of the interaction, so now we turn to momentum conservation, which we know holds because the only forces/torques acting are internal to the system.

Calling the mass of the child m and the radius of the merry-go-round R , we can write down the angular momentum referenced at the axis of the merry-go-round before and after, and invoking angular momentum conservation makes the rest easy:

$$\left. \begin{array}{l} \text{before: } L_{tot} = [I_{child} + I_{mgr}] \omega_o = \left[mR^2 + \frac{1}{2}(10m) R^2 \right] \omega_o = 6mR^2 \omega_o \\ \text{after: } L_{tot} = [I_{child} + I_{mgr}] \omega_o = \left[0 + \frac{1}{2}(10m) R^2 \right] \omega_o = 5mR^2 \omega_f \end{array} \right\} \Rightarrow \omega_f = \frac{6}{5} \omega_o \quad (6.3.5)$$

The rotation rate of the merry-go-round increases by 20%. It's interesting to note that there is no physical equivalent of this phenomenon in linear mechanics. That is, we don't see closed systems losing linear inertia (mass) and maintaining their momentum by compensating with a larger linear velocity.

It's also interesting to consider what happens to the kinetic energy of the system during this process. Like kinetic energy for linear motion, we can write it in terms of the momentum and inertia:

$$KE = \frac{p^2}{2m} \iff KE = \frac{L^2}{2I} \quad (6.3.6)$$

Given that the angular momentum doesn't change, the kinetic energy goes up in the same proportion that the rotational inertia goes down. Where does this increase in kinetic energy come from? Where is work done? When the child just stands at the edge of the merry-go-round, the static friction force acts toward the center of rotation, but it does no work, because it is acting perpendicular to the direction of the child's motion. But as the child starts moving inward, this static friction is doing work. In the end, the kinetic

energy of the merry-go-round equals its starting kinetic energy, plus the starting kinetic energy of the child, plus the work done by the static friction force.

It turns out that showing this for this case requires fancier integration to calculate the work than we want to do here, so let's try a simpler example. Let's let the mass of the merry-go-round be negligible compared to the mass of the child. Furthermore, we'll have the child walk halfway to the center of rotation (we can't let the child walk all the way in, or the massless merry-go-round will be spinning infinitely fast!).

First, let's compute the kinetic energy change of the system using angular momentum conservation (note that the merry-go-round doesn't contribute at all now, making things significantly easier):

$$L_{\text{before}} = L_{\text{after}} \Rightarrow I_o \omega_o = I_f \omega_f \Rightarrow mR^2 \omega_o = m \left(\frac{R}{2} \right)^2 \omega_f \Rightarrow \omega_f = 4\omega_o \quad (6.3.7)$$

As we saw above, the proportional increase in kinetic energy is the same as that of the rotational velocity, so the kinetic energy increase of the system is:

$$\Delta KE = KE_f - KE_o = 4KE_o - KE_o = \frac{3}{2} mR^2 \omega_o^2 \quad (6.3.8)$$

Okay, now let's see if we can calculate the work done by the static friction force. The force that keeps the child going in a circle equals the mass of the child multiplied by the centripetal acceleration, so a force barely exceeding this amount will get the child moving inward. We don't want the child to accelerate appreciably in the radial direction (the child stops at the new radius and it doesn't matter how long it takes to get there), so we can use this as the force that is doing work. The only trouble is, this force changes as the child moves inward, because the rotational speed and distance from the center are changing all the way. We can determine how the rotational speed varies with the child's distance from the center using angular momentum conservation, which allows us to write the force as a function of r as follows:

$$\left. \begin{aligned} F &= ma_c & \Rightarrow & F(r) = m[\omega(r)]^2 r \\ mr^2 \omega(r) &= mR^2 \omega_o & \Rightarrow & \omega(r) = \frac{R^2}{r^2} \omega_o \end{aligned} \right\} \Rightarrow F(r) = \frac{mR^4 \omega_o^2}{r^3} \quad (6.3.9)$$

Now we have only to do the work integral. The displacement is toward the center (r is getting smaller), so $dl = -dr$, and the force is in the direction of displacement, so $\vec{F} \cdot \vec{dl} = Fdl = -Fdr$. And the limits of integration are from $r = R$ to $r = \frac{R}{2}$:

$$W \left(R \rightarrow \frac{R}{2} \right) = \int_R^{\frac{R}{2}} -F(r) dr = -mR^4 \omega_o^2 \int_R^{\frac{R}{2}} \frac{dr}{r^3} = -mR^4 \omega_o^2 \left[-\frac{1}{2r^2} \right]_R^{\frac{R}{2}} = \frac{3}{2} mR^2 \omega_o^2 \quad (6.3.10)$$

Comparing this with Equation 6.3.8, we see that the work done in moving the child inward is precisely equal to the change in the system's kinetic energy.

Conceptual Question

A glass of ice water rests on the outer edge of a solid, uniform, rotating disk, which is spinning horizontally around its frictionless axle. The glass is a cylinder with a mass equal to one half the mass of the disk, and a radius that is one third the radius of the disk. Condensation at the bottom of the glass causes the coefficient of static friction on the top of the disk to go down, and the glass suddenly slides off. After this occurs, the rotational speed of the disk:

- slows down
- speeds up
- stays the same
- There is no way to tell from the information given

Solution

(c) Don't be fooled by all the details given – the rotational speed of the disk doesn't change! When the static friction force goes away, the system continues with its angular momentum, but you can't erase the glass of water from the system just because it slid away. Yes, the rotational inertia of what is going around the axis has changed, but as the glass slides away, it still has angular momentum (it will be a combination of its rotation about its center and movement relative to the axle, see

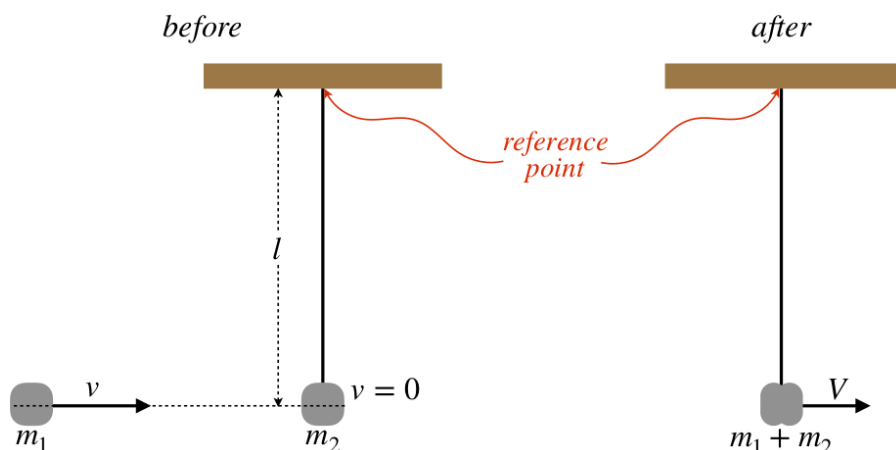
Equation 6.1.13 and Figure 6.1.2). In fact, the glass will continue to have the same angular momentum that it had right before it started sliding, since there is no net torque on it. If the whole system maintains its angular momentum, and the glass keeps the same angular momentum, then the disk must as well – it doesn't change speed at all.

Off-Center Collisions

Of all the problems that are solvable with angular momentum conservation, those that fall into the category of "off-center collisions" are the most interesting and complex. One reason is that unlike the cases of spinning collisions and changing rotational inertia, off-center collision problems often see cameo appearances from linear momentum conservation. Additionally, the fate of the system's mechanical energy becomes more interesting.

We begin with a problem that we are already familiar with from Section 4.6 – the ballistic pendulum. We were able to solve that problem by first solving the perfectly inelastic collision of the bullet & block to get their combined velocity, after which we used mechanical energy conservation to get the height to which the bullet & block swing. We will be more careful about extension in space (and the implications to rotational inertia) by replacing the bullet & block with two small clay balls that stick together. Also, we will not bother to look at the second half of the problem where the pendulum swings up, as the mechanical energy conservation portion of the problem is unchanged.

Figure 6.3.3 – Ballistic Pendulum with Two Small Clay Balls



If we choose the position where the string is attached to the ceiling as a reference point, we note that at the moment of the collision, the gravity and tension forces both act through the reference point, which means that there are no external torques on the system. The bullet and block exert torques on each other, but those are internal and cancel each other. Therefore, as an alternative to using linear momentum conservation, we can use angular momentum conservation.

Before the collision, the pendulum (the length of which we will call l) has no angular momentum relative to the reference point, but the bullet does, according to Equation 6.1.6:

$$L_{\text{before}} = m_1 v l \quad (6.3.11)$$

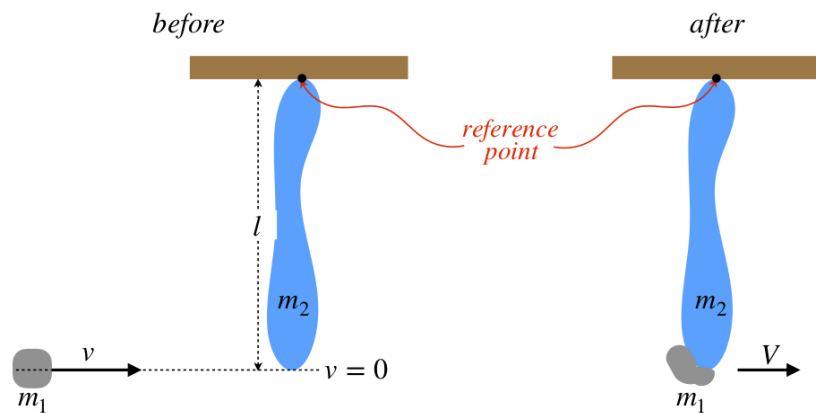
After the collision, the pendulum is rotating, and has a rotational inertia around the reference point, resulting in a final angular momentum of:

$$L_{\text{after}} = I\omega = [(m_1 + m_2) l^2] \left[\frac{V}{l} \right] = (m_1 + m_2) V l \quad (6.3.12)$$

And setting the initial angular momentum equal to the final gives the same result as when we used linear momentum (Equation 4.6.1).

Using angular momentum conservation is no longer optional – it is a requirement – when the target is not just a small ball at the end of a string, but is an extended object with a rotational inertia.

Figure 6.3.4 – Ballistic Pendulum with One Clay Ball



We know that the rotational inertia for this target around the reference point is less than it was when the target was a clay ball, since some of its mass is closer to the reference point. We will write the rotational inertia as some unknown fraction β multiplied by the rotational inertia of a small ball at the end of a string:

$$I_{\text{target}} = \beta m_2 l^2 \quad (6.3.13)$$

For example, if this target is a uniform thin rod, then Equation 5.2.7 applies, and $\beta = \frac{1}{3}$, or if the target is a uniform disk or cylinder pivoted about an axis perpendicular to its flat side and about its edge, then Equation 5.2.23 applies, with $R = \frac{1}{2}l$, giving $\beta = \frac{3}{8}$, and so on.

Applying angular momentum as we did above, we can find the final speed of the blob of clay and/or the rotational speed of the pendulum. The initial angular momentum is the same as before, so:

$$\begin{aligned} L_{\text{after}} = I\omega &= [m_1 l^2 + \beta m_2 l^2] \left[\frac{V}{l} \right] = (m_1 + \beta m_2) V l \Rightarrow V = \frac{m_1}{m_1 + \beta m_2} v \Rightarrow \omega = \frac{V}{l} \\ &= \left(\frac{m_1}{m_1 + \beta m_2} \right) \frac{v}{l} \end{aligned} \quad (6.3.14)$$

The claim was made above that we no longer have the option of using linear momentum conservation for this problem. Before we see *why* this must be true, let's show that it is true for the specific case of a uniform thin rod that has the same mass as the clay ($m_1 = m_2$). If we use linear momentum conservation, then when the clay is stuck on the end of the rod, the center of mass velocity of the rod + clay system is:

$$m_1 v = (m_1 + m_2) v_{\text{cm}} \Rightarrow v_{\text{cm}} = \frac{1}{2} v \quad (6.3.15)$$

The center of mass of the rod + clay system is halfway between the center of mass of the rod and the position of the clay, so it is a distance of $\frac{3}{4}l$ from the reference point. With a linear speed of $\frac{1}{2}v$, we get that the pendulum should have a rotational speed of:

$$\omega = \frac{\frac{1}{2}v}{\frac{3}{4}l} = \frac{2}{3} \frac{v}{l} \quad (6.3.16)$$

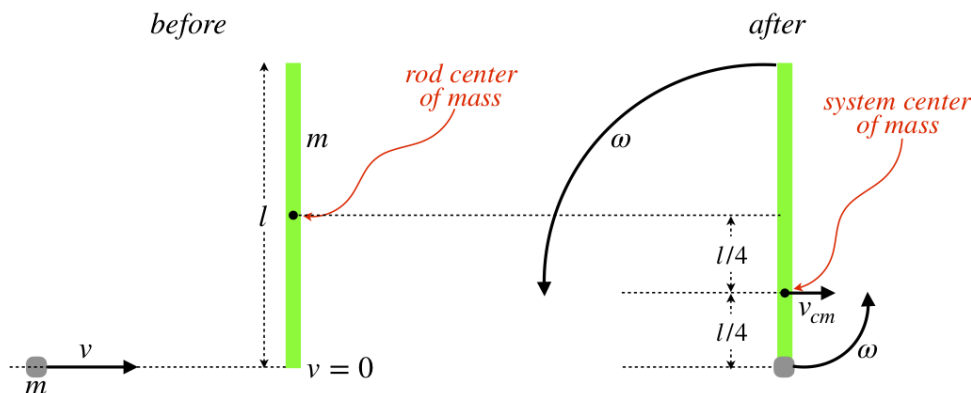
Let's check to see if this is right by plugging $\beta = \frac{1}{3}$ (for a thin rod rotated around its end) and $m_1 = m_2$ into Equation 6.3.14:

$$\omega = \left(\frac{m}{m + \frac{1}{3}m} \right) \frac{v}{l} = \frac{3}{4} \frac{v}{l} \quad (6.3.17)$$

So we see that using linear momentum conservation does not agree with using angular momentum conservation in this case. The reason is the presence of the pivot. The pivot will never exert a torque on the rod relative to the reference point, but it *will* exert a force on it, thereby ruining linear momentum conservation. But this brings up another puzzle: Whatever force the pivot exerts, it causes the speed of the center of mass to be *greater* after the collision than we found for conserved linear momentum, so the force on the rod by the pivot must be *forward*. That doesn't sound right – doesn't the pivot slow down the rod? To solve this puzzle, we get to look at yet another case – an off-center collision with no fixed pivot.

Let's do the same clay-hits-end-of-uniform-thin-rod-with-same-mass problem as above, this time free of any pivot (we'll also assume no gravity is present). First of all, we know that without a force coming from the pivot, the result we obtained in Equation 6.3.15 must be correct, as linear momentum must be conserved. Also we know that after the collision, with no forces on the clay + rod system, it must rotate around its center of mass. This calls for a fresh new diagram:

Figure 6.3.5 – Off-Center Perfectly Inelastic Collision



With the rod rotating counterclockwise, the bottom of the rod must be moving forward faster than the system's center of mass, while the top of the rod might actually be moving *backward*, depending upon the values of ω , l and v_{cm} . If this turns out to be the case, then it makes sense that a pivot could push the rod forward upon impact – the rod's rotation is fast enough compared to its linear motion that the top "tries" to move backward, but is prevented from doing so by a forward push from the pivot. Okay, let's see if this is the case mathematically.

At the moment of the impact, we have:

$$\begin{aligned} v_{bottom} &= v_{cm} + r\omega = \frac{1}{2}v + \frac{1}{4}l\omega \\ v_{top} &= v_{cm} - R\omega = \frac{1}{2}v - \frac{3}{4}l\omega \end{aligned} \quad (6.3.18)$$

Now we need to come up with ω . Even though linear momentum is conserved in this case, Equation 6.3.16 still isn't correct, as it assumes that the top end of the rod is held fixed. We need to use angular momentum conservation. Without a fixed pivot, what do we use as a reference point? The answer is *anywhere* – the angular momentum is conserved relative to every reference point! However, if we are carefree about this choice, we have to be extra careful when adding up the angular momentum after the collision. In the case of a fixed pivot, it was easy because we were able to use the rotational inertia around that point. When we have no fixed point on the object, we have to use Equation 6.1.13. So why not use the center of mass of the clay + rod system at the time of collision as the reference point, and get rid of that pesky second term from Equation 6.1.13?

$$\begin{aligned} L_{before} &= mvr_{\perp} \\ L_{after} &= I_{cm}\omega \end{aligned} \quad (6.3.19)$$

It's clear from the diagram that r_{\perp} is $\frac{1}{4}l$, but we need to do a little bit of work to determine the rotational inertia of the system around its center of mass. This will be the sum of the rotational inertia of the point mass clay and the rotational inertia of the rod about its center (Equation 5.2.19), offset (using the parallel-axis theorem) by $d = \frac{1}{4}l$:

$$I_{cm} = I_{clay} + I_{rod} = \left[m \left(\frac{1}{4}l \right)^2 \right] + \left[\frac{1}{12}ml^2 + m \left(\frac{1}{4}l \right)^2 \right] = \frac{5}{24}ml^2 \quad (6.3.20)$$

Invoking angular momentum conservation and plugging in for r_{\perp} and I_{cm} gives:

$$mvr_{\perp} = I_{cm}\omega \Rightarrow mv \left(\frac{1}{4}l \right) = \frac{5}{24}ml^2\omega \Rightarrow l\omega = \frac{6}{5}v \quad (6.3.21)$$

Plugging this back into Equation 6.3.18 confirms what we suspected, that without the top end fixed, its initial motion after the collision is *backward*, which is why the force by the pivot must be forward when it is attached:

$$\begin{aligned} v_{bottom} &= \frac{1}{2}v + \frac{1}{4}\left(\frac{6}{5}v\right) = \frac{4}{5}v \\ v_{top} &= \frac{1}{2}v - \frac{3}{4}\left(\frac{6}{5}v\right) = -\frac{2}{5}v \end{aligned} \quad (6.3.22)$$

This has been a long journey through off-center collisions, but we have one more stop – the fate of the system's mechanical energy. We explored perfectly inelastic head-on collisions in [Section 4.5](#), and found a simple relation between the starting and ending kinetic energies of the system – [Equation 4.5.7](#). Given that the final speed of the center of mass of the system has to be the same regardless of whether the collision is head-on or off-center, [Equation 4.5.7](#) clearly cannot work for off-center collisions, as these result in rotations, and as we know, the total kinetic energy is the sum of linear and rotational parts. This means that perfectly inelastic collisions that occur off-center do not lose as much mechanical energy as perfectly inelastic head-on collisions.

This actually makes some intuitive sense. Let's take as an example a bullet digging into a block of wood. The bullet is subject to a non-conservative force that does enough work to slow the bullet to the same speed as the region of the block of wood it is entering (i.e. the bullet stops inside the block). Now let's assume that the force exerted on the bullet by the wood is the same wherever it enters the wood (it is something like " $\mu_k N$," where the normal force is the wood squeezing the bullet). Whether the bullet enters the block at its center of mass or at its edge, the center of mass of the block reaches the same final speed – we'll call the moment when this final speed is reached " t_o ." If the bullet hits the center of mass, at t_o the bullet will have slowed to the same speed as the final speed of the center of mass. If the bullet hits the outer edge of the block and makes it spin, then the bullet is not slowed as much at t_o , because the edge of the block is moving faster than the final speed of center of mass of the block. If the bullet isn't slowed as much when it hits the edge, then not as much work is done on it (smaller change in its kinetic energy) by the non-conservative force, and less mechanical energy is converted to thermal.

Let's compute the fraction of kinetic energy that remains for the case above and compare it to the result if the collision occurs at the center of mass.

$$\begin{aligned} \text{collides with center:} \quad \frac{KE_f}{KE_o} &= \frac{m_1}{m_1 + m_2} = \frac{m}{m + m} = \frac{1}{2} \\ \text{collides with end:} \quad \frac{KE_f}{KE_o} &= \frac{\frac{1}{2}(2m)v_{cm}^2 + \frac{1}{2}I_{cm}\omega^2}{\frac{1}{2}mv^2} = \frac{m\left(\frac{1}{2}v\right)^2 + \frac{1}{2}\left(\frac{5}{24}ml^2\right)\left(\frac{6}{5}\frac{v}{l}\right)^2}{\frac{1}{2}mv^2} = \frac{4}{5} \end{aligned} \quad (6.3.23)$$

As you can see, less energy is lost when the clay sticks to the end and spins the rod than when it hits the center and doesn't spin it. Here is a nice demonstration of this phenomenon. First the puzzle:



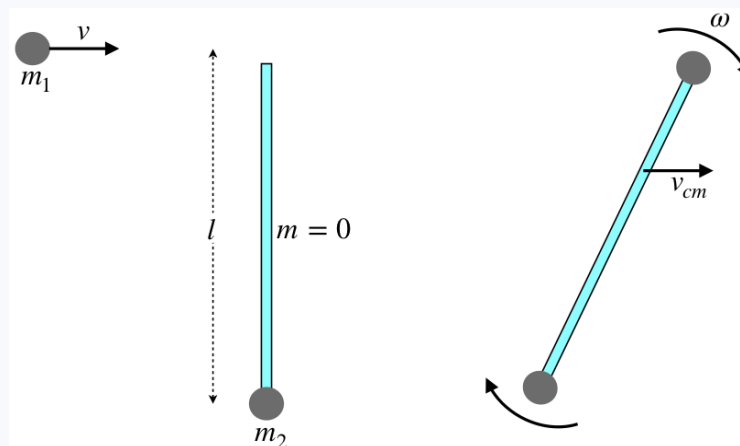
And now the experimental evidence:



Both blocks rise to the same height, because their upward linear velocities start off the same, due to conservation of linear momentum, which is identical for both blocks, independent of where the bullet strikes. Our analysis above resolves the "puzzle" of the difference in mechanical energies of the two systems.

Example 6.3.3

A massless magnetic rod has a small steel ball (which does have mass, but a negligible radius) attached to one end, and is at rest. Another small steel ball approaches the open end of this rod at a right-angle, and when it reaches the end of the rod, sticks to it. The dumbbell-looking combination continues forward, spinning as it goes (see the diagram). Show the surprising result that no kinetic energy is lost in this collision. The diagram provides labeling of quantities that you can use – you cannot make any assumptions about the relative values of m_1 and m_2 .



Solution

We are showing that kinetic energy is conserved, and the only principles that we can use are linear and angular momentum conservation. Let's start with linear momentum conservation. We have done this a hundred times – the incoming momentum equals the outgoing:

$$m_1 v = (m_1 + m_2) v_{cm} \Rightarrow v_{cm} = \frac{m_1}{m_1 + m_2} v$$

Now for conservation of angular momentum. Let's use the center of mass at the time of collision as the reference point. So we need to determine the perpendicular distance of the incoming ball from the center of mass. This is a straightforward calculation (e.g. use the incoming ball as the origin), which gives:

$$r_{\perp} = r_1 = \frac{m_2}{m_1 + m_2} l$$

For later reference, we also have for the distance of the other ball from the center of mass:

$$r_2 = \frac{m_1}{m_1 + m_2} l$$

With this we get the starting angular momentum, and with the rotational inertia of the dumbbell about the center of mass, we get an equation resulting from angular momentum conservation:

$$\left. \begin{aligned} L_o &= m_1 v r_{\perp} = \frac{m_1 m_2}{m_1 + m_2} v l \\ L_f &= I_{\text{dumbbell}} \omega = (m_1 r_1^2 + m_2 r_2^2) \omega = \frac{m_1 m_2}{m_1 + m_2} l^2 \omega \end{aligned} \right\} \Rightarrow L_o = L_f \Rightarrow \omega = \frac{v}{l}$$

Now all we have to do is construct the final kinetic energy:

$$KE_f = \frac{1}{2} (m_1 + m_2) v_{cm}^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} (m_1 + m_2) \left[\frac{m_1}{m_1 + m_2} v \right]^2 + \frac{1}{2} \left[\frac{m_1 m_2}{m_1 + m_2} l^2 \right] \left[\frac{v}{l} \right]^2 = \frac{1}{2} m_1 v^2 = KE_o$$

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