

## 4.2: Center of Mass

### Center of Mass Again

It should be clear by now how important the concept of center of mass is in classical mechanics. First it appeared in Newton's 2nd law, then in the discussion of internal energy, and now again in the topic of momentum. So far our only exposure to center of mass as a calculated quantity comes from [Equation 2.4.10](#), which we will repeat here:

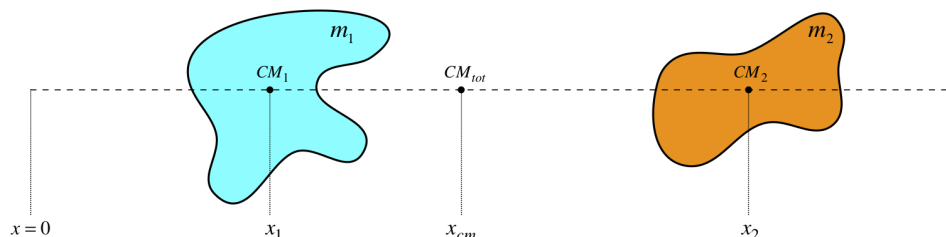
$$\begin{aligned}
 \vec{r}_{cm} &= x_{cm} \hat{i} + y_{cm} \hat{j} + z_{cm} \hat{k} \\
 &= \frac{[m_1 x_1 + m_2 x_2 + \dots] \hat{i} + [m_1 y_1 + m_2 y_2 + \dots] \hat{j} + [m_1 z_1 + m_2 z_2 + \dots] \hat{k}}{M} \\
 &= \frac{m_1 [x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}] + m_2 [x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}] + \dots}{M} \\
 &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots}{M}
 \end{aligned} \tag{4.2.1}$$

In this chapter, we will have a closer look at this quantity, to see how we can compute it for cases more general than a collection of a few point particles. In particular, we are going to look at objects that we treat as continuous distributions of mass, rather than collections of discrete particles. Of course, real matter is a collection of discrete particles, but a continuous model is much more practical to handle mathematically.

### Center of Mass of a Collection of Objects

Suppose now we want to know the center of mass of multiple extended objects, where all the heavy-lifting has already been done – the centers of mass of the objects are already known (see below for how to do this heavy-lifting). How do we determine the center of mass of such a system? It turns out to be pretty easy when you know the locations of the centers of mass of the two objects – just treat them as if they are point particles with all of their mass concentrated at their own centers of mass, and then do the calculation above.

**Figure 4.2.2 – Center of Mass for Two extended objects**



For proof of this, let's treat two extended objects (A and B) as collections of lots of point particles (atoms, if you like), and write down their centers of mass (measured from a common origin) in terms of the masses and positions of their atoms.

$$\left. \begin{aligned} \vec{r}_{cm A} &= \frac{m_{1A} \vec{r}_{1A} + m_{2A} \vec{r}_{2A} + \dots}{M_A} \\ \vec{r}_{cm B} &= \frac{m_{1B} \vec{r}_{1B} + m_{2B} \vec{r}_{2B} + \dots}{M_B} \end{aligned} \right\} \iff \vec{r}_{cm} = \frac{M_A \vec{r}_{cm A} + M_B \vec{r}_{cm B}}{M_A + M_B} \tag{4.2.2}$$

The left-hand side equations are those of the center of mass for each object in terms of its atoms' masses and positions. The right-hand side gives the center of mass of the two-object system in terms of the masses of the objects and the positions of their individual centers of mass. When the expressions for  $\vec{r}_{cm A}$  and  $\vec{r}_{cm B}$  from the left side are plugged into the right-hand side equation, then all the atoms of both objects are come together into a single center of mass formula, as if they were part of a single system with total mass  $M_A + M_B$ , proving the contention above.

#### Exercise

Two thin circular disks made from the same material lie flat on a horizontal surface, with their outer edges in contact with each other. One disk has a larger radius ( $R$ ) than the other ( $r$ ), and have equal thicknesses. Find how far the center of mass of the two-disk system lies from the center of the larger disk.

### Solution

The disks are made from the same uniform material, so they have equal mass densities. That means that the mass of the larger disk is larger than that of the smaller disk by the same factor as the ratio of their areas. That is, if the larger disk has twice the area of the smaller one, then it has twice as much mass. We therefore have the following relationship between the masses and radii of the disks:

$$\frac{M}{m} = \frac{\pi R^2}{\pi r^2} \Rightarrow M = \frac{R^2}{r^2} m$$

Let's choose the center of the larger disk as the origin, and have the center of the other disk lie on the  $+x$ -axis. The disks are uniform, so their individual centers of mass lie at their geometric centers, and we can compute the center of mass of the system by treating the disks as point masses located at these centers. The distance of the center of mass from the origin is what we are looking for, so:

$$x_{cm} = \frac{Mx_1 + mx_2}{M + m} = \frac{M(0) + m(R+r)}{M + m} = \frac{m(R+r)}{\frac{R^2}{r^2}m + m} = \frac{(R+r)r^2}{R^2 + r^2}$$

We can double-check this answer by looking at an obvious special case:  $R = r$ . If the disks are identical, then the center of mass must be halfway between their centers, which is the point where they are in contact, a distance  $R$  from the center of the larger disk. Plugging in  $R$  for  $r$  indeed gives this answer.

## Center of Mass of Continuous Objects

We now turn to the problem of computing the position of the center of mass of an object whose distribution of mass is known. What follows is pure math, but it is important math that returns over and over in physics.

### Alert

The important thing to gain from this discussion is to understand how the set-up process works. It culminates in an integral, but performing the integral is mere busywork compared to the task of setting it up. It's easy to be overwhelmed by the thought of the integral that is being constructed, but if you understand each step that leads up to it (and don't try to just jump to an answer that looks like something you have seen before), it will go fine.

We will keep this simple by restricting ourselves to objects for which the position of the center of mass in two of the three dimensions is obvious, which means we don't need to concern ourselves with the whole vector described in Equation 2.4.10 – just the  $x$ -component will do. A good model for this is a simple thin, cylindrical rod. This rod's mass distribution is completely cylindrically symmetric, which means that the center of mass lies on the axis passing through its center. But the mass distribution as a function of position on this axis may not be uniform. For example, it may be more dense on one end than on the other. Put another way, the particles located within the rod may be packed together more tightly in one region of the rod than in another, which means that the center of mass will not necessarily lie at the point halfway between the ends.

We need to say a few words about **mass density** before we proceed. Density is a measure of how closely-packed in space a quantity of something is. This quantity can be many different things. Here we will be considering mass, but in later physics classes you will deal with density of electric charge (and even, bizarrely, probability!). A **uniform density** for a region in space means that the quantity (whatever it happens to be) is evenly-distributed everywhere within that region. The way we define an average density for a region in space is to add up how much "stuff" is there, and then divide it by the total space it occupies. This gives an average density, but of course densities can vary from one point in space to another, in which case a **density function** is defined. We will deal with only the simplest variable densities here. As we will mainly be looking at thin rods for our examples, we will only consider densities that might vary along the length of the rod – this simplifies the process to a single dimension.

The mass density function in this case is a function of a single variable, has units of  $\frac{kg}{m}$ , and is called a **linear mass density**. This mass density function is typically denoted as  $\lambda(x)$ . If it is uniform, then the function is a constant  $\lambda$ , and the amount of mass  $m$

within a given length  $l$  is simply given by:

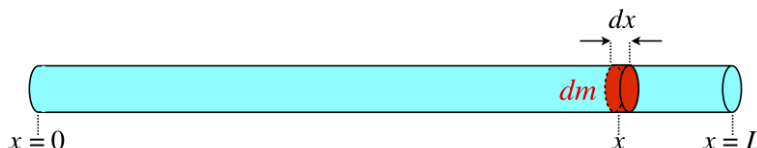
$$m = \lambda l \quad (4.2.3)$$

If the density is not uniform, then it is only a constant over an infinitesimal length  $dx$ , so the equation above can only apply to a tiny piece of mass  $dm$ , and the relationship is different at every position  $x$  because the density is different at every position:

$$dm = \lambda(x) dx \quad (4.2.4)$$

Now that we can write down how much mass is at every position, we are ready to do our calculation. We begin by drawing a diagram with the rod in a coordinate system along the  $x$ -axis such that one end is at the origin and the other is at  $x = L$ . The figure below provides a fully-labeled diagram that is very helpful for solving such problems.

**Figure 4.2.3 – Setup Diagram for Computations Involving Mass Density of a Thin Rod**



The center of mass is found by multiplying the amount of mass at each point by the  $x$ -coordinate of that mass, then adding up all of those products and dividing by the total mass. Of course, in this case we have an infinite number of point masses, so the sum is infinitely long, but the masses are infinitesimally small, so we solve this by converting the sum into an integral, in which we add up all the pieces from  $x = 0$  to  $x = L$ :

$$x_{cm} = \frac{dm_1 x_1 + dm_2 x_2 + \dots}{dm_1 + dm_2 + \dots} = \frac{\int_{x=0}^{x=L} dm x}{\int_{x=0}^{x=L} dm} \quad (4.2.5)$$

Now we plug in Equation 4.2.4 to give the following formula for center of mass (in one dimension) for a thin rod with a linear mass density that varies with  $x$ :

$$x_{cm} = \frac{\int_{x=0}^{x=L} \lambda(x) x dx}{\int_{x=0}^{x=L} \lambda(x) dx} \quad (4.2.6)$$

Okay, so let's do a couple of examples...

### A Uniform Rod

As was stated above, if the rod is uniform, then the density is a constant (which we will call simply  $\lambda$ ). Plugging this into Equation 4.2.7 leads to a simple calculation and an unsurprising result:

$$x_{cm} = \frac{\int_{x=0}^{x=L} \lambda x dx}{\int_{x=0}^{x=L} \lambda dx} = \frac{\cancel{\lambda} \int_{x=0}^{x=L} x dx}{\int_{x=0}^{x=L} dx} = \frac{[\frac{1}{2}x^2]_0^L}{[x]_0^L} = \frac{1}{2}L \quad (4.2.7)$$

So we have calculated what we already knew – that for a thin rod with a uniform mass density, the center of mass is at its center (which on our coordinate system lies at  $x = \frac{1}{2}L$ ).

## A Non-Uniform Rod

Next we'll look at an example of a rod which has a mass density that varies from one end to the other. This variable density is expressed in its density function:

$$\lambda(x) = \lambda_o \left( \frac{x}{L} + 1 \right) \quad (4.2.8)$$

Before we do the math, let's try to make sense of this function. The easiest way to do this is to consider the endpoints. At  $x = 0$ , the density equals the constant  $\lambda_o$ , while at  $x = L$  that density has grown to twice that much. This increase of density happens linearly with the variable  $x$ . What should we *expect* to see when we compute the center of mass? Well, the rod is more dense near the  $x = L$  end than the  $x = 0$  end, so the center of mass should be at an  $x$  value greater than  $L/2$ . Okay, so let's plug the density function into [Equation 4.2.6](#) and see what we get:

$$x_{cm} = \frac{\int_{x=0}^{x=L} \left[ \lambda_o \left( \frac{x}{L} + 1 \right) \right] x \, dx}{\int_{x=0}^{x=L} \left[ \lambda_o \left( \frac{x}{L} + 1 \right) \right] dx} = \frac{\cancel{\lambda_o} \int_{x=0}^{x=L} \left( \frac{x^2}{L} + x \right) dx}{\cancel{\lambda_o} \int_{x=0}^{x=L} \left( \frac{x}{L} + 1 \right) dx} = \frac{\left[ \frac{x^3}{3L} + \frac{x^2}{2} \right]_0^L}{\left[ \frac{x^2}{2L} + x \right]_0^L} = \frac{5}{9}L \quad (4.2.9)$$

Interestingly, the center of mass doesn't depend upon the density constant  $\lambda_o$ .

### Analyze This

Two identical rods of mass  $M$  and length  $L$  have the same non-uniform density profile. When one of these rods is placed along the  $x$ -axis with one end of the rod at the origin, the density as a function of  $x$  is proportional to the following function:

$$\lambda(x) \propto \left( \frac{x^2}{L^2} + 1 \right)$$

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- what we are given (perhaps translated from English to mathematics)
- what we can infer, if anything
- quantities we can compute (or almost compute!), if anything

### Analysis

The first thing we can do is determine the constant of proportionality in terms of the mass and length of the rods. Calling this constant  $\lambda_o$ , we have:

$$\lambda(x) = \lambda_o \left( \frac{x^2}{L^2} + 1 \right)$$

The total mass of the object is the density integrated over the whole length of the rod:

$$M = \int_{x=0}^{x=L} dm = \int_0^L \lambda(x) \, dx = \lambda_o \int_0^L \left( \frac{x^2}{L^2} + 1 \right) dx = \lambda_o \left[ \frac{x^3}{3L^2} + x \right]_0^L = \frac{4}{3} \lambda_o L \Rightarrow \lambda_o = \frac{3M}{4L}$$

Next we can compute the position of the center of mass of a rod with the lower-density end placed at the origin:

$$x_{cm} = \frac{\int_{x=0}^{x=L} x \, dm}{M} = \frac{\int_0^L x \lambda(x) \, dx}{\frac{4}{3} \lambda_o L} = \frac{3}{4 \lambda_o L} \int_0^L x \lambda_o \left( \frac{x^2}{L^2} + 1 \right) dx = \frac{3}{4L} \left[ \frac{x^4}{4L^2} + \frac{x^2}{2} \right]_0^L = \frac{9}{16}L$$

## Objects with More Dimensions

We have only discussed the simplest of continuous objects - thin rods that are more-or-less one-dimensional, and computing their centers of mass requires only an integral over a single variable. Real-world objects are three-dimensional, and computing their

centers of mass is more complicated in two ways. First, the density function can be a function of three variables, rather than just one. And second, integration of the mass elements requires a three-dimensional integral. We will not go into the details of these sorts of calculations here, as they are heavily steeped in mathematics with very little physics content. The reader can expect to start encountering these types of integrals (in a different context – not center of mass) when they get to Physics 9C.

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