University of California, Davis

UCD: Physics 9A - Classical Mechanics

Tom Weideman

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Detailed Licensing

Licensing

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Preface

Physics and Models

The whole idea of the study of physics is to understand how the universe operates. We cannot actually ever know for sure how this works, but we play a sort of game: We develop a model that explains why things happen the way they do, and then we test the effectiveness of that model when it comes to predicting how other things will unfold. If the model predicts accurately, it is a “good” model, and if it doesn’t, it is discarded.

Inherent to this description is the idea of “accuracy.” No model we have ever designed has ever predicted a result with 100% accuracy. Mainly this is because 100% accuracy requires 100% accurate measurements of both the starting conditions and of the results, and this simply isn’t possible. What we settle for instead is a sense of what sorts of problems our model is intended to solve. Some models are precise to an incredibly small dimension (like models that predict atomic behavior), but these are not useful for making predictions in the macroscopic world where trillions of trillions of atoms are involved. Conversely, we also make macroscopic models that breakdown when our measurements become too fine.

So all models come with them an understanding that they work “up to a point.” When I discuss a problem involving a “frictionless surface,” one can certainly argue that no such thing exists, but true as that statement is, it is not relevant. The model of the frictionless surface allows us to answer questions about situations where the amount of friction is small, and our coarse measurements can’t distinguish the effects of that small amount of friction. Further, this model can be used as a starting point, to which we can later append a friction effect to make a more inclusive model.

You will sometimes hear me (or future physics instructors) say that such-and-such is true if a certain quantity is “small.” This simply means that if the quantity is small enough, the coarseness of our measurements provide too much noise for us to really notice the effect of that small quantity.

Measurement and Units

While we can make some general predictions about the behavior of our universe, these are not usually particularly satisfying. The statement, “If I drop something, it will fall to Earth” can be considered a “theory of gravity,” but big deal. How long does it take the dropped object to fall some specified distance? How fast is it going when it lands? How do the motions of two different dropped objects differ from each other? All of these are questions we would like to answer as well, and they all require measurement. But if I measure the time for an object to fall and call it “3,” while you measure the same process and call it “17,” we are not going to get anywhere. We need a standardized system of units that we can agree upon so that we can compare results.

Many hundreds of years before Galileo, Aristotle sought to explain everything, but he did so descriptively. Galileo was among the first set out to do so mathematically. Galileo was studying the effects of gravity on motion (Aristotle simply said that things that are heavy fall, and things that are light rise), and did experiments where he rolled balls down ramps and timed their journeys. He started zeroing-in on a precise mathematical description, but every time he got close to accepting his results, the experiment would go haywire and his new results would disagree badly with the early ones. It turns out that the problem was that he was using his own heartbeat to time the motion of the ball, and when his predictions started coming true, he got excited, his heart beat faster, and the predictions began to fail.

There are many systems of units available to us. We could for example measure speed (which is a rate of distance covered over time) in units of furlongs per fortnight. But there is one system that we use in physics as the default, from which we only rarely stray. It is called the *Système Internationale d’Unites*, or *SI units* for short. The three most fundamental measurements we have in this system are *meters* (distance), *kilograms* (mass), and *seconds* (time). For this reason, this system of units is also often referred to as *mks units*.

First-time physics students often pay little attention to units when they are solving problems, thinking of them as more of a nuisance than a help. You should fight this tendency. If you are solving a problem to find a speed and you end up with an answer that (because you carefully carried units through the math) came out to be kilograms per meter, then that is an indication that you made a mistake somewhere. Many students plug in the numbers and then throw the proper units in at the end, and this provides them with no opportunity for catching mistakes.

CHAPTER OVERVIEW

0: Preliminaries

0.1: Why Physics?

0.2: Physics is Hard!

0.3: How to Learn Physics

0.4: Basics of Scientific Measurement

0

0.1: Why Physics?

Not Just for Physics Majors

Introductory physics like you find in the 9-series at UC Davis is not just offered to the relatively small number of students that have chosen to major in Physics. Indeed, the vast majority of these classes are populated by students majoring in some branch of engineering, with a few other STEM fields represented as well. This sometimes leads to questions of why the courses that don't closely relate to the majors are required at all. Here we attempt to explain why such a broad physics curriculum is required of otherwise very focused majors...

Human Intellectual Capital

Let's start with a decidedly non-ivory-tower perspective on higher education – the view of businesses and macroeconomics. Investopedia.com defines ***intellectual capital*** this way:

... the collection of all informational resources a company has at its disposal that can be used to drive profits, gain new customers, create new products or otherwise improve the business

The phrase "informational resources" is broad and very vague. Naturally it includes things like patents, secret formulas, and computer algorithms, but we will be focusing on the *human* element of intellectual capital – "brain power," if you will. And in particular, we will confine ourselves to brain power in STEM fields.

Knowledge vs. Understanding

It is useful to divide "informational resources" in the context of human intellectual capital into essentially two types – knowledge and understanding. Most people have some inkling of the difference between these two things, even if only vaguely. A simple example that clarifies this difference is the act of riding a bicycle. One can look up "how to ride a bicycle" in Google, and retrieve all of the relevant information:

1. what actions to take
   - hold the handle bars
   - sit on the seat
   - get the bike rolling
   - start pedaling
   - lean the bike in the direction you want it to turn
2. tricks for how to master riding
   - lower the seat so that your feet can easily reach the ground
   - roll the bike forward by pushing with your feet
   - raise your feet intermittently to get used to balancing while rolling
   - bring your feet up to pedals to continue the rolling, removing them to use ground to regain balance as needed

The first of these lists obviously falls squarely in the category of knowledge. The second list also constitutes knowledge, but in some sense it is *meta*-knowledge in that it gives you tips for achieving understanding. But most notably, *it doesn't directly give you that understanding* (measured by your ability to actually ride) – you absolutely have to gain this on your own by your own efforts. This is a general feature of understanding; humans cannot simply look something up to achieve understanding – we must deliberately immerse ourselves into the pursuit before we can reach that point.

When considering the difference between these two aspects of human intellectual capital, it should be clear which is more valuable. Engineering firms do not gain a lot of value by recruiting people that have information committed to memory, when that is merely a web search away. Instead, they are looking for people that have a deeper understanding, as that requires time and effort to acquire, and cannot be replicated in short order. Furthermore, understanding has a certain organic quality to it, in that understanding of one pursuit can often quickly be re-purposed for another task. Riding a motorcycle and ice skating both rely on the same basic method for turning as riding a bike, and though there are other skills involved with these two activities, having some core understanding of the turning process is useful.

Applying this to STEM

When it comes to human intellectual capital for STEM fields, it is clear that having an understanding of physical, chemical, and biological processes is far more valuable than essentially memorizing a database of example processes. Knowing *why* a certain chemical reaction occurs is more valuable than remembering the components and final product for a specific reaction. Understanding how to compute the stresses and strains of a system of beams is better than memorizing certain standard structures of beams that are used. And so on.

As clear as the value gap between memory and reasoning is for STEM fields, when it comes to STEM education, we do run into problems with emphasizing what is important. This is not the place to go into all of these, so here's the most dramatic example of such a problem: All education involves an evaluation process (grading). It is much easier to evaluate based on memory than on understanding. It is also much easier to study for exams that test memory rather than understanding. Unfortunately, this has led to ubiquitous testing based on students' ability to remember, rather than their ability to figure things out. This in turn introduces an incentive to study in the manner most effective for those tests, which means that many STEM students are not "practicing their bike riding" to gain understanding – they are essentially memorizing search engine results on how to ride a bike instead, because it is much easier and less time-consuming to do this than practice riding. Imagine teaching a class in bike riding, and at the end the exam consists of a multiple choice exam that covers the elements of riding a bike listed above, but *doesn't actually require that students get on a bike to demonstrate that they can now ride it.* This is unfortunately exactly how many STEM classes are structured.

Where Physics Comes In

This is an ongoing problem, and one that shows itself particularly clearly in physics, which is perhaps the subject where the value gap between understanding and knowing is the greatest. A physics class where exam problems are given that are similar to previously-assigned problems (in homework or "practice exams") incentivizes memorizing, and students taking such a class gain very little value from it, even though they may believe otherwise. Given that classes taught in this manner exist from elementary school, high school, and unfortunately even into college, it can be difficult to "right the ship" later – students have never trained themselves to study toward understanding rather than knowledge, and can become quite frustrated when they finally need to do so. Then student exasperation over this sudden change feeds back into instructor behavior – instructors want students to be happy, and exams based on knowledge are easier to write anyway, so changes creep in that can make the problem endemic in higher-education.

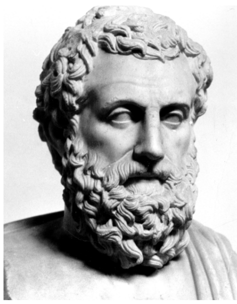
While this problem presents a serious challenge, the subject of physics, taught correctly, provides perhaps the best training ground for teaching students to reason rather than memorize. For this reason, far more than any other, introductory physics is required of virtually all STEM majors, even if physical principals are rarely, if ever, directly used in that field. The wisdom of this university policy is often lost on students, especially those majoring in STEM fields that seem to be the farthest removed from physical principles, like the life sciences. What complicates matters even more is that physics (again, when taught with a goal of understanding rather than knowledge) is a *very* challenging subject, a fact that is the subject of the next section.

0.2: Physics is Hard!

Some Historical Perspective

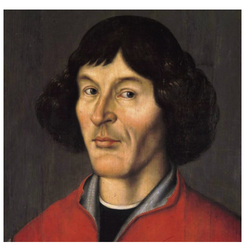
You are in luck! You are poised to study of one of the most difficult subjects ever undertaken by humankind. For some of you, this may not exactly conform to your definition of "luck." But for just about every career physicist out there, the inherent difficulty of this subject is the primary reason they like it so much. There is a mistaken notion out there that a professional physicist just naturally understands their subject, and that it is easy for them. Some physicists may even try to lead others to believe that this is true, because it makes them look smarter. But the fact is that this subject simply does not fit neatly into our brains, and we all struggle with it. Physicists get better at the basics of physics from constant practice, but this should not be taken as evidence that they understood it instantly the first time they saw it, while lesser mortals struggle with it. Physics is hard for *everyone*. Here's a short historical perspective on this fact, showing how even the greatest human minds in history struggled with physics...

Aristotle (384 - 322 BCE)



Considered one of the most brilliant minds in human history, when it came to physics (or "natural philosophy"), Aristotle got just about everything wrong. His erroneous approach to something as basic as the physics of celestial and terrestrial motion would dominate western thought for nearly two millennia! Aristotle was a great observer of the natural world. He was great at collecting knowledge. He *tried* to derive understanding from these observations, and failed terribly – the laws of nature were just too subtle for even his prodigious intellect.

Copernicus (1473 - 1543)



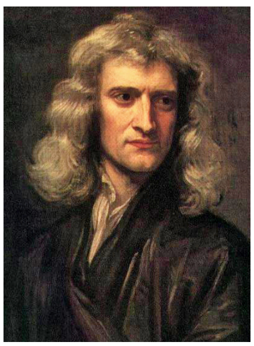
One of the greatest problems people have in figuring out physics comes from thinking they already understand it. Starting with Aristotle's notions of celestial motion that were later tweaked by Ptolemy in the second century, it took another *thirteen* centuries for someone like Copernicus to come along and suggest the simpler (and correct) explanation that the Earth is not the center of the universe and in fact the planets orbit the Sun.

Galileo (1564 - 1642) and Kepler (1571 - 1630)

After a century of everyone basically ignoring Copernicus, Galileo and Kepler arrived. Between Galileo's direct observations of Jupiter's moons (through a new device called a telescope) and Kepler's mathematical model of planetary orbits, one would think that the geocentric model would finally be discarded, but the Church would not allow that. Even when shown the right path, we humans are slow to embrace it! We begin to see a common problem here – it takes *understanding* to fully believe what these geniuses concluded. They acquired this understanding through many years of hard work, but the people they passed their conclusions to could only evaluate their claims on the basis of a *belief* (knowledge) system. This is the first clear demonstration of the value of understanding in human intellectual capital.

Newton (1642 - 1726)



Finally humanity spawned a remarkable genius that could complete assembling the celestial puzzle started by Copernicus, Galileo, and Kepler, through the brilliance of his theory of Universal Gravitation. It took 2000 years for someone to finally discard Aristotle's notion that motion in the heavens is governed by different laws than motion on the Earth.

It is worth noting that humanity was not struggling with esoteric mathematical constructs like quantum field theory for all these centuries. The topic in which these people left their marks was little more than basic motion (along with changing the mindset to a more universal notion of motion that applies to both the heavens and at the Earth's surface) – something Physics 9A students study in the very first week! For whatever reason, even the most elementary aspects of physics are hard for humans to understand.

Relating, not Memorizing

Adding to the problem of the difficulty of physics is something that was discussed in the previous section. People get better at pursuits the more they train for them. Partly due to a need for evaluating students with grades, and partly due to a lack of understanding about what true progress in physics education looks like, most students are academically trained to memorize. This skill is not one that is useful in physics, which is much more about building an understandable structure of interrelated ideas. So introductory physics is hard because students are not trained in the critical thinking skills needed. Indeed, physics classes turn out to *be the training* that students need for later. As with training atrophied muscles in the body, starting from scratch is always the most difficult, painful endeavor.

The Curse of Incentives

There is one last problem that conspires with those above to make physics classes very challenging, and that is the negative effects brought about by grade incentives. When one needs to "dig into" a subject to reach a level of understanding, it requires a great deal of self-motivation. If one is just trying to "get through" a class, or if the only motivation is to get a decent grade, with no interest in actually understanding the material, then shortcuts will be taken, and the hard work that is necessary to actually reach understanding will not be done. Grades work as a fine incentive for doing busy work, and for memorizing, but reaching actual understanding is a much messier business, and as the following video shows (from outside the context of physics), simply won't inspire people to do the outside-the-box thinking necessary to get the real message from physics instruction.

A physics class can be taught in two ways where the exam evaluations are analogous to the candle problem discussed in this video. The first is with the tacks in the box. This is a "hard" physics class, made harder by using grades to motivate students. The second is with the tacks outside the box. This is an "easy" physics class made easier by using grades to motivate students. The trouble is, students only take away a valuable educational experience (where the idea is to learn to think outside the box) in the first class. The second case can scarcely be called a physics class at all – it is little more than a series of mindless exercises.

So physics – *real* physics – is just hard.

0.3: How to Learn Physics

No Shortcuts, No Memorizing

In previous sections we discussed the importance of taking a class in physics, and why a true physics class is so challenging. Here we will try to put together what we learned there to come up with an effective way to get through this daunting task of learning physics.

The first most important principle we need to embrace is that there is no easy path. Trying to gain an understanding of physics (or really, anything) without engaging in it fully is a losing battle. If you want to build muscle by lifting weights, you won't do so by watching videos of other people lifting those weights – you have to get in there and do it yourself.

Students in physics classes tend to focus too much on having problem solutions to study from. Sample problems are only valuable for those that struggle with them. They are worthless if all they are used for is to "study" their solutions. In this case, the word "study" is just a euphemism for "memorize," and indeed physics study shortcuts all boil down to memorization. Going back to the weightlifting analogy, if the weight is lifted for you, and then handed to you to hold up, it may feel like you are accomplishing something (it is not trivial to hold it up), but this does not exercise your muscles. And a physics class, properly taught, seeks to test your "physics muscles," not your ability to regurgitate what you have seen before.

So in the way of study advice, the point is this: Avoid looking at solutions as much as possible! Derive the maximum benefit by putting in your own effort. Even those times you fail to completely figure something out (and you will – physics is hard!), you will learn more than you will by avoiding this discomfort by jumping to the solutions. This is not to say that the solutions are not helpful – but like a spotter that helps you when you lift weights, the less you rely on them, the better.

Deliberate Practice

In the last couple decades a lot of stuff has been written about effective training of athletes and the achievement of expertise in general. One idea that has really taken hold is the idea of *deliberate practice*. Perhaps you have heard the claim that to become elite in some pursuit, one needs to invest on the order of 10,000 hours of practice? The number seems to be what gets all the attention, but what often gets lost in discussion is the *form* that this practice needs to take. For example, a player learning to hit a baseball better will not achieve the goal of elite hitter simply by spending 10,000 hours in a batting cage. Rather, those hours need to include reducing proper hitting technique to its infinitesimal constituent parts: Hours need to be spent on getting the swing-plane of the bat to the correct angle, getting hip rotation right, achieving proper weight transfer, and so on. And many more hours are needed to combine these tiny pieces together properly into an integrated swing. This *deliberate* practice of components is what those 10,000 hours must be comprised of, and coaching from someone knowledgeable about these things is pretty much a necessity.

Now of course you will not have anything close to 10,000 hours available, but the goal here is to get a fruitful first exposure to physics, not become an elite practitioner. But the idea of deliberate practice is still an important one, no matter what the scale of the numbers of hours may be. Fortunately, you have an instructor, teaching assistants, and even this textbook to serve as "coaches" to help direct you through this practice. But the responsibility of maintaining the discipline necessary to do this right is on you, the student.

You will find that there are many tools or templates offered during this and future physics courses that are intended to get you through problem solving. To employ deliberate practice in the context of these tools and template means that you should spend some time getting good at these *without regard to your success in solving the problem itself*. For example, one of the most important tools in this class on mechanics is called the "free-body diagram." A short amount of time is dedicated to teaching you how to draw one of these, but they are a critical part of solving so many different kinds of problems. Most students invest a short time on learning to do these, and move on well before mastering them – they focus much more on getting the answer to the problem, and hardly at all on getting this step right. Often when a problem is stated, if it does not include an explicit step that says, "draw a careful free-body diagram," many students will not bother to do so, or if they do it will be very crude and incomplete – they jump straight to writing down equations, and inevitably get things wrong. Free-body diagrams are not just "busy work" at the start of a problem – drawing them requires deep conceptual thought that is necessary for avoiding misconceptions and modeling the math properly. Failing to master this tiny component to physics problem solving is like failing to master weight transfer in hitting a baseball, the grip of a golf club, or turning one's head to breath properly when swimming. Time must be spent on these components, or the whole pursuit fails.

When in doubt as to whether you are applying proper focus on components when doing practice physics problems, simply remind yourself that your goal needs to be to get better at the process, *not* to get the right answer.

Embrace the Big Picture

Suppose you watch dozens of scenes from a movie, and are later asked to answer some questions about them. Hopefully it is clear that putting them into the context of the movie plot makes them easier to understand and easier to recall details than if they are viewed in random order and out of context. This is a general feature of human understanding – we are better at remembering details and extrapolating conclusions when the information is organized into a contextual framework. The same is true about physics. It's not always easy to see the bigger picture when struggling to understand specific concepts and nuanced mathematical models, but striving to do so can itself help one to overcome those struggles with the details.

Put more succinctly: The best way to learn new ideas in physics is to relate them back to things you already understand. If some new topic seems utterly disjoint from what you have learned already, then you know there is something you are missing, and it is an indication that you need to delve in further. A good instructor will do their best to segue from one subject to the next, so that a big picture is developed. But if this doesn't happen, or if you are unable to grasp the connections the instructor is trying to make, then it is worth your time to go back and make those connections yourself.

Students that have a disjointed understanding of physics feel like a useful approach to studying the subject is to do as many different practice problems as they can. There goal is to commit to memory as many different "tricks" as they can find, in the hope that the exam will involve a trick they have memorized. This approach is utterly ineffective in a properly-taught class that emphasizes understanding over memorization. Practice problems are helpful, but the emphasis should not be on how they are different (the "tricks"), but how they are the *same*. Once it is clear what elements a wide variety of problems have in common, seeing the big picture is easier, and one can focus their deliberate practice of those common elements. This is the path to success – building a simple, understandable, mental big-picture, and mastering the fundamental tools that go with it.

Doing Sample Problems

One of the most common pieces of advice given for studying physics is to "carpet bomb" your brain by doing countless sample problems. This is only half-true, and the fact that most people that give this advice seem to think it is some sort of magic pill shows that they don't understand the importance of the word "deliberate" in "deliberate practice". What unfortunately typically plays out for the unsuspecting student who follows this advice is this:

- Finds a new problem to solve.
- EITHER:
    - Solves it quickly and correctly so it was of no benefit, as nothing new was learned.
- OR:
    - Gets stuck early, as it challenging.
    - Rather than struggling too much with the problem, decides not to "waste time" with blind alleys, and simply peeks at the solutions.
    - The solution makes sense, so after reading it, feels like they have learned something.
- Repeats cycle.

This cycle of "just doing lots of practice problems" is utterly useless for learning physics, and when the student that follows the advice in this way can't seem to solve any exam problems, they are understandably frustrated that so many hours of practice were not effective. That's the tragedy of this advice – it is so incomplete and deceptively simple that it frequently leads to lots of wasted hours of work that could have been better-spent.

One might ask, "If this is all true, why do so many people that have been successful in their study of physics give this advice?" The answer depends upon which category of two categories the adviser belongs to. If it is a practicing physicist, then it is likely in their nature not to quickly resort to looking at solutions. They don't mind "wasting their time" with blind alleys – getting unstuck by themselves is the "fun part".

If it is a fellow student that is claiming that this strategy got them through a class, and they are not of the same mentality described above for the practicing physicist, then the odds are that what made them successful was poor examinations. Some instructors intentionally write their exams so that they closely resemble some subset of a collection of problems given for practice. When this is done, "practicing" by studying the solutions to a large number of sample problems is effective for the test-taking, because memorization is key. Unfortunately, this teaching practice is more common than it should be, and leads to lots of students believing they have learned more than they really have. This does not lead to success down the road, when a robust understanding of physics (and physics problem-solving) is needed, and the short-term memorization from the previously-taken class is useless. The general rule is, if your physics course (and particularly the exams) are easy, then it's not because the instructor is good at teaching it to you (no one can actually "make physics easy") – it's because they are actually not asking you (with their exams) to demonstrate that you have learned anything of any value. And usually, when exams don't require that students learn anything, students don't take the appropriate steps to do so.

Using This Textbook's Sample Problems

This textbook provides you with sample problems to help you get in your deliberate practice, and they are constructed to help you avoid the "study the solutions" pitfall described above. These sample problems come in two parts. The first part (given in the body of a chapter) provides only the physical situation, where you are asked to simply "analyze". There is no question for you to answer here – just flex your muscles with extracting as much as you can from what has been given. The second part of the sample problem appears at the end of the chapter, and it is a question that accompanies the physical situation given earlier. The reason for this split will be explained below, but here is the most effective use of these sample problems:

1. Do as much of the analysis as you possibly can without looking at the analysis provided in the textbook. Write this analysis down on a piece of paper – don't just think this through in your head! This is very important.
2. Open the analysis window to see what analysis has been provided for you. See if the analysis you did matches what is there, and make notes on your piece of paper about where you went wrong. *Add notes to this piece of paper that describe in your own words* any parts of the analysis that you overlooked that was provided by the textbook.
3. Go to the question at the end of the chapter, and try to solve it, referring to the notes you have written on your piece of paper.

So why the split? In the analysis stage, without an end goal of a question to answer, you don't get stuck trying to find a specific path. This is not to say that you won't get stuck! At times you will find that you can hardly think of any information that can be extracted. But when one has a specific question in their heads, it is difficult to free-up their thinking to allow them to see important points of analysis. In essence, this method breaks down the task into smaller parts, and like a coach that provides drills, forces your practice to be more "deliberate".

0.4: Basics of Scientific Measurement

Things You Should Already Know

As a final section for the "Preliminaries" chapter, we will review a few things you should have encountered somewhere in a science class prior to enrolling in the university. Some things that fall into this category are covered by course prerequisites. For example, everyone entering Physics 9A should already have a solid working knowledge of trigonometry and basic calculus (differentiation and integration). This section will discuss other basics that are not explicitly in the course prerequisites, but will nevertheless be assumed to be understood by students entering the course.

Physical Dimensions and Units

When describing a physical quantity in the subject of mechanics (covered in Physics 9A), it can be broken down into a combination of three distinct *physical dimensions*. These dimensions are *distance*, *time*, and *mass*. Physical quantities can be measured in many different scales called *units* (distance can be measured in centimeters, yards, or light years, time measured in seconds, hours, or fortnights, and so on), but while two measurements may use different units, their physical dimensions are the same. For example, two speeds may be measured in meters per second and miles per hour, respectively, but they have in common the dimensions of "length per unit time."

Dimensional Analysis

When solving physics problems, it is often useful to check one's work at the end (or even while in progress), as it is possible to have made an algebraic error in the calculation. One way to make a quick check is to see if the dimensions of a quantity work out right. For example, in this class we will talk about a quantity called energy. It turns out that energy has units of: $mass \times \frac{length^2}{time^2}$. Suppose a problem asks for a computation of velocity, after some algebra you get:

$$v = \sqrt{\frac{2E}{m}}$$

While it is not a definitive check of whether the answer is right, it is possible to gain confidence in the answer or catch an error if it is wrong by plugging in the dimensions to see if they work out in the equality. The dimensions of velocity we know to be length-per-time, so if we plug in the dimensions for energy E and mass m, we can check to see if things work out. Note that the number 2 has no physical dimensions, so it can be ignored (the brackets around the variable v mean "dimensions of"):

$$[v] = \sqrt{\frac{mass \times \frac{length^2}{time^2}}{mass}} = \frac{length}{time}$$

The mass dimensions divide out, and the quare root of the squared length and time results in a confirmation of the dimensions.

Unit Conversion

While the dimensions of physical quantities are always the same, they may be measured differently. Occasionally it is desirable to convert a numerical value from one system of units to another. One reason might be that a problem is given where different quantities are measured in different systems of units, and the final answer should not contain both (say) inches and meters. There is a simple procedure for making this conversions...

It starts with knowing what a measurement of a dimensions in one system of units is in another system. For example, 1 inch is a length equal to 2.54 centimeters. Given that this is true, then the ratio of these two values (with either one in the numerator) must be equal to one:

$$\frac{2.54cm}{1in} = 1 = \frac{1in}{2.54cm}$$

This is useful to know, because a quantity can always be multiplied by 1 without affecting its value. So suppose we want to know how many centimeters there are in 3.5 inches. All we have to do is multiply 3.5 inches by 1, using the fraction with inches in the denominator so that the inches unit cancels, and the number of centimeters is left behind:

$$3.5in = (3.5in) \cdot 1 = \left(3.5\ \cancel{in}\right)\left(\frac{2.54cm}{1\ \cancel{in}}\right) = 8.89cm$$

If the quantity is more complicated than just a single length, the same procedure can be followed for each separate unit. For example, suppose we wish to know how fast a car is traveling in miles per hour when we are given that it is moving at a speed of 40 meters per second. Now we need to convert meters to miles, and seconds to hours. We can even use intermediate units like kilometers and minutes along the way. That is, suppose we know the conversion between kilometers and miles is $1.60km = 1mi$. Of course we know that there are 1000 meters in 1 kilometer, 60 seconds in a minute, and 60 minutes in an hour, so we form the following ratios with a value of 1 for the purposes of our conversion:

$$1 = \frac{1.61km}{1mi} = \frac{1mi}{1.61km}, \quad 1 = \frac{1000m}{1km} = \frac{1km}{1000m}, \quad 1 = \frac{60s}{1min} = \frac{1min}{60s}, \quad 1 = \frac{60min}{1hr} = \frac{1hr}{60min}$$

Now we take the original value and start multiplying it by as many 1's as we need to in order to replace the units as we want them:

$$40\frac{m}{s} = \left(40\frac{\cancel{m}}{\cancel{s}}\right)\left(\frac{1\ \cancel{km}}{1000\ \cancel{m}}\right)\left(\frac{1mi}{1.61\ \cancel{km}}\right)\left(\frac{60\ \cancel{s}}{1\ \cancel{min}}\right)\left(\frac{60\ \cancel{min}}{1hr}\right) = 89.4\frac{mi}{hr}$$

Significant Figures

Most (but certainly not all) physics problems that you will encounter in this course provide you with some numeric values associated with a physical system. The presumption is that those numbers were determined by a measuring device of some kind. Measuring devices have varying levels of precision. For example, if one measures a distance with a meter stick, one can expect the measurement to be accurate to within about a millimeter. If a measurement is taken using a microscope, the measurement may be accurate to within a micron (one thousandth of a millimeter). In all scientific disciplines, it is understood that the measuring device's precision is reflected in the number itself. So for example, a measurement with a meter stick (in meters) will show no more than three decimal places, because the fourth decimal place signifies fractions of millimeters. Conversely, whenever we see a number given in a problem that describes a length to three decimal places, we assume that the measuring device could only do that well (so maybe it was a meter stick with millimeter subdivisions). Note that if the result is a round number, then the number provided can still provide information about the true level of precision by including trailing zeros. So if a number $0.100m$ is given, then the implication is that it is accurate down to millimeters.

For larger numbers this can get a little weird. For example, suppose we want to express the number above in microns. Clearly the value is $10,000\mu m$, but now it seems that we have lost the information about precision (which is the same if we measured with the same device). The solution here is to use scientific notation, and only keep trailing zeros to the place where the precision ends. So the above number with the same precision would be expressed as: $1.00 \times 10^5 \mu m$.

Understanding the precision of given numbers is one thing, but ultimately these numbers are used in calculations to get new numbers, so the question becomes, "How do we express the precision of our calculated numbers?" The simple answer is that the calculated number's level of precision is only as good as the least precise number in the calculation. For example, suppose we want to know how far something travels when we are given its speed and the time it moves at this constant speed. The distance is obviously computed by multiplying these numbers. Suppose the speed is measured fairly crudely, no more precisely than 1 meter per second. Let's say the speed is $35\frac{m}{s}$. Let's also assume that the time is measured *very* precisely – with an atomic clock, and the time comes out to be $3.8933274s$ Simply multiplying these numbers together gives a distance traveled equal to $136.266459m$ But if we claim this is the answer, then someone reading it will assume that this distance value is accurate down to $10^{-6}m$ (microns). But suppose the actual speed value was $35.3\frac{m}{s}$ (this extra decimal place was not caught by our measuring device). Then the correct distance value would be over $137m$, and the accuracy implied by our 9-digit answer would be misleading. So rather than keep all 9 digits generated by the exact calculation, we round off that answer to the number of digits ("significant figures") in the less-precise number. The speed has only two digits, so we would round-off our final answer to only two significant figures, changing the answer from $136.266459m$ to $140m$.

There is one other thing to mention here regarding significant figures. If a measured value is multiplied by an exact number, then the exact number is not taken into the significant figure calculation. For example, a physics formula might include a factor of one-half. This number is not a measured value – it is exact – so even though you might think it is 0.5 and has only one significant figure, this is not correct. In fact it is 0.50000000000..., which means that it will not limit the number of significant figures of the final answer at all – only the measured value(s) in the calculation will. Another interesting example is the calculation of the circumference of a circle. Suppose the diameter is given to be $1.9 cm$, then the circumference is equal to $1.9\pi cm$. To express this as a decimal, you might think that the two significant figures for the diameter means that you only keep two significant figures for π. But π is an exact number (even if we can't express it completely as a decimal), so you need to keep more decimal places of that number. Using $\pi \approx 3.1$ gives an answer for the circumference (to two significant figures) of $5.9 cm$, while using more decimal places for pi raises the answer for the circumference to $6.0 cm$.

CHAPTER OVERVIEW

1: Motion

1.1: Vectors
1.2: Vector Multiplication
1.3: Straight-Line Motion
1.4: Kinematics
1.5: Graphing
1.6: Motion in Multiple Dimensions
1.7: Examples of 2-Dimensional Motion
1.8: Relative Motion
Sample Problems

1.1: Vectors

Definition of a Vector

Just being able to put numbers on physical quantities is not sufficient for describing nature. Very often physical quantities have directions. For example, a description of something's motion is incomplete if you merely state how fast it is going. [*Okay, so an asteroid is moving at 35,000 miles per hour, but is it headed for Earth?!*] We therefore have the following definition for physical quantities that exhibit both these properties:

> **Definition: Vector**
>
> *A vector is a quantity with both magnitude and direction.*

We will frequently represent a vector quantity with an arrow, where the direction of the vector is the direction that the arrow points, and the magnitude of the vector is represented by the length of the arrow. This is not to say that vectors *are* arrows – arrows just make a handy geometric representation. So while an arrow representing a vector might be $6cm$ long, that doesn't mean that the vector has a magnitude of $6cm$. The vector might represent the speed and direction of a moving object, for example, and then the vector's magnitude isn't even in units of cm. However, if we draw two arrow representations of the same sort of quantity, and one is twice as long as the other, the implication is that the longer arrow represents a vector with twice the magnitude of the vector represented by the shorter arrow.

> **Alert**
>
> *There is no way to compare magnitudes of different physical quantities. If a distance vector is drawn as an arrow on the same page as a velocity vector's arrow, the relative sizes of the two arrows are meaningless.*

There are a few other things that we should say about vectors and the arrows that represent them:

- *Where* the arrow representing a vector is positioned is not a distinct feature of the vector. That is, an arrow representing a vector can be moved at will, and so long as it isn't stretched, shrunk, or rotated, it will represent the same vector. Just changing an arrow's location does not change its magnitude or its direction if it is moved carefully.
- Vector directions (and therefore the directions of their representative arrows) can be reversed mathematically through multiplication by –1.
- Vector lengths can be expanded or shrunk (scaled) through multiplication by a regular number (called a scalar). If the number is greater than 1, the vector expands in length, and if it is less than 1, it contracts.

One other thing... When we write a symbol for a vector quantity, we will do so with a small arrow above the letter, like this: $\overrightarrow{A}$. Variables with the same letter as a defined vector that do not include an arrow, are assumed to represent the *magnitude* of that vector. So for example, when used in the same context, the variable A represents the magnitude of $\overrightarrow{A}$.

Vector Addition/Subtraction

For these mathematical quantities we call vectors to have any value to us, they have to allow for simple mathematical operations, such as addition. The directional nature of vectors makes addition much trickier than simply summing the two magnitudes. It turns out that a well-defined vector addition involves simple geometry. It goes like this: Transport one of the vectors (in a parallel fashion, so as not to change its direction) so that its tail is in contact with the head of the other vector. Then fashion a new vector such that its tail is at the open tail and its head is at the open head.

Figure 1.1.1 – Graphical Vector Addition

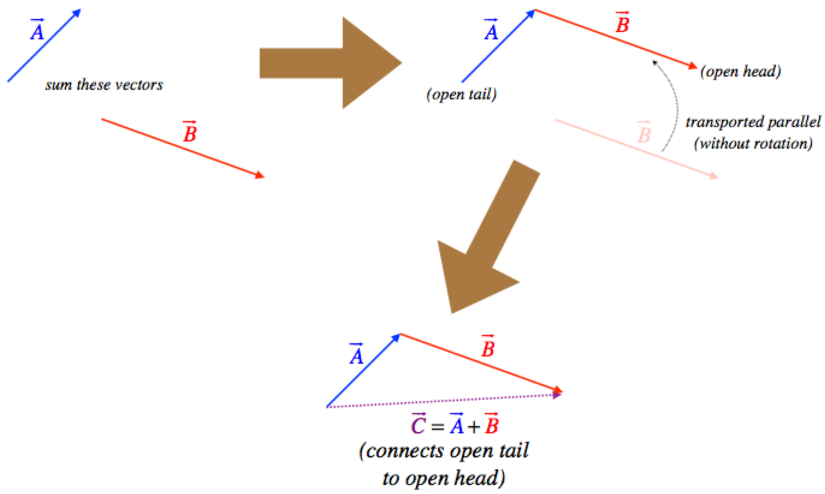
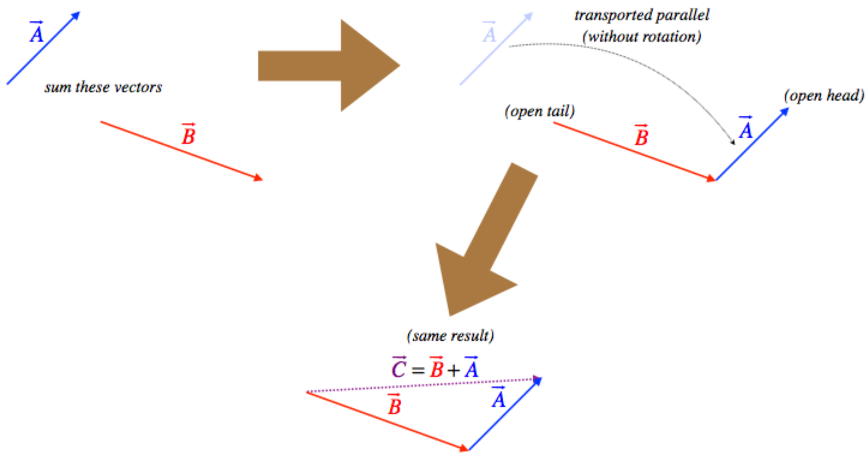


Figure 1.1.2 – Vector Addition is Commutative



What about subtracting two vectors? Well, we can do this by following the same method as for regular numbers: Whichever vector we wish to subtract we multiply by –1, and then add the result to the other vector, which we do in the manner described above. We already know that multiplying a vector by –1 reverses its direction (and leaves its magnitude unchanged), so this is a well-defined operation for us.

Vector Components

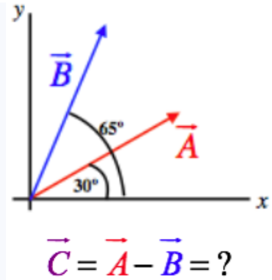
The graphical method of adding vectors are not always convenient. For example, we shouldn't have to actually measure the length of the new vector, we should be able to calculate it. Well, of course we can do this using some sophisticated knowledge of triangles. For example, given we know the lengths and directions of the two vectors we are adding, we can determine the length of the third leg of the triangle using the Law of Cosines:

$$C^2 = A^2 + B^2 - 2AB\cos\theta \tag{1.1.1}$$

With all of the lengths of the triangle legs and one of the angles (the one between A and B), we can get the other angles using the Law of Sines.

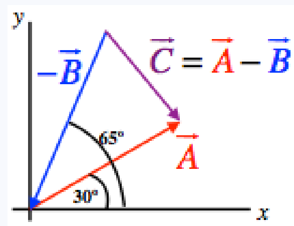
Exercise

The magnitudes of the two vectors shown in the diagram below are: $A = 132$ and $B = 145$. Find the magnitude and direction (angle made with the x-axis) of the vector that is the difference of these two vectors.



Solution

Using the fact that the negative of a vector is the same vector pointing in the opposite direction along with using tail-to-head vector addition, we get the following diagram for the three vectors:



The angle between $\overrightarrow{A}$ and $\overrightarrow{B}$ is obviously 65° – 30° = 35°, so for this triangle we have the lengths of two sides and the angle between them. We can therefore find the length of the third side ($\overrightarrow{C}$) from the law of cosines:

$$C^2 = A^2 + B^2 - 2AB\cos\theta \quad \Rightarrow \quad C = \sqrt{(132)^2 + (145)^2 - 2\,(132)\,(145)\cos(35^o)} = 84.2$$

Next we can determine the angle between $\overrightarrow{A}$ and $\overrightarrow{C}$ using the law of sines:

$$\frac{\sin 35^o}{C} = \frac{\sin\theta_{AC}}{B} \quad \Rightarrow \quad \theta_{AC} = \sin^{-1}\left[\frac{145}{84.2}\sin 35^o\right] = 81^o$$

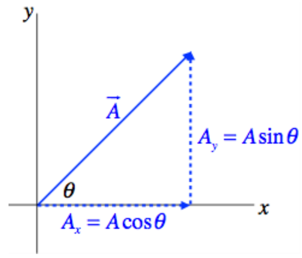
If we rotate $\overrightarrow{C}$ counterclockwise through this angle, it will be parallel to $\overrightarrow{A}$, if we then rotated it back clockwise by 30° (the angle $\overrightarrow{A}$ makes with the x-axis), then it will be parallel to the x-axis. Therefore the angle $\overrightarrow{C}$ makes with the x-axis is: –81° + 30° = –51° (below the x-axis). This answer certainly conforms with the diagram above, which shows $\overrightarrow{C}$ with a smaller magnitude than $\overrightarrow{A}$ and $\overrightarrow{B}$ and pointing down to the right.

While we can use these tools to mathematically solve for the sum of two vectors, it turns out that there is another way we can do it that doesn't require quite as much geometrical reasoning. This method exploits three simple facts:

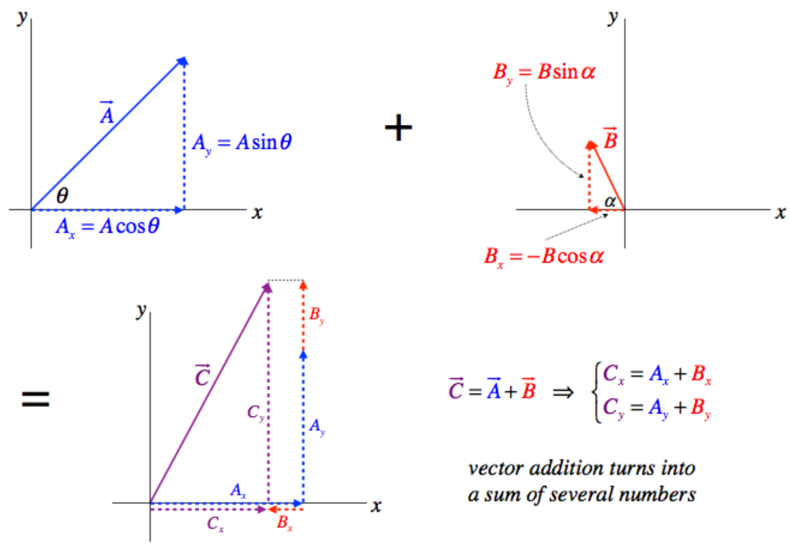
- We can replace any single vector as a sum of two (or more) vectors.
- It is easy to add two vectors that are parallel.
- If we use right triangles, trigonometry is easier to work with than with general triangles and the law of cosines/sines.

The trick is to select two (or three, if necessary) perpendicular axes (they do not have to be horizontal and vertical, they only need to be perpendicular to each other), and break up every vector involved into a sum of two perpendicular vectors parallel to these axes. The lengths of these perpendicular vectors are called the *components of the vector along those axes*. Going back to the list of advantages above, remember that we can add similar components like numbers, and we can determine these components easily using trigonometry.

Figure 1.1.3 – Vector Components



<u>Figure 1.1.4 – Summing Vectors Using Components</u>



The sums of components are like summing numbers, but only components along the same axes can be added. The results are then more components, which then have to be reconstructed into a vector.

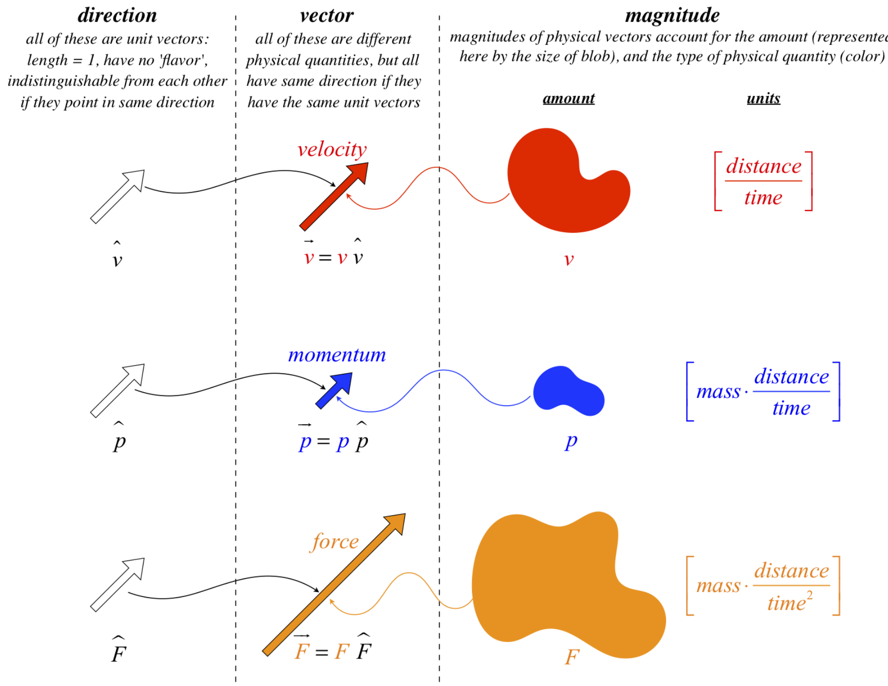
Unit Vectors

So we can use perpendicular coordinate systems to describe vectors in terms of their components. Essentially this means that to describe a vector in terms of a set of three axes, we need to know three numbers. But it might be useful to actually express these vectors as a single mathematical entity, and that's where the notion of the *unit vector* comes in. Vectors have magnitude and direction, and with unit vectors we can mathematically break up the vector into those two parts. The magnitude is just a number (with physical units) without direction, and a unit vector is a vector (without units) that has a length of 1, so that it can be scaled to any length without contributing anything to the magnitude. Therefore we can write a vector as a simple product:

$$\vec{A} = A\widehat{A} \tag{1.1.2}$$

where $\widehat{A}$ is the unit vector (usually pronounced "A-hat"). It is a unitless vector of length 1 that points in the direction of the vector $\vec{A}$. The value A is a number with physical units that equals the magnitude. The diagram below gives a graphic description of how this construction works for a few common physical vectors. The unit vectors provide a very basic template by defining the direction, and the magnitude fills in the template by contributing the girth and 'flavor' (physical units) of the vector.

<u>Figure 1.1.4 – Unit Vectors and Magnitudes</u>



If we combine this notion with components, we can write any vector as a sum of components multiplying unit vectors in the directions of the three spatial dimensions. By convention, we give these unit vectors the names $\hat{i}$, $\hat{j}$, and $\hat{k}$ for the axes x, y, and z, respectively. So specifically, we have:

$$\overrightarrow{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \tag{1.1.3}$$

Now we can just use this as a mathematical representation of vectors, and we do not have to appeal to geometry at all. For example,

$$\begin{aligned}\overrightarrow{C} &= \overrightarrow{A} + \overrightarrow{B} \\ &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= C_x\hat{i} + C_y\hat{j} + C_z\hat{k} \\ &= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}\end{aligned} \tag{1.1.4}$$

Giving us the same result as we got before for the components of the sum of two vectors.

Exercise

Repeat the calculation of the previous Exercise, this time using components.

Solution

Breaking the two vectors into their x and y components gives:

$$\begin{aligned}\overrightarrow{A} &= A_x\hat{i} + A_y\hat{j} = A\cos\theta\;\hat{i} + A\sin\theta\;\hat{j} = 132\cos 30^o\hat{i} + 132\sin 30^o\hat{j} = 114.3\;\hat{i} + 66.0\;\hat{j} \\ \overrightarrow{B} &= A_x\hat{i} + A_y\hat{j} = B\cos\theta\;\hat{i} + B\sin\theta\;\hat{j} = 145\cos 65^o\hat{i} + 145\sin 65^o\hat{j} = 61.3\;\hat{i} + 131.4\;\hat{j}\end{aligned}$$

Next we subtract $\overrightarrow{B}$ from $\overrightarrow{A}$ to get $\overrightarrow{C}$, then compute its magnitude (using the Pythagorean theorem) and direction (using trigonometry):

$$\overrightarrow{C} = \overrightarrow{A} - \overrightarrow{B} = 53.0\ \hat{i} - 65.4\ \hat{j}$$

$$\left|\overrightarrow{C}\right| = \sqrt{53.0^2 + 65.4^2} = 84.2$$

$$angle = \tan^{-1}\left(\frac{-65.4}{53.0}\right) = -51^o$$

This matches the answer found in in the previous exercise.

1.2: Vector Multiplication

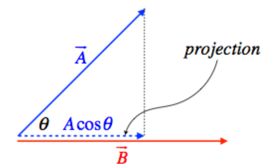
Now we know how to do some math with vectors, and the question arises, "If we can add and subtract vectors, can we also multiply them?" The answer is yes and no. It turns out that there is not one unique way to define a product of two vectors, but instead there are two…

Scalar Product

Soon we will be looking at how we can describe the effect that a force pushing on an object has on its speed as it moves from one position to another. The force is a vector, because it has a magnitude (the amount of the push) and a direction (the way the push is directed). And the movement of the object is also a vector (tail is at the object's starting point, and head is at its ending point). It will turn out that this effect is describable mathematically as the product of the amount of force and the amount of movement. This is simple to compute if the push is along the direction of movement, but what if it is not? It turns out that only the amount of push that *acts in the direction of the movement* will affect the object's speed.

We therefore would like to introduce the notion of the *projection* of one vector onto another. The best description of this is, "the amount of a given vector that points along the other vector." This could be imagined as the "shadow" one vector casts upon another vector:

Figure 1.2.1 – Projecting One Vector Onto Another



$\left(\text{amount of vector } \overrightarrow{A} \text{ that points along the direction of vector } \overrightarrow{B}\right) = A\cos\theta$

So if we want to multiply the length of a vector by the amount of a second vector that is projected onto it we get:

$$(\text{projection of } \overrightarrow{A} \text{ onto } \overrightarrow{B})(\text{magnitude of } \overrightarrow{B}) = (A\cos\theta)(B) = AB\cos\theta \tag{1.2.1}$$

This is the first of the two types of vector multiplication, and it is called a *scalar product*, because the result of the product is a scalar. We usually write the product with a dot (giving its alternative name of *dot product*):

$$\overrightarrow{A}\cdot\overrightarrow{B} \equiv AB\cos\theta, \qquad \theta = \textit{angle between } \overrightarrow{A} \textit{ and } \overrightarrow{B} \tag{1.2.2}$$

Exercise

The vector $\overrightarrow{A}$ has a magnitude of 120 units, and when projected onto $\overrightarrow{B}$, the projected portion has a value of 105 units. Suppose that $\overrightarrow{B}$ is now projected onto $\overrightarrow{A}$, and the projected length is 49 units. Find the magnitude of $\overrightarrow{B}$.

Solution

The factor that determines the length of the projection is $\cos\theta$. The angle between the two vectors is the same regardless of which vector is projected, so the factor is the same in both directions. The projection of $\overrightarrow{A}$ onto $\overrightarrow{B}$ is 7/8 of the magnitude of $\overrightarrow{A}$, so the magnitude of $\overrightarrow{B}$ must be 8/7 of its projection, which is 56 units. Note that when the projection of one vector is multiplied by the magnitude of the other, the same product results regardless of which way the projection occurs. That is, the scalar product is the same in either order (i.e. it is commutative).

Note that a scalar product of a vector with itself is the square of the magnitude of that vector:

$$\overrightarrow{A}\cdot\overrightarrow{A} = A^2\cos 0 = A^2 \tag{1.2.3}$$

It should be immediately clear what the scalar products of the unit vectors are. They have unit length, so a scalar product of a unit vector with itself is just 1.

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \tag{1.2.4}$$

They are also *mutually orthogonal*, so the scalar products with each other are zero:

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \tag{1.2.5}$$

This gives us an alternative way to look at components, which are projections of a vector onto the coordinate axes. Since the unit vectors point along the x, y, and z directions, the components of a vector can be expressed as a dot product. For example:

$$\begin{aligned} \overrightarrow{A}\cdot\hat{i} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot\hat{i} \\ &= A_x \cancelto{1}{\hat{i}\cdot\hat{i}} + A_y \cancelto{0}{\hat{j}\cdot\hat{i}} + A_z \cancelto{0}{\hat{k}\cdot\hat{i}} \end{aligned} \tag{1.2.6}$$

Unit vectors also show us an easy way to take the scalar product of two vectors whose components we know. Start with two vectors written in component form:

$$\overrightarrow{A} = A_x\hat{i} + A_y\hat{j}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j}$$

then just do "normal algebra," distributing the dot product as you would with normal multiplication:

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\hat{i}+A_y\hat{j})\cdot(B_x\hat{i}+B_y\hat{j}) \\ &= (A_xB_x)\ \hat{i}\cdot\cancelto{1}{\hat{i}} + (A_yB_x)\ \hat{j}\cdot\cancelto{0}{\hat{i}} + (A_xB_y)\ \hat{i}\cdot\cancelto{0}{\hat{j}} + (A_yB_y)\ \hat{j}\cdot\cancelto{1}{\hat{j}} \\ &= A_xB_x + A_yB_y\end{aligned} \tag{1.2.7}$$

If we didn't have this simple result, think about what we would have to do: We would need to calculate the angles each vector makes with (say) the x-axis. Then from those two angles, we need to figure out the angles between the two vectors. Then we would need to compute the magnitudes of the two vectors. Finally, with the magnitudes of the vectors and the angle between the vectors, we could finally plug into our scalar product equation.

Alert

With two different ways to compute a scalar product, it should be clear that the simplest method to use will depend upon what information is provided. If you are given (or can easily ascertain) the magnitudes of the vectors and the angle between them, then use Equation 1.2.2, but if you are given (or can easily ascertain) the components of the vectors, use Equation 1.2.7.

Exercise

The two vectors given below are perpendicular to each other. Find the unknown z-component.

$$\vec{A} = +5\hat{i} - 4\hat{j} - \hat{k} \qquad \vec{B} = +2\hat{i} + 3\hat{j} + B_z\hat{k}$$

Solution

The scalar product of two vectors is proportional to the cosine of the angle between them. This means that if they are orthogonal, the scalar product is zero. The dot product is easy to compute when given the components, so we do so and solve for B_z:

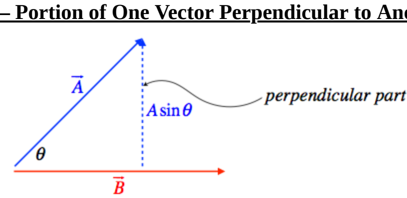
$$0 = \vec{A}\cdot\vec{B} = (+5)(+2) + (-4)(+3)\ + (-1)(B_z) \quad\Rightarrow\quad B_z = -2$$

The scalar product of two vectors in terms of column vectors works exactly how you would expect – simply multiply the similar components and sum all the products.

Vector Product

As mentioned earlier, there are actually two ways to define products of vectors. If the scalar product involves the amount of one vector that is *parallel* to the other vector, then it should not be surprising that our other product involves the amount of a vector that is *perpendicular* to the other vector.

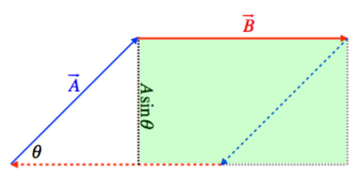
Figure 1.2.2 – Portion of One Vector Perpendicular to Another



$(amount\ of\ vector\ \vec{A}\ that\ points\ perpendicular\ to\ the\ direction\ of\ vector\ \vec{B}) = A\sin\theta$

If we take a product like before, multiplying this perpendicular piece by the magnitude of the other vector, we get an expression similar to what we got for the scalar product, this time with a sine function rather than a cosine. For reasons that will be clear soon, this type of product is referred to as a *vector product*. Because this is distinct from the scalar product, we use a different mathematical notation as well – a cross rather than a dot (giving it an alternative name of *cross product*). This has a simple (though not entirely useful, at least not in physics) geometric interpretation in terms of the parallelogram defined by the two vectors:

Figure 1.2.3 – Constructing a Vector Product (Magnitude)



$magnitude\ of\ \vec{A}\times\vec{B} = \left|\vec{A}\times\vec{B}\right| = (A\sin\theta)(B) = AB\sin\theta = area\ of\ parallelogram$

$$magnitude\ of\ \vec{A}\times\vec{B} = \left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta = AB\sin\theta \tag{1.2.8}$$

But there is another even bigger difference between the vector and scalar products. While the projection always lands parallel to the second vector, the perpendicular part implies an orientation, since the perpendicular part can point in multiple directions. Any quantity that has an orientation has the potential to be a vector, and in fact we will define a vector that results from this type of product as follows: If we follow the perimeter of the parallelogram above in the direction given by the two vectors, we get a clockwise orientation [*Would we get the same orientation if the product was in the opposite order:* $\overrightarrow{B} \times \overrightarrow{A}$ *?*]. We turn this circulation direction into a vector direction (which points in a specific direction in space) using a convention called the *right-hand-rule*:

Convention: Right-hand-rule

Point the fingers of your right hand in the direction of the first vector in the product, then orient your hand such that those fingers curl naturally into the direction of the second vector in the product. As your fingers curl, your extended thumb points in a direction that is perpendicular to both vectors in the product. This is the direction of the vector that results from the cross product.

If we perform the cross product with the vectors in the opposite order, our fingers curl in the opposite direction, which makes our thumb point in the opposite direction in space. This means that the cross product has an *anticommutative* property:

$$\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A} \tag{1.2.9}$$

A cross product of any vector with itself gives zero, since the part of the first vector that is perpendicular to the second vector is zero:

$$|\overrightarrow{A} \times \overrightarrow{A}| = A^2 \sin 0 = 0 \tag{1.2.10}$$

As with the scalar product, the vector product can be easily expressed with components and unit vectors. The vector products of the unit vectors with themselves are zero. Each of the unit vectors is at right angles with the other two unit vectors, so the magnitude of the cross product of two unit vectors is also a unit vector (since the sine of the angle between them is 1).

Convention: Right-handed Coordinate Systems

We will always choose a right-handed coordinate system, which means that using the right-hand-rule on the +x to +y axis yields the +z axis.

In terms of the unit vectors, we therefore have:

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \tag{1.2.11}$$

and

$$\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}, \qquad \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}, \qquad \hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j} \tag{1.2.12}$$

This allows us to do cross products purely mathematically (without resorting to the right-hand-rule) when we know the components, as we did for the scalar product. Again start with two vectors in component form:

$$\overrightarrow{A} = A_x \hat{i} + A_y \hat{j}$$

$$\overrightarrow{B} = B_x \hat{i} + B_y \hat{j}$$

then, as in the case of the scalar product, just do "normal algebra" (apart from the cross product being anti-commutative) by distributing the cross product, and applying the unit vector cross products above:

$$\overrightarrow{A} \times \overrightarrow{B} = (A_x \hat{i} + A_y \hat{j}) \times (B_x \hat{i} + B_y \hat{j}) \tag{1.2.13}$$

$$= (A_x B_x)\ \cancelto{0}{\hat{i} \times \hat{i}} + (A_y B_x)\ \cancelto{-\hat{k}}{\hat{j} \times \hat{i}} + (A_x B_y)\ \cancelto{+\hat{k}}{\hat{i} \times \hat{j}} + (A_y B_y)\ \cancelto{0}{\hat{j} \times \hat{j}} \tag{1.2.14}$$

$$= (A_x B_y - A_y B_x)\hat{k} \tag{1.2.15}$$

It is not obvious right now how we will use the dot and cross product in physics, but it is coming, so it's a good idea to get a firm grasp on these important tools now.

Exercise

Using the two vectors $\overrightarrow{A}$ *and* $\overrightarrow{B}$ *from the previous exercise...*

a. compute the vector product $\overrightarrow{A} \times \overrightarrow{B}$, and

b. use the result of the vector product to confirm that $\overrightarrow{A}$ and $\overrightarrow{B}$ are perpendicular to each other.

Solution

a. The vector product is a straightforward computation that only requires the distributive property and Equations 1.2.11 and 1.2.12:

$$\begin{aligned} \overrightarrow{A} \times \overrightarrow{B} &= \left(+5\hat{i} - 4\hat{j} - \hat{k}\right) \times \left(+2\hat{i} + 3\hat{j} - 2\hat{k}\right) \\ &= +10\ \cancelto{0}{\left(\hat{i} \times \hat{i}\right)} + 15\left(\hat{i} \times \hat{j}\right) - 10\left(\hat{i} \times \hat{k}\right) - 8\left(\hat{j} \times \hat{i}\right) - 12\ \cancelto{0}{\left(\hat{j} \times \hat{j}\right)} + 8\left(\hat{j} \times \hat{k}\right) - 2\left(\hat{k} \times \hat{i}\right) - 3\left(\hat{k} \times \hat{j}\right) + 2\ \cancelto{0}{\left(\hat{k} \times \hat{k}\right)} \\ &= +11\hat{i} + 8\hat{j} + 23\hat{k} \end{aligned}$$

b. We can compute the magnitudes of the three vectors $\overrightarrow{A}$, $\overrightarrow{B}$, *and* $\overrightarrow{C} = \overrightarrow{A} \times \overrightarrow{B}$ *, and use them to find the angle:*

$$\left.\begin{array}{l} \left|\overrightarrow{A}\right| = \sqrt{(+5)^2+(-4)^2+(-1)^2} = \sqrt{42} \\ \left|\overrightarrow{B}\right| = \sqrt{(+2)^2+(+3)^2+(-2)^2} = \sqrt{17} \\ \left|\overrightarrow{C}\right| = \sqrt{(+11)^2+(+8)^2+(23)^2} = \sqrt{714} \end{array}\right\} \quad \left|\overrightarrow{C}\right| = \left|\overrightarrow{A}\right|\left|\overrightarrow{B}\right|\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{714}}{\sqrt{42}\sqrt{17}} = 1 \quad \Rightarrow \quad \theta = 90^o$$

1.3: Straight-Line Motion

Start Simple

There is nothing more fundamental in the study of physics than motion. We will bring a lot of mathematics to bear on this subject (including the vectors we just learned about), but we are going to start as simply as possible – with motion of a single particle that remains on a straight line. This simplifies our task in a couple ways:

1. By restricting ourselves to single particles, we don't have to worry about the complicated motions of systems of particles, where each of the particles can move differently than the others.
2. By keeping the motion along a straight line, there are only two directions involved, and these directions can be characterized simply as "positive" and "negative" – there is no need for unit vectors.

Throughout this course, for new topics we will take this approach of starting as simply as we can in these two ways (single particle in one dimension), and only extend to more general results once the simpler case is understood. It should also be noted that it is possible to treat a system of particles as if it is a single particle when all of the particles follow the same motion. This is assured when the system of particles is a rigid object that does not rotate. Many of the examples we do for the time being will be of this variety, even if it is not explicitly stated that the object has these two properties of rigidity and zero rotation.

Displacement

In order for motion to occur for an object, obviously its *position* must change from one instant in time to another. We will refer to the coordinate position of the straight line on which the object moves as $x(t)$. A change in this position we call the *displacement*, and refer to it as a change in position:

$$\text{displacement} = \Delta x \equiv x_f - x_o \tag{1.3.1}$$

Alert

It's a good idea to get used to this now, as you will use it throughout the Physics 9-series: When referring to a time-dependent quantity, the "delta" (Δ) means "after minus before," or "final minus initial."

Notice that if the final position is a smaller number than the initial position, then the object has a negative displacement. Eventually we will treat displacement as a vector, but for our straight-line motion, the sign of the value provides all the information we need about the direction. In this text you will receive several warnings about the precise use of physics language, which is frequently at odds with how the same words are used in casual conversation. Here is the first example:

Alert

"Displacement" sounds a lot like "distance covered." Walking a mile to the store and back again is a two mile walk, but the displacement in this case is ***not*** *two miles. Displacement is a* *<u>vector</u>* *whose magnitude is the distance between the starting and ending points, and whose direction points from the starting point to the ending point.*

Average Velocity

Of course, there is more to motion than just displacement. We will generally also be interested in how fast that displacement occurs. We therefore define a rate called the *average velocity* thus:

$$\text{average velocity} = v_{ave} \equiv \frac{\Delta x}{\Delta t} = \frac{x_f - x_o}{t_f - t_o} \tag{1.3.2}$$

Since we know displacement is a vector (of course in our current simple 1-dimensional model it can only have two directions), then average velocity must be a vector as well.

Instantaneous Velocity

Just talking about the before and after gets pretty boring, so what do we do about during? That is, how do we define a velocity at a single moment in time – the instantaneous velocity? Well, we know the answer to this from calculus. We start with our idea of average velocity, and just shrink the time span down very small, until it vanishes:

$$\text{instantaneous velocity} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{1.3.3}$$

Average and Instantaneous Acceleration

We take our discussion of motion to one level more – we consider that things might speed up or slow down. Just as we defined average velocity in terms of before and after positions, we also define *average acceleration* in terms of before and after (instantaneous) velocities:

$$\text{average acceleration} = a_{ave} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_o}{t_f - t_o} \tag{1.3.4}$$

And, as before, we use calculus to extend this notion of average acceleration to instantaneous acceleration, which we describe as the amount that our object is speeding up or slowing down at a single moment in time:

$$\text{instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{1.3.5}$$

$$= \frac{d^2x}{dt^2} \tag{1.3.6}$$

Alert

Another language warning – In standard English parlance, we are used to reserving the word "acceleration" to mean only "speeding up." In physics it means specifically the rate of change of velocity, which for straight line motion includes both speeding up and slowing down (for multi-dimensional motion it gets even trickier).

Conceptual Question

If a moving object is slowing down, is it possible that the magnitude of its acceleration is increasing? If an object is speeding up, is it possible that the magnitude of its acceleration is decreasing? In either of these cases, can the magnitude of the acceleration be zero? Explain.

Solution

If an object is slowing down, then it is experiencing an acceleration in the direction opposite to its motion. If this acceleration increases in magnitude, then it slows down faster. So naturally it can be slowing down as the acceleration magnitude increases. Similarly, as an object is speeding up, it is experiencing an acceleration in the direction of its motion. If the magnitude of this acceleration decreases, then the rate at which it speeds up decreases, but it is still speeding up. If the object is either slowing down or speeding up, then its velocity must be changing, and the acceleration cannot be zero.

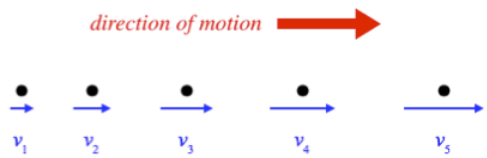
Motion Diagrams

A motion diagram starts as merely a series of collinear dots that represent the position of an object at different equally-spaced intervals of time. You can think of it as a time-lapse photograph using a strobe light. There is one other piece of information that goes with this starting diagram: the direction that the object is moving. An example of this starting point might be this:

Figure 1.3.1a – Creating a Motion Diagram

From this we need to somehow extract the instantaneous velocity (magnitude and direction, which may be changing) at each position, and the acceleration (magnitude and direction, assumed to be constant throughout) of the object. At this point we are only working qualitatively, so our goal is to sketch onto the diagram velocity vectors at each dot that have magnitudes and directions that approximately represent the velocities of the object at those points (v_1, v_2, etc.), keeping in mind that the time intervals between dots are all the same, and the acceleration is constant throughout. You can do this intuitively (it must be going faster if it covers more distance in the same time), or you can figure it out from Equation 1.3.2. Adding the instantaneous velocity vectors to the above diagram makes it look like this:

Figure 1.3.1b – Creating a Motion Diagram

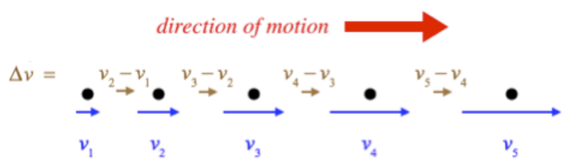


Now for acceleration. Since we are assuming constant acceleration (at least for the five-data-point interval we are considering), the average acceleration equals the instantaneous acceleration. With each dot being separated by the same time interval, the acceleration between dots is proportional to the velocity changes (magnitude and direction), and in this case of constant acceleration, is the same between every pair of dots:

$$a = constant = a_{ave} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{v_3 - v_2}{t_3 - t_2} = \ldots \tag{1.3.7}$$

Putting this into the diagram gives:

Figure 1.3.1c – Creating a Motion Diagram



Note that Δv is determined using the usual tail-to-head vector addition, which in one dimension just consists of keeping the signs straight.

If we didn't know whether or not the acceleration was constant, we could make a good guess by comparing the Δv's. Notice that we need two dots to determine the average velocity for a single time interval, since two dots gives us a displacement. But if we want to know how the speed is changing (i.e. the acceleration), we need three dots. If dots #1 and #2 are closer together than dots #2 and #3, we know the object has sped up, and if the first two dots are father apart, then the object is slowing down. So when we label our motion diagram, we can arbitrarily draw-in the first velocity vector on the first dot, but we can't add the velocity vector to the second dot if there is no third dot present to show us if the object is going faster, slower, or the same speed at the second dot. The more changes we want to consider (like if we want to know about a changing acceleration), the more dots we need.

This is in fact the nature of calculus – the change of a change of a change, etc, requires an additional measurement of position for each additional change computed. So the motion diagram only needs three dots if the acceleration is known (or assumed) to be constant, but to confirm that it is constant requires four dots.

Conceptual Question

Suppose we are given a motion diagram like the figure above, except that each velocity vector arrow is twice as long as the one before it. Does this diagram depict a constant acceleration? Explain.

Solution

Since every velocity vector is shown after the same time interval, the acceleration is only constant if the change in the velocity vector is the same each time. Let's call the magnitude of the first velocity vector "1 unit." This means that the second velocity vector ihas a magnitude of 2 units (since it doubled in length), and the one after that has a magnitude of 4 units. But this means that the change for the first time interval is 1 unit (from 1 to 2), and the change for the second interval is 2 units (from 2 ro 4). These changes are not equal – the object is speeding up more in the second interval than it did in the first – so this acceleration is not constant.

1.4: Kinematics

Equations of Motion

Okay, enough of the definitions. Let's see how these things all fit together, and how they can be used. What we will be looking at are called the *equations of motion*, and this topic is often referred to as *kinematics*. It is important to note that we are not yet dealing with causes for these motions, but only the motions themselves.

We will mostly only deal with constant accelerations (unless otherwise specified), and since instantaneous acceleration is the derivative of velocity, it is not difficult to integrate it to get the instantaneous velocity as a function of time:

$$\left.\begin{array}{l} a=\dfrac{dv}{dt} \quad \Rightarrow \quad v(t)=\int a\;dt=at+const \\ const=v(t=0)\equiv v_o \end{array}\right\} \quad v(t)=at+v_o \tag{1.4.1}$$

The constant of integration is found by plugging $t=0$ into Equation 1.4.1, which results in the velocity of the object at the starting time, which is typically designated as v_o.

We can play exactly the same game to obtain the equation of motion for position as a function of time, since we know how it relates to the instantaneous velocity:

$$\left.\begin{array}{l} v=\dfrac{dx}{dt} \quad \Rightarrow \quad x(t)=\int v dt=\int (at+v_o)\,dt=\frac{1}{2}at^2+v_ot+const \\ const=x(t=0)\equiv x_o \end{array}\right\} \quad x(t)=\frac{1}{2}at^2+v_ot+x_o \tag{1.4.2}$$

Notice that if we have all the details of this last equation, we can obtain the velocity equation above simply by taking a derivative. We cannot go in the opposite direction without also obtaining the starting position.

Analyze This

A particle moves along the x-axis with an acceleration that varies linearly with time.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

*First, we note that this is **not** a case of constant acceleration, so equations 1.4.1 and 1.4.2 do not apply. But the calculus we employed to get to these equations does apply, so we just need some sort of mathematical expression for acceleration to repeat that process. We are given that the acceleration varies linearly with time, so translating this into a mathematical expression gives:*

$$a(t)=\lambda t+\beta$$

where λ and β are unknown constants. Note that the acceleration at time $t=0$ is just β, so it is more descriptive to just call it a_o from this point on.

Without these constants (and some others), we cannot compute values like speeds and positions at different times, but we can still do some calculus in terms of the unknowns. The velocity of the particle at a time t in terms of the acceleration is:

$$v(t)=\int a(t)\,dt=\int(\lambda t+a_o)\,dt=\frac{1}{2}\lambda t^2+a_ot+\gamma$$

where γ is the constant of integration. We note that at $t=0$ the velocity just equals γ, so hereafter we'll just call that constant v_o.

We can repeat this process for the position of the particle as a function of time (noting that the constant of integration this time is the position at time $t=0$):

$$x(t)=\int v(t)\,dt=\int\left(\frac{1}{2}\lambda t^2+a_ot+v_o\right)dt=\frac{1}{6}\lambda t^3+\frac{1}{2}a_ot^2+v_ot+x_o$$

Let's make an accounting of all the numbers we can encounter in a constant-acceleration situation:

- independent variable: t
- dependent variables: x , v

- constants of the motion: x_o , v_o , a (acceleration is constant by assumption)

With six numbers to work with, you can imagine there are many ways to set up a problem to solve for something unknown. Everything you need to solve any such problem is provided in the above equations. However, it is often easier to put those equations together to form a new equation, to cut down on the algebra needs for certain types of problems. The most common useful re-combining of these variables involves eliminating time from the two equations, since you may be given velocities and positions. The algebra is straightforward:

$$\left.\begin{array}{l} v_f=at+v_o \;\;\Rightarrow\;\; t=\dfrac{v_f-v_o}{a} \\ x_f-x_o=\frac{1}{2}at^2+v_ot \end{array}\right\} \;\; x_f-x_o=\frac{1}{2}a\left(\frac{v_f-v_o}{a}\right)^2+v_o\left(\frac{v_f-v_o}{a}\right) \;\;\Rightarrow\;\; 2a\left(x_f-x_o\right)=v_f{}^2-v_o{}^2 \tag{1.4.3}$$

You can think of this equation as the "before/after" equation, because it deals only with starting and ending positions and velocities, and has eliminated time as an input variable.

While we are accumulating useful (though unnecessary) equations for motion with constant acceleration, we should also include the two equations that involve average velocity. The first is just a rewriting of the definition of average velocity, with the "final" position occurring at time t:

$$v_{ave}=\frac{x_f-x_o}{t}=\frac{x\left(t\right)-x_o}{t} \;\;\Rightarrow\;\; x\left(t\right)=v_{ave}t+x_o \tag{1.4.4}$$

The second equation is quite useful, though it applies *only* to motion involving constant acceleration:

$$v_{ave}=\frac{x_f-x_o}{t}=\frac{\frac{1}{2}at^2+v_ot}{t}=\frac{1}{2}at+v_o=\frac{1}{2}\left(v_f-v_o\right)+v_o \;\;\Rightarrow\;\; v_{ave}=\frac{v_o+v_f}{2} \tag{1.4.5}$$

For constant acceleration, the average velocity simply equals the arithmetic average of the starting and ending velocities. We will better see why it comes out this way when we start discussing graphing shortly.

Free-Fall

There is one type of straight-line motion that involves constant acceleration that we are all familiar with: free-fall.

We will look more closely at how to explain this in terms of forces in a future section, but assuming air resistance has a small effect (remember, we are devising a simplified model here), then it turns out (as shown by Galileo dropping stones from the Tower of Pisa, and more dramatically in the demonstration) that objects all accelerate at the same constant rate as they fall to Earth. This rate of acceleration is commonly given the symbol g, and it has the value:

$$\textit{acceleration due to gravity near the surface of the earth} = g = 9.8\,\frac{m}{s^2}$$

Note the units of distance-per-time-squared are the units of acceleration. This acceleration is of course always directed downward, and depending on our choice of coordinate system, this can be either positive or negative. Once the coordinate system is selected, the sign for g stays the same no matter which way the object is moving. If the positive direction is chosen to be upward, and the object is moving upward, then its velocity is positive and the negative value of g leads to a slowing of the object's motion. If it is moving down, then its velocity is negative, and the negative acceleration leads to the velocity becoming more negative (i.e. it is speeding up).

Analyze This

A ball is thrown vertically upward at the same instant that a second ball is dropped from rest directly above it.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Both balls are under the influence of the earth's gravity, and therefore both accelerate at a rate g downward. However, one starts with a velocity in the upward direction, while the other starts from rest. With both balls subject to the same velocity equation:

$$v(t) = -gt + v_o$$

their different values for v_o ensure that they will always have different velocities. In particular, when the ball thrown upward is eventually moving downward, then assuming there is no collision, its speed will always be less than the speed of the other ball. This means that if the two balls start sufficiently high above the Earth's surface, they are guaranteed to eventually collide.

Suppose the higher ball starts at a distance of y_o above the lower ball. Calling the initial speed of the lower ball v_o, and calling its starting height zero, we can write down the equations of motion of both balls:

$$\begin{array}{ll} upper: & y_{higher}(t) = -\frac{1}{2}gt^2 + y_o \\ lower: & y_{lower}(t) = -\frac{1}{2}gt^2 + v_o t \end{array}$$

Naturally, if we are interested in when the two balls collide, we simply set the two heights equal to each other. If we do this, this seemingly complicated situation reduces to something very simple, as the effects of the gravitational accelerations of the two balls cancel out, which means that they will collide when the lower ball traverses the separation with its initial velocity, ignoring the common acceleration:

$$y_{higher} = y_{lower} \quad \Rightarrow \quad v_o t_{collision} = y_o$$

1.5: Graphing

Interpreting Graphs

We conclude our discussion of straight-line motion by taking on the topic of representing motion with graphs. These graphs represent what is happening to the various dependent variables (x, v, and a) over time. There are three goals here:

1. To interpret a graph in terms of the physical motion of the object it represents.
2. To sketch a graph that represents the physical motion of an object, given a description of that motion.
3. To sketch a graph of one or two dependent variables based on the graph of another dependent variable.

Alert

These are not always easy tasks to perform, for two main reasons: First, our first instinct when we see a graph is to interpret it as a picture, rather than a plot of a quantity vs. time. The second problem (and this persists throughout the study of physics) is the tendency to confuse the change of a quantity for the value of that quantity. More precisely, we tend to lose sight of the fact that a variable's value at an instant and its rate of change are quite independent of each other.

For task #1, here are some of the questions we should be able to answer:

Q1: Where is the object (which side of the origin is it on)?

This would seem to be quite trivial (and it is): The position at any given time is the value on the vertical axis for the x vs. t graph. Where we run into trouble is thinking that we might have some idea of how to answer this question for the velocity and acceleration graphs. Those graphs only give us information about the object's changing position and changing speed, respectively, not where the object is at any given time. If we are separately given where the object is at some point in time (say at $t=0$), then we can determine its position at other times. One way to think of this is that the velocity graph gives us the *shape* of the position graph, but that shape could be located anywhere up-and-down the vertical axis. All of this is just repeating what we found in Section 1.4 – that when we integrate $v(t)$, we get an unknown constant x_o that must be provided separately.

Q2: Is the object at rest, or is it moving?

Another seemingly obvious question to answer, but again there are things to keep in mind. Although this is a property of velocity we *can* answer it using the position graph (we only get unknown constants when we integrate, not when we take derivatives). Mathematically, we know that the velocity is the slope of the position graph, so since "at rest" means zero velocity, the object is at rest when the tangent line to the x vs. t graph has zero slope. But we should strive to look at this *physically* as well. Obviously an object that is moving is one whose position is changing, so if the x value is changing, the object is moving. If we are given a v vs. t graph, we have to be careful not to use the same criterion as we did for the x vs. t graph. Instead, whether the object is moving or not is a simple matter of whether or not the value of v is zero. If we have the acceleration graph, then integrating it to get the velocity graph leaves an unknown constant (v_o). We know the shape of the v vs. t graph, but not where it is located up-and-down the vertical axis. This means that with just the acceleration graph we cannot know where the velocity graph crosses the horizontal axis, and therefore have no idea where the object is coming to rest.

Q3: Which way is the object moving?

The direction of motion of the object can also be obtained from both the position and velocity graphs. From the position graph, we know that the sign of the slope is the sign of the velocity (which is the direction of motion). On the velocity graph, we simply need to determine if the value of the velocity is positive or negative (i.e. is the graph below or above the horizontal axis). A common mistake is to confuse these two things. For example, the position graph being below the horizontal axis does not mean the object is moving in the $-x$ direction, and a positive slope of the velocity graph does not mean that the object is moving in the $+x$ direction. Once again, the acceleration graph does not – by itself – provide information about the direction of the object's motion, because the question of above-or-below the horizontal axis for the velocity graph cannot be answered when the acceleration graph only gives the v vs. t graph's shape.

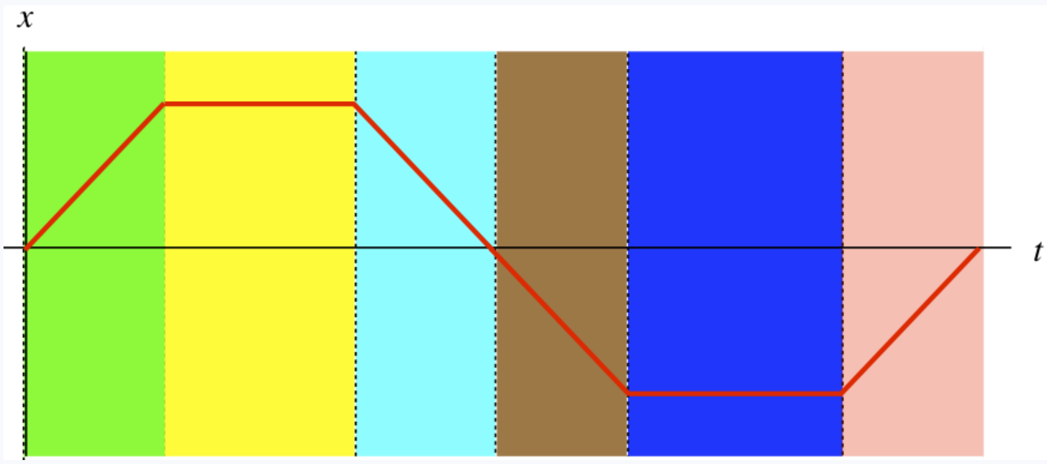
Q4: Is the object speeding-up or slowing down?

This is probably the trickiest question of all, because it doesn't have a direct correlation to the value or slope of any of the graphs. To make this determination, you actually need *two* pieces of information – the directions of both the velocity and the acceleration. This is because if the object is accelerating in the same direction that it is moving, then it is speeding up, and if it is accelerating in the opposite direction as the direction of motion, then it is slowing down. We therefore cannot determine the answer to this question from the acceleration graph alone, because that graph by itself does not provide the direction of motion (the function $v(t)$ associated with this acceleration could be above or below the horizontal axis anywhere). We can determine speeding-up/slowing-down from the v vs. t graph, by comparing the slope of the graph with the value of the graph at the same point. If they have the same sign, then the acceleration is in the same direction as the velocity, and it is speeding up. If they are opposite, then it is slowing down. But there is a simpler, physical way to make this determination: If the v vs. t graph at the point in question is heading closer to the horizontal axis, then its velocity is heading toward zero, and it is slowing down, while if it is heading away, it is speeding up. Naturally horizontal parts of the the v vs. t graph represent motion in which the object is neither speeding up nor slowing down.

Making this determination from the x vs. t graph is even more challenging. Clearly if a section of the x vs. t graph is a straight line, then the velocity is constant, and the object is neither speeding up nor slowing down. So what about when $x(t)$ is curved? The trick to use here is to determine if continuing this curve will eventually cause the graph to go horizontal (i.e. reach a max or a min), or vertical. If it is the former, then the object is slowing (a horizontal slope is stationary), and the latter is speeding up. Note that both of these can occur for either concave or convex curves, for positive or negative slopes, and above or below the horizontal axis.

Exercise

For the position vs. time graph of an object moving in one dimension given below answer each of the four questions given above for every segment of time indicated by the different colors.



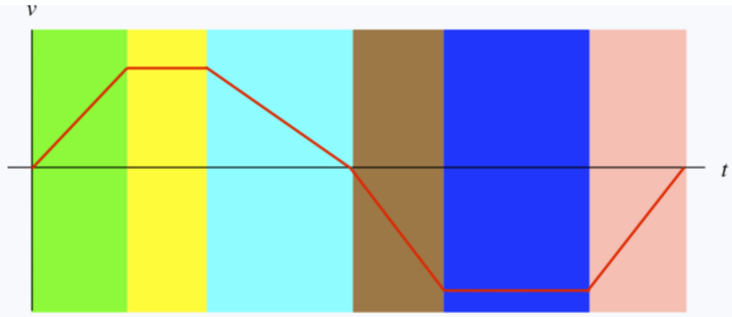
Solution

Where is the object?	*At rest or moving?*	*Moving which way?*	*Speeding up or slowing down?*
+ side of origin	***moving***	***+x direction***	***neither… constant speed***
+ side of origin	***at rest***	***not moving***	***neither… constant speed***
+ side of origin	***moving***	***–x direction***	***neither… constant speed***
– side of origin	***moving***	***–x direction***	***neither… constant speed***
– side of origin	***at rest***	***not moving***	***neither… constant speed***
– side of origin	***moving***	***+x direction***	***neither… constant speed***

Exercise

For the velocity vs. time graph of an object moving in one dimension given below:

a. *Answer each of the four questions given above for every segment of time indicated by the different colors.*
b. *Sketch the position vs. time and acceleration vs. time graphs associated with this same motion. Assume that the object was at the origin at time* $t=0$.

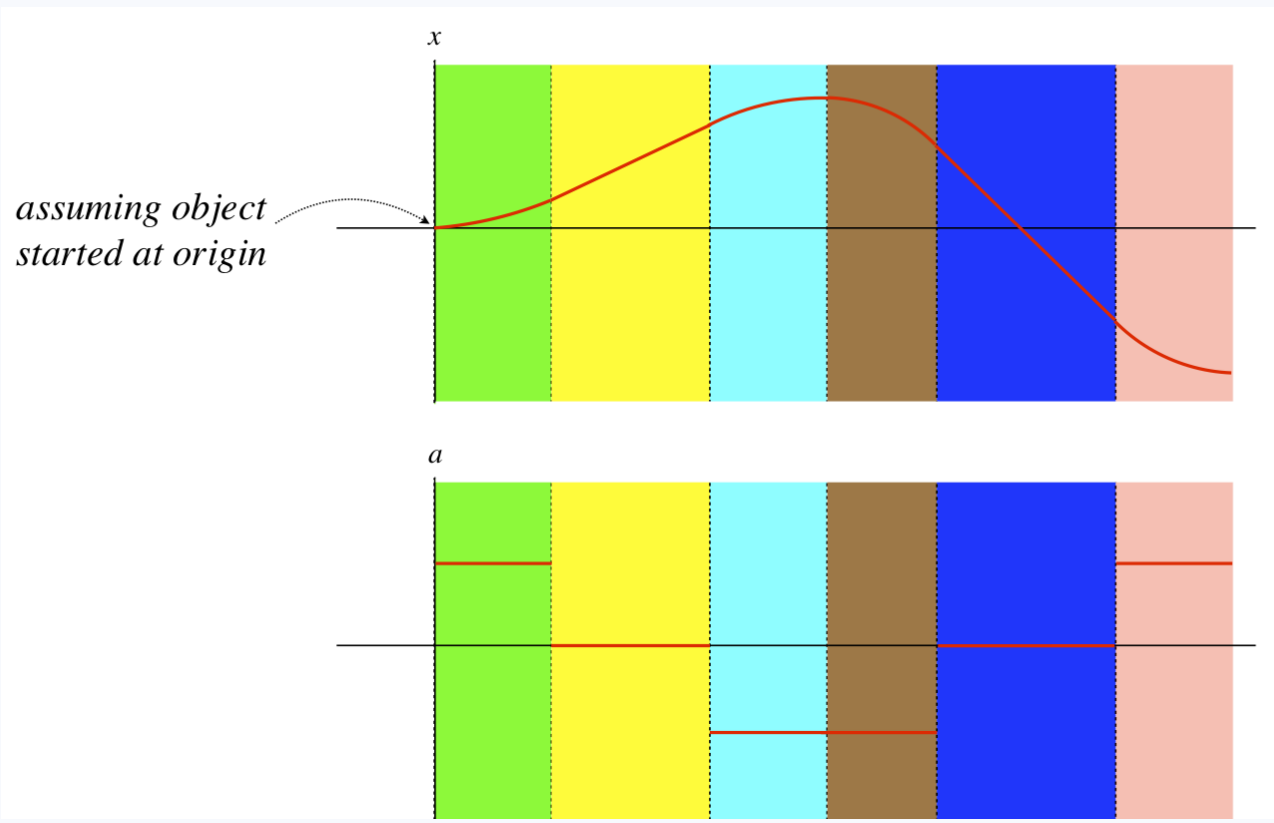


Solution

a.

Where is the object?	At rest or moving?	Moving which way?	Speeding up or slowing down?
don't know	moving	+x direction	speeding up
don't know	moving	+x direction	neither… constant speed
don't know	moving	+x direction	slowing down
don't know	moving	–x direction	speeding up
don't know	moving	–x direction	neither… constant speed
don't know	moving	–x direction	slowing down

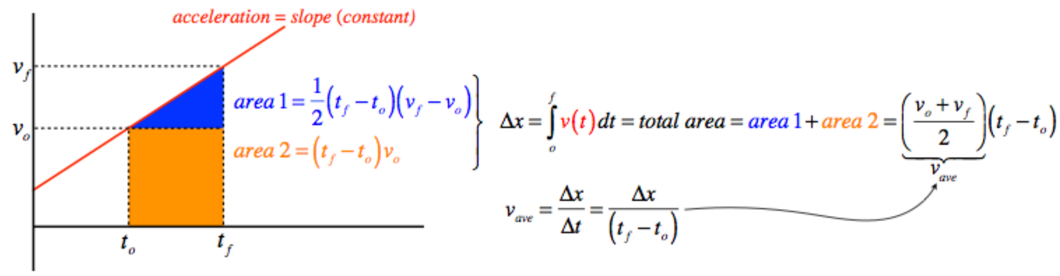
b.



Integrating Using Graphs

We have already seen that we can derive equations of motion for $v(t)$ and $x(t)$ by integrating their derivatives, and we know that integrals of functions equal the areas under the curves those functions represent, so we can use this knowledge to tie together these two facts. If we are given the graph of a motion, we can compute the area under the curve between the starting and ending points to get a definite integral, and therefore the change between the starting and ending values. So for example, if we again assume constant acceleration, a velocity-vs-time graph is a straight line whose slope is the acceleration. The area under this line from the starting time to the ending time will be the displacement between these two times (note: we still don't know the positions for these times, only the change in positions). This actually demonstrates the average velocity relation we found earlier:

Figure 1.5.1 – Area Under Velocity Curve Is Displacement



Notice that it is vital that the acceleration is constant for this formula for average velocity to come out, because the area under the curve involves the area of a triangle that requires a straight line on top. Of course, the average velocity could *accidentally* come out to equal the arithmetic average of the starting and ending velocities when the acceleration is not constant (if the area under the curved graph happens to equal the area under the straight line graph between the same two points), but we cannot rely on such coincidences when solving problems. Moreover, this means that we cannot assume the converse – if the arithmetic mean of a starting and ending velocity equals the average velocity, we cannot conclude that the acceleration was constant over that time interval.

1.6: Motion in Multiple Dimensions

Now that we have mastered the simplest form of motion, it's time to branch out to more general cases. We will continue to only consider the motion of individual particles (or equivalently rigid objects that don't rotate), but bo longer will the motion of objects be constrained to move along a straight line. Of course, this means that we can no longer allow simple positive and negative values to tell us about directions – we need to introduce vectors into the story. Fortunately, we have built our vector mathematics tools to the point where we can make use of them here.

Position and Displacement

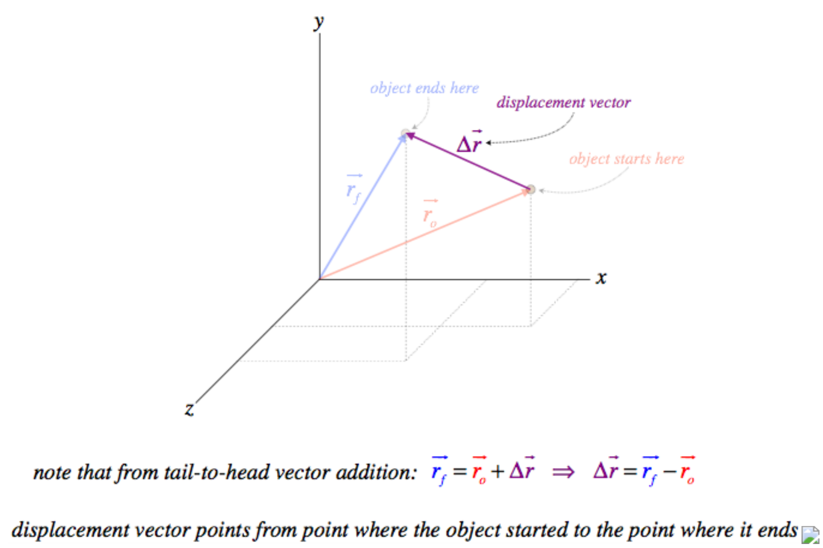
Without the luxury of being able to describe the position of an object with a single (positive or negative) value, we now have to do so in terms of something called a *position vector*. If we assume a coordinate system is in place, the position of the object can be described by its coordinates, x, y, and z. These also happen to be the components of the position vector, which we define as the vector that points from the origin to the point in space:

$$\overrightarrow{r}=x\hat{i}+y\hat{j}+z\hat{k} \tag{1.6.1}$$

If an object moves from one position to another, then clearly it is displaced, and we can describe this displacement in terms of the change of the position, as we did for straight-line motion. The only difference is that here we create a *displacement vector*:

$$\Delta\overrightarrow{r}=\overrightarrow{r}_f-\overrightarrow{r}_o=(x_f-x_o)\,\hat{i}+(y_f-y_o)\,\hat{j}+(z_f-z_o)\,\hat{k} \tag{1.6.2}$$

Figure 1.6.1 – Displacement Vector



Velocity

We follow the same process as we did with straight-line motion to determine average and instantaneous velocity vectors. Namely, we define the average and instantaneous velocities in terms of the rate of change of the displacement:

$$\overrightarrow{v}_{ave}=\frac{\Delta\overrightarrow{r}}{\Delta t}=\frac{\Delta x}{\Delta t}\hat{i}+\frac{\Delta y}{\Delta t}\hat{j}+\frac{\Delta z}{\Delta t}\hat{k} \tag{1.6.3}$$

$$\overrightarrow{v}=\frac{d\overrightarrow{r}}{dt}=\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k} \tag{1.6.4}$$

While this vector formula might appear to imply that the direction of the velocity vector is the same as the direction of the position vector, it's important to understand that in fact the direction of the velocity vector is the same as the direction of *the infinitesimal change* of the position vector. Let's look at an example that makes this clear...

Consider a particle moving in the $x-y$ plane. Its time-dependent position vector can be expressed in terms of its time-dependent components as:

$$\overrightarrow{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

For the sake of this example, let's suppose that the particle's position components have the following time dependences:

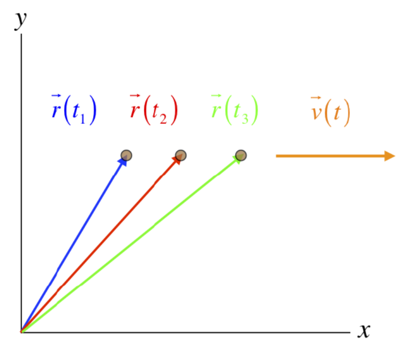
$$x(t) = \left(3.0\frac{m}{s}\right)t \qquad y(t) = 5.0m$$

Now we can calculate the velocity vector by taking the time derivative of the position vector. The unit vectors $\hat{i}$ and $\hat{j}$ don't change with time, so the derivative is simply:

$$\overrightarrow{v}(t) = \frac{d}{dt}\overrightarrow{r} = \left[\frac{d}{dt}x(t)\right]\hat{i} + \left[\frac{d}{dt}y(t)\right]\hat{j} = \left(3.0\frac{m}{s}\right)\hat{i}$$

The position vector changes in both magnitude and direction, while the velocity vector does neither. This demonstrates that the formula that relates the position and velocity vectors might appear to imply some kinship between these vectors, but in fact the presence of the derivative removes the possibility of generalizations like them pointing in the same direction. A diagram of this example for three different times should help visualize this difference:

Figure 1.6.2 – Comparison of a Position Vector and the Related Velocity Vector



Acceleration

Naturally acceleration works the same way as velocity in terms of the calculus:

$$\overrightarrow{a}_{ave} = \frac{\Delta\overrightarrow{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} + \frac{\Delta v_z}{\Delta t}\hat{k} \tag{1.6.5}$$

$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} \tag{1.6.6}$$

Alert

Notice that if we confine ourselves to motion in just one dimension (say the x-axis), then we get exactly the equations we obtained in Section 1.3. So what motion in three dimensions amounts to is additional bookkeeping – we have three separate sets of kinematic relations to keep track of, instead of only one.

Splitting Direction and Magnitude – Velocity

Alert

Students occasionally struggle with what follows, perhaps because the idea of a unit vector is still a bit abstract to them. If you find yourself in this situation, you should spend a little extra time to become comfortable with these ideas, as they are central to everything that follows in this course.

We know that whenever we take the derivative of a vector like position (to obtain velocity) or velocity (to obtain acceleration), a non-zero result comes about when that vector is changing in some way (magnitude, direction, or both). Let’s see what happens if we split these two vector properties up and treat them separately...

The unit vectors we have encountered to this point have been exclusively the *cartesian unit vectors* – those that point in the x, y, and z directions: $\hat{i}$, $\hat{j}$, and $\hat{k}$. When we first encountered unit vectors, we saw that a vector can be written as a product of its magnitude and the unit vector that points in its direction (Equation 1.1.2). If this vector happens to be changing direction over time, then unlike the cartesian unit vectors, this unit vector changes over time. As a first example, let's look at what this means for the position vector (the derivative of which is the velocity vector). We know how to express the position vector in terms of cartesian unit vectors (Equation 1.6.1), but in terms of its magnitude and directional unit vector, it is written in the same manner as Equation 1.1.2:

$$\overrightarrow{r} = r\,\hat{r} \tag{1.6.7}$$

Recall that when a vector's variable name (in this case, r) is written without the arrow over it, it refers to the magnitude of the vector. In this case, this magnitude is, in terms of the cartesian components:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{1.6.8}$$

Referring back to Figure 1.6.2, we see a case where both the magnitude and direction of the position vector are changing. Therefore when we compute the velocity vector, the derivative will act on both the magnitude and on the unit vector, and it turns out that the usual product rule works perfectly well here:

$$\overrightarrow{v} = \frac{d}{dt}\overrightarrow{r} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \tag{1.6.9}$$

Alert

This was just stated above, but it bears repeating... The cartesian unit vectors don't change with time (they always point in the x, y, and z directions), but other unit vectors (like $\hat{r}$) can and do change with time, so their derivatives don't automatically vanish. It is the derivative of this unit vector that determines how that vector's direction is changing.

Okay, so let's consider the following questions:

Conceptual Question

How is Equation 1.6.9 affected when the object happens to be moving either directly toward or directly away from the origin?

Solution

If the object is moving directly toward or away from the origin, then the position vector (whose tail is at the origin and head is at the object) is always pointing in the same direction, but never changes direction. Therefore the second term in that equation vanishes, leaving only the first term.

Conceptual Question

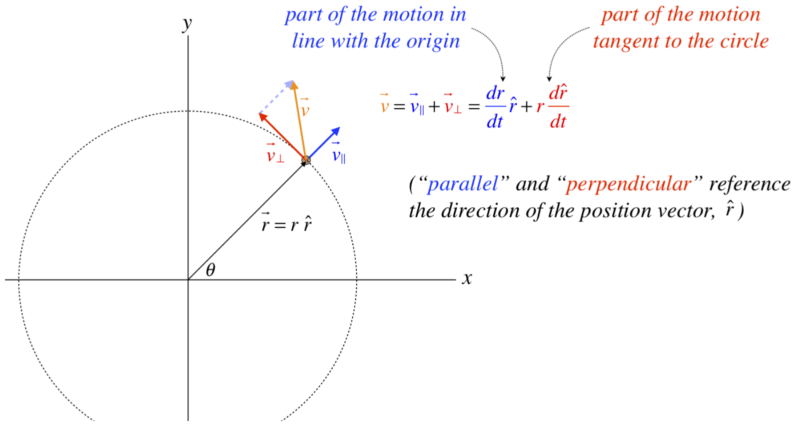
How is Equation 1.6.9 affected when the object happens to be moving such that its distance from the origin never changes?

Solution

If the object's distance from the origin never changes, then the magnitude of the position vector is not changing, which means that the first term vanishes. Clearly in order to move while staying the same distance from the origin, the direction of motion must be changing, so the second term is not zero.

But wait, an object moving such that its distance from a single point never changes must be traveling in a circle (assuming its motion remains in a plane). So this velocity vector is that of circular motion around the origin. A general velocity vector (one in which neither term from the product rule vanishes) can therefore be thought of as a vector sum of a velocity vector that points radially outward from an origin, and one that points tangent to a circle centered at that origin. Geometrically, it looks like this:

Figure 1.6.3 – Parallel and Perpendicular Components



Clearly the derivative of the position unit vector is a new vector that is perpendicular to the position unit vector. We can check to see if this is true, as well as make sense of all this by returning to the easier-to-work-with cartesian unit vector approach. We do this by writing the position vector in polar coordinates. Using θ as the angle in the diagram above, we use trigonometry to break the position vector into x and y components, written in terms of r and θ:

$$\overrightarrow{r} = r\cos\theta\;\hat{i}\; + r\sin\theta\;\hat{j} \tag{1.6.10}$$

Combining this with Equation 1.6.1 and Equation 1.6.7 gives us the position unit vector in terms of the cartesian unit vectors:

$$\hat{r} = \cos\theta\;\hat{i}\; + \sin\theta\;\hat{j} \tag{1.6.11}$$

It's easy to confirm that this unit vector has a length of 1, as it should. The claim above is that the time derivative of $\hat{r}$ is perpendicular to $\hat{r}$ itself, which we can now check directly, using our clever tool from Section 1.2 – the scalar product of these two vectors should vanish. Start by computing the derivative of the position unit vector. The cartesian unit vectors have zero derivative, but of course θ can be changing as the object moves, so:

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\cos\theta\;\hat{i}\; + \sin\theta\;\hat{j}\right) = -\sin\theta\left(\frac{d\theta}{dt}\right)\;\hat{i} + \cos\theta\left(\frac{d\theta}{dt}\right)\;\hat{j} = \frac{d\theta}{dt}\left(-\sin\theta\;\hat{i} + \cos\theta\;\hat{j}\right) \tag{1.6.12}$$

Now perform the dot product:

$$\hat{r}\cdot\frac{d\hat{r}}{dt} = \left(\cos\theta\;\hat{i}\; + \sin\theta\;\hat{j}\right)\cdot\left[\frac{d\theta}{dt}\left(-\sin\theta\;\hat{i} + \cos\theta\;\hat{j}\right)\right] = \frac{d\theta}{dt}\left(-\cos\theta\sin\theta + \sin\theta\cos\theta\right) = 0 \tag{1.6.13}$$

Exercise

Show that the time derivative of any unit vector is either zero (as in the case of $\hat{i}$) or is perpendicular to the unit vector itself (as in the case of $\hat{r}$). [Hint: The product rule for the derivative works for the scalar product.]

Solution

Naturally the derivative of the number 1 is zero, and this happens to be the result of a scalar product of a unit vector with itself, so applying the product rule:

$$0 = \frac{d}{dt}(1) = \frac{d}{dt}\left(\widehat{A}\cdot\widehat{A}\right) = \frac{d\widehat{A}}{dt}\cdot\widehat{A} + \widehat{A}\cdot\frac{d\widehat{A}}{dt} = 2\widehat{A}\cdot\frac{d\widehat{A}}{dt}$$

This can only equal zero if the vector resulting from the derivative is zero, or it is perpendicular to $\widehat{A}$.

Splitting Direction and Magnitude – Acceleration

Above we found that a velocity vector can be broken into two components – one parallel to the position vector and one perpendicular to it. The first accounts for velocity in line with the origin, and the second for velocity tangent to a circle around the origin. This has few applications in physics, because typically the choice of origin is arbitrary. But when we extend the use of this vector calculus machinery to acceleration, it gets more interesting and far more useful, as we will see.

If we replace the position vector with the velocity vector and follow the same procedure as above, we get for the acceleration:

$$\overrightarrow{a}=\frac{d}{dt}\overrightarrow{v}=\frac{d}{dt}(v\,\hat{v})=\frac{dv}{dt}\,\hat{v}+v\,\frac{d\hat{v}}{dt} \tag{1.6.14}$$

Once again we see that the product rule separates the derivative into a sum of two vectors: one that is parallel to the original velocity, and one that is perpendicular to it. For future reference, we'll right the two terms this way:

$$\overrightarrow{a}_{\parallel}\equiv\frac{dv}{dt}\,\hat{v}\qquad\overrightarrow{a}_{\perp}\equiv v\,\frac{d\hat{v}}{dt} \tag{1.6.15}$$

We already know that the acceleration vector is the rate of change of the velocity vector, and that the velocity vector includes the speed and direction of motion, so here we see that the acceleration breaks into two vectors: $\overrightarrow{a}_{||}$, which handles only the change of speed, and $\overrightarrow{a}_{\perp}$, which handles only the change of direction. If only the first term is non-zero, then the object is speeding up or slowing down in a straight line. If only the second term is non-zero, then the object is neither speeding-up, nor slowing down, but its direction of motion is changing. We will get a lot of mileage out of this division of labor in the chapters to come.

Analyze This

A particle moves through space with a velocity vector that varies with time according to:

$$\overrightarrow{v}(t)=\alpha\;\hat{i}-\beta t\;\hat{j}\;,$$

where α and β are positive constants.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

We know the velocity vector as a function of time, so a derivative with respect to time gives us the acceleration. The unit vectors $\hat{i}$ and $\hat{j}$ are unchanging, so their derivatives are zero, which means that the acceleration is constant and exclusively in the $-\hat{j}$ direction:

$$\overrightarrow{a}(t)=\frac{d}{dt}\overrightarrow{v}(t)=-\beta\;\hat{j}$$

With only one component of the velocity changing, and the other remaining fixed, the direction of motion must be changing. In addition, with the magnitude of the velocity equal to: $v=\left|\overrightarrow{v}\right|=\sqrt{v_x^2+v_y^2}$, and v_x constant while v_y changes, the magnitude of the velocity must also be changing. This means that if we were to express the velocity vector in terms of its magnitude and directional unit vector, then both terms of Equation 1.6.14 would be non-zero.

We can even determine both of these terms by doing some calculus. The first term is easier than the second to obtain directly, so we can do that and then just subtract it from the full acceleration (given above) to get the second term. The first term is $\frac{dv}{dt}\hat{v}$, where:

$$v=\sqrt{v_x^2+v_y^2}\quad\Rightarrow\quad\frac{dv}{dt}=\frac{d}{dt}\sqrt{\alpha^2+\beta^2t^2}=\frac{\beta^2t}{\sqrt{\alpha^2+\beta^2t^2}}$$

$$\hat{v}=\frac{\overrightarrow{v}}{v}=\frac{\alpha\hat{i}-\beta t\hat{j}}{\sqrt{\alpha^2+\beta^2t^2}}$$

Proceeding to the second term $v\frac{d\hat{v}}{dt}$:

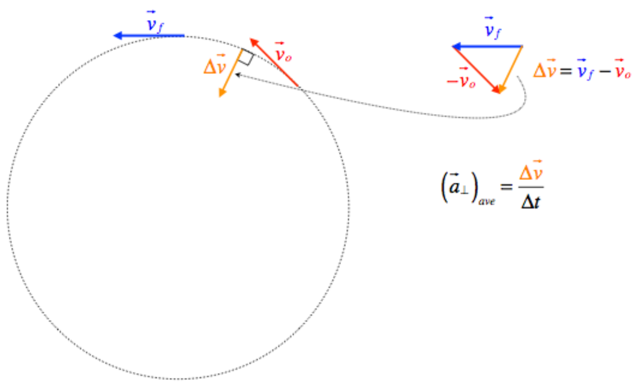
$$v\frac{d\hat{v}}{dt}=\overrightarrow{a}-\frac{dv}{dt}\hat{v}=-\beta\;\hat{j}-\beta^2t\frac{\alpha\hat{i}-\beta t\hat{j}}{\alpha^2+\beta^2t^2}$$

1.7: Examples of 2-Dimensional Motion

Circular Motion

Using what we just derived regarding the parallel and perpendicular components of acceleration, we turn now to the special case of an object traveling in a circle. The parallel part of the acceleration obviously always points tangent to the circle, which narrows it down to two directions at any given point on the circle. If the object is speeding up, then of course the tangent points in the direction of motion, and if it is slowing down, the tangent vector points in the direction opposite to the motion. The perpendicular part must be at a right angle to this tangent, which means it must be toward or away from the center of the circle. Unlike the tangent case, however, both directions are *not* possible. We can see this by considering the average perpendicular acceleration vector over two nearby moments in time:

Figure 1.7.1 – Direction of Centripetal Acceleration



this acceleration is called centripetal, which means "center seeking"

What is the magnitude of this *centripetal acceleration*? Well, it depends upon how fast the object is going (the faster it is moving, the more acceleration is required to turn in the same circle), and the radius of the circle (the acceleration is greater when the radius is smaller). Deriving the magnitude is left as an exercise, but the answer comes out to be:

$$|\overrightarrow{a}_c| = \frac{v^2}{r} \tag{1.7.1}$$

Sometimes circular motion is the result of something rotating. For example, a bug on the outer rim of a rotating turntable travels in a circle, and therefore experiences centripetal acceleration. Well, when we deal with rotating objects we often know only the rate of its rotation (say in units of revolutions per minute), and we have to translate into linear motion to know the speed. There is a simple and standard way to do this.

Digression: Radians

If we are talking about rotational motion, we need to discuss how we measure such motion. We clearly don't measure the speed of a spinning top or turntable in terms of meters per second, but rather how much it turns in a period of time. How does one measure the angle through which something turns? One way is to divide the full circle up into 360 equally-sized pie slices. The magnitude of one of these angles we call a 'degree.' But really this division is arbitrary. So why was 360º selected? Well, given that we often have to deal with slices of pie, we can avoid having to use fractional degrees if we select a number with lots of factors, and 360 certainly fits the bill – it is divisible by 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 120, and 180.

But there is no reason at all that divisibility needs to be our only criterion. In fact, we don't even need to divide the circle into an integral number of pieces. For example, we could divide it into 7.5 pieces and call the size of each piece "1 flibber." We can even translate between different systems of units: $1^f = 48^o$.

But there is another criterion that leads to a definition of a measurement of angles that is different from degrees. Suppose we want a simple translation from angle to arclength (the distance traveled along the circular curve subtended by that angle). We know that traveling around an entire circle requires a journey of a distance equal to 2π *times the radius of the circle. So going around some fraction of a circle requires a journey equal to that fraction times* $2\pi r$*. So if we divide the circle into an uneven* number of pieces such that 2π of these pieces fit into the circle, then in these units you can calculate the arclength by just multiplying the angle measured in those units (called *radians*) by the radius of the circle:

$$s = r\theta,\ \theta \text{ measured in radians} \tag{1.7.2}$$

If we want to translate between the speed that something is going around the circle to the angular speed at which the slice of the pie is being swept-out by this motion, we need only take a derivative:

$$v = \frac{ds}{dt} = \frac{d}{dt}(r\theta) = \cancelto{0}{\frac{dr}{dt}}\theta + r\frac{d\theta}{dt} = r\omega, \qquad \omega \equiv \frac{d\theta}{dt} = rate\ of\ rotation\ in\ \frac{rad}{s} \tag{1.7.3}$$

This gives us an alternative way of expressing the centripetal acceleration:

$$|\vec{a}_c| = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \tag{1.7.4}$$

Analyze This

A bead is threaded onto a circular hoop of wire which lies in a vertical plane. The bead starts at the bottom of the hoop from rest, and is pushed around the hoop such that it speeds up at a steady rate.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

The tangential motion of the bead is in one dimension, so we can use the usual kinematics equations to describe its motion along the circle. That is, equations like:

$$x(t) = x_o + v_o t + \frac{1}{2}at^2$$

can be used here, where the displacement $\Delta x = x - x_o$ is the distance measured around the arc of the circle. So if the bead goes all the way around, then Δx is just the circumference of the circle.

The fact that the bead "speeds up at a steady rate" means that the part of the bead's acceleration in its direction of motion (i.e. tangent to the circle) is a constant. This is not to say that the acceleration of the bead is constant, however. Its motion is changing direction, which means that there is a component of acceleration perpendicular to its velocity (in this case, centripetal). Given that the magnitude of centripetal acceleration depends upon the radius of the circle (which in this case doesn't change) and the speed of the object (which in this case does change), then this component of acceleration is changing in magnitude over time.

We can apply some general mathematics to this case. If we call the constant acceleration tangent to the circle a, then since it starts from rest, the speed of the bead tangent to the circle at any given time is just $v(t) = at$. If we call the radius of the circle R, then we can write down the centripetal acceleration as a function of time as:

$$a_c = \frac{v^2}{R} = \frac{a^2}{R}t^2$$

Projectile Motion

For circular motion, we have the components of velocity changing in tandem in a specific manner to keep the path circular. Another – actually simpler – form of motion involves only a single fixed coordinate component of velocity changing, while the other components involve no change in velocity. What I am alluding to here is *projectile motion*, which comes about because the vertical component of motion is subject to constant acceleration (as we discussed when we talked about free-fall), while the horizontal component is unaffected by gravity’s influence. This kind of motion is only one small step from the free-fall we are already familiar with, in that it includes a *totally independent* horizontal component of motion that incorporates (to the extent that air resistance can be ignored) no acceleration. As simple as this sounds, a couple of examples muddy the waters a bit, and sorting them out is very instructive:

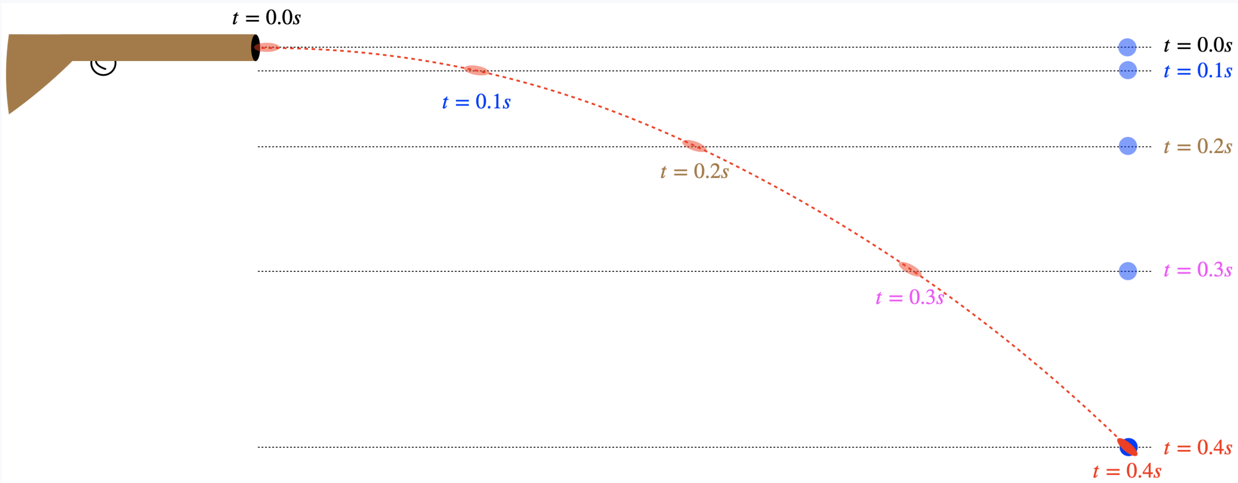
Conceptual Question

A hunter climbs a tree and fires a bullet directly at a monkey that is hanging from a branch of another tree at precisely the same height as the barrel of the hunter's gun. The instant the bullet leaves the gun, the monkey lets go of the branch. Ignoring air resistance (and the size of the monkey - assume it is very small), what is the fate of the monkey?

A. *The monkey will be hit by the bullet.*
B. *The bullet will pass beneath the monkey.*
C. *The bullet will fly over the monkey's head.*
D. *Whether the bullet flies over the monkey's head or passes beneath it depends upon how fast the bullet is moving when it leaves the barrel of the gun.*
E. *What kind of jerk shoots a monkey?*

Solution

A (and E). The vertical motion of the monkey is independent of the horizontal motion, so in equal time spans, the bullet falls the same distance as the monkey. Since they started at the same level, they remain at the same level at all times, including when the horizontal position of the bullet equals the horizontal position of the monkey.



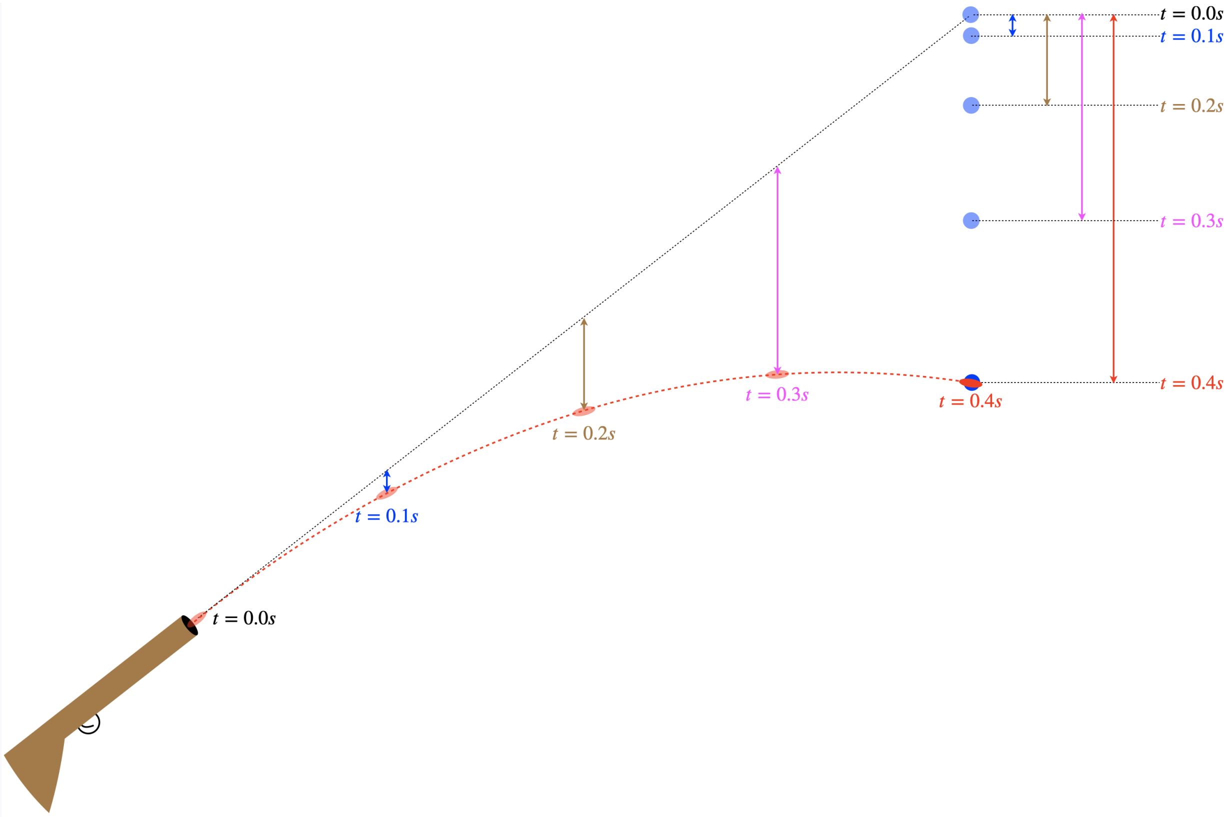
Conceptual Question

After falling out of the tree the last time he tried to shoot a monkey (his gun misfired), the hunter now decides to shoot a monkey from the ground. He is aiming upward at an angle, and is assuming the monkey will again let go of the branch just as the bullet is on its way. How should the hunter aim this time, if he is to bag his simian prize?

A. *He should aim right at the monkey.*
B. *He should aim above the monkey.*
C. *He should aim below the monkey.*
D. *Unlike the level-shot case, where he should aim this time does depend upon how fast the bullet is coming out of the gun.*
E. *The jerk should just aim at himself.*

Solution

A (and E). If there was no gravity, the bullet would follow a straight line to the monkey. With gravity acting straight down, the amount that the bullet drops below that straight line is the same that an object starting from rest on that straight line falls in an equal time. So the bullet and the monkey remain the same distance from the straight line at all times. When the bullet's horizontal position equals the monkey's horizontal position, they will coincide.



The only difference between this example and the previous one is that in the previous case, the line joining the barrel of the gun and the target was horizontal. Still, not everyone may be as convinced in this case as in the previous one, so let's do the math...

Suppose there is no gravity. The path that the bullet takes (y as a function of x) can be written down pretty easily. If the point where the bullet exits the barrel is chosen to be the origin, then the straight line to the monkey has a slope equal to the ratio of the vertical and horizontal components of velocity:

$$y = mx + b = \left(\frac{v_{oy}}{v_{ox}}\right) x$$

Now suppose there is gravity. We have separate horizontal and vertical equations of motion. Again, with the bullet starting at the origin, we have:

$$x = v_{ox} t$$
$$y = v_{oy} t - \frac{1}{2} g t^2$$

Now solve for t in the first equation and plug it into the first term of the second equation to get:

$$y = \frac{v_{oy}}{v_{ox}} x - \frac{1}{2} g t^2$$

Comparing this with the first equation above, we see that the y value would follow the same straight line if not for the second term, and the amount that the height of the bullet y is decreased from that line after a time t is exactly the same distance that the monkey falls from that line in the same time.

With the number of variables and constants involved in projectile motion problems, there are countless ways to construct problems. There is no substitute for independent thinking and creativity, but the steps given below provide a good starting point for solving these kinds of problems.

- Draw a picture, labeling it as completely as you can, using information you have been given. Then spend some time thinking about what is happening – put yourself into the situation.

Alert

*While this is given as a step for projectile problems, this is actually how you should start **every** physics problem!*

- Pick an x, y origin as well as $+x$ and $+y$ directions. Often for projectile problems up is chosen as the positive direction (making the acceleration due to gravity a negative value), but this is by no means required. What is important is that you use the positive direction consistently throughout the constants and variables in the equations.
- Break any initial velocities into components along the x and y directions.
- Write down the equations of motion, circling quantities that you know, and underlining the number you are looking for. If you have too many un-circled quantities for the number of equations available, you cannot yet do the algebra, so you'll need to review the statement of the problem for any values concealed in the language of the problem (if you just scan a problem for numbers without carefully reading it, you will miss these).
- Solve the algebra and reconstitute components back into vectors, if necessary.
- Briefly check to see if the answer makes sense.

Alert

*This is actually how you should end **every** physics problem!*

One thing in projectile motion that is a useful tool is known as the range equation. This was of particular use to military firing cannonballs or (farther back in history than that), catapults. This equation relates the distance that a projectile will fly assuming it lands at the same vertical position that it started, given the starting speed of the projectile and the starting angle. Let's treat finding this equation as if it was a projectile motion problem given to us, and follow the procedure outlined above

Analyze This

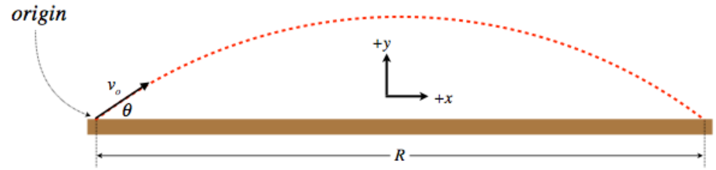
A cannonball is fired at an angle θ up from the horizontal at a speed of v_o, along level ground. Ignore air resistance in your analysis.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

The diagram below labels the origin and the + directions. It sure seems like knowing the initial angle and speed (as well as the fact that the ground is level) should be sufficient to determine how far the cannonball travels. We will therefore look for R (referred to as the "range") in terms of the initial speed v_o of the projectile and the launch angle θ, which we treat as known values. The rest of the procedure and algebra is given below.



$$R(v_o,\theta)=?$$

horizontal | *vertical*

$$v_{ox}=v_o\cos\theta \qquad v_{oy}=v_o\sin\theta$$

equations of motion

$$\begin{cases} v_x = v_{ox} + a_x t \\ \underline{x} = x_o + v_{ox}t + \frac{1}{2}a_x t^2 \end{cases} \qquad \begin{cases} v_y = v_{oy} + a_y t \\ y = y_o + v_{oy}t + \frac{1}{2}a_y t^2 \end{cases}$$

circled values we know:

$$\begin{cases} v_{ox}, v_{oy} \;\; (\text{see above}) \\ a_x=0, \;\; a_y=-g \\ x_o=0 \\ y_o=0, \;\; y=0 \;\; (\text{lands at same height where it starts}) \end{cases}$$

underlined value we are looking for: $x=R$

from second vertical equation: $0=0+\left(v_o\sin\theta\right)t-\frac{1}{2}gt^2 \;\Rightarrow\; t=\frac{2v_o\sin\theta}{g}$

plug into second horizontal equation: $R(v_o,\theta)=0+\left(v_o\cos\theta\right)\left(\frac{2v_o\sin\theta}{g}\right)+0 \;\Rightarrow\; \frac{v_o^2\left(2\sin\theta\cos\theta\right)}{g}=\frac{v_o^2}{g}\sin 2\theta$

It is interesting to note that for any given launch speed, there are in general two (complementary) angles that correspond to the same range (except for 45^o). For example, a cannonball fired at $\theta=50^o$ will land at the same place as a cannonball fired at the same speed at an angle of $\theta=40^o$.

It is also possible to determine the angle at which the range is maximized for a given velocity. Treating v_o as a constant and maximizing the function $R\left(\theta\right)$ gives:

$$0=\frac{dR}{d\theta}=2\frac{v_o^2}{g}\cos 2\theta \quad\Rightarrow\quad \theta=45^o$$

Analyze This

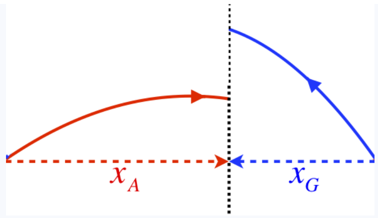
Two warlords aim identical catapults (i.e. they both release rocks at the same speed) at each other, with both of them being at the same altitude. The warlords have made the necessary computations to crush the other, and fire their catapults simultaneously. Amazingly, the two stones do not collide with each other in mid-air, but instead the stone Alexander fired passes well below the stone that Genghis shot.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

These are two projectiles fired with equal speeds to equal ranges, but using the result from the previous analysis box, we know they must have been fired at complimentary angles. While the ranges are the same, the times are not, since the time for the full flight is twice the time from the apex to the ground, which means that the projectile fired at the lower angle (which will reach a lower peak height) will reach its target first. Because Alexander's stone passes below Genghis's stone, it must have a greater x component of velocity, which means that when the stones pass by each other, the crossing point must be closer to Genghis than Alexander, and Alexander's stine is on the way down, while Genghi's is still on the way up. It looks something like this:



If we knew something about the times these projectiles take to hit their targets, we could relate the (constant) x-components of their velocities. Combining this with the knowledge that they have equal total speeds and are fired at complementary angles gives us a lot to work with.

1.8: Relative Motion

Reference Frames

Our last topic for motion in multiple dimensions relates what different observers of the same motion measure for velocities. Let's start with the following simple example:

Ann and Bob are traveling on a train together. The train is traveling north at 60 mph, and while Bob remains in place on the train, Ann runs south through the dining car at 10 mph. Bob sees Ann traveling south at 10 mph, but Ann & Bob's mutual friend Chu, who is off the train and looking through the windows, sees Ann moving *north* at 50 mph. Both Bob and Chu are witnessing the same event, but they are doing so from do distinctly different perspectives, which we call *reference frames*. As a result of being in different reference frames, Bob and Chu make different measurements of Ann's velocity vector (they disagree on both the magnitude and direction of her motion).

If we can clearly describe how the two reference frames are related to one another, we can translate between Bob's measurements and Chu's by doing the proper mathematics. In the example above the mathematics is intuitive, but we will want a systematic way of doing it for more complicated situations, such as when the motions are not along the same line. It shouldn't be surprising that the way to do this is to bring vectors into the conversation.

Relative Velocity Vectors

We begin by introducing some language. When an observer – who we will call "A" – in a given reference frame measures the velocity vector of an object (or another frame) – which we will call "B" – we express this vector in words and symbols in this way:

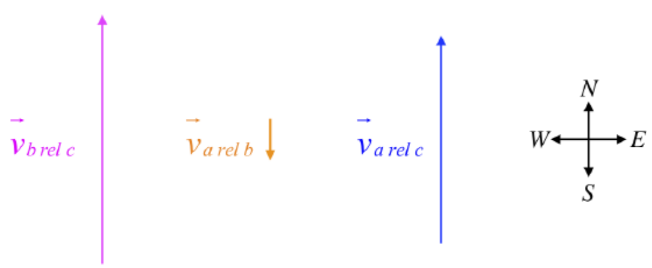
$$\text{" velocity of } B \text{ relative to } A\text{ "} \quad \Longleftrightarrow \quad \overrightarrow{v}_{B\;rel\;A} \tag{1.8.1}$$

Let's see if we can put the above example into this language. There are three entities here. Two are frames and on is a moving object. The moving object is Ann, and she is being observed by Bob, in the reference frame of the train, and Chu, in the reference frame of the earth. In the example, we expressed three different relative velocity vectors:

$$\begin{aligned} \text{" velocity of Bob relative to Chu "} &\Longleftrightarrow \overrightarrow{v}_{b\;rel\;c} = (60mph)\,\widehat{north} \\ \text{" velocity of Ann relative to Bob "} &\Longleftrightarrow \overrightarrow{v}_{a\;rel\;b} = (10mph)\,\widehat{south} \\ \text{" velocity of Ann relative to Chu "} &\Longleftrightarrow \overrightarrow{v}_{a\;rel\;c} = (50mph)\,\widehat{north} \end{aligned} \tag{1.8.2}$$

Let's represent these three vectors as arrows beside each other in a diagram:

<u>Figure 1.8.1 – Relative Velocity Vectors</u>



The first thing we notice when we look closely at these is that our intuitive understanding of the original statement of the situation can be represented as a vector addition. Placing the the tail of the first vector at the head of the second vector, we find that the third vector can connect the open tail to the open head. In other words, we can express the result of the above example as a vector addition:

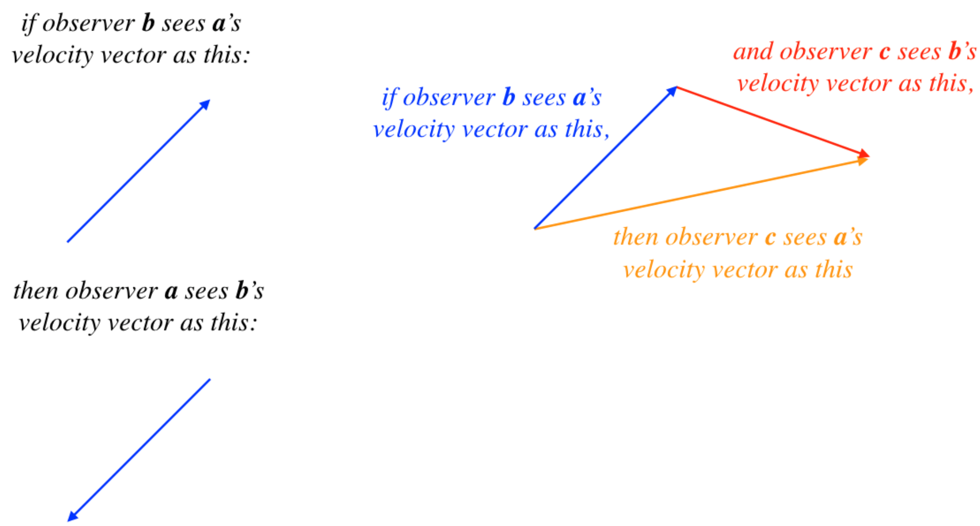
$$\overrightarrow{v}_{a\;rel\;b} + \overrightarrow{v}_{b\;rel\;c} = \overrightarrow{v}_{a\;rel\;c} \tag{1.8.3}$$

Note the ordering of the frames here is like a chain connecting Ann to Chu through Bob: Ann relative to Bob, then Bob relative to Chu, gives Ann relative to Chu. It turns out that this vector equation works not only when the velocities lie along a line, but also when they do not. For example, we can use the same vector equation if Ann were walking *across* the train (perpendicular to its motion.

There is one other feature of these relative velocity vectors that we will need, and that is reversing the perspective. In the case above, we have that Ann is moving 10 mph south relative to the Bob, but we can also talk about how Ann sees Bob moving *relative to her*. Bob starts off south of her, and as she runs by him, he ends up north of her. Therefore from Ann's perspective, Bob is moving north at 10 mph. So there is a simple way to alter a relative vector to reverse the perspective of reference frames: Switch the two frames in the subscript, and reverse the direction of the vector (i.e. multiply the original vector by –1). Here is a summary of these two rules:

Figure 1.8.2 – Summary of Relative Velocity Rules

$$\vec{v}_{a\ relative\ to\ b} = -\vec{v}_{b\ relative\ to\ a} \qquad \vec{v}_{a\ relative\ to\ b} + \vec{v}_{b\ relative\ to\ c} = \vec{v}_{a\ relative\ to\ c}$$



Analyze This

You stand on the bank of a river, contemplating swimming across, but the place where you hope to cross is just upstream of a dangerous waterfall. When you look at the speed of the river, you estimate that it is about the same speed as you are able to swim.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

This scenario involves three reference frames: the swimmer, the river, and the Earth (or equivalently, the river banks). This provides for several relative velocity combinations: swimmer relative to river, river relative to Earth, swimmer relative to Earth, and the reverse-order of all three of these. The trick to analyzing such situations is having a clear interpretation of what each of these means, and then following the two vector rules for relative motion given above. So let's start with interpretation...

The velocity of the swimmer relative to the river is related to the swimming ability – how well they can move through the water. This applies to both speed and distance. For example, if the swimmer in this case swims directly upstream, they get nowhere, because the river motion cancels the swimmer. But this doesn't mean the swimmer doesn't get a workout! The distance swam ***relative to the river*** *is not zero, and when this displacement is added to the displacement of the river relative to the Earth, then the result is zero displacement* ***relative to the Earth****. So any information given about how fast or how far the person can swim is incorporated in this "swimmer relative to river" vector.*

The velocity of the river relative to the Earth is self-explanatory - it is the magnitude and direction of the velocity of the river as you watch it go by while standing on its banks.

The velocity of the swimmer relative to the Earth is the motion of the swimmer that an observer on the riverbank sees, without regard to what the water is doing. If the person is swimming upstream, this velocity is zero in this case, and downstream it is twice as fast as the person can swim in still water. Across the stream, their are components parallel and perpendicular to the river flow.

These three velocity (or, if multiplied by a common time, displacement) vectors are related to each other according to the rules given above, with extra care being given to make sure that the order of "a relative to b" is correct, and including the proper signs where needed.

Problems such as this one often come down to using an upstream component of swimming velocity to slow or stop the rate at which the river sweeps the person downstream, while using a perpendicular component to make progress in crossing. If the swimmer (or boat) can move faster than the water, then it is possible to completely cancel downstream progress and still have some component of velocity to move across. In this case, this is not possible, since the river is flowing at the same speed as the swimmer can swim. So the swimmer is guaranteed, no matter what angle they take, to be swept downstream some amount if they want to get across.

Obviously quantities like the width of the river and how far upstream the swimmer starts will be important for most calculations related to this scenario. The specific swimming speed and river speed are unlikely to play a role, as they are given to be equal (though it is possible they could still be needed if the time of the swim is to be calculated).

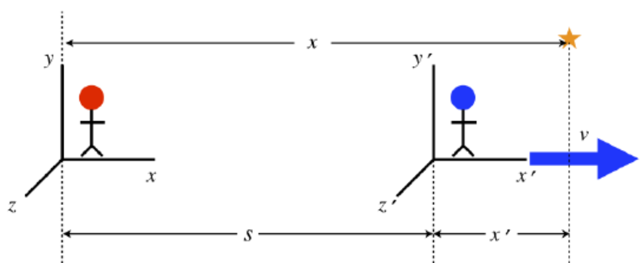
Galilean Transformation

Let's now consider two observers in difference reference frames that are moving at a constant speed relative to one another, which we will call v. We'll define the coordinate systems of these two observers such that their origins coincide at time $t=0$, and both observers agree on this starting time. Since the frames are moving relative to each other, this common origin only lasts for that one instant in time. We'll also define the coordinate systems such that they have common x, y, and z axes when their origins coincide, and have their relative motion be along their common x-axis. We will label position coordinates and time measured by the frame moving in the $+x$-direction with a prime, to distinguish it from the other frame.

Suppose both observers record the motion of the same object. One observer gets equations of motion of this object for its three spatial coordinates (x, y, z) as a function of time t, while the other observer gets equations of motion of the object for (x', y', z') as a function of time t'. The question we want to answer is, "Given what we know about how these frames are related to each other, what are the relations between the primed and unprimed coordinates?"

Let's start by noting that when the primed observer's origin has moved a distance s relative to the unprimed observer's origin, the x-component of an object's position measured in the unprimed frame will be greater than the same component measured in the primed frame by that amount:

Figure 1.8.3 – Relating Coordinates of Reference Frames



We defined the frames so that their origins coincided when each of them measured the time to be zero, so the distance s is simply equal to vt. The only difference in the two frames is in the x-direction, and the clocks are synchronized, so we have a complete translation of the two frames:

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned} \tag{1.8.4}$$

These are referred to as the *Galilean transformation equations*. They translate the coordinates of one frame into another that is moving relative to the first, with the restrictions indicated above regarding coinciding origins and so on. While this may not seem particularly interesting, keep in mind that these coordinates (when combined with cartesian unit vectors) compose the position vector, whose first derivative with respect to time is the velocity vector, etc. That is, every element of 3-dimensional kinematics – all the equations of motion of observed objects – can be translated into what they would be in another frame of reference through this transformation.

Exercise

Ann and Bob are observers from different reference frames in relative motion, with all of the conditions necessary for their coordinate systems to be related by the Galilean transformation given above (Bob is in the primed frame, moving in the x-direction relative to Ann at a speed v). Ann observes a toy rocket moving in the y-direction with a speed u. Show that the velocity vector of this same rocket as measured by Bob is the same as would be obtained using the method of relative velocity vectors described in the previous section.

Solution

Let's start by computing the velocity vector of the ball according to Bob using the Galilean transformation. Taking the derivative of the position components with respect to time gives the components of the velocity vector seen by Bob, so substituting for t' and x' in the derivative gives:

$$\left.\begin{array}{l} \dfrac{dx'}{dt'}=\dfrac{d\left(x-vt\right)}{dt}=\dfrac{dx}{dt}-v=-v \\ \dfrac{dy'}{dt'}=\dfrac{dy}{dt}=u \\ \dfrac{dz'}{dt'}=\dfrac{dz}{dt}=0 \end{array}\right\} \;\;\Rightarrow\;\; \overrightarrow{u}'=-v\hat{i}+u\hat{j}$$

Now let's use the tail-to-head relative velocity vector method from the previous section. The velocity of the rocket relative to Ann is $u\hat{j}$, and the velocity of Bob relative to Ann is $+v\hat{i}$. To get the velocity of the rocket relative to Bob, we need to form the "vector chain," which means we first need to get the velocity of Ann relative to Bob. Swapping the relative order requires only a minus sign, so doing this and putting together the vector chain gives:

$$\left.\begin{array}{l} \text{velocity of rocket relative to Bob}=\overrightarrow{u}' \\ \text{velocity of rocket relative to Ann}=u\hat{j} \\ \text{velocity of Ann relative to Bob}=-v\hat{i} \end{array}\right\} \;\;\Rightarrow\;\; \overrightarrow{u}'=u\hat{j}-v\hat{i}$$

Sample Problems

All of the problems below have had their basic features discussed in an "Analyze This" box in this chapter. This means that the solutions provided here are incomplete, as they will refer back to the analysis performed for information (i.e. the full solution is essentially split between the analysis earlier and details here). If you have not yet spent time working on (not simply reading!) the analysis of these situations, these sample problems will be of little benefit to your studies.

Problem 1.1

The acceleration of a particle moving along the x-axis is given by the equation:

$$a\left(t\right)=\left(0.300\dfrac{m}{s^3}\right)t+\left(2.40\dfrac{m}{s^2}\right)$$

The particle is at position $x=+4.60m$ and is moving in the $-x$ direction at a speed of $12.0\dfrac{m}{s}$ at time $t=0s$.

a. *Find the time at which the particle (briefly) comes to rest.*
b. *Find the position where the particle (briefly) comes to rest.*

Solution

a. From the analysis, we have an equation for the velocity of the particle at all times. Here we are given all the constants we need, namely:

$$\lambda=0.300\dfrac{m}{s^3},\quad a_o=2.40\dfrac{m}{s^2},\quad v_o=-12.0\dfrac{m}{s}$$

So all we need to do is plug these into the velocity equation, set the velocity equal to zero, and solve for the time in the quadratic equation:

$$v\left(t\right)=0=\dfrac{1}{2}\lambda t^2+a_ot+v_o\quad\Rightarrow\quad t=\dfrac{-a_o\pm\sqrt{a_o^2-4\left(\frac{1}{2}\lambda\right)v_o}}{2\left(\frac{1}{2}\lambda\right)}=4.00s$$

b. We just computed the time at which it comes to rest, and we already derived the equation for position in the analysis, so we can just plug the values in, noting that the position at time $t=0$ is given to be $x_o=+4.60m$:

$$x\left(t=4\right)=-21.0m$$

Problem 1.2

A ball is thrown vertically upward at the same instant that a second ball is dropped from rest directly above it. The two balls are $12.0m$ apart when they start their motion. Find the maximum speed at which the first ball can be thrown such that it doesn't collide with the second ball before it returns to its starting height. Treat the balls as being very small (i.e. ignore their diameters).

Solution

The balls will collide at the point in time derived in the analysis, with the starting difference in height being given as $y_o=12.0m$. The problem states that this time must be at least as long as it takes the lower ball to return to its starting point. In such a flight, the lower ball makes a total displacement of zero, so since we know its acceleration, we can solve for the time of travel in terms of the initial speed:

$$y\left(t\right)=0=-\dfrac{1}{2}gt^2+v_ot\quad\Rightarrow\quad t=\dfrac{2v_o}{g}$$

If we plug this into the equation found in the analysis that relates the starting speed to the time of collision, we will find the starting velocity for which the balls will collide exactly at the lower ball's starting height.:

$$v_ot_{collision}=v_o\left(\dfrac{2v_o}{g}\right)=y_o\quad\Rightarrow\quad v_o=\sqrt{\dfrac{gy_o}{2}}=7.67\dfrac{m}{s}$$

Clearly is the lower ball starts at any speed greater than this, then the balls will collide sooner, and they will have not yet fallen to the starting position of the lower ball.

Problem 1.3

A particle moves through space with a velocity vector that varies with time according to:

$$\overrightarrow{v}(t) = \alpha\ \hat{i} - \beta t\ \hat{j}\ ,$$

where α and β are positive constants. Find the rate at which the **speed** *of this particle is changing at time $t=0$. Does this rate remain the same for all later times?*

Solution

We already did all the math we need in the analysis. The rate of speed change is just $\frac{dv}{dt}$, which is computed in the analysis. Plugging-in $t=0$ gives a rate of speed change that equals zero! We see this does not remain true for all values of t, because it only vanishes at $t=0$.

The reason for this is that the acceleration vector is a constant, and is initially perpendicular to the velocity vector:

$$\overrightarrow{v}(0) = \alpha\hat{i} \qquad \overrightarrow{a}(0) = -\beta\hat{j}$$

So at that moment, the acceleration only changes the direction of motion (does not speed it up). But after $t=0$, the constant acceleration has not changed, and the particle is moving in a new direction, so the acceleration then does change the speed.

Problem 1.4

A bead is threaded onto a circular hoop of wire which lies in a vertical plane. The bead starts at the bottom of the hoop from rest, and is pushed around the hoop such that it speeds up at a steady rate. Find the angle that the bead's acceleration vector makes with the horizontal when it gets back to the bottom of the hoop.

Solution

As stated in the analysis, we can treat the motion tangent to the circle like any other 1-dimensional accelerated motion. In this case, the distance the bead travels is given, so the "no time" kinematics equation (Equation 1.4.3) is most applicable. Let's call the radius of the circle R and the final velocity v. The tangential acceleration is constant, the bead starts from rest, and the bead travels one circumference, so we get:

$$2a\Delta x = {v_f}^2 - {v_o}^2 \quad \Rightarrow \quad a_{\parallel} = \frac{\left(v^2 - 0^2\right)}{2\left(2\pi R\right)} = \frac{v^2}{4\pi R}$$

The centripetal acceleration is toward the center of the circle, so it points upward and its magnitude is simply:

$$a_{\perp} = \frac{v^2}{R}$$

The tangent of the angle that the full acceleration vector makes with the horizontal is the vertical component divided by the horizontal component, so:

$$\theta = \tan^{-1}\left(\frac{\frac{v^2}{R}}{\frac{v^2}{4\pi R}}\right) = \tan^{-1}(4\pi) = 85^o$$

Problem 1.5

A cannonball is fired at an angle θ up from the horizontal at a speed of v_o along level ground. A second cannonball is fired at the same speed, but at a different angle. Both cannonballs travel the same horizontal distance before landing, but one of the cannonballs takes twice as long to make the journey as the other. Find the two angles at which the cannonballs are launched.

Solution

In the analysis, we found that except for a 45^o launch angle, there are two values that correspond to the same range for a given launch velocity, and that these angles are complementary. From the vertical equation in the analysis, we have the following flight time for a given angle and launch speed:

$$t = \frac{2v_o \sin\theta}{g}$$

This applies to both cannonballs, so accumulating everything together we get:

$$\left.\begin{array}{l} t_1 = \dfrac{2v_o \sin\theta_1}{g} \\ t_2 = \dfrac{2v_o \sin\theta_2}{g} \\ \theta_1 = 90^o - \theta_2 \\ t_1 = 2t_2 \end{array}\right\} \quad \Rightarrow \quad \sin(90^o - \theta_2) = 2\sin\theta_2 \quad \Rightarrow \quad \cos\theta_2 = 2\sin\theta_2 \quad \Rightarrow \quad \theta_2 = \tan^{-1} 0.5 = 26.6^o, \quad \theta_1 = 63.4^o$$

Problem 1.6

Two warlords aim identical catapults (i.e. they both release rocks at the same speed) at each other, with both of them being at the same altitude. The warlords have made the necessary computations to crush the other, and fire their catapults simultaneously. Amazingly, the two stones do not collide with each other in mid-air, but instead the stone Alexander fired passes well below the stone that Genghis shot. Genghis is annihilated 8.0s after the catapults are fired, and Alexander only got to celebrate his victory for 4.0s before he too was destroyed.

a. *Find the maximum height reached by each of the rocks.*
b. *Find the amount of time that elapses from the launch to the moment that the rocks pass each other in the air.*
c. *Find the angles at which each warlord fires his rock.*

Solution

Conceptual analysis of this problem is found here.

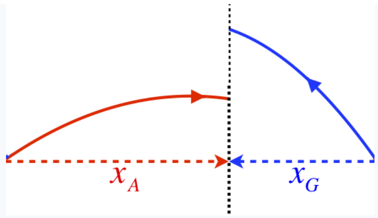
a. The time it takes a rock to travel to its peak height and back down again is equal to twice the time it takes to travel down from its peak height. Traveling down from its peak height, it starts with zero initial velocity, so we can calculate the height immediately for each rock:

$$h_A = \frac{1}{2}g\left(\frac{t_A}{2}\right)^2 = \frac{1}{2}\left(9.8\frac{m}{s^2}\right)\left(\frac{8.0s}{2}\right)^2 = 78.4m$$
$$h_G = \frac{1}{2}g\left(\frac{t_G}{2}\right)^2 = \frac{1}{2}\left(9.8\frac{m}{s^2}\right)\left(\frac{8.0s + 4.0s}{2}\right)^2 = 176.4m$$

b. The x–components of the velocities of the rocks never change, and since it takes 12s for Genghis's rock to travel the same horizontal distance as Alexander's rock traveled in 8s, Alexander's rock is traveling in the x-direction at a rate 1.5 times as great as Genghis's rock is traveling in the x–direction. When they are at the same x–position (passing each other), the distance each has traveled is each one's velocity times the time we are looking for, and we can express both of these distances in terms of the x–component of Genghis's rock using the ratio described above:

$$\begin{array}{l} x_A = v_{Ax}t, \quad v_{Ax} = 1.5v_{Gx} \quad \Rightarrow \quad x_A = 1.5v_{Gx}t \\ x_G = v_{Gx}t \end{array}$$

Since the rocks travel from both ends and are now at the same horizontal position, the sum of the distances they travel equals the total separation of the two warlords. This allows us to calculate the time:



$$x_A + x_G = x_{tot} = v_G t_{tot} \quad \Rightarrow \quad (1.5 v_G)\, t + v_G t = v_G\,(12.0 s) \quad \Rightarrow \quad t = \frac{12.0 s}{2.5} = 4.8 s$$

c. Clearly there are two different angles that will result in the rock traveling the same distance. One can see this from the range equation, but from a physical standpoint, this happens because one rock spends less time in the air but has a greater x–velocity, while the other spends more time in the air with a smaller x–velocity. To spend 1.5 times as long in the air, Genghis's rock needs to start with 1.5 times as much vertical component of velocity as Alexander's rock. This means that the ratios of the x and y components of the two rock velocities are inverses of one another, which means that the two angles are complimentary (i.e. $\theta_A = 90^o - \theta_G$ *). But the total speeds of the rocks are the same, so:*

$$\left.\begin{array}{l} v_{Ax} = v_o \cos\theta_A = v_o \cos(90^o - \theta_G) = v_o \sin\theta_G \\ v_{Gx} = v_o \cos\theta_G \end{array}\right\} \quad \Rightarrow \quad \frac{v_{Ax}}{v_{Gx}} = 1.5 = \frac{\sin\theta_G}{\cos\theta_G} \quad \Rightarrow \quad \left\{\begin{array}{l} \theta_G = \tan^{-1} 1.5 = 56.3^o \\ \theta_A = 90^o - \theta_G = 33.7^o \end{array}\right.$$

Problem 1.7

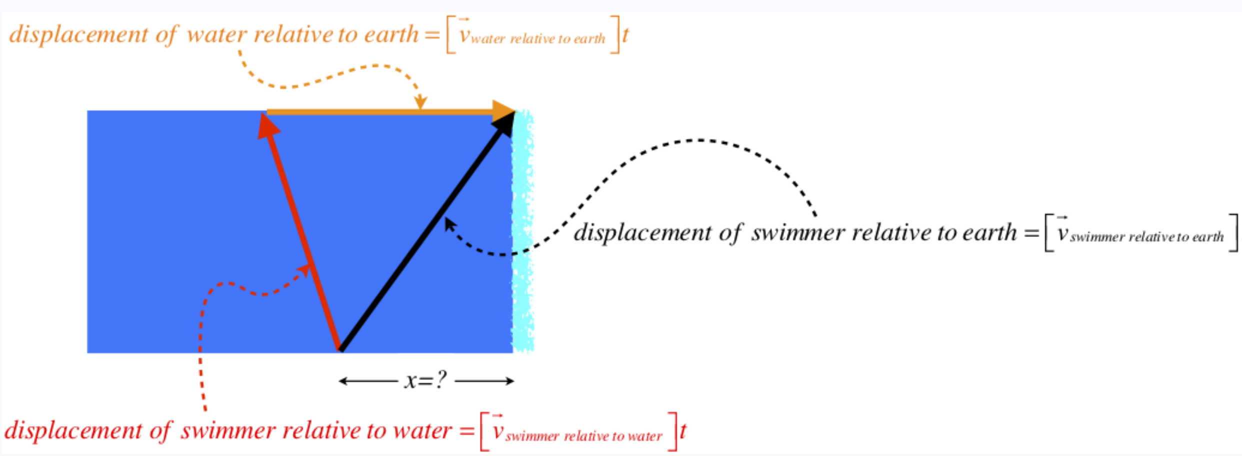
You stand on the bank of a river, contemplating swimming across, but the place where you hope to cross is just upstream of a dangerous waterfall. When you look at the speed of the river, you estimate that it is about the same speed as you are able to swim. You realize that you can only swim so far in the cold water at this speed before your muscles shut down, and in still water you estimate that this distance is about $100m$*. The width of the river is about* $80m$*.*

a. *Find the minimum distance that you must start upstream of the waterfall in order to not be swept over it.*
b. *If the river flows west-to-east and you start on its south shore, compute the direction in which you must swim in order to get safely across if you leave from the starting point computed in part (a).*

Solution

The analysis discusses the relevant reference frames in this problem: the river, the swimmer, and the Earth.

a. Clearly to minimize the distance upstream that you need to start, you must swim with a component of your velocity relative to the river being upstream. The more you are able to turn yourself upstream, the less you will float downstream, and the closer you can start to the waterfall. But there is a limit to how far you can swim relative to the water, so your angle with the river must be such that when you reach your limit relative to the river, you reach the other side. The velocities are all constant and the time spans are all equal, so they are proportional to the displacements, which we can draw:



We are given that the speed of the river relative to the earth is the same as the speed of the swimmer relative to the water, so we'll call that quantity v*, and the width of the river (which we know), we'll call* w*. From the Pythagorean theorem we can* *get the distance swum upstream against the current:*

$$distance\ swum\ against\ water = \sqrt{(vt)^2 - (w)^2}$$

The distance the water moves downstream relative to the earth is clearly vt, so the total distance the swimmer moves downstream is:

$$x = vt - \sqrt{(vt)^2 - (w)^2}$$

But we actually know the value of vt, because it is the maximum distance that the swimmer can go in the water. Plugging in all the values therefore gives our answer:

$$x = (100m) - \sqrt{(100m)^2 - (80m)^2} = 40m$$

b. The angle is easy to determine, since we know the length of the displacement vector of the swimmer relative to the water and the width of the river:

$$\cos\theta = \frac{w}{vt} = \frac{80m}{100m} \quad \Rightarrow \quad \theta = 37^o west\ of\ north$$

CHAPTER OVERVIEW

2: Force

2.1: Forces, Fundamental and Composite
2.2: Macroscopic Forces
2.3: Forces as Interactions
2.4: Effects of Force on Motion
2.5: Applications of Newton's Laws
2.6: Additional Twists - Constraints
Sample Problems

Thumbnail: A line drawing of two ice skaters demonstrating Newton's third law. Image used wtih permission (CC BY-SA 3.0 Unported; Benjamin Crowell).

2.1: Forces, Fundamental and Composite

Newton's First Law

We now understand how to handle motion in all its forms, but really we haven't done much in the way of physics, because we haven't explained what causes these different motions. In ancient times, Aristotle made the observation that eventually all things seem to come to rest, which led him to conclude that a stationary condition was "natural" for everything (well, everything on Earth – heavenly bodies never seemed to stop moving). He stated that keeping things moving requires constant pushing or pulling, or it would eventually settle into a state of rest.

This is a very intuitive way of describing the nature of things, and most people even today see the world this way. It wasn't until nearly 2000 years after Aristotle that a genius born on Christmas day would overturn that long-held belief. His name was Isaac Newton, and he claimed that in fact nature behaved in precisely the opposite manner. Newton claimed that it was not natural for objects to be at rest unless they were already at rest. If they were already moving, then it was natural for them to continue moving. He claimed that it was the fact that objects on Earth could not escape the slowing effects of pushes and pulls that accounted for them always coming to rest.

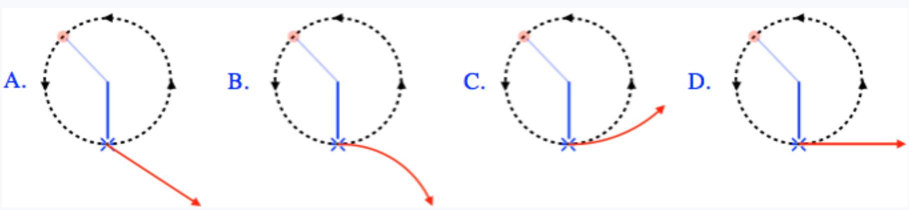
But Newton was more specific about this "natural state of motion." He stated that the only type of motion that would continue indefinitely if undisturbed by pushes or pulls was constant velocity (speed and direction) motion. That is, any motion that involved changes of speed or direction requires a push or pull.

Newton's 1st Law of Motion

Objects at rest or in motion at a constant speed in a straight line will remain in that state unless acted upon by an external influence.

Conceptual Question

A stone is swung in a horizontal circle while tied to a string, which suddenly breaks. Which of the paths below represents the motion the rock will follow? (these are paths viewed from above)



e. *It depends upon whether the rock is speeding up or slowing down at the moment the string broke.*

Solution

(d) The rock starts off moving in a circle, which means it was accelerating and according to the first law, it must have had a force on it. From the description of the motion, the only force contributing to that net force had to be the tension force by the string. At the instant that the string breaks, the force vanishes, which means that the rock can no longer accelerate. Zero acceleration means constant velocity, which means that whatever speed and direction the rock had at the moment that the string broke, it must maintain. Note that the rock has no memory whatsoever of the fact that it was accelerating just a moment before, so it neither continues accelerating for a short time, nor does it compensate for the previous acceleration by accelerating the other way.

Definition of Force

What we have been calling "pushes and pulls" or "external influences" is called *force* in physics. Most people have an intuitive idea of what force is, and like so many other physics concepts, this intuition is very likely *wrong*. We'll start by saying what force is *not*, then move on to its definition.

Alert

Force is not a quantity stored in, or possessed by, an object. Force cannot be transferred from one object to another, nor can one claim that one object "has" more force than another. This can be a hard notion to shake.

Definition: Force

Force is an interaction between two objects, which comes in the form of a push or a pull.

This simple definition belies some very difficult conceptual ideas that people (like Aristotle, and indeed every human since) struggle with, as we will soon see. The trick will be for us to develop some tools we can rely on that will help us get past our misconceptions. We will develop these tools in the sections to come, but here we will focus on the nature of forces that we encounter in everyday life. But one thing we can conclude from this is that pushes and pulls have definite directions, which means that we can conclude that *forces are vectors*.

Individual Particles vs. Systems of Particles

When we consider forces on and/or by individual particles, we find a couple of things. First, all such forces act "at a distance." That is, particles never actually touch each other – it is useful to think of particles as being merely points, with no extension in space, which makes them touching each other rather problematic. So these particles are somehow aware of each other's presence, and exert pushes and/or pulls on one another. The second thing we find is that these forces only come in a limited variety of just 4 types: gravitation, electromagnetic, and two different types of nuclear forces. It is believed that while these forces all manifest very differently (the forces depend upon different particle properties, and vary differently with particle separation), they ultimately are different manifestations of a single force. Indeed, it was once thought that the electric and magnetic forces were distinct, until it was shown quite conclusively that they are two sides of the same coin, and they are now referred to as a single force. It also happens that modern theorists have shown that one of the nuclear forces (called the "weak nuclear force") is just a different manifestation of the electromagnetic force. This combination is therefore often referred to as the "electroweak" force amongs physicists. This particular unification of seemingly disparate forces is much harder to describe to those not fluent in the languages of high-energy physics and advanced mathematics, and so the simpler (older) claim that there are four such forces lingers. These four (three) action-at-a-distance-between-individual-particles forces are called the *fundamental forces*.

Suppose now that we have two collections of particles, each of which we categorize as a "system", or more crudely, as an "object." These two objects exert forces on one another in the following way: Every particle in system #1 exerts a fundamental force on every particle in system #2. The sum of all these forces we can now call a single "force between the two objects." Clearly due to its cumulative nature, it is not "fundamental," but it is still nevertheless a force (we will use the word *composite* to describe these non-fundamental forces), in that it will cause the affected objects to no longer remain at rest or in motion in a straight line at constant speed. Given that much of what we will discuss are macroscopic systems where objects comprised of trillions of trillions of particles exert forces on each other, it makes sense to categorize some of the more common examples of these composite forces. But under it all, it is important to remember that all of these flavors of forces are just macroscopic special cases of just a few fundamental forces.

Digression: Quantum Mechanics vs. Classical Mechanics

All of the discussion here (and later in this textbook) about fundamental forces and individual particles assumes that we are employing a "classical" mechanical model for describing the universe. We have known for a long time that the realm of the very small (i.e. individual particles) does not actually function in this manner. Nevertheless, physics is about using whatever model we like that describes nature in a consistent manner that has predictive value for the conditions imposed on it. This classical model will not work if we take an extremely close look at what is happening to particles, but here we are only using this model to get a more general sense for what is happening macroscopically – the world where we look at enormous systems of particles ("objects" like chairs and bicycles) – and for this purpose, this model serves us very well.

Van der Waals Force

There is clearly a wide chasm that must be bridged in order to move a discussion from a fundamental force between two point-like particles to a force between two cars in a traffic accident. The most important step in this daunting journey boils down to a simple observation of what happens when two small clusters (systems you can call molecules, if you prefer) of particles are brought into proximity with each other...

- When they are at just the right separation, the clusters do not exert a net force on each other. Every one of the individual particles of one cluster exerts a fundamental force on every particle in the other cluster, but the sum total of these forces is zero.
- When the clusters are moved closer together than the "perfect separation" described in the previous bullet, a strange thing happens – the sum total of fundamental forces between individual particles no longer comes to zero. When they get too close,

the clusters *repel* each other.
- If the clusters are pulled slightly farther apart than the "perfect separation", then the composite force between the clusters becomes attractive.

This simple-yet-amazing property of this composite force (typically referred to as a *Van der Waals force*) resulting from many electromagnetic forces is what we have to thank for our very existence. If systems of particles could only (as in the case of individual particles) *only* exert either attractive or repulsive forces on each other, then all matter would either collapse in on itself or explode. We will not study the mathematics of this type of composite force for several chapters, but we will refer back to its characteristic properties frequently later as we describe even more crude composite forces that we will work with in the macroscopic world.

Alert

An astute reader that looks up "Van der Waals forces" (or who perhaps studied them already in a chemistry class) will undoubtedly find that this name is generally given specifically to forces between particles in a gas. Indeed, this is the specific phenomenon that Van der Waals studied. But it turns out that the property is much more robust than only applying to gases, so we are taking some license here and referring to all forces that behave similarly with this moniker. In a later chapter, we will discuss a mathematical model for this kind of force called the "Lennard-Jones potential." The point is that we should not get too worked-up about labels we give this physical behavior – it is the behavior itself that is important.

2.2: Macroscopic Forces

Macroscopic Composite Forces

As instructive as it is to discuss the microscopic underpinnings of forces, at some point we need to have some working knowledge of the macroscopic forces we will be dealing with in everyday mechanics problems. We undertake here to make an accounting of these very forces.

gravity

As mentioned in the previous section, most of our macroscopic composite forces will ultimately stem from Van der Waals forces, which themselves are a composite of just the electromagnetic force. But there is one exception – gravity. "But wait," you say, "isn't gravity a fundamental force?" Yes, when it is between two particles. But the Earth is comprised of many particles, and so is a hammer, so when a hammer is pulled downward by the Earth, the force is composite. If we were to look very closely at the details of the force between these two objects, we would find that we have to describe it somewhat differently from the fundamental force case.

Alert

In an attempt to maintain the distinction, we will refer to the fundamental force between particles (which we will study in Chapter 7) as "gravitation", and the composite force between objects (one of them almost always being the Earth) as "gravity."

What makes gravity so different as a macroscopic force from the others we will discuss here is that it doesn't display a Van der Waals sort of attract-if-pulled-apart-repel-if-pushed-together behavior between clusters. Unlike electromagnetism, there is no repulsive element of gravity, so each of the individual particles in one cluster only attracts the individual particles in the other cluster, resulting in only attractive forces between clusters.

Gravitation, like all fundamental forces, depends upon two things – a property of the gravitating particles (namely, their masses), and the separation of those particles. When we are talking about the composite gravity force on a stone at the Earth's surface, we assume that the stone never gets particularly far from that surface (even a mile above the Earth's surface is only about 1/4000th the radius of the Earth), so under the assumption that the stone never gets really far from the Earth (like outside its atmosphere), the gravity force remains only a function of the mass of the Earth, M_E, and the radius of the Earth, R_E, (both are fixed numbers), and the mass of the stone, m, (which can be different for different stones). This all boils down to a simple mathematical description of the gravity force on objects like stones: It acts downward – toward the Earth, because it is only attractive – and is proportional to the mass of the object. The constant of proportionality we will call "g", a symbol we have not coincidentally already used to represent the acceleration of a freely-falling object. It is this constant that depends on the mass and the radius of the Earth, $g = GM_E/R_E^2$, where G is the gravitational constant. We'll return to this when we cover gravitation in a later section. But for now:

$$\overrightarrow{F}_{gravity} = mg\left(-\hat{j}\right) \quad , \qquad g = 9.8\,\frac{m}{s^2} \tag{2.2.1}$$

The direction of this force (downward, toward the Earth) is expressed in the unit vector direction, $-\hat{j}$.

elastic (spring) force

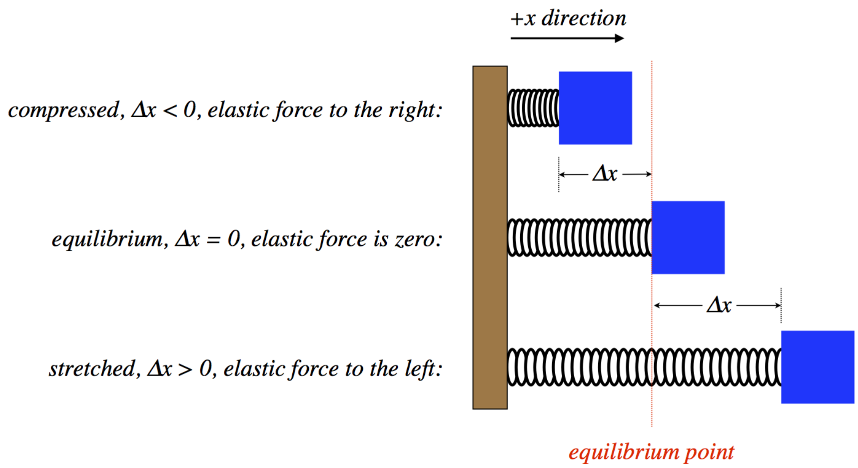
A good starting point for macroscopic manifestation of the Van der Waals effect is the force exerted by a spring, often referred to more generically as an *elastic force*. This is a macroscopic force that very closely mimics the behavior of Van der Waals forces, in that compressing the spring between two objects (moving them closer together) results in a force from the spring that seeks to push the objects apart, and stretching the spring results in a force that seeks to pull them together. There is also an "equilibrium" length of the spring at which no force is exerted at all. All forces of this nature are given the generic description of *restoring forces*, in that the force induced by making a change from equilibrium seeks to restore the equilibrium.

The similarity between the spring force and Van der Waals forces is so strong that physicists frequently use particles attached springs as a model for microscopic behavior. Naturally there are not any tiny little springs bonding molecules to each other, but the model allows for workable mathematics that yields remarkably accurate predictions. The simplified mathematics is apparent in the formula that accompanies the elastic force. The elastic force depends upon three things:

- the displacement from the equilibrium (the distance that the spring is stretched or compressed)

- the "stiffness" of the spring (usually referred to as the *spring constant*)
- whether the spring is compressed or stretched (this only affects the direction of the force, while the two previous items affect the magnitude)

Figure 2.1.1 – Elastic Restoring Force



Putting these properties together mathematically gives:

$$\overrightarrow{F} = k\left(-\overrightarrow{\Delta x}\right) \tag{2.2.2}$$

The value k is the spring constant (which is always a positive number, and measures how stiff the spring is), $\overrightarrow{\Delta x}$ is the displacement vector of the object on the spring from the equilibrium point, and the minus sign indicates the restoring nature of the force, as it always points in the opposite direction of the displacement. This formula is commonly known as *Hooke's Law*, named after a contemporary (and rival) of Newton's.

Alert

Note that the usage of the "Δ" in Hooke's law is different from how we have used it up to this point – here it refers to a difference in locations, rather than a change that occurs over a period of time.

tension

Suppose that a spring is attached to a fixed point (say a wall), and someone pulls on the other end with a certain amount of force. Naturally the spring will stretch until the Hooke's law force grows to the point where it balances the pulling force. At this point, the stretched spring also exerts the same Hooke's law force on the wall. In other words, the force the person exerts on the spring is "transmitted" all the way to the wall. If we don't care about the intermediate elements of this force (i.e. the amount the spring stretches), or equivalently, if the spring constant is so large that the stretch is negligible, then we have a simplified version of the elastic force called *tension*, which we will usually denote with the symbol "T".

There is no "formula" for tension, as we saw for gravity and springs, because it is really just a reactionary force – it is determined by other applied forces that are present. The amount that the person pulls on the string attached to the wall is the amount that the tension force pulls on the wall. This cannot be expressed as a formula involving quantities related to the string and wall.

An interesting element of tension has to do with how its transmission direction can be redirected. For example, one can pull on a rope attached to an object without the direction of the applied pull being the same as the pull on the object, if a pulley is involved. We will spend some time on the effects of pulleys in the sections to come.

Alert

One will often see the phrase "tension in the string" used in the context of physics problems. It is very important that one does not conceptually interpret this as force being stored within the string. A more accurate phrase in such cases would be "tension force exerted on the object by the string."

contact (normal) force

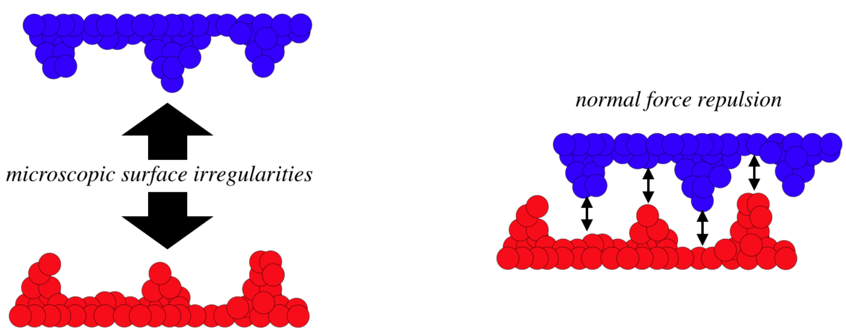
In the same way that tension is a simplified version of the elastic force in the case where a spring is stretched, *contact force* is a simplified version of the elastic force when the spring is compressed. The one major difference is that there is no intermediate object like a rope – it occurs when two objects are in direct contact with each other. Of course, what we call "contact" at the macroscopic level is really nothing of the kind microscopically. The clusters of particles at/near the outer surfaces of the two objects that are close to each other repel, thanks to Vans der Waals repulsion. This repulsion is a restoring force similar to that of a compressed spring. But like tension, for this force, we are not concerned with the details of the amount of compression or the stiffness of the springs, just that the compression ceases when the applied force is balanced by the elastic force. Like the tension force, this one is purely reactionary, and therefore has no formula that expresses it in terms of properties of the two objects in contact.

An important property of this force is its direction. In the case of tension the direction was easy – just look at which way the rope is pointing. For the contact force, the direction is always perpendicular ("normal") to the surfaces in contact. Note that the surfaces do not need to be flat – even a curved surface has a well-defined perpendicular at a given point. This property is the source for a perhaps more-commonly used name for this force, *normal force*, as well as for the most common symbol used to represent it, "N".

friction

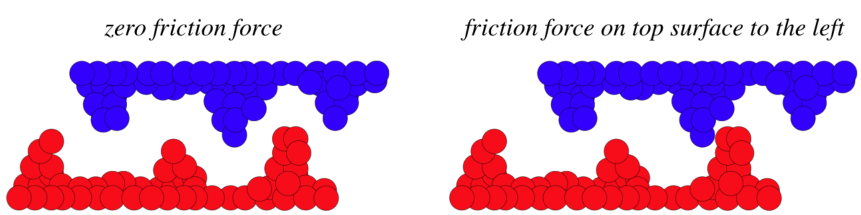
Another reason that the contact force between surfaces is referred to a normal force is that there is another force that results from two surfaces coming in contact. While the normal force is perpendicular to the surfaces, the *friction* force is the force between the surfaces that is parallel to those surfaces. As with the normal force, friction is a result of Van der Waals forces between clusters of particles on one object's surface and clusters of particles on the other object's surface. But a critical feature for friction is the microscopic irregularities that exist in the surfaces.

Figure 2.1.2a – Irregular Surfaces Pushed Together



As the surfaces are pushed together and the repulsive normal force starts to take effect, the irregularities naturally "mesh" with each other. This meshing causes attractive and repulsive forces to take effect between the peaks of the irregularities along the direction *parallel* to the surfaces (horizontally, in the diagram above). If no external force is applied to the surfaces, then these forces between peaks will balance themselves out (the objects will move very slightly across each other) to leave only the normal force between the surfaces. If, however, the surfaces are offset from this equilibrium, some of the peaks will get closer (resulting in a repulsive Van der Waals force), and some will get farther apart (resulting in an attractive Van der Waals force), and the net result is a force that opposes this displacement.

Figure 2.1.2b – Effect of Trying to Slide Surfaces Across Each Other



Considering this is a microscopic view of the surfaces, it is clear that they are not shifted very far for this friction force to take hold. Indeed, we would never even notice such a shift. Very much like the case of tension (where we do not observe the tiny stretch of a string), or normal force (where we do not notice the tiny compression of a surface), the small displacement of this friction force is unnoticeable and therefore appears purely reactionary – it occurs when we *try* to slide two surfaces across each other. This friction force is what is occurring when we try to slide a heavy box across a floor, and it won't move. The friction force comes from the restoring Van der Waals force from trillions of irregularities being displaced, and it exactly balances the applied force. Because it is not related to ongoing sliding of the surfaces across each other, this is called *static friction*.

Of course, we know that friction also occurs when the surfaces actually *do* slide across each other. The mechanism is essentially the same, just repeated over and over as bumps in one surface encounter new bumps in the other. While static friction opposed the *attempted* slide of the surfaces across each other, *kinetic friction* opposes the actual, ongoing, slide. Notice that when a surface is pushed too weakly across another to get them sliding, static friction is in effect, but as soon as that external push exceeds the ability of the static friction force to compensate, the sliding begins and kinetic friction takes over. This moment of sudden loss of the purely-reactionary static friction force is analogous to the tension force suddenly going away when the string breaks.

While the microscopic mechanism for the two types of friction are essentially the same, they do have some differences. For example, it is relevant to ask how the maximum static friction force compares to the kinetic friction force, for two surfaces under the same conditions. To answer this we consider the effect of the "depth" of meshing of the surface irregularities. If the surfaces are pushed closer together, then there are more particle clusters available to engage with each other (it's not just the tips of the peaks anymore), which should make either friction force stronger. If all else is equal, then two irregular surfaces sliding across each other is sort of "bouncing along," and the average depth of the meshing is a little less than if the surfaces are unmoving. We would therefore expect the maximum static friction force for two surfaces at rest with respect to each other to be slightly greater than the kinetic friction force when the surfaces are sliding.

So how can we express all this mathematically? First, we have already determined that the static friction force is reactionary, so there is no equation to express it. However, we also know that the magnitude of this force is limited for any given circumstance – pushing the two surfaces hard enough will get them to slide. So we can express static friction as an inequality:

$$f_{static} \leq f_{max} \tag{2.2.3}$$

Here f_{max} represents the maximum force that can exist between the surface irregularities parallel to the surfaces before they start sliding across each other. There are two factors that determine this maximum: how rough the surfaces are (how deep the pits in it go), and how far the surfaces are "meshed." The only reason they don't mesh fully is the repulsive Van der Waals forces that act perpendicularly – the normal force. When the surfaces are pushed harder against each other, increasing the normal force, they mesh more deeply, and the maximum static friction force rises. Experimentation shows that, to a good approximation, the maximum static friction force is actually proportional to that normal force, giving us:

$$f_{static} \leq \mu_s N \tag{2.2.4}$$

where μ_s is a dimensionless constant (usually less than 1) called the *coefficient of static friction*.

Once the surfaces are actually sliding across each other, the friction force is a fixed value (not less than or equal to some maximum). Once again, this fixed value experimentally is found to be approximately proportional to the normal force, giving us an equality that looks similar to the inequality above:

$$f_{kinetic} = \mu_k N \tag{2.2.5}$$

where μ_k is called the *coefficient of kinetic friction*.

Both coefficients of friction reflect properties of the surfaces. It is an oversimplification to say that they give a measure of how deep the jagged irregularities are, but this is not a terrible mental picture to have when thinking about these constants. It should also be noted that most physics problems that involve friction have wording that goes something like, "an object slides along a surface with coefficient of kinetic friction equal to...", but it is important to remember that the coefficient of friction for a single surface makes no sense – it can only really be defined in terms of both surfaces.

It is natural at this point to ask the following question: "How can the magnitude of the kinetic friction force (or the maximum static friction force) depend only upon the normal force and "roughness"? If all else is equal, wouldn't the surface area in contact also play a role? After all, more surface area means that more surface irregularities encounter each other. But more surface area also means there are more molecules repelling each other *perpendicular* to the surfaces. So suppose we increase the surface area without changing the normal force. To get the same normal force from more repulsing molecules, those molecules need to be farther apart, which means that the surfaces don't "mesh" as deeply. Less meshing means less friction force. So it turns out that the increase in the number of irregularity "encounters" that comes with more surface area is accompanied by less depth in meshing, and these two effects cancel each other out, making contact surface area (to a good approximation) an unimportant factor in calculating friction force.

drag

The final macroscopic force to add to our pantheon is called *drag*. This comes about whenever an object is moving through a fluid. If the fluid happens to be air, then this force is commonly referred to as *air resistance*. Drag is similar to kinetic friction in that its direction on a moving object is always opposite to that object's motion relative to the dragging fluid. It differs from kinetic friction in that the magnitude of the drag force varies with the speed of the object relative to the fluid, whereas kinetic friction remains approximately constant for all speed.

Microscopically, the drag force can be viewed as countless collisions of the moving object with the tiny particles in the fluid. Again, the particles in the moving object don't actually touch the fluid particles, but as the object moves through the fluid, the particles get close enough together to repel, and naturally that repulsion acting on the object is in the opposite direction to its motion through the gas.

The mathematics of drag turns out to be quite complicated, though three of the physical properties that factor in are fairly easy to enumerate:

- relative speed of the object and the fluid – By increasing the speed, there are more collisions with fluid particles per second, which increases the associated force.
- cross-sectional area of the object through the fluid – The number of particles that strike the object increases as the cross-sectional area (the area perpendicular to the direction of motion) is increased.
- density of the fluid – This is a measure of how close together the particles in the fluid are to each other. Increasing the density therefore increases the number of particles in the fixed space through which the object passes, and more particles means more collisions, which results in more force.

Interestingly, the *shape* of the object also has an effect, because the fluid flow around the object will also result in forces. That is, two objects can have the same cross-sectional area, but one can be more aerodynamic (for gases) or hydrodynamic (for liquids) than the other, and this plays a role in the amount of drag force.

Putting all this together, we get a not-quite-fully-formed formula that looks like:

$$F_{drag}=(constant)\rho Af\left(v\right) \tag{2.2.6}$$

where ρ is the fluid density (measured in mass per volume), A is the cross-sectional area of the object, and $f\left(v\right)$ is some unknown function of the speed of the object relative to the fluid that gets larger as v gets larger. This somewhat unsatisfying result is usually packaged in the following way: Choose a function of v that comes close for "typical" speeds, and then lump together all the other factors (two of which are the appropriate tweak to the velocity function and the shape of the object, but there are a few more) into what is called the *drag coefficient*, c_d (which is then determined experimentally), to give:

$$F_{drag}=\frac{1}{2}c_d\rho v^2A \tag{2.2.7}$$

Apart from a couple very basic applications (such as something called "terminal velocity", which we will discuss later), we will not typically complicate our physics discussions by incorporating the effects of drag. Phrases like "ignoring air resistance" will be quite commonplace going forward, to the point where they will be understood to be in effect unless explicitly stated otherwise.

2.3: Forces as Interactions

Stepping Back

From previous sections, we now have a definition of force (push or pull between two objects), a notion of fundamental (between particles) vs. composite (between clusters of particles), and a description of all the types of composite forces we are going to encounter in this class. In this section, we will back away from the specifics of the previous section, and explore more general features of force. We will find that our faulty human intuition and imprecise language when it comes to physical concepts will give us some challenges, but hopefully this section will provide some tools for overcoming them.

Newton's Third Law

As is implied by the name “first law,” Newton was not finished – he posited two other laws of motion as well. We’ll return to the second law in a future section, but first we will discuss the third law. You have almost certainly heard it before:

Newton's Third Law

For every action there is an equal and opposite reaction.

This is an extremely unfortunate use of language, and this law has been misinterpreted for hundreds of years as a result. It is often heard quoted in movies to essentially express how natural it is to seek retribution. Something like, if someone hits you, you will hit them back afterward.

ALERT

The idea of a “reaction” as we understand it in common parlance is that it is a consequence of a previous action, but this is not the way that Newton meant it.

Okay, then, so how did Newton mean it? Forces are *interactions*, and just as it is impossible for a single hand to clap, it is equally impossible for a single object to be the sole participant in a force interaction. So if one object experiences a force from another, there must be a reciprocal force also felt in the other direction at exactly the same moment, with precisely the same magnitude and in precisely the opposite direction (remember, forces are vectors). So for every force you can name, there exists an evil twin that acts in the opposite direction with equal magnitude. These "twins" we will refer to as *Newton’s third law force pairs*. It is important to note that while all third law pairs are forces equal in magnitude and opposite in direction, *not all pairs of forces equal in magnitude and opposite in direction to each other are third law pairs with each other*. That is, "equal-and-opposite" is a necessary (restrictive), but not sufficient (defining) condition. It is also worth noting that this law doesn't depend upon the force being fundamental between particles, or composite between objects – it is just a general feature of force itself.

What follows is a very useful tool in our study of force, and in particular for identification of third law pairs:

The "Force Phrase"

"$...\langle type\;of\;force\rangle$ **on** $\langle object\;experiencing\;force\rangle$ **by** $\langle object\;exerting\;force\rangle$ ..."

By "type of force", we mean one of the macroscopic forces listed in the previous section (gravity, contact, tension, etc.), and since two objects are always involved, both must be listed here. In addition, the *specific* force between the two objects must be indicated, meaning that the phrase specifies which object is doing the pushing or pulling, and which object is being pushed or pulled. Of course they are *both* pushed or pulled, but that just means there are two forces involved (one acting on each object), and this force phrase singles out one of them.

This phrase can be used to great effect to identify third law pairs. If one can correctly describe a force using the force phrase, then its third-law pair is the force whose phrase simply reverses the "on" and "by" objects. Naturally the type of force must be the same for both, as they are two halves of the same single interaction.

Conceptual Question

A child sits on a swing, swinging back-and forth. At the bottom of the swing, which of the following forces is the Newton’s 3rd law pair to the contact force the child exerts on the seat of the swing?

a. *the tension force on the seat by the chain of the swing*
b. *the gravitational force on the child by the Earth*

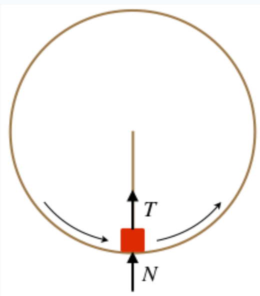
c. *the gravitational force on the Earth by the child*
d. *the centripetal force on the child by the seat*
e. *none of these*

Solution

(e) The contact force on the seat by the child is a result of the interaction of the seat and the child. The third law pair is therefore the contact force on the child by the seat. Forces (a)-(c) aren't the same interaction as the one given, and (d) is not even a type of force!

Conceptual Question

A block weighing 12lb travels in a circular path in a vertical plane. As the block does this, it slides along a frictionless circular track, and it is also attached to a string, the other end of which is attached to a fixed point at the center of the circle. When the block is at the bottom of its circular path, the contact force exerted on it by the track equals the tension force exerted on it by the string, and both are equal to 12lb. Which of the following forces is the Newton's 3rd Law pair corresponding to the gravity force on the block?



a. *the normal force on the block*
b. *the tension force on the block*
c. *Either (a) or (b) can be considered a third law pair for the gravity force.*
d. *the sum of (a) and (b)*
e. *None of the above is a third law pair to the gravity force on the block.*

Solution

(e) Don't let all the special information provided and coincidental numbers fool you! Just reverse the "on" and the "by" in the force phrase. The gravity force interaction is between the block and the earth, so the third law pair of the gravity force on the block by the earth is the gravity force on the earth by the block.

Free-Body Diagrams

Possibly our most powerful tool for analyzing forces and their effects on the motions of objects is the *free-body diagram* (or *FBD* for short). This is a diagram that consists of a single system (the "free-body," which can be a single object or a collection of objects with the same collective fate), with arrows representing force vectors drawn on it. There are a few rules to drawing an accurate FBD:

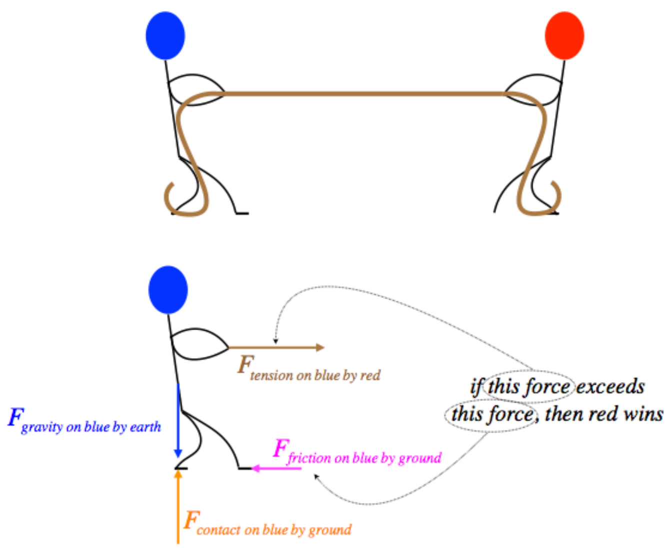
- It must include only an isolated "system." This system can consist of one object or many, but the analysis that follows applies to the system as a whole, and nothing outside this system – whatever its role in the physics – is included in the diagram.
- The force vectors must be "real" forces. If you can't name the force with one of the forces mentioned earlier, then you are probably trying to fix something that isn't broken by inventing a force. Also, no forces calculated from aggregates of other forces should be included – just separate, physically-describable forces.
- Only forces *on* the system can be included – never forces *by* the system. If every vector is labeled using the force phrase, there is no way to go wrong here.
- For now, where the force vectors are located on the system is not important, so the entire system can be reduced to a single dot for simplicity (we will discuss why we are allowed to do this for systems of particles in the next section). But

later this quarter the location where the force acts will become important, so it might be a good idea to try to place the force vectors properly right away. Since the type of force and its basic nature are related to where it acts on an system, this will also help confirm that you are dealing with the right forces, and are not trying to invent a force that doesn't exist.

Okay, let's consider the following puzzling question... Suppose two people engage in a tug-o-war. According to Newton's third law, the tension force on person #1 by person #2 equals the tension force on person #2 by person #1. This is inescapable. But then how does anyone ever *win*, if both always pull with the same force? Let's draw a FBD to see if we can see why.

The first step is to isolate a single object (the "free body") – we will therefore choose one of the two competitors. Then we need to think about the physical situation, and name all the forces *on* (not *by*!) that object, and add vector arrows to the diagram to represent those forces. Only then can we decide how the motion of the free body might be affected by these forces.

Figure 2.3.1 – Analyzing a Tug-o-War Using a FBD



The reason the question is confusing is that we think that the two forces that are equal-and-opposite must always cancel out, but how exactly do forces "cancel?" They *have to act on the same system to be added together and cancel*. By drawing a careful force diagram in which we only include the forces on the system in question (in this case, the blue-headed stick figure), we see that in fact the third law pairs that are equal and opposite are split between two FBDs, and therefore cannot cancel each other. The real determining factor of whether an individual wins the tug-o-war is whether that individual receives a friction force from the ground that is greater than or less than the tension force, unbalancing the total horizontal force on them.

Does this mean the person is at the whim of the ground, that either decides to provide a big or small friction force? Of course not! The friction force on our feet by the ground is equal-and-opposite to the friction force our feet exert on the ground, and we do this by leaning back and sliding (or push our foot forward as if to slide it) across the ground. We can see that this is the case, because even the strongest human in the world cannot win a tug-o-war against a small child if the strong person is on ice or on some rolling device that doesn't allow them to push horizontally (and thereby be pushed back the opposite way).

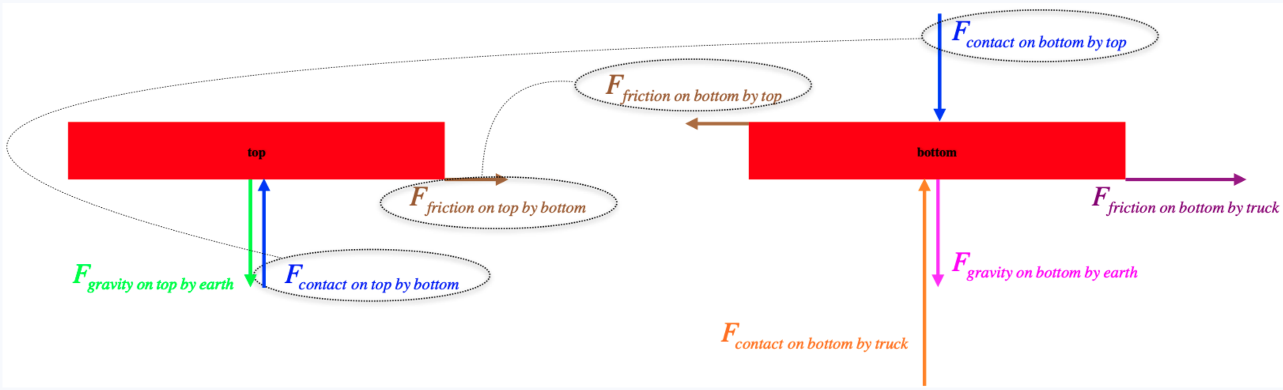
Exercise

A pickup truck with the tailgate down carries two identical sheets of plywood stacked in its bed as it accelerates horizontally. The plywood sheets do not slide within the bed or across each other.

a. Draw a free-body diagram for both sheets of plywood, labeling the vector arrows using the force phase.

b. Indicate which forces in your diagrams are third-law force pairs of each other.

Solution


$F_{\text{contact on bottom by top}}$
$F_{\text{friction on bottom by top}}$
top
bottom
$F_{\text{friction on top by bottom}}$
$F_{\text{friction on bottom by truck}}$
$F_{\text{gravity on top by earth}}$
$F_{\text{contact on top by bottom}}$
$F_{\text{gravity on bottom by earth}}$
$F_{\text{contact on bottom by truck}}$

2.4: Effects of Force on Motion

Simple Cases First

What follows is the most important step we need to take in our discussion of forces, in that it links what we have been learning in this chapter about forces to what we learned in the previous chapter about motion. Following the procedure we have established already, we will begin by considering the simplest cases first, and then expand what we learn there into more complex scenarios. To that end, we begin by considering the effect of forces on *particles only*, and will address the effects of force on collections of particles afterward.

Newton's Second Law

We have built some tools for analyzing situations where forces act on objects (force phrase, FBDs), and we know that there can only be accelerations when forces are present (first law). But we still are not yet able to describe the motion of a particle under the influence of one or more forces. That's because the first law only tells us *qualitatively* what is happening. In physics we seek to develop quantitative models, and that's where the second law comes in. It is really just a more detailed description of the first law, or alternatively, the first law is just a special case of the second law.

We know that force is related not to velocity (because the first law says that constant velocity exists in the absence of force), but rather the *change* of velocity. More specifically, the rate of change of the velocity – the acceleration. Newton defined force in the simplest possible fashion in terms of acceleration – with a linear relationship. He reasoned that pushing equal amounts on two particles of different masses resulted in different changes of motion, so he stated that the relationship between force and acceleration as a simple proportionality:

$$\text{acceleration of particle} = \frac{\text{force acting on particle}}{\text{mass of particle}} \tag{2.4.1}$$

The idea is that for a given force, the reaction of the particle (in the form of an acceleration) is inversely-proportional to the amount of mass the particle possesses. Let's take a moment to mention units:

$$[F] = \frac{kg \cdot m}{s^2} = \text{"Newtons"} \;\; (N)$$

There is much more detail lurking in here. First of all, acceleration and force are both vectors, while mass is a scalar, so the second law is actually a vector equation:

$$\overrightarrow{a} = \frac{\overrightarrow{F}}{m} \tag{2.4.2}$$

This means that the acceleration experienced by a particle is just a scaled vector of the force exerted on the particle. That is, the acceleration and the force always point in the same direction (mass is never negative). Of course, this scaling also changes the units.

ALERT

*Most people first encounter Newton's second law expressed as $\overrightarrow{F} = m\overrightarrow{a}$. While this is mathematically equivalent to what is above, it is very dangerous to write this way, as it encourages a very common misconception. We write it as we do above to emphasize the interpretation: "the **effect on the motion** (the acceleration) results from the **cause** (the force), moderated by a **property** of the object experiencing the effect (the mass)." The danger of using the other expression is that it reads like, "the force **of** the particle equals the mass **of** the particle multiplied by the acceleration **of** the particle." This turns the quantities of force and acceleration into properties of the particle, rather than cause and effect, and this leads to subtle-but-important misconceptions.*

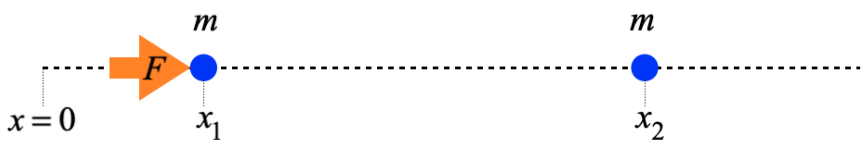
We aren't done modifying the second law to its proper form yet! A large number of forces can be on a particle at the same time, so which force is the one that causes the acceleration? All of them. Do we figure out the accelerations of each force and then add them up? That makes no sense physically – particles do not experience lots of accelerations at once. Instead, we take all of the forces together and add them as vectors to create a single composite force that we call the *net force*, and that is what goes into the equation:

$$\vec{a} = \frac{\vec{F}_{net}}{m} \tag{2.4.3}$$

Collections of Particles

Now that we have the basics of Newton's second law for particles, we would like to apply it to the many macroscopic forces we have discussed, but those forces all involve interactions with collections of particles, so we need to see if we can extend the reach of the second law. Here is the problem: If a force acts on a subset of particles in a collection (e.g. a normal force only acts on the particles at the surface of an object), then those particles will be accelerated, while other particles will not. In such situations where multiple particles in a collection are accelerating differently, how do we define the "acceleration of the full group"? There is actually a mathematical answer to this! Let's look at the simplest possible example first...

Figure 2.4.1 – Force Acts on One Particle in a Pair of Identical Particles



Here we have a force with magnitude F acting in the $+x$-direction one particle in a pair of particles. According to Newton's second law, we know how this particle reacts to this force:

$$a_1 = \frac{d^2x_1}{dt^2} = \frac{F}{m} \tag{2.4.4}$$

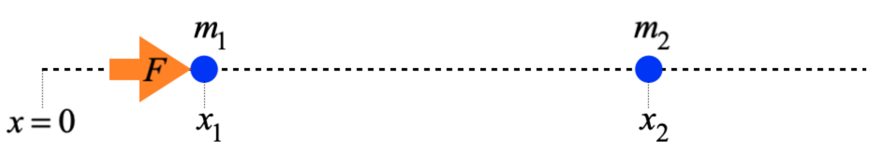
But we want to see if we can somehow apply the second law to the combination of the two particles, which would look like this:

$$a_{pair} = \frac{d^2x_{pair}}{dt^2} = \frac{F}{m_{pair}} = \frac{F}{2m} \tag{2.4.5}$$

The problem we have here is how do we define the position of the pair, x_{pair}? Given that the two masses were equal, it seems reasonable to define the halfway point between the particles to be the "pair's position". Will this work? Mathematically, we express this as $x_{pair} = \frac{x_1+x_2}{2}$, and plugging this in above, we see that in fact the answer is yes. Noting that the second derivative of x_2 is zero because that mass is unaffected by the force, we get agreement:

$$\frac{d^2}{dt^2}x_{pair} = \frac{d^2}{dt^2}\left(\frac{x_1+x_2}{2}\right) = \frac{F}{2m} \quad \Rightarrow \quad \frac{d^2x_1}{dt^2} = \frac{F}{m} \tag{2.4.6}$$

Figure 2.4.2 – Force Acts on One Particle from a Pair of Particles with Different Masses



This time the extension to the pair is a little different:

$$a_1 = \frac{d^2x_1}{dt^2} = \frac{F}{m_1} \qquad a_{pair} = \frac{d^2x_{pair}}{dt^2} = \frac{F}{m_{pair}} = \frac{F}{m_1+m_2} \tag{2.4.7}$$

Looking at what happened above, it's clear that picking the midway point between the two particles no longer works. If we think of the center point between two equal masses as the "average position" of the total mass of the pair, then when the masses are unequal, we would not expect the average position to be halfway between them. The simplest "try" is to choose an average location that is closer to the heavier particle, by an amount in proportion to their masses. The formula that accomplishes this is:

$$x_{pair} = \frac{m_1x_1+m_2x_2}{m_1+m_2} \tag{2.4.8}$$

So if the masses are equal ($m_1 = m_2 = m$), then we get the result above. If m_2 is twice as massive as m_1, then the "pair's location" is twice as far from m_1 as it is from m_2. So let's try this:

$$\frac{d^2}{dt^2}x_{pair} = \frac{d^2}{dt^2}\left(\frac{m_1x_1+m_2x_2}{m_1+m_2}\right) = \frac{m_1\frac{d^2x_1}{dt^2}+m_2\cancelto{0}{\frac{d^2x_2}{dt^2}}}{m_1+m_2} = \frac{F}{m_1+m_2} \tag{2.4.9}$$

It works! Just to recap what we have found here: If we have a pair of particles, and we define the "position of the pair" to be the precise point described by Equation 2.4.8, then the acceleration of the pair (defined as the second derivative of its position, of course) equals the force on the pair divided by the mass of the pair, *even though the force only acts on one of the particles in the pair*. But this definition of the location of the pair of particles works in far more general cases than this.

Exercise

Show that in the two particle example above, the definition of position of the pair given by Equation 2.4.8 gives the correct result for the case when a force F_1 acts on particle 1 and a force F_2 acts on particle 2 (both along the x-axis).

Solution

Both particles are now accelerated independently by different forces, and their accelerations are given by:

$$a_1 = \frac{d^2x_1}{dt^2} = \frac{F_1}{m_1} \qquad a_2 = \frac{d^2x_2}{dt^2} = \frac{F_2}{m_2}$$

Now let's look at the acceleration of the pair:

$$a_{pair} = \frac{d^2}{dt^2}x_{pair} = \frac{m_1\frac{d^2x_1}{dt^2}+m_2\frac{d^2x_2}{dt^2}}{m_1+m_2} = \frac{m_1a_1+m_2a_2}{m_1+m_2} = \frac{F_1+F_2}{m_1+m_2}$$

The sum F_1+F_2 is the net force on the pair (you have to add together all the forces acting on any particles to get the net force), which shows that the acceleration of the pair equals the net force on the pair divided by the pair's mass.

What we have been calling the "position of a collection of particles" is commonly referred to as the *center of mass* of that collection. Above we have restricted it to two particles along the x-axis, but it is easy enough to generalize. If the particles have y and/or z coordinates, then the center of mass in the y and/or z directions can be defined in the same way as it was for the x-direction. And if we want to add more particles, we just include each one's position multiplied by the mass in the numerator sum, and of course make the denominator the total mass of all the particles. Putting it all together, we can write the definition of the center of mass in terms of position vectors of all the particles:

$$\begin{aligned} \overrightarrow{r}_{cm} &= x_{cm}\,\hat{i}+y_{cm}\,\hat{j}+z_{cm}\,\hat{k} \\ &= \frac{[m_1x_1+m_2x_2+\dots]\,\hat{i}+[m_1y_1+m_2y_2+\dots]\,\hat{j}+[m_1z_1+m_2z_2+\dots]\,\hat{k}}{M} \\ &= \frac{m_1\left[x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right]+m_2\left[x_2\hat{i}+y_2\hat{j}+z_2\hat{k}\right]+\dots}{M} \\ &= \frac{m_1\overrightarrow{r}_1+m_2\overrightarrow{r}_2+\dots}{M} \end{aligned} \tag{2.4.10}$$

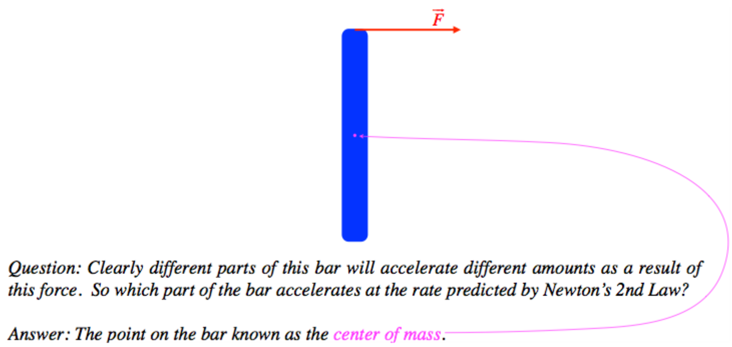
Using this as the position of the collection of particles, that collection's acceleration works perfectly with Newton's second law, no matter how the forces on the group are distributed amongst the particles. We therefore put it all together with "a_{cm}" referring to the acceleration of the group's center of mass:

Newton's Second Law of Motion

$$\overrightarrow{a}_{cm} = \frac{\overrightarrow{F}_{net}}{m} \tag{2.4.11}$$

So now we have a robust law we can use for real objects and the composite forces we enumerated in a previous section. While this law works for any collection of particles, in almost all of our applications, the group of particles will be rigid objects. In these cases, using the center of mass is significantly easier than above, where we have to add a bunch of terms representing all of the particles. For example, if the rigid object isn't rotating, then the motion of its center of mass is identical to the motion of every particle in the collection. Introductory physics classes spend a lot of time on examples like this, which explains why Newton's second law is frequently described with no mention of center of mass at all. But even the simplest of questions cannot be answered without this knowledge. Suppose that a force is applied perpendicularly to a rigid rod at its end. It will clearly start to spin as well as move forward, which means the particles comprising the rod are all accelerated different amounts.

Figure 2.4.3 – What Part of an Extended Object Accelerates According to the Second Law?



The case of a rigid object like this bar does raise an interesting question: What do we do with the forces that particles *within* the collection exert on each other? Such forces are necessary for the object to remain rigid. To answer this question, let's zoom-in on just two particles within the collection. One of the particles is pushed by the outside force, and as it starts to accelerate, it pulls or pushes on the other particle, to keep their structure rigid. Let's call this internal force on particle 2 by particle 1 "$\overrightarrow{F}_{int}$". According to Newton's third law, there must also be present within the collection an "evil twin" force acting on particle 1 by particle 2 exactly equal to $-\overrightarrow{F}_{int}$. So when we add up all the forces on the collection, these two forces must be included (since they both act on particles within the group), but as they are equal magnitude vectors in opposite directions, they just cancel in the sum. This will be true for any pair of particles within the collection that we care to name, so we can essentially ignore the internal forces altogether.

Second Law Misconceptions

Nearly everyone reading this textbook has encountered Newton's second law before, even if it was as far back as a science class in middle school. It's unlikely that the reader has seen the discussion of center of mass before, but when it comes to the final result, many feel like they "know it already." But knowing an equation is very different from understanding what it means, so we will take some time here to try to root out some common misconceptions about this law held by even the most dedicated physics students that have seen this before. We will find that many, if not all, of these misconceptions can be avoided by keeping the following two things in mind:

- The second law mathematically expresses a cause & effect relationship, with the cause being a combination of many forces on the object, and the effect being that object's (center of mass) acceleration (*not* its velocity!). It is not an equality that expresses a relationship between several properties of the object.
- It is not enough to describe a force that is present in a physical situation – to put it into Newton's second law, it must be acting *on* the collection of particles, and it must be caused by another entity that is *external* to that collection. Liberal use of the force phrase and free-body diagrams is very helpful in sorting this out.

There is no better way to demonstrate what the most common misconceptions are, or show how the two reminders above helps to sort them out than to look at examples. The reader is strongly urged to commit to an answer in each case before reading the solution – this provides maximal benefit.

First a question that addresses the problem of identifying what forces get plugged into the equation for Newton's second law:

Conceptual Question

A driver steps on the brake pedal of her car, slowing the car down, and her body experiences an acceleration as a result. Which of the following forces does Newton's second law include when determining her acceleration?

a. *normal force by driver's foot on the brake pedal*
b. *friction force by the car tires on the road*
c. *friction force by the road on the car tires*
d. *all of these*
e. *none of these*

Solution

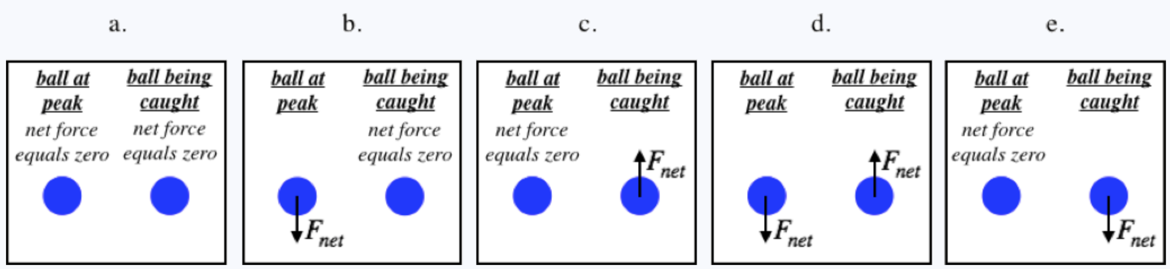
(e) One must be very precise when it comes to identifying forces, as ultimately they must be plugged into a mathematical formula. It is not enough that a force sets off a chain of events that leads to an acceleration, it must be the direct effect of that acceleration in order to be the force that is featured in the second law. In addition to being a direct force, it isn't even sufficient to isolate the correct interaction – the specific "twin" from the third law force pair must be identified. That is, the force must be ***on*** *the object in order to accelerate it. The problem asks what force slows down her body. The normal force on the brake pedal affects the motion of the brake pedal. The friction force on the road affects the motion of the road. The friction force on the tires (which are part of the car's "collection of particles") affects the motion of the car. While parts of the car (namely tension by the seatbelt, friction by the car seat, and normal force by the steering wheel) do slow her down, and the friction force on the tires slows down the car, this chain of events does not mean that the friction force slows her down. If you plug the friction force on the tires and her body's mass into the second law, the acceleration you calculate for her will not be correct.*

If all of that discussion still confuses you, consider what you would draw for a free-body diagram here. We are interested in the acceleration of an object (the driver), so that is the free-body that we need to draw. Next we need to add force vectors to this diagram, which, according to the force phrase, are acting ***on*** *the free-body. If the description of the force does not include the phrase "on the driver", then it does not belong on the FBD, and does not figure into the calculation of the driver's acceleration according to the second law.*

Next we consider what the acceleration in Newton's second law really means, and how its direction matches the direction of the net force:

Conceptual Question

A boy throws a ball straight up, and catches it when it returns. Which pair of diagrams best represents the directions of the net forces experienced by the ball when it hits the peak of its flight (i.e. when it isn't moving), and while the boy is catching it (i.e. not after he has caught it)?



Solution

(d) Gravity is always acting on the ball, no matter where it is. When it reaches its peak, there are no other forces on it (the boy's hand is no longer in contact with it), so the gravity force is the net force, and it points down. When the ball is in the process of being caught, gravity is still acting, but there is also a force up on it by the boy's hand. The ball is moving downward and is slowing, so it is acceleration vector points ***upward****, which means the force from the boy's hand exceeds the force of gravity and the net force is upward.*

This question addresses the role of mass in the second law, as well as the notion of "inertia":

Conceptual Question

A stone is dropped to the ground. As it falls, the stone accelerates (without air resistance) at $9.80\frac{m}{s^2}$ *downward. As a result of this gravity force interaction, what happens to the Earth?*

a. *It accelerates upward toward the stone at* $9.80\frac{m}{s^2}$.
b. *It accelerates upward toward the stone at a rate less than* $9.80\frac{m}{s^2}$.
c. *It doesn't accelerate - it only exerts a gravity force, it doesn't feel it.*
d. *It doesn't accelerate - it has too much inertia.*
e. *both (c) and (d)*

Solution

(b) The magnitude of the gravity force on the stone by the Earth equals (according to Newton's third law) the magnitude the gravity force on the Earth by the stone. Yes, the stone exerts a gravity force on the Earth, and it is equal to the force the Earth exerts on the stone! So why don't we feel the Earth lurch upward when we drop something? The magnitude of acceleration of an object with a force on it is given by the second law to be $a = \dfrac{F}{m}$. *The Earth and the stone feel equal forces, but the Earth's mass is much, much bigger, so it reacts to the same force with an acceleration is much, much less. This acceleration is much too small for us to feel (the Earth's mass is about 5,972,190,000,000,000,000,000,000 kilograms).*

Note that the (wrong) idea most people have of "inertia" is that it expresses a threshold. That is, they think that a certain minimum amount of force is required to "overcome" an object's inertia, and the more massive that object is, the higher that minimum force is. In the example above, this leads to the explanation that the Earth's mass is so great that there is no way that the gravity force from a stone can possibly equal the minimum force needed for acceleration. But this is *not* what the second law tells us! The Earth *does* accelerate, no matter how small the force might be.

Where does this notion of inertia come from? We see cases all the time in everyday life where this trait seems to be exhibited: We can easily get a chair to start sliding across a floor, but it takes a great deal more force to get a huge couch to start sliding. We internally reason that since the couch is more massive, this must be the property that accounts for the "inertia" that requires a threshold force to overcome. But from our earlier study of friction, we see that what is really causing this threshold effect is static friction. We must push harder than the maximum static friction force. This maximum is partly determined by the normal force between the surfaces. This normal force comes from the fact that gravity pulls the couch down to the floor. And the gravity force is greater when the object is more massive. So a more massive object exerts a greater normal force, which results in a greater maximum static friction force, which increases the threshold force needed to get it accelerating. Whew, now we see one reason why physics is so hard – one quantity (mass) can have an effect on another (minimum force to get something moving), but they are only related very indirectly under specific (but common) circumstances, and if we oversimplify the explanation ("inertia"), we get everything wrong when those circumstances are changed.

Here's a question with a bit of arithmetic:

Conceptual Question

A car with a mass of $2000kg$ *is moving in a straight line and has an acceleration vector of magnitude* $4\frac{m}{s^2}$ *pointing to the east just before it crashes into another car that is stationary. What force does the stationary car exert on the incoming car?*

a. $8000N$ *to the east*
b. $8000N$ *to the west*
c. $8000N$*, but the direction can only be determined if one knows whether the car was speeding up or slowing down*
d. $0N$ *- the equal-and opposite forces between the cars cancel out*
e. *The information given is not relevant to answering the question.*

Solution

(e) This is one of the most common misconceptions among people who first encounter Newton's second law. They think that an object "has" an acceleration, and it "has" a mass, so obviously it "has" a force equal to ma *as well, and when it hits something else, that is the force that it hits with. But acceleration is the* ***result*** *of a net force not a component of it. So the acceleration and mass of the incoming car tells us the net force on the car that is causing the acceleration* ***prior to the collision****. When the collision occurs, the circumstances (i.e. external forces on the car) change, which means that the car's acceleration changes – the car doesn't "remember" the acceleration it had before! Without being given this new acceleration, we can't use the car's mass to compute the net force on it.*

Okay, so there was no arithmetic to be done there after all. This misconception is so prevalent that the reader is strongly encouraged to not move on from here until they have sorted it out.

Here's a question that combines a couple of ideas:

Conceptual Question

Two cars are in contact with their front bumpers, and are pushing against each other. As they do, the red car is rolling forward and the blue car is rolling backward at a constant speed. Which of the following is true:

a. *The red car exerts a net force on the blue car.*
b. *We know that the net force on the two-car combination is zero, because the combination is not accelerating.*
c. *The pair of cars is not accelerating because the contact force of the red car on the blue car equals the contact force of the blue car on the red car.*
d. *Two of these statements are true.*
e. *All of these statements are true.*

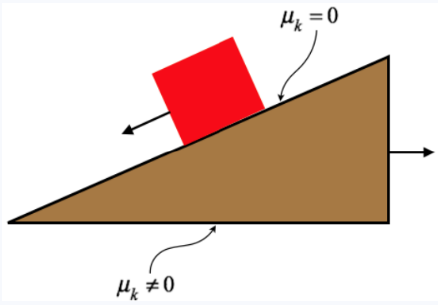
Solution

(b) This question again addresses the definition of acceleration (it is not the same as velocity!). If there was a net force on the blue car, it would be accelerating, but it is moving at a constant speed in a straight line, which is zero acceleration. The pair is also moving in a straight line at a constant speed, so its acceleration is zero, which tells us that this pair is experiencing no net force. However, the contact forces between the cars are ***always*** *going to be equal due to the third law, so these forces (and any others that are internal to the pair) have no say whatsoever in whether the pair accelerates. Only the outside forces (friction on the tires by the road) play a role.*

And finally, let's throw in a tough one that brings in the center of mass idea:

Conceptual Question

A block starts at rest on a wedge, which itself is at rest on a horizontal tabletop, as shown in the diagram. The contact between the block and wedge is frictionless, but the contact between the wedge and tabletop is not. When the block is released, the block begins sliding down the plane, and the static friction with the tabletop is insufficient to stop the wedge from sliding the opposite way (though it does experience kinetic friction). We don't know whether the block or the wedge has a greater mass. What can we say about the center of mass of the block + wedge combination?



a. *It accelerates straight down.*
b. *It accelerates down and to the right.*
c. *It accelerates down and to the left.*
d. *It accelerates down and in the same horizontal direction as the motion of the more massive object.*
e. *It accelerates down and in the same horizontal direction as the motion of the less massive object.*

Solution

(c) It is clear that the center of mass drops, since the block descends in height and the wedge remains at the same height. The center of mass starts at rest, so it must be accelerating downward. With the two objects moving in opposite directions, it is not immediately clear what happens to the center of mass of the combination in the horizontal direction. From Newton's second law, we know that the center of mass of a combination accelerates in the same direction as the net force on it. The net force in the second law only needs to take into account the forces on the combination that are external to it – we can ignore

all the internal forces. So what are these external forces? There is gravity down, a normal force up by the table (the combination of these resulting in the downward acceleration of the combination's center of mass), and a kinetic friction force by the table opposing the wedge's horizontal motion (i.e. pointing to the left). Since the only horizontal external force on the combination is to the left, that is the horizontal component of the direction that the center of mass accelerates.

A Summary of Concepts Related to Newton's Laws

Much of what we have discussed in this section and the one before it will be repeated below, but putting all of these ideas in one place may help the reader consolidate the ideas into a cogent "big picture."

1. Force is not a quantity contained within an object.
2. Forces are push or pull interactions between two objects. If one looks at the two individual forces that make up the interaction, then those two forces are always equal in magnitude and opposite in direction (Newton's third Law).
3. To avoid confusion, we learned the all-important "force phrase," which reminds us that the individual forces that make up the interaction force pairs always act *on* one object and *by* another.
4. Forces are the cause of accelerations. It is impossible to have one of these without the other. This means that forces (if the vectors don't all cancel each other out) speed up, slow down, or change the direction of an object's motion. And conversely, if an object's motion slows down, speeds up, or changes direction, then it must be experiencing a (net) force. (Newton's first Law)
5. Forces are vectors, which is to say that they have magnitude and direction.
6. The force vector that causes an object to accelerate is the *net* force on that object, that is, the vector sum of all of the individual forces exerted on the object. A net force is a combination of one or more real forces, but is not itself a type of force.
7. Only the forces *on* an object can contribute to its acceleration (i.e. added together to give the net force), never the forces *by* it. Forces by an object only affect the motions of the *other* objects that they act on.
8. The amount of net force on an object is proportional to the amount of acceleration it experiences, and the constant of proportionality is the mass, a measure of how much stuff is present in the object. (Newton's second Law)
9. The fact that net force and acceleration are proportional means that as vectors, they must point in the same direction, since mass is never negative.
10. Mass is sometimes called "inertia," which can be loosely thought of as resistance to acceleration. But this must not be confused with resistance to motion – the smallest net force will cause an acceleration of the largest mass. If a mass at rest doesn't start to move when a small individual force acts on it, it is because there is another force balancing it out, causing zero net force, not because the inertia of the object cannot be overcome.
11. The part of the object (or collection of particles) that experiences the acceleration described in Newton's second Law is the center of mass of the object, not the point on the object where the force is acting.
12. A useful tool for analyzing forces is the free-body diagram, which consists of isolating an object, followed by drawing in all the force vectors acting on it. Careful use of the force phrase helps us avoid putting incorrect forces on this diagram.

2.5: Applications of Newton's Laws

Consolidating What We Know So Far

We will now take a section to spend some time applying Newton's laws to some common circumstances. These applications will require a good understanding of what we have seen so far, but they are still fairly "basic" in the sense that they do not incorporate complications called "constraints," which we will put off discussing until next section. We'll start by tying up a couple of loose ends related to air resistance and friction, and then move on to some examples the reader can analyze to "grow their muscles" in understanding how Newton's laws are applied.

Air Resistance on Falling Objects

We already know something about gravity from our study of free-fall and projectile motion. We know that the acceleration is the same for objects of different masses. While we have used this as a model, it is a big step to claim that gravity fundamentally follows this rule. We know that a feather will experience the same acceleration due to gravity as a stone, if air resistance is removed. Now how do we put air resistance back into our model so that the reduced acceleration of the feather makes sense?

The effect of reduced acceleration is easy to show with a free-body diagram of two objects that are identical except for mass and are falling through the air and happen to be at the same speed. For these two objects the air resistance forces are equal, and the gravity force is greater on the heavier object. The net forces on the two objects are therefore different, giving the following accelerations:

$$\left.\begin{array}{l} a_{(heavy)}=\dfrac{F_{gravity\;(heavy)}-F_{drag}}{m_{(heavy)}} \\ a_{(light)}=\dfrac{F_{gravity\;(light)}-F_{drag}}{m_{(light)}} \\ F_{gravity}=mg \end{array}\right\} \Rightarrow \begin{array}{l} a_{(heavy)}=g-\dfrac{F_{drag}}{m_{(heavy)}} \\ a_{(light)}=g-\dfrac{F_{drag}}{m_{(light)}} \end{array} \tag{2.5.1}$$

So the reason the heavier mass accelerates more is simply that the effect that the air resistance force has on it is smaller. From these final equations, we see that in the special case of assuming zero drag, we find that the acceleration happens to equal the constant g for objects of any mass.

Alert

*It is important to understand that here g has a different meaning than it had when we were discussing motion involving gravity-caused acceleration. Here the g is a physical constant, which we use to determine the gravity force on an object with mass m. It does **not** mean that the object is accelerating at $9.8\frac{m}{s^2}$! When an object experiences no other force than gravity, the object's acceleration just happens to equal this constant, but the constant is present regardless of the state of acceleration of the object.*

An object accelerating in free-fall keeps moving faster with time, which means that the drag force due to the air keeps increasing (drag is a function of the speed through the fluid). This increase of speed cannot maintain the same rate forever, because eventually the speed will be great enough that the drag force will equal the gravity force. When this occurs, the two opposing forces cancel, and the second law tells us that acceleration must cease! What actually happens as the object falls is that the drag force gradually increases as the speed increases, gradually decreasing the falling object's acceleration. We will not go into the calculus that gives the resulting equation of motion, but instead will jump to the point where the acceleration diminishes to essentially zero. The speed at which this occurs is called *terminal velocity*. Clearly this velocity is determined by the many factors that go into the drag and gravitation forces.

Analyze This

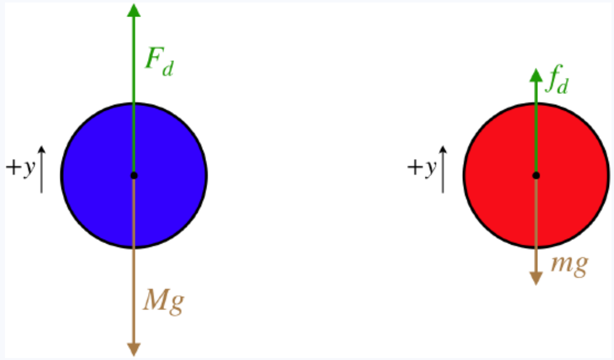
Two spherical objects of identical radii but different masses are dropped from different heights through the air. They both reach terminal velocity at the same moment in time, and at that moment, they are side-by-side.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Start with a free-body diagram:



At terminal velocity, the object is no longer accelerating, which means that the net force on the object is zero. The only forces on these objects are gravity and air drag, so these opposing forces must be equal. The heavier object has a greater gravity force on it, so it must be also experiencing a greater drag force. The two objects have the same cross-sectional areas drag coefficients, and they are falling through the same air. This means that the only quantity that can account for their different drag forces is their speeds. Applying Newton's second law and noting that with velocity being the only variable to account for the difference in forces, we have:

$$F_d = Mg \quad \Rightarrow \quad M = \frac{F_d}{g} \propto V_{terminal}^2 \;, \qquad f_d = mg \quad \Rightarrow \quad m = \frac{f_d}{g} \propto v_{terminal}^2$$

The most direct conclusion that we can derive from this is that the ratio of their terminal velocities in terms of the ratio of their masses is:

$$\frac{V_{terminal}}{v_{terminal}} = \sqrt{\frac{M}{m}}$$

It should also be noted that although these two spheres may be at the same height at the moment they each reach terminal velocity, they will not continue to remain side-by-side, since the heavier one is moving faster than the lighter one.

Slowing Motion with Friction

If a book is slid across a horizontal tabletop, it slows because there is a net force on it. The free-body diagram reveals that this net force comes from the kinetic friction force on the bottom of the book by the table surface. In our discussion of static vs. kinetic friction, we said that the maximum possible static friction force is greater than the kinetic friction force for the same two surfaces. That means that if the book we are sliding could somehow experience the maximum *static* friction force, then according to the second law, it would slow down faster. But how is one to accomplish this, given that the book must be sliding across the surface (the definition of kinetic friction) in order to be moving at all?

Okay, so maybe it is impossible to use static friction to slow a sliding book, but consider slowing an automobile (something we might be very interested in being able to slow as quickly as possible). What slows a car is the friction force on its tires by the road. Unlike the sliding book, the tires *roll*, unless we "lock up" the brakes. The interesting thing about rolling tires is that they are moving, but are not *sliding*. When breaks are applied, a friction force is introduced to the tires, but if they keep rolling and don't start skidding, then this force is static friction. The harder the brakes are applied, the greater this static friction force becomes. If the maximum value of static friction is exceeded, then the tires stop rolling and they start skidding across the surface. The friction force on the tires goes down when this occurs, because the kinetic friction is smaller than the maximum static friction. So anti-lock braking systems (ABS) common in today's automobiles automatically release the breaks briefly so that the tires again turn, restoring rolling and allowing the return of static friction. This would be like trying to push a heavy cardboard box across a floor in extremely short bursts – as soon as the box starts sliding (and gets easier to push), you stop and start over. Before the invention of ABS, drivers were told to "lightly pump their brakes" in slippery situations to create this same effect. ABS does the pumping for us, with a much greater frequency than we could manage, and to great effect.

Practice

What follows is a series of "Analyze This" boxes, intended to give the reader some practice employing Newton's laws (and especially the second law). These examples come in several types:

- direct applications of Newton's laws to draw conclusions about forces
- bridges from Newton's laws to topics we studied prior to this chapter, such as vector math and kinematics
- graphical representation and interpretation

One bit of advice regarding performing analysis on any problem in this chapter (as well as in most chapters to come): After getting a very basic sense of what the physical situation is about, *always* employ the most powerful tool in our arsenal – the free-body diagram. This is true even when the situation seems so simple that a FBD is not needed – those "simple" cases are often trickier than they appear! It is not an exaggeration to say that drawing accurate FBDs is *the most important skill you need to master for Physics 9A*. It is also a critical skill for future physics and engineering classes – it is not just one of those requirements you need to check-off on your way to your "real" STEM education. This is your chance to get off to the right start.

Of course once you have some FBDs, there is no need to stop there – take the analysis as far as it will go with the information you have! Here is a template of the analysis procedure you should follow:

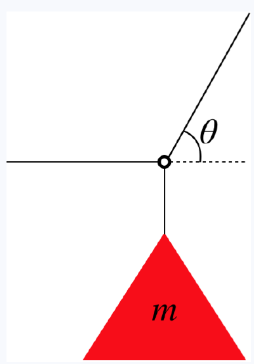
1. Briefly discuss the "big picture" of the problem, and point out whatever special features come to mind. Don't try to figure everything out right then – this is just to get your brain kick-started.
2. Draw careful, detailed free-body diagrams of the objects involved, following the guidelines you have been given for doing this. Until you are an expert at this (it will take awhile before you can make this claim), you should label the force vectors such that the "force phrase" description of the force is evident. If you are unable to do this, then your FBD is likely incorrect. Also, it is a good idea to include an indication of the coordinate system you are using, to ensure that you get the signs correct later.
3. Use the force diagrams to write down a mathematical expression for the net force on the object. If the force has two or three components, then write these components separately, like this: $F_{\text{net }x} = \ldots$, $F_{\text{net }y} = \ldots$
4. Employ Newton's second law by setting each net force vector component equal to the object's mass times its corresponding acceleration vector component. If you know the object is not accelerating in that direction, then you can set this equal to zero!

This template takes you well into the analysis, and prepares you very well to answer any question that may come along about that physical system.

This first case is a basic problem from a subfield of mechanics known as "statics."

Analyze This

A sign hangs from a wire that is attached to a ring that is also attached to two wires (one of which is horizontally-oriented), as shown in the diagram. The wires and ring have negligible mass.

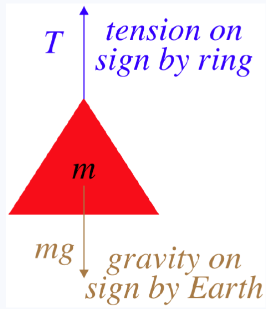


Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

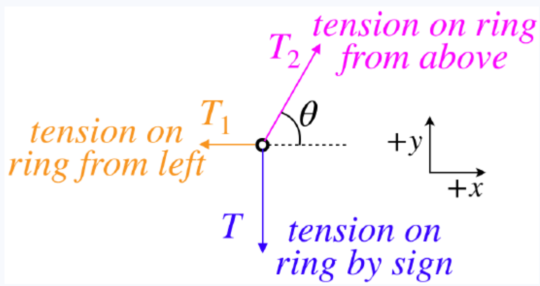
Analysis

Clearly the sign is being held motionless due to the tension force in the wire attached to it canceling-out the gravity force on it. As obvious as this is, we can confirm it with a free-body diagram and Newton's second law, since we know that the sign is *not accelerating:*



$$F_{\text{net}} = +T - mg = ma = 0 \quad \Rightarrow \quad T = mg$$

It is unlikely that this will be all that the problem is about, however, as it doesn't say anything about the other two wires or the angle θ. Clearly the FBD of the sign will not help us with those, so we turn to a FBD of the ring:



We see that now the force vectors are in two dimensions, which means we will have two equations that come from Newton's second law – one for each component of the net force. As we noted for the sign previously, the system remains at rest, which means that the acceleration vector is zero (and therefore both of its components are as well). Only the tension force T_2 has more than one component, so breaking that up and forming the equations from the second law gives:

$$F_{\text{net } x} = -T_1 + T_2 \cos\theta = ma_x = 0 \;, \quad F_{\text{net } y} = -T + T_2 \sin\theta = ma_y = 0$$

We can now put together the results of the two FBD's to obtain the tensions T_1 and T_2 in terms of the weight of the sign, if we note that Newton's third law tells us that the tension on the sign by the ring equals the tension on the ring by the sign (in anticipation of this, we have called both of these forces simply "T" in the free-body diagrams):

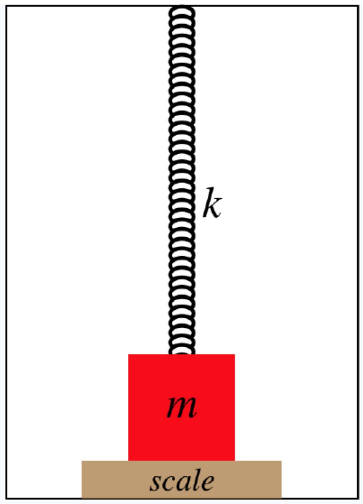
$$T = mg \quad \Rightarrow \quad -mg + T_2 \sin\theta = 0 \quad \Rightarrow \quad T_2 = \frac{mg}{\sin\theta} \;, \quad T_1 = T_2 \cos\theta = mg\cot\theta$$

To complete our analysis, we will take a quick look at our final results to see if they make sense. Suppose that the angle $\theta = 90^o$. Then the sign is essentially hanging straight down from wire #2, and sure enough, the tension force applied by that wire is the entire weight of the sign. Also, in this case, wire #1 should be doing nothing, and indeed T_1 comes out to be zero.

Our next example combines several forces and allows for possible acceleration.

Analyze This

A block is attached to a spring that stretches down from the ceiling of a stationary elevator that is capable of accelerating up or down. The block is then lifted slightly, and a bathroom scale is placed beneath it, so that the block rests on it. The spring is still stretched at this point, but not as fully as when the block was hanging from it.



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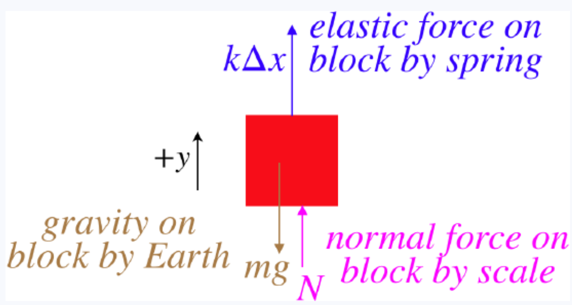
- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

There are several ideas in play here. First, there is a stretched spring, so we will have to include Hooke's law into our analysis. Second, there is a normal force between the block and the bathroom scale. Such scales are constructed to measure weight, but in reality all they can measure is the normal force applied to them. For example, a bathroom scale will give a reading if you put it against a wall and push on it, but that reading is clearly not your weight. And of course there is the gravity force on the block. We start by putting these three forces into a free-body diagram:



We take our usual next step, which is to apply Newton's second law:

$$F_{\text{net}} = +k\Delta x + N - mg = ma$$

This is as far as we can take this without more information, but we can imagine what happens if the elevator is accelerating...

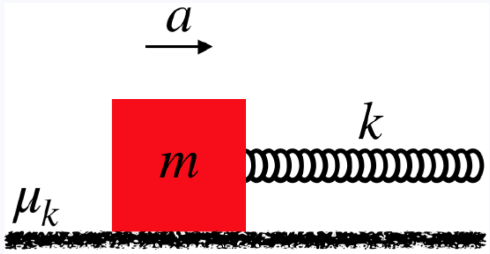
If it is accelerating up, then $a > 0$*. Clearly the value of* mg *cannot decrease to allow for the block's acceleration with the elevator, and the spring force cannot increase, because the scale is preventing the spring from stretching any further (and the spring constant can't change, of course). So an upward acceleration can only be accompanied by an increase in the scale's reading (its normal force).*

If the elevator is accelerating down, then what can we conclude? Well, either the elastic force or the scale force must decrease (again, mg *cannot change). But for the spring force to decrease, the spring must stretch less, which means the block must leave contact with the scale. This would make the normal force zero, that is, as long as the scale is registering any normal force, the spring's amount of stretch is unchanged. So for a very low acceleration, the normal force of the scale will come down, while the spring force remains unchanged, but if the acceleration is larger, then the scale's reading goes to zero, and the mass lifts off the scale, reducing the spring stretch.*

Let's get kinetic friction in on the fun...

Analyze This

A block is sped-up at a steady rate along a rough, horizontal surface by a stretched spring that pulls on it.



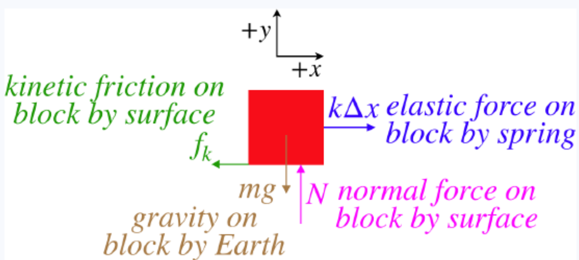
Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Our first observation as we ponder this is that as the block slides, the stretched spring is remaining stretched by the same amount. Presumably whatever is pulling on the other end of the spring must also be accelerating along. We know this because we are told that the acceleration is "at a steady rate". A steady acceleration means a constant net force, so the sum of the horizontal forces must remain constant, and since the friction force remains constant, the spring force must as well. This only occurs if the spring remains stretched the same amount.

Start with a force diagram of the block:



Now apply Newton's second law for both the x and y components, noting that there is no acceleration of the block in the vertical direction:

$$F_{\text{net }x}=-f_k+k\Delta x=ma_x\ ,\quad F_{\text{net }y}=N-mg=ma_y=0\quad\Rightarrow\quad N=mg$$

There is one other piece of information we can add to this analysis. We know something about kinetic friction – it is proportional to the normal force between the two rubbing surfaces. Putting this in the mix and calling the acceleration that only exists along the x-axis simply "a" gives:

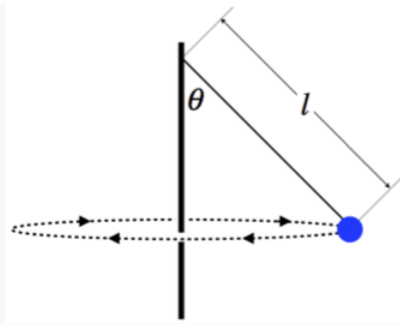
$$f_k=\mu_kN=\mu_kmg\quad\Rightarrow\quad -\mu_kmg+k\Delta x=ma$$

Okay, now let's start re-using themes, but include some complications that require a bit more thought. The key to getting through these "trickier" examples is to just "follow the method" – don't try to think too far ahead, or get stuck on a preconceived idea of what you expect to be the answer.

Here is an example in two-dimensions like the hanging sign problem above, but this time our old friend centripetal acceleration from Chapter 1 is included.

Analyze This

A tetherball swings around a pole, making a full circle at regular time intervals. The rope has negligible mass.

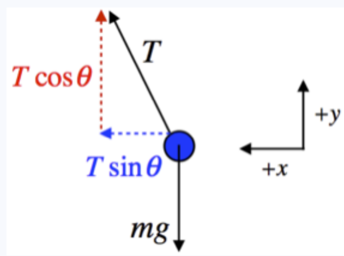


Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Start with a force diagram of the ball, including a coordinate system (we will dispense with the rather obvious force phrase descriptions here):



Next sum the forces along the x and y axes and apply Newton's second law:

$$F_{\text{net }x} = T\sin\theta = ma_x \;, \quad F_{\text{net }y} = T\cos\theta - mg = ma_y$$

The ball is not accelerating vertically, as it remains at the same constant height, so $a_y = 0$. In the horizontal direction, however, the motion of the ball is changing direction, which means that it must be accelerating. This motion is circular, so the acceleration is toward the center of the circle, and if the ball is moving at a speed v, this centripetal acceleration is:

$$a_x = \frac{v^2}{R}$$

where R is the radius of the circle. Putting these two accelerations in above gives:

$$T\sin\theta = m\frac{v^2}{R} \;, \quad T\cos\theta - mg = 0$$

The tension T can be eliminated from these equations (and the value of the mass m also cancels-out) to give:

$$\tan\theta = \frac{v^2}{gR}$$

We can take it even a little further than this. The description mentions "regular time intervals" for the tetherball's motion. If we call the interval for a full revolution "t", then we can relate the speed v to the radius of the circle and the time interval:

$$v = \frac{\text{circumference}}{t} = \frac{2\pi R}{t} \quad \Rightarrow \quad \tan\theta = \frac{4\pi^2 R}{gt^2}$$

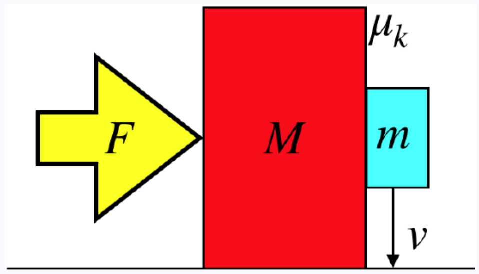
We can even take one more step. In the diagram given, the length of the rope is labeled as "l". The radius of the circle can easily be written in terms of this length and the angle θ, giving us a final result that relates the interval to the angle and length of rope:

$$R = l\sin\theta \quad \Rightarrow \quad \cos\theta = \frac{gt^2}{4\pi^2 l}$$

Another situation where kinetic friction comes into play, though it is a tough one to rely upon intuition for – just follow the method!

Analyze This

A large block is pushed along a horizontal, frictionless surface by an external fixed force. In contact with the rear vertical face of the large block is a smaller block, and as the two blocks are accelerated horizontally, the smaller block slides down the rough (not frictionless) face of the larger block at a constant speed.



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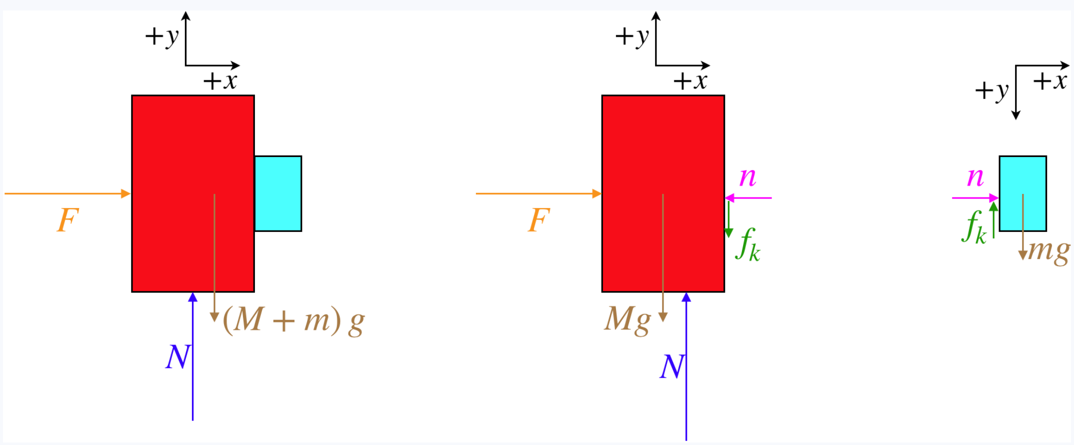
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Few problems demonstrate the importance of drawing free-body diagrams better than this one. One thing that is interesting here is that we have several choices for what is the "object" in the free-body diagram. We can choose the large block, the small block or the system of both blocks. As this is just analysis, and we don't know where we will eventually need to go, we'll just do all three:



These show nicely how the two third-law pair forces (the normal force between the blocks and the kinetic friction force between the blocks) appear when we split the blocks into separate systems, but are unneeded when they are internal to the system of two blocks. Next let's write down the equations that come from Newton's second law for each of these diagrams:

two blocks

$$F_{\text{net }x} = F = (M+m)\,a_x \ , \quad F_{\text{net }y} = N - (M+m)\,g = ma_y$$

large block

$$F_{\text{net }x} = F - n = Ma_x \ , \quad F_{\text{net }y} = N - Mg - f_k = ma_y = 0$$

small block

$$F_{\text{net }x} = n = ma_x \;, \qquad F_{\text{net }y} = mg - f_k = ma_y$$

Next we need to consider the accelerations. The acceleration of the big block in the vertical direction is clearly zero, as it never moves up or down. The values for the a_y's in the two block and small block cases are not as obvious, as the center of mass of each of these system is clearly falling. But we are given that the descent of the small block is at a constant speed, so while the center of mass is moving in the $-y$ direction, it is not accelerating in either case. We can therefore declare that both of the a_y's appearing in the equations above are zero. The acceleration in the $+x$-direction is obviously not zero, but whatever it is, with the blocks both moving together, it is the same for all three systems. We therefore can compute the acceleration for all three systems most easily from the two-block system:

$$a = \dfrac{F}{M+m}$$

Anything else we can extract from what we are given? Well, from the small block system, given that the vertical acceleration is zero, we know that the kinetic friction force equals the block's weight, but we can also write the kinetic friction force in terms of the coefficient of kinetic friction and the normal force between the surfaces:

$$mg - f_k = ma_y = 0 \quad \Rightarrow \quad mg = f_k = \mu_k n$$

We know the horizontal acceleration of the small block, and the only horizontal force on it is n, so we can plug in for n to finally get:

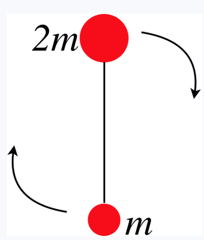
$$mg = \mu_k n = \mu_k ma = \mu_k m\dfrac{F}{M+m} \quad \Rightarrow \quad F = \dfrac{1}{\mu_k}\left(M+m\right)g$$

This tells us that the force that needs to be applied to allow the block to slide at a constant speed is greater than the weight of the two-block system by a factor of μ_k^{-1} (recall that coefficients of kinetic friction are generally less than 1). If the force is any greater than this, then the normal force between the blocks will be greater, making the friction force greater than the weight of the small block, and the small block's decent will actually slow. If the force is less than this amount, the friction force will be less than the weight of the small block, and its descent will speed up.

Here's an example to make sure you haven't forgotten about the role of center of mass in Newton's laws.

Analyze This

A system of two balls of different masses attached by a string are thrown horizontally through the air, and rotates at a steady rate about its center of mass as it goes. Air resistance is negligible for the system, and at the moment the balls are thrown, the larger ball is directly above the smaller ball, as in the diagram.



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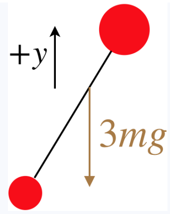
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Let's start with a free-body diagram of the system. As it is flying through the air, there is only one force on it – gravity. The internal tension forces between the balls are a third law pair, and can therefore be ignored. At some arbitrary moment during the flight, the FBD looks like:



Note that we strive to locate the total gravity force on the system at the center of mass for the system, and if we treat the two balls as though they are particles, the center of mass comes out to be two-thirds of the distance from the smaller mass m to the larger mass $2m$.

What is interesting here is that with this being the only force, the acceleration of this system is simple to compute:

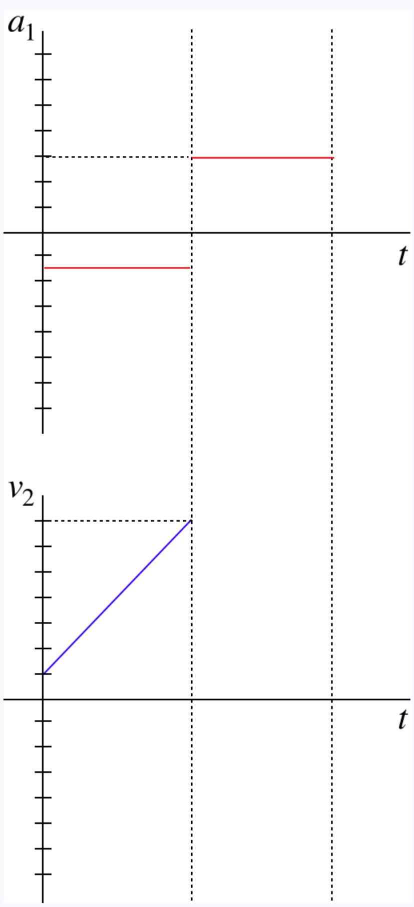
$$F_{\text{net } y} = -3mg = 3ma_y \;, \quad a_y = -g$$

In other words, this object is just a standard projectile! As such, we can use the projectile equations to describe its motion. But we have to be careful – this only describes the motion of the center of mass of the system, not the individual balls at the ends of the string. But if we know something about the rate of rotation of the two balls, then if we are given a time, we can use the projectile equations to locate the center of mass, and the rotation rate to locate the balls relative to that center of mass.

And finally, an example that combines understanding of graphs with Newton's laws.

Analyze This

Two particles, #1 and #2 interact only with each other. The acceleration of particle #1 is plotted on the graph below for a period of time. The velocity of particle #2 was plotted simultaneously, but the data for the second half of the time interval was lost, and its graph is also shown below.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

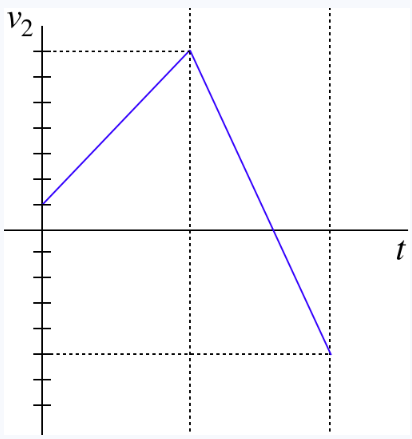
- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

This is one of those rare occasions when a free-body diagram is not called for, but Newton's laws are still very important to the analysis. We are given that these two graphs are for the motions of particles that only interact with each other. Given that these graphs represent accelerated motion, it must be that the particles exert forces on each other. From Newton's third law, these forces must be equal-and opposite.

We cannot extract the forces from these plots, we can only get the accelerations. But by Newton's second law, these are proportional to the forces. The acceleration of particle #1 can be read directly from the graph, and it is $-1.5\;units$ *for the first half. Particle #2's acceleration is the slope of its velocity graph, and that is* $+6.0\;units$ *for the first half. With 4 times as much acceleration and an equal force on it, we conclude that particle #2 has one fourth the mass of particle #1.*

The masses of the particles do not change for the second half of the time interval, so since we know the acceleration of particle #1 $+3\;units$*, we also immediately know the acceleration of particle #2 from Newtons's second and third laws – it must be* -4 *times particle #1's acceleration, or* $-12\;units$*. If we are asked to add the second half of the time interval to particle #2's velocity graph, it would have to be a straight line that drops 12 units during that second interval:*



2.6: Additional Twists - Constraints

Let's Spice Things Up!

If Newton's laws are the main ingredients of the mechanics recipes we have been cooking-up, then the time has come to add the spices. Actual physical systems are typically characterized by more than just the three laws of motion. Very often there are other restrictions – called *constraints* – that relate physical quantities to each other. Some of these are independent of Newton's laws of motion, while others are "rules of thumb" that ultimately come from Newton's laws.

We have actually seen a couple of examples of these already, in the examples we look at in the previous section, though we never referred to them in this way. One of them is the relationship between the kinetic friction force and the normal force for two surfaces. This relationship (equation) does not come from the invocation of Newton's second or third law, it is just an extra equation that we can use to solve problems that include the feature of rough surfaces sliding across each other. Another example we have seen is the formula for centripetal acceleration. Newton's second law tells us how to relate forces to accelerations, but this specific relationship between acceleration and velocity for the case of circular motion only applies to special cases.

We will add to our "spice rack" of possible constraints in this section, and in so doing, we will greatly multiply the variety of mechanics problems we can solve.

Constraints on Forces

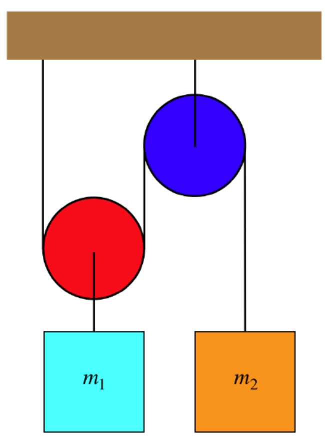
The two quantities in Newton's second law that can be constrained are force and acceleration. We will look at each possibility in turn, starting with force constraints.

Pulleys

One of the favorite devices for physics mechanics problems is the pulley. As usual, we will start with the simplest model, which in this case means we will assume that pulleys are massless and frictionless. As we will see more clearly later in the course, these two conditions are sufficient to ensure that the tension applied at one end of the rope is the same as the tension applied at the other, even though the intermediate section of the rope goes around one or more pulleys. This means the measured tension force can have different directions, depending upon where it is measured, but it always has the same magnitude. This is a constraint on the tension forces present in a physical system.

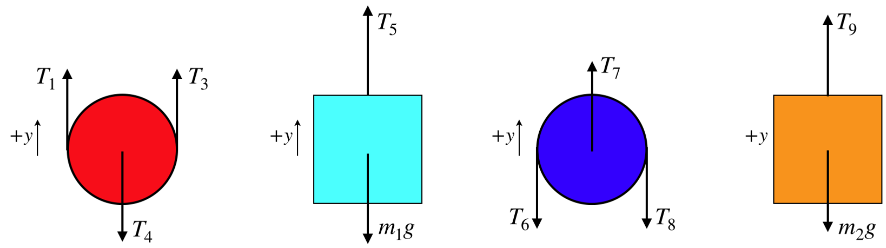
Pulleys get especially interesting in situations like the following example, where at least one of the pulleys is able to move. The two blocks remain at rest in the system of ropes and pulleys shown in the diagram. Given this information, can you conclude how the two masses compare?

Figure 2.6.1 – Blocks Hanging from Multiple Pulleys



By now we know that when it comes to analyzing the forces present in a system, there is no better tool than the free-body diagram. We begin there:

Figure 2.6.2 – Free-Body Diagrams of Blocks and Pulleys



Look at all those tension force vectors! There is one for every segment of rope pulling on an object. One might ask why there are two tension force vectors drawn for the same rope on each pulley. The simplest answer is to consider what you would feel if you grabbed the rope on both sides of the red pulley, and cut the rope above the two points where you are holding. Clearly you would feel the pulley pulling down on both ends of the rope. If you feel forces down by each end of the rope, then the pulley must feel forces up by each end as well, according to Newton's third law.

Here is where our pulley force constraint comes into play. Assuming the pulleys have negligible mass (or are static, and these are both), and assuming their axles are frictionless, then we can use the constraint that tension forces exerted by every segment of a single continuous piece of rope has the same magnitude. There are three ropes involved here, so this constraint in mathematical terms gives:

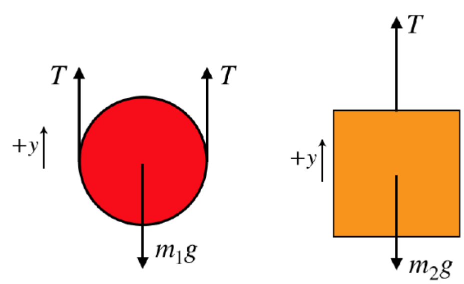
$$T_1 = T_3 = T_6 = T_8 = T_9\ , \quad T_4 = T_5 \tag{2.6.1}$$

Alert

*We must be careful not to equate too many of these tensions – this constraint only holds for a **single, continuous piece of rope**.*

This constraint, if used carefully (i.e. making certain that the conditions required are in place), allows us to greatly streamline our problem-solving process. For example, when drawing the FBDs, we can avoid 9 different tension force labels, and just label all the tensions from the long, common rope simply "T." Another simplification for this diagram is noting that if we put the (red) pulley that is free to move up and down into a single system with the (blue) block that moves with it, then we can ignore the internal tension force ($T_4 = T_5$) altogether, and just label the downward force with the weight of the block. Let's incorporate these shortcuts into a revised FBD before proceding with our analysis. If we note that the FBD of the blue pulley is of no value to our analysis, we get the following very efficient FBDs:

Figure 2.6.3 – Streamlined Free-Body Diagrams of Blocks and Pulleys



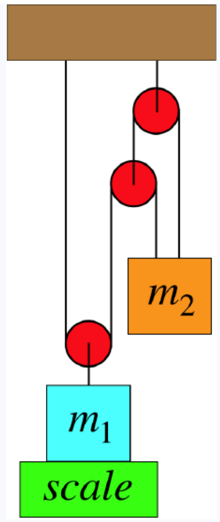
The next step in our analysis is to sum the forces for each object and apply Newton's second law, which in this case involves zero acceleration. In taking the sum of forces, we have to take care to correctly use our coordinate system:

$$\left.\begin{array}{l} 0 = a_1 = \dfrac{F_{net\ 1}}{m_1} = \dfrac{2T - m_1g}{m_1} \quad \Rightarrow \quad T = \dfrac{m_1g}{2} \\ 0 = a_2 = \dfrac{F_{net\ 2}}{m_2} = \dfrac{T - m_2g}{m_2} \quad \Rightarrow \quad T = m_2g \end{array}\right\} \quad \Rightarrow \quad m_1 = 2m_2 \tag{2.6.2}$$

Notice that the light weight m_2 holds up the heavier one because the placement of the pulley allows us to use the tension from the same rope twice on the heavier mass. This trick can actually be repeated as many times as we like (the pulley can have multiple tracks in it), and this enables us to lift very heavy weights with very little force. This invention is called a *block and tackle*. They are used for sailing ships (the heavy sails and boom can be pulled tighter), lifting engine blocks, and many other applications.

Analyze This

In the system shown below, the blue block remains at rest on the scale while it is attached to the pulley system as shown. All of the pulleys are massless and frictionless, and the rope is massless.



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- *what we are given (perhaps translated from English to mathematics)*
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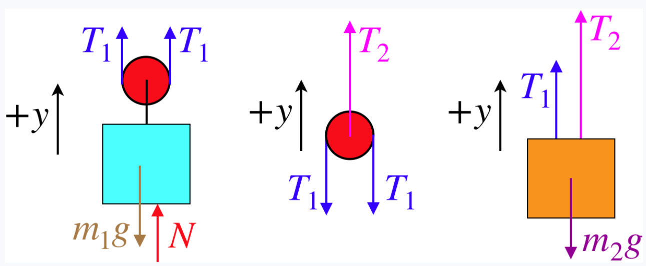
Analysis

As usual, the first step in analyzing such systems is to draw a FBD. In this case, we will employ whatever "shortcuts" indicated above that we can, but we need to be extra careful when multiple ropes are involved!



As noted in the text above, a FBD of the fixed (highest) pulley provides no useful information, since it only introduces the force by the ceiling pulling it up. If we are asked about this force, then we would have to add this FBD to the collection. For these three FBDs, we can construct the equations that result from Newton's second law, with zero acceleration:

$$F_{\text{net}} = +2T_1 + N - m_1 g = 0 \;, \quad F_{\text{net}} = T_2 - 2T_1 = 0 \;, \quad F_{\text{net}} = +T_1 + T_2 - m_2 g = 0$$

We can now (for example) eliminate the variables T_1 and T_2 from the equations and get the scale reading (the normal force) in terms of the blocks' masses:

$$T_2 = 2T_1 \quad \Rightarrow \quad 2T_1 = \frac{2}{3} m_2 g \quad \Rightarrow \quad N = \left(m_1 - \frac{2}{3} m_2 \right) g$$

It makes sense that the heavier m_2 is, the less force will be registered by the scale, since the orange block is pulling up on the blue block through the pulley system.

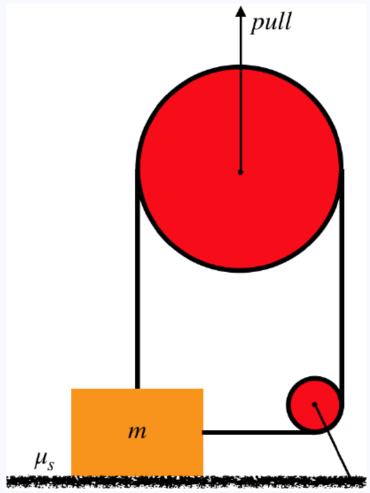
Friction

Another constraint on forces is one we have discussed previously – how the friction force between surfaces is related to the normal force between those surfaces. While the friction and normal forces appear in the equations that come from Newton's second law, the relationship between friction and normal force is "extra." We have already seen examples of this in action for kinetic friction in the previous section, so here we will direct our attention to the constraint for static friction.

When one has a system of equations, and then throws in an additional equation like $f_k = \mu_k N$, it is easy to incorporate into the algebra. Incorporating an inequality like $f_s \le \mu_s N$ is tougher, mainly because there is a conceptual element to it. Generally it shows itself in the statement of the question with language like, "Find the largest (or smallest) force for which...", or "Find the value of <whatever> at which the system starts to move". That is, there needs to be some mention of an *extremum*. This is because the extremum will trace its roots back to the maximum value of static friction, and when we are interested in the maximum value, our inequality becomes an algebraically-useful equality.

Analyze This

A rope is fastened to a block in two places and passes through a system of two massless, frictionless pulleys, as shown in the diagram below. The block rests on a rough horizontal surface. The bigger pulley can be pulled upward.



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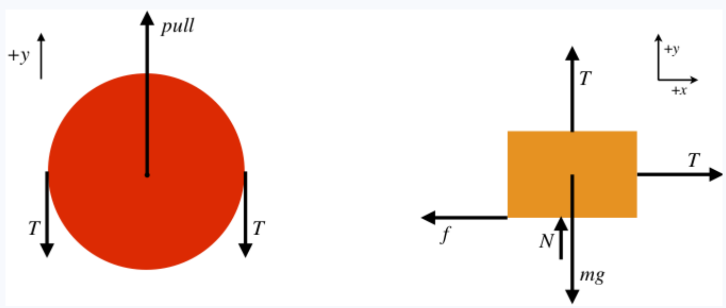
- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Start with the free-body diagrams and coordinate systems. The FBD of the smaller pulley will yield us nothing useful, so there are just two FBDs to draw. Note that the tension on the side of the block comes from the same rope as the tension on the top of the block, so thanks to the pulley constraint, they are equal, and we'll label them both the same "T".



The block is not accelerating at all (nor is the pulley), so the sum of the forces in each of the x and y directions comes out to zero.

$$F_{\text{net}} = +pull - 2T = 0; , \quad F_{\text{net}\ x} = +T - f_s = 0 , \quad F_{\text{net}\ y} = +T + N - mg = 0$$

Obviously if we pull up with sufficient force, then we'll lift the block off the horizontal surface. In particular, to lift the block the FBD of the block shows that the tension force pulling up needs to exceed the weight of the block. With two such tension forces pulling down on the pulley, we would have to pull up on the pulley with twice the weight of the block to get it to accelerate vertically. But the interesting part of this system lies with the horizontal motion...

As long as this system remains static, the friction force will only react to the other forces to hold the block in place horizontally. We can't say anything about this force without more information, but it is natural to ask, "How hard to we have to pull up on the larger pulley in order to get the block to start moving? We would expect that the force required is less than the force needed to left the block, computed above, but how do we find this force?

*If we have to pull "just hard enough" to get the block moving, then this occurs when the horizontal pull equals the **maximum** static friction force, which converts our static friction inequality into a constraint equation:*

$$f = \mu_s N$$

Note that the block will have to start sliding before it starts rising, because rising requires than the normal force goes to zero, and it will slide when the static friction force is small-but-non-zero. Now solve the equations simultaneously to get:

$$T = \mu_s N = \mu_s (mg - T) \quad \Rightarrow \quad pull = 2T = \frac{2\mu_s}{1+\mu_s} mg$$

Constraints on Acceleration

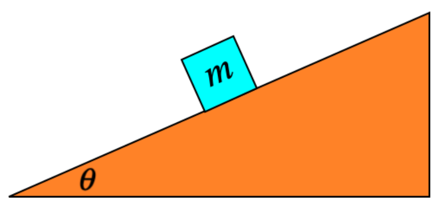
We now turn our attention to constraints on acceleration. These are "constraints" in a more literal sense than for forces, in that they are not shortcuts for more complicated cases. Acceleration constraints are purely mathematical (not physical) relationships, and therefore require fewer – if any – special conditions. These constraints on acceleration come in several varieties – from restrictions between components of acceleration for a single object, to accelerations of separate (connected) objects, to restrictions due to "special" motion.

Inclined Planes

In a large percentage of our examples so far, objects have been accelerating either horizontally or vertically. These have actually been "constraints" of a sort, because we know that when an object is accelerating along a horizontal surface (i.e. the object is "constrained to remain on the horizontal surface"), then we can immediately infer that its vertical acceleration is zero, and we can then plug this fact into the vertical equations for Newton's second law.

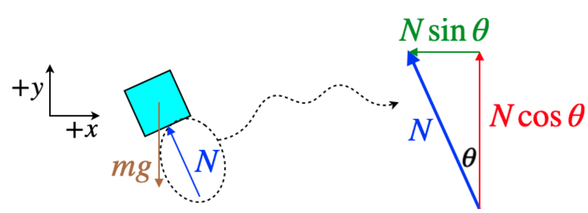
But now let's suppose that the object is confined to travel along a different surface. We will eventually talk about confinement to curved surfaces, but the simplest "new" case is a flat surface that makes an angle with the horizontal – a so-called *inclined plane*. This constraint forces the horizontal and vertical components of acceleration to have a specific relationship with each other. Namely, if the angle the plane makes with the horizontal is θ, then $\frac{a_y}{a_x} = \tan\theta$. So if we sum our forces along the horizontal (x) and vertical (y) axes, and apply Newton's second law, then we have an additional equation to throw into the mix thanks to the constraint that the object remains in contact with the inclined plane. To see this in action, let's look at the simplest possible case – a block on a frictionless inclined plane:

Figure 2.6.4 – Block on a Frictionless Inclined Plane



With no friction present, there are only two forces on the block. Drawing the free-body diagram and resolving the angled normal force into the usual horizontal and vertical components gives:

Figure 2.6.5 – FBD of Block on a Frictionless Inclined Plane



[*The reader is encouraged to do the geometry to confirm that the angle θ in the normal force resolution is the same as the angle of the inclined plane up from the horizontal.*]

Now we apply Newton's second law to this diagram, for both components of the net force:

$$F_{\text{net }x} = -N\sin\theta = ma_x \;, \quad F_{\text{net }y} = N\cos\theta - mg = ma_y \tag{2.6.3}$$

Now we can do a bit of algebra, and apply our acceleration component constraint to get (after some trig identities):

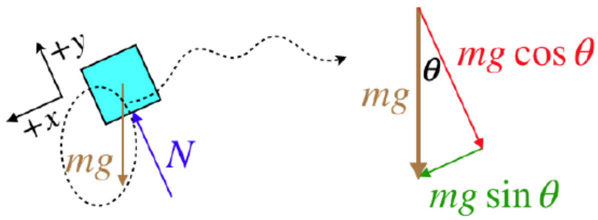
$$-\frac{N\cos\theta}{N\sin\theta} = \frac{ma_y + mg}{ma_x} \quad \Rightarrow \quad -\cot\theta = \tan\theta + \frac{g}{a_x} \quad \Rightarrow \quad a_x = -g\sin\theta\cos\theta \quad \Rightarrow \quad a_y = a_x\tan\theta = -g\sin^2\theta \tag{2.6.4}$$

The negative signs have appeared because the coordinate system chosen has to-the-right and upward as the positive directions, and this block accelerates to-the-left and down. We are interested in the motion of the block, which means its total acceleration along the inclined plane. We can get this from the components of acceleration:

$$a=\sqrt{a_x^2+a_y^2}=\sqrt{g^2\sin^2\theta\left(\cos^2\theta+\sin^2\theta\right)}=g\sin\theta \tag{2.6.5}$$

It turns out that we can save ourselves some of the algebra above when it comes to inclined planes by using a common trick. We have been treating our horizontal/vertical x, y coordinate system like it is sacred, but it is certainly not. We can choose *any axes we like*, as long as we stick with them throughout, and correctly reference the forces on those axes. Consider what happens if we choose the following coordinate system:

Figure 2.6.6 – Useful Coordinate System for an Inclined Plane



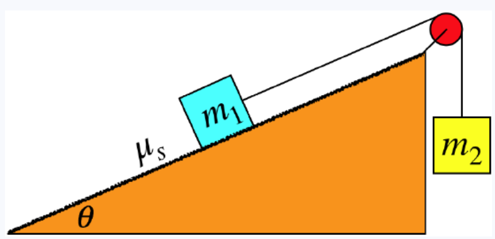
We have the same physical situation – same FBD – but this time the normal force is parallel to the y-axis, and it is the gravity force that we have to break into components. So what have we gained? The block is constrained to slide along the surface, so it *only* accelerates parallel to the x-axis. This makes the equations from Newton's second law easier to work with, giving the same answer as above for the acceleration immediately:

$$F_{\text{net }x}=mg\sin\theta=ma_x=ma \quad\Rightarrow\quad a=g\sin\theta\;,\quad F_{\text{net }y}=N-mg\cos\theta=ma_y=0 \tag{2.6.6}$$

This coordinate system choice is appreciated even more when one encounters a problem with other forces that are also parallel to the rotated axes (like friction, parallel to the surface). The fewer forces that need to be broken into components, the better.

Analyze This

The system shown in the diagram below remains at rest. The rope and pulley are massless, and the pulley is frictionless, but the inclined plane is not.



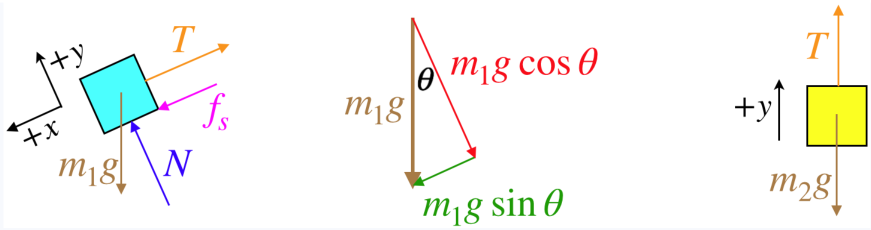
Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

There is obviously a lot going on here, but as always, our analysis just needs to take it one step at a time. We start as usual with free-body diagrams, but the FBD of the blue block poses a bit of a puzzle for us. With a component of the gravity force acting down the plane, and the tension force acting up the plane, we can't tell from looking at the diagram which of these two forces is greater. The static friction force only reacts to the other forces, so if the tension force is greater, then the friction force must point down the plane to keep the block from sliding up, and if the tension force is less, then the friction force points up the plane to keep the block from sliding down. (The friction force could even equal zero, if the magnitude of the tension force happens to be equal to $m_1g\sin\theta$.)

The answer to this conundrum is this: As an unknown, the static friction force must point either in the $+x$ or $-x$ direction. If we happen to draw the vector in the wrong direction, then after we do the math to obtain an answer for this force, then the answer will have the right magnitude, but will have a negative sign. This doesn't mean we have made a mistake – the FBD is there to help us solve the problem, which it did! It allowed us to compute the magnitude, and with the sign, it also told us the direction. So just draw the friction force on the FBD in either direction – it is a means to an end, not a declaration of which way you believe the direction to truly be.



Now for Newton's second law. With the system remaining at rest, we can immediately plug-in zero for the accelerations:

$$F_{\text{net }x} = +m_1 g \sin\theta + f_s - T = 0\ , \quad F_{\text{net }y} = +N - m_1 g \cos\theta = 0\ , \quad F_{\text{net}} = +T - m_2 g = 0$$

We can go no further without more information, which (for example) could trigger something like the static friction / normal force equation of constraint.

Pulleys

Another way that accelerations can be constrained involves ropes moving through pulleys. This constraint relates the motion of one object to that of another when they are connected through a system of pulleys. The only assumption required for this kind of constraint is that the rope does not stretch – however much it gets shorter in one segment, it gets longer by the same amount in another segment. Let us return to the system shown in Figure 2.6.1 and ask the following question: If the block m_1 rises a distance Δy, what happens to the block m_2?

First of all, it should be clear that m_2 drops as m_1 rises, so the only question is, how far? This may not be apparent at first, but think of it this way: When the pulley holding m_1 moves up 1 unit of distance, both segments of rope going up from that pulley get shorter by 1 unit. These two units of rope don't simply vanish, and in fact they are taken-up by the free end of the string, which is attached to m_2. This means that as m_1 rises a distance of Δy, m_2 must drop exactly twice that far: $2\Delta y$.

What does this say about the comparison of the speeds and accelerations of the two blocks? Well, they are required to move simultaneously, so every unit of length that m_1 rises is matched by a drop of m_2 that is twice as much, which means that m_2 always moves at twice the speed and accelerates twice as much as m_1. If this system is not balanced (as it was in the original example), then applying Newton's second law to both blocks includes two accelerations, but these are *constrained* to be related to each other by a factor of two, providing us with an additional constraint equation:

$$2\,|a_1| = |a_2| \tag{2.6.7}$$

What's with the absolute values, you ask? Well, these variables can have positive or negative values, and we must be careful when it comes to signs. In particular, we have to look at how our constraint relates to our choice of coordinate systems for the two blocks. In Figure 2.6.2, we chose "up" as the positive direction for both blocks. So we need to ask ourselves, "If one block experiences a positive displacement, what is the sign of the displacement of the other block?" In this case it's clear that the displacement of the two blocks have opposite signs. Therefore the constraint equation for the block accelerations is:

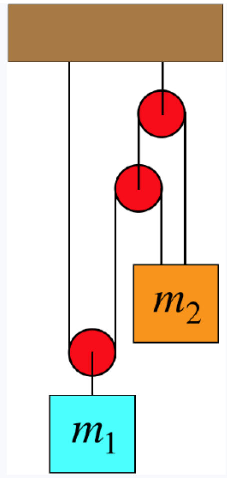
$$2a_1 = -a_2 \tag{2.6.8}$$

Note that it is perfectly fine to set up different coordinate systems for the two blocks – each FBD is entitled to its own individual coordinate system. How the coordinate systems relate to each other affects the equation of constraint. So for example, if we had instead chosen downward to be the $+y$-direction for block #2 (but left upward as positive for the other block), then there would be no need for the minus sign in the constraint equation – positive displacements of one block correspond to positive displacements of the other block. We see that there is therefore no "correct" choice of coordinate system, but we must take care when the time comes to combine the equations from the two FBDs, to ensure that the constraint equations relate the variables correctly.

With this constraint mechanism in mind, let's recycle a pulley system we analyzed above, this time without the condition that it remains static with the blue block resting on a scale:

Analyze This

In the system shown below, the blocks are free to accelerate. All of the pulleys are massless and frictionless, and the rope is massless.



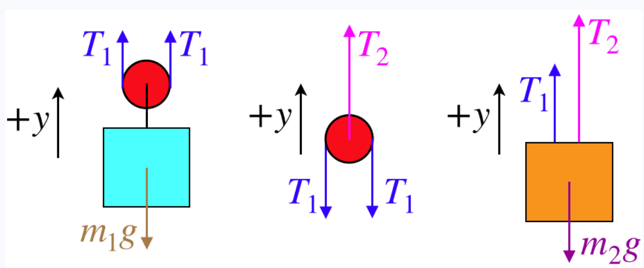
Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

[The analysis of this system teaches us a very important lesson about these types of problems: Even though this will clearly behave very differently from the case studied above, much of the early analysis is the same. This demonstrates why it is important to practice the little pieces of analysis, as they come up over and over.]

The FBDs for this system are the same as those above, with the exception of the normal force coming from the scale:



Okay, now we invoke Newton's second law. Unlike the previous case, we need to include non-zero accelerations:

$$F_{\text{net}} = +2T_1 - m_1 g = m_1 a_1 \;, \quad F_{\text{net}} = T_2 - 2T_1 = m_{pulley} a_{pulley} \;, \quad F_{\text{net}} = +T_1 + T_2 - m_2 g = m_2 a_2$$

Okay, now is where things get a bit different. First, we note that the equation of motion for the pulley includes the mass of the pulley, which we are given is zero. So the relationship between the two tensions is the same as before:

$$T_2 = 2T_1$$

And now the time has come to relate the constrained accelerations of the two blocks. [This is quite tricky – much trickier than you will encounter on your own in this class, but we'll slog through it in the hope that it helps you to do simpler examples on your own.] Start by defining the positions of the three moving objects, measured as the distance down from the highest pulley: y_1, y_p, *and* y_2. *The lengths of the long and short ropes (L and l, respectively) can be written in terms of these distances (plus a constant to account for the radii of the pulleys and in the case of the longer rope, the distance of the highest pulley from the ceiling):*

$$L = const + \text{left segment} + \text{middle segment} + \text{right segment} = const + y_1 + (y_1 - y_p) + (y_2 - y_p) = const + 2y_1 - 2y_p + y_2$$
$$l = const + \text{left segment} + \text{right segment} = const + y_p + y_2$$

We are interested in how the change in the height of the blue block is related to the change in height of the orange block, which means we are looking to relate dy_1 *to* dy_2. *The length of the rope doesn't change, nor does the constant, so:*

$$dL = 0 = 0 + 2dy_1 - 2dy_p + dy_2 \;, \quad dl = 0 = 0 + dy_p + dy_2 \quad \Rightarrow \quad dy_1 = -\frac{3}{2} dy_2$$

From this result, we conclude that the magnitude of the acceleration of the blue block is 1.5 times the acceleration of the orange block. Notice that we have chosen 'up' to the be positive direction for both blocks, but when one block goes down, the other must go up. So the accelerations must have opposite signs, and we conclude:

$$a_1 = -\frac{3}{2}a_2$$

From this and in equations above, one can compute the accelerations of the blocks in terms of their masses. Sparing the reader the algebra, it comes out to be:

$$a_1 = \frac{6m_2 - 9m_1}{9m_1 + 4m_2}g\,, \quad a_2 = \frac{6m_1 - 4m_2}{9m_1 + 4m_2}g$$

The tensions in the two ropes can also be computed in terms of the two masses:

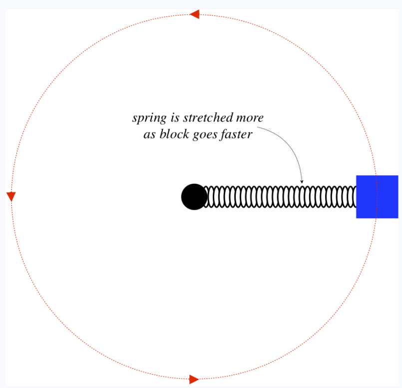
$$T_1 = \frac{5m_1m_2}{9m_1 + 4m_2}g\,, \quad T_2 = \frac{10m_1m_2}{9m_1 + 4m_2}g$$

Circular Motion

The last of the accelerated constraints involves knowing something specific about the acceleration of the one or more objects involved. A trivial case would be that you could be simply *given* the acceleration. Another possibility is that the object could be moving in a straight line and you could be given details about its motion (initial speed, distance traveled, etc.), and the acceleration could be computed using kinematics. But the most interesting and useful ways to constrain the acceleration of an object is to have it move in a circle, so that it experiences a centripetal acceleration.

Conceptual Question

Everyone knows that a spring can only be stretched if it is pulled from both ends – pulling from one end only moves the whole spring without stretching it. With this in mind, consider the following: A block is attached to one end of a massless spring, the other end of which is attached to a vertical fixed peg in a frictionless horizontal surface. The block is spun around a circle, and the spring stretches as a result of this motion, (which means that both ends are being pulled). In fact, the faster the motion, the more the spring stretches. Clearly the peg is pulling on one end of the spring as the block goes in the circle, but what force is pulling the block outward to stretch the spring?



Solution

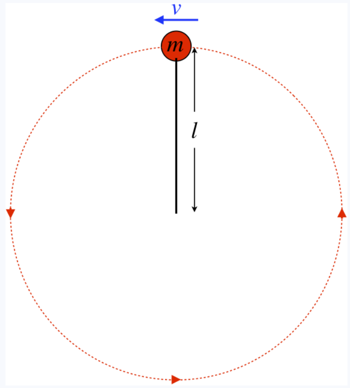
The block is ***not*** *pulled outward! It is only pulled inward (by the spring). It is not the block that needs to be pulled outward to stretch the spring, but rather the spring that needs to be pulled that way. The spring pulls the block inward (keeping it accelerating centripetally), and the third-law-pair force of the block on the spring is what pulls the spring outward. This is a fantastic example of how imprecise wording can get someone in trouble in physics discussions.*

This points out possibly better than any other example the importance of isolating objects with force diagrams. The block here is not a conduit for some mysterious force pulling out on the spring – it *is* the object pulling out on the spring. You thoroughly need to trust the third law here to get the force between the spring and the block, and you need to thoroughly trust the second law to realize that the block does not require another force on it outward to balance the spring force, because it is accelerating.

Now for an example that incorporates circular motion. What makes this problem interesting is the information that is hidden within the wording...

Analyze This

A rock on a string flies around (with negligible air drag) in a circle in a vertical plane (in the presence of the earth's gravity) such that it just barely gets by the top (the string remains at its full length at the top of the circle, just barely not going limp) as it continues in its circular path.



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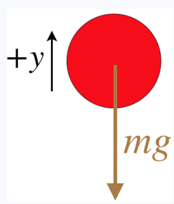
- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Start with a free-body diagram. Clearly there is gravity acting on the rock, so the only question is whether the string is contributing a tension force. While it is straight, we are told that the rock "barely gets by the top", which means that the string has gone limp at the peak, and the tension force is zero. That makes the FBD rather simple:



Though we have not stated so explicitly, we can think of the "tension force from string equals zero" as a force constraint. It is information given to us (in an obscure manner) that pertains to the value of one of the forces that is not derived from the application of Newton's laws, so it fits the description perfectly. Whether we call it a constraint or not isn't important – what matters is that we can extract this piece of physical data from the information given.

Next we turn to the acceleration constraint – the rock is traveling in a circle. If we call the speed of the rock at the top of the circle "v," and the length of the string "l," as indicated in the diagram, then the centripetal acceleration toward the other end of the string is:

$$a_c=\frac{v^2}{l}$$

Relating this acceleration to the only force present, we see that these vectors are in the same direction, as they should be, and Newton's law gives us a result that expresses the velocity of the rock in terms of only the length of the string!

$$F_{\text{net}}=ma \quad \Rightarrow \quad mg=ma_c=m\frac{v^2}{l} \quad \Rightarrow \quad v=\sqrt{gl}$$

Sample Problems

All of the problems below have had their basic features discussed in an "Analyze This" box in this chapter. This means that the solutions provided here are incomplete, as they will refer back to the analysis performed for information (i.e. the full solution is essentially split between the analysis earlier and details here). If you have not yet spent time working on (not simply reading!) the analysis of these situations, these sample problems will be of little benefit to your studies.

Problem 2.1

Two spherical objects of identical radii but different masses are dropped from different heights through the air. They both reach terminal velocity at the same moment in time, and at that moment, they are side-by-side. From this point on, the time that the heavier sphere takes to reach the ground is half the time that the lighter sphere takes.

a. *Find the ratio of the masses of the two spheres.*
b. *Treating the two spheres as a single system, the constant velocities of the individual spheres assures that the system's center of mass is also moving at a constant speed. In other words, the system is also at terminal velocity. The system as a whole does not have the same cross-sectional area or the same shape as the individual spheres, so it has a different drag coefficient C_d than that of the individual spheres c_d. Find C_d in terms of c_d.*

Solution

a. Both spheres are moving at constant speeds, and they are both covering the same distance, so the times they take to reach their destination are inversely-proportional to these speeds. The heavier mass takes half the time to get to the ground, so it must be moving twice as fast. Using the result from the analysis, we find the ratio of the masses to be:

$$\frac{2}{1}=\frac{V_{terminal}}{v_{terminal}}=\sqrt{\frac{M}{m}} \quad \Rightarrow \quad M=4m$$

b. The density of the air (ρ), cross-sectional area (A), and drag coefficient (c_d) are all the same for both cases, so given what we learned about the velocities, the drag forces on the two spheres are (capital letter symbols refer to the heavier sphere, lower-case symbols to the lighter sphere):

$$f_d=\frac{1}{2}c_d\rho Av^2\;, \quad F_d=\frac{1}{2}c_d\rho AV^2=\frac{1}{2}c_d\rho A(2v)^2=4f_d$$

For the system as a whole, the air density is unchanged, and the cross-sectional area is the sum of the areas of the two spheres ($A_{sys}=2A$). The terminal velocity of the system is the velocity of its center of mass (v_{cm}), so in terms of the new drag coefficient, the drag force on the system is:

$$F_d\left(\text{system}\right)=\frac{1}{2}C_d\rho A_{sys}v_{cm}^2$$

Now we just need to include two more things. First, the drag force on the system is just the sum of the drag forces on the spheres:

$$F_d\left(\text{system}\right)=F_d+f_d=5f_d$$

And second, we need the speed of the center of mass. We have the ratio of the masses from part (a), so:

$$v_{cm}=\frac{d}{dt}y_{cm}=\frac{d}{dt}\left(\frac{m_1y_1+m_2y_2}{m_1+m_2}\right)=\frac{MV+mv}{M+m}=\frac{(4m)\left(2v\right)+mv}{4m+m} \quad \Rightarrow \quad v=\frac{5}{9}v_{cm}$$

Now we put it all together:

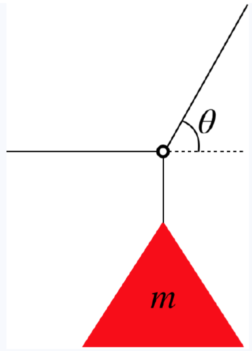
$$F_d\left(\text{system}\right)=5f_d=5\left[\frac{1}{2}c_d\rho Av^2\right]=5\left[\frac{1}{2}c_d\rho\left(\frac{1}{2}A_{sys}\right)\left(\frac{5}{9}v_{cm}\right)^2\right]$$

Comparing this to the equation above allows us to extract the drag coefficient:

$$C_d=\frac{1}{2}(5)\left(\frac{5}{9}\right)^2c_d\approx 0.77c_d$$

Problem 2.2

A sign hangs from a wire that is attached to a ring that is also attached to two wires (one of which is horizontally-oriented), as shown in the diagram. The wires and ring have negligible mass. The tension in one of the wires is three times as great as the tension in the other wire. Find the angle θ.



Solution

In the analysis we found, among other things:

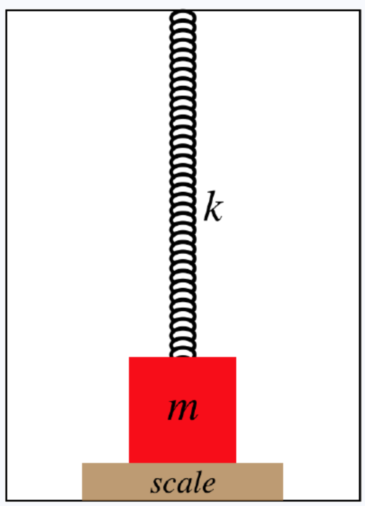
$$T_1 = T_2 \cos\theta$$

The cosine is always less than or equal to 1, so the force with the greater magnitude must be T_2. Plugging-in $T_2 = 3T_1$ and solving for θ gives:

$$\cos\theta = \frac{1}{3} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right) = 70.5^o$$

Problem 2.3

A block is attached to a spring that stretches down from the ceiling of a stationary elevator that is capable of accelerating up or down. The block is then lifted slightly, and a bathroom scale is placed beneath it, so that the block rests on it. The spring is still stretched at this point, but not as fully as when the block was hanging from it. The spring constant is $k = 24.0\frac{N}{m}$, the mass of the block is $m = 6.00kg$, and in the stationary elevator the scale reads $8.20N$.



a. *Find the direction and minimum magnitude of the acceleration of the elevator necessary to bring the scale reading to zero.*
b. *If the acceleration is double the value given in part (a), find how far above the scale the block rises.*

Solution

a. In the analysis for this situation, we found the relationship between the forces and the acceleration:

$$F_{\text{net}} = +k\Delta x + N - mg = ma$$

The reading of the scale is the normal force, and when the elevator is not accelerating we'll call it N_o, We therefore have:

$$a = 0 \quad \Rightarrow \quad N_o = mg - k\Delta x$$

The scale reads zero at the point when the normal force goes to zero, so setting N equal to zero in the original equation above, and noting that the spring stretch and weight of the block remain the same for the cases of when the normal force just goes to zero and $a = 0$, we get:

$$ma = +k\Delta x + 0 - mg = -N_o \quad \Rightarrow \quad a = -\frac{N_o}{m} = -\frac{8.20N}{6.00kg} = -1.37\frac{m}{s^2}$$

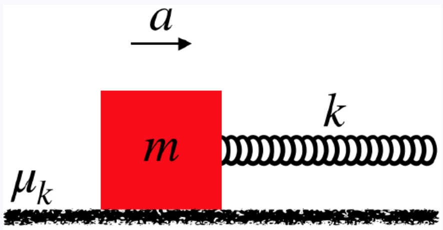
The minus sign indicates that the elevator must accelerate downward, as we would expect.

b. When the block leaves contact with the scale, the normal force remains zero, and at twice the acceleration we have a new spring stretch we'll call $\Delta x'$ Subtracting this stretch from the original stretch gives the height that the block rises, so:

$$+k\Delta x' + 0 - mg = m(2a) = 2(k\Delta x - mg) \quad \Rightarrow \quad k\Delta x - k\Delta x' = mg - k\Delta x = N_o \quad \Rightarrow \quad \Delta x - \Delta x' = \frac{N_o}{k} = \frac{8.20N}{24.0\frac{N}{m}} = 0.342m$$

Problem 2.4

A block is sped-up at a steady rate along a rough, horizontal surface by a stretched spring that pulls on it. The coefficient of kinetic friction between the block and the horizontal surface is $\mu_k = 0.300$. as the block is being pulled in this manner, it suddenly comes upon a smooth (frictionless) region of the horizontal surface, and its acceleration instantly increases to an amount that is 250% of the original acceleration. Find the before and after accelerations.



Solution

In the analysis we found this:

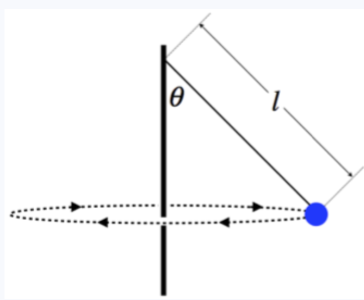
$$-\mu_k mg + k\Delta x = ma$$

Initially we have $\mu_k = 0.300$, and when the surface becomes smooth, this changes to $\mu_k = 0.0$ while a changes to $a_{after} = 2.50a$. The spring stretch doesn't have time to change when the block suddenly reaches the smooth area, so the value of $k\Delta x$ remains unchanged. This gives us two equations that we can solve simultaneously for the acceleration a:

$$0 + k\Delta x = m(2.50a) \quad \Rightarrow \quad -(0.300)mg + 2.50ma = ma \quad \Rightarrow \quad a = \frac{0.300}{1.50}g = 1.96\frac{m}{s^2}\ , \quad a_{after} = 4.90\frac{m}{s^2}$$

Problem 2.5

A tetherball swings around a pole, making a full circle at regular time intervals. The rope has negligible mass. A pebble is dropped from rest from the point where the rope is in contact with the pole. In the time it takes the pebble to cross the horizontal plane of the tetherball's circular motion, find the number of radians the tether ball has traversed of its circle.



Solution

The distance that the pebble travels is found from the right triangle formed by the rope and the pole (from the point where the rope is attached to the ball's rotational plane), and as it is in free-fall, this can be related to the time of the journey, starting from rest:

$$\Delta y = l\cos\theta = \frac{1}{2}gt_{drop}^2$$

In the analysis, we found the following relationship (where "t" is the time of one full revolution of the ball):

$$\cos\theta=\frac{gt^2}{4\pi^2 l} \quad\Rightarrow\quad l\cos\theta=\frac{1}{2\pi^2}\left(\frac{1}{2}gt^2\right)$$

Setting these equal, we can determine what fraction of the time of a tetherball revolution is the time the pebble takes to drop:

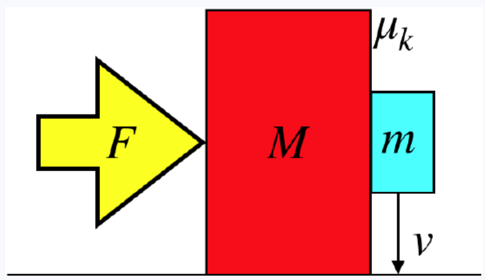
$$\frac{t_{drop}}{t}=\frac{1}{\sqrt{2}\,\pi}$$

In a full revolution, the tetherball sweeps out a full $2\pi\ radians$, so in the fraction of time, it travels:

$$\Delta\Omega=\left(\frac{t_{drop}}{t}\right)2\pi=\sqrt{2}\ rad$$

Problem 2.6

A large block is pushed along a horizontal, frictionless surface by an external fixed force. In contact with the rear vertical face of the large block is a smaller block, and as the two blocks are accelerated horizontally, the smaller block slides down the rough (not frictionless) face of the larger block at a constant speed. The coefficient of kinetic friction between the blocks is $\mu_k=0.40$. The large block begins at rest, and after $0.25s$, the small block hits the horizontal surface. At this moment, the large block has been displaced horizontally the same distance that the small block has fallen. Find the speed of the smaller block when it hits the horizontal surface.



Solution

The analysis gave us the force that must be applied to allow the kinetic friction force to equal; the weight of the small block (so that it doesn't accelerate vertically), and this gives us the horizontal acceleration of both blocks:

$$F=\frac{1}{\mu_k}(M+m)\,g \quad\Rightarrow\quad a=\frac{F}{M+m}=\frac{g}{\mu_k}$$

The big block starts at rest, so after a time t is moves a distance equal to:

$$\Delta x=\frac{1}{2}at^2=\frac{g}{2\mu_k}t^2$$

The distance the smaller block falls at a constant speed in the same time is $v_y t$, so ssince we are given that these distances are equal, we get:

$$\frac{g}{2\mu_k}t^2=v_y t \quad\Rightarrow\quad v_y=\frac{g}{2\mu_k}t=\frac{9.8\frac{m}{s^2}}{2\cdot 0.40}(0.25s)=3.1\frac{m}{s}$$

The horizontal component of the smaller block's velocity is found from its horizontal acceleration:

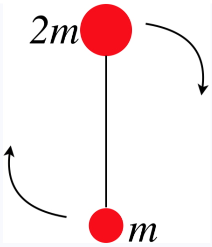
$$v_x=at=\frac{g}{\mu_k}t=2v_y=6.1\frac{m}{s}$$

And the block's total speed is therefore:

$$v=\sqrt{v_x^2+v_y^2}=6.8\frac{m}{s}$$

Problem 2.7

A system of two balls of different masses attached by a string are thrown horizontally through the air, and rotates at a steady rate about its center of mass as it goes. Air resistance is negligible for the system, and at the moment the balls are thrown, the larger ball is directly above the smaller ball, as in the diagram. The length of the string is $15cm$, and the two-ball system makes 2 full revolutions every second. Find the amount that the height of the heavier ball has changed 1.25 seconds after the balls are released.



Solution

This system as a whole (i.e. its center of mass) behaves like any other projectile as discussed in the analysis, and since it is launched horizontally, we can easily compute how far the center of mass falls:

$$\Delta y_{cm} = \frac{1}{2}gt^2 = 766cm$$

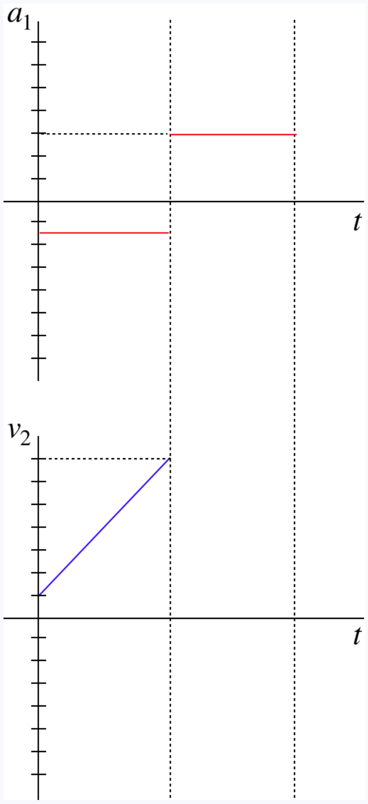
With the system rotating at a rate of 2 revolutions per second, after 1.25 seconds it has made 2.5 revolutions, which means that the heavier ball is now directly below the center of mass. With the two masses in a 2-to-1 ratio, the center of mass half as far from the heavier ball than it is from the lighter ball. This means that it started one third of the length of the string above the center of mass, and 1.25 seconds later it was one third of the length of the string below the center of mass. So the amount it has fallen is the distance the center of mass has fallen plus an additional amount of two-thirds of the string length:

$$\Delta y_{2m} = \frac{2}{3}l + \Delta y_{cm} = \frac{2}{3}(15cm) + 766cm = 776cm$$

Problem 2.8

Two particles, #1 and #2 interact only with each other. The acceleration of particle #1 is plotted on the graph below for a period of time. The velocity of particle #2 was plotted simultaneously, but the data for the second half of the time interval was lost, and its graph is also shown below. Particle #1 comes to rest end of its journey.

a. *Fill in the missing segment of the graph.*
b. *Plot the graph of the center of mass of this two-particle system.*



Solution

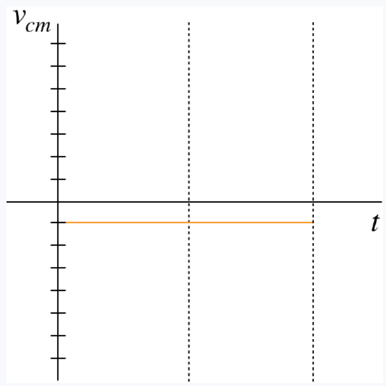
a. This was done in its entirety in the analysis! [It's often the case that the analysis can get us right up to the doorstep of the answer to any question, but occasionally the analysis can provide a complete answer.]

b. The particles are interacting only with each other, which means that there is no external force on this system. This means that its center of mass must maintain a constant velocity (Newton's first law). We are given that particle #1 comes to rest at the end of the journey, so we know the velocities of both particles at that moment in time, which means we can compute the unchanging velocity of the center of mass.

In the analysis, we used the second and third laws to determine the ratio of the masses of the particles, namely: $m_1 = 4m_2$*. This gives us the center of mass velocity:*

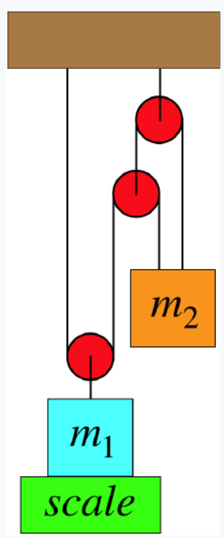
$$v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} = \frac{4\cancelto{0}{v_1} + 1v_2}{4+1} = \frac{-5units}{5} = -1unit$$

The graph of this (constant center) of mass velocity is this simple:



Problem 2.9

In the system shown below, the blue block remains at rest on the scale while it is attached to the pulley system as shown. All of the pulleys are massless and frictionless, and the rope is massless. In this setup, the scale reads $980N$*. The blue block is then removed from the pulleys and placed on the scale by itself, and the scale reads* $1470N$*. Find the mass* m_2 *of the orange block.*



Solution

In the analysis, we found:

$$N = \left(m_1 - \frac{2}{3}m_2\right)g$$

Doing some algebra to solve for what we are looking for gives:

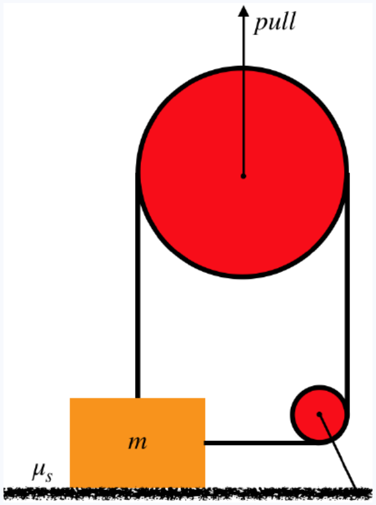
$$m_2 = \frac{3}{2}\frac{m_1 g - N}{g}$$

When the blue block is placed on the scale by itself, the normal force measured by the scale equals the weight of the block, $m_1 g$. Putting in this given value and the value of N when the pulleys are connected gives us the value of m_2:

$$m_2 = \frac{3}{2}\frac{1470N - 980N}{9.8\frac{m}{s^2}} = 75kg$$

Problem 2.10

A rope is fastened to a block in two places and passes through a system of two massless, frictionless pulleys, as shown in the diagram below. The block rests on a rough horizontal surface. The bigger pulley can be pulled upward. It is discovered that when the strength of the pull reaches one-half the weight of the block, the block just starts to slide. Find the coefficient of static friction between the block and the horizontal surface.



Solution

The analysis gave us the relationship between the pull force, the coefficient of static friction, and the weight of the block, for the case when the friction force is maximized:

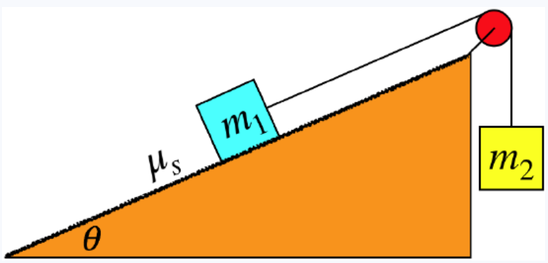
$$pull = \frac{2\mu_s}{1+\mu_s}mg$$

We are given that the sliding starts when the magnitude of the pull force is one-half the weight of the block, so $pull = \frac{1}{2}mg$. Plugging this in and solving for μ_s gives:

$$\mu_s = 0.33$$

Problem 2.11

The system shown in the diagram below remains at rest. The rope and pulley are massless, and the pulley is frictionless, but the inclined plane is not. The coefficient of static friction is 0.35, and the blue block has twice the mass of the yellow block. Find the maximum and minimum values that θ can have such that the system remains at rest.



Solution

We start by noting that if $theta$ is at its extreme maximum of 90^o, then the normal force between the blue block and the plane is zero, resulting in zero friction force, and greater mass of m_1 will cause the system to accelerate (the yellow block will rise). If θ is at its other extreme of 0^o, then the plane is flat, and the maximum static friction force is just $\mu_s = 0.35$ multiplied by the weight of the blue block. But this is less than the weight of the yellow block, so the system will accelerate in the other direction (the yellow block will fall). So there must be two extremes for θ, between which the system remains at rest.

We are looking for the extreme case, so we invoke the constraint that the friction force is its maximum:

$$f_s = \mu_s N$$

Putting this constraint together with the equations from the analysis and using the information given above that $m_1 = 2m_2$ gives:

$$0 = +m_1 g \sin\theta + f_s - T = +m_1 g \sin\theta + \mu_s N - T = +m_1 g \sin\theta + \mu_s m_1 g \cos\theta - m_2 g \quad \Rightarrow \quad \sin\theta = -\mu_s \cos\theta + \frac{1}{2}$$

We will find that this equation results in two possible values for θ. The smaller one gives a positive value for f_s (so the friction force points down the plane, as in the FBD from the analysis), and the larger one gives a negative value for f_s (indicating that the vector points up the plane, opposite to the FBD from which we derived these equations). The rest is algebra: Start by putting the equation in terms of only $\cos\theta$:

$$\sin^2\theta = \left(-\mu_s \cos\theta + \frac{1}{2}\right)^2 = \mu_s^2 \cos^2\theta - \mu_s \cos\theta + \frac{1}{4} = 1 - \cos^2\theta \quad \Rightarrow \quad 0 = \left(\mu_s^2 + 1\right)\cos^2\theta - \mu_s \cos\theta - \frac{3}{4}$$

Now solve for $\cos\theta$ using the quadratic formula:

$$\cos\theta = \frac{+\mu_s \pm \sqrt{\mu_s^2 - 4\left(\mu_s^2+1\right)\left(-\frac{3}{4}\right)}}{2\left(\mu_s^2+1\right)} = \frac{+\mu_s \pm \sqrt{4\mu_s^2 + 3}}{2\left(\mu_s^2+1\right)} = 0.988 \text{ or } -0.676$$

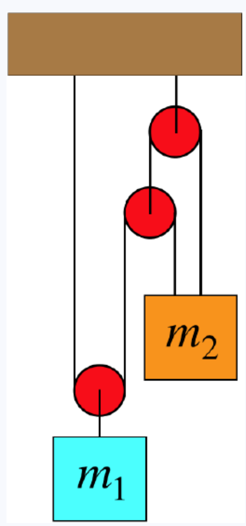
Taking the inverse-cosine gives these two angles:

$$\theta = 8.9^o \text{ or } 132.6^o$$

The larger of these numbers looks unusual, but it comes from the negative root of the quadratic equation, which represents the case when the frictional force points in the opposite direction than we diagrammed. We can just take the supplement of this angle (or ignore the minus sign on 0.676), and we get the more sensible answer of 47.4^o.

Problem 2.12

In the system shown below, the blocks are free to accelerate. All of the pulleys are massless and frictionless, and the rope is massless. The orange block starts at a height of $8.0m$ above the blue block, and they are released from rest. After $2.0s$, the blocks are at the same height. Find the ratio of the masses, $\frac{m_2}{m_1}$.



Solution

We computed expressions for the accelerations of the two blocks in the analysis. We can rewrite these in terms of the ratio we are looking for:

$$a_1=\frac{6m_2-9m_1}{9m_1+4m_2}g=\frac{6\frac{m_2}{m_1}-9}{9+4\frac{m_2}{m_1}}g\;,\quad a_2=\frac{6m_1-4m_2}{9m_1+4m_2}g=\frac{6-4\frac{m_2}{m_1}}{9+4\frac{m_2}{m_1}}g$$

These are the rates that they are accelerating relative to the earth, but we also know that they are in ***opposite directions****, with the blue block accelerating in the direction we chose to be positive in the FBD, and the orange block accelerating in the negative direction. So they are accelerating toward each other at a rate equal to the difference of these values. They started from rest, and we are given the distance they travel relative to each other, so we have:*

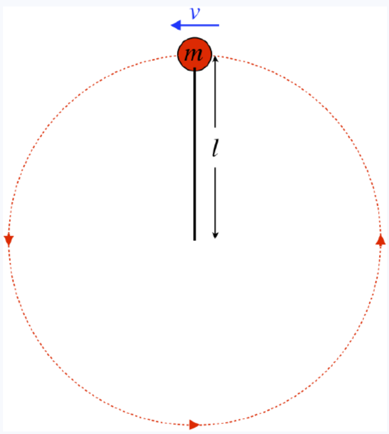
$$\Delta y=\frac{1}{2}a_{rel}t^2\quad\Rightarrow\quad a_{rel}=\frac{2\Delta y}{t^2}=\frac{2\cdot 8.0m}{(2.0s)^2}=4.0\frac{m}{s^2}$$

Setting the relative acceleration equal to a_1-a_2 and solving for the mass ratio gives our answer:

$$4.0\frac{m}{s^2}=a_{rel}=a_1-a_2=\frac{10\frac{m_2}{m_1}-15}{9+4\frac{m_2}{m_1}}g\quad\Rightarrow\quad\frac{m_2}{m_1}=2.2$$

Problem 2.13

A rock on a string flies around in a circle (with negligible air drag) in a vertical plane (in the presence of the earth's gravity) such that it just barely gets by the top (the string remains at its full length at the top of the circle, just barely not going limp) as it continues in its circular path. The string, which has a length of $120cm$, suddenly becomes detached from the rock at the point when the rock hits its peak, turning it into a projectile. Find how far the rock has traveled in the horizontal direction when it has fallen to a height level with the center of the circle.



Solution

We worked out the speed of the rock at the peak of the circle in the analysis: $v=\sqrt{gl}$. When the string releases it, there is no vertical component of velocity, so we can find the time it takes the rock to fall the distance l:

$$l=\frac{1}{2}gt^2\quad\Rightarrow\quad t=\sqrt{\frac{2l}{g}}$$

With a constant horizontal component of velocity, the distance traveled horizontally is just the product of that component of velocity and the time, so:

$$\Delta x=vt=\sqrt{gl}\cdot\sqrt{\frac{2l}{g}}=\sqrt{2}\,l=170cm$$

CHAPTER OVERVIEW

3: Work and Energy

3.1: The Work - Energy Theorem
3.2: Work and Energy for Collections of Particles
3.3: Conservative and Non-Conservative Forces
3.4: Energy Accounting with Conservative Forces: Potential Energy
3.5: Energy Accounting with Non-Conservative Forces: Thermal Energy
3.6: Mechanical Advantage and Power
3.7: Energy Diagrams
Sample Problems

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3.1: The Work - Energy Theorem

Ignoring Directional Changes

For a large number of applications in mechanics, we are not interested in how a force causes the direction of motion of an object to change. In these cases, we only care about how that force changes the speed of the object. By now we know how much of a pain vectors can be, so having an alternative to Newton’s second law to solve problems where only changes in speed are of interest is a welcome improvement. To see how we get to such a place, we need to go back to what we previously said about acceleration, and how it breaks into perpendicular parts – one that is parallel to the velocity (the “speeding-up/slowing-down” part), and the part that is perpendicular to the velocity (the “changing direction” part). We expressed this mathematically in Equation 1.6.14. We will now restrict our attention to the first term of that equation. Note that restricting ourselves to the part of the acceleration parallel to the direction of motion means we also restrict ourselves only to the component of the net force parallel to the motion. We will also begin, as we always do, by restricting our discussion to the motions of single particles, and will come back later to determine the consequences our findings have on systems of many particles.

Kinetic Energy and the Work-Energy Theorem

We have a neat trick that allows us to relate the change of the speed to the net force. The net force is proportional to the time derivative of the velocity vector, and we can use the product rule for derivatives of dot products of vectors, so let's take a derivative of the square of the velocity:

$$\frac{d}{dt}v^2=\frac{d}{dt}\left(\overrightarrow{v}\cdot\overrightarrow{v}\right)=\frac{d\overrightarrow{v}}{dt}\cdot\overrightarrow{v}+\overrightarrow{v}\cdot\frac{d\overrightarrow{v}}{dt}=2\overrightarrow{a}\cdot\overrightarrow{v} \tag{3.1.1}$$

To get to the net force, we multiply both sides by the mass of the particle and divide both sides by 2:

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right)=\left(m\overrightarrow{a}\right)\cdot\overrightarrow{v}=\overrightarrow{F}_{net}\cdot\overrightarrow{v} \tag{3.1.2}$$

This makes some sense. The rate of change on the left side of this equation only depends upon the rate at which the speed changes (it is insensitive to changes in direction), and the dot product on the right side ensures that only the projection of the net force along the direction of motion (i.e. the direction of the velocity) plays a role. The part of the net force that causes the particle to change direction is cast aside with the use of the dot product. We can take this a little bit further by expressing the velocity vector on the right side as a tiny displacement (which we will call $\overrightarrow{dl}$) divided by the tiny time interval. Multiplying both sides by dt then gives an equation that expresses a small change in the quantity $\frac{1}{2}mv^2$ (called the *kinetic energy*) due to a net force acting on the particle as it displaces a small amount $\overrightarrow{dl}$.

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right)=\overrightarrow{F}_{net}\cdot\frac{\overrightarrow{dl}}{dt}\quad\Rightarrow\quad d\left(\frac{1}{2}mv^2\right)=\overrightarrow{F}_{net}\cdot\overrightarrow{dl} \tag{3.1.3}$$

Suppose the particle now undergoes several displacements, so that the change in the kinetic energy is no longer infinitesimal. This is tricky business, as each displacement may be the same (if it moves in a straight line), or it may change direction (if it follows a curvy path). Also, the net force on the particle might change as it moves from one place to another. We express the sum of many infinitesimals as an integral, and since the sum of the right side of this equation depends upon the directions of many displacements, this particular type of integral is called a *line integral*. This does not mean that the displacements are along a straight line, however – here the word "line" is rather misleading – the word "trajectory" might be better.

Of course, the left side of this equation is simply a small number, and adding those up does not depend upon anything as complicated as a trajectory, so it ends up being just a change from the beginning of the path to the end. If we call the start of the journey A and the end B, then we can express the totals for the whole journey as:

$$\Delta\left(\frac{1}{2}mv^2\right)=\frac{1}{2}mv_B^2-\frac{1}{2}mv_A^2=\int\limits_A^B\overrightarrow{F}_{net}\cdot\overrightarrow{dl} \tag{3.1.4}$$

The line integral on the right side of this equation is called the work done (by the net force) going from the initial to final positions. We can (and later, will) discuss the work done by individual forces, and the work done by the net force is the total of all of those works. We will often write the above equation with the following abbreviated notation:

$$\Delta KE = W_{tot}\left(A \rightarrow B\right) \tag{3.1.5}$$

In words, this reads: "The change of an particle's kinetic energy when it changes its position from A to B equals the work done on it by all forces on it, computed over a well-defined path connecting those endpoints." This is known as the work-energy theorem. It does exactly what we set out to do – it expresses the effect forces have on the change in an particle's speed, with no regard to its directional changes. It doesn't solve any problem that can't be solved by Newton's second law, and in fact for some cases it isn't even any easier to work with. But for other cases is it *much easier* to work with, as we will see, and these are the cases for which this approach was invented.

These new quantities of kinetic energy and work have units of what we will more generically refer to as energy, and we give energy units their own name:

$$\left[KE\right] = \frac{kg\cdot m^2}{s^2} = \text{"}\;Joules\;\text{"} \quad \left(J\right)$$

Exercise

A single force which varies in magnitude and direction in space acts upon a particle, and is given by the equation below. Find the change in the particle's kinetic energy as it moves from the origin along the $+x$-axis a distance of 2m.

$$\overrightarrow{F}\left(x,y\right) = \left(\alpha x^2+\beta y^3\right)\hat{i} + \left(\beta x^3+\alpha y^2\right)\hat{j}, \quad where: \;\; \alpha = 2.4\frac{J}{m^2} \;\; and \;\; \beta = 4.5\frac{J}{m^3}$$

Solution

This is a direct application of the work-energy theorem, which means it consists entirely of computing a line integral. To do this, we first need to define the path mathematically, and all of the tiny displacements $\overrightarrow{dl}$ along that path. The path in this case is pretty simple – it is a straight line along the x-axis from the origin ($0m, 0m$) to the point ($2m, 0m$). Along this path, the value of y remains a constant zero. The direction of every infinitesimal displacement is the $+\hat{i}$ direction, and the magnitude of each displacement is simply dx. The work integral therefore becomes:

$$W\left(A \rightarrow B\right) = \int\limits_A^B \overrightarrow{F}\cdot\overrightarrow{dl} = \int\limits_{x=0m}^{x=2m} \overrightarrow{F}\cdot\left(dx\;\hat{i}\right)$$

Now we just need to plug in for the force. The force must be evaluated at each point on the path, and since the value of y is zero on the entire path, we can set $y=0$ in the force vector, simplifying things greatly:

$$\overrightarrow{F}\left(on\;the\;path\right) = \left(\alpha x^2\right)\hat{i} + \left(\beta x^3\right)\hat{j}$$

The dot product of this vector with the tiny displacement vector simplifies things even more:

$$\overrightarrow{F}\left(on\;the\;path\right)\cdot\overrightarrow{dl} = \left[\left(\alpha x^2\right)\hat{i} + \left(\beta x^3\right)\hat{j}\right]\cdot\left[dx\;\hat{i}\right] = \alpha x^2 dx$$

Finally, we just perform the integral and apply the work-energy theorem:

$$\Delta KE = \int\limits_{0m}^{2m} \alpha x^2 dx = \left[\frac{1}{3}\alpha x^3\right]_{0m}^{2m} = 6.4J$$

Line Integrals

As you can tell from the example above, the hardest part of using the work-energy theorem is setting up the line integral. There are several elements that need to be kept in mind:

1. **define a direction for the tiny displacement vectors for every point on the path**

The direction of the tiny displacement vectors (which we will assume to be in the (x, y) plane) will have components equal to the displacements in the x and y directions:

$$\overrightarrow{dl} = dx\ \hat{i} + dy\ \hat{j} \tag{3.1.6}$$

2. **write the magnitude of the tiny displacements in terms of the integration variable**

The displacement vector as written above doesn't tell us much. We also need to include the *path* for this to be useful. Since we are assuming that everything is in the (x, y) plane, the path can be expressed as a relationship between the variables x and y. For example, if the path is a straight line, then we can write $y = mx + b$. In this case, we can replace the dy in the displacement vector:

$$\frac{dy}{dx} = m \quad \Rightarrow \quad dy = m\ dx \quad \Rightarrow \quad \overrightarrow{dl} = \left[\hat{i} + m\hat{j}\right] dx \tag{3.1.7}$$

This puts the displacement vector in terms of a single variable (x) for integration (we could of course have instead chosen our integration variable to be y). More generally, the path could be a function: $y = f(x)$, in which case the m above would be replaced by the derivative of the function:

$$\overrightarrow{dl} = \left(\hat{i} + \frac{df}{dx}\ \hat{j}\right) dx \tag{3.1.8}$$

Note also that the path may not even be a function, since it could have multiple y values for each x value. [Suffice to say that path integrals have a lot more going on than we will cover in this course, and we'll leave coverage of the more nuanced details to a course in vector calculus.]

3. **evaluate the force vector at each point in the path**

The force vector will be in terms of x and y (i.e. it is defined at all points in space), but in the integral only its value along the path matters, so we can substitute the equation that defines the path (such as $y = mx + b$ in the case of a straight-line) into the force vector so that it is a function of only one variable, allowing us to do the integral.

4. **take the dot product**

We of course know how to do this by now, but it is important to remember that it must be done. This step goes back to the start of our discussion of this method. This dot product assures that we are only using the part of the force vector that lies along the tiny displacement, which means we are only using the part of the force vector that changes the speed of the particle.

Of course, much more complicated paths than straight lines are possible. The following example illustrates how this is handled.

Exercise

Compute the work done on a particle by the force given below, along a parabolic path $y = \lambda x^2$ connecting the origin to the point on the path with an x value of 1m, where $\lambda = 0.4m^{-1}$.

$$\overrightarrow{F}(x, y) = \alpha x^2\ \hat{i} + \beta y\ \hat{j}, \quad where: \quad \alpha = 1.5\frac{J}{m^2} \quad and \quad \beta = 3.0\frac{J}{m}$$

Solution

Start by determining the displacement vector as a function of x along the path:

$$\left.\begin{array}{l} \overrightarrow{dl} = dx\ \hat{i} + dy\ \hat{j} \\ y = \lambda x^2 \quad \Rightarrow \quad \dfrac{dy}{dx} = 2\lambda x \quad \Rightarrow \quad dy = 2\lambda x dx \end{array}\right\} \Rightarrow \overrightarrow{dl} = \left(\hat{i} + 2\lambda x\ \hat{j}\right) dx$$

Next write the force vector along the path only (in terms of x):

$$\left.\begin{array}{l} \overrightarrow{F}(x, y) = \alpha x^2\ \hat{i} + \beta y\ \hat{j} \\ y = \lambda x^2 \end{array}\right\} \quad \Rightarrow \quad \overrightarrow{F}_{on\ path}(x) = \alpha x^2\ \hat{i} + \beta \lambda x^2\ \hat{j}$$

Now for the dot product:

$$\vec{F}_{on\ path} \cdot \vec{dl} = \left[\left(\alpha x^2\right)(1) + \left(\beta \lambda x^2\right)(2\lambda x)\right] dx = \left(\alpha x^2 + 2\beta \lambda^2 x^3\right) dx$$

And finally integrate between the two endpoints, defined in terms of the x variable that we have put everything in terms of:

$$W(A \to B) = \int_{x=0m}^{x=1m} \left(\alpha x^2 + 2\beta\lambda^2 x^3\right) dx = \left[\frac{1}{3}\alpha x^3 + \frac{1}{2}\beta\lambda^2 x^4\right]_{x=0m}^{x=1m} = 0.74J$$

Lost Information

It is important to note that while the introduction of the work-energy theorem will simplify things for us with a subset of problems, we do sacrifice some information. By throwing out the part of the force that acts to change the direction of motion of the particle, we cannot use this method to determine which way the particle is moving after the force acts on it – we only know how fast it is going. Also, we lose information about the time element of the motion between the starting and ending points. This should not be surprising – just because we know how fast something is moving, if we don't know the directions it takes to get from start to finish, we still don't know anything about the elapsed time. For example, a projectile thrown into the air will reach the same speed at two different points of time – once on the way up and once on the way down. If we don't know anything about the direction of motion, we don't know which time we are looking at.

To see this another way, consider a situation we are very familiar with – a particle moving in a straight line, accelerating at a constant rate. We know that we can write its acceleration in terms of the starting and final velocities using Equation 1.4.3:

$$2a\Delta x = {v_f}^2 - {v_o}^2 \tag{3.1.9}$$

By Newton's second law, the acceleration here must have been caused by a (net) force in the same direction, so substituting the ratio of force/mass for the acceleration gives:

$$2\frac{F_{net}}{m}\Delta x = {v_f}^2 - {v_o}^2 \quad \Rightarrow \quad F_{net}\Delta x = \frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_o}^2 \tag{3.1.10}$$

This is once again the work-energy theorem (in one dimension, for a constant net force), and we see that it came directly from the kinematics equation from which the time variable had been eliminated.

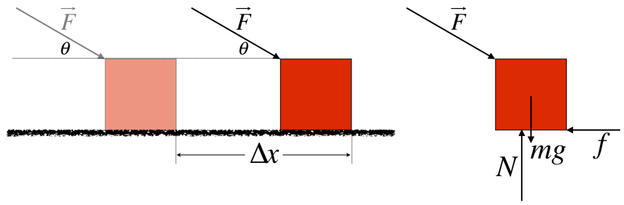
Work Contributions of Individual Forces

It probably isn't immediately clear what is to be gained from this work-energy approach. After all, one still has to determine the net force at each point in the path of the object's motion, so our attempt to escape the tyranny of vectors would appear to be a failure. But there is much more to this story. It begins with the recognition that total work done can be broken into a sum of works done by individual forces:

$$\begin{aligned} W_{tot}(A \to B) &= \int_A^B \vec{F}_{net} \cdot \vec{dl} \\ &= \int_A^B \left(\vec{F}_1 + \vec{F}_2 + \dots\right) \cdot \vec{dl} \\ &= \int_A^B \vec{F}_1 \cdot \vec{dl} + \int_A^B \vec{F}_2 \cdot \vec{dl} + \dots \\ &= W_1(A \to B) + W_2(A \to B) + \dots \end{aligned} \tag{3.1.11}$$

There are a number of advantages to this, but the one we can see immediately is that if one of the individual forces happens to be everywhere perpendicular to the path of the particle from A to B, then the work it contributes is zero, and we can simply ignore it – no need to do the vector addition to add it to the other forces. Consider the following example of pushing a block across a rough horizontal surface. [*As we have stated several times, rigid macroscopic objects that don't rotate behave like individual particles, so we are still following the restrictions we set above.]*] Figure 3.1.1 shows a diagram of what is happening and a FBD of the block.

Figure 3.1.1 – Pushing Block Across a Rough Horizontal Surface



The work done by the net force can be broken down into a sum of the works done by each individual force:

$$W_{applied\ force}=\int_A^B\vec{F}\cdot\vec{dl}=F\Delta x\cos\theta \tag{3.1.12}$$

$$W_{friction}=\int_A^B\vec{f}\cdot\vec{dl}=f\Delta x\cos 180^o=-f\Delta x \tag{3.1.13}$$

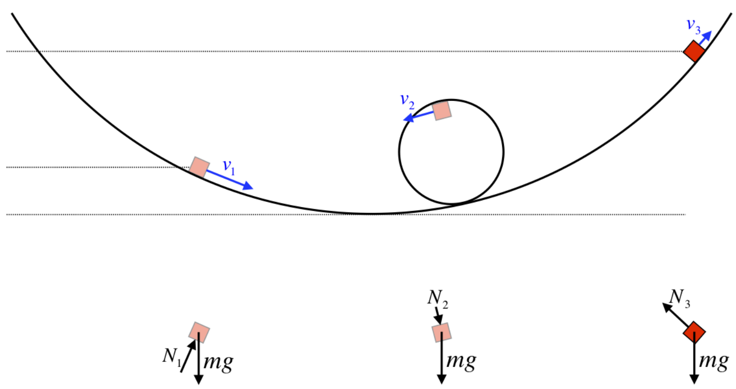
$$W_{gravity}=\int_A^B\left(-mg\,\hat{j}\right)\cdot\vec{dl}=mg\Delta x\cos 90^o=0 \tag{3.1.14}$$

$$W_{normal}=\int_A^B\left(N\,\hat{j}\right)\cdot\vec{dl}=N\Delta x\cos 90^o=0 \tag{3.1.15}$$

$$W_{tot}=W_{applied\ force}+W_{friction}+W_{gravity}+W_{normal}=(F\cos\theta-f)\,\Delta x=F_{net}\Delta x \tag{3.1.16}$$

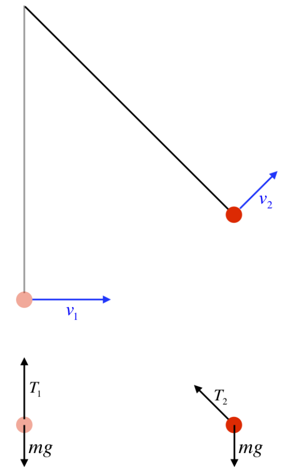
Just by looking at the physical situation it is clear that the gravity and contact forces will play no role in the total work done, as they are always perpendicular to the motion. This greatly reduces the number of forces (and vector addition) we would otherwise have to deal with. Let's look at some even more compelling examples:

Figure 3.1.2 – Loop-de-Loop



For a block sliding around a frictionless loop-de-loop track, the path it follows is quite complicated. The FBD of the block as it travels along the track includes only two forces – gravity and the normal force by the track. The motion of the block is parallel to the track everywhere, which means it is perpendicular to the normal force everywhere. That means that no matter what our starting and ending points are, the normal force does no work on the block! Of the two forces involved, the normal force is by far the hardest to deal with, since its direction and magnitude change everywhere on the track. but if we are only interested in the speed of the block, we only need to worry about the work done by the gravity force, which has a constant direction and magnitude. We'll come back to the simple result that comes from this shortly.

Figure 3.1.3 – Simple Pendulum



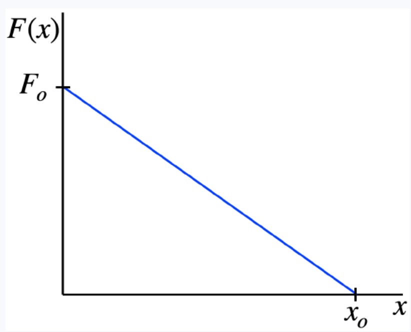
For the simple pendulum, we see the same result for the tension as we found for the normal force in the loop-de-loop example. The tension force remains at right angles to the motion of the bob at the end of the string, so there is no work done by the tension force. If all we care about is the speed of the bob, then we only need to compute the work done by gravity.

Analyze This

A toy train rolls along a straight, frictionless track, parallel to the x-axis. As it rolls, it experiences a force given by the equation:

$$\vec{F} = F\left(x\right)\left[0.600\ \hat{i} + 0.800\ \hat{j}\right]$$

The function $F\left(x\right)$ can be expressed by the graph below.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

In the equation for the force, the quantity in brackets is unitless and has a magnitude of 1: $\sqrt{0.60^2+0.80^2}=1$. This means that the function $F\left(x\right)$ is the magnitude of the force vector as a function of x.

This force has a component in the $+x$-direction, which is parallel to the direction of motion of the train. We don't know which way the train is moving, so we don't know whether the work done on it is positive or negative (whether the force is speeding up the train or slowing it down). The work done by the force on the train is therefore equal to plus-or-minus the x-component of the force, integrated over the displacement of the train. This will just equal 0.60 times the area under the $F\left(x\right)$ curve, so if the train moves from the origin to the position $x=x_o$, then the work done is just 0.60 times the area of the triangle shown, and since the motion in this case is in the $+x$-direction, the work done is positive.

If we are given the mass and starting speed of the train, we can use the work-energy theorem to compute its final speed.

3.2: Work and Energy for Collections of Particles

Internal Energy

We now wish to extend our results from the previous section beyond just single particles. We want to ultimately talk about macroscopic objects, but of course these are merely collections of particles. If we choose a model where this macroscopic collection of particles are held rigidly in place, without the ability to move independently from each other, then such an object can be treated in exactly the same way as an individual particle (assuming the object doesn't rotate). We will actually do this quite a lot in the coming pages, but for now we want to look at what happens if the particles in our collection *can* move independently of each other.

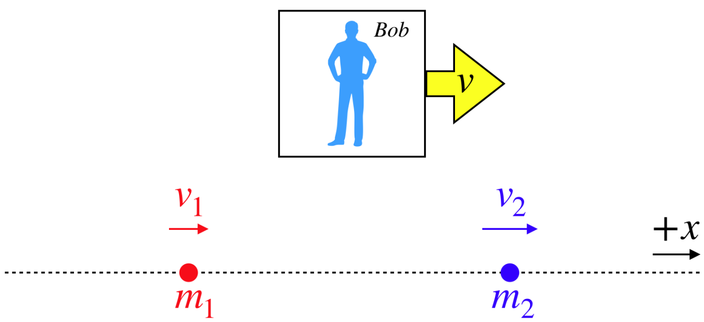
We will start by looking a little closer at kinetic energy. When we first talked about force, we made very clear that force was not a quantity possessed by an object – it is an interaction between two objects. Let's ask the same question of kinetic energy – is kinetic energy a property that "belongs" to a particle? At first blush, the answer would appear to be yes. The particle has a well-defined mass and speed, so since $KE=\frac{1}{2}mv^2$, it seems like we can attribute this quantity to the particle itself – there is no interaction or cause/effect here as there was for force.

But alas, there *is* a problem with attributing this quantity solely to the particle, and that has to do with the observer that *measures* the kinetic energy. If it was a property that belonged exclusively to the particle, then everyone that measures it for the same particle should get the same number. But clearly this is not the case. Imagine that Ann watches a particle with mass m and speed v fly by. She says, "That particle has a kinetic energy of $\frac{1}{2}mv^2$." But Bob, who is moving past Ann, traveling along with the particle, measures the speed of the particle to be zero, and claims that the particle's kinetic energy is zero! Kinetic energy is not a property we can attribute to the particle if different observers measure different values for it. Clearly there is always a frame (the one that moves along with the particle) for which the kinetic energy is zero.

Now let's suppose we have a collection two non-interacting particles. The total energy of this two-particle collection is just the sum of the kinetic energies of these particles. Does there exist a frame in which the energy of this collection is zero? In general, the answer is no! If the two particles happen to be moving at the same speed in the same direction, then yes, jumping over to the frame common to both of these particles will result in both particles being stationary. But in any case where the two particles are moving relative to each other, then no such "zero total energy" frame exists. If you jump on the frame of one particle, then it is stationary, but the other is moving.

Okay then, let's see if we can determine, for the simplest case of two particles, what frame results in the *lowest* amount of total energy, given that it is in general not equal to zero. It is probably not clear why we would care to know this yet, but hang in there...

Figure 3.2.1 – Two Particle Collection (Non-Interacting)



The figure shows a very simple case of two particles moving along a line at different speeds (shown from our perspective), and another observer (Bob) moving along the same direction. If Bob is moving slower than the slowest particle, then he will see speeds for the particles that are both slower than what we see from our perspective. Bob will measure a smaller energy for this two-particle collection than what we measure, since both particles are moving slower. It is easy to write mathematically what energy Bob sees. Particle #1 is moving (according to him) at a speed of v_1-v , and particle #2 at a speed of v_2-v , so:

$$E\left(\text{seen by Bob}\right)=\frac{1}{2}m_1\left(v_1-v\right)^2+\frac{1}{2}m_2\left(v_2-v\right)^2 \tag{3.2.1}$$

Now Bob (by changing speeds) can choose to look at these particles from any frame he likes, which means that the energy of the two-particle collection is a function of the speed v (relative to us) that he chooses. If we want to know the minimum energy one can measure for this two-particle combination, one only needs to minimize the function $E\left(v\right)$ with respect to v:

$$0=\frac{dE}{dv}=-m_1\left(v_1-v\right)-m_2\left(v_2-v\right) \quad\Rightarrow\quad v=\frac{m_1v_1+m_2v_2}{m_1+m_2} \tag{3.2.2}$$

This looks vaguely familiar... If we take a time derivative of the position of the center of mass of two particles aligned on the x-axis (Equation 2.4.8), we get the same expression:

$$\frac{d}{dt}x_{cm} = \frac{d}{dt}\left(\frac{m_1x_1+m_2x_2}{m_1+m_2}\right) = \frac{m_1v_1+m_2v_2}{m_1+m_2} \tag{3.2.3}$$

So this is the velocity of the center of mass of the two-particle combination! Bob measures the minimum energy for the collection when he is moving along with *the collection as a whole, i.e. in the same frame as its center of mass*. Put another way, from Bob's perspective, while he sees the individual particles moving, according to him, the combined two-particle group is not.

Let's answer another question: For those of us not in the center-of-mass frame of the collection like Bob, how much greater do we measure the collection's energy to be? We can calculate this simply by subtracting the energy Bob measures from the amount we measure:

$$\text{extra energy we measure above the minimum} = \left[\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2\right]-\left[\frac{1}{2}m_1\left(v_1-v_{cm}\right)^2+\frac{1}{2}m_2\left(v_2-v_{cm}\right)^2\right] \tag{3.2.4}$$

The reader is encouraged to plow through the algebra here to obtain the final result, which is surprisingly simple:

$$\text{extra energy we measure above the minimum} = \frac{1}{2}\left(m_1+m_2\right)v_{cm}^2 \tag{3.2.5}$$

But wait, this is just the kinetic energy equation for a single object of mass $M=m_1+m_2$ and speed v_{cm}. The details related to the fact that the "object" is made out of two moving particles is utterly ignored in this expression. This completes a very simple, intuitive picture for how we do the energy accounting for a collection of particles:

$$E_{tot}\left(\text{collection of particles}\right) = KE_{collection}+E_{internal}\ , \tag{3.2.6}$$

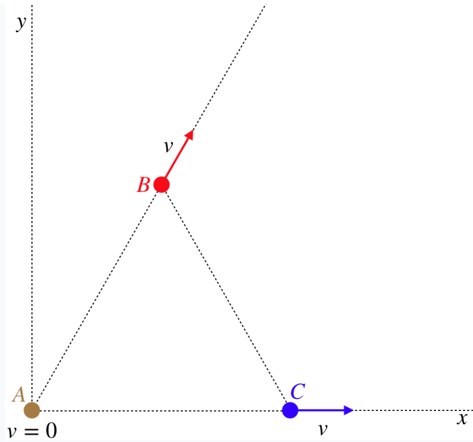
where $KE_{collection}$ is the kinetic energy of the collection as a single entity, calculated from its total mass and center of mass velocity, and $E_{internal}$ is the *internal energy* of the collection, defined as the sum of the kinetic energies of the particles comprising the collection, as measured in the center of mass frame of reference. Notice that the internal energy of a collection *is* a feature of the full group of particles as an entity, no matter who is looking at it. The total energy then only changes with observer because the whole group's KE changes when the perspective is changed.

A few additional comments need to be added here:

- This definition of internal energy depends upon the particles not interacting with each other. As such, it works pretty well for something we will call "ideal gases" when we get to Physics 9B. We will later make a small alteration to this definition when the particles within the collection push and pull on each other, but the idea of internal energy will endure.
- It should be clear that if we use a model for a macroscopic object that consists of particles which are rigidly arranged, this model allows for no internal energy. In these cases, the rigid object can be treated as if it was just a single particle, which simplifies calculations. This model is useful as an approximation (just like assuming pulleys are massless, surfaces are frictionless, or there is no air resistance), but in reality all objects are made of particles that can move, and therefore all objects possess some internal energy.
- While particles within the collection of a rigid object that rotates are moving relative to each other, in a future chapter we will introduce another type of kinetic energy (i.e. "rotational KE") for the object as a whole, which is not considered part of the internal energy.

Analyze This

Three identical particles, A, B, and C are positioned at the vertices of an equilateral triangle. Particle A remains at rest at the origin, while particles B and C move directly away from particle A at equal speeds along the lines defined by the triangle, as shown in the diagram.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

We can immediately determine the energy in the three-particle collection measured in the frame shown, as it is just the sum of the kinetic energies of the two moving particles. They are identical in mass, and are moving at equal speeds, so:

$$E_{tot\ ABC}=\frac{1}{2}m_Av_A^2+\frac{1}{2}m_Bv_B^2+\frac{1}{2}m_Cv_C^2=0+\frac{1}{2}mv^2+\frac{1}{2}mv^2=mv^2$$

The particles are moving relative to each other, so this collection must have some internal energy. But this internal energy must be less than mv^2, because the frame shown is not the center of mass frame. How do we know this? Well, the particles at this moment lie on the vertices of an equilateral triangle, and particles B and C continue moving along the lines defined by that triangle at equal speeds, so at any later time they must still lie on the vertices of an equilateral triangle (each is always the same distance from particle A as the other, and the angle at particle A remains 60^o). The center of mass for three equal-mass particles that form an equilateral triangle is obviously going to be the geometric center of that triangle. But this triangle is growing with time, so the geometric center is moving away from the origin, along a line that forms a 30^o angle with the x-axis. So the center of mass is moving relative to the frame defined by the diagram.

From the geometry given and the fact that all three particles have the same mass, we can determine the precise velocity of the center of mass measured in the diagramed frame in terms of v:

$$\begin{aligned}v_{cm\ x}&=\frac{m_Av_{Ax}+m_Bv_{Bx}+m_Cv_{Cx}}{m_A+m_B+m_C}=\frac{0+mv\cos 60^o+mv}{3m}=\frac{v}{2}\\ v_{cm\ y}&=\frac{m_Av_{Ay}+m_Bv_{By}+m_Cv_{Cy}}{m_A+m_B+m_C}=\frac{0+mv\sin 60^o+0}{3m}=\frac{v}{2\sqrt{3}}\end{aligned}$$

We can use this result to determine the speed of the particles relative to the center of mass of the collection. We have a cool trick at our disposal here. Clearly due to the symmetry, every particle is moving away from the center of mass at an equal speed (imagine standing at the center of mass, stationary in your frame, and watching all the particles move away from you as they remain in an equilateral triangle). But there is one particle whose speed relative to the center of mass is easy to compute: particle A. In the frame of the diagram, particle A is not moving, and the center of mass is moving away from it at some speed, so if we change to the rest frame of the center of mass, particle A must be moving away from the center of mass at the same speed. We therefore conclude that every particle is moving away from the center of mass at a speed of:

$$v_{cm\ frame}=\sqrt{v_x^2+v_y^2}=\frac{v}{\sqrt{3}}$$

And now that we know how fast every particle is moving in the center of mass frame, we know the internal energy of the group:

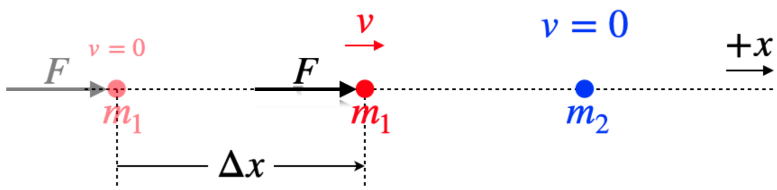
$$E_{internal}=3\left(\frac{1}{2}mv_{cm\ frame}^2\right)=\frac{1}{2}mv^2$$

The kinetic energy of the full collection in the diagrammed frame can be found either by subtracting the internal energy ($\frac{1}{2}mv^2$) from the total energy (mv^2), or by plugging the total mass ($3m$ and the center of mass velocity ($\frac{v}{\sqrt{3}}$) into the KE formula ($\frac{1}{2}m_{tot}v_{cm}^2$). Both clearly give the same answer.

Work Done on Collections of Particles

Next we will look at the effect of work on collections of particles. We once again choose a very simple model to start with – two particles start at rest on the x-axis, and a force is applied to one of them, speeding it up along that axis. The force remains constant in both magnitude and direction, and is applied until the particle is displaced by a distance of Δx.

Figure 3.2.2 – Work Done on Two Particle Group



We know exactly how much energy has been added to this collection. It started with zero, and the amount of work that was done equals the kinetic energy change of particle #1. Given that particle #1 started at rest and particle #2 remains at rest, its change of kinetic energy constitutes all of the energy given to the particle collection. But now we know that since the particles within the group are in relative motion, some of this energy given to the two-particle entity goes into internal energy, which means that not all the energy added to the collection by this work goes into its collective (center of mass) kinetic energy. Summarizing:

$$\text{energy added to collection} = F\Delta x = \frac{1}{2}m_1v^2 = KE_{collection} + E_{int} \tag{3.2.7}$$

So it would appear that the work-energy theorem only applies to individual particles, since the work done does not equal the change in the full group's kinetic energy. But appearances are deceiving! While $F\Delta x$ is the work done on one particle in the collection, it is not the work done on the group as a whole, because *the displacement Δx is not equal to the displacement of the group*. The group's displacement is the change in the position of its center of mass, and the position of the center of mass moves *less* than particle #1 does. This shorter displacement of the two-particle collection results in less work done on it than the same force does on the individual particle, and the difference is exactly equal to the internal energy. In other words, the work done in moving the center of mass of the group exactly equals the change in the group's kinetic energy.

Exercise

Show that the above statement is true (for the two-particle example above): The work done moving the group's center of mass equals the change in its kinetic energy, confirming the work-energy theorem for collection of particles.

Solution

The change of the position of the center of mass is:

$$\Delta x_{cm} = \frac{m_1\Delta x_1 + m_2\Delta x_2}{m_1+m_2} = \frac{m_1\Delta x + 0}{m_1+m_2} = \frac{m_1}{m_1+m_2}\Delta x$$

So the work done on the group is:

$$W\left(\text{on group}\right) = F\Delta x_{cm} = \left(\frac{m_1}{m_1+m_2}\right)F\Delta x = \left(\frac{m_1}{m_1+m_2}\right)\frac{1}{2}m_1v^2$$

Now let's calculate the kinetic energy change of the group (it changes from zero):

$$KE_{sys} = \frac{1}{2}\left(m_1+m_2\right)v_{cm}^2 = \frac{1}{2}\left(m_1+m_2\right)\left(\frac{m_1v+0}{m_1+m_2}\right)^2 = \left(\frac{m_1}{m_1+m_2}\right)\frac{1}{2}m_1v^2$$

They are indeed equal.

It is interesting to note that if equal forces act on each of the two particles in opposite directions, then they both speed up, but of course in this case the net force on the two-particle collection is zero, which means that the center of mass doesn't move. In this case, all of the energy goes into the internal energy. As a general rule, combinations of forces that stretch or compress objects (without accelerating their centers of mass) add to the internal energy of that object. This observation gives us a hint about how to deal with *internal* forces (forces between particles in the same object) later on.

While everything we have shown here has applied to a specific case of a force acting on a single particle of a two-particle group in one dimension, the results apply much more generally.

3.3: Conservative and Non-Conservative Forces

Path-Dependence for Work

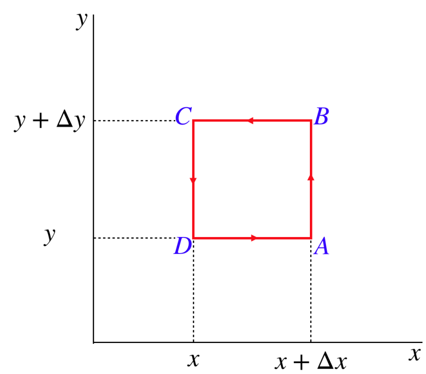
As we saw in our introduction to the work-energy theorem, the calculation of the work done by a force requires that we perform a line integral. In such an integral, the actual path that the particle takes from the start to the end is, in general, an important factor. There are some special cases, however, where the line integral calculation of the work done on a particle by a force as it moves from point *A* to point *B* results in a single, unique, value, no matter what path is taken between those points. A force that satisfies such a property is called a *conservative force*. Conversely, a force for which the work done on a particle depends upon the path it takes between two endpoints is called a *non-conservative force*.

Another way to characterize these two types of forces is to look at the work they do to a particle that follows a *closed path* – a journey that begins and ends at the same position in space. Suppose a particle starts at a point *A*, and is moved around for awhile while acted upon by a force until it finally returns to point *A*. If the force is conservative, then *every* path that brings it back around to its starting point results in the same work done by that force. One can imagine choosing paths that are shorter and shorter until finally the "path" we try is just not moving it at all. Clearly there is no work done in this case, and if it is true for one path (however trivial), it is true for all paths. The therefore conclude:

The work done by a conservative force around any closed path is equal to zero.

This fact gives us a nice mathematical trick for recognizing conservative forces. To see the source of this trick, we will (as usual) use a simple case, which we will then be able to generalize. Consider a rectangular closed path in the x-y plane with the sides of the rectangle parallel to the x and y axes:

Figure 3.3.1 – A Simple Closed Path



The work done by a force $\overrightarrow{F}(x,y) = F_x(x,y)\;\hat{i} + F_y(x,y)\;\hat{j}$ for the part of the journey from A to B only includes the y-component of the force, evaluated at x-position $x+\Delta x$:

$$W(A\rightarrow B) = \overrightarrow{F}\cdot\overrightarrow{\Delta l} = \left(F_x\;\hat{i} + F_y\;\hat{j}\right)\cdot\left(\Delta y\;\hat{j}\right) = F_y(x+\Delta x, y)\,\Delta y \tag{3.3.1}$$

The part of the journey from point C to D looks similar, except that the direction is negative of the $A\rightarrow B$ path, and the force is evaluated at x, rather than $x+\Delta x$:

$$W(C\rightarrow D) = \overrightarrow{F}\cdot\overrightarrow{\Delta l} = \left(F_x\;\hat{i} + F_y\;\hat{j}\right)\cdot\left(-\Delta y\;\hat{j}\right) = -F_y(x,y)\,\Delta y \tag{3.3.2}$$

The work done over the $B\rightarrow C$ and $D\rightarrow A$ paths are found similarly, with the x's and y's swapped:

$$W(B\rightarrow C) = -F_x(x+\Delta x, y)\,\Delta x \qquad W(D\rightarrow A) = F_x(x,y)\,\Delta x \tag{3.3.3}$$

Now we add all of these contributions together to get the work done around the closed loop, and if the force is conservative, we set this equal to zero, giving:

$$0 = \left[F_y(x+\Delta x, y) - F_y(x,y)\right]\Delta y - \left[F_x(x, y+\Delta y) - F_x(x,y)\right]\Delta x \tag{3.3.4}$$

Dividing both sides by $\Delta x\Delta y$ gives:

$$0 = \frac{F_y(x+\Delta x, y) - F_y(x,y)}{\Delta x} - \frac{F_x(x, y+\Delta y) - F_x(x,y)}{\Delta y} \tag{3.3.5}$$

We have taken a bit of license here, when we did not concern ourselves with (for example) how the y-component of the force might change during the path $A \rightarrow B$. But we can remedy this by allowing the dimensions to shrink to zero, $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$. This turns the two terms into derivatives. More specifically, it turns them into *partial derivatives*. A partial derivative is a derivative that acts on a function of multiple variables (in this case, x and y), but only measures the rate of change of the function with respect to one of the variables, *while holding the other one fixed.* The symbols used for partial derivatives are slightly different than those for regular derivatives:

$$\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \rightarrow 0} \frac{f\left(x+\Delta x, y\right)-f\left(x, y\right)}{\Delta x} \tag{3.3.6}$$

Here's an example:

$$f\left(x, y\right)=x^2 y^3+3 y+6 x^3 \quad \Rightarrow \quad \frac{\partial f}{\partial x}=2 x y^3+0+18 x^2\ , \quad \frac{\partial f}{\partial y}=3 x^2 y^2+3+0 \tag{3.3.7}$$

The condition we have for a force $\overrightarrow{F}\left(x, y\right)=F_x\left(x, y\right)\ \hat{i}+F_y\left(x, y\right)\ \hat{j}$ to be conservative is therefore most compactly written as:

$$0=\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y} \tag{3.3.8}$$

If the force has components in three dimensions, then of course this mathematical condition has to also apply to the y-z and x-z planes as well.

Digression: Curl

While it is beyond the scope of this course, a reader with more mathematical background than required for this class may recognize this condition for a conservative force as being the zero "curl" of the force vector field:

$$\text{curl of } \overrightarrow{F}\left(x, y, z\right)=\overrightarrow{\nabla} \times \overrightarrow{F} \equiv \left|\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F z \end{array}\right|=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right) \hat{i}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right) \hat{y}$$
$$+\left(\frac{\partial F_x}{\partial y}-\frac{\partial F_y}{\partial x}\right) \hat{k}=0$$

Analyze This

Consider the following forces that act on a particle as it moves in the (x, y) plane (α is a constant):

- $\overrightarrow{F}\left(x, y\right)=\alpha\left(x\ \hat{i}-y\ \hat{j}\right)$
- $\overrightarrow{F}\left(x, y\right)=\alpha\left(y\ \hat{i}-x\ \hat{j}\right)$

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

There are a few ways we can check whether this force is conservative or not. The simplest involves taking the partial derivatives of the components of the force:

$\overrightarrow{F}\left(x, y\right)=\alpha\left(x\ \hat{i}-y\ \hat{j}\right)\ :$

$$\frac{\partial}{\partial x} F_y=\frac{\partial}{\partial x}(-\alpha y)=0$$
$$\frac{\partial}{\partial y} F_x=\frac{\partial}{\partial y}(\alpha x)=0$$

With both of these derivatives vanishing, our check confirms that this force is conservative:

$$\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}=0-0=0$$

$\overrightarrow{F}(x,y) = \alpha\left(y\;\hat{i} - x\;\hat{j}\right)$:

$$\frac{\partial}{\partial x}F_y = \frac{\partial}{\partial x}(-\alpha x) = -\alpha$$
$$\frac{\partial}{\partial y}F_x = \frac{\partial}{\partial y}(\alpha y) = \alpha$$

For this case, our check shows that the force is non-conservative:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = -\alpha - \alpha = -2\alpha \neq 0$$

Another method would be to perform a work integral between two points over different paths. Unfortunately, this method will only work as a proof that the force is non-conservative if the two integrals come out different. If the two integrals come out the same, it does not prove that the force is conservative, since those two paths could ***coincidentally*** *result in the same work for a non-conservative force. As we cannot perform the work integral for* ***all*** *paths, we can't use that method to conclusively prove that the force is conservative.*

Examples of Conservative Forces

Let's look at a few important examples of conservative forces.

central forces

The first is very fundamental to all of physics, and it goes like this:

Any force on a particle that originates from a single point is conservative.

This can be proven mathematically with the tools we have, but we will refrain from doing so here. The way this is usually expressed by physicists is, "*central forces are conservative*," where a central force is one that originates from a single point (the "center"). Reminding ourselves that forces are interactions between two objects, a force that "originates" from a single point is one that is exerted by an object located at that single point. Put another way, this is a force from another particle! We labeled forces that act between individual particles as "fundamental forces," so we therefore conclude that *all fundamental forces are conservative*.

Digression: Spherical Sources

While all forces that have point particles as a source are certainly central forces, the converse is not true. It is possible to have a central force whose source is a collection of particles, if those particles are placed very symmetrically – in the shape of a sphere. This fact will become important later when we get to gravitation, because stars and planets very closely approximate spheres, which means we can treat the gravitational forces they exert as conservative to a very good approximation.

This means that to the extent that we see non-conservative forces in nature, the source of its non-conservative nature must be that the forces are exerted by objects that are collections of particles rather than by individual particles. This is not to say that the composite forces we have discussed (gravity, elastic, drag, etc.) are inherently non-conservative. It just means that they require just the right conditions to be "considered" conservative to a good approximation.

Alert

Resist the temptation to label a certain ***type*** *of force as either conservative or non-conservative. One cannot tell whether a type (gravity, tension, etc.) is conservative or not without more details of how this type of force is acting. It is perhaps better not to say that a given force is conservative, and instead say that the force is "being applied in a conservative manner." But this is not the language you will find elsewhere, so we will not use it here.*

gravity

As a composite force, we have approximated gravity as a constant force at all points in space – we assume that the region involved is small compared to the size of the Earth. Subject to the limits of this approximation, we can declare gravity to be conservative:

$$\overrightarrow{F}_{grav} = -mg\;\hat{j} \quad\Rightarrow\quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial}{\partial x}(mg) - \frac{\partial}{\partial y}(0) = 0 \tag{3.3.9}$$

It is enlightening to look at how much work is actually done by gravity as a particle moves around. The work done by gravity on a particle that moves from point A to point B (near the surface of the Earth, of course) is given by:

$$W(A\rightarrow B)=\int\limits_{A\rightarrow B}\overrightarrow{F}_{grav}\cdot\overrightarrow{dl} \tag{3.3.10}$$

As usual, we have that $\overrightarrow{dl}=dx\;\hat{i}+dy\;\hat{j}$, but as we will see, we will be able to determine the work done without specifying a path from A to B by expressing y as a function of x:

$$W(A\rightarrow B)=\int\limits_A^B\left(-mg\;\hat{j}\right)\cdot\left(dx\;\hat{i}+dy\;\hat{j}\right)=-mg\left(y_B-y_A\right) \tag{3.3.11}$$

The work done by gravity depends only upon the change in height of the particle. It is important to note a few things about this result:

- It doesn't depend upon the path taken from the starting height to the ending height – it could go straight up, or take many loop-de-loops that go above and below the starting and ending points, and the result is the same.
- The actual value of the starting and ending heights are not relevant, it is only the *difference* of the starting and ending heights that matters.
- According to the work-energy theorem, if gravity is the only force that does work on a particle (or if other forces that might be acting happen to cancel out), then the change of kinetic energy of a particle (and therefore its speed) can be calculated using nothing more than the change in the particle's height.

Exercise

A particle follows projectile motion free of air resistance. According to the work-energy theorem, its change in kinetic energy equals the work done on it by the net force, which in this case is only gravity, so:

$$W_{grav}=\Delta KE\quad\Rightarrow\quad -mg\Delta y=\Delta\left(\frac{1}{2}mv^2\right)$$

Confirm that this is true using tools we have from kinematics.

Solution

We will assume that the particle travels from point A to point B. For projectile motion, the x-component of velocity doesn't change, which means that:

$$v_{Ax}=v_{Bx}$$

The y-component does change, and since we know nothing about how long the particle is in the air, we will use the "no time equation" for vertical motion accelerated by gravity ($a_y=-g$):

$$v_{By}^2-v_{Ay}^2=2\left(-g\right)\left(y_B-y_A\right)$$

Using the result of the first equation, we can add $0=v_{Ax}^2-v_{Bx}^2$ to the left side of this equation without changing it, giving

$$\left(v_{Bx}^2+v_{By}^2\right)-\left(v_{Ax}^2+v_{Ay}^2\right)=-2g\left(y_B-y_A\right)$$

The quantities in parentheses are just the squares of the speeds at points B and A, respectively, so:

$$v_B^2-v_A^2=2g\left(y_B-y_A\right)\quad\Rightarrow\quad\Delta\left(v^2\right)=-2g\Delta y$$

Now multiplying both sides by the mass of the particle and dividing both sides by 2 reconstructs the work-energy theorem result given above.

Analyze This

A small block slides along a frictionless, horizontal surface into a frictionless vertical half-circle track, and it remains in contact with the track, until at least the $\theta=90^o$ point (with θ defined in the diagram).

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

With no friction present, the only forces acting on the block are gravity and the normal force. We can ask how much work each of these forces do on the block. We assume the gravitational force is a constant mg downward, so from our efforts above, we conclude that the work done by gravity is:

$$W_{grav} = -mg\Delta y$$

If we call the radius of the circle R, then we can express the change in the height in terms of R and θ:

$$W_{grav} = -mgR\left[1+\sin\left(\theta-90^o\right)\right] = -mgR\left[1-\cos\theta\right]$$

The work done by the normal force might seem complicated at first, since on the curved track it is constantly changing direction, but every infinitesimal displacement by the block $\overrightarrow{dl}$ is parallel to the track, while the normal force is always perpendicular to the track. Each contribution to the work done is $dW = \overrightarrow{N} \cdot \overrightarrow{dl}$, and since the normal force and displacement are perpendicular to each other at all times the work done by this force is zero.

Now that we know the total work done by all forces, we can apply the work-energy theorem to get the speed of the block after it climbs the angle θ:

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2 = -mgR\left(1-\cos\theta\right) \quad \Rightarrow \quad v_f = \sqrt{v_o^2 - 2gR\left(1-\cos\theta\right)}$$

There is one other thing that we can extract from this analysis. We know that the block is still in contact with the track when it reaches the angle θ. It has been slowing down as it gets higher, so the centripetal acceleration $\frac{v^2}{R}$ that keeps it going in a circle has gotten smaller, but it is not zero. The normal force combined with a component of the gravity force is what is maintaining this acceleration, and as it goes higher from its current position, the component of the gravity force acting toward the center of the circle will get larger. So this means that the decaying centripetal acceleration is due to a decreasing normal force. The normal force can only go to zero (it can't become negative), and if it does, then the block will start to lose contact with the track. We can therefore determine the minimum speed the block must have at angle θ to maintain contact with the track (anything less, and it falls off). The gravity force can be broken into components radial (toward the center of the circle) and tangential to the track (use a FBD and geometry to determine these components), and setting the normal force equal to zero gives:

$$N - mg\cos\theta = m\frac{v^2}{R} \quad \Rightarrow \quad v_{min} = \sqrt{-gR\cos\theta}$$

[Note that the minus sign is present because angle being used (the cosine of $\theta > 90^o$ is negative).]

elastic

In one way, the elastic force is an even easier case than that of gravity – the particle's displacement is entirely confined to one dimension. But it does include the twist that unlike gravity, it does not maintain a constant magnitude. Let's define the one dimension we are working in to be the x-axis. Let's further assume that the particle that is subject to the elastic force experiences zero force when it is located at

$x = x_o$. That is, when the particle is here, then the spring is at its equilibrium length. Then according to Hooke's law, the force on the particle is:

$$\overrightarrow{F}_{elastic} = -k\left(x - x_o\right)\hat{i} \tag{3.3.12}$$

When the particle is located in the region $x > x_o$ it experiences a force in the $-x$-direction (back toward x_o), and when it is in the region $x < x_o$, it experiences a force in the $+x$-direction (also back toward x_o). So this is the restoring force we expect for the elastic force. The partial derivative check once again demonstrates that this is a conservative force. As with the case of gravity, we can compute the work done by the spring on the particle. All the displacements in this case are confined to the x-axis, so we have simply that $\overrightarrow{dl} = dx\ \hat{i}$, giving:

$$W\left(A\rightarrow B\right) = \int\limits_{A\rightarrow B}\overrightarrow{F}_{elastic}\cdot\overrightarrow{dl} = \int\limits_{A}^{B}\left[-k\left(x-x_o\right)\hat{i}\right]\cdot\left(dx\ \hat{i}\right) = \left[-\frac{1}{2}kx^2 + kx_o x\right]_{x_A}^{x_B} = -\frac{1}{2}k\Delta\left(x^2\right) + kx_o\Delta x \tag{3.3.13}$$

We can simplify the look of this result a bit by defining the equilibrium point of the spring to be the origin $x_o = 0$, giving:

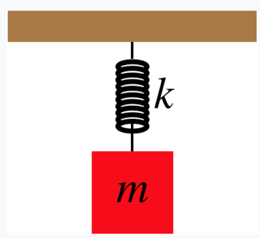
$$W\left(A\rightarrow B\right) = -\frac{1}{2}k\Delta\left(x^2\right) \tag{3.3.14}$$

We see that this result differs from that of gravity, in that the work done by gravity is proportional to how much the (vertical) position changes, while the work done by the elastic force is proportional to how much the *square* of the position changes. So if one particle changes its height by twice as much as another particle, gravity does twice as much work on it. But if a particle changes its position from the equilibrium point twice as much as another particle, the spring does four times as much work on it.

We have been careful to define all of these conservative forces in terms of the displacement of a particle, rather than an "object." But just as we saw in earlier sections, we can extend these results to non-rotating objects whose constituent particles remain rigidly in place within the object. If they do not, then an object that follows a closed path may end up with a different internal energy, which can have an effect on the amount of kinetic energy it has. And if the kinetic energy changes around a closed path, then the work done around that closed path is not zero, and the force on that object is not conservative.

Analyze This

A block is attached to a vertical spring, the other end of which is attached to the ceiling. The block is held stationary at a height where the spring is at its equilibrium length. The block is then released.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

There are two forces acting on the block when it is released. One is the force of gravity downward, and the other is the elastic force of the spring upward. Both of these conservative forces do work at the same time on the block, and since the block moves downward, the gravity force (which acts in the direction of motion) does positive work, while the elastic force (which acts opposite to the direction of motion does negative work.

One thing we can determine, if given all the numbers, is how far the block will fall before stopping and bouncing back up. If we are careless, we might conclude too quickly that this must be where the two forces cancel each other out – but this is wrong! As the block is falling, when it reaches the point where the forces cancel, then the net force is zero, which means it stops ***accelerating****. This does not mean it stops moving.*

The way we find the distance it falls before stopping is to use the work-energy theorem. The total work done by all forces equals the change in kinetic energy. The kinetic energy starts at zero, and we are interested in where the block once again has zero kinetic energy – when it stops falling. As stated above, gravity does positive work as the block falls, and the spring does negative work, so *we are looking for the distance where these cancel out to give zero total work. Calling the stretch of the spring from equilibrium at the point where the block stops $+\Delta y$, then the change of the block's height is $-\Delta y$, and we can use the work results for the gravity and elastic forces above to get:*

$$0 = W_{tot} = W_{spring} + W_{grav} = \left[-\frac{1}{2}k(\Delta y)^2\right] + [-mg(-\Delta y)] \quad \Rightarrow \quad \Delta y = \frac{2mg}{k}$$

The solution also included the possibility that $\Delta y = 0$ – naturally no total work is done if the block never moves, or (since the forces are conservative) if it comes back to the position where it started.

Also notice that if the block was hung stationary from the spring, then the forces would balance, giving $k\Delta y = mg$. The value of Δy in this case is half as much as that found above. So the zero-net-force position is exactly halfway between the maximum and minimum heights of the block.

Now for Some Non-Conservative Forces

Next we will discuss a couple of non-conservative forces. The first is fairly obvious, but the second may be surprising.

kinetic friction

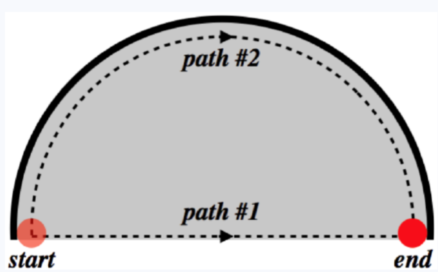
The kinetic friction force is quite unlike the others we have discussed. It's magnitude is simple enough – if the normal force and the surfaces remain unchanged, then the kinetic friction is constant – but the direction is quite another matter. The kinetic friction force on one object always acts in the direction opposite to the direction of motion of that object relative to the surface it is rubbing against. Consider a hockey puck sliding on a horizontal table. If it is sliding north when it is located at point (x, y), then the friction force points south. If, at a later time, it is sliding east at that same point (x, y), then the direction of the friction force points west. What this tells us is that the *direction of the force cannot be determined from the location alone* – one must know the direction of motion.

If we now look back at our partial derivative formula for determining whether a force is conservative, it requires that we know the x and y components of the force as a function of the position (x, y). We cannot get this information in the case of kinetic friction. The partial derivative equation is both a necessary and sufficient condition for the force to be conservative, so when it is unusable for a given force, that force is necessarily non-conservative.

Perhaps this is an unsatisfying explanation. Fair enough, let's do a more rigorous analysis of the case of kinetic friction for a specific case:

Analyze This

A puck is slid along a horizontal rough surface in a straight line along the diameter of a circle (the gravity, contact and friction forces are the only forces on the puck). The same puck is then slid on the same surface starting at the same speed along the circle defined by the diameter indicated in the first experiment (it slides around the inside surface of a frictionless circular wall). The figure shown depicts a top view of these two paths.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Solution

The kinetic friction force always acts in the direction directly opposite to the motion. So for every small displacement $\overrightarrow{dl}$ along the path taken, the direction of the friction force is always the same as the direction of $-\overrightarrow{dl}$. When we compute the work done by friction, we therefore have:

$$dW = \overrightarrow{f}_k \cdot \overrightarrow{dl} = \left|\overrightarrow{f}_k\right|\left|\overrightarrow{dl}\right| \cos 180^o = -f_k dl$$

The magnitude of the kinetic friction force is constant, as it is proportional to the unchanging normal force (the surface remains horizontal). So the line integral can pull the constant f_k out of the integral, leaving:

$$W\left(A \rightarrow B\right) = -\int_A^B f_k dl = f_k \int_A^B dl = -f_k \left(\text{length of path}\right)$$

The length of the path that follows a half-circle path (#2) is longer than the straight-line path (#1) by a factor of $\frac{\pi}{2}$, so the work done on the puck by friction is greater (more negative) for path #2 than for path #1 by that factor.

Notice that both paths begin and end at the same points, but the work done is different for the two paths, demonstrating that kinetic friction is non-conservative.

One might ask, "The frictional force between the particles on the surfaces is ultimately electric, which is a fundamental force, and therefore conservative, so what is the physical source of this force being non-conservative?" The answer is that by only interacting at their surfaces, some particles in the rubbing objects are accelerated while others are not. This results in changes in the internal energies of the rubbing objects. In the analysis example above, path #1 results in less energy being transferred into internal energy than path #2 (more surface particles are affected in the longer path). With the internal energies changed by different amounts, the kinetic energies change by different amounts. This can only occur if different amounts of work are performed, and since the two paths begin and end at the same place, and by definition this means that the force is non-conservative.

gravity

Wait... gravity? Didn't we just learn that gravity is a conservative force? Gravitation as a force between individual particles is conservative as all fundamental forces are, and gravity near the surface of the Earth is approximately constant, making it approximately conservative, but here we are going to look at an example where this force does not behave as a conservative force.

The Voyager probes were launched in 1977. Neither was given sufficient kinetic energy to escape the Sun's gravitational pull without "help." If we approximate a probe and the sun as individual particles (when they are far apart, this is reasonable), then when the gravitational pull of the Sun brings the probe back to the point where it began, no net work will be done, and the probe's kinetic energy goes back to its starting value. Now let's introduce a third particle – Jupiter. We will declare the Sun and Jupiter to be a two-particle object (we'll call it the "solar system"), and examine what can happen to the individual Voyager "particle" when it interacts with this system.

Some very bright minds at NASA launched the Voyager probes so that they would get gravitational "kicks" from flying close to planets, the result of which is increased speed for the Voyager probes that is sufficient to escape the Sun's gravitational pull. Now suppose NASA made a small miscalculation, and sent the probe (with its extra gravitational kick) back toward the Sun, instead of away from it. Then when it returns to the point where it started, it is moving *faster* than when it was launched. This means that the starting and ending kinetic energies at the same point in space are not equal, and the force on the probe by the solar system is not conservative. [*If you are curious about how this gravitational "kick" is accomplished, see the simplified explanation at the* *end of Section 7.3*.]

As with the case of friction, this result comes about because the force on the particle comes from different parts of the system acting independently, and those different parts moving relative to each other (as Jupiter moves around the Sun). The Sun and Jupiter "particles" each exert their own conservative gravitational force on the probe, and the aggregate of these forces comes out to be non-conservative.

3.4: Energy Accounting with Conservative Forces: Potential Energy

Internal Forces

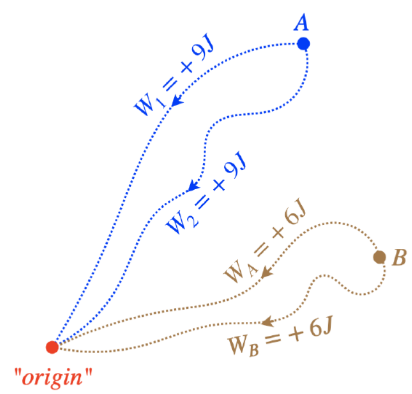
We have discussed how work done on a object (a collection of particles) can contribute to both the overall kinetic energy of that object (measured using its total mass and the speed of its center of mass) and the internal energy of the object. But our definition of internal energy as the sum of the kinetic energies of the particles in the center of mass frame is not very robust. For example, suppose that the particles within an object exert forces on each other, as they do in pretty much every object except for a container of ideal gas. This will change the kinetic energies of the particles, and according to our current definition of internal energy, it will change. Implicit in the name "*internal* energy" is the idea that it is a fixed quantity unless some outside influence changes it. It would be nice, therefore, to somehow account for the effects of the internal forces so that we have an intuitively-defined internal energy.

Conservative Forces Provide a Shortcut

One thing that the internal forces between individual particles have going for them is that they are conservative. We have learned that when forces with this property do work on a particle as it moves from point A to point B, it doesn't matter what path the particle takes in that journey. This actually allows us to introduce a significant shortcut when dealing with conservative forces. Consider the following situation...

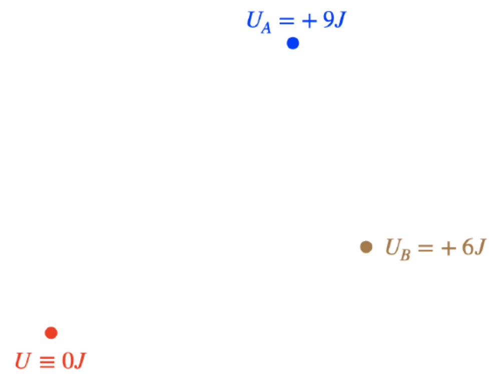
Suppose we have a conservative force that is well-defined by the location of the particle on which it acts. That is, at every point in space, there is a well-defined force vector $\vec{F}(x, y, z)$. Let's start by choosing, completely arbitrarily, a point in space that we call the "origin." Next, we will place a particle at some other position in space (position "A", and move it from that position to the origin through some arbitrary path, adding up the work done on the particle as we do (i.e. performing the line integral for the force from A to the origin). We know that since the force is conservative, if we had chosen a different path between these two points, we would have computed the same work done by the force. Let's repeat this for a new starting point (position "B"). Once again we find that we can follow any path from B to the origin and the work done would be the same.

Figure 3.4.1 – Work Done Coming to the Origin



Given that it doesn't matter what path is taken from any given point in space, we can save ourselves a lot of trouble by just labeling every point in space with the energy value that equals the work the force would do in moving the particle from that point to the origin. Then whenever a particle is moved from a point in space to the origin, we would immediately know how much its kinetic energy changed, without having to even think about doing a line integral. Note that the origin would automatically be labeled with a zero energy value.

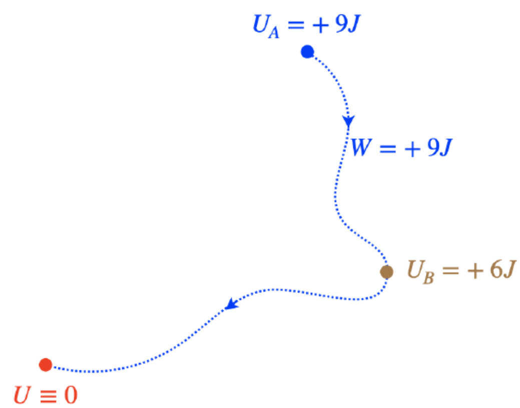
Figure 3.4.2 – Labeled Points in Space



Now whenever a particle is at one of these points in space, one might say that the force "provides the potential to change the particle's kinetic energy by an amount equal to the value at that point, if it is moved to the origin." Not coincidentally, these values are known as *potential energies*. As we have seen here, the value of a potential energy at a point in space depends upon two things: The conservative force that is involved, and the position chosen to be the origin (which we will hereafter refer to instead as the "position of zero potential energy", as the word "origin" has coordinate system implications).

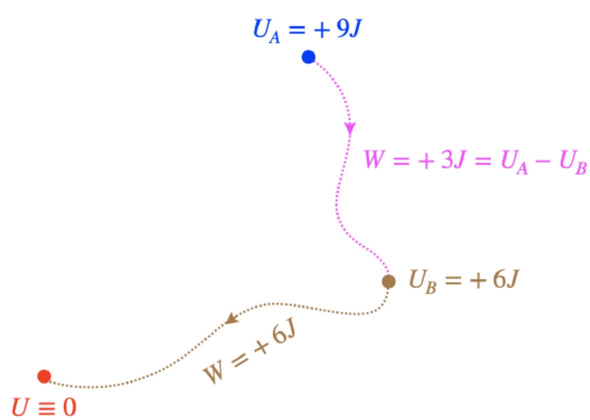
This would seem to have limited use, as it only applies to a very specific ending point of the journey of the particle. But actually this trick is more robust than this. Consider the case of a particle again moved from A to the origin above, but this time let's chose a path that passes through B.

Figure 3.4.3 – Path from A to Origin Through B



Like any other path to the origin, this one comes with a work done on the particle equal to the potential energy at point A. But the work done by the force during the part of the trip from B to the origin contributes an amount of work equal to the potential energy at point B, and this tells us something about the work done in going from point A to point B, neither of which is the origin.

Figure 3.4.4 – Work Done from A to B



It's also to important to note that the work done from point A to point B *doesn't depend at all upon where we placed the origin.* The second leg of the trip shown above from point B to the origin could have gone to any origin whatsoever, and the work done from A to B would still come out the same. We therefore claim, more generally, the following:

The work done by a conservative force acting on a particle that moves from point A to point B equals the amount that the potential at those points drops as a result of the trip.

$$W(A \rightarrow B) = U_A - U_B = -(U_B - U_A) = -\Delta U \tag{3.4.1}$$

Here we have adopted the usual meaning of "Δ" meaning "after minus before". There are a few important things to emphasize here:

- If the particle returns to where it started, then the difference is between the same two numbers, giving zero. As we have already seen, for a conservative force, the work done over a path that returns to its starting point is indeed zero.
- Once we have arbitrarily chosen an origin (zero potential energy) to be a given point in space, then the potential energy values of all the other points in space are well-defined. Actually, we do not need to even have zero potential be our starting point – fixing any value for potential energy at a specific point in space will define all of the potential energies just as well. When we introduced kinetic energy, we pointed out that we can't say that a particle or collection of particles "has" a specific kinetic energy, without first defining the reference frame of the observer of that kinetic energy. Potential energy has a similar ambiguity: We can't say that a particle "has" a specific potential energy without first clarifying what we have chosen as our arbitrary reference position & value for the potential energy.
- The potential energy values defined in space are *only for the specific conservative force in question*. The particle may have several forces on it, and some may not be conservative. This shortcut can still be used for the individual conservative force's work contribution, but not for all the forces combined (unless the combination of all the forces happens to make a conservative composite force). Non-conservative forces do not provide us this opportunity – there is simply no way to define a potential energy for forces like kinetic friction.

Digression: Fields

The idea of associating a value with every point in space is one that comes up frequently in physics. These values and the positions to which they are assigned are collectively referred to as a field. If the values associated with each point in space are simply numbers, as they are for potential energy, then it is called a scalar field. It's also possible to assign a vector at each point in space (say, for example, the velocity of the particle at every point in space filled by a gas), and this is called a vector field. We will deal much more with fields in future physics classes, especially Physics 9C.

When the particle moves from a higher potential energy to a lower one, positive work is done, which means the kinetic energy rises. Potential energy can be thought of as a bank balance - work has the effect of either "spending" the balance, moving that potential energy into the kinetic energy of the particle), or "storing" the balance, taking kinetic energy away and adding it to the potential energy balance. If we are talking about particles or objects on which there are only conservative forces acting, then every change in its total potential energy is balanced by an opposite change in its kinetic energy – the sum of these two numbers remains constant. Whenever we run across a quantity that remains fixed despite a change in circumstances, we can that this quantity is *conserved*. If only conservative forces are present, then:

$$0 = \Delta U + \Delta KE = \Delta(U + KE) \tag{3.4.2}$$

The quantity $U + KE$ is conserved (unchanging) provided that there are only conservative forces present, and that these forces are represented by potential energy functions. This quantity is commonly known as *mechanical energy*.

Potential Energy for Collections of Particles

It might seem like the use of potential energy would be very limited, given that it is only defined for conservative forces, and as we have seen, when many particles are involved, the force on a single particle can easily be non-conservative. In the previous section, we mentioned the example of a three "particle" collection of the Sun, Jupiter, and the Voyager probe where the latter "particle" experienced a combined force from the other two that was not conservative. The reason that the force on the voyager probe is non-conservative is that when the gravitational forces from the Sun and Jupiter do work on Voyager, those same forces also do work on the Sun and Jupiter! So the kinetic energy gained by the probe from the other two bodies is energy lost by those other bodies. Put another way, the Sun + Jupiter combination loses exactly the same amount that the kinetic energy that Voyager gains.

It turns out that if we consider all of the particles in the collection together, and we add up the kinetic and potential energies of all the particles combined, then this number for the grouping (assuming no external work comes in) remains fixed (conserved). The way we compute the potential energies is pairwise – every possible combination of two particles has a fundamental force between them, and therefore a well-defined potential energy. So the total potential energy of the Sun+Jupiter+Voyager combination is the sum of the gravitational potential energies of Sun+Voyager, Jupiter+Voyager, and Sun+Jupiter. When you also add in their kinetic energies, you get a total energy for the collection that remains conserved. The apparent weirdness comes from the fact that these three bodies can re-partition the energy in unexpected ways (so that Voyager gains a lot of KE, for example).

Because this scheme for adding potential energies pairwise between particles in a collection assures that the total energy of the grouping remains unchanged, we are able to make the following powerful claim:

The total energy of any isolated collection of particles remains conserved.

$$\Delta E_{tot} = \Delta \left(\sum KE_i + \sum U_i \right) = 0 \tag{3.4.3}$$

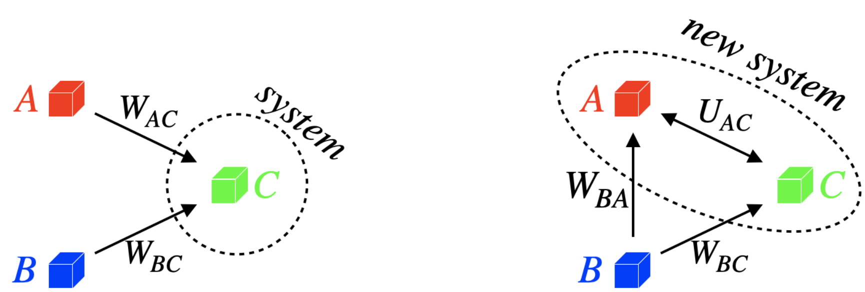
Naturally if the collection is not "isolated," external bodies can exert forces that do work on the particles, thereby adding energy to, or taking energy away from the collection. But the *internal* forces within the group of particles that do work only serve to exchange kinetic and potential energy between the particles within the group, thereby leaving the total energy unchanged.

There is one bit of clarification needed for the equation above. The index i is obviously intended to run over all of the particles in the collection. The quantity U_i, or "potential energy of the ith particle" really involves *all* of the particles, because the potential energy values are determined from pairwise forces between particles. So the value of U_i is really a sum of potential energies between particle i and particle 1, particle i and particle 2, and so on.

Energy Accounting For Conservative Forces

We now know two alternative ways to account for what changes the kinetic energy of a particle or collection of particles (macroscopic object). What differs between these choices is the definition of something we will call a *system*. A system is simply a well-defined collection of objects (in most cases we will be discussing from now on, these will not be single particles). Energy is transferred into or out of a system through work done by forces exerted by objects outside the system. Objects within the system that exert forces on each other keep the energy in the system, though it can change forms.

Figure 3.4.5 – Changing Energy Accounting for Conservative Forces



The figure above shows a physical situation where three objects are interacting with each other through conservative forces. On the left, we have chosen just object C as our system and the forces from the other two objects do work on this object, exchanging energy with the system. The left diagram employs exclusively the work-energy theorem – the work from the external forces changes the kinetic energy of object in the system.

One the right is the same physical situation, but we have changed our accounting method. We have extended what we have defined as a system to include one of the other two objects. As the forces involved were conservative, we can account for the interaction between objects A and C using potential energy. This energy is entirely contained within the system, where it can be exchanged freely with the kinetic energies of objects A and C. Object B has been left outside the system, so accounting for its contribution to the system's energy still relies upon computing work.

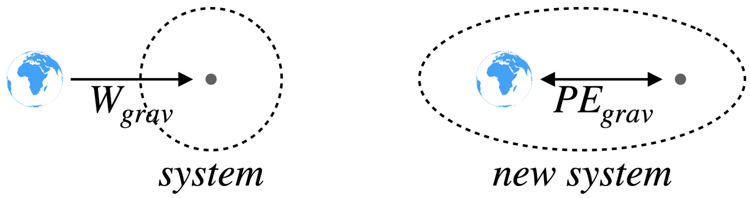
Notice that in our definition of system as only object C, we didn't care at all about the interaction between particles A and B in our accounting, but once object A was brought into the system, this interaction became relevant, because the energy exchange is now across the system border.

Given the difficulty of computing line integrals, we will generally pull *all* of the objects into the system that are exerting conservative forces, so that we can use the potential energy "shortcut." Let's look at a concrete example – gravity.

Potential Energy Function for Gravity

Even though it is not a force between two individual particles, the force of gravity by the Earth on an object like a stone is (to an extremely good approximation) conservative. As discussed above, we can choose to do the energy accounting for a falling stone by putting making the stone the system and computing the work done on the stone by the Earth, or we can include the Earth in the system and compute the potential energy change of the Earth/stone system as the stone falls.

Figure 3.4.6 – Energy Accounting for Gravity



In Equation 3.3.11, we computed the work done by the gravity force on a projectile, and found:

$$W_{grav}\left(A\rightarrow B\right)=-mg\left(y_B-y_A\right)=mgy_A-mgy_B \tag{3.4.4}$$

Above we said that the work done from point A to point B is the potential energy at point A minus the potential energy at point B, so it seems clear that we can define the potential energy function for gravity (near the Earth's surface) as:

$$U_{grav}\left(y\right)=mgy \tag{3.4.5}$$

But this definition implies that we have already determined a position where we have defined $y=0$. We can of course choose our arbitrary reference point to be whatever we like, so it can be useful to express this in the equation. We therefore have:

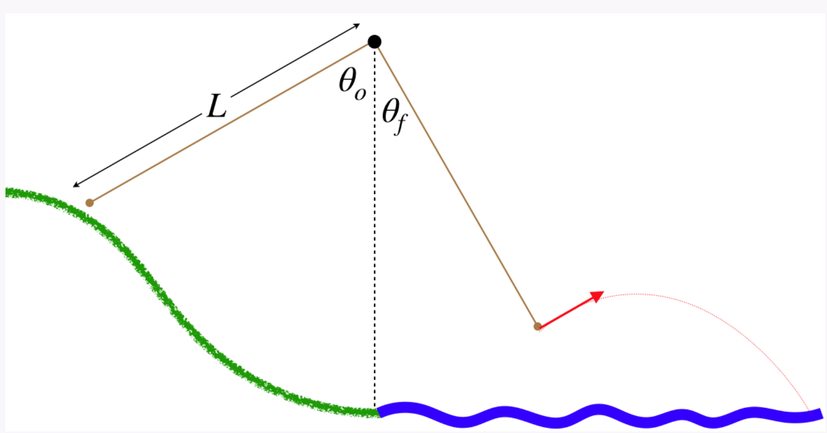
$$U_{grav}\left(y\right)=mgy+U_o, \tag{3.4.6}$$

where U_o is an arbitrary constant. Notice that if we take a difference of potential energies at two different altitudes, the constant drops out, showing that its actual value doesn't change the physics:

$$\Delta U_{grav}=U\left(y_B\right)-U\left(y_A\right)=\left(mgy_B+U_o\right)-\left(mgy_A+U_o\right)=mg\left(y_B-y_A\right)=mg\Delta y \tag{3.4.7}$$

Analyze This

There are few things as fun as swinging into a river from a rope swing tied to the limb of a tree on its banks. The person at the end of this rope starts at the top of a hill at one angle, then swings to another angle when they let go and fly into the water.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

The tension of the rope does no work here, as it acts perpendicular to your motion throughout. Ignoring air resistance, we therefore can use the total energy conservation of the swinger to determine the change in the swinger's speed from the point where they leave the ground to the point where they release the rope (the swinger is an object that is collection of many particles, but we assume that none of the total energy referred to here is going into the swinger's internal energy):

$$0 = \Delta U_{grav} + \Delta KE$$

The potential energy here is due to gravity, so we need to know the change in height from the start to the end. This requires some geometry to figure out. The simplest approach is to use the right triangles the rope makes with the vertical and horizontal directions, and measure the distances below the tree limb. Calling the height of the tree limb zero, we have:

$$\left.\begin{array}{ll}\text{start:} & y_o = -L\cos\theta_o \\ \text{end:} & y_f = -L\cos\theta_f\end{array}\right\} \quad \Rightarrow \quad \Delta U_{grav} = mg\left(y_f - y_o\right) = mgL\left(\cos\theta_o - \cos\theta_f\right)$$

Putting this into total energy conservation gives:

$$0 = \Delta U_{grav} + \Delta KE = mgL\left(\cos\theta_o - \cos\theta_f\right) + \left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2\right) \quad \Rightarrow \quad v_f = \sqrt{v_o^2 - 2gL\left(\cos\theta_o - \cos\theta_f\right)}$$

Potential Energy Function for Elastic Force

For the case of an object pushed or pulled by a spring, we can again choose only the object as a system and compute the work done by the spring force, or include the spring in the system and use the change in its potential energy. For the a stretched or compressed spring with the x variable defined to be zero when the spring is at its equilibrium length, we obtained Equation 3.3.14:

$$W_{spring}\left(A \rightarrow B\right) = -\frac{1}{2}k\Delta\left(x^2\right) = \frac{1}{2}kx_A^2 - \frac{1}{2}kx_B^2 \tag{3.4.8}$$

As with the case of gravity, this immediately implies a function for elastic potential energy:

$$U_{elastic}\left(x\right) = \frac{1}{2}kx^2 + U_o \tag{3.4.9}$$

If we return to the more general case of the position of the spring at equilibrium being x_o, then to get the proper potential energy function, we only need to make the substitution $x \rightarrow x - x_o$, giving:

$$U_{elastic}\left(x\right) = \frac{1}{2}k\left(x - x_o\right)^2 + U_o \tag{3.4.10}$$

The Formal Relationship Between Potential Energy and Force

Given the close ties between work done by a conservative force and the potential energy function for that force, we must be able to link force to potential energy more formally. If we take Equation 3.4.1 and rewrite it using the definition of work, we get:

$$U_B - U_A = -\int_A^B \overrightarrow{F} \cdot \overrightarrow{dl} \tag{3.4.11}$$

The whole idea of using potential energy is to be able to express it as a function of position, so the question that arises is, "Is there some way to 'reverse' this equation, so that if we are given the potential energy function, we can determine the force function?" Well, we already know the answer to this: The 'reverse' of an integral is a derivative! This is the Fundamental Theorem of Calculus. Reversing a line integral is a little bit trickier than doing it for the simple single-variable integrals we are used to from our basic calculus classes, but it can nevertheless be done. To see how this works, let's consider only a very tiny change in potential energy due to a very small displacement. This changes the left hand side of our equation to an infinitesimal, and the right hand side is no longer a sum of many pieces, but is instead only a single piece:

$$dU = -\overrightarrow{F} \cdot \overrightarrow{dl} \tag{3.4.12}$$

In three dimensions, the tiny displacement can be written as:

$$\vec{dl} = dx\;\hat{i} + dy\;\hat{j} + dz\;\hat{k} \tag{3.4.13}$$

This means that the dot product with the force vector is:

$$\vec{F} \cdot \vec{dl} = F_x dx + F_y dy + F_z dz \tag{3.4.14}$$

Suppose we make our tiny displacement only along the x-axis, so that dy and dz are zero. Then clearly all the work done by the force is given by the first term above, and we get that the small change in potential energy that occurs when the position changes a small amount in the x-direction is:

$$dU\left(x \rightarrow x+dx\right) = -F_x dx \quad \Rightarrow \quad F_x = -\frac{dU}{dx} \tag{3.4.15}$$

This is fine for a potential that changes only in the x-direction, but what happens if the potential energy is also a function of y and z? The answer is that we *treat y and z as though they are constants*, which means that $dy = dz = 0$, and our result above works. When we treat y and z as constants, we have to return to the partial derivatives we discussed in Section 3.3. We therefore have the following relationship between the potential energy function and the force components:

$$F_x = -\frac{\partial}{\partial x}U, \quad F_y = -\frac{\partial}{\partial y}U, \quad F_z = -\frac{\partial}{\partial z}U \tag{3.4.16}$$

Digression: Gradient

As was the case with the curl digression in Section 3.3, a reader that is further along in math than most will recognize this relationship between the scalar function $U\left(x,y,z\right)$ and the vector function $\vec{F}\left(x,y,z\right)$ as that of a (negative) "gradient":

$$\vec{F} = -\frac{\partial U}{\partial x}\;\hat{i} - \frac{\partial U}{\partial y}\;\hat{j} - \frac{\partial U}{\partial z}\;\hat{k} \equiv -\vec{\nabla}U = \text{negative gradient of } U\left(x,y,z\right)$$

Exercise

We know that a potential energy function can only exist for a force that is conservative. It is a mathematical fact that multiple partial derivatives of a single function can be performed in any order. That is, if the partial derivative of the function $f\left(x,y\right)$ with respect to x is taken, followed by a partial derivative of the result with respect to y, then the same result is obtained if the partial derivatives are performed in the opposite order. Use this mathematical property to show that Equation 3.4.17 produces a conservative force from the potential energy function.

Solution

We already have a check for a conservative force – Equation 3.3.8. If we apply this to the x and y-components of the force given by Equation 3.4.15, we get:

$$\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x = \frac{\partial}{\partial x}\left(-\frac{\partial U}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial U}{\partial x}\right) = 0$$

The mathematical property of the reversibility of partial derivatives was used in the final equality.

Exercise

Show that the partial derivative link between the potential energy function works for the two macroscopic conservative forces we have discussed:

a. *gravity:* $U_{grav}\left(x,y,z\right) = mgy + U_o$
b. *elastic:* $U_{elastic}\left(x,y,z\right) = \frac{1}{2}kx^2 + U_o$

Solution

Just plug-in and turn the crank in each case:

a. gravity:

$$\left.\begin{array}{l} F_x=-\frac{\partial}{\partial x}\left(mgy+U_o\right)=0 \\ F_y=-\frac{\partial}{\partial y}\left(mgy+U_o\right)=-mg \\ F_z=-\frac{\partial}{\partial z}\left(mgy+U_o\right)=0 \end{array}\right\} \quad \overrightarrow{F}=-mg\,\hat{j}$$

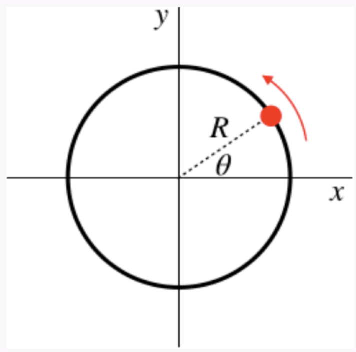
a. elastic:

$$\left.\begin{array}{l} F_x=-\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2+U_o\right)=-kx \\ F_y=-\frac{\partial}{\partial y}\left(\frac{1}{2}kx^2+U_o\right)=0 \\ F_z=-\frac{\partial}{\partial z}\left(\frac{1}{2}kx^2+U_o\right)=0 \end{array}\right\} \quad \overrightarrow{F}=-kx\,\hat{i}$$

Analyze This

A bead is threaded onto a frictionless circular loop that lies in the horizontal x-y plane, as shown in the diagram below. This bead is subjected to a conservative force that is characterized by the potential energy function:

$$U(x,y)=-\alpha\left(Rx+y^2\right) \qquad \alpha>0$$



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

The loop of wire is frictionless, so it can only produce a force that lies perpendicular to the circle (i.e. radially inward or outward). With the bead only allowed to move along the circle, this means that the loop can do no work on the bead. Also, with the loop being in a horizontal plane, the bead never changes elevation, so gravity will not do any work either. This means that the only force acting on the bead is the one that comes from the potential energy function given. The obvious next step is to determine this force, which we do with the partial derivatives:

$$F_x=-\frac{\partial U}{\partial x}=\alpha R \qquad F_y=-\frac{\partial U}{\partial y}=2\alpha y$$

So the component of the force along the x-direction remains constant, while the component along the y direction grows stronger as the magnitude of the y value gets larger. We can therefore immediately determine the positions on the loop where this potential field exerts the strongest force. It does this when the bead crosses the y-axis, and at these points, the magnitude of the force from the potential field is:

$$F_{max}=\sqrt{(\alpha R)^2+(2\alpha R)^2}=\sqrt{5}\,\alpha R$$

We know that since the applied force is conservative, when the bead does a complete circle around the loop, it must return to the same kinetic energy at which it started. This means that it can't continually speed up in the same direction, though the direction of the tangential acceleration can flip from clockwise to counterclockwise, or vice-versa. It must make such a flip in a smooth, continuous way, so there must be at least two positions on the loop where the (tangential) acceleration is zero *(one where it changes from clockwise acceleration to counterclockwise, and one where it changes back). At such positions, the entire acceleration vector must point radially (no component tangential). The loop already supplies a radial force, so we are looking for the positions where the potential field's force is also purely radial. So let's look for the positions (θ values) at which angles result in zero tangential force.*

The force from this potential field will point radially when the tangent of the angle it makes equals plus-or-minus the tangent of the angle the position vector makes. These tangents are simply the ratios of the y and x components, so we have:

$$\frac{F_y}{F_x}=\tan\theta \quad \Rightarrow \quad \tan\theta=\frac{2\alpha y}{\alpha R}=2\frac{y}{R}$$

The y-position divided by the radius of the circle equals $\sin\theta$, so:

$$\frac{\sin\theta}{\cos\theta}=2\sin\theta \quad \Rightarrow \quad \sin\theta=0 \; or \; \cos\theta=\frac{1}{2}$$

There are many angles for which one of these conditions exists. Solving these shows that it is true at $\theta=0^o,\ \pm 60^o,\ and\ 180^o$.

It should also be noted that wherever the force changes direction, the bead goes from a state of either speeding-up to slowing-down, or vice-versa, which means that the speed of the bead either hits a local maximum or minimum at these positions. Given that the total energy of the bead remains conserved, low points of potential energies must correspond to high point of kinetic energies (and therefore speeds), and vice-versa. So looking at each of these critical points we find:

$$U(\theta=0^o)=U(R,0)=-\alpha R^2$$

$$U(\theta=180^o)=U(-R,0)=+\alpha R^2$$

$$U(\theta=\pm 60^o)=U\left(\frac{1}{2}R,\pm\frac{\sqrt{3}}{2}R\right)=-\alpha R^2\left(\frac{1}{2}+\frac{3}{4}\right)=-\frac{5}{4}\alpha R^2$$

So it appears that the bead reaches its maximum speed at $\theta=\pm 60^o$, and its minimum speed at $\theta=180^o$.

Equipotential Surfaces

Suppose a particle is under the influence of a single conservative force. At a given point in space, the force exerted on the particle has a specific direction. If the particle is displaced slightly in a direction perpendicular to this force vector, then this force will do no work. If no work is done by the force, then the potential energy due to that conservative force could not have changed. Suppose now we keep displacing the particle from place-to-place, always moving in a direction perpendicular to the force. The region mapped out by all of this moving of the particle will be a surface throughout that region of space, and the force vector at every point on this surface will be perpendicular to it. The potential energy at every point on this surface is the same, and for this reason such a surface is called an *equipotential surface*. Here's an example:

$$U(x,y,z)=-\alpha\left(x^2+y^2+z^2\right) \tag{3.4.17}$$

The sum of the squares of the x, y, and z coordinates is just the square of the distance from the origin. So any sphere that we choose that is centered at the origin will have the same potential at every point on it – these spheres are equipotentials.

We claimed that the force vectors at every point on an equipotential surface is perpendicular to that surface, and we can check that for this example:

$$\overrightarrow{F}=\hat{i}\left(-\frac{\partial U}{\partial x}\right)+\hat{j}\left(-\frac{\partial U}{\partial y}\right)+\hat{k}\left(-\frac{\partial U}{\partial z}\right)=2\alpha\left(x\hat{i}+y\hat{j}+z\hat{k}\right) \tag{3.4.18}$$

Hopefully you recognize the part of this vector in parentheses. It is the position vector relative to the origin, Equation 1.6.1. This vector points directly to the point (x,y,z) from the origin, which means that it is perpendicular to the sphere centered at the origin that contains that point, confirming the general property that *the force vectors in space associated with a potential are perpendicular to the equipotential surfaces everywhere*.

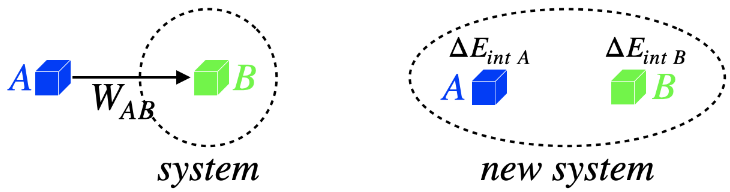
Notice that for the function $U(x,y,z)$ above, if $\alpha>0$, the potential energy gets *smaller* as one gets farther from the origin, and the force vector from this potential points away from the origin. This is also a general feature – *the force associated with a potential* *points in the direction from greater potential to lower potential*. It should be clear on many fronts why this must be the case. If an object moves from a region of higher potential to one of lower potential, this decrease in potential energy must be balanced by an increase in kinetic energy, which means the object speeds up. Objects speed up when the net force on them points in the same direction that they are moving, so the force must point from where the potential energy is higher to where it is lower.

3.5: Energy Accounting with Non-Conservative Forces: Thermal Energy

Energy Accounting for Non-Conservative Forces

Okay, so we have seen how we can pull conservative forces due to macroscopic objects into a system and change the accounting to using the simpler potential energies, but what about the case of work done by an external macroscopic object in the case where the force is not conservative? As we have seen already in such cases, we can still account for the *total* energy if the internal energies are included in the tally. The schematic diagram is only a bit different from the case of the conservative force.

Figure 3.5.1 – Changing Energy Accounting for Non-Conservative Forces



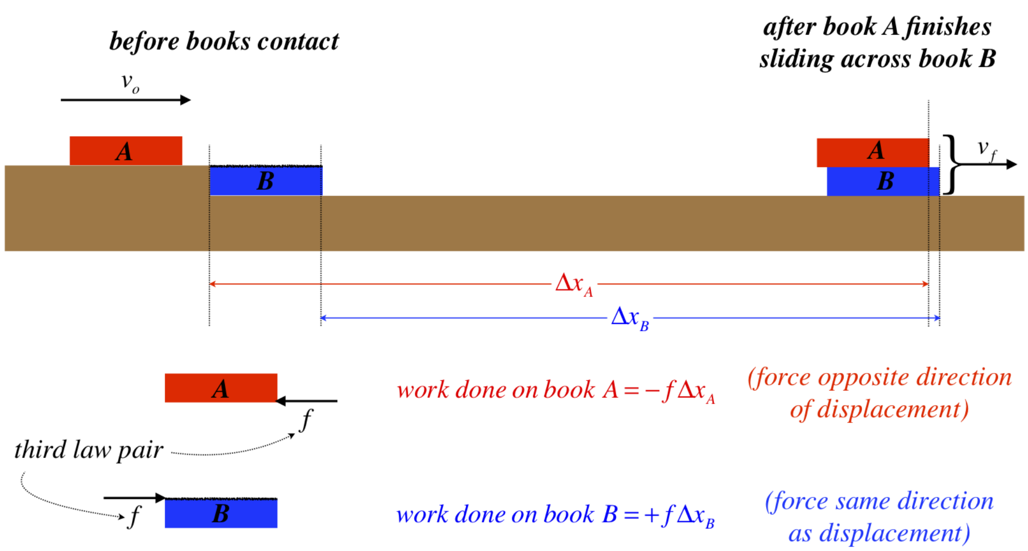
In this case, the external non-conservative force does not allow for redefining the system with a potential energy. Instead, the transfer of energy can only be accounted-for in terms of the internal energies of the objects involved. The change of accounting from the work-energy theorem to the new system therefore looks like:

$$W_{AB} = \Delta KE_B \quad \Rightarrow \quad \Delta KE_A + \Delta KE_B + \Delta E_{int\ A} + \Delta E_{int\ B} = 0 \tag{3.5.1}$$

This rather abstract description is difficult to wrap one's mind around without a clear example, as gravity was for the conservative force case. So we'll demonstrate how this accounting works with the most common example of a non-conservative force – kinetic friction. We want to keep this as simple as possible, so we will consider the following scenario...

The system in this example consists of two books. They are sliding across a horizontal tabletop, but the tabletop is frictionless, so it exerts no horizontal force on the books. The tabletop does exert a vertical force (the normal force) upward on the books, but as the books do not displace vertically, this force does no work. The Earth also exerts a force (gravity) on the books, but again, as they do not displace vertically, this force does no work either. So despite the presence of the tabletop and the Earth's gravity, the two books really are an isolated system, with no external work done on them (we also assume no air resistance during their motion). Below is a diagram of what occurs between the two books: At first they are separate, and then book A slides on top of stationary book B, the kinetic friction force between them causing book B to speed up and book A to slow down, until they eventually stop rubbing across each other and slide away together at the same speed.

Figure 3.5.2 – Energy Accounting for Kinetic Friction



If we treat book B as a system, then the (non-conservative) friction force exerted on it by book A does external work, leading to a change in the kinetic energy of the system:

$$W_{AB} = +f\Delta x_B = \Delta KE_B = KE_{Bf} - KE_{Bo} = \frac{1}{2}m_B v_f^2 - 0 \tag{3.5.2}$$

If we instead treat both books as a system, then we need to include both of their kinetic energies and their internal energies, and we get the same result as is expressed in Equation 3.5.1. The changes in the kinetic energies of the two books are equal to the works done on each, expressed in the diagram. There are two important things to note here. First, the change in the kinetic energy of book A is negative (it slows down), and this is evident from the work done on it by book B, since the kinetic friction force is in the opposite direction of book A's displacement. Second, the positive work done on book B (and therefore its kinetic energy change) is a smaller magnitude than the negative work done on book A (equal to its kinetic energy change), since they experience forces of equal magnitude (Newton's 3rd Law), and book A displaces farther than book B during the time that they are rubbing against each other. The upshot of these facts is that the quantity $\Delta KE_B + \Delta KE_A$ *is a negative number*. This in turn requires that the changes of internal energy of the two books is positive – they collectively gain internal energy. This is manifested microscopically as an increase in the kinetic and potential energies of the particles in the books, a phenomenon we will address shortly.

We can't compute the change in internal energy of an individual book, but the combined increase in their internal energies can be expressed in terms of what is given in the diagram:

$$\Delta E_{int\ B} + \Delta E_{int\ A} = -\left(\Delta KE_B + \Delta KE_A\right) = -f\Delta x_B + f\Delta x_A = f\left(\Delta x_A - \Delta x_B\right) \tag{3.5.3}$$

The quantity $\Delta x_A - \Delta x_B$ is the distance that book A slides across book B, and it is a positive number, which means that there is an increase in the internal energies of the two books. This makes sense, as it means that the amount of energy that goes into the internal energies of the books is a measure of the number of interactions between molecules at the surfaces of the books, and the distance that they slide *across each other* is directly related to the number of molecular interactions.

There is still much more we can do with this simple two-book model, and we will return to it in a future chapter.

Thermal Energy

There is one subset of the internal energy category that deserves special mention. In most macroscopic model calculations, the internal energy change that occurs in the system comes in the form of *random* motions of molecules in the participating objects. When two surfaces slide across each other with kinetic friction, the surface molecules push and pull against each other, and as they are bound to their respective objects, they react by vibrating. These vibrations are not in-sync – they are quite random, and the vibrations spread to their neighbor molecules as well. Such vibrating particles possess energy, both kinetic and potential (the latter due to the potential energy from the restoring Van der Waals forces), and this energy is within the objects themselves, so it is internal. This microscopic, random internal energy is so important that it is given its own name – *thermal energy*.

As is implied by the name, thermal energy is measured most easily through temperature – the two sliding books above become warmer. We will wait until Physics 9B to see how thermal energy and temperature are mathematically related to each other. For the purposes of this class, knowing that a change in temperature is an indication of a change in internal energy will be sufficient. One other important feature of thermal energy that we will not explain until Physics 9B is that with models like the sliding books, the conversion of some of a book's kinetic energy into thermal energy is a one-way trip. We would have a wait a long time if we ever wanted to see the two books cool off and spontaneously slide off each other. This comes from the fact that all (or a very large majority) of the randomly-moving momlecules would have to synchronize and move in the same direction at the same time, and this is improbable in the extreme.

Alert

While it will be some time before you encounter the idea of "heat" in Physics 9B, it is important to understand as early as possible that thermal energy is not the same as heat. Heat is not a form of internal energy.

To get some sense of how important thermal energy is to this discussion, it should be noted that it requires quite a contrived situation to produce an example in the mechanics of macroscopic objects where internal energy is *not* thermal energy. Indeed, for this reason, most textbooks don't even make the distinction, and instead go straight to the total energy conservation equation:

$$0 = \Delta KE + \Delta U + \Delta E_{\text{thermal}} \tag{3.5.4}$$

Thermal energy can have different intrinsic properties for different physical systems. For example, for a system that is a gas, the particles don't interact very much, which means there is very little potential energy included in such a system's internal/thermal energy. For a solid, on the other hand, the molecules are bound to each other, so the energy is divided pretty equally between the potential energies of the particle interactions and the kinetic energies of their motions.

Summary of Energy Accounting

In the last two sections, we have talked about a dizzying number of ways that we can keep track of the energy accounting for a physical system, so let's get away from all the conceptual discussion and summarize what is most useful for solving problems. As a general rule, we will be dealing with systems involving macroscopic objects whose internal energies are all thermal. Aside from these thermal energies, all of the remaining energy will be mechanical. This means that what we will mainly need for any problem are two things: The equation for total energy conservation, and the ability to compute a change in thermal energy by performing the work integral for the non-conservative (usually kinetic friction) force:

$$0=\Delta KE+\Delta U+\Delta E_{thermal} \;\;, \qquad \Delta E_{thermal}=-W_f=-\int\limits_A^B f\cdot dl \tag{3.5.5}$$

Some important/useful notes:

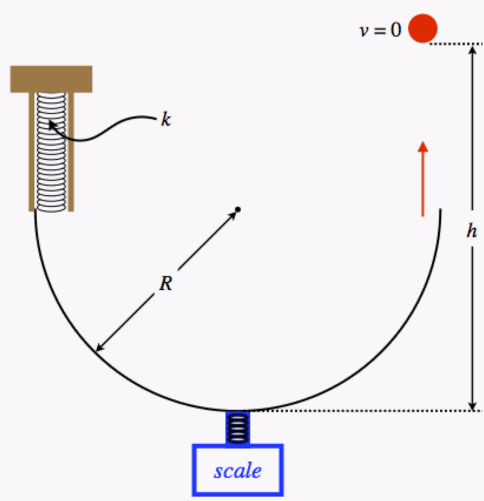
- There could be more than one type of potential energy present at the same time (such as a spring and gravity), so keep in mind that the "ΔU" in the formula above is a placeholder for all of them summed together.
- We are assuming here that the objects in the system are rigid and are not rotating, so that the kinetic energy for each object is simply $\frac{1}{2}mv^2$. This allows us to not be concerned about the specifics of each object's center of mass motion, as every part of it is moving at the same speed. We will see how to deal with the kinetic energy associated with rotational motion in a future chapter.
- It is often helpful to break down the Δ's into "before" and "after", rewriting the energy conservation equation this way:

$$KE_{before}+U_{before}=KE_{after}+U_{after}+\Delta E_{thermal} \tag{3.5.6}$$

The interpretation of this equation is a simple one: The system begins with some total energy that is in the form of mechanical energy. Later, the distribution of this total energy has changed. The amount of kinetic and potential has changed, and assuming there was also a non-conservative force present, some of that starting energy has become thermal energy, internal to the objects. Naturally the system started with some thermal energy, and we could similarly divide that between the two sides of the equation, but this equation is used for specific calculations, and while it is possible to compute the thermal energy change using a work integral, it is not possible (in Physics 9A) to compute the thermal energy of an object directly.

Analyze This

A ball is launched straight up into the air with the apparatus shown below. The ball is pushed upward so that it compresses the spring, and is released from rest. It then travels around a frictionless half-circle track, at the bottom of which is a scale that measures the contact force the ball exerts on the track at that point.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

The first thing we note here is that if we ignore air resistance, there are no non-conservative forces present, which means that $\Delta E_{thermal} = 0$, and the mechanical energy of the system doesn't change from one moment to the next:

$$KE_{before} + U_{before} = KE_{after} + U_{after}$$

Next, we can see that at various times in the motion of the ball, there are two different potential energies possible - one from the spring and one from gravity. We are told that the ball is launched by the spring, which means that the spring must have been compressed. If we call this amount of compression Δy, then at the moment the ball is released, the system contains an amount of spring potential energy equal to:

$$U_{before}(spring) = \frac{1}{2}k\Delta y^2$$

Determining the amount of gravitational potential energy stored in the system requires that we define a point of zero potential energy. Naturally we can choose anywhere we like as this zero-point, because ultimately only the change in the gravitational potential energy from "before" to "after" will matter, and that will be the same wherever we happen to call $U_{grav} = 0$. The diagram labels a height for the ball when it stops rising measured from the bottom of the track, so let's choose that position as zero gravity. Doing this, we can determine the gravitational potential energy of the ball just as it is launched. It's height above the bottom of the track is the radius of the track ***plus the amount the spring is compressed****, so we have:*

$$U_{before}(gravity) = mg(R + \Delta y)$$

Clearly the ball starts at rest, so it starts with zero kinetic energy. This completes a "before" picture that we can use to solve a problem. There are a number of "after" times we might be asked about, most of them presumably after the ball has left contact with the spring. At such times, the ball will have a new height and a new speed. The resulting gravitational potential energy and kinetic energy can then be put into the "after" side of the equation, and one can then solve for unknowns.

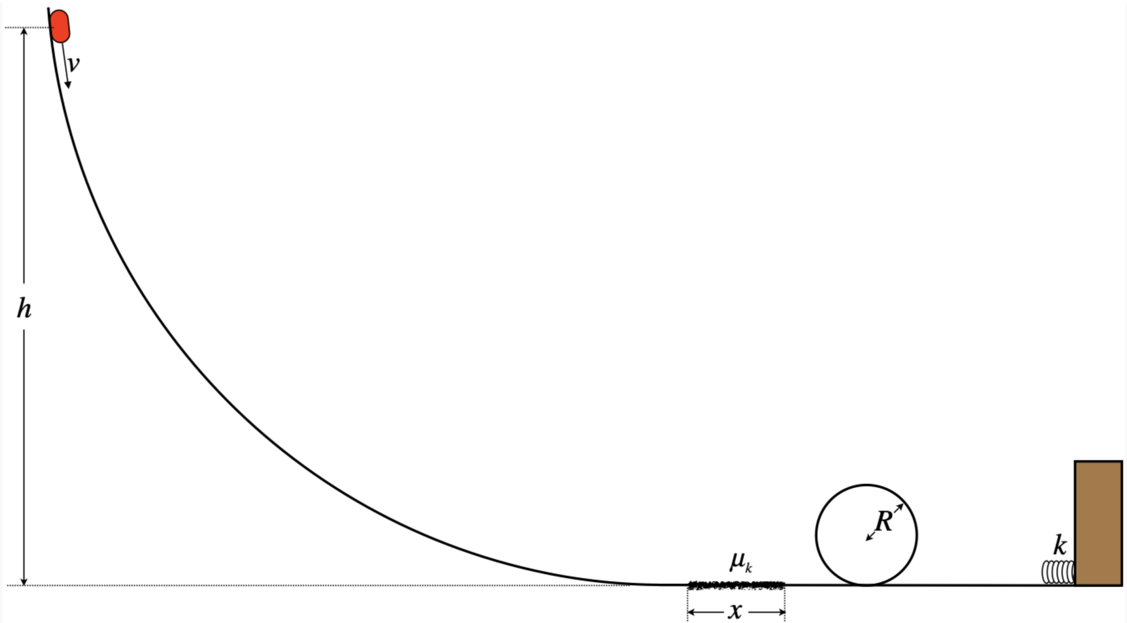
One item that we have not accounted for is the presence of the scale. It doesn't contribute to the energy of the ball in any way, so why is it even mentioned? Well, the scale provides a clever way to measure the speed of the ball (and therefore its kinetic energy) at the bottom of the track. When the ball passes directly above the scale, it is moving in a circle, and the force exerted by the track on the ball at that point (which is measured by the scale) is straight up toward the center of the circle. The gravity force is straight down, so the net force on the ball is toward the center of the circle, which means its acceleration is centripetal. We therefore have:

$$F_{scale} - mg = m\frac{v_{bottom}^2}{R}$$

So if we have the scale measurement (and obviously the mass of the ball), we can compute the kinetic energy of the ball at that bottom point from the radius of the track.

Analyze This

A puck slides down a frictionless track, over a short horizontal rough (frictional) patch, around a frictionless loop-de-loop, and into an ideal spring fixed to a wall, where it bounces off and goes back the other way, returning through the loop-de-loop, over the rough patch, and back up the ramp.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

We clearly have several modes for energy to change form here. The puck will change gravitational potential energy as if moves vertically, it will convert energy from mechanical to thermal when it experiences kinetic friction on the rough patch, potential energy in the spring will change when the puck compresses it, and of course the kinetic energy of the puck can change throughout.

The only weird part here is the loop-de-loop. Like the ramp, it introduces an opportunity for the gravitational potential energy to change, but what else can we say about it? Well, in a previous 'Analyze This' example, we determined a condition for the puck going around the loop to remain in contact. We tend to assume that the puck will make it around the loop to get to the spring, but in fact it has to have at least enough speed at the top of the loop so that it the gravitational force is barely enough to maintain centripetal acceleration. In particular, we found that the minimum speed it must have at the top of the loop is:

$$v_{min}=\sqrt{gR}$$

We can express this as a minimum kinetic energy that the puck must have at the top of the loop in order to maintain contact:

$$KE_{min}=\frac{1}{2}mv^2_{min}=\frac{1}{2}mgR$$

Given that the gravitational potential energy at the top of the loop is greater than at the bottom by an amount $mg\Delta y=2mgR$, it means that to make it around the loop the puck must have a kinetic energy of at least $\frac{5}{2}mgR$. If it came down the ramp from rest with a vertical drop of \(\frac{5}{2}R\), then that would be just enough to let it get around, but of course in this case, the starting point would need to be higher, since the puck loses some of it's kinetic energy to thermal energy thanks to work done by kinetic friction, which we address next.

If we know the coefficient of kinetic friction, then all we need is the normal force between the puck and the rough patch to get the force of kinetic friction, and as this force is constant, the work done by the kinetic friction force is easy to compute, and this energy goes into increasing the thermal energy of the system:

$$\Delta E_{thermal}=-W_f=-\int\vec{F}_f\cdot\overrightarrow{dl}=F_fx=\mu_kNx=\mu_kmgx$$

[The last equality comes from the fact that the surface where the rough patch lies is horizontal, so the normal force equals the puck's weight.]

There is one last important point to make in this analysis. As many places as are available for energy to turn forms, it is not usually necessary to include all of them in the process of solving a problem. Here's an example... Suppose we want to know how high the puck goes back up the ramp after it bounces back one time. We could track its motions through the process, going around the loop and bouncing off the spring, but assuming we know that it has enough energy to traverse the loop-de-loop (something we will need to check, using the criterion found above), we simply set the "before" time as the moment the puck is released and the "after" time when it comes to rest, then all we need to do is include the thermal energy resulting from two trips across the rough patch:

$$KE_{before} + U_{before} = KE_{after} + U_{after} + \Delta E_{thermal} \quad \Rightarrow \quad 0 + mgy_{before} = 0 + mgy_{after} + 2\mu_k mgx \quad \Rightarrow \quad y = h - 2\mu_k x$$

Analyze This

A block slides along a horizontal frictionless surface until it runs into a spring at its equilibrium length. Just as it starts to compress the spring, the surface becomes rough.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

With everything occurring on a horizontal surface, gravitational potential energy does not play a role. It's not that it isn't present, but it never changes, so it will not appear in our equations. The block will change kinetic energy, and the spring will gain potential energy as it is compressed. In addition, as the spring is being compressed, there is work being done by the kinetic friction force, so this will result in a change of thermal energy for the system. Our energy accounting therefore comes to:

$$0 = \Delta KE + \Delta U_{spring} + \Delta E_{thermal}$$

If we assume the friction force remains constant along the rough portion of the horizontal surface, then the work done by the constant kinetic friction force is just the negative of that force multiplied by the distance of the slide across the surface. The surface is horizontal, so the normal force equals the weight of the block. This distance is also the compression of the spring, so putting it all together, we have:

$$\Delta E_{thermal} = -W_f = -\left(-F_f \Delta x\right) = \mu_k N \Delta x = \mu_k mg\Delta x \quad \Rightarrow \quad 0 = \frac{1}{2}m\left(v_f^2 - v_o^2\right) + \left(\frac{1}{2}k\Delta x^2 - 0\right) + \mu_k mg\Delta x$$

If the roughness of the surface (measured by the coefficient of kinetic friction) does not remain constant along the slide of the block, then the problem becomes a bit tougher. The normal force is the same, but μ_k is now a function of position, complicating the work integral:

$$\Delta E_{thermal} = -W_f = \int_0^{\Delta x} F_f dx = \int_0^{\Delta x} \mu(x) N dx = mg\int_0^{\Delta x} \mu(x) dx$$

3.6: Mechanical Advantage and Power

Reconciling Work with Mechanical Advantage

Back when we first talked about pulleys, we said that the block-and-tackle device was useful for lifting heavy objects. The figure below shows two blocks lifted the same distance by pulling on a rope in a pulley system. We know that it requires less force to lift the same mass for the case on the left than the case on the right, but now let's compare the amount of work done by the pull force in the two cases. In both cases, the block is raised the same distance, and in both cases it starts and ends at rest.

Figure 3.6.1 – Work Done with Pulley Systems

The pull force acts downward on the end off the rope, and the direction the end of the rope moves is downward, so there is positive work done in both cases. With identical blocks, the force required to be applied to the rope for the left case is half as great as the force required to lift the block in the right case. However, in order to lift the block the same distance from where it started, the rope must be pulled *twice as far* in the right case than in the left case, thanks to the pulley ratio constraint. With half the force acting over twice the distance, the amount of work done is the same. The multiplicative factor by which the force needed for the load exceeds the force exerted on the system is called *mechanical advantage (MA)*:

$$MA \equiv \frac{force\ on\ load}{force\ applied} \tag{3.6.1}$$

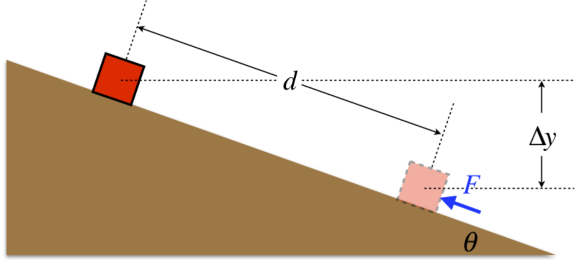
What we have found is that since the work done on the load is the same as the work done from outside, the mechanical advantage can also be expressed in terms of the ratio of the displacement of the object on which the applied force acts (e.g. the end of the rope in a block-and-tackle), and the displacement of the load:

$$W_{in} = W_{out} \quad \Rightarrow \quad F_{applied}\Delta x_{applied} = F_{on\ load}\Delta x_{load} \quad \Rightarrow \quad MA = \frac{\Delta x_{applied}}{\Delta x_{load}} \tag{3.6.2}$$

The mechanical advantage can of course be multiplied by including more pulleys (or mor loops around the same pulley) – the pulley constraint ratio discussed earlier is the same as the mechanical advantage in every case.

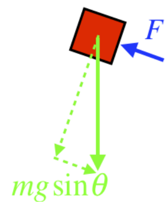
So it appears that the block & tackle (and simple machines more generally) trade effort (force) for displacement, such that the amount of work done remains the same. Let's see if we see a similar result for the case of the inclined plane.

Figure 3.6.2 – Work Done with an Inclined Plane



Let's assume that the inclined plane shown in the figure above is frictionless, that the force applied to the block is parallel to the plane, and that it is just enough so that the block moves at a constant speed up the plane. A FBD (without the irrelevant normal force which acts perpendicular to the motion) looks like:

Figure 3.6.3 – Partial FBD of Block on Plane



For the block to stay at a constant speed, the force up the plane must equal the force down the plane, which means:

$$F = mg\sin\theta \tag{3.6.3}$$

So the force required is reduced by the factor $\sin\theta$ compared to lifting the block straight up. But from trigonometry, the distance the block must be pushed is:

$$d = \frac{\Delta y}{\sin\theta} \tag{3.6.4}$$

When we multiply the force by the distance to get the work done, we get:

$$W = Fd = (mg\sin\theta)\left(\frac{\Delta y}{\sin\theta}\right) = mg\Delta y \tag{3.6.5}$$

This is the amount of work required to lift the block straight up at a constant speed, so once again the simple machine trades extra distance for less force to get the same work. This agrees with the result we have already obtained, using gravitational potential energy. [*Note: The change in gravitational potential energy equals the negative of the work done* ***by the gravity force****. In this case, we have computed the work done by us, and since the force we apply to do the work is in the opposite direction as the gravity component, we get a positive value.*]

Power

We take a moment now to introduce yet another physics word whose common usage in English is very different from its meaning in physics.

Definition: Power

Power is the rate at which work is performed.

Note that just like we can talk about the work done by an individual force or a collection of forces, we can also talk about the power "delivered" to a system by one or more forces. For example, if a car is moving at a constant speed on level ground, its kinetic energy is not changing over time, so no total work is being done on it. If no work is done on it over time, there is no net power being delivered to it. But clearly the engine of the car is doing *something*. So it is useful to break up the power delivered by separate sources if we want to isolate the rate at which the engine is doing work, without worrying about the rate at which air resistance and friction are are doing negative work on the car to bring the total to zero.

Mathematically, we therefore have for the power delivered by a given force is:

$$P \equiv \frac{dW}{dt} \tag{3.6.6}$$

Since the units of work is Joules, the units of power is Joules per second, which we rename as: *watts* (W).

One nice shortcut for power involves the force doing the work and the velocity of the object on which the work is being performed:

$$dW = \overrightarrow{F}\cdot\overrightarrow{dl} \quad\Rightarrow\quad P = \frac{dW}{dt} = \frac{\overrightarrow{F}\cdot\overrightarrow{dl}}{dt} = \overrightarrow{F}\cdot\frac{\overrightarrow{dl}}{dt} = \overrightarrow{F}\cdot\overrightarrow{v} \tag{3.6.7}$$

Note that this is the power delivered to the moving object (i.e. the rate at which energy is added to or taken away from the object) at the *instant* that the force and velocity are the vectors given above. If this is integrated from a starting time to a final time, the result is the total work done over that time span by the force.

Exercise

Two forces act on a moving object of mass $1.50kg$, causing its velocity to change over time. One of the two forces is given below, as is the velocity as a function of time.

$$\vec F_1 = (3.00N)\ \hat i + (4.00N)\ \hat j \qquad \vec v(t) = \left(0.800\frac{m}{s^2}t - 4.0\frac{m}{s}\right)\ \hat i + \left(1.8\frac{m}{s}\right)\ \hat j$$

At one moment in time, only the force not given above is delivering power to the object. Find the amount of power delivered by this force at this moment in time.

Solution

First we need to find the time at which this is occurring. We know that the force given above is not delivering any power at this moment, so its dot product with the velocity vector must vanish, giving:

$$0 = \vec F \cdot \vec v = (3.00N)\left(0.800\frac{m}{s^2}t - 4.0\frac{m}{s}\right) + (4.00N)\left(1.8\frac{m}{s}\right) \quad \Rightarrow \quad t = 2.00s$$

We can find the net force on the particle as a function of time by computing the acceleration from the velocity and using Newton's 2nd Law:

$$\vec F_{net} = m\frac{d\vec v}{dt} = (1.50kg)\left(0.800\frac{m}{s^2}\right)\hat i = (1.20N)\,\hat i$$

This results in a total power delivered to the object as a function of time that is:

$$P_{tot} = \vec F_{net} \cdot \vec v = (1.20N)\left(0.800\frac{m}{s^2}t - 4.0\frac{m}{s}\right)$$

At the time computed above, the force $\vec F_1$ is not delivering any power, so all of the total power must be coming from the second force. Thus:

$$\text{at } t = 2.00s: \quad P_2 = P_{tot} = -2.88W$$

Analyze This

A particle starts from rest and experiences a net force that has a constant magnitude. This force does change direction, however, such that the particle is made to move in a circular path.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

To get the particle moving, the force must be tangential to the circle around which the particle will eventually be traveling. But as the particle speeds up, in order to continue its motion in a circle, the component of the net force toward the center of the circle grows, and since the magnitude of the force remains fixed, there is a decreasing amount of tangential component of the force that goes into speeding up the motion of the particle.

For a given mass of particle and radius of circle, there is a limited speed at which the particle can travel – the speed at which the entire force acts to maintain the centripetal acceleration:

$$F = m\frac{v_{max}^2}{R} \quad \Rightarrow \quad v_{max} = \sqrt{\frac{FR}{m}}$$

3.7: Energy Diagrams

A New Tool for Energy Analysis: Energy Diagrams

An *energy diagram* provides us a means to assess features of physical systems at a glance. We will examine a couple of simple examples, and then show how it can be used for more advanced cases in physics and chemistry. It's important to understand that there is no new physics in here – what we have learned to this point is simply represented diagrammatically, making it easier in some cases to see the "big picture" of a physical system.

First of all, it should be noted that we will be confining ourselves to energy diagrams for 1-dimensional motion. This dimension will be represented by the horizontal axis, and the vertical axis will have units of energy. Secondly, the physical systems represented by energy diagrams will involve only one (conservative) force acting on an object.

Construction of an energy diagram entails first graphing the potential energy function for the conservative force on the axes. Note that potential energy function includes an arbitrary additive constant, which means that the entire graph can be moved up or down on the vertical axis as much as one likes without changing the physical system at all. There is one common convention that is followed regarding the height of the graph on the vertical axis, which we will see below, but it should be remembered that this is only a convention, and doesn't change any of the physical properties of the system.

The graph of the potential energy function could apply to any object under the influence of this conservative force. To represent a specific system, the diagram also needs to indicate the total mechanical energy of the system, and this is done with a horizontal line with the correct height on the vertical axis.

That's all there is to drawing these diagrams. The real value comes from interpreting them, which we will discuss in the context of a couple of simple examples.

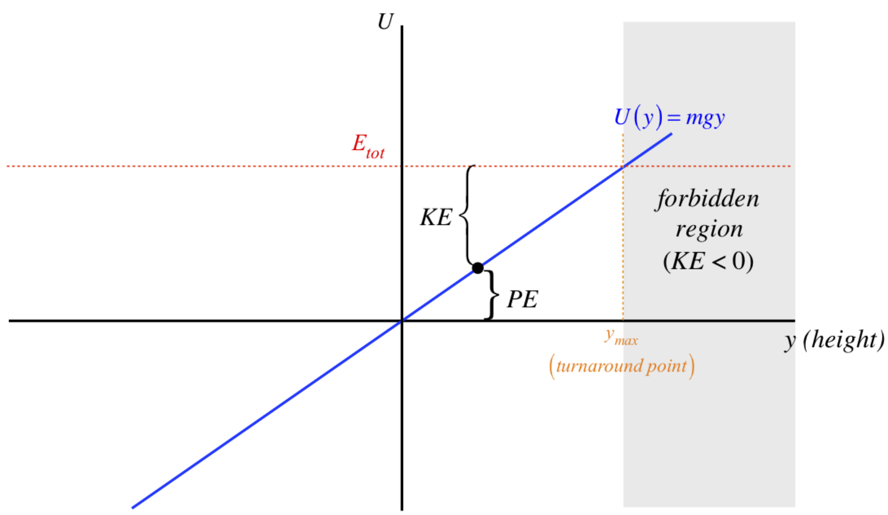
Two Simple Examples

Let's look at the energy diagrams for the two conservative forces we have dealt with so far... gravity and the elastic force.

Gravity

If we choose the arbitrary constant U_o for the gravitational potential energy to be zero, we have as a graph of the potential energy function a straight line that passes through the origin. We then include a horizontal line to represent the total energy of the particular system (which I will label as E_{tot}). Now for interpretation...

Figure 3.7.1 – Energy Diagram for Object Influenced by Gravity Near Earth's Surface



The position of the object (which in this case is the height above some defined zero point), is the value along the horizontal axis. For every position of the object, there is a corresponding value of its potential energy, given by the height of the $U(y)$ graph above (if positive) or below (if negative) the horizontal axis. The total (mechanical) energy of this system is conserved (i.e. it is the same for every position of the object), which explains why the total energy graph is a horizontal line. For a given position, the gap between the total energy line and the potential energy line equals the kinetic energy of the object, since the sum of this gap and the height of the potential energy graph is the total energy.

We can also interpret the intersection point of the total energy and the potential energy graphs. At this point, the total energy equals the potential energy, which means the object has no kinetic energy – i.e. the object is at rest at this position. How can an object under the influence of only gravity be at rest? It can be for just an instant, when it reaches the peak of its flight. Therefore the value on the horizontal axis corresponding to this intersection point is the highest elevation the object can reach. Note that for heights (horizontal axis values) greater than this, the potential energy is *greater* than the mechanical energy, which would require a negative kinetic energy. This is of course impossible, and we call this the *forbidden region* of the diagram, as we will never find the system in one of these states. As time passes, when the object reaches the intersection point, it must have done so from the allowed region, which means that when the object comes to rest here, it reverses its direction of motion. Consequently, this position is often referred to as the *turnaround point*.

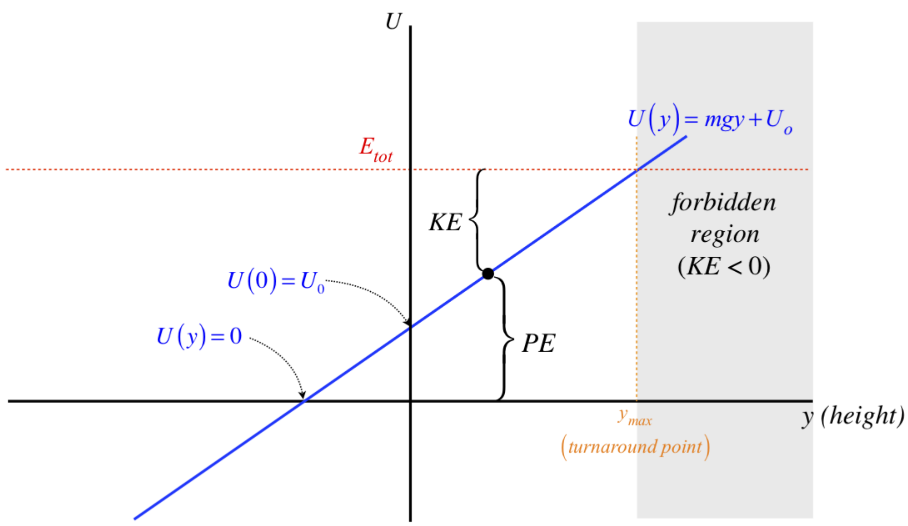
There is one other nugget of information we can extract from this diagram, though in this particular case it is fairly trivial. If we evaluate the negative of the slope of the potential energy graph at the point where the object is at some moment, we know the force acting on the object at that moment. In the case of gravity, the force is the same everywhere:

$$F_y = -\frac{dU}{dy} = -\frac{d}{dy}(mgy) = -mg \tag{3.7.1}$$

[*Note: There is no need for partial derivatives here, as we are only dealing with one-dimensional potential energy functions.*]

If we change the arbitrary constant, the only quantities that change in the entire picture are the potential energy and total energy. Every physically-observable quantity (kinetic energy, turnaround point, and force) remains unchanged. This may not be immediately apparent, but looking at the graph it is easy to see:

Figure 3.7.2 – Redefined Zero Point for Gravitational Potential Energy

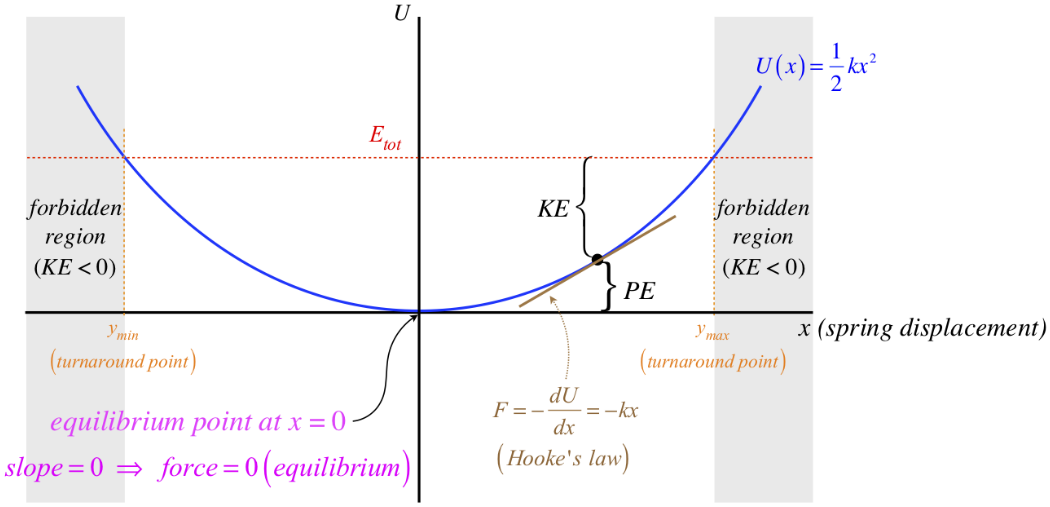


Interestingly, the fact that this potential is a straight line means that a shift of the graph up or down by an additive constant is equivalent to redefining the origin. This is easily seen by noting that this graph can also be viewed as the previous graph shifted to the left by y_o, where $mgy_o = U_o$. So for this simple case, changing the zero point of potential energy is equivalent to changing the position which we call the origin.

Elastic Force

We take precisely the same steps to draw the energy diagram for a mass on a spring, but there are some differences, such as two forbidden regions and a different slope for every position, and there is one additional feature for this potential that doesn't exist for the case of gravity: an *equilibrium point*.

Figure 3.7.3 – Energy Diagram for Object Influenced by Elastic Force



The two forbidden regions arise here because the spring has a maximum stretch and a maximum compression that result in potential energy equaling the total energy. Regions of potential energy confined by two turnaround points like this are often referred to as *potential wells*. Clearly the slope of the potential energy curve is different everywhere, which reflects the fact that the force by the spring is different for every position the mass can have.

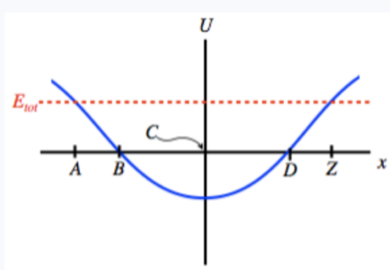
An equilibrium point occurs whenever the slope vanishes (at maxima, minima, and inflection points in the potential energy curve) – there are simply places where the force vanishes. For the spring, it is the position where the spring is neither stretched nor compressed from its natural length. This particular equilibrium is referred to as a *stable equilibrium* for the following reason: If the object is at rest at this point, and it is given a small nudge in either direction, the resulting force acts to bring the object back to its original position. We can see this here, because the slope on the (+) side of the equilibrium point is positive, which means the force is in the negative direction. The force on the (–) side of the equilibrium point similarly acts back toward the equilibrium. Forces that create stable equilibrium like this are called *restoring forces*.

It should be clear that any minimum in the potential energy curve will lead to a stable equilibrium, but what about maxima? In this case, the forces that result from small displacements away from the equilibrium point act to push or pull the object farther from its starting point. This type of equilibrium is therefore referred to as an *unstable equilibrium*. Inflection points lie between parts of the curve that are concave (stable) on one side and convex (unstable) on the other, and the resulting equilibrium is referred to as a *meta-stable equilibrium*.

As with the case of gravity, shifting the entire curve up or down by an amount U_o doesn't change any of the physics, including the position of the equilibrium point. Unlike the gravity case, shifting the entire curve left or right (changing the definition of the origin) is not equivalent to the addition of an additive constant to the potential energy curve. But in both cases, the physics is unchanged by the positioning of the curve relative to either of the axes.

Conceptual Question

An object is subjected to a one-dimensional conservative force whose potential energy curve is represented in the graph of U vs x below. The total energy of the object is indicated by the horizontal dashed line. As the object moves from position A to position Z, at which of the positions indicated is the power delivered to the object the greatest?



a. A
b. B
c. C

d. D
e. *The power delivered is constant during the entire journey.*

Solution

(b) The power delivered at any instant is the dot product of the force vector with the velocity vector. The force is the negative of the slope of the potential energy function. At point C this slope is zero, so there is no force, and no power is delivered. At point A, the object's potential energy equals its total energy, so it is not moving, and with zero velocity, no power is delivered. There is power delivered at points B and D, but only the power delivered at point B is positive, since in that case the force direction and velocity direction are both the same. At point D the particle is still moving in the $+x$-direction, but the force is the opposite way (slope is positive, force is negative), slowing the object down. The power delivered to the object is obviously greatest when the object is receiving the power, not when it is being sapped away.

Bound States of Two Particles

While our models of terrestrial gravity and the elastic force are useful, we have to keep in mind that force interactions are between two objects, and the energy diagrams we have drawn appear to involve only a single object (the other object is lurking in the background). The leap to discussing two particles is not a difficult one. Instead of being a position along the x or y axis, the one dimension of freedom becomes the *separation* of the two particles, which we represent with the variable r. If we can express the force as a function of the separation of the two particles interacting, then we can express the potential energy as a function of that variable as well, and voilà – we can draw an energy diagram. For the sake of interpreting such diagrams correctly, we have to keep in mind that the horizontal axis represents a separation, rather than a position, which leads to a big difference from the energy diagrams we created above – the horizontal axis has no negative values. It also should be remembered that these diagrams only relate motion between the particles along the line joining them. If we want to include motion *around* each other (as in the case of an orbit), we require more information than we can get from the energy diagram.

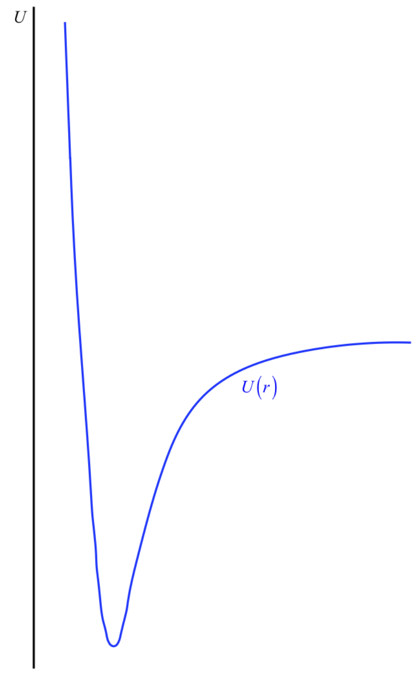
When we look at the universe in both the microscopic and macroscopic realm, we see countless examples of forces holding systems together. Gravity interactions between the sun and the planets keeps them in orbit. Electromagnetic interactions between protons and electrons holds them together in atoms, and electromagnetic interactions between atoms bind them into molecules. Systems of two bodies that possess too little total energy to escape each other's attractive force are in what is called a *bound state*. For the purposes of this section, we will confine ourselves to discussing bound states between microscopic particles, and save the discussion of orbits of celestial bodies for the chapter on gravitation.

Bound states between atoms and molecules in our universe are quite extraordinary. We have already talked a bit about these Van der Waals forces, and the time has come to look at them a bit more closely. Here is how the famous Nobel laureate Richard Feynman put it:

> *If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? All things are made of atoms—little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another. In that one sentence ... there is an enormous amount of information about the world.*

From our earlier discussion, we know that this means that forces between atoms (and between molecules, which are clusters of atoms, which we will also just treat as "particles," ignoring the possibility that they can acquire internal energy) must be *restoring* in nature, with an equilibrium separation. Indeed this does tell us a lot about the shape of the potential energy curve – it must contain a local minimum. But we know even more than this. Clearly a separation distance of zero is not possible (the particles can't occupy the same space), and this is assured by an ever-increasing force as they continue to get closer. This means that the potential energy curve gets steeper and approaches infinity as the separation distance gets closer to zero (i.e. the graph gets closer to the vertical axis). We know one other thing: The force between the two particles gets weaker as they get farther apart, and drops to zero in the limit as their separation goes to infinity. This means that the potential energy curve "flattens out" to a horizontal line as the distance from the vertical axis goes to infinity. Amazingly, this seemingly insignificant amount of information gives us a general shape for the interaction potential of two particles.

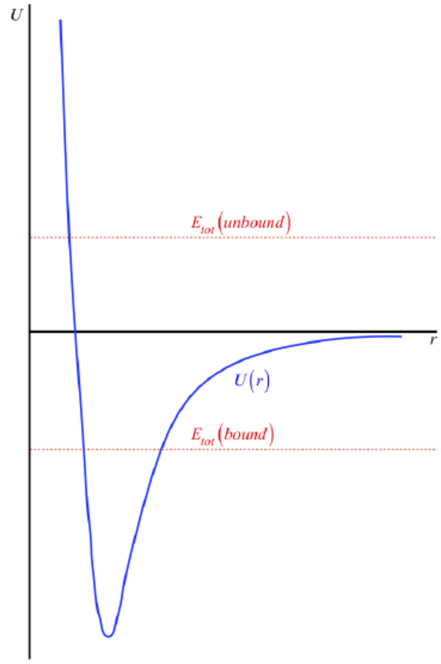
Figure 3.7.4 – Shape of Inter-Particle Interaction Potentials



The horizontal axis has been intentionally left out here, because this is not something we can determine. Recall that this potential energy curve can be raised or lowered an arbitrary amount without changing the physics, so we are actually free to place the horizontal axis (the point we call "zero energy") wherever we like. It is here that we come to the convention alluded-to at the start of this section: It is generally agreed to place the horizontal axis at the position that the potential energy curve reaches as $r\rightarrow\infty$. That is, it is usually agreed that the potential energy of the interaction vanishes when the particles are separated by an infinite distance (see the figure below).

One nice consequence of the $U\left(r\rightarrow\infty\right)\rightarrow 0$ is that it gives us a simple rule-of-thumb to determine whether or not the two particles in this potential are bound to each other. If the total energy of the system is positive (i.e. the horizontal line representing the total energy is above the r-axis), then that means that when the two particles are moving away from each other, the graphs never intersect to give a "forbidden region," and they just keep moving apart – they are not bound. If the total energy is negative, then the total energy horizontal line intersects the potential energy graph in two places, giving two turnaround points, keeping the particles within a range of separations.

Figure 3.7.5 – Bound and Unbound States



All of the physical interpretations we came up with above apply to this function as well, though it might be a bit confusing at first, since for much of the graph the potential energy is negative. But note:

- The kinetic energy for a specific position is still always positive, and equals the gap between the point on the curve and the total energy line.
- The force between the particles is still the negative of the slope, which means it is *attractive (seeks to make r smaller) when the slope is positive, and is repulsive (seeks to make r bigger) when the slope is negative*.
- The equilibrium and turnaround points are defined as the bottom of the dip and the intersection points of the total energy line and potential energy curve, respectively.

Alert

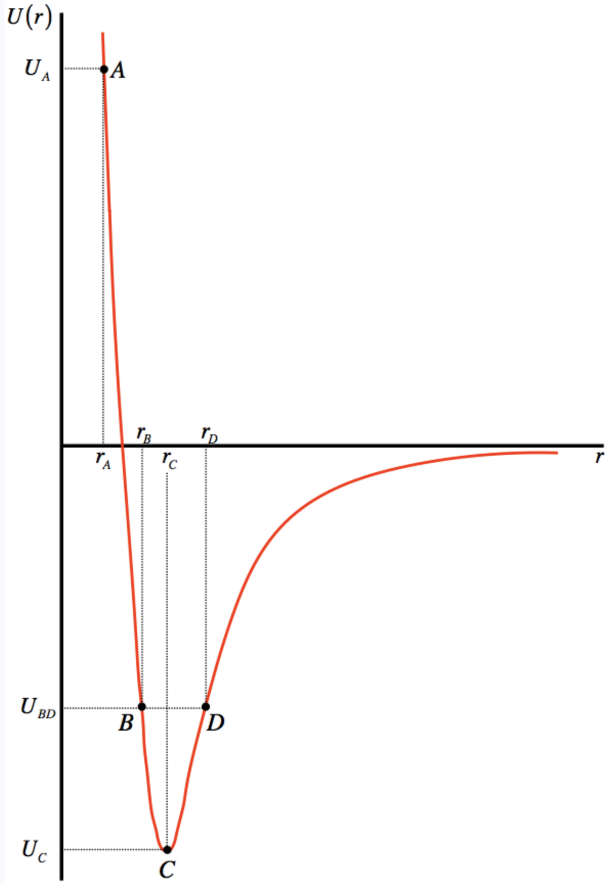
When two particles are bound to each other, to break the bond, energy must be added to them. That is, the total energy line must be moved up until it is above the r-axis. Collections of particles must ***receive additional energy*** *to break their chemical bonds, and energy only comes out of chemical reactions in which new bonds are formed. For some reason, belief of the exact reverse is a very common misconception.*

A model of the potential energy that works very well for two neutrally-charged atoms or molecules was constructed in 1924 by John Lennard-Jones. This *Lennard-Jones potential* has a simple-but-surprising form (yes, those powers are not typos!):

$$U(r) = \epsilon\left[\left(\frac{r_o}{r}\right)^{12} - 2\left(\frac{r_o}{r}\right)^{6}\right] \tag{3.7.2}$$

Exercise

Two molecules separated by a small distance interact with each other such that a potential energy curve that looks like the one below is the result. The particles start at rest in state A. A short time later, they are bound to each other, stuck within the range of separation from r_B to r_D.



In terms of the labels provided in the diagram, answer the following:

a. *How much energy was added to or taken away from (specify which) this two-particle system in the process of going from the starting state to the bound state?*
b. *What was the average inter-particle force as particles went from a separation of r_A to r_B? Was this force attractive or repulsive?*
c. What is the sum of the kinetic energies of the two molecules when they are separated by the distance r_C?
d. How much energy needs to be added to the bound particle system in order to completely separate them?

Solution

a. The difference in energy between the starting unbound state and the ending bound state is:

$$\Delta E = U_{BD} - U_A < 0$$

(Energy is taken away.)

b. The inter-particle force is given by the negative of the slope of the potential energy curve. The part of the curve between points A and B is not quite straight, but we are only asked for the average force, so we determine the slope using the endpoints:

$$F_{\text{between particles}} = -\frac{\Delta U}{\Delta r} = -\frac{U_{BD} - U_A}{r_B - r_A}$$

The slope of the curve is negative, so the force is positive, which means it is repulsive.

c. The kinetic energy is the gap between the total energy line (in this case, at U_{BC}) and the specific state in question, so the kinetic energy is just $U_{BD} - U_C$.

d. The "binding energy" is the amount of energy that needs to be added to the two particles to just barely (with no kinetic energy leftover) separate them to $r = \infty$. The system has a total energy of $U_B D$, which is a negative number, so it needs $-U_B D$ to be added to it to get its energy to zero – just enough to break the bond.

Modeling Bonds as Springs

It is quite common to model chemical bonds as springs, but this seems like a strange practice, given that the potential energy function looks like the equation above, which is nothing like the potential energy function of a spring. Surprisingly, a spring can nevertheless act as a reasonable replacement for the Lennard-Jones potential when the total energy is close to the lowest point of the curve. Certainly the curve at the bottom of the well *resembles* the parabolic curve of the elastic potential, but one might argue that the bottom of *any* concave curve will resemble the elastic potential energy curve. It turns out that this argument is completely correct – *every smooth concave curve can be approximated by the parabolic curve of the elastic potential energy*!

How well this approximation works depends upon the range of values we confine ourselves to. If we look at the whole curve, then obviously the approximation of the Lennard-Jones potential with a parabola breaks down badly. But as we narrow-down our view to a smaller and smaller range near the bottom of the potential well, this approximation gets better. The reason for this is related to some amazing mathematics: Any smooth function of a single variable can be written as a series (often infinite) of powers of that variable, with each term multiplied by a different constant:

$$f(x)=a_o+a_1x+a_2x^2+a_3x^3+\ldots \tag{3.7.3}$$

From this perspective, it's clear that *almost every* function has a bit of the x^2 function in it. For the Lennard-Jones potential, we can define x in the expansion above as $\left(1-\frac{r}{r_o}\right)$, which is a measure of how far the particle separation is from the equilibrium. For example, if the separation distance (r) is 90% of the equilibrium distance (r_o), then $\left(1-\frac{r}{r_o}\right)=0.1$, and for r equal to 99% of r_o, $\left(1-\frac{r}{r_o}\right)=0.01$.

In the spring model, it is precisely the distance that the spring is stretched or compressed from the equilibrium that determines the potential energy. So if we write the Lennard-Jones potential as an expansion in powers of $\left(1-\frac{r}{r_o}\right)$, we can more easily compare it to a spring potential energy:

$$\epsilon\left[\left(\frac{r_o}{r}\right)^{12}-2\left(\frac{r_o}{r}\right)^6\right]=U(r)=a_o+a_1\left(1-\frac{r}{r_o}\right)+a_2\left(1-\frac{r}{r_o}\right)^2+a_3\left(1-\frac{r}{r_o}\right)^3+\ldots \tag{3.7.4}$$

To use a spring to model this function, we want the series to look like a quadratic, so we need the contributions of the terms in the series with powers greater than 2 to be small compared to the terms before them. Well, if we restrict ourselves to values of r that are close to r_o (i.e. consider particles that are separated by a distance close to the equilibrium separation, which means the total energy is quite low), then the difference $\left(1-\frac{r}{r_o}\right)$ is a small number less than 1. The more times we multiply this small number by itself, the smaller the result, so higher powers provide ever-smaller contributions to the sum of the series.

All that remains if we are going to use the elastic potential energy to approximate the Lennard-Jones (or any other) potential is to figure out how to determine the equivalent "spring constant." Fortunately we have a nice trick for doing this. We can compare the way in which the spring constant enters the potential energy function to the series expansion, and determine what number we need from the series expansion to find the *effective spring constant* k_{eff}:

$$\frac{1}{2}k_{\text{eff}}x^2=a_2x^2 \quad \Rightarrow \quad k_{\text{eff}}=2a_2 \tag{3.7.5}$$

Okay, so we know where to find the effective spring constant – we express the actual potential as an infinite series, then look at the constant that multiplies the quadratic term in the expansion, and multiply it by two. But this still means we need to come up with the series... or does it? Consider the following useful mathematical trick: First, take two derivatives of Equation 3.7.3:

$$f(x)=a_o+a_1x+a_2x^2+a_3x^3+\ldots \quad\Rightarrow\quad \frac{df}{dx}=0+a_1+2a_2x+3a_3x^2+\ldots \quad\Rightarrow\quad \frac{d^2f}{dx^2}=0+0+2a_2+6a_3x+\ldots \tag{3.7.6}$$

Now evaluate this second derivative at the point $x=0$. All of the terms after the constant term then vanish, leaving what we are looking for:

$$\left.\frac{d^2f}{dx^2}\right|_{x=0} = 2a_2 = k_{\text{eff}} \tag{3.7.7}$$

Exercise

Find the effective spring constant for the Lennard-Jones potential given in Equation 3.7.2 in terms of the constants ϵ and r_o.

Solution

Let's start by noting that taking derivatives of a potential function $U\left(r\right)$ with respect to $x \equiv 1-\frac{r}{r_o}$ and evaluating at $x=0$ is no different from taking derivatives with respect to r and evaluating at $r=r_o$. The former method is nice for explaining how this works, but the latter is easier in practice. So we just follow the method given:

$$\frac{d}{dr}U\left(r\right) = \frac{d}{dr}\epsilon\left[\left(\frac{r_o}{r}\right)^{12} - 2\left(\frac{r_o}{r}\right)^{6}\right] = 12\epsilon r_o\left[\left(\frac{r_o}{r}\right)^{11} - \left(\frac{r_o}{r}\right)^{5}\right]$$

$$\frac{d^2}{dr^2}U\left(r\right) = \frac{d}{dr}12\epsilon r_o\left[\left(\frac{r_o}{r}\right)^{11} - \left(\frac{r_o}{r}\right)^{5}\right] = 12\epsilon r_o^2\left[11\left(\frac{r_o}{r}\right)^{10} - 5\left(\frac{r_o}{r}\right)^{4}\right]$$

Evaluating at $r=r_o$ gives the answer:

$$k_{\text{eff}} = \frac{d^2U}{dr^2}|_{r=r_o} = 72\epsilon r_o^2$$

Sample Problems

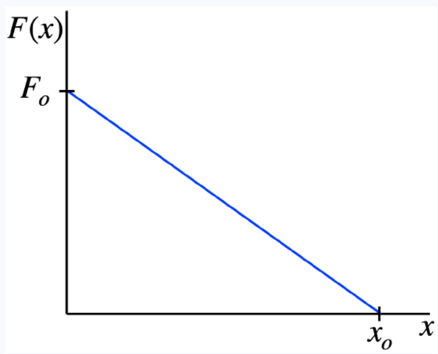
All of the problems below have had their basic features discussed in an "Analyze This" box in this chapter. This means that the solutions provided here are incomplete, as they will refer back to the analysis performed for information (i.e. the full solution is essentially split between the analysis earlier and details here). If you have not yet spent time working on (not simply reading!) the analysis of these situations, these sample problems will be of little benefit to your studies.

Problem 3.1

A toy train rolls along a straight, frictionless track, parallel to the x-axis. As it rolls, it experiences a force given by the equation:

$$\vec{F} = F\left(x\right)\left[0.600\ \hat{i} + 0.800\ \hat{j}\right]$$

The function $F\left(x\right)$ can be expressed by the graph below.



The train starts at the position $x_o = 12m$ with a speed of $3.0\frac{m}{s}$, moving in the $-x$ direction, and the force stops it when it gets halfway to the origin. If the same train is then picked up and placed on the track at the origin, find its acceleration.

Solution

As we saw in the analysis, the work done by this force on the train when it moves along the x-axis is 0.60 times the area under the graph given. In this case, the displacement of the train is from x_o to $\frac{1}{2}x_o$, so we use the area for that interval to compute the work (and since the displacement is in the $-x$-direction, this work is negative). The maximum force reached is $\frac{1}{2}F_o$, so area of this triangle is:

$$\text{area under curve} = \frac{1}{2}bh = \frac{1}{2}\left(\frac{1}{2}x_o\right)\left(\frac{1}{2}F_o\right) = \frac{1}{8}F_o x_o$$

The work done by the force during the train's journey is therefore:

$$W = -0.600\left(\text{area under graph}\right) = -0.60\left(\frac{1}{8}F_o x_o\right) = -F_o\left(0.90m\right)$$

This work results in a change of the train's kinetic energy, according to the work-energy theorem. The final kinetic energy is zero (the train stops), so:

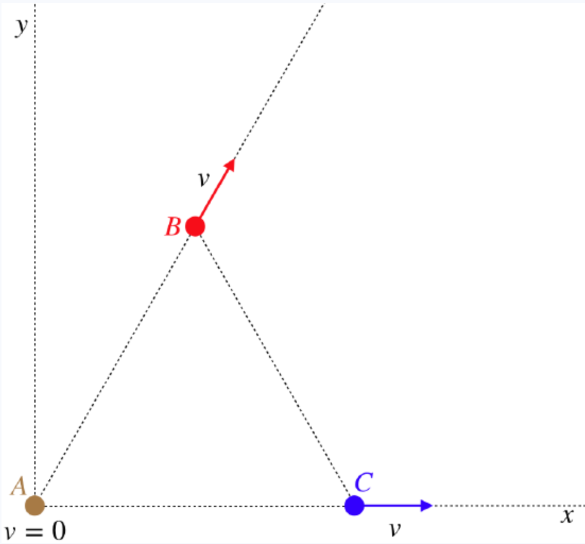
$$W = \Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2 - \quad\Rightarrow\quad -F_o\left(0.9m\right) = 0 - \frac{1}{2}m\left(3.0\frac{m}{s}\right)^2 \quad\Rightarrow\quad F_o = m_o\left(5.0\frac{m}{s^2}\right)$$

If the train is now placed at the origin, then the net force by the train + tracks will be in the $+x$ direction (the tracks exert a force on the train that won't let it move any other way), so only the x-component of the force will contribute to the acceleration. The train therefore feels a net force of $0.60F_o$, and dividing this by the mass of the train gives its acceleration, according to Newton's 2nd Law. The equation above gives us this ratio:

$$a = \frac{0.60F_o}{m} = 3.0\frac{m}{s^2}$$

Problem 3.2

Three identical particles, A, B, and C are positioned at the vertices of an equilateral triangle. Particle A remains at rest at the origin, while particles B and C move directly away from particle A at equal speeds along the lines defined by the triangle, as shown in the diagram.



The kinetic energy of each particle is $10J$.

a. Find the internal energy of the three-particle system.
b. Find the internal energy of the two-particle system AB.
c. Find the internal energy of the two-particle system BC.

Solution

a. We anticipated this question in the analysis, so we will not reproduce the work here. There we found that the internal energy of the three-particle system was $\frac{1}{2}mv^2$, which is the same as the kinetic energy of a single particle (measured in the diagramed frame of reference), which here is 10J.

b. The internal energy is found by subtracting the system's collective kinetic energy from its total energy, the latter of which can be found using the kinetic energies of the individual particles. Clearly, therefore, the total energy of the AB system in the reference frame given is $10J$. The system's collective kinetic energy comes from the motion of its center of mass. With just two particles of equal mass involved, the center of mass speed is:

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{mv + 0}{m + m} = \frac{1}{2}v$$

The kinetic energy of the AB system is therefore:

$$KE_{system\ AB} = \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}(2m)\left(\frac{1}{2}v^2\right) = \frac{1}{2}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}KE_B = 5J$$

Subtracting this from the total energy of the two-particle system gives us its internal energy:

$$E_{int\ AB} = E_{tot\ AB} - KE_{system\ AB} = (KE_A + KE_B) - KE_{system\ AB} = (0 + 10J) - 5J = 5J$$

c. If we think about what the arrangement of particles looks like at later times, we conclude that it remains an equilateral triangle, but just a bigger one. This means that if we looked at the motion of C from the perspective of B, we would see exactly the same thing as viewing B's motion from A's perspective. The internal energy is independent of the frame from which we view it, so the internal energy of the BC system must be the same as it is for the AB system. We can of course confirm this with lots of velocity vector math. Sigh, okay, here it is... Place A at the origin, and C moves along the -axis, while B moves away from the origin in a direction 60º above the -axis. The velocity vectors are therefore:

$$\vec{v}_B = v\left(\cos 60^o\,\hat{i} + \sin 60^o\,\hat{j}\right) = \frac{1}{2}v\left(\hat{i} + \sqrt{3}\hat{j}\right) \;, \qquad \vec{v}_C = v\hat{i}$$

The center of mass velocity of the system BC is:

$$\vec{v}_{cm} = \frac{m\vec{v}_B + m\vec{v}_C}{m+m} = \frac{1}{4}v\left(3\hat{i} + \sqrt{3}\hat{j}\right)$$

This makes the system's collective kinetic energy equal to:

$$KE_{system\ BC} = \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}(2m)\left(\frac{1}{4}v\right)^2\left(3^3 + \sqrt{3}^2\right) = \frac{3}{4}mv^2$$

The system's total energy is the sum of the kinetic energies of the two particles B and C, which is just $E_{tot;BC} = mv^2$. So the internal energy is:

$$E_{int\ BC} = E_{tot\ BC} - KE_{system\ AB} = mv^2 - \frac{3}{4}mv^2 = 5J$$

This confirms that it is the same internal energy as the system AB.

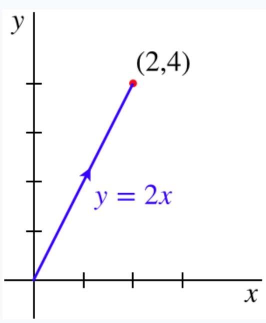
Problem 3.3

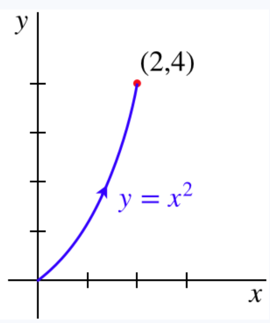
Consider the following forces that act on a particle as it moves in the (x,y) plane (α is a constant):

- $\overrightarrow{F}(x,y) = \alpha\left(x\;\hat{i} - y\;\hat{j}\right)$
- $\overrightarrow{F}(x,y) = \alpha\left(y\;\hat{i} - x\;\hat{j}\right)$

Check each of these forces to determine if it is conservative, in each of the following ways:

a. *Use the partial derivative check (Equation 3.3.8).*
b. *Compute the work done by each force on a particle that moves from the origin to the point $(2,4)$ in the (x,y) plane over the following two paths:*





Solution

a. We already did this in the analysis.

b. This method is quite a bit tougher (which makes us thankful for the partial derivative method). You may want to review the discussion of line integrals for more guidance on how this works. We have two line integrals to perform for each force – one following the linear path and one following the parabolic path. In every case, the infinitesimal displacement vector can be written as:

$$\overrightarrow{dl} = dx\;\hat{i} + dy\;\hat{j} = \left(\hat{i} + \frac{dy}{dx}\hat{j}\right)dx$$

Evaluating this vector on the specific path being used is then just a simple matter of knowing the derivative for that path, and the integral can then be performed over x between the endpoints of the path (for both of these paths, that will be $x: 0 \rightarrow 2$). The thing to always keep in mind is that anything we evaluate must be specific to the path, and we are converting everything to the single variable x so we can perform the integral. This will become clear below.

$\overrightarrow{F}(x,y) = \alpha\left(x\;\hat{i} - y\;\hat{j}\right)$

$$W = \int\limits_{lin} \vec{F}\cdot\overrightarrow{dl} = \int\limits_{x=0}^{x=2} \alpha\left(x\;\hat{i} - y\;\hat{j}\right)\cdot\left(\hat{i} + \frac{dy}{dx}\hat{j}\right)dx = \alpha\int\limits_{x=0}^{x=2}\left(x - y\frac{dy}{dx}\right)dx$$

Linear path, so $y = 2x$ *and* $\frac{dy}{dx} = 2$:

$$W = \alpha\int\limits_{x=0}^{x=2}\left[x - (2x)(2)\right]dx = \left[-\frac{3}{2}\alpha x^2\right]_0^2 = -6\alpha$$

Parabolic path, so $y = x^2$ *and* $\frac{dy}{dx} = 2x$:

$$W = \alpha\int\limits_{x=0}^{x=2}\left[x - \left(x^2\right)(2x)\right]dx = \alpha\left[\frac{1}{2}x^2 - \frac{1}{2}x^4\right]_0^2 = -6\alpha$$

The two line integrals along different paths between the same endpoints yield the same answer, as they should for this force that we have already determined to be conservative. Now for the second force:

$\overrightarrow{F}(x,y) = \alpha\left(y\;\hat{i} - x\;\hat{j}\right)$

$$W = \int\limits_{lin} \vec{F}\cdot\overrightarrow{dl} = \int\limits_{x=0}^{x=2} \alpha\left(y\;\hat{i} - x\;\hat{j}\right)\cdot\left(\hat{i} + \frac{dy}{dx}\hat{j}\right)dx = \alpha\int\limits_{x=0}^{x=2}\left(y - x\frac{dy}{dx}\right)dx$$

Linear path, so $y = 2x$ *and* $\frac{dy}{dx} = 2$:

$$W = \alpha\int\limits_{x=0}^{x=2}\left[2x - (x)(2)\right]dx = 0$$

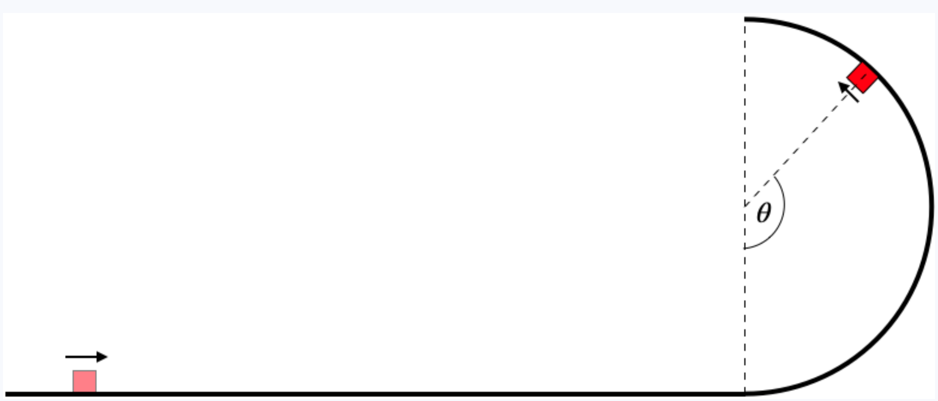
Parabolic path, so $y = x^2$ *and* $\frac{dy}{dx} = 2x$:

$$W = \alpha\int\limits_{x=0}^{x=2}\left[x^2 - (x)(2x)\right]dx = \alpha\left[-\frac{1}{3}x^3\right]_0^2 = -\frac{8}{3}\alpha$$

These two line integrals are not equal, which confirms that this force in non-conservative.

Problem 3.4

A small block slides along a frictionless, horizontal surface into a frictionless vertical half-circle track, and it remains in contact with the track, until at least the $\theta = 90^o$ *point (with* θ *defined in the diagram).*



Find the fraction of kinetic energy lost by the block in getting to the top of the half-circle, if it just barely maintains contact with the track at the top.

Solution

In the analysis, we found a relationship between the block's initial and final speeds, in terms of the angle θ. In this case, we are interested in its speed at the top, so $\theta = 180^o$, giving:

$$v_f^2 = v_o^2 - 2gR\left(1 - \cos 180^o\right) = v_o^2 - 4gR$$

We also found in the analysis an expression for the minimum velocity needed for the block to remain in contact with the track, as a function of the angle θ. As we are given that the block barely remains in contact at the top of the half-circle, we must assume that it has the minimum speed it can have at this point. This speed is the "final" speed, so we have:

$$v_f^2 = -gR\cos 180^o = gR$$

So we have v_f^2 in terms of R, and combining these two equations allows us to do the same for v_o^2:

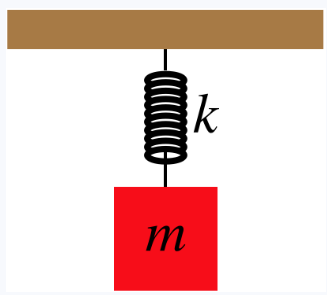
$$gR = v_o^2 - 4gR \quad \Rightarrow \quad v_o^2 = 5gR$$

Now all we have to do is construct the kinetic energy ratio requested:

$$\frac{KE_{lost}}{KE_o} = \frac{KE_o - KE_f}{KE_o} = \frac{\frac{1}{2}mv_o^2 - \frac{1}{2}mv_f^2}{\frac{1}{2}mv_o^2} = \frac{v_o^2 - v_f^2}{v_o^2} = \frac{5gR - gR}{5gR} = 0.8$$

Problem 3.5

A block is attached to a vertical spring, the other end of which is attached to the ceiling. The block is held stationary at a height where the spring is at its equilibrium length. The block is then released.



The block falls a distance of $120cm$ before finally stopping and bouncing back up. Find the distance it has fallen when it reaches its maximum speed.

Solution

With some thought, the analysis actually gives us the answer immediately. We found that the position where the spring force cancels the gravity force is exactly halfway between the top and the bottom, which we also found to be the position where the spring and gravity forces are equal. The gravity force is unchanging, so slightly above this midway position, the spring stretch is not quite enough to produce a force as great as the force of gravity. This means that everywhere above this midway point there is a net force downward. So the block continues speeding up in its descent until it reaches the midway point. After crossing the midway point, the spring force is greater than the gravity force, so the block slows down. Therefore the block reaches a maximum speed after dropping $60cm$.

Okay, now let's do it the "cool" mathematical way, using the work-energy theorem. If we call the starting point of the block $y = 0$ and treat downward as the $+y$-direction, then from the equation we found in the analysis, we have the total work done on the block in terms of y, and we can set it equal to the kinetic energy at the position y (yes, the work done equals the change in kinetic energy, but the block starts from rest, so the change equals the kinetic energy itself):

$$KE\left(y\right) = W\left(0 \rightarrow y\right) = -\frac{1}{2}ky^2 + mgy$$

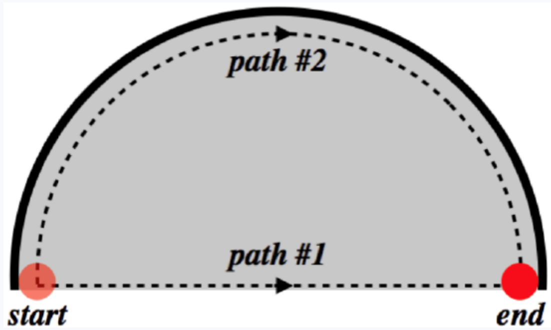
We want to know the value of y where the speed of the block is a maximum. Well, obviously this occurs where the kinetic energy is a maximum (the block's mass doesn't change). So maximizing the $KE(y)$ function gives:

$$0=\frac{d}{dy}KE(y)=-ky+mg \quad \Rightarrow \quad y=\frac{mg}{k}$$

We already found in the analysis that the maximum stretch of the spring is twice this value, so once again we find that the y value where the block is moving fastest is $60cm$.

Problem 3.6

A puck is slid along a horizontal rough surface in a straight line along the diameter of a circle (the gravity, contact and friction forces are the only forces on the puck). The same puck is then slid on the same surface starting at the same speed along the circle defined by the diameter indicated in the first experiment (it slides around the inside surface of a frictionless circular wall). The figure shown depicts a top view of these two paths.



The puck is pushed at the start point so that it slides around path #2, and it just comes to rest at the end point. If it is pushed at the same speed from the start point along path #1, find the fraction of initial speed that it has lost when it reaches the end point.

Solution

In the analysis we found that the work done by friction over path #2 is greater than the work done by friction over path #1 by a factor of $\frac{\pi}{2}$. From the work-energy theorem, this means that the changes in kinetic energy for these two paths are related in the same way:

$$\Delta KE_2=\frac{\pi}{2}\Delta KE_1 \quad \Rightarrow \quad 0-\frac{1}{2}mv_o^2=\frac{\pi}{2}\left(\frac{1}{2}mv_f^2-\frac{1}{2}mv_o^2\right)$$

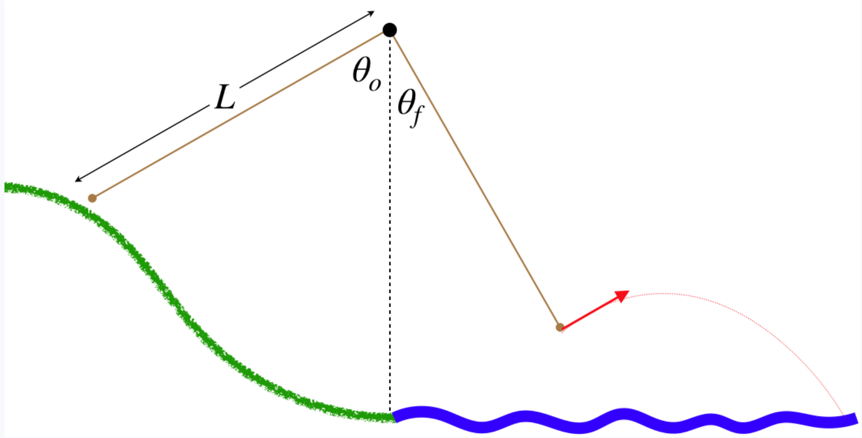
Solving for the final velocity in terms of the initial velocity gives:

$$v_f=\sqrt{1-\frac{2}{\pi}}\,v_o=0.60v_o$$

So the puck loses about 40% of its speed along path #1.

Problem 3.7

There are few things as fun as swinging into a river from a rope swing tied to the limb of a tree on its banks. The person at the end of this rope starts at the top of a hill at one angle, then swings to another angle when they let go and fly into the water.



The length of the rope is $6.0m$, and the starting and ending angles are $\theta_o = 60^o$ and $\theta_f = 30^o$, respectively. The end of the rope when it hangs vertically at the shoreline is $2.4m$ above the water level. You start from rest (don't get a running start).

a. *Find your speed at the point when you release the rope.*
b. *Find the distance above the water that you reach at the peak of your flight.*

Solution

a. We solved this in the analysis, so all that remains is to plug in the numbers. The initial speed is zero, and the length of the rope and angles are given, so:

$$v_f = \sqrt{v_o^2 - 2gL\left(\cos\theta_o - \cos\theta_f\right)} = \sqrt{0 - 2\left(9.8\frac{m}{s^2}\right)(6.0m)\left(\cos 60^o - \cos 30^o\right)} = 6.6\frac{m}{s}$$

b. We know that at the point of release, the velocity vector makes a 30° angle with the horizontal, so the horizontal component of this velocity (which never changes) is:

$$v_x = v\cos 30^o = 5.7\frac{m}{s}$$

When you hit your peak height, you will have a zero y-component of velocity, so the quantity above will be your total speed. Given we are ignoring air resistance, the only work is being done throughout is by gravity, which means we can use the same method as in the analysis – whether the rope is involved throughout or not is irrelevant. So again calling the height of the tree limb $y = 0$, we have:

$$-\Delta KE = \Delta U_{grav} = mg\left(y_f - y_o\right) = mgy_f - mg\left(-L\cos 60^o\right) \quad \Rightarrow \quad y_f = -\frac{1}{2g}v_f^2 - L\cos 60^o$$

Plugging in for L, and v_x for v_f, we get:

$$y_f = -\frac{\left(5.7\frac{m}{s}\right)^2}{2\left(9.8\frac{m}{s^2}\right)} - (6.0m)\cos 60^o = -4.7m$$

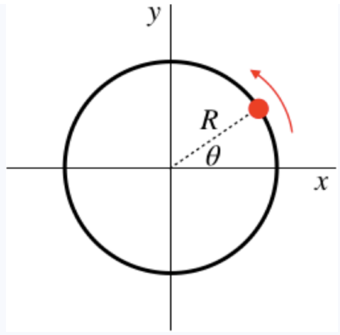
This is the distance below the tree branch, and since we know how high the tree branch is above the water, we have our answer:

$$h = 6.0m + 2.4m - 4.7m = 3.7m$$

Problem 3.8

A bead is threaded onto a frictionless circular loop that lies in the horizontal x-y plane, as shown in the diagram below. This bead is subjected to a conservative force that is characterized by the potential energy function:

$$U(x,y) = -\alpha\left(Rx + y^2\right) \qquad \alpha > 0$$



The physical values for this set up are as follows: $\alpha = 0.05\frac{kg}{s^2}$, $R = 16cm$, *mass of bead* $= 4.0g$. *The bead travels counterclockwise around the loop, and makes it all the way around the circle without stopping, with its minimum speed measured to be:* $90\frac{cm}{s}$. *Find the maximum speed attained by the bead.*

Solution

We determined in the analysis the maximum and minimum potential energies, and these correspond to the minimum and maximum kinetic energies, respectively. We are also given the minimum speed of the bead, so if we apply the conservation of energy to the change that occurs between the maxes and mins, we get:

$$0 = \Delta KE + \Delta U \quad \Rightarrow \quad KE_{max} = KE_{min} + U_{max} - U_{min}$$

Putting in the expressions for the kinetic energies and the results from the analysis for the potential energies, we get:

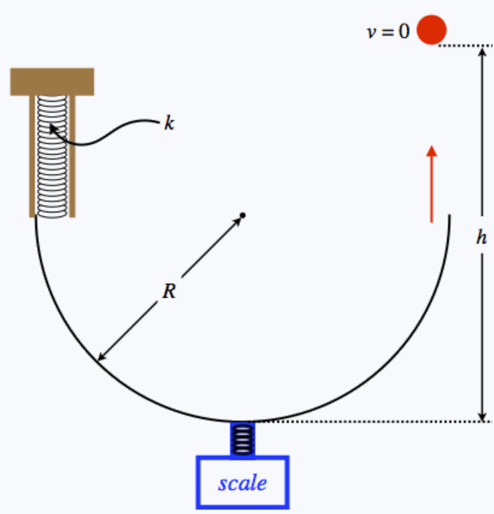
$$\frac{1}{2}mv_{max}^2 = \frac{1}{2}mv_{min}^2 + \left(+\alpha R^2\right) - \left(-\frac{5}{4}\alpha R^2\right) \quad \Rightarrow \quad v_{max} = \sqrt{v_{min}^2 + \frac{9\alpha R^2}{2m}}$$

And finally, plugging in the values gives:

$$v_{max} = \sqrt{\left(0.9\frac{m}{s}\right)^2 + \frac{9\left(0.05\frac{kg}{s^2}\right)(0.16m)^2}{2\,(0.004kg)}} = 1.5\frac{m}{s}$$

Problem 3.9

A ball is launched straight up into the air with the apparatus shown below. The ball is pushed upward so that it compresses the spring, and is released from rest. It then travels around a frictionless half-circle track, at the bottom of which is a scale that measures the contact force the ball exerts on the track at that point.



The mass of the ball is $m = 0.400kg$, *the stiffness of the spring is* $k = 22.0\frac{N}{m}$, *the radius of the track is* $R = 1.60m$, *and at the moment the ball is above the scale, the scale reads* $F_{scale} = 18.5N$.

a. *Find the speed of the ball as it passes the scale.*
b. *Find the height h reached by the ball when it comes to rest.*
c. *Find the amount that the spring was compressed before the ball was released.*

Solution

a. In the analysis, we found the relationship between the speed of the ball and the scale reading. We now know the ball's mass and the radius of the track, so:

$$F_{scale} - mg = ma_c = m\frac{v^2}{R} \quad \Rightarrow \quad v = \sqrt{R\left(\frac{N}{m} - g\right)} = \sqrt{(1.60m)\left(\frac{18.5N}{0.400kg} - 9.8\frac{m}{s^2}\right)} = 7.64\frac{m}{s}$$

b. There is no friction force by the track, and the contact force it exerts is perpendicular to the motion, so it does no work, which means that mechanical energy is conserved. With the speed of the ball at the bottom, we can therefore compute the height it reaches, where it comes to rest:

$$0 = \Delta KE + \Delta U_{grav} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2 + mgh \quad \Rightarrow \quad h = \frac{v_o^2}{2g} = \frac{\left(7.64\frac{m}{s}\right)^2}{2\left(9.8\frac{m}{s^2}\right)} = 2.98m$$

c. We know the total energy in the system, either from the KE at the scale, or the PE at the peak height:

$$E_{tot} = mgh = (0.400kg)\left(9.8\frac{m}{s^2}\right)(2.98m) = 11.7J$$

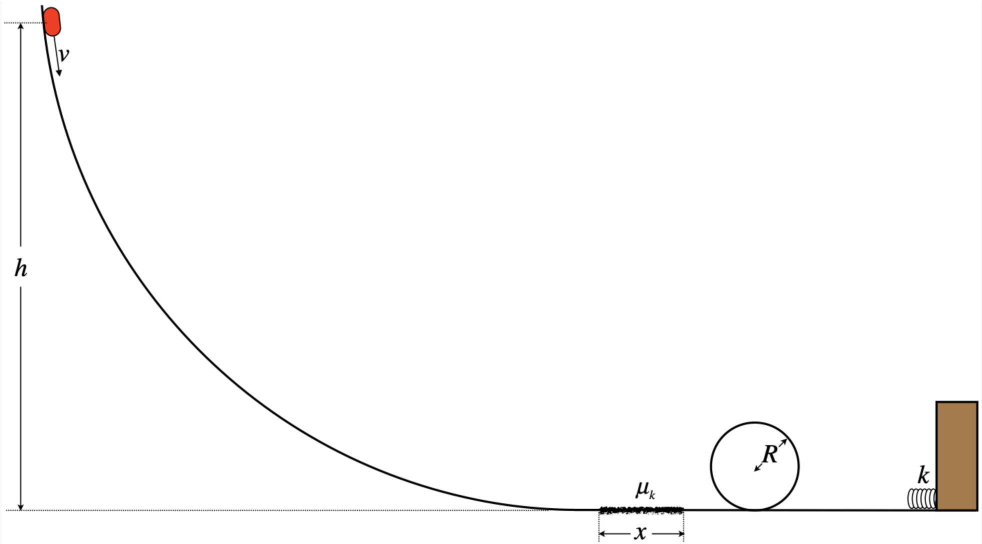
When the ball compressed the spring, it had no KE, so all of this energy was stored in the PE of gravity and the elastic PE of the spring. Calling the compression Δy, then the height of the ball at the start is $R + \Delta y$. Summing the two PE's and setting the sum equal to the total energy gives a quadratic equation, which we then solve for Δy:

$$E_{tot} = mg(R + \Delta y) + \frac{1}{2}k(\Delta y)^2 \quad \Rightarrow \quad \Delta y = \frac{-mg \pm \sqrt{(mg)^2 - 2k(mgR - E_{tot})}}{k}$$

$$\Delta y = \frac{-(0.400kg)\left(9.8\frac{m}{s^2}\right) \pm \sqrt{(0.400kg)^2\left(9.8\frac{m}{s^2}\right)^2 - 2\left(22.0\frac{N}{m}\right)\left((0.400kg)\left(9.8\frac{m}{s^2}\right)(1.60m) - 11.7J\right)}}{22.0\frac{N}{m}} = 0.55m$$

Problem 3.10

A puck slides down a frictionless track, over a short horizontal rough (frictional) patch, around a frictionless loop-de-loop, and into an ideal spring fixed to a wall, where it bounces off and goes back the other way, returning through the loop-de-loop, over the rough patch, and back up the ramp.



The coefficient of kinetic friction for the rough patch is 0.30, and the length of that patch is $0.80cm$. The puck bounces off the spring one time, and eventually comes to rest, without ever falling off the loop-de-loop.

a. *Find the maximum value possible for the radius the loop-de-loop.*
b. *If the radius of the loop-de-loop is the maximum value found in part (a), find the height at which the puck was released from rest.*

Solution

a. The analysis shows that the amount of mechanical energy converted to thermal energy by a single trip across the patch is equal to $\mu_k mgx$. The puck bounces off the spring once, and makes it around the loop, but never makes it back to the loop after passing over the patch, going up the ramp, and then reentering the patch. The most kinetic energy it can have after passing through the loop-de-loop the second time is therefore the amount of energy converted to thermal after two full trips across the patch:

$$\text{kinetic energy at bottom of loop after bouncing off the spring } = 2\mu_k mgx$$

The puck must also get around the loop, and the larger the radius of the loop is, the more total energy the puck must have. But we have an upper-limit on the puck's kinetic energy at the bottom of the loop, so this gives us an upper limit on the radius of the loop. From the analysis we found that the puck barely makes it around the loop if its kinetic energy at the bottom of the loop is $\frac{5}{2}mgR$. Setting this equal to the quantity found above gives us our answer:

$$\frac{5}{2}mgR_{max} = 2\mu_k mgx \quad \Rightarrow \quad R_{max} = \frac{4}{5}\mu_k x = \frac{4}{5}(0.30)(80cm) = 19.2cm$$

b. In order for the situation above to occur, the puck must come to rest after passing over the patch exactly three times (once coming down the ramp the first time, then two more times, as described above). Therefore all of the potential energy of the unmoving puck at the release point is converted to thermal energy, and the solution is quick:

$$U_{start} = \Delta E_{thermal} \quad \Rightarrow \quad mgh = 3\mu_k mgx \quad \Rightarrow \quad h = 3(0.3)(80cm) = 72cm$$

Problem 3.11

A block slides along a horizontal frictionless surface until it runs into a spring at its equilibrium length. Just as it starts to compress the spring, the surface becomes rough.

The mass of the block is $1.2kg$, and it approaches the spring at a speed of $6.5\frac{m}{s}$. The spring stiffness is $2.2\frac{N}{m}$, and the coefficient of kinetic friction is a constant value of 0.3. The block eventually comes to rest and remains there.

a. Find the distance that the spring is compressed when the block comes to rest.
b. Find the fraction of the block's kinetic energy that has become thermal by the time the block comes to rest.
c. Find the minimum coefficient of static friction that the rough surface can have.

Solution

a. In the analysis we derived an expression that relates the compression of the spring to all the other quantities:

$$0=\frac{1}{2}m\left(v_f^2-v_o^2\right)+\left(\frac{1}{2}k\Delta x^2-0\right)+\mu_k mg\Delta x$$

The final speed is $v_f=0$ and this is a quadratic equation in the value we are looking for (Δx), so the answer is immediate:

$$k\Delta x^2+2\mu_k mg\Delta x-mv_o^2=0 \quad \Rightarrow \quad \Delta x=\frac{-2\mu_k mg\pm\sqrt{4\mu_k^2m^2g^2+4kmv_o^2}}{2k}=\frac{\mu_k mg}{k}\left(-1\pm\sqrt{1+\frac{kv_o^2}{\mu_k^2mg^2}}\right)$$

Plugging in the numbers:

$$\Delta x=\left(\frac{(0.3)(1.2kg)\left(9.8\frac{m}{s^2}\right)}{2.2\frac{N}{m}}\right)\left(-1\pm\sqrt{1+\frac{\left(2.2\frac{N}{m}\right)\left(6.5\frac{m}{s}\right)^2}{(0.3)^2(1.2kg)\left(9.8\frac{m}{s^2}\right)^2}}\right)=3.46m$$

b. The thermal energy is the work done by friction:

$$E_{thermal}=\mu_k mg\Delta x=(0.3)(1.2kg)\left(9.8\frac{m}{s^2}\right)(3.46m)=12.2J$$

We can also compute the incoming kinetic energy:

$$KE_o=\frac{1}{2}mv_o^2=\frac{1}{2}(1.2kg)\left(6.5\frac{m}{s}\right)^2=25.4$$

So the fraction of energy converted to thermal is:

$$f=\frac{12.2J}{25.4J}=0.48$$

c. The fact that the block has stopped and has not started sliding back means that the net force on the block is zero, which means that the static friction force is equal to the push of the spring. We can compute this force:

$$F=k\Delta x=\left(2.2\frac{N}{m}\right)(3.46m)=7.61N$$

The maximum static friction force must be this much, so setting the maximum equal to this number gives us the minimum possible coefficient of static friction:

$$\mu_sN=\mu_smg\geq 7.61N \quad \Rightarrow \quad \mu_s\geq\frac{7.61N}{(1.2kg)\left(9.8\frac{m}{s^2}\right)}=0.65$$

Problem 3.12

A particle starts from rest and experiences a net force that has a constant magnitude. This force does change direction, however, such that the particle is made to move in a circular path. Find the power delivered to the particle when it reaches half of its maximum speed, in terms of its mass m, the net force magnitude F, and the radius R of the circle to which the particle is confined.

Solution

We have an expression for the maximum velocity of the particle from the analysis, so half this amount is:

$$v = \frac{1}{2}\sqrt{\frac{FR}{m}}$$

At this speed, the force component that is radially inward to maintain circular motion is:

$$F_{radial} = \frac{mv^2}{R} = \frac{1}{4}F$$

The part of the net force that is tangential can then be found using the Pythagorean theorem, as the radial and tangential parts are the two components of the total:

$$F^2 = F_{radial}^2 + F_{tangential}^2 \quad \Rightarrow \quad F_{tangential} = \sqrt{F^2 - F_{radial}^2} = \sqrt{F^2 - \frac{1}{16}F^2} = \frac{\sqrt{15}}{4}F$$

The power delivered is the force component in the direction of motion multiplied by the speed, so:

$$P = \vec{F}\cdot\vec{v} = F_{tangential}\,v = \left(\frac{\sqrt{15}}{4}F\right)\left(\frac{1}{2}\sqrt{\frac{FR}{m}}\right) = \sqrt{\frac{15F^3R}{64m}}$$

CHAPTER OVERVIEW

4: Linear Momentum

4.1: Repackaging Newton's Second Law
4.2: Center of Mass
4.3: Momenta of Systems
4.4: Momentum and Energy
4.5: More Collisions
Sample Problems

Thumbnail: A pool break-off shot. Image used with permission (CC-SA-BY; No-w-ay).

4.1: Repackaging Newton's Second Law

Definition of Impulse

In chapter 2, we made a point of emphasizing that force is not possessed by objects – it is an interaction between them. One way we know this is that if the same force is exerted on identical objects that start at rest, the two objects are not necessarily moving the same afterward. There is an important ingredient missing here – the *duration* that the force acts. Since the force causes an acceleration, the longer it acts, the more the velocity is affected. So multiplying the force by the amount of time it acts may provide us with a useful quantity. The force may be changing magnitude or direction while it acts, but over a very short time this product is:

$$d\overrightarrow{J} = \overrightarrow{F}\,dt \tag{4.1.1}$$

If we want to know the totality of this quantity over a finite time interval, we need to add up all these little contributions. We give this quantity the name *impulse*.

Definition: Impulse

$$\overrightarrow{J}_{tot}\left(t_A \rightarrow t_B\right) \equiv \int_{t_A}^{t_B} \overrightarrow{F}_{net}\,dt$$

This quantity is the sum of the product of the forces and the times over which those forces act. This certainly sounds very similar to work, which takes a product of forces and displacements. Also, impulse will have an impact on the motion of the object, as work did. But there are also many differences between these two quantities.

The first difference between impulse and work is that they obviously represent different physical quantities, because they have different units. While work has units of energy which we measure in Joules (or Newton-meters), impulse has units of force-times-time, measured in Newton-seconds. A second difference is that the impulse integral (mercifully) is not a line integral – there is no "path" to concern ourselves with when computing impulse. And third, because there is no dot product involved with the impulse integral, the result is a vector, in contrast to work, which is a scalar.

Definition of Momentum

The definition of impulse is not the end of the story, any more than the definition of work was. It needs to be related to the effect it has on the motion of the object. In the case of work, this relationship was expressed as the work-energy theorem:

$$\begin{aligned} \text{actions of pushes and pulls} \quad &= \quad W_{tot}\left(A \rightarrow B\right) \\ &= \quad \int_A^B \overrightarrow{F}_{net} \cdot \overrightarrow{dl} \\ &= \quad \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2 \\ &= \quad \Delta KE \\ &= \quad \text{effect of pushes and pulls} \end{aligned} \tag{4.1.2}$$

For the case of impulse, we find this relationship again by coming back to Newton's second law, and noting that the integral of acceleration is velocity:

$$\begin{aligned}
\text{actions of pushes and pulls} &= \overrightarrow{J}_{tot}\left(t_A \rightarrow t_B\right) \\
&= \int\limits_{t_A}^{t_B} \overrightarrow{F}_{net} dt \\
&= \int\limits_{t_A}^{t_B} \left[m \overrightarrow{a}_{cm}\right] dt \\
&= \left[m \overrightarrow{v}_{cm}\right]_A^B \\
&= \Delta\left(m \overrightarrow{v}_{cm}\right) \equiv \Delta \overrightarrow{p}_{cm} \\
&= \text{effect of pushes and pulls}
\end{aligned} \tag{4.1.3}$$

We call the quantity $\overrightarrow{p}_{cm}$ the *momentum* of the collection of particles on which the net force is acting, defined in terms of the total mass of the collection of particles and the velocity (magnitude *and* direction) of the collection's center of mass. As we saw in chapter 2, Newton's 2nd law works for any collection of particles, whether they form a solid object or are completely non-interacting, like a gas. So since this result is derived from the 2nd law, the same is true here. While this definition applies to a collection of particles, it is useful to define this same quantity for individual particles as well.

Definition: Momentum

$$\overrightarrow{p} \equiv m \overrightarrow{v}$$

It turns out that if we do happen to have a collection of particles, each with their own individual momentum, then computing the collective momentum $\overrightarrow{p}_{cm}$ is a simple matter of summing the individual momenta:

$$\vec{p}_1 + \vec{p}_2 + \dots = m_1\vec{v}_1 + m_2\vec{v}_2 + \dots = \left(m_1 + m_2 + \dots\right) \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \dots}{\left(m_1 + m_2 + \dots\right)} = M_{tot}\vec{v}_{cm} \tag{4.1.4}$$

Equation 4.1.3 is known as the *impulse-momentum theorem*. Like kinetic energy, momentum is related to the motion of the object (and the mass), but besides being a different function of mass and velocity than kinetic energy, it is also different in that momentum is a vector while kinetic energy is not. This means that the total impulse can lead to a change in the magnitude or direction (or both) of the momentum vector.

The astute reader will undoubtedly realize that all we have really done here is to introduce a new vector, and use it to repackage Newton's 2nd law. Indeed, we can rewrite the 2nd law thus:

$$\overrightarrow{F}_{net} = m \overrightarrow{a}_{cm} = m \frac{d}{dt} \overrightarrow{v}_{cm} = \frac{d}{dt}\left(m \overrightarrow{v}_{cm}\right) = \frac{d}{dt} \overrightarrow{p}_{cm} \tag{4.1.5}$$

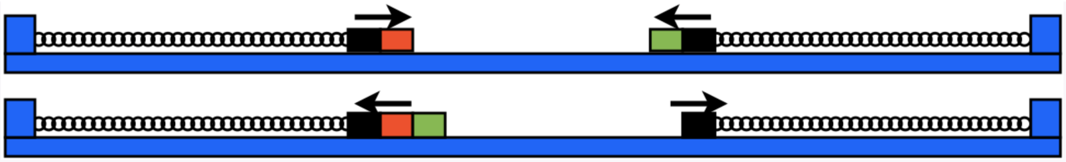
The Link to Internal Energy

Given that we are making comparisons of work with impulse and momentum with kinetic energy, it is useful to point out a direct mathematical relationship, which not only points out the difference between the two quantities, but will also be quite useful later on. For an individual particle we have:

$$KE = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m} = \frac{\overrightarrow{p} \cdot \overrightarrow{p}}{2m} \tag{4.1.6}$$

Analyze This

Two pairs of identical blocks on identical springs are side-by-side as shown in the diagram below. They are set into motion such that just as they reach their (equal) maximum displacements toward each other, they barely come into contact (there is no collision – their springs stop them just as they touch). When they contact, one of the blocks is transferred to the other, and their motion continues.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

Let's start by defining two separate systems here: One that includes the left spring and the blocks it is moving, and one that includes the right spring and the block(s) it is moving. At the moment the blocks touch, all of the potential energy of both systems is stored in the stretch of their respective springs. Losing or adding a block has no effect on this energy, because at the time of the exchange the blocks are not moving, and therefore have no kinetic energy.

So the left side has the same total energy with three blocks as it previously had with two blocks, and the right side has the same total energy that it started with as well. What does that tell us about how the motions of the two systems change, if at all, after the block exchange? We can start by asking about the maximum speeds the blocks attain on each side, before and after. First of all, this occurs when the spring reaches its equilibrium position, because at that moment the potential energy is as low as it can be. Both before and after, this maximum kinetic energy is the same, but the masses are different, so the speeds will not be the same. In order for the kinetic energy to be the same with three blocks as with two, the ratio of the velocities must be:

$$\begin{array}{llll}\text{left spring:} & \frac{1}{2}(2m)\,v^2_{\text{max before}} = \frac{1}{2}(3m)\,v^2_{\text{max after}} & \Rightarrow & v_{\text{max after}} = \sqrt{\frac{2}{3}}\;v_{\text{max before}} \\ \text{right spring:} & \frac{1}{2}(2m)\,v^2_{\text{max before}} = \frac{1}{2}(1m)\,v^2_{\text{max after}} & \Rightarrow & v_{\text{max after}} = \sqrt{2}\;v_{\text{max before}}\end{array}$$

Note that this means that the left side is moving slower overall than the right side, and they will not complete a full cycle at the same time. Even though each side will reach far enough at their maximum spring stretched for the blocks to once again touch, the motions are no longer synchronized, so the blocks will not touch again.

One can also compare the momentum of the left side blocks at the equilibrium point to the momentum of the right side blocks at their equilibrium point. The kinetic energies are the same, so since the left side has 3 times the mass, we have:

$$\text{at equilibrium: } KE_{\text{left}} = KE_{\text{right}} \quad\Rightarrow\quad \frac{p^2_{\text{left}}}{2\,(3m)} = \frac{p^2_{\text{right}}}{2\,(1m)} \quad\Rightarrow\quad p_{\text{left}} = \sqrt{3}\;p_{\text{right}}$$

Consider now a collection of particles, each with their own momentum (for simplicity, we will assume that the collection is an ideal gas – the particles are not interacting with each other). Above, we showed that the total momentum of the collection is just the sum of the momenta of the individual particles. But in the previous chapter, we found that the sum of the individual particle kinetic energies is *not* equal to the collection's (center of mass) kinetic energy. This is easiest to see in the center of mass reference frame. In this frame, $\vec{v}_{cm} = 0$, which means that the collection's total momentum is zero. But the individual particles can be moving in this frame, so the sum of their kinetic energies is not zero. The difference between adding kinetic energies and momenta is that the kinetic energy of every particle is *positive*, but momentum vectors of multiple particles add like vectors, and can cancel to zero.

The short summary of this is that in the center of mass frame there is zero momentum for the collection, but there is still non-zero internal energy. Let's see how it works mathematically with a simple two-particle system, like we discussed in Section 3.2. We start by defining the momentum of the particle as $\vec{p}_1$ and $\vec{p}_2$. The total momentum of the two-particle collection is just the sum of these:

$$\vec{p}_{cm} = \vec{p}_1 + \vec{p}_2 \tag{4.1.7}$$

The *total* energy of this group of particles is just the sum of their kinetic energies:

$$E_{tot} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \tag{4.1.8}$$

The kinetic energy of the group (defined in terms of its total mass and center-of-mass speed) is:

$$KE = \frac{p_{cm}^2}{2\left(m_1+m_2\right)} = \frac{\left(\vec{p}_{cm}\cdot\vec{p}_{cm}\right)}{2\left(m_1+m_2\right)} = \frac{\left(\vec{p}_1+\vec{p}_2\right)\cdot\left(\vec{p}_1+\vec{p}_2\right)}{2\left(m_1+m_2\right)} = \frac{\left(p_1^2+p_2^2+2\vec{p}_1\cdot\vec{p}_2\right)}{2\left(m_1+m_2\right)} \tag{4.1.9}$$

The internal energy is the total energy minus the group's (center of mass) kinetic energy, so using the last two equations we get, after some algebra:

$$E_{int} = E_{tot} - KE = \frac{1}{2\left(m_1+m_2\right)}\left[\frac{m_2}{m_1}p_1^2 + \frac{m_1}{m_2}p_2^2 - 2\vec{p}_1\cdot\vec{p}_2\right] \tag{4.1.10}$$

The general features of this equation match what we already know about internal energy for a two-particle combination. For example, if the particles are moving in the same direction, then they are moving slower in the center of mass frame, and the internal energy is smaller than if they happened to be moving the same speeds toward each other. This fact is reflected in the last term above – if $\vec{p}_1$ and $\vec{p}_2$ are pointing in the same direction, then the dot product is positive, and the minus sign indicates that the internal energy is reduced. If the two momentum vector directions are in opposite directions, then the dot product is negative, and the last term increases the internal energy.

One thing that is not obvious is whether this equation allows for a negative internal energy for some choice of momentum vectors. This is of course impossible physically, as the smallest the internal energy can be is zero, when the particles are stationary relative to each other. Let's check this in the following way: Start with one particle having a momentum of $\vec{p}_1$, and then let the internal energy be a function of the momentum $\vec{p}_2$ that we choose for the second particle. Then minimize the internal energy function with respect to this variable. Obviously the minimum will occur when the negative term is as large as possible, and this occurs when $\vec{p}_1\cdot\vec{p}_2 = p_1p_2$, so we have:

$$E_{int}\left(p_2\right) = \frac{1}{2\left(m_1+m_2\right)}\left[\frac{m_2}{m_1}p_1^2 + \frac{m_1}{m_2}p_2^2 - 2p_1p_2\right] \tag{4.1.11}$$

Taking a derivative of this function with respect to p_2 (remember, p_1 was selected at the beginning, so it is not varying) and setting it equal to zero gives:

$$0 = \frac{dE_{int}}{dp_2} = \frac{1}{2\left(m_1+m_2\right)}\left[2\frac{m_1}{m_2}p_2 - 2p_1\right] \quad \Rightarrow \quad \frac{m_1}{m_2}p_2 = p_1 \tag{4.1.12}$$

Noting that $\frac{p}{m} = v$, this result gives that the internal energy is a minimum when $v_1 = v_2$ – both particles moving at the same speed in the same direction. We can confirm by plugging this back in that this gives us the expected result of zero internal energy, so indeed it can never be negative.

Systems and Momentum Conservation

Let's return to following the trajectory of our discussion of work-energy, by revisiting the notion of a system. As before, we define a system as an arbitrarily-grouped collection of objects (which themselves can have internal energies), that can experience forces between themselves, or from outside the system. Previously we said that forces between objects within the system were responsible for internal work and forces exerted on objects within the system from outside provide external work. We will now similarly define internal impulse as coming from forces between objects within the system, and external impulse as coming from objects outside the system.

When it comes to forces between objects within our defined system, we know that the work done on one object does not cancel the work done on the other object. If the internal force is conservative, then the non-zero total work done between the objects can be accounted-for through a change of potential energy. If the internal force is non-conservative, then the non-zero total work done between the objects can be accounted-for through a change of thermal energy. Is there an analogous process for impulse?

To answer this question, we need to determine whether impulses internal to a system don't cancel out, as in the case of work. We again start with Newton's 3rd law, which ensures that the two forces involved in creating the pair of impulses are equal-and-opposite. Impulse vectors have the same directions as their associated force vectors, so the 3rd law pair of forces results in a pair of impulses that are in opposite directions. But what about the magnitudes? Well, the force magnitudes are equal thanks to the third law, so all that remains is the time interval. There is never a moment when a force is acting that its third law pair isn't also acting, so the time intervals are the same. This leads to the following very important result: *All of the impulses internal to a system cancel each other out*. This means that there is no momentum analog to potential or thermal energy within a system. There is only momentum, and if the system experiences no external impulses, then momentum is conserved for the system. Comparing to what we got for energy, it looks like this:

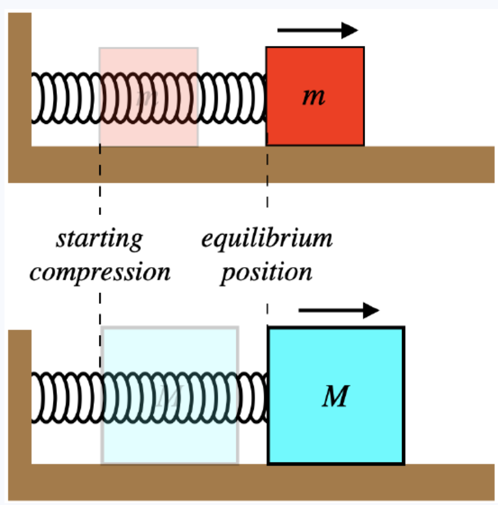
$$\begin{array}{cc} \textbf{work-energy} & \textbf{impulse-momentum} \\ W_{ext} = \Delta KE \underbrace{- W_{cons} - W_{non-cons}}_{from\ internal\ forces} & \overrightarrow{J}_{ext} = \Delta \overrightarrow{p}_{cm} \underbrace{- \overrightarrow{J}_{cons} - \overrightarrow{J}_{non-cons}}_{from\ internal\ forces} \\ W_{ext} = \Delta KE + \Delta PE + \Delta E_{thermal} & \overrightarrow{J}_{ext} = \Delta \overrightarrow{p}_{cm} + 0 \quad + 0 \end{array} \tag{4.1.13}$$

There are two important features of this result:

1. ***It doesn't matter what forces are acting internally***. The result we obtained made no mention of whether the internal force was conservative or non-conservative – all forces satisfy Newton's third law, and the pairs act for equal periods of time, so the impulses cancel regardless of the nature of the force.
2. ***The quantity (momentum) that is conserved within the closed system is a vector***. This means that adding up all of the momentum vectors of a system at one point in time, then doing so again at another point in time, will give the same total vector in both cases, if the system is isolated from external impulses. This means that the total magnitude and direction don't change, or equivalently that the components measured in a given coordinate system don't change.

Conceptual Question

Two blocks of different masses are attached to identical springs that are horizontal to the frictionless surface on which the block rests. If the springs are compressed the same distance and the blocks are released from rest, how do the following quantities compare for the two blocks when they reach the equilibrium point?



a. *kinetic energy*
b. *momentum*
c. *velocity*

Solution

a. The springs are stretched an equal amount, which means they both store the same potential energy. That means that when they get to the equilibrium point where they both have zero potential energy, they must have the same kinetic energy, since the mechanical energy is conserved.

b. We can determine the difference in momenta for the two blocks in two ways. First, we can consider the impulse given to each block by the spring. In the case of the more massive block, the spring force will accelerate it less, which means it will take longer to get to the equilibrium point. At every point during their journeys, the two blocks experience the same amount of force, but since the time interval for the heavier block is longer, it must experience the greater impulse. Therefore the *heavier block gains more momentum, and since both blocks started with zero momentum, the heavier block must have more momentum at the equilibrium point. The second solution is much simpler: We already know that the two blocks end with the same KE, so since $KE = p^2/2m$, the block with the greater mass must have more momentum.*

c. With the same kinetic energy, using $KE = \frac{1}{2}mv^2$, we see that the block with the greater mass must have the lower velocity.

The moral of this story: Although we tend to use kinetic energy, momentum, and velocity as proxies for motion, they are all quite different quantities.

Partial Momentum Conservation

We have to give some extra thought to what we mean by a conserved vector. Since a vector has both magnitude and direction, then to be conserved, both of those properties must remain unchanged. An equivalent way of saying this is that for the vector to be conserved, every component of that vector must be individually conserved. If the full momentum vector is not conserved, it is still possible for one or two of its components to be conserved, if the components of the external impulse in those directions is zero. So for example, a projectile (with no air resistance) conserves momentum in the two horizontal directions, but not in the vertical direction. This allows us to use momentum conservation to solve a much broader range of problems than if we can only consider complete momentum conservation.

4.2: Center of Mass

Center of Mass Again

It should be clear by now how important the concept of center of mass is in classical mechanics. First it appeared in Newton's 2nd law, then in the discussion of internal energy, and now again in the topic of momentum. So far our only exposure to center of mass as a calculated quantity comes from Equation 2.4.10, which we will repeat here:

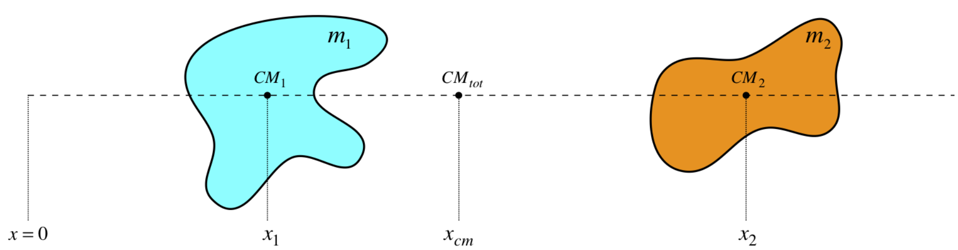
$$\begin{aligned} \overrightarrow{r}_{cm} &= x_{cm}\,\hat{i} + y_{cm}\,\hat{j} + z_{cm}\,\hat{k} \\ &= \frac{\left[m_1x_1+m_2x_2+\ldots\right]\hat{i}+\left[m_1y_1+m_2y_2+\ldots\right]\hat{j}+\left[m_1z_1+m_2z_2+\ldots\right]\hat{k}}{M} \\ &= \frac{m_1\left[x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right]+m_2\left[x_2\hat{i}+y_2\hat{j}+z_2\hat{k}\right]+\ldots}{M} \\ &= \frac{m_1\overrightarrow{r}_1+m_2\overrightarrow{r}_2+\ldots}{M} \end{aligned} \tag{4.2.1}$$

In this chapter, we will have a closer look at this quantity, to see how we can compute it for cases more general than a collection of a few point particles. In particular, we are going to look at objects that we treat as continuous distributions of mass, rather than collections of discrete particles. Of course, real matter *is* a collection of discrete particles, but a continuous model is much more practical to handle mathematically.

Center of Mass of a Collection of Objects

Suppose now we want to know the center of mass of multiple extended objects, where all the heavy-lifting has already been done – the centers of mass of the objects are already known (see below for how to do this heavy-lifting). How do we determine the center of mass of such a system? It turns out to be pretty easy when you know the locations of the centers of mass of the two objects – just treat them as if they are point particles with all of their mass concentrated at their own centers of mass, and then do the calculation above.

Figure 4.2.2 – Center of Mass for Two extended objects



For proof of this, let's treat two extended objects (A and B) as collections of lots of point particles (atoms, if you like), and write down their centers of mass (measured from a common origin) in terms of the masses and positions of their atoms.

$$\left.\begin{aligned} \overrightarrow{r}_{cm\ A} &= \frac{m_{1A}\overrightarrow{r}_{1A}+m_{2A}\overrightarrow{r}_{2A}+\ldots}{M_A} \\ \overrightarrow{r}_{cm\ B} &= \frac{m_{1B}\overrightarrow{r}_{1B}+m_{2B}\overrightarrow{r}_{2B}+\ldots}{M_B} \end{aligned}\right\} \quad\Longleftrightarrow\quad \overrightarrow{r}_{cm} = \frac{M_A\overrightarrow{r}_{cm\ A}+M_B\overrightarrow{r}_{cm\ B}}{M_A+M_B} \tag{4.2.2}$$

The left-hand side equations are those of the center of mass for each object in terms of its atoms' masses and positions. The right-hand side gives the center of mass of the two-object system in terms of the masses of the objects and the positions of their individual centers of mass. When the expressions for $\overrightarrow{r}_{cm\ A}$ and $\overrightarrow{r}_{cm\ B}$ from the left side are plugged into the right-hand side equation, then all the atoms of both objects are come together into a single center of mass formula, as if they were part of a single system with total mass M_A+M_B, proving the contention above.

Exercise

Two thin circular disks made from the same material lie flat on a horizontal surface, with their outer edges in contact with each other. One disk has a larger radius (R) than the other (r), and have equal thicknesses. Find how far the center of mass of the two-disk system lies from the center of the larger disk.

Solution

The disks are made from the same uniform material, so they have equal mass densities. That means that the mass of the larger disk is larger than that of the smaller disk by the same factor as the ratio of their areas. That is, if the larger disk has twice the area of the smaller one, then it has twice as much mass. We therefore have the following relationship between the masses and radii of the disks:

$$\frac{M}{m}=\frac{\pi R^2}{\pi r^2} \quad \Rightarrow \quad M=\frac{R^2}{r^2}\,m$$

Let's choose the center of the larger disk as the origin, and have the center of the other disk lie on the $+x$-axis. The disks are uniform, so their individual centers of mass lie at their geometric centers, and we can compute the center of mass of the system by treating the disks as point masses located at these centers. The distance of the center of mass from the origin is what we are looking for, so:

$$x_{cm}=\frac{Mx_1+mx_2}{M+m}=\frac{M\left(0\right)+m\left(R+r\right)}{M+m}=\frac{m\left(R+r\right)}{\frac{R^2}{r^2}m+m}=\frac{\left(R+r\right)r^2}{R^2+r^2}$$

We can double-check this answer by looking at an obvious special case: $R=r$. If the disks are identical, then the center of mass must be halfway between their centers, which is the point where they are in contact, a distance R from the center of the larger disk. Plugging in R for r indeed gives this answer.

Center of Mass of Continuous Objects

We now turn to the problem of computing the position of the center of mass of an object whose distribution of mass is known. What follows is pure math, but it is important math that returns over and over in physics.

Alert

The important thing to gain from this discussion is to understand how the set-up process works. It culminates in an integral, but performing the integral is mere busywork compared to the task of setting it up. It's easy to be overwhelmed by the thought of the integral that is being constructed, but if you understand each step that leads up to it (and don't try to just jump to an answer that looks like something you have seen before), it will go fine.

We will keep this simple by restricting ourselves to objects for which the position of the center of mass in two of the three dimensions is obvious, which means we don't need to concern ourselves with the whole vector described in Equation 2.4.10 – just the x-component will do. A good model for this is a simple thin, cylindrical rod. This rod's mass distribution is completely cylindrically symmetric, which means that the center of mass lies on the axis passing through its center. But the mass distribution as a function of position on this axis may not be uniform. For example, it may be more dense on one end than on the other. Put another way, the particles located within the rod may be packed together more tightly in one region of the rod than in another, which means that the center of mass will not necessarily lie at the point halfway between the ends.

We need to say a few words about *mass density* before we proceed. Density is a measure of how closely-packed in space a quantity of something is. This quantity can be many different things. Here we will be considering mass, but in later physics classes you will deal with density of electric charge (and even, bizarrely, probability!). A *uniform density* for a region in space means that the quantity (whatever it happens to be) is evenly-distributed everywhere within that region. The way we define an average density for a region in space is to add up how much "stuff" is there, and then divide it by the total space it occupies. This gives an average density, but of course densities can vary from one point in space to another, in which case a *density function* is defined. We will deal with only the simplest variable densities here. As we will mainly be looking at thin rods for our examples, we will only consider densities that might vary along the length of the rod – this simplifies the process to a single dimension.

The mass density function in this case is a function of a single variable, has units of $\frac{kg}{m}$, and is called a *linear mass density*. This mass density function is typically denoted as $\lambda\left(x\right)$. If it is uniform, then the function is a constant λ, and the amount of mass m

within a given length l is simply given by:

$$m = \lambda l \tag{4.2.3}$$

If the density is not uniform, then it is only a constant over an infinitesimal length dx, so the equation above can only apply to a tiny piece of mass dm, and the relationship is different at every position x because the density is different at every position:

$$dm = \lambda\left(x\right)dx \tag{4.2.4}$$

Now that we can write down how much mass is at every position, we are ready to do our calculation. We begin by drawing a diagram with the rod in a coordinate system along the x-axis such that one end is at the origin and the other is at $x = L$. The figure below provides a fully-labeled diagram that is very helpful for solving such problems.

Figure 4.2.3 – Setup Diagram for Computations Involving Mass Density of a Thin Rod

The center of mass is found by multiplying the amount of mass at each point by the x-coordinate of that mass, then adding up all of those products and dividing by the total mass. Of course, in this case we have an infinite number of point masses, so the sum is infinitely long, but the masses are infinitesimally small, so we solve this by converting the sum into an integral, in which we add up all the pieces from $x = 0$ to $x = L$:

$$x_{cm} = \dfrac{dm_1x_1 + dm_2x_2 + \dots}{dm_1 + dm_2 + \dots} = \dfrac{\int\limits_{x=0}^{x=L} dm\ x}{\int\limits_{x=0}^{x=L} dm} \tag{4.2.5}$$

Now we plug in Equation 4.2.4 to give the following formula for center of mass (in one dimension) for a thin rod with a linear mass density that varies with x:

$$x_{cm} = \dfrac{\displaystyle\int\limits_{x=0}^{x=L} \lambda\left(x\right)x\,dx}{\displaystyle\int\limits_{x=0}^{x=L} \lambda\left(x\right)dx} \tag{4.2.6}$$

Okay, so let's do a couple of examples...

A Uniform Rod

As was stated above, if the rod is uniform, then the density is a constant (which we will call simply λ). Plugging this into Equation 4.2.7 leads to a simple calculation and an unsurprising result:

$$x_{cm} = \dfrac{\displaystyle\int\limits_{x=0}^{x=L} \lambda x\ dx}{\displaystyle\int\limits_{x=0}^{x=L} \lambda\ dx} = \dfrac{\cancel{\lambda}\displaystyle\int\limits_{x=0}^{x=L} x\ dx}{\cancel{\lambda}\displaystyle\int\limits_{x=0}^{x=L} dx} = \dfrac{\left[\frac{1}{2}x^2\right]_0^L}{\left[x\right]_0^L} = \dfrac{1}{2}L \tag{4.2.7}$$

So we have calculated what we already knew – that for a thin rod with a uniform mass density, the center of mass is at its center (which on our coordinate system lies at $x = \frac{1}{2}L$).

A Non-Uniform Rod

Next we'll look at an example of a rod which has a mass density that varies from one end to the other. This variable density is expressed in its density function:

$$\lambda\left(x\right)=\lambda_o\left(\frac{x}{L}+1\right) \tag{4.2.8}$$

Before we do the math, let's try to make sense of this function. The easiest way to do this is to consider the endpoints. At $x=0$, the density equals the constant λ_o, while at $x=L$ that density has grown to twice that much. This increase of density happens linearly with the variable x. What should we *expect* to see when we compute the center of mass? Well, the rod is more dense near the $x=L$ end than the $x=0$ end, so the center of mass should be at an x value greater than $L/2$. Okay, so let's plug the density function into Equation 4.2.6 and see what we get:

$$x_{cm}=\frac{\int\limits_{x=0}^{x=L}\left[\lambda_o\left(\frac{x}{L}+1\right)\right]x\,dx}{\int\limits_{x=0}^{x=L}\left[\lambda_o\left(\frac{x}{L}+1\right)\right]dx}=\frac{\cancel{\lambda_o}\int\limits_{x=0}^{x=L}\left(\frac{x^2}{L}+x\right)dx}{\cancel{\lambda_o}\int\limits_{x=0}^{x=L}\left(\frac{x}{L}+1\right)dx}=\frac{\left[\frac{x^3}{3L}+\frac{x^2}{2}\right]_0^L}{\left[\frac{x^2}{2L}+x\right]_0^L}=\frac{5}{9}L \tag{4.2.9}$$

Interestingly, the center of mass doesn't depend upon the density constant λ_o.

Analyze This

Two identical rods of mass M and length L have the same non-uniform density profile. When one of these rods is placed along the x-axis with one end of the rod at the origin, the density as a function of x is proportional to the following function:

$$\lambda\left(x\right)\propto\left(\frac{x^2}{L^2}+1\right)$$

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

The first thing we can do is determine the constant of proportionality in terms of the mass and length of the rods. Calling this constant λ_o, we have:

$$\lambda\left(x\right)=\lambda_o\left(\frac{x^2}{L^2}+1\right)$$

The total mass of the object is the density integrated over the whole length of the rod:

$$M=\int\limits_{x=0}^{x=L}dm=\int\limits_0^L\lambda\left(x\right)dx=\lambda_o\int\limits_0^L\left(\frac{x^2}{L^2}+1\right)dx=\lambda_o\left[\frac{x^3}{3L^2}+x\right]_0^L=\frac{4}{3}\lambda_oL\quad\Rightarrow\quad\lambda_o=\frac{3M}{4L}$$

Next we can compute the position of the center of mass of a rod with the lower-density end placed at the origin:

$$x_{cm}=\frac{\int\limits_{x=0}^{x=L}x\,dm}{M}=\frac{\int\limits_0^L x\lambda\left(x\right)dx}{\frac{4}{3}\lambda_oL}=\frac{3}{4\lambda_oL}\int\limits_0^L x\lambda_o\left(\frac{x^2}{L^2}+1\right)dx=\frac{3}{4L}\left[\frac{x^4}{4L^2}+\frac{x^2}{2}\right]_0^L=\frac{9}{16}L$$

Objects with More Dimensions

We have only discussed the simplest of continuous objects - thin rods that are more-or-less one-dimensional, and computing their centers of mass requires only an integral over a single variable. Real-world objects are three-dimensional, and computing their centers of mass is more complicated in two ways. First, the density function can be a function of three variables, rather than just one. And second, integration of the mass elements requires a three-dimensional integral. We will not go into the details of these sorts of calculations here, as they are heavily steeped in mathematics with very little physics content. The reader can expect to start encountering these types of integrals (in a different context – not center of mass) we they get to Physics 9C.

4.3: Momenta of Systems

Using Momentum Conservation

When we examined the work-energy theorem, we found that it was not much more than a reformulation of Newton's 2nd Law for cases where we are only interested in speed (not direction) changes. As such, it had only limited usefulness. But when we went a little deeper, we found that this theorem spawned a very useful "shortcut" (the principle of energy conservation) that allowed us to solve certain types of problems much more easily than we could otherwise. We have already expressed a conservation principle for momentum, but let's do so again here, comparing it to the familiar counterpart in energy.

Energy: In isolated system (one where there is not external work being done on any of the objects in it), the total energy of the system remains constant.

$$\underbrace{KE+PE+E_{thermal}}_{\text{before}}=\underbrace{KE+PE+E_{thermal}}_{\text{after}} \tag{4.3.1}$$

Momentum: In isolated system (one where there is not external impulse delivered to any of the objects in it), the total (vector) momentum of the system remains constant.

$$\vec{p}_{cm}\,(\text{before})=\vec{p}_{cm}\,(\text{after}) \quad \Rightarrow \quad \underbrace{\vec{p}_1+\vec{p}_2+\dots}_{\text{before}}=\underbrace{\vec{p}_1+\vec{p}_2+\dots}_{\text{after}} \tag{4.3.2}$$

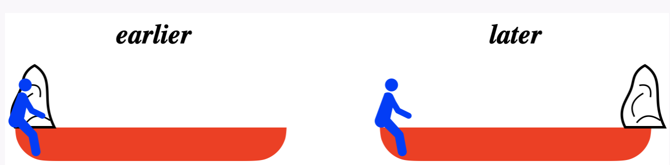
The sum on each side is over the several objects in the system. So adding up the momentum vectors of all the objects before some event, and then doing it again after the event gives the same vector. This of course assumes that the "event" does not involve an external impulse, though it can include as many internal impulses as you like.

Alert

It is important to remember that this equation does not mean that each of the terms remains unchanged. Rather, they change in such a way that the changes all compensate for each other, and the vector sum of the all-new momentum vectors comes out to the same that came out before.

Analyze This

A child sits on the rear end of a sled (whose mass is uniformly-distributed along its length) with a block of frozen snow at rest in her lap. The sled is sliding forward on the horizontal, frictionless snow at constant a speed, when the child suddenly shoves the block forward in the sled (she remains firmly planted on the sled). After a period of time, the block comes to rest in the front of the sled.



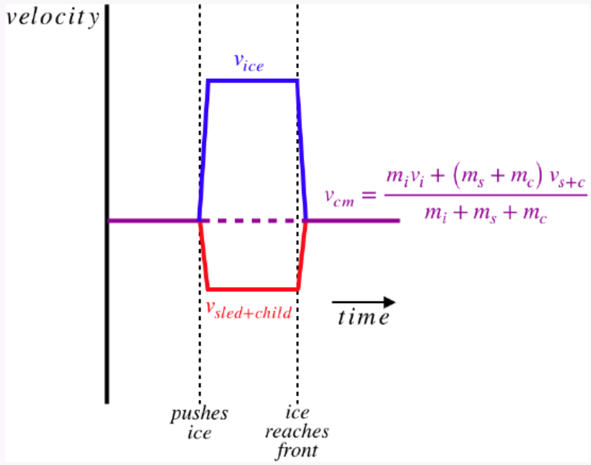
Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

The forces between the girl, the block of snow, and the sled are all internal to that system of those three objects. With no friction coming from the snow, this means that there are no external forces on this system, and its total momentum remains unchanged. This means that the center of mass of the system of the child, the sled, and the ice, continues sliding at the same constant rate as before. This does not mean that the sled+child combination slides at the same rate throughout this process, because the increased speed of the ice means that the remaining mass of the system must change speed to keep the center of mass speed unchanged.

Once the ice reaches the front of the sled, however, the whole system is moving at the same speed again, which means that it returns to the speed it had before the ice was pushed. Without knowing the masses of the parts of this system, we don't know the specific effect of the sliding ice – it could just slow the sled+child, stop the sled+child, or even cause them to move backward. The graph of speeds as a function of time below expresses this well:



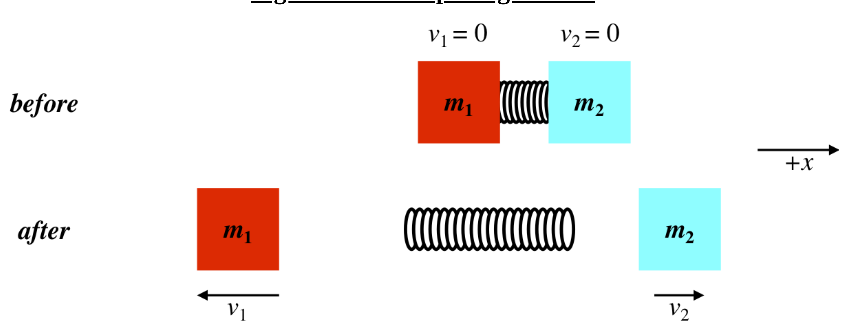
A few things to note here:

- *The graph assumes that the mass of the sled+child is greater than the mass of the ice, because the internal force between them delivers the same impulse to both, which means that they change their momenta by the same amount. The graph shows the speed of the ice changing more, so to experience the same change of momentum, it must have less mass.*
- *Even though the ice and sled+child have different speeds for a short period of time, even during that time, the speed of the center of mass doesn't change (depicted by the dotted purple line).*
- *We don't know the actual speeds, so we can't place the time axis on the graph. If it happens to coincide with the horizontal red line segment, then the sled+child come to rest while the ice slides forward. If the time axis is above this horizontal red line (it must be below the purple line, as we have defined the starting velocity to be in the positive direction), then the sled+child actually moves backward while the ice slides forward.*

Using Center of Mass

Let's look at an example of how we can use what we know about center of mass the analyze a case of two blocks of different masses that squeeze a (massless) spring between them until they are released from rest.

Figure 4.3.1 – Repelling Masses



Intuitively one can probably tell that for this situation $m_2 > m_1$. When a light object pushes off a heavy one (a flea jumping off a dog, a bullet leaving a gun, etc.), the lighter object's motion is always affected more. With our physics training, we can explain it with Newton's second and third laws: The blocks push on each other with equal forces (third law), and with equal forces, the block with less mass will accelerate more. They both start from rest and are pushed for equal periods of time, so the one with the greater acceleration will be going faster when they separate, sending it a greater distance in the same time period.

Okay, now let's look at it from the perspective of momentum conservation. Treating the two blocks as a single system, the spring force produces only internal impulses, which means that the momentum of the system is conserved. The momentum before the spring unloads is zero, so it must be zero afterward. If v_1 and v_2 are the speeds of the two blocks (i.e. these are positive numbers), then we have for our conservation equation:

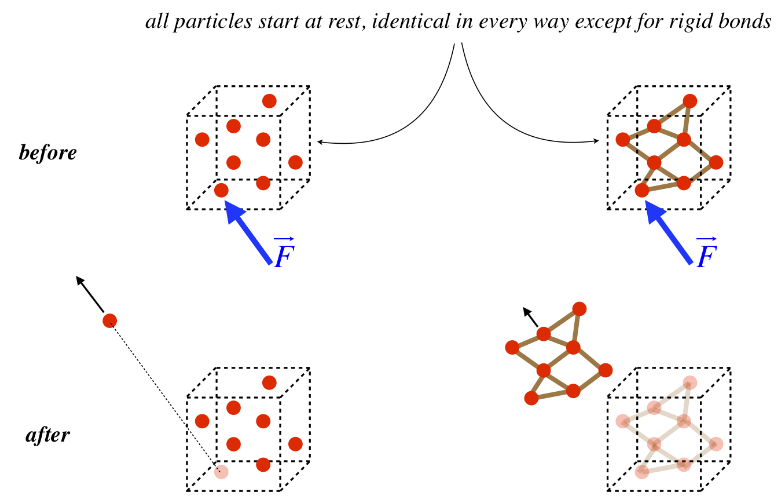
$$\text{momentum before} = \text{momentum after} \quad \Rightarrow \quad 0 = m_1v_1\left(-\hat{i}\right) + m_2v_2\left(+\hat{i}\right) \quad \Rightarrow \quad v_1 = \frac{m_2}{m_1}v_2 \tag{4.3.3}$$

Since it's clear from the diagram that $v_1 > v_2$, it must be that $m_2 > m_1$. We can also use what we know about center of mass here. The system experiences no external net impulse and its center of mass is stationary, so it must remain stationary! We don't know exactly where the center of mass is before the repulsion, but since it stays put, we can draw a vertical line down into the second diagram to find where it is after the repulsion. This clearly results in the center of mass being closer to m_2, which means that is the larger mass.Center of Mass Acceleration

Let's see if we can incorporate what we have learned about center of mass to make sense of Newton's second law. Consider the two systems shown in Figure 4.3.2. Each consists of a collection of 8 identical particles in close proximity to each other (the boxes shown are just used as a reference for later motion – they are not physical objects). In the left system, the particles are floating freely (there is no gravity or other forces), while in the right diagram, the particles are bound together with rigid, massless rods. The two systems are identical in every way except for the presence of these rods – the particle all have the same positions and masses as their counterparts, and are all at rest.

Now for the experiment: Suppose we exert the same force on the same particle in both systems. Clearly the reaction is different in the two cases – in the left case, only the particle given the push accelerates away, while in the right cases the entire group of particles accelerates. The question is, in which case does the center of mass of the system of particles accelerate *more*?

Figure 4.3.2 – Forces on Free and Rigid Systems

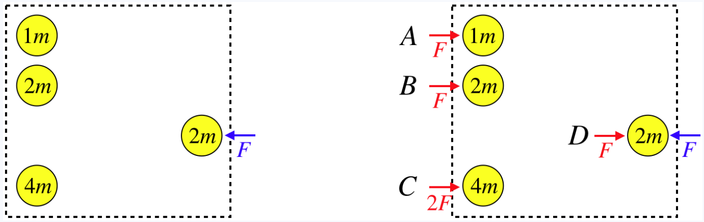


Here is the short answer: The forces that are (or are not) between the particles defining the system are internal, and therefore have no effect on the velocity of the system's center of mass. The only external force on each system is $\overrightarrow{F}$, and each system has the same mass, so Newton's second law says that both systems should react with the same acceleration of their center of mass.

But that is unconvincing when we see only one particle move in one case, and the whole conglomerate move in the other! Let's suppose the forces act for some small period of time. The acceleration of the single particle will be eight times greater than that of the conglomerate, so in the same time interval it will move eight times as far as the conglomerate. Let's call the initial position of the center of mass the origin. The seven particles left behind experience no change in their position relative to this origin, and the one particle's position relative to the origin travels eight units of distance, while all eight of the particles in the other system travel just one unit from their original positions relative to the origin. Treating the direction of motion as the $+x$ direction, and plugging the masses and distances into Equation 4.2.1, it should be immediately clear that both centers of mass move by the same amount. As strange as it sounds, Newton's second law works for any system of particles, whether they bond together to form a solid object, or are completely independent of each other, like particles in a gas.

Conceptual Question

A system of four balls of varying relative masses is shown in the left diagram below, and there is a force exerted on one of the balls as indicated. In the right diagram are a few options for other forces that can be exerted on balls in this system. Which of *these forces will assure that the center of mass of this system does not accelerate? The forces shown are the only forces present (i.e. there is no gravity or other forces to worry about here).*



a. *D only*
b. *B or D only*
c. *B, C, or D only*
d. *A, B, or D only*
e. *There needs to be a force F exerted to the right on every ball at at the same time.*

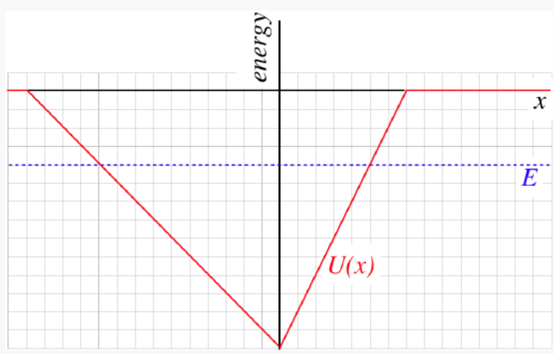
Solution

d

For the center of mass to not accelerate, the net force on the system must be zero. This means that a force must be applied to the system in the opposite direction. It doesn't matter where in the system this force is applied.

Analyze This

Two different particles are confined by the same potential, shown in the diagram. Both particles have the same total energy, also depicted in the diagram. At one moment the particles pass each other precisely at the origin, with one particle moving in the $-x$-direction and the other moving in the $+x$-direction.



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- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

There is a lot unpack here. We'll start by labeling the particles: The particle moving in the $-x$ direction we'll call "particle A", and the one moving the other way "particle B". Clearly at the moment that they cross each other at the origin, the center of mass is at the origin. But does it remain there as they continue moving away from each other?

To answer this question, we consider the net force on the two particle system. If it is zero, then the center of mass does not accelerate. The system experiences two forces, one on each particle. These forces can be computed from the slope of the potential energy function $U(x)$. The left side of the potential curve affects particle A (pushing it in the $+x$-direction), and the right side affects particle B (pushing it in the $-x$-direction). The slopes of the two sides are not equal, so the forces on the particles are not equal, which means that there is in fact a net force on the two-particle system, and the center of mass is accelerating. With more force being applied in the $-x$-direction, the center of mass is accelerating in that direction. We even have enough information to determine the ratio of these two forces, thanks to the grid lines:

$$\left.\begin{array}{l}\text{force on particle A} = F_A = -\text{slope of left segment} = 1 \; unit \\ \text{force on particle B} = F_B = -\text{slope of right segment} = -2 \; units\end{array}\right\} \quad \left|\frac{F_B}{F_A}\right| = 2$$

Does this mean that a short time later the center of mass is on the $-x$ side of the origin? No! We don't know which way the center of mass is moving when it is at the origin. If it is stationary or moving in the $-x$-direction, then of course the center of mass speed is increasing in the $-x$-direction and the center of mass will later be on the $-x$ side. But the center of mass may be moving in the $+x$-direction, which would mean that it is slowing down, but will still be on the $+x$ side a short time later.

Is there any way to know the direction of the center of mass motion when both particles are at the origin? Another way to ask this is, in which direction is the momentum of the system? One thing we do know is that both particles have the same total energy, and when they are both at the origin, they have the same potential energy as well. This means that at the moment they pass each other, they have equal kinetic energies. We know a relationship between kinetic energy (Equation 4.1.6):

$$KE = \frac{p^2}{2m} \quad \Rightarrow \quad p = \sqrt{2m(KE)}$$

With both particles having the same kinetic energy, the particle with more mass is the one with more momentum, and when these momenta are summed, the direction of motion of the particle with more mass is the direction in which the center of mass is moving.

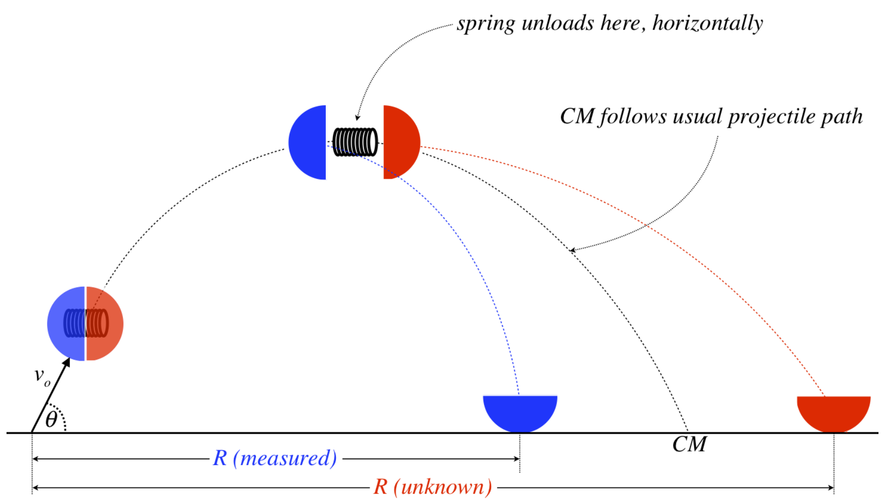
Center of Mass Frame

Sometimes analysis of problems that involve multiple objects interacting with each other is simplified by using what is called the *center of mass frame* of reference. Here’s an example.

A child's toy called a "hot potato" consists of two hemispherical shells that close on a spring and are held together by a latch on a timer. When the time expires, the latch is released and the spring is allowed to expand, shooting the two shells in opposite directions, exposing the toy company to a product liability lawsuit from the family of the child that holds the hot potato when it goes off.

Let's suppose a child throws this hot potato through the air, and the peak of its projectile motion, it explodes so that the two shells are propelled horizontally, as shown in Figure 4.3.3. The landing point of the shell that lands closest to where the toy was thrown is noted, but the other shell flies off into some tall grass and is lost. Naturally the child knows the starting speed they gave the toy as well as the exit angle, and she can easily measure the distance that the closer shell travels form the launch point. From this information and her vast knowledge of physics, she conceives of a plan to find the other shell that is far more elegant than searching for it in the tall grass.

Figure 4.3.3 – Exploding Projectile

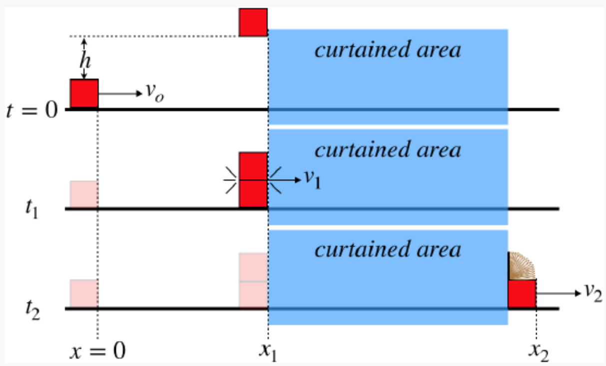


The forces on the shells by the spring are internal to the two-shell system, so assuming air resistance is negligible, the center of mass of the system will behave exactly as it would if the internal forces didn't exist. With the starting angle and speed known, the child can use the range equation (see Example 1.7.4) to calculate the landing point of the center of mass of the system. Then with the actual landing point of one piece of the toy, she can use the center of mass formula to compute the landing point of the other piece.

You might think we can do the same even if the spring unloads in an orientation that is other than horizontal, but this is not the case. The center of mass motion still follows the same parabolic trajectory, but naturally the center of mass is always between the two shells. In the case above, the shells land simultaneously (they both start with zero vertical component of velocity when the explosion occurs), so the center of mass lands at the same time, between the shells. When the explosion is not horizontal, one shell lands before the other, then friction stops is horizontal motion while the other shell keeps moving horizontally. This makes calculating the landing point of the center of mass using the usual range equation impossible.

Analyze This

A block slides along a frictionless horizontal surface at a speed v, starting at position $x = 0$ and time $t = 0$. An identical block dropped from rest lands directly on top of it. The surfaces of the blocks are sticky, so the top block adheres to the bottom block when it lands on it, and they continue along together. The blocks slide together into a curtained-off area, during which a spring noise and a "thud" are heard. At a later time, the bottom block emerges from the curtain without the top block on it, after apparently having its top lid sprung open from within.



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Analysis

The collision of the falling block and the sliding block is an example of a case where the momentum of the a system is only conserved for one component. The blocks experience internal forces (normal force and static friction), and these have no effect on the two-block system momentum, but the horizontal surface pushing up on the bottom block is external, so although the two-block system had downward momentum just before the landing, the external force provides an impulse to take it away. However, the surface is frictionless, so there is no external force along the x-direction, and momentum is conserved along that direction. We can apply momentum conservation along that direction to write their combined speed v_1 in terms of the bottom block's initial speed v_o:

$$mv_o + m(0) = 2mv_1 \quad \Rightarrow \quad v_1 = \frac{1}{2}v_o$$

We don't know the details of what happens behind the curtain – we don't know when the lid of the lower box sprung open, or where ((x\) position) it happened. But assuming that there are no external forces occurring behind the curtain, we can assume that the velocity of the center of mass of the two-block system along the x-direction is unchanged – the internal force from the spring-loaded box does not affect the center of mass motion.

We actually know the speed of the center of mass of the system behind the curtain, since both blocks are moving at the same speed as they enter. It is v_1, computed in terms of v_o above:

$$v_{cm} = \frac{1}{2}v_o$$

If we happen to know the speed v_2 at which the bottom block emerges from the curtain, then we can use the known center of mass speed and speed of the bottom block to derive information about the other block.

$$v_{cm} = \frac{mv_{top} + mv_2}{2m} \quad \Rightarrow \quad v_{top} = 2v_{cm} - v_2 = v_o - v_2$$

If we are given more information like the times t_1 and t_2 and the positions x_1 and x_2, we can derive more than just relationships between speeds. For example, with the whole system located at x_1 at t_1, and knowing its center of mass speed, we know where the center of mass is at the later time t_2. And combining this knowledge with the position of the bottom block (x_2) allows us to locate the position of the top block, even though it is hidden behind the curtain.

Rocketry

While we are on the topic of two parts of a system going their separate ways by pushed off each other, this brings us to the topic of rocketry. A rocket that is stationary in space somehow is able to accelerate itself by firing its engines. How can the center of mass of the rocket system accelerate without any external forces acting on it? Well, it can't of course, but the rocket (or rather, its fuselage) is not an isolated system. It expels fuel (in the form of very hot gas) backward. If we include the fuel as part of the system, then the center of mass of the system doesn't accelerate at all! All that matters in the end is that the fuselage of the rocket is propelled forward. Note also that the rocket has more mass than the fuel, but the ignited fuel sends particles away at very high speeds, and this momentum balances the momentum of the fuselage in the opposite direction (which has more mass and lower velocity).

4.4: Momentum and Energy

Collisions

For the remainder of this chapter we will focus on what is the most common application of momentum conservation – collisions. A collision event can be a very complicated process, with lots of different objects taking part and interacting with each other with all sorts of different forces. Momentum conservation is very useful in these cases, because if every object involved is included in the system, the total incoming and outgoing momenta are equal, since the complicated *internal* forces don't affect the system's total momentum. This allows us to focus only on "before" and "after" conditions, as we can ignore what goes on in the intervening time.

In studying collisions, we will naturally not start with wildly complex cases. As always, we will start as simply as possible – with a head-on collision between one moving "projectile" object and another stationary "target" object. Because the collision is head-on, the motions of the objects will remain along the same line as the original motion of the projectile object. Later, we will allow both objects to move as they enter the collision, and we will extend the geometry of the collisions to include motions in two dimensions.

If the collisions are between two individual particles, then when they collide there is no way to move any of the original energy in the form of kinetic energy of the particles into internal energy (an object must consist of more than one particle to have internal energy). This means that the sum of the kinetic energies of the particles before the collision must equal the sum of their kinetic energies after the collision. If the collision is between two objects (collections of particles), then there is no way to avoid introducing internal energy to these objects, but if the particles in these objects are held in their positions very rigidly, then very little internal energy is introduced into the objects, and to a very good approximation the same "$\sum KE_{in} = \sum KE_{out}$" property that applies to particles also applies to macroscopic objects. We call such collisions *elastic*. Collisions between objects where this is not the case we call *inelastic*.

System Rest Frame Energy

In Section 3.2, we discussed the concept of internal energy for a collection of particles. We defined this as the total energy (kinetic and potential) of a group of particles as measured in the reference frame of the collection. We eventually noted that this internal energy virtually always manifests itself as thermal energy – energy randomly and unpredictably distributed amongst a large number of particles. Now that we are talking about collections (which we now call "systems") of larger objects, we will find it useful to introduce a concept similar to internal energy for these systems. If we define a group of objects to be a closed system, then we can define this system's *rest frame energy* as the total energy of that system measured in the reference frame where the system's center of mass is at rest.

Clearly this definition is identical to that of internal energy for particles, but we give it a new name to distinguish it from the case where we don't ever actually look at the detailed motions and potential energies between the constituent particles. The reason for defining this quantity at all will become obvious in its usefulness shortly. The main thing we need to keep in mind is that this quantity can only change if forces act on the system from outside – interactions between objects in the system change the forms of energy within the system, but never the amount of rest frame energy. In this regard it has much in common with momentum, and we will next see how intertwined these two physical properties are.

"Perfectly" Inelastic Collisions

Let's solve a simple collision problem using conservation of momentum, and make an accounting of what happens to the energy as a result of this collision. This collision consists of two clay balls. Initially ball #1 is moving toward ball #2, which is stationary. The collision is direct, and the two lumps of clay deform and mash together into one mass of clay, moving-off together. Collisions of this kind where the two objects end up with the same velocity at the end are called *perfectly* (or *totally*) *inelastic*.

Figure 4.4.1 – Perfectly Inelastic Collision

If we know the two masses m_1 and m_2 and the incoming velocity v_1, momentum conservation gives us the final velocity V:

$$\underbrace{p_1 + p_2}_{\text{before}} = p_{\text{after}} \quad \Rightarrow \quad m_1 v_1 + 0 = MV \quad \Rightarrow \quad V = \frac{m_1}{m_1+m_2} v_1 \tag{4.4.1}$$

Comparing this to Equation 3.2.3 (with $v_2 = 0$), we see that V is just the center of mass velocity. This is actually not that surprising. With no external forces, the center of mass of this two-object system should not change speed, and since the two balls combined are moving at a single speed, that speed would have to be the center of mass velocity of the system before the collision, and this is exactly what we recognize it to be. Next let's look at what happens to the kinetic energy:

$$\Delta KE = KE_{after} - KE_{before} = \frac{1}{2} MV^2 - \left(\frac{1}{2} m_1 v_1^2 + 0 \right) = \frac{1}{2} (m_1 + m_2) \left(\frac{m_1}{m_1+m_2} v_1 \right)^2 - \frac{1}{2} m_1 v_1^2 = \\ - \left(\frac{m_2}{m_1+m_2} \right) \frac{1}{2} m_1 v_1^2 \tag{4.4.2}$$

The negative sign indicates that kinetic energy is lost during the collision, and the last quantity in parentheses is the original kinetic energy, so the fractional amount of kinetic energy lost is:

$$\text{fraction of KE lost} = \frac{m_2}{m_1+m_2} \tag{4.4.3}$$

Where does this kinetic energy go? Well, the two-object system is isolated, so it cannot simply vanish - it can only change form. In particular, it has to go into the internal energy of the two clay balls. The particles that comprise the clay balls start vibrating faster in a random fashion, i.e. the thermal energy of the clay balls goes up.

Okay, now let's approach the same collision from the perspective of the rest frame energy of the system. The first thing we need is the velocities of the two clay balls before the collision as measured in the center of mass frame. We know the velocity of the center of mass already, so we just need to subtract this from the velocities of the two clay balls to get their velocities in the new frame:

$$\begin{array}{c} \text{starting velocity in cm frame of } m_1 = v_1 - v_{cm} = v_1 - \left(\frac{m_1}{m_1+m_2} v_1 \right) = \frac{m_2}{m_1+m_2} v_1 \\ \text{starting velocity in cm frame of } m_2 = 0 - v_{cm} = -\frac{m_1}{m_1+m_2} v_1 \end{array} \tag{4.4.4}$$

The rest frame energy of this system is the sum of the kinetic energies derived from these velocities (ignoring for now the contribution of whatever thermal energy is in the clay balls prior to the collision):

$$\text{rest frame kinetic energy} = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2} v_1 \right)^2 + \frac{1}{2} m_2 \left(-\frac{m_1}{m_1+m_2} v_1 \right)^2 = \left(\frac{m_2}{m_1+m_2} \right) \frac{1}{2} m_1 v_1^2 \tag{4.4.5}$$

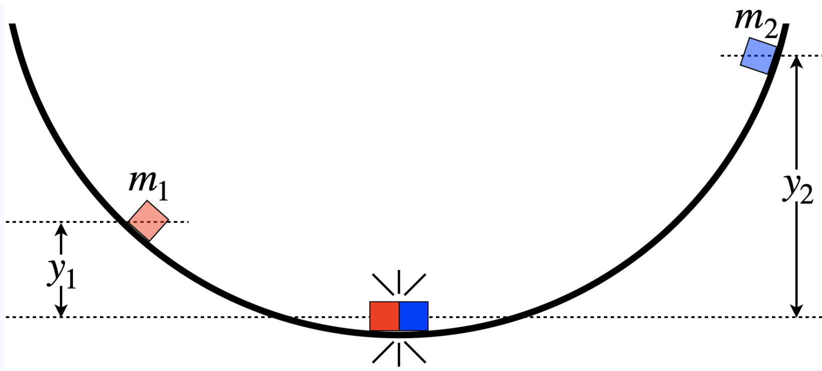
This is exactly the amount of lost kinetic energy that we calculated above. We interpret this result as follows:

1. The system in the "before" state possesses a certain amount of rest frame energy. In this case, this rest frame energy is the sum of the kinetic energies of the clay balls (measured in the center of mass frame) as well as their starting thermal energies.
2. When objects within a closed system collide, the forces involved in the collision reshape the form this energy, but since the system is closed, the total energy remains the same. In this case, the two clay balls stick together and their amalgamation remains at rest (again, in the center of mass reference frame). There is no longer any kinetic energy present – it has all be reshaped into thermal energy, which is added to whatever thermal energy the clay balls started with.
3. In the previous calculation, performed in the frame where clay ball #2 was stationary (often referred to as the *laboratory frame*), the system *still* has the same rest frame energy (this quantity, like internal energy for collections of particles, is intrinsic to the system), and since all of this is converted into thermal energy, we see the total kinetic energy in the laboratory frame drop by exactly this much.

In short, perfectly inelastic collisions simply have the effect of converting all of the rest frame energy from easily observable kinetic energy of macroscopic objects into random, microscopic kinetic and potential energies of particles (thermal energy). One can compute this kinetic energy "loss" by either computing the rest frame energy (by changing reference frames), *or* by using momentum conservation. It is a subtle thing, but these ultimately boil down to the same physical principle.

Analyze This

Two blocks slide down opposite sides of a frictionless curved ramp from different heights, colliding at the exact bottom, as shown in the diagram below. Upon colliding, they stick together, and move as a single entity thereafter (if they move at all).



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Analysis

Although we know the blocks collide at the bottom, we are not given any information about the timing here. They travel different distances to get to the bottom, so they may be released from rest at different times, or m_2 may be given more initial downward speed than m_1, or any number of other possibilities could account for the blocks meeting at the bottom. But we can nevertheless say a few things about what happens here.

The first thing we can say is that if we define the bottom of ramp as $U_{grav}=0$, then all of the energy in the two-block system is kinetic. The total system energy is therefore:

$$E_{tot}=KE_1+KE_2=\frac{p_1^2}{m_1}+\frac{p_2^2}{m_2}$$

We also know that the momentum is conserved for the collision, so the momentum of the two-block combination (which is moving as one mass after the collision) is the sum of the momenta of the two blocks. Calling rightward the $+x$-direction, the value of p_2 below is a negative number:

$$p_f=p_1+p_2$$

If the momentum of the left block at the bottom is greater than the momentum of the right block, then $p_1+p_2>0$ and the combination will continue to the right, and if the right block's momentum is greater, then $p_1+p_2<0$ and it will continue to the left. We can determine how high on the ramp the two-block combination will go (in either direction) by using mechanical energy conservation. At the moment of the collision, all of the double-block's energy is kinetic, and when it stops on the ramp, it is all potential, so conservation demands:

$$KE_o=U_f \quad \Rightarrow \quad \frac{p_f^2}{2M}=Mgy_f \quad \Rightarrow \quad y_f=\frac{(p_1+p_2)^2}{2gM^2}$$

We can speculate about some possible extensions to this problem. One that comes to mind is having the two blocks coming to a dead stop upon colliding. In this case, the final momentum is zero, which means that the blocks have equal magnitudes of momenta at the bottom: $p_1=p_2$. If this is true, then it means that the ratio of the their collision kinetic energies (and therefore their total energies at earlier times is related to the ratios of their masses:

$$\frac{KE_1}{KE_2}=\frac{\frac{p_1^2}{2m_1}}{\frac{p_2^2}{2m_2}}=\frac{m_2}{m_1}$$

So for the blocks to stop dead at the center, if m_1 is n times as much mass as m_2, then block #1 must enter the collision with $\frac{1}{n}^{th}$ as much kinetic energy as block #2. If they both happen to start from rest (at different times, so that they reach the *bottom at the same time), then the kinetic energy comes is directly related to the heights from which they start (since they start with energy that is entirely potential), which gives us a ratio of the starting heights in terms of the mass ratio:*

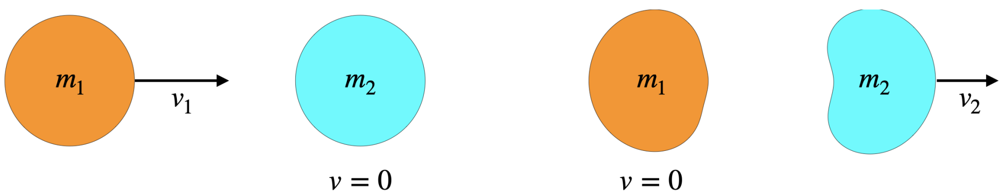
$$\frac{KE_1}{KE_2}=\frac{U_1}{U_2}=\frac{m_1gy_1}{m_2gy_2}=\frac{m_2}{m_1}\quad\Rightarrow\quad\frac{y_1}{y_2}=\left(\frac{m_2}{m_1}\right)^2$$

We can of course speculate about different scenarios as well, such as knowing the starting heights and calculating the final height if the blocks do not stop dead, or perhaps giving the blocks some initial speeds.

Other Inelastic Collisions

As we noted above, all collisions involving objects (as opposed to particles) are to some degree inelastic, as the particles in the object can never be held perfectly rigidly in place. Let's have a look at inelastic collisions where the objects don't stick together and have the same speed at the end. In these cases only *some* of the system rest energy is converted from kinetic energy into thermal energy. Some of it remains in the form of kinetic energy of the objects. This is best demonstrated with a simple example. Consider a collision between two unequal masses where a projectile object hits a target (stationary) object head-on, and the result is that the projectile stops entirely, while the target moves off.

Figure 4.4.2 – Another Inelastic Collision



Let's start by solving for the kinetic energy converted to thermal energy in this collision using momentum conservation. Setting the before and after momenta equal, we get:

$$\underbrace{p_1+p_2}_{\text{before}}=\underbrace{p_1+p_2}_{\text{after}}\quad\Rightarrow\quad m_1v_1+0=0+m_2v_2\quad\Rightarrow\quad v_2=\frac{m_1}{m_2}v_1 \tag{4.4.6}$$

The change in the system's kinetic energy is therefore:

$$\begin{aligned}\Delta KE=KE_f-KE_o=\frac{1}{2}m_2v_2^2-\frac{1}{2}m_1v_1^2=\frac{1}{2}m_2\left(\frac{m_1}{m_2}v_1\right)^2-\frac{1}{2}m_1v_1^2=\left(\frac{m_1}{m_2}-1\right)\left(\frac{1}{2}m_1v_1^2\right)\\=\left(\frac{m_1}{m_2}-1\right)KE_o\end{aligned} \tag{4.4.7}$$

This result tells us a couple of things. First, if the two masses happen to be equal ($m_1=m_2$), then when the incoming object stops and the other continues, all of the energy remains kinetic, as the change equals zero (the collision is elastic). Of course, with the diagram showing deformation of the two objects requires that internal energy is given to the two objects (the particles in the objects are pushed closer together, making them interact differently with each other).

Second, this kind of collision can only have a *loss* of kinetic energy if $m_1<m_2$, so that the change is negative. If there is a spring compressed on object #2, waiting to be triggered by the collision, then it is possible for the energy stored in that spring to go into the kinetic energies of the two objects, making the change in kinetic energy positive. In this case, to have the projectile stop and the target move away, we would need $m_1>m_2$. Most collisions we encounter in the real world do not involve stored potential energy waiting to be released, so we will continue to focus on the one described above.

Okay, so how does this fit with the system rest energy description? Well, we first note that in this case the kinetic energy in the rest frame of the center of mass is non-zero both before and after the collision. So unlike the perfectly inelastic case, not *all* of this rest frame kinetic energy is converted to thermal. With a bit of math, we can confirm that once again the kinetic energy lost from the system rest energy is the amount converted to thermal. Noting that the starting rest frame kinetic energy before the collision is the same as before, given by Equation 4.4.5 above. Next, we can avoid some math by noting that after the collision, the situation is the similar to before, with m_1 swapped with m_2 and v_2 in place of v_1, so the rest frame kinetic energy can be just written-down by making these swaps. The change in the rest frame kinetic energy is therefore:

$$\left.\begin{array}{l} \text{system rest frame KE before} = \left(\frac{m_2}{m_1+m_2}\right)\frac{1}{2}m_1v_1^2 \\ \text{system rest frame KE after} = \left(\frac{m_1}{m_2+m_1}\right)\frac{1}{2}m_2v_2^2 \end{array}\right\} \quad \Delta KE_{\text{system rest frame}} = \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)\left(v_2^2 - v_1^2\right) \tag{4.4.8}$$

Plugging-in Equation 4.4.6 gives:

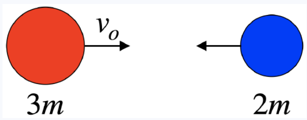
$$\Delta KE_{\text{system rest frame}} = \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)\left[\left(\frac{m_1}{m_2}v_1\right)^2 - v_1^2\right] = \left(\frac{m_1}{m_2} - 1\right)\left(\frac{1}{2}m_1v_1^2\right) \tag{4.4.9}$$

So we see that the amount of kinetic energy that is lost in the lab frame all comes from the kinetic energy in the rest frame. This makes perfect sense, given that the collision doesn't change the center of mass velocity, which means the system's center of mass kinetic energy $\frac{1}{2}m_{tot}v_{cm}^2$ doesn't change.

In summary, we find that in any collision of two objects, the energy converted from kinetic to thermal comes from the system's rest frame kinetic energy, and the fraction of that energy converted depends upon the details of the collision itself. It can range from zero (when the two objects are very rigid, so the collision is elastic) to *all* of the rest frame kinetic energy (when the two objects merge, and the collision is perfectly inelastic), but the amount converted can never exceed the rest frame kinetic energy.

Analyze This

The diagram below depicts a moment just before a collision of two balls made of bouncetech™, a material made by an engineering firm that develops new materials. This experiment was set up as a head-on collision in the center of mass reference frame of the balls. The company's goal is to lose as little kinetic energy as possible to thermal energy in the bounce. To their absolute horror, the two balls stick together! They determine the kinetic energy converted to thermal in this collision to be E_o. They re-check their bouncetech™ formula, and realize that they left out an important ingredient, bounconium. When they repeated the experiment with the corrected mix, they got a much better result.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

This experiment takes place in the center of mass frame, so there is a lot we can extract from what is given. First of all, when the balls stick together in the first experiment, they stop moving altogether, because by definition the center of mass cannot be moving in this frame, and the collision doesn't change this fact, so when they are stuck together, they cannot be moving.

The center of mass is not moving before the collision, so we can determine the speed of the blue ball:

$$v_{cm} = 0 = \frac{3mv_o - 2mv_{blue}}{3m+2m} \quad \Rightarrow \quad v_{blue} = \frac{3}{2}v_o$$

Given that they stop, all of the initial kinetic energy is converted to thermal, so in terms of these values, we can compute the thermal energy:

$$E_o = \frac{1}{2}(3m)v_o^2 + \frac{1}{2}(2m)\left(\frac{3}{2}v_o\right)^2 = \frac{15}{4}mv_o^2$$

What if bouncetech™ achieved the impossible, and the collision cam out perfectly elastic? What would the motion look like after the collision? Well, first of all, in this center of mass frame, the balls would have to be going in opposite directions, or the center of mass could not remain stationary. Second, the ratios of the speeds of the two balls would have to remain the same as when they approached each other in order for the center of mass to remain stationary. And finally, the total kinetic energy has to add up to the same. Without even doing the math, there is a simple solution that satisfies all these criteria: The balls bounce back at the same speeds at which they came in.

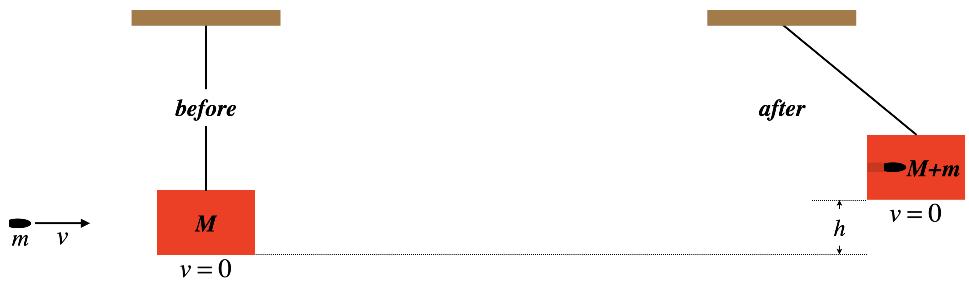
The analysis of the elastic collision gives us an upper limit on the speeds at which we can expect to see the balls moving after the collision. Anything more is impossible, as the kinetic energy would have to increase. But anything less (all the way down to zero) is possible.

The Ballistic Pendulum

When it comes to perfectly inelastic collisions, there is one problem that stands out as a classic – the ballistic pendulum. The idea is to measure the muzzle velocity of a gun, and it goes like this:

A bullet is fired into a block of wood that is hanging by a string from the ceiling. The mass of the bullet and the block are given, as is the height to which the block rises. Find the incoming velocity of the bullet.

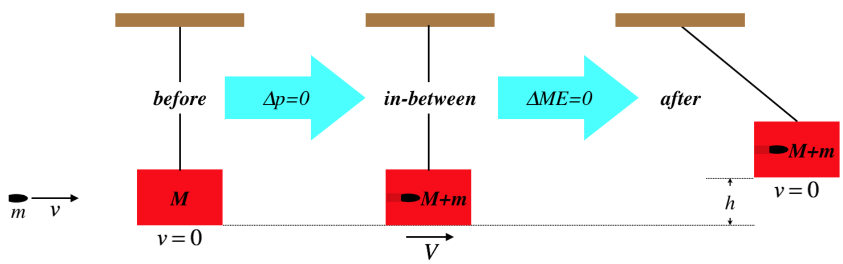
Figure 4.4.3 – The Ballistic Pendulum



If we break down the motion of the bullet and the block during the short span of time that the bullet is working its way into the block, things get very complicated, because the block begins to swing before the bullet comes to rest inside it, which means that the center of mass of the system is not quite moving in a circle yet. This is a problem because we will assume that the tension force does no work here, and we can only assume that if it acts perpendicular to the block+bullet system's motion, which must therefore be in a circle. Thankfully, the bullet is moving very fast, and gets imbedded into the block in a very short time, leaving very little time for the non-conservative part of the tension force to do damage to our results. We therefore neglect the time that the bullet takes to get into the block, and treat the block/bullet interaction as instantaneous.

Clearly this is an inelastic collision, and we can do this calculation in two parts. The first part consists of the momentum conservation problem that derives the speed of the block+bullet system immediately after the collision in terms of the incoming speed of the bullet and the two masses. Then the second part involves the mechanical energy conservation of the bullet+block swinging up to a new height and coming to rest.

Figure 4.4.4 – The Usual Breakdown of the Ballistic Pendulum



This is the way you will see this problem solved in virtually every textbook that covers this problem. But there is another way, which doesn't require these steps. The block+bullet system starts with some rest frame kinetic energy, and because their collision is perfectly inelastic, all of this energy goes into thermal. This leaves behind the energy of the system's center of mass, and since we are assuming that the tension does negligible work on the system, all of this goes from kinetic into potential. So:

$$U_f = KE_o \quad \Rightarrow \quad (M+m)\,gh = \frac{1}{2}(M+m)\,v_{cm}^2 = \frac{1}{2}(M+m)\left(\frac{mv+0}{M+m}\right)^2 = \left(\frac{m^2}{M+m}\right)v^2 \tag{4.4.10}$$

Solving for v gives:

$$v = \left(\frac{M+m}{m}\right)\sqrt{2gh} \tag{4.4.11}$$

There is nothing magical about this approach. It simply avoids re-solving the perfectly inelastic collision case, by noting that all the rest frame kinetic energy becomes thermal, and jumping straight to the center of mass kinetic energy. Both of these methods involve the same physical principals of momentum and energy conservation.

Analyze This

A large sled is at rest on a horizontal, frictionless sheet of ice, when a heavy rock is thrown onto it from behind. The rock is moving purely horizontally when it comes into contact with the sled, and it skids across the rough top surface of the sled until it and the sled are moving forward together at the same speed.

Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

While we have significantly more detail about the force interaction of the two colliding objects, this collision is still perfectly inelastic, with a stationary target, just like the two clay balls discussed at the start of this section. We can therefore use the same conclusion about the fraction of initial KE converted into thermal energy that we found then:

$$\frac{E_{thermal}}{E_o}=\frac{m_2}{m_1+m_2}=\frac{m_{sled}}{m_{rock}+m_{sled}}$$

Suppose we are given the mass of the rock and the coefficient of kinetic friction between the rock and the sled. We then know the amount of the kinetic friction force between the two colliding objects. From here, we can be given one of two pieces of information about the rock's trip across the surface of the sled. If we know how long it takes the rock to stop sliding, then from the constant friction force and the time, we know the impulse the rock delivers to the sled, and from that the new momentum of the sled, and if we know the mass of the sled, then we also know its speed at the end.

$$J_{\text{on sled}}=\Delta p \quad \Rightarrow \quad \left(\mu_k m_{rock} g\right)\Delta t=m_{sled}\Delta v=m_{sled}v_f$$

The final speed of the rock is the same as the final speed of the sled, and the rock experiences the same impulse as the sled (except that it is negative), so we can compute the incoming speed of the rock:

$$J_{\text{on rock}}=-J_{\text{on sled}} \quad \Rightarrow \quad -\left(\mu_k m_{rock} g\right)\Delta t=m_{rock}\left(v_f-v_o\right)$$

The other thing we can be given regarding the rock's trip across the sled is the distance it skids, Δx. In this case, we can use what we previously found when we examined the case shown in Figure 3.5.2: The work done by kinetic friction as the rock slides across the sled is the amount of energy converted to thermal. We know from the very first example above with a stationary target, the fraction of the total energy this is:

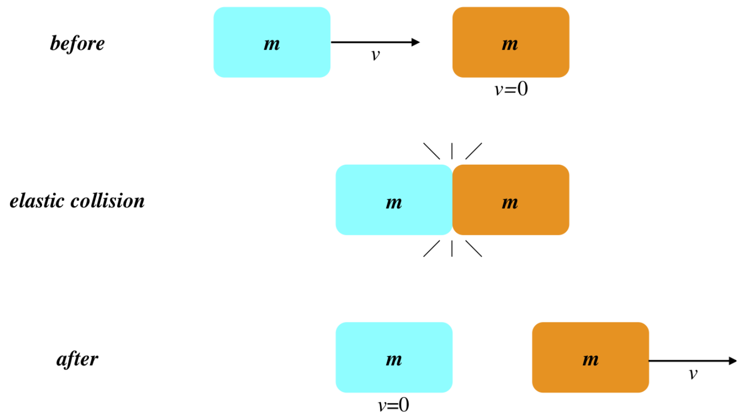
$$\text{work done by friction}=f\Delta x=\mu_k m_{rock} g\Delta x=E_{thermal}=\frac{m_2}{m_1+m_2}E_{total}=\left(\frac{m_{sled}}{m_{rock}+m_{sled}}\right)\left(\frac{1}{2}m_{rock}v_o^2\right)$$

4.5: More Collisions

Elastic Collisions

In the previous section, we focused on inelastic collisions. Here we will look at elastic collisions, where the kinetic energy of the system remains unchanged, meaning none of the rest frame kinetic energy is converted into thermal energy. This kind of collision is standard between particles, but between macroscopic objects, this is really only an approximation. When we are told that a given collision is elastic (or at least can be approximated as such), then that gives us an additional condition that we can use to solve the problem. We'll go through a few examples of elastic collisions in one dimension below. In each case, the diagram will show the experimental result, which we will then show mathematically using the combination of momentum and kinetic energy conservation.

Figure 4.5.1 – Elastic Collision of Equal Masses, Target Stationary

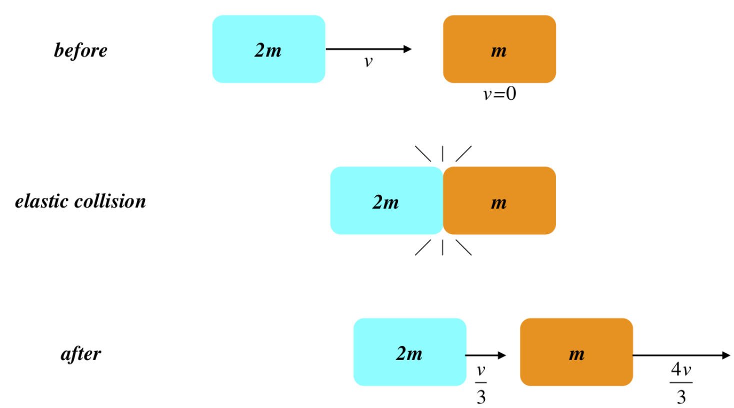


We see that the incoming cart stops completely and the target cart moves off with the same velocity as the original cart (note that the center of mass continues moving at a constant speed, as it should). We now show this mathematically... Dropping the vector arrows, since the motion is in one dimension, and choosing to the right as the (+) direction, we have:

$$\left.\begin{array}{lll} p_o = p_f: & mv + 0 = mv_1 + mv_2 & \Rightarrow \quad v = v_1 + v_2 \\ KE_o = KE_f: & \frac{1}{2}mv^2 + 0 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 & \Rightarrow \quad v^2 = v_1^2 + v_2^2 \end{array}\right\} \Rightarrow \quad v_1 = 0, \;\; v_2 = v \quad or \quad v_1 = v, \;\; v_2 = 0 \tag{4.5.1}$$

Wait, why do we get two solutions? That is, why can *either* velocity equal zero? Well, if the incoming cart were to *miss the target cart*, then that too is an elastic "collision," inasmuch as the momentum and kinetic are both conserved, so the math takes into account that as a possibility.

Figure 4.5.2 – Elastic Collision of Unequal Masses, Target Lighter and Stationary



The algebra is only a little tougher this time:

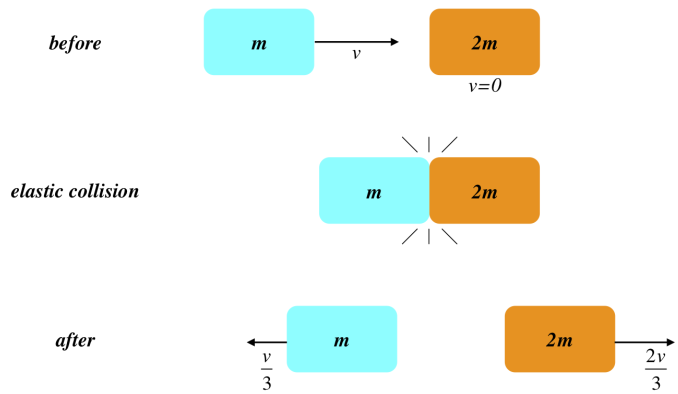
$$\left.\begin{array}{lll} p_o = p_f: & 2mv + 0 = 2mv_1 + mv_2 & \Rightarrow \quad 2v = 2v_1 + v_2 \\ KE_o = KE_f: & \frac{1}{2}2mv^2 + 0 = \frac{1}{2}2mv_1^2 + \frac{1}{2}mv_2^2 & \Rightarrow \quad 2v^2 = 2v_1^2 + v_2^2 \end{array}\right\} \Rightarrow \quad 4v_1 = v_2 \quad \Rightarrow \quad v_1 = \frac{v}{3}, \;\; v_2 = \frac{4v}{3} \tag{4.5.2}$$

Both carts continue forward, the lighter one at 4 times the speed of the heavier one. Note that once again $v_1=v, \;\; v_2=0$ is a solution (the incoming cart misses the target).

Let's consider an application of this in the real world. Suppose we are passengers in one of two vehicles involved in a head-on collision. Which vehicle would we rather be in, the lighter one or the heavier one? Intuitively we know we would rather be in the heavier vehicle, but why? Well, we would want to experience as little force as possible (force is what breaks bones). The force that our dashboard or steering column exerts on us is going to equal our mass times our acceleration (as it constitutes our net horizontal force), and we are constrained to experience the same acceleration as our car. So compare the accelerations of the two carts here. The heavier cart goes from a speed v down to a speed of v/3, for a change of 2v/3. The lighter cart's velocity changes from 0 to 4v/3 in the same period of time, which means it experiences twice the acceleration. More acceleration for our car means more acceleration for us, which means more force on us, which is bad.

Lastly, we look at the lighter object bouncing off the heavier one:

Figure 4.5.3 – Elastic Collision of Unequal Masses, Target Heavier and Stationary



The math:

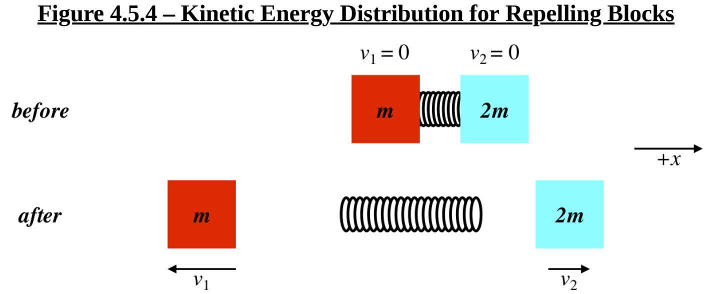
$$\left.\begin{array}{lll} p_o=p_f: & mv+0=mv_1+2mv_2 & \Rightarrow \quad v=v_1+2v_2 \\ KE_o=KE_f & \frac{1}{2}mv^2+0=\frac{1}{2}mv_1^2+\frac{1}{2}2mv_2^2 & \Rightarrow \quad v^2=v_1^2+2v_2^2 \end{array}\right\} \Rightarrow \quad 2v_1=-v_2 \quad \Rightarrow \quad v_1=-\frac{v}{3}, \;\; v_2=\frac{2v}{3} \tag{4.5.3}$$

The lighter cart bounces off the heavier one at half the speed that the heavier one continues forward (or the incoming cart misses the target). There is actually a clever way we could have solved this case more quickly by using the solution of the previous case and what we know about relative motion. If we move along with the incoming block and declare ourselves to be "stationary," then we see the heavier mass coming toward us at a speed v, which is exactly the same physical situation as we had above. After the collision, we will see the heavier mass continuing in the same direction at a speed of $v/3$, while the target block moves in the same direction at a speed of $4v/3$. That is what *we see*. Going back to the original frame, these two speeds change by v, which means the heavy object is not going left at $v/3$ – it is going *right* at $v-v/3=2v/3$, while the smaller block is moving left at a speed of $4v/3-v=v/3$.

Kinetic Energy Distribution Within a System

Let's return once again to an example we looked at in the previous section (Figure 4.3.1), and ask a new question about it (the example has been simplified slightly by giving one block exactly twice the mass of the second block).

Figure 4.5.4 – Kinetic Energy Distribution for Repelling Blocks



The spring stored some potential energy when it was compressed, and it gave this energy to the kinetic energy of the two blocks. What fraction of this energy is given to each of the blocks? One might be inclined to believe that since the spring exerts equal forces on both blocks, they both get equal amounts of kinetic energy. But by now we know better! They only get the same amount of energy if the spring does the same amount of work on both, and it's clear here that the lighter mass is pushed a longer distance before losing contact with the spring than the heavier mass, so with equal forces acting on each, more work is done on the lighter mass. Specifically, the lighter mass is accelerated twice as much by the equal force, so it displaces twice as far, and therefore gets twice as much energy as the heavier block.

Another way to see it is to note that both blocks must have the same magnitude of momentum after the spring expands (since the momenta must cancel to equal zero and remain conserved), so using Equation 4.1.6 we can compare their kinetic energies:

$$KE_1 = \frac{p_1^2}{2m_1} = \frac{p^2}{2m} \quad \Rightarrow \quad KE_2 = \frac{p_2^2}{2m_2} = \frac{p^2}{2\,(2m)} = \frac{1}{2}KE_1 \tag{4.5.4}$$

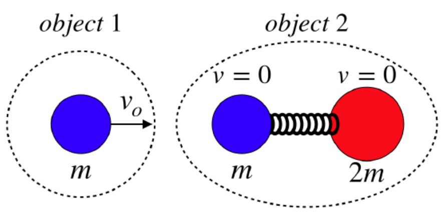
This confirms what we reasoned above.

Now we can see more clearly why we are able to refer to the gravitational potential energy of the system of a small stone and the earth as simply the gravitational potential energy "of the stone," ignoring the fact that the earth is also involved. This is because when the potential energy is converted to kinetic energy, virtually all of the kinetic energy goes to the stone, and none of it to the earth (imagine the heavier block above being *much* heavier).

Source of Inelasticity

We have said more than once that all collisions between particles are elastic, while collisions between objects are not. But objects are *made* of particles, which means that when they collide, it involves the collision of particles. How is this not a contradiction? Let's see if we can sort this out with the simplest possible example imaginable. Let's look at a collision between object made up of a *single* particle, and another object made up of two particles. The latter object we will model as two particles of different masses, bound together by a spring. We will further simplify things by assuming that the two particles are separated by the equilibrium length of the spring, and they are not vibrating (i.e. this two-particle object has zero internal energy). A diagram of the collision is shown in the figure below.

Figure 4.5.5 – Microscopic View of Two Object Collision



When the blue particles collide, they will do so elastically. As we saw in the example above, when a projectile collides head-on elastically with a stationary target of equal mass, the incoming particle stops, and the target particle continues forward at the same speed that the incoming particle had. This means that object 1 will stop, and one of the particles in object 2 will start moving toward the other particle. This will compress the spring, which will cause the other particle to also move to the right. That is, object 2 as a whole will start moving to the right. Its particles will also vibrate back-and-forth within the object – the object will have internal energy. This is precisely the recipe for an inelastic collision. In the case of bigger objects with trillions upon trillions of particles, this internal energy is spread throughout the particles, and their motions are randomly-distributed, which is to say that this internal energy is thermal. The simplicity of this example, by contrast, allows us to precisely track this energy. Let's do that.

The motion of object 2 after the collision is measured by its center of mass velocity, which the spring will not affect, as it provides only an internal force within the collection of particles. Therefore the center of mass velocity can simply be computed using the initial condition when the blue particle is moving and the red particle is not:

$$v_{cm} = \frac{mv_o + 0}{m + 2m} = \frac{1}{3}v_o \tag{4.5.5}$$

The total energy of the two-object system is unchanged, and initially it was just the kinetic energy of the incoming particle, so it is:

$$E_{system} = \frac{1}{2}mv_o^2 \tag{4.5.6}$$

The kinetic energies of the two objects after the collision are: Zero for object 1 (which stops), and for object 2:

$$KE_2 = \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}(3m)\left(\frac{1}{3}v_o\right)^2 = \frac{1}{3}\left(\frac{1}{2}mv_o^2\right) = \frac{1}{3}E_{system} \tag{4.5.7}$$

So we see that this collision between two objects is inelastic, because only one-third of the original kinetic energy remains in the objects. The remaining two thirds is stored in the internal energy of object 2 as its particles vibrate.

Exercise

Suppose the same two objects as above collide again, but this time the incoming object strikes the other side. They are the same two objects, so would you expect the result to be the same? Confirm or refute your intuition mathematically.

Solution

Repeating the process above, we have the same total system energy as before. The elastic collision between the incoming blue particle and the larger red particle gives a different result, however. We calculated earlier what happens when an object collides elastically with another object twice as heavy. The incoming object bounces-back with one-third its incoming speed, and the heavier one moves forward with two thirds the incoming speed.

The center of mass velocity of the two-particle object can be computed from the red particle's new speed:

$$v_{cm}=\frac{2m\left(\frac{2}{3}v_o\right)+0}{2m+m}=\frac{4}{9}v_o$$

Now we can use the reflected speed of the incoming object and the center of mass speed of the target object to determine the kinetic energy in the system after the collision:

$$KE_{sys}=KE_1+KE_2=\frac{1}{2}m\left(\frac{1}{3}v_o\right)^2+\frac{1}{2}(3m)\left(\frac{4}{9}v_o\right)^2=\frac{19}{27}\left(\frac{1}{2}mv_o^2\right)=\frac{19}{27}E_{system}$$

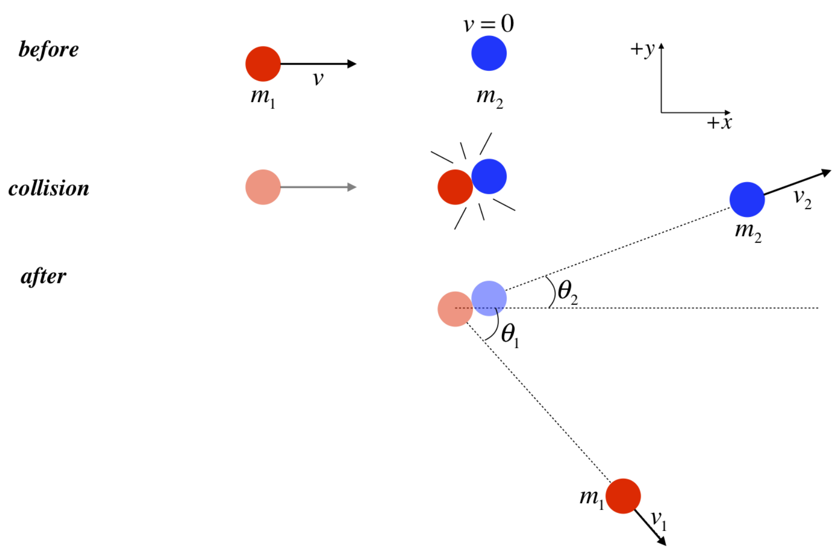
So it appears that considerably less energy goes into the internal energy of the two-particle object in this case than when the smaller particles collide.

General Two-Dimensional Collisions

We have been saying for awhile now that one of the big differences between momentum conservation and energy conservation is the fact that momentum is a vector while energy is not. This means that there are actually three momentum quantities that are equal before and after (if the full momentum vector is conserved). Here we will look at what this entails.

Let's look at a standard two-dimensional collision. In this example, we will have a stationary ball struck by another. The two balls have different masses, and they collide off-center, so that they emerge from the collision in directions angled off the original direction of motion. We'll set up the geometry and label all the known and unknown variables with a diagram, and then do the physics:

Figure 4.5.6 – General Two-Dimensional Collision in the Target Frame



Now we need to apply momentum conservation. Since momentum is a conserved vector, each of its components are individually conserved, which means that momentum conservation provides us two separate equations to work with. In the "before" case, we have an x-component of momentum that is simply the incoming mass times the incoming velocity (m_1v), while the y-component of momentum is zero. in the "after" case, we need to resolve the momenta into components. Setting before equal to after gives:

$$\begin{array}{ll} x-direction: & m_1v=m_1v_1\cos\theta_1+m_2v_2\cos\theta_2 \\ y-direction: & 0=-m_1v_1\sin\theta_1+m_2v_2\sin\theta_2 \end{array} \tag{4.5.8}$$

You'll note the minus sign for the component in the $-y$-direction. This is not strictly necessary, as this negative sign could be absorbed into θ_1, but it is generally less confusing to put the signs in explicitly, and let all the angle values be positive.

Let's consider what would be required to solve a problem that looks like this. We have two equations, and seven distinct variables. If this is all we know about the collision, then to completely unravel this physical situation, we need to know five of these quantities. So for example, we could be given the two masses, the incoming speed, and the outgoing speed and direction of one of the balls, and we can solve for the outgoing speed and direction of the other ball. If we also provided the target ball a starting velocity, or a y-component to the incoming ball's velocity, then there would be even more unknowns. But we can quickly reduce this problem back to the one above, by first rotating our coordinate system so that the incoming velocity is once again in the x direction, and then changing the reference frame to the rest frame of the target ball. It is also sometimes useful to change to the center of mass reference frame.

Notice that once such a problem is solved, once can then check to see if the collision is elastic, by comparing the kinetic energy before and after the collision:

$$KE_{before} = \frac{1}{2}m_1 v^2 \qquad KE_{after} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \tag{4.5.9}$$

This comparison could be a difference (determining how much kinetic energy is lost), or a fraction (determining the percentage of kinetic energy remaining or the percentage lost). Note that a collision *can* result in an increase of kinetic energy, but this can only happen if there is some potential energy stored within the colliding objects that is unleashed by the collision. This is such an uncommon occurrence (the circumstances need to be quite contrived), that it is safe to assume that a collision is either elastic (conserves kinetic energy) or is inelastic such that kinetic energy is lost.

Not all problems are posed with five of the seven variables given. The energy condition can be given instead, which provides a third equation, requiring only four of the seven variables in the statement of the problem. Needless to say, these problems can require a lot of tedious algebra, but getting the equations set up using momentum conservation and the fate of the system's kinetic energy is where the physics is.

Elastic Two-Dimensional Collisions

As daunting as the full-blown problem shown above can be, there are cases where shortcuts or simplifications exist. We look first at the case of elastic collisions. If we want to know all the information shown above, we have no choice but to go through the algebra involved. But we can achieve an interesting result without recourse to the coordinate system at all. Namely, it turns out that the ratios of the masses of the colliding objects and their outgoing speeds completely determine the angle *between* the outgoing velocity vectors, $\theta_1 + \theta_2$. To get this result, we will use Equation 4.1.5 extensively...

Let's call the incoming momentum $\overrightarrow{p}$ and the mass of the incoming object m_1. Then the kinetic energy of the system (in the frame where the target is stationary) is:

$$KE_{before} = \frac{1}{2m_1}\overrightarrow{p}\cdot\overrightarrow{p} \tag{4.5.10}$$

Now let's define the outgoing momenta of the two objects as $\overrightarrow{p}_1$ and $\overrightarrow{p}_2$, with the latter being for the target object after collision. The kinetic energy after the collision is therefore:

$$KE_{after} = \frac{1}{2m_1}\overrightarrow{p}_1\cdot\overrightarrow{p}_1 + \frac{1}{2m_2}\overrightarrow{p}_2\cdot\overrightarrow{p}_2 \tag{4.5.11}$$

Now we apply momentum conservation:

$$\overrightarrow{p} = \overrightarrow{p}_1 + \overrightarrow{p}_2 \quad\Rightarrow\quad KE_{before} = \frac{1}{2m_1}\left(\overrightarrow{p}_1 + \overrightarrow{p}_2\right)\cdot\left(\overrightarrow{p}_1 + \overrightarrow{p}_2\right) = \frac{1}{2m_1}\overrightarrow{p}_1\cdot\overrightarrow{p}_1 + \frac{1}{2m_1}\overrightarrow{p}_2\cdot\overrightarrow{p}_2 + \frac{1}{m_1}\overrightarrow{p}_1\cdot\overrightarrow{p}_2 \tag{4.5.12}$$

Applying kinetic energy conservation (remember, we are assuming an elastic collision):

$$KE_{before} = KE_{after} \quad\Rightarrow\quad \cancel{\frac{1}{2m_1}\overrightarrow{p}_1\cdot\overrightarrow{p}_1} + \frac{1}{2m_1}\overrightarrow{p}_2\cdot\overrightarrow{p}_2 + \frac{1}{m_1}\overrightarrow{p}_1\cdot\overrightarrow{p}_2 = \cancel{\frac{1}{2m_1}\overrightarrow{p}_1\cdot\overrightarrow{p}_1} + \frac{1}{2m_2}\overrightarrow{p}_2\cdot\overrightarrow{p}_2 \tag{4.5.13}$$

Now multiply through by m_1 and rearrange things a bit to get:

$$\overrightarrow{p}_1\cdot\overrightarrow{p}_2 = \frac{1}{2}\left(\frac{m_1}{m_2} - 1\right)\overrightarrow{p}_2\cdot\overrightarrow{p}_2 \tag{4.5.14}$$

Now write the dot products in terms of the magnitudes of the vectors and the angles between them:

$$p_1 p_2 \cos\theta = \frac{1}{2}\left(\frac{m_1}{m_2} - 1\right) p_2^2 \tag{4.5.15}$$

The angle θ is of course the angle between the two outgoing velocity vectors (which point the same direction as the momentum vectors). The $p_2 = 0$ solution to this corresponds to the case of the incoming object missing the target entirely (because the target remains stationary), so assuming the target is not missed, we can divide both sides by p_2 and if we also plug in $p_1 = m_1 v_1$ and $p_2 = m_2 v_2$, we get the promised relationship of the *scattering angle* in terms of the masses and outgoing speeds:

$$\theta = \cos^{-1}\left[\frac{1}{2}\left(1 - \frac{m_2}{m_1}\right)\frac{v_2}{v_1}\right] \tag{4.5.16}$$

We can extract some interesting information from this result:

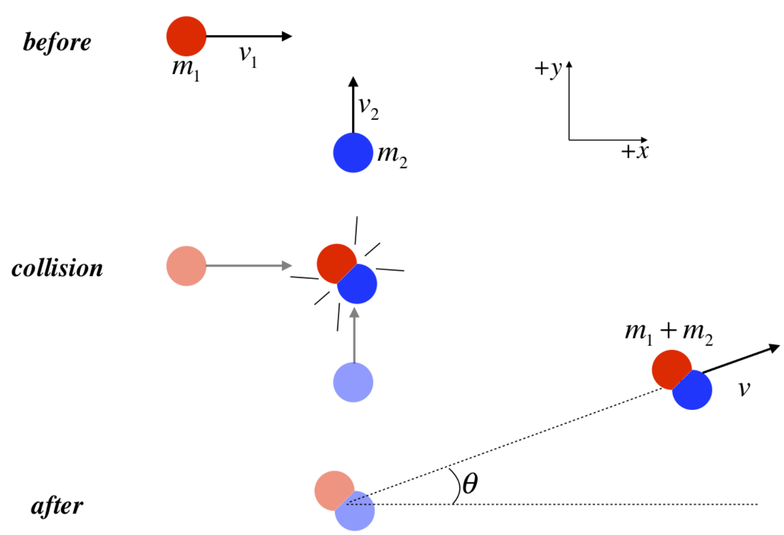
- We see that if the masses are equal, then the scattering angle is precisely 90^o, since the cosine of this angle vanishes. In this case, the scattering angle doesn't depend at all on how off-center the collision is (except that a direct head-on hit naturally leads to an angle of 0^o or 180^o). The degree of how off-center the collision is (which is measured by a quantity known as the *impact parameter*) *does* effect the angles θ_1 and θ_2 in Figure 4.5.6, but not the sum of those angles. If the masses are not equal, then the impact parameter does play a role in the scattering angle, because it has a say in the ratio of the outgoing speeds.
- If $m_2 > m_1$, the argument of the inverse cosine is negative, so the angle must be greater than 90^o. This makes sense, because if the target mass is greater than the incoming mass, the incoming mass "bounces back," rather than "plowing through" (a result we found for the one-dimensional elastic collisions we examined above), and since the target mass has a forward component to its final velocity, the angle is greater than 90^o.
- The argument of the inverse cosine can never be larger than +1 or smaller than –1, which places limits on the outgoing speeds given the masses. For example, if the incoming mass m_1 is twice the target mass m_2, then the largest possible ratio of the two outgoing velocities is 4. This ratio occurs when $\theta = 0$, and indeed we have seen this result already above (Equation 4.5.2).

It should be noted that this result could also be achieved using the formulas resulting from Figure 4.5.5, but it would require an unnatural desire to slog through trigonometric identities.

Perfectly Inelastic Two-Dimensional Collisions

As much as we were able to do with elastic collisions, perfectly inelastic collisions are even easier to handle. This is because the outgoing motions of the two objects are constrained to be the same (i.e. they stick together and have the same final speed and direction). This constraint means that if we are given all of the incoming conditions (the masses of the two objects, and their incoming velocity vectors), we can determine the result completely. That is, the amount of energy lost in the collision does not need to be given – it is unique and can in fact be computed. The figure below is a diagram for an example of a perfectly inelastic collision. [*This is somewhat simplified by having the incoming objects approach each other at right angles, but not as simple as the case of looking at it from the target frame, which makes the collision one-dimensional!*]

Figure 4.5.7 – A Perfectly Inelastic Two-Dimensional Collision



We follow the same procedure as we did for Figure 4.5.6, this time with the simplification that we have a single outgoing momentum:

$$\begin{aligned} x-direction: \quad & m_1 v_1 = (m_1 + m_2)\, v\cos\theta \\ y-direction: \quad & m_2 v_2 = (m_1 + m_2)\, v\sin\theta \end{aligned} \tag{4.5.17}$$

The amount of energy converted to thermal from this collision equals the loss of kinetic energy from the system, and as we saw in the one-dimensional case, this amount doesn't depend upon the details of the internal non-conservative force. It only matters that eventually (after the two objects end their tumultuous collision) settle into moving off together with a common velocity. The amount of energy converted is:

$$\Delta E_{thermal} = KE_{before} - KE_{after} = \left[\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right] - \left[\frac{1}{2}(m_1+m_2)\,v^2\right] \tag{4.5.18}$$

For the case above where the two incoming objects have velocities are right angles to each other, we can turn this into an equation that includes only the masses and incoming speeds. Sparing the reader the algebra, the result is:

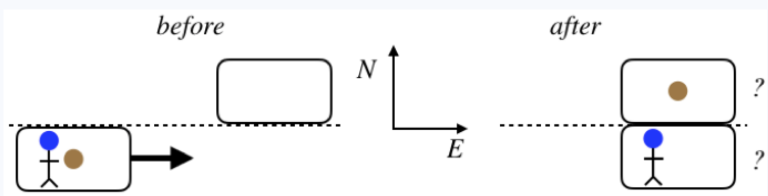
$$E_{thermal} = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(v_1^2+v_2^2\right) \tag{4.5.19}$$

Notice that since the two velocities are perpendicular, the sum of their squares is actually the square of their *relative* velocity. This is not a surprising result, and in fact will translate into collisions at any angle (though the equation will look different), because we would not expect the post-collision blob to be any hotter when the collision is viewed in one frame as opposed to another. As mentioned above, we can always view this collision from the target frame, making the collision one-dimensional, and the total kinetic energy of the system before the collision is a function of the relative velocity. In that case, we can use Equation 4.4.3 to compute the energy converted to thermal.

So suppose we drop a ball of clay to the ground. Viewing this from the earth's rest frame, the earth becomes the stationary target with mass m_2, and essentially all of the clay's incoming kinetic energy is converted to thermal (because $m_2 \approx m_1 + m_2$), and the clay's (and earth's) temperature goes up a bit. If we view it from the clay's rest frame, then the kinetic energy of the earth is enormous (same relative speed, much larger mass), and after the collision we might therefore expect the temperatures to go up a lot, but making the clay the stationary target now makes the target mass m_2 very small compared to $m_1 + m_2$, which makes the fraction multiplying the earth's kinetic energy very small – exactly small enough to give the same energy change as before.

Analyze This

A cart slides along a frictionless surface in an easterly direction. The cart contains a person and a medicine ball. The cart slides past an identical (but empty) stationary cart, also on the frictionless surface. When the carts are side-by-side, the person throws the medicine ball into the other cart by pushing the ball in the north direction.



Your goal in the analysis is to extract everything you can from what has been given. At a minimum, every analysis should include these items:

- *what we are given (perhaps translated from English to mathematics)*
- *what we can infer, if anything*
- *quantities we can compute (or almost compute!), if anything*

Analysis

This situation is a bit more complicated than a simple two-object collision, but the principles behind it are the same. Treating both carts, the person, and the medicine ball together as a single system, there are not external forces on the system, and the total momentum vector is conserved – the x and y components of momentum are each separately conserved.

Let's call the magnitude of the initial easterly velocity of the cart v_o. This gives us the initial momentum (which will remain unchanged):

$$p_x = (m_{cart} + m_{person} + m_{ball})\,v_o$$
$$p_y = 0$$

Next we need to think about the "after" situation. Now we have two separate components to deal with, as the objects involved will have some y components to their motion. Calling the incoming cart #1, and the other cart #2, we have:

$$x\text{-direction:}\quad p_x = (m_{cart} + m_{person})\,v_{1x} + (m_{cart} + m_{ball})\,v_{2x}$$
$$y\text{-direction:}\quad p_y = (m_{cart} + m_{person})\,v_{1y} + (m_{cart} + m_{ball})\,v_{2y}$$

Invoking momentum conservation gives:

$$\begin{array}{lll} x\text{-direction:} & (m_{cart}+m_{person}+m_{ball})\,v_o & =(m_{cart}+m_{person})\,v_{1x}+(m_{cart}+m_{ball})\,v_{2x} \\ y\text{-direction:} & 0 & =(m_{cart}+m_{person})\,v_{1y}+(m_{cart}+m_{ball})\,v_{2y} \end{array}$$

With the push being in the north direction, the x-component of person + cart 1 is not changed, which means we can put in $v_{1x}=v_o$, giving:

$$m_{ball}v_o=(m_{cart}+m_{ball})\,v_{2x}$$

This is all fine if we are given the velocities of the carts after the ball has been transferred, but it seems clear that in such a problem the details of the ball transfer itself might be given. If this is the case, then really we have two momentum problems to solve – the problem of the first cart ejecting the ball, and the second of the second car receiving the ball.

*We in fact already know that the ball is thrown northward by the person. Does this mean we can write the momentum of the ball as being purely in the y-direction? No! The person throws the ball northward, but it **was already moving eastward** when it was thrown, so its momentum actually has both x and y components. So for the ball-is-thrown half of the problem, the ball, person, and cart #1 will not change their speeds in the x-direction at all.*

*How to compute the effects on the y-components of the velocities of the ball and cart #1 depends upon what is given about the medicine ball's motion. If the northward component of its velocity relative to the ground is given, then things are pretty straightforward. Just use this quantity and the mass of the ball to compute its y-component of momentum, and that same amount of momentum is what cart #1 + person must have in the southerly direction, to conserve the momentum in the y-direction (which was initially zero). But if the velocity of the medicine ball **as seen by the person** is given, then it gets considerably trickier, because this is the speed of the medicine ball north **relative to** the person + cart #1, which will recoil south.*

Calling the northward component of the ball's velocity relative to the person v_{bp}, the ball relative to earth v_b, and the southward component of the person + cart #1 after the ball is released v_p (which has a negative value), we can use the y-components of the usual relative motion formula (Equation 1.8.3) to get:

$$v_b=v_{bp}+v_p$$

We then just need to use the quantities v_b and v_p in the y-component parts of our momentum conservation equations. We already noted that the x-component for the throwing part is not very interesting, so the y-component part looks like:

$$0=m_{ball}v_b+(m_{cart}+m_{person})\,v_p$$

For the second part, where cart #2 receives the ball, both the x and y components of the ball's motion are important, and the momentum conservation equations are:

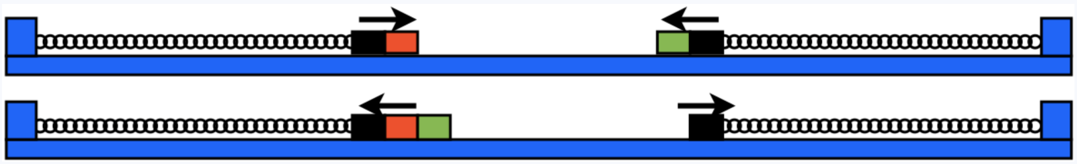
$$\begin{array}{ll} x\text{-direction:} & m_{ball}v_o+0=(m_{ball}+m_{cart})\,v_{2x} \\ y\text{-direction:} & m_{ball}v_b+0=(m_{ball}+m_{cart})\,v_{2y} \end{array}$$

Sample Problems

All of the problems below have had their basic features discussed in an "Analyze This" box in this chapter. This means that the solutions provided here are incomplete, as they will refer back to the analysis performed for information (i.e. the full solution is essentially split between the analysis earlier and details here). If you have not yet spent time working on (not simply reading!) the analysis of these situations, these sample problems will be of little benefit to your studies.

Problem 4.1

Two pairs of identical blocks on identical springs are side-by-side as shown in the diagram below. They are set into motion such that just as they reach their (equal) maximum displacements toward each other, they barely come into contact (there is no collision – their springs stop them just as they touch). When they contact, one of the blocks is transferred to the other, and their motion continues.



*Explain what happens to the momentum of the **four block** system shortly after the blocks separate into three and one.*

Solution

The momentum of the four block system starts at zero, and at the point when the blocks separate, the two spring forces on the system are equal and in opposite directions, so one might think that the momentum remains zero, because there is no net force on the system. But this doesn't last for long, because the single block starts moving faster than the three blocks, which means that a short time later, the spring of the single block is not stretched as much as the spring for the three blocks. At this moment in time, there is a net force on the 4-block system (to the left), so the momentum will be increasing to the left, due to this net impulse.

Without doing any math, we can see that the forces that act on each side one their way from full stretch to the equilibrium point are the same (they only depend upon the stretch of the spring), but since the three block group takes longer to get to the equilibrium point, is experiences more impulse, which is why the analysis shows that it experiences a greater change in momentum than the single block.

Problem 4.2

Two identical rods of mass M and length L have the same non-uniform density profile. When one of these rods is placed along the x-axis with one end of the rod at the origin, the density as a function of x is proportional to the following function:

$$\lambda\left(x\right)\propto\left(\frac{x^2}{L^2}+1\right)$$

The two rods are laid end-to-end. Describe the possible locations of the center of mass of the two-rod system.

Solution

If the ends of the rods that are connected are of the same density (either both are the less-dense ends, or both are the most-dense ends), then the mass is distributed symmetrically about opposite sides of the connection point of the rods, and the center of mass is right at the connection point. But if the more-dense end of one rod (which we will call rod "A") is connected to the less-dense end of the other rod (rod "B"), then the total mass is not symmetrically-placed across the connection point. In the analysis, we found that the center of mass of each rod is 9/16 of its length from the less-dense end. This means that the center of mass of rod A is at $x_A=\frac{9}{16}L$, and the center of mass of rod B is at $x_B=L+\frac{9}{16}L=\frac{25}{16}L$. These two centers of mass are for objects of equal mass, so the center of mass of the system is simply halfway between their centers, which is at:

$$x_{cm}\left(both\;rods\right)=\frac{1}{2}\left(x_A+x_B\right)=\frac{17}{16}L$$

So the center of mass is 1/16th of the length of a rod from the connection point, located on the rod whose less-dense end it at the connection point.

Problem 4.3

A child sits on the rear end of a sled (whose mass is uniformly-distributed along its length) with a block of frozen snow at rest in her lap. The sled is sliding forward on the horizontal, frictionless snow at constant a speed, when the child suddenly shoves the block forward in the sled (she remains firmly planted on the sled). After a period of time, the block comes to rest in the front of the sled.

Assume that the mass of the sled is uniformly distributed along its length. Here are the physical properties associated with this situation:

$$\begin{array}{lll} \text{mass of child} = 30.0kg & \text{mass of sled} = 12.0kg & \text{mass of ice} = 18.0kg \\ \text{length of sled} = 3.5m & \text{initial speed} = 1.6\frac{m}{s} & \text{time that ice slides} = 2.5s \end{array}$$

a. *Find the speed of the sled after the ice block stops sliding forward.*
b. *Find the position of the center of mass of the child+ sled + block before the block is pushed. Reference this position from the rear of the sled.*
c. *Find the distance (relative to the ground) that the center of mass of the child + sled + block moves during the period of time that the block slides forward.*
d. *Find the distance that the sled moves during the period of time that the block slides forward.*

Solution

a. As we stated in the analysis, when all three objects are again moving together, they must have the same speed that they started with, so $v_f = 1.6\frac{m}{s}$.

b. This is just the usual center of mass calculation, and the positions of the child :

$$x_{cm} = \frac{m_c x_c + m_i x_i + m_s x_s}{m_c + m_i + m_s} = \frac{0 + 0 + (12.0kg)(1.75m)}{30.0kg + 18.0kg + 12.0kg} = 0.350m$$

c. The speed of the system remains unchanged during this time (as indicated in part a), so the distance that the center of mass moves during this period is just the speed times the time:

$$\Delta x_{cm} = v_{cm} t = \left(1.6\frac{m}{s}\right)(2.5s) = 4.0m$$

d. However far the sled travels, the child travels the same distance, and the snow travels that distance plus the length of the sled. Plugging in all these changes into the equation for the change of center of mass (of the whole system) gives:

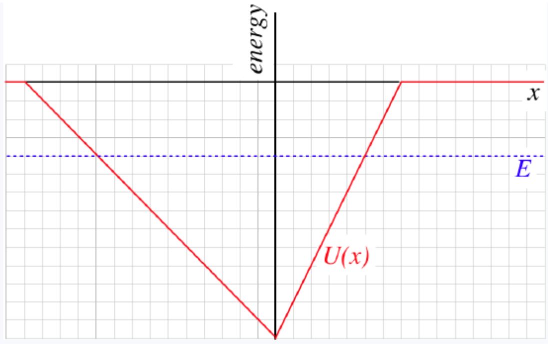
$$\Delta x_{cm} = \frac{m_c \Delta x_c + m_i \Delta x_i + m_s \Delta x_s}{m_c + m_s + m_s} = \frac{m_c \Delta x_c + m_i(\Delta x_s + L) + m_s \Delta x_s}{M}$$

Solving for the displacement of the sled gives:

$$\Delta x_s = \Delta x_{cm} - \frac{m_i L}{M} = 4.0m - \frac{(18kg)(3.5m)}{60kg} = 2.95m$$

Problem 4.4

Two different particles are confined by the same potential, shown in the diagram. Both particles have the same total energy, also depicted in the diagram. At one moment the particles pass each other precisely at the origin, with one particle moving in the $-x$-direction and the other moving in the $+x$-direction.



A short time later, both particles come to rest at the same instant. Find the location of the center of mass of the two particles at the moment that they come to rest. Express the answer in terms of the units defined by the grid lines in the diagram.

Solution

In the analysis, we found the ratio of the forces on the two particles. We are told here that the particles take the same period of time to come to rest. If the forces are applied for the same period of time, then the force that is twice as great imparts twice as much impulse on particle B as the lesser force imparts on particle A. Experiencing twice the impulse, the change of particle B's momentum is twice as great. Both particles come to rest, so particle B must start with twice as much momentum as particle A when they are at the origin: $p_B = 2p_A$ *.*

In the analysis we also noted the relationship between momentum and kinetic energy, and reasoned that the particles have the same kinetic energy when they are at the origin. We can therefore draw a conclusion about their masses:

$$KE_A = KE_B \quad \Rightarrow \quad \frac{p_A^2}{2m_A} = \frac{p_B^2}{2m_B} \quad \Rightarrow \quad \frac{m_B}{m_A} = \frac{p_B^2}{p_A^2} = 4$$

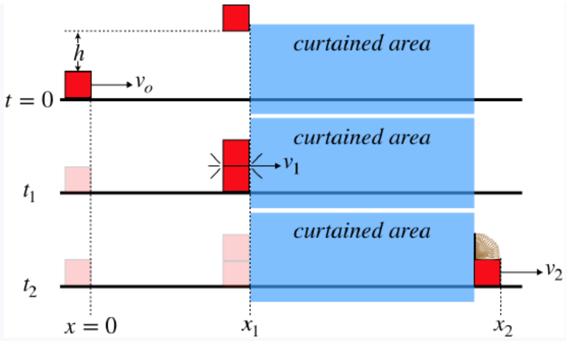
With the relative masses of the two particles now known, all we need is their positions when they come to rest. We can determine this by realizing that at rest they have no kinetic energy, so their total energy equals their potential energy – the points of intersection between the horizontal total energy line and the potential energy curve (which we have previously called the "turnaround points"). So when they come to rest, particle A is at the position $x = -10$ *and particle B is at the position* $x = +5$*. Now we just plug into the center of mass formula:*

$$x_{cm} = \frac{m_A x_A + m_B x_B}{m_A + m_B} = \frac{m_A(-10) + (4m_A)(+5)}{m_A + 4m_A} = +2$$

So the center of mass moves two units in the $+x$*-direction in the time that the particles move from the origin to the turnaround points.*

Problem 4.5

A block slides along a frictionless horizontal surface at a speed v*, starting at position* $x = 0$ *and time* $t = 0$*. An identical block dropped from rest lands directly on top of it. The surfaces of the blocks are sticky, so the top block adheres to the bottom block when it lands on it, and they continue along together. The blocks slide together into a curtained-off area, during which a spring noise and a "thud" are heard. At a later time, the bottom block emerges from the curtain without the top block on it, after apparently having its top lid sprung open from within.*



The falling block is dropped at $t = 0$ from a height of $19.6m$ above the top of the sliding block (the diagram distances are not to scale). The labeled positions are $x_1 = 9.6m$ and $x_2 = 18.0m$, and the bottom block emerges at time $t_2 = 5.0s$. Find the location of the top block at the moment the bottom block emerges from behind the curtain.

Solution

In the analysis we found the velocity of the center of mass of the two block system in terms of the initial velocity of the sliding block, but we are not given that here. We know that the falling block was released from rest at $t = 0$, and how far it drops, so we can compute t_1:

$$h = \frac{1}{2}gt_1^2 \quad \Rightarrow \quad t_1 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2\,(19.6m)}{9.8\frac{m}{s^2}}} = 2.0s$$

The sliding block travels a known distance in this time, so we can compute its speed:

$$v_o = \frac{x_1}{t_1} = \frac{9.6m}{2.0s} = 4.8\frac{m}{s}$$

The result from the analysis gives us the center of mass speed:

$$v_{cm} = \frac{1}{2}v_o = 2.4\frac{m}{s}$$

With the speed of the center of mass of the system we can compute the position of the center of mass when the bottom block emerges. The bottom block spends a time $t_2 - t_1 = 3.0s$ behind the curtain. At the start of this time span, the center of mass is at $x_{cm} = x_1 = 9.6m$, so at the end its position is:

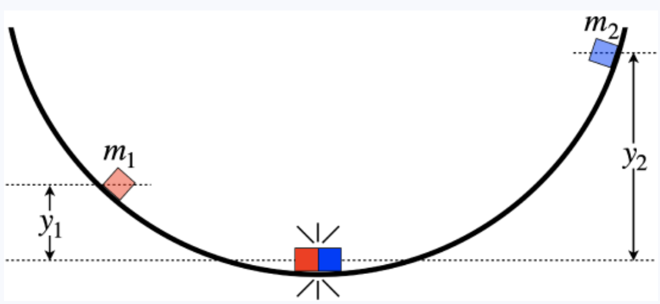
$$x_{cm} = x_1 + v_{cm}t_2 = 9.6m + \left(2.4\frac{m}{s}\right)(3.0s) = 16.8m$$

The blocks are equal masses, so their center of mass is halfway between them, giving us the locaton of the top block:

$$x_{cm} = \frac{x_{top} + x_{bottom}}{2} \quad \Rightarrow \quad x_{top} = 2x_{cm} - x_{bottom} = 2\,(16.8m) - 18.0m = 15.6m$$

Problem 4.6

Two blocks slide down opposite sides of a frictionless curved ramp from different heights, colliding at the exact bottom, as shown in the diagram below. Upon colliding, they stick together, and move as a single entity thereafter (if they move at all).



The masses of the blocks are equal, and both blocks are released from rest. After they collide, their center of mass returns to the starting height of the left block.

a. *Find how many times higher the right block is released than the left block.*
b. *Find the fraction of the original mechanical energy is converted into thermal energy.*

Solution

a. The analysis showed that the height the double-block rises before stopping is related to the momentum of the two-block system, which can be determined from the individual momenta of the blocks. So calling the equal masses just "m", the starting height of the left block "h", and choosing rightward as the positive direction (so the starting momentum of the right block and final momentum of the two blocks are negative):

$$\left.\begin{array}{l} p_1=\sqrt{2m_1KE_1}=\sqrt{2m_1U_1}=m\sqrt{2gh} \\ p_2=-\sqrt{2m_2KE_2}=-\sqrt{2m_2U_2}=-m\sqrt{2gy_2} \\ p_f=-\sqrt{2\left(m_1+m_2\right)KE_f}=-\sqrt{4mU_f}=-2m\sqrt{gh} \\ p_f=p_1+p_2 \end{array}\right\} \quad -2m\sqrt{gh}=m\sqrt{2gh}-m\sqrt{2gy_2} \quad \Rightarrow \quad y_2=9h$$

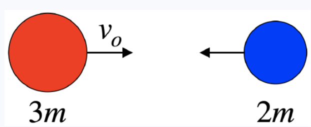
b. The original mechanical energy is $U_1+U_2=mgh+9mgh=10mgh$. After the double-block comes to rest, the mechanical energy of the system is $U_f=2mgh$. Therefore the percentage of mechanical energy converted to thermal in the collision is:

$$\text{fraction converted}=\frac{10mgh-2mgh}{10mgh}=0.8$$

This is a perfectly inelastic collision, so the reader may be confused about why this fraction is equal to 0.5, according to Equation 4.4.3. Be careful! That equation applies specifically to the frame where m_1 is the projectile and m_2 is a stationary target. If we wanted to do lots of extra work, we could change to the rest frame of the left block (making the right block the projectile, which in this frame has all of the system's incoming energy), and use Equation 4.4.3 to obtain the same result.

Problem 4.7

The diagram below depicts a moment just before a collision of two balls made of bouncetech™, a material made by an engineering firm that develops new materials. This experiment was set up as a head-on collision in the center of mass reference frame of the balls. The company's goal is to lose as little kinetic energy as possible to thermal energy in the bounce. To their absolute horror, the two balls stick together! They determine the kinetic energy converted to thermal in this collision to be E_o. They re-check their bouncetech™ formula, and realize that they left out an important ingredient, bounconium. When they repeated the experiment with the corrected mix, they got a much better result.



The red ball bounces back with two-thirds the speed at which it came into the collision.

a. *Find the fraction of kinetic energy that is lost to thermal.*
b. *To please shareholders, the company unscrupulously decides to report to the fraction of kinetic energy lost to thermal as measured in the lab frame where the blue ball was a stationary target. Compute this fraction.*

Solution

a. In the analysis we found that the total kinetic energy in the center of mass frame is $\frac{15}{4}mv^2$. The red ball slows to two-thirds of its initial speed, and since the speeds of the red and blue balls must have the same ratio of speeds to maintain zero system momentum in this frame, it too must bounce away with two-thirds of its initial speed. The kinetic energies of the balls are proportional to the squares of their speeds, so the kinetic energy of both (and therefore their sum) must drop to $\left(\frac{2}{3}\right)^2=\frac{4}{9}$ of its original value. Thus $\frac{5}{9}$ of the rest frame kinetic energy is converted to thermal.

b. The actual amount of energy converted to thermal can be computed from the fraction found above:

$$E_{therm} = \frac{5}{9}E_o = \frac{5}{9}\left(\frac{15}{4}mv_o^2\right) = \frac{75}{36}mv_o^2$$

The thermal energy converted is the same in any frame, but the combined kinetic energy of the balls in frames other than the center of mass frame is greater, so the ratio of energy converted to thermal is smaller. We determined in the analysis that the blue ball is moving at a speed of $\frac{3}{2}v_o$*, so changing frames to one where it is stationary changes the speed of the red ball to* $v_o + \frac{3}{2}v_o = \frac{5}{2}v_o$ *, moving to the right. The blue ball contributes no kinetic energy in this frame, so the system's total kinetic energy in this frame is:*

$$KE_{lab} = \frac{1}{2}(3m)\left(\frac{5}{2}v_o\right)^2 = \frac{75}{8}mv_o^2$$

The fraction of energy converted in this reference frame is thus:

$$\frac{E_{therm}}{KE_{lab}} = \frac{\frac{75}{36}}{\frac{75}{8}} = \frac{2}{9}$$

Losing 22% of its kinetic energy sure looks a lot better in shareholder reports than losing 56% of it!

Problem 4.8

A large sled is at rest on a horizontal, frictionless sheet of ice, when a heavy rock is thrown onto it from behind. The rock is moving purely horizontally when it comes into contact with the sled, and it skids across the rough top surface of the sled until it and the sled are moving forward together at the same speed.

The mass of the rock is 6.5kg, and it is moving at a speed of $1.8\frac{m}{s}$ *when it lands on the* $13.0kg$ *sled.*

a. *Find the amount of energy converted to thermal from kinetic.*
b. *Find the ratio of the distance the sled slides on the ice to the length of the rock's skid-mark on the top of the sled.*

Solution

a. From the analysis, we have the fraction of the energy converted in this stationary target case, and since we have the rock's mass and initial speed, we know the starting energy, so:

$$E_{thermal} = \left(\frac{m_2}{m_1+m_2}\right)E_o = \left(\frac{m_{sled}}{m_{rock}+m_{sled}}\right)\left(\frac{1}{2}m_{rock}v_o^2\right) = \left(\frac{13.0kg}{6.5kg+13.0kg}\right)\left(\frac{1}{2}(6.5kg)\left(1.8\frac{m}{s}\right)^2\right) = 7.02J$$

b. The friction force exerted on the rock is the same magnitude as the friction force exerted on the sled (Newton's 3rd law). The energy converted to thermal is the work done on the rock by friction over the distance of the skid-mark:

$$E_{thermal} = f\Delta x_{skid}$$

The work done on the sled by the friction force is its change in kinetic energy, and it starts from rest, so:

$$\frac{1}{2}m_{sled}v_f^2 = f\Delta x_{sled}$$

Dividing these two equations causes the friction force to cancel out, leaving the ratio we are looking for:

$$\frac{\Delta x_{sled}}{\Delta x_{skid}} = \frac{\frac{1}{2}m_{sled}v_f^2}{E_{thermal}}$$

So the ratio we are looking for is the same as the ratio of the thermal energy created to the final kinetic energy of the sled. We can put both of these quantities in terms of the initial energy carried-in by the rock. The relationship between the thermal *energy and the initial energy is given above. For the sled's kinetic energy, we use momentum conservation for the perfectly inelastic case that relates the final velocity in terms of initial:*

$$v_f = \frac{m_{rock}}{m_{rock}+m_{sled}}v_o \quad \Rightarrow \quad \frac{1}{2}m_{sled}v_f^2 = \frac{m_{sled}m_{rock}}{\left(m_{rock}+m_{sled}\right)^2}\left(\frac{1}{2}m_{rock}v_o^2\right) = \frac{m_{sled}m_{rock}}{\left(m_{rock}+m_{sled}\right)^2}E_o$$

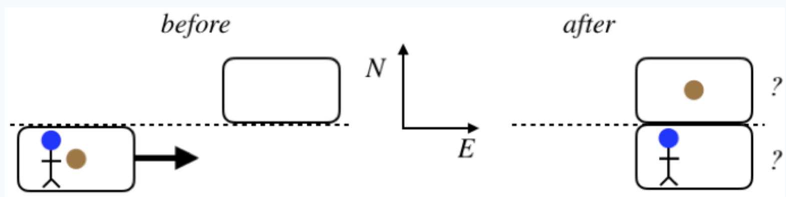
Now form the ratio, and perform algebra:

$$\frac{\Delta x_{sled}}{\Delta x_{skid}} = \frac{\frac{m_{sled}m_{rock}}{\left(m_{rock}+m_{sled}\right)^2}E_o}{\left(\frac{m_{sled}}{m_{rock}+m_{sled}}\right)E_o} = \frac{m_{rock}}{m_{rock}+m_{sled}} = \frac{1}{3}$$

So the sled slides along the ice a distance that is one third the length of the skid-mark.

Problem 4.9

A cart slides along a frictionless surface in an easterly direction. The cart contains a person and a medicine ball. The cart slides past an identical (but empty) stationary cart, also on the frictionless surface. When the carts are side-by-side, the person throws the medicine ball into the other cart by pushing the ball in the north direction.



Both carts have masses of $100kg$, the person's mass is $60.0kg$, and the medicine ball has a mass of $15.0kg$. The speed of the cart before is $2.40\frac{m}{s}$. The person sees the ball move away from them at a speed of $3.20\frac{m}{s}$, and it comes to rest inside the other cart. Find the speed and direction of both carts after the medicine ball has been exchanged. Express the directions as angles that are north or south (indicate which) of east.

Solution

We'll start with what we know immediately – the x-component of the velocity of cart #1 never changes, since the ball is thrown in the y-direction:

$$v_{1x} = v_o = 2.40\frac{m}{s}$$

We anticipated this in the analysis – the relative motion of the person and medicine ball are given. We'll start by computing the x-component of the motion of cart #2 + ball. We can do this because we know that the ball does not change its x-component of velocity after being pushed:

$$m_{ball}v_o = \left(m_{ball}+m_{cart}\right)v_{2x} \quad \Rightarrow \quad v_{2x} = \left(\frac{m_{ball}}{m_{ball}+m_{call}}\right)v_o = \left(\frac{15.0kg}{15.0kg+100kg}\right)\left(2.40\frac{m}{s}\right) = 0.313\frac{m}{s}$$

As we found in the analysis:

$$0 = m_{ball}v_b + \left(m_{cart}+m_{person}\right)v_p$$

We are not given v_b (which is the y-component of the ball's velocity relative to the Earth), but we can write it in terms of person's y-component of velocity v_p and the relative speed v_{bp}, as we did in the analysis:

$$v_b = v_{bp} + v_p$$

Plugging this in above gives us the y-component of the person + cart 1:

$$0 = m_{ball}\left(v_{bp}+v_p\right) + \left(m_{cart}+m_{person}\right)v_p$$
$$\Rightarrow \quad v_{1y} = v_p = -\left(\frac{m_{ball}}{m_{ball}+m_{person}+m_{cart}}\right)v_{bp} = -\left(\frac{15.0kg}{15.0kg+60.0kg+100kg}\right)\left(3.20\frac{m}{s}\right) = -0.274\frac{m}{s}$$

With the two components of the final velocity for cart 1 now known, we can write its magnitude and direction:

$$v_1=\sqrt{v_{1x}^2+v_{1y}^2}=\sqrt{\left(2.40\frac{m}{s}\right)^2+\left(-0.274\frac{m}{s}\right)^2}=2.42\frac{m}{s}$$

$$\theta_1=\tan^{-1}\left(\frac{-0.274}{2.40}\right)=6.51^o\text{ south of east}$$

Now we turn to cart 2 + ball. We have already solved for the y-component of velocity for cart 1 + person, and since the total system started with no total momentum in the y-direction, cart 2 + ball must have a momentum that cancels that of cart 1 + person:

$$\left(m_{cart}+m_{ball}\right)v_{2y}=\left(m_{cart}+m_{person}\right)v_{1y}\quad\Rightarrow\quad v_{2y}=\left(\frac{100kg+60.0kg}{100kg+15.0kg}\right)\left(0.274\frac{m}{s}\right)=0.381\frac{m}{s}$$

We already have the velocity of cart 2 in the x-direction, so we put it together with this y direction result to get our answers:

$$v_2=\sqrt{v_{2x}^2+v_{2y}^2}=\sqrt{\left(0.313\frac{m}{s}\right)^2+\left(0.381\frac{m}{s}\right)^2}=0.493\frac{m}{s}$$

$$\theta_2=\tan^{-1}\left(\frac{0.381}{0.313}\right)=50.6^o\text{ north of east}$$

CHAPTER OVERVIEW

5: Rotations and Rigid Bodies

5.1: Rotational Kinematics

5.2: Rotational Inertia

5.3: Dynamics of Rotating Objects

5.4: Torque

5.5: Static Equilibrium

Sample Problems

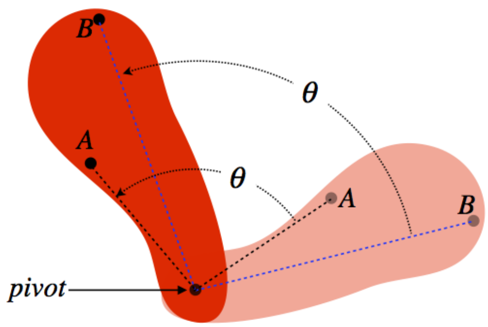
5.1: Rotational Kinematics

Our first foray into linear motion was with kinematics, and we start our discussion of rotation with the same topic.

Rigid Body Rotation

Whenever we talk about "rotation," there is something that is generally implied – we are not talking about a point mass or a collection of independently-moving point masses. Instead, we are generally referring to the rotation of a rigid object. A rigid object is nothing more than a collection of particles that are confined to stay at specific positions relative to each other. When we talk about rotation, all these particles follow different paths and travel different distances, but they all have something in common.

Figure 5.1.1 – Motion of Two Points on a Rotating Rigid Body



Drawing a straight line from the fixed point (called the *pivot*) to two different points on the object, we see that the angles through which these straight lines sweep are the same, and indeed this is true for *every* point on the object. So as we talk about rigid body rotation, our old language of linear motion (displacement, velocity, acceleration) that is based on units of distance and time, will have to give way to a new language for rotational motion, based on the units of radians (the most common unit of angular measure) and time. This language will be very similar to what we used for the linear case, usually with the word "angular" or "rotational" appended in front of the usual words.

Just because we are going to a new language, it doesn't mean we throw out the physical principles we have learned so far. But to apply them in our new area of study, we need to develop some way to translate between the two. Back in Section 1.7, in our discussion of circular motion, we came up with a translation between the arclength traveled by an object in circular motion and the angle is motion sweeps out. Certainly the points A and B in the figure above are following a circular path (they remain a fixed distance from the pivot), so this relation applies to them. If a given point on a rigid body is a distance r from the pivot, then the relationship between the distance it travels along the arclength and the angle measured in radians is given by Equation 1.7.2, and the relationship between its linear speed and the rate at which the angle is changing (in radians per second) is given by Equation 1.7.3, both of which we'll reiterate here:

$$s = R\theta, \qquad v = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \tag{5.1.1}$$

While s and v are different for every point on the rigid object, we see that θ and ω are common to all of them. We therefore embrace these as our *angular displacement* and *angular velocity* measurements, respectively, for the rigid body as a whole. We can similarly define an *angular acceleration* (α) in terms of the change of the linear speed of a spot on the rotating object:

$$a = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{5.1.2}$$

While each point mass comprising the rigid object may have its own linear velocity/acceleration, they all share a common angular velocity/acceleration. We therefore can simplify our discussion of rigid body rotation from tracking the many different motions of all of the individual parts of the object to one simple parameter common to all of them. We therefore (for the moment) step away from the translation between linear and angular motion – which we have already discussed in earlier sections – and instead focus on purely rotational motion, following exactly the same path as we did for linear motion. You'll note that as a rule the convention for rotational motion, we stick with Greek variables, in contrast to the Latin variables we used for linear motion.

Alert

Whenever the word "acceleration" is combined with circular motion, one naturally thinks of centripetal acceleration. Be careful not to make that association here! The link between linear acceleration and angular acceleration is through the component of acceleration responsible for speeding up the spot on the rigid object, not the acceleration responsible for changing its direction of motion (which is centripetal acceleration). So for example, an object rotating at a constant rate has no point on it that is speeding up (and has zero angular acceleration), but every point on it (except at the pivot) experiencing a centripetal acceleration. Conversely, a rotating object that slows down, stops, and reverses its direction of motion is experiencing angular acceleration at all times, including the moment it stops, but the centripetal acceleration of points on the object is zero at the moment that it stops.

We can fully clarify the role of angular and centripetal acceleration mathematically. For a point on the object, its acceleration has two components:

$$\overrightarrow{a} = \overrightarrow{a}_\perp + \overrightarrow{a}_\parallel, \quad where: \begin{cases} a_\perp = a_c = r\omega^2 \\ a_\parallel = r\alpha = r\dfrac{d\omega}{dt} \end{cases} \tag{5.1.3}$$

Rotational Equations of Motion

We define the following angular (rotational) versions of what we studied previously in kinematics:

$$\begin{aligned} position & : \theta(t) \\ displacement & : \Delta\theta = \theta_2 - \theta_1 \\ average\ velocity & : \omega_{ave} = \frac{\Delta\theta}{\Delta t} \\ instantaneous\ velocity & : \omega(t) = \frac{d\theta}{dt} \\ average\ acceleration & : \alpha_{ave} = \frac{\Delta\omega}{\Delta t} \\ instantaneous\ acceleration & : \alpha(t) = \frac{d\omega}{dt} \end{aligned} \tag{5.1.4}$$

The calculus that leads to the equations of motion works out exactly the same way as before (we have only changed the variable names), giving us:

$$\begin{aligned} \theta(t) = & \quad \theta_o + \omega_o t + \tfrac{1}{2}\alpha t^2 \\ \omega(t) = & \quad \omega_o + \alpha t \\ \omega_f^2 - \omega_o^2 = & \quad 2\alpha\left(\Delta\theta\right) \\ \omega_{ave} = & \quad \frac{\omega_o + \omega_f}{2} \quad (if\ \alpha = constant) \end{aligned} \tag{5.1.5}$$

Note that like the case of one-dimensional linear motion, we need to define at the outset a "positive" direction, but for rotation, this means choosing clockwise or counterclockwise from a specific perspective.

Analyze This

A bug stands on the outer edge of a turntable as it begins to spin, accelerating rotationally in the horizontal plane from rest at a constant rate. The bug is held on the turntable by static friction, but as the turntable spins ever faster, this will not remain the case forever.

Analysis

The static friction force is responsible for the bug's acceleration, which can be broken into two components – radial and tangential. These acceleration components are shown in Equation 5.1.3. The bug will slide off the turntable when the static friction force is insufficient to maintain this acceleration. The maximum static friction force is the coefficient of static friction multiplied by the normal force between the turntable and the bug, and since the turntable is horizontal and not accelerating up or down, this normal force equals the weight of the bug. We therefore can say that the bug will start to fall off the turntable when the following holds:

$$\left|\vec{F}_{net}\right| = \mu_s mg$$

The magnitude of the net force can be written in terms of the magnitude of the acceleration, so:

$$\left|\vec{F}_{net}\right| = m\left|\vec{a}\right| \quad \Rightarrow \quad \mu_s g = \sqrt{a_\perp^2 + a_\parallel^2} = \sqrt{\left(r\omega^2\right)^2 + \left(r\alpha\right)^2} = r\sqrt{\omega^4 + \alpha^2}$$

And finally, we should note that the angular acceleration and angular velocity are related. The turntable starts from rest, so putting this into the usual kinematics equations gives:

$$\omega = \alpha t$$
$$2\alpha\theta = \omega^2$$

Whichever of these relationships is more useful can then be plugged back in above to reduce the number of unknowns.

Directions of Rotational Kinematics Vectors

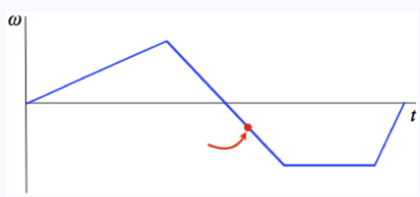
When we did all of this previously, we found it was easy to keep track of directions in one dimension, simply by checking the sign of the value, but when we had to go to more dimensions, we needed to treat these quantities like vectors. How can we do that for this rotational vectors?

The answer comes from all the way back in Chapter 1 – the Right Hand Rule! It goes like this: curl the fingers of your right hand (in their natural finger-curling manner) in the direction that the object is rotating, and your thumb points the direction of the vector. The direction is *perpendicular to the plane of rotation*.

This direction applies to all of the angular motion vectors – displacement, velocity, and acceleration. But be careful about the acceleration vector! Just as in the linear case, the acceleration vector points in the direction of the *changing* velocity vector, not the direction of the velocity vector itself. So if a rotating object is slowing down, the angular acceleration vector points in the opposite direction as the angular velocity vector.

Conceptual Question

The graph below depicts the rotational velocity of a merry-go-round as a function of time, where the positive direction is defined to be downward (into the surface of the Earth). You are standing near the merry-go-round, watching children go by. At the point indicated in the graph, which of the following are you seeing?



a. *The kids closest to you are moving to the right and are speeding up.*
b. *The kids closest to you are moving to the right and are slowing down.*
c. *The kids closest to you are moving to the left and are speeding up.*
d. *The kids closest to you are moving to the left and are slowing down.*
e. *The kids closest to you are moving to the left, but their speed is not changing.*

Solution

(a) From the RHR, we determine that the positive rotational direction is clockwise as you look at the merry-go-round from above (the kids on the merry-go-round are wondering why you are apparently giving their ride a thumbs-down!). Looking at it from ground level, this means that rotation in a positive direction results in seeing the nearest kids go by from right-to-left. At the point in question, the sign of the rotational velocity is negative, which means the kids are going by left-to-right. A short time later, the rotational velocity will be more negative, which means they are speeding up.

5.2: Rotational Inertia

Rotational Kinetic Energy and Rotational Inertia

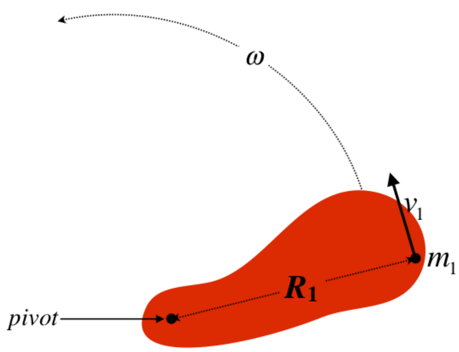
While our first approach to studying dynamics for linear motion was Newton's laws (forces cause accelerations), we will find it easier to examine rotational dynamics from a standpoint of energy first. Consider an object that is rotating around a stationary center of mass. Does such an object possess kinetic energy? We might be inclined to say that it does, but with the center of mass not moving, its momentum is zero, which would make the quantity $\frac{p^2}{2m}$ also equal to zero.

Rigid objects are collections of multiple particles, and when they are rotating, all those particles (except those right at the pivot point) are moving, which means they all have kinetic energy. At any given moment, there are particles moving in opposite directions, and if the center of mass of the object is stationary, these opposite momenta (which are vectors) cancel, Their kinetic energies, on the other hand, are not vectors, and are all positive numbers, so they can never cancel out.

In some sense, the particles comprising a rotating object can be thought of as contributing to the "internal" energy of the object as we discussed back in Section 3.2. But doing this runs contrary to the main reason for the introduction of the mechanical/internal energy idea, which was to separate the kinetic energy of the system *that we can clearly see* from the kinetic energy that is *concealed from us* inside the confines of the system. We can clearly see rotational motion of an object, so we choose to include rotational kinetic energy in the category of "mechanical energy."

Okay, so a rotating object does possess kinetic energy. Our task now is to express that kinetic energy in terms of the rotation variables we have already defined, but all we know about kinetic energy is the linear version. In the figure below we consider the motion of a single particle within a rigid rotating object.

Figure 5.2.1 – Motion of a Single Particle in a Rotating Rigid Body



This is particle #1 – one of many within the rigid object. We can write down its kinetic energy, and in fact we can express it in terms of a rotational variable and the particle's distance from the pivot:

$$KE_1=\frac{1}{2}m_1v_1^2=\frac{1}{2}m_1(R_1\omega)^2=\frac{1}{2}m_1R_1^2\omega^2 \tag{5.2.1}$$

If we want the total kinetic energy of the object, we need to add up the kinetic energy of all the particles. Thanks to our definition of angular velocity, we can factor that part out of all the terms:

$$KE_{whole\ object}=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2+\dots=\frac{1}{2}m_1R_1^2\omega^2+\frac{1}{2}m_2R_2^2\omega^2+\dots=\frac{1}{2}\left[m_1R_1^2+m_2R_2^2+\dots\right]\omega^2 \tag{5.2.2}$$

Notice that the quantity in brackets in the final equality is determined solely by the distribution of mass throughout the object. That is, it is an intrinsic property of the object and the choice of pivot, not dependent upon how it is moving. We generally abbreviate this quantity with an I, which gives us a familiar form for the kinetic energy formula:

$$KE_{rotating\ rigid\ body}=\frac{1}{2}I\omega^2,\qquad where:\ \ I\equiv m_1R_1^2+m_2R_2^2+\dots \tag{5.2.3}$$

This looks just like the linear kinetic energy formula, with the angular speed replacing the linear speed, and I replacing the mass. This quantity certainly contains some information about the mass of the object, but it is more complicated than just the mass, and is called the *rotational inertia*, or more commonly (and less descriptively), the *moment of inertia*. Notice that this “inertia” depends not only upon the amount of stuff (mass), but also where that mass is. This means that two different objects can actually weigh exactly the same amount, but when they are rotated at equal speeds, one of them has more KE than the other. As you might guess, this occurs when more of the mass is concentrated farther from the pivot for the former object than the latter.

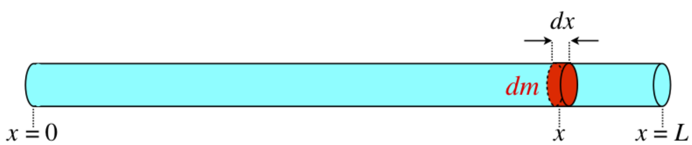
Alert

It is important to note that we will only be considering rotations around ***axes****, not points. In our two-dimensional figures, an axis that is perpendicular to the plane of the figure is indistinguishable from a single point, but we will not discuss motion that involves an object's motion changing its plane of rotation. So rotational inertia for three-dimensional objects involves the distances of the tiny masses from a common axis, not a common point.*

Calculating Rotational Inertia for Continuous Objects

Our task is to compute the rotational inertia, for which the formula in terms of masses and their positions is different from the one for center of mass (see Section 4.2), but the procedure is exactly the same. We start with the same picture (Figure 4.2.3, which is reproduced below), and convert the sums into integrals, as before.

Figure 5.2.2 – Setup Diagram for Computations Involving Mass Density of a Thin Rod



$$I = dm_1x_1^2 + dm_2x_2^2 + \dots = \int\limits_{x=0}^{x=L} dm\; x^2 \tag{5.2.4}$$

Note that the rotational inertia is calculated around a specific pivot point, which we have chosen to be our origin for the calculation.

As before, we replace the dm with $\lambda\left(x\right)dx$, and we have our formula for the rotational inertia along the x-axis around the pivot point at the origin:

$$I = \int\limits_{x=0}^{x=L} \lambda\left(x\right)x^2dx \tag{5.2.5}$$

Let's return to the cases for which we computed the centers of mass in Section 4.2 – the uniform and non-uniform rod. Unlike the case of center of mass, where the answer is a location on the rod, the final answer for the rotational inertia will have units of $kg\cdot m^2$, and the formula for it will involve the total mass of the rod and its length. Also it is important to remember that while the center of mass is a location that doesn't depend upon where we put our coordinate system to calculate it, the rotational inertia is only defined relative to a specific pivot point.

A Uniform Rod of Mass M and Length L, Pivoted About an End

Plugging the constant λ into Equation 5.2.5 and performing the integral gives:

$$I = \int\limits_{x=0}^{x=L} \lambda x^2dx = \lambda\left[\frac{1}{3}x^3\right]_0^L = \frac{1}{3}\lambda L^3 \tag{5.2.6}$$

We are not finished yet, because this answer is not in terms of the rod's mass. Since this rod is uniform, the mass is simply the (constant) density multiplied by its length, which gives:

$$I = \frac{1}{3}\left(\frac{M}{L}\right)L^3 = \frac{1}{3}ML^2 \tag{5.2.7}$$

We will find that every rotational inertia we encounter has this basic form: A constant (usually written as a fraction) multiplied by the mass of the object and the square of some natural length dimension of the object. In this case it is the length of the rod, but it may also be something like the radius of a disk or sphere.

A Non-Uniform Rod of Mass M and Length L, Pivoted About Its Lighter End

Now we repeat the process for the non-uniform density function for which we computed the center of mass in Section 4.2:

$$\lambda\left(x\right)=\lambda_o\left(\frac{x}{L}+1\right) \tag{5.2.8}$$

Note that unlike the uniform case, the results should not come out the same for both ends of the rod, since more of the mass is concentrated near the end at $x=L$. We are calculating this rotational inertia about the lighter end, since all of the x values in the integral are measured from that end.

$$I=\int\limits_{x=0}^{x=L}\lambda_o\left(\frac{x}{L}+1\right)x^2dx=\lambda_o\left[\frac{1}{4L}x^4+\frac{1}{3}x^3\right]_0^L=\frac{7}{12}\lambda_oL^3 \tag{5.2.9}$$

We are not done yet, because we are given the mass of the rod, not the constant λ_o. We therefore need to compute the total mass in terms of this constant. We do this by integrating density function over the length of the rod:

$$M=\int\limits_{x=0}^{x=L}\lambda_o\left(\frac{x}{L}+1\right)dx=\lambda_o\left[\frac{1}{2L}x^2+x\right]_0^L=\frac{3}{2}\lambda_oL \quad\Rightarrow\quad \lambda_o=\frac{2}{3}\frac{M}{L} \tag{5.2.10}$$

Plugging this back in above gives our answer:

$$I=\frac{7}{12}\left(\frac{2}{3}\frac{M}{L}\right)L^3=\frac{7}{18}ML^2 \tag{5.2.11}$$

Exercise

Find the rotational inertia of the non-uniform rod of mass M and length L whose mass density function is given by Equation 5.2.8, when rotated about its heavier end ($x=L$).

Solution

The difference between this calculation and the one above is that the variable x that appears in Equation 5.3.5 doesn't match the x that appears in the density formula. The density formula is referenced to our coordinate system, but the x in the rotational inertia integral represents the distance of each tiny piece of mass dm from the pivot point at $x=L$. So we need to make a change in the integral so that the x variable that appears in it matches the x in the density function. Making the substitution $x\rightarrow L-x$ (so $dx\rightarrow -dx$), into the integral does the trick, because then the integrand is zero at the pivot point ($x=L$) as it should be:

$$\begin{aligned} I_{heavy\ end} &=\int\limits_{x=L}^{x=0}dm\ \left(x-L\right)^2=\int\limits_{x=L}^{x=0}\lambda\left(x\right)\left(x-L\right)^2\left(-dx\right)=\int\limits_{x=0}^{x=L}\lambda_o\left(\frac{x}{L}+1\right)\left(x-L\right)^2dx \\ &=\lambda_o\int\limits_{x=0}^{x=L}\left(\frac{x^3}{L}-x^2-xL+L^2\right)dx=\lambda_o\left[\frac{x^4}{4L}-\frac{x^3}{3}-\frac{x^2L}{2}+xL^2\right]_0^L \\ &=\tfrac{5}{12}\lambda_oL^3 \end{aligned}$$

We need to plug in for λ_o (which was computed above) to get our final answer:

$$I_{heavy\ end}=\frac{5}{12}\left(\frac{2}{3}\frac{M}{L}\right)L^3=\frac{5}{18}ML^2$$

Principal Axes

It’s clear that the choice of the pivot is important to the calculation of the rotational inertia, but so is the axis. Real objects are 3-dimensional, so they actually have 3 independent rotation axes, each of which has its own rotational inertia around it. These axes are called the *principal axes*. The origin of these axes is located at – what else? – the center of mass of the object. The principal axes are only easy to identify for objects with some degree of symmetry. Some objects are so symmetric that more than one set of axes will work. For example, a uniform sphere has so much symmetry that any set of three mutually perpendicular axes whose origin coincides with the center of the sphere will work, and of course the rotational inertias around all these axes are the same.

The reason it is natural to define the origin of the principal axes to be at the center of mass is that if an object is rotating freely in space with no forces on it, its axis of rotation *must* pass through its center of mass (though it doesn't need to be around one of the principal axes). This is actually surprisingly easy to prove. Suppose an object was rotating around an axis that does not pass through the center of mass. This would mean that the center of mass is moving in a circle around the axis of rotation. But circular motion is accelerated motion. According to Newton's second law, the center of mass cannot be accelerating if there are no forces on the object, which contradicts our assumption.

Computing Rotational Inertia Without Integration

Throughout our study of mechanics, our goal has been to develop shortcut tools to help us deal with physical systems in simpler ways. We developed work-energy so that we could solve problems that pay no attention to direction or time without slogging through Newton's laws (such as speed at a given height on a loop-de-loop). We developed impulse-momentum so that we could more easily solve problems involving systems in which the internal forces are complicated (such as collisions). Now we are developing a tools related to rigid body rotations so that we don't have to track the linear motions of all the particles in the system. With this very practical mindset, it is not surprising that physicists have developed tools for computing rotational inertia that avoid the ugliness of always having to perform integrals. The first such shortcut is simply a collection of rotational inertias that are associated with common symmetric geometries, such as rods, disks, and spheres. Our collection is given at the end of the section. There are two tools that we can combine with our collection of rotational inertias that will allow us to "bootstrap" our way to determining many more.

Additivity Around a Common Axis

Suppose we know the rotational inertias of two separate objects around a common axis. If these two objects are attached so that they rotate together rigidly around that common axis, then the rotational inertia of the combined object is simply the sum of their rotational inertias. This is evident from the formula for rotational inertia: Each object has its own sum of mx^2 terms, and when the objects are combined such that their x axes are common, then the new sum of mx^2 terms is simply the combination of the two individual sums. To summarize:

$$I = I_1 + I_2 \tag{5.2.12}$$

Exercise

Use the additive property of rotational inertia and the result given by Equation 5.2.7 to find the rotational inertia of a uniform thin rod of mass M and length L about its center of mass.

Solution

We can treat a rod rotated around an axis through its center as if it is two separate half-rods of half the mass and half the length, attached at their ends. The axis that passes though the center of the rod passes through the ends of these two half-rods, and we know the rotational inertia of each half-rod. The additivity property then gives us the rotational inertia of the whole rod about its center:

$$I_{uniform\ thin\ rod\ about\ its\ center} = 2I_{half-rod\ about\ end} = 2\left[\frac{1}{3}\left(\frac{M}{2}\right)\left(\frac{L}{2}\right)^2\right] = \frac{1}{12}ML^2$$

Parallel Axis Theorem

As we have seen multiple times already, just changing the axis around which an object is rotated will result in a different rotational inertia. Suppose we calculate the rotational inertia of an object about an axis, then slide that axis in a parallel fashion on the object, and calculate the new rotational inertia, then do it over and over, recording the new values each time. One might ask, "Where is the axis (parallel to the original one) for which the rotational inertia is the *smallest*?" Is there any way to guess where this might be, and is it unique, or might there be multiple places where the rotational inertia hits a minimum?

To answer this question, let's look at a one-dimensional object that lies along the x-axis, and consider its rotational inertia around the y-axis. Writing it as a sum rather than an integral, it is:

$$I = m_1x_1^2 + m_2x_2^2 + \ldots \tag{5.2.13}$$

Now let's suppose we decide to change where we place the origin, moving it a distance $+x$ along the x-axis. When we do this, the distance from the axis to mass m_1 changes from x_1 to $x_1 - x$. Also, since the original axis went through the origin, this new axis is no longer the y-axis – now it intersects the x-axis at x. The new rotational inertia is, therefore:

$$I = m_1 (x_1 - x)^2 + m_2 (x_2 - x)^2 + \ldots \tag{5.2.14}$$

We can consider this to be a function of x. That is, this formula provides the rotational inertia of the object about the axis located at x. We can now answer our question about where the rotational inertia is a minimum by using calculus. The value of x for which the function $I(x)$ is a minimum satisfies:

$$0 = \frac{dI}{dx} = -2m_1 (x_1 - x) - 2m_2 (x_2 - x) + \ldots \tag{5.2.15}$$

Solving for x here provides a familiar result:

$$x = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots} \tag{5.2.16}$$

The rotational inertia of an object for all axes parallel to each other is a minimum for the axis that passes through the center of mass! Actually, this should not be too surprising. The rotational inertia of an object will be minimized around an axis that is as close as possible to as much of the object's mass as possible, and the center of mass is the "average location of mass," so it makes sense that this would be "as close to as much of the object's mass as possible."

Given this information, we can write the rotational inertia of an object around an axis parallel to an axis passing through the center of mass a positive-valued "adjustment" to the rotational inertia around the center of mass. It turns out (we will not prove it here) that this adjustment is quite simple – it is just the mass of the object multiplied by the square of the offset distance between the new axis and the axis through the center of mass. This is called the *parallel axis theorem*:

$$I_{new} = I_{cm} + Md^2, \tag{5.2.17}$$

where d is the distance separating the new axis and the center of mass.

Exercise

Use the parallel axis theorem and the result given by Equation 5.2.7 to find the rotational inertia of a uniform thin rod of mass M and length L about its center of mass.

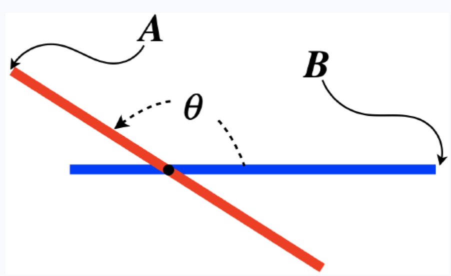
Solution

The distance that the end of the rod is separated from the rod's center of mass is $d = L/2$. Plugging this into the parallel axis theorem gives our answer, which agrees with what we got in the previous Exercise:

$$I_{new} = I_{cm} + Md^2 \quad \Rightarrow \quad I_{cm} = I_{new} - Md^2 = \frac{1}{3}ML^2 - M\left(\frac{L}{2}\right)^2 = \left(\frac{1}{3} - \frac{1}{4}\right)ML^2 = \frac{1}{12}ML^2$$

Conceptual Question

Two straight metal rods with equal lengths but differing masses are firmly welded together at their centers of mass so that they make an angle θ, as in the diagram below. The mass of one of the rods is uniformly-distributed along its length, while the other has a non-uniform mass distribution. If this rigid object is now rotated around one of the points A or B at some fixed rotational speed ω, around which point will the object have the most kinetic energy?



a. *A*

b. *B*
c. *It could be either A or B, depending upon which rod has more mass.*
d. *It could be either A or B, depending upon the value of θ.*
e. *Both (c) and (d) are true.*

Solution

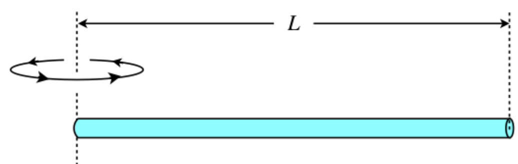
(b) The centers of mass of the two rods coincide, so the center of mass of the two rod system is at the same place (where they are welded together). The rotational inertia around some pivot is (by the parallel axis theorem) equal to the rotational inertia around the center of mass plus Md^2, where M is the mass of the object and d is the distance from the center of mass to the pivot. So for this object, the rotational inertia will be greater around the point that is farther from the center of mass. In this case, that point is B. It doesn't matter which rod has more mass, as the total mass is the same in either case, and it doesn't matter what the angle is, as the two points A and B don't change their distance from the center of mass when the angle is changed.

Rotational Inertias of Some Common Geometries

In all of the cases indicated below, the mass of the object is M, and the material making up the object has uniform density. The reader is encouraged (as an exercise) to navigate their way between various relations using the additivity and parallel axes theorem tools. [*Note: When it comes to rotating two-dimensional objects such as rings and disks, we will confine our studies to axes perpendicular to the two-dimensional planes in which these objects lie. For rotations around axes parallel to this plane, one would need yet another useful tool, known as the perpendicular axes theorem.*]

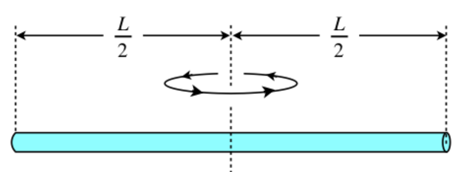
Thin Rods

Figure 5.2.3 – Thin Straight Rod Rotated About One end



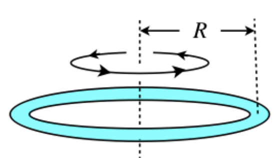
$$I = \frac{1}{3}ML^2 \tag{5.2.18}$$

Figure 5.2.4 – Thin Straight Rod Rotated About Center



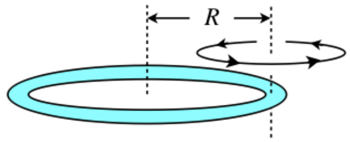
$$I = \frac{1}{12}ML^2 \tag{5.2.19}$$

Figure 5.2.5 – Thin Circular Ring (or Thin Cylindrical Shell) Rotated About Center



$$I = MR^2 \tag{5.2.20}$$

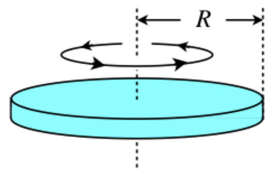
Figure 5.2.6 – Thin Circular Ring (or Thin Cylindrical Shell) Rotated About Edge



$$I=2MR^2 \tag{5.2.21}$$

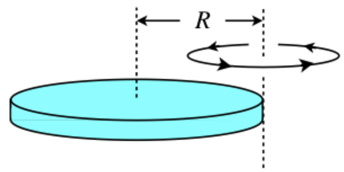
Disks (or Cylinders)

Figure 5.2.7 – Solid Disk (or Cylinder) Rotated About Center



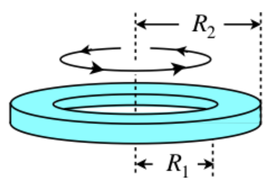
$$I=\frac{1}{2}MR^2 \tag{5.2.22}$$

Figure 5.2.8 – Solid Disk (or Cylinder) Rotated About Edge



$$I=\frac{3}{2}MR^2 \tag{5.2.23}$$

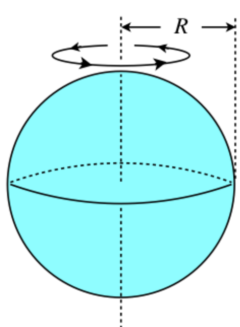
Figure 5.2.9 – Hollow Disk (or Cylinder) Rotated About Center



$$I=\frac{1}{2}M\left(R_1^2+R_2^2\right) \tag{5.2.24}$$

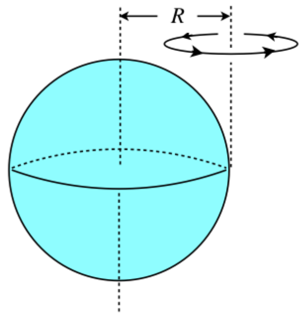
Spheres

Figure 5.2.10 – Solid Sphere Rotated About Center



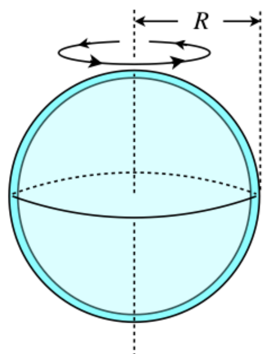
$$I = \frac{2}{5} MR^2 \tag{5.2.25}$$

Figure 5.2.11 – Solid Sphere Rotated About Edge



$$I = \frac{7}{5} MR^2 \tag{5.2.26}$$

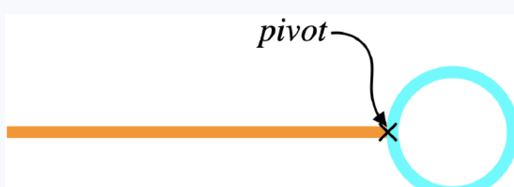
Figure 5.2.12 – Thin Spherical Shell Rotated About Center



$$I = \frac{2}{3} MR^2 \tag{5.2.27}$$

Exercise

The frame of a badminton racquet is constructed from two identical thin aluminum rods of uniform density, mass M, and length L. One of the rods is bent into a circle and is welded to the end of the other rod. Find the moment of inertia of the racquet around the axis that passes through the welded spot perpendicular to the plane of the raquet.



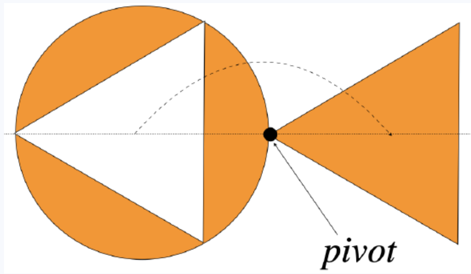
Solution

The circumference of the loop is L, so its radius is this number divided by 2π.We can use the shortcuts given above to determine the moments of inertia of the rod about its end and the loop around its edge, and then use the additive property of moment of inertia to get the total:

$$I = I_1 + I_2 = \frac{1}{3} ML^2 + 2MR^2 = \frac{1}{3} ML^2 + 2M \left(\frac{L}{2\pi} \right)^2 = \left(\frac{1}{3} + \frac{1}{2\pi^2} \right) ML^2$$

Exercise

An equilateral triangle is cut out of a circular piece of metal with a mass M and radius R, and the pieces are welded back together as shown below. Find the moment of inertia about the axis perpendicular to the plane of the object at the pivot point indicated.



Solution

Let's call the mass and rotational inertia around the CM of the triangle M_t and I_t respectively, and the mass and rotational inertia around the CM of the Circle with the hole M_c and I_c respectively. From the parallel axis theorem and the property of additivity at a common axis, we have that the rotational inertia around the pivot (which is a distance R from both CMs) is:

$$I_{tot} = \left(I_t + M_t R^2\right) + \left(I_c + M_c R^2\right) = \left(I_t + I_c\right) + \left(M_t R^2 + M_c R^2\right)$$

When the metal disk was whole, the two pieces shared their CM, and the sum of their CM rotational inertias was the rotational inertia of the full disk, so:

$$I_t + I_c = I_{disk} = \frac{1}{2} M R^2$$

Also, the masses of of the two pieces sum to the mass of the full disk:

$$M_t + M_c = M$$

Putting it together gives:

$$I_{tot} = \frac{3}{2} M R^2$$

5.3: Dynamics of Rotating Objects

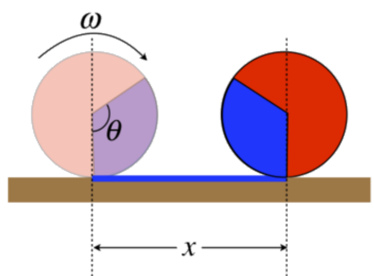
Rolling Without Slipping and Pulleys

A very large number of the mechanical energy conservation problems we will do involve the relation we discussed previously that relates rotational motion to linear motion. Specifically we will apply this to what is referred to as *rolling without slipping*, or *perfect rolling*. There are two reasons this is an important condition to understand:

- When two surfaces slip across each other, thermal energy is the result. So when an object rolls without slipping, there may be static friction present, but there is no kinetic friction, which means that no thermal energy is produced and mechanical energy is conserved.
- When a round object rolls perfectly, the distance it travels in a straight line is directly related to the angle through which it rotates.

We'll keep the first observation in mind for later, but right now let's focus on the second condition:

Figure 5.3.1 – Perfect Rolling



The linear distance traveled equals the arclength of the shaded region if the wheel is rolling without slipping, so we have:

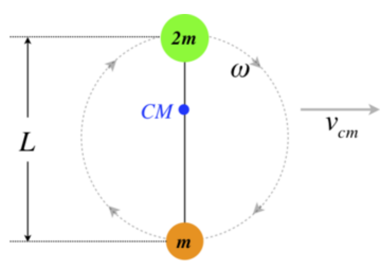
$$x = arclength = R\theta \quad \Rightarrow \quad v = \frac{dx}{dt} = R\frac{d\theta}{dt} = R\omega \tag{5.3.1}$$

Imagine now that instead of this being a wheel, it is a spool that is unwinding. Then the blue line represents string that is coming off the spool. We can therefore conclude that the relation $v = R\omega$ also applies to the linear speed of a rope that is either unraveling from a rotating spool or passing over a turning pulley.

Total Kinetic Energy as a Sum of Linear and Rotational

It's time we considered the case of an object whose center of mass is moving while it rotates. Let's start with a simple case of two rocks of different masses attached by a string:

Figure 5.3.2 – Unbalanced Dumbell Spinning as It Moves



This system is rotating as its center of mass moves in a straight line (assume there is no gravity present). We are given its rotational speed ω and the velocity of its center of mass, and wish to answer the question, "How much kinetic energy does this system possess at the moment depicted in the diagram?"

We could easily answer this question if we knew the speeds of the two rocks, but we are not given those numbers. We have to extract them from what is given, and this requires some thought. We know three things that get us to this answer:

- The velocity of a rock relative to us equals its velocity relative to the center of mass, plus the velocity of the center of mass (see Section 1.8 for a refresher).
- The center of mass lies at the point two-thirds of the distance from m to $2m$.
- The rotational velocities of both rocks are the same, but the linear velocities relative to the center of mass depend upon their distances from the center of mass according to the usual $v = r\omega$.

Let us label the bottom rock as #1, and the top rock as #2. Putting the first and third conditions together first gives us:

$$v_1 = v_{cm} - r_1\omega \qquad v_2 = v_{cm} + r_2\omega \tag{5.3.2}$$

The sign of the second term in each equation is determined by whether the rotational motion adds to or takes away from the linear motion of the center of mass. Next we invoke the second condition. The fact that the center of mass is two-thirds of the distance from m to $2m$ means:

$$r_1 = \frac{2}{3}L \qquad r_2 = \frac{1}{3}L \tag{5.3.3}$$

Putting all of the above into the kinetic energy of the system gives an expression for the total kinetic energy in terms of the values given. Collecting terms proportional to the squares of center of mass velocity and angular velocity gives:

$$\begin{aligned} KE_{tot} &= KE_1 + KE_2 \\ &= \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}(2m)\,v_2^2 \\ &= \tfrac{1}{2}m\left[v_{cm} - r_1\omega\right]^2 + \tfrac{1}{2}(2m)\left[v_{cm} + r_2\omega\right]^2 \\ &= \tfrac{1}{2}m\left[v_{cm} - \left(\tfrac{2}{3}L\right)\omega\right]^2 + \tfrac{1}{2}(2m)\left[v_{cm} + \left(\tfrac{1}{3}L\right)\omega\right]^2 \\ &= \tfrac{1}{2}(3m)\,v_{cm}^2 + \tfrac{1}{2}\left(\tfrac{2}{3}mL^2\right)\omega^2 \end{aligned} \tag{5.3.4}$$

The $3m$ in the first term is the total mass of the system, so the first term is the kinetic energy of system if was not spinning. That means that the second term is the amount of kinetic energy added to the system by virtue of its spinning. The part of the second term in parentheses looks suspiciously like a rotational inertia, and in fact it equals the rotational inertia of the system about its center of mass:

$$I_{cm} = m_1r_1^2 + m_2r_2^2 = (m)\left(\frac{2}{3}L\right)^2 + (2m)\left(\frac{1}{3}L\right)^2 = \frac{2}{3}mL^2 \tag{5.3.5}$$

This turns out to be a completely general rule for the kinetic energy of an object that is rotating as its center of mass moves:

$$KE_{tot} = KE_{lin} + KE_{rot} = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \tag{5.3.6}$$

Exercise

Show the same result (Equation 5.3.6) for two general point masses m_1 and m_2 separated by an unknown distance (call their distances from the center of mass r_1 and r_2), this time using the moment in time when m_1 is directly in front of m_2 (i.e. the line joining them is horizontal).

Solution

At the moment when the two masses form a horizontal line, their linear motions due to rotation are perpendicular to the center of mass motion. Determining their total speeds is therefore a simple application of the Pythagorean theorem, and the result follows surprisingly quickly:

$$\begin{aligned} v_1^2 &= v_{cm}^2 + (r_1\omega)^2 \\ v_2^2 &= v_{cm}^2 + (r_2\omega)^2 \end{aligned}$$

Now plug this into the kinetic energy for the system as the sum of the kinetic energies of the two masses:

$$\begin{aligned} KE_{tot} &= \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 \\ &= \tfrac{1}{2}m_1\left(v_cm^2 + r_1^2\omega^2\right) + \tfrac{1}{2}m_2\left(v_{cm}^2 + r_2^2\omega^2\right) \\ &= \tfrac{1}{2}(m_1 + m_2)\,v_{cm}^2 + \tfrac{1}{2}\left(m_1r_1^2 + m_2r_2^2\right)\omega^2 \end{aligned}$$

While the above equation is generally true for any object, if the object is rotating about a fixed point, the expression for total KE can be simpler to write. Specifically, it is what we have written before, in terms of the rotational inertia about the fixed point:

$$KE_{tot} = \frac{1}{2}I_{fixed\ point}\omega^2 \tag{5.3.7}$$

It's not hard to show that this is equivalent to Equation 5.3.6. Assuming the fixed point is not the center of mass (or the assertion is proved trivially), then let's call the distance from the center of mass to the fixed point "d." The center of mass is following a circular path of radius d around the fixed point, which means we can relate the linear velocity of the center of mass to its angular velocity around the fixed point:

$$v_{cm} = \omega d \tag{5.3.8}$$

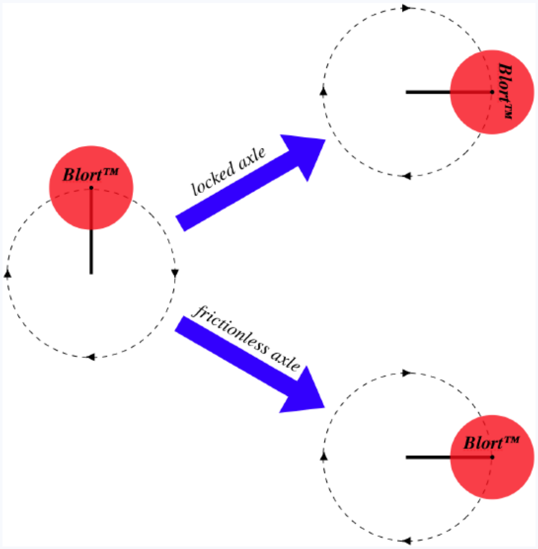
Putting this into our center-of-mass energy equation gives:

$$KE_{tot} = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}m(\omega d)^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}\underbrace{\left(md^2 + I_{cm}\right)}_{I_{fixed\ point}}\omega^2 \tag{5.3.9}$$

Where in the final step we employed the parallel-axis theorem.

Analyze This

The Blort Corporation makes a special widget that consists of a uniform disk pivoted around an axle at the end of a rod of negligible mass, which in turn rotates about its other end. This widget has two settings: It can be set in the "locked" position so that the disk does not rotate around its axle, or the "free" position so that the disk rotates frictionlessly about the axle. The difference these settings have on the motion of the disk as the rod rotates is depicted in the figure below.



Analysis

If we call the mass of the disk M, the radius of the disk R, the length of the rod L, and the rate at which the rod is rotating ω, we can compute the kinetic energies of these two settings. In the case of the free setting, the disk is simply moving in a circle without rotating, so it has only the linear component of kinetic energy:

$$KE_{free} = \frac{1}{2}mv_{cm}^2 + 0 = \frac{1}{2}M(L\omega)^2 = \frac{1}{2}ML^2\omega^2$$

For the locked axle case, we can find the energy two ways. The first is to treat the rod + disk as a single rigid object (which of course it is), with a fixed point for the rotation. The rod has no mass, but we can find the moment of inertia of this rigid object using the parallel-axis theorem:

$$KE_{locked} = \frac{1}{2}I_{fixed}\,\omega^2 = \frac{1}{2}\left(\frac{1}{2}MR^2 + ML^2\right)\omega^2 = \frac{1}{4}M\left(R^2 + 2L^2\right)\omega^2$$

Alternatively, we can use the linear + rotational form of kinetic energy, and the same result as this is attained. To use this method, one needs to figure out the rotational speed of the disk. It's not immediately obvious that it is the same as the rotational speed of the rod, so consider this: In one full revolution of the rod with the locked axle, the disk also makes exactly one full revolution. So the rotational rate of the disk is also ω.

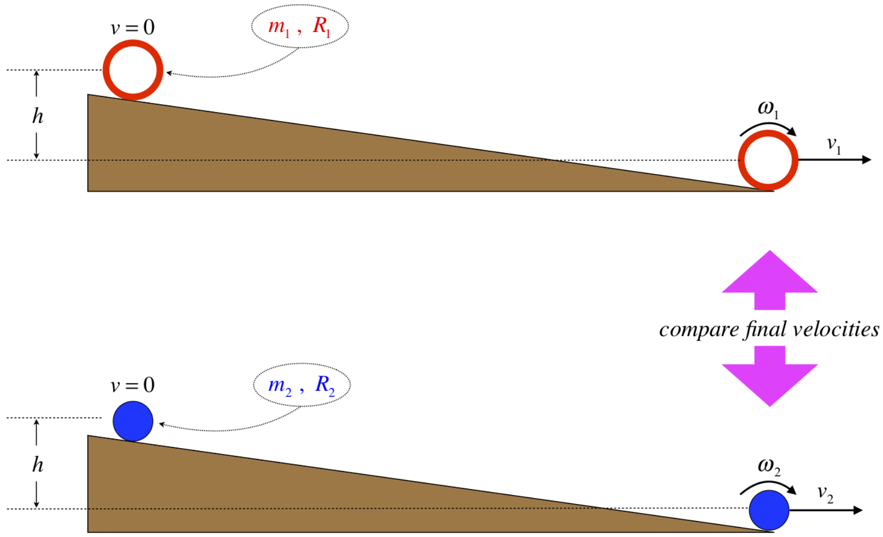
What is clear about this result is that for the same rotational speed for the rod, a different amount of kinetic energy is in the system for the two settings. The way we put energy into a system is to do work on it, so it appears that to achieve the same rotational speed of the rod, more work is required for the locked setting than for the free setting.

Mechanical Energy Conservation with Perfect Rolling

Let's put together what we have concluded so far in this section. We begin by noting that two cylinders with equal masses do not possess the same rotational inertia about their central axes if one is hollow and the other is solid. Now imagine rolling both of these cylinders (without slipping) down an inclined plane. Can you guess which one would reach the bottom of the incline with the greater speed? The main point to be made here is that the energy that comes from gravitational PE goes into KE, but now the KE has two different forms: linear and rotational. The linear and angular speeds are directly related through the "no slipping" condition, so the energy will convert into the two forms of kinetic energy in a fixed ratio. We will soon see how the rotational inertia affects the ratio, but it seems clear that the hollow cylinder puts more of its energy into rotation (for the same velocity) than the solid cylinder. This would seem to indicate that the hollow will have the same kinetic energy as the solid cylinder only if it is turning (and therefore moving) more slowly.

It's easy to trick oneself in such situations, so let's solve the math carefully to be sure.

Figure 5.3.3 – Comparing Hollow and Solid Cylinder Rolling Dynamics



We will work both problems in parallel, to make the difference more evident. Start with mechanical energy conservation from the top of the plane to the bottom. We can invoke this because without slipping there is no rubbing, which means no mechanical energy is converted to thermal energy.

$$\Delta KE + \Delta PE_{grav} = 0 \quad \Rightarrow \quad KE_o + PE_o = KE_f + PE_f \tag{5.3.10}$$

If we choose the zero point of potential energy to be the bottom of the incline, the initial and final potential energies in both cases are mgh and zero, respectively. The initial kinetic energy is zero in both cases, and the final kinetic energy is the sum of the linear and rotational kinetic energies (Equation 5.3.6):

$$\begin{array}{rll} & hollow\;cylinder & solid\;cylinder \\ \\ energy\;conservation: & m_1gh = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}I_1\omega_{1f}^2 & m_2gh = \frac{1}{2}m_2v_{2f}^2 + \frac{1}{2}I_2\omega_{2f}^2 \\ perfect\;rolling\,(v=R\omega): & m_1gh = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}I_1\left(\frac{v_{1f}}{R_1}\right)^2 & m_2gh = \frac{1}{2}m_2v_{2f}^2 + \frac{1}{2}I_2\left(\frac{v_{2f}}{R_2}\right)^2 \\ rotational\;inertia\;of\;cylinders: & m_1gh = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}\left(m_1R_1^2\right)\left(\frac{v_{1f}}{R_1}\right)^2 & m_1gh = \frac{1}{2}m_2v_{2f}^2 + \frac{1}{2}\left(\frac{1}{2}m_2R_2^2\right)\left(\frac{v_{2f}}{R_2}\right)^2 \\ algebra: & v_{1f} = \sqrt{gh} & v_{2f} = \sqrt{\frac{4gh}{3}} \end{array} \tag{5.3.11}$$

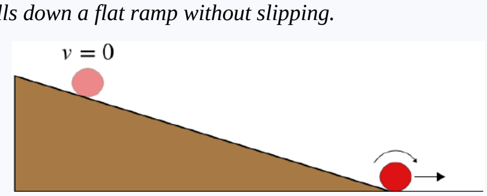
So in fact the solid cylinder is moving faster than the hollow one, as we predicted. What is especially interesting is that with the perfect rolling condition in place, the masses and radii of the cylinders are irrelevant! We are used to final speeds of objects accelerated by gravity being independent of the mass, but here we see that when we impose perfect rolling, the radius also plays no role, but the distribution of the mass within the cylinder is all that matters.

Alert

As we are discussing mechanical energy conservation again, it is a good time to remind ourselves that our conclusions only tell us how to compare speeds before and after – what goes on between these two moments and direction of motion are lost bits of information. This is as true now that rotation is involved as it was when it wasn't. For example, if we were to race the two cylinders down identical ramps, then naturally the solid cylinder would get to the bottom first, since they both start at rest and accelerate at constant rates. The object with the faster final speed must have taken less time to get to the bottom because it had a greater average velocity. The math shown above doesn't take into account the paths the two cylinders take, so if the ramps are not identical (but still result in the same height change), the conclusion about speeds at the bottom is the same as before, but the winner of the race may not be the solid cylinder!

Analyze This

A solid uniform sphere starts from rest and rolls down a flat ramp without slipping.



Analysis

Calling the height that the sphere descends h, we can compute its final speed using mechanical energy conservation. Following the usual method of including both the linear and rotational kinetic energy, we get:

$$KE_{before}+U_{before}=KE_{after}+U_{after} \quad \Rightarrow \quad mgh+0=\left(\frac{1}{2}mv_{cm}^2+\frac{1}{2}I\omega^2\right)+0$$

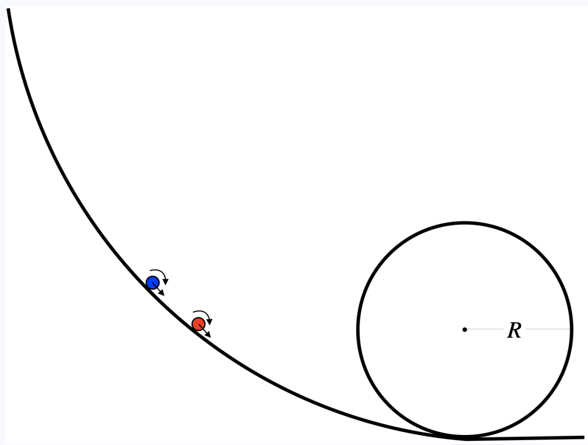
The moment of inertia of a solid sphere is $\frac{2}{5}mR^2$, so putting this in and noting that perfect rolling means $v=R\omega$, we have:

$$mgh=\frac{1}{2}mv^2+\frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2=\frac{7}{10}mv^2 \quad \Rightarrow \quad v=\sqrt{\frac{10}{7}gh}$$

Another thing we can note in the analysis is that the free-body diagram of the sphere never changes during its time on the ramp, so its acceleration must remain constant. With a constant linear acceleration, and the starting and ending speeds, we can possibly extract more information from kinematics equations.

Analyze This

A solid and a hollow sphere roll without slipping simultaneously (one behind the other) down a ramp and around a loop-de-loop. The radii of the spheres are negligible compared to the radius of the loop.



Analysis

The rolling-without-slipping condition relates the linear speeds of the spheres to their rotational speeds according to $v=R\omega$. This results in an amount of kinetic energy for each sphere that is different for a given linear speed, because they have different moments of inertia:

$$KE_{solid}=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2=\frac{1}{2}mv^2+\frac{1}{2}\left(\frac{2}{5}mR^2\right)\omega^2=\frac{7}{10}mv^2$$

$$KE_{hollow}=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2=\frac{1}{2}mv^2+\frac{1}{2}\left(\frac{2}{3}mR^2\right)\omega^2=\frac{5}{6}mv^2$$

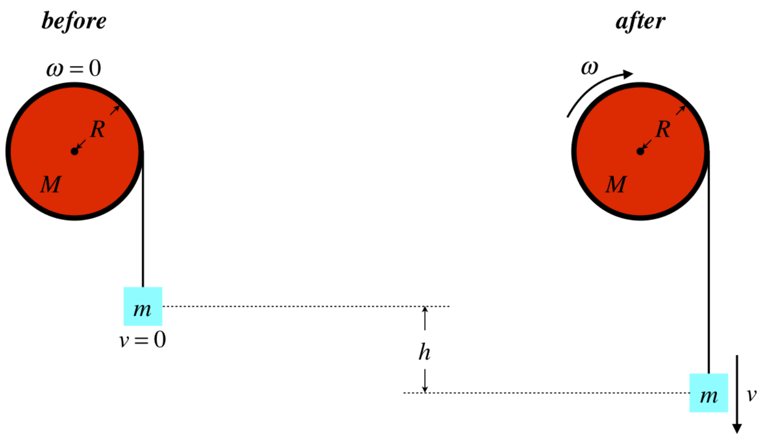
The typical thing to think about in cases where a loop-de-loop is involved whether the object has enough speed to make it around. With the spheres having very small radii, they are essentially moving in a circular path with a radius equal to that of the loop as they go around. In order to make it around, they have to be barely moving fast enough that the gravitational force is providing all the pull necessary to keep them going in a circle at the top of the loop – any faster and there would be normal force from the loop, and any slower and the sphere would fall off the loop. This condition therefore requires that the sphere has speed of:

$$\frac{v^2}{R}=g \quad \Rightarrow \quad v=\sqrt{gR}$$

Massive Spools

Another example that falls into this same category of mechanical energy conservation with perfect rolling is a falling mass unwinding a massive spool. Let's assume the spool is frictionless and is a uniform disk, and determine the speed of the falling block after it has dropped a known distance. We are also assuming – as always – that the string is massless, but we should also point out that it is very thin, so that its departure from the spool does not reduce the radius of the spool.

Figure 5.3.4 – Falling Block Unwinds Spool

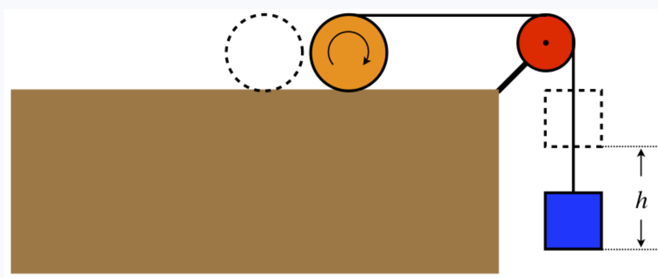


Once again, we can solve this using mechanical energy conservation, as there are no non-conservative forces present. What is new here is that some of the potential energy lost by the block as it drops goes into the rotational kinetic energy of the spool. The math is strikingly similar to the rolling cylinder case above:

$$\begin{aligned} energy\ conservation: &\quad mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 \\ perfect\ rolling\,(v = R\omega): &\quad mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\left(\frac{v}{R}\right)^2 \\ rotational\ inertia\ of\ spool: &\quad mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 \\ algebra: &\quad v = \sqrt{\frac{4mgh}{2m+M}} \end{aligned} \tag{5.3.12}$$

Analyze This

One end of a massless rope is wound around a uniform solid cylinder, while the other end passes over a massless, frictionless pulley and is attached to a hanging block, as in the diagram below. The block is released from rest, pulling the cylinder along the horizontal surface such that it rolls without slipping.



Analysis

The amount that the block falls equals the distance traveled by the cylinder plus the length of rope that unwinds from it. Since the cylinder rolls without slipping, the amount that unwinds is also equal to the distance it travels, so the sum of the distance traveled by the cylinder and the rope unwound is just double the distance that the cylinder travels. Therefore, the speed of the block is at all times twice the linear speed of the cylinder.

With no non-conservative forces present, the mechanical energy of the system is conserved, so subscripting the masses and velocities with 'b' for block, and 'c' for cylinder, we get:

$$m_bgh = \frac{1}{2}m_bv_b^2 + \frac{1}{2}m_cv_c^2 + \frac{1}{2}I\omega^2$$

We know the rotational inertia of the cylinder in terms of its mass and radius, that the block moves twice as fast as the cylinder, and that the cylinder rolls without slipping. Putting all of these constraints into the equation above gives us our answer:

$$\left.\begin{array}{l} I = \frac{1}{2}m_cR^2 \\ v_b = 2v_c \\ R\omega = v_c \end{array}\right\} \quad \Rightarrow \quad m_bgh = \frac{1}{2}m_b(2v_c)^2 + \frac{1}{2}m_cv_c^2 + \frac{1}{2}\left(\frac{1}{2}m_cR^2\right)\omega^2 = \frac{1}{2}\left(4m_b + \frac{3}{2}m_c\right)v_c^2$$

Solving for the speed of the cylinder (and the speed of the block is twice this much):

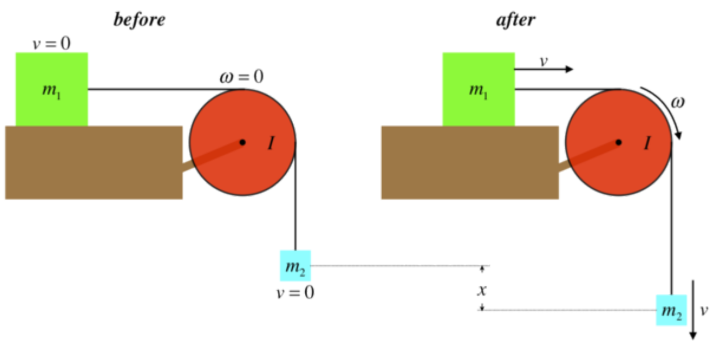
$$v_c = \sqrt{\frac{4m_b}{8m_b + 3m_c} gh}$$

The linear acceleration of the block and spool are constant, so knowing the final velocity allows us to use kinematics equations if we are given additional information.

Massive Pulleys

The result for this example may remind you of an assumption we made long ago regarding pulleys. We have always assumed that they were frictionless and massless. We said that the result of these assumptions was that the tension for the rope was the same everywhere (namely on both sides of the pulley). We are now equipped to look at what happens if the pulley has mass. We'll do so with a simple model physical system. In Figure 5.3.5, the hanging block accelerates as it falls, linearly accelerating the block on the frictionless horizontal surface and rotationally accelerating the pulley in the process.

Figure 5.3.5 – Effect of a Massive Pulley on Rope Tension



We are interested in comparing the tension force by the rope on both sides of this pulley, so let's use the work-energy theorem, which takes into account the forces. Treating each block as a separate system, on which the tension in each end of the rope performs work (and gravity does work on block #2 as well), and noting that both move at the same speed at all times, we have:

$$\begin{array}{lcr} W_1 = \Delta KE_1 & \Rightarrow & T_1 \cdot x = \frac{1}{2}m_1 v^2 \\ W_2 = \Delta KE_2 & \Rightarrow & (m_2 g - T_2) \cdot x = \frac{1}{2}m_2 v^2 \end{array} \tag{5.3.13}$$

Let's compare the two tensions by computing the difference:

$$(T_1 - T_2) \cdot x = \underbrace{\frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2}_{\Delta KE_{blocks}} \underbrace{- m_2 g x}_{\Delta PE_{grav}} \tag{5.3.14}$$

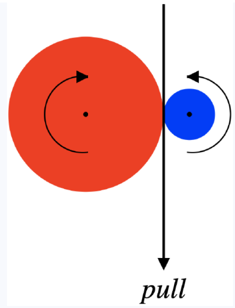
For the tensions to be equal, all of the gravitational potential energy lost by the falling block must go into the two blocks. But we now know that a massive pulley will have kinetic energy. Let's add the pulley's increase in kinetic energy to both sides of the equation, and invoke mechanical energy conservation:

$$(T_1 - T_2) \cdot x + \left[\frac{1}{2}I\omega^2\right] = \Delta KE_{blocks} + [\Delta KE_{pulley}] + \Delta PE_{grav} = 0 \quad \Rightarrow \quad T_1 - T_2 = \frac{\frac{1}{2}I\omega^2}{x} \tag{5.3.15}$$

The tensions can only be equal when the rotational inertia of the pulley is zero, which means it must be massless.

Analyze This

Two disks are cut out of the same material, as shown in the diagram below. They are pivoted around stationary axles such that the two disks lie in the vertical plane, with their outer rims pinching a massless rope between them. The rope is pulled downward, causing both disks to turn without the rope slipping.



Analysis

Let's call the radii of the large disk R and of the small disk r. We are given that they are made of the same uniform material, so their masses are proportional to their areas, which means their masses have the following ratio:

$$\frac{M}{m}=\frac{\pi R^2}{\pi r^2}=\left(\frac{R}{r}\right)^2$$

The ratio of their moments of inertia can therefore also be written in terms of their radii:

$$\frac{I_{large}}{I_{small}}=\frac{\frac{1}{2}MR^2}{\frac{1}{2}mr^2}=\left(\frac{M}{m}\right)\left(\frac{R^2}{r^2}\right)=\left(\frac{R}{r}\right)^4$$

When the rope is pulled, the disks don't slip, which means that their edges are moving at the same linear speed. Because of the perfect rolling condition, this means that they do not rotate at the same angular speed, and the ratios of these speeds is also expressible in terms of the ratio of radii:

$$v_{rope}=R\omega_{large}=r\omega_{small} \quad \Rightarrow \quad \frac{\omega_{large}}{\omega_{small}}=\frac{r}{R}$$

Digression: Energy Storage

One of the big issues today with green energy like solar and wind-generated electricity, is storage. The advantage to fossil fuel production of electricity is that we can produce it whenever we like, but for solar and wind power, we are at the mercy of when the sun shines or the wind blows. So storing the energy generated from these green sources is of paramount importance. Batteries are coming along, but they have their own environmental issues (lithium mining, waste when they degrade, etc.), so other means of storage are sought.

There are many ideas that have been put forth, such as using spare electricity to pump water above a dam so that it can be released when needed; pressurizing tanks of air with spare electricity, then allowing the pressurized air to drive a generator later; and using spare electricity to desalinate water, followed by using the osmotic pressure between the new fresh water and the salty water to drive a generator. But possibly the best idea (which has been around a long time) is to simply store the energy in the form of kinetic energy – spin a flywheel. A flywheel is just a disk created for the sole purpose of spinning so that it holds kinetic energy until it can be used later. The idea is for the spare electricity to get this thing spinning (with as little friction as possible), so that later when we need the energy back, the flywheel can be connected to a generator and the kinetic energy can be converted back into electrical energy. The beauty of this idea is in its simplicity – it is inexpensive and scalable. And reducing the friction to a very low value is something we can do quite well with today's technology (think maglev and evacuated chambers). In order to be as efficient with our use of space as possible, and so that we don't reach rotational speeds that are insanely high, we will of course want flywheels with very large rotational inertias.

Swinging Around Fixed Points

There is one other common physical situation involving mechanical energy conservation and rotation that needs to be addressed. If a rigid extended object is pivoted around a fixed point that is not the center of mass, and it is allowed to swing around that pivot under the influence of gravity, then how do we use mechanical energy conservation to describe its motion? Specifically, as the object swings, some points of the object may move upward (increasing gravitational potential energy), while others may swing downward (decreasing gravitational potential energy). How can we deal with the overall change in gravitational potential energy in such a case?

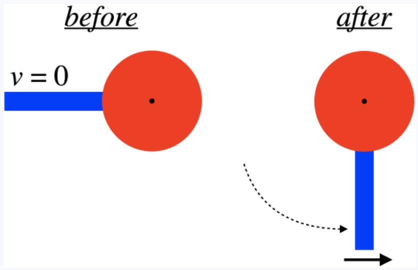
The answer will likely be unsurprising. Write the change of potential energy of the whole object as the sum of the potential energies of each tiny mass that makes up the object, and the result follows immediately:

$$\begin{aligned}\Delta U\,(whole\ object) &= \Delta U_1+\Delta U_2+\ldots \\ &= m_1 g\Delta y_1+m_2 g\Delta y_2+\ldots \\ &= \frac{Mg}{M}\left(m_1\Delta y_1+m_2\Delta y_2+\ldots\right) \\ &= Mg\frac{m_1\Delta y_1+m_2\Delta y_2+\ldots}{M} = Mg\Delta y_{cm}\end{aligned} \tag{5.3.16}$$

where $M=m_1+m_2+\ldots$ is the mass of the whole object and Δy_{cm} is the change in height of the center of mass of the object.

Analyze This

One end of a uniform metal thin rod is welded to the outer edge of a metal disk. The masses of these two objects are the same, and the length of the rod is equal to the diameter of the disk. The disk is suspended on a frictionless axle positioned at its center, and the rod is released from rest from a horizontal orientation and allowed to swing down to the vertical position.



Analysis

This system experiences a loss of gravitational potential energy during this swing, which can be measured in two different ways. First, we can just use the method above, where we find the center of mass of the whole system before and after, and use its full mass and that drop. In this case, with the length of rod equaling the diameter of the disk, the center of mass of the rod is a distance R (the radius of the disk) from the weld point. The center of mass of the disk is also this distance from the weld point, and since the masses of the disk and rod are equal, the weld point must be its center of mass. This point drops a distance of R, so the loss of potential energy (with m defined as the mass of each object) is just:

$$\Delta U=-2mgR$$

A second way to do this, which may be a bit simpler to use, is to note that the center of mass of the rod drops a distance $2R$, while the center of mass of the disk does not change at all, giving the same potential energy change:

$$\Delta U=\Delta U_{rod}+\Delta U_{disk}=-mg\left(2R\right)+0=-2mgR$$

This potential energy becomes kinetic energy. The object swings around a fixed point, so we can compute the moment of inertia about the fixed point and use the usual formula. To get this moment of inertia, we'll need the additivity of moments of inertia and the parallel axis theorem. The disk about the axle has a well-known moment of inertia. To get the contribution of the rod, we use its moment of inertia about its center of mass, and displace it a distance of $2R$ using the parallel axis theorem:

$$I_{tot}=I_{disk}+I_{rod}=\frac{1}{2}mR^2+\left[\frac{1}{12}m(2R)^2+m(2R)^2\right]=\frac{29}{6}mR^2$$

Putting this into the energy conservation equation and solving for the angular speed of the swing at the bottom of the arc gives:

$$KE_f=-\Delta U \quad\Rightarrow\quad \frac{1}{2}\left(\frac{29}{6}mR^2\right)\omega^2=2mgR \quad\Rightarrow\quad \omega=\sqrt{\frac{24g}{29R}}$$

5.4: Torque

Rotational Newton's Second Law

As we saw for linear motion, we can only go so far with energy conservation. If we want to analyze aspects of motion such as elapsed time and direction of motion, we need more than mechanical energy conservation to work with. In the linear case, we found that this meant that we had to use Newton's Second Law. We now seek the rotational equivalent of that law.

The rotational equivalent of the Newton's Second Law must relate the reaction of the system (rotational acceleration) to an external influence (rotational force), with the degree of this effect being determined by an internal property of the system (rotational mass). That is, we need a rotational substitute for all of the participants of this formula:

$$\overrightarrow{a}_{cm} = \frac{\overrightarrow{F}_{net}}{m} \tag{5.4.1}$$

We already found a rotational version of acceleration in our discussion of rotational kinematics – it is the angular acceleration. We even defined a direction for this vector using the right-hand rule. The center of mass qualification in the case above is unneeded for the rotational case, because the angular acceleration is the same about every point on a rigid object.

We have also determined an appropriate candidate for the "rotational mass" – the rotational inertia. This is certainly a reasonable choice, for a couple of reasons. First, from our direct experience we know that it is easier to swing an object (e.g. a baseball bat) when holding the heavier end than when holding the lighter end, so the degree to which an extended object "resists" angular acceleration is determined by the distribution of mass. Second, if the physics is to remain consistent, why would the quantity that plays the role of mass in kinetic energy be different from the quantity that plays the role of mass for the second law?

With those two quantities established, we can now get a glimpse into what the "rotational force" is by examining the units:

$$[\alpha] = \frac{[rotational\ force]}{[I]} \quad \Rightarrow \quad [rotational\ force] = \left[\frac{rad}{s^2}\right][kg \cdot m^2] = \frac{kg \cdot m^2}{s^2} \tag{5.4.2}$$

This is weird... These are units of energy! We'll need to chalk this up to coincidence, since clearly the vector quantity of rotational force cannot be a measure of energy. One way to see the difference is to remember the presence of radians in the numerator, even though they are not physical units. We will soon see the source of this coincidence, and it shouldn't take long before the apparent ambiguity between this quantity and energy fades away.

Alert

While the physical units are the same as energy, we ***never*** *refer to the SI units of this quantity as "Joules." Using this term implies that we are talking about energy, which we are not. Generally we stick to "Newton-meters."*

We can't continue calling this vector "rotational force" forever, so we will henceforth refer to it by its proper name: *torque*. In keeping with our tradition of using Greek variables for rotational quantities, we will represent torque with $\overrightarrow{\tau}$, giving as our rotational Newton's second law:

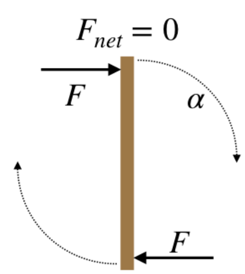
$$\overrightarrow{\alpha} = \frac{\overrightarrow{\tau}_{net}}{I} \tag{5.4.3}$$

Torque

In the cases of acceleration and inertia, we found a direct relationship between the linear and rotational quantities, so we would expect there to be a similar relationship between force and torque. Furthermore, since the linear/rotational bridge for acceleration and inertia both require a point of reference (the pivot), we would expect the same to be true for the bridge between force and torque.

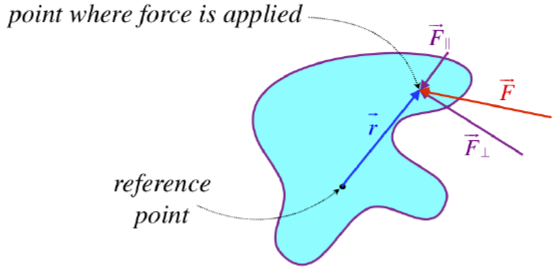
The first thing we notice is that an object can experience no net force and yet still experience a nonzero rotational acceleration:

Figure 5.4.1 – Zero Net Force Can Accelerate Rotationally



If the two forces shown in the figure above are moved so that act at the same point on the object, then it's clear that they also cancel rotationally. So apparently the place *where* the force acts is important to computing torque. If we choose a reference point (we will refer to this as a "pivot" in cases when it happens to be a fixed point, but in general it does not), then the application point of a force can be described by a position vector $\overrightarrow{r}$ that points from the reference point to the point where the force is applied. But there is still more that we have to worry about here. If two forces with the same magnitudes as those in the figure above were applied at the ends of the bar, but were pointing vertically, then no angular acceleration would result. Let's put all this together...

Figure 5.4.2 – Parts of a Force that Cause Angular Acceleration



The force vector can be decomposed into two perpendicular vectors – one that is parallel to the position vector, and one perpendicular to it. When it comes to causing the object to accelerate its rotation around the pivot, it's clear that the part of the force that is parallel to the position vector $\overrightarrow{F}_{\parallel}$ will have no effect, while the perpendicular part of the force $\overrightarrow{F}_{\perp}$ will.

If we were to perform experiments to test the effects of various force magnitudes, we would find that the angular acceleration is proportional to the magnitude of the force – push twice as hard in the same direction at the same point on the object, and its angular acceleration is twice as great around the same pivot. If we were to perform further experiments to test the effects of applying the force at different positions, we would find that the angular acceleration is proportional to the magnitude of the position vector – extend the position vector in the same direction to twice its original length and apply the same force in the same direction, and the angular acceleration is once again twice as great around the same pivot. Mathematically, we express the results of these experiments this way:

$$\left|\overrightarrow{\tau}\right| \sim \left|\overrightarrow{r}\right| \left|\overrightarrow{F}\right| \tag{5.4.4}$$

Notice that the units of this product work out correctly, so all we need to do is incorporate the "only the perpendicular part of $\overrightarrow{F}$ has an effect" into the math. If we call the angle between the position vector and the force vector θ, then the perpendicular component is $F\sin\theta$. Assuming there are no other constants involved (and there aren't any), we get, for the magnitude of the torque:

$$\left|\overrightarrow{\tau}\right| = \left|\overrightarrow{r}\right| \left|\overrightarrow{F}\right| \sin\theta \tag{5.4.5}$$

This looks familiar – we actually saw something just like it, way back in Equation 1.2.8. Torque *is* a vector that is derived from the product of two other vectors. Is it possible that it is simply a cross-product of these two vectors? The magnitude works, but what about direction? In Figure 5.4.2, the force will accelerate the rotation counterclockwise, which means that according to the right-hand-rule, the acceleration vector points out of the page. If we perform a cross-product of the position vector (up to the right) and the force vector (up to the left), the right-hand-rule results in a vector that also points out of the page. We therefore write:

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F} \tag{5.4.6}$$

Exercise

A rigid object is pivoted around the origin. The force vector given below acts on this object at the position also indicated below. Find the torque vector exerted on the object due to this force.

$$\overrightarrow{F} = 1.5N\;\hat{i} + 0.80N\;\hat{j} - 2.4N\;\hat{k}, \qquad position:\;(x,\;y,\;z) = (3.0m,\;0.0m,\;-2.0m)$$

Solution

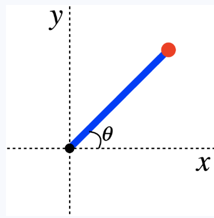
This is a straightforward calculation of a cross product:

$$\begin{aligned}\overrightarrow{\tau} &= \overrightarrow{r} \times \overrightarrow{F} \\ &= \left(3.0m\;\hat{i} + 0.0m\;\hat{j} - 2.0m\;\hat{k}\right) \times \left(1.5N\;\hat{i} + 0.80N\;\hat{j} - 2.4N\;\hat{k}\right) \\ &= \left[(3.0m)(0.80N) - (0.0m)(1.50N)\right]\hat{k} + \left[(0.0m)(-2.40N) - (-2.0m)(0.80N)\right]\hat{i} + \left[(-2.0m)(1.50N) - (3.0m)(-2.4N)\right]\hat{j} \\ &= 1.6Nm\;\hat{i} + 4.2Nm\;\hat{j} + 2.4Nm\;\hat{k}\end{aligned}$$

Analyze This

A small marble is attached to the end of a thin rigid rod with an equal mass, whose other end is held fixed at the origin. The rod starts at rest in the $x-y$ plane, and makes an angle θ up from the x-axis, as shown in the diagram. There is no gravity present, but the marble (not the rod) is subjected to a force from a potential energy field given by:

$$U\left(x,y\right)=\beta xy+U_o\;,\quad \beta=constant>0$$



Analysis

We can use the potential energy function to determine the force at every point in space:

$$\vec F=-\frac{\partial U}{\partial x}\;\hat i-\frac{\partial U}{\partial y}\;\hat j=-\beta\left(y\;\hat i+x\;\hat j\right)$$

The torque exerted relative to the origin at the point (x,y) is the cross-product of the position vector there and the force vector there:

$$\vec\tau=\vec r\times\vec F=\left(x\;\hat i+y\;\hat j\right)\times\left[-\beta\left(y\;\hat i+x\;\hat j\right)\right]=\beta\left(y^2-x^2\right)\hat k$$

Let's call the length of the rod L. Then the coordinates of the marble in terms of L and θ are:

$$x=L\cos\theta\quad y=L\sin\theta$$

Plugging these in gives us the torque on the object in terms of θ.

$$\vec\tau=\beta L^2\left(\sin^2\theta-\cos^2\theta\right)\hat k=-\beta L^2\cos 2\theta\;\hat k$$

If we call the mass of the marble (and rod) m, we can also compute the moment of inertia of the object, and combine it with the torque to obtain its angular acceleration at any angle θ. When doing so, it is important to remember that the reference point for the moment of inertia must be the same as for the torque, which in this case is the origin. Here we have a rod pivoted about its end and a point mass a known distance from the pivot, so the moment of inertia is the sum of these contributions:

$$I=I_{rod}+I_{marble}=\frac{1}{3}mL^2+mL^2=\frac{4}{3}mL^2$$

So now, from Newton's 2nd Law for rotations:

$$\vec\alpha=\frac{\vec\tau}{I}=\frac{-\beta L^2\cos 2\theta\;\hat k}{\frac{4}{3}mL^2}=-\frac{3\beta\cos 2\theta}{4m}\;\hat k$$

The math is correct, and we have obtained a formula, but the exploration of our "analysis" can go much further. For example, we note that at $\theta=45^o$, this acceleration is zero – there is no torque on the object. Does this make sense? Well, this angle occurs when $y=x>0$, and plugging this back into the force vector that we found reveals that the force on the marble is not zero, but its direction in space is parallel to $-\hat i-\hat j$. When the marble is at $\theta=45^o$, that direction is pointed directly at the origin. A force exerted on the marble that points directly through the rod is not going to cause the rod to accelerate rotationally, so this makes sense!

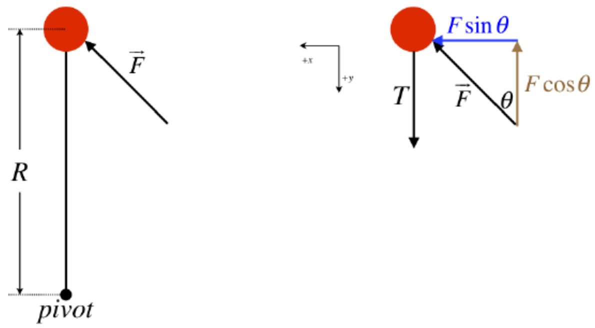
Another thing we note is that if the rod is turned slightly clockwise from $\theta=45^o$, then there is a torque in the clockwise direction (the torque vector points in the $-\hat k$ direction, which is into the page, and from the RHR is clockwise). So a small clockwise nudge from $\theta=45^o$ will cause the rotation to speed up. We similarly find that a small nudge from $\theta=45^o$ in a counterclockwise direction results in a counterclockwise torque, speeding it up in that direction as well. Back in Section 3.7 we called a position like $\theta=45^o$ a point of unstable equilibrium.

Finally, we note that since the sign of $\cos 2\theta$ changes between positive and negative as θ goes around a full circle, the torque alternates direction. It makes sense that the torque would not be in s single direction, because if the object makes a full turn in the direction of this torque and returns to where it started, it would have to be turning faster, as the single-direction torque keeps speeding it up. But this is impossible, because that would mean it gained KE, while the potential energy for a full 360^o turn comes back to its original value. Indeed we can therefore conclude that any potential energy function whatsoever that we care to use must result in torques that alternate between clockwise and counterclockwise!

Linking Rotational and Linear

Let’s do a sanity check on our definition of torque and its role in the rotational second law. We can do it very simply by choosing a single point mass tied to a string whose other end is held as a fixed pivot (we'll leave gravity out of this). We'll start with the linear version of Newton’s second law, and translate it into the rotational version.

Figure 5.4.3 – A Simple System Solved Two Ways



The forces in the x and y directions provide two equations through Newton's second law:

$$a_x=\frac{\sum F_x}{m} \quad\Rightarrow\quad a_{\parallel}=\frac{F\sin\theta}{m} \tag{5.4.7}$$

$$a_y=\frac{\sum F_y}{m} \quad\Rightarrow\quad a_{\perp}=\frac{T-F\cos\theta}{m} \tag{5.4.8}$$

Now we translate to rotational motion by first converting the parallel part of the acceleration into angular acceleration:

$$a_{\parallel}=R\alpha \tag{5.4.9}$$

Then convert mass into rotational inertia:

$$m=\frac{I}{R^2} \tag{5.4.10}$$

Plugging Equation 5.4.9 and Equation 5.4.10 into Equation 5.4.7 gives:

$$R\alpha=\frac{F\sin\theta}{\frac{I}{R^2}} \quad\Rightarrow\quad \alpha=\frac{FR\sin\theta}{I}=\frac{\tau}{I} \tag{5.4.11}$$

One important thing to note here is that while the torque and rotational inertia depend upon the pivot point (i.e. they are different values if we use a new reference point), the translation between the angular acceleration and linear acceleration exactly balances this difference. For example, if we replace the pivot defined above with a new one that is a distance $2R$ from the object, all of the math works out exactly the same. That is, the torque is twice as great and the rotational inertia is four times as great, resulting in a rotational acceleration that is half as large as before, but when it is multiplied by twice the radius to get the linear acceleration, the same result occurs, as it must.

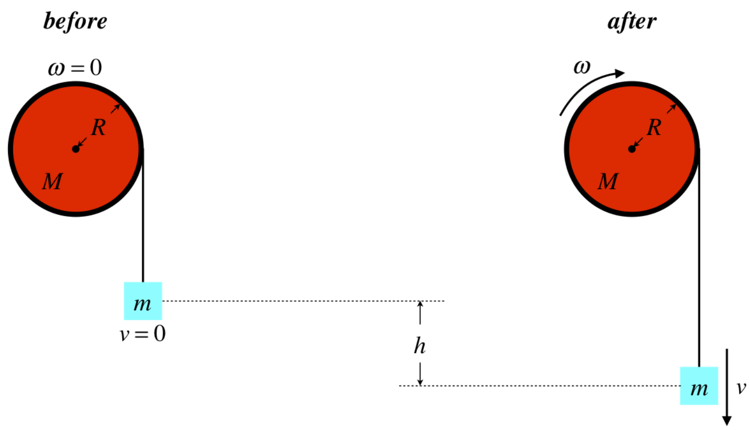
Solving Problems

Now we can do a whole set of problems involving torque causing rotational acceleration. There are many similarities with solving problems involving linear forces and accelerations, but here are some differences:

- Free-body diagrams now require that forces be placed appropriately on the objects, since torque depends upon force placement (no more using dots to represent the object).
- There usually is no need to resolve the torque vector into components. In fact, most problems can limit torque (and angular acceleration) to just "clockwise" and "counterclockwise" – the direction of the torque vector can be left until the end.
- One must either know or be able to calculate the rotational inertia of the object on which the torques acts.
- The perfect rolling condition is sometimes applied.

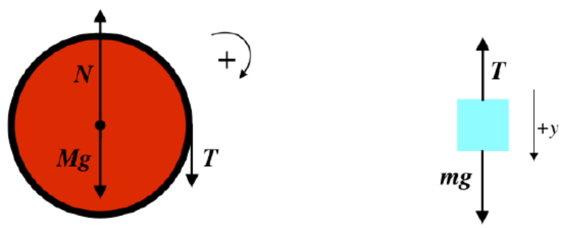
To get an idea of the process, we'll re-work the problem of the falling block unwinding a spool, this time using rotational second law instead of energy conservation:

Figure 5.4.4 – Falling Block Unwinds Spool (Redux)



Start with free-body diagrams:

Figure 5.4.5 – FBD's of Block and Spool



Next we need to right down the equations for Newton's second law for each object. The block is moving in a straight line, so we are already familiar with that one:

$$a_y = \frac{\sum F_y}{m} \quad \Rightarrow \quad a = \frac{mg - T}{m} = g - \frac{T}{m} \tag{5.4.12}$$

The spool is rotating, so we need to use the rotational version for it. Before we can sum the torques for the spool, we need to select a reference point, and its axle is a pretty obvious choice. The length of the position vector from this reference point to the where the gravity and normal forces act is zero, so those forces produce no torque around the axle (which makes sense – pushing on an axle should not cause something to spin around it). This leaves only the tension force. It acts tangent to the spool, so this force is perpendicular to the position vector connecting the pivot to the point where the force acts, which makes the magnitude of torque it produces equal to simply the product of the tension and the radius of the spool. The direction of this torque is positive, since it causes a clockwise acceleration and our FBD defines that as the positive direction. As this is the only torque, it is the net torque, and we have:

$$\alpha = \frac{\tau_{net}}{I} \quad \Rightarrow \quad \alpha = \frac{T \cdot R}{I} \tag{5.4.13}$$

Now we have to incorporate our constraints (our "other information"). We know that the spool is a uniform solid disk with mass M, giving us its rotational inertia. Also, we know that the rate at which the string exits the spool is related to the rotation rate of the spool according to the usual "no slipping" condition, so we have an equation relating the block's linear acceleration a to the spool's angular acceleration α:

$$I = \frac{1}{2}MR^2, \qquad \alpha = \frac{a}{R} \tag{5.4.14}$$

Putting these constraints into Equation 5.4.13 and combining this with Equation 5.4.12 gives:

$$\left.\begin{array}{l} \dfrac{a}{R} = \dfrac{T \cdot R}{\frac{1}{2}MR^2} \quad \Rightarrow \quad T = \dfrac{Ma}{2} \\ a = g - \dfrac{T}{m} \end{array}\right\} \quad \Rightarrow \quad a = \frac{2m}{2m+M}g \tag{5.4.15}$$

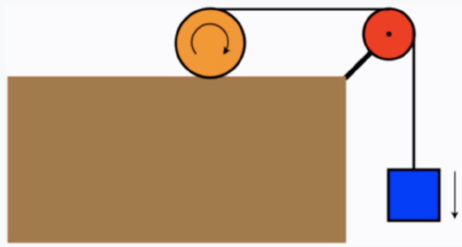
We see that the acceleration of the block is constant, so we can use a kinematics equation to determine the velocity after displacing a distance h from rest:

$$v_f^2 - v_o^2 = 2a\Delta y \quad \Rightarrow \quad v = \sqrt{\frac{4mgh}{2m+M}} \tag{5.4.16}$$

This agrees with our previous answer.

Analyze This (Again!)

Now that we have some new tools to work with, we can return to a physical system that we previously analyzed with energy conservation and re-analyze it using what we now know. As before, the spool rolls without slipping as the block descends, and here we will immediately assume that the mass of the block equals the mass of the spool.



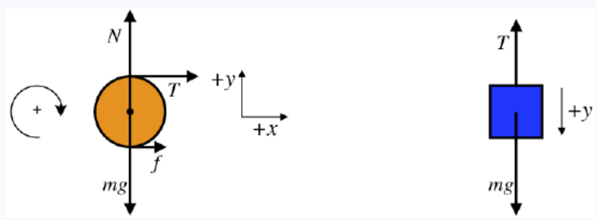
Analysis

We will only include in this analysis information we did not already obtain when we looked at this case previously. Since we are now interested in the effects of individual forces on linear and rotational motion, we can point out that rolling without slipping is only possible if static friction is acting between the horizontal surface and the spool. This introduces a limiting factor on the friction force, based on the coefficient of friction and the normal force between the surfaces. In particular, the maximum static friction force is given by:

$$f_{max} = \mu_s N = \mu_s mg$$

The final equality here comes from the fact that the surface is horizontal, so the normal force and gravitational forces must be exactly equal, with zero vertical acceleration.

Okay, now let's tackle the equations that come from Newton's second law. We of course start with force diagrams:



You might ask how we know that the friction force points in the direction indicated in the diagram. Technically, we don't yet know this, but we don't have to. If, in the course of our calculations, we find that the only way a solution can work out is if the value of f is negative, then the friction force must point the other way. We will see shortly that the direction on the diagram is in fact the only direction it can point.

Remembering from the previous time we analyzed this that the block at all times moves twice as fast as the spool (and therefore accelerates twice as much), there are three equations that come from Newton's second law for the cylinder (the horizontal and vertical linear net force equations, and the net torque equation), and there is one equation that comes out for the block:

$$\begin{array}{lll} & cylinder & block \\ \\ x-direction: & a=\dfrac{T+f}{m} & \\ y-direction: & 0=N-mg & 2a=\dfrac{mg-T}{m} \\ torques: & \alpha=\dfrac{TR-fR}{I} & \end{array}$$

Plugging in for the rotational inertia and the angular acceleration gives:

$$\frac{a}{R}=\frac{TR-fR}{\frac{1}{2}mR^2} \quad \Rightarrow \quad \frac{a}{2}=\frac{T-f}{m}$$

Adding this equation to the x-direction equation for the cylinder gives:

$$\frac{3}{2}a=\frac{T+f}{m}+\frac{T-f}{m} \quad \Rightarrow \quad T=\frac{3}{4}ma$$

Now combine this result with the y-direction equation for the block to get:

$$2a=\frac{mg-\frac{3}{4}ma}{m} \quad \Rightarrow \quad a=\frac{4}{11}g$$

In our previous analysis, if the masses of the spool and block were equal, we would find that after the block falls a distance h, the final velocity of the cylinder is:

$$v_c = \sqrt{\frac{4}{11}gh}$$

From the kinematic equation we have:

$$v_f^2 - v_o^2 = 2a\Delta x \quad \Rightarrow \quad v_c = \sqrt{2a\Delta x}$$

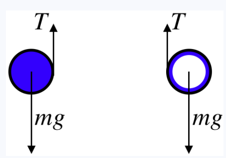
It appears at first blush that if we plug in the acceleration found above, that it disagrees with this answer by a factor of two within the radical. But there is one other thing to remember: The quantity Δx is the displacement of the accelerated object (in this case, the cylinder), but the drop of the block h is twice as great as the displacement of the cylinder, so putting in $\Delta x = \frac{1}{2}h$ confirms the answer.

Analyze This

Two ends of a massless rope are wound around two spools with equal masses and radii. One of the spools is a solid, uniform disk, while the other is a thin, hollow cylinder. The rope between them goes over a massless, frictionless pulley in a vertical plane. The spools are released from rest from the same height, and the rope does not slip over the pulley.

Analysis

There are a lot of potentially "moving parts" here (both spools and the pulley are free to turn), and keeping the motion constraints straight can be a bit daunting. At first blush, one might think that the solid spool will unwind faster than the hollow one, causing it to fall faster, just as it rolls faster down a slope. But making such generalizations is dangerous without careful analysis, so let's forge ahead with that, starting as we so often do, with free-body diagrams. The pulley is massless, which means the tension in the rope is the same on both sides, and since the pulley is held in place by the axle, a free-body diagram of it will not prove particularly useful. The two spools are another matter. Calling their common mass m, we get:



Well this is interesting – both spools have the same FBD's! With the same mass, and same starting conditions (starting at rest), Newton's 2nd law for linear motion ensures that they will have identical motions. Therefore they must remain side-by-side as they fall.

But wait, we know that both spools also experience the same torques about their centers, because the only force that exerts any torque about the centers is the tension, and this is the same for both spools, as is the radius of each spool. But equal torques will not result in equal angular accelerations, because they have different moments of inertia – the hollow spool (with the higher moment of inertia) will have to be unwinding slower than the solid spool at any given moment in time. How is this possible, if they have the same radius and are falling linearly at the same rate?

The answer is that the rope is moving! The solid spool is giving up rope faster than the hollow one, but the amount of rope between each of these and the pulley is the same. So some of the rope given up by the solid spool must be passing over the pulley to the other side – the pulley is rotating clockwise.

We have already gained a lot of insight into the physics of this situation, but the power of analysis is not to be underestimated - let's see if we can take it further...

First, let's apply Newton's 2nd law to the FBDs above:

$$mg - T = ma$$

Next we need to think about the rope constraints. We'll call the radius of the pulley R and the radii of the spools r. When the solid spool unwinds and angle $\Delta\theta_{solid}$, it gives up an amount of rope equal to $r\Delta\theta_{solid}$. Similarly, for the hollow spool, we know that it gives up a length of rope equal to $r\Delta\theta_{hollow}$. The sum of these quantities is the additional rope put into the system. The spools both drop equal distances at the same time, so on each side of the pulley, the rope gets longer by half this total:

$$\Delta y = \frac{1}{2}\left(r\Delta\theta_{solid} + r\Delta\theta_{hollow}\right)$$

Two time derivatives of the y value is the linear acceleration of the spools (the same acceleration that appears in the first equation above). Two derivatives of the angles the spools rotate through are the angular accelerations of the spools, so we have:

$$a = \frac{1}{2}r\left(\alpha_{solid} + \alpha_{hollow}\right)$$

These angular accelerations come from torques exerted on the spools about their centers by the tension force (the gravity force acts through their centers, so it provides no torque about that axis). So from Newton's 2nd Law for rotational motion, and using what we know about their moments of inertia, we get:

$$\alpha = \frac{\tau}{I} \quad \Rightarrow \quad \begin{cases} \alpha_{solid} = \frac{Tr}{\frac{1}{2}mr^2} = \frac{2T}{mr} \\ \alpha_{hollow} = \frac{Tr}{mr^2} = \frac{T}{mr} \end{cases}$$

Plugging these values back in above gives:

$$a = \frac{1}{2}r\left(\frac{2T}{mr} + \frac{T}{mr}\right) = \frac{3T}{2m} \quad \Rightarrow \quad T = \frac{2}{3}ma$$

And using this in the first equation gives us the exact linear acceleration of the falling spools:

$$mg - \frac{2}{3}ma = ma \quad \Rightarrow \quad a = \frac{3}{5}g$$

Remarkable that we know the exact numerical answer without knowing the mass of a spool or the radii of the spools or the pulley.

Rotational Work

We have now discussed the rotational version of energy conservation and Newton's second law, so the link between these two topics – the work-energy theorem – should follow naturally. Rather than provide a derivation (which would really just resemble what we have done before for the linear case), we'll just write down the answer that makes sense from following our linear/rotational parallel.

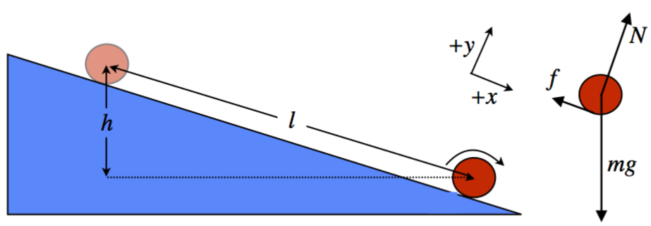
$$\begin{aligned} W_{A\rightarrow B}\,(linear) \quad &= \int_A^B \overrightarrow{F}\cdot\overrightarrow{dl} \quad = \Delta KE \quad = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2 \\ W_{A\rightarrow B}\,(rotational) \quad &= \int_A^B \overrightarrow{\tau}\cdot\overrightarrow{d\theta} \quad = \Delta KE \quad = \tfrac{1}{2}I\omega_B^2 - \tfrac{1}{2}I\omega_A^2 \end{aligned} \tag{5.4.17}$$

If we were so inclined, we could do the same unwinding-the-spool problem for a third time, this time with the rotational work-energy theorem. The approach looks slightly different, but when you actually sit down to do it, you see the same things come out of it as before. This time instead of relating the accelerations, we would relate the distance the mass drops to the angle the spool rotates.

Back when we discussed objects rolling down an inclined plane without slipping, we avoided talking about one potentially confusing point that we are now equipped to deal with. For a ball or cylinder to roll down, there has to be a friction force (otherwise it would merely slide). This friction force can only be static friction, because we are assuming there is no slipping, and we said that without any rubbing, the mechanical energy must be conserved. But this friction force acts *up the plane while the object moves down it*, which means that it does negative work on the object. This would seem to imply that mechanical energy should not be conserved, so how were we able to make the assumption that it is conserved?

The answer is, "Because the static friction force also does *positive* rotational work which adds energy to the object in rotational form, and this addition exactly balances the loss in linear form." This is not hard to prove. Start with a diagram and a FBD:

Figure 5.4.6 – Work Done on Cylinder by Static Friction as It Rolls Down Plane



Computing the work done by static friction for linear motion is very simple, since the friction force is constant and the motion is in a straight line:

$$W\left(x_1 \rightarrow x_2\right) = \int_{x_1}^{x_2} \overrightarrow{F}\cdot\overrightarrow{dx} = -f\cdot l \tag{5.4.18}$$

As expected, this work takes energy out of the cylinder system. Next we compute the rotational work done on the cylinder. The torque is a constant equal to fR, and is acting in the same direction as the rotational displacement, so

$$W\left(\theta_1 \rightarrow \theta_2\right)=\int_{\theta_1}^{\theta_2} \overrightarrow{\tau} \cdot \overrightarrow{d\theta}=+(fR) \cdot \theta \tag{5.4.19}$$

Putting these together gives us the total work done on the cylinder by the static friction force. Note that since it rolls without slipping, the linear distance it travels is related to the angle through which it rotates by the usual relation:

$$W_{static\ friction}=W\left(x_1 \rightarrow x_2\right)+W\left(\theta_1 \rightarrow \theta_2\right)=-f \cdot l+fR\theta=f(-l+R\theta)=0 \tag{5.4.20}$$

So we see that in fact the work done by static friction here only serves to convert linear kinetic energy into rotational kinetic energy, and our understanding of how thermal energy is generated remains intact.

Rotational Power

We spoke before about how sometimes we are interested in the rate at which work is done, calling this value “power.” Well, as with everything else we studied in linear motion, there is of course a rotational version:

$$P=\frac{dW}{dt}=\overrightarrow{\tau} \cdot \overrightarrow{\omega} \tag{5.4.21}$$

You sometimes hear the silly “debate” about torque vs. horsepower for car & truck engines. This should make it clear what the difference is. Power delivered to the wheels is directly related to torque exerted on them, but it is dependent upon how fast they are turning. Engines that can still produce a lot of torque at high speeds are powerful. To get an idea of why it might be hard to maintain torque at high speeds, imagine pedaling a bike downhill – when you get going fast enough, it’s difficult to push hard on (provide torque to) your pedals. So generally the effectiveness of an engine is defined by torque at low speeds and power at high speeds. If you want fast acceleration off the line or the ability to pull a stump out of the ground, you want torque. If you want to go fast or tow a heavy trailer up a hill at a steady speed, you want power.

5.5: Static Equilibrium

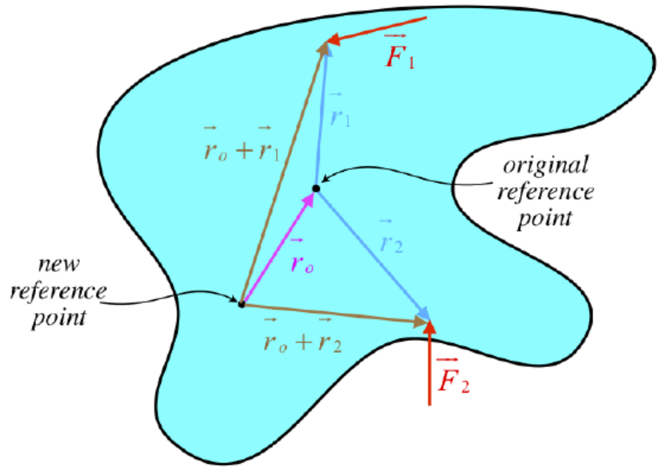
Pivots and Torque Reference Points

The definition of torque (Equation 5.4.6) includes the position vector $\overrightarrow{r}$, which points from a reference point to the point where the force is applied. When we are interested in how the torque is accelerating the object rotationally around a fixed point ("pivot"), it is convenient to choose the reference point to be that fixed point. This is because the forces applied at that fixed point (to keep it fixed) provide zero torque when referenced there, and those forces are generally not known. We explore here the effect of changing the reference point in the particular case when there is *no net force*, though perhaps there could be a net torque. The net torque around a given reference point is:

$$\overrightarrow{\tau}_{net} = \overrightarrow{r}_1 \times \overrightarrow{F}_1 + \overrightarrow{r}_2 \times \overrightarrow{F}_2 + \ldots \tag{5.5.1}$$

The reference point is located at the tails of the $\overrightarrow{r}_i$ vectors, but suppose we want to change that reference point. We can do this by simply adding the same constant vector $\overrightarrow{r}_o$ to every position vector. This has the effect of shifting the reference point from the point of $\overrightarrow{r}_o$ to its tail, as shown in the figure below [*Note: The figure shows only two of the many forces applied.*]

Figure 5.5.1 – Changing the Reference Point



The net torque around this new reference point is:

$$\begin{aligned} \overrightarrow{\tau}_{net}(new) &= \left(\overrightarrow{r}_o + \overrightarrow{r}_1\right) \times \overrightarrow{F}_1 + \left(\overrightarrow{r}_o + \overrightarrow{r}_2\right) \times \overrightarrow{F}_2 + \ldots \\ &= \left[\overrightarrow{r}_1 \times \overrightarrow{F}_1 + \overrightarrow{r}_2 \times \overrightarrow{F}_2 + \ldots\right] + \overrightarrow{r}_o \times \left[\overrightarrow{F}_1 + \overrightarrow{F}_2 + \ldots\right] \\ &= \overrightarrow{\tau}_{net}(original) + \overrightarrow{r}_o \times \overrightarrow{F}_{net} \end{aligned} \tag{5.5.2}$$

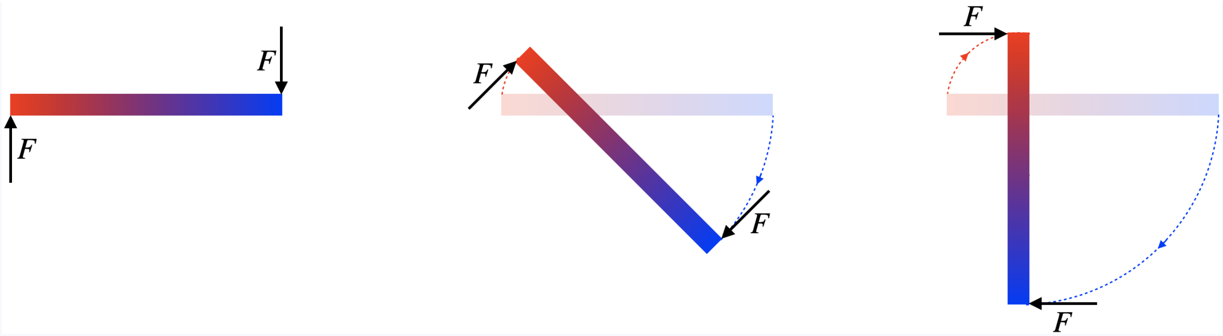
But we assumed that the net force was zero, so we get the remarkable result that the *net torque is the same around every reference point*!

Alert

As amazing as this result is, be careful not to mistake it for too general of a result. The net torque on an object by a collection of forces is only independent of the reference point if those forces result in zero net force.

Analyze This

A board starts at rest and is free of any attachments (it is not pivoted on anything). It is pushed in opposite directions on both of its ends with forces of equal magnitude, at right angles to the board. The forces continue to be applied at right angles with the same magnitude, causing the board to rotate in the manner depicted in the diagram until the board has rotated by 90^o.



Analysis

The first thing we notice here is that at all times the board experiences two equal-and-opposite forces. This means that the net force on the board is always zero, and thanks to Newton's 2nd Law, we know that the center of mass of the board cannot accelerate. The board started at rest, so its center of mass in fact never moves. This means that the fixed point on the board (around which the board appears to pivot) is in fact its center of mass. The mass of the board is therefore not uniformly-distributed, and its red end in the diagram is more dense than the blue end.

Suppose we only know the length of the board: L*. We cannot determine the torques about the center of mass exerted by each force, because we don't know how far from the ends of the board the center of mass is. But we can determine the* ***sum*** *of these torques. From the result derived above, the zero net force allows us to measure the net torque around any point. Choosing one end of the board, the force applied there provides zero torque, and the force on the other end provides a torque of* $\tau = FLsin90^o = FL$ *. If one wanted to do more work, this same result could be obtained less "cleverly" by calling the distance from one end to the center of mass* d*, making the distance from the other end to the center of mass* $L-d$ *. Then multiplying each by* F *and adding them together (the torques are in the same direction) gives the same result.*

Static Equilibrium

We have spent a great deal of time studying motion in all its forms, but now we're going to step back and look at something called *static equilibrium*. Simply put, this means unmoving (static), and not about to move (equilibrium). This is a particularly important subject for engineers who aspire to build things that won't (easily) fall down. From Newton's laws for linear and rotational motion, we have two conditions for the equilibrium part of this condition:

- net force on object is zero
- net torque on object is zero

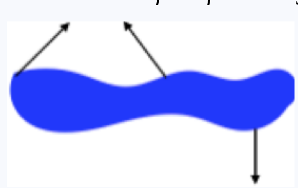
We are quite familiar with the net force part of this, but we need to do a bit of work on net torque. We know the formal definition of torque, but there is more we need to understand in order to apply this to static equilibrium problems. The first tool that we can immediately add to our toolbox for solving such problems is the result we got above. If the object is in static equilibrium, then it is experiencing zero net force, which means that no matter what reference point we choose, the net torque will be the same. But the net torque is *zero* for equilibrium, so we will have the following condition to work with:

For objects in static equilibrium, the net torque calculated around ***any*** *reference point whatsoever is zero.*

We will find the flexibility to choose any point we like as a reference to point to be very useful in what is to come.

Conceptual Question

For the force diagram below, the force vectors are drawn in the proper locations on the object, and are pointing in the proper directions, but the lengths of the vectors are not to scale. Which of the following statements are true about the effects these forces can have on the motion of this object? Assume that none of the force magnitudes can be set to zero.



a. *The force magnitudes can be set so that the object will not accelerate rotationally, while at the same time its center of mass does not accelerate linearly.*
b. *There is no way to set the force magnitudes to prevent either linear or rotational acceleration.*
c. *The force magnitudes can be set so that either there is no linear acceleration of the object's center of mass, or there is no rotational acceleration of the object, but both cannot be achieved at the same time.*
d. *The force magnitudes can be set so that the object's center of mass will not accelerate linearly, but there is no way to prevent its rotational acceleration.*
e. *The force magnitudes can be set so that the object will not accelerate rotationally, but there is no way to prevent linear acceleration of its center of mass.*

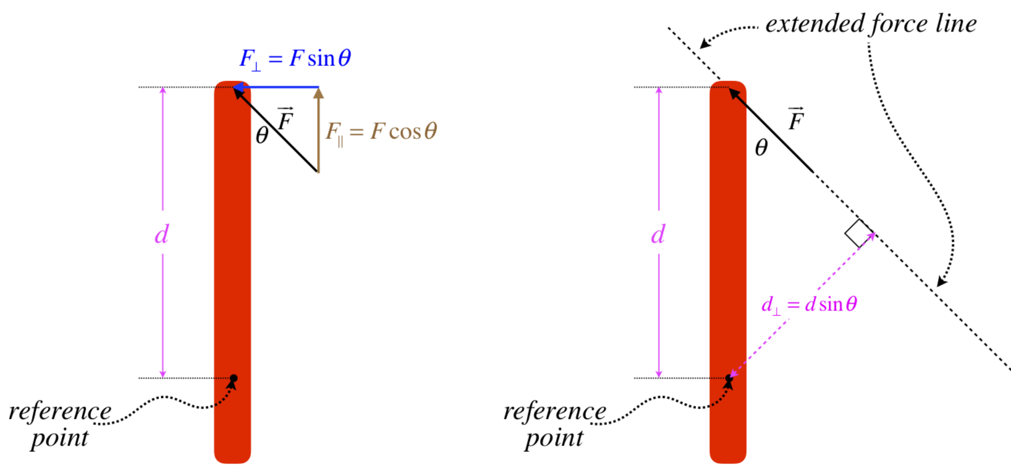
Solution

(d)
The two force vectors can be adjusted relative to each other so that their horizontal components cancel. Then both of their magnitudes can be adjusted in the same proportions so that the horizontal net force remains zero, while their combined vertical component of force cancels the other force vector. So zero net force is achievable. However, if we consider a reference point where the middle force acts on the object (giving that middle force zero contribution to torque), the torque of the other two forces will never cancel, no matter what adjustments are made to the force magnitudes. With no way to make the torque vanish, there is no way to prevent rotational acceleration.

Using Geometry to Determine Torque

Our definition of torque is all well-and-good, but in practice we rarely define a position vector and take a cross product. Instead, we tend to use the concept behind torque, and then some geometry. The figure below shows two ways to geometrically get to the same torque due to an applied force.

Figure 5.5.2 – Alternative Methods of Computing Torque



The left version consists of taking only the component of force that is perpendicular to the line joining the reference point and the point where the force is applied, giving the torque magnitude calculation:

$$\tau = F_\perp d = (F\sin\theta)\, d \tag{5.5.3}$$

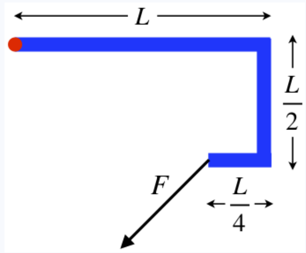
The right side of the figure shows another useful way to compute the same torque magnitude. Rather than finding the part of $\overrightarrow{F}$, it involves finding the perpendicular part of $\overrightarrow{r}$. This is done by extending the line of force and then geometrically determining the perpendiular distance from the reference point to that line. The result is the same as above:

$$\tau = Fd_\perp = F\,(d\sin\theta) \tag{5.5.4}$$

The perpendicular distance from the reference point to the line of force is often referred to as the *moment-arm*, or *lever-arm*. We will find this to often be the method of choice of computing torques when it comes to solving problems.

Conceptual Question

What can you say about the torque applied to the object due to the force F about the red pivot in the diagram?



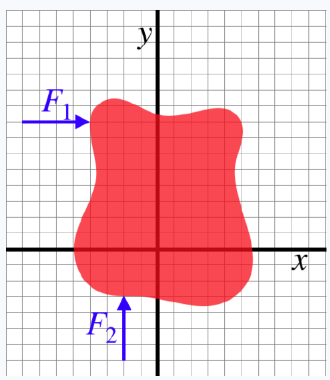
a. *it equals $\frac{1}{2}FL$*
b. *it equals $\frac{1}{4}FL$*
c. *it is greater than $\frac{1}{2}FL$*
d. *it is less than $\frac{1}{4}FL$, but greater than zero*
e. *net torques always sum to zero*

Solution

(c)
There are a couple of ways to answer this. This first is to extend the force line along F and look at the perpendicular distance from the pivot to that line (this is the moment arm). It should be clear from the geometry that this moment arm exceeds $\frac{1}{2}L$, which means the torque must be greater than $\frac{1}{2}FL$. Another way to see it is to break F into two separate vectors, one pointing left and the other pointing down. Both of these forces produce clockwise torques, and the horizontal force has a moment arm of $\frac{1}{2}L$, while the vertical force has a moment arm of $\frac{3}{4}L$. Since the sum of these two force components exceeds the magnitude of the original force, and since one of them has a moment arm larger than $\frac{1}{2}L$, then the combined torques must exceed $\frac{1}{2}FL$.

Analyze This

The blob in the figure below is rigid and in static equilibrium. The two forces shown are two of the total of three forces exerted on the object.



Analysis

The two conditions of equilibrium require that the net force and net torque equal zero. It doesn't take much to find the third force's magnitude and direction from what is given, as it just needs to make a sum that equals zero, which means the third force must be:

$$\vec{F}_3 = -\left(\vec{F}_1 + \vec{F}_2\right)$$

With the two forces at right angles to each other, the magnitude of their sum is:

$$\left|\vec{F}_3\right| = \sqrt{F_1^2 + F_2^2}$$

We also easily get the angle. Clearly the balancing force $\vec{F}_3$ points to-the-left-and-down to cancel these forces. Measuring the angle down from the x-axis:

$$\tan\theta = \frac{F_2}{F_1}$$

But of course knowing the force vector is not the whole story. What about where it is applied? Well, clearly it can be applied at many points on the object and still provide the same cancelling torque. Once the force's direction is known, the slope of the "force line" is determined. This line then just needs to be shifted so that the resulting torque cancels the others. And since the net force is already known to be zero, any reference point for this torque can be used.

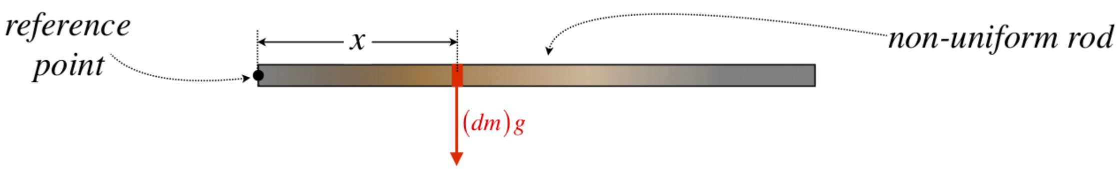
Center of Gravity

Up to now, whenever we have drawn a force diagram of an object, we have always placed the force vector for gravity at its center, while other forces are placed wherever they happen to act on the object. Gravity is somewhat special in that the force actually acts on every single atom in the object, but we can't draw all of those individual force vectors. Drawing it at the center of mass makes sense from the standpoint of Newton's second law, since if gravity is the only force, then it accelerates the object, and the part of the object that accelerates is the center of mass.

Wherever it happens to be appropriate to locate a single gravity force vector on a free-body diagram, it is called the object's *center of gravity*. We are currently dealing with torque, and the position at which a force acts has become quite important, so we need to examine more closely whether we can declare the center of mass of an object to be its center of gravity.

We choose as our test subject a horizontal non-uniform rod of length L, and select one of its ends as a reference point. The plan is to add up all of the infinitesimal torques that occur about this reference point due to gravity acting on every particle in the rod, and see if this total torque can be replaced by the entire gravity force acting at a single point (so that we can draw our free-body diagrams with only one gravity force vector!). An arbitrary piece of the rod will be a distance x from the reference point, and the torque exerted there will be the weight of that piece multiplied by x:

Figure 5.5.3 – Center of Gravity of a Non-Uniform Rod



$$d\tau = (dm\ g)\,x \quad \Rightarrow \quad \tau = \int_0^L dm\ gx = Mg\left[\frac{1}{M}\int_0^L dm\ x\right] = Mgx_{cm} \tag{5.5.5}$$

Sure enough, we get the same torque around the reference point if we put a single force vector with magnitude Mg (the object's full weight) acting at the object's center of mass.

Alert

It should be mentioned that there was a rather subtle assumption made in the above discussion – the gravity force is assumed to be the same at all points on the rod. If the gravity force can somehow vary from one end of the rod to the other, then the positions of these two centers will not coincide. If you are wondering how this can ever be the case, the answer is that the scale of distances must be very large, so that there are measurable differences in the gravity force from one point on the object to the other. This will not be an issue for our typically terrestrially-constrained studies, but can arise when talking about orbits of large bodies like moons.

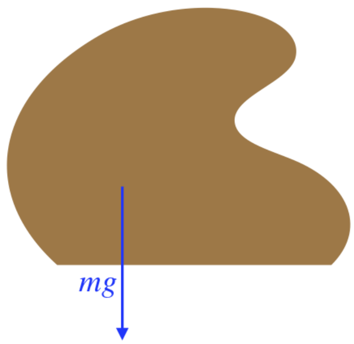
Note that like center of mass, the center of gravity of an object does not have to lie on the object. For example, a hoop's center of gravity is located in the empty space at its center. We now know how to locate the position of the gravity force on an object, and locating most other forces will be fairly intuitive (with one notable exception, which we will address next). This will enable us to use torque to analyze a whole range of real-world problems.

Placement of the Normal Force

Like the gravity force, the normal force can act at many places at once. When two surfaces come into contact, all of the particles at one surface repel all of the particles at the other. So once again we have the problem of where to draw a single force vector, this time for the normal force. The normal force is different from the gravity force, in one important way – it just *compensates* for other forces. That is, it adjusts according to other circumstances. Let's use what we know about static equilibrium to see how to place the normal force properly.

Consider the oddly-shaped object shown in figure below. We'll assume that the object sits at rest on a horizontal surface, the density of this object is not uniform, and that the center of gravity is at the position indicated in the diagram.

Figure 5.5.4 – Deducing the Normal Force Placement Balancing Only Gravity



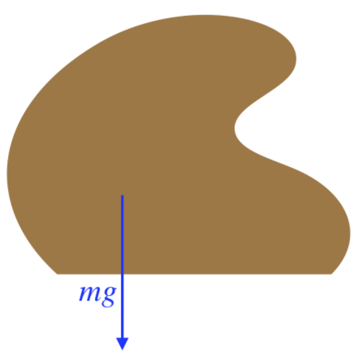
A (rather unsystematic) process for locating the position of the normal force goes like this:

1. Note that the object is in static equilibrium, which means that the normal force is equal to the weight (net force is zero), and that the net torque around any reference point we care to choose is also zero.
2. Try various positions for the normal force, and if we can prove that there must be a non-zero net torque around a reference point, then throw that position out.
3. Repeat step (2) for various positions until one is found that cannot be ruled-out.

For this object, we could try a normal force vector acting at the center of the base of the object. But then if we choose a reference point between the normal force vector and the weight vector, see see that those two forces must produce a non-zero counter-clockwise torque. We can similarly rule out any position to the right of the weight vector. If we try a position to the left of the weight vector, we get a similar result, this time with the torque being clockwise. We therefore conclude that for this case the normal force vector must be applied exactly where the weight vector intersects the base. No matter where we choose a reference point in that case, the two forces result in equal-and-opposite torques.

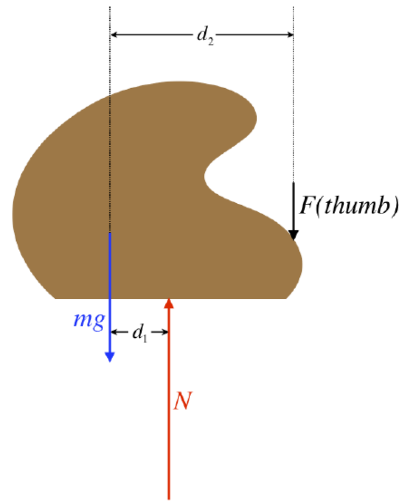
Let's complicate matters some by introducing a second force to our object – suppose we push down on the right side of the object with our thumb, as shown in the figure below.

Figure 5.5.5a – Deducing the Normal Force Placement Balancing Two Forces



Let's try the same position for the normal force as before – in line with the gravity force. If we choose as a reference a point in line with these two forces, then they create no torque between the two of them, and the added force by the thumb creates a net clockwise torque. This isn't possible for an object in static equilibrium, so the normal force placement has moved from its original placement as a result of the added thumb force. It's easy to see that the normal force hasn't moved left, as placing the reference point at the normal force results in both the weight and the thumb force producing clockwise torques. So the normal force must move right, but how far? Perhaps it moves into line with the thumb force? No... We can choose the reference point to be in line with these two forces (so they both create zero torque), and the gravity force would yield a net counterclockwise torque.

Figure 5.5.5b – Deducing the Normal Force Placement Balancing Two Forces



So we conclude that the normal force must act somewhere between the gravity and thumb forces. If we know the magnitudes of these two forces, then we know the magnitude of the normal force (the net force is zero), and in fact we can also determine precisely where a single normal force is acting on the object. Calling the distance between the weight and normal force vector placements d_1 and the distance between the normal force and thumb force vector placements d_2, we can sum the torques around a reference point where the normal force acts (so it contributes no torque) to get:

$$0 = \tau_{net} = -mgd_1 + Fd_2 \quad \Rightarrow \quad \frac{d_1}{d_2} = \frac{F}{mg} \tag{5.5.6}$$

[*Note: In the torque sum, clockwise was chosen as the positive direction.*]

In the diagram, the weight is shown to be greater than the thumb force, making the ratio less than 1, which means the placement of the normal force is closer to the placement of the weight vector than the thumb force vector. If the thumb pushes down more, then the normal force placement moves to the right. Note also that the same result arises if the reference point is chosen elsewhere. For example, if the reference point is chosen to be where the weight force acts, then the net torque equation gives zero contribution from the weight, and contributions from both the normal force (counterclockwise), and the thumb force (clockwise). The normal force can then be written in terms of the weight and thumb force (the net force is zero), giving:

$$0 = \tau_{net} = -Nd_1 + F\left(d_1 + d_2\right) \quad \Rightarrow \quad 0 = -\left(mg + \cancel{F}\right)d_1 + \cancel{Fd_1} + Fd_2 \quad \Rightarrow \quad \frac{d_1}{d_2} = \frac{F}{mg} \tag{5.5.7}$$

Conceptual Question

Two different blocks are at rest on opposite ends of a smooth uniform wooden plank, which balances at a point that is two-thirds of the length of the plank from one end, as shown in the diagram. A force of F_1 is applied to the block farthest from the balance point, and a force of F_2 is applied to the other block. Both forces are horizontal and point toward the center of the plank. As the blocks accelerate due to their respective forces (without friction from the plank), the plank remains balanced. Which of the following can be concluded about the magnitudes of the two forces?

a. $F_1 = F_2$
b. $F_1 = 2F_2$
c. $2F_1 = F_2$

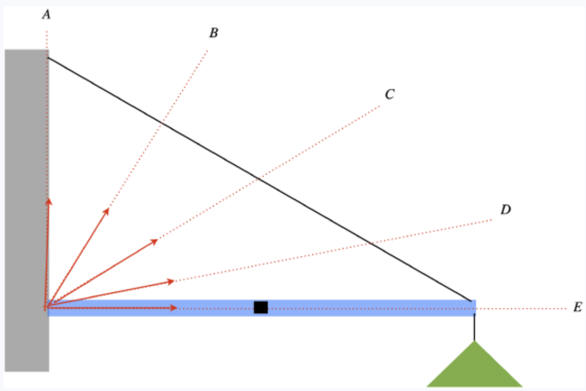
d. $F_1 > F_2$ *(the exact proportion depends upon the mass of the plank)*
e. $F_1 < F_2$ *(the exact proportion depends upon the mass of the plank)*

Solution

(a)
The system starts at static equilibrium, which means the center of mass of the two blocks + plank system lies directly above the balance point. For the center of mass of this system to not accelerate away from this location, the net force on the system must be zero. This is achieved when the two forces (which are in opposite directions) have equal magnitudes.

Conceptual Question

Below is a diagram of a sign hanging from a wall with a boom and a support wire. If the boom is uniform in density (its center of mass represented by the black dot) and weighs about the same as the sign, which of the force vectors shown most closely approximates the direction of the total force on the boom by the wall?



a. *A*
b. *B*
c. *C*
d. *D*
e. *E*

Solution

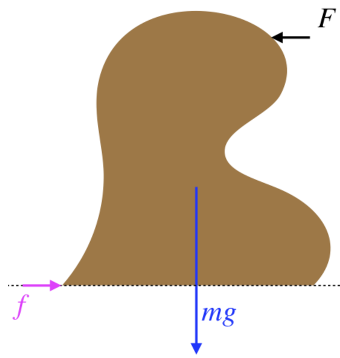
(d)

The mass of the boom and sign are equal, so their combined center of gravity is three-quarters of the boom length from the wall. We can replace those two weights with a single gravity force acting at that center of gravity. If we choose as our pivot the point of intersection between the line of that gravity force and the tension force, then they both contribute zero torque. The only force left is that of the wall, and for it to not create a net torque around that pivot, it must also pass through that point. It does this if it points in direction D.

Conditions for Tipping

Let's make a slight change to the situation just described. Suppose I push horizontally on the top of the object. What happens to the normal force position as the magnitude of the push increases? Assuming there is a static friction force to prevent the object from sliding, we have a free-body diagram (missing the normal force) that looks like this:

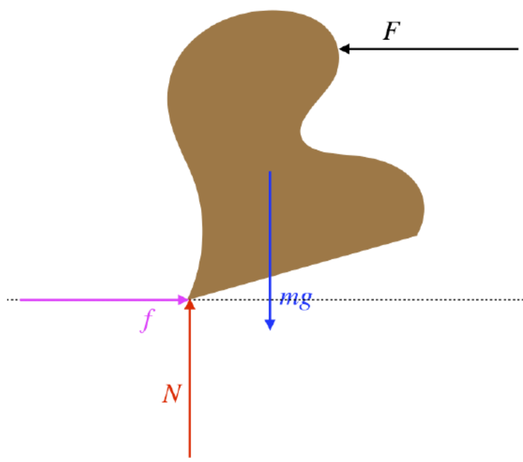
Figure 5.5.5c – Deducing the Normal Force Placement Balancing Two Forces



Choosing a pivot point at the intersection point of the gravity and friction forces, we see that the push force exerts a net counterclockwise torque. For the normal force to counteract this (and given that it must push straight up), we find that it must be placed to the left of the center of gravity.

Let's take a moment to consider the magnitudes of these forces. So long as the object doesn't slide, the static friction force must equal the push. The object doesn't accelerate up or down, so the normal force must have the same magnitude as the gravity force. Both of these conditions are important when we consider what happens when the push force is increased. The friction force also increases until it hits its maximum, at which point the object starts sliding. If we suppose the static friction force doesn't hit its maximum, how is the increased torque by the push compensated by the normal force, if it can't change magnitude? It must move left. But it can only move left for so long, and when it has gone as far as it can go, any added push results in angular acceleration – the object tips.

Figure 5.5.6 – Tipping



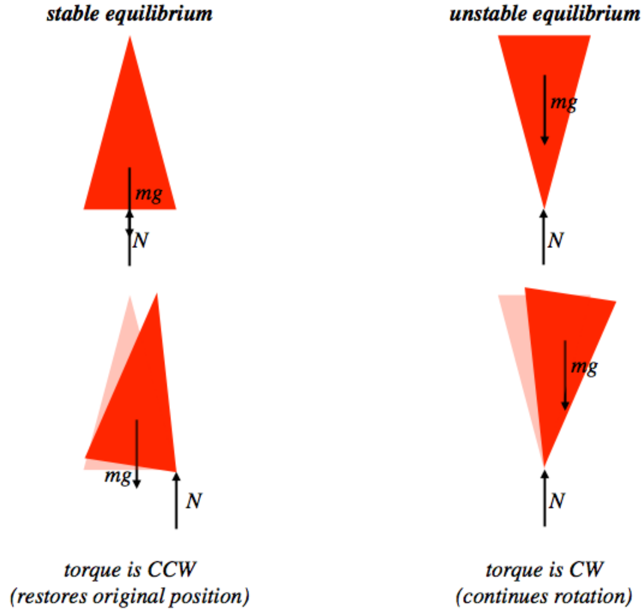
Suppose you want to push an object across the floor without tipping it over. To get it to slide, you have to push with a force at least equal to the static friction force, so to avoid tipping, this given amount of force needs to provide as little torque as possible – push at a point close to the bottom. With very little torque from the push force, the normal force can easily remain inside of the edge of the object, and the object won't tip before it slides.

Stable/Unstable Equilibrium

If the object is oddly-shaped and the only forces acting on it are gravity and the normal force, then this analysis gives us an answer as to whether the object falls over – if the normal force can be directly beneath the center of gravity, then it will stand up. By "can," of course we mean that some part of the base that is in contact with the surface (where the normal force acts) must be below the center of gravity.

In Section 3.7 we discussed stable and unstable equilibrium from the perspective of energy diagrams, and the concept of whether an equilibrium is stable or unstable was first addressed. The idea is that if the system is moved slightly from its equilibrium state, do the forces (or, in our current case, torques) act to return the system to equilibrium (stable), or to continue moving the system away from equilibrium (unstable). How these definitions apply to tipped objects is clear from the free-body diagrams, as shown in the figure below. We can also define a *degree* of stability to a standing object. We define it as the angle through which we can rotate it such that if we let it go, restoring torques act to return it to its original position.

Figure 5.5.7 – Stable and Unstable Equilibrium



Note that this definition of stability matches with what we saw in energy diagrams. Recall that an equilibrium was a point where the potential energy function has zero slope, and the equilibrium is stable if the potential energy grows on both sides of the equilibrium, and is unstable if the potential energy falls off on both sides. Consider what happens to the gravitational potential energy of the object in both cases shown above. In the stable case, tipping the object slightly *raises* the center of mass of the object (increasing its gravitational potential energy), while in the unstable case a slight tip *lowers* the center of gravity (decreasing its gravitational potential energy).

Problem-Solving

Problems involving static equilibrium can be approached in a very systematic way, the steps of which are outlined below:

1. Determine the object in static equilibrium you need to analyze and isolate it in a force diagram. This can sometimes be easier said than done. Sometimes the problem involves more than one extended object in contact with each other. In this case, determining the object you choose (or perhaps the combination of both objects) depends upon what you are asked to solve for (usually a force). You can't really go wrong here, though – if you choose an object that will not give you the answer you need, it should occur to you as you draw the force diagram. Also, you may find that a "wrong" choice of object may simply make your task a bit longer (more simultaneous equations) – annoying, but no real harm done.
2. Define a linear (x, y) coordinate system for force components, and a rotational coordinate system (positive rotation direction) for torques.
3. Extend each force vector with a dotted line as far as it will go on the page in both directions.
4. Choose a reference point. For now we won't worry about choosing a "good" one, choose any – but stick with it for the remaining steps. When you get better at these problems (which you can only achieve by doing them, especially if you do the same problem in multiple ways), you will get better at choosing convenient reference points. Please note that not all static equilibrium problems involve hinges or other "natural" pivots – The reference point doesn't need to be one of these!
5. Ignoring distractions like the object itself, use geometry to find the perpendicular distance from every force line to the pivot point (i.e. all the moment arms). Do not worry about what angle you use to find these (i.e. it doesn't have to be the angle from the torque equation $\tau = rf\sin\theta$) – just use geometry.
6. Multiply the moment arm by the magnitude of each force, and this is the magnitude of the torque due to each force.
7. Determine whether each torque is clockwise or counterclockwise, and give each the appropriate sign when summing the torques and setting that sum equal to zero for the equilibrium torque condition. Note that you could simply alternatively add up all the

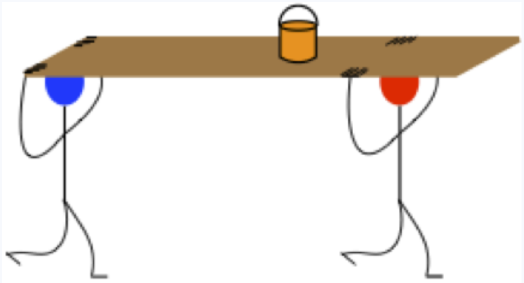
CW torques, place them on one side of the equation, and set them equal to the sum of the CCW torques on the other side of the equation. This is easier to implement, but loses the "flavor" of what equilibrium is (zero net torque), so I describe it both ways.

8. If you are lucky (or were clever at the outset), this equation may be all you need in order to find what you are looking for. If it isn't, you have two alternatives from here...
   - Write out the sum of the forces in the x and y directions (or just one of those directions, if that is all you need), and set the net forces equal to zero (another condition for equilibrium). These additional equations should be all you need to find what you are looking for.
   - Choose a new reference point and repeat the torque method described above. Recall that the torques should sum to zero around any point, so this is completely valid. The thing to keep in mind is that wherever you choose your reference point, if a force line goes through it, then that force won't appear in the torque equation because the moment arm for it is zero. Therefore you can choose your reference point at a spot through which lines for unknown forces pass, eliminating the need to eliminate them using simultaneous equations later. Whatever you do, don't choose a reference point that lies along a line of the force that you are actually looking for – it doesn't give you an equation that includes that force!

What follows is a set of several physical examples of static equilibrium for you to analyze. While they all look quite different, they can all be effectively analyzed in the manner described above. While analysis always goes quite far in solving a problem without even knowing what the question is, this is especially true of these types of problems, so understanding the analysis portion is even more critical for these types of problems than usual.

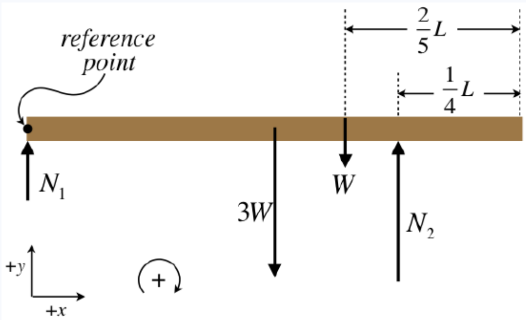
Analyze This

Two painters carry a plank of plywood that they use for scaffolding over their heads on their way to the job site. The plank has a uniform mass distribution. Atop the plank is a can of paint weighing one third as much as the plank. The painter in the rear is holding the plank at the very end and the painter in front is holding the plank one quarter of the the plank length from the front. The can of paint is two-fifths of the plank length from the front. The plank remains horizontal as they carry it.



Analysis

We start by identifying the object in equilibrium (the plank), and drawing a free-body diagram for it (we'll call the length of the plank L). We will choose the pivot to be the back of the plank, and will refer to the weights of the can of paint and plank as W and 3W, respectively. Also we have chosen an (x, y) coordinate system and the positive direction of rotation to be clockwise, as shown in the diagram.



Next apply the conditions of equilibrium. Clearly the x-direction forces are not meaningful, and the y-direction force equation and torque equations are:

$$\begin{array}{ll} vertical\ forces: & 0 = N_1 - 3W - W + N_2 \\ torques: & 0 = N_1(0) + 3W\left(\frac{1}{2}L\right) + W\left(\frac{3}{5}L\right) - N_2\left(\frac{3}{4}L\right) \end{array}$$

The L's cancel out of the torque equation, resulting in a relation between the force exerted by the front painter and the weight of the can:

$$N_2 = \frac{14}{5}W$$

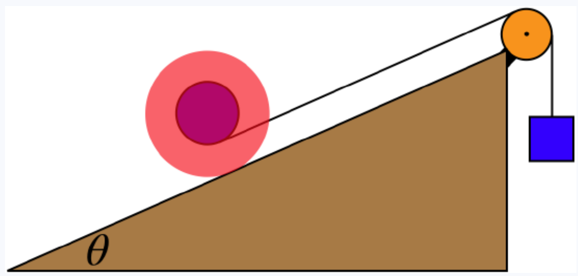
The total weight carried by the two painters is found from the force equation (or from common sense), and equals $4W$. So we can compute the percentage of the total load carried by each painter.

$$\frac{N_2}{4W} = 0.7$$

The front painter carries 70% of the load, and the rear painter 30%.

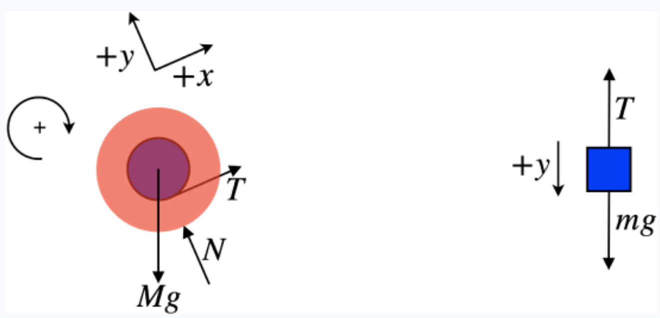
Analyze This

The diagram below depicts a yo-yo on an inclined plane with its string over a massless pulley and attached to a hanging block. The whole system is in static equilibrium.

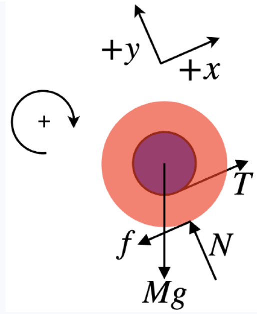


Analysis

We start, as always, with a FBD. We are not told about the frictional condition of the surface, so we will leave off a friction force on the yo-yo for now, and see what happens...



Let's take a close look at the FBD of the yo-yo. If we choose its center as a reference point, we see that the gravity force and the normal force don't produce torques, but the tension force does. So this FBD is not correct, as the yo-yo cannot be in static equilibrium with a net torque. There must therefore be a static friction force acting on its outer edge. We even know the direction. About the center of the yo-yo, the torque from the tension is counterclockwise, and to produce a balancing clocwise torque, the static friction force must act down the plane. The new, corrected FBD for the yo-yo is:



Next we apply the conditions of equilibrium – the sum of the forces and torques add up to zero. Clearly, doing this for the block gives us that the tension equals the weight of the block. The FBD of the yo-yo gives three equations. Calling the radius of the yo-yo's hub r and the radius of the yo-yo's outer rim R, we have:

$$\begin{aligned} F_x: &\quad T - f - Mg\sin\theta = 0 \\ F_y: &\quad N - Mg\cos\theta \\ \tau \text{ (about center)}: &\quad fR - Tr = 0 \end{aligned}$$

With static friction in play, we can also write down the constraint:

$$f \leq \mu_s N$$

As usual, this relation becomes useful if we are told that the system is at some extreme, so that the friction force is maximized. We can now start solving simultaneous equations, but with no knowledge of what we are looking for, we don't know what variables to start eliminating with the algebra, so we will end the anlysis here.

Sample Problems

All of the problems below have had their basic features discussed in an "Analyze This" box in this chapter. This means that the solutions provided here are incomplete, as they will refer back to the analysis performed for information (i.e. the full solution is essentially split between the analysis earlier and details here). If you have not yet spent time working on (not simply reading!) the analysis of these situations, these sample problems will be of little benefit to your studies.

Problem 5.1

A bug stands on the outer edge of a turntable as it begins to spin, accelerating rotationally in the horizontal plane from rest at a constant rate. The bug is held on the turntable by static friction, but as the turntable spins ever faster, this will not remain the case forever. The turntable, which has a radius of $0.40m$, has its rotational speed increase at a steady rate from rest, and reaches a speed of $\omega_1 = 1.2\frac{rad}{s}$ after its first full revolution.

a. *Find the linear speed of the bug after the turntable makes n full revolutions.*
b. *Find the coefficient of static friction if the bug falls off after the turntable makes n full revolutions.*

Solution

a. In the analysis, we indicated a kinematics relationship between α, θ, and ω. The angular acceleration is constant, so it does not depend on the number of revolutions, and for n revolutions, $\theta = 2\pi n$, so the angular velocity after n revolutions is:

$$\omega_n^2 = 2\alpha\left(2\pi n\right) = \omega_1^2 n \quad \Rightarrow \quad \omega_n = \omega_1\sqrt{n}$$

The linear speed is the angular speed multiplied by the radius, so:

$$v_n = r\omega_n = r\omega_1\sqrt{n} = (0.40m)\left(1.2\frac{rad}{s}\right)\sqrt{n} = \left(0.48\frac{m}{s}\right)\sqrt{n}$$

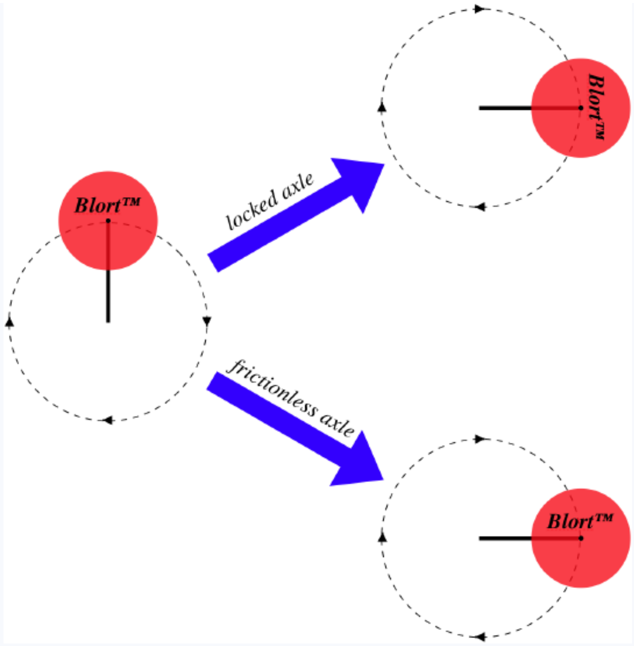
b. In the analysis we derived a formula for the coefficient of static friction in terms of the angular velocity and acceleration. Incorporating what we found above in terms of the number of full revolutions, we get:

$$\mu_s = \frac{r\sqrt{\omega_n^4+\alpha^2}}{g} = \frac{r\sqrt{n^2\omega_1^4+\left(\frac{\omega_1^2}{4\pi}\right)^2}}{g} = \frac{r\omega_1^2}{g}\sqrt{n^2+\frac{1}{16\pi^2}} = 0.059\sqrt{n^2+0.0063}$$

Notice that unless the angular acceleration is very large so that the bug falls off very quickly – in much less than a full revolution ($n \ll 1$) – virtually all of the static friction force goes into maintaining the centripetal acceleration, and only a very tiny fraction of the total static friction is involved with speeding up the bug.

Problem 5.2

The Blort Corporation makes a special widget that consists of a uniform disk pivoted around an axle at the end of a rod of negligible mass, which in turn rotates about its other end. This widget has two settings: It can be set in the "locked" position so that the disk does not rotate around its axle, or the "free" position so that the disk rotates frictionlessly about the axle. The difference these settings have on the motion of the disk as the rod rotates is depicted in the figure below.



An engineer for a company that uses the Blort widgets in their manufacturing wants to make sure that the power output of the motor that turns the rod automatically adjusts so that the rod's rotation is the same whether the axle is in the fixed or free setting. The specifications of the widget indicates that the rod's length is equal to the diameter of the disk. By what factor must the power output of the motor be adjusted between these two settings?

Solution

In the analysis, we calculated the kinetic energies for a given rotational speed for each of these settings. The power from the motor goes directly into the kinetic energy of the widget, so the ratio of the kinetic energies will match the ratio of the power outputs:

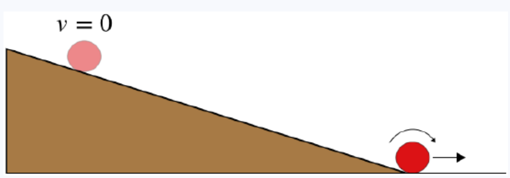
$$\frac{P_{locked}}{P_{free}} = \frac{KE_{locked}}{KE_{free}} = \frac{\frac{1}{4}M\left(R^2+2L^2\right)\omega^2}{\frac{1}{2}ML^2\omega^2} = 1+\frac{1}{2}\left(\frac{R}{L}\right)^2$$

The length of the rod is the diameter of the disk, so it is twice the radius, giving:

$$\frac{P_{locked}}{P_{free}} = 1.125$$

Problem 5.3

A solid uniform sphere starts from rest and rolls down a flat ramp without slipping.



The sphere descends a vertical distance of $3.6m$ *by the time it reaches the bottom, and it takes* $6.6s$ *to make the journey. Find the angle that the ramp makes with the horizontal.*

Solution

We know the final speed of the sphere from our analysis:

$$v_f = \sqrt{\frac{10}{7}gh}$$

The sphere started from rest, and as noted in the analysis, the acceleration is constant, so the average velocity is:

$$v_{ave} = \frac{v_o + v_f}{2} = \sqrt{\frac{5}{14} gh}$$

This number, along with the time, gives us the straight-line (along the ramp) distance traveled by the ball:

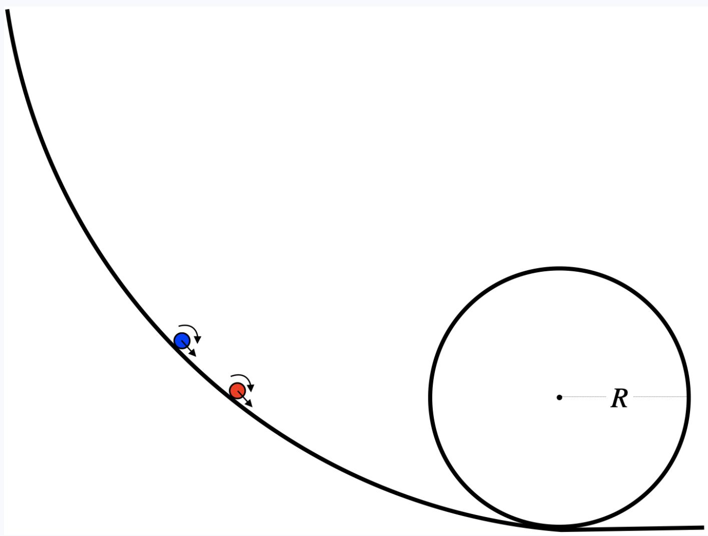
$$d = v_{ave} t = \sqrt{\frac{5}{14} gh}\ t$$

We now have the hypothenuse and opposite side of a right triangle, so we can get the angle:

$$\theta = \sin^{-1}\left(\frac{h}{d}\right) = \sin^{-1}\left(\sqrt{\frac{14h}{5gt^2}}\right) = \sin^{-1}\left(\sqrt{\frac{14\,(3.6m)}{5\left(9.8\frac{m}{s^2}\right)(6.6s)^2}}\right) = 8.8^o$$

Problem 5.4

A solid and a hollow sphere roll without slipping simultaneously (one behind the other) down a ramp and around a loop-de-loop. The radii of the spheres are negligible compared to the radius of the loop.



Both spheres are released simultaneously from rest, and both barely make it around the loop. Find which sphere is in front of the other, and the ratio of their starting heights.

Solution

In the analysis we found that if they have the same velocity, the two spheres will have different kinetic energies. We also found the speed either sphere must have in order to get all the way around the loop. Plugging this value into the kinetic energies of the spheres tells us how much kinetic energy they must have to make it around:

$$\left.\begin{array}{l} KE_{solid} = \frac{7}{10} mv^2 \\ v = \sqrt{gR} \end{array}\right\} \quad KE_{solid} = \frac{7}{10} mgR$$

$$\left.\begin{array}{l} KE_{hollow} = \frac{5}{6} mv^2 \\ v = \sqrt{gR} \end{array}\right\} \quad KE_{solid} = \frac{5}{6} mgR$$

Referencing zero gravitational potential energy at the bottom of the loop, at the top of the loop, the spheres alsoi have a potential energy of $2mgR$. Given that they start from rest, they start with only potential energy, so that equals their total energy:

$$E_{tot\ hollow} = mgh_{hollow} = mg\left(2R + \frac{5}{6}R\right) = \frac{17}{6}mgR$$

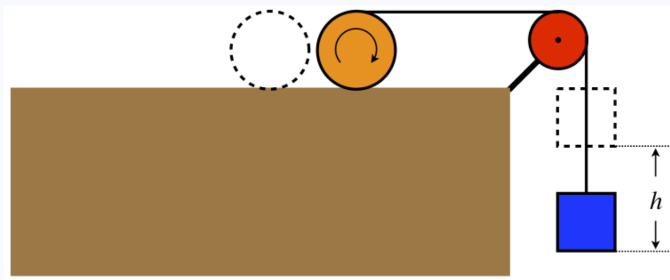
$$E_{tot\ solid} = mgh_{solid} = mg\left(2R + \frac{7}{10}R\right) = \frac{27}{10}mgR$$

The hollow sphere needs to start higher, and since they start simultaneously from rest, the leading sphere is the solid one. The ratio of their starting heights is:

$$\frac{h_{hollow}}{h_{solid}} = \frac{85}{81}$$

Problem 5.5

One end of a massless rope is wound around a uniform solid cylinder, while the other end passes over a massless, frictionless pulley and is attached to a hanging block, as in the diagram below. The block is released from rest, pulling the cylinder along the horizontal surface such that it rolls without slipping.



The cylinder and block are both weigh $22N$. Find the tension in the string.

Solution

In the analysis, we found the final velocity of the cylinder after the block drops a distance h. Plugging in equal masses for the block and cylinder gives, and solving for the velocity of the block using, $v_b = 2v_c$:

$$v_f = 2\sqrt{\frac{4m}{8m+3m}gh} = \sqrt{\frac{16}{11}gh}$$

The acceleration of the block is constant, so from the kinematics equation with no time variable, we can get the acceleration:

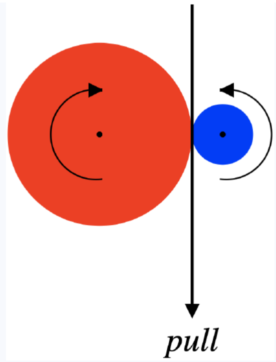
$$v_f^2 - v_o^2 = 2ay \quad \Rightarrow \quad a = \frac{v_f^2}{2h} = \frac{8}{11}g$$

The acceleration is a result of the net force, which is the vector sum of the downward gravitational force and the upward tension force, so setting upward as the positive direction (making the acceleration negative), we get:

$$T - mg = ma \quad \Rightarrow \quad T = m(g+a) = m\left(g - \frac{8}{11}g\right) = \frac{3}{11}mg = \frac{3}{11}(22N) = 6N$$

Problem 5.6

Two disks are cut out of the same material, as shown in the diagram below. They are pivoted around stationary axles such that the two disks lie in the vertical plane, with their outer rims pinching a massless rope between them. The rope is pulled downward, causing both disks to turn without the rope slipping.



The smaller disk has one-third the radius of the larger disk. As the rope is pulled, power is delivered to the two-disk system. Find the fraction of the total power delivered to the larger disk.

Solution

To find the fraction of the power delivered, we only need to figure out the ratio of the energy of the small disk to the large disk at a given speed. This ratio is:

$$\frac{KE_{small}}{KE_{large}} = \frac{\frac{1}{2}I_{small}\omega_{small}^2}{\frac{1}{2}I_{large}\omega_{large}^2} = \left(\frac{I_{small}}{I_{large}}\right)\left(\frac{\omega_{small}}{\omega_{large}}\right)^2$$

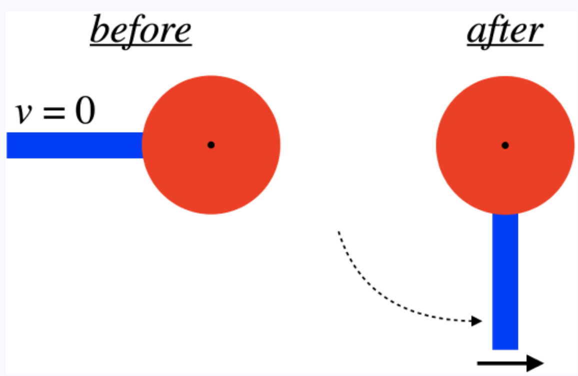
We found these ratios in terms of the ratios of the radii of the disks in the analysis.

$$\left.\begin{array}{l}\frac{I_{small}}{I_{large}} = \left(\frac{r}{R}\right)^4 \\ \frac{\omega_{small}}{\omega_{large}} = \frac{R}{r}\end{array}\right\} \quad \frac{KE_{small}}{KE_{large}} = \left(\frac{r}{R}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

Nine times as much power goes to the large disk as the small disk, which means that 90% of the total power delivered by the pulled rope is going to the large disk.

Problem 5.7

One end of a uniform metal thin rod is welded to the outer edge of a metal disk. The masses of these two objects are the same, and the length of the rod is equal to the diameter of the disk. The disk is suspended on a frictionless axle positioned at its center, and the rod is released from rest from a horizontal orientation and allowed to swing down to the vertical position.



The linear speed of the open end of the rod at the bottom of the swing is measured to be $3.0\frac{m}{s}$*. The pendulum is then removed from the axle and is swung in the same manner (from rest horizontally) with the open end of the rod now attached to the axle (so the disk is swinging down). Find the linear speed of the bottom edge of the disk when it gets to the bottom of the swing.*

Solution

Let's start by using our result from the analysis to determine what is given. We know that the linear speed of the bottom of the rod is the angular velocity multiplied by the distance to the axle, so:

$$v_{rod\ end} = r\omega \quad \Rightarrow \quad 3.0\frac{m}{s} = 3R\omega_{rod\ swings}$$

We can solve for the value of R here, but as we will soon see, this is not necessary. We do, however, have to follow all the same steps from the analysis for the new setup. First, when pivoted at the open end of the rod, the center of mass of the whole object descends a distance of $2R$ (twice as far as the previous case), giving:

$$\Delta U = -2mg\,(2R) = -4mgR$$

The moment of inertia also changes. This time we have the rod about its end and the disk extended by the parallel axis theorem:

$$I_{tot} = I_{rod} + I_{disk} = \frac{1}{3}m(2R)^2 + \left[\frac{1}{2}mR^2 + m(3R)^2\right] = \frac{65}{6}mR^2$$

Putting this into the energy conservation gives:

$$KE_f = -\Delta U \quad \Rightarrow \quad \frac{1}{2}\left(\frac{65}{6}mR^2\right)\omega^2 = 4mgR \quad \Rightarrow \quad \omega = \sqrt{\frac{48g}{65R}}$$

This is a bit slower than the previous case. The ratio of the two angular velocities are:

$$\frac{\omega_{disk\ swings}}{\omega_{rod\ swings}} = \sqrt{\frac{58}{65}}$$

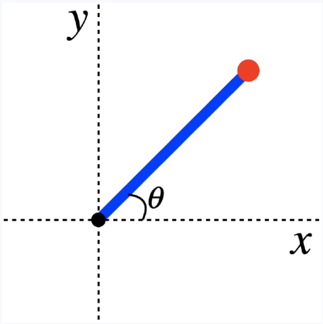
But we are interested in the linear velocity of the bottom of the disk. This is a distance of $4R$ from the axle, so using the result above, its linear velocity is:

$$v_{disk\ bottom} = 4R\omega_{disk\ swings} = 4R\left(\sqrt{\frac{58}{65}}\omega_{rod\ swings}\right) = \frac{4}{3}\sqrt{\frac{58}{65}}\,(3R\omega_{rod\ swings}) = \sqrt{\frac{928}{585}}\left(3.0\frac{m}{s}\right) = 3.8\frac{m}{s}$$

Problem 5.8

A small marble is attached to the end of a thin rigid rod with an equal mass, whose other end is held fixed at the origin. The rod starts at rest in the $x-y$ plane, and makes an angle θ up from the x-axis, as shown in the diagram. There is no gravity present, but the marble (not the rod) is subjected to a force from a potential energy field given by:

$$U\,(x,y) = \beta xy + U_o\ , \quad \beta = constant > 0$$



The values of the variables given above are:

$$\theta = 30^o\ , \quad \text{mass of marble} = 0.45kg\ , \quad \beta = 1.2\frac{J}{m^2}$$

a. *Find the magnitude and direction of the angular acceleration when the rod is released. Express the direction of this acceleration both as a unit vector and as either clockwise or counterclockwise from the perspective of this diagram.*
b. *Find the maximum angular velocity attained by the rod, and the orientation angle of the rod when this maximum is reached.*

Solution

a. As usual, most of the heavy-lifting in this problem was already done in the analysis. Using the expression for the angular acceleration derived in the analysis, we have:

$$\vec{\alpha}=-\frac{3\beta\cos 2\theta}{4m}\hat{k}=-\frac{3\left(1.2\frac{J}{m^2}\right)\cos 60^o}{4(0.45kg)}\hat{k}=-1.0\frac{rad}{s^2}\hat{k}$$

This vector is in the $-\hat{k}$ direction, which points into the page, as we are using a right-handed coordinate system. From the RHR, this direction is clockwise from the perspective looking at the diagram.

b. Note that the linear velocity is a maximum when the rotational velocity is a maximum, so if we write the rotational velocity as a function of θ, we just need to do calculus to find where the maximum occurs:

$$0=\frac{d\omega}{d\theta}=\frac{d\omega}{dt}\frac{dt}{d\theta}=\frac{\alpha}{\omega}\quad\Rightarrow\quad\alpha=0$$

From the result for the angular acceleration, we see that the extrema occur at $\cos 2\theta=0$, so $\theta=\pm45^o$ or $\theta=\pm135^o$. All of these angles satisfy $|x|=|y|$. Next we need to determine which ones correspond to the maximum speed. Clearly the maximum kinetic energy occurs when the potential energy is a minimum, and looking at the potential energy function, this occurs when either x or y (but not both!) is negative. These two cases correspond to $\theta=-45^o$ and $\theta=135^o$. The other two angles correspond to maximum potential energies, but since the marble starts with zero kinetic energy at a lower potential energy, it can never reach these points. Therefore the only place where the marble (which starts at $\theta=30^o$ and accelerates clockwise) can reach a maximum speed is $\theta=-45^o$.

Let's call the length of the rod L, as in the analysis. When the marble gets to $\theta=-45^o$, the potential energy is:

$$x=-y=\frac{1}{\sqrt{2}}L;\quad\Rightarrow\quad U=-\frac{1}{2}\beta L^2+U_o$$

Initially we had:

$$\left.\begin{array}{l}x=L\cos 30^o=\frac{\sqrt{3}}{2}L\\ y=L\sin 30^o=\frac{1}{2}L\end{array}\right\}\quad U=\frac{\sqrt{3}}{4}\beta L^2+U_o$$

The object starts from rest, so its final kinetic energy is just equal to the amount of potential energy lost:

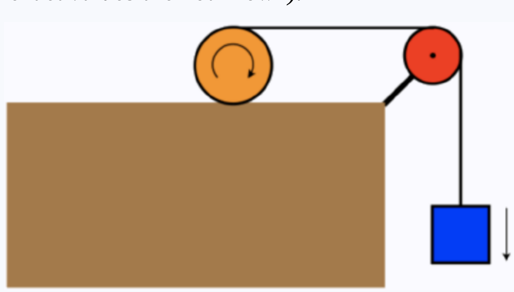
$$KE=-\Delta U\quad\Rightarrow\quad\frac{1}{2}I_{tot}\omega^2=\left(\frac{\sqrt{3}}{4}\beta L^2+U_o\right)-\left(-\frac{1}{2}\beta L^2+U_o\right)=\left(\frac{\sqrt{3}+2}{4}\right)\beta L^2$$

Plugging in for the moment of inertia of the whole object as found in the analysis, we get:

$$\frac{1}{2}\left(\frac{4}{3}mL^2\right)\omega^2=\left(\frac{\sqrt{3}+2}{4}\right)\beta L^2\quad\Rightarrow\quad\omega=\sqrt{\frac{3}{8}(\sqrt{3}+2)\left(\frac{1.2\frac{J}{m^2}}{0.45kg}\right)}=1.9\frac{rad}{s}$$

Problem 5.9

Returning to the physical system of problem 5.6, we consider a new question. As before, the cylinder rolls without slipping, and the masses are equal (though here their exact values are not known).



Find the minimum coefficient of static friction between the cylinder and the horizontal surface that will allow for this perfect rolling to occur.

Solution

In the analysis we wrote down the equations that come from Newton's 2nd Law (for linear and rotational motion), and found the accelerations of the cylinder and block. The if the cylinder is just barely rolling because the coefficient of friction is as low as it can be, that means that the largest possible static friction is occurring:

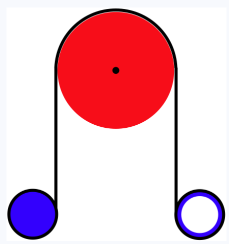
$$f=\mu_s N=m_s mg$$

Putting this into the x-direction equation for the cylinder from Newton's 2nd Law, and plugging in the tension in the rope and the acceleration (both found in the analysis) gives:

$$f+T=ma \quad\Rightarrow\quad \mu_s mg=-\frac{3}{4}ma+ma=\frac{1}{4}ma=\frac{1}{4}m\left(\frac{4}{11}g\right) \quad\Rightarrow\quad \mu_s=\frac{1}{11}$$

Problem 5.10

Two ends of a massless rope are wound around two spools with equal masses and radii. One of the spools is a solid, uniform disk, while the other is a thin, hollow cylinder. The rope between them goes over a massless, frictionless pulley in a vertical plane. The spools are released from rest from the same height, and the rope does not slip over the pulley.



The radius of the pulley is $25cm$. Find the angle through which the pulley has turned when $2.0s$ has elapsed.

Solution

From the result in the analysis, we can compute how much additional rope is in play. It is simply twice the distance that the spools fall:

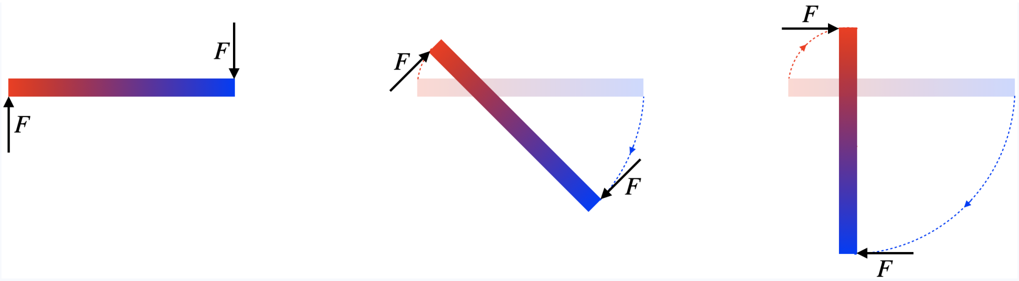
$$\Delta L=2\Delta y=2\left(\frac{1}{2}at^2\right)=\left(\frac{3}{5}g\right)t^2=\frac{3}{5}\left(9.8\frac{m}{s^2}\right)(2.0s)^2=23.5m$$

The torques are the same on both spools, but the solid spool has one-half the moment of inertia of the hollow one, so it has twice the angular acceleration, which means that over the same time period, it loses twice as much rope. Let's call the amount of rope lost by the hollow spool l, so the amount lost by the solid spool is $2l$. The rotation of the pulley makes up for this difference, which means it takes $\frac{1}{2}l$ from the side where the solid spool is, and places it on the side of the hollow spool. Noting that the total rope lost by both spools combined is $3l=\Delta L$, and using the no-slipping condition, we have:

$$s=R\theta \quad\Rightarrow\quad \theta=\frac{s}{R}=\frac{\frac{1}{2}l}{R}=\frac{\Delta L}{6R}=\frac{23.52m}{6\,(0.25m)}=15.7rad$$

Problem 5.11

A board starts at rest and is free of any attachments (it is not pivoted on anything). It is pushed in opposite directions on both of its ends with forces of equal magnitude, at right angles to the board. The forces continue to be applied at right angles with the same magnitude, causing the board to rotate in the manner depicted in the diagram until the board has rotated by 90^o.



The time it takes the board to rotate the 90^o *is* t*. Derive an expression for the moment of inertia of this board in terms of* t*, the length of board* L*, and the force* F*.*

Solution

The analysis showed us that wherever the center of mass happens to be, the torque applied is still equal to FL*. Newton's 2nd Law for rotations gives:*

$$\tau = I\alpha \quad \Rightarrow \quad I = \frac{\tau}{\alpha} = \frac{FL}{\alpha}$$

The board starts from rest, and the torque remains constant, so it accelerates rotationally at a constant rate, which means its motion satisfies the usual kinematics equation:

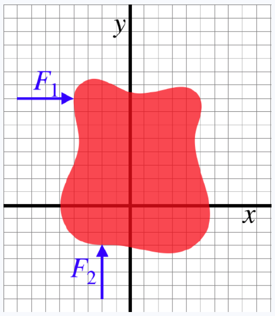
$$\Delta\theta = \frac{1}{2}\alpha t^2$$

We know that the rod rotates through an angle of $\frac{\pi}{2}$ *radians, and we know the time elapsed is* t*, so we can solve for* α *and this gives our final answer:*

$$\alpha = \frac{2\Delta\theta}{t^2} = \frac{\pi}{t^2} \quad \Rightarrow \quad I = \frac{FLt^2}{\pi}$$

Problem 5.12

The blob in the figure below is rigid and in static equilibrium. The two forces shown are two of the total of three forces exerted on the object.



The magnitude of F_1 *is three-quarters the magnitude of* F_2*. Find the equation of the line along which the third force acts.*

Solution

For this object to remain in equilibrium, the net torque about every point in space must vanish. Consider the reference point $x=-2,\, y=+8$*. Both of the given forces act through this point, so neither of them provides a torque around this reference point. There is only one more force present, and for the net torque around that reference point to be zero, the third force must also contribute zero torque, and this is only possible if that third force passes through the reference point.*

In the analysis we found the tangent of the angle the force vector makes with the x*-axis. This is the slope of the force line, and we are given the ratio of these two force magnitudes:*

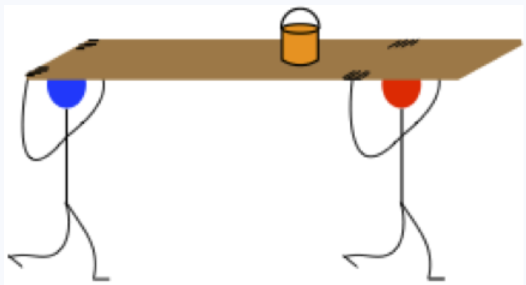
$$slope = \frac{F_2}{F_1} = \frac{4}{3}$$

Now we know one point on the line and its slope, and these two quantities completely define it. Putting an arbitrary (x, y) point and the reference point into the slope equation gives us the equation of the line:

$$\frac{4}{3} = slope = \frac{y - y_{ref}}{x - x_{ref}} = \frac{y-8}{x+2} \quad \Rightarrow \quad y = \frac{4}{3}x + \frac{32}{3}$$

Problem 5.13

Two painters carry a plank of plywood that they use for scaffolding over their heads on their way to the job site. The plank has a uniform mass distribution. Atop the plank is a can of paint weighing one third as much as the plank. The painter in the rear is holding the plank at the very end and the painter in front is holding the plank one quarter of the the plank length from the front. The can of paint is two-fifths of the plank length from the front. The plank is horizontal as they carry it.



The can of paint weighs $50N$. Find how much force each painter is exerting on the plank.

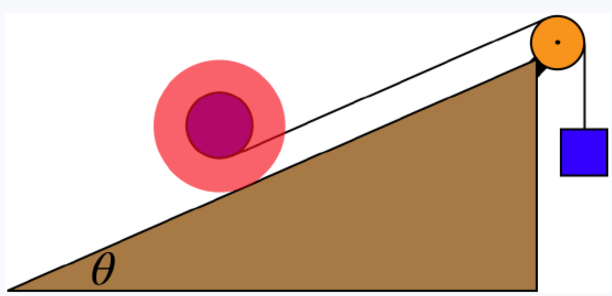
Solution

As is typical for these types of problems, the analysis often solves the whole thing. In this case, we know that the plank weighs three times as much as the paint, so the total weight carried is $200N$. We found in the analysis how the load is distributed, so we have our answers already:

$$rear\ painter = 0.30\,(200N) = 60N\ , \quad front\ painter = 0.70\,(200N) = 140N$$

Problem 5.14

The diagram below depicts a yo-yo on an inclined plane with its string over a massless pulley and attached to a hanging block. The whole system is in static equilibrium.



The inner radius of the yo-yo is half the outer radius, and the coefficient of friction is 0.40.

a. *Find the maximum angle θ for which this system can be at static equilibrium (assume that the hanging mass can be adjusted to whatever is necessary).*
b. *The mass of the yo-yo is $0.35kg$. If the angle θ is at its maximum, find the hanging weight.*

Solution

a. We have the equations of equilibrium from the analysis already. As the angle gets larger, the F_y equation shows that the normal force on the yo-yo by the plane gets smaller. This reduces the value of the maximum static friction force available.

When the angle is so great that the new maximum friction force equals the actual friction force, then making the angle any larger would cause the yo-yo to slip. So the maximum angle without slippage means that $f=\mu_sN$. *Now we do some algebra.*

Eliminate the tension first:

$$T=\frac{R}{r}f \quad \Rightarrow \quad \left(\frac{R}{r}-1\right)f-Mg\sin\theta=0$$

Next eliminate the friction force:

$$f=\mu_sN \quad \Rightarrow \quad \left(\frac{R}{r}-1\right)\mu_sN-Mg\sin\theta=0$$

And now the normal force:

$$N=Mg\cos\theta \quad \Rightarrow \quad \left(\frac{R}{r}-1\right)\mu_sMg\cos\theta-Mg\sin\theta=0$$

Plugging-in the given $R=2r$ *and solving for* θ *gives:*

$$\theta=\tan^{-1}\mu_s=\tan^{-1}(0.4)=21.8^o$$

b. Now to find the hanging mass, we need the tension. From the three equations above, we see that we can get the normal force from the mass of the yo-yo, the friction force from the normal force, and the tension from the friction:

$$mg=T=\frac{R}{r}f=\frac{R}{r}\mu_sN=\frac{R}{r}\mu_sMg\cos\theta \quad \Rightarrow \quad m=(2)\,(0.4)\,(0.35kg)\cos(21.8^o)=0.26kg$$

CHAPTER OVERVIEW

6: Angular Momentum

6.1: Linking Linear and Angular Momentum
6.2: Effects of Torque
6.3: Applications of Angular Momentum Conservation
Sample Problems

Thumbnail: A gyroscope is a device used for measuring or maintaining orientation and angular velocity. It is a spinning wheel or disc in which the axis of rotation (spin axis) is free to assume any orientation by itself. When rotating, the orientation of this axis is unaffected by tilting or rotation of the mounting, according to the conservation of angular momentum. Image used with permission (Public Domain; LucasVB).

6.1: Linking Linear and Angular Momentum

Rotational Impulse-Momentum Theorem

By now we have a very good sense of how to develop the formalism for rotational motion in parallel with what we already know about linear motion. We turn now to momentum. Replacing the mass with rotational inertia and the linear velocity with angular velocity, we get:

$$\overrightarrow{p} \equiv m\overrightarrow{v} \quad\Longleftrightarrow\quad \overrightarrow{L} \equiv I\overrightarrow{\omega} \tag{6.1.1}$$

The vector L is called *angular momentum*, and it has units of:

$$[L]=\frac{kg\cdot m^2}{s}=J\cdot s$$

Continuing the parallel with the linear case, the momentum is relates to the force through the impulse-momentum theorem, which is:

$$\int\limits_{t_A}^{t_B}\overrightarrow{F}_{net}dt=\Delta\overrightarrow{p}_{cm} \quad\Longleftrightarrow\quad \int\limits_{t_A}^{t_B}\overrightarrow{\tau}_{net}dt=\Delta\overrightarrow{L} \tag{6.1.2}$$

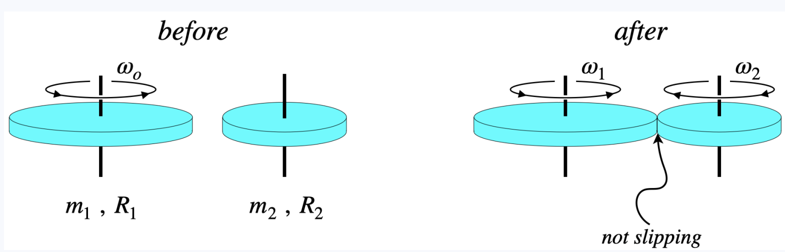
While there is no need to append "*cm*" to the angular momentum as we do with the linear momentum, we do have to keep in mind that all of the quantities in the rotational case must be referenced to the same point. That is, the net torque requires a reference point, and the angular momentum contains a rotational inertia, which also requires a reference point.

Recall that the impulse-momentum theorem is just a repackaging of Newton's second law, and so it is with the rotational case, though there is a twist, as we will see shortly:

$$\overrightarrow{F}_{net}=\frac{d\overrightarrow{p}_{cm}}{dt}=\frac{d\left(m\overrightarrow{v}_{cm}\right)}{dt}=m\overrightarrow{a}_{cm} \quad\Longleftrightarrow\quad \overrightarrow{\tau}_{net}=\frac{d\overrightarrow{L}}{dt}=\frac{d\left(I\overrightarrow{\omega}\right)}{dt}=I\overrightarrow{\alpha} \tag{6.1.3}$$

Analyze This

Two uniform disks are free to rotate frictionlessly around vertical axles. Initially one of the disks is rotating, while the other is not. They are then brought together so that their outer edges rub against each other. Kinetic friction between the two rubbing surfaces slows down disk #1, while speeding up disk #2. This continues until their rotational speeds are such that no slipping occurs between the two surfaces. With kinetic friction no longer present, they continue with constant rotational motion from this point forward.



Analysis

The surfaces stop slipping when the outer edges of the two disks are moving at the same linear (tangential) speed. We therefore have the "after" constraint:

$$v_1=v_2 \quad\Rightarrow\quad R_1\omega_1=R_2\omega_2$$

The friction force exerts torques on both disks. Newton's 3rd law ensures that each disk experiences the same magnitude of kinetic friction, and for the same period of time, but the torques about their axles are different, because the moment-arms are not equal (the disks have different radii). So the two disks experience different magnitudes of rotational impulse, and the ratio of the magnitudes of these impulses is:

$$\frac{|\tau_1|\,\Delta t}{|\tau_2|\,\Delta t}=\frac{f_kR_1\Delta t}{f_kR_2\Delta t}=\frac{R_1}{R_2}$$

These rotational impulses equal the changes in the angular momenta of their respective disks, which we can write in terms of the before & after values:

$$|\Delta L_1| = I_1\,|\Delta\omega_1| = \frac{1}{2}m_1R_1^2\,(\omega_o - \omega_1)$$

$$|\Delta L_2| = I_2\,|\Delta\omega_2| = \frac{1}{2}m_2R_2^2\,(\omega_2 - 0)$$

Plugging these angular momentum changes into the impulse ratios above gives:

$$\frac{|\Delta L_1|}{|\Delta L_2|} = \frac{|\tau_1|\,\Delta t}{|\tau_2|\,\Delta t} \;\Rightarrow\; \frac{m_1}{m_2}\frac{R_1^2}{R_2^2}\left(\frac{\omega_o - \omega_1}{\omega_2}\right) = \frac{R_1}{R_2} \;\Rightarrow\; \frac{m_1}{m_2}\left(\frac{\omega_o - \omega_1}{\omega_2}\right) = \frac{R_2}{R_1}$$

We can now use our constraint on the two "after" angular speeds to solve for each of them:

$$\frac{m_1}{m_2}\left(\frac{\omega_o - \omega_1}{\frac{R_1}{R_2}\omega_1}\right) = \frac{R_2}{R_1} \;\Rightarrow\; \omega_1 = \left(\frac{m_1}{m_1+m_2}\right)\omega_o$$

$$\omega_2 = \frac{R_1}{R_2}\omega_1 \;\Rightarrow\; \omega_2 = \frac{R_1}{R_2}\left(\frac{m_1}{m_1+m_2}\right)\omega_o$$

Link Between Angular and Linear Momentum

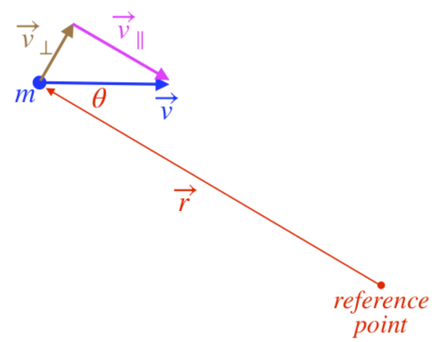
When there are several particles in a system, we find the momentum of the system by adding the momenta of the particles:

$$\overrightarrow{p}_{cm} = \overrightarrow{p_1} + \overrightarrow{p_2} + \dots \tag{6.1.4}$$

We have a definition for the angular momentum of a rigid object, but can we define the angular momentum of a single particle, and then add up all of the angular momenta of the particles to get the angular momentum of the system, in the same way that we do it for linear momentum? The answer is yes, but we have to be careful about our reference point. That is, to add the angular momentum of every particle together to get a total angular momentum, the individual angular momenta must be measured around the same reference.

So how do we define the angular momentum of an individual particle around a certain reference point? Let's look at a picture of the situation. The particle has a mass m, a velocity $\overrightarrow{v}$, and is located at a position $\overrightarrow{r}$ with the tail of that position vector at the reference point.

Figure 6.1.1a – Angular Momentum of a Point Particle

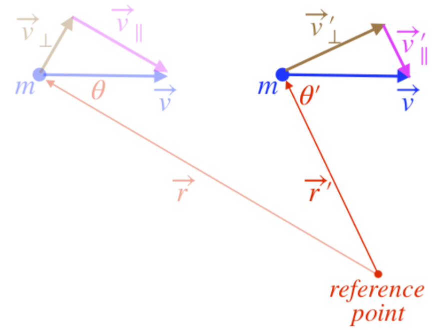


If this particle was a part of a rigid body rotating around the reference point, the parallel component of the velocity vector would be zero. So it makes sense to exclude that part of the velocity vector when defining the angular momentum of this particle. We know the rotational inertia of the point particle, and the relation between $v_\perp$ and ω, so we get for the magnitude of the angular momentum:

$$L_{single\ particle} = I\omega = \left[mr^2\right]\left[\frac{v_\perp}{r}\right] = mrv_\perp = mrv\sin\theta \tag{6.1.5}$$

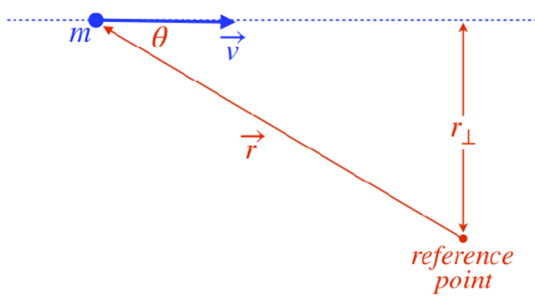
Suppose the particle continues moving free of any forces. What happens to its angular momentum? Let's look at what happens to the picture:

Figure 6.1.1b – Angular Momentum of a Point Particle



What a mess! The mass and velocity vector remain the same, but everything else changes. How can we determine what happens to the angular momentum? Well, have a look at Equation 6.1.2. With no force on the particle, there can't be any torque on the system, so the angular momentum must remain unchanged. It turns out there is a simpler way to look at the angular momentum, to see why this must be the case.

Figure 6.1.1c – Angular Momentum of a Point Particle



We can define the quantity $r_\perp$ in a manner similar to how we defined moment arm – it is the perpendicular distance from the reference point to the line along which the particle is moving. Doing this gives us an alternative way of writing the magnitude of the particle's angular momentum. Using the fact that $r_\perp = r\sin\theta$, we have:

$$L_{single\ particle} = mrv_\perp = mrv\sin\theta = mvr_\perp \tag{6.1.6}$$

Now it is quite easy to see that the angular momentum of the particle doesn't change while it moves – it keeps the same mass and speed, and stays on the same line, so $r_\perp$ doesn't change either.

Angular momentum is a vector, so what direction does it have here? Going back to the idea of this particle being part of a rigid object, it's clear that this object would be rotating clockwise around the reference, so from the right hand rule, the vector must point into the page. We would like a mathematical expression of this, and as with the case of torque, it comes from the cross product. The two vectors involved are the position vector and the velocity vector, and indeed we see that the following cross product results in the correct direction, and takes care of the $\sin\theta$ contribution as well:

$$\vec{L}_{single\ particle} = m\vec{r} \times \vec{v} = \vec{r} \times \vec{p} \tag{6.1.7}$$

This is a nice, compact expression of the relation between the linear momentum of a particle and its angular momentum around a reference point. To see this relation come full circle, imagine that a force is exerted on the particle. This would cause the momentum to change. It would also result in a torque on the system about the reference point, causing the angular momentum to change. Taking the derivative of Equation 6.1.7 with respect to time gives:

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F} \tag{6.1.8}$$

The velocity vector is parallel to the momentum vector, so the cross product in the first term is zero, leaving us with a relation between torque and force that we have seen before (Equation 5.4.6).

Now that we can deal with the angular momentum of a single particle relative to some reference point, we can simply add the contributions of many such particles within a system, relative to the same reference point:

$$\vec{L}_{system} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + \ldots \tag{6.1.9}$$

Note that these particles may be part of a rigid object, or may not be bound to each other at all. If they happen to be bound into a single rigid object rotating around a fixed point on the object, then the result is more easily expressed in terms of the rigid object's rotational inertia and angular velocity (Equation 6.1.1):

$$\begin{aligned} \overrightarrow{L}_{rigid\ object} &= \overrightarrow{r}_1 \times m_1 \overrightarrow{v}_1 + \overrightarrow{r}_2 \times m_2 \overrightarrow{v}_2 + \dots \\ &= [r_1 \hat{r}_1] \times m_1 [v_1 \hat{v}_1] + [r_2 \hat{r}_2] \times m_2 [v_2 \hat{v}_2] + \dots \\ &= [r_1 \hat{r}_1] \times m_1 [r_1 \omega \hat{v}_1] + [r_2 \hat{r}_2] \times m_2 [r_2 \omega \hat{v}_2] + \dots \\ &= m_1 r_1^2 [\omega \hat{r}_1 \times \hat{v}_1] + m_2 r_2^2 [\omega \hat{r}_2 \times \hat{v}_2] + \dots \\ &= m_1 r_1^2 [\omega \widehat{\omega}] + m_2 r_2^2 [\omega \widehat{\omega}] + \dots \\ &= I \overrightarrow{\omega} \end{aligned} \tag{6.1.10}$$

Consider next an extended object that is not rotating, but is moving in a straight line relative to some reference point. Despite the fact that it is not rotating, it can have angular momentum relative to that reference point. Writing the angular momentum of the whole object as a sum of the angular momenta of its particles, we get:

$$\overrightarrow{L}_{not-rotating\ extended\ object} = m_1 \overrightarrow{r}_1 \times \overrightarrow{v}_1 + m_2 \overrightarrow{r}_2 \times \overrightarrow{v}_2 + \dots \tag{6.1.11}$$

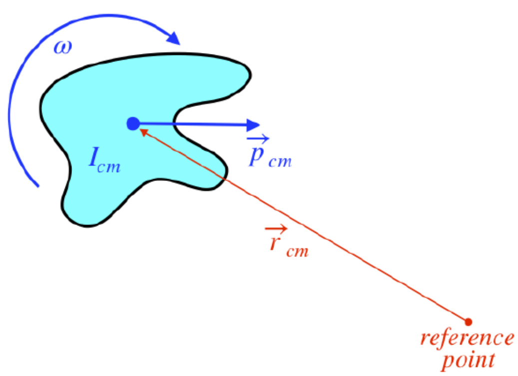
With the object not rotating and all the particles held rigidly in place, every particle has the same velocity, which equals the velocity of the object's center of mass, so this can be factored out of all the cross products, giving:

$$\overrightarrow{L}_{not-rotating\ extended\ object} = \left(m_1 \overrightarrow{r}_1 + m_2 \overrightarrow{r}_2 + \dots\right) \times \overrightarrow{v}_{cm} = \left(\frac{m_1 \overrightarrow{r}_1 + m_2 \overrightarrow{r}_2 + \dots}{M}\right) \times \left(M \overrightarrow{v}_{cm}\right) = \overrightarrow{r}_{cm} \times \overrightarrow{p}_{cm} \tag{6.1.12}$$

What this means is that an extended object moving in a straight line has the same angular momentum relative to a reference point as a point particle located at the object's center of mass, with the same mass and velocity.

If the extended object has both its center of mass moving at a constant velocity relative to the reference point and it is also rotating around an axis through its own center of mass, then things get complicated. We won't into the details of the most general case, but it is not unreasonable to consider the case of the linear velocity lying in the plane perpendicular to the rotation vector (e.g. an object moving within this screen while rotating around an axis perpendicular to this screen – see Figure 6.1.2).

Figure 6.1.2 – Total Angular Momentum



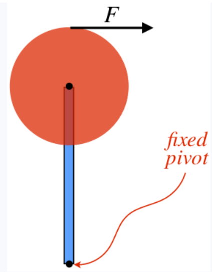
The total angular momentum comes out to be reminiscent of the parallel-axis theorem and of the kinetic energy being the sum of linear and rotational parts:

$$\overrightarrow{L}_{tot} = \overrightarrow{L}_{rotation\ around\ cm} + \overrightarrow{L}_{cm\ moving\ by\ reference\ point} = I_{cm} \overrightarrow{\omega} + \overrightarrow{r}_{cm} \times \overrightarrow{p}_{cm} \tag{6.1.13}$$

An interesting and important consequence of this is that an object that is only rotating around its center of mass (but not moving linearly) has the same angular momentum measured relative to *every* reference point.

Analyze This

The center of a uniform solid disk is threaded onto an axle at the end of a thin uniform rod. The rod and the disk have equal masses, and the radius of the disk is one-third the length of the rod. The rod is attached to a fixed pivot point at its other end, around which it is free to rotate. With the rod and disk both starting from rest, a force of constant magnitude is exerted tangent to the edge of the disk at the point farthest from the pivot for a short time. There is no gravity present.



Analysis

We'll start by creating a couple of labels. The mass of the disk and the rod are the same, and we will call this mass m. The radius of the disk we will call R, and the rod is three times this long, so its length is $3R$.

The disk + rod system is given a rotational impulse about the pivot point, so its angular momentum will change. The force exerted is a distance $4R$ from the pivot, and is directed perpendicular to the line joining the pivot and the point where it acts, so it delivers a total torque of $\tau = 4RF$. Multiplying this constant torque by the time span over which it acts gives the total impulse, and therefore the total angular momentum of the system.

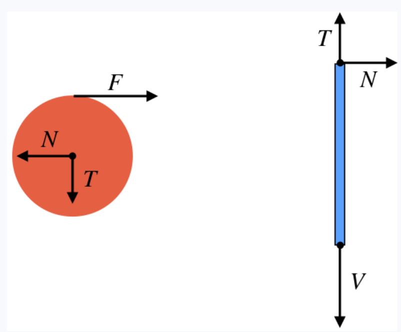
This angular momentum is manifested in three ways: 1. The disk rotates around its center, 2. The disk moves in a circular path around the pivoted end of the rod, 3. The rod rotates around its pivoted end. The rotational speeds for the last two cases are the same, and we'll call this speed ω_1. The rotational speed of the disk about its center we'll call ω_2. With these definitions in place, the magnitude of the angular momentum of the system about the fixed pivot is:

$$L_{system} = L_{disk} + L_{rod} = \left(I_{disk}\omega_2 + mr^2\omega_1\right) + I_{rod}\omega_1$$

Plugging-in $r = 3R$ (the center of mass of the disk is one rod-length from the pivot) and the moments of inertia of the rod and disk, we get:

$$L_{system} = \frac{1}{2}mR^2\omega_2 + m(3R)^2\omega_1 + \frac{1}{3}m(3R)^2\omega_1 = mR^2\left(12\omega_1 + \frac{1}{2}\omega_2\right)$$

We know this equals the impulse delivered by the torque, but the problem is well-specified, so we should be able to to do more than write the answer in terms of two angular speeds – they should be related to each other somehow. To work this out, we have to deal with the disk and rod as separate systems. A couple of free-body diagrams are therefore called-for:



A quick explanation of these FBDs: N is the normal force by the axle on the disk, reacting to the applied force F. T is the "tension" force keeping the disk moving in a circle. V is the vertical force by the pivot that makes sure there is a net force which keeps the center of mass of the rod moving in a circle. Neither T nor V contribute to any torques. We'll say that F and N act for a time Δt to contribute to their respective rotational impulses.

The net torque on the rod about the pivot is $3RN$. Multiply this by the time it acts (and remembering that it starts from rest), we have, from the impulse-momentum theorem:

$$3RN\Delta t = I_{rod}\omega_1 = \frac{1}{3}m(3R)^2\omega_1 \quad \Rightarrow \quad N\Delta t = mR\omega_1$$

The net torque on the disk about its center is FR, so:

$$FR\Delta t = I_{disk}\omega_2 = \frac{1}{2}mR^2\omega_2 \quad \Rightarrow \quad F\Delta t = \frac{1}{2}mR\omega_2$$

[Note: While it might appear as though this rotational impulse determines the rotational motion of the disk relative to the rod, it does not. The resulting motion is the total angular velocity (relative to the lab). If this force was zero, and the rod is made to turn without any torque on the disk (i.e. the force is applied to the rod instead of the disk), the disk would maintain its orientation relative to the lab as the rod rotates, turning relative to the rod in the opposite direction at the same rate that the rod rotates. In this case, ω_2 would be zero, which matches the zero value of F.]

We need one more equation, and it comes from the linear impulse-momentum theorem for the disk. From the FBD, we see that the net force on the disk is $F-N$, and this results in a change of (tangential) momentum of mv_{cm}. The final linear velocity of the disk's center of mass is directly related to ω_1 (it moves with the end of the rod) so:

$$(F-N)\,\Delta t = mv_{cm} = m\,(3R\omega_1)$$

Putting these last three equations together gives a relationship between ω_1 and ω_2:

$$\frac{1}{2}mR\omega_2 - mR\omega_1 = 3mR\omega_1 \quad \Rightarrow \quad \omega_2 = 8\omega_1$$

This can be put back into the equation for the system's total angular momentum to get:

$$L_{system} = 16mR^2\omega_1 = 2mR^2\omega_2$$

6.2: Effects of Torque

Gyroscopic Precession

Back in Section 1.6 and Section 1.7, we discussed circular motion at constant speed as motion that occurs because a net force pulling an object toward a central point causes the object's velocity vector to only change direction, and not magnitude. At the time, we hadn't yet discussed momentum, but clearly we can now replace "velocity vector" in the previous sentence with "momentum vector." We can write Newton's second law (Equation 4.1.4) in terms of the changing magnitude and direction of the momentum:

$$\overrightarrow{F}_{net} = \frac{d}{dt}\overrightarrow{p} = \frac{d}{dt}(p\hat{p}) = \frac{dp}{dt}\hat{p} + p\frac{d\hat{p}}{dt} \tag{6.2.1}$$

Circular motion at a constant speed would exhibit no change in the magnitude of momentum – the first term in Equation 6.2.1 is zero – while all of the force would go into changing the direction of momentum. As we saw back in Section 1.6, the two terms in Equation 6.2.1 are always perpendicular to each other, which means that the net force on an object going in a circle at a constant speed is always perpendicular to the momentum vector.

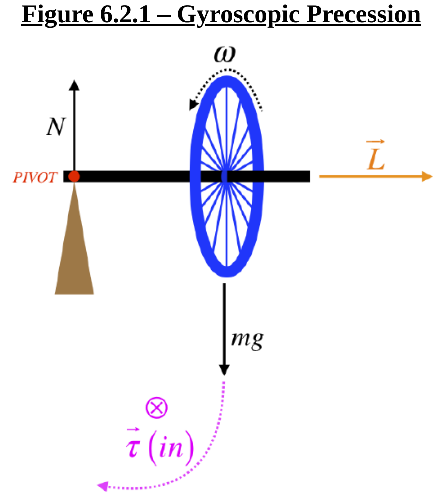
None of this is new to us, but as we have been doing for the last two chapters, we will now look at the rotational equivalent of this behavior. Switching Equation 6.2.1 to the rotational equivalent gives:

$$\overrightarrow{\tau}_{net} = \frac{d}{dt}\overrightarrow{L} = \frac{d}{dt}\left(L\widehat{L}\right) = \frac{dL}{dt}\widehat{L} + L\frac{d\widehat{L}}{dt} \tag{6.2.2}$$

We are already aware of how a net torque can change the magnitude of an object's angular momentum – speeding up and slowing down rotation is something we have already looked at in detail. But what if we insist that the magnitude remain constant (the object maintains the same rotational inertia and keeps spinning at a constant rate), while only the the direction of motion changes? That is, what if the first term in Equation 6.2.2 is zero, while the second term is not? How can we construct a physical system that behaves this way? Answering this last question will require quite a lot of facility with the right hand rule, but here goes...

We start with a rotating object. We'll use as our model a bike wheel turning around an axle. The angular momentum vector will point along the axis of the wheel according to the right hand rule. Now we need a net torque that points perpendicular to the angular momentum. We can achieve this by placing on end of the wheel's axle on a support and allowing the weight of the wheel to pull it down as the support pushes up.

Figure 6.2.1 – Gyroscopic Precession

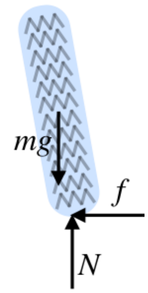


There is clearly a net torque around the pivot at the point of support, trying to turn the wheel clockwise in Figure 6.2.1. But the *direction of the torque vector*, according to the right hand rule, is into the page. This torque vector points in a direction that is perpendicular to the angular momentum vector, exactly as we required above.

The next question is, how does this system behave? In the case of the object going in a circle at a constant speed, the momentum vector of the object never changed length, it only rotated its direction. The way that it rotated was the change of the linear velocity vector pointed in the direction of the force (see Figure 1.7.1). To follow this same behavior, the point of the angular momentum vector must turn in the direction of the torque vector, which is into the page. That is, to behave Newton's second law for rotations, this wheel should not fall down, but instead should *rotate into the page*! In fact it does exactly this, rotating around the pivot in a counterclockwise direction as viewed from above. This phenomenon is known as *gyroscopic precession*.

While this is certainly a striking phenomenon to witness first-hand, it is quite ubiquitous in everyday life. Everyone who rides a bicycle knows that getting it to turn is a matter of *leaning*, not turning the handle bars. To see how this is an example of gyroscopic precession, consider the following force diagram of a tilted wheel:

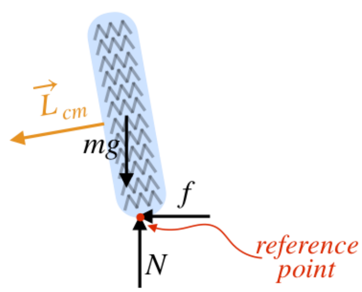
Figure 6.2.2 – Free-Body Diagram of a Tilted Wheel (Forces, End View)



The fact that there is only one force in the horizontal direction (the friction force) means that this wheel's center of mass must be accelerating to the left. If this wheel is stationary, then this is certainly happening as the wheel falls over to the left. On the other hand, if the wheel is rolling forward (into the page), and this tilt is a result of turning, then it is not falling over, but its center is mass is still accelerating to the left (centripetally, toward the center of the turn).

This explains how the tilt results in a change of the wheel's direction of motion, but not its change of *orientation*. That is, why does the wheel *turn* as its center of mass changes direction? For an explanation of this, we can look at the torques and the angular momentum vector. Again assuming that the wheel in the diagram is rolling into the page, the angular momentum vector measured around the center of mass points to the left (and slightly downward).

Figure 6.2.3 – Free-Body Diagram of a Tilted Wheel (Torques, Rear View)

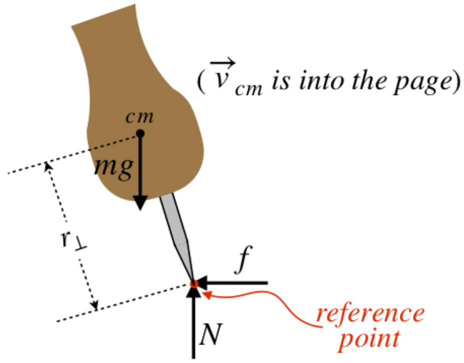


The direction of the net torque around the center of mass is difficult to determine directly. Gravity provides no torque, and the friction and normal forces give opposing torques. But if we choose a reference point where the wheel contacts the road, it is clear that about that point the torque is counterclockwise (only gravity contributes). If the forces act to torque the wheel counterclockwise around that reference point, clearly the net torque around the center of mass will be counterclockwise as well. [*Note: It won't be as large of a torque around the center of mass as there is around the road contact point. The weight and normal forces are equal (the wheel is not accelerating vertically), and the moment arm for gravity around the road contact is the same as the moment arm for the normal force around the center of mass, so those provide equal counterclockwise torques. But for the center of mass reference point there is an additional torque applied by the friction force, and it is clockwise, reducing the net counterclockwise torque.*]

A counterclockwise net torque gives (through the right hand rule) a torque vector that points out of the page. This causes a change in the angular momentum vector out of the page, according to the second law. For $\overrightarrow{L}_{cm}$ to gain a component pointing out of the page, the wheel has to turn left.

It turns out, however, that the change of angular momentum of the spinning wheel about its center is not the only way that this phenomenon is manifest here. In fact, it's not even the greatest contributor to turning-by-leaning. Consider ice skaters. They too change direction by leaning, but they don't have rotating wheels to perform this gyroscopic effect for them. In this case, the diagram is the same (to avoid sketching an entire ice skater, we'll draw just the ice skate):

Figure 6.2.4 – Free-Body Diagram of a Tilted Ice Skate (Rear View)



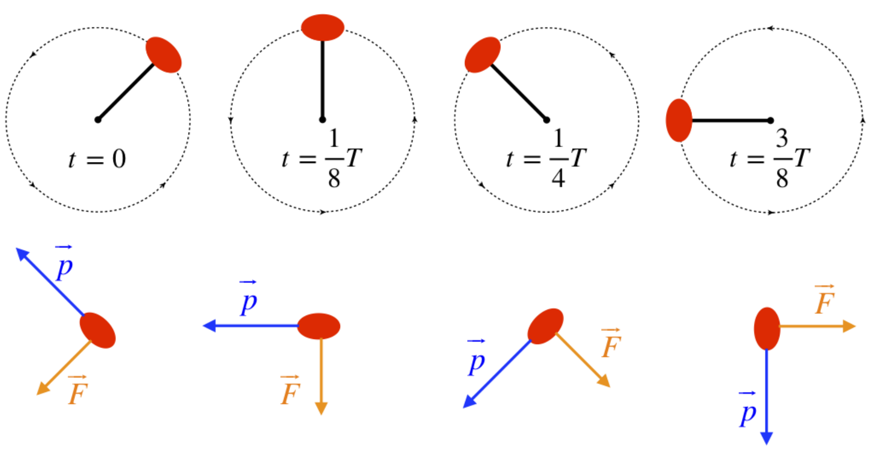
While there is no rotating wheel here, there *is* angular momentum relative to the reference point. The ice skate is moving into the page, and as we know, objects moving in straight lines do have angular momentum relative to reference points that don't lie along the line of linear motion. The magnitude of this angular momentum is $mv_{cm}r_\perp$ about the reference point indicated, but we are more interested in its direction. The position vector points from the reference point to the center of mass of the skate, and the momentum vector points into the page, so $\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$ points to the left and slightly down, *just like it was for the bike wheel's rotation*. The result is the same – the torque about the reference point causes this angular momentum vector to turn in a direction that is out of the page, which means the skate turns left.

Note that the bicycle has this same thing going on – the center of mass of the bike + rider has an angular momentum relative to the point where the wheel contacts the ground, and leaning to one side or the other will turn the entire system, not just the wheel.

Let's return to the original example of the precession of a wheel pivoted about an end of its axle. The torque remains constant in magnitude, which means that the angular momentum vector changes at a constant rate – the wheel therefore precesses around the pivot at a constant rate. Let's see if we can determine the rate of this precession, i.e. its rotational velocity around the *vertical* axis (not around the axle of the wheel - we already know that). Rather than try to slog through all the vector calculus, let's do this by following our circular motion analogy.

Figure 6.2.5 depicts a stone tied to a string going in a counterclockwise circle at a constant speed. The four diagrams are snapshots of the motion at four different times, separated by one-eight of a rotational period. Below each diagram is a depiction of the net force vector on the stone and its linear momentum vectors at that moment in time (note these vectors are always perpendicular, as they should be).

Figure 6.2.5 – Force and Momentum Vectors for Stone in Circular Motion

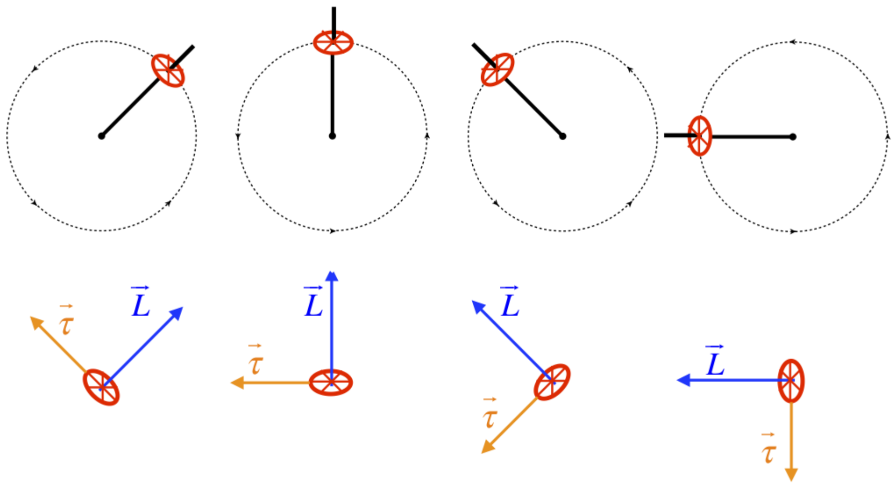


Let's compute the angular rate (which we will call Ω to avoid later confusion with the ω for the wheel) of the rock going around the circle *in terms of the magnitudes of* $\overrightarrow{F}$ and $\overrightarrow{p}$. The magnitude of the force is the mass of the rock multiplied by the centripetal acceleration:

$$F = ma_c = m\frac{v^2}{R} = mv\left(\frac{v}{R}\right) = p\Omega \quad \Rightarrow \quad \Omega = \frac{F}{p} \tag{6.2.3}$$

That's quite compact! Okay, let's follow precisely this path for the precessing wheel. We start with a similar diagram of a top view (gravity is acting into the page).

Figure 6.2.6 – Torque and Angular Momentum Vectors for Precessing Wheel



The main difference that jumps out between Figure 6.2.5 and Figure 6.2.6 is that for the stone, the momentum is tangent to the circle while the force is radial, while for the wheel, the angular momentum is radial and the torque is tangent. Other than that, however, these follow each other exactly – the force/torque is perpendicular to and "leads" the momentum, as they both revolve together. Given that they follow the same behavior and have exactly the same differential relationship (the force/torque is the time derivative of the momentum), it's perfectly reasonable to expect that the rotational frequency would have the same relationship. Namely:

$$\Omega = \frac{\tau}{L} \tag{6.2.4}$$

If you are not comfortable with this "derivation by analogy," then you can reach the same result quickly using Equation 6.2.2. The rate at which the wheel is spinning about its axle doesn't change, which means that the magnitude of its angular momentum remains constant, and the first term is zero. Solving the remainder of the equation for the rate at which the direction is changing requires only a division of both sides by the magnitude of the angular momentum, giving the same result as above.

We can write this in terms of the rotational inertia of the wheel about its axis I, the rotational speed of the wheel ω, the length of the axle l (the distance from the pivot to the center of mass of the wheel + axle), and the mass of the wheel M. The torque can be quickly calculated with a quick look at Figure 6.2.1, and plugging in for the angular momentum of the wheel, we get the precession speed:

$$\Omega = \frac{Mgl}{I\omega} \tag{6.2.5}$$

It should be noted that when the wheel precesses, the system is now also rotating it a horizontal plane, which gives it a component of angular momentum vector in the upward direction, and this angular momentum is also affected by the torque. This effect becomes progressively easier to ignore for a given setup as the wheel spins faster about its axis, because according to Equation 6.2.5, the faster the wheel spins about its axis, the slower it precesses, and the slower it precesses, the smaller the upward component of angular momentum. In any case, this secondary effect manifests itself as a bobbing up-and-down of the wheel as it precesses, and is known as *nutation*.

Conceptual Question

A wheel whose axis is vertical (i.e. the plane of the wheel is parallel to the ground) rotates clockwise as viewed by someone looking down at it. If a small nudge is given to the top of the axis of this wheel toward the south, which way does the top of the axis move in its immediate response to this nudge?

a. *south*

b. *north*
c. *east*
d. *west*
e. *It does not move, the gyroscopic effect prevents it from moving at all.*

Solution

(d) Using the RHR, the clockwise rotation when looking down on it corresponds to an angular momentum vector that points downward. Imagine you are facing the spinning wheel from the north side (i.e. you are facing south), and push it at its top with the fingers of your right hand pointing upward. Your right hand will curl such that your thumb is pointing to the left, and that is the direction of the torque you are imparting on the wheel. The angular momentum vector that points down will change in the direction of the torque, so the axis of the wheel will tilt such that its bottom rotates in the direction of your thumb, which means the top of the axis will rotate the other way (to your right). If you are facing south, then a tilt to the right is toward the west.

Central Forces

Consider a flat disk rotating around its center. Every particle in this object is following a circular path, and so every particle is experiencing a net force. We might therefore ask how angular momentum can be conserved – with net forces on every particle, the forces on them are not cancelling-out. The answer is that net force is not the same thing as net torque. Measured from the fixed reference point, the force vector on every particle in the object points parallel to the position vector, which means the torque (the cross product of position and force) on every particle is zero. We can in fact elevate this idea to a very general rule, which first requires a definition:

Definition: Central Force

A central force is a force (which can act on many objects) that is directed directly toward or directly away from a single fixed point in space.

And from the preceding discussion, we saw that whenever the force vector is parallel to the position vector, if we choose the "source" of the force to be our reference point, we conclude:

Central forces do not exert torques (relative to the central point) on the objects they influence, and therefore angular momentum around the central reference point is conserved.

Analyze This

Consider this position-dependent force:

$$\overrightarrow{F}(x,y)=\lambda\left[x\;\hat{i}+y\;\hat{j}\right]$$

A rock is tied to a string whose other end is held fixed at the origin, and is then set into circular motion in the $x-y$ plane (there is no gravity present). While the rock is moving at a constant speed in a circle, the force described above is turned on. The string later breaks and the rock eventually crosses the x-axis.

Analysis

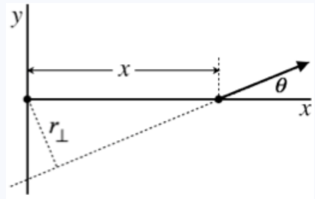
This force can be rewritten in terms of the position vector relative to the origin as:

$$\overrightarrow{F}=\lambda\overrightarrow{r}$$

Clearly this force points directly away from a single point (the origin), which makes it a central force. This means that it exerts no torque around the origin, which also means that it cannot change the angular momentum relative to this point. When the force is turned on, it therefore has no effect on the speed of the rock while the rock remains attached to the string. When the string breaks, the rock will neither follow a straight line, nor will it maintain a constant speed, but with only the central force acting on it, the angular momentum remains constant. We can easily compute the angular momentum it started with in terms of its speed while on the string, its mass, and the length of the string. It is:

$$L_o=mv_or_\perp=mv_oR$$

After the string breaks, when the rock reaches the x-axis, its position on the axis and the angle its velocity vector makes with the axis are related to the perpendicular distance $r_\perp$, as can be seen in this diagram:



$$\sin\theta = \frac{r_\perp}{x}$$

The angular momentum at the x-axis is $mv_fr_\perp$, so setting this equal to the starting angular momentum, we can solve for θ:

$$L_o = L_f \quad \Rightarrow \quad mv_oR = mv_fr_\perp \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{v_oR}{v_fx}\right)$$

6.3: Applications of Angular Momentum Conservation

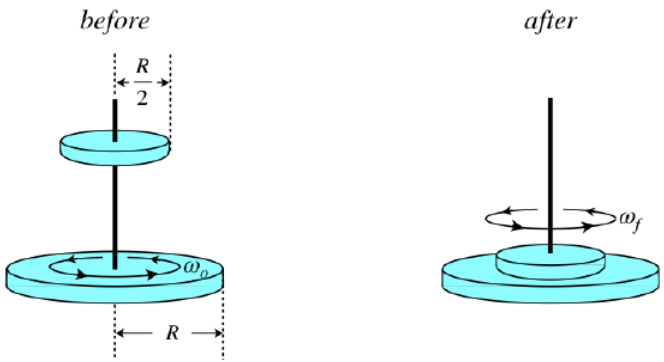
If we continue to follow the trail we blazed in linear motion, our next step is to consider what happens when we choose a system for which there are no external rotational impulses. For such a system, we can declare the angular momentum to be conserved before and after any event, however complicated the internal interactions might be.

In the linear case, we saw that the primary application of momentum conservation was related to collisions, because it was useful to ignore the complicated forces that come about between the colliding objects. What sorts of problems might angular momentum conservation be useful for solving? There are actually three basic varieties that commonly arise in classical mechanics, and we will look each one in turn.

Spinning Collisions

Two uniform solid disks with small holes in their centers, are threaded onto the same frictionless vertical cylindrical rod. One of the disks lies flat on a frictionless horizontal surface and is rotating at a speed ω_o around the rod, while the other disk is held at rest directly above it. Both disks are made from the same material, and have the same thickness, but the spinning disk has twice the radius of the stationary disk. The smaller disk is then dropped on top of the larger one, and after a short time the kinetic friction force between the two disks brings them both to the same rotational speed, which is a fraction of the larger disk's original speed. Find this fraction, and the fraction of the original kinetic energy still left the system afterward (it loses some from work done by kinetic friction).

Figure 6.3.1 – Rotating Disk Inelastic Collision



This is clearly the rotational version of a perfectly inelastic collision, as both of the objects end up moving together. We solve it in the same way that we solve the linear counterpart – by noting that the only torques involved are internal to the two-disk system, which means that the total angular momentum is the same before and after the collision.

$$\left.\begin{array}{ll} before: & L_{tot}=I_1\omega_o+I_2\left(0\right) \\ after: & L_{tot}=\left(I_1+I_2\right)\omega_f \end{array}\right\} \;\;\Rightarrow\;\; \omega_f=\frac{I_1}{I_1+I_2}\omega_o \tag{6.3.1}$$

Because it has the same thickness and is made from the same material, the ratio of the two disks' masses equals the ratio of their areas. With one-half the radius, the smaller disk therefore has one-fourth the mass, and we get:

$$I_1=\frac{1}{2}MR^2 \;\;\Rightarrow\;\; I_2=\frac{1}{2}\left(\frac{1}{4}M\right)\left(\frac{1}{2}R\right)^2=\frac{1}{16}I_1 \;\;\Rightarrow\;\; \omega_f=\frac{I_1}{I_1+\frac{1}{16}I_1}\omega_o=\frac{16}{17}\omega_o \tag{6.3.2}$$

Now for the fraction of kinetic energy leftover:

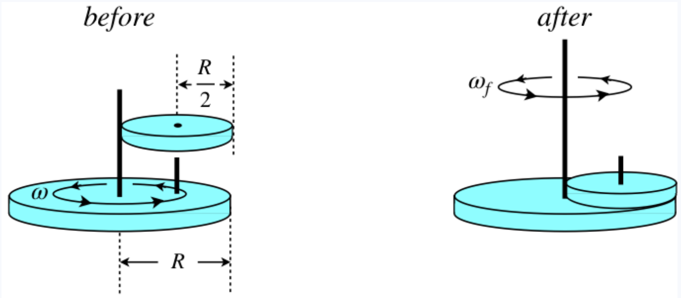
$$\left.\begin{array}{l} KE_o=\frac{1}{2}I_1\omega_o^2 \\ KE_f=\frac{1}{2}\left(I_1+I_2\right)\omega_f^2 \end{array}\right\} \;\;\Rightarrow\;\; \frac{KE_f}{KE_o}=\frac{\frac{1}{2}\left(I_1+I_2\right)\omega_f^2}{\frac{1}{2}I_1\omega_o^2}=\frac{\left(\frac{17}{16}I_1\right)\left(\frac{16}{17}\omega_o\right)^2}{I_1\omega_o^2}=\frac{16}{17} \tag{6.3.3}$$

We can actually achieve this last answer even more easily using $L_o=L_f$:

$$\frac{KE_f}{KE_o}=\frac{\frac{1}{2}I_f\omega_f^2}{\frac{1}{2}I_o\omega_o^2}=\frac{I_o}{I_f}\frac{\left(I_f\omega_f\right)^2}{\left(I_o\omega_o\right)^2}=\frac{I_o}{I_f}\cancel{\frac{L_f^2}{L_o^2}}=\frac{I_1}{I_1+I_2} \tag{6.3.4}$$

Analyze This

A large uniform-density disk of radius R rotates in a horizontal plane around a frictionless axle with a rotational speed ω. This disk includes a vertical post located a distance half its radius from its axle, and onto this axle is placed (very suddenly) a second disk with half the radius of the larger disk. The second disk is made of the same uniform material as the larger disk, and has the same thickness.



Analysis

The angular momentum about the axle of the two-disk system is conserved, as there are no external torques introduced. The starting angular momentum is clear: The smaller disk starts with zero angular momentum, and the larger disk's angular momentum is:

$$L_{before} = L_{large} + L_{small} = I_{large}\omega + 0 = \frac{1}{2}MR^2\omega$$

Expressing the angular momentum after depends upon whether the two disks become a single "rigid body". A rigid body is an object for which all of its parts remain at their same relative positions. If there is no friction between the two disks, then as the large disk rotates, the smaller disk retains its same orientation relative to the Earth. This is because without friction, the large disk can only affect the motion of the smaller disk through its interaction at the axle, and this cannot produce any torque with which to start the smaller disk rotating. On the other hand, if there is friction between the two disks, then eventually they have no relative sliding, and the two are effectively behaving like a single rigid body. Let's address each case separately...

no friction

In this case, after the small disk is dropped onto the axle, it has only orbital angular momentum, as it does not spin on its axis. If we call its mass m, then this orbital angular momentum is found using its speed and the radius of its orbit, $r = \frac{R}{2}$:

$$L_{small} = mv_\perp r = m\omega_f\left(\frac{R}{2}\right)^2 = \frac{1}{4}mR^2\omega_f$$

With half the radius, the area of the smaller disk is one quarter the area of the larger disk, which means it has one quarter the mass of the larger disk. Putting $m = \frac{1}{4}M$ into the angular momentum of the smaller disk and adding it to the angular momentum of the larger disk gives the "after" angular momentum:

$$L_{after} = \frac{1}{2}MR^2\omega_f + \frac{1}{4}\left(\frac{1}{4}M\right)R^2\omega_f = \frac{9}{16}MR^2\omega_f$$

Invoking angular momentum conservation gives the final angular velocity in terms of the initial angular velocity:

$$L_{after} = L_{before} \quad \Rightarrow \quad \omega_f = \frac{8}{9}\omega$$

friction

If the two disks experience kinetic friction after they make contact, and we wait until they stop skidding across each other, then the "after" situation is a rigid body. We can solve this in two equivalent ways: 1. We can compute the moment of inertia of the two-disk system using the parallel axis theorem and the additivity property, then multiply this my the final angular speed to get the final angular momentum. 2. We can continue to treat the disks separately, and add-in the spinning portion of the smaller disk's angular momentum to the "after" tally. We will do method #2 here.

Note that once the disks are no longer sliding, the rotation rate of the smaller disk about its axis in the Earth frame is the same rate as the larger disk – it reaches the same orientation relative to the Earth that it started with after one full revolution. The spin portion of the angular momentum is therefore the moment of inertia of the small disk about its center times the final angular speed:

$$L_{spin} = I_{center}\omega_f = \frac{1}{2}mr^2\omega_f = \frac{1}{2}\left(\frac{1}{4}M\right)\left(\frac{1}{2}R\right)^2\omega_f = \frac{1}{32}MR^2\omega_f$$

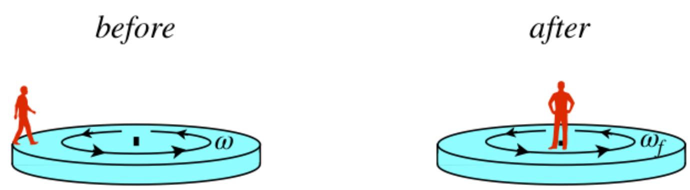
Adding this into the L_{after} and recomputing the final rotational speed gives:

$$L_{after} = \frac{19}{32}MR^2\omega_f \quad \Rightarrow \quad \omega_f = \frac{16}{19}\omega$$

Changing Rotational Inertia

A child stands on the outer edge of a merry-go-round, which is spinning around a fixed axle on a horizontal frictionless surface. The merry-go-round is a solid, uniform disk with ten times the mass of the child, and is spinning at a rotational speed ω_o. The child then slowly walks to the center of the merry-go-round. What, if anything, happens to the rotational speed of the merry-go-round?

Figure 6.3.2 – System Changes Rotational Inertia While Rotating



Before we invoke angular momentum conservation and launch into the mathematics, it might help to think about this in a "less evolved" manner – let's think about the *internal* interactions in the child + merry-go-round system. When the child takes a step, toward the center, they are moving from a faster moving part of the merry-go-round to a slower part. This means that the merry-go-round will exert a static friction force on the feet of the child tangent to the circular motion, acting to slow them down. There is, of course, a Newton's third law pair friction force on the merry-go-round by the feet of the child in the opposite direction, which results in a torque that acts to speed up the merry-go-round's rotation. So we would expect the linear speed of the child to slow with every step, as the merry-go-round's rotational speed increases. The details of these changes are hard to work out using the details of the interaction, so now we turn to momentum conservation, which we know holds because the only forces/torques acting are internal to the system.

Calling the mass of the child m and the radius of the merry-go-round R, we can write down the angular momentum referenced at the axis of the merry-go-round before and after, and invoking angular momentum conservation makes the rest easy:

$$\left.\begin{array}{ll} before: & L_{tot} = \left[I_{child} + I_{mgr}\right]\omega_o = \left[mR^2 + \frac{1}{2}(10m)R^2\right]\omega_o = 6mR^2\omega_o \\ after: & L_{tot} = \left[I_{child} + I_{mgr}\right]\omega_o = \left[\;\;0\;\; + \frac{1}{2}(10m)R^2\right]\omega_o = 5mR^2\omega_f \end{array}\right\} \quad \Rightarrow \quad \omega_f = \frac{6}{5}\omega_o \tag{6.3.5}$$

The rotation rate of the merry-go-round increases by 20%. It's interesting to note that there is no physical equivalent of this phenomenon in linear mechanics. That is, we don't see closed systems losing linear inertia (mass) and maintaining their momentum by compensating with a larger linear velocity.

It's also interesting to consider what happens to the kinetic energy of the system during this process. Like kinetic energy for linear motion, we can write it in terms of the momentum and inertia:

$$KE = \frac{p^2}{2m} \quad \Longleftrightarrow \quad KE = \frac{L^2}{2I} \tag{6.3.6}$$

Given that the angular momentum doesn't change, the kinetic energy goes up in the same proportion that the rotational inertia goes down. Where does this increase in kinetic energy come from? Where is work done? When the child just stands at the edge of the merry-go-round, the static friction force acts toward the center of rotation, but it does no work, because it is acting perpendicular to the direction of the child's motion. But as the child starts moving inward, this static friction *is* doing work. In the end, the kinetic energy of the merry-go-round equals its starting kinetic energy, plus the starting kinetic energy of the child, plus the work done by the static friction force.

It turns out that showing this for this case requires fancier integration to calculate the work than we want to do here, so let's try a simpler example. Let's let the mass of the merry-go-round be negligible compared to the mass of the child. Furthermore, we'll have the child walk halfway to the center of rotation (we can't let the child walk all the way in, or the massless merry-go-round will be spinning infinitely fast!).

First, let's compute the kinetic energy change of the system using angular momentum conservation (note that the merry-go-round doesn't contribute at all now, making things significantly easier):

$$L_{before} = L_{after} \quad \Rightarrow \quad I_o\omega_o = I_f\omega_f \quad \Rightarrow \quad mR^2\omega_o = m\left(\frac{R}{2}\right)^2\omega_f \quad \Rightarrow \quad \omega_f = 4\omega_o \tag{6.3.7}$$

As we saw above, the proportional increase in kinetic energy is the same as that of the rotational velocity, so the kinetic energy increase of the system is:

$$\Delta KE = KE_f - KE_o = 4KE_o - KE_o = \frac{3}{2}mR^2\omega_o^2 \tag{6.3.8}$$

Okay, now let's see if we can calculate the work done by the static friction force. The force that keeps the child going in a circle equals the mass of the child multiplied by the centripetal acceleration, so a force barely exceeding this amount will get the child moving inward. We don't want the child to accelerate appreciably in the radial direction (the child stops at the new radius and it doesn't matter how long it takes to get there), so we can use this as the force that is doing work. The only trouble is, this force changes as the child moves inward, because the rotational speed and distance from the center are changing all the way. We can determine how the rotational speed varies with the child's distance from the center using angular momentum conservation, which allows us to write the force as a function of r as follows:

$$\left.\begin{array}{lll} F = ma_c & \Rightarrow & F(r) = m[\omega(r)]^2 r \\ mr^2\omega(r) = mR^2\omega_o & \Rightarrow & \omega(r) = \dfrac{R^2}{r^2}\omega_o \end{array}\right\} \quad \Rightarrow \quad F(r) = \frac{mR^4\omega_o^2}{r^3} \tag{6.3.9}$$

Now we have only to do the work integral. The displacement is toward the center (r is getting smaller), so $dl = -dr$, and the force is in the direction of displacement, so $\vec{F}\cdot\vec{dl} = Fdl = -Fdr$. And the limits of integration are from $r = R$ to $r = \frac{R}{2}$:

$$W\left(R \rightarrow \frac{R}{2}\right) = \int_R^{\frac{R}{2}} -F(r)\,dr = -mR^4\omega_o^2\int_R^{\frac{R}{2}}\frac{dr}{r^3} = -mR^4\omega_o^2\left[-\frac{1}{2r^2}\right]_R^{\frac{R}{2}} = \frac{3}{2}mR^2\omega_o^2 \tag{6.3.10}$$

Comparing this with Equation 6.3.8, we see that the work done in moving the child inward is precisely equal to the change in the system's kinetic energy.

Conceptual Question

A glass of ice water rests on the outer edge of a solid, uniform, rotating disk, which is spinning horizontally around its frictionless axle. The glass is a cylinder with a mass equal to one half the mass of the disk, and a radius that is one third the radius of the disk. Condensation at the bottom of the glass causes the coefficient of static friction on the top of the disk to go down, and the glass suddenly slides off. After this occurs, the rotational speed of the disk:

a. *slows down*
b. *speeds up*
c. *stays the same*
d. *There is no way to tell from the information given*

Solution

(c) Don't be fooled by all the details given – the rotational speed of the disk doesn't change! When the static friction force goes away, the system continues with its angular momentum, but you can't erase the glass of water from the system just because it slid away. Yes, the rotational inertia of what is going around the axis has changed, but as the glass slides away, it still has angular momentum (it will be a combination of its rotation about its center and movement relative to the axle, see

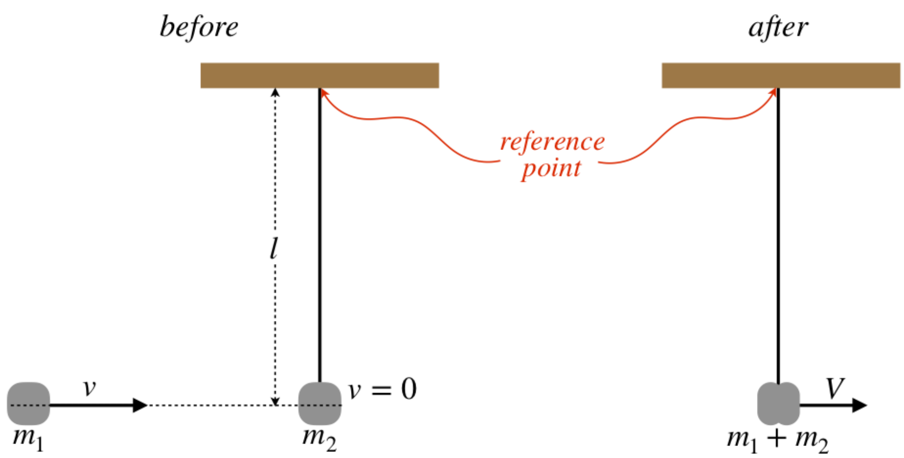
Equation 6.1.13 *and* Figure 6.1.2*). In fact, the glass will continue to have the same angular momentum that it had right before it started sliding, since there is no net torque on it. If the whole system maintains its angular momentum, and the glass keeps the same angular momentum, then the disk must as well – it doesn't change speed at all.*

Off-Center Collisions

Of all the problems that are solvable with angular momentum conservation, those that fall into the category of "off-center collisions" are the most interesting and complex. One reason is that unlike the cases of spinning collisions and changing rotational inertia, off-center collision problems often see cameo appearances from linear momentum conservation. Additionally, the fate of the system's mechanical energy becomes more interesting.

We begin with a problem that we are already familiar with from Section 4.6 – the ballistic pendulum. We were able to solve that problem by first solving the perfectly inelastic collision of the bullet & block to get their combined velocity, after which we used mechanical energy conservation to get the height to which the bullet & block swing. We will be more careful about extension in space (and the implications to rotational inertia) by replacing the bullet & block with two small clay balls that stick together. Also, we will not bother to look at the second half of the problem where the pendulum swings up, as the mechanical energy conservation portion of the problem is unchanged.

Figure 6.3.3 – Ballistic Pendulum with Two Small Clay Balls



If we choose the position where the string is attached to the ceiling as a reference point, we note that at the moment of the collision, the gravity and tension forces both act through the reference point, which means that there are no external torques on the system. The bullet and block exert torques on each other, but those are internal and cancel each other. Therefore, as an alternative to using linear momentum conservation, we can use angular momentum conservation.

Before the collision, the pendulum (the length of which we will call l) has no angular momentum relative to the reference point, but the bullet does, according to Equation 6.1.6:

$$L_{before} = m_1 v l \tag{6.3.11}$$

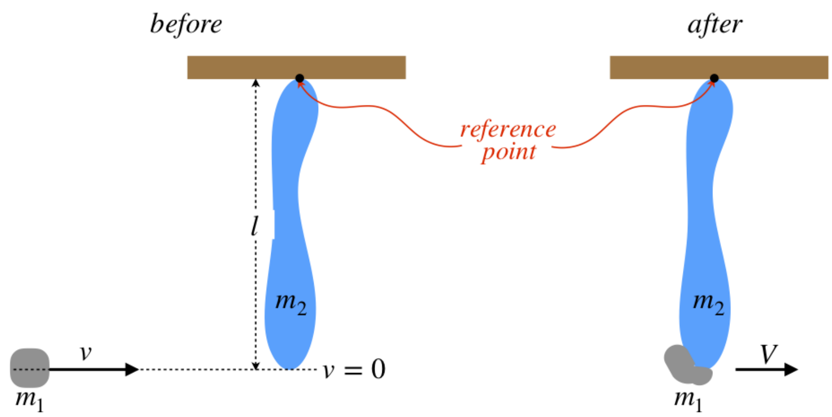
After the collision, the pendulum is rotating, and has a rotational inertia around the reference point, resulting in a final angular momentum of:

$$L_{after} = I\omega = \left[\left(m_1 + m_2\right) l^2\right] \left[\frac{V}{l}\right] = \left(m_1 + m_2\right) V l \tag{6.3.12}$$

And setting the initial angular momentum equal to the final gives the same result as when we used linear momentum (Equation 4.6.1).

Using angular momentum conservation is no longer optional – it is a requirement – when the target is not just a small ball at the end of a string, but is an extended object with a rotational inertia.

Figure 6.3.4 – Ballistic Pendulum with One Clay Ball



We know that the rotational inertia for this target around the reference point is less than it was when the target was a clay ball, since some of its mass is closer to the reference point. We will write the rotational inertia as some unknown fraction β multiplied by the rotational inertia of a small ball at the end of a string:

$$I_{target} = \beta m_2 l^2 \tag{6.3.13}$$

For example, if this target is a uniform thin rod, then Equation 5.2.7 applies, and $\beta = \frac{1}{3}$, or if the target is a uniform disk or cylinder pivoted about an axis perpendicular to its flat side and about its edge, then Equation 5.2.23 applies, with $R = \frac{1}{2}l$, giving $\beta = \frac{3}{8}$, and so on.

Applying angular momentum as we did above, we can find the final speed of the blob of clay and/or the rotational speed of the pendulum. The initial angular momentum is the same as before, so:

$$L_{after} = I\omega = \left[m_1 l^2 + \beta m_2 l^2\right]\left[\frac{V}{l}\right] = (m_1 + \beta m_2)\, Vl \quad\Rightarrow\quad V = \frac{m_1}{m_1 + \beta m_2} v \quad\Rightarrow\quad \omega = \frac{V}{l} \tag{6.3.14}$$
$$= \left(\frac{m_1}{m_1 + \beta m_2}\right)\frac{v}{l}$$

The claim was made above that we no longer have the option of using linear momentum conservation for this problem. Before we see *why* this must be true, let's show that it is true for the specific case of a uniform thin rod that has the same mass as the clay ($m_1 = m_2$). If we use linear momentum conservation, then when the clay is stuck on the end of the rod, the center of mass velocity of the rod + clay system is:

$$m_1 v = (m_1 + m_2)\, v_{cm} \quad\Rightarrow\quad v_{cm} = \frac{1}{2} v \tag{6.3.15}$$

The center of mass of the rod + clay system is halfway between the center of mass of the rod and the position of the clay, so it is a distance of $\frac{3}{4}l$ from the reference point. With a linear speed of $\frac{1}{2}v$, we get that the pendulum should have a rotational speed of:

$$\omega = \frac{\frac{1}{2}v}{\frac{3}{4}l} = \frac{2}{3}\frac{v}{l} \tag{6.3.16}$$

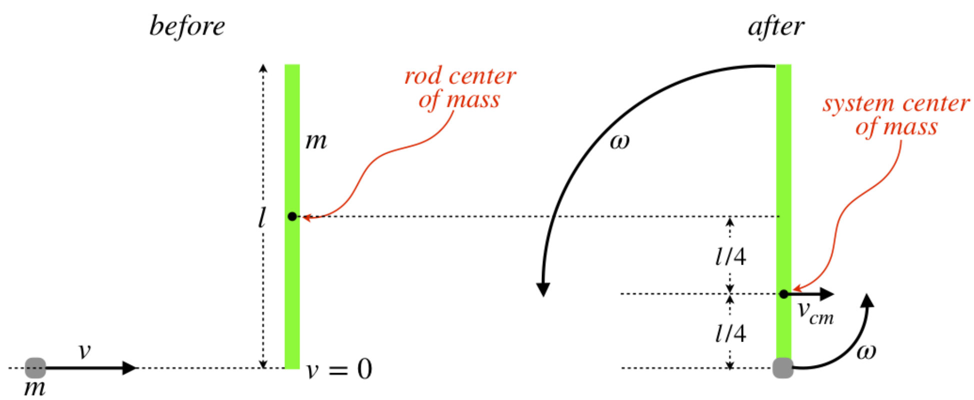
Let's check to see if this is right by plugging $\beta = \frac{1}{3}$ (for a thin rod rotated around its end) and $m_1 = m_2$ into Equation 6.3.14:

$$\omega = \left(\frac{m}{m + \frac{1}{3}m}\right)\frac{v}{l} = \frac{3}{4}\frac{v}{l} \tag{6.3.17}$$

So we see that using linear momentum conservation does not agree with using angular momentum conservation in this case. The reason is the presence of the pivot. The pivot will never exert a torque on the rod relative to the reference point, but it *will* exert a force on it, thereby ruining linear momentum conservation. But this brings up another puzzle: Whatever force the pivot exerts, it causes the speed of the center of mass to be *greater* after the collision than we found for conserved linear momentum, so the force on the rod by the pivot must be *forward*. That doesn't sound right – doesn't the pivot slow down the rod? To solve this puzzle, we get to look at yet another case – an off-center collision with no fixed pivot.

Let's do the same clay-hits-end-of-uniform-thin-rod-with-same-mass problem as above, this time free of any pivot (we'll also assume no gravity is present). First of all, we know that without a force coming from the pivot, the result we obtained in Equation 6.3.15 must be correct, as linear momentum must be conserved. Also we know that after the collision, with no forces on the clay + rod system, it must rotate around its center of mass. This calls for a fresh new diagram:

Figure 6.3.5 – Off-Center Perfectly Inelastic Collision



With the rod rotating counterclockwise, the bottom of the rod must be moving forward faster than the system's center of mass, while the top of the rod might actually be moving *backward*, depending upon the values of ω, l and v_{cm}. If this turns out to be the case, then it makes sense that a pivot could push the rod forward upon impact – the rod's rotation is fast enough compared to its linear motion that the top "tries" to move backward, but is prevented from doing so by a forward push from the pivot. Okay, let's see if this is the case mathematically.

At the moment of the impact, we have:

$$\begin{aligned} v_{bottom} &= v_{cm} + r\omega = \frac{1}{2}v + \frac{1}{4}l\omega \\ v_{top} &= v_{cm} - R\omega = \frac{1}{2}v - \frac{3}{4}l\omega \end{aligned} \tag{6.3.18}$$

Now we need to come up with ω. Even though linear momentum is conserved in this case, Equation 6.3.16 still isn't correct, as it assumes that the top end of the rod is held fixed. We need to use angular momentum conservation. Without a fixed pivot, what do we use as a reference point? The answer is *anywhere* – the angular momentum is conserved relative to every reference point! However, if we are carefree about this choice, we have to be extra careful when adding up the angular momentum after the collision. In the case of a fixed pivot, it was easy because we were able to use the rotational inertia around that point. When we have no fixed point on the object, we have to use Equation 6.1.13. So why not use the center of mass of the clay + rod system at the time of collision as the reference point, and get rid of that pesky second term from Equation 6.1.13?

$$\begin{aligned} L_{before} &= mvr_{\perp} \\ L_{after} &= I_{cm}\omega \end{aligned} \tag{6.3.19}$$

It's clear from the diagram that $r_{\perp}$ is $\frac{1}{4}l$, but we need to do a little bit of work to determine the rotational inertia of the system around its center of mass. This will be the sum of the rotational inertia of the point mass clay and the rotational inertia of the rod about its center (Equation 5.2.19), offset (using the parallel-axis theorem) by $d = \frac{1}{4}l$:

$$I_{cm} = I_{clay} + I_{rod} = \left[m\left(\frac{1}{4}l\right)^2\right] + \left[\frac{1}{12}ml^2 + m\left(\frac{1}{4}l\right)^2\right] = \frac{5}{24}ml^2 \tag{6.3.20}$$

Invoking angular momentum conservation and plugging in for $r_{\perp}$ and I_{cm} gives:

$$mvr_{\perp} = I_{cm}\omega \quad \Rightarrow \quad mv\left(\frac{1}{4}l\right) = \frac{5}{24}ml^2\omega \quad \Rightarrow \quad l\omega = \frac{6}{5}v \tag{6.3.21}$$

Plugging this back into Equation 6.3.18 confirms what we suspected, that without the top end fixed, its initial motion after the collision is *backward*, which is why the force by the pivot must be forward when it is attached:

$$\begin{aligned} v_{bottom} &= \frac{1}{2}v + \frac{1}{4}\left(\frac{6}{5}v\right) = \frac{4}{5}v \\ v_{top} &= \frac{1}{2}v - \frac{3}{4}\left(\frac{6}{5}v\right) = -\frac{2}{5}v \end{aligned} \tag{6.3.22}$$

This has been a long journey through off-center collisions, but we have one more stop – the fate of the system's mechanical energy. We explored perfectly inelastic head-on collisions in Section 4.5, and found a simple relation between the starting and ending kinetic energies of the system – Equation 4.5.7. Given that the final speed of the center of mass of the system has to be the same regardless of whether the collision is head-on or off-center, Equation 4.5.7 clearly cannot work for off-center collisions, as these result in rotations, and as we know, the total kinetic energy is the sum of linear and rotational parts. This means that perfectly inelastic collisions that occur off-center do not lose as much mechanical energy as perfectly inelastic head-on collisions.

This actually makes some intuitive sense. Let's take as an example a bullet digging into a block of wood. The bullet is subject to a non-conservative force that does enough work to slow the bullet to the same speed as the region of the block of wood it is entering (i.e. the bullet stops inside the block). Now let's assume that the force exerted on the bullet by the wood is the same wherever it enters the wood (it is something like "$\mu_k N$," where the normal force is the wood squeezing the bullet). Whether the bullet enters the block at its center of mass or at its edge, the center of mass of the block reaches the same final speed – we'll call the moment when this final speed is reached "t_o." If the bullet hits the center of mass, at t_o the bullet will have slowed to the same speed as the final speed of the center of mass. If the bullet hits the outer edge of the block and makes it spin, then the bullet is not slowed as much at t_o, because the edge of the block is moving faster than the final speed of center of mass of the block. If the bullet isn't slowed as much when it hits the edge, then not as much work is done on it (smaller change in its kinetic energy) by the non-conservative force, and less mechanical energy is converted to thermal.

Let's compute the fraction of kinetic energy that remains for the case above and compare it to the result if the collision occurs at the center of mass.

$$\begin{aligned} collides\ with\ center: \quad \frac{KE_f}{KE_o} &= \frac{m_1}{m_1+m_2} &&= \frac{m}{m+m} &&= \tfrac{1}{2} \\ collides\ with\ end: \quad \frac{KE_f}{KE_o} &= \frac{\frac{1}{2}(2m)\,v_{cm}^2 + \frac{1}{2}I_{cm}\omega^2}{\frac{1}{2}mv^2} &&= \frac{m\left(\frac{1}{2}v\right)^2 + \frac{1}{2}\left(\frac{5}{24}ml^2\right)\left(\frac{6}{5}\frac{v}{l}\right)^2}{\frac{1}{2}mv^2} &&= \tfrac{4}{5} \end{aligned} \tag{6.3.23}$$

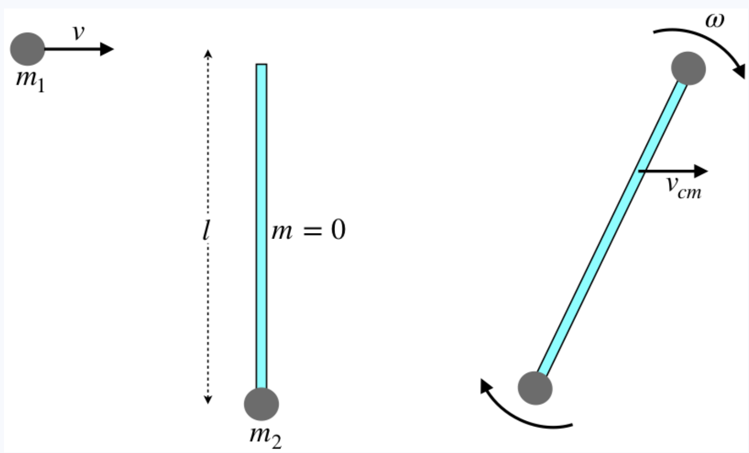
As you can see, less energy is lost when the clay sticks to the end and spins the rod than when it hits the center and doesn't spin it. Here is a nice demonstration of this phenomenon. First the puzzle:

And now the experimental evidence:

Both blocks rise to the same height, because their upward linear velocities start off the same, due to conservation of linear momentum, which is identical for both blocks, independent of where the bullet strikes. Our analysis above resolves the "puzzle" of the difference in mechanical energies of the two systems.

Example 6.3.3

A massless magnetic rod has a small steel ball (which does have mass, but a negligible radius) attached to one end, and is at rest. Another small steel ball approaches the open end of this rod at a right-angle, and when it reaches the end of the rod, sticks to it. The dumbbell-looking combination continues forward, spinning as it goes (see the diagram). Show the surprising result that no kinetic energy is lost in this collision. The diagram provides labeling of quantities that you can use – you cannot make any assumptions about the relative values of m_1 *and* m_2.



Solution

We are showing that kinetic energy is conserved, and the only principles that we can use are linear and angular momentum conservation. Let's start with linear momentum conservation. We have done this a hundred times – the incoming momentum equals the outgoing:

$$m_1 v = (m_1 + m_2)\, v_{cm} \quad \Rightarrow \quad v_{cm} = \frac{m_1}{m_1 + m_2} v$$

Now for conservation of angular momentum. Let's use the center of mass at the time of collision as the reference point. So we need to determine the perpendicular distance of the incoming ball from the center of mass. This is a straightfoward calculation (e.g. use the incoming ball as the origin), which gives:

$$r_\perp = r_1 = \frac{m_2}{m_1 + m_2} l$$

For later reference, we also have for the distance of the other ball from the center of mass:

$$r_2 = \frac{m_1}{m_1 + m_2} l$$

With this we get the starting angular momentum, and with the rotational inertia of the dumbbell about the center of mass, we get an equation resulting from angular momentum conservation:

$$\left.\begin{array}{l} L_o = m_1 v r_\perp = \dfrac{m_1 m_2}{m_1 + m_2} v l \\ L_f = I_{dumbbell}\,\omega = \left(m_1 r_1^2 + m_2 r_2^2\right)\omega = \dfrac{m_1 m_2}{m_1 + m_2} l^2 \omega \end{array}\right\} \quad \Rightarrow \quad L_o = L_f \quad \Rightarrow \quad \omega = \frac{v}{l}$$

Now all we have to do is construct the final kinetic energy:

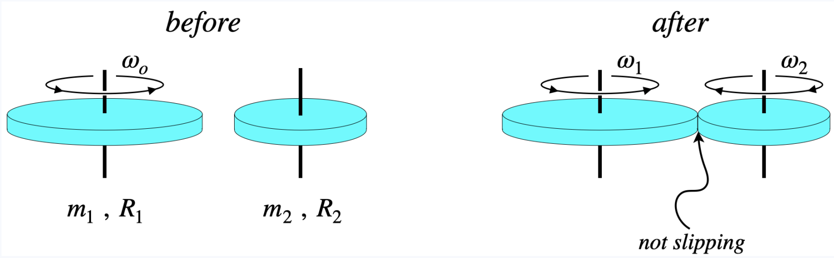
$$KE_f = \frac{1}{2}(m_1 + m_2)\, v_{cm}^2 + \frac{1}{2} I \omega^2 = \frac{1}{2}(m_1 + m_2)\left[\frac{m_1}{m_1 + m_2} v\right]^2 + \frac{1}{2}\left[\frac{m_1 m_2}{m_1 + m_2} l^2\right]\left[\frac{v}{l}\right]^2 = \frac{1}{2} m_1 v^2 = KE_o$$

Sample Problems

All of the problems below have had their basic features discussed in an "Analyze This" box in this chapter. This means that the solutions provided here are incomplete, as they will refer back to the analysis performed for information (i.e. the full solution is essentially split between the analysis earlier and details here). If you have not yet spent time working on (not simply reading!) the analysis of these situations, these sample problems will be of little benefit to your studies.

Problem 6.1

Two uniform disks are free to rotate frictionlessly around vertical axles. Initially one of the disks is rotating, while the other is not. They are then brought together so that their outer edges rub against each other. Kinetic friction between the two rubbing surfaces slows down disk #1, while speeding up disk #2. This continues until their rotational speeds are such that no slipping occurs between the two surfaces. With kinetic friction no longer present, they continue with constant rotational motion from this point forward.



The disks are the same thickness and are made from the same material. Disk #1 has a radius that is twice that of disk #2. Find the percentage of the starting kinetic energy that is converted to thermal during the period that the edges of the disks rub against each other.

Solution

With twice the radius, the larger disk has four times the horizontal surface area, and with equal thicknesses, then means that there is four times as much mass for the larger disk than the smaller one. Calling the smaller disk's radius "R" and mass "m", we have:

$$R_1 = 2R\,,\;\;R_2 = R\;\;\;m_1 = 4m\,,\;\;m_2 = m$$

To find the thermal energy, we just need to subtract the "after" kinetic energy from the "before" kinetic energy. From the analysis, we have the "after" rotational speeds of the disks in terms of initial speed of disk #1, so we have:

$$KE_{1\,after} = \frac{1}{2}I_1\omega_1^2 = \frac{1}{2}\left[\frac{1}{2}m_1R_1^2\right]\left(\frac{m_1}{m_1+m_2}\right)^2\omega_o^2 = \frac{1}{2}\left[\frac{1}{2}(4m)(2R)^2\right]\left(\frac{4m}{4m+m}\right)^2\omega_o^2 = \frac{64}{25}mR^2\omega_o^2$$

$$KE_{2\,after} = \frac{1}{2}I_2\omega_2^2 = \frac{1}{2}\left[\frac{1}{2}m_2R_2^2\right]\left(\frac{R_1}{R_2}\right)^2\left(\frac{m_1}{m_1+m_2}\right)^2\omega_o^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)(2)^2\left(\frac{4m}{4m+m}\right)^2\omega_o^2 = \frac{16}{25}mR^2\omega_o^2$$

$$KE_{total\,after} = KE_{1\,after} + KE_{2\,after} = \frac{16}{5}mR^2\omega_o^2$$

The "before" kinetic energy is:

$$KE_{total\,before} = \frac{1}{2}\left(\frac{1}{2}m_1R_1^2\right)\omega_o^2 = \frac{1}{4}(4m)(2R)^2\omega_o^2 = 4mR^2\omega_o^2$$

The thermal energy created is therefore:

$$\Delta E_{thermal} = KE_{total\,before} - KE_{total\,after} = \frac{4}{5}mR^2\omega_o^2$$

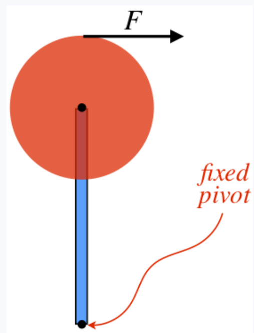
And as a fraction of the total initial kinetic energy this is:

$$\frac{\Delta E_{thermal}}{KE_{total\,before}} = \frac{\frac{4}{5}}{4} = \frac{1}{5}$$

So 20% of the initial kinetic energy becomes thermal.

Problem 6.2

The center of a uniform solid disk is threaded onto an axle at the end of a thin uniform rod. The rod and the disk have equal masses, and the radius of the disk is one-third the length of the rod. The rod is attached to a fixed pivot point at its other end, around which it is free to rotate. With the rod and disk both starting from rest, a force of constant magnitude is exerted tangent to the edge of the disk at the point farthest from the pivot for a short time. There is no gravity present.



After the applied force is removed, the disk and rod are spinning freely. Of the full system's angular momentum, determine what fraction of it comes exclusively from the motion of the rod.

Solution

From the analysis, the total angular momentum of the system in terms of the mass of the rod and its rotational speed is:

$$L_{system} = 16mR^2\omega_1$$

The angular momentum of the rod only is its moment of inertia about the fixed pivot multiplied by its rotational speed:

$$L_{rod} = I_{rod}\omega_1 = \frac{1}{3}m(3R)^2 = 3mR^2\omega_1 \tag{1}$$

The fraction of the system's angular momentum about the pivot that comes from just the rod's motion is therefore $\frac{3}{16}$.

Problem 6.3

Consider this position-dependent force:

$$\overrightarrow{F}(x,y) = \lambda\left[x\;\hat{i} + y\;\hat{j}\right]$$

A rock is tied to a string whose other end is held fixed at the origin, and is then set into circular motion in the $x-y$ plane (there is no gravity present). While the rock is moving at a constant speed in a circle, the force described above is turned on. The string later breaks and the rock eventually crosses the x-axis.

The string has a length of $0.50m$, the rock has a mass of $0.30kg$, and rock revolves around at a speed of $4.0\frac{m}{s}$. After the string breaks, when the rock crosses the x-axis at $x = 2.5m$, its does so moving at an angle of 7.9^o with that axis. Find the work done by the force $\vec{F}$ from the point when the string breaks to the point where the rock crosses the x-axis.

Solution

We can determine the final speed of the rock using the result from the analysis:

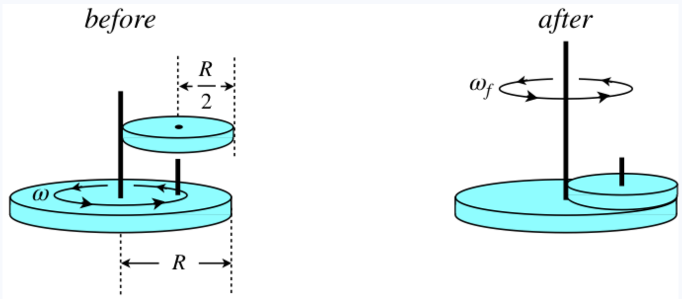
$$v_f = \frac{v_oR}{x\sin\theta} = \frac{\left(4.0\frac{m}{s}\right)(0.50m)}{(2.5m)\sin 7.9^o} = 5.82\frac{m}{s}$$

With the final speed of the rock, we can compute its kinetic energy, and the amount of kinetic energy gained from the point when the string breaks equals the work done by the only force acting, $\vec{F}$:

$$W=\Delta KE=\frac{1}{2}m\left[v_f^2-v_o^2\right]=\frac{1}{2}(0.30kg)\left[\left(5.82\frac{m}{s}\right)^2-\left(4.0\frac{m}{s}\right)^2\right]=2.68J$$

Problem 6.4

A large uniform-density disk of radius R rotates in a horizontal plane around a frictionless axle with a rotational speed ω. This disk includes a vertical post located a distance half its radius from its axle, and onto this axle is placed (very suddenly) a second disk with half the radius of the larger disk. The second disk is made of the same uniform material as the larger disk, and has the same thickness.



Kinetic friction between the disks acts until the smaller disk no longer spins on the lower disk, and instead turns with the larger disk as if their surfaces were glued. The work done by this kinetic friction equals the increase in the thermal energy of the two disks. Find this increase in thermal energy for the system as a fraction of its initial kinetic energy.

Solution

In the analysis we found the final rotational velocity in terms of the initial velocity for the "friction" case:

$$\frac{\omega_f}{\omega}=\frac{16}{19}$$

The angular momentum of the system remains conserved, and the kinetic energy can be written as:

$$KE_{rot}=\frac{1}{2}I\omega^2=\frac{1}{2}(I\omega)\,\omega=\frac{1}{2}L\omega$$

Using this expression, we can compute the ratio of the kinetic energy after to the kinetic energy before:

$$\frac{KE_{after}}{KE_{before}}=\frac{\frac{1}{2}L\omega_f}{\frac{1}{2}L\omega}=\frac{\omega_f}{\omega}=\frac{16}{19}$$

The remaining fraction of the energy is what goes to thermal, so:

$$\frac{\Delta E_{thermal}}{KE_{before}}=\frac{3}{19}$$

CHAPTER OVERVIEW

7: Gravitation

7.1: Universal Gravitation

7.2: Kepler's Laws

7.3: Energy in Gravitational Systems

Thumbnail: A simple swinging pendulum. Image used with permission (Public domain; Lucas V. Barbosa (Kieff)).

7.1: Universal Gravitation

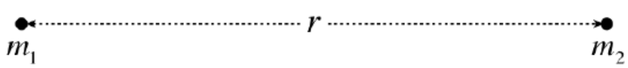
Newton Again

We return to a topic we have discussed only in the simplest of terms, but which has a great deal more depth. Of Newton's many achievements, one of his greatest has to have been the amazing realization that the gravity force is not simply a terrestrial phenomenon. Until he came along, people thought that objects "naturally" fall when they are near the Earth, and that heavenly bodies "naturally" do their little dance. To make the connection that the motions of planets could be explained using the very same paradigm that explains why things fall to Earth is truly a great achievement in human thought. Newton (apocryphally after seeing an apple fall from a tree) called this his *law of universal gravitation*, with emphasis on "universal," as it points out that the law applies both on Earth and in the heavens.

The key to Newton's idea is that the gravitational force actually exists between two objects and depends upon the masses of each and their separation in space. The Earth is no more special than the apple – both attract each other with equal force (which we know from the third law already), and the magnitude of that force depends upon their masses and their separations.

This actually does not fit well with our current understanding of the gravity force. In particular, we have been saying that the force equals mg, even as the height (distance from the Earth's surface) changes, so how is this dependent upon separation? First of all, it turns out that it is not the separation of the outer surfaces of the objects that matters, but rather their centers. In fact, it is even more complicated than that, so to simplify it, let's first just assume that the two gravitating objects are very small (point masses), so that their separation is well-defined:

Figure 7.1.1 – Two Point Masses Separated by *r*



In this case, the basis for Newton's law of universal gravitation can be described as follows:

- the force is exclusively attractive – experimentally, we only see gravity act as a "pull."
- the strength of the force grows linearly with the amount of each mass – experimentally, we find that the force doubles when we double either of the two masses involved, triples when either mass is tripled, and so on.
- the strength of the force varies in inverse proportion with the square of the separation – experimentally, we find that doubling the separation of the two objects reduces the force by a factor of four, tripling the separation reduces the force by a factor of nine, and so on.

Assuming these are the only factors that come into play for gravity (for example, the relative motions of the two objects doesn't affect the force), then we can write a proportionality for the magnitude of the gravity force between two point masses:

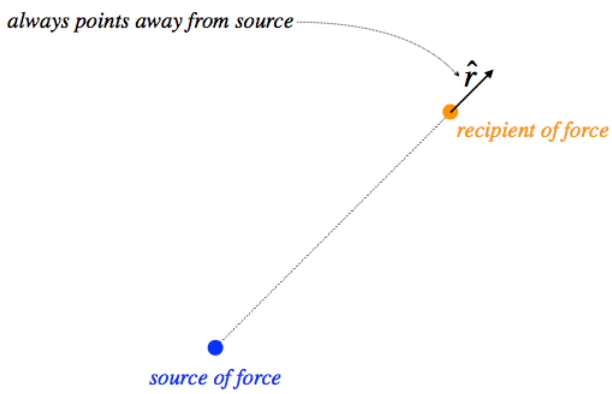
$$F_{gravity} \sim \frac{m_1 m_2}{r^2} \tag{7.1.1}$$

This satisfies all the criteria given above. All that remains is to turn it into an equality by inserting a multiplicative constant that turns it into units of force, with the correct observed magnitude:

$$F_{gravity} = \frac{G m_1 m_2}{r^2}, \qquad G \equiv 6.67 \times 10^{-11} \frac{Nm^2}{kg^2} \tag{7.1.2}$$

With the strength of the force, and the knowledge that it is attractive in nature, we have Newton's law of gravity. As usual, we would like to write this in a compact way that included the direction – as a vector equation. To do this, we temporarily discard the "equal partner" view, and treat one of the point masses as the source of the force (the object that the force is "by"), and the other as the recipient (the object the force is "on"). As an object's motion is determined by the forces on it, we treat the source of the force as the "origin," and define the position vector as pointing from the origin to the object on which the force acts. Therefore the unit vector of the position vector at the affected object always points away from the source of the gravity force.

Figure 7.1.2 – Defining Position Unit Vector for Gravity



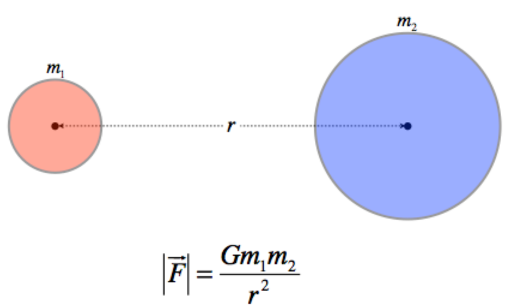
With this defined, we see that the attractive force on the recipient points in the $-\hat{r}$ direction, giving us a nice, compact vector equation for Newton's law of gravity:

$$\vec{F}_{gravity} = \frac{Gm_1m_2}{r^2}(-\hat{r}) \tag{7.1.3}$$

Spherical Bodies

Now of course we really aren't especially interested in gravity between point masses, when everything we see has some extension in space. So really what we have to do is treat two bodies as collections of point particles, all of which are attracting every other point particle. But this is quite cumbersome, and leads to all sorts of integral calculus. For our purposes, we will simply state the result that in the case of spherically-symmetric objects, they can be treated (in terms of gravitation) as if all of their mass were concentrated at their centers.

Figure 7.1.3 – Spheres Gravitate Like Point Masses at Their Centers



[*Note: This "spherical symmetry" does not require that the density of the spheres be uniform – the density can still vary radially. So the spheres can (for example) be more dense near their centers than near their surfaces, but the density cannot vary with the polar or azimuthal angles. That is, sampling the density throughout the sphere must reveal the same density everywhere that the distance from the sphere's center is held constant.*]

This turns out to be a convenient consequence of the inverse-square law, as you will no doubt examine in greater detail in more advanced math & physics classes. This result is something we will exploit greatly (at least as an approximation), since planets and stars are very close to being spherical.

Gravity at the Earth's Surface

Imagine a very small object (which can be effectively treated as a point object) was pulled toward a large spherical body, and stopped when it reached its surface. In that case, the gravitational force would be calculated using the radius of the large spherical object. Now we'll let that large spherical object be the Earth, and let the small object be a human (you).

Figure 7.1.4 – Universal Gravitation at the Earth's Surface

[*Note that the separation R_E points to the center of the Earth, not to a point on the Earth's surface just off the coast of Florida.*]

We can use now Newton's law of gravity to compute the force exerted on you. All we need is the radius of the Earth ($6.38 \times 10^6 m$), and the mass of the Earth ($5.98 \times 10^{24} kg$):

$$your\ weight = \frac{Gm_1m_2}{R_E^2} = \frac{\left(6.67 \times 10^{-11} \frac{Nm^2}{kg^2}\right)\left(5.98 \times 10^{24} kg\right) m}{\left(6.38 \times 10^6 m\right)^2} = \left(9.80 \frac{m}{s^2}\right) m = mg \tag{7.1.4}$$

So now you know where our constant g comes from. Now you might be concerned that our projectile calculations have not been accurate, because g is only correct at the surface of the earth, and projectiles might go quite high. Let's look at an example – how much does the gravitational force decrease when we go high up in the sky in a commercial airline? Commercial flights typically fly at an altitude of about $10,000m$(about $33,000ft$), so making the adjustment to the gravitational force gives:

$$your\ weight\ in\ airplane = \frac{\left(6.67 \times 10^{-11} \frac{Nm^2}{kg^2}\right)\left(5.98 \times 10^{24} kg\right) m}{\left(6.38 \times 10^6 m + 1 \times 10^4 m\right)^2} = \left(9.77 \frac{m}{s^2}\right) m \tag{7.1.5}$$

It's hardly noticeable. If you weighed yourself on earth and were 150 lbs, then in the plane the scale would read 149.5lbs. Okay, so let's go to a place where we know the distance makes a big difference – all the way into outer space to the international space station (ISS). The altitude in this case is about $400,000m$

$$your\ weight\ in\ ISS = \frac{\left(6.67 \times 10^{-11} \frac{Nm^2}{kg^2}\right)\left(5.98 \times 10^{24} kg\right) m}{\left(6.38 \times 10^6 m + 4 \times 10^5 m\right)^2} = \left(8.68 \frac{m}{s^2}\right) m \tag{7.1.6}$$

Wait just a minute… How can those people be floating around their space station if they have only lost about 11% of their weight?

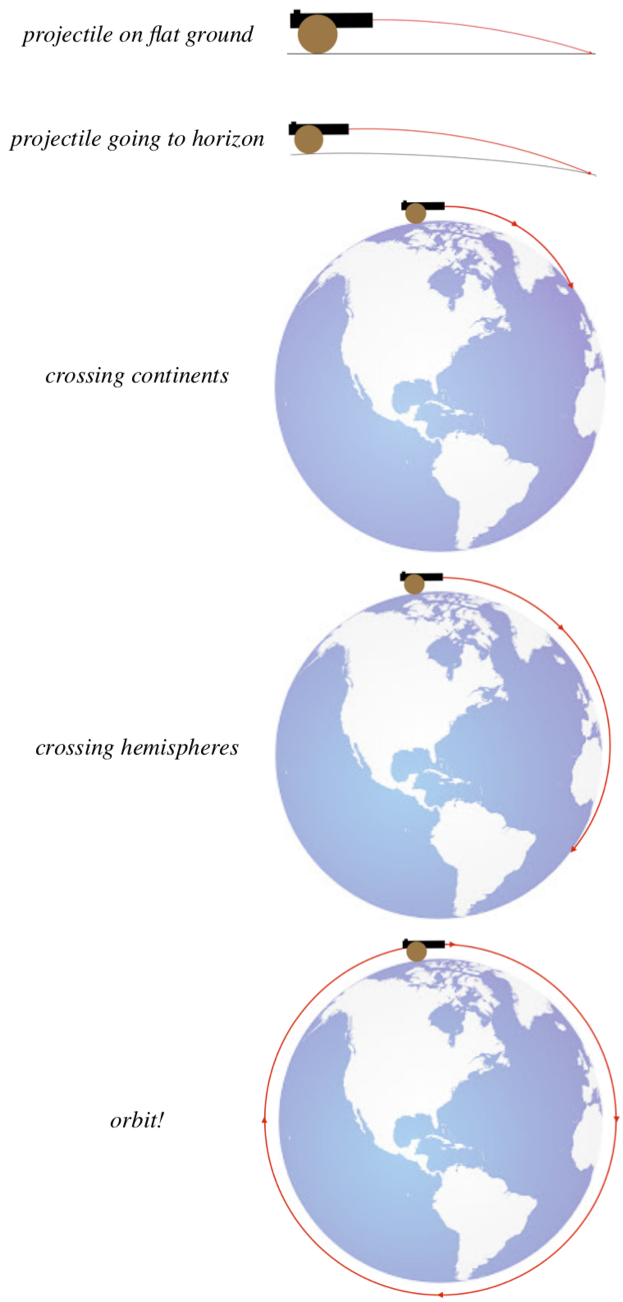
Free-Fall and Orbits

Suppose you are at the top floor of a skyscraper in an elevator when suddenly, tragically, the cable breaks. Assuming you could see past what I can only assume would be your abject terror, what would you see going on around you? The other screaming people around you, the hat on your head, and the penny that was on the floor would all be accelerating at the same rate, g. Since nothing is accelerating faster than anything else, if you hold out your pencil and release it, it doesn't drop to the floor of the elevator – it just "floats" there in front of you. In effect, the entire contents of the elevator is experiencing zero gravity.

Even conceding that being in a room in free-fall is equivalent to zero gravity, the space station is not plummeting to Earth, so how does it apply? Well, we know from our study of projectile motion that the horizontal motion of an object doesn't take away from the fact that it is in free-fall vertically. Indeed, there exist companies that fly planes in parabolic projectile trajectories so that the passengers can experience weightlessness for a couple minutes (before they have to pull out of the dive). So in fact if our elevator were a projectile, we would have the same zero gravity experience. Newton knew this, and came up with the following incredibly clever thought experiment:

If we fire a cannon horizontally, the cannonball follows the usual parabolic path, landing some distance away. If we increase the muzzle velocity, it goes farther before landing. Increase the muzzle velocity even more, and the landing point approaches the horizon. As we keep going this way, the projectile "falls over" the curvature of the Earth, and when the speed is finally fast enough, it never actually lands! Orbits are just the most extreme case of projectile motion.

Figure 7.1.5 – Newton's Cannon



So if being inside a container in free-fall is equivalent to being weightless, then a mouse inside a hollow cannonball fired by Newton's cannon would conclude that there is "no gravity," because the orbiting cannonball is a projectile in free-fall at all times. The astronauts on the space station experience weightlessness not because the Earth's gravitational influence is zero out there, but because that influence is the same on everything in the station, and everything is therefore in free-fall at the same rate. Indeed, if there was no gravity out there, then there would be no force to keep the space station moving in a circle, and it would fly away from Earth!

To understand how important this argument was in the context of his time, Newton used it to explain how a single phenomenon (gravity) could explain both terrestrial (projectile) motion and heavenly (orbital) motion at the same time. What is more, he backed it all up with mathematics! His law of universal gravitation predicted to very high precision the motions of the planets, even as it predicts the motions of cannonballs.

7.2: Kepler's Laws

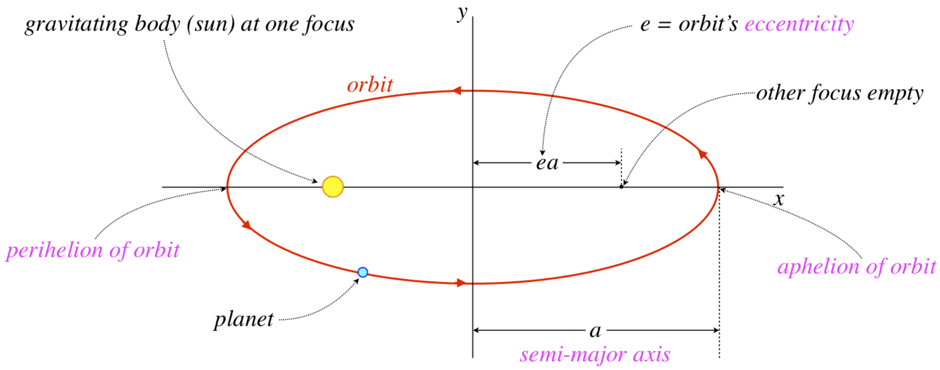
More than 20 years before Newton was born, a fellow named Johannes Kepler took a shot at explaining the orbits of the planets. He too posited that physical laws might be able to explain the motions, but didn't possess the tools (mathematical and physical) at Newton's disposal decades later (though admittedly, Newton did develop these tools for himself). Instead, what Kepler had were the remarkably detailed and accurate measurements of planetary motions made by an astronomer named Tycho Brahe, which he used to look for patterns in the motions. Amazingly, he found that the planets indeed moved with mathematical precision, and published his three laws of celestial motion, all of which are in exact accordance with the law of universal gravitation. While reading about his three laws, consider what a monumental accomplishment this was. Tycho Brahe's data detailed the motions of the planets *as he viewed them from Earth* (which itself is orbiting the sun).

Kepler's First Law

Kepler's First Law: The paths of bodies trapped in orbits form closed ellipses, with the gravitating body at one of the foci.

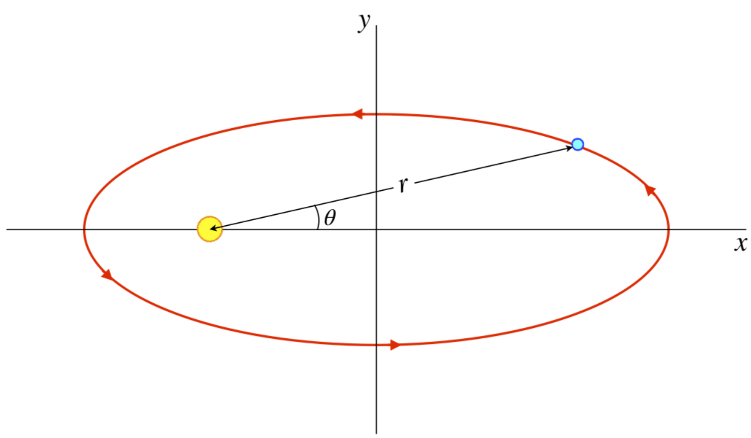
The many elements of an ellipse and how an orbit fits into the picture are expressed in Figure 7.2.1.

Figure 7.2.1 – Elliptical Orbit



There are many ways to describe such an orbit mathematically. A common way is to write the distance between the two bodies as a function of the angle that the line between them makes with the major (longer) axis:

Figure 7.2.2 – Polar Coordinate Description of Ellipse



The formula for the ellipse in these coordinates is:

$$r(\theta) = \frac{a\left(1-e^2\right)}{1-e\cos\theta} \tag{7.2.1}$$

[*Note: It is also possible to measure the angle in the opposite direction (with $\theta = 0$ corresponding to the perihelion), in which case the denominator is a sum rather than a difference.*]

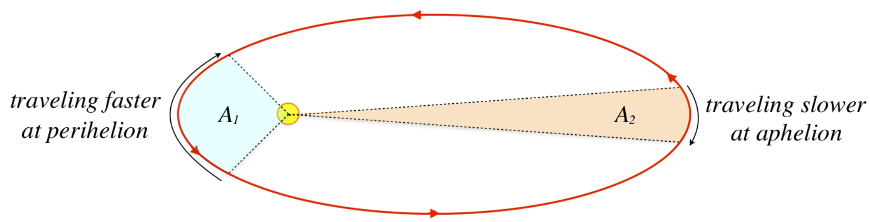
Circles are special cases of ellipses (eccentricity equal to zero), so naturally circular orbits are possible. Notice that if we plug in $e=0$ above, we get the simple orbit equation $r=a$. With a great deal of mathematics (first surmounted by Isaac Newton), one can can show that in fact this is a natural consequence of the inverse-square force law we already stated for gravitation. While we won't go quite so far as to perform this derivation, below we will make a closer examination of features of the ellipse (which we have expressed as a purely mathematical object here) in terms of physical quantities.

Kepler’s Second Law

Kepler's Second Law: An orbiting planet sweeps out equal areas in equal times during its orbit.

Kepler noticed that the planet moved faster when it was near the perihelion than when it was near the aphelion, and through painstaking examination of the data determined that in fact the amount of area the orbit sweeps out in a given period of time is the same everywhere in the orbit.

Figure 7.2.3 – Equal Areas in Equal Times



Note that while the diagram compares areas swept out through the perihelion and aphelion, the result applies to any part of the orbit – if we wait the same period of time, the area swept out will be the same: $A_1=A_2$. We will soon look into the physical aspects of gravitational orbits that lead to this result, but again one can't help but marvel at what it must have taken to derive this remarkable discovery from the raw data.

Kepler’s Third Law

Kepler's Third Law: For every object orbiting the same gravitational source, the ratio of the cube of the semi-major axis of the orbital ellipse and the square of the orbital period is the same constant: $\frac{a^3}{T^2}=constant$.

While the first law makes a general statement about all gravitational orbits, and the second law relates two different parts of a specific gravitational orbit, the third law gives a way of comparing different orbits of the same gravitating object (in Kepler's case, this gravitating object was the sun). Of the three laws, this one has the greatest practical value, because it means that even without knowing anything about the law of gravitation, one can make a prediction about one orbiting body based on observations of another orbiting body, if both are going around the same gravitating object. We will see what this mysterious constant is in terms of physical details, but to solve certain problems, knowing the actual value of the constant isn't necessary.

Example 7.2.1

An astronomer notices that an asteroid is positioned such that the Earth is directly between it and the Sun. It has a roughly circular orbit (like the Earth), which is in the same plane and in the same direction as the Earth. This asteroid is far away from the Earth – about 8 times farther from the Earth than the distance separating the Earth and the Sun. These are all approximations, but about how long will this astronomer have to wait see these three bodies reach these same positions?

Solution

For a circular orbit, the semi-major axis of the orbit is simply the radius of the orbit, and since the asteroid and Earth are both orbiting the same gravitational source (the sun), then Kepler's third law results in the same constant for both:

$$\frac{R^3_{earth}}{T^2_{earth}}=constant=\frac{R^3_{asteroid}}{T^2_{asteroid}}\quad\Rightarrow\quad T_{asteroid}=\left(\frac{R_{asteroid}}{R_{earth}}\right)^{\frac{3}{2}}T_{earth}$$

The period of the Earth's orbit is exactly 1 year, and the asteroid is 9 times farther from the sun as the Earth, so the time it will take the asteroid to come back to the same spot will be:

$$T_{asteroid}=27years$$

Of course, in 27 years, the Earth will also be in the spot where it started, so this is our answer.

Note that if the question asked for the time that elapses before the sun, Earth, and asteroid are all aligned again (which is different from reaching their original positions), the answer is different: When they realign for the first time, the Earth will have completed one orbit plus a bit more, while the asteroid will have completed a fraction of an orbit. Let's call the angle that the asteroid moves through in that first year $\Delta\theta$. The Earth catches up to it after a full revolution, so the angle the Earth moves through is:

$$\theta_{earth} = 2\pi + \Delta\theta$$

The Earth is moving 27 times as fast as the asteroid, so in this equal time frame the Earth has moved through an angle 27 times as great as the asteroid, which gives:

$$2\pi + \Delta\theta = 27\Delta\theta \quad \Rightarrow \quad \Delta\theta = \frac{2\pi}{26}$$

That is, the asteroid completes 1/26th of its orbit at the point when the Earth catches up to it. The asteroid's orbit takes 27 years, so the first alignment occurs at:

$$\Delta T = \frac{27 years}{26} = 1 year, 14 days$$

A nice application of Kepler's 3rd law involves man-made satellites that orbit the Earth. Telecommunications satellites we like to remain at a single position in the sky, so that we don't have to turn our satellite dishes to find them – we just point them in the right direction and leave it. To accomplish this, we need two things: The satellite has to be directly above the equator, and it has to be orbiting the Earth in the same direction that the Earth is rotating, with an orbital period of exactly 1 day. This is known as a geostationary orbit. We can use Kepler's 3rd law to determine how high off the Earth's surface this satellite needs to be.

Reconciling Kepler's Laws with Universal Gravitation

There are a couple of things we can say about the physics of gravitational orbits. First, gravity is a conservative force, which means that the mechanical energy of the system is conserved. We don't yet know how to describe the potential energy due to the gravitational force (hopefully it is clear that our old "$U(y) = mgy + U_o$ " treatment is no longer adequate, since this results in a constant force), but we will look at this in Section 7.3. The point is that the mechanical energy is a "constant of the motion," which we can use, for example, to describe the speed of the orbiting body as a function of the distance from the gravitational source.

Consider the Earth + sun system. The fraction of this system's mass that belongs to the Earth is about 3×10^{-6} , which means that the distance from the center of the sun to the center of mass of the system is this fraction multiplied by the (on average) 92 million miles separating the two bodies. The distance from the center of the sun to the center of mass of the system is therefore:

$$r_{cm} = \left(3 \times 10^{-6}\right)\left(92 \times 10^{6}\ miles\right) = 276\ miles \tag{7.2.2}$$

The radius of the sun is about 432,000 miles, so the center of mass of the system lies less than one tenth of one percent of the sun's radius from the sun's center. It's therefore a pretty good approximation to treat the center of the sun as a fixed point. [*Note: Even the center of mass of the Jupiter + sun system barely lies outside the sun's radius, even though Jupiter is much more massive than Earth, and is much farther away.*] The gravitational force is directed at this fixed point, so it constitutes a central force. As we found in Section 6.2, central forces have the property of conserving the angular momentum of the system (since they produce no torque). We therefore conclude that like the mechanical energy, the angular momentum of an orbiting body is also a constant of the motion, when the gravitating body is significantly more massive than the orbiting body. [*Note: The angular momentum of the whole system is always conserved, even when the two masses are comparable, but in that case we can't treat one object as orbiting another stationary one, which means we have to consider the motion of both objects, complicating our picture.*]

Kepler's First Law

Showing that elliptical orbits are a direct result of the law of universal gravitation is a mathematical exercise that is somewhat beyond the scope of this work. While this derivation could nevertheless be included here, the value of doing so is minimal, and will therefore be left for the reader to explore in an upper-division treatment of classical mechanics. Instead, we will look at only small – but very instructive – aspects of Kepler's first law.

Never mind that we have no reason to expect that orbits will be elliptical... Why would we even expect them to be *closed*? That is, it certainly isn't clear that when the polar angle θ changes by 2π, that the orbiting body's distance from the gravitating body will be the same. It turns out, however, that this element of gravitational orbits is not hard to demonstrate. Start by applying Newton's second law to the orbiting body on which a net force due to gravity is acting:

$$m\frac{d\overrightarrow{v}}{dt}=-\frac{GMm}{r^2}\hat{r} \tag{7.2.3}$$

We can rewrite this in terms of how the velocity changes with the angle θ by using the chain rule:

$$m\frac{d\overrightarrow{v}}{d\theta}\frac{d\theta}{dt}=-\frac{GMm}{r^2}\hat{r} \tag{7.2.4}$$

The angular momentum of the orbiting body can be written in terms of its mass, its angular velocity $\omega=\frac{d\theta}{dt}$, and its distance from the fixed point:

$$L=mr^2\omega \tag{7.2.5}$$

This angular momentum is a constant of the motion (i.e. it is conserved throughout the orbit), so we find that the rate at which the velocity vector changes with respect to θ is a vector with a constant magnitude:

$$\frac{d\overrightarrow{v}}{d\theta}=-\frac{GMm}{L}\hat{r} \tag{7.2.6}$$

We can write the position unit vector in terms of the angle relative to a cartesian coordinate system, as we did in Equation 1.6.11:

$$\frac{d\overrightarrow{v}}{d\theta}=-\frac{GMm}{L}\left(\cos\theta\;\hat{i}\;+\sin\theta\;\hat{j}\right) \tag{7.2.7}$$

Integrating over the angle gives the velocity vector as a function of θ (and an undetermined constant of integration $\overrightarrow{v}_o$):

$$\overrightarrow{v}(\theta)=-\frac{GMm}{L}\left(\sin\theta\;\hat{i}-\cos\theta\;\hat{j}\right)+\overrightarrow{v}_o \tag{7.2.8}$$

From this result we can conclude that the magnitude and direction of the velocity return to the same value every time θ changes by 2π. This means that the kinetic energy returns to its same value periodically as well. But the mechanical energy of this system is conserved, so the potential energy also returns to its value with the same periodicity. But (as we will see in the next section), the potential energy is defined by the separation of the two masses, so this separation also returns every time θ changes by 2π. Well, if every time the angle changes by 2π the orbiting body is the same distance away from the gravitating body, is moving at the same speed, and is moving in the same direction, then clearly its motion is being repeated – the orbit is closed.

Notice that if dependence on r in the law of gravitation was anything other than inverse-square, then the r's would not cancel as they did in reaching Equation 7.2.6 (the angular momentum would still be the same constant of the motion, as the force would still be central), which would give an equation for the velocity that depends on both r and θ. This ruins the argument above, and the orbit would not be closed.

Digression: Orbits Are Not Quite Closed After All

As amazing as Newton's accomplishment was with his theory of gravity, roughly 230 years later, a fellow named Albert Einstein provided an improved theory, called the General Theory of Relativity. It had been known for some time that observations of the orbit of Mercury indicated that its orbit was in fact not closed. For a long time it was thought that the discrepancy was the result of an unseen planet or other gravitating body that was pulling Mercury off course, but Einstein's theory showed that the inverse-square theory of Newton, while a very good approximation, is not quite right, and his new theory predicted Mercury's motion perfectly.

While we are skipping the mathematics detailing how we get to the equation of the ellipse, we can still extract some information from Equation 7.2.8 that will be useful to us later. To simplify the discussion that follows, we will assume that we already know that the elliptical orbit is the result.

We haven't yet defined the *orientation* of the (x,y) coordinate system used in Equation 7.2.8 – so far we have only required that the origin be at the gravitating body. Let's choose the x-axis to lie along the major axis such that the point of maximum separation (aphelion) lies on the positive side of the x-axis (giving us Figure 7.2.2). With these axes, it is clear that at the point $\theta=0$, the velocity of the orbiting body is in the $+\hat{j}$ direction. Looking at Equation 7.2.8, we see that this means that the constant vector $\overrightarrow{v}_o$ must point parallel to the minor axis, otherwise it would give the body a component of velocity along the x direction. But which way does this constant vector point, $+\hat{j}$ or $-\hat{j}$?

Consider the velocity of the orbiting body at the perihelion ($\theta=\pi$). In this case, the object is now moving in the $-\hat{j}$ direction, but because it is closer to the reference point at the gravitating body, it must be moving faster to conserve angular momentum. For the constant vector $\overrightarrow{v}_o$ to make the orbiting body faster at ($\theta=\pi$) than at ($\theta=0$), we must have:

$$\overrightarrow{v}_o=v_o\left(-\hat{j}\right) \quad \Rightarrow \quad \begin{cases} \theta=0 \quad (aphelion) & v_{min}=\dfrac{GMm}{L}-v_o \\ \theta=\pi \quad (perihelion) & v_{max}=\dfrac{GMm}{L}+v_o \end{cases} \tag{7.2.9}$$

We can relate the maximum and minimum speeds of the orbit using angular momentum conservation. Looking at Figure 7.2.1, we see that the value of $r_\perp$ for the aphelion is: $a+ea=a(1+e)$. At the perihelion: $r_\perp=a(1-e)$. Setting equal the angular momenta at these two positions in the orbit gives:

$$L_{aphelion}=L_{perihelion} \quad \Rightarrow \quad mv_{min}\left[a(1+e)\right]=mv_{max}\left[a(1-e)\right] \quad \Rightarrow \quad v_{min}=v_{max}\left(\frac{1-e}{1+e}\right) \tag{7.2.10}$$

Using this result and Equations 7.2.9, we can eliminate the troublesome v_o, and get:

$$\begin{aligned} v_{max}&=(1+e)\frac{GMm}{L} \\ v_{min}&=(1-e)\frac{GMm}{L} \end{aligned} \tag{7.2.11}$$

It will also be useful to have an expression for the angular momentum in terms of the masses, eccentricity, and major axis. To get this, multiply the first of the Equations 7.2.11 by the orbiting body's mass and $r_\perp$ [for v_{max} this is: $a(1-e)$] to get:

$$L=mv_{max}r_\perp=m\left[(1+e)\frac{GMm}{L}\right]\left[a(1-e)\right] \quad \Rightarrow \quad L^2=a\left(1-e^2\right)GMm^2 \tag{7.2.12}$$

Putting this back into Equation 7.2.1 simplifies it a bit in terms of physical constants, putting in a form that will be useful later:

$$r(\theta)=\left(\frac{L^2}{GMm^2}\right)\frac{1}{1-e\cos\theta} \tag{7.2.13}$$

One last thing to note before moving on to Kepler's second law. Looking at Equation 7.2.8, we see that if v_o happens to equal zero, then the velocity vector has a constant magnitude, and its direction is always perpendicular to $\hat{r}$ (which can be confirmed quickly by performing a dot product). So a non-zero constant of integration is responsible for making an otherwise circular orbit eccentric.

Example 7.2.2

Show that the distance of closest approach (the perihelion distance) is given by:

$$r_{min}=\frac{L^2}{GMm^2}\left(\frac{1}{1+e}\right)$$

Do this in two different ways:

a. *Using calculus and Equation 7.2.13. [Note: There are two angles that result in extrema, but only one gives a minimum.]*
b. *Using the equation for angular momentum at r_{min} and one of the Equations 7.2.11.*

Solution

a. We seek the minimum value of $r(\theta)$, so we start by finding the value of θ where this minimum occurs:

$$0=\left(\frac{L^2}{GMm^2}\right)\frac{d}{d\theta}\left(\frac{1}{1-e\cos\theta}\right)\quad\Rightarrow\quad 0=\frac{\sin\theta}{1-e\cos\theta}\quad\Rightarrow\quad \theta=0\ \ or\ \ \pi$$

Note that $\theta=0$ minimizes the denominator, so it gives a maximum for r, not a minimum. On the other hand, $\theta=\pi$ maximizes the denominator, and therefore gives a minimum. This result makes sense when we look at how $theta$ is defined in Figure 7.2.2. Plugging $\theta=\pi$ into Equation 7.2.13 gives the desired answer.

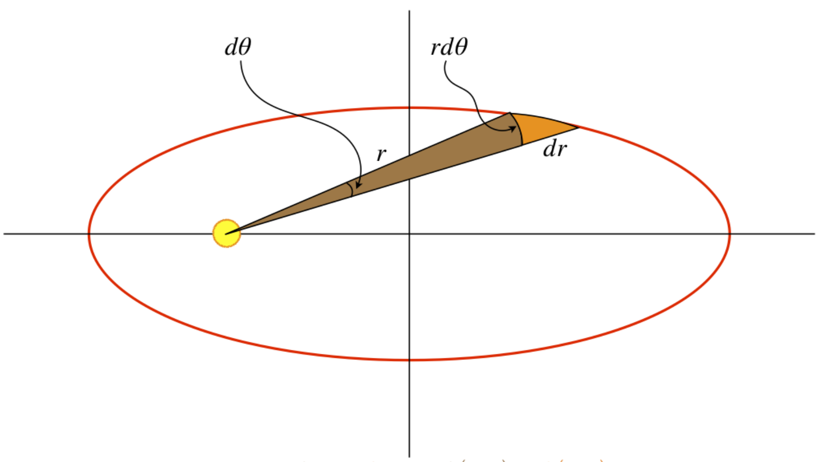
b. The angular momentum at the point closest approach will involve the maximum speed, since it is conserved throughout the orbit, so using the expression for the maximum speed in Equations 7.2.11 we get:

$$L=mv_{max}r_{min}\quad\Rightarrow\quad r_{min}=\frac{L}{mv_{max}}=\frac{L}{m\left((1+e)\frac{GMm}{L}\right)}=\frac{L^2}{GMm^2}\left(\frac{1}{1+e}\right)$$

Kepler's Second Law

We look next at Kepler's equal-areas-swept-out-in-equal-times law. The geometry of measuring areas swept out of ellipses is impossibly difficult to do mathematically, but an *infinitesimal* amount of area swept out in an infinitesimal time period is something we can do. Figure 7.2.4 shows how we can mathematically describe the area swept out in an infinitesimal period of time. The amount swept out can be broken into two triangles. Okay, so the sides of the triangles are curved, but when the curves are infinitesimal in length, the amount they differ from straight lines is insignificant. The area of the two triangles are color-coded in the diagram. We notice that while both have infinitesimal areas, the orange triangle includes a product of *two* infinitesimal quantities. When the area is then divided by a small time span and the limit is taken as the time span goes to zero, the ratio of $d\theta$ and dt approaches a finite value (specifically, the angular velocity ω at that moment in time), but in the second term there is still another infinitesimal dr that goes to zero in the limit. In other words, the orange triangle contributes nothing to the area swept out in the infinitesimal time span dt. With a little mathematical manipulation, we see that the rate at which area is swept out is the angular momentum of the orbiting body divided by twice its mass. We know that both the angular momentum and the mass remain constant for the orbit, so the rate at which area is swept out also remains constant. Kepler's second law is equivalent to angular momentum conservation.

<u>Figure 7.2.4 – Kepler's Second Law Expresses Angular Momentum Conservation</u>



$$dA=dA_1+dA_2=\tfrac{1}{2}b_1h_1+\tfrac{1}{2}b_2h_2=\tfrac{1}{2}\left(rd\theta\right)r+\tfrac{1}{2}\left(rd\theta\right)dr$$

$$\frac{dA}{dt}=\underbrace{\frac{1}{2}r^2\frac{d\theta}{dt}}_{finite}+\underbrace{\cancel{\frac{1}{2}rdr\frac{d\theta}{dt}}}_{infinitesimal}=\frac{1}{2}r^2\omega=\frac{mr^2\omega}{2m}=\frac{L}{2m}=constant$$

Kepler's Third Law

It is quite straightforward to show that Kepler's third law holds for circular orbits, so let's do that first. The speed is constant for the entire orbit, it equals the circumference of the orbit divided by the time it takes for a full orbit:

$$v = \frac{2\pi R}{T} \tag{7.2.14}$$

We can use the fact that the gravitational force is causing centripetal acceleration to get the following expression for the square of the constant speed of the orbiting body:

$$\frac{GMm}{R^2} = m\frac{v^2}{R} \quad \Rightarrow \quad v^2 = \frac{GM}{R} \tag{7.2.15}$$

Plugging Equation 7.2.14 into Equation 7.2.15 and doing some algebra gives Kepler's third law, with the semi-major axis equaling the radius of the circular orbit (zero eccentricity):

$$\left(\frac{2\pi R}{T}\right)^2 = \frac{GM}{R} \quad \Rightarrow \quad \frac{R^3}{T^2} = \frac{GM}{4\pi^2} = constant \tag{7.2.16}$$

This gives us not only that the ratio is a constant, but specifically what the constant is. As we can now confirm, this constant depends only upon the mass of the gravitating body.

It's quite remarkable that this law holds equally well for elliptical orbits, where R is replaced by a. We can show this by starting with a result we found from Kepler's second law. The rate at which area is swept out is constant, so the total area of the ellipse is this rate multiplied by the time of a full orbit. Reviewing our conic sections, we plug in the area of an ellipse, and get:

$$area\ of\ ellipse = \pi\ ab = \frac{dA}{dt}T = \frac{L}{2m}T \tag{7.2.17}$$

The quantity b is the length of the semi-minor axis, which is related to the length of the semi-major axis in terms of the eccentricity:

$$b^2 = a^2\left(1 - e^2\right) \tag{7.2.18}$$

Squaring Equation 7.2.17 and eliminating b^2 using Equation 7.2.18 gives us:

$$\pi^2 a^4\left(1 - e^2\right) = \frac{L^2 T^2}{4m^2} \tag{7.2.19}$$

We had the good foresight to derive Equation 7.2.12, so plugging L^2 from that equation into Equation 7.2.19 gives us our answer:

$$\pi^2 a^4\left(1 - e^2\right) = \left[a\left(1 - e^2\right) GMm^2\right]\frac{T^2}{4m^2} \quad \Rightarrow \quad \frac{a^3}{T^2} = \frac{GM}{4\pi^2} \tag{7.2.20}$$

Example 7.2.3

Attractive central forces that are not inverse-square do not produce closed orbits except when the orbit happens to be circular. Whenever there is a closed orbit, a result like Kepler's third law (which relates the orbit period to the radius of the circular orbit) will be the result. Derive the effective Kepler third law for an attractive central force that varies as the inverse-cube of the separation.

Solution

Following the process used above for a gravitational circular orbit, we get:

$$\left.\begin{array}{l} \dfrac{k}{R^3} = m\dfrac{v^2}{R} \\ v = \dfrac{2\pi R}{T} \end{array}\right\} \quad \Rightarrow \quad \boxed{\frac{R^4}{T^2} = \frac{k}{4\pi^2 m} = constant}$$

7.3: Energy in Gravitational Systems

Gravitational Potential Energy

We showed in Section 3.2 that our terrestrial model of gravity is a conservative force, and it certainly seems reasonable to assume that universal gravitation is as well, but really we should check to see if this is the case. In Section 3.6 we outlined a procedure for determining whether a force is conservative or not – basically it consists of trying to construct a potential energy function whose gradient equals the force, and if we succeed, then the force is conservative. If we can show that it is impossible to do this, then the force is non-conservative. Let's see what happens when we bring this to bear on gravitation.

Let's start by writing the gravity force in cartesian coordinates:

$$\overrightarrow{F}(x,y,z) = -\frac{GMm}{r^2}\hat{r} = -\frac{GMm}{r^3}\overrightarrow{r} = -\frac{GMm}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\left(x\hat{i}+y\hat{j}+z\hat{k}\right) \tag{7.3.1}$$

Consider next the following partial derivative:

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2x) = \frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \tag{7.3.2}$$

This is precisely the x-component of the gravitational force. Obviously partial derivatives with respect to y and z yield similar results – the y and z components of the gravitational force. This means we can immediately define a potential energy function whose negative gradient is the force:

$$U(x,y,z) = \frac{-GMm}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}} + U_o \quad \Rightarrow \quad U(r) = \frac{-GMm}{r} + U_o \tag{7.3.3}$$

We could have saved ourselves a lot of trouble if we happened to know a useful fact from vector calculus: The gradient of a function that is purely a function of r can be written as:

$$\overrightarrow{\nabla}U(r) = \frac{d}{dr}U(r)\hat{r} \tag{7.3.4}$$

So:

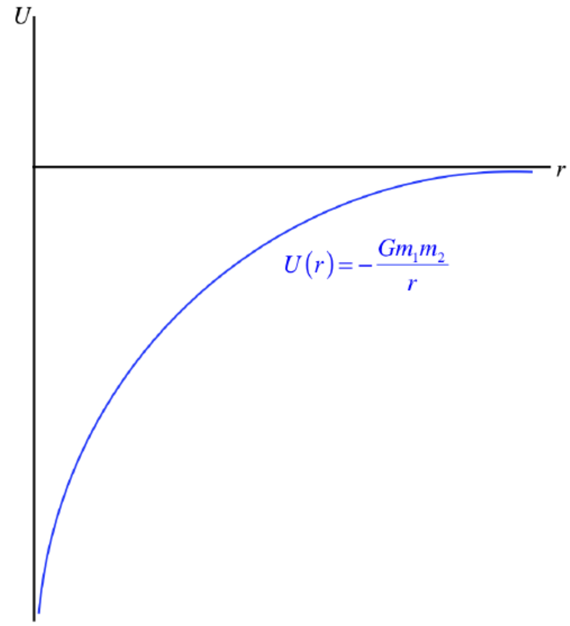
$$\overrightarrow{F} = -\overrightarrow{\nabla}U(r) = -\frac{d}{dr}\left[\frac{-GMm}{r}+U_o\right]\hat{r} = -\frac{GMm}{r^2}\hat{r} \tag{7.3.5}$$

As with other potential energy functions for another physical system that we have discussed (the intermolecular forces described by the Lennard-Jones potential), we typically choose our arbitrary constant of integration such that the potential energy falls off to zero at infinity. In the case of our gravitational potential energy, this gives what we will use for the potential energy function henceforth:

$$U_{grav}(r) = \frac{-GMm}{r} \tag{7.3.6}$$

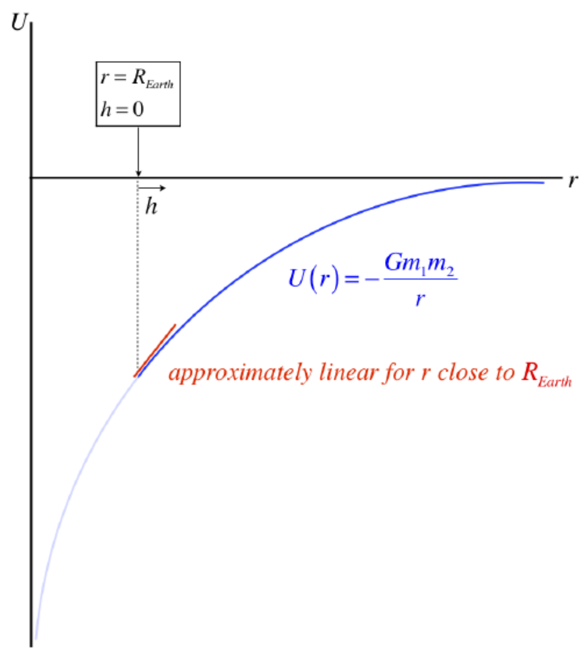
A graph of this function looks like this:

Figure 7.3.1 – Gravitational Potential



This function is significantly different from the "mgy" that we have been using up to now. To reconcile these two models, we need to make a note of the restriction of our terrestrial model of gravity – it holds for a region very close to the surface of the Earth, $r = R_{Earth}$. Calling "h" the height of the object from the ground, the gravitational potential function above is cut-off at ground level. If we restrict ourselves to r values close to this point, the curve above is very close to a straight line:

Figure 7.3.2 – Gravitational Potential Near Earth's Surface



The negative slope of the potential energy curve is the force, so the slope of the straight line approximating the curve near $r = R_{Earth}$ is the constant force in that region. Taking the negative derivative gives Newton's gravitational force, and when evaluated at $r = R_{Earth}$, this force comes out to be mg, as we saw in Equation 7.1.4.

Bound and Unbound Gravitational Systems

With a graph of the potential energy which goes to zero at infinity, we are naturally drawn again to energy diagrams. There is a problem with jumping straight into this, however. Energy diagrams require 1-dimensional motion, and while $U(r)$ is a function of a single variable, the motion is not 1-dimensional. To see where things break down, consider a closed elliptical orbit. As the planet moves toward the perihelion, the value of r gets smaller. If Figure 7.3.1 were the potential curve for an energy diagram, then when the planet is moving toward smaller values of r, it would keep speeding up indefinitely as it approaches $r=0$. Okay, so in practice it would hit the surface of the sun before it could "accelerate indefinitely," but still this does not represent orbital motion. In short, there is no part of this potential energy graph that represents the turnaround point that is the perihelion.

Fortunately, we have a nice trick to take care of this shortcoming. When we draw energy diagrams, the kinetic energy comes from the motion along the direction parallel to the one dimension. In this case of measuring energy along the radius, the kinetic energy for the energy diagram can only come from the part of the velocity that is *radial*. The total kinetic energy is the sum of the radial and tangential parts:

$$KE_{tot} = KE_{radial} + KE_{tangential} \tag{7.3.7}$$

The tangential speed multiplied by the mass and the distance from the center is the angular momentum, which is a constant of the motion, so we have:

$$\left.\begin{array}{l} KE_{tangential} = \frac{1}{2}mv^2_{tangential} \\ L = mv_{tangential}\,r \end{array}\right\} \quad \Rightarrow \quad KE_{tangential} = \frac{L^2}{2mr^2} \tag{7.3.8}$$

If we now construct the total energy of the system, we have:

$$E_{tot} = KE_{radial} + KE_{tangential} + U = \frac{1}{2}mv^2_{radial} + \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{7.3.9}$$

Note that this can now be *treated* as 1-dimensional system, by combining the last two terms into a single function of r that we call an *effective potential*:

$$U_{eff}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{7.3.10}$$

Graphing this allows us to work exactly as before, though we have to keep in mind that whatever we determine the kinetic energy to be is really only a fraction of the kinetic energy. So, for example, at the turnaround points (where we have said the kinetic energy is zero), the kinetic energy is in fact the tangential term. This makes sense for closed orbits, since the orbiting body never actually stops moving entirely – it only stops moving *radially*. The graph of this effective potential has a rather familiar shape, and including the total system energy makes it an energy diagram.

Figure 7.3.3 – Energy Diagram of a Gravitating System

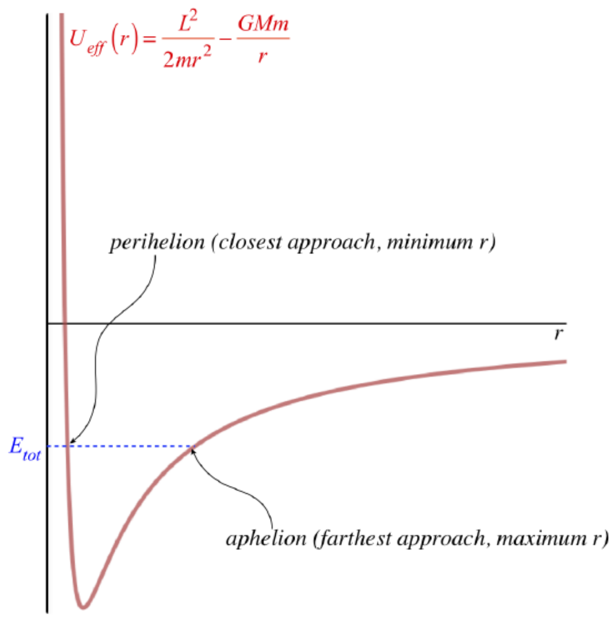


Figure 7.3.3 labels the turnaround points as the perihelion and aphelion, but there is much more we can extract from this diagram:

- **circular motion**

If the total energy lies at the bottom of the dip, then according to what we know about about energy diagrams, the kinetic energy is zero. But in this case, it means that the contribution to the kinetic energy by the radial component of velocity is zero. In other words, the orbit is circular. This fits with the fact that there is only one turnaround point, making the perihelion and aphelion the same distance.

- **semi-major axis**

The semi-major axis is the average of the perihelion and aphelion distances, so it is the value of r halfway between the two turnaround points on the graph.

- **eccentricity**

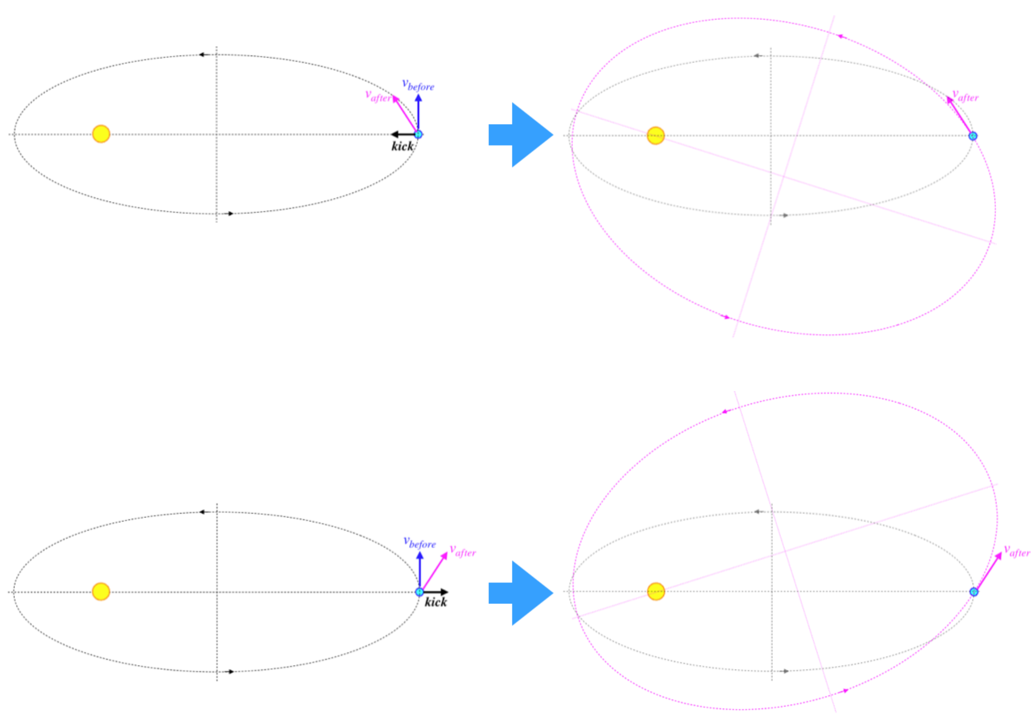
Looking at Figure 7.2.1, we can write the perihelion and aphelion distances in terms of the eccentricity and the semi-major axis, which we can then invert to get the eccentricity in terms of the perihelion and aphelion distances and the semi-major axis. Doing this gives:

$$e=\frac{r_{max}-r_{min}}{2a} \tag{7.3.11}$$

Looking at Figure 7.3.3, we see that raising the total energy (but keeping it negative, and doing it without increasing the angular momentum, which would change the graph of the effective potential) increases the length of the semi-major axis, but much more of this change comes from the increase of the aphelion distance than from the decrease of the perihelion distance (especially close to the horizontal axis). This results in an increase of eccentricity. How does one increase the total energy without increasing the angular momentum? By giving the orbiting body a "kick" that points radially (inward or outward). This exerts no torque (the force is parallel to the position vector), so the angular momentum doesn't change, but the push increases the total energy, because it adds a radial component of velocity without changing the tangential component.

When a radial kick is given, a polar angle of zero in the same coordinate system will no longer correspond to the aphelion of the orbit – the orbital ellipse will be rotated. If the kick is outward, then at the position of the kick the orbiting body is still getting farther from the gravitating body, which means it is heading for the new aphelion, and the orbit has rotated in the direction of the orbit. That is, if the orbit was clockwise, then the major axis rotates clockwise. For a radially inward kick, the orbiting body is now getting closer to the gravitating body, which means it is coming from the aphelion, so the major has rotated the direction opposite to the orbit direction.

Figure 7.3.4 – Effect of Radial "Kicks" to Orbits



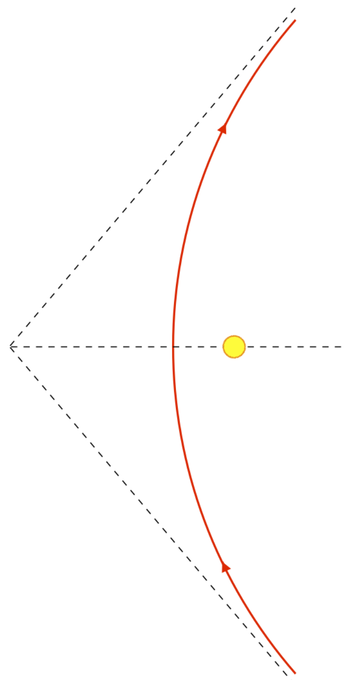
A tangential kick in the direction of motion would also raise the total energy, but it would also increase the angular momentum, increasing the positive term in the effective potential. This has the effect of raising the bottom of the curve, and if this is done properly (at the aphelion), the bottom of the curve comes up faster than the energy line goes up, bringing the eccentricity down. The eccentricity can even be brought all the way down to zero (i.e. a circular orbit) in this way (see Example 7.3.1).

- **hyperbolic trajectory – unbounded**

If the total energy is greater than zero, the orbiting body is not "bound" to the gravitating body. That is, after the orbiting body makes its closest approach, it zooms away, and although it slows down as it departs, it never stops moving away from the gravitating body. The height of the total energy line above the horizontal axis represents the finite kinetic energy of the orbiting body when it gets very far away, since the potential vanishes there. While the tangential part of the kinetic energy also goes to zero very far away, the angular momentum does not vanish – it remains conserved. With a finite speed and a non-zero angular momentum, the motion of the orbiting body must be asymptotically approaching a line that passes by the gravitating body, and we get what is called a *hyperbolic trajectory*. [*The word "trajectory" is generally preferred over "orbit," because the latter typically implies that the affected body is trapped by the gravitating body.*]

The orbit equation that we found for the elliptical orbit (Equation 7.2.13) still works, but for the hyperbolic trajectory the eccentricity is greater than 1.

Figure 7.3.5 – Hyperbolic Trajectory



- **parabolic trajectory – barely (un)bounded**

If the total energy of the system equals exactly zero, then the orbiting body just barely runs out of velocity as it reaches infinity. It technically is neither bound nor unbound – it is at the borderline between the two. In this case, the eccentricity equals exactly 1.

Example 7.3.1

An orbiting body is in an orbit where it is four times farther from the gravitating body at its aphelion than at its perihelion. By what percentage must its speed be increased at the aphelion to make its orbit circular?

Solution

Since orbits are closed, if we instantaneously speed up the orbiting body at its aphelion, it will return to that same point with the same speed and moving in the same direction (i.e. its motion will still be perpendicular to the radius). That means that when the orbiting body returns, it will either be once again at its aphelion (if the speed was not increased substantially), or it will be at its perihelion (if the speed was increased a great deal). There is also an "in-between" increase whereby the *perihelion and aphelion are the same – a circular orbit. We can compute the required increase in speed by comparing the speed at the aphelion of an eccentric orbit to the speed of a circular orbit whose radius is the same as the semi-major axis of the eccentric orbit. Starting with* Equation 7.2.11*, we have:*

$$v_{min}=(1-e)\,\frac{GMm}{L}=(1-e)\,\frac{GMm}{mv_{min}r_{max}}=\left(\frac{1-e}{1+e}\right)\frac{GM}{v_{min}\,a}\quad\Rightarrow\quad v_{min}=\sqrt{\left(\frac{1-e}{1+e}\right)\frac{GM}{a}}$$

For a circular orbit, the eccentricity is zero, which makes the ratio of the minimum velocity of the eccentric and circular orbits:

$$\frac{v_{circular}}{v_{min}}=\frac{\sqrt{\frac{GM}{R}}}{\sqrt{\left(\frac{1-e}{1+e}\right)\frac{GM}{a}}}=\sqrt{\left(\frac{1+e}{1-e}\right)\frac{a}{R}}$$

The new circular orbit must have a radius equal to the aphelion distance of the previous orbit, so:

$$\frac{v_{circular}}{v_{min}}=\sqrt{\left(\frac{1+e}{1-e}\right)\frac{a}{(1+e)\,a}}=\sqrt{\frac{1}{1-e}}$$

Now we need to know the eccentricity for an orbit where the aphelion distance is 4 times the perihelion distance. Writing these distances in terms of the semi-major axis gives:

$$\left.\begin{array}{l} r_{min}=(1-e)\,a \\ r_{max}=(1+e)\,a \end{array}\right\}\quad\Rightarrow\quad 4=\frac{r_{max}}{r_{min}}=\frac{1+e}{1-e}\quad\Rightarrow\quad e=\frac{3}{5}$$

Plugging this in above gives us that the velocity must be increased by a factor of $\sqrt{\frac{5}{2}}$ *at the aphelion of the eccentric orbit to turn it into a circular orbit. This corresponds to a percentage increase of:*

$$\%\ increase=\left(\sqrt{\frac{5}{2}}-1\right)\times 100\%=\boxed{58\%}$$

Suppose an orbiting body is bound by the gravitational attraction of a gravitating body that is a distance R away. If it is bound, it must be that it possesses insufficient kinetic energy such that when it is added to the (negative) gravitational potential the total energy makes it to zero. We define *escape velocity* as the minimum speed that an orbiting body must have in order to (barely) go infinitely far away from a gravitational source. Setting the total energy equal to zero, for the case of the object moving at escape velocity, we have:

$$\frac{1}{2}mv_{escape}^2+U(R)=0\quad\Rightarrow\quad v_{escape}=\sqrt{-\frac{2U(R)}{m}}=\sqrt{\frac{2GM}{R}}\tag{7.3.12}$$

Notice that whether or not an object can escape a gravitational attraction doesn't depend upon the would-be escaper's *mass*, but only its *velocity*. This makes for an interesting discussion when it comes to light. Light has no mass, so we would think that Newton's law of gravitation would indicate that it is unaffected by gravity. But when it comes to escape velocity, the mass does not factor in at all, so is the conclusion that light is unaffected by gravity incorrect?

It turns out that in fact light *is* affected by gravity (and specifically how this happens requires Einstein's improved theory of gravity), but what is more, we can compute whether light can escape a gravitating body. The speed of light is a well-defined constant, so the question of whether light can escape boils down to how close the origination of the light is to the gravitating body, and of course the mass of that body. The distance from which light will not escape is known as the *Schwarzschild radius*, and is found by plugging the speed of light (typically designated as c) into the escape velocity formula:

$$c=\sqrt{\frac{2GM}{R}}\quad\Rightarrow\quad R_S=\frac{2GM}{c^2}\tag{7.3.13}$$

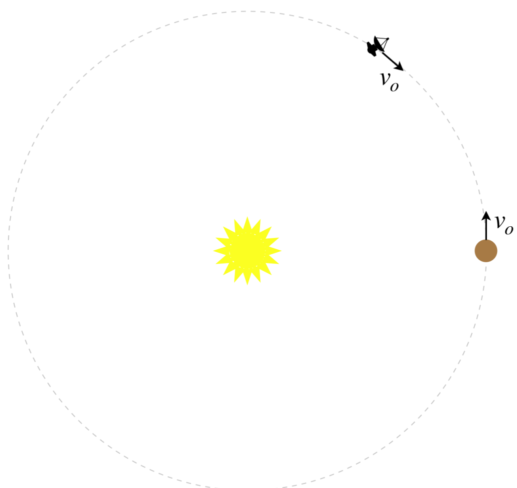
The Earth's mass is $5.97 \times 10^{24} kg$, and the speed of light is $3.0 \times 10^{8} \frac{m}{s}$. Its Schwarzschild radius therefore comes out to equal about 8.8 millimeters. Okay, that is quite small, and since the distance is measured from the center of the Earth, it is clear that we will not witness the phenomenon of light being trapped by the Earth's gravity. For light to be trapped by gravity, the gravitating body must fall *inside* the Schwarzschild radius. This requires that the gravitating body be incredibly dense, and since the only force available to pull the matter into that kind of density is the same gravity force, generally a very large amount of mass is required. Assuming this occurs, the object thus created is called a *black hole*. This is a nicely descriptive name, as "black" indicates that light does not escape its gravitational influence. Explaining how appropriate the word "hole" is in this name requires some knowledge of Einstein's theory, which is unfortunately beyond the scope of this work.

Gravitational Slingshots

Let's look at a simple model that demonstrates the process of a gravitational slingshot. This is the phenomenon that was exploited by NASA scientists to get the Voyager probes past to escape velocity for the Sun, despite not being launched with sufficient kinetic energy at the outset. It is also thought to be the likely candidate for so-called "rogue planets" that once orbited the sun, and are now moving through interstellar space, no longer held in orbit.

Start with three gravitating particles, two of which initially are in orbit around the third, in opposite directions, roughly in the same (circular) orbital path. One of these orbiting bodies (Jupiter) is much more massive than the other (Voyager), and of course the Sun is much more massive than both of these. At this point Voyager, being in a closed orbit, does not have sufficient energy to escape the Sun.

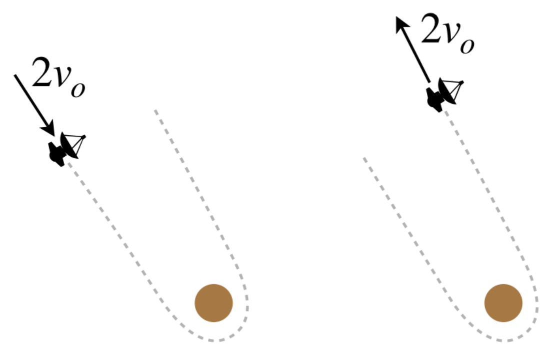
Figure 7.3.6 – Voyager Far from Jupiter (Sun's Perspective)



At this stage, voyager is quite far from Jupiter (say roughly the same distance as from the Sun), and since the mass of the Sun is so much greater than that of Jupiter (about 1000 times greater), Voyager's motion is, to a very good approximation, governed only by the gravitational force of the Sun. So it merrily continues along its orbital path, and since it is in roughly the same orbit as Jupiter, it is moving at the same speed, which in the diagram we have called "v_o."

When voyager comes significantly closer to Jupiter, then even though Jupiter has much less mass than the Sun, the much closer proximity to Jupiter makes it such that Voyager's motion is to a very good approximation governed entirely by Jupiter's gravitational force. From Jupiter's perspective, Voyager is coming toward it at a relative speed of $2v_o$ (before Jupiter's gravity starts to speed it up significantly).

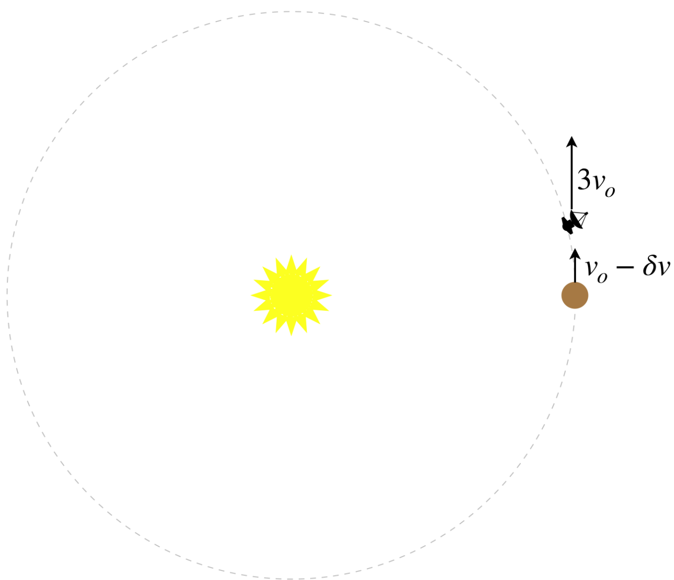
Figure 7.3.7 – Voyager Close to Jupiter (Jupiter's Perspective)



Naturally Voyager is not in *exactly* the same orbit as Jupiter, or it would crash into it. The slight offset results in Voyager slinging around the planet and coming out the opposite direction. Since Voyager essentially only "knows" about Jupiter's presence, (approximate) conservation of energy in this two-body system results in Voyager emerging with the same speed it had on its way in, but in the opposite direction (this is essentially the same as an elastic collision between two objects, where one has much, much less mass than the other).

So now Voyager is moving at the same speed as before relative to Jupiter, but in the opposite direction. As it gets farther away, Jupiter's gravitational force wanes, and attention again returns to the Sun. But now Voyager has *three times* the speed it had before relative to the Sun ($2v_o$ relative to Jupiter, plus Jupiter's v_o, for a total of $3v_o$), and it is the same distance away from the Sun as before.

Figure 7.3.8 – Voyager After Jupiter's "Kick" (Sun's Perspective)



Jupiter naturally loses a little of its speed (δv – a completely negligible amount, due to the vast difference in mass it has with Voyager) in this exchange. But Voyager's new speed is now more than enough to escape the Sun's gravitational pull. The overall energy of the system remains conserved, but it is redistributed so that voyager is no longer in a bound state. In the language of Chapter 3, we can say that the total force exerted on Voyager by the solar system (Sun + Jupiter system) is non-conservative, because when Voyager returns to its original position relative to this system, it has more kinetic energy than before, so the work done for that closed path was not zero. The internal energy of the solar system goes down (Jupiter's motion slows), and it goes to the kinetic energy of Voyager.

Exercise

Show that tripling the speed of any satellite in a circular orbit is enough to allow it to escape the gravitating body.

Solution

From Kepler's Third Law for a circular orbit (Equation 7.2.16), we can compute the speed of a satellite in a circular orbit of radius R. It is the circumference divided by the period of the orbit, so:

$$\frac{R^3}{T^2}=\frac{GM}{4\pi^2} \quad \Rightarrow \quad v=\frac{2\pi R}{T}=\sqrt{\frac{GM}{R}}$$

Comparing this to the formula for escape velocity (Equation 7.3.12), we see that escape velocity is only greater than the circular orbital speed by a factor of $\sqrt{2}$. So increasing the orbital speed by a factor of 3 is more than enough to get the job done.

CHAPTER OVERVIEW

8: Small Oscillations

8.1: Simple Harmonic Motion

8.2: Other Restoring Forces

Sample Problems

8.1: Simple Harmonic Motion

Equation of Motion for an Elastic Force

We have discussed the idea of a restoring force a few times already. If such a force counteracts displacements in both directions of one-dimensional motion, then it can cause the object to move back-and forth across the equilibrium point: An object subject to a restoring force is displaced from its equilibrium point and released. It accelerates toward the equilibrium point thanks to the restoring force. Upon arrival at the equilibrium point, it doesn't stop, because the restoring force is zero there. As it continues past, the restoring force acts to slow down and eventually stop the object, whereupon the object accelerates back toward the equilibrium point and the motion repeats in the opposite direction. This is called *oscillatory motion*, and it results from all two-way one-dimensional restoring forces.

The most common sort of restoring force we study is the elastic force. Indeed, other restoring forces occurring in nature (such as those between particles exhibiting a Lennard-Jones potential energy, as discussed at the end of Section 3.7) are often modeled as masses on springs. The oscillatory motion induced by the elastic restoring force is quite special, as we will see, and is called *simple harmonic motion*. We seek here the equation that relates the position of the mass as a function of time (with the equilibrium point being the origin), usually referred to as the *equation of motion* for this force.

Start (naturally) with Newton's second law, where the net force is simply that of a spring (Hooke's law). As we are working in one dimension, we once again have the luxury of treating our vector directions as simply (+) or (–):

$$F_{net}=-kx=ma \quad \Rightarrow \quad \frac{d^2x}{dt^2}+\frac{k}{m}x=0 \tag{8.1.1}$$

We seek to determine the function $x\left(t\right)$ that satisfies this differential equation. This is actually simpler than it might at first appear, if thought about in the following way: First, let's imagine that the ratio $\frac{k}{m}$ is just the number 1. Can we think of a function that after two derivatives becomes the negative of itself? We don't know a whole lot of special functions, but amazingly, there are actually a couple that do satisfy this: sine and cosine. The derivative of sine is cosine, and then a second derivative brings it back to negative sine.

There are lots of different features we can include with a sine (or cosine) function, so let's write one out in all its glory:

$$x\left(t\right)=A\sin(\omega t+\phi) \tag{8.1.2}$$

Two derivatives of this function gives:

$$\frac{d^2}{dt^2}\left[A\sin(\omega t+\phi)\right]=\frac{d}{dt}\left[\omega A\cos(\omega t+\phi)\right]=-\omega^2A\sin(\omega t+\phi) \tag{8.1.3}$$

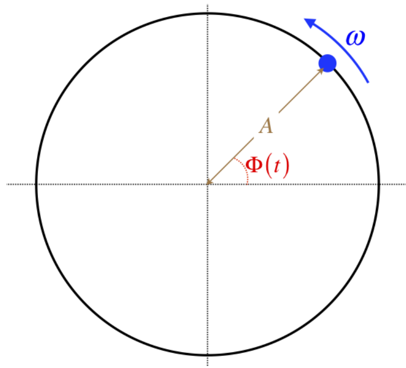
Plugging this into the differential equation gives a solution if we have:

$$\omega=\sqrt{\frac{k}{m}} \tag{8.1.4}$$

Total Phase

It may seem crazy at this point to introduce the greek letter ω as a constant here when we so recently used it as a measure of angular velocity, and there is no rotational motion going on here. But there is a good reason to do this. Consider a bead moving at a constant speed on a circular loop of wire of radius A.

Figure 8.1.1 – Bead Moving on a Circular Loop of Wire



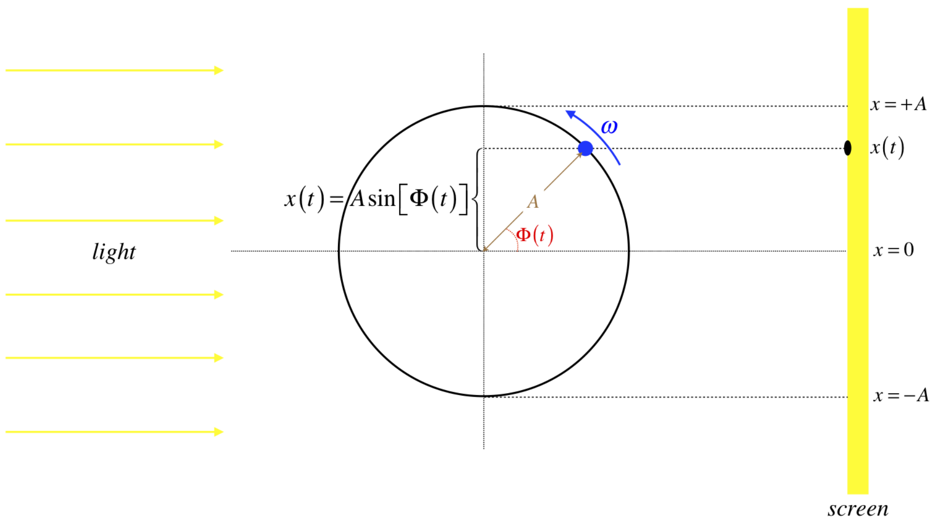
The equation describing its motion is something we are quite familiar with:

$$\Phi(t) = \omega t + \phi \tag{8.1.5}$$

We have changed notation a bit here from what we used previously for circular or rotational motion: $\Phi(t)$ represents the total angle traversed, in place of our previous $\theta(t)$, and ϕ represents the starting angle at $t = 0$, in place of our previous θ_o.

Next let's imagine placing a light source to the left of this loop, and a screen (the plane of which is perpendicular to the plane of the loop) to the right of the loop. The light would project a shadow of the bead onto the screen, and the motion of this shadow can be described mathematically:

<u>Figure 8.1.2 – Motion of a Projection of a Bead moving on a Circular Loop</u>



The motion of the shadow is simply a component of the motion of the bead – if we know the angle the bead makes on the circule, we know the height of the shadow on the screen. In terms of the starting angle and angular velocity of the bead, the motion of the shadow can be written explicitly as:

$$x(t) = A\sin(\omega t + \phi) \tag{8.1.6}$$

This is precisely the same as the equation of motion of a mass on a spring, Equation 8.1.2. That is, if we placed a mass on a spring at the screen with the equilibrium position at $x = 0$, pulled the spring to a maximum stretch (or pushed it to a maximum compression) of $x = A$, and then waited for the shadow to land on the mass before releasing it, the shadow would remain on the mass as it moves, *if* the angular velocity of the bead happens to equal $\sqrt{k/m}$.

We can now discard our bead-on-circular-loop model, but we keep the mathematical structure it leaves behind. The argument of the sine function $\Phi(t)$ is called the *total phase* of the harmonic motion of the mass-on-spring. The maximum expansion/compression of the spring A is called the *amplitude* of the harmonic motion. The constant ω is referred to as the *angular frequency* (not angular velocity – we've left the bead model behind), and still is expressed in units of radians per second. This constant sometimes gives way to a frequency f that is measured in cycles per second (or *hertz*), with the translation between the two being:

$$\omega = 2\pi f \tag{8.1.7}$$

The *period of oscillation* is the time it takes the system to come all the way back to where it started, and as the time per cycle, it is the inverse of the frequency:

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{8.1.8}$$

Alert

The period is the time required for the system to complete a full cycle, which is not the same as the time required for the mass to return to a previous position. The mass must return to the same position and it must be ***moving in the same direction****. In other words, the total phase Φ must change by 2π.*

And finally, the constant ϕ is called the *phase constant*, and it carries the information of where the mass is at time $t = 0$.

Kinematics of Harmonic Motion

Once we have a formula for the position of an object following simple harmonic motion, we can use the usual calculus tools to determine the velocity and acceleration at various times as well. The velocity as a function of time is:

$$v(t) = \frac{d}{dt}x(t) = \frac{d}{dt}[A\sin(\omega t + \phi)] = A\omega\cos(\omega t + \phi) \tag{8.1.9}$$

We note a couple of features of this result. First, since the cosine function never exceeds 1, we have the maximum speed of the object:

$$v_{max} = A\omega = A\sqrt{\frac{k}{m}} \tag{8.1.10}$$

And second, this maximum speed is achieved at $x = 0$ (the equilibrium point), which makes sense, since the spring was accelerating it toward that point, and immediately after passing it, the spring starts slowing it down.

The acceleration of the mass as a function of time we get from another derivative:

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}[A\omega\cos(\omega t + \phi)] = -A\omega^2\sin(\omega t + \phi) \tag{8.1.11}$$

The fact that this result is only different from the position function by a factor of $\left(-\frac{k}{m}\right)$ brings us back to what started all of this, Equation 8.1.1.

Mechanical Energy

We already know that the elastic force is conservative, so mechanical energy is conserved during simple harmonic motion. At any given time during the motion, the mass will have kinetic and potential energy, with its total energy remaining constant. It's easy to write an expression for the total energy in this system by choosing a convenient point in the motion – when the mass is stationary. This occurs when it reaches its maximum separation from the equilibrium point, i.e. when the displacement equals the amplitude:

$$E_{tot} = \frac{1}{2}kx_{max}^2 = \frac{1}{2}kA^2 \tag{8.1.12}$$

We can double-check this result by looking at the moment in time when there is zero potential energy and all of the mechanical energy is kinetic – at the equilibrium point. Using Equation 8.1.9, we get:

$$E_{tot} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}m\left(A\sqrt{\frac{k}{m}}\right)^2 = \frac{1}{2}kA^2 \tag{8.1.13}$$

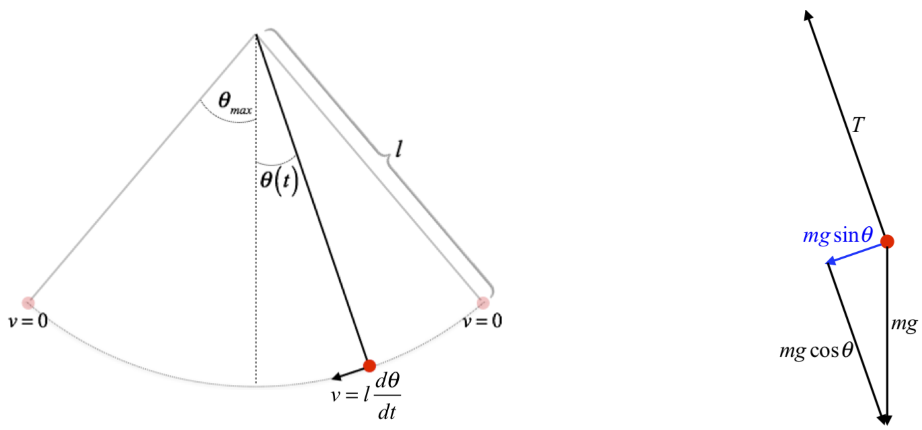
8.2: Other Restoring Forces

Pendulums

A mass on a spring is not the only physical system that exhibits simple harmonic motion. Another example is – at least to a good approximation for small amplitudes – *pendulums*. To say that a pendulum has a restoring force is imprecise – a pendulum is characterized by angular motion, and therefore it is affected by a restoring *torque*. The first type of pendulum we will consider is the *simple pendulum*. This is exactly as it sounds – it consists of a point mass under the influence of gravity at the end of a massless string which is attached to a fixed point.

It's clear that if one defines the motion of the simple pendulum in terms of angular position, the motion is oscillatory – gravity keeps producing a torque that seeks to restore vertical alignment. But is it simple harmonic motion? We need to do the analysis to figure it out. Figure 8.2.1 gives a diagram with lots of labeling, along with a free-body diagram.

Figure 8.2.1 – The Simple Pendulum



We wish to describe the motion of the pendulum, which means finding the function $\theta(t)$. We do this using Newton's second law, as we did with the mass-on-spring. We can use either the linear or the rotational form of Newton's second law – naturally both lead to the same result. Let's use the rotational version, as we will need to do later when the pendulum is not "simple." Choosing counterclockwise as the positive direction (so the pendulum to the right of the vertical is in the positive region), we see that the torque for the diagram above is negative – the restoring force has the opposite sign of the displacement, as it must.

$$\left.\begin{array}{l}\tau=-rF_\perp=-l\left(mg\sin\theta\right)\\ \tau=I\alpha=\left(ml^2\right)\alpha=\left(ml^2\right)\dfrac{d^2\theta}{dt^2}\end{array}\right\}\quad\Rightarrow\quad\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0 \tag{8.2.1}$$

This is not the same as the differential equation for the mass-on-spring, given in Equation 8.1.1. However, if we assume the pendulum exhibits a *small amplitude*, then the value of $\sin\theta$ is very close to the value of θ, when measured in radians. For example, for a 30^o angle, the sine is 0.5000, and in radians this angle is $\frac{\pi}{6}$ = 0.5236, for a deviation of less than 5%.

Alert

*If you are not comfortable with this $\sin\theta\approx\theta$ approximation, start getting used to it – it is used **all the time** in physics!*

Applying this approximation, we get the differential equation:

$$\frac{d^2\theta}{dt^2}+\frac{g}{l}\theta=0 \tag{8.2.2}$$

This does match the differential equation we found for the mass-on-spring, so the solution is the same (simple harmonic motion):

$$\theta(t)=\theta_{max}\sin(\omega t+\phi),\qquad where\;\;\omega=\sqrt{\frac{g}{l}} \tag{8.2.3}$$

Alert

*Note that the ω used in the argument of the sine function above is the angular frequency of the motion, and is a constant value. It is **not** the angular velocity of the pendulum $\frac{d\theta}{dt}$, which is constantly changing.*

All of the same things that we followed with for the mass-on-spring follows here, such as the velocity and acceleration functions and their maximum values, as well as the energy stored in the system. This last item bears some examination before we move on. The energy in the system is the gravitational potential energy stored at the maximum angle (measured relative to the bottom of the swing). Doing some geometry, we can get the height of the mass above the bottom of the swing, and from it the total energy:

$$y = l\left(1-\cos\theta_{max}\right) \quad \Rightarrow \quad E_{tot} = U_{max} = mgy = mgl\left(1-\cos\theta_{max}\right) \tag{8.2.4}$$

Okay, now we get to use the approximation of the cosine function for small angles:

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2 \tag{8.2.5}$$

Plugging this in above gives:

$$E_{tot} = \frac{1}{2}mgl\theta_{max}^2 \tag{8.2.6}$$

The reader will note that this bears a resemblance to Equation 8.1.12, which also indicates that the total energy in the system is proportional to the square of the amplitude.

The leap to other pendulums (those that are not "simple") is not a difficult one to take, if the restoring torque is also based on gravity (i.e. the pendulum swings). All this requires is replacing the point mass's rotational inertia of ml^2 with whatever the pendulum's rotational inertia around the fixed point happens to be, and computing the restoring torque based on wherever the center of mass happens to be. Calling the rotational inertia I and the distance of the center of mass from the fixed point d, and following the same procedure as above, we get for the differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{mgd}{I}\theta = 0 \tag{8.2.7}$$

The angular frequency is still the square root of the coefficient of θ, so the period of oscillation for this more general pendulum is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}} \tag{8.2.8}$$

Example 8.2.1

A thin, uniform rod with a length of $40cm$ is suspended vertically from one end, from which it is free to rotate without friction. If it is rotated a small angle from the vertical and released from rest, find how long it takes to reach a vertical orientation.

Solution

The rod is displaced a small angle, so we can treat it as a pendulum. When it starts from rest, it starts at its maximum angular displacement, and are asked to find the time it takes to get to its equilibrium point. This constitutes exactly one fourth of a cycle (it takes the same time to swing up to its maximum angular displacement from the equilibrium point), so all we need to compute is one fourth of a period. The rotational inertia is that of a rod about its end, and the center of mass is at the middle of the rod, so:

$$\frac{T}{4} = \frac{2\pi}{4}\sqrt{\frac{\frac{1}{3}ml^2}{mg\left(\frac{1}{2}l\right)}} = \pi\sqrt{\frac{l}{6g}} = \boxed{0.26s}$$

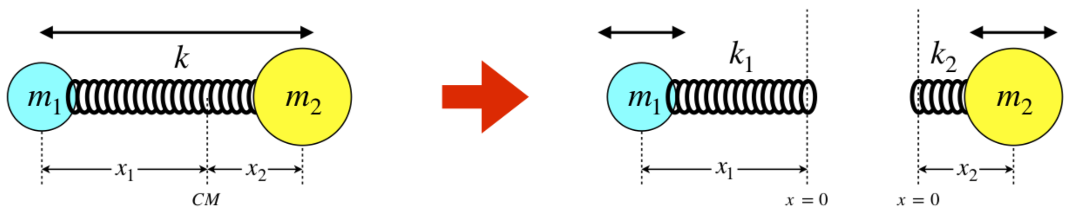
Potential Wells

At the end of Section 3.7, we looked at how we can model chemical bonds as springs. A program was outlined there which provided a way to derive an effective spring constant for any potential with a local minimum. It turns out that the "natural" vibration frequencies of these bonds are quite important when it comes to things like wavelengths of light that the material will absorb or emit, so being able to derive an effective spring constant from the potential function gives us a lot of information about how the material will behave.

There is one complication that arises with these bonds-as-springs models, however. The two molecules attached by the spring are *both* moving. How do we even fit this into our model where a single mass is oscillating through an equilibrium point? It turns out that there is a nice trick we can use for all two-body problems like this by considering the center of mass frame. [*Note: This trick is also used for gravitation when the orbiting body and gravitating body have comparable masses.*]

Consider a spring with masses at both ends. There is no net external force on the system, so as they vibrate, the center of mass remains at rest (we are assuming it started off at rest). We can therefore break this into two separate mass-on-spring systems, with the center of mass being a fixed point for each of them.

Figure 8.2.2 – Two Masses Connected By a Spring



There are a number of things we can say about this model. First, there is a relationship between the variables x_1 and x_2 and the amount each of these changes. They are both measured from the center of mass in the left diagram, and are both positive values in the right diagram, giving:

$$m_1x_1 = m_2x_2 \quad \Rightarrow \quad m_1\Delta x_1 = m_2\Delta x_2 \quad \Rightarrow \quad \Delta x_2 = \frac{m_1}{m_2}\Delta x_1 \tag{8.2.9}$$

The amount of force exerted on each mass by the spring is the same at every moment (Newton's third law), and the magnitude of this force is determined by the stretch (or compression) of the full spring according to Hooke's law. The stretch/compression of the full spring is equal to the sum of the stretches/compressions of the two springs in the separated diagram, so:

$$F = k\Delta x = k\left(\Delta x_1 + \Delta x_2\right) = k\Delta x_1\left(1+\frac{m_1}{m_2}\right) \tag{8.2.10}$$

But looking at this force from the perspective of just m_1 in the right diagram, the force exerted on it is due to its own spring and its displacement. Making this comparison gives us k_1 in terms of k:

$$F = k\Delta x_1\left(1+\frac{m_1}{m_2}\right) = k_1\Delta x_1 \quad \Rightarrow \quad k_1 = \left(\frac{m_1+m_2}{m_2}\right)k \tag{8.2.11}$$

The angular frequency of oscillation for m_1 is determined by its mass and the spring constant of the elastic force acting on it:

$$\omega = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k}{\mu}}, \qquad where: \;\; \mu \equiv \frac{m_1m_2}{m_1+m_2} \tag{8.2.12}$$

The motion of m_2 mirrors that of m_1, except with a different (smaller) amplitude. We know this because the two masses have to reach their maximum and minimum displacements at the same time to keep the center of mass stationary. So the angular frequency of oscillation for m_2 should come out to be the same as it is for m_1, and sure enough, it does (repeat all of the steps above with the subscripts 1 and 2 reversed).

The quantity μ has units of mass, and is commonly referred to as the *reduced mass* of the system. Its use is a common shortcut for reducing two-body problems to one-body problems. The angular frequency here takes on the usual form for a one-dimensional simple harmonic oscillator, and all that needs to be done is to calculate the reduced mass from the two masses involved and use the full spring constant (possibly computed from a potential function with a local minimum).

Sample Problems

All of the problems below have had their basic features discussed in an "Analyze This" box in this chapter. This means that the solutions provided here are incomplete, as they will refer back to the analysis performed for information (i.e. the full solution is essentially split between the analysis earlier and details here). If you have not yet spent time working on (not simply reading!) the analysis of these situations, these sample problems will be of little benefit to your studies.

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Problem 8.3

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