4.4: Examples

Whiteboard Problem 4.4.1

Two blocks, of mass m and $3m$, are compressed on either side of a spring and tied together with a rope. They are sitting at rest on a frictionless surface, as shown in the figure.

1. The rope breaks and the larger block flies away from the smaller one at a speed of 2.00 m/s. If m=0.450 kg, what is the speed of the smaller block?
2. Now retie the rope and perform the experiment again, but slide the blocks along the surface with an initial speed of 3.5 m/s to the right. The rope breaks and the spring acts in the same way on the blocks. What is the speed of the smaller block?
3. For both parts (a) and (b): What is the speed of the center of mass of the system both before and after the rope breaks?

Example 4.4.2: Center of Mass of the Earth-Moon System

Using data from text appendix, determine how far the center of mass of the Earth-moon system is from the center of Earth. Compare this distance to the radius of Earth, and comment on the result. Ignore the other objects in the solar system.

Strategy

We get the masses and separation distance of the Earth and moon, impose a coordinate system, and use Equation 4.3.1 with just N = 2 objects. We use a subscript “e” to refer to Earth, and subscript “m” to refer to the moon.

Solution

Define the origin of the coordinate system as the center of Earth. Then, with just two objects, Equation 4.3.1 becomes

$$R=\frac{m_c r_c+m_m r_m}{m_c+m_m}. \tag{4.4.1}$$

We can find the values of the distances and masses from the Internet,

$$m_c=5.97\times 10^{24}\ kg \tag{4.4.2}$$

$$m_m=7.36\times 10^{22}\ kg \tag{4.4.3}$$

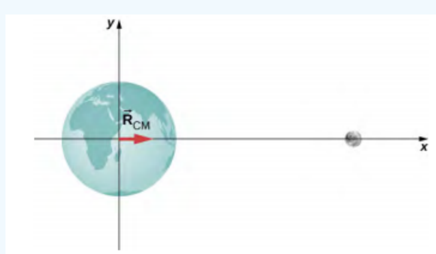
$$r_m=3.82\times 10^{5}\ m. \tag{4.4.4}$$

We defined the center of Earth as the origin, so r_e = 0 m. Inserting these into the equation for R gives

$$\begin{aligned} R &= \frac{(5.97\times 10^{24}\ kg)(0\ m)+(7.36\times 10^{22}\ kg)(3.82\times 10^{8}\ m)}{(5.98\times 10^{24}\ kg)+(7.36\times 10^{22}\ kg)} \\ &= 4.64\times 10^{6}\ m. \end{aligned}$$

Significance

The radius of Earth is 6.37 x 10^6 m, so the center of mass of the Earth-moon system is (6.37 − 4.64) x 10^6 m = 1.73 x 10^6 m = 1730 km (roughly 1080 miles) **below** the surface of Earth. The location of the center of mass is shown (not to scale).



Exercise 4.4.3

Suppose we included the sun in the system. Approximately where would the center of mass of the Earth-moon-sun system be located? (Feel free to actually calculate it.)

Example 4.4.4: Center of Mass of a Salt Crystal

Figure 4.4.3 shows a single crystal of sodium chloride—ordinary table salt. The sodium and chloride ions form a single unit, NaCl. When multiple NaCl units group together, they form a cubic lattice. The smallest possible cube (called the unit cell) consists of four sodium ions and four chloride ions, alternating. The length of one edge of this cube (i.e., the bond length) is 2.36×10^{-10} m. Find the location of the center of mass of the unit cell. Specify it either by its coordinates ($r_{CM,x}$, $r_{CM,y}$, $r_{CM,z}$), or by r_{CM} and two angles.

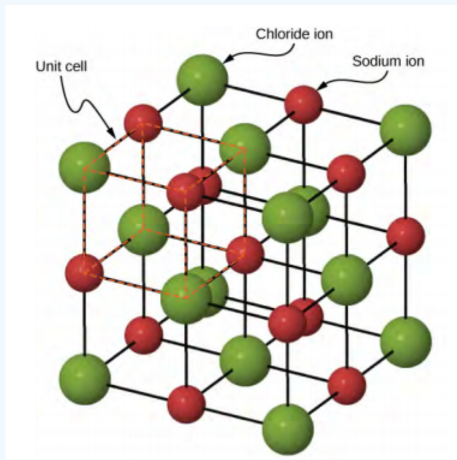


Figure 4.4.3: A drawing of a sodium chloride (NaCl) crystal.

Strategy

We can look up all the ion masses. If we impose a coordinate system on the unit cell, this will give us the positions of the ions. We can then apply Equation 4.3.1 in each direction (along with the Pythagorean theorem).

Solution

Define the origin to be at the location of the chloride ion at the bottom left of the unit cell. Figure 4.4.4 shows the coordinate system.

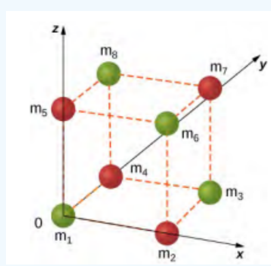


Figure 4.4.4: A single unit cell of a NaCl crystal.

There are eight ions in this crystal, so N = 8:

$$\vec{r}_{CM} = \frac{1}{M} \sum_{j=1}^{8} m_j \vec{r}_j. \tag{4.4.5}$$

The mass of each of the chloride ions is

$$35.453u \times \frac{1.660 \times 10^{-27}\ kg}{u} = 5.885 \times 10^{-26}\ kg \tag{4.4.6}$$

so we have

$$m_1 = m_3 = m_6 = m_8 = 5.885 \times 10^{-26}\ kg. \tag{4.4.7}$$

For the sodium ions,

$$m_2 = m_4 = m_5 = m_7 = 3.816 \times 10^{-26}\ kg. \tag{4.4.8}$$

The total mass of the unit cell is therefore

$$M = (4)(5.885 \times 10^{-26}\ kg) + (4)(3.816 \times 10^{-26}\ kg) = 3.880 \times 10^{-25}\ kg. \tag{4.4.9}$$

From the geometry, the locations are

$$\begin{aligned}
\vec{r}_1 &= 0 \\
\vec{r}_2 &= (2.36 \times 10^{-10}\ m)\hat{i} \\
\vec{r}_3 &= r_{3x}\hat{i} + r_{3y}\hat{j} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{j} \\
\vec{r}_4 &= (2.36 \times 10^{-10}\ m)\hat{j} \\
\vec{r}_5 &= (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_6 &= r_{6x}\hat{i} + r_{6z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_7 &= r_{7x}\hat{i} + r_{7y}\hat{j} + r_{7z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{j} + (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_8 &= r_{8y}\hat{j} + r_{8z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{j} + (2.36 \times 10^{-10}\ m)\hat{k}.
\end{aligned}$$

Substituting:

$$\begin{aligned}
|\vec{r}_{CM,x}| &= \sqrt{r^2_{CM,x} + r^2_{CM,y} + r^2_{CM,z}} \\
&= \frac{1}{M}\sum_{j=1}^{8} m_j (r_x)_j \\
&= \frac{1}{M}(m_1 r_{1x} + m_2 r_{2x} + m_3 r_{3x} + m_4 r_{4x} + m_5 r_{5x} + m_6 r_{6x} + m_7 r_{7x} + m_8 r_{8x}) \\
&= \frac{1}{3.8804 \times 10^{-25}\ kg}\Big[(5.885 \times 10^{-26}\ kg)(0\ m) + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) \\
&\quad + (5.885 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + 0 + 0 \\
&\quad + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + 0\Big] \\
&= 1.18 \times 10^{-10}\ m.
\end{aligned}$$

Similar calculations give $r_{CM,y} = r_{CM,z} = 1.18 \times 10^{-10}$ m (you could argue that this must be true, by symmetry, but it's a good idea to check).

Significance

As it turns out, it was not really necessary to convert the mass from atomic mass units (u) to kilograms, since the units divide out when calculating r_{CM} anyway.

To express r_{CM} in terms of magnitude and direction, first apply the three-dimensional Pythagorean theorem to the vector components:

$$\begin{aligned}
r_{CM} &= \sqrt{r^2_{CM,x} + r^2_{CM,y} + r^2_{CM,z}} \\
&= (1.18 \times 10^{-10}\ m)\sqrt{3} \\
&= 2.044 \times 10^{-10}\ m.
\end{aligned}$$

Since this is a three-dimensional problem, it takes two angles to specify the direction of $\vec{r}_{CM}$. Let ϕ be the angle in the x,y-plane, measured from the +x-axis, counterclockwise as viewed from above; then:

$$\phi = \tan^{-1}\left(\frac{r_{CM,y}}{r_{CM,x}}\right) = 45^o. \tag{4.4.10}$$

Let θ be the angle in the y,z-plane, measured downward from the +z-axis; this is (not surprisingly):

$$\theta = \tan^{-1}\left(\frac{R_z}{R_y}\right) = 45^o. \tag{4.4.11}$$

Thus, the center of mass is at the geometric center of the unit cell. Again, you could argue this on the basis of symmetry

? Exercise 4.4.5

Suppose you have a macroscopic salt crystal (that is, a crystal that is large enough to be visible with your unaided eye). It is made up of a **huge** number of unit cells. Is the center of mass of this crystal necessarily at the geometric center of the crystal?

Two crucial concepts come out of these examples:

1. 1. As with all problems, you must define your coordinate system and origin. For center-of-mass calculations, it often makes sense to choose your origin to be located at one of the masses of your system. That choice automatically defines its distance in Equation 4.3.1 to be zero. However, you must still include the mass of the object at your origin in your calculation of M, the total mass Equation 4.3.1. In the Earth-moon system example, this means including the mass of Earth. If you hadn't, you'd have ended up with the center of mass of the system being at the center of the moon, which is clearly wrong.
2. In the second example (the salt crystal), notice that there is no mass at all at the location of the center of mass. This is an example of what we stated above, that there does not have to be any actual mass at the center of mass of an object.
