3.1: Coordinate Systems and Components of a Vector (Part 1)

Vectors are usually described in terms of their components in a coordinate system. Even in everyday life we naturally invoke the concept of orthogonal projections in a rectangular coordinate system. For example, if you ask someone for directions to a particular location, you will more likely be told to go 40 km east and 30 km north than 50 km in the direction 37° north of east.

In a rectangular (Cartesian) xy-coordinate system in a plane, a point in a plane is described by a pair of coordinates (x, y). In a similar fashion, a vector $\vec{A}$ in a plane is described by a pair of its vector coordinates. The x-coordinate of vector $\vec{A}$ is called its x-component and the y-coordinate of vector $\vec{A}$ is called its y-component. The vector x-component is a vector denoted by $\vec{A}_x$. The vector y-component is a vector denoted by $\vec{A}_y$. In the Cartesian system, the x and y **vector components** of a vector are the orthogonal projections of this vector onto the x- and y-axes, respectively. In this way, following the parallelogram rule for vector addition, each vector on a Cartesian plane can be expressed as the vector sum of its vector components:

$$\vec{A} = \vec{A}_x + \vec{A}_y. \tag{3.1.1}$$

As illustrated in Figure $3.1.1$, vector $\vec{A}$ is the diagonal of the rectangle where the x-component $\vec{A}_x$ is the side parallel to the x-axis and the y-component $\vec{A}_y$ is the side parallel to the y-axis. Vector component $\vec{A}_x$ is orthogonal to vector component $\vec{A}_y$.

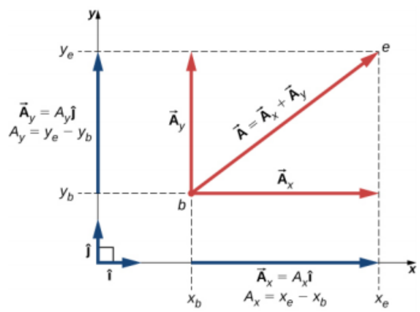

Figure $3.1.1$: Vector $\vec{A}$ in a plane in the Cartesian coordinate system is the vector sum of its vector x- and y-components. The x-vector component $\vec{A}_x$ is the orthogonal projection of vector $\vec{A}$ onto the x-axis. The y-vector component $\vec{A}_y$ is the orthogonal projection of vector $\vec{A}$ onto the y-axis. The numbers A_x and A_y that multiply the unit vectors are the scalar components of the vector.

It is customary to denote the positive direction on the x-axis by the unit vector $\hat{i}$ and the positive direction on the y-axis by the unit vector $\hat{j}$. Unit vectors of the axes, $\hat{i}$ and $\hat{j}$, define two orthogonal directions in the plane. As shown in Figure $3.1.1$, the x- and y-components of a vector can now be written in terms of the unit vectors of the axes:

$$\begin{cases} \vec{A}_x = A_x\hat{i} \\ \vec{A}_y = A_y\hat{j} \end{cases} \tag{3.1.2}$$

The vectors $\vec{A}_x$ and $\vec{A}_y$ defined by 3.1.2 are the vector components of vector $\vec{A}$. The numbers A_x and A_y that define the vector components in Equation 3.1.2 are the **scalar components** of vector $\vec{A}$. Combining Equation 3.1.1 with Equation 3.1.2, we obtain **the component form of a vector**:

$$\vec{A} = A_x\hat{i} + A_y\hat{j}. \tag{3.1.3}$$

If we know the coordinates $b(x_b, y_b)$ of the origin point of a vector (where b stands for "beginning") and the coordinates e(x_e, y_e) of the end point of a vector (where e stands for "end"), we can obtain the scalar components of a vector simply by subtracting the origin point coordinates from the end point coordinates:

$$\begin{cases} A_x = x_e - x_b \\ A_y = y_e - y_b. \end{cases} \tag{3.1.4}$$

Example 3.1.1: Displacement of a Mouse Pointer

A mouse pointer on the display monitor of a computer at its initial position is at point (6.0 cm, 1.6 cm) with respect to the lower left-side corner. If you move the pointer to an icon located at point (2.0 cm, 4.5 cm), what is the displacement vector of the pointer?

Strategy

The origin of the xy-coordinate system is the lower left-side corner of the computer monitor. Therefore, the unit vector $\hat{i}$ on the x-axis points horizontally to the right and the unit vector $\hat{j}$ on the y-axis points vertically upward. The origin of the displacement vector is located at point b(6.0, 1.6) and the end of the displacement vector is located at point e(2.0, 4.5). Substitute the coordinates of these points into Equation 3.1.4 to find the scalar components D_x and D_y of the displacement vector $\vec{D}$. Finally, substitute the coordinates into Equation 3.1.3 to write the displacement vector in the vector component form.

Solution

We identify $x_b = 6.0$, $x_e = 2.0$, $y_b = 1.6$, and $y_e = 4.5$, where the physical unit is 1 cm. The scalar x- and y-components of the displacement vector are

$$D_x = x_e - x_b = (2.0 - 6.0)\ cm = -4.0\ cm, \tag{3.1.5}$$

$$D_y = y_e - y_b = (4.5 - 1.6)\ cm = +2.9\ cm. \tag{3.1.6}$$

The vector component form of the displacement vector is

$$\vec{D} = D_x\ \hat{i} + D_y\ \hat{j} = (-4.0\ cm)\ \hat{i} + (2.9\ cm)\ \hat{j} = (-4.0\ \hat{i} + 2.9\ \hat{j})\ cm. \tag{3.1.7}$$

This solution is shown in Figure 3.1.2.

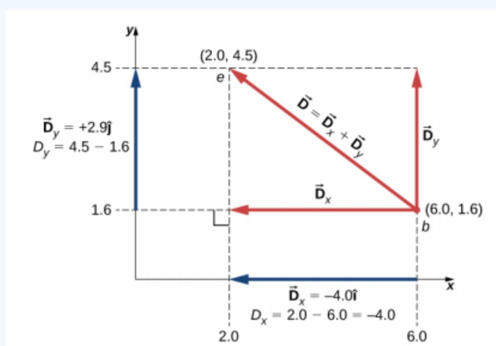

Figure 3.1.2: The graph of the displacement vector. The vector points from the origin point at b to the end point at e.

Significance

Notice that the physical unit—here, 1 cm—can be placed either with each component immediately before the unit vector or globally for both components, as in Equation 3.1.7. Often, the latter way is more convenient because it is simpler.

The vector x-component $\vec{D}_x = -4.0\ \hat{i} = 4.0(-\hat{i})$ of the displacement vector has the magnitude $|\vec{D}_x| = |-4.0||\hat{i}| = 4.0$ because the magnitude of the unit vector is $|\hat{i}| = 1$. Notice, too, that the direction of the x-component is $-\hat{i}$, which is antiparallel to the direction of the +x-axis; hence, the x-component vector $\vec{D}_x$ points to the left, as shown in Figure 3.1.2. The scalar x-component of vector $\vec{D}$ is $D_x = -4.0$. Similarly, the vector y-component $\vec{D}_y = +2.9\hat{j}$ of the displacement vector has magnitude $|\vec{D}_y| = |2.9||\hat{j}| = 2.9$ because the magnitude of the unit vector is $|\hat{j}| = 1$. The direction of the y-component is $+\hat{j}$, which is parallel to the direction of the +y-axis. Therefore, the y-component vector $\vec{D}_y$ points up, as seen in Figure 3.1.2. The scalar y-component of vector $\vec{D}$ is $D_y = +2.9$. The displacement vector $\vec{D}$ is the resultant of its two vector components.

The vector component form of the displacement vector Equation 3.1.7 tells us that the mouse pointer has been moved on the monitor 4.0 cm to the left and 2.9 cm upward from its initial position.

Exercise 3.1.2

A blue fly lands on a sheet of graph paper at a point located 10.0 cm to the right of its left edge and 8.0 cm above its bottom edge and walks slowly to a point located 5.0 cm from the left edge and 5.0 cm from the bottom edge. Choose the rectangular coordinate system with the origin at the lower left-side corner of the paper and find the displacement vector of the fly. Illustrate your solution by graphing.

When we know the scalar components A_x and A_y of a vector $\vec{A}$, we can find its magnitude A and its direction angle θ_A. The **direction angle**—or direction, for short—is the angle the vector forms with the positive direction on the x-axis. The angle θ_A is measured in the counterclockwise direction from the +x-axis to the vector (Figure 3.1.3). Because the lengths A, A_x, and A_y form a right triangle, they are related by the Pythagorean theorem:

$$A^2 = A_x^2 + A_y^2 \Leftrightarrow A = \sqrt{A_x^2 + A_y^2}. \tag{3.1.8}$$

This equation works even if the scalar components of a vector are negative. The direction angle θ_A of a vector is defined via the tangent function of angle θ_A in the triangle shown in Figure 3.1.3:

$$\tan\theta = \frac{A_y}{A_x} \Rightarrow \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right). \tag{3.1.9}$$

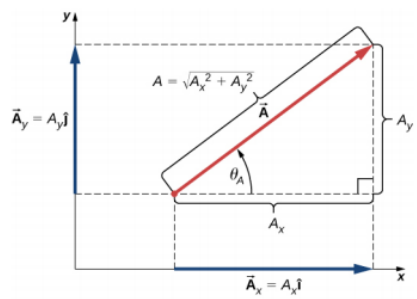

Figure 3.1.3: For vector $\vec{A}$, its magnitude A and its direction angle θ_A are related to the magnitudes of its scalar components because A, A_x, and A_y form a right triangle.

When the vector lies either in the first quadrant or in the fourth quadrant, where component A_x is positive (Figure 3.1.4), the angle θ in Equation 3.1.9) is identical to the direction angle θ_A. For vectors in the fourth quadrant, angle θ is negative, which means that for these vectors, direction angle θ_A is measured clockwise from the positive x-axis. Similarly, for vectors in the second quadrant, angle θ is negative. When the vector lies in either the second or third quadrant, where component A_x is negative, the direction angle is $\theta_A = \theta + 180°$ (Figure 3.1.4).

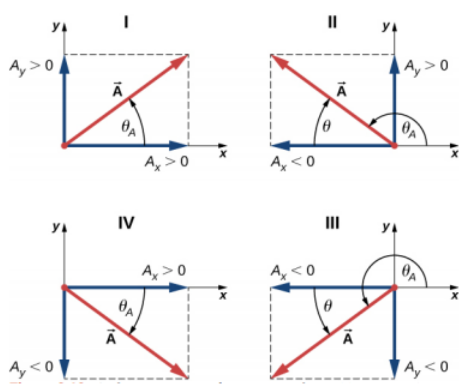

Figure 3.1.4: Scalar components of a vector may be positive or negative. Vectors in the first quadrant (I) have both scalar components positive and vectors in the third quadrant have both scalar components negative. For vectors in quadrants II and III, the direction angle of a vector is $\theta_A = \theta + 180°$.

Example 3.1.3: Magnitude and Direction of the Displacement Vector

You move a mouse pointer on the display monitor from its initial position at point (6.0 cm, 1.6 cm) to an icon located at point (2.0 cm, 4.5 cm). What is the magnitude and direction of the displacement vector of the pointer?

Strategy

In Example 3.1.1, we found the displacement vector $\vec{D}$ of the mouse pointer (see Equation 3.1.7). We identify its scalar components D_x = −4.0 cm and D_y = + 2.9 cm and substitute into Equation 3.1.8 and Equation 3.1.9 to find the magnitude D and direction θ_D, respectively.

Solution

The magnitude of vector $\vec{D}$ is

$$D=\sqrt{D_x^2+D_y^2}=\sqrt{(-4.0\ cm)^2+(2.9\ cm)^2}=\sqrt{(4.0)^2+(2.9)^2}\ cm=4.9\ cm. \tag{3.1.10}$$

The direction angle is

$$\tan\theta=\frac{D_y}{D_x}=\frac{+2.9\ cm}{-4.0\ cm}=-0.725 \Rightarrow \theta=\tan^{-1}(-0.725)=-35.9^o. \tag{3.1.11}$$

Vector $\vec{D}$ lies in the second quadrant, so its direction angle is

$$\theta_D=\theta+180^o=-35.9^o+180^o=144.1^o. \tag{3.1.12}$$

Exercise 3.1.4

If the displacement vector of a blue fly walking on a sheet of graph paper is $\vec{D}=(-5.00\ \hat{i}-3.00\ \hat{j})$ cm, find its magnitude and direction.

In many applications, the magnitudes and directions of vector quantities are known and we need to find the resultant of many vectors. For example, imagine 400 cars moving on the Golden Gate Bridge in San Francisco in a strong wind. Each car gives the bridge a different push in various directions and we would like to know how big the resultant push can possibly be. We have already gained some experience with the geometric construction of vector sums, so we know the task of finding the resultant by drawing the vectors and measuring their lengths and angles may become intractable pretty quickly, leading to huge errors. Worries like this do not appear when we use analytical methods. The very first step in an analytical approach is to find vector components when the direction and magnitude of a vector are known.

Let us return to the right triangle in Figure 3.1.3. The quotient of the adjacent side A_x to the hypotenuse A is the cosine function of direction angle θ_A, A_x/A = cos θ_A, and the quotient of the opposite side A_y to the hypotenuse A is the sine function of θ_A, A_y/A = sin θ_A. When magnitude A and direction θ_A are known, we can solve these relations for the scalar components:

$$\begin{cases} A_x=A\cos\theta_A \\ A_y=A\sin\theta_A. \end{cases} \tag{3.1.13}$$

When calculating vector components with Equation 3.1.13, care must be taken with the angle. The direction angle θ_A of a vector is the angle measured **counterclockwise** from the positive direction on the x-axis to the vector. The clockwise measurement gives a negative angle.

Example 3.1.5: Components of Displacement Vectors

A rescue party for a missing child follows a search dog named Trooper. Trooper wanders a lot and makes many trial sniffs along many different paths. Trooper eventually finds the child and the story has a happy ending, but his displacements on various legs seem to be truly convoluted. On one of the legs he walks 200.0 m southeast, then he runs north some 300.0 m. On the third leg, he examines the scents carefully for 50.0 m in the direction 30° west of north. On the fourth leg, Trooper goes directly south for 80.0 m, picks up a fresh scent and turns 23° west of south for 150.0 m. Find the scalar components of Trooper’s displacement vectors and his displacement vectors in vector component form for each leg.

Strategy

Let's adopt a rectangular coordinate system with the positive x-axis in the direction of geographic east, with the positive y-direction pointed to geographic north. Explicitly, the unit vector $\hat{i}$ of the x-axis points east and the unit vector $\hat{j}$ of the y-axis points north. Trooper makes five legs, so there are five displacement vectors. We start by identifying their magnitudes and direction angles, then we use Equation 3.1.13 to find the scalar components of the displacements and Equation 3.1.3 for the displacement vectors.

Solution

On the first leg, the displacement magnitude is L_1 = 200.0 m and the direction is southeast. For direction angle θ_1 we can take either 45° measured clockwise from the east direction or 45° + 270° measured counterclockwise from the east direction. With the first choice, θ_1 = −45°. With the second choice, θ_1 = + 315°. We can use either one of these two angles. The components are

$$L_{1x} = L_1 \cos\theta_1 = (200.0\ m)\cos 315^o = 141.4\ m, \tag{3.1.14}$$

$$L_{1y} = L_1 \sin\theta_1 = (200.0\ m)\sin 315^o = -141.4\ m, \tag{3.1.15}$$

The displacement vector of the first leg is

$$\vec{L}_1 = L_{1x}\ \hat{i} + L_{1y}\ \hat{j} = (14.4\ \hat{i} - 141.4\ \hat{j})\ m. \tag{3.1.16}$$

On the second leg of Trooper's wanderings, the magnitude of the displacement is L_2 = 300.0 m and the direction is north. The direction angle is θ_2 = + 90°. We obtain the following results:

$$L_{2x} = L_2 \cos\theta_2 = (300.0\ m)\cos 90^o = 0.0, \tag{3.1.17}$$

$$L_{2y} = L_2 \sin\theta_2 = (300.0\ m)\sin 90^o = 300.0\ m, \tag{3.1.18}$$

$$\vec{L}_2 = L_{2x}\ \hat{i} + L_{2y}\ \hat{j} = (300.0\ m)\ \hat{j}. \tag{3.1.19}$$

On the third leg, the displacement magnitude is L_3 = 50.0 m and the direction is 30° west of north. The direction angle measured counterclockwise from the eastern direction is θ_3 = 30° + 90° = + 120°. This gives the following answers:

$$L_{3x} = L_3 \cos\theta_3 = (50.0\ m)\cos 120^o = -25.0\ m, \tag{3.1.20}$$

$$L_{3y} = L_3 \sin\theta_3 = (50.0\ m)\sin 120^o = +43.3\ m, \tag{3.1.21}$$

$$\vec{L}_3 = L_{3x}\ \hat{i} + L_{3y}\ \hat{j} = (-25.0\ \hat{i} + 43.3\ \hat{j})\ m. \tag{3.1.22}$$

On the fourth leg of the excursion, the displacement magnitude is L_4 = 80.0 m and the direction is south. The direction angle can be taken as either θ_4 = −90° or θ_4 = + 270°. We obtain

$$L_{4x} = L_4 \cos\theta_4 = (80.0\ m)\cos(-90^o) = 0, \tag{3.1.23}$$

$$L_{4y} = L_4 \sin\theta_4 = (80.0\ m)\sin(-90^o) = -80.0\ m, \tag{3.1.24}$$

$$\vec{L}_4 = L_{4x}\ \hat{i} + L_{4y}\ \hat{j} = (-80.0\ m)\ \hat{j}. \tag{3.1.25}$$

On the last leg, the magnitude is L_5 = 150.0 m and the angle is θ_5 = −23° + 270° = + 247° (23° west of south), which gives

$$L_{5x} = L_5 \cos\theta_5 = (150.0\ m)\cos 247^o = -58.6\ m, \tag{3.1.26}$$

$$L_{5y} = L_5 \sin\theta_5 = (150.0\ m)\sin 247^o = -138.1\ m, \tag{3.1.27}$$

$$\vec{L}_5 = L_{5x}\ \hat{i} + L_{5y}\ \hat{j} = (-58.6\ \hat{i} - 138.1\ \hat{j})\ m. \tag{3.1.28}$$

Exercise 3.1.6

If Trooper runs 20 m west before taking a rest, what is his displacement vector?

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