8.3: Universal Gravity

Up to this point, all we have said about gravity is that, near the surface of the Earth, the gravitational force exerted by the Earth on an object of mass m is $F^G = mg$, which corresponds to a potential energy of interaction $U_g = mgh$. This is, indeed, a pretty good approximation, but it does not really tell you anything about what the gravitational force is where other objects or distances are involved.

The first comprehensive theory of gravity, formulated by Isaac Newton in the late 17th century, postulates that any two "particles" with masses m_1 and m_2 will exert an attractive force (a "pull") on each other, whose magnitude is proportional to the product of the masses, and inversely proportional to the square of the distance between them. Mathematically, we write

$$F_{12}^G = \frac{Gm_1 m_2}{r_{12}^2}. \tag{8.3.1}$$

Here, r_{12} is just the magnitude of the position vector of particle 2 relative to particle 1 (so r_{12} is, indeed, the distance between the two particles), and G is a constant, known as "Newton's constant" or the *gravitational constant*, which at the time of Newton still had not been determined experimentally. You can see from Equation (8.3.1) that G is simply the magnitude, in newtons, of the attractive force between two 1-kg masses a distance of 1 m apart. This turns out to have the ridiculously small value G = 6.674×10^{-11} N m^2/kg^2 (or, as is more commonly written, $m^3kg^{-1}s^{-2}$). It was first measured by Henry Cavendish in 1798, in what was, without a doubt, an experimental tour de force for that time. As you can see, gravity as a force between any two ordinary objects is absolutely insignificant, and it takes the mass of a planet to make it into something you can feel.

There is a potential energy corresponding to this force, *a la* mgh:

$$U_G = -\frac{Gm_1 m_2}{r_{12}} \tag{8.3.2}$$

The variables are the same as in the force equation (notice the *negative derivative of the potential gives you the force* - that's a universal rule, which we will explore later). These two equations together constitute what is often called Universal Gravity, because it's the law of gravity that applies to any two objects in the universe. This should be contrasted with the more familiar $F_g = mg$ and $U_g = mgh$, which only apply near the surface of the Earth.

Since U_G should apply everywhere, it should be true that we can derive U_g from U_G, so let's show how that is done. The situation we want to understand is the gravitational interaction near the Earth - in fact, very near the Earth, so that we can write the height of the object from the surface h is much smaller than the radius of the Earth, $h << R_E$. Setting the two masses in equation (8.3.2) to the Earth M_E and the mass of the object in question m, we can write

$$U_G = -\frac{GM_E m}{R_E + h} = -\frac{GM_E m}{R_E(1 + h/R_E)} \tag{8.3.3}$$

Notice in going to either side of the equation sign, I have simply pulled out R_E from the expression in the denominator, those two expressions are exactly equal. But now we have that quantity h/R_E, and we've already said $h << R_E$, which means h/R_E is a very small value. Recall from your calculus class that for a small value x, you can write

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \ldots \tag{8.3.4}$$

(This is called a Taylor Expansion, and you can check our work on Wolfram Alpha.) We have to be careful to write this expression as an approximation now, since it's not exact, but it should be a good approximation for things that actually are close to the surface of the Earth - perhaps as high as Everest, for example.

Anyway, let's truncate the series after two terms (keep more if you want it to be more exact!) and write

$$U_G \approx -\frac{GM_E m}{R_E}\left(1 - \frac{h}{R_E}\right) = -\frac{GM_E m}{R_E} + \frac{GM_E m}{R_E^2} h. \tag{8.3.5}$$

Notice in this last expression, the first term $-GM_E m/R_E$ is a particular constant which stays the same for a particular planet (Earth) and object of mass m - let's call that C. The second term is similar, but depends linearly on the height h above the surface of the Earth. In fact, if we define a new constant $g = GM_E/R_E^2$, we can write that second term as mgh, and we have found the relationship between the two different kinds of gravitational potential energy:

$$U_G \approx C + U_g. \tag{8.3.6}$$

That's quite satisfying, but what is going on with the constant? To understand that, we can go back to something we discussed last chapter - for closed systems that do not dissipate energy, we can write $\Delta K = -\Delta U$. Notice here that's it's just the *change in energy that matters, and not the total value*. For example, if I wanted to find the change in Universal Gravitational energy I could write

$$\Delta U_G = U_{G,f} - U_{G,i} \approx (C + U_{g,f}) - (C + U_{g,i}) = U_{g,f} - U_{g,i} = \Delta U_g. \tag{8.3.7}$$

In other words, if we consider the change in energy, the constant doesn't matter - in fact, we usually just set this constant to be 0 in all of our problems. The conceptual understanding of that is essentially that it does not matter where we set our $h = 0$ point to be - if we drop an object 10 m, we have to get the same answer for the final speed if we consider the $h = 0$ point to be either at the bottom of the motion or the top.

It's important to keep in mind this approximation technique - it's a very powerful tool that is used all over the place in the physics. Sometimes it's to make our work easier, while having essentially no impact on the outcome of calculations, like in this case. In other cases, the exact answer is so difficult to calculate that we are forced to make approximations to get an answer at all. One example of this which is just beyond what we will study is air drag, another example which is far beyond what we will study is the use of Feynman diagrams in quantum field theory.