16.4: Examples

? Whiteboard Problem 16.4.1: The Range Formula

The range of a projectile which starts and ends at the same altitude can be found with

$$R = \frac{2v_0^2 \sin\theta \cos\theta}{g}, \tag{16.4.1}$$

where v_0 is the initial speed and θ is the launch angle.

1. Derive this formula from the kinematic equations for projectile motion.
2. Find the angle at which this range will be maximized. *Hint:* Use the trig identify

$$\sin 2\theta = 2\sin\theta\cos\theta. \tag{16.4.2}$$

✔ Example 16.4.2: A Fireworks Projectile Explodes high and away

During a fireworks display, a shell is shot into the air with an initial speed of 70.0 m/s at an angle of 75.0° above the horizontal, as illustrated in Figure 16.4.3. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. (a) Calculate the height at which the shell explodes. (b) How much time passes between the launch of the shell and the explosion? (c) What is the horizontal displacement of the shell when it explodes? (d) What is the total displacement from the point of launch to the highest point?

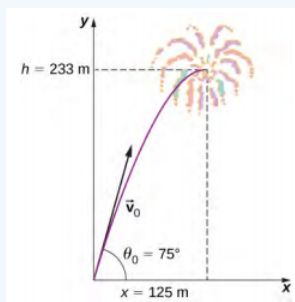

Figure 16.4.3: The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, which is found to be at a height of 233 m and 125 m away horizontally.

Strategy

The motion can be broken into horizontal and vertical motions in which $a_x = 0$ and $a_y = -g$. We can then define x_0 and y_0 to be zero and solve for the desired quantities.

Solution

a. By "height" we mean the altitude or vertical position y above the starting point. The highest point in any trajectory, called the apex, is reached when $v_y = 0$. Since we know the initial and final velocities, as well as the initial position, we use the following equation to find y: $v_y^2 = v_{0y}^2 - 2g(y - y_0)$. Because y_0 and v_y are both zero, the equation simplifies to $0 = v_{0y}^2 - 2gy$. Solving for y gives $y = \frac{v_{0y}^2}{2g}$. Now we must find v_{0y}, the component of the initial velocity in the y direction. It is given by $v_{0y} = v_0 \sin\theta_0$, where v_0 is the initial velocity of 70.0 m/s and $\theta_0 = 75°$ is the initial angle. Thus $v_{0y} = v_0 \sin\theta = (70.0\ m/s)\sin 75^o = 67.6\ m/s$ and y is $y = \frac{(67.6\ m/s)^2}{2(9.80\ m/s^2)}$. Thus, we have $y = 233\ m$. Note that because up is positive, the initial vertical velocity is positive, as is the maximum height, but the acceleration resulting from gravity is negative. Note also that the maximum height depends only on the vertical component of the initial velocity, so that any projectile with a 67.6-m/s initial vertical component of velocity reaches a maximum height of 233 m (neglecting air resistance). The numbers in this example are reasonable for large fireworks displays, the shells of which do reach such heights before exploding. In practice, air resistance is not completely negligible, so the initial velocity would have to be somewhat larger than that given to reach the same height.

b. As in many physics problems, there is more than one way to solve for the time the projectile reaches its highest point. In this case, the easiest method is to use $v_y = v_{0y} - gt$. Because $v_y = 0$ at the apex, this equation reduces $0 = v_{0y} - gt$ or $t = \frac{v_{0y}}{g} = \frac{67.6\ m/s}{9.80\ m/s^2} = 6.90\ s$. This time is also reasonable for large fireworks. If you are able to see the launch of fireworks, notice that several seconds pass before the shell explodes. Another way of finding the time is by using $y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$. This is left for you as an exercise to complete.

c. Because air resistance is negligible, $a_x = 0$ and the horizontal velocity is constant, as discussed earlier. The horizontal displacement is the horizontal velocity multiplied by time as given by $x = x_0 + v_x t$, where x_0 is equal to zero. Thus, $x = v_x t$, where v_x is the x-component of the velocity, which is given by $v_x = v_0 \cos\theta = (70.0\ m/s)\cos 75^o = 18.1\ m/s$. Time t for both motions is the same, so x is $x = (18.1\ m/s)(6.90\ s) = 125\ m$. Horizontal motion is a constant velocity in the absence of air resistance. The horizontal displacement found here could be useful in keeping the fireworks fragments from falling on spectators. When the shell explodes, air resistance has a major effect, and many fragments land directly below.

d. The horizontal and vertical components of the displacement were just calculated, so all that is needed here is to find the magnitude and direction of the displacement at the highest point: $\vec{s} = 125\hat{i} + 233\hat{j} |\vec{s}| = \sqrt{125^2 + 233^2} = 264\ m$ $\theta = \tan^{-1}\left(\frac{233}{125}\right) = 61.8^o$. Note that the angle for the displacement vector is less than the initial angle of launch. To see why this is, review Figure 16.4.1, which shows the curvature of the trajectory toward the ground level. When solving Example 4.7(a), the expression we found for y is valid for any projectile motion when air resistance is negligible. Call the maximum height y = h. Then, $h = \frac{v_{0y}^2}{2g}$. This equation defines the **maximum height of a projectile above its launch position** and it depends only on the vertical component of the initial velocity.

? Exercise 16.4.3

A rock is thrown horizontally off a cliff 100.0 m high with a velocity of 15.0 m/s. (a) Define the origin of the coordinate system. (b) Which equation describes the horizontal motion? (c) Which equations describe the vertical motion? (d) What is the rock's velocity at the point of impact?

✓ Example 16.4.4: Calculating projectile motion- Tennis Player

A tennis player wins a match at Arthur Ashe stadium and hits a ball into the stands at 30 m/s and at an angle 45° above the horizontal (Figure 16.4.4). On its way down, the ball is caught by a spectator 10 m above the point where the ball was hit. (a) Calculate the time it takes the tennis ball to reach the spectator. (b) What are the magnitude and direction of the ball's velocity at impact?

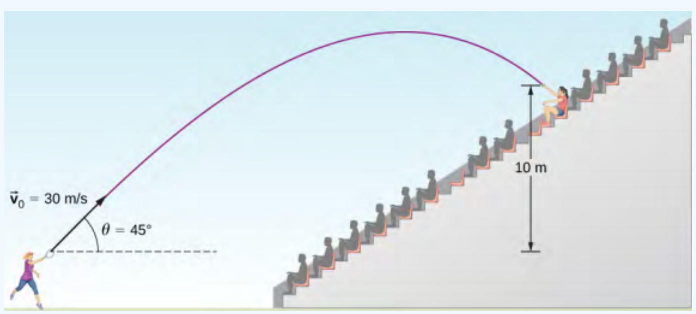

Figure 16.4.4: The trajectory of a tennis ball hit into the stands.

Strategy

Again, resolving this two-dimensional motion into two independent one-dimensional motions allows us to solve for the desired quantities. The time a projectile is in the air is governed by its vertical motion alone. Thus, we solve for t first. While the ball is rising and falling vertically, the horizontal motion continues at a constant velocity. This example asks for the final velocity. Thus, we recombine the vertical and horizontal results to obtain $\vec{v}$ at final time t, determined in the first part of the example.

Solution

a. While the ball is in the air, it rises and then falls to a final position 10.0 m higher than its starting altitude. We can find the time for this by using the third equation in 16.2.5: $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$. If we take the initial position y_0 to be zero, then the final position is y = 10 m. The initial vertical velocity is the vertical component of the initial velocity: $v_{0y} = v_0 \sin\theta_0 = (30.0\ m/s)\sin 45^o = 21.2\ m/s.$. Substituting into our kinematic equation for y gives us $10.0\ m = (21.2\ m/s)t - (4.90\ m/s^2)t^2$ Rearranging terms gives a quadratic equation in t: $(4.90\ m/s^2)t^2 - (21.2\ m/s)t + 10.0\ m = 0$.Use of the quadratic formula yields t = 3.79 s and t = 0.54 s. Since the ball is at a height of 10 m at two times during its trajectory—once on the way up and once on the way down—we take the longer solution for the time it takes the ball to reach the spectator: $t = 3.79\ s$.The time for projectile motion is determined completely by the vertical motion. Thus, any projectile that has an initial vertical velocity of 21.2 m/s and lands 10.0 m above its starting altitude spends 3.79 s in the air.

b. We can find the final horizontal and vertical velocities v_x and v_y with the use of the result from (a). Then, we can combine them to find the magnitude of the total velocity vector $\vec{v}$ and the angle θ it makes with the horizontal. Since v_x is constant, we can solve for it at any horizontal location. We choose the starting point because we know both the initial velocity and the initial angle. Therefore, $v_x = v_0 \cos\theta_0 = (30\ m/s)\cos 45^o = 21.2\ m/s.$. The final vertical velocity is given by the last equation in 16.2.5: $v_y = v_{0y} - gt$. Since v_{0y} was found in part (a) to be 21.2 m/s, we have $v_y = 21.2\ m/s - (9.8\ m/s^2)(3.79s) = -15.9\ m/s$.The magnitude of the final velocity $\vec{v}$ is $v = \sqrt{v_x^2 + v_y^2} = \sqrt{(21.2\ m/s)^2 + (-15.9\ m/s)^2} = 26.5\ m/s$. The direction θ_v is found using the inverse tangent: $\theta_v = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{21.2}{-15.9}\right) = -53.1^o$.

Significance

a. As mentioned earlier, the time for projectile motion is determined completely by the vertical motion. Thus, any projectile that has an initial vertical velocity of 21.2 m/s and lands 10.0 m above its starting altitude spends 3.79 s in the air.

b. The negative angle means the velocity is 53.1° below the horizontal at the point of impact. This result is consistent with the fact that the ball is impacting at a point on the other side of the apex of the trajectory and therefore has a negative y component of the velocity. The magnitude of the velocity is less than the magnitude of the initial velocity we expect since it is impacting 10.0 m above the launch elevation.

✔ Example 16.4.5: Comparing golf shots

A golfer finds himself in two different situations on different holes. On the second hole he is 120 m from the green and wants to hit the ball 90 m and let it run onto the green. He angles the shot low to the ground at 30° to the horizontal to let the ball roll after impact. On the fourth hole he is 90 m from the green and wants to let the ball drop with a minimum amount of rolling after impact. Here, he angles the shot at 70° to the horizontal to minimize rolling after impact. Both shots are hit and impacted on a level surface. (a) What is the initial speed of the ball at the second hole? (b) What is the initial speed of the ball at the fourth hole? (c) Write the trajectory equation for both cases. (d) Graph the trajectories.

Strategy

We see that the range equation (see example problem 16.4.1) has the initial speed and angle, so we can solve for the initial speed for both (a) and (b). When we have the initial speed, we can use this value to write the trajectory equation.

Solution

a. $R = \frac{v_0^2 \sin 2\theta_0}{g} \Rightarrow v_0 = \sqrt{\frac{Rg}{\sin 2\theta_0}} = \sqrt{\frac{(90.0\ m)(9.8\ m/s^2)}{\sin(2(30^o))}} = 31.9\ m/s$

b. $R = \frac{v_0^2 \sin 2\theta_0}{g} \Rightarrow v_0 = \sqrt{\frac{Rg}{\sin 2\theta_0}} = \sqrt{\frac{(90.0\ m)(9.8\ m/s^2)}{\sin(2(70^o))}} = 37.0\ m/s$

c. $y = x\left[\tan\theta_0 - \frac{g}{2(v_0\cos\theta_0)^2}x\right]$ Second hole: $y = x\left[\tan 30^o - \frac{9.8\ m/s^2}{2[(31.9\ m/s)(\cos 30^o)]^2}x\right] = 0.58x - 0.0064x^2$ Fourth hole: $y = x\left[\tan 70^o - \frac{9.8\ m/s^2}{2[(37.0\ m/s)(\cos 70^o)]^2}x\right] = 2.75x - 0.0306x^2$

d. Using a graphing utility, we can compare the two trajectories, which are shown in Figure 16.4.6.

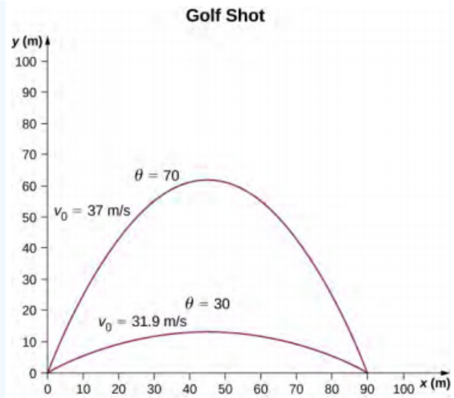

Figure 16.4.6: Two trajectories of a golf ball with a range of 90 m. The impact points of both are at the same level as the launch point.

Significance

The initial speed for the shot at 70° is greater than the initial speed of the shot at 30°. Note from Figure 16.4.6 that two projectiles launched at the same speed but at different angles have the same range if the launch angles add to 90°. The launch angles in this example add to give a number greater than 90°. Thus, the shot at 70° has to have a greater launch speed to reach 90 m, otherwise it would land at a shorter distance.

Exercise 16.4.6

If the two golf shots in Example 4.9 were launched at the same speed, which shot would have the greatest range?

Simulation

At PhET Explorations: Projectile Motion, learn about projectile motion in terms of the launch angle and initial velocity.