

10.2: Work on a Single Particle

Consider a particle that undergoes a displacement Δx while a constant force F acts on it. In one dimension, the *work* done by the force on the particle is defined by

$$W = F\Delta x \quad (\text{constant force}) \quad (10.2.1)$$

and it is positive if the force and the displacement have the same sign (that is, if they point in the same direction), and negative otherwise.

In three dimensions, the force will be a vector \vec{F} with components (F_x, F_y, F_z) , and the displacement, likewise, will be a vector $\Delta\vec{r}$ with components $(\Delta x, \Delta y, \Delta z)$. The work will be defined then as

$$W = F_x\Delta x + F_y\Delta y + F_z\Delta z. \quad (10.2.2)$$

This expression is an instance of what is known as the *dot product* (or *inner product*, or *scalar product*) of two vectors. Given two vectors \vec{A} and \vec{B} , their dot product is defined, in terms of their components,

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z. \quad (10.2.3)$$

This can also be expressed in terms of the vectors' magnitudes, $|\vec{A}|$ and $|\vec{B}|$, and the angle they make, in the following form:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \phi \quad (10.2.4)$$

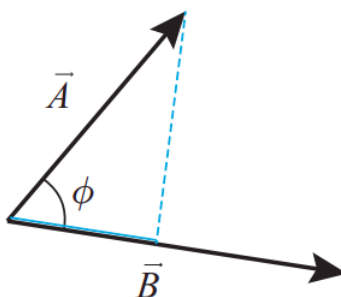


Figure 10.2.1: Illustrating the angle ϕ to be used when calculating the dot product of two vectors by the formula (10.2.4). One way to think of this formula is that you take the projection of vector \vec{A} onto vector \vec{B} (indicated here by the blue lines), which is equal to $|\vec{A}| \cos \phi$, then multiply that by the length of \vec{B} (or vice-versa, of course).

Figure 10.2.1 shows what we mean by the angle ϕ in this expression. The equality of the two definitions, Eqs. (10.2.3) and (10.2.4), is pretty easy to see; consider a coordinate system in which the vector \vec{B} is completely along the x-axis. Then our vectors can be written

$$\vec{A} = A \cos(\phi)\hat{x} + A \sin(\phi)\hat{y}, \quad \vec{B} = B\hat{x}. \quad (10.2.5)$$

And now if we perform the dot product as shown in Equation (10.2.3),

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = AB \cos(\phi), \quad (10.2.6)$$

which is exactly what Equation (10.2.4) says! As a sidenote, this is an important demonstration of a basic principle in the study of physics - if you can prove something in one coordinate system, it must be true in all others. This is not just a calculational trick, but a fundamental truth we would hope our theory can conform with - the coordinate system is a choice humans make to perform a calculation, but can't have any real physics meaning. In fact, the advantage of Equation (10.2.4) is that it is independent of the choice of a system of coordinates.

Using the dot product notation, the work done by a constant force can be written as

$$W = \vec{F} \cdot \Delta\vec{r}. \quad (10.2.7)$$

Equation (10.2.4) then shows that, as I mentioned in the introduction, when the force is perpendicular to the displacement ($\phi = 90^\circ$) the work it does is zero. You can also see this directly from Equation (10.2.2), by choosing the x axis to point in the

direction of the force (so $F_y = F_z = 0$), and the displacement to point along any of the other two axes (so $\Delta x = 0$): the result is $W = 0$.

If the force is not constant, again we follow the standard procedure of breaking up the total displacement into pieces that are short enough that the force may be taken to be constant over each of them, calculating all those (possibly very small) “pieces of work,” and adding them all together. In one dimension, the final result can be expressed as the integral

$$W = \int_{x_i}^{x_f} F(x) dx \quad (\text{variable force}). \quad (10.2.8)$$

So the work is given by the “area” under the F -vs- x curve. In more dimensions, we have to write a kind of multivariable integral known as a *line integral*. That is advanced calculus, so we will not go there this semester.

Work Done by the Net Force, and the Work-Energy Theorem

So much for the math and the definitions. Where does the energy come in? Let us suppose that F is either the only force or the *net force* on the particle—the sum of all the forces acting on the particle. Again, for simplicity we will assume that it is constant (does not change) while the particle undergoes the displacement Δx . We started this chapter with the understanding that the work done on the system changes the energy of the system (we will prove that mathematically in a later chapter). If we now consider a system in which all of the interactions can be described by forces doing work, from the principle of conservation of energy we can write

$$W_{\text{net}} = \Delta K. \quad (10.2.9)$$

In words, *the work done by the net force acting on a particle as it moves equals the change in the particle’s kinetic energy in the course of its displacement*. This result is often referred to as *the Work-Energy Theorem*.

As you may have guessed from our calling it a “theorem,” the result (10.2.9) is very general. It holds in three dimensions, and it holds also when the force isn’t constant throughout the displacement—you just have to use the correct equation to calculate the work in those cases. It would apply to the work done by the net force on an extended object, also, provided it is OK to treat the extended object as a particle—so basically, a rigid object that is moving as a whole and not doing anything fancy such as spinning while doing so.

Another possible direction in which to generalize (10.2.9) might be as follows. By definition, a “particle” has *no* other kind of energy, besides (translational) kinetic energy. Also, and for the same reason (namely, the absence of internal structure), it has no “internal” forces—all the forces acting on it are external. So—for this very simple system—we could rephrase the result (10.2.9) by saying that the work done by the net *external* force acting on the system (the particle in this case) is equal to the change in its *total* energy. It is in fact in this form that we will ultimately generalize (10.2.9) to deal with arbitrary systems.

Before we go there, however, I would like to take a little detour to explore another “reasonable” extension of the result (10.2.9), as well as its limitations.

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