9.3: Equilibrium and Turning Points

Let's continue our discussion of potential energy graphs by introducing some new terms - turning points and equilibrium points. We already saw **turning points** in the last section - these were the point in the motion at which the object stopped moving and turned around to go the other direction. **Equilibrium points** are the points in the motion where the object *could be* at rest (note the object does not have to actually be at rest at that point, but under some conditions could be).

To get a better understanding of these terms, we'll look at two specific examples. First, let's look at an object, freely falling vertically, near the surface of Earth, in the absence of air resistance. The mechanical energy of the object is conserved, E = K + U, and the potential energy, with respect to zero at ground level, is U(y) = mgy, which is a straight line through the origin with slope mg. In the graph shown in Figure $9.3.1$, the x-axis is the height above the ground y and the y-axis is the object's energy.

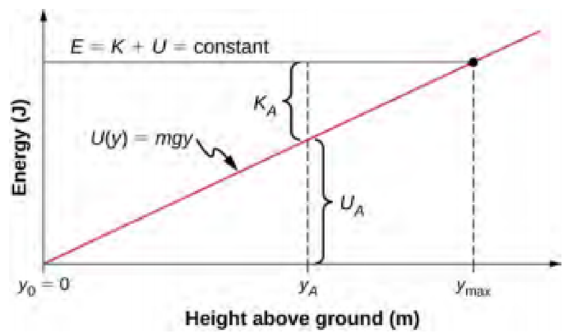

Figure $9.3.1$: The potential energy graph for an object in vertical free fall, with various quantities indicated.

The line at energy E represents the constant mechanical energy of the object, whereas the kinetic and potential energies, K_A and U_A, are indicated at a particular height y_A. You can see how the total energy is divided between kinetic and potential energy as the object's height changes. First, let's note that *the kinetic energy of an object can never be negative*. This is true because there is nothing negative in the formula $1/2mv^2$...nature says mass can never be negative, and mathematics says nothing squared can ever be negative (no imaginary speeds allowed!).

Since kinetic energy can never be negative, there is a maximum potential energy and a maximum height, which an object with the given total energy cannot exceed:

$$K = E - U \geq 0, \tag{9.3.1}$$

$$U \leq E. \tag{9.3.2}$$

If we use the gravitational potential energy reference point of zero at y_0, we can rewrite the gravitational potential energy U as mgy. Solving for y results in

$$y \leq \frac{E}{mg} = y_{max}. \tag{9.3.3}$$

We note in this expression that the quantity of the total energy divided by the weight (mg) is located at the maximum height of the particle, or y_{max}. At the maximum height, the kinetic energy and the speed are zero, so if the object were initially traveling upward, its velocity would go through zero there, and y_{max} would be a turning point in the motion. At ground level, $y_0 = 0$, the potential energy is zero, and the kinetic energy and the speed are maximum:

$$U_0 = 0 = E - K_0, \tag{9.3.4}$$

$$E = K_0 = \frac{1}{2}mv_0^2, \tag{9.3.5}$$

$$v_0 = \pm\sqrt{\frac{2E}{m}}. \tag{9.3.6}$$

The maximum speed $\pm v_0$ gives the initial velocity necessary to reach y_{max}, the maximum height, and $-v_0$ represents the final velocity, after falling from y_{max}. You can read all this information, and more, from the potential energy diagram we have shown. *Notice that the turning point occured where the total energy and the potential energy intersected* in the graph - that's the point with zero kinetic energy. This system does not have an equilibrium point, because there is nowhere the falling object could be at rest - it's always going to be trying to move downwards under the gravitational interaction.

For a second examples, consider a mass-spring system on a frictionless, stationary, horizontal surface, so that gravity and the normal contact force do no work and can be ignored (Figure 9.3.2). This is like a one-dimensional system, whose mechanical energy E is a constant and whose potential energy, with respect to zero energy at zero displacement from the spring's unstretched length, x = 0, is $U(x) = \frac{1}{2}kx^2$.

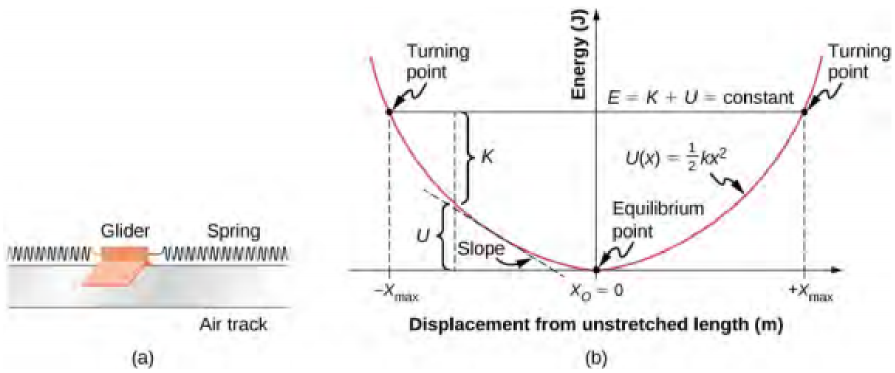

Figure 9.3.2: (a) A glider between springs on an air track is an example of a horizontal mass-spring system. (b) The potential energy diagram for this system, with various quantities indicated.

You can read off the same type of information from the potential energy diagram in this case, as in the case for the body in vertical free fall, but since the spring potential energy describes a variable force, you can learn more from this graph. As for the object in vertical free fall, you can deduce the physically allowable range of motion and the maximum values of distance and speed, from the limits on the kinetic energy, $0 \leq K \leq E$. Therefore, *K = 0 and U = E at a turning point*, of which there are two for the elastic spring potential energy,

$$x_{max} = \pm\sqrt{\frac{2E}{k}}. \tag{9.3.7}$$

The glider's motion is confined to the region between the turning points, $-x_{max} \leq x \leq x_{max}$. This is true for any (positive) value of E because the potential energy is unbounded with respect to x. For this reason, as well as the shape of the potential energy curve, U(x) is called an infinite potential well. At the bottom of the potential well, x = 0, U = 0 and the kinetic energy is a maximum, K = E, so $v_{max} = \pm\sqrt{\frac{2E}{m}}$.

However, this potential has another special point, at $x = 0$. *This is the equilibrium point*, because if the object had zero kinetic energy at that point, it would not move. Notice that an object bouncing back and forth between the two turning points doesn't stop at this equilibrium, because it doesn't have $K = 0$ there. Note that on either side of the equilibrium point, the potential energy increases - another way of defining the equilibrium point is "the point which is a (local) minimum of potential energy". No matter where an object starts, it will be driven towards the equilibrium point.

Finally, we should add that the description we just gave ("local minimum of potential") is actually for a *stable* equilibrium point - there are also unstable equilibrium points. For example, consider turning the spring potential upside-down (alternatively, look at the "bumps" in Figure 9.2.3). We could then place an object with $K = 0$ right at that point, and it would technically not move since there is no slope in the potential energy function. However, the moment we give it any kind of bump one way or the other, it will immediately go in that direction. In other words, an unstable equilibrium is a local *maximum* of potential energy, where an object placed there will move away from it if displaced.