Conservation Laws, Newton's Laws, and Kinematics

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Edited by Christopher Duston

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CHAPTER OVERVIEW

1: C1) Abstraction and Modeling

This first section is going to serve two purposes: first, introduce some of the key important concepts that we will need right from the beginning, like units and significant figures. Second, we want to open up the question: "what is physics"? Like many such open-ended questions, there are many ways to answer it. It is not going to be our intention to answer this fully in any sense - we just want to answer it enough so that as a student, you know why this book exists and what you should be getting out of it. To get right to the point: **physics is the study of the abstraction of the physical world**.

So that's a pretty obscure statement, so let's break it down a bit. When we say "abstraction", that's basically a way of replacing all the real physical objects out there that we want to understand (balls, cars, airplanes, etc...objects!) with mathematical representations of those objects. These representations are simple - sometimes they are just points, or maybe box-shaped things with wheels to represent cars, etc. The point of the abstraction is both to simplify the problem (points are easier to deal with than real life planes), and also to make it mathematically precise. "how a car moves" is actually a question far too complicated for a simple introduction to the world of physics to handle. There are literally millions (way more!) of interactions taking place in the motion of a car - things like the engine, the wheels, and the transmission - but even little things like the air molecules hitting the outside of the car and slowing it down. By replacing the entire thing with "a point" or "a box with wheels", we make the problem simple enough for us to actually perform some calculations to understand it's motion.

We should be clear what the kind of mathematical precision we are talking about here as well. We don't mean "calculating the exact value of the speed of the car to 5 decimal places" (we'll see in this chapter that kind of precision is not actually what we are after). We mean something more like "numerically well-defined". For example, if we said "The position of the car is 5 m from the end of the race", we can't actually perform any calculations with that information, because *which part of the car are we talking about*? For sure, you could say "the front", or "the back", or "30 cm from the driver", but now we might need to know even more things about the car - how big is it, or how big the seats are (to determine the position of the driver relative to the finish line). The abstraction avoids all this by the replacement of real objects with simple representations of them. If the car is represented as a point, then "5 m from the end of the race" becomes very well-defined, because it can only refer to one location! Similarly for the representation of the car as a box with wheels - more information is required (dimensions of the box and the wheels?), but it's still far less then what might be required for the real, physical car.

Now that we've covered "abstraction", what does "study of the abstraction" mean? Well, it's often said that physics is "the study of the fundamental interactions" - that's what we mean, but we mean it specifically *within the abstraction*. We aren't actually studying how the fundamental interactions of the real world influence real physical objects - we are representing the world with an abstraction, and studying the abstraction. The force of gravity is actually enormously complicated in the real world (it's everywhere, it acts between each individual molecule and each other individual molecule....you'd be calculating for your entire life!), but in our abstraction, it's just a simple force acting between two points. By proposing simple, fundamental interactions between objects we are hoping to model the real world.

Of course, we claim that physics actually is the study of the real world, but you aren't ever going to see that connection by using solely the abstraction - you have to go into the laboratory and perform experiments to verify that our abstraction is an "accurate and precise" representation of the real world. We aren't going to talk much about that in this text, since it is primarily designed to give you an introduction to the prediction and quantitative aspects of the field - hopefully, your theoretical experience with this text will be coupled with a laboratory experience as well, so that you can see how well physics actually does at describing phenomena and events in the real world.

1.1: About this Text

This text has been written with several goals in mind:

- Break sections into short chucks appropriate for reading before each class meeting, specifically for use in "studio physics" classrooms.
- Present conservation laws before kinematics.
- Acknowledge that students may be co-enrolled in their first calculus course.

Before proceeding with the text, we will motivate and explain each of these goals. Being an open source resource, this text is constantly in a state of improvement. Feel free to write to the editor, Professor Christopher Duston (dustonc at merrimack dot edu) if you find significant errors that you think should be corrected.

This text has been written to support a so-called "studio" approach to teaching physics. There are many flavors of this pedagogy (for example, SCALE-UP from North Carolina State, Studio Physics from Florida State, and Studio Physics at The Massachusetts Institute of Technology), but an essential element is that class time is significantly devoted to problem solving over lecture time. To support this, it is necessary for students to be exposed to the material at some level before even entering the classroom. There are several ways to achieve that (this is the so-called "flipped classroom model"), but it is our feeling that student success is maximized by designing this initial exposure around the studio model, rather then using a traditional textbook. A traditional and carefully constructed 1000-page physics textbook is a beautiful thing, but if the material is broken into sections appropriate for the content, rather then the delivery system, there will be a fundamental mismatch between the goals of the students and the goals of the textbook. So the first goal of this text is to break the content into sections appropriate for single class meetings. At Merrimack College (where this textbook was designed to be used), there are two class meetings per week, each class meeting is about two hours long, and each meeting tackles one section of this text.

This text was also designed to be the initial source of exposure to the content. Another challenge when using a traditional physics textbook in the studio model is that the first thing a novice reads about might be a completely well-thought out and well-motivated description of a physical law, but a student is likely not prepared for that level of description. The process of learning is not linear and hierarchical, but chaotic and varied. Of course, in the end we want students to be experts *a la* Bloom's Taxonomy, but starting from the bottom and explaining how they get to the top will not service their own internalized development of the material. To that end, the goal of this text is primarily to provide students with enough material to "soak the sponge", so that when they enter the class they are prepared to learn (and make mistakes!), rather then prepared to solve problems on an exam. To achieve this, the material is presented in a streamlined matter, with simplified motivations and only the first few initial steps spelled out. For sure, this means more complicated material is occasionally only briefly touched upon, and in some cases left out completely. However, we would argue that as a fundamental science, more complicated material in the field of physics can always be added in by instructor demand, in a way that their own students can handle it. And, given that anything can be found on the internet, collecting special topics in textbooks seems like something which has already been done by any number of a myriad of authors. If that's what you want, this textbook is not for you!

The content order of this textbook (conservation laws before Newton's laws) is inspired by the wonderful texts of Eric Mazur and Thomas Moore (in fact, the current version of this book blatantly steals the chapter order from Moore's *Six Ideas In Physics*, *Units C* and *N*). For more well-thought out motivations, one can consult those admirable texts - for our part, we will simply motivate this in two ways. First, in many ways conservation laws are more fundamental then Newton's laws. Indeed, the argument about the invariance of specific quantities under time and space translations is so fundamental that it boarders on the Philosophical nature of the Universe, rather then our particular approach to modeling it. More directly, equations like $\Delta E=0$ are scalar and discrete, and therefore typically easier for students to deal with then vector expressions that imply continuous change, like $\Sigma\vec{F}=m\vec{a}$. Second, it is our experience that students have already been exposed to Newton's laws and kinematics - and in some cases have built up significant misunderstandings that are difficult to break down. (A trivial example might be the statement that "the force of gravity is negative!", a viewpoint strongly held by many students. This concept is so rife with intellectual inconsistency that I regret even bringing it up...) By starting students with new, consistently correct ideas about modeling and problem solving in the context of conservation laws, we prepare them to tear down their previous knowledge and rebuild it on a more solid foundation.

Finally, this textbook is written with the knowledge that some (many?) students are going to be stepping into their first physics class while simultaneously taking their first calculus class. This breaks the traditional "calculus before physics" mantra that has been a part of our pedagogy for decades. The source of this change will not be covered here, but we are highly motivated to deal with it rather then preach about its evils from the rooftops. Of course, as a textbook that primarily covers mechanics, it's absolutely true that most of what can be found in this textbook is actually directly derivable from simple ideas based in calculus. However, that is also not usually where students coming to the field are, and it would be an injustice to ask them to climb that mountain before demonstrating the joy of understanding basic concepts in the field. More practically, it is our experience that in so-called "calculus-based physics classes", the biggest challenges for students is not the calculus itself, but the vector algebra and analysis which is utilized in nearly no other field as extensively as physics. Therefore, we do not view the loss of a semester of calculus before this class as a significant barrier, and do not start by assuming students have it. By the second half of the book (Newton's laws), it is assumed that students have a grasp of all the basic notations from calculus.

Open Source Resources and the Construction of This Text

From the outset, we must acknowledge that this text fails in all the goals presented above. Writing a text from scratch is a lengthy business, and we have students to teach! However, since the advent of free and open source educational resources, another avenue has been opened - the "collect, evaluate, and modify" approach. Open source solutions have been available for some time (University Physics by OpenStax is perhaps the most well-known example, as well as Calculus-Based Physics by Jeffrey Schnick), but these followed the more traditional order of material. However, once Julio Gea-Banacloche published his fantastic University Physics I: Classical Mechanics, the possibility of satisfying our goals via open source publication became realizable. However, rearranging and breaking apart a textbook comes with it's own dangers, and we are squarely in the middle of those dangers with this text, which was constructed in the following way:

1. Rearrange the Gea-Banacloche text to match the specific order of material, which closely follows Thomas Moore's book.
2. Supplement material which is lacking from OpenStax.
3. Write new material to fill in some gaps.

As such, we owe a great deal of gratitude to both Gea-Banacloche and the OpenStax organization, for making such an approach feasible. However, the responsibility is now all on our shoulders, to continue to modify and develop this book so that eventually we can satisfy the lofty pedagogical and practical goals we outlined at the beginning of this section.

Whiteboard Problems

In our studio classroom, over half the time is spent solving problems on whiteboards in groups. During this time, the instructor (and typically one or two learning assistants) are on hand and available to assist if a group gets stuck. Over the years, we've developed many more of these style problems then we will typically use (3-4 per class period, generally), so we've included the extras in this text. At the end of each chapter we will have an "Example" section, and these whiteboard problems will be presented there (along with some good examples from the two main source texts we discussed above). These can be used identically to example problems, or they could be used as actual whiteboard problems for a studio classroom. In the future, we are planning on adding our entire set of whiteboard problems to these sections - this will correspond with a generally shortening of the actual texts of the chapter, to match our goal of being accessible as a pre-class reading assignment.

1.2: Modeling in Physics

Classical mechanics is the branch of physics that deals with the study of the motion of anything (roughly speaking) larger than an atom or a molecule. That is a lot of territory, and the methods and concepts of classical mechanics are at the foundation of any branch of science or engineering that is concerned with the motion of anything from a star to an amoeba—fluids, rocks, animals, planets, and any and all kinds of machines. Moreover, even though the accurate description of processes at the atomic level requires the (formally very different) methods of quantum mechanics, at least three of the basic concepts of classical mechanics that we are going to study this semester, namely, momentum, energy, and angular momentum, carry over into quantum mechanics as well, with the last two playing, in fact, an essential role.

Particles in Classical Mechanics

In the study of motion, the most basic starting point is the concept of the *position* of an object. Clearly, if we want to describe accurately the position of a macroscopic object such as a car, we may need a lot of information, including the precise shape of the car, whether it is turned this way or that way, and so on; however, if all we want to know is how far the car is from Fort Smith or Fayetteville, we do not need any of that: we can just treat the car as a dot, or mathematical point, on the map—which is the way your GPS screen will show it, anyway. When we do this, we say that are describing the car (or whatever the macroscopic object may be) as a **particle**.

In classical mechanics, an "ideal" particle is an object with no appreciable size—a mathematical point. In one dimension (that is to say, along a straight line), its position can be specified just by giving a single number, the distance from some reference point, as we shall see in a moment (in three dimensions, of course, three numbers are required). In terms of energy (which is perhaps the most important concept in all of physics, and which we will introduce properly in due course), an ideal particle has only one kind of energy, what we will later call *translational kinetic energy*; it cannot have, for instance, rotational kinetic energy (since it has "no shape" for practical purposes), or any form of internal energy (elastic, thermal, etc.), since we assume it is too small to have any internal structure in the first place.

The reason this is a useful concept is not just that we can often treat extended objects as particles in an approximate way (like the car in the example above), but also, and most importantly, that if we want to be more precise in our calculations, *we can always treat an extended object (mathematically) as a collection of "particles."* The physical properties of the object, such as its energy, momentum, rotational inertia, and so forth, can then be obtained by adding up the corresponding quantities for all the particles making up the object. Not only that, but the interactions between two extended objects can also be calculated by adding up the interactions between all the particles making up the two objects. This is how, once we know the form of the gravitational force between two particles (which is fairly simple, as we will see in Section 8.7), we can use that to calculate the force of gravity between a planet and its satellites, which can be fairly complicated in detail, depending, for instance, on the relative orientation of the planet and the satellite.

The mathematical tool we use to calculate these "sums" is *calculus*—specifically, integration—and you will see many examples of this... in your calculus courses. Calculus I is only a corequisite for this course, so we will not make a lot of use of it here, and in any case you would need multidimensional integrals, which are an even more advanced subject, to do these kinds of calculations. But it may be good for you to keep these ideas on the back of your mind. Calculus was, in fact, invented by Sir Isaac Newton precisely for this purpose, and the developments of physics and mathematics have been closely linked together ever since.

Anyway, back to particles, the plan for this semester is as follows: we will start our description of motion by treating every object (even fairly large ones, such as cars) as a "particle," because we will only be concerned at first with its translational motion and the corresponding energy. Then we will progressively make things more complex: by considering systems of two or more particles, we will start to deal with the *internal energy* of a system. Then we will move to the study of *rigid bodies*, which are another important idealization: extended objects whose parts all move together as the object undergoes a translation or a rotation. This will allow us to introduce the concept of rotational kinetic energy. Eventually we will consider *wave motion*, where different parts of an extended object (or "medium") move relative to each other. So, you see, there is a logical progression here, with most parts of the course building on top of the previous ones, and energy as one of the main connecting themes. The technical word for the process being described in the preceding paragraphs is **abstraction**; the essence of which is to take the physical world and model it with abstract mathematical quantities. We can then use these quantities to construct physical theories, make predictions, and verify them with experiments. In this text we will primarily be interested in the middle section of this process - *taking physical laws and making predictions about how objects in the real world will behave*.

Aside: The Atomic Perspective

As an aside, it should perhaps be mentioned that the building up of classical mechanics around this concept of ideal particles had nothing to do, initially, with any belief in “atoms,” or an atomic theory of matter. Indeed, for most 18th and 19th century physicists, matter was supposed to be a continuous medium, and its (mental) division into particles was just a mathematical convenience.

The atomic hypothesis became increasingly more plausible as the 19th century wore on, and by the 1920’s, when quantum mechanics came along, physicists had to face a surprising development: matter, it turned out, was indeed made up of “elementary particles,” but these particles could not, in fact, be themselves described by the laws of classical mechanics. One could not, for instance, attribute to them simultaneously well-defined positions and velocities. Yet, in spite of this, most of the conclusions of classical mechanics remain valid for macroscopic objects, because, most of the time, it is OK to (formally) “break up” extended objects into chunks that are small enough to be treated as particles, but large enough that one does not need quantum mechanics to describe their behavior.

Quantum properties were first found to manifest themselves at the macroscopic level when dealing with thermal energy, because at one point it really became necessary to figure out where and how the energy was stored at the truly microscopic (atomic) level. Thus, after centuries of successes, classical mechanics met its first failure with the so-called *problem of the specific heats*, and a completely new physical theory—quantum mechanics—had to be developed in order to deal with the newly-discovered atomic world. But all this, as they say, is another story, and for our very brief dealings with thermal physics—the last chapter in this book—we will just take specific heats as given, that is to say, something you measure (or look up in a table), rather than something you try to calculate from theory.

1.3: Units and Standards

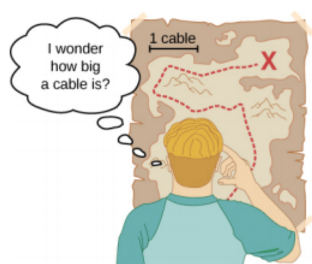
Learning Objectives

- Describe how SI base units are defined.
- Describe how derived units are created from base units.
- Express quantities given in SI units using metric prefixes.

As we saw previously, the range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of Earth, from the tiny sizes of subnuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist. Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than qualitative descriptions alone. To comprehend these vast ranges, we must also have accepted units in which to express them. We shall find that even in the potentially mundane discussion of meters, kilograms, and seconds, a profound simplicity of nature appears: all physical quantities can be expressed as combinations of only seven base physical quantities.

We define a **physical quantity** either by specifying how it is measured or by stating how it is calculated from other measurements. For example, we might define distance and time by specifying methods for measuring them, such as using a meter stick and a stopwatch. Then, we could define average speed by stating that it is calculated as the total distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way (Figure 1.3.1).



Distances given in unknown units are maddeningly useless.

Figure 1.3.1: Distances given in unknown units are maddeningly useless.

Two major systems of units are used in the world: **SI units** (for the French **Système International d'Unités**), also known as the **metric system**, and **English units** (also known as the **customary** or **imperial system**). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States. English units may also be referred to as the **foot–pound–second** (fps) system, as opposed to the **centimeter–gram–second** (cgs) system. You may also encounter the term **SAE units**, named after the Society of Automotive Engineers. Products such as fasteners and automotive tools (for example, wrenches) that are measured in inches rather than metric units are referred to as **SAE fasteners** or **SAE wrenches**.

Virtually every other country in the world (except the United States) now uses SI units as the standard. The metric system is also the standard system agreed on by scientists and mathematicians.

SI Units: Base and Derived Units

In any system of units, the units for some physical quantities must be defined through a measurement process. These are called the **base quantities** for that system and their units are the system's **base units**. All other physical quantities can then be expressed as algebraic combinations of the base quantities. Each of these physical quantities is then known as a **derived quantity** and each unit is called a **derived unit**. The choice of base quantities is somewhat arbitrary, as long as they are independent of each other and all other quantities can be derived from them. Typically, the goal is to choose physical quantities that can be measured accurately to a high precision as the base quantities. The reason for this is simple. Since the derived units can be expressed as algebraic combinations of the base units, they can only be as accurate and precise as the base units from which they are derived.

Based on such considerations, the International Standards Organization recommends using seven base quantities, which form the International System of Quantities (ISQ). These are the base quantities used to define the SI base units. Table 1.3.1 lists these seven ISQ base quantities and the corresponding SI base units.

Table 1.3.1: ISQ Base Quantities and Their SI Units

ISQ Base Quantity	SI Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electrical Current	ampere (A)
Thermodynamic Temperature	kelvin (K)
Amount of Substance	mole (mol)
Luminous Intensity	candela (cd)

You are probably already familiar with some derived quantities that can be formed from the base quantities in Table 1.3.1. For example, the geometric concept of area is always calculated as the product of two lengths. Thus, area is a derived quantity that can be expressed in terms of SI base units using square meters (m x m = m^2). Similarly, volume is a derived quantity that can be expressed in cubic meters (m^3). Speed is length per time; so in terms of SI base units, we could measure it in meters per second (m/s). Volume mass density (or just density) is mass per volume, which is expressed in terms of SI base units such as kilograms per cubic meter (kg/m^3). Angles can also be thought of as derived quantities because they can be defined as the ratio of the arc length subtended by two radii of a circle to the radius of the circle. This is how the radian is defined. Depending on your background and interests, you may be able to come up with other derived quantities, such as the mass flow rate (kg/s) or volume flow rate (m^3/s) of a fluid, electric charge (A • s), mass flux density [$kg/(m^2 \cdot s)$], and so on. We will see many more examples throughout this text. For now, the point is that every physical quantity can be derived from the seven base quantities in Table 1.3.1, and the units of every physical quantity can be derived from the seven SI base units.

For the most part, we use SI units in this text. Non-SI units are used in a few applications in which they are in very common use, such as the measurement of temperature in degrees Celsius (°C), the measurement of fluid volume in liters (L), and the measurement of energies of elementary particles in electron-volts (eV). Whenever non-SI units are discussed, they are tied to SI units through conversions. For example, 1 L is 10^{-3} m^3.

Check out a comprehensive source of information on SI units at the National Institute of Standards and Technology (NIST) Reference on Constants, Units, and Uncertainty.

Units of Time, Length, and Mass: The Second, Meter, and Kilogram

The initial chapters in this textmap are concerned with mechanics, fluids, and waves. In these subjects all pertinent physical quantities can be expressed in terms of the base units of length, mass, and time. Therefore, we now turn to a discussion of these three base units, leaving discussion of the others until they are needed later.

The Second

The SI unit for time, the **second** (abbreviated s), has a long history. For many years it was defined as 1/86,400 of a mean solar day. More recently, a new standard was adopted to gain greater accuracy and to define the second in terms of a nonvarying or constant physical phenomenon (because the solar day is getting longer as a result of the very gradual slowing of Earth's rotation). Cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. In 1967, the second was redefined as the time required for 9,192,631,770 of these vibrations to occur (Figure 1.3.2). Note that this may seem like more precision than you would ever need, but it isn't—GPSs rely on the precision of atomic clocks to be able to give you turn-by-turn directions on the surface of Earth, far from the satellites broadcasting their location.

Figure $1.3.2$: An atomic clock such as this one uses the vibrations of cesium atoms to keep time to a precision of better than a microsecond per year. The fundamental unit of time, the second, is based on such clocks. This image looks down from the top of an atomic fountain nearly 30 feet tall. (credit: Steve Jurvetson)

The Meter

The SI unit for length is the **meter** (abbreviated m); its definition has also changed over time to become more precise. The meter was first defined in 1791 as 1/10,000,000 of the distance from the equator to the North Pole. This measurement was improved in 1889 by redefining the meter to be the distance between two engraved lines on a platinum–iridium bar now kept near Paris. By 1960, it had become possible to define the meter even more accurately in terms of the wavelength of light, so it was again redefined as 1,650,763.73 wavelengths of orange light emitted by krypton atoms. In 1983, the meter was given its current definition (in part for greater accuracy) as the distance light travels in a vacuum in 1/299,792,458 of a second (Figure $1.3.3$). This change came after knowing the speed of light to be exactly 299,792,458 m/s. The length of the meter will change if the speed of light is someday measured with greater accuracy.

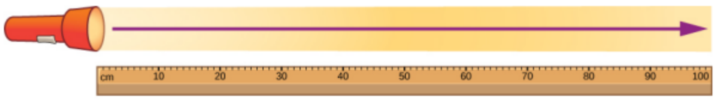


Figure $1.3.3$: The meter is defined to be the distance light travels in 1/299,792,458 of a second in a vacuum. Distance traveled is speed multiplied by time.

The Kilogram

The SI unit for mass is the **kilogram** (abbreviated kg); From 1795–2018 it was defined to be the mass of a platinum–iridium cylinder kept with the old meter standard at the International Bureau of Weights and Measures near Paris. However, this cylinder has lost roughly 50 micrograms since it was created. Because this is the standard, this has shifted how we defined a kilogram. Therefore, a new definition was adopted in May 2019 based on the Planck constant and other constants which will never change in value. We will study Planck's constant in quantum mechanics, which is an area of physics that describes how the smallest pieces of the universe work. The kilogram is measured on a Kibble balance (see $1.3.4$). When a weight is placed on a Kibble balance, an electrical current is produced that is proportional to Planck's constant. Since Planck's constant is defined, the exact current measurements in the balance define the kilogram.

Figure 1.3.4: Redefining the SI unit of mass. The U.S. National Institute of Standards and Technology's Kibble balance is a machine that balances the weight of a test mass with the resulting electrical current needed for a force to balance it.

Metric Prefixes

SI units are part of the **metric system**, which is convenient for scientific and engineering calculations because the units are categorized by factors of 10. Table 1.3.1 lists the metric prefixes and symbols used to denote various factors of 10 in SI units. For example, a centimeter is one-hundredth of a meter (in symbols, 1 cm = 10^{-2} m) and a kilometer is a thousand meters (1 km = 10^3 m). Similarly, a megagram is a million grams (1 Mg = 10^6 g), a nanosecond is a billionth of a second (1 ns = 10^{-9} s), and a terameter is a trillion meters (1 Tm = 10^{12} m).

Table 1.3.2: Metric Prefixes for Powers of 10 and Their Symbols

Prefix	Symbol	Meaning	Prefix	Symbol	Meaning
yotta-	Y	10^{24}	yocto-	Y	10^{-24}
zetta-	Z	10^{21}	zepto-	Z	10^{-21}
exa-	E	10^{18}	atto-	E	10^{-18}
peta-	P	10^{15}	femto-	P	10^{-15}
tera-	T	10^{12}	pico-	T	10^{-12}
giga-	G	10^{9}	nano-	G	10^{-9}
mega-	M	10^{6}	micro-	M	10^{-6}
kilo-	k	10^{3}	milli-	k	10^{-3}
hecto-	h	10^{2}	centi-	h	10^{-2}
deka-	da	10^{1}	deci-	da	10^{-1}

The only rule when using metric prefixes is that you cannot "double them up." For example, if you have measurements in petameters (1 Pm = 10^{15} m), it is not proper to talk about megagigameters, although 10^6 x 10^9 = 10^{15}. In practice, the only time this becomes a bit confusing is when discussing masses. As we have seen, the base SI unit of mass is the kilogram (kg), but metric prefixes need to be applied to the gram (g), because we are not allowed to "double-up" prefixes. Thus, a thousand kilograms (10^3 kg) is written as a megagram (1 Mg) since

$$10^3 \ kg = 10^3 \times 10^3 \ g = 10^6 \ g = 1 \ Mg. \tag{1.3.1}$$

Incidentally, 10^3 kg is also called a **metric ton**, abbreviated t. This is one of the units outside the SI system considered acceptable for use with SI units.

As we see in the next section, metric systems have the advantage that conversions of units involve only powers of 10. There are 100 cm in 1 m, 1000 m in 1 km, and so on. In nonmetric systems, such as the English system of units, the relationships are not as simple—there are 12 in in 1 ft, 5280 ft in 1 mi, and so on.

Another advantage of metric systems is that the same unit can be used over extremely large ranges of values simply by scaling it with an appropriate metric prefix. The prefix is chosen by the order of magnitude of physical quantities commonly found in the task at hand. For example, distances in meters are suitable in construction, whereas distances in kilometers are appropriate for air travel, and nanometers are convenient in optical design. With the metric system there is no need to invent new units for particular applications. Instead, we rescale the units with which we are already familiar.

✔ Example 1.3.1: Using Metric Prefixes

Restate the mass 1.93 x 10^{13} kg using a metric prefix such that the resulting numerical value is bigger than one but less than 1000.

Strategy

Since we are not allowed to "double-up" prefixes, we first need to restate the mass in grams by replacing the prefix symbol k with a factor of 10^3 (Table $1.3.2$). Then, we should see which two prefixes in Table $1.3.2$ are closest to the resulting power of 10 when the number is written in scientific notation. We use whichever of these two prefixes gives us a number between one and 1000.

Solution

Replacing the k in kilogram with a factor of 10^3, we find that

$$1.93 \times 10^{13} \ kg = 1.93 \times 10^{13} \times 10^3 \ g = 1.93 \times 10^{16} \ g.$$

From Table $1.3.2$, we see that 10^{16} is between "peta-" (10^{15}) and "exa-" (10^{18}). If we use the "peta-" prefix, then we find that 1.93 × 10^{16} g = 1.93 × 10^1 Pg, since 16 = 1 + 15. Alternatively, if we use the "exa-" prefix we find that 1.93 x 10^{16} g = 1.93 x 10^{-2} Eg, since 16 = −2 + 18. Because the problem asks for the numerical value between one and 1000, we use the "peta-" prefix and the answer is 19.3 Pg.

Significance

It is easy to make silly arithmetic errors when switching from one prefix to another, so it is always a good idea to check that our final answer matches the number we started with. An easy way to do this is to put both numbers in scientific notation and count powers of 10, including the ones hidden in prefixes. If we did not make a mistake, the powers of 10 should match up. In this problem, we started with 1.93 x 10^{13} kg, so we have 13 + 3 = 16 powers of 10. Our final answer in scientific notation is 1.93 x 10^1 Pg, so we have 1 + 15 = 16 powers of 10. So, everything checks out.

If this mass arose from a calculation, we would also want to check to determine whether a mass this large makes any sense in the context of the problem. For this, Figure 1.4 might be helpful.

? Exercises 1.3.1

Restate 4.79 x 10^5 kg using a metric prefix such that the resulting number is bigger than one but less than 1000.

Answer

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1.4: Unit Conversion

Learning Objectives

- Use conversion factors to express the value of a given quantity in different units.

It is often necessary to convert from one unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you may need to convert units of feet or meters to miles.

Let's consider a simple example of how to convert units. Suppose we want to convert 80 m to kilometers. The first thing to do is to list the units you have and the units to which you want to convert. In this case, we have units in meters and we want to convert to kilometers. Next, we need to determine a conversion factor relating meters to kilometers. A **conversion factor** is a ratio that expresses how many of one unit are equal to another unit. For example, there are 12 in. in 1 ft, 1609 m in 1 mi, 100 cm in 1 m, 60 s in 1 min, and so on. Refer to Appendix B for a more complete list of conversion factors. In this case, we know that there are 1000 m in 1 km. Now we can set up our unit conversion. We write the units we have and then multiply them by the conversion factor so the units cancel out, as shown:

$$80\ \cancel{m} \times \frac{1\ km}{1000\ \cancel{m}} = 0.080\ km. \tag{1.4.1}$$

Note that the unwanted meter unit cancels, leaving only the desired kilometer unit. You can use this method to convert between any type of unit. Now, the conversion of 80 m to kilometers is simply the use of a metric prefix, as we saw in the preceding section, so we can get the same answer just as easily by noting that

$$80\ m = 8.0 \times 10^1\ m = 8.0 \times 10^{-2}\ km = 0.080\ km, \tag{1.4.2}$$

since "kilo-" means 10^3 and 1 = −2 + 3. However, using conversion factors is handy when converting between units that are not metric or when converting between derived units, as the following examples illustrate.

Example 1.4.1: Converting Nonmetric Units to Metric

The distance from the university to home is 10 mi and it usually takes 20 min to drive this distance. Calculate the average speed in meters per second (m/s). (**Note**: Average speed is distance traveled divided by time of travel.)

Strategy

First we calculate the average speed using the given units, then we can get the average speed into the desired units by picking the correct conversion factors and multiplying by them. The correct conversion factors are those that cancel the unwanted units and leave the desired units in their place. In this case, we want to convert miles to meters, so we need to know the fact that there are 1609 m in 1 mi. We also want to convert minutes to seconds, so we use the conversion of 60 s in 1 min.

Solution

1. Calculate average speed. Average speed is distance traveled divided by time of travel. (Take this definition as a given for now. Average speed and other motion concepts are covered in later chapters.) In equation form,

$$Average\ speed = \frac{Distance}{Time}.$$

2. Substitute the given values for distance and time:

$$Average\ speed = \frac{10\ mi}{20\ min} = 0.50\ \frac{mi}{min}.$$

3. Convert miles per minute to meters per second by multiplying by the conversion factor that cancels miles and leave meters, and also by the conversion factor that cancels minutes and leave seconds:

$$0.50 \frac{\cancel{mile}}{\cancel{min}} \times \frac{1609\ m}{1\ \cancel{mile}} \times \frac{1\ \cancel{min}}{60\ s} = \frac{(0.50)(1609)}{60} m/s = 13\ m/s.$$

Significance

Check the answer in the following ways:

1. Be sure the units in the unit conversion cancel correctly. If the unit conversion factor was written upside down, the units do not cancel correctly in the equation. We see the "miles" in the numerator in 0.50 mi/min cancels the "mile" in the denominator in the first conversion factor. Also, the "min" in the denominator in 0.50 mi/min cancels the "min" in the numerator in the second conversion factor.
2. Check that the units of the final answer are the desired units. The problem asked us to solve for average speed in units of meters per second and, after the cancelations, the only units left are a meter (m) in the numerator and a second (s) in the denominator, so we have indeed obtained these units.

? Exercise 1.4.1

Light travels about 9 Pm in a year. Given that a year is about 3 x 10^7 s, what is the speed of light in meters per second?

Answer

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✓ Example 1.4.2: Converting between Metric Units

The density of iron is 7.86 g/cm^3 under standard conditions. Convert this to kg/m^3.

Strategy

We need to convert grams to kilograms and cubic centimeters to cubic meters. The conversion factors we need are 1 kg = 10^3 g and 1 cm = 10^{-2}m. However, we are dealing with cubic centimeters (cm^3 = cm x cm x cm), so we have to use the second conversion factor three times (that is, we need to cube it). The idea is still to multiply by the conversion factors in such a way that they cancel the units we want to get rid of and introduce the units we want to keep.

Solution

$$7.86 \frac{\cancel{g}}{\cancel{cm^3}} \times \frac{kg}{10^3\ \cancel{g}} \times \left(\frac{\cancel{cm}}{10^{-2}\ m}\right)^3 = \frac{7.86}{(10^3)(10^{-6})} kg/m^3 = 7.86 \times 10^3\ kg/m^3$$

Significance

Remember, it's always important to check the answer.

1. Be sure to cancel the units in the unit conversion correctly. We see that the gram ("g") in the numerator in 7.86 g/cm^3 cancels the "g" in the denominator in the first conversion factor. Also, the three factors of "cm" in the denominator in 7.86 g/cm^3 cancel with the three factors of "cm" in the numerator that we get by cubing the second conversion factor.
2. Check that the units of the final answer are the desired units. The problem asked for us to convert to kilograms per cubic meter. After the cancelations just described, we see the only units we have left are "kg" in the numerator and three factors of "m" in the denominator (that is, one factor of "m" cubed, or "m^3"). Therefore, the units on the final answer are correct.

? Exercise 1.4.2

We know from Figure 1.4 that the diameter of Earth is on the order of 10^7 m, so the order of magnitude of its surface area is 10^{14} m^2. What is that in square kilometers (that is, km^2)? (Try doing this both by converting 10^7 m to km and then squaring it and then by converting 10^{14} m^2 directly to square kilometers. You should get the same answer both ways.)

Answer

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Unit conversions may not seem very interesting, but not doing them can be costly. One famous example of this situation was seen with the **Mars Climate Orbiter**. This probe was launched by NASA on December 11, 1998. On September 23, 1999, while attempting to guide the probe into its planned orbit around Mars, NASA lost contact with it. Subsequent investigations showed a piece of software called SM_FORCES (or “small forces”) was recording thruster performance data in the English units of pound-seconds (lb • s). However, other pieces of software that used these values for course corrections expected them to be recorded in the SI units of newton-seconds (N • s), as dictated in the software interface protocols. This error caused the probe to follow a very different trajectory from what NASA thought it was following, which most likely caused the probe either to burn up in the Martian atmosphere or to shoot out into space. This failure to pay attention to unit conversions cost hundreds of millions of dollars, not to mention all the time invested by the scientists and engineers who worked on the project.

Exercise 1.4.3

Given that 1 lb (pound) is 4.45 N, were the numbers being output by SM_FORCES too big or too small?

1.5: Significant Figures

Learning Objectives

- Determine the correct number of significant figures for the result of a computation.
- Describe the relationship between the concepts of accuracy, precision, uncertainty, and discrepancy.
- Calculate the percent uncertainty of a measurement, given its value and its uncertainty.
- Determine the uncertainty of the result of a computation involving quantities with given uncertainties.

Figure 1.5.1 shows two instruments used to measure the mass of an object. The digital scale has mostly replaced the double-pan balance in physics labs because it gives more accurate and precise measurements. But what exactly do we mean by **accurate and precise**? Aren't they the same thing? In this section we examine in detail the process of making and reporting a measurement.

(a) (b)

Figure 1.5.1: (a) A double-pan mechanical balance is used to compare different masses. Usually an object with unknown mass is placed in one pan and objects of known mass are placed in the other pan. When the bar that connects the two pans is horizontal, then the masses in both pans are equal. The "known masses" are typically metal cylinders of standard mass such as 1 g, 10 g, and 100 g. (b) Many mechanical balances, such as double-pan balances, have been replaced by digital scales, which can typically measure the mass of an object more precisely. A mechanical balance may read only the mass of an object to the nearest tenth of a gram, but many digital scales can measure the mass of an object up to the nearest thousandth of a gram. (credit a: modification of work by Serge Melki; credit b: modification of work by Karel Jakubec)

Accuracy and Precision of a Measurement

Science is based on observation and experiment—that is, on measurements. **Accuracy** is how close a measurement is to the accepted reference value for that measurement. For example, let's say we want to measure the length of standard printer paper. The packaging in which we purchased the paper states that it is 11.0 in. long. We then measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in. These measurements are quite accurate because they are very close to the reference value of 11.0 in. In contrast, if we had obtained a measurement of 12 in., our measurement would not be very accurate. Notice that the concept of accuracy requires that an accepted reference value be given.

The **precision** of measurements refers to how close the agreement is between repeated independent measurements (which are repeated under the same conditions). Consider the example of the paper measurements. The precision of the measurements refers to the spread of the measured values. One way to analyze the precision of the measurements is to determine the range, or difference, between the lowest and the highest measured values. In this case, the lowest value was 10.9 in. and the highest value was 11.2 in. Thus, the measured values deviated from each other by, at most, 0.3 in. These measurements were relatively precise because they did not vary too much in value. However, if the measured values had been 10.9 in., 11.1 in., and 11.9 in., then the measurements would not be very precise because there would be significant variation from one measurement to another. Notice that the concept of precision depends only on the actual measurements acquired and does not depend on an accepted reference value.

The measurements in the paper example are both accurate and precise, but in some cases, measurements are accurate but not precise, or they are precise but not accurate. Let's consider an example of a GPS attempting to locate the position of a restaurant in a city. Think of the restaurant location as existing at the center of a bull's-eye target and think of each GPS attempt to locate the restaurant as a black dot. In Figure $1.5.1a$, we see the GPS measurements are spread out far apart from each other, but they are all relatively close to the actual location of the restaurant at the center of the target. This indicates a low-precision, high-accuracy measuring system. However, in Figure $1.5.1b$, the GPS measurements are concentrated quite closely to one another, but they are far away from the target location. This indicates a high-precision, low-accuracy measuring system.

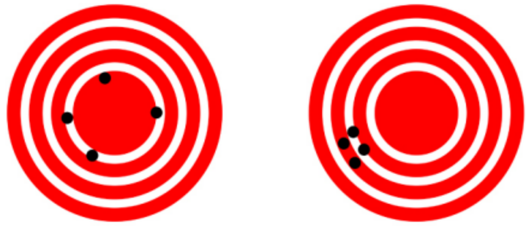

Figure $1.5.2$: A GPS attempts to locate a restaurant at the center of the bull's-eye. The black dots represent each attempt to pinpoint the location of the restaurant. (a) The dots are spread out quite far apart from one another, indicating low precision, but they are each rather close to the actual location of the restaurant, indicating high accuracy. (b) The dots are concentrated rather closely to one another, indicating high precision, but they are rather far away from the actual location of the restaurant, indicating low accuracy. (credit a and credit b: modification of works by Dark Evil)

Accuracy, Precision, Uncertainty, and Discrepancy

The precision of a measuring system is related to the **uncertainty** in the measurements whereas the accuracy is related to the **discrepancy** from the accepted reference value. Uncertainty is a quantitative measure of how much your measured values deviate from one another. There are many different methods of calculating uncertainty, each of which is appropriate to different situations. Some examples include taking the range (that is, the biggest less the smallest) or finding the standard deviation of the measurements. Discrepancy (or "measurement error") is the difference between the measured value and a given standard or expected value. If the measurements are not very precise, then the uncertainty of the values is high. If the measurements are not very accurate, then the discrepancy of the values is high.

Recall our example of measuring paper length; we obtained measurements of 11.1 in., 11.2 in., and 10.9 in., and the accepted value was 11.0 in. We might average the three measurements to say our best guess is 11.1 in.; in this case, our discrepancy is 11.1 – 11.0 = 0.1 in., which provides a quantitative measure of accuracy. We might calculate the uncertainty in our best guess by using the range of our measured values: 0.3 in. Then we would say the length of the paper is 11.1 in. plus or minus 0.3 in. The uncertainty in a measurement, A, is often denoted as δA (read "delta A"), so the measurement result would be recorded as A ± δA. Returning to our paper example, the measured length of the paper could be expressed as 11.1 ± 0.3 in. Since the discrepancy of 0.1 in. is less than the uncertainty of 0.3 in., we might say the measured value agrees with the accepted reference value to within experimental uncertainty.

Some factors that contribute to uncertainty in a measurement include the following:

- Limitations of the measuring device
- The skill of the person taking the measurement
- Irregularities in the object being measured
- Any other factors that affect the outcome (highly dependent on the situation)

In our example, such factors contributing to the uncertainty could be the smallest division on the ruler is 1/16 in., the person using the ruler has bad eyesight, the ruler is worn down on one end, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be calculated to quantify its precision. If a reference value is known, it makes sense to calculate the discrepancy as well to quantify its accuracy.

Percent uncertainty

Another method of expressing uncertainty is as a percent of the measured value. If a measurement A is expressed with uncertainty δA, the percent uncertainty is defined as

$$Percent\ uncertainty = \frac{\delta A}{A} \times 100\% \tag{1.5.1}$$

Example 1.5.1: Calculating Percent Uncertainty: A Bag of Apples

A grocery store sells 5-lb bags of apples. Let's say we purchase four bags during the course of a month and weigh the bags each time. We obtain the following measurements:

- Week 1 weight: 4.8 lb
- Week 2 weight: 5.3 lb
- Week 3 weight: 4.9 lb
- Week 4 weight: 5.4 lb

We then determine the average weight of the 5-lb bag of apples is 5.1 ± 0.3 lb. What is the percent uncertainty of the bag's weight?

Strategy

First, observe that the average value of the bag's weight, A, is 5.1 lb. The uncertainty in this value, δA, is 0.3 lb. We can use the following equation to determine the percent uncertainty of the weight:

$$Percent\ uncertainty = \frac{\delta A}{A} \times 100\% \tag{1.5.2}$$

Solution

Substitute the values into the equation:

$$Percent\ uncertainty = \frac{\delta A}{A} \times 100\% = \frac{0.3\ lb}{5.1\ lb} \times 100\% = 5.9\% \approx 6\% \tag{1.5.3}$$

Significance

We can conclude the average weight of a bag of apples from this store is 5.1 lb ± 6%. Notice the percent uncertainty is dimensionless because the units of weight in δA = 0.3 lb canceled those inn A = 5.1 lb when we took the ratio.

Exercises 1.5.1

A high school track coach has just purchased a new stopwatch. The stopwatch manual states the stopwatch has an uncertainty of ±0.05 s. Runners on the track coach's team regularly clock 100-m sprints of 11.49 s to 15.01 s. At the school's last track meet, the first-place sprinter came in at 12.04 s and the second-place sprinter came in at 12.07 s. Will the coach's new stopwatch be helpful in timing the sprint team? Why or why not?

Uncertainties in Calculations

Uncertainty exists in anything calculated from measured quantities. For example, the area of a floor calculated from measurements of its length and width has an uncertainty because the length and width have uncertainties. How big is the uncertainty in something you calculate by multiplication or division? If the measurements going into the calculation have small uncertainties (a few percent or less), then the **method of adding percents** can be used for multiplication or division. This method states **the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation**. For example, if a floor has a length of 4.00 m and a width of 3.00 m, with uncertainties of 2% and 1%, respectively, then the area of the floor is 12.0 m^2 and has an uncertainty of 3%. (Expressed as an area, this is 0.36 m^2 [12.0 m^2 x 0.03], which we round to 0.4 m^2 since the area of the floor is given to a tenth of a square meter.)

Precision of Measuring Tools and Significant Figures

An important factor in the precision of measurements involves the precision of the measuring tool. In general, a precise measuring tool is one that can measure values in very small increments. For example, a standard ruler can measure length to the nearest millimeter whereas a caliper can measure length to the nearest 0.01 mm. The caliper is a more precise measuring tool because it can measure extremely small differences in length. The more precise the measuring tool, the more precise the measurements.

When we express measured values, we can only list as many digits as we measured initially with our measuring tool. For example, if we use a standard ruler to measure the length of a stick, we may measure it to be 36.7 cm. We can't express this value as 36.71 cm because our measuring tool is not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices the stick length seems to be somewhere in between 36.6 cm and 36.7 cm, and he or she must estimate the value of the last digit. Using the method of **significant figures**, the rule is that **the last digit written down in a measurement is the first digit with some uncertainty**. To determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value 36.7 cm has three digits, or three significant figures. Significant figures indicate the precision of the measuring tool used to measure a value.

Zeros

Special consideration is given to zeros when counting significant figures. The zeros in 0.053 are not significant because they are placeholders that locate the decimal point. There are two significant figures in 0.053. The zeros in 10.053 are not placeholders; they are significant. This number has five significant figures. The zeros in 1300 may or may not be significant, depending on the style of writing numbers. They could mean the number is known to the last digit or they could be placeholders. So 1300 could have two, three, or four significant figures. To avoid this ambiguity, we should write 1300 in scientific notation as 1.3 x 10^3, 1.30 x 10^3, or 1.300 x 10^3, depending on whether it has two, three, or four significant figures. **Zeros are significant except when they serve only as placeholders**.

Significant Figures in Calculations

When combining measurements with different degrees of precision, **the number of significant digits in the final answer can be no greater than the number of significant digits in the least-precise measured value**. There are two different rules, one for multiplication and division and the other for addition and subtraction.

1. **For multiplication and division, the result should have the same number of significant figures as the quantity with the least number of significant figures entering into the calculation.** For example, the area of a circle can be calculated from its radius using A = πr^2. Let's see how many significant figures the area has if the radius has only two—say, r = 1.2 m. Using a calculator with an eight-digit output, we would calculate

$$A = \pi r^2 = (3.1415927...) \times (1.2\ m)^2 = 4.5238934\ m^2. \tag{1.5.4}$$

But because the radius has only two significant figures, it limits the calculated quantity to two significant figures, or

$$A = 4.5\ m^2. \tag{1.5.5}$$

although π is good to at least eight digits.

2. **For addition and subtraction, the answer can contain no more decimal places than the least-precise measurement**. Suppose we buy 7.56 kg of potatoes in a grocery store as measured with a scale with precision 0.01 kg, then we drop off 6.052 kg of potatoes at your laboratory as measured by a scale with precision 0.001 kg. Then, we go home and add 13.7 kg of potatoes as measured by a bathroom scale with precision 0.1 kg. How many kilograms of potatoes do we now have and how many significant figures are appropriate in the answer? The mass is found by simple addition and subtraction:

$$\begin{array}{l} \ \ 7.56\ kg \\ -6.052\ kg \\ \underline{+13.7\ kg\qquad\qquad} \\ 15.208\ kg = 15.2\ kg. \end{array}$$

Next, we identify the least-precise measurement: 13.7 kg. This measurement is expressed to the 0.1 decimal place, so our final answer must also be expressed to the 0.1 decimal place. Thus, the answer is rounded to the tenths place, giving us 15.2 kg.

Significant Figures in This Text

In this text, most numbers are assumed to have three significant figures. Furthermore, consistent numbers of significant figures are used in all worked examples. An answer given to three digits is based on input good to at least three digits, for example. If the input has fewer significant figures, the answer will also have fewer significant figures. Care is also taken that the number of significant figures is reasonable for the situation posed. In some topics, particularly in optics, more accurate numbers are needed and we use more than three significant figures. Finally, if a number is exact, such as the two in the formula for the circumference of a circle, C = $2\pi r$, it does not affect the number of significant figures in a calculation. Likewise, conversion factors such as 100 cm/1 m are considered exact and do not affect the number of significant figures in a calculation.

CHAPTER OVERVIEW

2: C2) Particles and Interactions

In this chapter we will study particles and interactions. We will make our first step into the abstraction of the real world by **replacing any objects we want to describe with points**. As we talked about in the last chapter, this is to make things not only 1) simpler, but more importantly 2) mathematical well-defined. The way this works with points is dead simple - take a real object, made up of many (essentially an infinite number of) points, and pick one of it's point to represent the entire thing. This is now well-defined because we can now talk about the position $\vec{r}$ of the object as being exactly the position of the point we chose. And it's far simpler, since we are only dealing with a single point instead of innumerably many.

So after we have a bunch of points, what then? Well, we want to understand the **interactions between the points** (remember, that was one of the ways we described what physics is - the study of these interactions). While we might be tempted to call these interactions "forces", in many cases the forces acting on a particular object are relatively complicated. So first, we are actually going to choose the simplist possible interaction between two objects we can consider - a collision. This is simply when one object comes into contact with another, changing the motion of both.

So what can happen in such a collision? Well, the position $\vec{r}$ and the velocity $\vec{v}$ of either object can change - and that's kind of it, since our objects are just points! The points might have masses, for sure, but for now we are going to assume these masses don't change. (Consider a two-car collision - for sure, some mass is moved back and forth if the collision is bad enough, but generally the two cars keep all their points within each other and don't exchange them.) But how does (for example) the velocity change when the objects interact? It turns out that the velocity is actually not the thing that tells us what happens when two objects collide, it's actually the product of the mass and the velocity, the momentum $\vec{p}=m\vec{v}$. It's pretty intuitive that both the mass and the velocity have to play a role here, since a heavy object hitting a light object is going to be different from two light objects hitting each other. This brings us to our first real principle of physics we are going to study:

Principle of Momentum

Objects interact by transferring momentum

Although this statement seems quite trivial, it actually allows us to perform our first calculations. Consider a collision between two points, each of mass 1 kg, and let's say one of the objects is at rest, while the other is moving towards the first at a speed of 10 m/s. That means the first object has a momentum of 10 kg m/s, while the second has zero (since it's not moving). After they collide, let's say the second object moved away at a speed of 2 m/s. That means it gained 2 kg m/s of momentum...and based on our principle of momentum transfer, the first object must have lost that same 2 kg m/s, leaving it with 8 kg m/s. But that also means we know how fast the object is moving after the collision - since it's mass is 1 kg, it's moving at 8 m/s! This simple example gives us all the essential concepts we are going to be studying not just for this chapter, but for the entire first half of the text.

A brief final note - we didn't really consider anything about the directions that the two objects moved in that example, because it was pretty clear they were moving in a straight line. However, some quantities in physics carry direction, like velocity $\vec{v}$ and position $\vec{r}$ - what about momentum? Well, we defined momentum with a particular formula, $m\vec{v}$. Velocity carries direction, but mass does not, so the momentum will be in the same direction as the velocity. This makes perfect sense with our momentum transfer principle above as well - an object with a particular momentum $\vec{p}$ in a particular direction will transfer that same momentum to the other object, in the same direction.

This chapter is dedicated to a conceptual understanding of this principle - not that there are no calculations, but we are going to be a little careless with things like vectors and directions. We first want to be sure we get some of the conceptual ideas surrounding momentum transfer, and in later chapters we will add in the mathematical formalism required to handle the directions - and all kinds of other interesting collisions!

2.1: Inertia

In everyday language, we speak of something or someone "having a large inertia" to mean, essentially, that they are very difficult to set in motion. We do know, from experience, that lighter objects are easier to set in motion than heavier objects, but most of us probably have an intuition that gravity (the force that pulls an object towards the earth and hence determines its weight) is not involved in an essential way here. Imagine, for instance, the difference between slapping a volleyball and a bowling ball. It is not hard to believe that the latter would hurt as much if we did it while floating in free fall in the space station (in a state of effective "weightlessness") as if we did it right here on the surface of the earth. In other words, it is not (necessarily) how heavy something feels, but just how *massive* it is.

But just what is this "massiveness" quality that we associate intuitively with a large inertia? Is there a way (other than resorting to the weight again) to assign to it a numerical value?

Relative Inertia and Collisions

One possible way to determine the *relative* inertias of two objects, conceptually, at least, is to try to use one of them to set the other one in motion. Most of us are familiar with what happens when two identical objects (presumably, therefore, having the same inertia) collide: if the collision is head-on (so the motion, before and after, is confined to a straight line), they basically exchange velocities. For instance, a billiard ball hitting another one will stop dead and the second one will set off with the same speed as the first one. The toy sometimes called "Newton's balls" or "Newton's cradle" also shows this effect. Intuitively, we understand that what it takes to stop the first ball is exactly the same as it would take to set the second one in motion with the same velocity.

But what if the objects colliding have different inertias? We expect that the change in their velocities as a result of the collision will be different: the velocity of the object with the largest inertia will not change very much, and conversely, the change in the velocity of the object with the smallest inertia will be comparatively larger. A velocity vs. time graph for the two objects might look somewhat like the one sketched in Figure $2.1.1$.

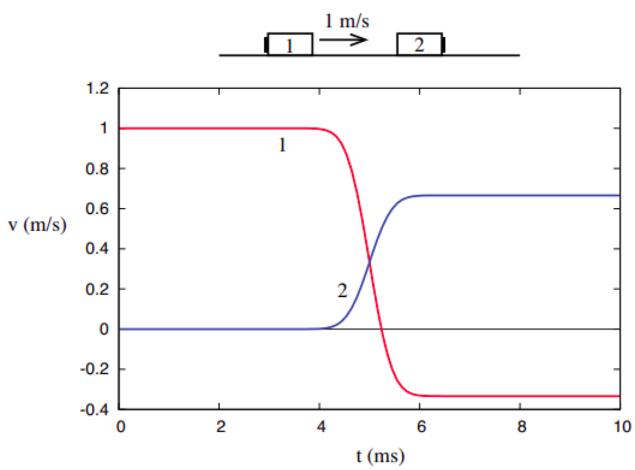

Figure $2.1.1$: An example of a velocity vs. time graph for a collision of two objects with different inertias.

In this picture, object 1, initially moving with velocity v_{1i} = 1 m/s, collides with object 2, initially at rest. After the collision, which here is assumed to take a millisecond or so, object 1 actually bounces back (notice it's velocity goes negative in the plot), so its final velocity is v_{1f} = −1/3 m/s, whereas object 2 ends up moving to the right with velocity v_{2f} = 2/3 m/s. So the change in the velocity of object 1 is $\Delta v_1 = v_{1f} - v_{1i}$ = −4/3 m/s, whereas for object 2 we have $\Delta v_2 = v_{2f} - v_{2i}$ = 2/3 m/s.

It is tempting to use this ratio, $\Delta v_1/\Delta v_2$, as a measure of the *relative inertia* of the two objects, only we'd want to use it upside down and with the opposite sign: that is, since $\Delta v_2/\Delta v_1$ = −1/2 we would say that object 2 has *twice* the inertia of object 1. But then we have to ask: is this a reliable, repeatable measure? Will it work for any kind of collision (within reason, of course: we clearly need to stay in one dimension, and eliminate external influences such as friction), and for any initial velocity?

To begin with, we have reason to expect that it does not matter whether we shoot object 1 towards object 2 or object 2 towards object 1, since *t*he relative motion is the same in both cases. Consider, for instance, what the collision in Figure $2.1.1$ appears like to a hypothetical observer moving along with object 1, at 1 m/s. To him, object 1 appears to be at rest, and it is object 2 that is coming towards him, with a velocity of −1 m/s. To see what the outcome of the collision looks like to him, just add the same −1 m/s to the final velocities we obtained before: object 1 will end up moving at v_{1f} = −4/3 m/s, and object 2 would move at v_{2f} = −1/3 m/s, and we would have a situation like the one shown in Figure $2.1.2$, where both curves have simply been shifted down by 1 m/s:

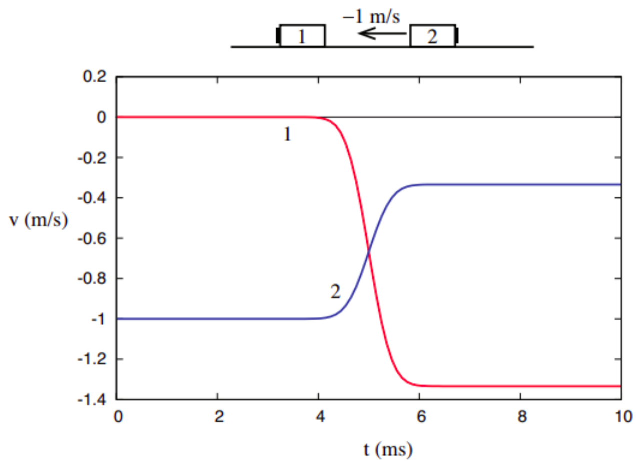

Figure $2.1.2$: Another example (really the same collision as in Figure $2.1.1$, only as seen by an observer initially moving to the right at 1 m/s).

But then, this is exactly what we should expect to find also in our laboratory if we actually did send the second object at 1 m/s towards the first one sitting at rest. All the individual velocities have changed relative to Figure $2.1.1$, but the *velocity changes*, Δv_1 and Δv_2, are clearly still the same, and therefore so is our (tentative) measure of the objects' relative inertia.

Clearly, the same argument can be used to conclude that the same result will be obtained when both objects are initially moving towards each other, as long as their *relative velocity* is the same as in these examples, namely, 1 m/s. However, unless we do the experiments we cannot really predict what will happen if we increase (or decrease) their relative velocity. In fact, we could imagine smashing the two objects at very high speed, so they might even become seriously mangled in the process. Yet, experimentally (and this is not at all an obvious result!), we would still find the same value of −1/2 for the ratio $\Delta v_2/\Delta v_1$, at least as long as the collision is not so violent that the objects actually break up into pieces.

Perhaps the most surprising result of our experiments would be the following: imagine that the objects have a "sticky" side (for instance, the small black rectangles shown in the pictures could be strips of Velcro), and we turn them around so that when they collide they will end up stuck to each other. In this case (which, as we shall see later, is termed a **completely inelastic** collision), the v-vs-t graph might look like Figure $2.1.3$ below.

Now the two objects end up moving together to the right, fairly slowly: $v_{1f} = v_{2f}$ = 1/3 m/s. The velocity changes are Δv_1 = −2/3 m/s and Δv_2 = 1/3 m/s, both of which are different from what they were before, in Figs. $2.1.1$ and $2.1.2$: yet, the ratio $\Delta v_2/\Delta v_1$ is still equal to −1/2, just as in all the previous cases.

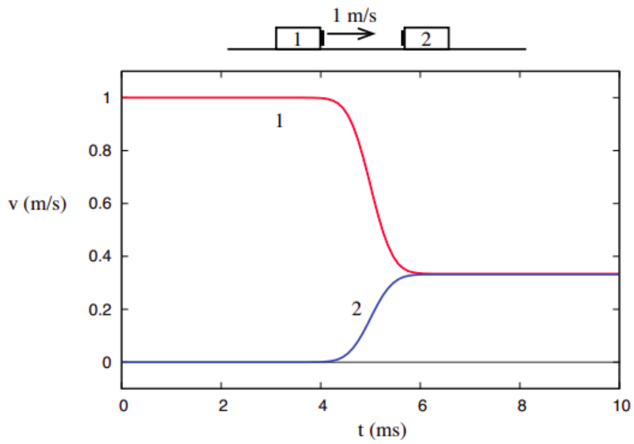

Figure 2.1.3 became stuck together when they collided.

Inertial Mass: Definition and Properties

At this point, it would seem reasonable to assume that this ratio, $\Delta v_2/\Delta v_1$, is, in fact, telling us something about an *intrinsic* property of the two objects, what we have called above their "relative inertia." It is easy, then, to see how one could assign a value to the inertia of any object (at least, conceptually): choose a "standard" object, and decide, arbitrarily, that its inertia will have the numerical value of 1, in whichever units you choose for it (these units will turn out, in fact, to be kilograms, as you will see in a minute). Then, to determine the inertia of another object, which we will label with the subscript 1, just arrange a one-dimensional collision between object 1 and the standard, under the right conditions (basically, no net external forces), measure the velocity changes Δv_1 and Δv_s, and take the quantity $-\Delta v_s/\Delta v_1$ as the numerical value of the ratio of the inertia of object 1 to the inertia of the standard object. In symbols, using the letter m to represent an object's inertia,

$$\frac{m_1}{m_s} = -\frac{\Delta v_s}{\Delta v_1} \tag{2.1.1}$$

But, since $m_s = 1$ by definition, this gives us directly the numerical value of m_1.

The reason we use the letter m is, as you must have guessed, because, in fact, the inertia defined in this way turns out to be identical to what we have traditionally called "mass." More precisely, the quantity defined this way is an object's *inertial mass*. The remarkable fact, mentioned earlier, that the force of gravity between two objects turns out to be proportional to their inertial masses, allows us to determine the inertial mass of an object by the more traditional procedure of simply weighing it, rather than elaborately staging a collision between it and the standard kilogram on an ice-hockey rink. But, in principle, we could conceive of the existence of two different quantities that should be called "inertial mass" and "gravitational mass," and the identity (or more precisely, the—so far as we know—exact proportionality) of the two is a rather mysterious experimental fact[1].

In any case, by the way we have constructed it, the inertial mass, defined as in Equation (2.1.1), does capture, in a quantitative way, the concept that we were trying to express at the beginning of the chapter: namely, how difficult it may be to set an object in motion. In principle, however, other experiments would need to be conducted to make sure that it does have the properties we have traditionally associated with the concept of mass. For instance, suppose we join together two objects of mass m. Is the mass of the resulting object $2m$? Collision experiments would, indeed, show this to be the case with great accuracy in the macroscopic world (with which we are concerned this semester), but this is a good example of how you cannot take anything for granted: at the microscopic level, it is again a fact that the inertial mass of an atomic nucleus is a little *less* than the sum of the masses of all its constituent protons and neutrons[2].

Probably the last thing that would need to be checked is that *the ratio of inertias is independent of the standard*. Suppose that we have two objects, to which we have assigned masses m_1 and m_2 by arranging for each to collide with the "standard object" independently. If we now arrange for a collision between objects 1 and 2 directly, will we actually find that the ratio of their velocity changes is given by the ratio of the separately determined masses m_1 and m_2? We certainly would need that to be the case, in order for the concept of inertia to be truly useful; but again, we should not assume anything until we have tested it! Fortunately, the tests would indeed reveal that, in every case, the expected relationship holds[3]

$$-\frac{\Delta v_2}{\Delta v_1} = \frac{m_1}{m_2}. \tag{2.1.2}$$

At this point, we are not just in possession of a useful definition of inertia, but also of a veritable *law of nature*, as we will explain next.

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[1] This fact, elevated to the category of a principle by Einstein (the *equivalence principle*) is the starting point of the general theory of relativity.

[2] And this is not just an unimportant bit of trivia: all of nuclear power depends on this small difference.

[3] Equation 2.1.2 actually is found to hold also at the microscopic (or *quantum*) level, although there we prefer to state the result by saying that conservation of momentum holds (see the following section).

2.2: Momentum

Inspired by Equation (2.1.2), let's define an objects momentum as

$$p = mv, \tag{2.2.1}$$

with mass m and velocity v, moving in one dimension (the choice of the letter p for momentum is apparently related to the Latin word “im*p*etus”).

We can think of momentum as a sort of extension of the concept of inertia, from an object at rest to an object in motion. When we speak of an object's inertia, we typically think about what it may take to get it moving; when we speak of its momentum, we typically think of that it may take to stop it (or perhaps deflect it). So, both the inertial mass m and the velocity v are involved in the definition.

We may also observe that what looks like inertia in some reference frame may look like momentum in another. For instance, if you are driving in a car towing a trailer behind you, the trailer has only a large amount of inertia, but no momentum, relative to you, because its velocity relative to you is zero; however, the trailer definitely has a large amount of momentum (by virtue of both its inertial mass and its velocity) relative to somebody standing by the side of the road.

Conservation of Momentum; Isolated Systems

For a system of objects, we treat the momentum as an *additive* quantity. So, if two colliding objects, of masses m_1 and m_2, have initial velocities v_{1i} and v_{2i}, we say that the total initial momentum of the system is $p_i = m_1 v_{1i} + m_2 v_{2i}$, and similarly if the final velocities are v_{1f} and v_{2f}, the total final momentum will be $p_f = m_1 v_{1f} + m_2 v_{2f}$.

We then assert that *the total momentum of the system is not changed by the collision*. Mathematically, this means

$$p_i = p_f \tag{2.2.2}$$

or

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}. \tag{2.2.3}$$

But this last equation, in fact, follows directly from Equation (2.1.2): to see this, move all the quantities in Equation (2.2.3) having to do with object 1 to one side of the equal sign, and those having to do with object 2 to the other side. You then get

$$\begin{aligned} m_1 \left(v_{1i} - v_{1f}\right) &= m_2 \left(v_{2f} - v_{2i}\right) \\ -m_1 \Delta v_1 &= m_2 \Delta v_2 \end{aligned} \tag{2.2.4}$$

which is just another way to write Equation (2.1.2). Hence, the result (2.1.2) ensures the conservation of the total momentum of a system of any two interacting objects (“particles”), regardless of the form the interaction takes, as long as there are no external forces acting on them.

Momentum conservation is one of the most important principles in all of physics, so let's take a little time to explain how we got here and elaborate on this result. First, as just mentioned, we have been more or less implicitly assuming that the two interacting objects form an *isolated* system, by which we mean that, throughout, they interact with nothing other than each other. (Equivalently, there are no external forces acting on them.)

It is pretty much impossible to set up a system so that it is *really* isolated in this strict sense; instead, in practice, we settle for making sure that the external forces on the two objects *cancel out*. This is what happens in the car collisions we have been considering so far: gravity is acting on the carts, but that force is balanced out by the upwards push of the track. A system on which there is no *net* external force is as good as isolated for practical purposes, and we will refer to it as such. (It is harder, of course, to completely eliminate friction and drag forces, so we just have to settle for approximately isolated systems in practice.)

Secondly, we have assumed so far that the motion of the two objects is restricted to a straight line—one dimension. In fact, momentum is a *vector* quantity (just like velocity is), so in general we should write

$$\vec{p} = m\vec{v}$$

and conservation of momentum, in general, holds as a vector equation for any isolated system in three dimensions:

$$\vec{p}_i = \vec{p}_f. \tag{2.2.5}$$

What this means, in turn, is that each separate component (x, y and z) of the momentum will be separately conserved (so Equation (2.2.5) is equivalent to three scalar equations, in three dimensions). When we get to study the vector nature of forces, we will see an interesting implication of this, namely, that it is possible for one component of the momentum vector to be conserved, but not another—depending on whether there is or there isn't a net external force in that direction or not. For example, anticipating things a bit, when you throw an object horizontally, as long as you can ignore air drag, there is no horizontal force acting on it, and so that component of the momentum vector is conserved, but the vertical component is changing all the time because of the (vertical) force of gravity.

Thirdly, although this may not be immediately obvious, for an isolated system of two colliding objects the momentum is truly conserved throughout the whole collision process. It is not just a matter of comparing the initial and final velocities: at any of the times shown in Figures 2.1.1, 2.1.2, or 2.1.3, if we were to measure v_1 and v_2 and compute $m_1v_1 + m_2v_2$, we would obtain the same result. In other words, the total momentum of an isolated system is *constant*: it has the same value at all times.

Finally, all these examples have involved interactions between only two particles. Can we really generalize this to conclude that the total momentum of an isolated system of any number of particles is constant, even when all the particles may be interacting with each other simultaneously? Here, again, the experimental evidence is overwhelmingly in favor of this hypothesis[4], but much of our confidence on its validity comes in fact from a consideration of the nature of the internal interactions themselves. It is a mathematical fact that all of the interactions so far known to physics have the property of conserving momentum, whether acting individually or simultaneously. No experiments have ever suggested the existence of an interaction that does not have this property.

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[4]For an important piece of indirect evidence, just consider that any extended object is in reality a collection of interacting particles, and the experiments establishing conservation of momentum almost always involve such extended objects. See the following section for further thoughts on this matter.

2.3: Force and Impulse

In the previous section we looked at the quantity $m\Delta v$, but notice that if mass is now changing, this is the same as the change in momentum of the object itself, Δp. That makes Equation (2.2.4) look like $\Delta p_1 = -\Delta p_2$, which is the statement that if one object gains $\Delta\vec{p}$, the object that it's interacting with must have lost that same amount of momentum. This phenomena known as **a conservation law**.

But of course we know that this is not the only way to describe interacting objects - in fact, perhaps the most intuitive way to describe two objects interacting is actually using a **force between them**. We are going to talk a lot more about forces in the second half of the this book, but right now we just want to acknowledge that if interactions can be described with either forces or momentum transfer, there must be some relationship between these two quantities. In fact, the relationship can be made concrete once you know the **time period $\boldsymbol{\Delta t}$** over which the interaction occurs.

Specifically, if you have a change in momentum $\Delta\vec{p}$ resulting from an interaction that happens over a time period Δt, we can associate a force with this interaction via

$$\boxed{\vec{F}_{ave} = \frac{\Delta\vec{p}}{\Delta t}} \tag{2.3.1}$$

Notice that this definition only really works for an *average* force $\vec{F}_{ave}$, since we are talking about the changes in time being possibly relatively large. If the change in time is very small (in the sense of an infinitesimal dt from calculus), we can actually write

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{2.3.2}$$

for the *instantaneous* force, which is the usual notion of force we are used to (this relationship can also be derived from Newton's second law).

Finally, we want to mention one more definition that comes up in this topic, **the impulse $\boldsymbol{\vec{J}}$,** which is the amount of momentum delivered by a particular interaction. In many ways, this is simply a renaming of the change in momentum $\Delta\vec{p}$, and in fact the concrete mathematical definition of the impulse demonstrates this:

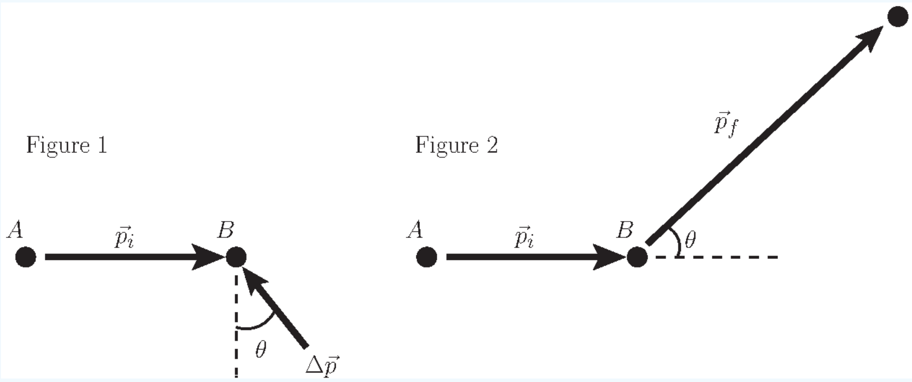
$$\boxed{\vec{J} = \Delta\vec{p}}. \tag{2.3.3}$$

All of the above is a perfectly fine description of a force that does not depend on time, which is often what we are dealing with. However, if a force does depend on time (think of throwing a baseball, or a "real" car crash - the force over the time period of those interactions may not be constant at all), we can still find the impulse delivered by performing an integration,

$$\vec{J} = \int \vec{F} dt. \tag{2.3.4}$$

2.4: Examples

Whiteboard Problem 2.4.1: Hockey Puck Scale Drawings



Consider an extra-heavy hockey puck, of mass 5.0 kg, traveling between points A and B in a straight line at a speed of 10.0 m/s. At point B, it gets hit by a hockey stick which delivers an impulse of $\Delta\vec{p}$ to the puck. Using meter sticks and protractors, make scale drawings of this process on the whiteboards, and answer the following two questions:

1. In figure 1, the amount of impulse delivered is Δp=30 kg m/s at an angle of 30○, as shown.
    1. What is the final momentum of the hockey puck, p_f?
2. In figure 2, we measure the final momentum p_f to be 100 kg m/s, moving at an angle of 40°, as shown.
    1. What was the impulse transferred to the hockey puck, Δp?
    2. At what angle did the impulse act on the puck?

Whiteboard Problem 2.4.2: Momentum Cart Transfer, Part II

Suppose a cart with mass m is moving to the right at a speed of v_0. It collides with a cart of mass $2m$, which is initially at rest. After the collision, the first cart rebounds to the left with a speed of $\frac{1}{3}v_0$.

1. How much impulse did the first cart deliver to the second? Use v_f as the final speed of the second cart.
2. How much impulse did the second cart deliver to the first?
3. Assuming that these two impulse transfers are the same, determine the final speed v_f in terms of the initial speed v_0.

Whiteboard Problem 2.4.3: Bouncing Ball

I drop a golf ball of mass $m = 46$ g onto the floor, and right before it hits it has a velocity of $v_{yi} = -2.2$ m/s.

1. What is the momentum of the ball right before it hits the ground?
2. If the speed of the ball after it bounces, v_{yf} , is 75\% of the speed right before it hits, what was the impulse the floor delivered to the ball?
3. If the ball interacts with the floor for a time period Δt, find an expression for the ratio between the average force the floor acted on the ball with to the weight of the ball mg, in terms of v_{yi} , v_{yf} , g, and Δt.
4. Calculate this ratio using Δt = 150 ms and the other parameters of the problem.

Example 2.4.1: Reading a Collision Graph

The graph shows a collision between two carts (possibly equipped with magnets so that they repel each other before they actually touch) on an air track. The inertia (mass) of cart 1 is 1 kg. Note: this is a *position vs. time* graph!

a. What are the initial velocities of the carts?
b. What are the final velocities of the carts?
c. What is the mass of the second cart?

d. Does the air track appear to be level? Why? (Hint: does the graph show any evidence of acceleration, for either cart, outside of the collision region?)
e. At the collision time, is the change in velocity (Δv)of the first cart positive or negative? How about the second cart? (Justify your answers.)
f. For the system consisting of the two carts, what is its initial (total) momentum? What is its final momentum?
g. Imagine now that one of the magnets is reversed, so when the carts collide they stick to each other. What would then be the final momentum of the system? What would be its final velocity?

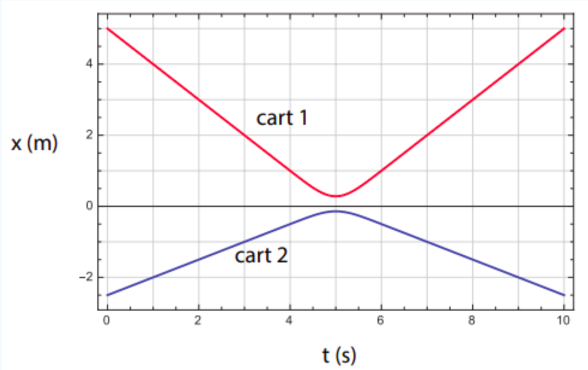


Figure $2.4.1$: A collision between two carts

Solution

(a) All the velocities are to be calculated by picking an easy straight part of each curve and calculating

$$v=\frac{\Delta x}{\Delta t}$$

for suitable intervals. In this way one gets

$$v_{1i}=-1\ \frac{\mathrm{m}}{\mathrm{s}}$$
$$v_{2i}=0.5\ \frac{\mathrm{m}}{\mathrm{s}}$$

(b) Similarly, one gets

$$v_{1f}=1\ \frac{\mathrm{m}}{\mathrm{s}}$$
$$v_{2f}=-0.5\ \frac{\mathrm{m}}{\mathrm{s}}$$

(c) Use this equation, or equivalent (conservation of momentum is OK)

$$\frac{m_2}{m_1}=-\frac{\Delta v_1}{\Delta v_2}$$
$$\frac{m_2}{m_1}=-\frac{1-(-1)}{-0.5-0.5}=2$$

so the mass of the second cart is 2 kg.

(d) Yes, the track appears to be level because the carts do not show any evidence of acceleration outside of the collision region (the position vs. time curves are straight lines outside of the region approximately given by 4.5 s < t < 5.5 s).

(e) The change in velocity of the first cart is positive. You can see this either graphically (the curve is like a parabola that opens upwards, i.e., concave), or algebraically (the cart’s velocity increases, going from −1 m/s to 1 m/s)

Similarly, the change in velocity of the second cart is negative. The curve is like a parabola that opens downwards, i.e., convex; or, algebraically, the cart’s velocity decreases, going from 0.5 m/s to −0.5 m/s.

(f) The initial momentum of the system is

$$p_i = m_1 v_{1i} + m_2 v_{2i} = (1\ \mathrm{kg}) \times \left(-1\ \frac{\mathrm{m}}{\mathrm{s}}\right) + (2\ \mathrm{kg}) \times \left(0.5\ \frac{\mathrm{m}}{\mathrm{s}}\right) = 0$$

The final momentum is

$$p_f = m_1 v_{1f} + m_2 v_{2f} = (1\ \mathrm{kg}) \times \left(1\ \frac{\mathrm{m}}{\mathrm{s}}\right) + (2\ \mathrm{kg}) \times \left(-0.5\ \frac{\mathrm{m}}{\mathrm{s}}\right) = 0$$

You could also just say that the final momentum should be the same as the initial momentum, since the system appears to be isolated.

(g) The momentum should be conserved in this case as well, so p_f = 0. The velocity would be

$$v_f = \frac{p_f}{m_1 + m_2} = 0.$$

CHAPTER OVERVIEW

3: C3) Vector Analysis

The language of physics is mathematics. This is not just a thing we tell ourselves, but is a basic fact that separates the physical sciences from the social sciences. Although we may speak and write about physics using a spoken and written language (English, in this case), this is just for descriptive convenience. Anytime we actually want to do anything with physics, we have to cast it into mathematics. Part of this is a desire for precision - if I say "the acceleration due to gravity is 9.81 m/s^2", I know exactly how precise that number is (typically we assume the uncertainty is in the last decimal point, so that implies $\pm 0.01\, m/s^2$ on that number). However, it's also a necessity, since no amount of words can actually be used to exactly describe any physical law, and words don't inherently have precise meaning. Take even the most basic of physical laws - Newton's first law "An object in motion remains in motion until acted on by an external force." What is "motion"? What does it mean to be "acted on"? What is "force"? No amount of language is going to answer these questions, only mathematics is going to give us careful definitions of these concepts[1].

So, once we are convinced that we need to use mathematics to do physics, we know that we need to understand how to associate physics quantities ("mass", "speed", "force") with mathematical variables. In some cases, this is very easy - for example, the temperature of something is just a number, call it T. Or the mass of something is a number m. Single numbers are called **scalars**. Sometimes these numbers have constraints on them that come from reality - for example, mass must be positive, and temperature has a minimum value (absolute zero). But there are also quantities which need more than just a single variable, or additional conditions on that variable - they actually must carry more information. The most important of these are **vectors**, which carry both a number (the *magnitude*, or length of the vector), and a *direction*.

The length of a vector is a pretty straightforward idea - vectors are arrows, and the size of the arrow is the length. But direction is a little bit more confusing, because it can be specified in several different ways. For example, I can be driving at a speed of 32 mph, in the direction "north". Or maybe in the direction "towards you", or "90° from East". Because of this uncertainty, we will often choose to represent vectors *using components* - that is, an xy-Cartesian coordinate system. Of course, this coordinate system is something we created out of thin air to help us solve the problem, so which coordinate system we pick should not influence our calculation in the slightest.

For a concrete example, consider a vector of length 5, that is pointed 45° above the horizontal (see the figure on bottom left). We can choose to represent this vector in a number of ways. In a coordinate system fixed to the horizontal, the vector has components

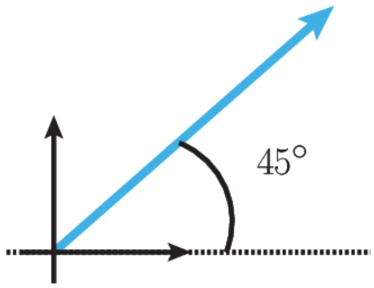
$$v_x = 5\cos(45) \simeq 3.53, v_y = 5\sin(45) \simeq 3.53. \tag{3.1}$$

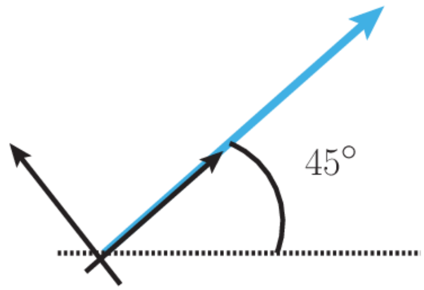
Alternatively, if you pick a coordinate system with the x-axis along the vector itself (figure on bottom middle), the components would be

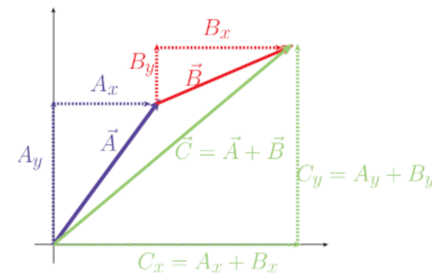
$$v_x = 5, v_y = 0. \tag{3.2}$$

In some cases one of these might be preferable to the other, but there is something important here: *although the coordinate representation changed, the magnitude and direction did not*. No choice of coordinate system can change the magnitude and direction of a physical variable like a vector, because coordinates are just helping tools to do physics.

There is one last point to make about vectors in coordinate systems, which is summed up by the last figure: performing mathematics like addition on vectors means the same mathematics applies to the components. This makes working with components very convenient - we don't actually need to go around making pictures of arrows and measuring them, we can just use the components and get the results algebraically.







[1]If you're interested, the mathematical statement of Newton's first law is something like "$\vec{a} = 0 \supset \Sigma\vec{F}_{ext} = 0$ ", using the symbol $\supset$ for "if".

3.1: Coordinate Systems and Components of a Vector (Part 1)

Vectors are usually described in terms of their components in a coordinate system. Even in everyday life we naturally invoke the concept of orthogonal projections in a rectangular coordinate system. For example, if you ask someone for directions to a particular location, you will more likely be told to go 40 km east and 30 km north than 50 km in the direction 37° north of east.

In a rectangular (Cartesian) xy-coordinate system in a plane, a point in a plane is described by a pair of coordinates (x, y). In a similar fashion, a vector $\vec{A}$ in a plane is described by a pair of its vector coordinates. The x-coordinate of vector $\vec{A}$ is called its x-component and the y-coordinate of vector $\vec{A}$ is called its y-component. The vector x-component is a vector denoted by $\vec{A}_x$. The vector y-component is a vector denoted by $\vec{A}_y$. In the Cartesian system, the x and y **vector components** of a vector are the orthogonal projections of this vector onto the x- and y-axes, respectively. In this way, following the parallelogram rule for vector addition, each vector on a Cartesian plane can be expressed as the vector sum of its vector components:

$$\vec{A} = \vec{A}_x + \vec{A}_y. \tag{3.1.1}$$

As illustrated in Figure $3.1.1$, vector $\vec{A}$ is the diagonal of the rectangle where the x-component $\vec{A}_x$ is the side parallel to the x-axis and the y-component $\vec{A}_y$ is the side parallel to the y-axis. Vector component $\vec{A}_x$ is orthogonal to vector component $\vec{A}_y$.

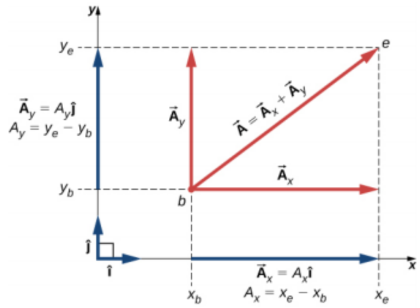


Figure $3.1.1$: Vector $\vec{A}$ in a plane in the Cartesian coordinate system is the vector sum of its vector x- and y-components. The x-vector component $\vec{A}_x$ is the orthogonal projection of vector $\vec{A}$ onto the x-axis. The y-vector component $\vec{A}_y$ is the orthogonal projection of vector $\vec{A}$ onto the y-axis. The numbers A_x and A_y that multiply the unit vectors are the scalar components of the vector.

It is customary to denote the positive direction on the x-axis by the unit vector $\hat{i}$ and the positive direction on the y-axis by the unit vector $\hat{j}$. Unit vectors of the axes, $\hat{i}$ and $\hat{j}$, define two orthogonal directions in the plane. As shown in Figure $3.1.1$, the x- and y-components of a vector can now be written in terms of the unit vectors of the axes:

$$\begin{cases} \vec{A}_x = A_x\hat{i} \\ \vec{A}_y = A_y\hat{j} \end{cases} \tag{3.1.2}$$

The vectors $\vec{A}_x$ and $\vec{A}_y$ defined by 3.1.2 are the vector components of vector $\vec{A}$. The numbers A_x and A_y that define the vector components in Equation 3.1.2 are the **scalar components** of vector $\vec{A}$. Combining Equation 3.1.1 with Equation 3.1.2, we obtain **the component form of a vector**:

$$\vec{A} = A_x\hat{i} + A_y\hat{j}. \tag{3.1.3}$$

If we know the coordinates $b(x_b, y_b)$ of the origin point of a vector (where b stands for "beginning") and the coordinates e(x_e, y_e) of the end point of a vector (where e stands for "end"), we can obtain the scalar components of a vector simply by subtracting the origin point coordinates from the end point coordinates:

$$\begin{cases} A_x = x_e - x_b \\ A_y = y_e - y_b. \end{cases} \tag{3.1.4}$$

Example 3.1.1: Displacement of a Mouse Pointer

A mouse pointer on the display monitor of a computer at its initial position is at point (6.0 cm, 1.6 cm) with respect to the lower left-side corner. If you move the pointer to an icon located at point (2.0 cm, 4.5 cm), what is the displacement vector of the pointer?

Strategy

The origin of the xy-coordinate system is the lower left-side corner of the computer monitor. Therefore, the unit vector $\hat{i}$ on the x-axis points horizontally to the right and the unit vector $\hat{j}$ on the y-axis points vertically upward. The origin of the displacement vector is located at point b(6.0, 1.6) and the end of the displacement vector is located at point e(2.0, 4.5). Substitute the coordinates of these points into Equation 3.1.4 to find the scalar components D_x and D_y of the displacement vector $\vec{D}$. Finally, substitute the coordinates into Equation 3.1.3 to write the displacement vector in the vector component form.

Solution

We identify $x_b = 6.0$, $x_e = 2.0$, $y_b = 1.6$, and $y_e = 4.5$, where the physical unit is 1 cm. The scalar x- and y-components of the displacement vector are

$$D_x = x_e - x_b = (2.0 - 6.0)\ cm = -4.0\ cm, \tag{3.1.5}$$

$$D_y = y_e - y_b = (4.5 - 1.6)\ cm = +2.9\ cm. \tag{3.1.6}$$

The vector component form of the displacement vector is

$$\vec{D} = D_x\ \hat{i} + D_y\ \hat{j} = (-4.0\ cm)\ \hat{i} + (2.9\ cm)\ \hat{j} = (-4.0\ \hat{i} + 2.9\ \hat{j})\ cm. \tag{3.1.7}$$

This solution is shown in Figure 3.1.2.

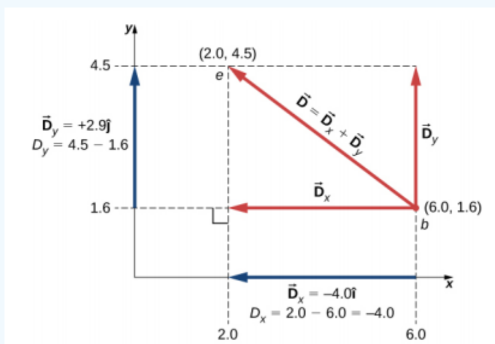


Figure 3.1.2: The graph of the displacement vector. The vector points from the origin point at b to the end point at e.

Significance

Notice that the physical unit—here, 1 cm—can be placed either with each component immediately before the unit vector or globally for both components, as in Equation 3.1.7. Often, the latter way is more convenient because it is simpler.

The vector x-component $\vec{D}_x = -4.0\ \hat{i} = 4.0(-\hat{i})$ of the displacement vector has the magnitude $|\vec{D}_x| = |-4.0||\hat{i}| = 4.0$ because the magnitude of the unit vector is $|\hat{i}| = 1$. Notice, too, that the direction of the x-component is $-\hat{i}$, which is antiparallel to the direction of the +x-axis; hence, the x-component vector $\vec{D}_x$ points to the left, as shown in Figure 3.1.2. The scalar x-component of vector $\vec{D}$ is $D_x = -4.0$. Similarly, the vector y-component $\vec{D}_y = +2.9\hat{j}$ of the displacement vector has magnitude $|\vec{D}_y| = |2.9||\hat{j}| = 2.9$ because the magnitude of the unit vector is $|\hat{j}| = 1$. The direction of the y-component is $+\hat{j}$, which is parallel to the direction of the +y-axis. Therefore, the y-component vector $\vec{D}_y$ points up, as seen in Figure 3.1.2. The scalar y-component of vector $\vec{D}$ is $D_y = +2.9$. The displacement vector $\vec{D}$ is the resultant of its two vector components.

The vector component form of the displacement vector Equation 3.1.7 tells us that the mouse pointer has been moved on the monitor 4.0 cm to the left and 2.9 cm upward from its initial position.

Exercise 3.1.2

A blue fly lands on a sheet of graph paper at a point located 10.0 cm to the right of its left edge and 8.0 cm above its bottom edge and walks slowly to a point located 5.0 cm from the left edge and 5.0 cm from the bottom edge. Choose the rectangular coordinate system with the origin at the lower left-side corner of the paper and find the displacement vector of the fly. Illustrate your solution by graphing.

When we know the scalar components A_x and A_y of a vector $\vec{A}$, we can find its magnitude A and its direction angle θ_A. The **direction angle**—or direction, for short—is the angle the vector forms with the positive direction on the x-axis. The angle θ_A is measured in the counterclockwise direction from the +x-axis to the vector (Figure 3.1.3). Because the lengths A, A_x, and A_y form a right triangle, they are related by the Pythagorean theorem:

$$A^2 = A_x^2 + A_y^2 \Leftrightarrow A = \sqrt{A_x^2 + A_y^2}. \tag{3.1.8}$$

This equation works even if the scalar components of a vector are negative. The direction angle θ_A of a vector is defined via the tangent function of angle θ_A in the triangle shown in Figure 3.1.3:

$$\tan\theta = \frac{A_y}{A_x} \Rightarrow \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right). \tag{3.1.9}$$

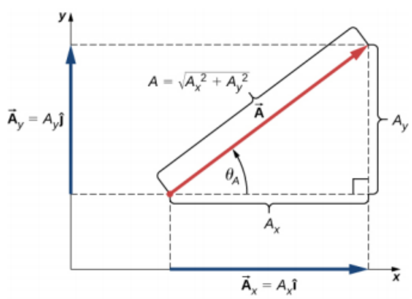


Figure 3.1.3: For vector $\vec{A}$, its magnitude A and its direction angle θ_A are related to the magnitudes of its scalar components because A, A_x, and A_y form a right triangle.

When the vector lies either in the first quadrant or in the fourth quadrant, where component A_x is positive (Figure 3.1.4), the angle θ in Equation 3.1.9) is identical to the direction angle θ_A. For vectors in the fourth quadrant, angle θ is negative, which means that for these vectors, direction angle θ_A is measured clockwise from the positive x-axis. Similarly, for vectors in the second quadrant, angle θ is negative. When the vector lies in either the second or third quadrant, where component A_x is negative, the direction angle is $\theta_A = \theta + 180°$ (Figure 3.1.4).

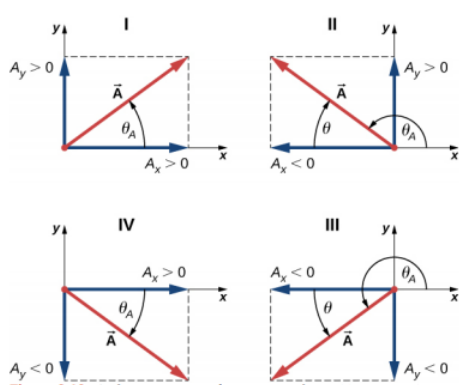


Figure 3.1.4: Scalar components of a vector may be positive or negative. Vectors in the first quadrant (I) have both scalar components positive and vectors in the third quadrant have both scalar components negative. For vectors in quadrants II and III, the direction angle of a vector is $\theta_A = \theta + 180°$.

Example 3.1.3: Magnitude and Direction of the Displacement Vector

You move a mouse pointer on the display monitor from its initial position at point (6.0 cm, 1.6 cm) to an icon located at point (2.0 cm, 4.5 cm). What is the magnitude and direction of the displacement vector of the pointer?

Strategy

In Example 3.1.1, we found the displacement vector $\vec{D}$ of the mouse pointer (see Equation 3.1.7). We identify its scalar components D_x = −4.0 cm and D_y = + 2.9 cm and substitute into Equation 3.1.8 and Equation 3.1.9 to find the magnitude D and direction θ_D, respectively.

Solution

The magnitude of vector $\vec{D}$ is

$$D=\sqrt{D_x^2+D_y^2}=\sqrt{(-4.0\ cm)^2+(2.9\ cm)^2}=\sqrt{(4.0)^2+(2.9)^2}\ cm=4.9\ cm. \tag{3.1.10}$$

The direction angle is

$$\tan\theta=\frac{D_y}{D_x}=\frac{+2.9\ cm}{-4.0\ cm}=-0.725\Rightarrow\theta=\tan^{-1}(-0.725)=-35.9^o. \tag{3.1.11}$$

Vector $\vec{D}$ lies in the second quadrant, so its direction angle is

$$\theta_D=\theta+180^o=-35.9^o+180^o=144.1^o. \tag{3.1.12}$$

Exercise 3.1.4

If the displacement vector of a blue fly walking on a sheet of graph paper is $\vec{D}=(-5.00\ \hat{i}-3.00\ \hat{j})$ cm, find its magnitude and direction.

In many applications, the magnitudes and directions of vector quantities are known and we need to find the resultant of many vectors. For example, imagine 400 cars moving on the Golden Gate Bridge in San Francisco in a strong wind. Each car gives the bridge a different push in various directions and we would like to know how big the resultant push can possibly be. We have already gained some experience with the geometric construction of vector sums, so we know the task of finding the resultant by drawing the vectors and measuring their lengths and angles may become intractable pretty quickly, leading to huge errors. Worries like this do not appear when we use analytical methods. The very first step in an analytical approach is to find vector components when the direction and magnitude of a vector are known.

Let us return to the right triangle in Figure 3.1.3. The quotient of the adjacent side A_x to the hypotenuse A is the cosine function of direction angle θ_A, $A_x/A = \cos\theta_A$, and the quotient of the opposite side A_y to the hypotenuse A is the sine function of θ_A, $A_y/A = \sin\theta_A$. When magnitude A and direction θ_A are known, we can solve these relations for the scalar components:

$$\begin{cases} A_x=A\cos\theta_A \\ A_y=A\sin\theta_A. \end{cases} \tag{3.1.13}$$

When calculating vector components with Equation 3.1.13, care must be taken with the angle. The direction angle θ_A of a vector is the angle measured **counterclockwise** from the positive direction on the x-axis to the vector. The clockwise measurement gives a negative angle.

Example 3.1.5: Components of Displacement Vectors

A rescue party for a missing child follows a search dog named Trooper. Trooper wanders a lot and makes many trial sniffs along many different paths. Trooper eventually finds the child and the story has a happy ending, but his displacements on various legs seem to be truly convoluted. On one of the legs he walks 200.0 m southeast, then he runs north some 300.0 m. On the third leg, he examines the scents carefully for 50.0 m in the direction 30° west of north. On the fourth leg, Trooper goes directly south for 80.0 m, picks up a fresh scent and turns 23° west of south for 150.0 m. Find the scalar components of Trooper's displacement vectors and his displacement vectors in vector component form for each leg.

Strategy

Let's adopt a rectangular coordinate system with the positive x-axis in the direction of geographic east, with the positive y-direction pointed to geographic north. Explicitly, the unit vector $\hat{i}$ of the x-axis points east and the unit vector $\hat{j}$ of the y-axis points north. Trooper makes five legs, so there are five displacement vectors. We start by identifying their magnitudes and direction angles, then we use Equation 3.1.13 to find the scalar components of the displacements and Equation 3.1.3 for the displacement vectors.

Solution

On the first leg, the displacement magnitude is L_1 = 200.0 m and the direction is southeast. For direction angle θ_1 we can take either 45° measured clockwise from the east direction or 45° + 270° measured counterclockwise from the east direction. With the first choice, θ_1 = −45°. With the second choice, θ_1 = + 315°. We can use either one of these two angles. The components are

$$L_{1x} = L_1 \cos\theta_1 = (200.0\ m)\cos 315^o = 141.4\ m, \tag{3.1.14}$$

$$L_{1y} = L_1 \sin\theta_1 = (200.0\ m)\sin 315^o = -141.4\ m, \tag{3.1.15}$$

The displacement vector of the first leg is

$$\vec{L}_1 = L_{1x}\ \hat{i} + L_{1y}\ \hat{j} = (14.4\ \hat{i} - 141.4\ \hat{j})\ m. \tag{3.1.16}$$

On the second leg of Trooper's wanderings, the magnitude of the displacement is L_2 = 300.0 m and the direction is north. The direction angle is θ_2 = + 90°. We obtain the following results:

$$L_{2x} = L_2 \cos\theta_2 = (300.0\ m)\cos 90^o = 0.0, \tag{3.1.17}$$

$$L_{2y} = L_2 \sin\theta_2 = (300.0\ m)\sin 90^o = 300.0\ m, \tag{3.1.18}$$

$$\vec{L}_2 = L_{2x}\ \hat{i} + L_{2y}\ \hat{j} = (300.0\ m)\ \hat{j}. \tag{3.1.19}$$

On the third leg, the displacement magnitude is L_3 = 50.0 m and the direction is 30° west of north. The direction angle measured counterclockwise from the eastern direction is θ_3 = 30° + 90° = + 120°. This gives the following answers:

$$L_{3x} = L_3 \cos\theta_3 = (50.0\ m)\cos 120^o = -25.0\ m, \tag{3.1.20}$$

$$L_{3y} = L_3 \sin\theta_3 = (50.0\ m)\sin 120^o = +43.3\ m, \tag{3.1.21}$$

$$\vec{L}_3 = L_{3x}\ \hat{i} + L_{3y}\ \hat{j} = (-25.0\ \hat{i} + 43.3\ \hat{j})\ m. \tag{3.1.22}$$

On the fourth leg of the excursion, the displacement magnitude is L_4 = 80.0 m and the direction is south. The direction angle can be taken as either θ_4 = −90° or θ_4 = + 270°. We obtain

$$L_{4x} = L_4 \cos\theta_4 = (80.0\ m)\cos(-90^o) = 0, \tag{3.1.23}$$

$$L_{4y} = L_4 \sin\theta_4 = (80.0\ m)\sin(-90^o) = -80.0\ m, \tag{3.1.24}$$

$$\vec{L}_4 = L_{4x}\ \hat{i} + L_{4y}\ \hat{j} = (-80.0\ m)\ \hat{j}. \tag{3.1.25}$$

On the last leg, the magnitude is L_5 = 150.0 m and the angle is θ_5 = −23° + 270° = + 247° (23° west of south), which gives

$$L_{5x} = L_5 \cos\theta_5 = (150.0\ m)\cos 247^o = -58.6\ m, \tag{3.1.26}$$

$$L_{5y} = L_5 \sin\theta_5 = (150.0\ m)\sin 247^o = -138.1\ m, \tag{3.1.27}$$

$$\vec{L}_5 = L_{5x}\ \hat{i} + L_{5y}\ \hat{j} = (-58.6\ \hat{i} - 138.1\ \hat{j})\ m. \tag{3.1.28}$$

Exercise 3.1.6

If Trooper runs 20 m west before taking a rest, what is his displacement vector?

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3.2: Coordinate Systems and Components of a Vector (Part 2)

Polar Coordinates

To describe locations of points or vectors in a plane, we need two orthogonal directions. In the Cartesian coordinate system these directions are given by unit vectors $\hat{i}$ and $\hat{j}$ along the x-axis and the y-axis, respectively. The Cartesian coordinate system is very convenient to use in describing displacements and velocities of objects and the forces acting on them. However, it becomes cumbersome when we need to describe the rotation of objects. When describing rotation, we usually work in the **polar coordinate system**.

In the polar coordinate system, the location of point P in a plane is given by two **polar coordinates** (Figure 3.2.1). The first polar coordinate is the **radial coordinate** r, which is the distance of point P from the origin. The second polar coordinate is an angle φ that the radial vector makes with some chosen direction, usually the positive x-direction. In polar coordinates, angles are measured in radians, or rads. The radial vector is attached at the origin and points away from the origin to point P. This radial direction is described by a unit radial vector $\hat{r}$. The second unit vector $\hat{t}$ is a vector orthogonal to the radial direction $\hat{r}$. The positive $+\hat{t}$ direction indicates how the angle φ changes in the counterclockwise direction. In this way, a point P that has coordinates (x, y) in the rectangular system can be described equivalently in the polar coordinate system by the two polar coordinates (r, φ). Equation 3.2.13 is valid for any vector, so we can use it to express the x- and y-coordinates of vector $\vec{r}$. In this way, we obtain the connection between the polar coordinates and rectangular coordinates of point P:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}. \tag{3.2.1}$$

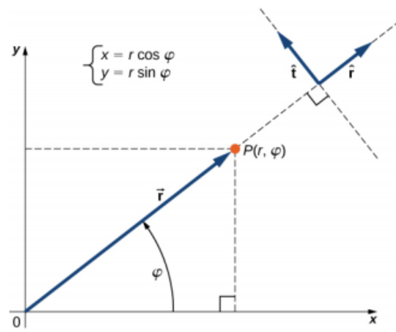


Figure 3.2.1: Using polar coordinates, the unit vector $\hat{r}$ defines the positive direction along the radius r (radial direction) and, orthogonal to it, the unit vector $\hat{t}$ defines the positive direction of rotation by the angle φ.

Example 3.2.1: Polar Coordinates

A treasure hunter finds one silver coin at a location 20.0 m away from a dry well in the direction 20° north of east and finds one gold coin at a location 10.0 m away from the well in the direction 20° north of west. What are the polar and rectangular coordinates of these findings with respect to the well?

Strategy

The well marks the origin of the coordinate system and east is the +x-direction. We identify radial distances from the locations to the origin, which are r_S = 20.0 m (for the silver coin) and r_G = 10.0 m (for the gold coin). To find the angular coordinates, we convert 20° to radians: 20° = $\frac{\pi\ 20}{180}$ = $\frac{\pi}{9}$. We use Equation 3.2.1 to find the x- and y-coordinates of the coins.

Solution

The angular coordinate of the silver coin is $\varphi_S = \frac{\pi}{9}$, whereas the angular coordinate of the gold coin is $\varphi_G = \pi - \frac{\pi}{9} = \frac{8\pi}{9}$. Hence, the polar coordinates of the silver coin are (r_S, φ_S) = (20.0 m, $\frac{\pi}{9}$) and those of the gold coin are (r_G, φ_G) = (10.0 m, $\frac{8\pi}{9}$). We substitute these coordinates into Equation 3.2.1 to obtain rectangular coordinates. For the gold coin, the coordinates are

$$\begin{cases} x_G = r_G\cos\varphi_G = (10.0\ m)\cos\frac{8\pi}{9} = -9.4\ m \\ y_G = r_G\sin\varphi_G = (10.0\ m)\sin\frac{8\pi}{9} = 3.4\ m \end{cases} \Rightarrow (x_G, y_G) = (-9.4\ m, 3.4\ m). \tag{3.2.2}$$

For the silver coin, the coordinates are

$$\left\{\begin{array}{l} x_S = r_S \cos\varphi_S = (20.0\ m)\cos\frac{\pi}{9} = 18.9\ m \\ y_S = r_S \sin\varphi_S = (20.0\ m)\sin\frac{\pi}{9} = 6.8\ m \end{array}\right. \Rightarrow (x_S, y_S) = (18.9\ m, 6.8\ m). \tag{3.2.3}$$

Vectors in Three Dimensions

To specify the location of a point in space, we need three coordinates (x, y, z), where coordinates x and y specify locations in a plane, and coordinate z gives a vertical positions above or below the plane. Three-dimensional space has three orthogonal directions, so we need not two but three unit vectors to define a three-dimensional coordinate system. In the Cartesian coordinate system, the first two unit vectors are the unit vector of the x-axis $\hat{i}$ and the unit vector of the y-axis $\hat{j}$. The third unit vector $\hat{k}$ is the direction of the z-axis (Figure $3.2.2$). The order in which the axes are labeled, which is the order in which the three unit vectors appear, is important because it defines the orientation of the coordinate system. The order x-y-z, which is equivalent to the order $\hat{i}$ - $\hat{j}$ - $\hat{k}$, defines the standard right-handed coordinate system (positive orientation).

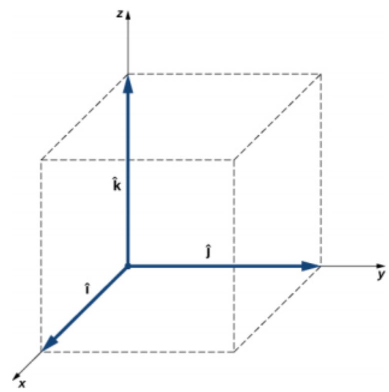


Figure $3.2.2$: Three unit vectors define a Cartesian system in three-dimensional space. The order in which these unit vectors appear defines the orientation of the coordinate system. The order shown here defines the right-handed orientation.

In three-dimensional space, vector $\vec{A}$ has three vector components: the x-component $\vec{A}_x$ = A_x $\hat{i}$, which is the part of vector $\vec{A}$ along the x-axis; the y-component $\vec{A}_y$ = A_y $\hat{j}$, which is the part of $\vec{A}$ along the y-axis; and the z-component $\vec{A}_z$ = A_z $\hat{k}$, which is the part of the vector along the z-axis. A vector in three-dimensional space is the vector sum of its three vector components (Figure $3.2.3$):

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}. \tag{3.2.4}$$

If we know the coordinates of its origin b(x_b, y_b, z_b) and of its end e(x_e y_e, z_e), its scalar components are obtained by taking their differences: A_x and A_y are given by

$$\left\{\begin{array}{l} A_x = x_e - x_b \\ A_y = y_e - y_b. \end{array}\right.$$

and the z-component is given by

$$A_z = z_e - z_b. \tag{3.2.5}$$

Magnitude A is obtained by generalizing Equation 2.4.8 to three dimensions:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{3.2.6}$$

This expression for the vector magnitude comes from applying the Pythagorean theorem twice. As seen in Figure $3.2.3$, the diagonal in the xy-plane has length $\sqrt{A_x^2 + A_y^2}$ and its square adds to the square $A_z{}^2$ to give A^2. Note that when the z-component is zero, the vector lies entirely in the xy-plane and its description is reduced to two dimensions.

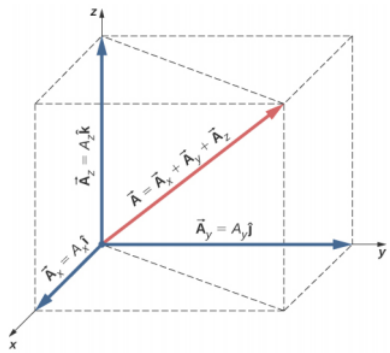


Figure 3.2.3: A vector in three-dimensional space is the vector sum of its three vector components.

Example 3.2.2: Takeoff of a Drone

During a takeoff of IAI Heron (Figure 3.2.4), its position with respect to a control tower is 100 m above the ground, 300 m to the east, and 200 m to the north. One minute later, its position is 250 m above the ground, 1200 m to the east, and 2100 m to the north. What is the drone's displacement vector with respect to the control tower? What is the magnitude of its displacement vector?

Figure 3.2.4: The drone IAI Heron in flight. (credit: SSgt Reynaldo Ramon, USAF)

Strategy

We take the origin of the Cartesian coordinate system as the control tower. The direction of the +x-axis is given by unit vector $\hat{i}$ to the east, the direction of the +y-axis is given by unit vector $\hat{j}$ to the north, and the direction of the +z-axis is given by unit vector $\hat{k}$, which points up from the ground. The drone's first position is the origin (or, equivalently, the beginning) of the displacement vector and its second position is the end of the displacement vector.

Solution

We identify b(300.0 m, 200.0 m, 100.0 m) and e(480.0 m, 370.0 m, 250.0m), and Equation 3.2.5 to find the scalar components of the drone's displacement vector:

$$\begin{cases} D_x = x_e - x_b = 1200.0\ m - 300.0\ m = 900.0\ m, \\ D_y = y_e - y_b = 2100.0\ m - 200.0\ m = 1900.0\ m, \\ D_z = z_e - z_b = 250.0\ m - 100.0\ m = 150\ m. \end{cases} \tag{3.2.7}$$

We substitute these components into Equation 3.2.4 to find the displacement vector:

$$\vec{D} = D_x\ \hat{i} + D_y\ \hat{j} + D_z\ \hat{k} = 900.0\ \hat{i} + 1900.0\ \hat{j} + 150.0\ \hat{k} = (0.90\ \hat{i} + 1.90\ \hat{j} + 0.15\ \hat{k})\ km. \tag{3.2.8}$$

We substitute into Equation 3.2.6 to find the magnitude of the displacement:

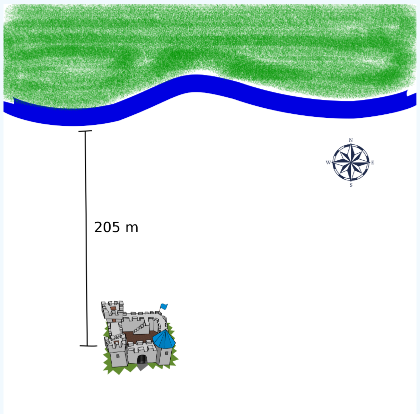
$$D = \sqrt{D_x^2 + D_y^2 + D_z^2} = \sqrt{(0.90\ km)^2 + (1.90\ km)^2 + (0.15\ km)^2} = 4.44\ km. \tag{3.2.9}$$

Exercise 3.2.3

If the average velocity vector of the drone in the displacement in Example 2.7 is $\vec{u}$ = (15.0 $\hat{i}$ + 31.7 $\hat{j}$ + 2.5 $\hat{k}$) m/s, what is the magnitude of the drone's velocity vector?

3.3: Examples

? Whiteboard Problem 3.3.1



You have escaped from your cell in a dungeon underneath the castle shown in the figure. If you can run through the tunnels and find an exit on the other side of the river (30 m wide and 205 m north, as shown), you'll escape from the evil wizard!

1. First you take a tunnel 180 m long traveling 30° north of east.
2. Then you make a right angle turn to the left, and travel another 180 m.

At the end of this tunnel, you see a hatch in the ceiling. Will you escape to the other side of the river if you take this hatch?

? Whiteboard Problem 3.3.2

Solve the following two problems using vector components.

1. Sandra drives her car directly North for 2.0 km, and then turns 20 degrees to the East and drives another 4.0 km. Find the vector (magnitude and direction) from her initial location to her final location.
2. Joe and Max shake hands and say goodbye. Joe walks east 1.00 km to a coffee shop. Max flags a cab and rides 5.00 km in a direction 30 degrees north of east to a bookstore. Find the vector (magnitude and direction) from the coffee shop (Joe) to the bookstore (Max).

Example 3.3.3: Analytical Computation of a Resultant

Three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in a plane are specified by their magnitudes A = 10.0, B = 7.0, and C = 8.0, respectively, and by their respective direction angles with the horizontal direction α = 35°, β = −110°, and γ = 30°. The physical units of the magnitudes are centimeters. Resolve the vectors to their scalar components and find the following vector sums:

a. $\vec{R} = \vec{A} + \vec{B} + \vec{C}$,
b. $\vec{D} = \vec{A} - \vec{B}$, and
c. $\vec{S} = \vec{A} - 3\,\vec{B} + \vec{C}$.

Strategy

First, we use Equation 3.2.13 to find the scalar components of each vector and then we express each vector in its vector component form given by $\overrightarrow{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}$. Then, we use analytical methods of vector algebra to find the resultants.

Solution

We resolve the given vectors to their scalar components:

$$\left\{\begin{array}{l} A_x = A\cos\alpha = (10.0\ cm)\cos 35^o = 8.19\ cm \\ A_y = A\sin\alpha = (10.0\ cm)\sin 35^o = 5.73\ cm \end{array}\right. \tag{3.3.1}$$

$$\begin{cases} B_x = B\cos\beta = (7.0\ cm)\cos(-110^o) = -2.39\ cm \\ B_y = B\sin\beta = (7.0\ cm)\sin(-110^o) = -6.58\ cm \end{cases} \tag{3.3.2}$$

$$\begin{cases} C_x = C\cos\gamma = (8.0\ cm)\cos(30^o) = 6.93\ cm \\ C_y = C\sin\gamma = (8.0\ cm)\sin(30^o) = 4.00\ cm \end{cases} \tag{3.3.3}$$

For (a) we may substitute directly into the given expression to find the scalar components of the resultant:

$$\begin{cases} R_x = A_x + B_x + C_x = 8.19\ cm - 2.39\ cm + 6.93\ cm = 12.73\ cm \\ R_y = A_y + B_y + C_y = 5.73\ cm - 6.58\ cm + 4.00\ cm = 3.15\ cm \end{cases} \tag{3.3.4}$$

Therefore, the resultant vector is $\vec{R} = R_x\hat{i} + R_y\hat{j} = (12.7\hat{i} + 3.1\hat{j})$ cm. For (b), we may want to write the vector difference as

$$\vec{D} = \vec{A} - \vec{B} = (A_x\hat{i} + A_y\hat{j}) - (B_x\hat{i} + B_y\hat{j}) = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j}. \tag{3.3.5}$$

Hence the difference vector is $\vec{D} = D_x\hat{i} + D_y\hat{j} = (10.6\hat{i} + 12.3\hat{j})$ cm.

For (c), we can write vector $\vec{S}$ in the following explicit form:

$$\vec{S} = \vec{A} - 3\vec{B} + \vec{C} = (A_x\hat{i} + A_y\hat{j}) - 3(B_x\hat{i} + B_y\hat{j}) + (C_x\hat{i} + C_y\hat{j}) = (A_x - 3B_x + C_x)\hat{i} + (A_y - 3B_y + C_y)\hat{j}. \tag{3.3.6}$$

Then, the scalar components of $\vec{S}$ are

$$\begin{cases} S_x = A_x - 3B_x + C_x = 8.19\ cm - 3(-2.39\ cm) + 6.93\ cm = 22.29\ cm \\ S_y = A_y - 3B_y + C_y = 5.73\ cm - 3(-6.58\ cm) + 4.00\ cm = 29.47\ cm \end{cases} \tag{3.3.7}$$

The vector is $\vec{S} = S_x\hat{i} + S_y\hat{j} = (22.3\hat{i} + 29.5\hat{j})$ cm.

Significance

Having found the vector components, we can illustrate the vectors by graphing or we can compute magnitudes and direction angles, as shown in Figure 3.3.1.

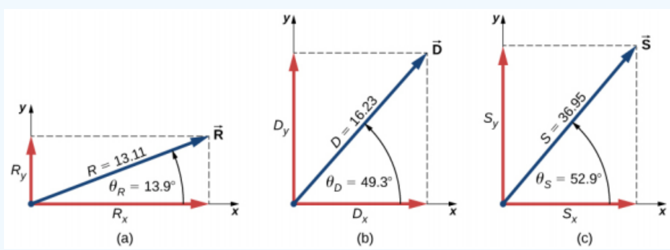


Figure 3.3.1: Graphical illustration of the solutions obtained analytically.

Exercise 3.3.4

Three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{F}$ are specified by their magnitudes A = 10.00, B = 7.00, and F = 20.00, respectively, and by their respective direction angles with the horizontal direction α = 35°, β = −110°, and φ = 110°. The physical units of the magnitudes are centimeters. Use the analytical method to find vector $\vec{F}$ = $\vec{A}$ + 2 $\vec{B}$ − $\vec{F}$. Verify that G = 28.15 cm and that θ_G = −68.65°.

Example 3.3.5: The Tug-of-War Game

Four dogs named Astro, Balto, Clifford, and Dug play a tug-of-war game with a toy (Figure 3.3.2). Astro pulls on the toy in direction α = 55° south of east, Balto pulls in direction β = 60° east of north, and Clifford pulls in direction γ = 55° west of north. Astro pulls strongly with 160.0 units of force (N), which we abbreviate as A = 160.0 N. Balto pulls even stronger than Astro with a force of magnitude B = 200.0 N, and Clifford pulls with a force of magnitude C = 140.0 N. When Dug pulls on the toy in such a way that his force balances out the resultant of the other three forces, the toy does not move in any direction. With how big a force and in what direction must Dug pull on the toy for this to happen?

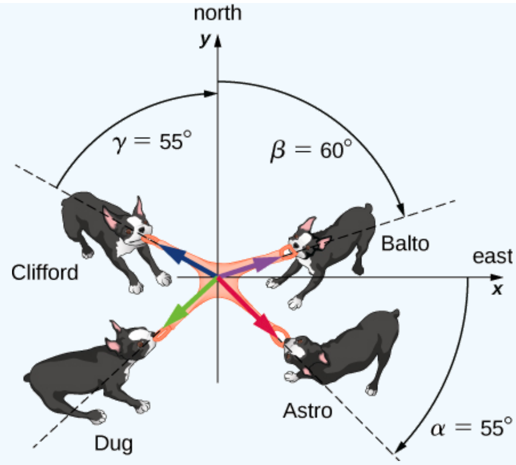


Figure 3.3.2: Four dogs play a tug-of-war game with a toy.

Strategy

We assume that east is the direction of the positive x-axis and north is the direction of the positive y-axis. As in Example 3.3.1, we have to resolve the three given forces — $\vec{A}$ (the pull from Astro), $\vec{B}$ (the pull from Balto), and $\vec{C}$ (the pull from Clifford)—into their scalar components and then find the scalar components of the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C}$. When the pulling force $\vec{D}$ from Dug balances out this resultant, the sum of $\vec{D}$ and $\vec{R}$ must give the null vector $\vec{D} + \vec{R} = \vec{0}$. This means that $\vec{D} = -\vec{R}$ so the pull from Dug must be antiparallel to $\vec{R}$.

Solution

The direction angles are $\theta_A = -\alpha = -55°$, $\theta_B = 90° - \beta = 30°$, and $\theta_C = 90° + \gamma = 145°$, and substituting them into Equation 2.4.13 gives the scalar components of the three given forces:

$$\begin{cases} A_x = A\cos\theta_A = (160.0\ N)\cos(-55^o) = +91.8\ N \\ A_y = A\sin\theta_A = (160.0\ N)\sin(-55^o) = -131.1\ N \end{cases} \tag{3.3.8}$$

$$\begin{cases} B_x = B\cos\theta_B = (200.0\ N)\cos 30^o = +173.2\ N \\ B_y = B\sin\theta_B = (200.0\ N)\sin 30^o = +100.0\ N \end{cases} \tag{3.3.9}$$

$$\begin{cases} C_x = C\cos\theta_C = (140.0\ N)\cos 145^o = -114.7\ N \\ C_y = C\sin\theta_C = (140.0\ N)\sin 145^o = +80.3\ N \end{cases} \tag{3.3.10}$$

Now we compute scalar components of the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C}$:

$$\begin{cases} R_x = A_x + B_x + C_x = +91.8\ N + 173.2\ N - 114.7\ N = +150.3\ N \\ R_y = A_y + B_y + C_y = -131.1\ N + 100.0\ N + 80.3\ N = +49.2\ N \end{cases} \tag{3.3.11}$$

The antiparallel vector to the resultant $\vec{R}$ is

$$\vec{D} = -\vec{R} = -R_x\hat{i} - R_y\hat{j} = (-150.3\hat{i} - 49.2\hat{j})N. \tag{3.3.12}$$

The magnitude of Dug's pulling force is

$$D = \sqrt{D_x^2 + D_y^2} = \sqrt{(-150.3)^2 + (-49.2)^2}N = 158.1\ N. \tag{3.3.13}$$

The direction of Dug's pulling force is

$$\theta = \tan^{-1}\left(\frac{D_y}{D_x}\right) = \tan^{-1}\left(\frac{-49.2\ N}{-150.3\ N}\right) = \tan^{-1}\left(\frac{49.2}{150.3}\right) = 18.1^o. \tag{3.3.14}$$

Dug pulls in the direction 18.1° south of west because both components are negative, which means the pull vector lies in the third quadrant.

Exercise 3.3.6

Suppose that Balto in Example 3.3.2 leaves the game to attend to more important matters, but Astro, Clifford, and Dug continue playing. Astro and Clifford's pull on the toy does not change, but Dug runs around and bites on the toy in a different place. With how big a force and in what direction must Dug pull on the toy now to balance out the combined pulls from Clifford and Astro? Illustrate this situation by drawing a vector diagram indicating all forces involved.

Example 3.3.7: Vector Algebra

Find the magnitude of the vector $\vec{C}$ that satisfies the equation $2\vec{A} - 6\vec{B} + 3\vec{C} = 2\hat{j}$, $\vec{A} = \hat{i} - 2\hat{k}$ and $\vec{B} = -\hat{j} + \frac{\hat{k}}{2}$.

Strategy

We first solve the given equation for the unknown vector $\vec{C}$. Then we substitute $\vec{A}$ and $\vec{B}$; group the terms along each of the three directions $\hat{i}$, $\hat{j}$, and $\hat{k}$; and identify the scalar components C_x, C_y, and C_z. Finally, we substitute into Equation 3.3.6 to find magnitude C.

Solution

$$\begin{aligned} 2\vec{A} - 6\vec{B} + 3\vec{C} &= 2\hat{j} \\ 3\vec{C} &= 2\hat{j} - 2\vec{A} + 6\vec{B} \\ \vec{C} &= \frac{2}{3}\hat{j} - \frac{2}{3}\vec{A} + 2\vec{B} \\ &= \frac{2}{3}\hat{j} - \frac{2}{3}(\hat{i} - 2\hat{k}) + 2\left(-\hat{j} + \frac{\hat{k}}{2}\right) \\ &= \frac{2}{3}\hat{j} - \frac{2}{3}\hat{i} + \frac{4}{3}\hat{k} - 2\hat{j} + \hat{k} \\ &= -\frac{2}{3}\hat{i} + \left(\frac{2}{3} - 2\right)\hat{j} + \left(\frac{4}{3} + 1\right)\hat{k} \\ &= -\frac{2}{3}\hat{i} - \frac{4}{3}\hat{j} + \frac{7}{3}\hat{k} \end{aligned}$$

The components are $C_x = -\frac{2}{3}$, $C_y = -\frac{4}{3}$, and $C_z = \frac{7}{3}$, and substituting into Equation 3.3.6 gives

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2} = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(-\frac{4}{3}\right)^2 + \left(\frac{7}{3}\right)^2} = \sqrt{\frac{23}{3}}. \tag{3.3.15}$$

Example 3.3.8: Displacement of a Skier

Starting at a ski lodge, a cross-country skier goes 5.0 km north, then 3.0 km west, and finally 4.0 km southwest before taking a rest. Find his total displacement vector relative to the lodge when he is at the rest point. How far and in what direction must he ski from the rest point to return directly to the lodge?

Strategy

We assume a rectangular coordinate system with the origin at the ski lodge and with the unit vector $\hat{i}$ pointing east and the unit vector $\hat{j}$ pointing north. There are three displacements: $\vec{D}_1$, $\vec{D}_2$, and $\vec{D}_3$. We identify their magnitudes as $D_1 = 5.0$ km, $D_2 = 3.0$ km, and $D_3 = 4.0$ km. We identify their directions are the angles $\theta_1 = 90°$, $\theta_2 = 180°$, and $\theta_3 = 180° + 45° = 225°$. We resolve each displacement vector to its scalar components and substitute the components into Equation 3.2.13 to obtain the scalar components of the resultant displacement $\vec{D}$ from the lodge to the rest point. On the way back from the rest point to the lodge, the displacement is $\vec{B} = -\vec{D}$. Finally, we find the magnitude and direction of $\vec{B}$.

Solution

Scalar components of the displacement vectors are

$$\begin{cases} D_{1x} = D_1 \cos\theta_1 = (5.0\ km)\cos 90^o = 0 \\ D_{1y} = D_1 \sin\theta_1 = (5.0\ km)\sin 90^o = 5.0\ km \end{cases} \tag{3.3.16}$$

$$\begin{cases} D_{2x} = D_2 \cos\theta_2 = (3.0\ km)\cos 180^o = -3.0\ km \\ D_{2y} = D_2 \sin\theta_2 = (3.0\ km)\sin 180^o = 0 \end{cases} \tag{3.3.17}$$

$$\begin{cases} D_{3x} = D_3 \cos\theta_3 = (4.0\ km)\cos 225^o = -2.8\ km \\ D_{3y} = D_3 \sin\theta_3 = (4.0\ km)\sin 225^o = -2.8\ km \end{cases} \tag{3.3.18}$$

Scalar components of the net displacement vector are

$$\begin{cases} D_x = D_{1x} + D_{2x} + D_{3x} = (0 - 3.0 - 2.8)km = -5.8\ km \\ D_y = D_{1y} + D_{2y} + D_{3y} = (5.0 + 0 - 2.8)km = +2.2\ km \end{cases} \tag{3.3.19}$$

Hence, the skier's net displacement vector is $\vec{D}$ = $D_x\ \hat{i}$ + $D_y\ \hat{j}$ = (−5.8 $\hat{i}$ + 2.2 $\hat{j}$)km . On the way back to the lodge, his displacement is $\vec{B}$ = − $\vec{D}$ = −(−5.8 $\hat{i}$+ 2.2 $\hat{j}$)km = (5.8 $\hat{i}$ − 2.2 $\hat{j}$)km. Its magnitude is B = $\sqrt{B_x^2 + B_y^2}$ = $\sqrt{(5.8)^2 + (-2.2)^2}$ km = 6.2 km and its direction angle is θ= $\tan^{-1}\left(\frac{-2.2}{5.8}\right)$ = −20.8°. Therefore, to return to the lodge, he must go 6.2 km in a direction about 21° south of east.

Significance

Notice that no figure is needed to solve this problem by the analytical method. Figures are required when using a graphical method; however, we can check if our solution makes sense by sketching it, which is a useful final step in solving any vector problem.

Example 3.3.9: Displacement of a Jogger

A jogger runs up a flight of 200 identical steps to the top of a hill and then runs along the top of the hill 50.0 m before he stops at a drinking fountain (Figure 3.3.3). His displacement vector from point A at the bottom of the steps to point B at the fountain is $\vec{D}_{AB}$ = (−90.0 $\hat{i}$ + 30.0 $\hat{j}$)m. What is the height and width of each step in the flight? What is the actual distance the jogger covers? If he makes a loop and returns to point A, what is his net displacement vector?

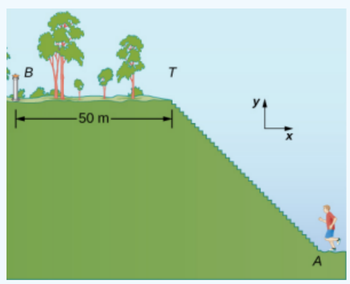


Figure 3.3.3: A jogger runs up a flight of steps.

Strategy

The displacement vector $\vec{D}_{AB}$ is the vector sum of the jogger's displacement vector $\vec{D}_{AT}$ along the stairs (from point A at the bottom of the stairs to point T at the top of the stairs) and his displacement vector $\vec{D}_{RB}$ on the top of the hill (from point T at the top of the stairs to the fountain at point B). We must find the horizontal and the vertical components of $\vec{D}_{TB}$. If each step has width w and height h, the horizontal component of $\vec{D}_{TB}$ must have a length of 200w and the vertical component must have a length of 200h. The actual distance the jogger covers is the sum of the distance he runs up the stairs and the distance of 50.0 m that he runs along the top of the hill.

Solution

In the coordinate system indicated in Figure 3.3.3, the jogger's displacement vector on the top of the hill is $\vec{D}_{RB}$ = (−50.0 m) $\hat{i}$. His net displacement vector is

$$\vec{D}_{AB} = \vec{D}_{AT} + \vec{D}_{TB}.$$

Therefore, his displacement vector $\vec{D}_{TB}$ along the stairs is

$$\begin{aligned} \vec{D}_{AT} &= \vec{D}_{AB} - \vec{D}_{TB} = (-90.0\hat{i} + 30.0\hat{j})m - (-50.0m)\hat{i}) = [(-90.050.0)\hat{i} + 30.0\hat{j})]m \\ &= (-40.0\hat{i} + 30.0\hat{j})m. \end{aligned}$$

Its scalar components are D_{ATx} = −40.0 m and D_{ATy} = 30.0 m. Therefore, we must have

$$200w = |-40.0|\text{m and } 200h = 30.0\text{ m}.$$

Hence, the step width is w = $\frac{40.0\ m}{200}$ = 0.2 m = 20 cm, and the step height is w = $\frac{30.0\ m}{200}$ = 0.15 m = 15 cm. The distance that the jogger covers along the stairs is

$$\vec{D}_{AT} = \sqrt{\vec{D}^2_{ATx} + \vec{D}^2_{ATy}} = \sqrt{(-40.0)^2 + (30.0)^2} m = 50.0\ m.$$

Thus, the actual distance he runs is D_{AT} + D_{TB} = 50.0 m + 50.0 m = 100.0 m. When he makes a loop and comes back from the fountain to his initial position at point A, the total distance he covers is twice this distance,or 200.0 m. However, his net displacement vector is zero, because when his final position is the same as his initial position, the scalar components of his net displacement vector are zero.

In many physical situations, we often need to know the direction of a vector. For example, we may want to know the direction of a magnetic field vector at some point or the direction of motion of an object. We have already said direction is given by a unit vector, which is a dimensionless entity—that is, it has no physical units associated with it. When the vector in question lies along one of the axes in a Cartesian system of coordinates, the answer is simple, because then its unit vector of direction is either parallel or antiparallel to the direction of the unit vector of an axis. For example, the direction of vector $\vec{d}$ = -5 m $\hat{i}$ is unit vector $\vec{d}$ = -$\hat{i}$. The general rule of finding the unit vector $\vec{V}$ of direction for any vector $\vec{V}$ is to divide it by its magnitude *V*:

$$\widehat{\mathbf{V}} = \frac{\overrightarrow{\mathbf{V}}}{V}. \tag{3.3.20}$$

We see from this expression that the unit vector of direction is indeed dimensionless because the numerator and the denominator in Equation 3.3.20 have the same physical unit. In this way, Equation 3.3.20 allows us to express the unit vector of direction in terms of unit vectors of the axes. The following example illustrates this principle.

Example 3.3.10: The Unit Vector of Direction

If the velocity vector of a military convoy is $\vec{v}$ = (4.000 $\hat{i}$ + 3.000 $\hat{j}$ + 0.100 $\hat{k}$)km/h, what is the unit vector of its direction of motion.

Strategy

The unit vector of the convoy's direction of motion is the unit vector $\hat{v}$ that is parallel to the velocity vector. The unit vector is obtained by dividing a vector by its magnitude, in accordance with Equation 3.3.20.

Solution

The magnitude of the vector $\vec{v}$ is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{4.000^2 + 3.000^2 + 0.100^2}\ km/h = 5.001\ km/h.$$

To obtain the unit vector $\hat{v}$, divide $\vec{v}$ by its magnitude:

$$\begin{aligned} \hat{v} = \frac{\vec{v}}{v} &= \frac{(4.000\hat{i} + 3.00\hat{j} + 0.100\hat{k})km/h}{5.001\ km/h} \\ &= \frac{(4.000\hat{i} + 3.000\hat{j} + 0.1100\hat{k})}{5.001} \\ &= \frac{4.000}{5.001}\hat{i} + \frac{3.000}{5.001}\hat{j} + \frac{0.100}{5.001}\hat{k} \\ &= (79.98\hat{i} + 59.99\hat{j} + 2.00\hat{k}) \times 10^{-2}. \end{aligned}$$

Significance

Note that when using the analytical method with a calculator, it is advisable to carry out your calculations to at least three decimal places and then round off the final answer to the required number of significant figures, which is the way we performed calculations in this example. If you round off your partial answer too early, you risk your final answer having a huge numerical error, and it may be far off from the exact answer or from a value measured in an experiment.

Exercise 3.3.11

Verify that vector $\hat{v}$ obtained in Example 3.3.3 is indeed a unit vector by computing its magnitude. If the convoy in Example 2.6.1 was moving across a desert flatland—that is, if the third component of its velocity was zero—what is the unit vector of its direction of motion? Which geographic direction does it represent?

CHAPTER OVERVIEW

4: C4) Systems and The Center of Mass

There are two basic concepts we are going to cover in this chapter: what a *system* is, and *the center of mass*.

A system is a very generic word ("the system of the Earth and the Moon", "a fridge is a closed system","my computer system") that we are going to make precise so that we can use it to describe our abstraction of the real world. Simply put, a **system is a collection of objects**. When we turn our objects into points, we are going to describe specific collections of those points as the systems. For a given problem, there may be multiple possible systems. For a simple example, consider the collision of two blocks sliding along the ground. We replace the blocks with points, and say "the system is the two blocks." Well that's a fine system if we're going to ignore friction, but if the blocks are experiencing friction between themselves and the ground, we had better include the ground in our system as well. So maybe system A is the blocks, but system B is the blocks *and* the ground, and if we want to include the effects of friction, we have to use system B. A simple idea, but really important for us as we start to think about how we model interactions, and which interactions are present in any given physical problem.

The center of mass, in comparison, is a definite mathematical concept. It's the answer to "when we replace our objects with points, what point do we actually use?" We use the center of mass because it's the balance point of the system, and therefore satisfies some important properties that random other points in our system don't satisfy. For example, consider the motion of a runner. If we replaced the runner with a point, and used the tip of their fingers as the point, we would have a very hard time confirming any of our calculations, because a real runners fingers are flying back and forth as they run! However, if we described their motion using their center of mass (a point inside their chest, at the center of their body), their motion would essentially be just a straight line - much easier to understand and use in our calculations.

The specific mathematical definition of the center of mass is

$$\vec{r}_{CM}=\frac{m_1\vec{r}_1+m_2\vec{r}_2+\ldots}{m_1+m_2+\ldots}=\frac{\sum m_i\vec{r}_i}{\sum m_i} \tag{4.1}$$

In this formula, each m_i is the mass of an object, and each $\vec{r}_i$ is the position of the mass. If we were going to use this definition to find the center of mass of the runner, we would break up their body into a bunch of little masses and find all their vector positions - it would take a long time[1]! In this course, we will generally do things more like "find the center of mass between two objects of mass m_1 and m_2, separated by a distance d". So what are $\vec{r}_1$ and $\vec{r}_2$ in this situation? They are position vectors, so you first have to decide where the origin of your system is - where you are measuring the position relative to. It's often easy to pick one of the masses as the origin, say the first one, so that $r_1=0$, $r_2=d$, and the center of mass is

$$r_{CM}=\frac{m_1\cdot 0+m_2\cdot d}{m_1+m_2}. \tag{4.2}$$

Notice that I got rid of the vector signs in this example - we will generally be more careful and keep track of both x- and y-directions separately, but here it's easy to see that the center of mass will be in a line between the two masses. If our numbers were $m_1=1$ kg, $m_2=10$ kg, and $d=1$ m, the answer is $r_{CM}\simeq 0.91$ m. This makes sense as "the balance point", because it's much closer to the larger mass.

[1]Of course, if we really wanted to do that calculation, we would probably use an integral!

4.1: The Law of Inertia

There is something funny about motion with constant velocity: it is *indistinguishable from rest*. Of course, you can usually tell whether you are moving *relative to something else*. But if you are enjoying a smooth airplane ride, without looking out the window, you have no idea how fast you are moving, or even, indeed (if the flight is exceptionally smooth) whether you are moving at all.

If I were to drop something from such an airplane, I know from experience that it would fall on a straight line—relative to me, that is. If it falls from my hand it will land at my feet, just as if we were all at rest. But we are *not* at rest. In the half second or so it takes for the object to fall, the airplane has moved forward 111 meters relative to the ground. Yet the (hypothetical) object I drop does not land 300 feet behind me! It moves forward with me as it falls, even though I am not touching it. *It keeps its initial forward velocity*, even though it is no longer in contact with me or anything connected to the airplane.

This remarkable observation is one of the most fundamental principles of physics, which we call the **law of inertia**. It can be stated as follows: in the absence of any external influence (or *force*) acting on it, an object at rest will stay at rest, while an object that is already moving with some velocity will keep that same velocity (speed and direction of motion)—at least until it is, in fact, acted upon by some force.

Please let that sink in for a moment, before we start backtracking, which we have to do now on several accounts. First, we used repeatedly the term "force," but we have not defined it properly. What if we just said that forces are precisely any "external influences" that may cause a change in the velocity of an object? That will work until it is time to explore the concept in more detail, a few chapters from now.

Next, I need to draw your attention to the fact that the object I (hypothetically) dropped did not actually keep its *total* initial velocity: it only kept its initial *forward* velocity. In the downward direction, it was speeding up from the moment it left my hand, as would any other falling object (and as we shall see later in this chapter). But this actually makes sense in a certain way: there was no forward force, so the forward velocity remained constant; there was, however, a vertical force acting all along (the force of gravity), and so the object did speed up in that direction. This observation is, in fact, telling us something profound about the world's geometry: namely, that forces and velocities are *vectors*, and laws such as the law of inertia will typically apply to the vector as a whole, as well as to each component separately (that is to say, each dimension of space). This anticipates, in fact, the way we will deal, later on, with motion in two or more dimensions; but we do not need to worry about that for a few chapters still.

Finally, it is worth spending a moment reflecting on how radically the law of inertia seems to contradict our intuition about the way the world works. What it seems to be telling us is that, if we throw or push an object, it should continue to move forever with the same speed and in the same direction with which it set out—something that we know is certainly not true. But what's happening in "real life" is that, just because we have left something alone, it doesn't mean the world has left it alone. After we lose contact with the object, all sorts of other forces will continue to act on it. A ball we throw, for instance, will experience air resistance or drag, and that will slow it down. An object sliding on a surface will experience friction, and that will slow it down too. Perhaps the closest thing to the law of inertia in action that you may get to see is a hockey puck sliding on the ice: it is remarkable (perhaps even a bit frightening) to see how little it slows down, but even so the ice does a exert a (very small) frictional force that would bring the puck to a stop eventually.

This is why, historically, the law of inertia was not discovered until people started developing an appreciation for frictional forces, and the way they are constantly acting all around us to oppose the relative motion of any objects trying to slide past each other.

This mention of relative motion, in a way, brings us full circle. Yes, relative motion is certainly detectable, and for objects in contact it actually results in the occurrence of forces of the frictional, or drag, variety. But *absolute* (that is, without reference to anything external) motion with constant velocity is fundamentally undetectable. And in view of the law of inertia, it makes sense: if no force is required to keep me moving with constant velocity, it follows that as long as I am moving with constant velocity I should not be feeling any net force acting on me; nor would any other detection apparatus I might be carrying with me.

This is the next very interesting fact about the physical world that we are about to discover: forces cause accelerations, or changes in velocity, but they do so in different degrees for different objects; and, moreover, the ultimate change in velocity *takes time*. The first part of this statement has to do with the concept of *inertial mass*, to be introduced in the next chapter; the second part we are going to explore right now, after a brief detour to define *inertial reference frames*.

Inertial Reference Frames

The example we just gave you of what happens when a plane in flight experiences turbulence points to an important phenomenon, namely, that there may be times where the law of inertia may not *seem* to apply in a certain reference frame. By this we mean that an object left at rest, like the water in a cup, may suddenly start to move—relative to the reference frame coordinates—even though nothing and nobody is acting on it. More dramatically still, if a car comes to a sudden stop, the passengers may be "projected forward"—they were initially at rest relative to the car frame, but now they find themselves moving forward (always in the car reference frame), to the point that, if they are not wearing seat belts, they may end up hitting the dashboard, or the seat in front of them.

Again, nobody has pushed on them, and in fact what we can see in this case, from outside the car, is nothing but the law of inertia at work: the passengers were just keeping their initial velocity, when the car suddenly slowed down under and around them. So there is nothing wrong with the law of inertia, but *there is a problem with the reference frame*: if one wants to describe the motion of objects in a reference frame like a plane being shaken up or a car that is speeding up or slowing down, we need to allow for the fact that objects may move—always relative to that frame—in an *apparent* violation of the law of inertia.

The way we deal with this in physics is by introducing the very important concept of an *inertial reference frame*, by which we mean a reference frame in which all objects will, at all times, be observed to move (or not move) in a way fully consistent with the law of inertia. In other words, the law of inertia has to hold *when we use that frame's own coordinates to calculate the objects' velocities*. This, of course, is what we always do instinctively: when I am on a plane I locate the various objects around me relative to the plane frame itself, not relative to the distant ground.

To ascertain whether a frame is inertial or not, we start by checking to see if the description of motion using that frame's coordinates obeys the law of inertia: does an object left at rest on the counter in the laboratory stay at rest? If set in motion, does it move with constant velocity on a straight line? The Earth's surface, as it turns out, is *not* quite a perfect inertial reference frame, but it *is* good enough that it made it possible for us to discover the law of inertia in the first place!

What spoils the inertial-ness of an Earth-bound reference frame is the Earth's rotation, which, as we shall see later, is an example of *accelerated motion*. In fact, if you think about the grossly non-inertial frames I have introduced above—the bouncy plane, the braking car—they all have this in common: that their velocities are changing; they are *not* moving with constant speed on a straight line.

So, once you have found an inertial reference frame, to decide whether another one is inertial or not is simple: if it is moving with constant velocity (relative to the first, inertial frame), then it is itself inertial; if not, it is not. We will show you how this works, formally, in a little bit (Chapter 15), after we get around to properly introducing the concept of acceleration.

It is a fundamental principle of physics that *the laws of physics take the same form in all inertial reference frames*. The law of inertia is, of course, an example of such a law. Since all inertial frames are moving with constant velocity relative to each other, this is another way to say that absolute motion is undetectable, and all motion is ultimately relative. Accordingly, this principle is known as the **principle of relativity**.

4.2: Extended Systems and Center of Mass

Consider a collection of particles with masses m_1, m_2,..., and located, at some given instant, at positions x_1, x_2.... (We are still, for the time being, considering only motion in one dimension, but all these results generalize easily to three dimensions.) The **center of mass** of such a system is a mathematical point whose position coordinate is given by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots}. \tag{4.2.1}$$

Clearly, the denominator of (4.2.1) is just the total mass of the system, which we may just denote by M. If all the particles have the same mass, the center of mass will be somehow "in the middle" of all of them; otherwise, it will tend to be closer to the more massive particle(s). The "particles" in question could be spread apart, or they could literally be the "parts" into which we choose to subdivide, for computational purposes, a single extended object.

If the particles are in motion, the position of the center of mass, x_{cm}, will in general change with time. For a solid object, where all the parts are moving together, the displacement of the center of mass will just be the same as the displacement of any part of the object. In the most general case, we will have (by subtracting x_{cmi} from x_{cmf})

$$\Delta x_{cm} = \frac{1}{M}\left(m_1 \Delta x_1 + m_2 \Delta x_2 + \ldots\right). \tag{4.2.2}$$

Dividing Equation (4.2.2) by Δt and taking the limit as $\Delta t \to 0$, we get the instantaneous velocity of the center of mass:

$$v_{cm} = \frac{1}{M}\left(m_1 v_1 + m_2 v_2 + \ldots\right). \tag{4.2.3}$$

But this is just

$$v_{cm} = \frac{p_{sys}}{M}. \tag{4.2.4}$$

where $p_{sys} = m_1 v_1 + m_2 v_2 + \ldots$ is the total momentum of the system.

Center of Mass Motion for an Isolated System

Equation (4.2.4) is a very interesting result. Since the total momentum of an isolated system is constant, it tells us that the center of mass of an isolated system of particles moves at constant velocity, regardless of the relative motion of the particles themselves or their possible interactions with each other. As indicated above, this generalizes straightforwardly to more than one dimension, so we can write $\vec{v}_{cm} = \vec{p}_{sys}/M$. Thus, we can say that, for an isolated system in space, not only the speed, but also the direction of motion of its center of mass does not change with time.

Clearly this result is a sort of generalization of the law of inertia. For a solid, extended object, it does, in fact, provide us with the precise form that the law of inertia must take: in the absence of external forces, *the center of mass* will just move on a straight line with constant velocity, whereas the object itself may be moving in any way that does not affect the center of mass trajectory. Specifically, the most general motion of an isolated rigid body is a straight line motion of its center of mass at constant speed, combined with a possible rotation of the object as a whole around the center of mass.

For a system that consists of separate parts, on the other hand, the center of mass is generally just a point in space, which may or may not coincide at any time with the position of any of the parts, but which will nonetheless move at constant velocity as long as the system is isolated. This is illustrated by Figure $4.2.1$, where the position vs. time curves have been drawn for the colliding objects of Figure 2.1.1. I have assumed that object 1 starts out at x_{1i} = −5 mm at t = 0, and object 2 starts at x_{2i} = 0 at t = 0. Because object 2 has twice the inertia of object 1, the position of the center of mass, as given by Equation (4.2.1), will always be

$$x_{cm} = x_1/3 + 2x_2/3$$

that is to say, the center of mass will always be in between objects 1 and 2, and its distance from object 2 will always be half its distance to object 1:

$$|x_{cm} - x_1| = \frac{2}{3}|x_1 - x_2|$$
$$|x_{cm} - x_2| = \frac{1}{3}|x_1 - x_2|$$

Figure $4.2.1$ shows that this simple prescription does result in motion with constant velocity for the center of mass (the green straight line), even though the x-vs-t curves of the two objects themselves look relatively complicated. (Please check it out! Take a ruler to Figure $4.2.1$ and verify that the center of mass is, at every instant, where it is supposed to be.)

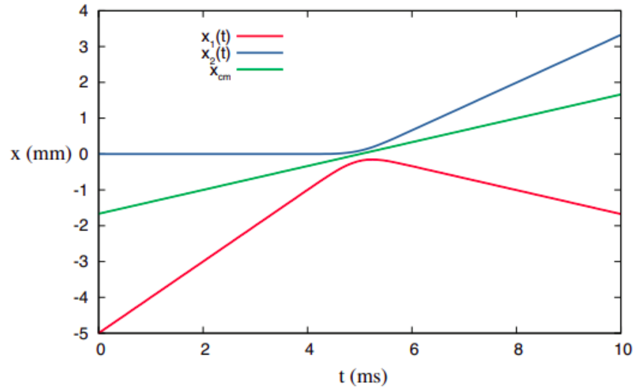


Figure $4.2.1$. The green line shows the position of the center of mass as a function of time.

The concept of center of mass gives us an important way to simplify the description of the motion of potentially complicated systems. We will make good use of it in forthcoming chapters.

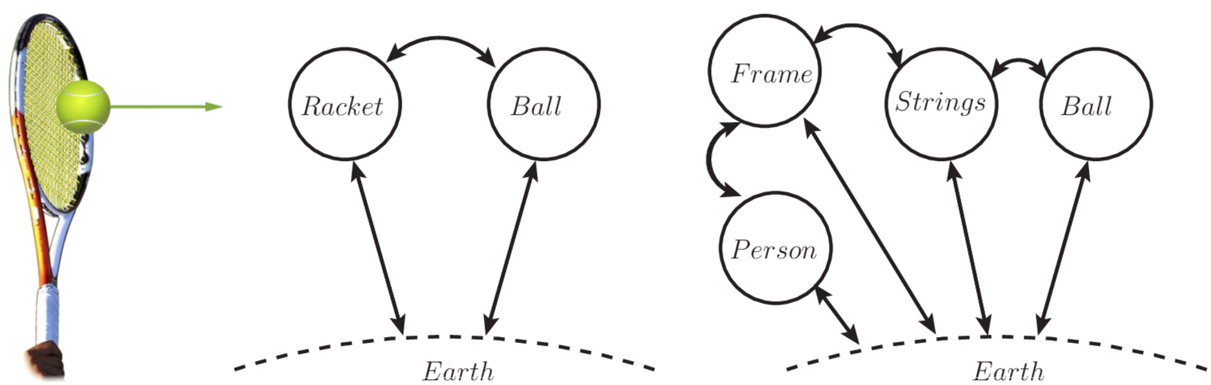
A very nice demonstration of the motion of the center of mass in two-body one-dimensional collisions can be found at https://phet.colorado.edu/sims/collision-lab/collision-lab_en.html (you need to check the “center of mass” box to see it).

Defining a System is not Unique

Although we will generally be studying relatively simple systems in the course of this class ("two carts", "a person pulling a sled", etc), it turns out that specifying exactly what things are in the system, and how those things interact, will involve some choices. We hope that these choices do not impact the answers we get, but in some cases they might determine which questions we can actually ask. It will be easiest to illustrate this with a definite example, so we will try to determine the system in the figure shown below, of a tennis racket hitting a ball.

The situation is straightforeward - a person is hitting a ball with a tennis racket. The figure in the center defines this system in what is probably considered the "simplest" fashion, and is generally the approach we will take in this text. The system has two objects ("Ball" and "Racket"). The two objects are interacting with each other, and since both have mass, are also interacting with the Earth via the gravitational force. This will allow us to answer basic questions like "how much force does the ball feel from the racket?" and "how far does the ball travel after it is hit?"

However, consider the figure on the right - it's an alterative, completely correct, definition for this system. In this case, we've broken the racket into three separate objects, "Strings", "Frame", and "Person", each with their own gravitational interactions with the Earth. This is more complicated, but allows us to answer more interesting questions - like "how much force did the frame absorb from the strings?" or, "how much force did the person deliver to the ball?" The original questions are still answerable, but with this more complicated system, we've increased the number of questions we can answer about this seemingly simple system.



As we said, we will generally be focused on the simpler of these systems - the middle one, with a single interaction between the racket and the ball. However, it's worth it to keep in mind that there are tyically a great many other interesting questions that can be answered with the same physical system, it's just a matter of altering the definition of the system. We will sometimes refer to this type of thing as **different models for the same physical situation**, and we pick which model we want depending on what we want to study about it.

4.3: Reference Frame Changes and Relative Motion

Everything up to this point assumes that we are using a fixed, previously agreed upon **reference frame**. Basically, this is just an origin and a set of axes along which to measure our coordinates.

There are, however, a number of situations in physics that call for the use of different reference frames, and, more importantly, that require us to *convert* various physical quantities from one reference frame to another. For instance, imagine you are on a boat on a river, rowing downstream. You are moving with a certain velocity relative to the water around you, but the water itself is flowing with a different velocity relative to the shore, and your actual velocity relative to the shore is the sum of those two quantities. Ships generally have to do this kind of calculation all the time, as do airplanes: the “airspeed” is the speed of a plane relative to the air around it, but that air is usually moving at a substantial speed relative to the earth.

The way we deal with all these situations is by introducing two reference frames, which here I am going to call A and B. One of them, say A, is “at rest” relative to the earth, and the other one is “at rest” relative to something else—which means, really, moving along with that something else. (For instance, a reference frame at rest “relative to the river” would be a frame that’s moving along with the river water, like a piece of driftwood that you could measure your progress relative to.)

In any case, graphically, this will look as in Figure $4.3.1$, which we have drawn for the two-dimensional case:

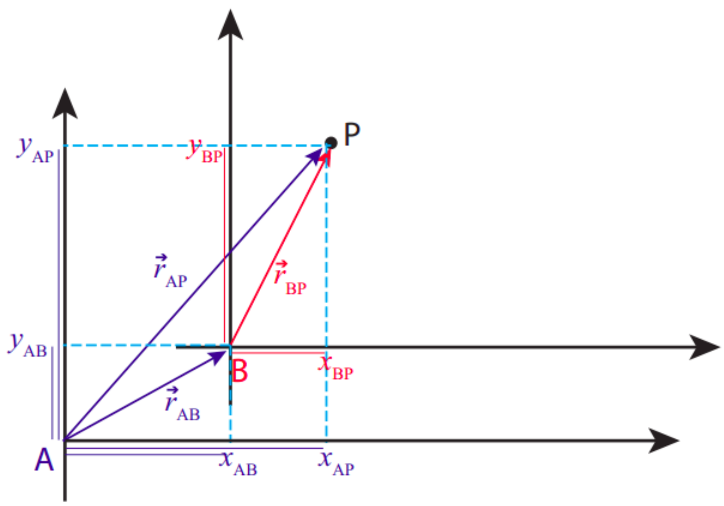


Figure $4.3.1$: Position vectors and coordinates of a point P in two different reference frames, A and B.

In the reference frame A, the point P has position coordinates (x_{AP}, y_{AP}). Likewise, in the reference frame B, its coordinates are (x_{BP}, y_{BP}). As you can see, the notation chosen is such that every coordinate in A will have an “A” as a first subscript, while the second subscript indicates the object to which it refers, and similarly for coordinates in B.

The coordinates (x_{AB}, y_{AB}) are special: they are the coordinates, in the reference frame A, of the origin of reference frame B. This is enough to fully locate the frame B in A, as long as the frames are not rotated relative to each other.

The thin colored lines I have drawn along the axes in Figure $4.3.1$ are intended to make it clear that the following equations hold:

$$\begin{aligned} x_{AP} &= x_{AB} + x_{BP} \\ x_{AP} &= x_{AB} + x_{BP} \end{aligned} \tag{4.3.1}$$

Although the figure is drawn for the easy case where all these quantities are positive, you should be able to convince yourself that Eqs. (4.3.1) hold also when one or more of the coordinates have negative values.

All these coordinates are also the components of the respective position vectors, shown in the figure and color-coded by reference frame (so, for instance, $\vec{r}_{AP}$ is the position vector of P in the frame A), so the equations (4.3.1) can be written more compactly as the single vector equation

$$\vec{r}_{AP} = \vec{r}_{AB} + \vec{r}_{BP}. \tag{4.3.2}$$

From all this you can see how to add vectors: algebraically, you just add their components separately, as in Eqs. (4.3.1); graphically, you draw them so the tip of one vector coincides with the tail of the other (we call this "tip-to-tail"), and then draw the sum vector from the tail of the first one to the tip of the other one.

Of course, I showed you already how to *subtract* vectors with Figure 1.2.3: again, algebraically, you just subtract the corresponding coordinates, whereas graphically you draw them with a common origin, and then draw the vector from the tip of the vector you are subtracting to the tip of the other one. If you read the previous paragraph again, you can see that Figure 1.2.3 can equally well be used to show that $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$, as to show that $\vec{r}_f = \vec{r}_i + \Delta\vec{r}$.

In a similar way, you can see graphically from Figure $4.3.1$ (or algebraically from Equation (4.3.2)) that the position vector of P in the frame B is given by $\vec{r}_{BP} = \vec{r}_{AP} - \vec{r}_{AB}$. The last term in this expression can be written in a different way, as follows. If I follow the convention I have introduced above, the quantity x_{BA} (with the order of the subscripts reversed) would be the x coordinate of the origin of frame A in frame B, and algebraically that would be equal to $-x_{AB}$, and similarly $y_{BA} = -y_{AB}$. Hence the vector equality $\vec{r}_{AB} = -\vec{r}_{BA}$ holds. Then,

$$\vec{r}_{BP} = \vec{r}_{AP} - \vec{r}_{AB} = \vec{r}_{AP} + \vec{r}_{BA}. \tag{4.3.3}$$

This is, in a way, the "inverse" of Equation (4.3.2): it tells us how to get the position of P in the frame B if we know its position in the frame A.

Let's show next how all this extends to displacements and velocities. Suppose the point P indicates the position of a particle at the time t. Over a time interval Δt, both the position of the particle and the relative position of the two reference frames may change. We can add yet another subscript, i or f, (for initial and final) to the coordinates, and write, for example,

$$\begin{aligned} x_{AP,i} &= x_{AB,i} + x_{BP,i} \\ x_{AP,f} &= x_{AB,f} + x_{BP,f} \end{aligned} \tag{4.3.4}$$

Subtracting these equations gives us the corresponding displacements:

$$\Delta x_{AP} = \Delta x_{AB} + \Delta x_{BP}. \tag{4.3.5}$$

Dividing Equation (4.3.5) by Δt we get the average velocities[1], and then taking the limit $\Delta t \to 0$ we get the instantaneous velocities. This applies in the same way to the y coordinates, and the result is the vector equation

$$\vec{v}_{AP} = \vec{v}_{BP} + \vec{v}_{AB}. \tag{4.3.6}$$

We have rearranged the terms on the right-hand side to (hopefully) make it easier to visualize what's going on. In words: the velocity of the particle P relative to (or *measured in*) frame A is equal to the (vector) sum of the velocity of the particle as measured in frame B, plus the velocity of frame B relative to frame A.

The result (4.3.6) is just what we would have expected from the examples mentioned at the beginning of this section, like rowing in a river or an airplane flying in the wind. For instance, for the airplane $\vec{v}_{BP}$ could be its "airspeed" (only it has to be a vector, so it would be more like its "airvelocity": that is, its velocity relative to the air around it), and $\vec{v}_{AB}$ would be the velocity of the air relative to the earth (the wind velocity, at the location of the airplane). In other words, A represents the earth frame of reference and B the air, or wind, frame of reference. Then, $\vec{v}_{AP}$ would be the "true" velocity of the airplane relative to the earth. You can see how it would be important to add these quantities as vectors, in general, by considering what happens when you fly in a cross wind, or try to row across a river, as in Figure $4.3.2$ below.

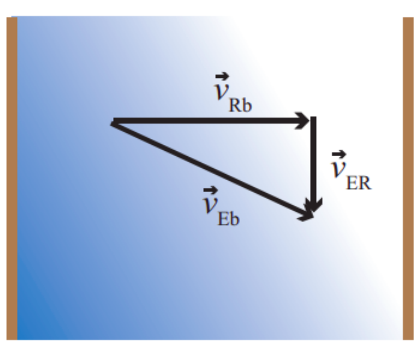


Figure $4.3.2$: Rowing across a river. If you head "straight across" the river (with velocity vector $\vec{v}_{Rb}$ in the moving frame of the river, which is flowing with velocity $\vec{v}_{ER}$ in the frame of the earth), your actual velocity relative to the shore will be the vector $\vec{v}_{Eb}$. This is an instance of Equation ($4.3.6$), with frame A being E (the earth), frame B being R (the river), and "b" (for "boat") standing for the point P we are tracking.

As you can see from this couple of examples, Equation ($4.3.6$) is often useful as it is written, but sometimes the information we have is given to us in a different way: for instance, we could be given the velocity of the object in frame A $(\vec{v}_{AP})$, and the velocity of frame B as seen in frame A $(\vec{v}_{AB})$, and told to calculate the velocity of the object as seen in frame B. This can be easily accomplished if we note that the vector $\vec{v}_{AB}$ is equal to $-\vec{v}_{BA}$; that is to say, the velocity of frame B as seen from frame A is just the opposite of the velocity of frame A as seen from frame B. Hence, Equation ($4.3.6$) can be rewritten as

$$\vec{v}_{AP}=\vec{v}_{BP}-\vec{v}_{BA}. \tag{4.3.7}$$

For most of the next few chapters we are going to be considering only motion in one dimension, and so we will write Equation ($4.3.6$) (or ($4.3.7$)) without the vector symbols, and it will be understood that v refers to the component of the vector $\vec{v}$ along the coordinate axis of interest.

A quantity that will be particularly important later on is the *relative velocity* of two objects, which we could label 1 and 2. The velocity of object 2 relative to object 1 is, by definition, the velocity which an observer moving along with 1 would measure for object 2. So it is just a simple frame change: let the earth frame be frame E and the frame moving with object 1 be frame 1, then the velocity we want is v_{12} ("velocity of object 2 in frame 1"). If we make the change A $\rightarrow$ 1, B $\rightarrow$ E, and P $\rightarrow$ 2 in Equation ($4.3.7$), we get

$$v_{12}=v_{E2}-v_{E1}. \tag{4.3.8}$$

In other words, the velocity of 2 relative to 1 is just the velocity of 2 minus the velocity of 1. This is again a familiar effect: if you are driving down the highway at 50 miles per hour, and the car in front of you is driving at 55, then its velocity relative to you is 5 mph, which is the rate at which it is moving away from you (in the forward direction, assumed to be the positive one).

It is important to realize that all these velocities are *real* velocities, each in its own reference frame. Something may be said to be truly moving at some velocity in one reference frame, and just as truly moving with a different velocity in a different reference frame. I will have a lot more to say about this in the next chapter, but in the meantime you can reflect on the fact that, if a car moving at 55 mph collides with another one moving at 50 mph in the same direction, the damage will be basically the same as if the first car had been moving at 5 mph and the second one had been at rest. For practical purposes, where you are concerned, another car's velocity relative to yours is that car's "real" velocity.

Resources

A good app for practicing how to add vectors (and how to break them up into components, magnitude and direction, etc.) may be found here:https://phet.colorado.edu/en/simulation/vector-addition.

Perhaps the most dramatic demonstration of how Equation ($4.3.6$) works in the real world is in this episode of *Mythbusters*: https://www.youtube.com/watch?v=BLuI118nhzc. (If this link does not work, do a search for "Mythbusters cancel momentum.") They shoot a ball from the bed of a truck, with a velocity (relative to the truck) of 60 mph backwards, while the truck is moving forward at 60 mph. I think the result is worth watching.

A very old, but also very good, educational video about different frames of reference is this one: https://www.youtube.com/watch?v=sS17fCom0Ns. You should try to watch at least part of it. Many things will be relevant to later parts of the course, including projectile motion.

[1] We have made a very natural assumption, that the time interval Δt is the same for observers tracking the particle's motion in frames A and B, respectively (where each observer is understood to be moving along with his or her frame). This, however, turns out to be *not* true when any of the velocities involved is close to the speed of light, and so the simple addition of velocities formula ($4.3.6$) does not hold in Einstein's relativity theory.

- **1.3: Reference Frame Changes and Relative Motion** by Julio Gea-Banacloche is licensed CC BY-SA 4.0. Original source: https://scholarworks.uark.edu/oer/3.

4.4: Examples

? Whiteboard Problem 4.4.1

Two blocks, of mass m and $3m$, are compressed on either side of a spring and tied together with a rope. They are sitting at rest on a frictionless surface, as shown in the figure.

1. The rope breaks and the larger block flies away from the smaller one at a speed of 2.00 m/s. If m=0.450 kg, what is the speed of the smaller block?
2. Now retie the rope and perform the experiment again, but slide the blocks along the surface with an initial speed of 3.5 m/s to the right. The rope breaks and the spring acts in the same way on the blocks. What is the speed of the smaller block?
3. For both parts (a) and (b): What is the speed of the center of mass of the system both before and after the rope breaks?

Example 4.4.2: Center of Mass of the Earth-Moon System

Using data from text appendix, determine how far the center of mass of the Earth-moon system is from the center of Earth. Compare this distance to the radius of Earth, and comment on the result. Ignore the other objects in the solar system.

Strategy

We get the masses and separation distance of the Earth and moon, impose a coordinate system, and use Equation 4.3.1 with just N = 2 objects. We use a subscript “e” to refer to Earth, and subscript “m” to refer to the moon.

Solution

Define the origin of the coordinate system as the center of Earth. Then, with just two objects, Equation 4.3.1 becomes

$$R=\frac{m_c r_c+m_m r_m}{m_c+m_m}. \tag{4.4.1}$$

We can find the values of the distances and masses from the Internet,

$$m_c=5.97\times 10^{24}\ kg \tag{4.4.2}$$

$$m_m=7.36\times 10^{22}\ kg \tag{4.4.3}$$

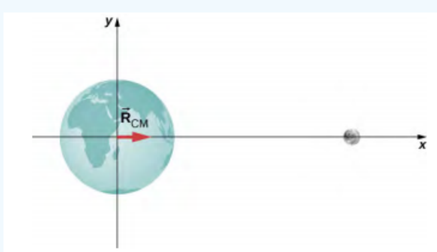
$$r_m=3.82\times 10^{5}\ m. \tag{4.4.4}$$

We defined the center of Earth as the origin, so r_e = 0 m. Inserting these into the equation for R gives

$$\begin{aligned} R &= \frac{(5.97\times 10^{24}\ kg)(0\ m)+(7.36\times 10^{22}\ kg)(3.82\times 10^{8}\ m)}{(5.98\times 10^{24}\ kg)+(7.36\times 10^{22}\ kg)} \\ &= 4.64\times 10^{6}\ m. \end{aligned}$$

Significance

The radius of Earth is 6.37 x 10^6 m, so the center of mass of the Earth-moon system is (6.37 − 4.64) x 10^6 m = 1.73 x 10^6 m = 1730 km (roughly 1080 miles) **below** the surface of Earth. The location of the center of mass is shown (not to scale).



? Exercise 4.4.3

Suppose we included the sun in the system. Approximately where would the center of mass of the Earth-moon-sun system be located? (Feel free to actually calculate it.)

✓ Example 4.4.4: Center of Mass of a Salt Crystal

Figure 4.4.3 shows a single crystal of sodium chloride—ordinary table salt. The sodium and chloride ions form a single unit, NaCl. When multiple NaCl units group together, they form a cubic lattice. The smallest possible cube (called the unit cell) consists of four sodium ions and four chloride ions, alternating. The length of one edge of this cube (i.e., the bond length) is 2.36 x 10^{-10} m. Find the location of the center of mass of the unit cell. Specify it either by its coordinates ($r_{CM,x}$, $r_{CM,y}$, $r_{CM,z}$), or by r_{CM} and two angles.

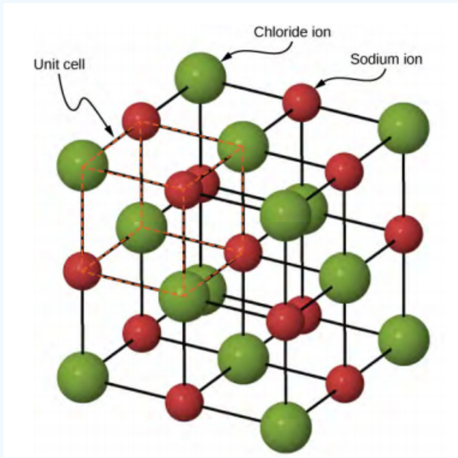


Figure 4.4.3: A drawing of a sodium chloride (NaCl) crystal.

Strategy

We can look up all the ion masses. If we impose a coordinate system on the unit cell, this will give us the positions of the ions. We can then apply Equation 4.3.1 in each direction (along with the Pythagorean theorem).

Solution

Define the origin to be at the location of the chloride ion at the bottom left of the unit cell. Figure 4.4.4 shows the coordinate system.

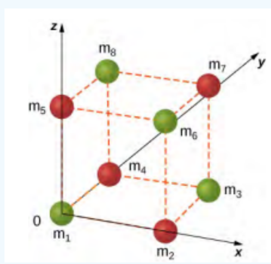


Figure 4.4.4: A single unit cell of a NaCl crystal.

There are eight ions in this crystal, so N = 8:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{j=1}^{8} m_j \vec{r}_j. \tag{4.4.5}$$

The mass of each of the chloride ions is

$$35.453u \times \frac{1.660 \times 10^{-27}\ kg}{u} = 5.885 \times 10^{-26}\ kg \tag{4.4.6}$$

so we have

$$m_1 = m_3 = m_6 = m_8 = 5.885 \times 10^{-26}\ kg. \tag{4.4.7}$$

For the sodium ions,

$$m_2 = m_4 = m_5 = m_7 = 3.816 \times 10^{-26}\ kg. \tag{4.4.8}$$

The total mass of the unit cell is therefore

$$M = (4)(5.885 \times 10^{-26}\ kg) + (4)(3.816 \times 10^{-26}\ kg) = 3.880 \times 10^{-25}\ kg. \tag{4.4.9}$$

From the geometry, the locations are

$$\begin{aligned}
\vec{r}_1 &= 0 \\
\vec{r}_2 &= (2.36 \times 10^{-10}\ m)\hat{i} \\
\vec{r}_3 &= r_{3x}\hat{i} + r_{3y}\hat{j} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{j} \\
\vec{r}_4 &= (2.36 \times 10^{-10}\ m)\hat{j} \\
\vec{r}_5 &= (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_6 &= r_{6x}\hat{i} + r_{6z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_7 &= r_{7x}\hat{i} + r_{7y}\hat{j} + r_{7z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{j} + (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_8 &= r_{8y}\hat{j} + r_{8z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{j} + (2.36 \times 10^{-10}\ m)\hat{k}.
\end{aligned}$$

Substituting:

$$\begin{aligned}
|\vec{r}_{CM,x}| &= \sqrt{r_{CM,x}^2 + r_{CM,y}^2 + r_{CM,z}^2} \\
&= \frac{1}{M}\sum_{j=1}^{8} m_j (r_x)_j \\
&= \frac{1}{M}(m_1 r_{1x} + m_2 r_{2x} + m_3 r_{3x} + m_4 r_{4x} + m_5 r_{5x} + m_6 r_{6x} + m_7 r_{7x} + m_8 r_{8x}) \\
&= \frac{1}{3.8804 \times 10^{-25}\ kg}\Big[(5.885 \times 10^{-26}\ kg)(0\ m) + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) \\
&\quad + (5.885 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + 0 + 0 \\
&\quad + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + 0\Big] \\
&= 1.18 \times 10^{-10}\ m.
\end{aligned}$$

Similar calculations give $r_{CM,y} = r_{CM,z} = 1.18 \times 10^{-10}$ m (you could argue that this must be true, by symmetry, but it's a good idea to check).

Significance

As it turns out, it was not really necessary to convert the mass from atomic mass units (u) to kilograms, since the units divide out when calculating r_{CM} anyway.

To express r_{CM} in terms of magnitude and direction, first apply the three-dimensional Pythagorean theorem to the vector components:

$$\begin{aligned}
r_{CM} &= \sqrt{r_{CM,x}^2 + r_{CM,y}^2 + r_{CM,z}^2} \\
&= (1.18 \times 10^{-10}\ m)\sqrt{3} \\
&= 2.044 \times 10^{-10}\ m.
\end{aligned}$$

Since this is a three-dimensional problem, it takes two angles to specify the direction of $\vec{r}_{CM}$. Let ϕ be the angle in the x,y-plane, measured from the +x-axis, counterclockwise as viewed from above; then:

$$\phi = \tan^{-1}\left(\frac{r_{CM,y}}{r_{CM,x}}\right) = 45^o. \tag{4.4.10}$$

Let θ be the angle in the y,z-plane, measured downward from the +z-axis; this is (not surprisingly):

$$\theta = \tan^{-1}\left(\frac{R_z}{R_y}\right) = 45^o. \tag{4.4.11}$$

Thus, the center of mass is at the geometric center of the unit cell. Again, you could argue this on the basis of symmetry

Exercise 4.4.5

Suppose you have a macroscopic salt crystal (that is, a crystal that is large enough to be visible with your unaided eye). It is made up of a **huge** number of unit cells. Is the center of mass of this crystal necessarily at the geometric center of the crystal?

Two crucial concepts come out of these examples:

1. 1. As with all problems, you must define your coordinate system and origin. For center-of-mass calculations, it often makes sense to choose your origin to be located at one of the masses of your system. That choice automatically defines its distance in Equation 4.3.1 to be zero. However, you must still include the mass of the object at your origin in your calculation of M, the total mass Equation 4.3.1. In the Earth-moon system example, this means including the mass of Earth. If you hadn't, you'd have ended up with the center of mass of the system being at the center of the moon, which is clearly wrong.
2. In the second example (the salt crystal), notice that there is no mass at all at the location of the center of mass. This is an example of what we stated above, that there does not have to be any actual mass at the center of mass of an object.

CHAPTER OVERVIEW

5: C5) Conservation of Momentum

In this chapter we are going to combine what we've done in Chapter 2 and Chapter 3, and introduce the **law of conservation of momentum**, which is the first real "rule" of nature that we are going to study in this book. In Chapter 2 we used this law in the form "interactions transfer momentum". That is the core concept, but it turns out that really implementing this is challenging without some mathematical tools - primarily that of vectors. Now that we have a better understanding of (and some practice doing) vector analysis, we are ready to state the full law of conservation of momentum: the *momentum of an isolated system does not change in time*. This is a seemingly simple statement, but it turns out to be extraordinarily powerful, because you can use it to make predictions about the systems you are studying - the final velocity of an object in a collision, for instance.

Of course, to make quantitative predictions, we need cast this law into mathematics, which is the following:

Conservation of Momentum

$$\Delta\vec{p} = 0 \Leftrightarrow \vec{p}_f - \vec{p}_i = 0 \Leftrightarrow \vec{p}_f = \vec{p}_i \tag{5.1}$$

(You should convince yourself that all three of those statements say exactly the same thing, using the definition of the delta and basic algebra!)

We should take note that this is a vector equation, which means there are actually two parts to it: $\Delta p_x = 0$ and $\Delta p_y = 0$. These two equations are independent, in the sense that they always both true at the same time, but aren't directly related.

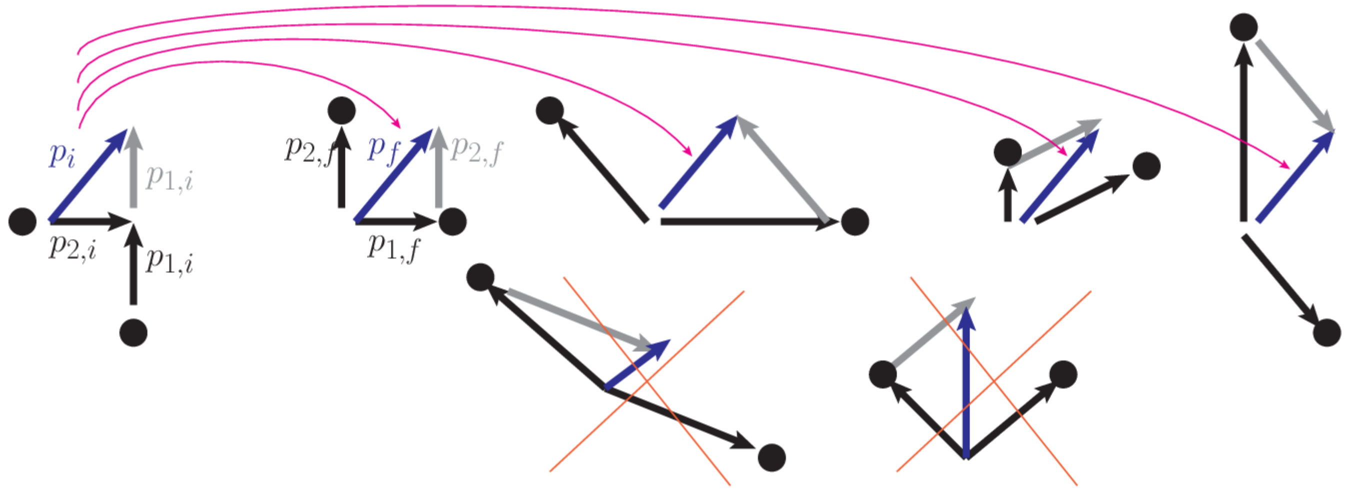
Let's also just briefly note how we should understand this mathematical statement in conjunction with our old "interactions transfer momentum" concept. If a system is isolated, there are no interactions between the system and the outside world, so there is no momentum transfer into or out of the system. That's exactly the same information contained in the equation above, since that says the change in momentum of an isolated system is zero.

Let's try some examples. Say you have a hockey puck (mass 0.5 kg) sliding on ice at a speed of 10 m/s, and it collides head on into another puck moving at 14 m/s. After the collision, the second puck moves away at 6 m/s, and we can use our laws to determine the final speed of the first. The equation says "final equals initial", so let's try to calculate the initial. The first puck has a momentum of 5 kg m/s, the second has 7 kg m/s - but it's moving the opposite direction, so the initial momentum in the x-direction is $p_{i,x} = 5\text{ kg m/s} - 7\text{ kg m/s} = -2\text{ kg m/s}$. We can also try to calculate the final momentum - the second puck has 3 kg m/s, in the positive direction, but we don't know the final speed of the first. However, using the equation above we can write

$$-2\text{ kg m/s} = (0.5\text{ kg})v_{1,f} + 3\text{ kg m/s}. \tag{5.2}$$

This equation can be solved for the unknown variable to get $v_{1,f} = -10\text{ m/s}$, which is negative because apparently the first puck bounces off the second and moves backward after the collision.

That example was pretty simple, since the collision only took place in a single direction (we chose x). However, our conservation of momentum law tells us that the vector (magnitude and direction) of the momentum must remain unchanged during a collision. The figure below illustrates that for the collision of two objects of similar masses (note the grey arrows are just helping us to make sure we add the vectors head to tail). The final momentum must be equal to the initial, and even though some of the collisions below look totally reasonable, they are prohibited by the law of conservation of momentum.


p_i
$p_{1,i}$
$p_{2,i}$
$p_{1,i}$
$p_{2,f}$
p_f
$p_{2,f}$
$p_{1,f}$

5.1: Conservation of Linear Momentum

Learning Objectives

- Explain the meaning of "conservation of momentum"
- Correctly identify if a system is, or is not, closed
- Define a system whose momentum is conserved
- Mathematically express conservation of momentum for a given system
- Calculate an unknown quantity using conservation of momentum

Recall what we learned about momentum transfer in Chapter 2: when objects interact with each other (like in a collision), they do so by *transferring momentum between each other*. What's more, this momentum is conserved - in that chapter, we expressed this conservation by noting the amount that one object lost was the same that the other object gained. Mathematically, we could write this as

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2, \tag{5.1.1}$$

for two objects. If this collison happens over some time period Δt, we can write this change as rate of change,

$$\frac{\Delta\vec{p}_1}{\Delta t} = -\frac{\Delta\vec{p}_2}{\Delta t}. \tag{5.1.2}$$

So now one object gains or looses momentum as the same rate that the other looses or gains it. Now performing some simple manipulations on this expression:

$$\frac{\Delta\vec{p}_1}{\Delta t} = -\frac{\Delta\vec{p}_2}{\Delta t} \rightarrow \frac{\Delta\vec{p}_1}{\Delta t} + \frac{\Delta\vec{p}_2}{\Delta t} = 0 \rightarrow \frac{\Delta\vec{p}_1 + \Delta\vec{p}_2}{\Delta t} = 0. \tag{5.1.3}$$

Using the definition of Δ, the top of this expression can be written as

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = (\vec{p}_{1f} - \vec{p}_{1i}) + (\vec{p}_{2f} - \vec{p}_{2i}) = (\vec{p}_{1f} + \vec{p}_{2f}) - (\vec{p}_{1i} + \vec{p}_{2i}). \tag{5.1.4}$$

Now looking at this expression, if we now define the total momentum of the system to be $\vec{P}_{sys} = \vec{p}_1 + \vec{p}_2$, we can see that what we just wrote was

$$\vec{P}_{sys,f} - \vec{P}_{sys,i}, \tag{5.1.5}$$

and combined with equation 5.1.3, we see this gives us

$$\vec{P}_{sys,f} - \vec{P}_{sys,i} = 0 \rightarrow \Delta\vec{P}_{sys} = 0. \tag{5.1.6}$$

Or, in other words, the **total momentum of the system does not change in time**. This is the best statement of conservation of momentum we have gotten so far, and is the best one to remember going forward. It does not matter "when" the initial and final states happen, all that matters is the amount of momentum does not change between those initial and final states.

Since momentum is a vector, both the magnitude and direction of this momentum must be conserved, as shown in Figure 5.1.1, the total momentum of the system before and after the collision remains the same, in both magnitude and direction.

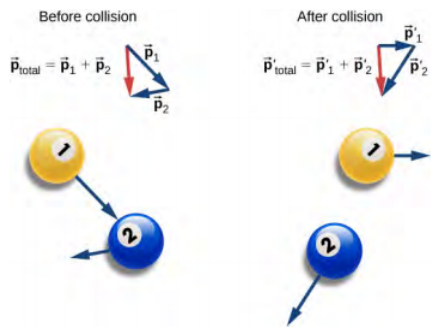


Figure 5.1.1: Before the collision, the two billiard balls travel with momenta $\vec{p}_1$ and $\vec{p}_2$. The total momentum of the system is the sum of these, as shown by the red vector labeled $\vec{p}_{total}$ on the left. After the collision, the two billiard balls travel with different momenta $\vec{p}\,'_1$ and $\vec{p}\,'_2$. The total momentum, however, has not changed, as shown by the red vector arrow $\vec{p}\,'_{total}$ on the right.

Generalizing this result to N objects, we obtain

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_N = constant \tag{5.1.7}$$

$$\sum_{j=1}^{N} \vec{p}_j = constant. \tag{5.1.8}$$

Equation 5.1.8 is the definition of the total (or net) momentum of a system of N interacting objects, along with the statement that the total momentum of a system of objects is constant in time—that is, *momentum is conserved*.

Conservation Laws

If the value of a physical quantity is constant in time, we say that the quantity is conserved.

Requirements for Momentum Conservation

There is a complication, however. A system must meet two requirements for its momentum to be conserved:

1. **The mass of the system must remain constant during the interaction.** As the objects interact (apply forces on each other), they may transfer mass from one to another; but any mass one object gains is balanced by the loss of that mass from another. The total mass of the system of objects, therefore, remains unchanged as time passes:

$$\left[\frac{dm}{dt}\right]_{system} = 0. \tag{5.1.9}$$

2. **The net external force on the system must be zero**. As the objects collide, or explode, and move around, they exert forces on each other. However, all of these forces are internal to the system, and thus each of these internal forces is balanced by another internal force that is equal in magnitude and opposite in sign. As a result, the change in momentum caused by each internal force is cancelled by another momentum change that is equal in magnitude and opposite in direction. Therefore, internal forces cannot change the total momentum of a system because the changes sum to zero. However, if there is some external force that acts on all of the objects (gravity, for example, or friction), then this force changes the momentum of the system as a whole; that is to say, the momentum of the system is changed by the external force. Thus, for the momentum of the system to be conserved, we must have

$$\vec{F}_{ext} = \vec{0}. \tag{5.1.10}$$

A system of objects that meets these two requirements is said to be a **closed system** (also called an isolated system). Another way to state equation 5.1.6 is

Law of Conservation of Momentum

The total momentum of a closed system is conserved:

$$\sum_{j=1}^{N} \vec{p}_j = constant. \tag{5.1.11}$$

This statement is called the **Law of Conservation of Momentum**. Along with the conservation of energy, it is one of the foundations upon which all of physics stands. All our experimental evidence supports this statement: from the motions of galactic clusters to the quarks that make up the proton and the neutron, and at every scale in between. In **a closed system, the total momentum never changes**.

Note that there absolutely **can** be external forces acting on the system; but for the system's momentum to remain constant, these external forces have to cancel, so that the **net** external force is zero. Billiard balls on a table all have a weight force acting on them, but the weights are balanced (canceled) by the normal forces, so there is no net force.

The Meaning of 'System'

A **system** (mechanical) is the collection of objects in whose motion (kinematics and dynamics) you are interested. If you are analyzing the bounce of a ball on the ground, you are probably only interested in the motion of the ball, and not of Earth; thus, the ball is your system. If you are analyzing a car crash, the two cars together compose your system (Figure $5.1.2$).

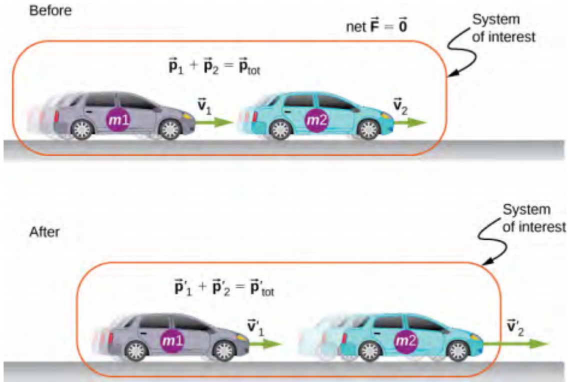


Figure $5.1.2$: The two cars together form the system that is to be analyzed. It is important to remember that the contents (the mass) of the system do not change before, during, or after the objects in the system interact.

5.2: The Problem Solving Framework

Solving problems in physics can be daunting task. Some solutions appear to be insights from the great beyond, or algebraic tricks that you'd never be able to reproduce on your own. Of course, the reason this appears to be the case is that the field of physics was developed by a bunch of experts on the natural world, who spent their entire lives studying it. They developed both analytical techniques and a general intuitive sense that can almost look like magic to mere mortals. However, the great thing about physics is that it's no mystery where these skills actually came from - the only thing we have to work with are the basic laws of the universe, and there are actually not many of those. For example, one could argue that in this textbook the only laws of the universe we are actually dealing with are the following:

- Conservation of Momentum
- Conservation of Energy
- Newton's three laws
- Kinematic Motion

(In fact, the last example on that list is really just an application of calculus, and perhaps should not be included at all. However, it will end up being a lot of what we do in the second half of this class, so we'll leave it there.)

Such a short list of things to learn! If all of mechanics is covered by that list, why does physics appear to be so hard? Well, that's because we are often applying these laws in situations which are new to us - if they weren't new, we could simply look up the answer, and that doesn't demonstrate any understanding of the physical world. To this end, we'd like to present the **problem solving framework**, which can tackle difficult (and easy!) problems we come across when trying to apply the physical laws we listed above.

The problem-solving framework is a 4-step process to get you from a problem statement to a solution, and is as follows[1]:

1. **Translate:** You might think about this step as "list the knowns and unknows": Take the words on the page and translate them into symbols, as appropriate. For example, "A train of mass 45 kg is traveling at a speed of 67 m/s", means that you can call the variable m the mass, v the speed, and they have particular values as part of the given information. This translation is necessary both because the language of physics is mathematical, and also it will help us organize our solution. This step also includes drawing the situation presented in the problem, and is a critical step that is often skipped by students. *A drawing will always help your thinking* - this is maybe even more true for a bad drawing, because it will be a sign that you do not yet understand the set up of the problem! Walk by the office of any physicist in the world and you will see sketches on a blackboard of whatever they were last working on.
2. **Model:** Decide what physical law you want to use to solve the problem. Sometimes, this is simply picking an equation to use ("this problem is asking us to find the center of mass, so I'm going to use the center of mass formula..."), but more often this step means picking one of the physical laws from the list above to use to solve the problem.
3. **Solve:** Solve! Perform whatever mathematical manipulations needs to be done to find the quantity that you can use to answer the question. Note that in many cases, this can be the longest and most difficult step, but *it also contains no physics!* The physics is done already in step 2, now we are just using math.
4. **Check:** It's critical to check your answer - at least to make sure it *could* be physically reasonable. Sometimes this is just "I found that this car is traveling at 800 mph, is that reasonable?" (no, you did something wrong!) A better way to do this step would be to pick an alternative solution to verify you get the same answer. The classic way to do this is to use Newton's laws if you solved it first with a conservation law, or vice versa. But since this is physics, we are always calculating something about the real world, and we should always be able to see if our calculations roughly match our expectations.

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[1]This framework was inspired by Unit C of Thomas Moore's textbook series *Six Idea That Shaped Physics.*

5.3: Examples

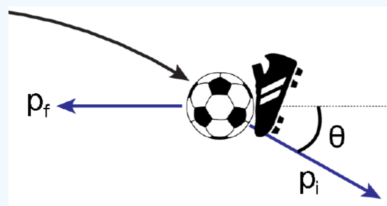
❓ Whiteboard Problem 5.3.1: System Definitions

For each of the following sets of objects, define *two systems each*; one that is isolated and one that is not.

1. The Earth, Sun, and Moon.
2. A football flying through the air, during the time after it is thrown and before it is caught. (you may ignore air resistance here)
3. Two colliding carts, ignoring friction and air resistance.
4. Two colliding carts, ignore air resistance but including the friction between the ground and the carts.

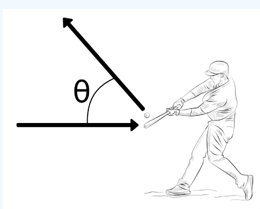
❓ Whiteboard Problem 5.3.2: Impulsive Shooter

A soccer player receives a pass in the air from a teammate and immediately fires the ball along the ground into the goal. The initial speed of the soccer ball is 15.0 m/s at an angle of 40.0° below the horizontal. The mass of the ball is 435 g.



If this is David Beckham, he can get the shot moving with a speed of of 30.0 m/s, moving straight along the ground. What impulse (magnitude and direction) did his foot transfer to the soccer ball?

❓ Whiteboard Problem 5.3.3: Impulsive Batter



A baseball player is hitting a baseball as shown in the figure. The baseball has a mass of 145 g, and is traveling at the batter at a speed of 35 m/s (that's an 80-mph fastball). The baseball player can deliver an impulse of 5.5 kg m/s to the baseball when they hit it.

1. The baseball player hits the ball at an angle of 15°, as shown in the figure. How much impulse does the baseball player give to the ball in the x- and y-directions?
2. What is the final speed of the ball?
3. How much impulse was transferred to the player (through the bat) when they hit the ball?

✓ Example 5.3.4: Collision, Center of Mass, and Recoil[1]

An 80-kg hockey player (call him player 1), moving at 3 m/s to the right, collides with a 90-kg player (player 2) who was moving at 2 m/s to the left. For a brief moment, they are stuck sliding together as they grab at each other.

a. What is their joint velocity as they slide together?
b. What was the velocity of their center of mass before and after the collision?

Solution

(a) Call the initial velocities v_{1i} and v_{2i}, the joint final velocity v_f, and assume the two players are an isolated system for practical purposes. Then conservation of momentum reads

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f. \tag{5.3.1}$$

Solving for the final velocity, we get

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}. \tag{5.3.2}$$

Substituting the values given, we get

$$v_f = \frac{80 \times 3 - 90 \times 2}{170} = 0.353\ \frac{\mathrm{m}}{\mathrm{s}}. \tag{5.3.3}$$

(b) According to Equation (3.3.3), the velocity of the center of mass, v_{cm}, is just the same as what we just calculated (Equation 5.3.2) above). This makes sense: after the collision, if the players are moving together, their system's center of mass has to be moving with them. Also, if the system is isolated, the center of mass velocity should be the same before and after the collision. So the answer is $v_{cm} = v_f$ = 0.353 m/s.

[1]For a variation of this problem that studies the relative velocity of this system with respect to another frame, check out Example 3.5.2 in University Physics 1 - Classical Mechanics, by Gea-Banacloche.

5.4: More Examples

❓ Problem-Solving Strategy: Conservation of Momentum

Using conservation of momentum requires four basic steps. The first step is crucial:

1. Identify a closed system (total mass is constant, no net external force acts on the system).
2. Write down an expression representing the total momentum of the system before the “event” (explosion or collision).
3. Write down an expression representing the total momentum of the system after the “event.”
4. Set these two expressions equal to each other, and solve this equation for the desired quantity

✓ Example $5.4.1$: Colliding Carts

Two carts in a physics lab roll on a level track, with negligible friction. These carts have small magnets at their ends, so that when they collide, they stick together (Figure $5.4.1$). The first cart has a mass of 675 grams and is rolling at 0.75 m/s to the right; the second has a mass of 500 grams and is rolling at 1.33 m/s, also to the right. After the collision, what is the velocity of the two joined carts?

Figure $5.4.1$: Two lab carts collide and stick together after the collision.

Strategy

We have a collision. We’re given masses and initial velocities; we’re asked for the final velocity. This all suggests using conservation of momentum as a method of solution. However, we can only use it if we have a closed system. So we need to be sure that the system we choose has no net external force on it, and that its mass is not changed by the collision.

Defining the system to be the two carts meets the requirements for a closed system: The combined mass of the two carts certainly doesn’t change, and while the carts definitely exert forces on each other, those forces are internal to the system, so they do not change the momentum of the system as a whole. In the vertical direction, the weights of the carts are canceled by the normal forces on the carts from the track.

Solution

Conservation of momentum is

$$\vec{p}_f = \vec{p}_i.$$

Define the direction of their initial velocity vectors to be the +x-direction. The initial momentum is then

$$\vec{p}_i = m_1 v_1\ \hat{i} + m_2 v_2\ \hat{i}.$$

The final momentum of the now-linked carts is

$$\vec{p}_f = (m_1 + m_2)\vec{v}_f.$$

Equating:

$$\begin{aligned}(m_1+m_2)\vec{v}_f &= m_1 v_1\ \hat{i} + m_2 v_2\ \hat{i} \\ \vec{v}_f &= \left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right)\hat{i}.\end{aligned}$$

Substituting the given numbers:

$$\begin{aligned}\vec{v}_f &= \left[\frac{(0.675\ kg)(0.75\ m/s) + (0.5\ kg)(1.33\ m/s)}{1.175\ kg}\right]\hat{i} \\ &= (0.997\ m/s)\hat{i}.\end{aligned}$$

Significance

The principles that apply here to two laboratory carts apply identically to all objects of whatever type or size. Even for photons, the concepts of momentum and conservation of momentum are still crucially important even at that scale. (Since they are massless, the momentum of a photon is defined very differently from the momentum of ordinary objects. You will learn about this when you study quantum physics.)

Exercise 5.4.2

Suppose the second, smaller cart had been initially moving to the left. What would the sign of the final velocity have been in this case?

Example 5.4.3: Ice Hockey 1

Two hockey pucks of identical mass are on a flat, horizontal ice hockey rink. The red puck is motionless; the blue puck is moving at 2.5 m/s to the left (Figure 5.4.3). It collides with the motionless red puck. The pucks have a mass of 15 g. After the collision, the red puck is moving at 2.5 m/s, to the left. What is the final velocity of the blue puck?

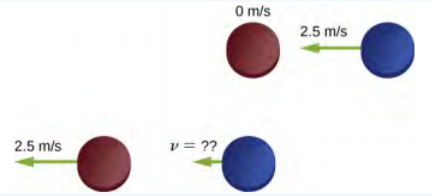


Figure 5.4.3: Two identical hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We're told that we have two colliding objects, we're told the masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy. Define the system to be the two pucks; there's no friction, so we have a closed system.

Before you look at the solution, what do you think the answer will be?

The blue puck final velocity will be:

a. zero
b. 2.5 m/s to the left
c. 2.5 m/s to the right
d. 1.25 m/s to the left
e. 1.25 m/s to the right
f. something else

Solution

Define the +x-direction to point to the right. Conservation of momentum then reads

$$\vec{p_f} = \vec{p_i}$$
$$mv_{r_f}\ \hat{i} + mv_{b_f}\ \hat{i} = mv_{r_i}\ \hat{i} - mv_{b_i}\ \hat{i}.$$

Before the collision, the momentum of the system is entirely and only in the blue puck. Thus,

$$mv_{r_f}\ \hat{i} + mv_{b_f}\ \hat{i} = -mv_{b_i}\ \hat{i}$$
$$v_{r_f}\ \hat{i} + v_{b_f}\ \hat{i} = -v_{b_i}\ \hat{i}.$$

(Remember that the masses of the pucks are equal.) Substituting numbers:

$$-(2.5\ m/s)\hat{i} + \vec{v}_{b_f} = -(2.5\ m/s)\hat{i}$$
$$\vec{v}_{b_f} = 0.$$

Significance

Evidently, the two pucks simply exchanged momentum. The blue puck transferred all of its momentum to the red puck. In fact, this is what happens in similar collision where $m_1 = m_2$.

Exercise 5.4.4

Even if there were some friction on the ice, it is still possible to use conservation of momentum to solve this problem, but you would need to impose an additional condition on the problem. What is that additional condition?

Example 5.4.5: Philae

On November 12, 2014, the European Space Agency successfully landed a probe named **Philae** on Comet 67P/Churyumov/Gerasimenko (Figure 5.4.4). During the landing, however, the probe actually landed three times, because it bounced twice. Let's calculate how much the comet's speed changed as a result of the first bounce.

Figure 5.4.4: An artist's rendering of Philae landing on a comet. (credit: modification of work by "DLR German Aerospace Center"/Flickr)

Let's define upward to be the +y-direction, perpendicular to the surface of the comet, and y = 0 to be at the surface of the comet. Here's what we know:

- The mass of Comet 67P: $M_c = 1.0 \times 10^{13}$ kg
- The acceleration due to the comet's gravity: $\vec{a} = -(5.0 \times 10^{-3}\ m/s^2)\ \hat{j}$
- **Philae**'s mass: $M_p = 96$ kg
- Initial touchdown speed: $\vec{v}_1 = -(1.0\ m/s)\ \hat{j}$
- Initial upward speed due to first bounce: $\vec{v}_2 = (0.38\ m/s)\ \hat{j}$
- Landing impact time: $\Delta t = 1.3$ s

Strategy

We're asked for how much the comet's speed changed, but we don't know much about the comet, beyond its mass and the acceleration its gravity causes. However, we are told that the **Philae** lander collides with (lands on) the comet, and bounces off of it. A collision suggests momentum as a strategy for solving this problem.

If we define a system that consists of both **Philae** and Comet 67/P, then there is no net external force on this system, and thus the momentum of this system is conserved. (We'll neglect the gravitational force of the sun.) Thus, if we calculate the change of momentum of the lander, we automatically have the change of momentum of the comet. Also, the comet's change of velocity is directly related to its change of momentum as a result of the lander "colliding" with it.

Solution

Let $\vec{p}_1$ be **Philae**'s momentum at the moment just before touchdown, and $\vec{p}_2$ be its momentum just after the first bounce. Then its momentum just before landing was

$$\vec{p}_1 = M_p\vec{v}_1 = (96\ kg)(-1.0\ m/s\ \hat{j}) = -(96\ kg \cdot m/s)\hat{j}$$

and just after was

$$\vec{p}_2 = M_p \vec{v}_2 = (96\ kg)(+0.38\ m/s\ \hat{j}) = (36.5\ kg \cdot m/s)\hat{j}.$$

Therefore, the lander's change of momentum during the first bounce is

$$\begin{aligned}\Delta \vec{p} &= \vec{p}_2 \vec{p}_1 \\ &= (36.5\ kg \cdot m/s)\hat{j} - (-96.0\ kg \cdot m/s\ \hat{j}) \\ &= (133\ kg \cdot m/s)\hat{j}\end{aligned}$$

Notice how important it is to include the negative sign of the initial momentum.

Now for the comet. Since momentum of the system must be conserved, the comet's momentum changed by exactly the negative of this:

$$\Delta \vec{p}_c = -\Delta \vec{p} = -(133\ kg \cdot m/s)\hat{j}.$$

Therefore, its change of velocity is

$$\Delta \vec{v}_c = \frac{\Delta \vec{p}_c}{M_c} = \frac{-(133\ kg \cdot m/s)\hat{j}}{1.0 \times 10^{13}\ kg} = -(1.33 \times 10^{-11}\ m/s)\hat{j}.$$

Significance

This is a very small change in velocity, about a thousandth of a billionth of a meter per second. Crucially, however, it is **not** zero.

? Exercise 5.4.6

The changes of momentum for **Philae** and for Comet 67/P were equal (in magnitude). Were the impulses experienced by **Philae** and the comet equal? How about the forces? How about the changes of kinetic energies?

CHAPTER OVERVIEW

6: C6) Conservation of Angular Momentum I

In this chapter we are going to move away from linear motion and start talking about angular motion. Fortunately, the *physics* here is exactly the same - angular momentum is conserved in precisely the way that linear momentum is conserved. However, angular momentum is often more confusing for students to deal with. This is actually very understandable, since it requires a few extra notions, as well as being something that we don't experience quite as often in real life. So in this introduction we are going to focus on some of the basic variables used to describe circular motion and momentum, and leave details about the conservation laws to later in the chapter.

Just like we measured linear motion with a change in linear position $\Delta\vec{r}$, we'd like to describe rotational motion with a change in angular position, $\Delta\vec{\theta}$. You should be familiar with how to measure an angle θ (see the left picture below), but the units we use turn out to be important. The physical (S.I.) unit that corresponds to an angle measure is the radian, and is defined (again, see the figure) as

$$\theta = \frac{s}{r}, \tag{6.1}$$

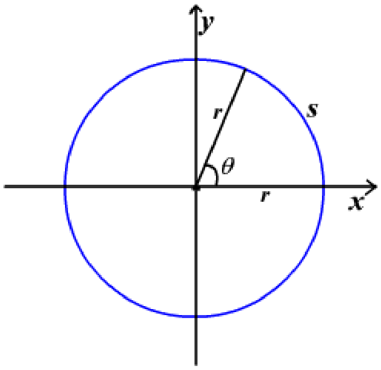
where s is the arclength and r is the radius of the circle in question. It's easy to see then how many radians are in an entire circle, since that corresponds to an arclength of $s = 2\pi r$, so $\theta = 2\pi r/r = 2\pi$. Of course, there is no problem with saying "an object is rotating at 3 revolutions per minute" - that's still a valid angular speed, it's just not in SI units. If we were going to calculate something, we would want to convert that into radians per second; let's do that real quick as an example:

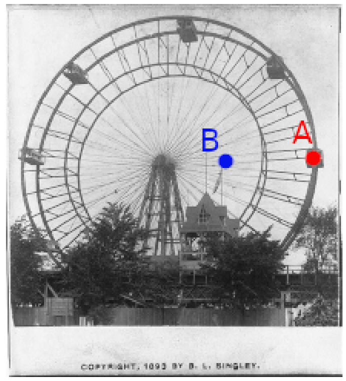
$$\frac{3\text{ rev}}{1\text{ min}}\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ sec}}\right) \simeq 0.314\text{ rad/s}. \tag{6.2}$$

The conversion factors here are represented as factors that you multiply the initial value by - there are 2π radians in one revolution, and 60 seconds in one minute.

So now that we know how to measure the angular position, how do we find the angular version of linear velocity, $\vec{v} = \Delta\vec{r}/\Delta t$? That's simple, since we are now just measuring the displacements in angles, and we get *angular velocity*[1] $\vec{\omega} = \Delta\vec{\theta}/\Delta t$. The *rotational speed* ω is defined the same way as the linear speed, as the magnitude of this vector quantity.

So that seems easy enough, but the challenge comes when we try to go back and forth between linear and rotational quantities. Let's try to do this with the ferris wheel shown in the figure on the right. This is "The Great Ferris Wheel", built for the 1893 World's Fair, and is 140 feet (43 m) in radius. When we say "the wheel is moving at an angular speed of 1 rotation a minute", that *applies to the entire object* - specifically, points A and B (which is halfway out to the edge) have the same angular speed (*why?*). The same is not true of the linear speeds of different points on the wheel. For example, over one rotation, point A travels a distance $2\pi(43\text{ m}) \sim 270\text{ m}$, while point B travels $2\pi(43/2\text{ m}) \sim 135\text{ m}$. Therefore, *the linear speed of A is greater then the linear speed of B, because A is traveling a longer distance!* We picked one rotation for convenience, but it would apply equally to any time period you chose.





[1]Notice that we haven't talked about how to assign a direction to this velocity - all velocities have directions! You do this with "the right hand rule", which we will talk about later in this chapter.

6.1: Angular Momentum

Back in Chapter 2 we introduced the momentum of an object moving in one dimension as $p = mv$, and found that it had the interesting property of being conserved in collisions between objects that made up an isolated system. Critical to that discussion was the idea that objects are represented by points, so they have well-defined positions. Of course, in real life, objects are not made up a single points, but are *extended*. We now want to move the discussion into studying these extended objects, that have real size, and understand how they transfer momentum differently than simple points. The biggest difference between points and extended objects it that *extended objects can rotate, as well as move through space*.

We want to define a quantity that measures this rotation, in the same way linear momentum measured something about how particles translated in space. This quantity is going to be angular momentum $\vec{L}$, which we define as

$$\vec{L} = I\vec{\omega} \tag{6.1.1}$$

The variables here are the rotational analogues of the linear quantities; $\vec{\omega}$ is angular velocity (like linear velocity $\vec{v}$), and I is called the moment of inertia, and is analogous to the mass m. In this section we will build up an understanding of each of these quantities, and how they can be used to solve problems dealing with the rotation of extended objects.

The equation above is a vector, so we need to understand it's direction. It's tempting to say "the object rotates counterclockwise" or the like, but is there a vector direction associated to "counterclockwise"? There is, but it's probably not what you're thinking! The direction of counterclockwise (or clockwise) rotation comes from **the right hand rule**. The most basic version of the right hand rule is shown in the figure below, and the way you use it is:

1. Take your right hand...
2. ...curl the fingers of your right hand in the direction of rotation...
3. ...your thumb points in the direction of $\vec{\omega}$.

Figure $6.1.1$: From the Wikipedia page on this topic. In this case, the arrow along the hand shows the direction of $\vec{\omega}$, and we might say "the angular velocity is directed up and to the right".

The angular velocity $\vec{\omega}$ is the exact analog to the linear velocity $\vec{v}$, although it deserves a little bit of attention. The unit for the angular velocity is angular distance per unit time, and in SI units that's radians per second. What's interesting about this is that for a single object, every point on the object will have the same angular velocity. It really has to be that way - if a hoop is spinning at two revolutions per second, asking what angular speed the outside of the hoop is spinning at had better be the same as every other point. Said another way, if any point on the object is spinning at a different rate (say, one revolution per second), that means the object is tearing itself apart! Those kinds of objects are more complicated, so we will always assume our object are *rigid*, meaning the entire object rotates at the same rate.

Now let's look at the newest aspect of this formula, the moment of inertia I. Like the mass, this quantity encodes the inertia of an object, but includes information about it's shape. Specifically, for an object made up of a bunch of masses m_i which are located at distances r_i from an axis of rotation, the moment of inertia is

$$I = \sum_i m_i r_i^2. \tag{6.1.2}$$

That looks like a simple formula, but if the shape is complicated, that sum might actually be very difficult or impossible to do[1]. Fortunately, there are simplier expressions for many of the shapes we often encounter in nature. Many of these shapes are shown in the figure below, and many more can be found in Wikipedia.

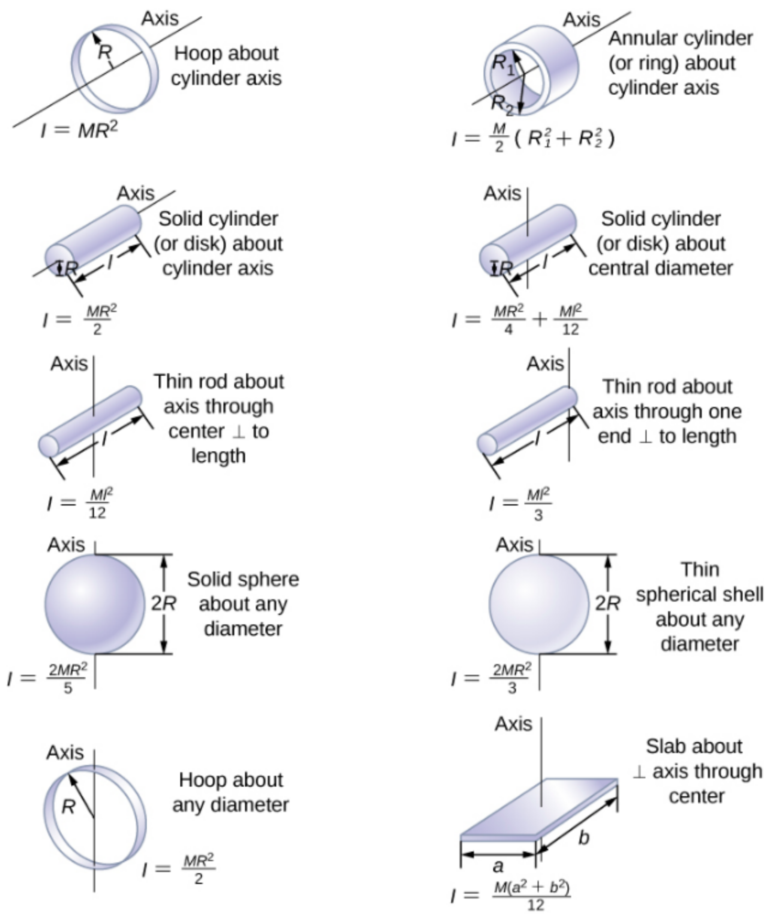


Table 6.1.1

In nearly all of these examples (save those that have multiple different dimensions, like the annulus and the slab), the moment of inertia can be written like

$$I = \alpha M R^2, \tag{6.1.3}$$

where $\alpha = 1/2$ for a cylinder (or disk), $\alpha = 1$ for a hoop, etc.

Just like with linear momentum, we have a conservation principle associated with angular momentum:

Law of Conservation of Angular Momentum

The total angular moment of a closed system is conserved:

$$\sum_j \vec{L}_j = constant. \leftrightarrow \Delta\vec{L} = 0 \tag{6.1.4}$$

In addition, we have all the conditions around using this law that we had for conservation of linear momentum; namely, that the system must be isolated.

Adding Moments of Inertia

Just like the case of linear momentum, the total angular momentum of a system can be found by simply adding up the angular momenta of the individual parts of a system. For example, if object 1 is a spinning hoop (of mass M and radius R, spinning at ω_1), and object 2 is a spinning cylinder (of mass m and radius r, spinning at ω_2), then looking at the figure above we can write the total angular momentum of that system as

$$L_{tot} = L_1 + L_2 = MR^2\omega_1 + \frac{1}{2}mr^2\omega_2. \tag{6.1.5}$$

Here, we are assuming the two objects are not moving together at all. But, it often happens that we want to know the angular momentum of a single object that is made up of shapes found in our figure. For example, see figure 6.3.3 in the examples section at the end of this chapter, which is two disks rotating together. In that case the two disks have the same radius R, but one has three times the mass of the other (M and $3M$). If those two disks are rotating at the same rate, their angular momentum would be

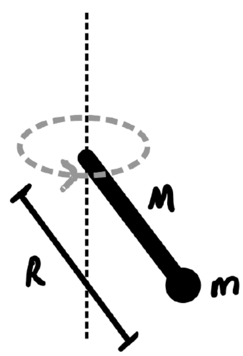
$$L_{tot} = L_1 + L_2 = \frac{1}{2}MR^2\omega + \frac{1}{2}3MR^2\omega = \frac{1}{2}(M+3M)R^2\omega. \tag{6.1.6}$$

Notice this is the same result we would have gotten if we had simply added their moments of inertia together first:

$$I_{tot} = I_1 + I_2 = \frac{1}{2}MR^2 + \frac{1}{2}3MR^2 = \frac{1}{2}(M+3M)R^2 \rightarrow L_{tot} = I_{tot}\omega = \frac{1}{2}(M+3M)R^2\omega. \tag{6.1.7}$$

The lesson here is *when adding moments of inertia, do not add their masses or radii together alone - add their moments of inertia*.

For another concrete example, consider finding the angular momentum of a rod with a mass on the edge (shown below).



The total moment of inertia of this object is tricky, because the objects are different shapes. The rod has a moment of inertia of $I_r = \frac{1}{3}MR^2$, while the point mass[2] had $I_p = mR^2$. It's temping to just add the masses together, $m+M$, but what fraction do you use out front, $\frac{1}{3}$ or 1? The answer is that you must add the moments of inertia themselves:

$$I_{tot} = I_r + I_p = \frac{1}{3}MR^2 + mR^2 = (\frac{1}{3}M + m)R^2. \tag{6.1.8}$$

Angular Speed vs Linear Speed

When objects rotate at a particular angular speed (in say, revolutions per second), the entire object moves at the same angular speed. This is because these objects are assumed to be rigid, so when one point on the objects goes around once, so does every other point. However, that doesn't mean that each of these points moves at the same *linear speed* (in say, meters per second). In fact, each point on the object moves at a different linear speed, depending on how far away it is from the center of rotation.

It's relatively easy to determine how these two quantities are related to each other, and this will turn out to be a very important formula. First, let's think about how fast a point a distance r from the center of an rotating object moves. If it takes a time period T for the object to rotate, such a point moves with speed

$$v = \frac{\text{distance traveled}}{\text{time interval}} = \frac{2\pi r}{T}. \tag{6.1.9}$$

Now, let's calculate the angular speed of this point (keeping in mind this is actually the angular speed of the entire object!). Over one rotation, it's angular speed will be

$$\omega = \frac{\text{angular distance traveled}}{\text{time interval}} = \frac{2\pi}{T}. \tag{6.1.10}$$

The 2π here is just the number of radians the point travels for each rotation. These two formula are valid, but not that interesting because we don't know what the particular time period is here - so let's use the two of them to eliminate the time interval T. Solving the second gives $T = 2\pi/\omega$, so let's plug that into the first:

$$v = \frac{2\pi r}{2\pi/\omega} = \frac{2\pi r\omega}{2\pi} \rightarrow v = r\omega. \tag{6.1.11}$$

This very simple formula will be very helpful later; for now, it can give us some insight into how these point move. For example, at a particular angular speed ω, objects closer to the axis of rotation (smaller r) will move slower (have smaller v), while objects farther from the axis of rotation (larger r) will move faster (have larger v). This seems counterintuitive, but can be understood by seeing that objects far from the axis of rotation have further to travel then objects closer to the center, and have to do it over the same period of time.

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[1] In your calculus class you will learn how to calculate this for a wide variety of shapes.

[2] The moment of inertia of a point mass is not found in the table above, but it's easy to see the answer by looking at equation 6.1.2; with only one point in the sum it becomes mr^2. This can also be seen by thinking about what a point mass actually looks like when it's rotating - a hoop, which is in the table above.

6.2: Angular Momentum and Torque

So far, we have seen that angular momentum is conserved in the same way that linear momentum is conserved. Specifically, in a system which can be modeled as isolated, angular momentum is conserved,

$$\Delta \vec{L} = 0, \tag{6.2.1}$$

in the same way that linear momentum was, $\Delta \vec{p} = 0$. There is another similarity between these two cases - when the system was *not* isolated, we could model the interactions between the system and the environment by a force $\vec{F}$ acting over a time period Δt, so that

$$\vec{F} = \frac{\Delta \vec{p}}{\Delta t} \rightarrow \Delta \vec{p} = \vec{F} \Delta t. \tag{6.2.2}$$

In other words, if the system is not isolated we can still work with $\Delta \vec{p}$, we just have to set it equal to the (net) force times the time interval, rather than simply zero.

There is a similar expression for angular momentum - when a system experiences a change in angular momentum $\Delta \vec{L}$ over a time period Δt due to an external interaction of some kind, we can model that interaction as delivering a torque $\vec{\tau}$ to the system, defined as

$$\vec{\tau} = \frac{\Delta \vec{L}}{\Delta t}. \tag{6.2.3}$$

From this equation, we can tell that the torque is in the same direction as the *change in the* angular momentum. In many cases, this will simply be along the axis of rotation (see the example below). Later in this course, we will study the kinematics of both linear and rotational motion to determine the time evolution of the position and velocity (both linear and angular) of objects that experience either forces or torques - but for now, we can use these two expressions to determine the final velocities after specific time periods.

Example 6.2.1

As a very simple example, consider the bike wheel in the figure below, with a moment of inertia of 10 kg m^2 shown spinning at a rate of 3 rev/s. It would appear this wheel is spinning freely, but of course we know that there is some kind of friction between the axle and the wheel that is slowing it down. If we modeled this friction as torque, and said for example that the size of this torque is 1.0 Nm, we could determine how long it would take for the wheel to stop using the equation above.

Solution

Following the problem solving framework from an earlier section of this textbook:

1. **Translate**: We will use the following variables:

$$\omega_i = 3 \text{ rev/s } = 18.8 \text{ rad/s}, \quad \omega_j = 0, \quad \Delta t = ?, \quad I = 10 \text{ kg m}^2, \quad \tau = 1.0 \text{ Nm}. \tag{6.2.4}$$

Notice a few things - first, we have converted the initial speed from revolutions per second to radians per second. Although this is not always necessary, here it's important because the torque is in units of Newton-meters, which is SI. We've set the final speed to be zero, specified our time as the unknown variable, and set our torque equal to the variable τ (if you are worried about the sign of that variable - good catch! We'll deal with that later...). Finally, we've indicated that the direction of the torque is in the z-coordinate, and that it is negative. Choosing the z-coordinate along the axis of rotation is rather arbitrary, but a common standard. We've included a negative sign to be clear we know it's slowing down relative to the direction of the angular momentum, which is in the direction of $\vec{\omega}$ here, since $\vec{L} = I\vec{\omega}$.

2. **Model**: We are going to use conservation of angular momentum - specifically the equation from above, $\vec{\tau} = \frac{\Delta \vec{L}}{\Delta t}$.
3. **Solve**: First, we specify the given equation into it's component form. Since the initial speed is just along the axis of rotation, let's take that to be the z-direction and then we just need that component::

$$\vec{\tau} = \frac{\Delta \vec{L}}{\Delta t} \rightarrow \tau_z = \frac{L_{zf} - L_{zi}}{\Delta t}. \tag{6.2.5}$$

Now we plug in the variables above, and solve for the unknown:

$$\rightarrow -\tau = \frac{0 - I\omega_i}{\Delta t} \rightarrow \Delta t = \frac{I\omega_i}{\tau} \simeq 188 \text{ s}. \tag{6.2.6}$$

Notice in this step we have indicated that the torque (in the z-direction) should be negative - that's because we declared that the angular velocity ω_i was positive, so if the object is going to *loose* momentum, the torque has to be negative. That also cancelled the negative sign on the other side of the equation.

4. **Check**: There is not a lot we can do to check this, but we can easily imagine a bike wheel spinning this fast might take ~3 minutes to slow down to a stop!

6.3: Examples

Law of Conservation of Angular Momentum

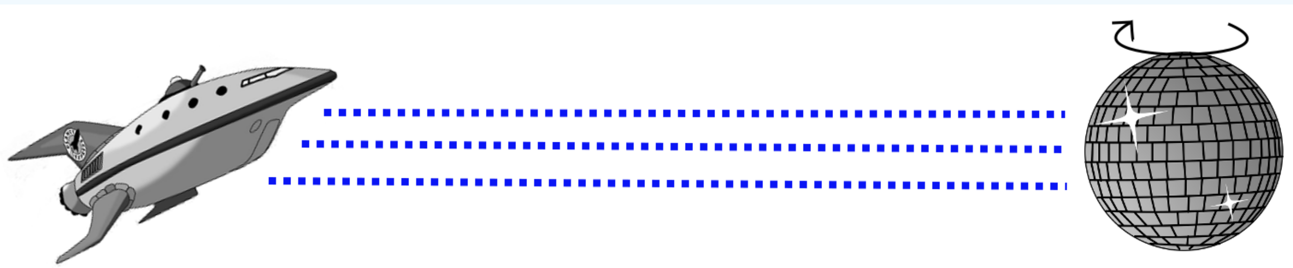
The angular momentum of a system of particles around a point in a fixed inertial reference frame is conserved if there is no net external torque around that point:

$$\frac{d\vec{L}}{dt} = 0 \tag{6.3.1}$$

or

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots + \vec{l}_N = constant. \tag{6.3.2}$$

Whiteboard Problem 6.3.1: Space Rescue



The crew of a spaceship has come across an abandoned circular probe, and they want to grab onto it with their tractor beam. The problem is that the crew doesn't know what will happen to them if they do this, because the probe is spinning, at a rate of once every 2 seconds. In this problem, model the spacecraft as being a point of mass 10,000 kg, and the probe as a hollow sphere radius 10 m and a similar mass of 10,000 kg.

1. Assume that turning on the tractor beam to grab the probe has the same effect as attaching the two ships with a massless rigid rod. Describe what would happen to the ships when the tractor beam is turned on.
2. If they are separated by a distance of 100 m when they do this, calculate the final *linear* speed of the spaceship.

Whiteboard Problem 6.3.2: Spinning Platform

I am standing on a frictionless platform holding a spinning bicycle wheel. I'm holding the bicycle wheel so that the axis of rotation is vertical.

1. As I turn the axis of rotation of the bicycle wheel, I start to spin on the platform. Why?
2. What is the moment of inertia of the wheel, relative to the axis passing through it's center? It has a mass of 4.1 kg and radius 32 cm.
3. Estimate the moment of inertia of my body as I spin on the platform, relative to the vertical axis passing through the center of the platform.
4. Notice that the wheel can spin around it's own axis *and* around the axis going through the center of the platform. What is the moment of inertia of the center of mass of the wheel as it spins around the axis of the platform? (You will have to estimate how far away from the center I am holding it).
5. Initially, the wheel spins 3 times a second and I am at rest on the platform. Then, I turn the wheel completely around, so my left hand is where my right hand used to be. How fast am I spinning on the platform?

Whiteboard Problem 6.3.3: MY FAVORITE PROBLEM

A ``pulsar" is a celestial object which ``blinks" on and off very quickly (in radio wavelengths). I propose that such objects are formed when rotating stars collapse without losing any mass, and since they conserve angular momentum, they are left spinning very quickly. I propose that the blinking is caused by a single hot spot on their surface, so it appears to blink as the pulsar spins. *(General Hint: This problem is easier if you use scientific notation for the large numbers, and/or use symbols for as much of it as you can!)*

1. What would the radius of a pulsar be if it's a solid sphere, blinking once a second, and formed from the collaspe of the Sun? The Sun rotates at about 15°/day, and has a radius of 696,000 km.
2. The fastest pulsar (PSR J1748-2442ad, discovered in 2005) spins at 716 times a second. What is the radius of this object, if it formed from a star with the mass of the Sun by my collapse theory?
3. What is the density of this object? Does this density sound reasonable? (maybe Google the density of some objects you know...) You will need the mass of the Sun for this part, 1.99×10^{30} kg.

Whiteboard Problem 6.3.4: Calcy Centrifuge

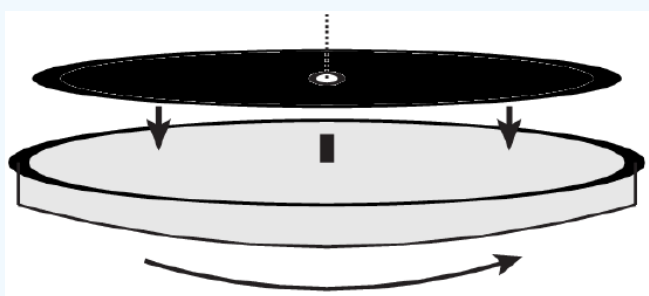
A particular centrifuge has an angular speed that depends on time like

$$\omega(t) = -C \exp^{-(t-t_0)} + \omega_0, \tag{6.3.3}$$

where t_0 and ω_0 are parameters that can be set by the user, and the constant $C = 1.0$ rad/s. The radius of this centrifuge is 15.5 cm.

1. If we want our centrifuge to start at $\omega = 0$ when $t = 0$ and travel at a maximum angular speed of 105 rad/s as $t \to \infty$, what do the constants t_0 and ω_0 have to be?
2. Draw a sketch of the function $\omega(t)$.
3. What is the torque as a function of time, $\tau(t)$, that the motor must act on this centrifuge with? Treat the centrifuge as a solid disk of mass 200 g.
4. Draw a sketch of the function $\tau(t)$.

Whiteboard Problem 6.3.5: Angular DJ



The figure shows a record being dropped onto a turntable. The turntable is initially spinning freely at 0.5 rev/s, and can be modeled as a disk with radius 20 cm, mass 0.50 kg, and moment of inertia $I = \frac{1}{2}MR^2$.

1. What is the angular speed of the turntable and the record together, after the record is dropped with zero initial angular speed? The record is also a disk, with a radius of 18 cm and mass of 100 g. *Note: turntables generally have a motor to keep them spinning at the same rate, but this one is spinning freely and does not!*
2. The turntable and record can be stopped by applying a brake to the turntable. This brake applies a small torque of 0.03 Nm. How long does it take this brake to slow the turntable to a stop?

Whiteboard Problems 6.3.6: A Real Life Skater

A figure skater is spinning at 2 rev/s with their arms held out from their body. The mass of this figure skater is 65 kg, and you can assume their body has a moment of inertia of $I = \frac{1}{4}MR^2$, where R is the length of their arms. *Quick reality check: most of this figure skater's mass is near the center of the rotational axis, so that's why their moment of inertia is less than that of a disk!*

1. If their arms are extended to a radius of 50 cm while they are spinning, how big is their moment of inertia?
2. Now they pull their arms in to a distance of 30 cm. How fast are they rotating now, assuming there is no friction between their skates and the ice?
3. Now repeat part (b), assuming friction is acting on their skates with a torque 10.0 Nm. If it takes 3 seconds for them to pull their arms in, how fast will they be rotating after they do so?

Example 6.3.7: Coupled Flywheels

A flywheel rotates without friction at an angular velocity ω_0 = 600 rev/min on a frictionless, vertical shaft of negligible rotational inertia. A second flywheel, which is at rest and has a moment of inertia three times that of the rotating flywheel, is dropped onto it (Figure 6.3.3). Because friction exists between the surfaces, the flywheels very quickly reach the same rotational velocity, after which they spin together.

a. Use the law of conservation of angular momentum to determine the angular velocity ω of the combination.
b. What fraction of the initial kinetic energy is lost in the coupling of the flywheels?

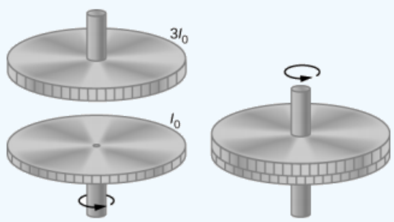


Figure 6.3.3: Two flywheels are coupled and rotate together.

Strategy

Part (a) is straightforward to solve for the angular velocity of the coupled system. We use the result of (a) to compare the initial and final kinetic energies of the system in part (b).

Solution

a. No external torques act on the system. The force due to friction produces an internal torque, which does not affect the angular momentum of the system. Therefore conservation of angular momentum gives

$$I_0\omega_0 = (I_0 + 3I_0)\omega,$$

$$\omega = \frac{1}{4}\omega_0 = 150\ rev/min = 15.7\ rad/s.$$

b. Before contact, only one flywheel is rotating. The rotational kinetic energy of this flywheel is the initial rotational kinetic energy of the system, $\frac{1}{2}I_0\omega_0^2$. The final kinetic energy is

$$\frac{1}{2}(4I_0)\omega^2 = \frac{1}{2}(4I_0)\left(\frac{\omega_0}{4}\right)^2 = \frac{1}{8}I_0\omega_0^2.$$

Therefore, the ratio of the final kinetic energy to the initial kinetic energy is

$$\frac{\frac{1}{8}I_0\omega_0^2}{\frac{1}{2}I_0\omega_0^2} = \frac{1}{4}.$$

Thus, 3/4 of the initial kinetic energy is lost to the coupling of the two flywheels.

Significance

Since the rotational inertia of the system increased, the angular velocity decreased, as expected from the law of conservation of angular momentum. In this example, we see that the final kinetic energy of the system has decreased, as energy is lost to the coupling of the flywheels. Compare this to the example of the skater in Figure 6.3.1 doing work to bring her arms inward and adding rotational kinetic energy.

? Exercise 6.3.8

A merry-go-round at a playground is rotating at 4.0 rev/min. Three children jump on and increase the moment of inertia of the merry-go-round/children rotating system by 25%. What is the new rotation rate?

✓ Example 6.3.9: Dismount from a High Bar

An 80.0-kg gymnast dismounts from a high bar. He starts the dismount at full extension, then tucks to complete a number of revolutions before landing. His moment of inertia when fully extended can be approximated as a rod of length 1.8 m and when in the tuck a rod of half that length. If his rotation rate at full extension is 1.0 rev/s and he enters the tuck when his center of mass is at 3.0 m height moving horizontally to the floor, how many revolutions can he execute if he comes out of the tuck at 1.8 m height? See Figure 6.3.4.

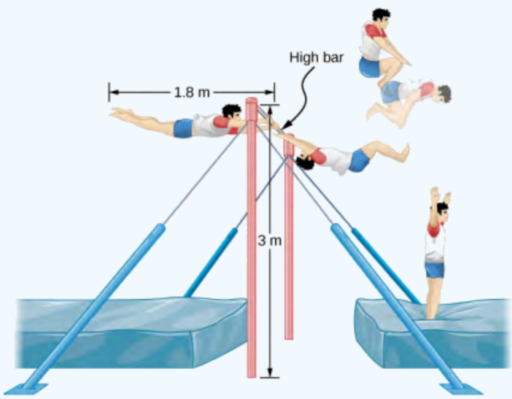


Figure 6.3.4: A gymnast dismounts from a high bar and executes a number of revolutions in the tucked position before landing upright.

Strategy

Using conservation of angular momentum, we can find his rotation rate when in the tuck. Using the equations of kinematics, we can find the time interval from a height of 3.0 m to 1.8 m. Since he is moving horizontally with respect to the ground, the equations of free fall simplify. This will allow the number of revolutions that can be executed to be calculated. Since we are using a ratio, we can keep the units as rev/s and don't need to convert to radians/s.

Solution

The moment of inertia at full extension is

$$I_0 = \frac{1}{12}mL^2 = \frac{1}{12}(80.0\ kg)(1.8\ m)^2 = 21.6\ kg \cdot m^2.$$

The moment of inertia in the tuck is

$$I_f = \frac{1}{12}mL_f^2 = \frac{1}{12}(80.0\ kg)(0.9\ m)^2 = 5.4\ kg \cdot m^2.$$

Conservation of angular momentum:

$$I_f\omega_f = I_0\omega_0 \Rightarrow \omega_f = \frac{I_0\omega_0}{I_f} = \frac{(21.6\ kg \cdot m^2)(1.0\ rev/s)}{5.4\ kg \cdot m^2} = 4.0\ rev/s.$$

Time interval in the tuck:

$$t=\sqrt{\frac{2h}{g}}=\sqrt{\frac{2(3.0-1.8)m}{9.8\ m/s}}=0.5\ s.$$

In 0.5 s, he will be able to execute two revolutions at 4.0 rev/s.

Significance

Note that the number of revolutions he can complete will depend on how long he is in the air. In the problem, he is exiting the high bar horizontally to the ground. He could also exit at an angle with respect to the ground, giving him more or less time in the air depending on the angle, positive or negative, with respect to the ground. Gymnasts must take this into account when they are executing their dismounts.

✓ Example 6.3.10: Conservation of Angular Momentum of a Collision

A bullet of mass m = 2.0 g is moving horizontally with a speed of 500.0 m/s. The bullet strikes and becomes embedded in the edge of a solid disk of mass M = 3.2 kg and radius R = 0.5 m. The cylinder is free to rotate around its axis and is initially at rest (Figure 6.3.5). What is the angular velocity of the disk immediately after the bullet is embedded?

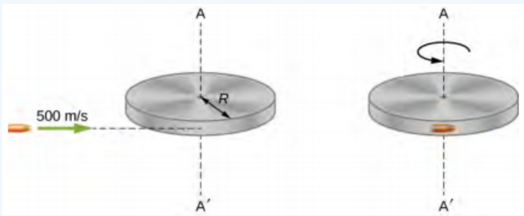


Figure 6.3.5: A bullet is fired horizontally and becomes embedded in the edge of a disk that is free to rotate about its vertical axis.

Strategy

For the system of the bullet and the cylinder, no external torque acts along the vertical axis through the center of the disk. Thus, the angular momentum along this axis is conserved. The initial angular momentum of the bullet is mvR, which is taken about the rotational axis of the disk the moment before the collision. The initial angular momentum of the cylinder is zero. Thus, the net angular momentum of the system is mvR. Since angular momentum is conserved, the initial angular momentum of the system is equal to the angular momentum of the bullet embedded in the disk immediately after impact.

Solution

The initial angular momentum of the system is

$$L_i=mvR.$$

The moment of inertia of the system with the bullet embedded in the disk is

$$I=mR^2+\frac{1}{2}MR^2=\left(m+\frac{M}{2}\right)R^2.$$

The final angular momentum of the system is

$$L_f=I\omega_f.$$

Thus, by conservation of angular momentum, $L_i = L_f$ and

$$mvR=\left(m+\frac{M}{2}\right)R^2\omega_f.$$

Solving for ω_f,

$$\omega_f=\frac{mvR}{\left(m+\frac{M}{2}\right)R^2}=\frac{(2.0\times10^{-3}\ kg)(500.0\ m/s)}{(2.0\times10^{-3}\ kg+1.6\ kg)(0.50\ m)}=1.2\ rad/s.$$

Significance

The system is composed of both a point particle and a rigid body. Care must be taken when formulating the angular momentum before and after the collision. Just before impact the angular momentum of the bullet is taken about the rotational axis of the disk.

CHAPTER OVERVIEW

7: C7) Conservation of Angular Momentum II

In the previous chapter, we dealt with objects having real size for the first time, and we learned how to calculate the angular momentum for extended objects as $\vec{L}=I\vec{\omega}$. In this chapter, we have to acknowledge that if extended objects can have angular momenta, *singular points can also (!), if we use a point other then their location as the reference.* This sounds very counter-intuitive, so let's first make sure we understand why that must be - essentially, objects are made up of points, and we believe angular momentum should be additive (two objects with angular momentum L_1 and L_2 have total angular momentum L_1+L_2 ...sounds like something we want, right?). If that's true, then when you think about an extended object as a collection of points, each of those points should have individual angular momenta L_i so that we can add them all to get the total $L=\sum_i L_i$.

Of course, those individual points themselves are not spinning, they are moving around the axis of the object (see the leftmost figure below). But how do we determine the angular momenta of these individual points? The answer is to use a cross-product:

$$\vec{L}=\vec{r}\times\vec{p}. \tag{7.1}$$

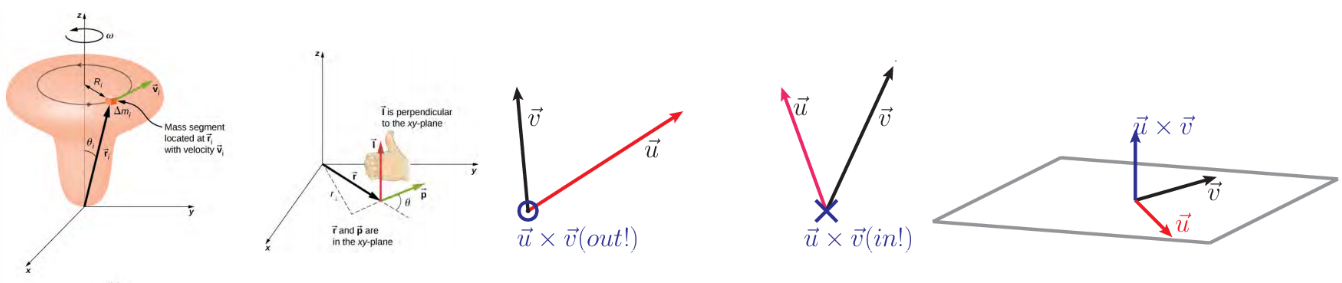
In this formula, the $\vec{r}$ is the position of the point and $\vec{p}$ is the momentum. We will go into this equation in some detail in the rest of the chapter, but the important thing to notice now is that it is a vector, so it has magnitude and direction. The magnitude can be found with the formula

$$|L|=|r||p|\sin\phi, \tag{7.2}$$

where ϕ is the angle between $\vec{r}$ and $\vec{p}$. The direction of any cross product $\vec{u}\times\vec{v}$ can be found with the **right hand rule**. The right hand rule (see second figure to the right) says:

1. Point (the fingers on your right hand!) in the direction of the first vector in $\vec{u}\times\vec{v}$.
2. Curl your fingers into the second vector in $\vec{u}\times\vec{v}$ (you may need to flip you hand around...)
3. Your thumb is now pointing in the direction of $\vec{u}\times\vec{v}$.

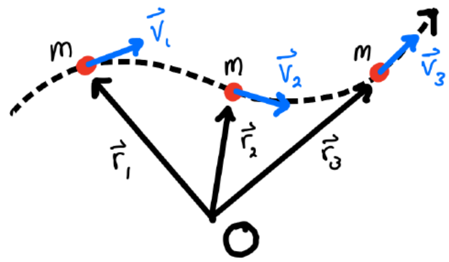
"This" right hand rule is actually related to the "first" right hand rule we learned in the last chapter for the direction of $\vec{L}=I\vec{\omega}$, but the one presented here works for any cross product, whereas the other one is just for the angular momentum of an object. The figure below (three images on the right) illustrates a few examples of these directions - check that you can get the right answers!



(images from Open Stax!)

7.1: The Angular Momentum of a Point and The Cross Product

In the previous section we studied the rotation of extended, rigid objects, arguing that such objects rotate by transfering angular momentum, in a similar way that point-like objects transfer linear momentum. In this section, we start by acknowledging that it's not *only* extended rigid objects that have angular momentum! Since such objects are made up of individual points, it must be that we can assign angular momentum to the individual points of extended objects. Put another way, it must be possible to determine the angular momentum of a point moving along any path, as shown in the following figure.



So there must be some way to encode the angular momentum $\vec{L}$ of this particle as it moves - and in fact, however we define it, if this system is isolated it also had better not change as it moves along this path! It turns out the correct way to do this the following:

$$\vec{L} = \vec{r} \times \vec{p}. \tag{7.1.1}$$

In this expression $\vec{r}$ is the position of the point, and $\vec{p}$ is it's momentum. The product here is called the cross product, which we first describe in pure mathematical terms before attempting to understand it physically.

The cross, or vector, product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \times \vec{B}$. It is defined as a vector perpendicular to both $\vec{A}$ and $\vec{B}$ (that is to say, to the plane that contains them both), with a magnitude given by

$$|\vec{A} \times \vec{B}| = AB \sin\theta \tag{7.1.2}$$

where A and B are the magnitudes of $\vec{A}$ and $\vec{B}$, respectively, and θ is the angle between $\vec{A}$ and $\vec{B}$, when they are drawn either with the same origin or tip-to-tail.

Since the result of $\vec{A} \times \vec{B}$ is a vector, we can also write this product in components. You might recognize this formula from your calculus courses, where you can define it as a determinant (for more information on where this formula comes from, check out Wikipedia). We will just copy the result down here:

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} - (A_x B_z - A_z B_y)\hat{y} + (A_x B_y - A_y B_x)\hat{z}. \tag{7.1.3}$$

Note that the magnitude of the formula we just wrote had better be equivalent to the version above, $|A||B|\sin(\theta)$. It's not too hard to show that, you just need to relate the components to the angle θ by computing (7.1.3) in a particular coordinate system.

The specific direction of $\vec{A} \times \vec{B}$ depends on the relative orientation of the two vectors. Basically, if $\vec{B}$ is counterclockwise from $\vec{A}$, when looking down on the plane in which they lie, assuming they are drawn with a common origin, then $\vec{A} \times \vec{B}$ points *upwards* from that plane; otherwise, it points downward (into the plane). One can also use the so-called *right-hand rule*, illustrated in Figure 7.1.1 to figure out the direction of $\vec{A} \times \vec{B}$. Note that, by this definition, the direction of $\vec{A} \times \vec{B}$ is the *opposite* of the direction of $\vec{B} \times \vec{A}$ (as also illustrated in Figure 7.1.1). Hence, *the cross-product is non-commutative*: the order of the factors makes a difference.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \tag{7.1.4}$$

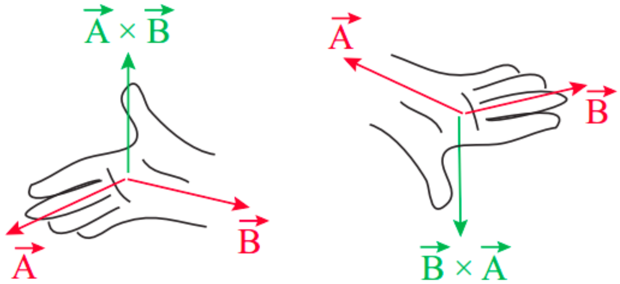


Figure 7.1.1: The “right-hand rule” to determine the direction of the cross product. Line up the first vector with the fingers, and the second vector with the flat of the hand, and the thumb will point in the correct direction. In the first drawing, we are looking at the plane formed by $\vec{A}$ and $\vec{B}$ from above; in the second drawing, we are looking at the plane from below, and calculating $\vec{B} \times \vec{A}$.

It follows from Equation (7.1.4) that the cross-product of any vector with itself must be zero. In fact, according to Equation (7.1.2), the cross product of any two vectors that are parallel to each other is zero, since in that case $\theta = 0$, and $\sin 0 = 0$.

Besides not being commutative, the cross product also does not have the associative property of ordinary multiplication: $\vec{A} \times (\vec{B} \times \vec{C})$ is different from $(\vec{A} \times \vec{B}) \times \vec{C}$. You can see this easily from the fact that, if $\vec{A} = \vec{B}$, the second expression will be zero, but the first one generally will be nonzero (since $\vec{A} \times \vec{C}$ is not parallel, but rather perpendicular to $\vec{A}$).

Of course, now that we have another definition of the angular momentum, we had better check that it matches the previous one, $\vec{L} = I\vec{\omega}$. Since that one only applies to rigid objects, let's calculate it for a single particle, and demonstrate that we get the same answer using the equation presented at the beginning of this section, (7.1.1). Consider a particle moving in the $x-y$ plane, shown in the figure below. Finding the angular momentum of this particle as a rigid object is easy,

$$L = I\omega = (mr^2)\omega. \tag{7.1.5}$$

(Recall that the moment of inertia of a single point is the same as for a hoop). Now, let's use our new formula. In the magnitude form (7.1.2), this is

$$L = |r||p|sin\theta = r(mv)sin(90^{\circ}). \tag{7.1.6}$$

Here the 90° comes from the fact that the vector $\vec{r}$ and the vector $\vec{p}$ are perpendicular to each other. To make these expressions match, first notice that $sin(90^{\circ}) = 1$. Then, recall the formula $v = \omega r$ (Equation 6.1.11). If we put both of these into our last expression we find:

$$L = r(mv)sin(90^{\circ}) = rm(\omega r) = mr^2\omega, \tag{7.1.7}$$

exactly as we expected. For our purposes, this provides enough evidence that we can use (7.1.1) as the correct expression for the angular momentum of single particles.

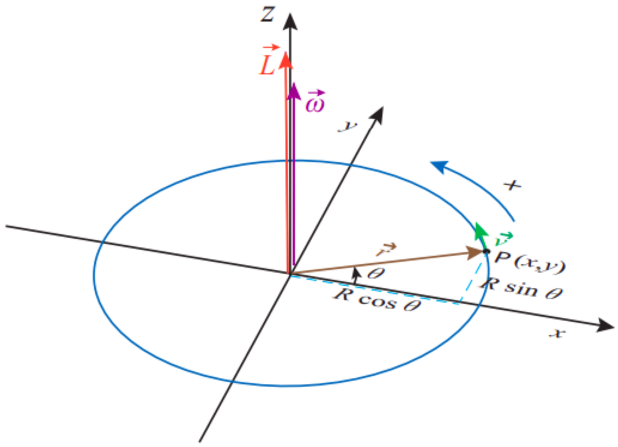


Figure $7.1.2$: A particle moving on a circle in the x-y plane. For the direction of rotation shown, the vectors $\vec{L} = m\vec{r} \times \vec{v}$ and $\vec{\omega}$ lie along the z axis, in the positive direction.

There are some other neat things we can do with $\vec{\omega}$ as defined above. Consider the cross product $\vec{\omega} \times \vec{r}$. Inspection of Figure $7.1.2$ and of Equation (8.4.12) shows that this is nothing other than the ordinary velocity vector, $\vec{v}$:

$$\vec{v} = \vec{\omega} \times \vec{r}. \tag{7.1.8}$$

We can also take the derivative of $\vec{\omega}$ to obtain the **angular acceleration** vector $\vec{\alpha}$:

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\vec{\omega}(t+\Delta t) - \vec{\omega}(t)}{dt} = \frac{d\vec{\omega}}{dt}. \tag{7.1.9}$$

Notice this "new" equation is just another example of the correspondence between angular and linear quantities - the angular acceleration is the derivative of the angular position with respect to time. For the motion depicted in Figure $7.1.2$, the vector $\vec{\alpha}$ will point along the positive z axis if the vector $\vec{\omega}$ is growing (which means the particle is speeding up), and along the negative z axis if $\vec{\omega}$ is decreasing.

One important property the cross product does have is the *distributive property* with respect to the sum:

$$(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}. \tag{7.1.10}$$

This, it turns out, is all that's necessary in order to be able to apply the product rule of differentiation to calculate the derivative of a cross product; you just have to be careful not to change the order of the factors in doing so. We can then take the derivative of both sides of Equation (7.1.8) to get an expression for the acceleration vector:

$$\begin{aligned} \vec{a} = \frac{d\vec{v}}{dt} &= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} \\ &= \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}. \end{aligned} \tag{7.1.11}$$

The first term on the right-hand side, $\vec{\alpha} \times \vec{r}$, lies in the x-y plane, and is perpendicular to $\vec{r}$; it is, therefore, *tangential* to the circle. In fact, looking at its magnitude, it is clear that this is just the *tangential acceleration vector*.

As for the second term in (7.1.11), $\vec{\omega} \times \vec{v}$, noting that $\vec{\omega}$ and $\vec{v}$ are always perpendicular, it is clear its magnitude is $|\omega||\vec{v}| = R\omega^2 = v^2/R$ (making use of Equation (6.1.11) again). This called the **centripetal acceleration,** which we will explore in more detail when we study rotational motion. Also, using the right-hand rule in Figure $7.1.2$, you can see that $\vec{\omega} \times \vec{v}$ always points inwards, towards the center of the circle; that is, along the direction of $-\vec{r}$. Putting all of this together, we can write this vector as just $-\omega^2\vec{r}$, and the whole acceleration vector as the sum of a tangential and a centripetal (radial) component, as follows:

$$\begin{aligned} \vec{a} &= \vec{a}_t + \vec{a}_c \\ \vec{a}_t &= \vec{\alpha} \times \vec{r} \\ \vec{a}_c &= -\omega^2 \vec{r}. \end{aligned} \tag{7.1.12}$$

To conclude this section, let me return to the angular momentum vector, and ask the question of whether, in general, the angular momentum of a rotating system, defined as the sum of the angular momentum over all the particles that make up the system, will or not satisfy the vector equation $\vec{L} = I\vec{\omega}$. We have seen that this indeed works for a particle moving in a circle. It will, therefore, also work for any object that is essentially flat, and rotating about an axis perpendicular to it, since in that case all its parts are just moving in circles around a common center.

However, if the system is a three-dimensional object rotating about an arbitrary axis, the result $\vec{L} = I\vec{\omega}$ does not generally hold. The reason is, mathematically, that the moment of inertia I is defined (Equation (6.1.2)) in terms of the distances of the particles to an *axis*, whereas the angular momentum involves the particle's distance to a *point*. For particles at different "heights" along the axis of rotation, these quantities are different. It can be shown that, in the general case, all we can say is that $L_z = I\omega_z$, if we call z the axis of rotation and calculate $\vec{L}$ relative to a point on that axis.

On the other hand, if the axis of rotation is an axis of symmetry of the object, then $\vec{L}$ has *only* a z component, and the result $\vec{L} = I\vec{\omega}$ holds as a vector equation. Most of the systems we will consider this semester will be covered under this clause, or under the "essentially flat" clause mentioned above.

In what follows we will generally assume that I has only a z component, and we will drop the subscript z in the equation $L_z = I\omega_z$, so that L and ω will not necessarily be the magnitudes of their respective vectors, but numbers that could be positive or negative, depending on the direction of rotation (clockwise or counterclockwise). This is essentially the same convention we used for vectors in one dimension, such as $\vec{a}$ or $\vec{p}$, in the early chapters; it is fine for all the cases in which the (direction of the) axis of rotation does not change with time, which are the only situations we will consider this semester.

7.2: Torque

We are finally in a position to answer the question, when is angular momentum conserved? To do this, we will simply take the derivative of $\vec{L}$ with respect to time, and use Newton's laws to find out under what circumstances it is equal to zero.

Let us start with a particle and calculate

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(m\vec{r}\times\vec{v}) = m\frac{d\vec{r}}{dt}\times\vec{v} + m\vec{r}\times\frac{d\vec{v}}{dt}. \tag{7.2.1}$$

The first term on the right-hand side goes as $\vec{v}\times\vec{v}$, which is zero. The second term can be rewritten as $m\vec{r}\times\vec{a}$. But, according to Newton's second law, $m\vec{a} = \vec{F}_{net}$. So, we conclude that

$$\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}_{net}. \tag{7.2.2}$$

So the angular momentum, like the ordinary momentum, will be conserved if the net force on the particle is zero, but also, and this is an important difference, when the net force is parallel (or antiparallel) to the position vector.

The quantity $\vec{r}\times\vec{F}$ is called the *torque* of a force around a point (the origin from which $\vec{r}$ is calculated, typically a pivot point or center of rotation) - we've seen this quantity already in Section 6.2. It is denoted with the Greek letter τ, "tau":

$$\vec{\tau} = \vec{r}\times\vec{F}. \tag{7.2.3}$$

For an extended object or system, the rate of change of the angular momentum vector would be given by the sum of the torques of all the forces acting on all the particles. For each torque one needs to use the position vector of the particle on which the force is acting.

The torque of a force around a point is basically a measure of how effective the force would be at causing a rotation around that point. Since $|\vec{r}\times\vec{F}| = rF\sin\theta$, you can see that it depends on three things: the magnitude of the force, the distance from the center of rotation to the point where the force is applied, and the angle at which the force is applied. All of this can be understood pretty well from Figure $7.2.1$ below, especially if you have ever had to use a wrench to tighten or loosen a bolt:

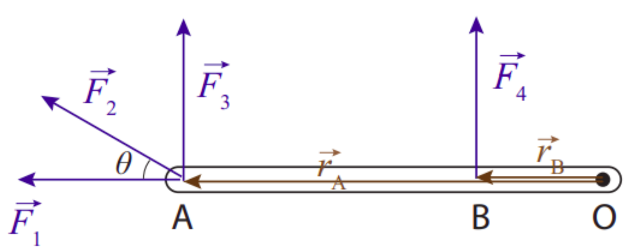


Figure $7.2.1$: The torque around the point O of each of the forces shown is a measure of how effective it is at causing the rod to turn around that point.

Clearly, the force $\vec{F}_1$ will not cause a rotation at all, and accordingly its torque is zero (since it is parallel to $\vec{r}_A$). Of all the forces shown, the most effective one is $\vec{F}_3$: it is applied the farthest away from O, for the greatest leverage (again, think of your experiences with wrenches). It is also perpendicular to the rod, for maximum effect ($\sin\theta$ = 1). The force $\vec{F}_2$, by contrast, although also applied at the point A is at a disadvantage because of the relatively small angle it makes with $\vec{r}_A$. If you imagine breaking it up into components, parallel and perpendicular to the rod, only the perpendicular component (whose magnitude is $F_2\sin\theta$) would be effective at causing a rotation; the other component, the one parallel to the rod, would be wasted, like $\vec{F}_1$.

In order to calculate torques, then, we basically need to find, for every force, the component that is perpendicular to the position vector of its point of application. Clearly, for this purpose we can no longer represent an extended body as a mere dot, as we have done previously. What we need is a more careful sketch of the object, just detailed enough that we can tell how far from the center of rotation and at what angle each force is applied. That kind of diagram is called an *extended free-body diagram*.

Figure $7.2.1$ could be an example of an extended free-body diagram, for an object being acted on by four forces. Typically, though, instead of drawing the vectors $\vec{r}_A$ and $\vec{r}_B$ we would just indicate their lengths on the diagram (or maybe even leave them out altogether, if we do not want to overload the diagram with detail). We will show a couple of examples of extended free-body diagrams in the next section.

Coming back to Equation (7.2.2), the main message of this section (other, of course, than the definition of torque itself), is that the rate of change of an object or system's angular momentum is equal to the net torque due to the external forces. Two special results follow from this one. First, if the net external torque is zero, angular momentum will be conserved. For example, consider the collision of a particle and a rod pivoting around one end. The only external force is the force exerted on the rod, at the pivot point, by the pivot itself, but the torque of that force around that point is obviously zero, since $\vec{r}=0$, so our assumption that the total angular momentum around that point was conserved is legitimate.

Finally, note that situations where the moment of inertia of a system, I, changes with time are relatively easy to arrange for any deformable system. Especially interesting is the case when the external torque is zero, so L is constant, and a change in I therefore brings about a change in $\omega = L/I$: this is how, for instance, an ice-skater can make herself spin faster by bringing her arms closer to the axis of rotation (reducing her I), and, conversely, slow down her spin by stretching out her arms. This can be done even in the absence of a contact point with the ground: high-board divers, for instance, also spin up in this way when they curl their bodies into a ball. Note that, throughout the dive, the diver's angular momentum around its center of mass is constant, since the only force acting on him (gravity, neglecting air resistance) has zero torque about that point.

Resources

Unfortunately, we will not really have enough time this semester to explore further the many interesting effects that follow from the vector nature of Equation (7.2.3), but you are at least subconsciously familiar with some of them if you have ever learned to ride a bicycle! A few interesting Internet references (some of which could perhaps inspire a good Honors project!) are the following:

- Walter Lewin's lecture on gyroscopic motion (and rolling motion):
  https://www.youtube.com/watch?v=N92FYHHT1qM
- A "Veritasium" video on "antigravity":
  https://www.youtube.com/watch?v=GeyDf4ooPdo
  https://www.youtube.com/watch?v=tLMpdBjA2SU
- And the old trick of putting a gyroscope (flywheel) in a suitcase:
  https://www.youtube.com/watch?v=zdN6zhZSJKw
  If any of the above links are dead, try googling them. (You may want to let me know, too!)

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[3]The additional assumption is that the force between any two particles lies along the line connecting the two particles (which means it is parallel or antiparallel to the vector $\vec{r}_1 - \vec{r}_2$). In that case, $\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = 0$. Most forces in nature satisfy this condition.

[4]Actually, friction forces and normal forces may be "spread out" over a whole surface, but, if the object has enough symmetry, it is usually OK to have them "act" at the midpoint of that surface. This can be proved along the lines of the derivation for gravity that follows.

7.3: Examples

Whiteboard Problem 7.3.1: Nutmeg

In soccer, a ``nutmeg" is when you kick the ball through the legs of an opponent, thouroughly humiliating them. In the figure, Ronaldo nutmegs Puyol by kicking the ball (with a mass of 435 g and radius of 11.5 cm) at a velocity of 1.75 m/s.

1. If the ball is rolling along the ground without slipping, what is the angular momentum of the ball about its center of mass?
2. When the ball passes through Puyol's legs, what is the angular momentum of the center of mass of the ball relative to his head? Puyol is 5'10" (1.78 m) tall.
3. Puyol spins around after the nutmeg and sees the ball a distance 1.15 m along the ground behind him, still rolling with the same velocity. What is the angular momentum of the center of mass of the ball now?

Example 7.3.2: Calculating Torque

Four forces are shown in Figure 7.3.4 at particular locations and orientations with respect to a given xy-coordinate system. Find the torque due to each force about the origin, then use your results to find the net torque about the origin.

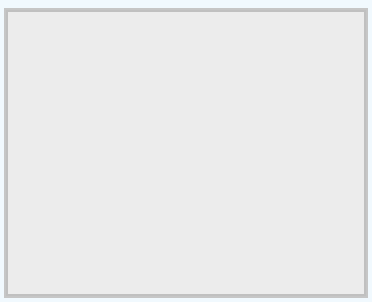

Figure 7.3.4: Four forces producing torques.

Strategy

This problem requires calculating torque. All known quantities—forces with directions and lever arms—are given in the figure. The goal is to find each individual torque and the net torque by summing the individual torques. Be careful to assign the correct sign to each torque by using the cross product of $\vec{r}$ and the force vector $\vec{F}$.

Solution

Use $|\vec{r}| = r_{\perp} F = rF\sin\theta$ to find the magnitude and $\vec{r} = \vec{r} \times \vec{F}$ to determine the sign of the torque.

The torque from force 40 N in the first quadrant is given by (4)(40)sin 90° = 160 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page, positive.

The torque from force 20 N in the third quadrant is given by −(3)(20)sin 90° = − 60 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is into the page, so it is negative.

The torque from force 30 N in the third quadrant is given by (5)(30)sin 53° = 120 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page, positive.

The torque from force 20 N in the second quadrant is given by (1)(20)sin 30° = 10 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page.

The net torque is therefore $\tau_{net} = \sum_i |\tau_i|$ = 160 − 60 + 120 + 10 = 230 N • m.

Significance

Note that each force that acts in the counterclockwise direction has a positive torque, whereas each force that acts in the clockwise direction has a negative torque. The torque is greater when the distance, force, or perpendicular components are greater.

Example 7.3.3: Calculating Torque on a rigid body

Figure 7.3.5 shows several forces acting at different locations and angles on a flywheel. We have $|\vec{F}_1|$ = 20 N, $|\vec{F}_2|$ = 30 N, $|\vec{F}_3|$ = 30 N, and r = 0.5 m. Find the net torque on the flywheel about an axis through the center.

Figure 7.3.5: Three forces acting on a flywheel.

Strategy

We calculate each torque individually, using the cross product, and determine the sign of the torque. Then we sum the torques to find the net torque. Solution We start with $\vec{F}_1$. If we look at Figure 7.3.5, we see that $\vec{F}_1$ makes an angle of 90° + 60° with the radius vector $\vec{r}$. Taking the cross product, we see that it is out of the page and so is positive. We also see this from calculating its magnitude:

$$|\vec{\tau}_1| = rF_1 \sin 150^o = (0.5\ m)(20\ N)(0.5) = 5.0\ N \cdot m. \tag{7.3.1}$$

Next we look at $\vec{F}_2$. The angle between $\vec{F}_2$ and $\vec{r}$ is 90° and the cross product is into the page so the torque is negative. Its value is

$$|\vec{\tau}_2| = -rF_2 \sin 90^o = (-0.5\ m)(30\ N) = -15.0\ N \cdot m. \tag{7.3.2}$$

When we evaluate the torque due to $\vec{F}_3$, we see that the angle it makes with $\vec{r}$ is zero so $\vec{r} \times \vec{F}_3$ = 0. Therefore, $\vec{F}_3$ does not produce any torque on the flywheel.

We evaluate the sum of the torques:

$$\tau_{net} = \sum_i |\tau_i| = 5 - 15 = -10\ N \cdot m. \tag{7.3.3}$$

Significance

The axis of rotation is at the center of mass of the flywheel. Since the flywheel is on a fixed axis, it is not free to translate. If it were on a frictionless surface and not fixed in place, $\vec{F}_3$ would cause the flywheel to translate, as well as $\vec{F}_1$. Its motion would be a combination of translation and rotation.

? Exercise 7.3.4: Ship Run Aground

A large ocean-going ship runs aground near the coastline, similar to the fate of the **Costa Concordia**, and lies at an angle as shown below. Salvage crews must apply a torque to right the ship in order to float the vessel for transport. A force of 5.0 x 10^5 N acting at point A must be applied to right the ship. What is the torque about the point of contact of the ship with the ground (Figure 7.3.6)?

Figure 7.3.6: A ship runs aground and tilts, requiring torque to be applied to return the vessel to an upright position.

CHAPTER OVERVIEW

8: C8) Conservation of Energy- Kinetic and Gravitational

Starting in this chapter we are going to move away from our first basic principle of physics (conservation of momentum), and onto **conservation of energy**. As another conservation law, the essentials are the same - there is a quantity in the system that is not changing, and by keeping track of that quantity, we can learn things about unknown aspects of the system. The things we are looking for will be similar (final speeds, for example), but since the quantity we are keeping track of is different (energy instead of momentum), the details will be different.

One major difference is that unlike momentum, *energy takes many different forms*. Momentum was only ever mass times velocity; even if you have a collection of points that could be more easily described by angular momentum, $\vec{L} = I\vec{\omega}$, that was just a convenience of adding up all the individual point's $m\vec{v}$. But energy can take several different forms, and most notably can be stored without motion - something momentum cannot do[1]. Here we will quickly outline two different forms that energy can take.

The first is kinetic energy, or energy of motion. This is the energy stored in a moving object, and all moving objects have it. The nice thing about kinetic energy is that there is a simple formula to describe it (kind of like for momentum!):

$$K = \frac{1}{2}mv^2. \tag{8.1}$$

Here m is the mass of the object, and v is the speed. Although this shares some of the same intuition as momentum ("more mass and speed, more energy"), it's important to recognize one difference - unlike momentum, *kinetic energy is a scalar and does not have direction*. So it doesn't matter what direction $\vec{v}$ is pointed in, what matters is the square of the magnitude, $|\vec{v}|^2$. Also similar to momentum, there is a rotational version of this as well that we will use when we encounter extended objects.

The second form of energy we want to immediately introduce is potential energy, or *energy of interaction*. This new concept describes how objects interact, and allows us to study interactions at a more fundamental level than momentum. Objects interact by storing and moving energy around to different parts of the system. For example, when you stretch a spring, you are transferring energy from your arms into the spring, which stays there until you let the spring return to it's original length. Every interaction has a different way of storing energy, although sometimes we may not know how to easily describe it (friction is an example of this).

Our first example of potential energy will be gravitational potential energy near the surface of the Earth. You already know about this - when you drop something it falls! The interaction of gravity allowed whoever lifted the object to store energy in the system, and you can release it by dropping it. In this case, the energy turns into the kind we discussed before, kinetic energy. The formula for gravitational potential energy near the Earth is

$$U_g = mgh, \tag{8.2}$$

where m is the mass of the object, $g = 9.81\ \mathrm{m/s}^2$ is the acceleration due to gravity, and h is the height above the reference point (often taken to be the ground).

We are going to spend a lot of time understanding these two kinds of energy, but let's start by considering the simple example of dropping an object we just mentioned. Assuming the object is not moving before you drop it, the initial energy is completely made up of U_g. As the object falls, it converts the energy from U_g into K, increasing the speed v while the height h decreases. That allows us to construct the formula,

$$mgh = \frac{1}{2}mv^2 \to v = \sqrt{2gh}, \tag{8.3}$$

which tells us the speed of the object when it falls from a height h. That's already a new result for us, but we should also point out that the only thing that changes how fast the object is moving is how high it's dropped from; the mass doesn't matter, and also we have no knowledge of the direction of motion (ok, yes we know in this case it's "downwards", but generally energy cannot tell us that!). This is typical for equations in energy problems - they are often pretty simple, but we have to be careful to know exactly what they can and can't tell us.

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[1]We have to be a little careful here - momentum can be stored in certain fields, like the electromagnetic and gravitational fields. However, understanding how that works takes a greater knowledge of the fundamental interactions then we can give here!

8.1: Kinetic Energy

For a long time in the development of classical mechanics, physicists were aware of the existence of two different quantities that one could define for an object of inertia m and velocity v. One was the momentum, mv, and the other was something proportional to mv^2. Despite their obvious similarities, these two quantities exhibited different properties and seemed to be capturing different aspects of motion.

When things got finally sorted out, in the second half of the 19th century, the quantity $\frac{1}{2}mv^2$ came to be recognized as a form of *energy*—itself perhaps the most important concept in all of physics. *Kinetic energy*, as this quantity is called, may be the most obvious and intuitively understandable kind of energy, and so it is a good place to start our study of the subject.

We will use the letter K to denote kinetic energy, and, since it is a form of energy, we will express it in the units especially named for this purpose, which is to say joules (J). 1 joule is 1 kg·m^2/s^2. In the definition

$$K=\frac{1}{2}mv^2 \tag{8.1.1}$$

the letter v is meant to represent the *magnitude* of the velocity vector, that is to say, the *speed* of the particle. Hence, unlike momentum, *kinetic energy is not a vector, but a scalar* : there is no sense of direction associated with it. In three dimensions, one could write

$$K=\frac{1}{2}m\left(v_x^2+v_y^2+v_z^2\right) \tag{8.1.2}$$

There is, therefore, some amount of kinetic energy associated with each component of the velocity vector, but in the end they are all added together in a lump sum.

For a system of particles, we will treat kinetic energy as an additive quantity, just like we did for momentum, so the total kinetic energy of a system will just be the sum of the kinetic energies of all the particles making up the system. Note that, unlike momentum, this is a scalar (not a vector) sum, and most importantly, that kinetic energy is, by definition, always positive, so there can be no question of a “cancellation” of one particle’s kinetic energy by another, again unlike what happened with momentum. Two objects of equal mass moving with equal speeds in opposite directions have a total momentum of zero, but their total kinetic energy is definitely nonzero. Basically, the kinetic energy of a system can never be zero as long as there is any kind of motion going on in the system.

Kinetic Energy in Collisions

To gain some further insights into the concept of kinetic energy, and the ways in which it is different from momentum, it is useful to look at it in the same setting in which we “discovered” momentum, namely, one-dimensional collisions in an isolated system. If we look again at the collision represented in Figure 2.1.1 of Chapter 2, reproduced below,

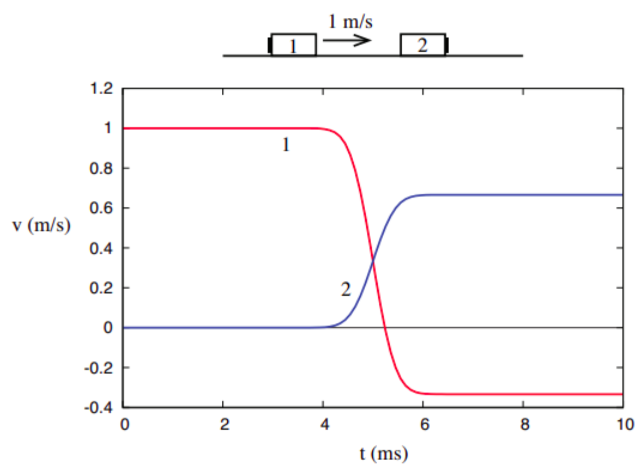

Figure 8.1.1

we can use the definition (8.1.1) to calculate the initial and final values of K for each object, and for the system as a whole. Remember we found that, for this particular system, $m_2=2m_1$, so we can just set m_1= 1 kg and m_2 = 2 kg, for simplicity. The initial and final velocities are v_{1i} = 1 m/s, v_{2i} = 0, v_{1f} = −1/3 m/s, v_{2f} = 2/3 m/s, and so the kinetic energies are

$$K_{1i} = \frac{1}{2}\text{ J}, K_{2i} = 0; \quad K_{1f} = \frac{1}{18}\text{ J}, K_{2f} = \frac{4}{9}\text{ J}.$$

Note that 1/18 + 4/9 = 9/18 = 1/2, and so

$$K_{sys,i} = K_{1i} + K_{2i} = \frac{1}{2}\text{ J} = K_{1f} + K_{2f} = K_{sys,f}.$$

In words, we find that, in this collision, the final value of the total kinetic energy is the same as its initial value, and so it looks like we have “discovered” *another* conserved quantity (besides momentum) for this system.

This belief may be reinforced if we look next at the collision depicted in Figure 2.1.2, again reproduced below. In this collision, the second object is now moving towards the first, which is stationary.

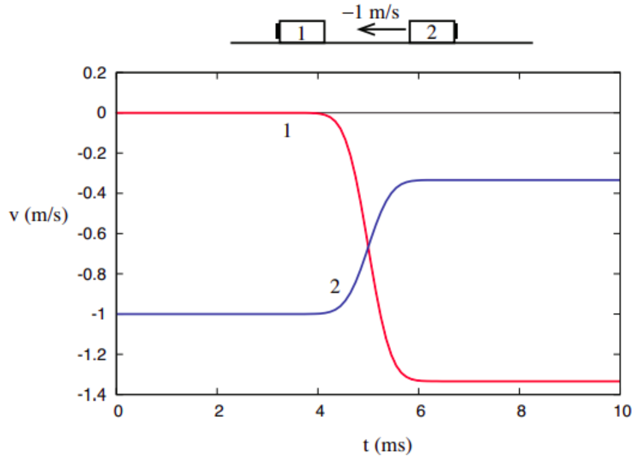

Figure 8.1.2.

The corresponding kinetic energies are, accordingly, K_{1i} = 0, K_{2i} = 1 J, $K_{1f} = \frac{8}{9}$ J, $K_{2f} = \frac{1}{9}$ J. These are all different from the values we had in the previous example, but note that once again the total kinetic energy after the collision equals the total kinetic energy before—namely, 1 J in this case.

Things are, however, very different when we consider the third collision example shown in Chapter 2, namely, the one where the two objects are stuck together after the collision.

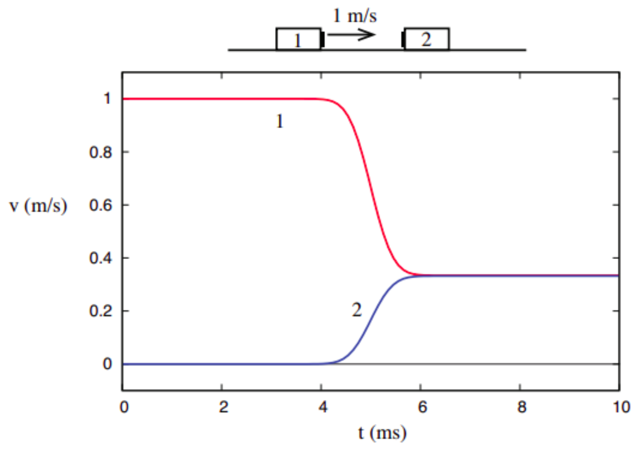

Figure 8.1.3

Their joint final velocity, consistent with conservation of momentum, is $v_{1f} = v_{2f}$ = 1/3 m/s. Since the system starts as in Figure 8.1.1, its kinetic energy is initially $K_{sys,i} = \frac{1}{2}$ J, but after the collision we have only

$$K_{sys,f} = \frac{1}{2}(3\text{ kg})\left(\frac{1}{3}\frac{\text{m}}{\text{s}}\right)^2 = \frac{1}{6}\text{ J}.$$

What this shows, however, is that unlike the total momentum of a system, which is completely unaffected by internal interactions, the total kinetic energy does depend on the details of the interaction, and thus conveys some information about its nature. We can then refine our study of collisions to distinguish two kinds: the ones where the initial kinetic energy is recovered after the collision, which we will call **elastic**, and the ones where it is not, which we call **inelastic**. A special case of inelastic collision is the one called *totally inelastic*, where the two objects end up stuck together, as in Figure 8.1.3. As we shall see later, the kinetic energy "deficit" is largest in that case.

Since whatever ultimately happens depends on the details and the nature of the interaction, we will be led to distinguish between "conservative" interactions, where kinetic energy is reversibly stored as some other form of energy somewhere, and "dissipative" interactions, where the energy conversion is, at least in part, irreversible. Clearly, elastic collisions are associated with conservative interactions and inelastic collisions are associated with dissipative interactions. This preliminary classification of interactions will have to be reviewed a little more carefully, however, in the next chapter.

8.2: Conservative Interactions

Let's summarize the physical concepts and principles we have encountered so far in our study of classical mechanics. We have "discovered" one important quantity, the inertia or inertial mass of an object, and introduced two different quantities based on that concept, the momentum $m\vec{v}$ and the kinetic energy $\frac{1}{2}mv^2$. We found that these quantities have different but equally intriguing properties. The total momentum of a system is insensitive to the interactions between the parts that make up the system, and therefore it stays constant in the absence of external influences (a more general statement of the law of inertia, the first important principle we encountered). The total kinetic energy, on the other hand, changes while any sort of interaction is taking place, but in some cases it may actually return to its original value afterwards. This chapter deals with interactions from an energy point of view, whereas later chapters will deal with them from a force point of view.

In the previous chapter I suggested that what was going on in an elastic collision could be interpreted, or described (perhaps in a figurative way) more or less as follows: as the objects come together, the total kinetic energy goes down, but it is as if it was being temporarily stored away somewhere, and as the objects separate, that "stored energy" is fully recovered as kinetic energy. Whether this does happen or not in any particular collision (that is, whether the collision is elastic or not) depends, as we have seen, on the kind of interaction ("bouncy" or "sticky," for instance) that takes place between the objects.

We are going to take the above description literally, and use the name *conservative interaction* for any interaction that can "store and restore" kinetic energy in this way. The "stored energy" itself—which is *not* actually kinetic energy while it remains stored, since it is not given by the value of $\frac{1}{2}mv^2$ at that time—we are going to call *potential energy*. Thus, conservative interactions will be those that have a "potential energy" associated with them, and vice-versa.

Potential Energy

Perhaps the simplest and clearest example of the storage and recovery of kinetic energy is what happens when you throw an object straight upwards, as it rises and eventually falls back down. The object leaves your hand with some kinetic energy; as it rises it slows down, so its kinetic energy goes down, down... all the way down to zero, eventually, as it momentarily stops at the top of its rise. Then it comes down, and its kinetic energy starts to increase again, until eventually, as it comes back to your hand, it has very nearly the same kinetic energy it started out with (exactly the same, actually, if you neglect air resistance).

The interaction responsible for this change in the object's kinetic energy is, of course, the gravitational interaction between it and the Earth, so we are going to say that the "missing" kinetic energy is temporarily stored as *gravitational potential energy* of the system formed by the Earth and the object. This potential energy takes a simple mathematical form,

$$U_g = mgy, \tag{8.2.1}$$

where y is the height of the object above the Earth. We have carefully called this height the coordinate y to emphasize that is can be negative if y is negative. It turns out that with this definition of potential energy, the total energy in freefall is

$$K + U^G = \text{constant}. \tag{8.2.2}$$

This is a statement of conservation of energy under the gravitational interaction. For any interaction that has a potential energy associated with it, the quantity $K+U$ is called the (total) *mechanical energy*.

Figure $8.2.1$ shows how the kinetic and potential energies of an object thrown straight up change with time (don't worry where this plot comes from yet, you will learn about that when we study projectile motion in detail). I have arbitrarily assumed that the object has a mass of 1 kg and an initial velocity of 2 m/s, and it is thrown from an initial height of 0.5 m above the ground. Note how the change in potential energy exactly mirrors the change in kinetic energy (so $\Delta U^G = -\Delta K$, mathematically), and the total mechanical energy remains equal to its initial value of 6.9 J throughout.

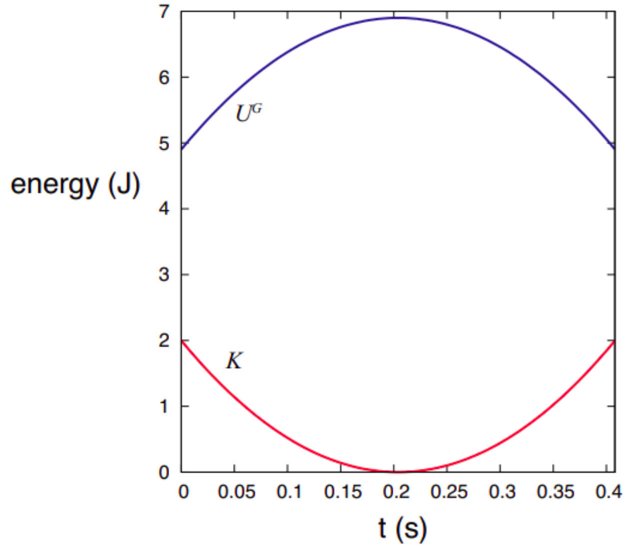

Figure 8.2.1: Potential and kinetic energy as a function of time for a system consisting of the earth and a 1-kg object sent upwards with v_i = 2 m/s from a height of 0.5 m.

There is something about potential energy that probably needs to be mentioned at this point. Because I have chosen to launch the object from 0.5 m above the ground, and I have chosen to measure y from the ground, I started out with a potential energy of mgy_i = 4.9 J. This makes sense, in a way: it tells you that if you simply dropped the object from this height, it would have picked up an amount of kinetic energy equal to 4.9 J by the time it reached the ground. But, actually, where I choose the vertical origin of coordinates is arbitrary. I could start measuring y from any height I wanted to—for instance, taking the initial height of my hand to correspond to y = 0. This would shift the blue curve in Figure 8.2.1 down by 4.9 J, but it would not change any of the physics. The only important thing I really want the potential energy for is to calculate the kinetic energy the object will lose or gain *as it moves from one height to another*, and for that only *changes* in potential energy matter. I can always add or subtract any (constant) number to or from U, and it will still be true that $\Delta K = -\Delta U$.

Closed Systems

Today, physics is pretty much founded on the belief that the energy of a closed system (defined as one that does not exchange energy with its surroundings—more on this in a minute) is always *conserved*: that is, internal processes and interactions will only cause energy to be "converted" from one form into another, but the total, after all the forms of energy available to the system have been carefully accounted for, will not change. This belief is based on countless experiments, on the one hand, and, on the other, on the fact that all the fundamental interactions that we are aware of do conserve a system's total energy.

Of course, recognizing whether a system is "closed" or not depends on having first a complete catalogue of all the ways in which energy can be stored and exchanged—to make sure that there is, in fact, no exchange of energy going on with the surroundings. Note, incidentally, that a "closed" system is not necessarily the same thing as an "isolated" system: the former relates to the total energy, the latter to the total momentum. A parked car getting hotter in the sun is not a closed system (it is absorbing energy all the time) but, as far as its total momentum is concerned, it is certainly fair to call it "isolated." Hopefully all these concepts will be further clarified when we introduce the additional auxiliary concepts of force, work, and heat.

For a closed system, we can state the principle of conservation of energy (somewhat symbolically) in the form

$$K+U+E_{\text{source}}+E_{\text{diss}}=\text{constant} \tag{8.2.3}$$

where K is the total, macroscopic, kinetic energy; U the sum of all the applicable potential energies associated with the system's *internal* interactions; E_{source} is any kind of internal energy (such as chemical energy) that is *not* described by a potential energy function, but can increase the system's mechanical energy; and E_{diss} stands for the contents of the "dissipated energy reservoir"—typically thermal energy. As with the potential energy U, the absolute value of E_{source} and E_{diss} does not (usually) really matter: all we are interested in is how much they change in the course of the process under consideration.

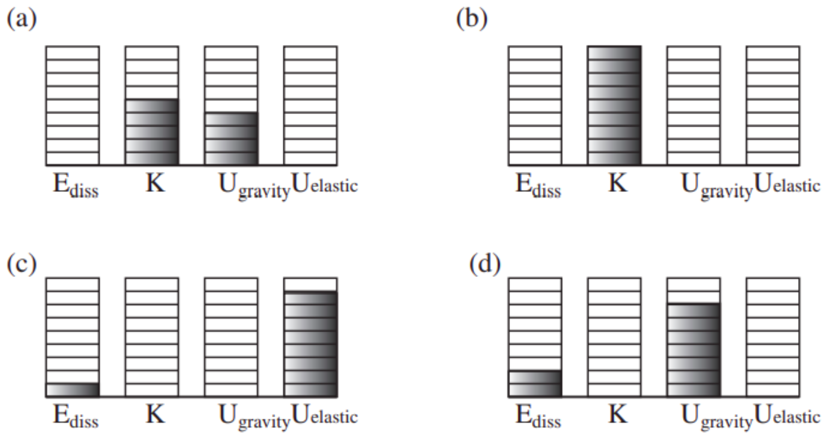

Figure 8.2.1: Energy bar diagrams for a system formed by the earth and a ball thrown downwards. (a) As the ball leaves the hand. (b) Just before it hits the ground. (c) During the collision, at the time of maximum compression. (d) At the top of the first bounce. The total number of energy “units” is the same in all the diagrams, as required by the principle of conservation of energy.

Figure 8.2.1 above is an example of this kind of “energy accounting” for a ball bouncing on the ground. If the ball is thrown down, the system formed by the ball and the earth initially has both gravitational potential energy, and kinetic energy (diagram (a)). So all the kinetic energy that we have is the kinetic energy of the ball. As the ball falls, gravitational potential energy is being converted into kinetic energy, and the ball speeds up. As it is about to hit the ground (diagram (b)), the potential energy is zero and the kinetic energy is maximum. During the collision with the ground, all the kinetic energy is temporarily converted into other forms of energy, which are essentially elastic energy of deformation (like the energy in a spring) and some thermal energy (diagram (c)). When it bounces back, its kinetic energy will only be a fraction e^2 of what it had before the collision (where e is the coefficient of restitution). This kinetic energy is all converted into gravitational potential energy as the ball reaches the top of its bounce (diagram (d)). Note there is more dissipated energy in diagram (d) than in (c); this is because I have assumed that dissipation of energy takes place both during the compression and the subsequent expansion of the ball.

Note that in this closed system, some energy was transferred to heat, but the total energy stayed the same, and was therefore conserved. The real problem appears if you can't actually *keep track of* that energy that gets transferred to heat, then the system would appear *not to* conserve energy. Mostly, we will assume no transfer to heat in our problems, but we should acknowledge this is not a problem with physics, it is rather a problem with our ability to keep track of all the sources of energy in a system. As you move through the field of physics, you will learn how we keep track of more of these sources of energy, and the problems you solve will become better and better approximations to "the real world".

8.3: Universal Gravity

Up to this point, all we have said about gravity is that, near the surface of the Earth, the gravitational force exerted by the Earth on an object of mass m is $F^G = mg$, which corresponds to a potential energy of interaction $U_g = mgh$. This is, indeed, a pretty good approximation, but it does not really tell you anything about what the gravitational force is where other objects or distances are involved.

The first comprehensive theory of gravity, formulated by Isaac Newton in the late 17th century, postulates that any two "particles" with masses m_1 and m_2 will exert an attractive force (a "pull") on each other, whose magnitude is proportional to the product of the masses, and inversely proportional to the square of the distance between them. Mathematically, we write

$$F_{12}^G = \frac{Gm_1m_2}{r_{12}^2}. \tag{8.3.1}$$

Here, r_{12} is just the magnitude of the position vector of particle 2 relative to particle 1 (so r_{12} is, indeed, the distance between the two particles), and G is a constant, known as "Newton's constant" or the *gravitational constant*, which at the time of Newton still had not been determined experimentally. You can see from Equation (8.3.1) that G is simply the magnitude, in newtons, of the attractive force between two 1-kg masses a distance of 1 m apart. This turns out to have the ridiculously small value G = 6.674×10^{-11} N m^2/kg^2 (or, as is more commonly written, $m^3kg^{-1}s^{-2}$). It was first measured by Henry Cavendish in 1798, in what was, without a doubt, an experimental tour de force for that time. As you can see, gravity as a force between any two ordinary objects is absolutely insignificant, and it takes the mass of a planet to make it into something you can feel.

There is a potential energy corresponding to this force, *a la* mgh:

$$U_G = -\frac{Gm_1m_2}{r_{12}} \tag{8.3.2}$$

The variables are the same as in the force equation (notice the *negative derivative of the potential gives you the force* - that's a universal rule, which we will explore later). These two equations together constitute what is often called Universal Gravity, because it's the law of gravity that applies to any two objects in the universe. This should be contrasted with the more familiar $F_g = mg$ and $U_g = mgh$, which only apply near the surface of the Earth.

Since U_G should apply everywhere, it should be true that we can derive U_g from U_G, so let's show how that is done. The situation we want to understand is the gravitational interaction near the Earth - in fact, very near the Earth, so that we can write the height of the object from the surface h is much smaller than the radius of the Earth, $h << R_E$. Setting the two masses in equation (8.3.2) to the Earth M_E and the mass of the object in question m, we can write

$$U_G = -\frac{GM_Em}{R_E + h} = -\frac{GM_Em}{R_E(1 + h/R_E)} \tag{8.3.3}$$

Notice in going to either side of the equation sign, I have simply pulled out R_E from the expression in the denominator, those two expressions are exactly equal. But now we have that quantity h/R_E, and we've already said $h << R_E$, which means h/R_E is a very small value. Recall from your calculus class that for a small value x, you can write

$$\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + \ldots \tag{8.3.4}$$

(This is called a Taylor Expansion, and you can check our work on Wolfram Alpha.) We have to be careful to write this expression as an approximation now, since it's not exact, but it should be a good approximation for things that actually are close to the surface of the Earth - perhaps as high as Everest, for example.

Anyway, let's truncate the series after two terms (keep more if you want it to be more exact!) and write

$$U_G \approx -\frac{GM_Em}{R_E}\left(1 - \frac{h}{R_E}\right) = -\frac{GM_Em}{R_E} + \frac{GM_Em}{R_E^2}h. \tag{8.3.5}$$

Notice in this last expression, the first term $-GM_Em/R_E$ is a particular constant which stays the same for a particular planet (Earth) and object of mass m - let's call that C. The second term is similar, but depends linearly on the height h above the surface of the Earth. In fact, if we define a new constant $g = GM_E/R_E^2$, we can write that second term as mgh, and we have found the relationship between the two different kinds of gravitational potential energy:

$$U_G \approx C + U_g. \tag{8.3.6}$$

That's quite satisfying, but what is going on with the constant? To understand that, we can go back to something we discussed last chapter - for closed systems that do not dissipate energy, we can write $\Delta K = -\Delta U$. Notice here that's it's just the *change in energy that matters, and not the total value*. For example, if I wanted to find the change in Universal Gravitational energy I could write

$$\Delta U_G = U_{G,f} - U_{G,i} \approx (C + U_{g,f}) - (C + U_{g,i}) = U_{g,f} - U_{g,i} = \Delta U_g. \tag{8.3.7}$$

In other words, if we consider the change in energy, the constant doesn't matter - in fact, we usually just set this constant to be 0 in all of our problems. The conceptual understanding of that is essentially that it does not matter where we set our $h = 0$ point to be - if we drop an object 10 m, we have to get the same answer for the final speed if we consider the $h = 0$ point to be either at the bottom of the motion or the top.

It's important to keep in mind this approximation technique - it's a very powerful tool that is used all over the place in the physics. Sometimes it's to make our work easier, while having essentially no impact on the outcome of calculations, like in this case. In other cases, the exact answer is so difficult to calculate that we are forced to make approximations to get an answer at all. One example of this which is just beyond what we will study is air drag, another example which is far beyond what we will study is the use of Feynman diagrams in quantum field theory.

8.4: Other Forms of Energy

Thermal Energy

From all the foregoing, it is clear that when an interaction can be completely described by a potential energy function we can define a quantity, which we have called the total mechanical energy of the system, $E_{mech} = K + U$, that is constant throughout the interaction. However, we already know from our study of collisions that this is rarely the case. Essential to the concept of potential energy is the idea of "storage and retrieval" of the kinetic energy of the system during the interaction process. When kinetic energy simply disappears from the system and does not come back, a full description of the process in terms of a potential energy is not possible.

Processes in which some amount of mechanical energy disappears (that is, it cannot be found anywhere anymore as either macroscopic kinetic or potential energy) are called *dissipative*. Mysterious as they may appear at first sight, there is actually a simple, intuitive explanation for them. All macroscopic systems consist of a great number of small parts that enjoy, at the microscopic level, some degree of independence from each other and from the body to which they belong. Macroscopic motion of an object requires all these parts to move together as a whole, at least on average; however, a collision with another object may very well "rattle" all these parts and leave them in a more or less disorganized state. If the total energy is conserved, then after the collision the object's atoms or molecules may be, on average, vibrating faster or banging against each other more often than before, but they will do so in random directions, so this increased "agitation" will not be perceived as macroscopic motion of the object as a whole.

This kind of random agitation at the microscopic level that I have just introduced is what we know today as *thermal energy*, and it is by far the most common "sink" or reservoir where macroscopic mechanical energy is "dissipated." In our example of an inelastic collision, the energy the objects had is not gone from the universe, in fact it is still right there inside the objects themselves; it is just in a disorganized or incoherent state from which, as you can imagine, it would be pretty much impossible to retrieve it, since we would have to somehow get all the randomly-moving parts to get back to moving in the same direction again.

We will have a lot more to say about thermal energy in a later chapter, but for the moment you may want to think of it as essentially *noise*: it is what is left (the residual motional or configurational energy, at the microscopic level) after you remove the average, macroscopically-observable kinetic or potential energy. So, for example, for a solid object moving with a velocity v_{cm}, the kinetic part of its thermal energy would be the sum of the kinetic energies of all its microscopic parts, calculated *in its center of mass* (or zero-momentum) *reference frame*; that way you remove from every molecule's velocity the quantity v_{cm}, which they all must have in common—on average (since the body as a whole is moving with that velocity).

In order to establish conservation of energy as a fact (which was one of the greatest scientific triumphs of the 19th century) it was clearly necessary to show experimentally that a certain amount of mechanical energy lost always resulted in the same predictable increase in the system's thermal energy. Thermal energy is largely "invisible" at the macroscopic level, but we detect it indirectly through an object's *temperature*. The crucial experiments to establish what at the time was called the "mechanical equivalent of heat" were carried out by James Prescott Joule in the 1850's, and required exceedingly precise measurements of temperature (in fact, getting the experiments done was only half the struggle; the other half was getting the scientific establishment to believe that he could measure changes in temperature so accurately!)

Fundamental Interactions

At the most fundamental (microscopic) level, physicists today believe that there are only four (or three, depending on your perspective) basic interactions: gravity, electromagnetism, the strong nuclear interaction (responsible for holding atomic nuclei together), and the weak nuclear interaction (responsible for certain nuclear processes, such as the transmutation of a proton into a neutron[1] and vice-versa). In a technical sense, at the quantum level, electromagnetism and the weak nuclear interactions can be regarded as separate manifestations of a single, consistent quantum field theory, so they are sometimes referred to as "the electroweak interaction."

All of these interactions are conservative, in the sense that for all of them one can define the equivalent of a "potential energy function" (generalized, as necessary, to conform to the requirements of quantum mechanics and relativity), so that for a system of elementary particles interacting via any one of these interactions the total kinetic plus potential energy is a constant of the motion. For gravity (which we do not really know how to "quantize" anyway!), this function immediately carries over to the macroscopic domain without any changes, as we shall see in a later chapter, and the gravitational potential energy function I introduced earlier in this chapter is an approximation to it valid near the surface of the earth (gravity is such a weak force that the gravitational interaction between any two earth-bound objects is virtually negligible, so we only have to worry about gravitational energy when one of the objects involved is the earth itself).

As for the strong and weak nuclear interactions, they are only appreciable over the scale of an atomic nucleus, so there is no question of them directly affecting any macroscopic mechanical processes. They are responsible, however, for various nuclear reactions in the course of which *nuclear energy* is, most commonly, transformed into electromagnetic energy (X- or gamma rays) and thermal energy.

All the other forms of energy one encounters at the microscopic, and even the macroscopic, level have their origin in electromagnetism. Some of them, like the electrostatic energy in a capacitor or the magnetic interaction between two permanent magnets, are straightforward enough scale-ups of their microscopic counterparts, and may allow for a potential energy description at the macroscopic level (and you will learn more about them next semester!). Many others, however, are more subtle and involve quantum mechanical effects (such as the exclusion principle) in a fundamental way.

Among the most important of these is *chemical energy*, which is an extremely important source of energy for all kinds of macroscopic processes: combustion (and explosions!), the production of electrical energy in batteries, and all the biochemical processes that power our own bodies. However, the conversion of chemical energy into macroscopic mechanical energy is almost always a dissipative process (that is, one in which some of the initial chemical energy ends up irreversibly converted into thermal energy), so it is generally impossible to describe them using a (macroscopic) potential energy function (except, possibly, for electrochemical processes, with which we will not be concerned here).

For instance, consider a chemical reaction in which some amount of chemical energy is converted into kinetic energy of the molecules forming the reaction products. Even when care is taken to “channel” the motion of the reaction products in a particular direction (for example, to push a cylinder in a combustion engine), a lot of the individual molecules will end up flying in the “wrong” direction, striking the sides of the container, etc. In other words, we end up with a lot of the chemical energy being converted into *disorganized microscopic agitation*—which is to say, *thermal energy*.

Electrostatic and quantum effects are also responsible for the elastic properties of materials, which *can* sometimes be described by macroscopic potential energy functions, at least to a first approximation. They are also responsible for the adhesive forces between surfaces that play an important role in friction, and various other kinds of what might be called “structural energies,” most of which play only a relatively small part in the energy balance where macroscopic objects are involved.

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[1] Plus a positron and a neutrino

8.5: Relative Velocity and the Coefficient of Restitution

An interesting property of elastic collisions can be disclosed from a careful study of figures 8.5.1 and 8.5.2. In both cases, as you can see, the *relative velocity* of the two objects colliding has the same magnitude (but opposite sign) before and after the collision. In other words: *in an elastic collision, the objects end up moving apart at the same rate as they originally came together*.

Recall that, in Chapter 4, we defined the velocity of object 2 relative to object 1 as the quantity

$$v_{12} = v_2 - v_1 \tag{8.5.1}$$

(compare Equation (4.3.8); and similarly the velocity of object 1 relative to object 2 is $v_{21} = v_1 - v_2$. With this definition you can check that, indeed, the collisions shown in Figs. (8.1.1) and (8.1.2) satisfy the equality

$$v_{12,i} = -v_{12,f} \tag{8.5.2}$$

(note that we could equally well have used v_{21} instead of v_{12}). For instance, in Figure (8.1.1), $v_{12,i} = v_{2i} - v_{1i}$ = −1 m/s, whereas $v_{12,f}$ = 2/3 − (−1/3) = 1 m/s. So the objects are initially moving towards each other at a rate of 1 m per second, and they end up moving apart just as fast, at 1 m per second. Visually, you should notice that the distance between the red and blue curves is the same before and after (but not during) the collision; the fact that they cross accounts for the difference in sign of the relative velocity, which in turns means simply that before the collision they were coming together, and afterwards they are moving apart.

It takes only a little algebra to show that Equation (8.5.2) follows from the joint conditions of conservation of momentum and conservation of kinetic energy. The first one ($p_i = p_f$) clearly has the form

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{8.5.3}$$

whereas the second one ($K_i = K_f$) can be written as

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2. \tag{8.5.4}$$

We can cancel out all the factors of 1/2 in Equation (8.5.4)[2], then rearrange it so that quantities belonging to object 1 are on one side, and quantities belonging to object 2 are on the other. We get

$$\begin{gathered} m_1\left(v_{1i}^2 - v_{1f}^2\right) = -m_2\left(v_{2i}^2 - v_{2f}^2\right) \\ m_1\left(v_{1i} - v_{1f}\right)\left(v_{1i} + v_{1f}\right) = -m_2\left(v_{2i} - v_{2f}\right)\left(v_{2i} + v_{2f}\right) \end{gathered} \tag{8.5.5}$$

(using the fact that $a^2 - b^2 = (a+b)(a-b)$). Note, however, that Equation (8.5.3) can also be rewritten as

$$m_1\left(v_{1i} - v_{1f}\right) = -m_2\left(v_{2i} - v_{2f}\right).$$

This immediately allows us to cancel out the corresponding factors in Eq (8.5.5), so we are left with $v_{1i} + v_{1f} = v_{2i} + v_{2f}$, which can be rewritten as

$$v_{1f} - v_{2f} = v_{2i} - v_{1i} \tag{8.5.6}$$

and this is equivalent to (8.5.2)

So, in an elastic collision the speed at which the two objects move apart is the same as the speed at which they came together, whereas, in what is clearly the opposite extreme, in a totally inelastic collision the final relative speed is *zero*—the objects do not move apart at all after they collide. This suggests that we can quantify how inelastic a collision is by the ratio of the final to the initial magnitude of the relative velocity. This ratio is denoted by e and is called the *coefficient of restitution*. Formally,

$$e = -\frac{v_{12,f}}{v_{12,i}} = -\frac{v_{2f} - v_{1f}}{v_{2i} - v_{1i}}. \tag{8.5.7}$$

For an elastic collision, e = 1, as required by Equation (8.5.2). For a totally inelastic collision, like the one depicted in Figure (8.1.3), e = 0. For a collision that is inelastic, but not totally inelastic, e will have some value in between these two extremes. This knowledge can be used to "design" inelastic collisions (for homework problems, for instance!): just pick a value for e, between 0 and 1, in Equation (8.5.7), and combine this equation with the conservation of momentum requirement (8.5.3). The two equations then allow you to calculate the final velocities for any values of m_1, m_2, and the initial velocities. Figure 8.5.4 below, for example, shows what the collision in Figure 8.5.1 would have been like, if the coefficient of restitution had been 0.6 instead of 1. You can check, by solving (8.5.3) and (8.5.7) together, and using the initial velocities, that v_{1f} = −1/15 m/s = −0.0667 m/s, and v_{2f} = 8/15 m/s = 0.533 m/s.

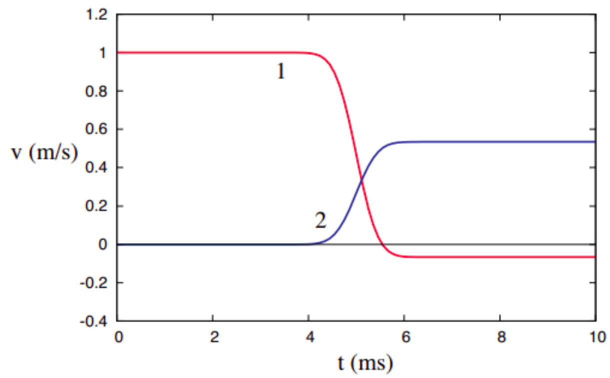

Figure 8.5.1.

Although, as I just mentioned, for most "normal" collisions the coefficient of restitution will be a positive number between 1 and 0, there can be exceptions to this. If one of the objects passes through the other (like a bullet through a target, for instance), the value of e will be negative (although still between 0 and 1 in magnitude). And e can be greater than 1 for so-called "explosive collisions," where some amount of extra energy is released, and converted into kinetic energy, as the objects collide. (For instance, two hockey players colliding on the rink and pushing each other away.) In this case, the objects may well fly apart faster than they came together.

An extreme example of a situation with $e > 0$ is an *explosive separation*, which is when the two objects are initially moving together and then fly apart. In that case, the denominator of Equation (8.5.7) is zero, and so e is formally infinite. This suggests, what is in fact the case, namely, that although explosive processes are certainly important, describing them through the coefficient of restitution is rare, even when it would be formally possible. In practice, use of the coefficient of restitution is mostly limited to the elastic-to-completely inelastic range, that is, $0 \leq e \leq 1$.

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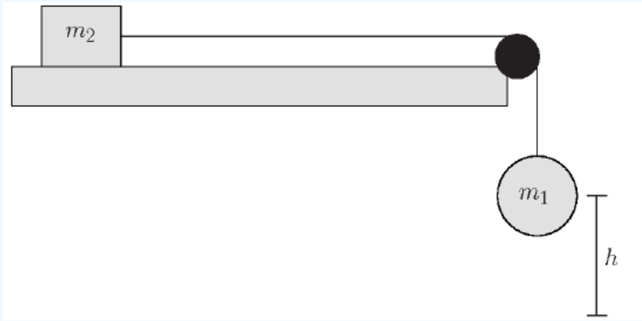
[2]You may be wondering, just why do we define kinetic energy with a factor 1/2 in front, anyway? There is no good answer at this point. Let's just say it will make the definition of "potential energy" simpler later, particularly as regards its relationship to *force*.

8.6: Examples

❓ Whiteboard Problem 8.6.1: Half-Atwood Machine



The figure shows a Half-Atwood machine, with a block of mass m_2 = 6.0 kg sliding across a frictionless surface, attached to a hanging mass m_1 = 350 g by a string going over a pulley.

1. What is the speed of the sliding block when the hanging mass falls 73 cm? Assume the system starts at rest.
2. Now assume that this system is not actually isolated, but is losing energy due to the contact interaction between the ramp and the block. This energy loss depends on the mass of the block, the distance d that the block slides, and a proportionality constant γ, like

$$\Delta E = -\gamma m d \qquad \begin{pmatrix} \text{where did} \\ \text{I get this?} \end{pmatrix} \tag{8.6.1}$$

If this constant is experimentally found to be $\gamma = 0.35$ m/s^2, how fast is the block moving if it falls the same distance?

❓ Whiteboard Problem 8.6.2: Jupiter Collision

In 1994, comet Shoemaker-Levy struck the planet Jupiter, causing explosions that were seen from Earth (above image from Wikipedia). The estimated speed of the comet before impact was 60 km/s, and based on the brightness of the explosion, it was estimated that around 4×10^{22} J of energy was released. Use this information to estimate the *size* of the comet. Unfortunately, we don't know what the comet was made of, so do this calculation twice:

1. First assume the comet was made of rock, and had a density of $3000\ kg/m^3$.
2. Now repeat the calculation assuming the comet was made of ice, with a density of $920 kg/m^3$.

❓ Whiteboard Problem 8.6.3: Destroy the Moon!

A mad scientist wants to destroy the Moon! He constructs a cannon that fires 1500 kg cannonballs, and needs to determine how fast the cannon must be fired to hit the Moon.

1. For his first calculation, he uses the potential energy due to gravity as $U_g = mgh$, and determines the minimum launch speed of the cannonballs to reach the Moon, at a distance of $d_M = 3.84 \times 10^8$ m. What speed does he calculate?

2. What will the initial kinetic energy of the cannonball be?
3. His assistant realizes that since the cannonball is going to be traveling far from the Earth, they must use

$$U_G = -G\frac{m_1 m_2}{r} \tag{8.6.2}$$

for the potential energy due to gravity. When the assistant redoes the calculation, what speed do they get?
4. What will the initial kinetic energy of the cannonball actually have to be?

? Example 8.6.4: Inelastic Collision in the middle of a swing

Tarzan swings on a vine to rescue a helpless explorer (as usual) from some attacking animal or another. He begins his swing from a branch a height of 15 m above the ground, grabs the explorer at the bottom of his swing, and continues the swing, upwards this time, until they both land safely on another branch. Suppose that Tarzan weighs 90 kg and the explorer weighs 70, and that Tarzan doesn't just drop from the branch, but pushes himself off so that he starts the swing with a speed of 5 m/s. How high a branch can he and the explorer reach?

Solution

Let us break this down into parts. The first part of the swing involves the conversion of some amount of initial gravitational potential energy into kinetic energy. Then comes the collision with the explorer, which is completely inelastic and we can analyze using conservation of momentum (assuming Tarzan and the explorer form an isolated system for the brief time the collision lasts). After that, the second half of their swing involves the complete conversion of their kinetic energy into gravitational potential energy.

Let m_1 be Tarzan's mass, m_2 the explorer's mass, h_i the initial height, and h_f the final height. We also have three velocities to worry about (or, more properly in this case, speeds, since their direction is of no concern, as long as they all point the way they are supposed to): Tarzan's initial velocity at the beginning of the swing, which we may call v_{top}; his velocity at the bottom of the swing, just before he grabs the explorer, which we may call v_{bot1}, and his velocity just after he grabs the explorer, which we may call v_{bot2}. (If you find those subscripts confusing, I am sorry, they are the best I could do; please feel free to make up your own.)

- *First part: the downswing*. We apply conservation of energy, in the form Equation (8.2.2), to the first part of the swing. The system we consider consists of Tarzan and the earth, and it has kinetic energy as well as gravitational potential energy. We ignore the source energy and the dissipated energy terms, and consider the system closed despite the fact that Tarzan is holding onto a vine (as we shall see in a couple of chapters, the vine does no "work" on Tarzan—meaning, it does not change his energy, only his direction of motion—because the force it exerts on Tarzan is always perpendicular to his displacement):

$$K_{top} + U^G_{top} = K_{bot1} + U^G_{bot}. \tag{8.6.3}$$

In terms of the quantities I introduced above, this equation becomes:

$$\frac{1}{2}m_1 v^2_{top} + m_1 g h_i = \frac{1}{2}m_1 v^2_{bot1} + 0$$

which can be solved to give

$$v^2_{bot1} = v^2_{top} + 2gh_i \tag{8.6.4}$$

Substituting, we get

$$v_{bot1} = \sqrt{\left(5\ \frac{\mathrm{m}}{\mathrm{s}}\right)^2 + 2\left(9.8\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)\times(15\ \mathrm{m})} = 17.9\ \frac{\mathrm{m}}{\mathrm{s}}$$

- *Second part: the completely inelastic collision*. The explorer is initially at rest (we assume he has not seen the wild beast ready to pounce yet, or he has seen it and he is paralyzed by fear!). After Tarzan grabs him they are moving together with a speed v_{bot2}. Conservation of momentum gives

$$m_1 v_{bot1} = (m_1 + m_2)\, v_{bot2} \tag{8.6.5}$$

which we can solve to get

$$v_{bot2} = \frac{m_1 v_{bot1}}{m_1 + m_2} = \frac{(90\ \mathrm{kg}) \times (17.9\ \mathrm{m/s})}{160\ \mathrm{kg}} = 10\ \frac{\mathrm{m}}{\mathrm{s}}$$

- *Third part: the upswing*. Here we use again conservation of energy in the form

$$K_{bot2} + U^G_{bot} = K_f + U^G_f \tag{8.6.6}$$

where the subscript *f* refers to the very end of the swing, when they both safely reach their new branch, and all their kinetic energy has been converted to gravitational potential energy, so K_f = 0 (which means that is as high as they can go, unless they start climbing the vine!). This equation can be rewritten as

$$\frac{1}{2}(m_1 + m_2)\, v^2_{bot2} + 0 = 0 + (m_1 + m_2)\, g h_f$$

and solving for h_f we get

$$h_f = \frac{v^2_{bot2}}{2g} = \frac{(10\ \mathrm{m/s})^2}{2 \times 9.8\ \mathrm{m/s}^2} = 5.15\ \mathrm{m}$$

✔ Example 8.6.5: Momentum and Energy Review: A Bouncing Superball

A superball of mass 0.25 kg is dropped from rest from a height of h = 1.50 m above the floor. It bounces with no loss of energy and returns to its initial height (Figure 8.6.2).

a. What is the superball's change of momentum during its bounce on the floor?
b. What was Earth's change of momentum due to the ball colliding with the floor?
c. What was Earth's change of velocity as a result of this collision?

(This example shows that you have to be careful about defining your system.)

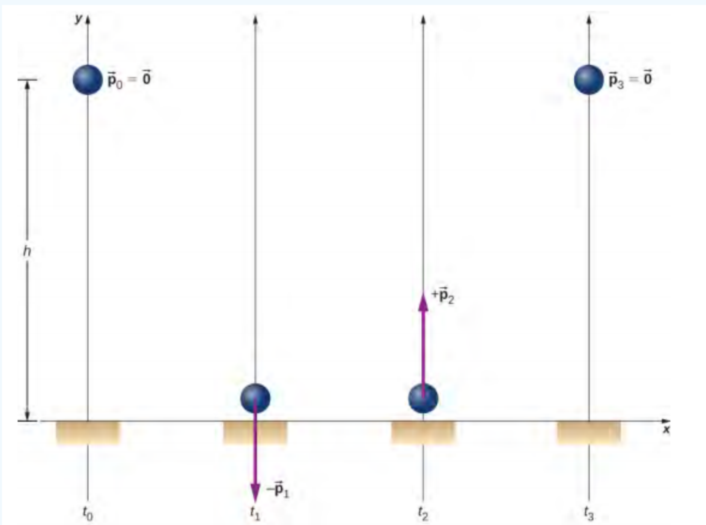


Figure 8.6.2: A superball is dropped to the floor (t_0), hits the floor (t_1), bounces (t_2), and returns to its initial height (t_3).

Strategy

Since we are asked only about the ball's change of momentum, we define our system to be the ball. But this is clearly not a closed system; gravity applies a downward force on the ball while it is falling, and the normal force from the floor applies a force during the bounce. Thus, we cannot use conservation of momentum as a strategy. Instead, we simply determine the ball's momentum just before it collides with the floor and just after, and calculate the difference. We have the ball's mass, so we need its velocities.

Solution

a. Since this is a one-dimensional problem, we use the scalar form of the equations. Let:
   - p_0 = the magnitude of the ball's momentum at time t_0, the moment it was released; since it was dropped from rest, this is zero.
   - p_1 = the magnitude of the ball's momentum at time t_1, the instant just before it hits the floor.
   - p_2 = the magnitude of the ball's momentum at time t_2, just after it loses contact with the floor after the bounce.

The ball's change of momentum is

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_2 - \vec{p}_1 \\ &= p_2\ \hat{j} - (-p_1\ \hat{j}) \\ &= (p_2 + p_1)\hat{j}.\end{aligned}$$

Its velocity just before it hits the floor can be determined from either conservation of energy or kinematics. We use kinematics here; you should re-solve it using conservation of energy and confirm you get the same result.

We want the velocity just before it hits the ground (at time t_1). We know its initial velocity $v_0 = 0$ (at time t_0), the height it falls, and its acceleration; we don't know the fall time. We could calculate that, but instead we use

$$\vec{v}_1 = -\hat{j}\sqrt{2gy} = -5.4\ m/s\ \hat{j}.$$

Thus the ball has a momentum of

$$\begin{aligned}\vec{p}_1 &= -(0.25\ kg)(-5.4\ m/s\ \hat{j}) \\ &= -(1.4\ kg \cdot m/s)\hat{j}.\end{aligned}$$

We don't have an easy way to calculate the momentum after the bounce. Instead, we reason from the symmetry of the situation.

Before the bounce, the ball starts with zero velocity and falls 1.50 m under the influence of gravity, achieving some amount of momentum just before it hits the ground. On the return trip (after the bounce), it starts with some amount of momentum, rises the same 1.50 m it fell, and ends with zero velocity. Thus, the motion after the bounce was the mirror image of the motion before the bounce. From this symmetry, it must be true that the ball's momentum after the bounce must be equal and opposite to its momentum before the bounce. (This is a subtle but crucial argument; make sure you understand it before you go on.) Therefore,

$$\vec{p}_2 = -\vec{p}_1 = +(1.4\ kg \cdot m/s)\hat{j}.$$

Thus, the ball's change of momentum during the bounce is

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_2 - \vec{p}_1 \\ &= (1.4\ kg \cdot m/s)\hat{j} - (-1.4\ kg \cdot m/s)\hat{j} \\ &= +(2.8\ kg \cdot m/s)\hat{j}.\end{aligned}$$

b. What was Earth's change of momentum due to the ball colliding with the floor? Your instinctive response may well have been either "zero; the Earth is just too massive for that tiny ball to have affected it" or possibly, "more than zero, but utterly negligible." But no—if we re-define our system to be the Superball + Earth, then this system is closed (neglecting the gravitational pulls of the Sun, the Moon, and the other planets in the solar system), and therefore the total change of momentum of this new system must be zero. Therefore, Earth's change of momentum is exactly the same magnitude: $\Delta \vec{p}_{Earth} = -2.8\; kg\; \cdotp m/s\; \hat{j} \ldotp$

c. What was Earth's change of velocity as a result of this collision? This is where your instinctive feeling is probably correct:

$$\begin{aligned}\Delta \vec{v}_{Earth} &= \frac{\Delta \vec{p}_{Earth}}{M_{Earth}} \\ &= -\frac{2.8\ kg \cdot m/s}{5.97 \times 10^{24}\ kg}\hat{j} \\ &= -(4.7 \times 10^{-25}\ m/s)\hat{j}.\end{aligned}$$

This change of Earth's velocity is utterly negligible

Significance

It is important to realize that the answer to part (c) is not a velocity; it is a change of velocity, which is a very different thing. Nevertheless, to give you a feel for just how small that change of velocity is, suppose you were moving with a velocity of 4.7 x 10^{-25} m/s. At this speed, it would take you about 7 million years to travel a distance equal to the diameter of a hydrogen atom.

? Exercise 8.6.6

Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)? Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)?

CHAPTER OVERVIEW

9: C9) Potential Energy- Graphs and Springs

This chapter is a continuation of our study of energy. Here we want to introduce another important source of energy - the potential energy of a spring. Although it's "just for springs", it turns out that many systems in the world can be modeled with springs (molecules, for instance, and nearly anything that exhibits oscillatory behavior). In fact, it will be the last of the three kinds of potential energy we are going to study in this book (the other two being due to the gravitational interaction, U_g and U_G). So to round out this chapter, we are also going to introduce some graphs of more complicated potential energy functions, and how we can extract information from them.

A spring is an example of something that interacts *elastically*, meaning that it stores energy if you either stretch it *or* compress it. How much energy it stores depends on the spring, so each spring has "a spring constant" k, that describes how much energy is stored in it when you compress it a given distance. The actual formula is

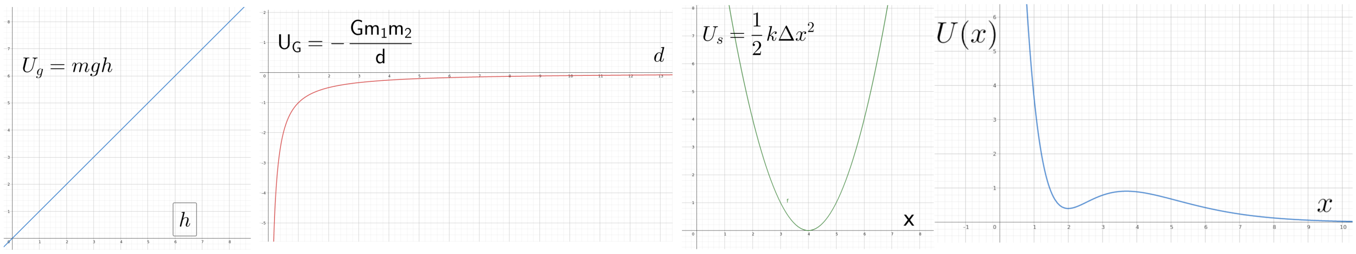
$$U_s = \frac{1}{2}k\Delta x^2, \tag{9.1}$$

where Δx is the amount the end of the spring is moved from it's equilibrium position - we will often write this as $\Delta x = x - x_0$, where x_0 is the equilibrium position of the spring. The fact that it is squared is what tells us that it doesn't matter what direction the change in length is; both compressing and stretching store energy.

To start talking about graphs, let's start with graphs of the three interactions we have seen in this book so far (below). Now think about how objects in each of these interactions behaves. For example, consider two asteroids separated by a distance d, using the potential energy for gravity (second graph). These two asteroids move towards each other right? What happens to the energy in the graph as they do? You can see that the potential energy decreases when that happens. In fact, that's true for the other form of gravitational potential as well. For the spring, something similar happens, but what direction the motion happens in depends on where you start - if you start on the right side of the equilibrium, decreasing energy means moving to smaller positions (closer to the equilibrium). If you're on the left side of the equilibrium, decreasing energy means increasing the x-coordinate (again, back to equilibrium).

What does all that mean? Well, it turns that nature wants to decrease the potential energy in a system! Unless something is actively preventing the motion, the potential will decrease. This means moving to the origin for the gravitational interaction, and towards the equilibrium point for the spring. We actually have special names for interactions that pull things together (**attractive**) as compared to push things apart (**repulsive**). So, just based on their plots, we can see that *gravity is an attractive interaction*. The *elastic interaction (springs) is either attractive or repulsive*, depending on which side of the equilibrium you are on.

Let's consider just one more interaction, one that's more complicated (last figure on the right). Playing the same game (nature wants to lower the energy!), we can see that if you start inside of the dip, you experience a spring-like interaction, either attractive or repulsive depending on where you start. If you start outside of the dip, the lowest energy is at higher and higher x-coordinate, so that interaction is repulsive out there. We can actually describe the qualitative motion of this system without ever looking at any equations, just with these simple considerations of "energy flow". This is only the first half of the story about potential energy graphs, which we will take up in more detail in the second half of this chapter.



$U_g = mgh$
h
$U_G = -\frac{Gm_1m_2}{d}$
d
$U_s = \frac{1}{2}k\Delta x^2$
x
$U(x)$
x

9.1: Potential Energy of a System

Types of Potential Energy

For each type of interaction present in a system, you can label a corresponding type of potential energy. The total potential energy of the system is the sum of the potential energies of all the types. Let's look at some particular examples of two important types of potential energy.

Gravitational Potential Energy Near Earth's Surface

The system of interest consists of our planet, Earth, and one or more particles near its surface (or bodies small enough to be considered as particles, compared to Earth). The gravitational force on each particle (or body) is just its weight *mg* near the surface of Earth, acting vertically down. As the Earth pulls downward on the object, the object is also pulling upwards on the Earth (that's Newton's third law, which we will cover in the second half of this text). However, this force of the object on the Earth is very small, so we will generally ignore it. Therefore, we consider this system to be a group of single-particle systems, subject to the uniform gravitational force of Earth.

As we've seen in the previous chapter, the gravitational potential energy function, near Earth's surface, is

$$U_g(y) = mgy \tag{9.1.1}$$

A particularly important aspect of this formula is the choice of $y = 0$. This is a coordinate system choice, but it's more significant because it means you can *choose were the zero of gravitational potential energy is*. We can see this is true by considering the following rewriting of the conservation of energy formula:

$$E_f - E_i = 0 \rightarrow (K_f + U_f) - (K_i + U_i) = 0 \rightarrow (K_f - K_i) + (U_f - U_i) = 0 \rightarrow \Delta K + \Delta U = 0 \rightarrow \Delta K = -\Delta U. \tag{9.1.2}$$

That last expression tells us what we care about - the change of the kinetic energy only depends on the change in the potential, not the absolute value. An object loosing 10 J of potential with always gain 10 J of kinetic, no matter if it started with 10 J and went to 0 J or started with 1600 J and went to 1590 J. Since this is a choice you can make, usually it's possible to make an "easy" choice; this is often when "the height of the object is zero", leading to the specific form of the expression 9.1.1.

Example 9.1.2: Gravitational PotentIAL Energy of a hiker

The summit of Great Blue Hill in Milton, MA, is 147 m above its base and has an elevation above sea level of 195 m (Figure 9.1.2). (Its Native American name, *Massachusett*, was adopted by settlers for naming the Bay Colony and state near its location.) A 75-kg hiker ascends from the base to the summit. What is the gravitational potential energy of the hiker-Earth system with respect to zero gravitational potential energy at base height, when the hiker is (a) at the base of the hill, (b) at the summit, and (c) at sea level, afterward?

Figure 9.1.2: Sketch of the profile of Great Blue Hill, Milton, MA. The altitudes of the three levels are indicated.

Strategy

First, we need to pick an origin for the *y*-axis and then determine the value of the constant that makes the potential energy zero at the height of the base. Then, we can determine the potential energies from Equation 9.1.1, based on the relationship between the zero potential energy height and the height at which the hiker is located.

Solution

a. Let's choose the origin for the *y*-axis at base height, where we also want the zero of potential energy to be. This choice makes the constant equal to zero and

$$U(\text{ base }) = U(0) = 0$$

b. At the summit, y = 147 m, so

$$U(\text{ summit }) = U(147 \text{ m}) = mgh = (75 \times 9.8 \text{ N})(147 \text{ m}) = 108 \text{ kJ}.$$

c. At sea level, y = (147 - 195) m = -48 m, so

$$U\text{ (sea-level) } = (75 \times 9.8\text{N})(-48\text{m}) = -35.3\text{kJ}.$$

Significance

Besides illustrating the use of Equation 9.1.1, the values of gravitational potential energy we found are reasonable. The gravitational potential energy is higher at the summit than at the base, and lower at sea level than at the base. The numerical values of the potential energies depend on the choice of zero of potential energy, but the physically meaningful differences of potential energy do not.

Exercise 9.1.2

What are the values of the gravitational potential energy of the hiker at the base, summit, and sea level, with respect to a sea-level zero of potential energy?

Elastic Potential Energy

The next form of potential energy we are going to look at is the energy stored in a spring, or the elastic potential energy. A spring is an object which be compressed a distance x (from equilibrium) or stretched a distance x (again, from equilibrium). In either case, the potential energy stored in such an object is

$$U_s(x) = \frac{1}{2}kx^2 \tag{9.1.3}$$

If the spring force is the only force acting, it is simplest to take the zero of potential energy at x = 0, when the spring is at its unstretched length - this has actually been done in the previous expression.

Example 9.1.3: Spring Potential Energy

A system contains a perfectly elastic spring, with an unstretched length of 20 cm and a spring constant of 4 N/cm. (a) How much elastic potential energy does the spring contribute when its length is 23 cm? (b) How much more potential energy does it contribute if its length increases to 26 cm?

Strategy

When the spring is at its unstretched length, it contributes nothing to the potential energy of the system, so we can use Equation 9.1.3 with the constant equal to zero. The value of *x* is the length minus the unstretched length. When the spring is expanded, the spring's displacement or difference between its relaxed length and stretched length should be used for the *x*-value in calculating the potential energy of the spring.

Solution

a. The displacement of the spring is x = 23 cm − 20 cm = 3 cm, so the contributed potential energy is $U = \frac{1}{2}kx^2 = \frac{1}{2}(4\text{ N/cm})(3\text{ cm})^2 = 0.18\text{ J}$.
b. When the spring's displacement is x = 26 cm − 20 cm = 6 cm, the potential energy is $U = \frac{1}{2}kx^2 = \frac{1}{2}(4\text{ N/cm})(6\text{ cm})^2 = 0.72\text{ J}$, which is a 0.54-J increase over the amount in part (a).

Significance

Calculating the elastic potential energy and potential energy differences from Equation 9.1.3 involves solving for the potential energies based on the given lengths of the spring. Since *U* depends on x^2, the potential energy for a compression (negative *x*) is the same as for an extension of equal magnitude.

Exercise 9.1.3

When the length of the spring in Example 8.2.3 changes from an initial value of 22.0 cm to a final value, the elastic potential energy it contributes changes by −0.0800J. Find the final length.

Gravitational and Elastic Potential Energy

A simple system embodying both gravitational and elastic types of potential energy is a one-dimensional, vertical mass-spring system. This consists of a massive particle (or block), hung from one end of a perfectly elastic, massless spring, the other end of which is fixed, as illustrated in Figure 9.1.1.

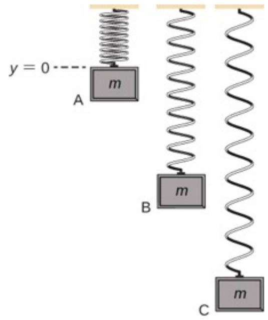


Figure 9.1.1: A vertical mass-spring system, with the positive *y*-axis pointing upward. The mass is initially at an unstretched spring length, point A. Then it is released, expanding past point B to point C, where it comes to a stop.

First, let's consider the potential energy of the system. We need to define the constant in the potential energy function of Equation 9.1.1. Often, the ground is a suitable choice for when the gravitational potential energy is zero; however, in this case, the highest point or when y = 0 is a convenient location for zero gravitational potential energy. Note that this choice is arbitrary, and the problem can be solved correctly even if another choice is picked.

We must also define the elastic potential energy of the system and the corresponding constant, as detailed in Equation 9.1.3. This is where the spring is unstretched, or at the y = 0 position.

If we consider that the total energy of the system is conserved, then the energy at point A equals point C. The block is placed just on the spring so its initial kinetic energy is zero. By the setup of the problem discussed previously, both the gravitational potential energy and elastic potential energy are equal to zero. Therefore, the initial energy of the system is zero. When the block arrives at point C, its kinetic energy is zero. However, it now has both gravitational potential energy and elastic potential energy. Therefore, we can solve for the distance y that the block travels before coming to a stop:

$$\begin{aligned} K_{\mathrm{A}} + U_{A} &= K_{C} + U_{\mathrm{C}} \\ 0 &= 0 + mgy_{C} + \frac{1}{2}k(y_{C})^{2} \\ y_{\mathrm{C}} &= \frac{-2mg}{k} \end{aligned}$$

Figure 9.1.4: A bungee jumper transforms gravitational potential energy at the start of the jump into elastic potential energy at the bottom of the jump.

Example 9.1.4: Potential energy of a vertical mass-spring system

A block weighing 1.2 N is hung from a spring with a spring constant of 6.0 N/m, as shown in Figure 9.1.3. (a) What is the maximum expansion of the spring, as seen at point C? (b) What is the total potential energy at point B, halfway between A and C? (c) What is the speed of the block at point B?

Strategy

In part (a) we calculate the distance y_C as discussed in the previous text. Then in part (b), we use half of the y value to calculate the potential energy at point B using equations Equation 9.1.1 and Equation 9.1.3. This energy must be equal to the kinetic energy, Equation 8.1.1, at point B since the initial energy of the system is zero. By calculating the kinetic energy at point B, we can now calculate the speed of the block at point B.

Solution

a. Since the total energy of the system is zero at point A as discussed previously, the maximum expansion of the spring is calculated to be:

$$\begin{aligned} y_C &= \frac{-2mg}{k} \\ y_C &= \frac{-2(1.2\text{ N})}{(6.0\text{ N/m})} = -0.40\text{ m} \end{aligned} \tag{9.1.4}$$

b. The position of y_B is half of the position at y_C or -0.20 m. The total potential energy at point B would therefore be:

$$\begin{aligned} U_B &= mgy_B + \left(\frac{1}{2}ky_B\right)^2 \\ U_B &= (1.2\text{ N})(-0.20\text{ m}) + \frac{1}{2}(6\text{ N/m})(-0.20\text{ m})^2 \\ U_B &= -0.12\text{ J} \end{aligned}$$

c. The mass of the block is the weight divided by gravity.

$$m = \frac{F_w}{g} = \frac{1.2\text{ N}}{9.8\text{ m/s}^2} = 0.12\text{ kg}$$

The kinetic energy at point B therefore is 0.12 J because the total energy is zero. Therefore, the speed of the block at point B is equal to

$$\begin{aligned} K &= \tfrac{1}{2}mv^2 \\ v &= \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(0.12\text{ J})}{(0.12\text{ kg})}} = 1.4\text{ m/s} \end{aligned} \tag{9.1.5}$$

Significance

Even though the potential energy due to gravity is relative to a chosen zero location, the solutions to this problem would be the same if the zero energy points were chosen at different locations.

? Exercise 9.1.4

Suppose the mass in Equation 9.1.5 is doubled while keeping the all other conditions the same. Would the maximum expansion of the spring increase, decrease, or remain the same? Would the speed at point B be larger, smaller, or the same compared to the original mass?

Simulation

View this simulation to learn about conservation of energy with a skater! Build tracks, ramps and jumps for the skater and view the kinetic energy, potential energy and friction as he moves. You can also take the skater to different planets or even space!

A sample chart of a variety of energies is shown in Table 9.1.1 to give you an idea about typical energy values associated with certain events. Some of these are calculated using kinetic energy, whereas others are calculated by using quantities found in a form of potential energy that may not have been discussed at this point.

Table 9.1.1: Energy of Various Objects and Phenomena

Object/phenomenon	Energy in joules
Big Bang	10^{68}
Annual world energy use	4.0×10^{20}

Object/phenomenon	Energy in joules
Large fusion bomb (9 megaton)	3.8×10^{16}
Hiroshima-size fission bomb (10 kiloton)	4.2×10^{13}
1 barrel crude oil	5.9×10^{9}
1 ton TNT	4.2×10^{9}
1 gallon of gasoline	1.2×10^{8}
Daily adult food intake (recommended)	1.2×10^{7}
1000-kg car at 90 km/h	3.1×10^{5}
Tennis ball at 100 km/h	22
Mosquito (10^{-2} g at 0.5 m/s)	1.3×10^{-6}
Single electron in a TV tube beam	4.0×10^{-15}
Energy to break one DNA strand	10^{-19}

9.2: Potential Energy Functions

It turns out that we can learn quite a lot about some pretty complicated interactions by looking at the functional form of their potential energies. In this section we will look at this, staring with one of the potentials we introducted in the last section, the elastic potential energy.

Imagine that we have two carts collide on an air track, and one of them, let us say cart 2, is fitted with a spring. As the carts come together, they compress the spring, and some of their kinetic energy is "stored" in it as elastic potential energy. In physics, we use the following expression for the potential energy stored in what we call an *ideal spring*[2]:

$$U_s(x) = \frac{1}{2}k(x - x_0)^2 \tag{9.2.1}$$

where k is something called the spring constant; x_0 is the "equilibrium length" of the spring (when it is neither compressed nor stretched); and x its actual length, so $x > x_0$ means the spring is stretched, and $x < x_0$ means it is compressed. For the system of the two carts colliding, we can take the potential energy to be given by Equation (9.2.1) if the distance between the carts is less than x_0, and 0 (corresponding to a relaxed spring) otherwise. If we put cart 1 on the left and cart 2 on the right, then the distance between them is $x_2 - x_1$, and so we can write, for the whole interaction

$$\begin{aligned} U(x_2 - x_1) = \frac{1}{2}k(x_2 - x_1 - x_0)^2 \qquad & \text{if } x_2 - x_1 < x_0 \\ 0 \qquad & \text{otherwise} \end{aligned} \tag{9.2.2}$$

This is enough to solve for the motion of the two carts, given the initial conditions. To see how, look in the "Examples" section at the end of this chapter. Here, I will just give you the result.

For the calculation, shown in Figure $9.2.2$ below, I have chosen cart 1 to have a mass of 1 kg, an initial position (at t = 0) of x_{1i} = −5 cm and an initial velocity of 1 m/s, whereas cart 2 has a mass of 2 kg and starts at rest at x_{2i} = 0. I have assumed the spring has a length of x_0 = 2 cm and a spring constant k = 1000 J/m^2 (which sounds like a lot but isn't really). The collision begins at $t_c = (x_{2i} - x_0 - x_{1i})/v_{1i}$ = 0.03 s, which is the time it takes cart 1 to travel the 3 cm separating it from the end of the spring. Prior to that point, the total kinetic energy K_{sys} = 0.5 J, and the total potential energy U = 0.

As a result of the collision, the spring compresses and undergoes "half a cycle" of oscillation with an "angular frequency" $\omega = \sqrt{k/\mu}$ (where μ is the "reduced mass" of the system, $\mu = m_1 m_2/(m_1 + m_2)$). That is, the spring is compressed and then pushes out until it gets back to its equilibrium length[3]. This lasts from $t = t_c$ until $t = t_c + \pi/\omega$, during which time the potential and kinetic energies of the system can be written as

$$\begin{aligned} U(t) &= \frac{1}{2}\mu v_{12,i}^2 \sin^2[\omega(t - t_c)] \\ K(t) &= K_{cm} + \frac{1}{2}\mu v_{12,i}^2 \cos^2[\omega(t - t_c)] \end{aligned} \tag{9.2.3}$$

(don't worry, all this will make a lot more sense after we get to Chapter 11 on simple harmonic motion, I promise!). After $t = t_c + \pi/\omega$, the interaction is over, and K and U go back to their initial values.

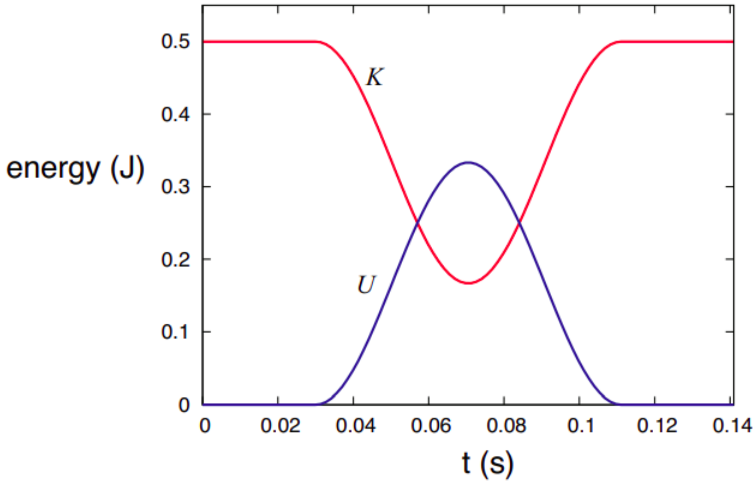


Figure 9.2.2: Potential and kinetic energy as a function of time for a system of two carts colliding and compressing a spring in the process.

If you compare Figure 9.2.2 with Figure 8.2.1 of Chapter 8, you'll see some similarities. The total energy is always the same, but it might be stored in different forms - motion, springs, or gravity.

---

[2]An "ideal spring" is basically defined, mathematically, by this expression, or by the corresponding force equation (which goes by the name of *Hooke's law*); usually, we also require that the spring be "massless" (by which we mean that its mass should be negligible compared to all the other masses involved in any given problem). Of course, for Equation (9.2.1) to hold for $x < x_0$, it must be possible to compress the spring as well as stretch it, which is not always possible with some springs.

[3]As noted earlier, we shall always assume our springs to be "massless," that is, that their inertia is negligible. In turn, negligible inertia means that the spring does not "keep going": it stops stretching as soon as it is back to its original length.

Potential Energy Functions and "Energy Landscapes"

The potential energy function of a system, as illustrated in the above examples, serves to let us know how much energy can be stored in, or extracted from, the system by changing its *configuration*, that is to say, the positions of its parts relative to each other. We have seen this in the case of the gravitational force (the "configuration" in this case being the distance between the object and the earth), and just now in the case of a spring (how stretched or compressed it is). In all these cases we should think of the potential energy as being a property of the system as a whole, not any individual part; it is, very loosely speaking, something akin to a "stress" in the system that can be turned into motion under the right conditions.

It is a consequence of the principle of conservation of momentum that, if the interaction between two particles can be described by a potential energy function, this should be a function only of their relative position, that is, the quantity $x_1 - x_2$ (or $x_2 - x_1$), and not of the individual coordinates, x_1 and x_2, separately[4]. The example of the spring in the previous section illustrates this, whereas the gravitational potential energy example shows how this can be simplified in an important case: in Equation (9.1.1), the height y of the object above the ground is really a measure of the distance between the object and the earth, something that we could write, in full generality, as $|\vec{r}_o - \vec{r}_E|$ (where $\vec{r}_o$ and $\vec{r}_E$ are the position vectors of the Earth and the object, respectively). However, since we do not expect the Earth to move very much as a result of the interaction, we can take its position to be constant, and only include the position of the object explicitly in our potential energy function, as we did above[5].

Generally speaking, then, we can identify a large class of problems where a "small" object or "particle" interacts with a much more massive one, and it is a good approximation to write the potential energy of the whole system as a function of only the position of the particle. In one dimension, then, we have a situation where, once the initial conditions (the particle's initial position and velocity) are known, the motion of the particle can be completely determined from the function $U(x)$, where x is the particle's position at any given time. This can be done using calculus (namely, let $v = \pm\sqrt{2m(E - U(x))}$ and solve the resulting differential equation); but it is also possible to get some pretty valuable insights into the particle's motion without using any calculus at all, through a mostly *graphical* approach that I would like to show you next.

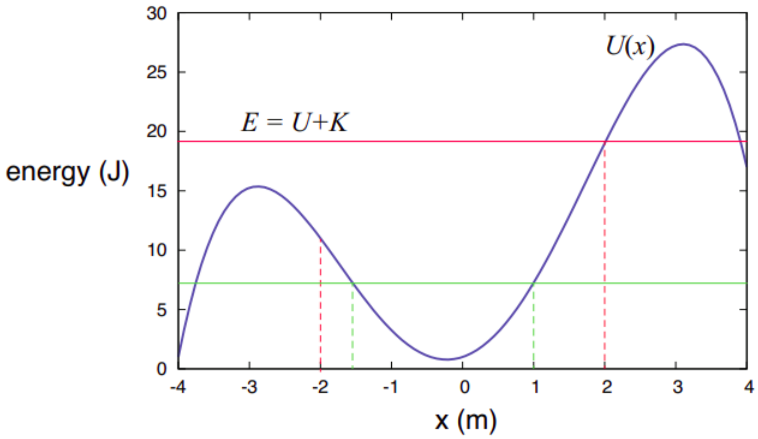


Figure 9.2.3: A hypothetical potential energy curve for a particle in one dimension. The horizontal red line shows the total mechanical energy under the assumption that the particle starts out at x = −2 m with K_i = 8 J. The green line assumes the particle starts instead from rest at x = 1 m.

In Figure 9.2.3 above I have assumed, as an example, that the potential energy of the system, as a function of the position of the particle, is given by the function $U(x) = -x^4/4 + 9x^2/2 + 2x + 1$ (in joules, if x is given in meters). Consider then what happens if the particle has a mass m = 4 kg and is found initially at x_i = −2 m, with a velocity v_i = 2 m/s. (This scenario goes with the red lines in Figure 9.2.3, so please ignore the green lines for the time being.) Its kinetic energy will then be K_i = 8 J, whereas the potential energy will be $U(-2)$ = 11 J. The total mechanical energy is then E = 19 J, as indicated by the red horizontal line.

Now, as the particle moves, the total energy remains constant, so as it moves to the right, its potential energy goes down at first, and consequently its kinetic energy goes up—that is, it accelerates. At some point, however (around x = −0.22 m) the potential energy starts to go up, and so the particle starts to slow down, although it keeps going, because $K = E - U$ is still nonzero. However, when the particle eventually reaches the point x = 2 m, the potential energy $U(2)$ = 19 J, and the kinetic energy becomes zero.

At that point, the particle stops and turns around, just like an object thrown vertically upwards. As it moves “down the potential energy hill,” it recovers the kinetic energy it used to have, so that when it again reaches the starting point x = −2 m, its speed is again 2 m/s, but now it is moving in the opposite direction, so it just passes through and over the next “hill” (since it has enough total energy to do so), and eventually moves outside the region shown in the figure.

As another example, consider what would have happened if the particle had been released at, say, x = 1 m, but with zero velocity. (This is illustrated by the green lines in Figure 9.2.3.) Then the total energy would be just the potential energy $U(1)$ = 7.25 J. The particle could not possibly move to the right, since that would require the total energy to go up. It can only move to the left, since in that direction $U(x)$ decreases (initially, at first), and that means K can increase (recall K is always positive as long as the particle is in motion). So the particle speeds up to the left until, past the point x = −0.22 m, $U(x)$ starts to increase again and K has to go down. Eventually, as the figure shows, we reach a point (which we can calculate to be x = −1.548 m) where $U(x)$ is once again equal to 7.25 J. This leaves no room for any kinetic energy, so the particle has to stop and turn back. The resulting motion consists of the particle oscillating back and forth forever between x = −1.548 m and x = 1 m.

At this point, you may have noticed that the motion I have described as following from the $U(x)$ function in Figure 9.2.3 resembles very much the motion of a car on a roller-coaster having the shape shown, or maybe a ball rolling up and down hills like the ones shown in the picture. In fact, the correspondence can be made *exact*—if we substitute sliding for rolling, since rolling motion has complications of its own. Given an arbitrary potential energy function $U(x)$ for a particle of mass m, imagine that you build a “landscape” of hills and valleys whose height y above the horizontal, for a given value of the horizontal coordinate x, is given by the function $y(x) = U(x)/mg$. (Note that mg is just a constant scaling factor that does not change the shape of the curve.) Then, for an object of mass m sliding without friction over that landscape, under the influence of gravity, the gravitational potential energy at any point x would be $U^G(x) = mgy = U(x)$, and therefore its speed at any point will be precisely the same as that of the original particle, if it starts at the same point with the same velocity

This notion of an “energy landscape” can be extended to more than one dimension (although they are hard to visualize in three!), or generalized to deal with configuration parameters other than a single particle’s position. It can be very useful in a number of disciplines (not just physics), to predict the ways in which the configuration of a system may be likely to change.

[4]We will see why in the next chapter!

[5]This will change in Chapter 13, when we get to study gravity over a planetary scale.

9.3: Equilibrium and Turning Points

Let's continue our discussion of potential energy graphs by introducing some new terms - turning points and equilibrium points. We already saw **turning points** in the last section - these were the point in the motion at which the object stopped moving and turned around to go the other direction. **Equilibrium points** are the points in the motion where the object *could be* at rest (note the object does not have to actually be at rest at that point, but under some conditions could be).

To get a better understanding of these terms, we'll look at two specific examples. First, let's look at an object, freely falling vertically, near the surface of Earth, in the absence of air resistance. The mechanical energy of the object is conserved, E = K + U, and the potential energy, with respect to zero at ground level, is U(y) = mgy, which is a straight line through the origin with slope mg. In the graph shown in Figure $9.3.1$, the x-axis is the height above the ground y and the y-axis is the object's energy.

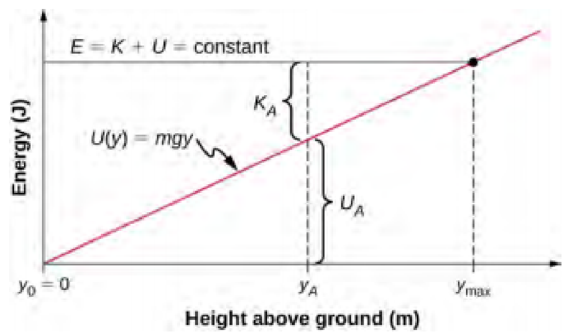


Figure $9.3.1$: The potential energy graph for an object in vertical free fall, with various quantities indicated.

The line at energy E represents the constant mechanical energy of the object, whereas the kinetic and potential energies, K_A and U_A, are indicated at a particular height y_A. You can see how the total energy is divided between kinetic and potential energy as the object's height changes. First, let's note that *the kinetic energy of an object can never be negative*. This is true because there is nothing negative in the formula $1/2mv^2$...nature says mass can never be negative, and mathematics says nothing squared can ever be negative (no imaginary speeds allowed!).

Since kinetic energy can never be negative, there is a maximum potential energy and a maximum height, which an object with the given total energy cannot exceed:

$$K = E - U \geq 0, \tag{9.3.1}$$

$$U \leq E. \tag{9.3.2}$$

If we use the gravitational potential energy reference point of zero at y_0, we can rewrite the gravitational potential energy U as mgy. Solving for y results in

$$y \leq \frac{E}{mg} = y_{max}. \tag{9.3.3}$$

We note in this expression that the quantity of the total energy divided by the weight (mg) is located at the maximum height of the particle, or y_{max}. At the maximum height, the kinetic energy and the speed are zero, so if the object were initially traveling upward, its velocity would go through zero there, and y_{max} would be a turning point in the motion. At ground level, y_0 = 0, the potential energy is zero, and the kinetic energy and the speed are maximum:

$$U_0 = 0 = E - K_0, \tag{9.3.4}$$

$$E = K_0 = \frac{1}{2}mv_0^2, \tag{9.3.5}$$

$$v_0 = \pm\sqrt{\frac{2E}{m}}. \tag{9.3.6}$$

The maximum speed $\pm v_0$ gives the initial velocity necessary to reach y_{max}, the maximum height, and $-v_0$ represents the final velocity, after falling from y_{max}. You can read all this information, and more, from the potential energy diagram we have shown. *Notice that the turning point occured where the total energy and the potential energy intersected* in the graph - that's the point with zero kinetic energy. This system does not have an equilibrium point, because there is nowhere the falling object could be at rest - it's always going to be trying to move downwards under the gravitational interaction.

For a second examples, consider a mass-spring system on a frictionless, stationary, horizontal surface, so that gravity and the normal contact force do no work and can be ignored (Figure 9.3.2). This is like a one-dimensional system, whose mechanical energy E is a constant and whose potential energy, with respect to zero energy at zero displacement from the spring's unstretched length, x = 0, is $U(x) = \frac{1}{2}kx^2$.

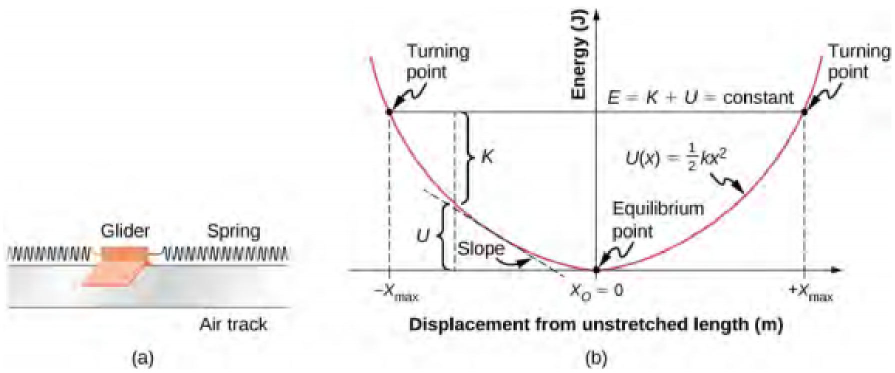


Figure 9.3.2: (a) A glider between springs on an air track is an example of a horizontal mass-spring system. (b) The potential energy diagram for this system, with various quantities indicated.

You can read off the same type of information from the potential energy diagram in this case, as in the case for the body in vertical free fall, but since the spring potential energy describes a variable force, you can learn more from this graph. As for the object in vertical free fall, you can deduce the physically allowable range of motion and the maximum values of distance and speed, from the limits on the kinetic energy, $0 \le K \le E$. Therefore, *K = 0 and U = E at a turning point*, of which there are two for the elastic spring potential energy,

$$x_{max} = \pm\sqrt{\frac{2E}{k}}. \tag{9.3.7}$$

The glider's motion is confined to the region between the turning points, $-x_{max} \le x \le x_{max}$. This is true for any (positive) value of E because the potential energy is unbounded with respect to x. For this reason, as well as the shape of the potential energy curve, U(x) is called an infinite potential well. At the bottom of the potential well, x = 0, U = 0 and the kinetic energy is a maximum, K = E, so $v_{max} = \pm\sqrt{\frac{2E}{m}}$.

However, this potential has another special point, at $x = 0$. *This is the equilibrium point*, because if the object had zero kinetic energy at that point, it would not move. Notice that an object bouncing back and forth between the two turning points doesn't stop at this equilibrium, because it doesn't have $K = 0$ there. Note that on either side of the equilibrium point, the potential energy increases - another way of defining the equilibrium point is "the point which is a (local) minimum of potential energy". No matter where an object starts, it will be driven towards the equilibrium point.

Finally, we should add that the description we just gave ("local minimum of potential") is actually for a *stable* equilibrium point - there are also unstable equilibrium points. For example, consider turning the spring potential upside-down (alternatively, look at the "bumps" in Figure 9.2.3). We could then place an object with $K = 0$ right at that point, and it would technically not move since there is no slope in the potential energy function. However, the moment we give it any kind of bump one way or the other, it will immediately go in that direction. In other words, an unstable equilibrium is a local *maximum* of potential energy, where an object placed there will move away from it if displaced.

9.4: Advanced Application- Springs and Collisions

Two Carts Colliding and Compressing a Spring

Unlike the discussion in the previous section, which considered a stationary spring and asked only questions about initial and final states, this example is intended to show you how one can use "energy methods" to solve for the actual motion of a relatively complicated system as a function of time. The system is the two carts colliding, one of them fitted with a spring, considered in Section 9.3. Although all the physics involved is straightforward, the math is at a higher level than you will be using this semester, so I'm presenting this here as a "curiosity" only.

First, recall the total kinetic energy for a collision problem can be written as $K=K_{cm}+K_{conv}$, where (if the system is isolated) K_{cm} remains constant throughout. Then, the total mechanical energy $E=K+U=K_{cm}+K_{conv}+U$. This is also constant, and before the interaction happens, when U = 0, we have $E=K_{cm}+K_{conv,i}$, so setting these two things equal and canceling out K_{cm} we get

$$K_{\text{conv}}=K_{\text{conv},i}-U \tag{9.4.1}$$

where the potential energy function is given by Equation (9.2.2). Introducing the relative coordinate $x_{12}=x_2-x_1$, and the relative velocity v_{12}, Equation (9.4.1) becomes

$$\frac{1}{2}\mu v_{12}^2=\frac{1}{2}\mu v_{12,i}^2-\frac{1}{2}k(x_{12}-x_0)^2 \tag{9.4.2}$$

an equation that must hold while the interaction is going on. We can solve this for v_{12}, and then notice that both x_{12} and v_{12} are functions of time, and the latter is the derivative with respect to time of the former, so

$$v_{12}=\pm\sqrt{v_{12,i}^2-(k/\mu)(x_{12}-x_0)^2} \tag{9.4.3}$$

$$\frac{dx_{12}}{dt}=\pm\sqrt{v_{12,i}^2-(k/\mu)(x_{12}(t)-x_0)^2}. \tag{9.4.4}$$

(The "$\pm$" sign means that the quantity on the right-hand side has to be negative at first, when the carts are coming together, and positive later, when they are coming apart. This is because I have assumed cart 1 starts to the left of cart 2, so going in cart 2, as seen from cart 1, appears to be moving to the left.)

Equation (9.4.4) is what is known, in calculus, as a differential equation. The problem is to find a function of t, $x_{12}(t)$, such that when you take its derivative you get the expression on the right-hand side. If you know how to calculate derivatives, you can check that the solution is in fact

$$x_{12}(t)=x_0-\frac{v_{12,i}}{\omega}\sin[\omega(t-t_c)] \quad \text{for } t_c\le t\le t_c+\pi/\omega \tag{9.4.5}$$

where the quantity $\omega=\sqrt{k/\mu}$, and the time t_c is the time cart 1 first makes contact with the spring: $t_c=(x_{2i}-x_0-x_{1i})/v_{1i}$. The solution (9.4.5) is valid for as long as the spring is compressed, which is to say, for as long as $x_{12}(t)<x_0$, or $\sin[\omega(t-t_c)]>0$, which translates to the condition on t shown above.

Having a solution for x_{12}, we could now obtain explicit results for $x_1(t)$ and $x_2(t)$ separately, using the fact that $x_1=x_{cm}-m_2x_{12}/(m_1+m_2)$, and $x_2=x_{cm}+m_1x_{12}/(m_1+m_2)$ (compare Eqs. (4.2.1), in chapter 4), and finding the position of the center of mass as a function of time is a trivial problem, since it just moves with constant velocity.

We do not, however, need to do any of this in order to generate the plots of the kinetic and potential energy shown in Figure 8.2.1: the potential energy depends only on x_2-x_1, which is given explicitly by Equation (9.4.5), and the kinetic energy is equal to $K_{cm}+K_{conv}$, where K_{cm} is constant and K_{conv} is given by Equation (9.4.2), which can also be easily rewritten in terms of Equation (9.4.5). The results are Eqs. (9.2.3) in the text.

Getting the Potential Energy Function from Collision Data

Consider the collision illustrated in Figure 4.3.1. Can we tell what the potential energy function is for the interaction between the two carts?

At first sight, it may seem that all the information necessary to "reconstruct" the function $U(x_1 - x_2)$ is available already, at least in graphical form: From Figure 4.3.1 you could get the value of $x_2 - x_1$ at any time t; then from Figure 8.5.1 you can get the value of K (in the elastic-collision scenario) for the same value of t; and then you could plot $U = E - K$ (where E is the total energy) as a function of $x_2 - x_1$.

But there is a catch: Figure 4.3.1 shows that the colliding objects never get any closer than $x_2 - x_1 \simeq 0.28$ mm, so we have no way to tell what $U(x_2 - x_1)$ is for smaller values of $x_2 - x_1$. This is essentially the problem faced by particle physicists when they use collisions (which they do regularly) to try to determine the precise nature of the interactions between the particles they study!

You can check this for yourself. The functions I used for $x_1(t)$ and $x_2(t)$ in Figure 4.3.1 are

$$\begin{aligned} x_1(t) &= \frac{1}{3}\left((2t-10)\,\mathrm{erf}(10-2t) + 10\,\mathrm{erf}(10) + t - \frac{e^{-4(t-5)^2}}{\sqrt{\pi}}\right) - 5 \\ x_2(t) &= \frac{1}{3}\left((5-t)\,\mathrm{erf}(10-2t) - 5\,\mathrm{erf}(10) + t + \frac{e^{-4(t-5)^2}}{2\sqrt{\pi}}\right) \end{aligned} \tag{9.4.6}$$

Here, "erf" is the so-called "error function," which you can find in any decent library of mathematical functions. This looks complicated, but it just gives you the shapes you want for the velocity curves. The derivative of the above is

$$\begin{aligned} v_1(t) &= \frac{1}{3}(1 + 2\,\mathrm{erf}(10-2t)) \\ v_2(t) &= \frac{1}{3}(1 - \mathrm{erf}(10-2t)) \end{aligned} \tag{9.4.7}$$

and you may want to try plotting these for yourself; the result should be Figure 4.3.1.

Now, assume (as I did for Figure 8.5.1) that m_1 = 1 kg, and m_2 = 2 kg, and use these values and the results of Eq (9.4.7) (assumed to be in m/s) to calculate K_{sys} as a function of t. Then $U = E_{sys} - K_{sys}$, with E_{sys} = 1/2 J:

$$U = \frac{1}{2} - \frac{1}{2}m_1 v_1^2(t) - \frac{1}{2}m_2 v_2^2(t) = \frac{1}{3}\left(1 - \mathrm{erf}^2(10-2t)\right) \tag{9.4.8}$$

and now do a parametric plot of U versus $x_2 - x_1$, using t as a parameter. You will end up with a figure like the one below:

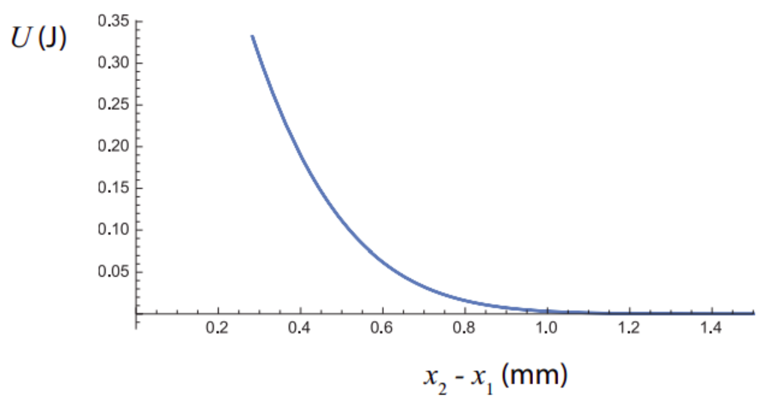


Figure 9.4.1. No information can be gathered from those figures (nor from the explicit expressions (9.4.6) and (9.4.7) above) on the values of U for $x_2 - x_1 < 0.28$ mm, the distance of closest approach of the two carts.

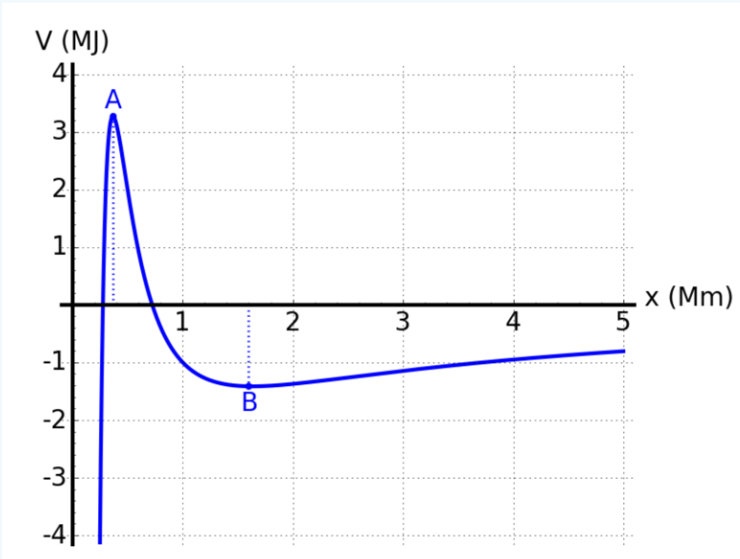
9.5: Examples

✓ Whiteboard Problem 9.5.1: Trippy Springy

I have a spring with a spring constant 1750 N/m and a mass 13.5 kg attached to the end. They are oriented horizontally so the mass is sliding across a frictionless surface.

1. I pull the spring a distance 5.4 cm from equilibrium. How much energy did I add on the system?
2. I let the mass go; what velocity is it traveling with when it crosses the equilibrium point?
3. The mass travels through the equilibrium point and stops at the maximum compression point of the spring. How far is the mass from the equilibrium point?
4. How much energy did the mass transfer to the spring as it moved from equilibrium to the maximum compression point?

? Whiteboard Problem 9.5.2: Planetary Energy Graph



The figure above shows the gravitational potential energy interaction between two massive planets orbiting around each other. At x = ∞, this potential is zero.

1. Indicate the sign of the force associated to this potential for each of the three regions in the following table.

	Between 0 and A	Between A and B	Between B and ∞
F positive or Negative?			

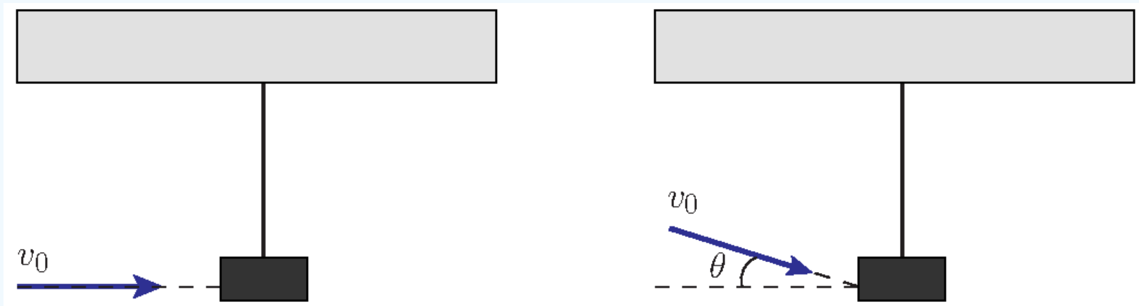
2. If the two planets are initially separated by a large distance, but are moving towards each other with a kinetic energy of 1.0 MJ, about how close will they get to each other?

? Whiteboard Problem 9.5.3: Safety Elevator

You are designing a back-up safety system for an elevator, that will catch the elevator on a spring if the cable breaks. The mass of the elevator is 750 kg, and the system needs to be able to stop an elevator that dropped a distance 10 m before hitting the spring, but only compressing a distance 2 m.

What does the spring constant have to be for this safety system to work as designed?

Whiteboard Problem 9.5.4: Ballistic Pendulum



A ballistic pendulum can be used to determine the muzzle speed of a gun. The bullet is fired into a hanging pendulum, and by measuring how far into the air the pendulum swings you can figure out how fast the bullet was traveling.

1. First determine the mass of the pendulum: I fire a bullet of mass 70 g straight into the pendulum block (left figure), at a speed of 500 m/s, and I measure that the block rises 62 cm into the air.
2. Now I take another gun which fires the same size bullet, and find the pendulum rises a height of 75 cm. How fast did the second gun fire the bullet?
3. What if I wasn't very careful when I aimed the bullet? If I fired at a 7.5° angle with respect to the horizontal (right figure), what was the true muzzle speed of the gun?

Example 9.5.5: Quartic and Quadratic Potential Energy Diagram

The potential energy for a particle undergoing one-dimensional motion along the x-axis is $U(x) = 2(x^4 - x^2)$, where U is in joules and x is in meters. The particle is not subject to any non-conservative forces and its mechanical energy is constant at $E = -0.25$ J. (a) Is the motion of the particle confined to any regions on the x-axis, and if so, what are they? (b) Are there any equilibrium points, and if so, where are they and are they stable or unstable?

Strategy

First, we need to graph the potential energy as a function of x. The function is zero at the origin, becomes negative as x increases in the positive or negative directions (x^2 is larger than x^4 for $x < 1$), and then becomes positive at sufficiently large $|x|$. Your graph should look like a double potential well, with the zeros determined by solving the equation $U(x) = 0$, and the extremes determined by examining the first and second derivatives of U(x), as shown in Figure 9.5.3.

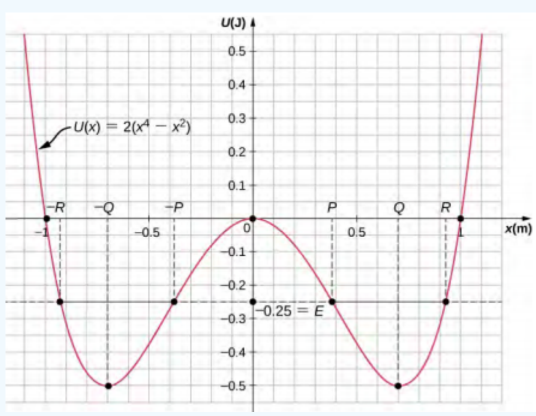


Figure 9.5.3: The potential energy graph for a one-dimensional, quartic and quadratic potential energy, with various quantities indicated.

You can find the values of (a) the allowed regions along the x-axis, for the given value of the mechanical energy, from the condition that the kinetic energy can't be negative, and (b) the equilibrium points and their stability from the properties of the force (stable for a relative minimum and unstable for a relative maximum of potential energy). You can just eyeball the graph to reach qualitative answers to the questions in this example. That, after all, is the value of potential energy diagrams.

You can see that there are two allowed regions for the motion (E > U) and three equilibrium points (slope $\frac{dU}{dx}$ = 0), of which the central one is unstable $\left(\frac{d^2U}{dx^2} < 0\right)$, and the other two are stable $\left(\frac{d^2U}{dx^2} > 0\right)$.

Solution

a. To find the allowed regions for x, we use the condition $K = E - U = -\frac{1}{4} - 2(x^4 - x^2) \geq 0$. If we complete the square in x 2 , this condition simplifies to $2\left(x^2 - \frac{1}{2}\right)^2 \leq \frac{1}{4}$, which we can solve to obtain $\frac{1}{2} - \sqrt{\frac{1}{8}} \leq x^2 \leq \frac{1}{2} + \sqrt{\frac{1}{8}}$ .This represents two allowed regions, $x_p \leq x \leq x_R$ and $-x_R \leq x \leq -x_p$, where x_p = 0.38 and x_R = 0.92 (in meters).
b. To find the equilibrium points, we solve the equation $\frac{dU}{dx} = 8x^3 - 4x = 0$ and find x = 0 and $x = \pm x_Q$, where $x_Q = \frac{1}{\sqrt{2}}$ = 0.707 (meters). The second derivative $\frac{d^2U}{dx^2} = 24x^2 - 4$ is negative at x = 0, so that position is a relative maximum and the equilibrium there is unstable. The second derivative is positive at $x = \pm x_Q$, so these positions are relative minima and represent stable equilibria.

Significance

The particle in this example can oscillate in the allowed region about either of the two stable equilibrium points we found, but it does not have enough energy to escape from whichever potential well it happens to initially be in. The conservation of mechanical energy and the relations between kinetic energy and speed, and potential energy and force, enable you to deduce much information about the qualitative behavior of the motion of a particle, as well as some quantitative information, from a graph of its potential energy.

? Exercise 9.5.6

Repeat Example 9.5.1 when the particle's mechanical energy is +0.25 J.

A block of mass m is sliding on a frictionless, horizontal surface, with a velocity v_i. It hits an ideal spring, of spring constant k, which is attached to the wall. The spring compresses until the block momentarily stops, and then starts expanding again, so the block ultimately bounces off.

a. In the absence of dissipation, what is the block's final speed?
b. By how much is the spring compressed?

Solution

This is a simpler version of the problem considered in Section 5.1, and in the next example. The problem involves the conversion of kinetic energy into elastic potential energy, and back. In the absence of dissipation, Equation (5.4.1), specialized to this system (the spring and the block) reads:

$$K + U^{spr} = \text{constant} \tag{9.5.1}$$

For part (a), we consider the whole process where the spring starts relaxed and ends relaxed, so $U_i^{spr} = U_f^{spr}$ = 0. Therefore, we must also have $K_f = K_i$, which means the block's final speed is the same as its initial speed. As explained in the chapter, this is characteristic of a conservative interaction.

For part (b), we take the final state to be the instant where the spring is maximally compressed and the block is momentarily at rest, so all the energy in the system is spring (which is to say, elastic) potential energy. If the spring is compressed a distance d (that is, $x - x_0 = -d$ in Equation (5.1.5)), this potential energy is $\frac{1}{2}kd^2$, so setting that equal to the system's initial energy we get:

$$K_i + 0 = 0 + \frac{1}{2}kd^2 \tag{9.5.2}$$

or

$$\frac{1}{2}mv_i^2 = \frac{1}{2}kd^2$$

which can be solved to get

$$d = \sqrt{\frac{m}{k}}v_i.$$

CHAPTER OVERVIEW

10: C10) Work

In the previous chapters, we have been considering the phenomena of *conservation of energy*, which is the idea that the total energy of a closed system does not change in time. We utilized this idea for solving problems by identifying all the interactions in the system, associating them with sources of energy, and setting the initial energy equal to the final energy. This idea is very powerful, but it certainly requires you to know how to associate a potential energy to a particular interaction. So far, we've only done this with two interactions - the force of gravity ($U_g = mgh$ or $U_G = -GmM/r$) and springs ($U_s = \frac{1}{2}m\Delta x^2$). So what happens when you have an interaction whose potential you don't know - or even worse, that doesn't even exist? (That's called a *nonconservative force*, we will cover those in Chapter N8).

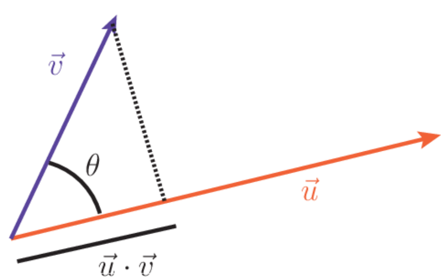
We certainly know that forces can be used to store energy, since that's what both the gravitational force and the spring force do. But what about some other force - say, a force pushing a box along the floor? Does this store energy? It's pretty easy to see that it does, because that box might go from moving to not moving, demonstrating that it now has kinetic energy, which it clearly got from your pushing. It turns out to be pretty easy to describe how much energy is transferred into the system from this force - **that quantity is called work**.

For a constant force, the work is a simple formula,

$$W = \vec{F} \cdot \Delta\vec{r}, \tag{10.1}$$

where $\vec{F}$ is the force and $\Delta\vec{r}$ is the change in position of the center of mass of the object. But notice the mathematical operation being performed here - it's a dot product between two vectors. For any two vectors $\vec{u} = u_x\hat{x} + u_y\hat{y}$ and $\vec{v} = v_x\hat{x} + v_y\hat{y}$, the dot product can be written in two different ways:

$$\vec{u} \cdot \vec{v} = \|\vec{v}\|\|\vec{u}\|\cos\theta, \text{ or } \vec{u} \cdot \vec{v} = u_x v_x + u_y v_y. \tag{10.2}$$



We have to be very careful with that first formula - the angle θ must be *the angle between the vectors $\vec{u}$ and $\vec{v}$ when they are arranged tail to tail (see the figure)*. Generally, if we have a picture, this first formula is the one we want to use. However, sometimes we might just have the components, and then it's much easier to use the second formula. In either case, notice carefully that the result is a scalar - in fact, the result is *the projection of the second vector onto the first*. We won't need that fact much in this class, but it's true geometrically.

So let's consider a few simple examples, focusing on the first of the two formula above. Let's act on our block with a force of 10 N.

1. If we push the block horizontally, a distance 10 m, we get $W = (10\ N)(10\ M)\cos(0) = 100$ J. (Notice the energy units!) The angle was zero here because we are *pushing in the same direction as the motion of the block*.
2. Let's push against the motion of the block - so it's moving forwards 10 m, but we are pushing the other way. Then we get $W = (10\ N)(10\ m)\cos(180^\circ) = -100$ J. Notice how the angle changed - the force vector $\vec{F}$ was 180° from the displacement vector $\Delta\vec{r}$, and the work done was negative.
3. Finally, let's push straight downwards on the block as it moves the same 10 m. The work is now $W = (10\ N)(10\ m)\cos(90^\circ) = 0$. The work done is zero here, because if you push straight downwards (and there is no friction), you can't change the energy of the block!

10.1: Introduction- Work and Impulse

In physics, “work” (or “doing work”) is what we call the process through which a force changes the energy of an object it acts on (or the energy of a system to which the object belongs). It is, therefore, a very technical term with a very specific meaning that may seem counterintuitive at times.

For instance, as it turns out, in order to change the energy of an object on which it acts, the force needs to be at least partly in line with the displacement of the object during the time it is acting. A force that is perpendicular to the displacement does no work—it does not change the object’s energy.

Imagine a satellite in a circular orbit around the earth. The earth is constantly pulling on it with a force (gravity) directed towards the center of the orbit at any given time. This force is always perpendicular to the displacement, which is along the orbit, and so it does no work: the satellite moves always at the same speed, so its kinetic energy does not change.

This can be contrasted with what is going on with forces and momentum - the gravitational force bends the satellite's trajectory, changing the direction (although not the magnitude) of the momentum vector. Of course, it is obvious that a force must change an object’s momentum, because that is pretty much how we defined force anyway. Recall Equation (2.3.1) for the average force on an object: $\vec{F}_{av} = \Delta\vec{p}/\Delta t$. We can rearrange this to read

$$\Delta\vec{p} = \vec{F}_{av}\Delta t. \tag{10.1.1}$$

For a constant force, the product of the force and the time over which it is acting is called the *impulse*, usually denoted as $\vec{J}$

$$\vec{J} = \vec{F}\Delta t. \tag{10.1.2}$$

Clearly, the impulse given by a force to an object is equal to the change in the object’s momentum (by Equation (10.1.1)), as long as it is the only force (or, alternatively, the net force) acting on it. If the force is not constant, we break up the time interval Δt into smaller subintervals and add all the pieces. Formally this results in an integral:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt. \tag{10.1.3}$$

Graphically, the x component of the impulse is equal to the area under the curve of F_x versus time, and similarly for the other components. We will see this basic effect as we study work as well, although the complications associated with the direction of force as compared to the direction of motion must be taken account of.

There is not a whole lot more to be said about impulse. The main lesson to be learned from Equation (10.1.1) is that one can get a desired change in momentum—bring an object to a stop, for instance—either by using a large force over a short time, or a smaller force over a longer time. It is easy to see how different circumstances may call for different strategies: sometimes you may want to make the force as small as possible, if the object on which you are acting is particularly fragile; other times you may just need to make the time as short as possible instead.

Of course, to bring something to a stop you not only need to remove its momentum, but also its (kinetic) energy. If the former task takes time, the latter, it turns out, takes *distance*. Work is a much richer subject than impulse, not only because, as indicated above, the actual work done depends on the relative orientation of the force and displacement vectors, but also because there is only one kind of momentum, but many different kinds of energy, and one of the things that typically happens when work is done is the *conversion* of one type of energy into another.

So there is a lot of ground to cover, but we’ll start small, in the next section, with the simplest kind of system, and the simplest kind of energy.

10.2: Work on a Single Particle

Consider a particle that undergoes a displacement Δx while a constant force F acts on it. In one dimension, the *work* done by the force on the particle is defined by

$$W = F\Delta x \quad \text{(constant force)} \tag{10.2.1}$$

and it is positive if the force and the displacement have the same sign (that is, if they point in the same direction), and negative otherwise.

In three dimensions, the force will be a vector $\vec{F}$ with components (F_x, F_y, F_z), and the displacement, likewise, will be a vector $\Delta\vec{r}$ with components $(\Delta x, \Delta y, \Delta z)$. The work will be defined then as

$$W = F_x\Delta x + F_y\Delta y + F_z\Delta z. \tag{10.2.2}$$

This expression is an instance of what is known as the *dot product* (or *inner product*, or *scalar product*) of two vectors. Given two vectors $\vec{A}$ and $\vec{B}$, their dot product is defined, in terms of their components,

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z. \tag{10.2.3}$$

This can also be expressed in terms of the vectors' magnitudes, $|\vec{A}|$ and $|\vec{B}|$, and the angle they make, in the following form:

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\phi \tag{10.2.4}$$

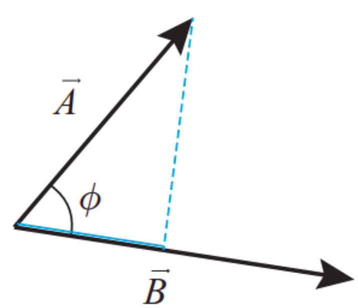


Figure 10.2.1: Illustrating the angle ϕ to be used when calculating the dot product of two vectors by the formula (10.2.4). One way to think of this formula is that you take the projection of vector $\vec{A}$ onto vector $\vec{B}$ (indicated here by the blue lines), which is equal to $|\vec{A}|\cos\phi$, then multiply that by the length of $\vec{B}$ (or vice-versa, of course).

Figure $10.2.1$ shows what we mean by the angle ϕ in this expression. The equality of the two definitions, Eqs. (10.2.3) and (10.2.4), is pretty easy to see; consider a coordinate system in which the vector $\vec{B}$ is completely along the x-axis. Then our vectors can be written

$$\vec{A} = A\cos(\phi)\hat{x} + A\sin(\phi)\hat{y}, \qquad \vec{B} = B\hat{x}. \tag{10.2.5}$$

And now if we perform the dot product as shown in Equation (10.2.3),

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y = AB\cos(\phi), \tag{10.2.6}$$

which is exactly what Equation (10.2.4) says! As a sidenote, this is an important demonstration of a basic principle in the study of physics - if you can prove something in one coordinate system, it must be true in all others. This is not just a calculational trick, but a fundamental truth we would hope our theory can conform with - the coordinate system is a choice humans make to perform a calculation, but can't have any real physics meaning. In fact, the advantage of Equation (10.2.4) is that it is independent of the choice of a system of coordinates.

Using the dot product notation, the work done by a constant force can be written as

$$W = \vec{F}\cdot\Delta\vec{r}. \tag{10.2.7}$$

Equation (10.2.4) then shows that, as I mentioned in the introduction, when the force is perpendicular to the displacement ($\phi = 90^\circ$) the work it does is zero. You can also see this directly from Equation (10.2.2), by choosing the x axis to point in the direction of the force (so $F_y = F_z = 0$), and the displacement to point along any of the other two axes (so $\Delta x = 0$): the result is $W = 0$.

If the force is not constant, again we follow the standard procedure of breaking up the total displacement into pieces that are short enough that the force may be taken to be constant over each of them, calculating all those (possibly very small) "pieces of work," and adding them all together. In one dimension, the final result can be expressed as the integral

$$W=\int_{x_i}^{x_f} F(x) d x \quad (\text { variable force }). \tag{10.2.8}$$

So the work is given by the "area" under the F-vs-x curve. In more dimensions, we have to write a kind of multivariable integral known as a *line integral*. That is advanced calculus, so we will not go there this semester.

Work Done by the Net Force, and the Work-Energy Theorem

So much for the math and the definitions. Where does the energy come in? Let us suppose that F is either the only force or the *net force* on the particle—the sum of all the forces acting on the particle. Again, for simplicity we will assume that it is constant (does not change) while the particle undergoes the displacement Δx. We started this chapter with the understanding that the work done on the system changes the energy of the system (we will prove that mathematically in a later chapter). If we now consider a system in which all of the interactions can be described by forces doing work, from the principle of conservation of energy we can write

$$W_{net}=\Delta K. \tag{10.2.9}$$

In words, *the work done by the net force acting on a particle as it moves equals the change in the particle's kinetic energy in the course of its displacement*. This result is often referred to as *the Work-Energy Theorem*.

As you may have guessed from our calling it a "theorem," the result (10.2.9) is very general. It holds in three dimensions, and it holds also when the force isn't constant throughout the displacement— you just have to use the correct equation to calculate the work in those cases. It would apply to the work done by the net force on an extended object, also, provided it is OK to treat the extended object as a particle—so basically, a rigid object that is moving as a whole and not doing anything fancy such as spinning while doing so.

Another possible direction in which to generalize (10.2.9) might be as follows. By definition, a "particle" has *no* other kind of energy, besides (translational) kinetic energy. Also, and for the same reason (namely, the absence of internal structure), it has no "internal" forces—all the forces acting on it are external. So—for this very simple system—we could rephrase the result (10.2.9) by saying that the work done by the net *external* force acting on the system (the particle in this case) is equal to the change in its *total* energy. It is in fact in this form that we will ultimately generalize (10.2.9) to deal with arbitrary systems.

Before we go there, however, I would like to take a little detour to explore another "reasonable" extension of the result (10.2.9), as well as its limitations.

10.3: The "Center of Mass Work"

We should be a little bit careful when using work in the form of Equation (10.2.1). It works perfectly fine if the particle is either *pointlike or rigid*. In those two cases, following the principle of "the motion of the object is the same as the motion of the center of mass" from Section 4.3, we can consider the entire formula to apply to the center of mass, like:

$$F_{\text{ext,net}} \Delta x_{cm} = \Delta K_{cm} \tag{10.3.1}$$

where K_{cm}, the translational kinetic energy, is, as usual, $K_{cm} = \frac{1}{2} M v_{cm}^2$, and Δx_{cm} is the displacement of the center of mass. We are also assuming the force is constant in this case. The result (10.3.1) holds for an arbitrary system, as long as $F_{ext,net}$ is constant, and can be generalized by means of an integral (as in Equation (10.2.6)) when it is variable.

So it seems that we could define the left-hand side of Equation (10.3.1) as "the work done on the center of mass," and take that as the natural generalization to a system of the result (10.2.8) for a particle. In fact, it's a bit more subtle than that, so we want to discuss that subtly in this section.

First, it seems that it is essential to the notion of work that one should multiply the force by the displacement *of the object on which it is acting*. More precisely, in the definition (10.2.1), we want the displacement *of the point of application of the force*[1]. But there are many examples of systems where there is nothing at the precise location of the center of mass, and certainly no force acting precisely there.

This is not necessarily a problem in the case of a rigid object which is not doing anything funny, just moving as a whole so that every part has the same displacement, because then the displacement of the center of mass would simply stand for the displacement of any point at which an external force might actually be applied. But for many *deformable* systems, this would not be case. In fact, for such systems one can usually show that $F_{ext,net} \Delta x_{cm}$ is actually *not* the work done on the system by the net external force. A simple example of such a system is shown below, in Figure 10.3.1.

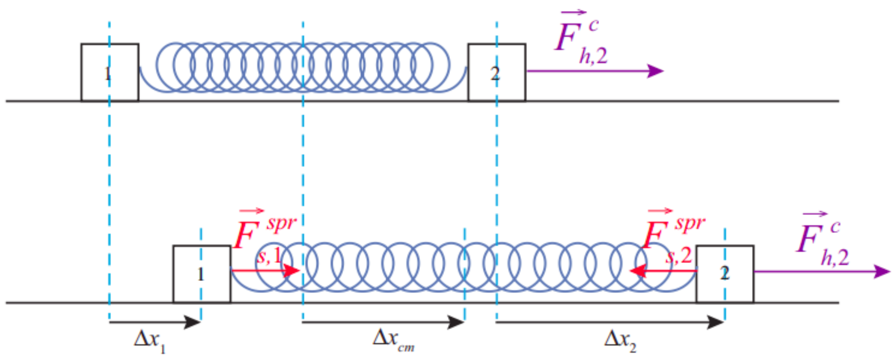


Figure 10.3.1: A system of two blocks connected by a spring. A constant external force, $\vec{F}^{\,c}_{h,2}$, is applied to the block on the right. Initially the spring is relaxed, but as soon as block 2 starts to move it stretches, pulling back on block 2 and pulling forward on block 1. Because of the stretching of the spring, the displacements Δx_1, Δx_{cm} and Δx_2 are all different, and the work done by the external force, $F^c_{h,2} \Delta x_2$, is different from the "center of mass work" $F^c_{h,2} \Delta x_{cm}$.

In this figure, the two blocks are connected by a spring, and the external force is applied to the block on the right (block 2). If the blocks have the same mass, the center of mass of the system is a point exactly halfway between them. If the spring starts in its relaxed state, it will stretch at first, so that the center of mass will lag behind block 2, and $F^c_{h,2} \Delta x_2$, which is the quantity that we should properly call the "work done by the net external force" will *not* be equal to $F^c_{h,2} \Delta x_{cm}$.

The best way to understand what's happening here is to think about all the sources of energy in the system - if you pull on that block-spring system, you are certainly going to add energy to it, but think about what happens if you follow this picture for a while. Eventually, you will have both blocks moving together, but the spring will be oscillating in some kind of consistent manner, therefore storing energy in the form of $\frac{1}{2} k x^2$. So while Equation (10.3.1) might not be literally true, it's reasonable to think we could write a variation of it that looks like

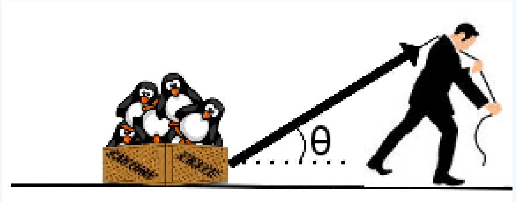
$$\text{Work done by external forces} = \text{Change of internal energy of the system.} \tag{10.3.2}$$

In fact, we will study conservation of energy in these cases in Chapter N8. For the moment, if we just assume our systems are rigid, with no internal energy, we can freely use Equation (10.3.1).

[1]As the name implies, this is the precise point at which the force is applied. For contact forces (other than friction; see later), this is easily identified. For gravity, a sum over all the forces exerted on all the particles that make up the object may be shown to be equivalent to a single resultant force acting at a point called the *center of gravity*, which, for our purposes (objects in uniform or near-uniform gravitational fields) will be the same as the center of mass.

10.4: Examples

✓ Whiteboard Problem 10.4.1: Box 'O Penguins



I am dragging a box full of 9 Emperor penguins along the ground as shown in the figure. Each of the penguins has a mass of 33.0 kg, and you can assume the box has zero mass. The rope which I am pulling the box with is making an angle of 22° with respect to the ground, and I pull the penguins a total distance of 15 m.

1. If the tension in the rope is 50 N, how much work did I do on the box during this process?
2. If the force due to friction between the box and ground is 25 N, how much work did friction do on the box during this process?
3. How much work did gravity do on the penguins in the box during this process?
4. Assuming the box of penguins starts from rest, how fast is the box moving after I pull it 15 m?

✓ Example 10.4.2: Braking

Suppose you are riding your bicycle and hit the brakes to come to a stop. Assuming no slippage between the tire and the road:

a. Which force is responsible for removing your momentum? (By "you" I mean throughout "you and the bicycle.")
b. Which force is responsible for removing your kinetic energy?

Solution

(a) According to what we saw in previous chapters, for example, Equation (2.3.1)

$$\frac{\Delta p_{\text{sys}}}{\Delta t} = F_{\text{ext,net}} \tag{10.4.1}$$

the total momentum of the system can only be changed by the action of an external force, and the only available external force is the force of friction between the tire and the road. So it is this force that removes the forward momentum from the system. The stopping distance, Δx_{cm}, and the force, can be related using Equation (10.3.1):

$$F^s_{r,t}\Delta x_{cm} = \Delta K_{cm}. \tag{10.4.2}$$

(b) Now, here is an interesting fact: the force of friction, although fully responsible for stopping your center of mass motion *does no work in this case*. That is because the point where it is applied—the point of the tire that is momentarily in contact with the road—is also momentarily at rest relative to the road: it is, precisely, *not slipping* (This in fact means the kind of friction here is *static* friction), so Δx in the equation $W = F\Delta x$ is zero. By the time that bit of the tire has moved on, so you actually have a nonzero Δx, you no longer have an F: the force of static friction is no longer acting on that bit of the tire, it is acting on a different bit—on which it will, again, do no work, for the same reason.

So, as you bring your bicycle to a halt the work $W_{ext,sys}$ = 0, and it follows that the total energy of your system is, in fact, conserved: *all* your initial kinetic energy is converted to thermal energy by the brake pad rubbing on the wheel.

CHAPTER OVERVIEW

11: C11) Rotational Energy

In this chapter we are going to continue our study of conservation laws by moving into a new kind of energy conservation - rotational energy. Just like we first studied conservation of linear momentum ($\Delta\vec{p}=0$) before moving onto rotational momentum ($\Delta\vec{L}=0$), we are going to move from linear kinetic energy ($1/2mv^2$) to learn how to include rotational energy into our energy conservation law, $\Delta E=0$. It turns out this is going to be easy, because rotational motion is just another kind of kinetic motion, so rotational energy is just another kind of kinetic energy.

First we should quickly recall what we already know about rotational motion (Chapter 6 and Chapter 7). A rotating object has an angular velocity $\vec{\omega}$, and the "rotational inertia" of the object is the moment of inertia, I. Moments of inertia are different for each object, but generally look like $I=\alpha mr^2$, where α is 1 for a point or a hollow cylinder, 1/2 for a disk, etc. We might not easily remember this coefficient, but we can always get a sense of the moment of inertia by going back to the original definition of $I=\sum m_i r_i^2$, which tells us that "the more masses m_i that are closer to the axis, the smaller the moment of inertia will be".

So how do we use this for rotational energy? Using the correspondance principle between linear and rotational quantities (remember, that's how we got from $\vec{p}=m\vec{v}$ to $\vec{L}=I\vec{\omega}$), we get from linear ("center-of-mass") kinetic energy $K_{cm}=\frac{1}{2}mv^2$ to rotational kinetic energy,

$$K_{rot}=\frac{1}{2}I\omega^2. \tag{11.1}$$

Note that this equation satisfies a lot of the same conceptual framework that $\frac{1}{2}mv^2$ does - the larger moment of inertia or angular speed, the more rotational energy is being stored in the system. It's also worthwhile here to note what kinds of objects have large and small moments of inertia - objects with lots of mass near the axis of rotation are easy to rotate, and therefore have small moments of inertia, and objects with lots of mass far away from the axis of rotation are hard to rotate, and have large moments of inertia. So while you could store the same amount of energy in two different objects, the object with the smaller I will be spinning faster (have larger ω), to keep K_{rot} constant.

We are going to use this in the exact same way that we use other sources of energy - but unlike linear vs rotational momentum, which are separately conserved, we only have one conservation of energy law. So, if our system has rotational energy, we are just going to write:

$$\Delta E=E_f-E_i=(K_{cm,f}+K_{rot,f}+U_f)-(K_{cm,i}+K_{rot,i}+U_i)=0. \tag{11.2}$$

(Naturally, we can have more than one source of potential energy U_f and U_i as well.)

11.1: Rotational Kinetic Energy, and Moment of Inertia

If a particle of mass m is moving on a circle of radius R, with instantaneous speed v, then its kinetic energy is

$$K_{rot} = \frac{1}{2}mv^2 = \frac{1}{2}mR^2\omega^2 \tag{11.1.1}$$

using $|\vec{v}| = R|\omega|$, Equation (6.1.11). Note that, at this stage, there is no real reason for the subscript "rot": equation (11.1.1) is all of the particle's kinetic energy. The distinction will only become important later in the chapter, when we consider extended objects whose motion is a combination of translation (of the center of mass) and rotation (around the center of mass).

Now, consider the kinetic energy of an extended object that is rotating around some axis. We may treat the object as being made up of many "particles" (small parts) of masses m_1, m_2.... If the object is rigid, all the particles move together, in the sense that they all rotate through the same angle in the same time, which means they all have the same angular velocity. However, the particles that are farther away from the axis of rotation are actually moving faster—they have a larger v, according to Equation (6.1.11). So the expression for the total kinetic energy in terms of all the particles' speeds is complicated, but in terms of the (common) angular velocity is simple:

$$\begin{aligned} K_{rot} &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots \\ &= \frac{1}{2}\left(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\right)\omega^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned} \tag{11.1.2}$$

where r_1, r_2,... represent the distance of the 1st, 2nd... particle to the axis of rotation, and on the last line I have introduced the quantity

$$I = \sum_{\text{all particles}} mr^2 \tag{11.1.3}$$

which is usually called the *moment of inertia* of the object about the axis considered (we have seen this quantity already in Chapter C6.1). In general, the expression (11.1.3) is evaluated as an integral, which can be written symbolically as $I = \int r^2 dm$; the "mass element" dm can be expressed in terms of the local density as ρdV, where V is a volume element. The integral is a multidimensional integral that may require somewhat sophisticated calculus skills, so we will not be calculating any of these this semester; rather, we will rely on the tabulated values for I for objects of different, simple, shapes. For instance, for a homogeneous cylinder of total mass M and radius R, rotating around its central axis, $I = \frac{1}{2}MR^2$; for a hollow sphere rotating through an axis through its center, $I = \frac{2}{3}MR^2$, and so on (see Table 6.1.1 for a list of these moments of inertia).

As you can see, the expression (11.1.2) for the kinetic energy of a rotating body, $\frac{1}{2}I\omega^2$, parallels the expression $\frac{1}{2}mv^2$ for a moving particle, with the replacement of v by ω, and m by I. This suggests that I is some sort of measure of a solid object's rotational inertia, by which we mean the resistance it offers to being set into rotation about the axis being considered. When we study the rotational version of Newton's Second Law later (in Chapter N9), we will see clearly that this interpretation is correct.

It should be stressed that the moment of inertia depends, in general, not just on the shape and mass distribution of the object, but also on the axis of rotation (again, see Table 6.1.1). In general, the formula (11.1.3) shows that, the more mass you put farther away from the axis of rotation, the larger I will be. Thus, for instance, a thin rod of length l has a moment of inertia $I = \frac{1}{12}Ml^2$ when rotating around a perpendicular axis through its midpoint, whereas it has the larger $I = \frac{1}{3}Ml^2$ when rotating around a perpendicular axis through one of its endpoints.

11.2: Rolling Motion

When considering objects that can move both translationally (up, down, left, right, in, out!) *and* rotationally, we are immediately confronted with a problem: these two motions can be either completely independent, completely related to each other, or something even in-between. Take for example a baseball thrown by a pitcher. It follows a particular path from the mound to the plate (its translational motion), but it might also have some spin (it's rotational motion). Those seem pretty independent, but in fact spinning balls like that can swerve back in forth because of something called The Magnus Effect (check it out on Wikipedia). Of course, that's why pitchers make the ball do that, is that motion makes it more complicated for the batter to hit the ball. The problem is that coupling between the air and the ball is *weak* in some sense - most of the energy of the ball is used for it's forward motion, only a small amount is used to make the ball move back and forth through the air.

Naturally, we would like to understand those kinds of interesting situations, but they end up being rather complicated. We would like to start with something simpler - motion in which the translation and rotation are either completely independent, and can be solved completely separately, or they are easily related to each other. If they are independent, you can solve them based on what you know already! However, there is a case in which the motions are completely related to each other, and is very common, called **rolling without slipping**. In this case, the translational motion and the rotational motion are coupled, and the relevant velocities are related to each other:

$$|\vec{v}_{cm}| = R|\omega|. \tag{11.2.1}$$

Here R is the radius of the entire object. Notice how this expression relates to Equation (6.1.1) from a previous section - they look very similar, but they are actually saying two somewhat different things. Equation (6.1.1) told us the relationship between the angular and linear speed of a particular point on an object, while the expression above tells us how to related the angular and linear speed *of the entire object* (remember that the center of mass is the location at which the object is, if the object was a single point).

The origin of the condition (11.2.1) is fairly straightforward. You can imagine an object that is rolling without slipping as "measuring the surface" as it rolls (or vice-versa, the surface measuring the circumference of the object as its different points are pressed against it in succession). So, after it has completed exactly one revolution (2π radians), it should have literally "covered" a distance on the surface equal to $2\pi R$, that is, advanced a distance $2\pi R$. But the same has to be true, proportionately, for any rotation angle $\Delta\theta$ other that 2π: since the length of the corresponding arc is $s = R|\Delta\theta|$, in a rotation over an angle $|\Delta\theta|$ the center of mass of the object must have advanced a distance $|\Delta x_{cm}| = s = R|\Delta\theta|$. Dividing by Δt as $\Delta t \to 0$ then yields Equation (11.2.1).

Figure 11.2.1: Left: illustrating the rolling without slipping condition. The cyan line on the surface has the same length as the cyan-colored arc, and will be the distance traveled by the disk when it has turned through an angle θ. Right: velocities for four points on the edge of the disk. The pink arrows are the velocities in the center of mass frame. In the Earth reference frame, the velocity of the center of mass, $\vec{v}_{cm}$, in green, has to be added to each of them. The resultant is shown in blue for two of them.

Considering a single rolling objects, the total kinetic energy can now be written as two terms,

$$K = K_{rot} + K_{cm} = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}^2 \tag{11.2.2}$$

combining this with the condition of rolling without slipping (11.2.1), we see that the ratio of the translational to the rotational kinetic energy is

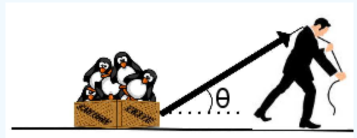
$$\frac{K_{cm}}{K_{rot}} = \frac{mv_{cm}^2}{I\omega^2} = \frac{mR^2}{I}. \tag{11.2.3}$$

The amount of energy available to interact with the object is whatever potential energy is running around the system, and that has to be split between translational and rotational in the proportion (11.2.3). An object with a proportionately larger I is one that, for a given angular velocity, needs more rotational kinetic energy, because more of its mass is away from the rotation axis. This leaves less energy available for its translational motion.

11.3: Examples

As you work through these examples, make sure to refer to the list of moments of inertia for different shaped objects, Table 6.1.1.

Whiteboard Problem 11.3.1: Box O' Penguins



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4. Assuming the box of penguins starts from rest, how fast is the box moving after I pull it 15 m?

Example 11.3.2: Moment of Inertia of a system of particles

Six small washers are spaced 10 cm apart on a rod of negligible mass and 0.5 m in length. The mass of each washer is 20 g. The rod rotates about an axis located at 25 cm, as shown in Figure $11.3.3$. (a) What is the moment of inertia of the system? (b) If the two washers closest to the axis are removed, what is the moment of inertia of the remaining four washers? (c) If the system with six washers rotates at 5 rev/s, what is its rotational kinetic energy?

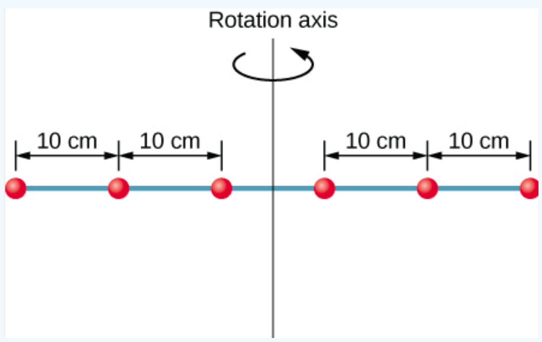

Figure $11.3.3$: Six washers are spaced 10 cm apart on a rod of negligible mass and rotating about a vertical axis.

Strategy

a. We use the definition for moment of inertia for a system of particles and perform the summation to evaluate this quantity. The masses are all the same so we can pull that quantity in front of the summation symbol.
b. We do a similar calculation.
c. We insert the result from (a) into the expression for rotational kinetic energy.

Solution

a. $I = \sum m_j r_j^2 = (0.02 \text{ kg}) \left(2 \times (0.25 \text{ m})^2 + 2 \times (0.15 \text{ m})^2 + 2 \times (0.05 \text{ m})^2\right) = 0.0035 \text{ kg} \cdot \text{m}^2$
b. $I = \sum_j m_j r_j^2 = (0.02 \text{ kg}) \left(2 \times (0.25 \text{ m})^2 + 2 \times (0.15 \text{ m})^2\right) = 0.0034 \text{ kg} \cdot \text{m}^2$
c. $K = \frac{1}{2} I \omega^2 = \frac{1}{2}\left(0.0035 \text{ kg} \cdot \text{m}^2\right)(5.0 \times 2\pi \text{ rad/s})^2 = 1.73 \text{ J}$

Significance

We can see the individual contributions to the moment of inertia. The masses close to the axis of rotation have a very small contribution. When we removed them, it had a very small effect on the moment of inertia.

Applying Rotational Kinetic Energy

Now let's apply the ideas of rotational kinetic energy and the moment of inertia table to get a feeling for the energy associated with a few rotating objects. The following examples will also help get you comfortable using these equations. First, let's look at a general problem-solving strategy for rotational energy.

? PROBLEM-SOLVING STRATEGY: ROTATIONAL ENERGY

1. Determine that energy or work is involved in the rotation.
2. Determine the system of interest. A sketch usually helps.
3. Analyze the situation to determine the types of work and energy involved.
4. If there are no losses of energy due to friction and other nonconservative forces, mechanical energy is conserved, that is, $K_i + U_i = K_f + U_f$.
5. If nonconservative forces are present, mechanical energy is not conserved, and other forms of energy, such as heat and light, may enter or leave the system. Determine what they are and calculate them as necessary.
6. Eliminate terms wherever possible to simplify the algebra.
7. Evaluate the numerical solution to see if it makes sense in the physical situation presented in the wording of the problem.

Example 11.3.3: Calculating helicopter energies

A typical small rescue helicopter has four blades: Each is 4.00 m long and has a mass of 50.0 kg (Figure $11.3.5$). The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades.

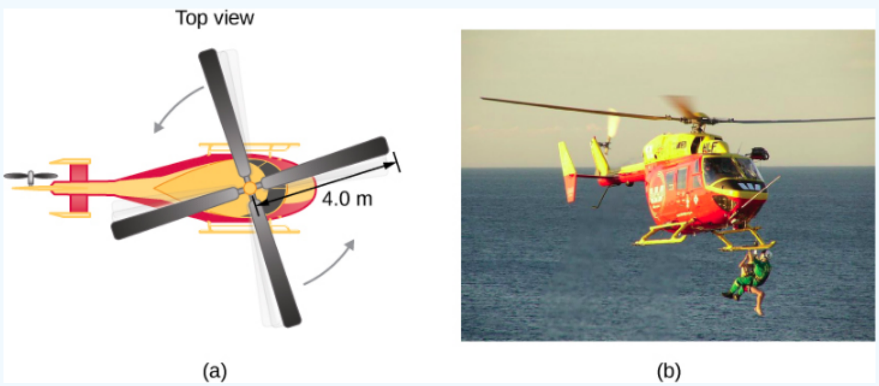

Figure 11.3.5: (a) Sketch of a four-blade helicopter. (b) A water rescue operation featuring a helicopter from the Auckland Westpac Rescue Helicopter Service. (credit b: modification of work by "111 Emergency"/Flickr)

Strategy

Rotational and translational kinetic energies can be calculated from their definitions. The wording of the problem gives all the necessary constants to evaluate the expressions for the rotational and translational kinetic energies.

Solution

a. The rotational kinetic energy is

$$K = \frac{1}{2} I \omega^2$$

We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find K. The angular velocity ω is

$$\omega = \frac{300 \text{ rev}}{1.00 \text{ min}} \frac{2\pi \text{ rad}}{1 \text{ rev}} \frac{1.00 \text{ min}}{60.0 \text{ s}} = 31.4 \frac{\text{rad}}{\text{s}}.$$

The moment of inertia of one blade is that of a thin rod rotated about its end, listed in Figure 11.3.4. The total I is four times this moment of inertia because there are four blades. Thus,

$$I = 4\frac{Ml^2}{3} = 4 \times \frac{(50.0\ \text{kg})(4.00\ \text{m})^2}{3} = 1067.0\ \text{kg} \cdot \text{m}^2.$$

Entering ω and I into the expression for rotational kinetic energy gives

$$K = 0.5\left(1067\ \text{kg} \cdot \text{m}^2\right)(31.4\ \text{rad/s})^2 = 5.26 \times 10^5\ \text{J}.$$

b. Entering the given values into the equation for translational kinetic energy, we obtain

$$K = \frac{1}{2}mv^2 = (0.5)(1000.0\ \text{kg})(20.0\ \text{m/s})^2 = 2.00 \times 10^5\ \text{J}.$$

To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is

$$\frac{2.00 \times 10^5\ \text{J}}{5.26 \times 10^5\ \text{J}} = 0.380.$$

Significance

The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades.

Example 11.3.4: Energy in a boomerang

A person hurls a boomerang into the air with a velocity of 30.0 m/s at an angle of 40.0° with respect to the horizontal (Figure 11.3.6). It has a mass of 1.0 kg and is rotating at 10.0 rev/s. The moment of inertia of the boomerang is given as $I = \frac{1}{12}mL^2$ where L = 0.7 m. (a) What is the total energy of the boomerang when it leaves the hand? (b) How high does the boomerang go from the elevation of the hand, neglecting air resistance?

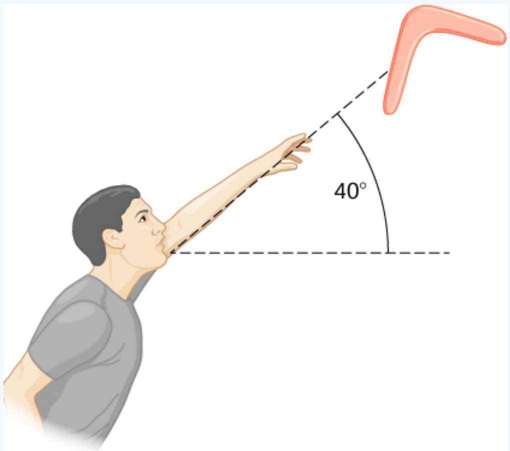

Figure 11.3.6: A boomerang is hurled into the air at an initial angle of 40°.

Strategy

We use the definitions of rotational and linear kinetic energy to find the total energy of the system. The problem states to neglect air resistance, so we don't have to worry about energy loss. In part (b), we use conservation of mechanical energy to find the maximum height of the boomerang.

Solution

a. Moment of inertia: $I = \frac{1}{12}mL^2 = \frac{1}{12}(1.0\ \text{kg})(0.7\ \text{m})^2 = 0.041\ \text{kg} \cdot \text{m}^2$.

Angular Velocity: $\omega = (10.0\ \text{rev/s})(2\pi) = 62.83\ \text{rad/s}$

The rotational kinetic energy is therefore

$$K_R = \frac{1}{2}\left(0.041 \text{ kg} \cdot \text{m}^2\right)(62.83 \text{ rad/s})^2 = 80.93 \text{ J}$$

The translational kinetic energy is

$$K_{\text{T}} = \frac{1}{2}mv^2 = \frac{1}{2}(1.0 \text{ kg})(30.0 \text{ m/s})^2 = 450.0 \text{ J}$$

Thus, the total energy in the boomerang is

$$K_{\text{Total}} = K_R + K_T = 80.93 + 450.0 = 530.93 \text{ J}.$$

b. We use conservation of mechanical energy. Since the boomerang is launched at an angle, we need to write the total energies of the system in terms of its linear kinetic energies using the velocity in the *x*- and *y*-directions. The total energy when the boomerang leaves the hand is

$$E_{\text{Before}} = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}I\omega^2$$

The total energy at maximum height is

$$E_{\text{Final}} = \frac{1}{2}mv_x^2 + \frac{1}{2}I\omega^2 + mgh$$

By conservation of mechanical energy, $E_{Before} = E_{Final}$ so we have, after canceling like terms,

$$\frac{1}{2}mv_y^2 = mgh.$$

Since v_y = 30.0 m/s ($\sin 40^\circ$) = 19.28 m/s, we find

$$h = \frac{(19.28 \text{ m/s})^2}{2\left(9.8 \text{ m/s}^2\right)} = 18.97 \text{ m}$$

Significance

In part (b), the solution demonstrates how energy conservation is an alternative method to solve a problem that normally would be solved using kinematics. In the absence of air resistance, the rotational kinetic energy was not a factor in the solution for the maximum height.

? Exercise 11.3.5

A nuclear submarine propeller has a moment of inertia of 800.0 kg • m^2. If the submerged propeller has a rotation rate of 4.0 rev/s when the engine is cut, what is the rotation rate of the propeller after 5.0 s when water resistance has taken 50,000 J out of the system?

CHAPTER OVERVIEW

12: C12) Collisions

In this chapter we are going to drill down a bit more carefully on the topic of collisions. We've already studied collisions in some detail when we first introduced momentum transfer (Chapter 2) as well as "full" conservation of momentum (Chapter 5), but now we are going to make the process of collisions more realistic by looking at conditions under which both momentum and energy are conserved.

The very first thing we want to do is to clarify what we mean when we say things like "energy is not being conserved in this interaction". If we consider our entire Universe as the system, **energy and momentum are both conserved in every single interaction in that system**. So when we state that a particular interaction is "not conserving energy", we are basically admitting that we didn't pick the system "correctly". Sometimes we don't have the knowledge to do so, or sometimes it's actually simpler to consider a system which transfers energy to the outside world, but we should be aware it's a problem with the choice of system, not about the interaction or phenomena itself. An immediate example of this is friction - we often say "energy is not conserved in systems that contain friction", but that's incorrect - the energy in friction is being converted from motion to heat, and we don't always keep track of that because it's often too difficult, particularly when our focus is on mechanics.

With that clarification, we can start to classify collisions depending on if they conserve energy or momentum - one, both, or neither. The first thing to say outright is that *momentum is conserved in all the collisions we are going to consider*. The reason is actually pretty simple - we've only studied two possible sources of momentum, either motion ($\vec{p} = m\vec{v}$) or impulse ($\Delta\vec{p} = \vec{F}\Delta t$). We can remove impulse by saying "we are only going to consider collisions in which there is no external forces acting on the system". So since we've gotten rid of impulse, there can only be momentum in the form of motion - we can't "hide" momentum anywhere except in all the mvs running around the system. Since momentum is only in the form of mvs, we can always track it, and thus ensure that it is conserved.

The situation is different for energy (which is why we talked about it in the friction example). Energy comes in many different forms - kinetic (linear and rotational), potential (gravity, springs, and a myriad of other interactions we haven't talked about), and work ($W = \vec{F} \cdot \Delta\vec{r}$). If we are throwing out external forces (like we did above), we can throw out work, but energy can be stored in many other places. Thermal (heat) energy is typically the best example of something we can't track in mechanics, but the world of physics is full of other interactions that store energy without needing motion[1], and any of those interactions could be a source of the missing energy.

It turns out that the situation in the preceding paragraph is the most common situation - it's typically hard / impossible to track all the sources of energy in our systems, so all we have to go on is conservation of momentum; collisions involving those kinds of systems are called **inelastic**. However, there are some very special kinds of collisions in which we actually can track all the energy (in the form of motion, $K = \frac{1}{2}mv^2$), and so energy is **also** conserved, and these are called **elastic** collisions. Examples of elastic collisions are bouncing superballs, collisions of billard balls, or collisions involving springs.

On one hand, you are going to have to memorize the names of these two collisions. On the other hand, we can get a good physical picture for each of them - for example, a car crash is a good example of an inelastic collision. There's "no bouncing", and energy is clearly being "lost" in the form of sound and mechanical deformation. On the other hand, if a cart runs into a spring and shoots backwards, it's "very bouncy", and not at all clear where lost energy could be stored - the kinetic energy went into the spring, and then back into the cart! So you need to keep these two kinds of collisions separate in your head, but it's also important to remember that **most collisions are inelastic, only very special collisions are elastic**.

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[1]In physics 2, you will study electromagnetism, and learn about energy storage in electric and magnetic fields. That covers a lot of interactions in the world, but there are still nuclear interactions (the strong and weak force) that can store energy and evade even those considerations.

12.1: Types of Collisions

Although momentum is conserved in all interactions, not all interactions (collisions or explosions) are the same. The possibilities include:

- A single object can explode into multiple objects (explosions).
- Multiple objects can collide and stick together, forming a single object (inelastic).
- Multiple objects can collide and bounce off of each other, remaining as multiple objects (elastic). If they do bounce off each other, then they may recoil at the same speeds with which they approached each other before the collision, or they may move off more slowly.

It's useful, therefore, to categorize different types of interactions, according to how the interacting objects move before and after the interaction.

Explosions

The first possibility is that a single object may break apart into two or more pieces. An example of this is a firecracker, or a bow and arrow, or a rocket rising through the air toward space. These can be difficult to analyze if the number of fragments after the collision is more than about three or four; but nevertheless, the total momentum of the system before and after the explosion is identical.

Note that if the object is initially motionless, then the system (which is just the object) has no momentum and no kinetic energy. After the explosion, the net momentum of all the pieces of the object must sum to zero (since the momentum of this closed system cannot change). However, the system **will** have a great deal of kinetic energy after the explosion, although it had none before. Thus, we see that, although the momentum of the system is conserved in an explosion, the kinetic energy of the system most definitely is not; it increases. This interaction—one object becoming many, with an increase of kinetic energy of the system—is called an **explosion**.

Where does the energy come from? Does conservation of energy still hold? Yes; some form of potential energy is converted to kinetic energy. In the case of gunpowder burning and pushing out a bullet, chemical potential energy is converted to kinetic energy of the bullet, and of the recoiling gun. For a bow and arrow, it is elastic potential energy in the bowstring.

Inelastic

The second possibility is the reverse: that two or more objects collide with each other and stick together, thus (after the collision) forming one single composite object. The total mass of this composite object is the sum of the masses of the original objects, and the new single object moves with a velocity dictated by the conservation of momentum. However, it turns out again that, although the total momentum of the system of objects remains constant, the kinetic energy doesn't; but this time, the kinetic energy decreases. This type of collision is called **inelastic**.

Any collision where the objects stick together will result in the maximum loss of kinetic energy (i.e., K_f will be a minimum). Such a collision is said to be **perfectly inelastic**. In the extreme case, multiple objects collide, stick together, and remain motionless after the collision. Since the objects are all motionless after the collision, the final kinetic energy is also zero; therefore, the loss of kinetic energy is a maximum.

- If $0 < K_f < K_i$, the collision is inelastic.
- If K_f is the lowest energy, or the energy lost by both objects is the most, the collision is perfectly inelastic (objects stick together).
- If $K_f = K_i$, the collision is elastic.

Elastic

The extreme case on the other end is if two or more objects approach each other, collide, and bounce off each other, moving away from each other at the same relative speed at which they approached each other. In this case, the total kinetic energy of the system is conserved. Such an interaction is called **elastic**.

In any interaction of a closed system of objects, the total momentum of the system is conserved ($\vec{p}_f = \vec{p}_i$) but the kinetic energy may not be:

- If $0 < K_f < K_i$, the collision is inelastic.

- If $K_f = 0$, the collision is perfectly inelastic.
- If $K_f = K_i$, the collision is elastic.
- If $K_f > K_i$, the interaction is an explosion.

The point of all this is that, in analyzing a collision or explosion, you can use both momentum and kinetic energy.

Dimensions and Equation Counting

It's worth pointing out how many equations and unknown variables we are dealing with when it comes to collision problems, because it is quite predictable and can give us some insight into how hard a particular problem might be before we get started on it. As discussed above, momentum is conserved in every collision, so

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = 0. \tag{12.1.1}$$

Since this is a vector equation, it actually contains a *number of linear independent equations equal to the dimension of the problem* (typically 1 or 2 for us, but generally 3). Since these linear equations can only be solved if there are an equal number of unknown variables and equations, we can only solve problems that have the same number of unknowns as dimensions (for example, a 1D problem can only ask one question - "what is the final velocity?" or "what was the mass of the first object?" - never "what was the final velocity AND the mass of the first object?"). This is the complete story for inelastic collisions - the number of unknowns has to match the dimension.

For elastic collisions, we have one more relationship, conservation of energy:

$$\Delta E = E_f - E_i = 0. \tag{12.1.2}$$

This is a scalar equation, and represents one further constraint on our system. However, that extra relationship means we can leave one further quantity unspecified - it is no longer free to be set, but must satisfy the extra equation from conservation of energy. This makes elastic collisions generally more complicated than inelastic problems, because we have an extra equation and unknown to deal with. To take the example from above, in 1D we have now two equations that govern our collision:

$$p_{f,x} - p_{i,x} = 0, \qquad E_f - E_i = 0. \tag{12.1.3}$$

So we can have two unknowns - the question "what was the final velocity AND the mass of the first object?" actually is well-posed and can be answered. We do this in the next example:

Example 12.1.1: Inelastic vs Elastic collision in 1D

We want this example to be as simple as possible - a cart of mass m moving with an initial speed v_0 towards a cart of mass $3m$, which is initially stationary. They collide, and we want to consider two possible situations:

1. If the collision was inelastic, what was the final speed of the first cart? Here, we will assume the second cart moves off with a speed of $v_0/4$.

2. If the collision was elastic, what was the final speeds of *both carts*?

Solution

1. In the inelastic case, just momentum is conserved, so we have a fairly simple conservation of momentum problem:

$$\Delta p_x = p_{x,f} - p_{x,i} = 0 \rightarrow (mv_{1,f} + 3m\frac{v_0}{4}) - (mv_0) = 0. \tag{12.1.4}$$

The first step to solving this is recognizing that there is an m in every term, so we can divide by that. Physically, that means *the mass does not contribute to the physics at all* - the solution will be the same no matter what the mass is. Solving this for the final velocity gets us

$$v_{1,f} = \frac{v_0}{4}. \tag{12.1.5}$$

So, the first cart moves at one quarter the speed, no matter what it's initial mass is. *(Note that although the mass does not matter, the relative sizes of the masses do matter. If their ratio was anything besides $m_2/m_1 = 3$, we would get a different answer here. The same goes for the speeds - if we picked a final speed of something besides $v_0/4$, we would get a different final answer.)*

2. For the elastic case, we have the exact same conservation of momentum equation (now with the speed of the second cart not yet known!)

$$\Delta p_x = p_{x,f} - p_{x,i} = 0 \rightarrow (mv_{1,f} + 3mv_{2,f}) - (mv_0) = 0. \tag{12.1.6}$$

Further, we have the following conservation of energy equation,

$$\Delta E = E_f - E_i = 0 \rightarrow \left(\frac{1}{2}mv_{1,f}^2 + \frac{1}{2}(3m)v_{2,f}^2\right) - \frac{1}{2}mv_0^2 = 0. \tag{12.1.7}$$

These two expressions must be solving simultaneously, since we do not know what $v_{2,f}$ is! The first step is to eliminate the 1/2 and the m from the conservation of energy equation, and the mass from the momentum equation:

$$v_{1,f}^2 + 3v_{2,f}^2 - v_0^2 = 0 \qquad (v_{1,f} + 3v_{2,f}) - v_0 = 0 \tag{12.1.8}$$

We can proceed in several different ways - probably the easiest is to solve the momentum equation for the first speed:

$$v_{1,f} = v_0 - 3v_{2,f}, \tag{12.1.9}$$

and plug it into the energy equation:

$$(v_0 - 3v_{2,f})^2 + 3v_{2,f}^2 - v_0^2 = 0 \rightarrow v_0^2 - 6v_0v_{2,f} + 9v_{2,f}^2 + 3v_{2,f}^2 - v_0^2 = 0 \rightarrow 12v_{2,f}^2 - 6v_0v_{2,f} = 0. \tag{12.1.10}$$

Here we notice that we can divide by 6, as well as $v_{2,f}$, and find $v_{2,f} = v_0/2$. The final speed is different, but notice we have less freedom to pick the initial conditions - we can't choose how fast the second cart moves after the collision, it's always $v_0/2$.

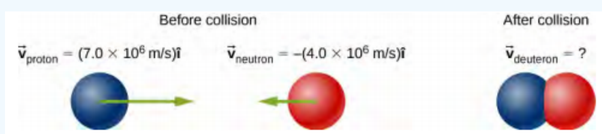
We still need to solve for the first cart - we can do that by going back to the solution for it's speed and plugging in our solution for the second:

$$v_{1,f} = v_0 - 3\frac{v_0}{2} = -\frac{v_0}{2}. \tag{12.1.11}$$

So in this case, the first cart bounced backwards, and moved at half the original speed.

✓ Example 12.1.2: Formation of a deuteron

A proton (mass 1.67 x 10^{-27} kg) collides with a neutron (with essentially the same mass as the proton) to form a particle called a deuteron. What is the velocity of the deuteron if it is formed from a proton moving with velocity 7.0 x 10^6 m/s to the left and a neutron moving with velocity 4.0 x 10^6 m/s to the right?



Strategy

Define the system to be the two particles. This is a collision, so we should first identify what kind. Since we are told the two particles form a single particle after the collision, this means that the collision is perfectly inelastic. Thus, kinetic energy is not conserved, but momentum is. Thus, we use conservation of momentum to determine the final velocity of the system.

Solution

Treat the two particles as having identical masses M. Use the subscripts p, n, and d for proton, neutron, and deuteron, respectively. This is a one-dimensional problem, so we have

$$Mv_p - Mv_n = 2Mv_d. \tag{12.1.12}$$

The masses divide out:

$$\begin{aligned} v_p - v_n &= 2v_d \\ (7.0 \times 10^6\ m/s) - (4.0 \times 10^6\ m/s) &= 2v_d \\ v_d &= 1.5 \times 10^6\ m/s. \end{aligned}$$

The velocity is thus $\vec{v}_d = (1.5 \times 10^6\ m/s)\hat{i}$.

Significance

This is essentially how particle colliders like the Large Hadron Collider work: They accelerate particles up to very high speeds (large momenta), but in opposite directions. This maximizes the creation of so-called “daughter particles.”

Example 12.1.3: Ice hockey 2

(This is a variation of an earlier example.)

Two ice hockey pucks of different masses are on a flat, horizontal hockey rink. The red puck has a mass of 15 grams, and is motionless; the blue puck has a mass of 12 grams, and is moving at 2.5 m/s to the left. It collides with the motionless red puck (Figure 12.1.1). If the collision is perfectly elastic, what are the final velocities of the two pucks?

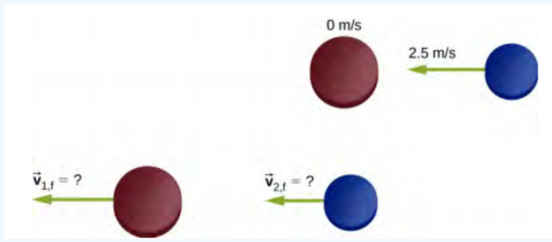


Figure 12.1.1: Two different hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We’re told that we have two colliding objects, and we’re told their masses and initial velocities, and one final velocity; we’re asked for both final velocities. Conservation of momentum seems like a good strategy; define the system to be the two pucks. There is no friction, so we have a closed system. We have two unknowns (the two final velocities), but only one equation. The comment about the collision being perfectly elastic is the clue; it suggests that kinetic energy is also conserved in this collision. That gives us our second equation.

The initial momentum and initial kinetic energy of the system resides entirely and only in the second puck (the blue one); the collision transfers some of this momentum and energy to the first puck.

Solution

Conservation of momentum, in this case, reads

$$\begin{aligned} p_i &= p_f \\ m_2 v_{2,i} &= m_1 v_{1,f} + m_2 v_{2,f}. \end{aligned}$$

Conservation of kinetic energy reads

$$\begin{aligned} K_i &= K_f \\ \frac{1}{2} m_2 v_{2,i}^2 &= \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2. \end{aligned}$$

There are our two equations in two unknowns. The algebra is tedious but not terribly difficult; you definitely should work it through. The solution is

$$v_{1,f} = \frac{(m_1 - m_2) v_{1,i} + 2 m_2 v_{2,i}}{m_1 + m_2} \tag{12.1.13}$$

$$v_{2,f} = \frac{(m_2 - m_1) v_{2,i} + 2 m_1 v_{1,i}}{m_1 + m_2} \tag{12.1.14}$$

Substituting the given numbers, we obtain

$$v_{1,f} = 2.22\ m/s \tag{12.1.15}$$

$$v_{2,f} = -0.28\ m/s. \tag{12.1.16}$$

Significance

Notice that after the collision, the blue puck is moving to the right; its direction of motion was reversed. The red puck is now moving to the left.

? Exercise 12.1.4

There is a second solution to the system of equations solved in this example (because the energy equation is quadratic): $v_{1,f}$ = −2.5 m/s, $v_{2,f}$ = 0. This solution is unacceptable on physical grounds; what's wrong with it?

✔ Example 12.1.5: Thor vs. iron man

The 2012 movie "The Avengers" has a scene where Iron Man and Thor fight. At the beginning of the fight, Thor throws his hammer at Iron Man, hitting him and throwing him slightly up into the air and against a small tree, which breaks. From the video, Iron Man is standing still when the hammer hits him. The distance between Thor and Iron Man is approximately 10 m, and the hammer takes about 1 s to reach Iron Man after Thor releases it. The tree is about 2 m behind Iron Man, which he hits in about 0.75 s. Also from the video, Iron Man's trajectory to the tree is very close to horizontal. Assuming Iron Man's total mass is 200 kg:

a. Estimate the mass of Thor's hammer
b. Estimate how much kinetic energy was lost in this collision

Strategy

After the collision, Thor's hammer is in contact with Iron Man for the entire time, so this is a perfectly inelastic collision. Thus, with the correct choice of a closed system, we expect momentum is conserved, but not kinetic energy. We use the given numbers to estimate the initial momentum, the initial kinetic energy, and the final kinetic energy. Because this is a one-dimensional problem, we can go directly to the scalar form of the equations.

Solution

a. First, we posit conservation of momentum. For that, we need a closed system. The choice here is the system (hammer + Iron Man), from the time of collision to the moment just before Iron Man and the hammer hit the tree. Let:
   - M_H = mass of the hammer
   - M_I = mass of Iron Man
   - v_H = velocity of the hammer before hitting Iron Man
   - v = combined velocity of Iron Man + hammer after the collision

Again, Iron Man's initial velocity was zero. Conservation of momentum here reads:

$$M_H v_H = (M_H + M_I)v. \tag{12.1.17}$$

We are asked to find the mass of the hammer, so we have

$$\begin{aligned} M_H v_H &= M_H v + M_1 v \\ M_H(v_H - v) &= M_I v \\ M_H &= \frac{M_I v}{v_H - v} \\ &= \frac{(200\ kg)\left(\frac{2\ m}{0.75\ s}\right)}{10\ m/s - \left(\frac{2\ m}{0.75\ s}\right)} \\ &= 73\ kg. \end{aligned}$$

Considering the uncertainties in our estimates, this should be expressed with just one significant figure; thus, $M_H = 7 \times 10^1$ kg.

b. The initial kinetic energy of the system, like the initial momentum, is all in the hammer: $

$$\begin{aligned} K_i &= \frac{1}{2} M_H v_H^2 \\ &= \frac{1}{2}(70\ kg)(10\ m/s)^2 \\ &= 3500\ J. \end{aligned}$$

$After the collision, $

$$\begin{aligned} K_f &= \frac{1}{2}(M_H + M_I)v^2 \\ &= \frac{1}{2}(70\ kg + 200\ kg)(2.67\ m/s)^2 \\ &= 960\ J. \end{aligned}$$

$Thus, there was a loss of 3500 J − 960 J = 2540 J.

Significance

From other scenes in the movie, Thor apparently can control the hammer's velocity with his mind. It is possible, therefore, that he mentally causes the hammer to maintain its initial velocity of 10 m/s while Iron Man is being driven backward toward the tree. If so, this would represent an external force on our system, so it would not be closed. Thor's mental control of his hammer is beyond the scope of this book, however.

? Exercise 12.1.6

Suppose there had been no friction (the collision happened on ice); that would make μ_k zero, and thus $v_{c,f} = \sqrt{2\mu_k g d} = 0$, which is obviously wrong. What is the mistake in this conclusion?

Subatomic Collisions and Momentum

Conservation of momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

At the beginning of the twentieth century, there was considerable interest in, and debate about, the structure of the atom. It was known that atoms contain two types of electrically charged particles: negatively charged electrons and positively charged protons. (The existence of an electrically neutral particle was suspected, but would not be confirmed until 1932.) The question was, how were these particles arranged in the atom? Were they distributed uniformly throughout the volume of the atom (as J.J. Thomson proposed), or arranged at the corners of regular polygons (which was Gilbert Lewis' model), or rings of negative charge that surround the positively charged nucleus—rather like the planetary rings surrounding Saturn (as suggested by Hantaro Nagaoka), or something else?

The New Zealand physicist Ernest Rutherford (along with the German physicist Hans Geiger and the British physicist Ernest Marsden) performed the crucial experiment in 1909. They bombarded a thin sheet of gold foil with a beam of high energy (that is, high-speed) alpha-particles (the nucleus of a helium atom). The alpha-particles collided with the gold atoms, and their subsequent velocities were detected and analyzed, using conservation of momentum and conservation of energy.

If the charges of the gold atoms were distributed uniformly (per Thomson), then the alpha-particles should collide with them and nearly all would be deflected through many angles, all small; the Nagaoka model would produce a similar result. If the atoms were arranged as regular polygons (Lewis), the alpha-particles would deflect at a relatively small number of angles.

What **actually** happened is that nearly **none** of the alpha-particles were deflected. Those that were, were deflected at large angles, some close to 180° —those alpha-particles reversed direction completely (Figure 12.1.2). None of the existing atomic models could explain this. Eventually, Rutherford developed a model of the atom that was much closer to what we now have—again, using conservation of momentum and energy as his starting point.

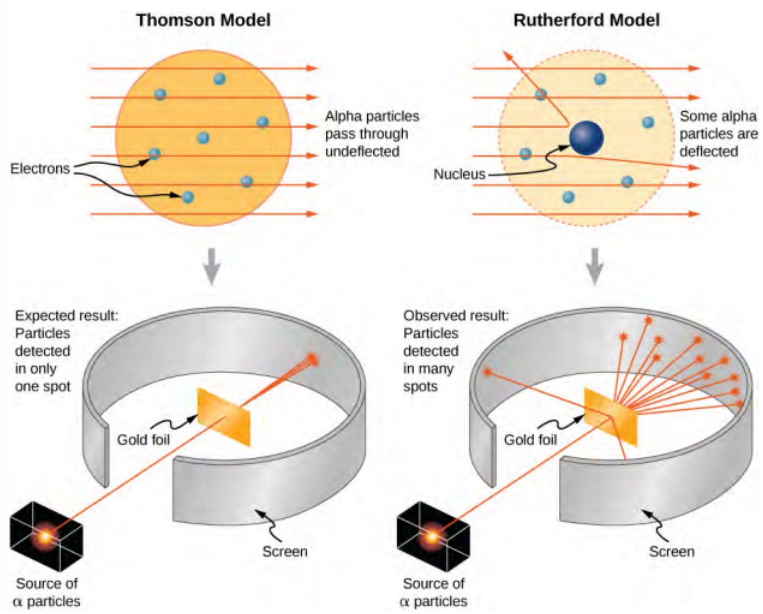
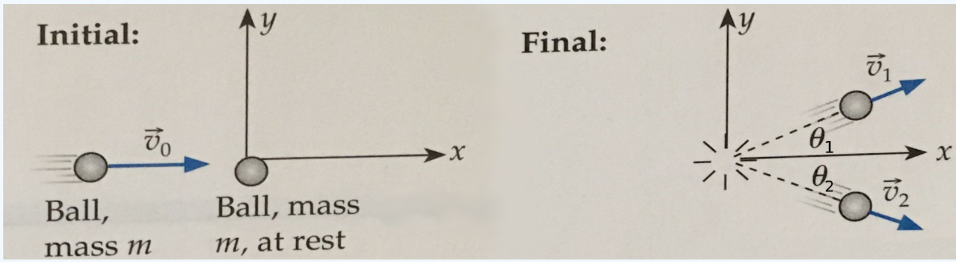


Figure 12.1.2: The Thomson and Rutherford models of the atom. The Thomson model predicted that nearly all of the incident alpha-particles would be scattered and at small angles. Rutherford and Geiger found that nearly none of the alpha particles were scattered, but those few that were deflected did so through very large angles. The results of Rutherford's experiments were inconsistent with the Thomson model. Rutherford used conservation of momentum and energy to develop a new, and better model of the atom—the nuclear model.

12.2: Examples

Whiteboard Problem 12.2.1: Croquet Collision



A Croquet player hits a wooden croquet ball, of mass $m = 0.450$ kg, at a speed of $v_0 = 3.5$ m/s, which collides, off-center, with a stationary ball of the same mass. After the collision, the first ball moves off with a speed of $v_1 = 3.0$ m/s at an angle of $\theta_1 = 10^\circ$ with respect to the x-axis, as shown in the figure. You may assume that this collision is *inelastic.*

1. At what angle, θ_2, relative to the x-axis does the second ball move away from the collision at?
2. What is the final speed of the second ball, v_2?
3. What was the change in energy of this collision? Can you explain your answer?

Whiteboard Problem 12.2.2: Pool Shark

A billiard ball moving at 3.06 m/s strikes a second billiard ball, of the same mass (150 g) and initially at rest, in a perfectly elastic collision. After the collision, the first ball moves away with a speed 2.65 m/s.

1. What is the initial kinetic energy of the system?
2. What is the final speed of the second ball?
3. If the first ball leaves the collision at an angle of 30° with respect to the original direction of motion, what angle does the second ball leave the collision at?

Whiteboard Problem 12.2.3: Red Light!

I am responsibly driving my 1360-kg Subaru Impreza through an intersection. I am traveling west, my light is green, so I proceed through at 45 mph. An irresponsible driver in a brand new Audi R8 (of mass 1678 kg) with Florida plates blows through the red light going north. He smashes into me, and our cars stick together after the collision, traveling at an angle of 65 $^\circ$ north of west.

1. Determine both (which can be done in either order),
   1. How fast was he traveling before he hit me?
   2. How fast are the two cars moving together after the collision?
2. What fraction of the total energy of the system was lost during this collision?

Whiteboard Problem 12.2.4: The Space Goo:

You are flying through empty space in your rocket, at 2000 m/s, far away from any stars, planets, or other massive bodies. However, you aren't paying very careful attention so you don't notice that there is a giant cube of ``Space-goo" on a collision course with you! It is moving at a speed of 500 m/s in a direction perpendicular to your own, and has a mass of 3000 kg.

Assuming the total mass of you and your personal rocket is 1500 kg, and you get stuck in the space-goo when you collide, what is the magnitude and direction of your final velocity after the collision?

Example 12.2.5: Collision Graph revisited

Look again at the collision graph from Example 2.4.1 from the point of view of the kinetic energy of the two carts.

a. What is the initial kinetic energy of the system?
b. How much of this is in the center of mass motion, and how much of is convertible?
c. Does the convertible kinetic energy go to zero at some point during the collision? If so, when? Is it fully recovered after the collision is over?
d. What kind of collision is this? (Elastic, inelastic, etc.) What is the coefficient of restitution?

Solution

(a) From the solution to Example 3.5.1 we know that

$$\begin{aligned} v_{1i} &= -1 \ \frac{\mathrm{m}}{\mathrm{s}} & v_{2i} &= 0.5 \ \frac{\mathrm{m}}{\mathrm{s}} \\ v_{1f} &= 1 \ \frac{\mathrm{m}}{\mathrm{s}} & v_{2f} &= -0.5 \ \frac{\mathrm{m}}{\mathrm{s}} \end{aligned}$$

and m_1 = 1 kg and m_2 = 2 kg. So the initial kinetic energy is

$$K_{sys,i} = \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = 0.5 \text{ J} + 0.25 \text{ J} = 0.75 \text{ J} \tag{12.2.1}$$

(b) To calculate $K_{cm} = \frac{1}{2}(m_1 + m_2) v_{cm}^2$, we need v_{cm}, which in this case is equal to

$$v_{cm} = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} = \frac{-1 + 2 \times 0.5}{3} = 0$$

so K_{cm} = 0, which means all the kinetic energy is convertible. We can also calculate that directly:

$$K_{\text{conv},i} = \frac{1}{2} \mu v_{12,i}^2 = \frac{1}{2} \left(\frac{1 \times 2}{1+2} \text{ kg} \right) \times \left(0.5 \ \frac{\mathrm{m}}{\mathrm{s}} - (-1) \ \frac{\mathrm{m}}{\mathrm{s}} \right)^2 = \frac{1.5^2}{3} \text{ J} = 0.75 \text{ J} \tag{12.2.2}$$

(c) If we look at figure 2.4.1, we can see that the carts do not pass through each other, so their relative velocity must be zero at some point, and with that, the convertible energy. In fact, the figure makes it quite clear that *both* v_1 and v_2 are zero at t = 5 s, so at that point also v_{12} = 0, and the convertible energy K_{conv} = 0. (And so is the total K_{sys} = 0 at that time, since K_{cm} = 0 throughout.)

On the other hand, it is also clear that K_{conv} is fully recovered after the collision is over, since the relative velocity just changes sign:

$$\begin{aligned} v_{12,i} &= v_{2i} - v_{1i} = 0.5 \ \tfrac{\mathrm{m}}{\mathrm{s}} - (-1) \ \tfrac{\mathrm{m}}{\mathrm{s}} = 1.5 \ \tfrac{\mathrm{m}}{\mathrm{s}} \\ v_{12,f} &= v_{2f} - v_{1f} = -0.5 \ \tfrac{\mathrm{m}}{\mathrm{s}} - 1 \ \tfrac{\mathrm{m}}{\mathrm{s}} = -1.5 \ \tfrac{\mathrm{m}}{\mathrm{s}} \end{aligned} \tag{12.2.3}$$

Therefore

$$K_{\text{conv},f} = \frac{1}{2} \mu v_{12,f}^2 = \frac{1}{2} \mu v_{12,i}^2 = K_{\text{conv},i}$$

(d) Since the total kinetic energy (which in this case is only convertible energy) is fully recovered when the collision is over, the collision is elastic. Using equation (12.2.3), we can see that the coefficient of restitution is

$$e = -\frac{v_{12,f}}{v_{12,i}} = -\frac{-1.5}{1.5} = 1$$

as it should be.

- **4.4: Examples** by Julio Gea-Banacloche is licensed CC BY-SA 4.0. Original source: https://scholarworks.uark.edu/oer/3.

CHAPTER OVERVIEW

13: Application - Orbits and Kepler's Laws

13.1: Orbits

Types of Orbits Under an Inverse-Square Force

Consider a system formed by two particles (or two perfect, rigid spheres) interacting only with each other, through their gravitational attraction. Conservation of the total momentum tells us that the center of mass of the system is either at rest or moving with constant velocity. Let us assume that one of the objects has a much greater mass, M, than the other, so that, for practical purposes, its center coincides with the center of mass of the whole system. This is not a bad approximation if what we are interested in is, for instance, the orbit of a planet around the sun. The most massive planet, Jupiter, has only about 0.001 times the mass of the sun.

Accordingly, we will assume that the more massive object does not move at all (by working in its center of mass reference frame, if necessary—note that, by our assumptions, this will be an inertial reference frame to a good approximation), and we will be concerned only with the motion of the less massive object under the force $F = GMm/r^2$, where r is the distance between the centers of the two objects. Since this force is always pulling towards the center of the more massive object (it is what is often called a *central force*), its torque around that point is zero, and therefore the angular momentum, $\vec{L}$, of the less massive body around the center of mass of the system is constant. This is an interesting result: it tells us, for instance, that the motion is *confined to a plane*, the same plane that the vectors $\vec{r}$ and $\vec{v}$ defined initially, since their cross-product cannot change.

In spite of this simplification, the calculation of the object's trajectory, or *orbit*, requires some fairly advanced mathematical techniques, except for the simplest case, which is that of a circular orbit of radius R. Note that this case requires a very precise relationship to hold between the object's velocity and the orbit's radius, which we can get by setting the force of gravity equal to the centripetal force:

$$\frac{GMm}{R^2} = \frac{mv^2}{R}. \tag{13.1.1}$$

So, if we want to, say, put a satellite in a circular orbit around a central body of mass M and at a distance R from the center of that body, we can do it, but only provided we give the satellite an initial velocity $v = \sqrt{GM/R}$ in a direction perpendicular to the radius. But what if we were to release the satellite at the same distance R, but with a different velocity, either in magnitude or direction? Too much speed would pull it away from the circle, so the distance to the center, r, would temporarily increase; this would increase the system's potential energy and accordingly reduce the satellite's velocity, so eventually it would get pulled back; then it would speed up again, and so on.

You may experiment with this kind of thing yourself using the PhET demo at this link:

https://phet.colorado.edu/en/simulation/gravity-and-orbits

You will find that, as long as you do not give the satellite—or planet, in the simulation—too much speed (more on this later!) the orbit you get is, in fact, a closed curve, the kind of curve we call *an ellipse*. I have drawn one such ellipse for you in Figure $13.1.3$.

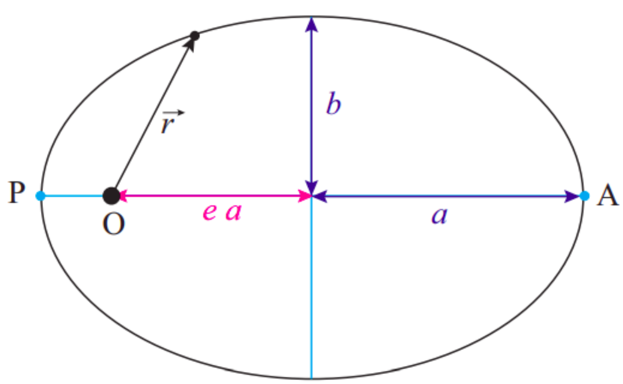


Figure $13.1.3$: An elliptical orbit. The semimajor axis is a, the semiminor axis is b, and the eccentricity $e = \sqrt{1 - b^2/a^2} = 0.745$ in this case.. The "center of attraction" (the sun, for instance, in the case of a planet's or comet's orbit) is at the point O.

As a geometrical curve, any ellipse can be characterized by a couple of numbers, a and b, which are the lengths of the *semimajor* and *semiminor* axes, respectively. These lengths are shown in the figure. Alternatively, one could specify a and a parameter known as the *eccentricity*, denoted by e (do not mistake this "e" for the coefficient of restitution of Chapter 4!), which is equal to $e=\sqrt{1-b^2/a^2}$. If a = b, or e = 0, the ellipse becomes a circle.

The most striking feature of the elliptical orbits under the influence of the $1/r^2$ gravitational force is that the "central object" (the sun, for instance, if we are interested in the orbit of a planet, asteroid or comet) is *not* at the geometric center of the ellipse. Rather, it is at a special point called the *focus* of the ellipse (labeled "O" in the figure, since that is the origin for the position vector of the orbiting body). There are actually two foci, symmetrically placed on the horizontal (major) axis, and the distance of each focus to the center of the ellipse is given by the product ea, that is, the product of the eccentricity and the semimajor axis. (This explains why the "eccentricity" is called that: it is a measure of how "off-center" the focus is.)

For an object moving in an elliptical orbit around the sun, the distance to the sun is minimal at a point called the perihelion, and maximal at a point called the aphelion. Those points are shown in the figure and labeled "P" and "A", respectively. For an object in orbit around the earth, the corresponding terms are perigee and apogee; for an orbit around some unspecified central body, the terms periapsis and apoapsis are used. There is some confusion as to whether the distances are to be measured from the surface or from the center of the central body; here I will assume they are all measured from the center, in which case the following relationships follow directly from Figure $13.1.3$:

$$\begin{aligned} r_{\max} &= (1+e)a \\ r_{\min} &= (1-e)a \\ r_{\min}+r_{\max} &= 2a \\ e &= \frac{r_{\max}-r_{\min}}{2a}. \end{aligned} \tag{13.1.2}$$

The ellipse I have drawn in Figure $13.1.3$ is actually way too eccentric to represent the orbit of any planet in the solar system (although it could well be the orbit of a comet). The planet with the most eccentric orbit is Mercury, and that is only e = 0.21. This means that b = 0.978a, an almost imperceptible deviation from a circle. I have drawn the orbit to scale in Figure $13.1.4$, and as you can see the only way you can tell it is an ellipse is, precisely, because the sun is not at the center.

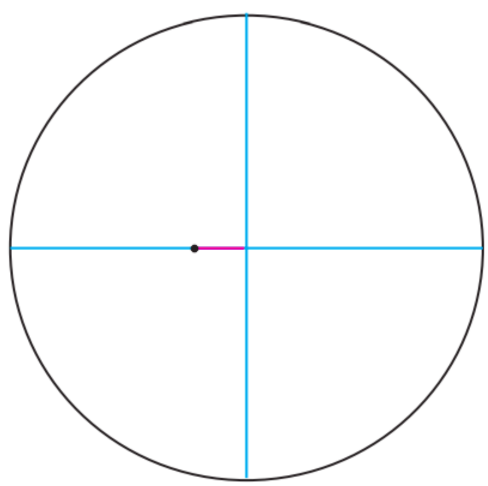


Figure $13.1.4$: Orbit of Mercury, with the sum approximately to scale

Since an ellipse has only two parameters, and we have two constants of the motion (the total energy, E, and the angular momentum, L), we should be able to determine what the orbit will look like based on just those two quantities. Under the assumption we are making here, that the very massive object does not move at all, the total energy of the system is just

$$E=\frac{1}{2}mv^2-\frac{GMm}{r}. \tag{13.1.3}$$

For a circular orbit, the radius R determines the speed (as per Equation (13.1.1)), and hence the total energy, which is easily seen to be $E=-\frac{GMm}{2R}$. It turns out that this formula holds also for elliptical orbits, if one substitutes the semimajor axis a for R:

$$E=-\frac{GMm}{2a}. \tag{13.1.4}$$

Note that the total energy (13.1.4) is *negative*. This means that we have a *bound* orbit, by which I mean, a situation where the orbiting object does not have enough kinetic energy to fly arbitrarily far away from the center of attraction. Indeed, since $U^G \rightarrow 0$ as $r \rightarrow \infty$, you can see from Equation (13.1.3) that if the two objects could be infinitely far apart, the total energy would eventually have to be positive, for any nonzero speed of the lighter object. So, if $E<0$, we have bound orbits, which are ellipses (of which a circle is a special case), and conversely, if $E>0$ we have "unbound" trajectories, which turn out to be hyperbolas[4]. These trajectories just pass near the center of attraction once, and never return.

The special borderline case when E = 0 corresponds to a *parabolic* trajectory. In this case, the particle also never comes back: it has just enough kinetic energy to make it "to infinity," slowing down all the while, so $v \rightarrow 0$ as $r \rightarrow \infty$. The initial speed necessary to accomplish this, starting from an initial distance r_i, is usually called the "escape velocity" (although it really should be called the escape speed), and it is found by simply setting Equation (13.1.3) equal to zero, with $r=r_i$, and solving for v:

$$v_{esc}=\sqrt{\frac{2GM}{r_i}}. \tag{13.1.5}$$

In general, you can calculate the escape speed from any initial distance r_i to the central object, but most often it is calculated from its surface. Note that v_{esc} does not depend on the mass of the lighter object (always assuming that the heavier object does not move at all). The escape velocity from the surface of the earth is about 11 km/s, or 1.1×10^4 m/s; but this alone would not be enough to let you leave the attraction of the sun behind. The escape speed from the sun starting from a point on the earth's orbit is 42 km/s.

To summarize all of the above, suppose you are trying to put something in orbit around a much more massive body, and you start out a distance r away from the center of that body. If you give the object a speed smaller than the escape speed at that point, the result will be E < 0 and an elliptical orbit (of which a circle is a special case, if you give it the precise speed $v=\sqrt{GM/r}$ in the right direction). If you give it precisely the escape speed (13.1.5), the total energy of the system will be zero and the trajectory of the object will be a parabola; and if you give it more speed than v_{esc}, the total energy will be positive and the trajectory will be a hyperbola. This is illustrated in Figure 13.1.5 below.

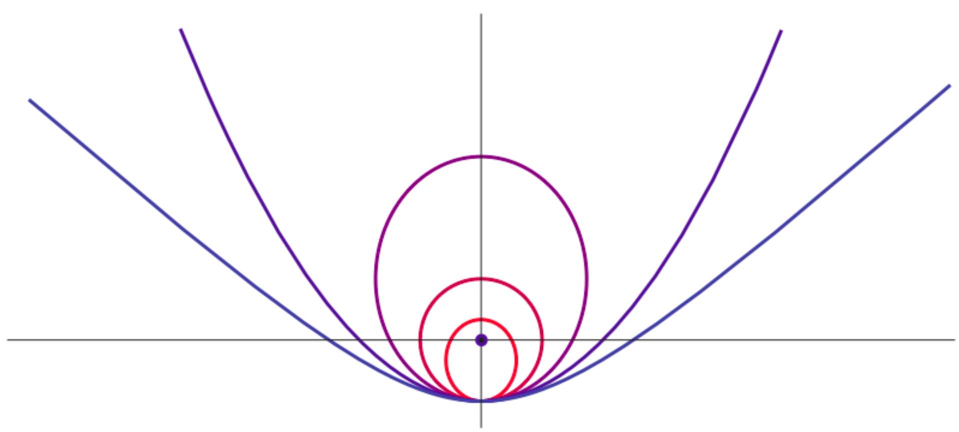


Figure 13.1.5: Possible trajectories for an object that is "released" with a sideways velocity at the lowest point in the figure, under the gravitational attraction of a large mass represented by the black circle. Each trajectory corresponds to a different value of the object's initial kinetic energy: if K_{circ} is the kinetic energy needed to have a circular orbit through the point of release, the figure shows the cases $K_i=0.5K_{circ}$ (small ellipse), $K_i=K_{circ}$ (circle), $K_i=1.5K_{circ}$ (large ellipse), $K_i=2K_{circ}$ (escape velocity, parabola), and $K_i=2.5K_{circ}$ (hyperbola).

Note that all the trajectories shown in Figure 13.1.5 have the same potential energy at the "point of release" (since the distance from that point to the center of attraction is the same for all), so increasing the kinetic energy at that point also means increasing the total energy (13.1.3) (which is constant throughout). So the picture shows different orbits in order of increasing total energy.

For a given total energy, the total angular momentum does not change the fundamental nature of the orbit (bound or unbound), but it can make a big difference on the orbit's shape. Generally speaking, for a given energy the orbits with less angular momentum will be "narrower," or "more squished" than the ones with more angular momentum, since a smaller initial angular momentum at the point of insertion means a smaller sideways velocity component. In the extreme case of zero initial angular momentum (no sideways velocity at all), the trajectory, regardless of the total energy, reduces to a straight line, either straight towards or straight away from the center of attraction.

For elliptical orbits, one can prove the result

$$e=\sqrt{1-\frac{L^2}{aGMm^2}} \tag{13.1.6}$$

which shows how the eccentricity increases as L decreases, for a given value of a (which is to say, for a given total energy). I should at least sketch how to obtain this result, since it is a variant of a procedure that you may have to use for some homework problems this semester. You start by writing the angular momentum as $L=mr_Pv_P$ (or mr_Av_A), where A and P are the special points shown in Figure $13.1.3$, where $\vec{v}$ and $\vec{r}$ are perpendicular. Then, you note that $r_P=r_{min}=(1-e)a$ (or, alternatively, $r_A=r_{max}=(1+e)a$), so $v_P=L/[m(1-e)a]$. Then substitute these expressions for r_P and v_P in Equation (13.1.3), set the result equal to the total energy (13.1.4), and solve for e.

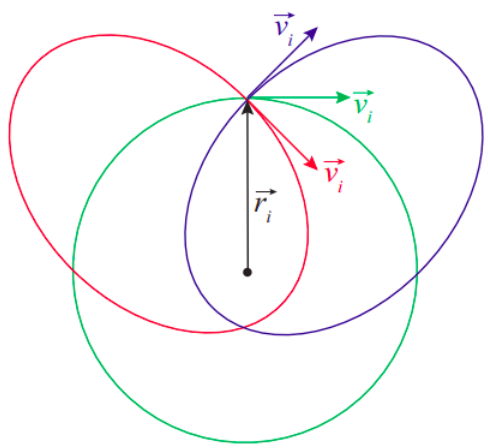


Figure $13.1.6$: Effect of the "angle of insertion" on the orbit.

Figure $13.1.6$ illustrates the effect of varying the angular momentum, for a given energy. All the initial velocity vectors in the figure have the same magnitude, and the release point (with position vector $\vec{r}_i$) is the same for all the orbits, so they all have the same energy; indeed, you can check that the semimajor axis of the two ellipses is the same as the radius of the circle, as required by Equation (13.1.4). The difference between the orbits is their total angular momentum. The green orbit has the maximum angular momentum possible at the given energy, since the green velocity vector is perpendicular to $\vec{r}_i$. Note that this (maximizing L for a given $E<0$) always results in a circle, in agreement with Equation (13.1.5): the eccentricity is zero when $L=L_{circ}\equiv\sqrt{aGMm^2}$, which is the largest value of L allowed in Equation (13.1.5).

For the other two orbits, $\vec{v}_i$ and $\vec{r}_i$ make angles of 45° and 135°, and so the angular momentum L has magnitude $L=L_{\text{circ}}\sin 45^\circ=L_{\text{circ}}/\sqrt{2}$. The result are the red and blue ellipses, with eccentricities $e=\sqrt{1-\sin^2(45^\circ)}=0.707$.

[4]There is apparently a way to describe a hyperbola as an ellipse with eccentricity $e>1$, but I'm definitely not going to go there.

13.2: Kepler's Laws

Kepler's Laws

The first great success of Newton's theory was to account for the results that Johannes Kepler had extracted from astronomical data on the motion of the planets around the sun. Kepler had managed to find a number of regularities in a mountain of data (most of which were observations by his mentor, the Danish astronomer Tycho Brahe), and expressed them in a succinct way in mathematical form. These results have come to be known as *Kepler's laws*, and they are as follows:

1. The planets move around the sun in elliptical orbits, with the sun at one focus of the ellipse
2. (Law of areas) A line that connects the planet to the sun (the planet's position vector) sweeps equal areas in equal times.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of its orbit (the same proportionality constant holds for all the planets).

I have discussed the first "law" at length in the previous section, and also pointed out that the math necessary to prove it is far from trivial. The second law, on the other hand, while it sounds complicated, turns out to be a straightforward consequence of the conservation of angular momentum. To see what it means, consider Figure $13.2.7$.

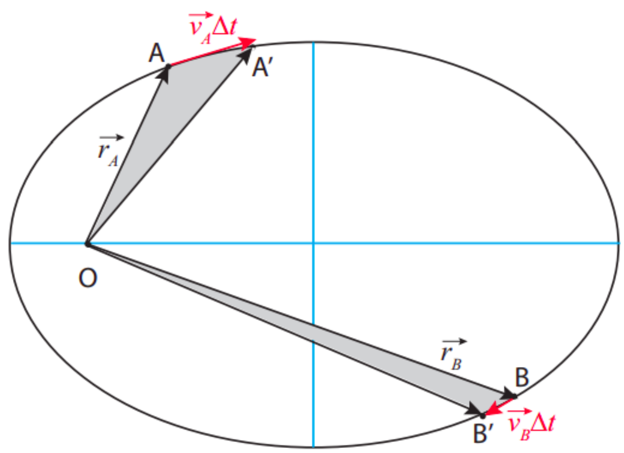


Figure $13.2.7$: Illustrating Kepler's law of areas. The two gray "curved triangles" have the same area, so the particle must take the same time to go from A to A$'$ as it does to go from B to B$'$.

Suppose that, at some time t_A, the particle is at point A, and a time Δt later it has moved to A$'$. The area "swept" by its position vector is shown in grey in the figure, and Kepler's second law states that it must be the same, for the same time interval, at any point in the trajectory; so, for instance, if the particle starts out at B instead, then in the same time interval Δt it will move to a point B$'$ such that the area of the "curved triangle" OBB$'$ equals the area of OAA$'$.

Qualitatively, this means that the particle needs to move more slowly when it is farther from the center of attraction, and faster when it is closer. Quantitatively, this actually just means that its angular momentum is constant! To see this, note that the straight distance from A to A$'$ is the displacement vector $\Delta\vec{r}_A$, which, for a sufficiently short interval Δt, will be approximately equal to $\vec{v}_A\Delta t$. Again, for small Δt, the area of the curved triangle will be approximately the same as that of the straight triangle OAA$'$. It is a well-known result in trigonometry that the area of a triangle is equal to 1/2 the product of the lengths of any two of its sides times the sine of the angle they make. So, if the two triangles in the figures have the same areas, we must have

$$|\vec{r}_A|\,|\vec{v}_A|\,\Delta t \sin\theta_A = |\vec{r}_B|\,|\vec{v}_B|\,\Delta t \sin\theta_B \tag{13.2.1}$$

and we recognize here the condition $\left|\vec{L}_{\boldsymbol{A}}\right| = \left|\vec{L}_B\right|$, which is to say, conservation of angular momentum. (Once the result is established for infinitesimally small Δt, we can establish it for finite-size areas by using integral calculus, which is to say, in essence, by breaking up large triangles into sums of many small ones.)

As for Kepler's third result, it is easy to establish for a circular orbit, and definitely not easy for an elliptical one. Let us call T the orbital period, that is, the time it takes for the less massive object to go around the orbit once. For a circular orbit, the angular velocity ω can be written in terms of T as $\omega = 2\pi/T$, and hence the regular speed $v = R\omega = 2\pi R/T$. Substituting this in Equation (???), we get $GM/R^2 = 4\pi^2 R/T^2$, which can be simplified further to read

$$T^2 = \frac{4\pi^2}{GM} R^3. \tag{13.2.2}$$

Again, this turns out to work for an elliptical orbit if we replace R by a.

Note that the proportionality constant in Equation (13.2.2) depends only on the mass of the central body. For the solar system, that would be the sun, of course, and then the formula would apply to any planet, asteroid, or comet, with the same proportionality constant. This gives you a quick way to calculate the orbital period of anything orbiting the sun, if you know its distance (or vice-versa), based on the fact that you know what these quantities are for the Earth.

More generally, suppose you have two planets, 1 and 2, both orbiting the same star, at distances R_1 and R_2, respectively. Then their orbital periods T_1 and T_2 must satisfy $T_1^2 = \left(4\pi^2/GM\right) R_1^3$ and $T_2^2 = \left(4\pi^2/GM\right) R_2^3$. Divide one equation by the other, and the proportionality constant cancels, so you get

$$\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{R_2}{R_1}\right)^3. \tag{13.2.3}$$

From this, some simple manipulation gives you

$$T_2 = T_1 \left(\frac{R_2}{R_1}\right)^{3/2}. \tag{13.2.4}$$

Note you can express R_1 and R_2 in any units you like, as long as you use the same units for both, and similarly T_1 and T_2. For instance, if you use the Earth as your reference "planet 1," then you know that T_1 = 1 (in years), and R_1 = 1, in AU (an AU, or "astronomical unit," is the distance from the Earth to the sun). A hypothetical planet at a distance of 4 AU from the sun should then have an orbital period of 8 Earth-years, since $4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8$.

A formula just like (13.2.2), but with a different proportionality constant, would apply to the satellites of any given planet; for instance, the myriad of artificial satellites that orbit the Earth. Again, you could introduce a "reference satellite" labeled 1, with known period and distance to the Earth (the moon, for instance?), and derive again the result (13.2.4), which would allow you to get the period of any other satellite, if you knew how its distance to the earth compares to the moon's (or, conversely, the distance at which you would need to place it in order to get a desired orbital period).

For instance, suppose I want to place a satellite on a "geosynchronous" orbit, meaning that it takes 1 day for it to orbit the Earth. I know the moon takes 29 days, so I can write Equation (13.2.4) as $1 = 29(R_2/R_1)^{3/2}$, or, solving it, $R_2/R_1 = (1/29)^{2/3} = 0.106$, meaning the satellite would have to be approximately 1/10 of the Earth-moon distance from (the center of) the Earth.

In hindsight, it is somewhat remarkable that Kepler's laws are as accurate, for the solar system, as they turned out to be, since they can only be mathematically derived from Newton's theory by making a number of simplifying approximations: that the sun does not move, that the gravitational force of the other planets has no effect on each planet's orbit, and that the planets (and the sun) are perfect spheres, for instance. The first two of these approximations work as well as they do because the sun is so massive; the third one works because the sizes of all the objects involved (including the sun) are much smaller than the corresponding orbits. Nevertheless, Newton's work made it clear that Kepler's laws could only be approximately valid, and scientists soon set to work on developing ways to calculate the corrections necessary to deal with, for instance, the trajectories of comets or the orbit of the moon.

Of the main approximations I have listed above, the easiest one to get rid of (mathematically) is the first one, namely, that the sun does not move. Instead, what one finds is that, as long as the sun and the planet are still treated as an isolated system, they will both revolve around the system's center of mass. Of course, the sun's motion (a slight "wobble") is very small, but not completely negligible. You can even see it in the simulation I mentioned earlier, at

phet.colorado.edu/en/simulation/gravity-and-orbits.

What is much harder to deal with, mathematically, is the fact that none of the planets in the solar system actually forms an isolated system with the sun, since all the planets are really pulling gravitationally on each other all the time. Particularly, Jupiter and Saturn have a non-negligible influence on each other's orbits, and on the orbits of every other planet, which can only be perceived over centuries. Basically, the orbits still look like ellipses to a very good degree, but the ellipses rotate very, very slowly (so they fail to exactly close in on themselves). This effect, known as orbital precession, is most dramatic for Mercury, where the ellipse's axes rotate by more than one degree per century.

Nevertheless, the Newtonian theory is so accurate, and the calculation techniques developed over the centuries so sophisticated, that by the early 20th century the precession of the orbits of all planets *except* Mercury had been calculated to near exact agreement with the best observational data. The unexplained discrepancy for Mercury amounted only to 43 seconds of arc per century, out of 5600 (an error of only 0.8%). It was eventually resolved by Einstein's general theory of relativity.

13.3: Weight, Acceleration, and the Equivalence Principle

Whether we write it as mg or as GMm/r^2, the force of gravity on an object of mass m has the remarkable property—not shared by any other known force—of being proportional to the inertial mass of the object. This means that, if gravity is the only force acting on a system made up of many particles, when you divide the force on each particle by the particle's mass in order to find the particle's acceleration, you get the same value of a for every particle (at least, assuming that they are all at about the same distance from the object exerting the force in the first place). Thus, all the parts making up the object will accelerate together, as a whole.

Suppose that you are holding an object, while in free fall (remember that "free fall" means that gravity is the only force acting on you), and you let go of it, as in Figure 13.3.1 below. Since gravity will give you and the object the same acceleration, you'll find that it does not "fall" relative to you—that is, it will not fall any faster nor more slowly than yourself. From your own reference frame, you will just see it hovering motionless in front of you, in the same position (relative to you) that it occupied before you let go of it. This is exactly what you see in videos shot aboard the International Space Station. The result is an impression of weightlessness, or "zero gravity"—even though gravity is very much nonzero; the space station, and everything inside it, is constantly "falling" to the earth, it just does not hit it because it has some sideways velocity (or angular momentum) to begin with, and the earth's pull just bends its trajectory around enough to keep it moving in a circle. But gravity is the only force acting on it, and on everything in it (at least until somebody pushes himself against a wall, or something like that).

So, the kind of acceleration you get from gravity is, paradoxically, such that, if you give in to it completely, you feel like there is *no* gravity.

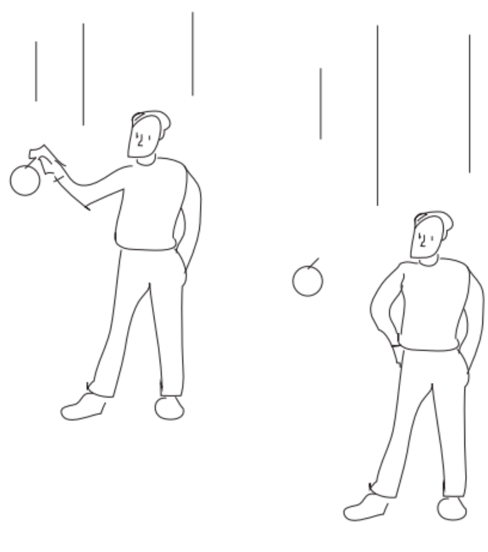


Figure 13.3.1: If you are holding something while in free fall (a) and let go, since you are all accelerating at the same rate, it stays in the same position relative to you (b), so it appears to be weightless.

The familiar sensation of weight, on the other hand, comes precisely from *not* giving in, and rather, enlisting other forces to fight against gravity. When you do this—when you simply stand on the surface of the earth, for instance—your feet are supported by the ground below you, but every other part of your body is supported by some other part of your body, immediately above or below it, that you can think of as a sort of spring that is either somewhat stretched or somewhat compressed. It is primarily your skeleton, and mostly your spine, that bears most of the compressive load. (See Figure 10.9, next page.) The sensation of weight is your response to this load. Interestingly, even though this constant squishing may actually result in your losing a little height in the course of a day (which you recover at night, when you lie horizontally), it is not a bad thing, rather the contrary: your bones have evolved so that they *need* this constant pressure to grow and replace the mass that they would otherwise lose in a "weightless" environment.

On the other hand, as shown in Figure 13.3.2(c), the *same* compression (or extension—for instance, for the muscles in your arms, as they hang by your side) would result from a situation in which you were, say, standing motionless inside a rocket that is accelerating upwards with $a = g$, but very far away from any gravity source. In Figure 13.3.2(b), the "spring" that represents your skeleton needs to be compressed so it can exert an upward force $F^{spr} = m_u g$ to support the weight of your upper body (simplified here as just a single mass m_u). In Figure 13.3.2(c), it needs to be compressed by the same amount, so it can exert the upward force $F^{spr} = m_u a$ needed to give your upper body an acceleration $a = g$. The equality of the two expressions is a direct consequence of the fact that the force of gravity is proportional to an object's inertial mass (since the second expression is just Newton's second law).

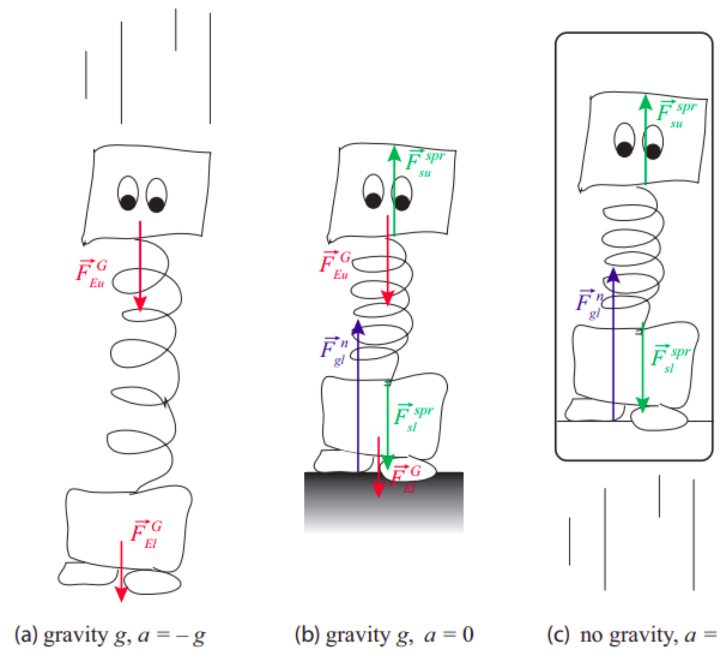


Figure 13.3.2: (a) In free fall, your skeleton (represented here by a relaxed spring) does not need to support your upper body, so there is no sensation of weight. When standing on the ground motionless under the influence of gravity, however (b), every part of your body needs to compress a little in order to support the weight of the parts above it (as shown here by the compressed spring). The same compression, and hence the same subjective sensation of weight, results if you are moving upwards with an acceleration $a = g$, but in the absence of gravity (c). (The subscripts u and l on the forces stand for "upper" and "lower" body, respectively.)

In general, then, when your whole body is subjected to an upward acceleration a, it feels like your weight is increased by an amount ma. The same thing holds regardless of direction—a forward acceleration a on a jet pilot's body feels like a "weight" ma pushing her against her seat. This is why these "effective forces" (or, more precisely, the accelerations that cause them) are measured in g's: a "force" of, say, $5g$, means that the pilot feels pushed against her seat with a "force" equal to 5 times her weight. What's really happening, of course, is the opposite—her seat is pushing her *forward*, but her internal organs are being compressed (in order to provide that same forward acceleration) the way they would under a gravity force five times stronger than at the earth's surface.

The parallels between being in a constantly accelerating frame of reference and being at rest under the influence of a constant gravity force go beyond the subjective sensation of weight. Figure 13.3.3 illustrates what happens when you drop something while traveling in the upwardly accelerating rocket, in the absence of gravity. From an inertial observer's point of view, the object you drop merely keeps the upward velocity it had the moment it left your hand; but, since you are in contact with the rocket, your own velocity is constantly increasing, and as a result of that you see the object fall—relative to you.

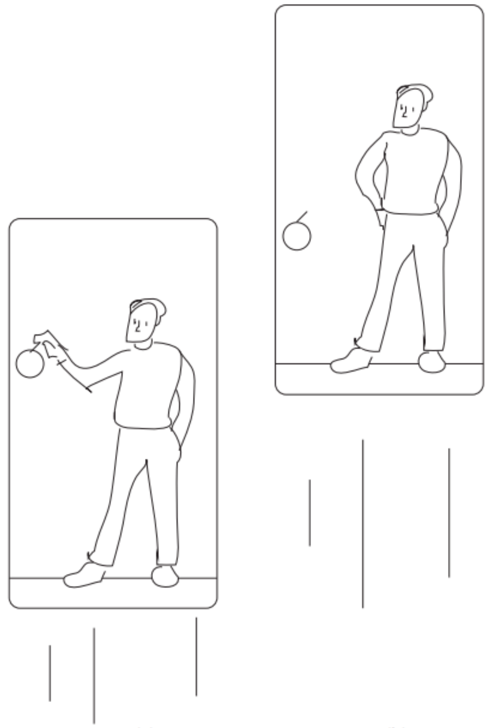


Figure 13.3.3: "Dropping" an object inside a constantly accelerating rocket, away from any gravity.

From a practical point of view, this suggests a couple of ways to provide an "artificial gravity" for astronauts who might one day have to spend a long time in space, either under extremely weak gravity (for instance, during a trip to Mars), or, what amounts to essentially the same thing, in free fall (as in a space station orbiting a planet). The one most often seen in movies consists in having the space station (or spaceship) constantly spin around an axis with some angular velocity ω. Then any object that is moving with the station, a distance R away from the axis, will experience a centripetal acceleration of magnitude $\omega^2 R$, which will feel like a gravity force $m\omega^2 R$ directed in the opposite direction, that is to say, away from the center. People would then basically "walk on the walls" (that is to say, sideways as seen from above, with their feet away from the rotation axis and their heads towards the rotation axis). If somebody let go of something they were holding, the object would "fall towards the wall." Unfortunately, while the idea might work for a space station, it would probably be impractical for a spaceship, since one would need a fairly large R and/or a fairly large rotation rate to get $\omega^2 R \simeq g$. (On the other hand, probably even something like $\frac{1}{5} g$ is better than nothing, so who knows...)

On a fundamental level, the equivalence between a constantly accelerated reference frame, and an inertial frame with a uniform gravitational field (such as, approximately, the surface of the earth), was elevated by Einstein to a basic principle of physics, which became the foundation of his general theory of relativity. This *equivalence principle* asserts that it is absolutely impossible to distinguish, by any kind of physics experiment, between the two situations just mentioned: a constantly accelerated reference frame is postulated to be completely equivalent in every way to an inertial frame with a uniform gravitational field.

A remarkable consequence of the equivalence principle is that light, despite having technically "zero rest mass," must bend its trajectory under the influence of gravity. This can be seen as follows. Imagine shooting a projectile horizontally inside the rocket in Figure 13.3.3. Although an inertial observer, looking from the outside, would see the projectile travel in a straight line, the observer inside the rocket would see its path bend down, just as for the projectiles we studied back in Chapter 8. This is for the same reason he would see the object fall, relative to him, in Figure 13.3.3: the projectile has a constant velocity, so it travels the same distance in every equal time interval, but the rocket is accelerating, so the distance it travels in equal time intervals is constantly increasing. In basically the same way, then, a beam of light sent horizontally inside the rocket, and traveling with constant velocity (and, therefore, in a straight line) in an inertial frame, would be seen as bending down in the rocket's reference frame.

However, if the equivalence principle is true, and physical phenomena look the same in a constantly accelerating frame as in an inertial frame with a constant gravitational field, it follows that light must also bend its path in the latter system, in much the same way as a projectile would. (I say "much the same way" because the effect is not just as simple as giving light an "effective mass";

there are other relativistic effects, such as space contraction and time dilation, that must also be reckoned with.) This gravitational bending was one of the most important early predictions of Einstein's General Relativity theory, and certainly the most spectacular. Since one needs the light rays to pass vary close to a large mass to get an observable effect, the way the prediction was verified was by looking at the apparent position of the stars that can be seen close to the edge of the sun's disk during a solar eclipse. The slight (apparent) shift in position predicted by Einstein was observed by Sir Arthur Eddington during the solar eclipse of 1919 (two expeditions were sent to remote corners of the earth for this purpose), and it was primarily responsible for Einstein's sudden fame among the general public of his day.

Today, with modern telescopes, this so-called "gravitational lensing" effect has become an important tool in astronomy, allowing us to interpret the pictures taken of distant galaxies, which are often shifted and/or distorted by the gravity of the galaxies that lie in between them and us.

It has even become possible to imagine an object so dense that it would "capture" light, attracting it so strongly that it could not leave the object's neighborhood. Such an object has come to be called a *black hole*. If you set the escape velocity of Equation (10.1.15) equal to the speed of light in vacuum, c, and solve for r_i, you obtain what is called the *Schwarzschild radius*, r_s, for a black hole of mass M; the idea being that, in order to *be* a black hole, the object has to be so dense that all its mass M is inside a sphere of radius smaller than r_s. Physicists today believe in the existence (and even what one might call the ubiquity) of black holes, of which the Schwarzschild solution was only the first calculated example. Note that r_s does not define the actual, physical surface of the object; it does, however, locate what is known as the black hole's *event horizon*. Nothing can be known, through observation, about anything that might happen closer to the black hole's center than the distance r_s, since no information can be transmitted faster than light, and no light can escape from a distance $r_i < r_s$.

13.4: Examples

Example 13.4.1: Orbital dynamics

In the early days of space flight, astronauts sometimes mentioned the counterintuitive aspects of orbital flight. For example, if, from a circular orbit around the Earth, they wanted to move to a lower orbit, the way to do it was to slow down their capsule (by firing a thruster in the direction opposite their motion). This would take them to a lower orbit, but then the capsule would start *speeding up*, on its own.

Use the concepts introduced in this chapter to explain what is going on in this scenario. Let R be the radius of the initial orbit. For simplicity, assume the thruster is on only for a very short time, so you can neglect the motion of the capsule during this time. In other words, treat it as an instantaneous reduction in velocity, and discuss:

a. What happens to the system's potential and kinetic energy, and angular momentum?
b. Is the new orbit circular or elliptic? How do you know? What is the new orbit's r_{max} (maximum distance to the center of the Earth)?
c. Why does the capsule speed up in its new orbit?
d. If the new orbit is not circular, what would the astronauts need to do to make it so? (Without getting any closer to the Earth, that is, keeping r_{min} the same.)

Make sure to draw a diagram of the situation. Make it as accurate as you can.

Solution

(a) Under the assumption that the capsule barely changes position during the thruster firing, the potential energy of the system, which is equal to $U^G = -GMm/R$, will not change: $U_f^G = U_i^G$.

The kinetic energy, on the other hand, will go down, since the capsule's speed is reduced: $K_f < K_i$. Hence, the total mechanical energy of the system, $E = K + U^G$, will decrease: $E_f < E_i$.

The angular momentum will go down, since v goes down.

(b) The new orbit has to be elliptical, since to have a circular orbit at a distance R requires a precise velocity (given just below Equation (10.1.11) by $v = \sqrt{GM/R}$), and now we have changed that.

However, since the orbit must still be a closed curve, it will contain the starting point, which is, by our assumption, a distance R away from the Earth. Also, if the direction of the velocity vector does not change as a result of the thruster firing (only the magnitude of v is supposed to change), it follows that at this point the velocity and the position vectors are perpendicular. For a circular orbit, this is the case everywhere. For an elliptical orbit, this is only true at the two extreme points labeled P and A in Figure 10.1.3 (the perigee and apogee, respectively). So, the initial position of the capsule becomes either the perigee or the apogee of the new orbit. Which is it?

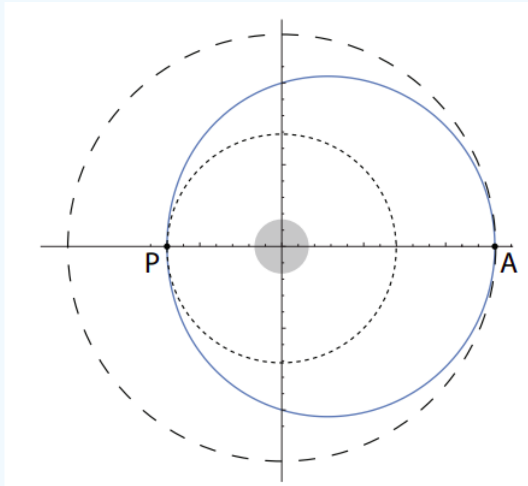
To get the answer, recall that we found in (a) that the total mechanical energy E has gone down. But, since E is a negative number, this means the *magnitude* of E has gone *up*. Then, in the formula (10.1.14),

$$E = -\frac{GMm}{2a}$$

the semimajor axis a must have gone *down*. For the original circular orbit, we had $a = R$; now, we must have $a < R$. This means that the starting point, a distance R away from the (center of the) Earth, cannot be the perigee (the point of closest approach), since at that point $r = r_{min}$, and r_{min} is always less than a (check again Figure 10.1.3, or Eqs. (10.1.12)). Instead, the starting point has to be the apogee of the new orbit, and therefore the distance at that point is also the maximum distance: $r_{max} = R$.

(c) The capsule speeds up in its new orbit because, as we just saw, it starts as far away from the Earth as it's going to get; therefore, as it moves it will start getting closer to the Earth, and we know from Kepler's second law that as it gets closer it has to speed up. (You can also say that, as it gets closer, the gravitational potential energy of the system will go down, and therefore its kinetic energy must increase.)

(d) The easiest way to change the new orbit to a circular orbit with radius r_{min} would be to perform another speed-reduction maneuver, but this time at perigee. At perigee, the distance to the Earth is already r_{min}, which is what you want it to be, but the capsule is moving too fast to stay on a circular orbit (put differently, the gravitational force of the Earth at that point is too weak to bend the orbit into a circle): that is why it eventually ends up "overshooting" the Earth on the other side. Reducing v will further reduce E and, by the same argument as above, it will result in an orbit with a smaller a, which is what you want (since, at the moment, $a > r_{min}$, and you want the new a to be equal to r_{min}).



The diagram of the situation is above (previous page). The long-dash circle is the original orbit; the solid line is the elliptical orbit resulting from the speed reduction at point A; the short-dash circle is the circular orbit that would result from another speed reduction at the point P. Note: the size of the orbits is greatly exaggerated compared to those in the early space flights, which were much closer to the Earth!

The way to draw this kind of figure is to first draw an accurate ellipse, making sure you know where the focus is; then draw the circles centered at the focus and touching the ellipse at the right points. An ellipse's equation in polar form, with the origin at one focus, is $r = a + ae\cos\phi$.

Example 13.4.2: orbital data from observations- halley's comet

Halley's comet follows an elliptical orbit around the sun. At its closest approach, it is a distance of 0.59 AU from the sun (an astronomical unit, AU, is defined as the average distance from the earth to the sun: 1 AU = 1.496 × 10^{11} m), and it is moving at 5.4 × 10^4 m/s. We know its period is approximately 76 years. Ignoring the forces exerted on the comet by the other solar system objects (a rather rough approximation):

a. Use the appropriate Kepler law to infer the value of a (the semimajor axis) for the comet's orbit.
b. What is the eccentricity of the comet's orbit?
c. Using the result in (a) and conservation of angular momentum, find the speed of the comet at aphelion (the point in its orbit when it is farthest away from the sun).

Solution

(a) The "appropriate Kepler law" here is the third one. For any two objects orbiting, for instance, the sun, the square of their orbital periods is proportional to the cube of their orbits' semimajor axes, with the same proportionality constant ($4\pi/GM_{sun}$; see Equation (10.1.18)). We do not even need to calculate the proportionality constant; we can divide the equation for Halley's comet by the equation for the earth, and get

$$\frac{T^2_{\text{Halley}}}{T^2_{\text{earth}}} = \frac{a^3_{\text{Halley}}}{a^3_{\text{earth}}} \tag{13.4.1}$$

where T^2_{earth} = 1 yr^2, and a^3_{earth} = 1 AU^3, so we get immediately

$$a_{\text{Halley}} = \left(76^2\right)^{1/3}\text{AU} = 17.9\text{ AU} \tag{13.4.2}$$

(b) We can get this one from a look at Figure 10.1.3: the product ea, plus the minimum distance between the comet and the sun (0.59 AU) is equal to a. (This is just what the second of the equations (10.1.12) says as well.). So we have

$$e=\frac{a-r_{\min}}{a}=1-\frac{r_{\min}}{a}=1-\frac{0.59}{17.9}=0.967 \tag{13.4.3}$$

Note that we did not even have to convert AU to kilometers. In these types of problems, particularly, where you have to manipulate very large numbers, it really pays off to do all the calculations symbolically and not substitute the numbers in until the very end, to see if something cancels out, and to prevent mistakes when copying large numbers form one line to the next; and sometimes, like here, you do not even have to convert to other units!

(c) At the point of closest approach (perihelion), the velocity and the position vector of the comet are perpendicular, and so the magnitude of the comet's angular momentum is just equal to $L=mrv$. The same happens at the farthest point in the orbit (aphelion), and since angular momentum is conserved for the Kepler problem, we can write

$$mr_{\min}v_{\max}=mr_{\max}v_{\min} \tag{13.4.4}$$

(the reason for this choice of subscripts is that we know that when r is maximum, v is minimum, and vice-versa). Solving for v_{min}, the speed at aphelion, we get

$$v_{\min}=\frac{r_{\min}}{r_{\max}}v_{\max}=\frac{r_{\min}}{2a-r_{\min}}v_{\max}=\frac{0.59}{2\cdot 17.9-0.59}5.4\times 10^4\ \frac{\mathrm{m}}{\mathrm{s}}=905\ \frac{\mathrm{m}}{\mathrm{s}}. \tag{13.4.5}$$

Here again the equation I used to find r_{max} can be derived directly from Figure 10.1.3 (and it is also one of the equations (10.1.12): $r_{min}+r_{max}=2a$). Once again, I was able to use AU throughout, since the units of distance cancel out in the fraction r_{min}/r_{max}.

CHAPTER OVERVIEW

14: N1) Newton's Laws

Newton's Laws

Up to this point in the semester, we've been studying the interaction between objects by modeling the interactions with energy and momentum. The transfer and conservation of these quantities allowed us to determine their motion. Now, we would like to describe the interactions between objects using **forces**. In some ways, this description of the physical world is more intuitive; forces push and pull on objects, much like how we interact with objects in our everyday lives. Of course, momentum and energy is still being transferred around, but the force description gives us a different perspective, and intuition about the motion is often more direct. On the other hand, since forces are vectors, it requires more mathematical sophistication and care then when dealing with energy.

Forces Are Vectors

When you push or pull on an object, it matters what direction you are pushing or pulling it. This is very natural; if you push in one direction and your friend pushes just as hard in the opposite direction, the object will not move. But what happens if you push in one direction and your friend pushes just as hard, but not *quite* in the opposite direction? The object might move in some other direction, and that's what we want to know about. All these various direction-and-magnitude complexities can be easily dealt with by **modeling all forces as vectors**. They must be written as $\vec{F} = F_x\hat{x} + F_y\hat{y}$ in the (x, y)-coordinate system you specify in order to get any answers right!

Newton's First Law

Newton's first law can be summarized as "an object in motion tends to stay in motion unless acted on by a net external force". The converse is also true; "an object at rest tends to stay at rest unless acted on by a net external force". An important word here is *net*, which means *sum of all*. A hockey puck sliding across the ice will continue to slide forever if there is no friction, but it *does have external forces acting on it* (gravity and the normal force, in this case). But these forces balance out, so there is no net force on the hockey puck. Newton's first law does not really help us solve problems, but rather it helps with our modeling process. It tells us when we should expect objects to exhibit motion.

Newton's Second Law

Newton's second law is the primary tool we will use to determine the motion of an object given some forces acting on it. We usually remember it as

$$\sum \vec{F} = m\vec{a}, \tag{14.1}$$

where the $\sum$ symbol means "add up all the forces". This is an important thing to remember - an object can have several forces acting on it, but a single object only ever has one acceleration $\vec{a}$. A very common mistake is to think "each force makes an acceleration F/m, and I will add them all up to get the acceleration of the object", but that is incorrect. A single object has only a single path in space, and therefore only has a single acceleration.

We can actually derive Newton's second law from the definition of force and momentum we have already encountered, namely

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}. \tag{14.2}$$

(Recall that "net" means "sum of all", which is mathematically the same thing as the symbol Σ.) To do this, we just use the definition of momentum, $\vec{p}=m\vec{v}$, and assume the mass is constant in time (as it often is). Then we get:

$$\vec{F}_{net}=\frac{d}{dt}(m\vec{v})=m\frac{d\vec{v}}{dt}=m\vec{a}. \tag{14.3}$$

One can write Newton's second law in a slightly different way,

$$\boxed{\vec{a}=\frac{\sum\vec{F}}{m}}, \tag{14.4}$$

which mathematically identical, but reads more like "the acceleration is the sum of the forces divided by the mass", which is more like how we use Newton's second law.

Finally, since this is a vector equation, it actually contains several independent equations inside it, one for each direction. For instance, if you are doing a 2D Newton's second law problem, you will actually have components in each direction,

$$a_x=\frac{\sum F_x}{m}, \qquad a_y=\frac{\sum F_y}{m}, \tag{14.5}$$

and you will have to solve for these independently.

Newton's Third Law

Newton's third law is "for every action there is an equal and opposite reaction". In this case, our "actions" are forces. The typical example of this is "I push on the wall with a force $\vec{F}$, so the wall pushes on me with a force $-\vec{F}$". Mathematically, if we have a force $\vec{F}_{AB}$ acting from object A to object B, Newton's third law tells us that we know there must be a force $\vec{F}_{BA}$ acting from object B to object A. The magnitudes of these forces are equal, and their directions are opposite:

$$|\vec{F}_{AB}|=|\vec{F}_{BA}|, \qquad \vec{F}_{AB}=-\vec{F}_{BA}. \tag{14.6}$$

Observe the way we've notated this - each force corresponds with *a pair of objects*, one that creates the force and one that experiences it. All forces have both - you push on a wall (*you* and *wall* are the objects), the force of the floor pushing up on you, etc. In the case of the forces above, we're writing $\vec{F}_{AB}$ to mean "the force created by A, acting on B", or "the force from A to B". The order of these subscripts is not always that important (since Newton's third law tells us that $|F_{AB}|=|F_{BA}|$), but we will try to be careful when we are writing them.

14.1: Forces and Newton's Three Laws

14.2: Details on Newton's First Law

According to Newton’s first law (the law of inertia), there must be a cause for any change in velocity (a change in either magnitude or direction) to occur. Inertia is related to an object’s mass. If an object’s velocity relative to a given frame is constant, then the frame is inertial and Newton’s first law is valid. A net force of zero means that an object is either at rest or moving with constant velocity; that is, it is not accelerating.

14.3: Details on Newton's Second Law

Newton’s second law of motion says that the net external force on an object with a certain mass is directly proportional to and in the same direction as the acceleration of the object. Newton’s second law can also describe net force as the instantaneous rate of change of momentum. Thus, a net external force causes nonzero acceleration.

14.4: Details on Newton’s Third Law

Newton’s third law of motion represents a basic symmetry in nature, with an experienced force equal in magnitude and opposite in direction to an exerted force. Action-reaction pairs include a swimmer pushing off a wall, helicopters creating lift by pushing air down, and an octopus propelling itself forward by ejecting water from its body. Choosing a system is an important analytical step in understanding the physics of a problem and solving it.

14.5: Free-Body Diagrams

14.6: Vector Calculus

14.7: Examples

14.1: Forces and Newton's Three Laws

Newton's Laws

Up to this point in the semester, we've been studying the interaction between objects by modeling the interactions with energy and momentum. The transfer and conservation of these quantities allowed us to determine their motion. Now, we would like to describe the interactions between objects using **forces**. In some ways, this description of the physical world is more intuitive; forces push and pull on objects, much like how we interaction with objects in our everyday lives. Of course, momentum and energy is still being transferred around, but the force description gives us a different perspective, and intuition about the motion is often more direct. On the other hand, since forces are vectors, it requires more mathematical sophistication and care then when dealing with energy.

Forces Are Vectors

When you push or pull on an object, it matters what direction you are pushing or pulling it. This is very natural; if you push in one direction and your friend pushes just as hard in the opposite direction, the object will not move. But what happens if you push in one direction and your friend pushes just as hard, but not *quite* in the opposite direction? The object might move in some other direction, and that's what we want to know about. All these various direction-and-magnitude complexities can be easily dealt with by **modeling all forces as vectors**. They must be written as $\vec{F}=F_x\hat{x}+F_y\hat{y}$ in the (x,y)-coordinate system you specify in order to get any answers right!

Newton's First Law

Newton's first law can be summarized as ``an object in motion tends to stay in motion unless acted on by a net external force". The converse is also true; ``an object at rest tends to stay at rest unless acted on by a net external force". An important word here is *net*, which means *sum of all*. A hockey puck sliding across the ice will continue to slide forever if there is no friction, but it *does have external forces acting on it* (gravity and the normal force, in this case). But these forces balance out, so there is no net force on the hockey puck. Newton's first law does not really help us solve problems, but rather it helps with our modeling process. It tells us when we should expect objects to exhibit motion.

Newton's Second Law

Newton's second law is the primary tool we will use to determine the motion of an object given some forces acting on it. We usually remember it as

$$\sum\vec{F}=m\vec{a}, \tag{14.1.1}$$

where the $\sum$ symbol means ``add up all the forces". This is an important thing to remember - an object can have several forces acting on it, but a single object only ever has one acceleration $\vec{a}$. A very common mistake is to think ``each force makes an acceleration F/m, and I will add them all up to get the acceleration of the object", but that is incorrect. A single object has only a single path in space, and therefore only has a single acceleration.

We can actually derive Newton's second law from the definition of force and momentum we have already encountered, namely

$$\vec{F}_{net}=\frac{d\vec{p}}{dt}. \tag{14.1.2}$$

(Recall that "net" means "sum of all", which is mathematically the same thing as the symbol Σ.)To do this, we just use the definition of momentum, $\vec{p}=m\vec{v}$, and assume the mass is constant in time (as it often is). Then we get:

$$\vec{F}_{net}=\frac{d}{dt}(m\vec{v})=m\frac{d\vec{v}}{dt}=m\vec{a}. \tag{14.1.3}$$

I typically write Newton's second law in a slightly different way,

$$\boxed{\vec{a}=\frac{\sum\vec{F}}{m}}, \tag{14.1.4}$$

which mathematically identical, but reads more like ``the acceleration is the sum of the forces divided by the mass", which is more like how we use Newton's second law.

Finally, since this is a vector equation, it actually contains several independent equations inside it, one for each direction. For instance, if you are doing a 2d Newton's second law problem, you will actually have components in each direction,

$$a_x = \frac{\sum F_x}{m}, \qquad a_y = \frac{\sum F_y}{m}, \tag{14.1.5}$$

and you will have to solve for these independently.

Newton's Third Law

Newton's third law is ``for every action there is an equal and opposite reaction". In this case, our ``actions" are forces. The typical example of this is ``I push on the wall with a force $\vec{F}$, so the wall pushes on me with a force $-\vec{F}$". Mathematically, if we have a force $\vec{F}_{AB}$ acting from object A to object B, Newton's third law tells us that we know there must be a force $\vec{F}_{BA}$ acting from object B to object A. The magnitudes of these forces are equal, and their directions are opposite:

$$|\vec{F}_{AB}| = |\vec{F}_{BA}|, \qquad \vec{F}_{AB} = -\vec{F}_{BA}. \tag{14.1.6}$$

Notice the way we've notated this - each force corresponds with *a pair of objects*, one that creates the force and one that experiences it. All forces have both - you push on a wall (*you* and *wall* are the objects), the force of the floor pushing up on you, etc. In the case of the forces above, we're writing $\vec{F}_{AB}$ to mean "the force created by A, acting on B", or "the force from A to B". The order of these subscripts is not always that important (since Newton's third law tells us that $|F_{AB}| = |F_{BA}|$), but we will try to be careful when we are writing them.

14.2: Details on Newton's First Law

Experience suggests that an object at rest remains at rest if left alone and that an object in motion tends to slow down and stop unless some effort is made to keep it moving. However, Newton’s first law gives a deeper explanation of this observation.

Newton’s First Law of Motion

A body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force.

Note the repeated use of the verb “remains.” We can think of this law as preserving the status quo of motion. Also note the expression “constant velocity;” this means that the object maintains a path along a straight line, since neither the magnitude nor the direction of the velocity vector changes. We can use Figure $14.2.1$ to consider the two parts of Newton’s first law.

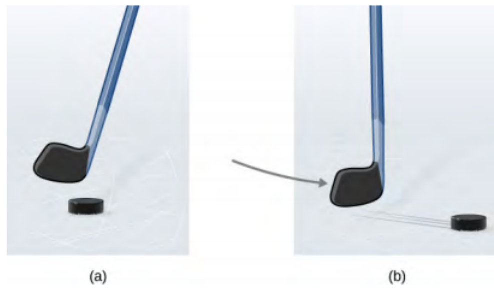


Figure $14.2.1$: (a) A hockey puck is shown at rest; it remains at rest until an outside force such as a hockey stick changes its state of rest; (b) a hockey puck is shown in motion; it continues in motion in a straight line until an outside force causes it to change its state of motion. Although it is slick, an ice surface provides some friction that slows the puck.

Rather than contradicting our experience, Newton’s first law says that there must be a cause for any change in velocity (a change in either magnitude or direction) to occur. This cause is a net external force, which we discussed in the introduction to this chapter. An object sliding across a table or floor slows down due to the net force of friction acting on the object. If friction disappears, will the object still slow down?

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface and ignoring air resistance, we can imagine the object sliding in a straight line indefinitely. Friction is thus the cause of slowing (consistent with Newton’s first law). The object would not slow down if friction were eliminated.

Gravitation and Inertia

Regardless of the scale of an object, whether a molecule or a subatomic particle, two properties remain valid and thus of interest to physics: gravitation and inertia. Both are connected to mass. Roughly speaking, **mass** is a measure of the amount of matter in something. **Gravitation** is the attraction of one mass to another, such as the attraction between yourself and Earth that holds your feet to the floor. The magnitude of this attraction is your weight, and it is a force.

Mass is also related to **inertia**, the ability of an object to resist changes in its motion—in other words, to resist acceleration. Newton’s first law is often called the **law of inertia**. As we know from experience, some objects have more inertia than others. It is more difficult to change the motion of a large boulder than that of a basketball, for example, because the boulder has more mass than the basketball. In other words, the inertia of an object is measured by its mass. The relationship between mass and weight is explored later in this chapter.

Inertial Reference Frames

Earlier, we stated Newton’s first law as “A body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force.” It can also be stated as “Every body remains in its state of uniform motion in a straight line unless it is compelled to change that state by forces acting on it.” To Newton, “uniform motion in a straight line” meant constant velocity, which includes the case of zero velocity, or rest. Therefore, the first law says that the velocity of an object remains constant if the net force on it is zero.

Newton's first law is usually considered to be a statement about reference frames. It provides a method for identifying a special type of reference frame: the **inertial reference frame**. In principle, we can make the net force on a body zero. If its velocity relative to a given frame is constant, then that frame is said to be inertial. So by definition, an inertial reference frame is a reference frame in which Newton's first law is valid. Newton's first law applies to objects with constant velocity. From this fact, we can infer the following statement.

Inertial Reference Frame

A reference frame moving at constant velocity relative to an inertial frame is also inertial. A reference frame accelerating relative to an inertial frame is not inertial.

Are inertial frames common in nature? It turns out that well within experimental error, a reference frame at rest relative to the most distant, or "fixed," stars is inertial. All frames moving uniformly with respect to this fixed-star frame are also inertial. For example, a nonrotating reference frame attached to the Sun is, for all practical purposes, inertial, because its velocity relative to the fixed stars does not vary by more than one part in 10^{10}. Earth accelerates relative to the fixed stars because it rotates on its axis and revolves around the Sun; hence, a reference frame attached to its surface is not inertial. For most problems, however, such a frame serves as a sufficiently accurate approximation to an inertial frame, because the acceleration of a point on Earth's surface relative to the fixed stars is rather small ($< 3.4 \times 10^{-2}\ m/s^2$). Thus, unless indicated otherwise, we consider reference frames fixed on Earth to be inertial.

Finally, no particular inertial frame is more special than any other. As far as the laws of nature are concerned, all inertial frames are equivalent. In analyzing a problem, we choose one inertial frame over another simply on the basis of convenience.

Newton's First Law and Equilibrium

Newton's first law tells us about the equilibrium of a system, which is the state in which the forces on the system are balanced. Consider an object with two forces acting on it, $\vec{F}_1$ and $\vec{F}_2$, which combine to form a resultant force, or the net external force: $\vec{F}_R = \vec{F}_{net} = \vec{F}_1 + \vec{F}_2$. To create equilibrium, we require a balancing force that will produce a net force of zero. This force must be equal in magnitude but opposite in direction to $\vec{F}_R$, which means the vector must be $-\vec{F}_R$.

Newton's first law is deceptively simple. If a car is at rest, the only forces acting on the car are weight and the contact force of the pavement pushing up on the car (Figure 14.2.3). It is easy to understand that a nonzero net force is required to change the state of motion of the car. However, if the car is in motion with constant velocity, a common misconception is that the engine force propelling the car forward is larger in magnitude than the friction force that opposes forward motion. In fact, the two forces have identical magnitude.

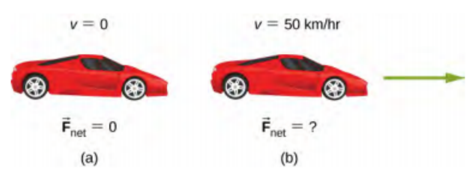


Figure 14.2.3: A car is shown (a) parked and (b) moving at constant velocity. How do Newton's laws apply to the parked car? What does the knowledge that the car is moving at constant velocity tell us about the net horizontal force on the car?

Example 5.1: When Does Newton's First Law Apply to Your Car?

Newton's laws can be applied to all physical processes involving force and motion, including something as mundane as driving a car.

a. Your car is parked outside your house. Does Newton's first law apply in this situation? Why or why not?
b. Your car moves at constant velocity down the street. Does Newton's first law apply in this situation? Why or why not?

Strategy

In (a), we are considering the first part of Newton's first law, dealing with a body at rest; in (b), we look at the second part of Newton's first law for a body in motion.

Solution

a. When your car is parked, all forces on the car must be balanced; the vector sum is 0 N. Thus, the net force is zero, and Newton’s first law applies. The acceleration of the car is zero, and in this case, the velocity is also zero.
b. When your car is moving at constant velocity down the street, the net force must also be zero according to Newton’s first law. The car’s engine produces a forward force; friction, a force between the road and the tires of the car that opposes forward motion, has exactly the same magnitude as the engine force, producing the net force of zero. The body continues in its state of constant velocity until the net force becomes nonzero. Realize that **a net force of zero means that an object is either at rest or moving with constant velocity, that is, it is not accelerating**. What do you suppose happens when the car accelerates? We explore this idea in the next section.

Significance

As this example shows, there are two kinds of equilibrium. In (a), the car is at rest; we say it is in **static equilibrium**. In (b), the forces on the car are balanced, but the car is moving; we say that it is in **dynamic equilibrium**. Again, it is possible for two (or more) forces to act on an object yet for the object to move. In addition, a net force of zero cannot produce acceleration.

? Exercise 5.2

A skydiver opens his parachute, and shortly thereafter, he is moving at constant velocity. (a) What forces are acting on him? (b) Which force is bigger?

Simulation

Engage in this simulation to predict, qualitatively, how an external force will affect the speed and direction of an object’s motion. Explain the effects with the help of a free-body diagram. Use free-body diagrams to draw position, velocity, acceleration, and force graphs, and vice versa. Explain how the graphs relate to one another. Given a scenario or a graph, sketch all four graphs.

14.3: Details on Newton's Second Law

Newton's second law is closely related to his first law. It mathematically gives the cause-and-effect relationship between force and changes in motion. Newton's second law is quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton's second law as a simple equation that gives the exact relationship of force, mass, and acceleration, we need to sharpen some ideas we mentioned earlier.

Force and Acceleration

First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is acceleration. Newton's first law says that a net external force causes a change in motion; thus, we see that a **net external force causes nonzero acceleration**.

We defined external force as force acting on an object or system that originates outside of the object or system. Let's consider this concept further. An intuitive notion of **external** is correct—it is outside the system of interest. For example, in Figure $14.3.1a$, the system of interest is the car plus the person within it. The two forces exerted by the two students are external forces. In contrast, an internal force acts between elements of the system. Thus, the force the person in the car exerts to hang on to the steering wheel is an internal force between elements of the system of interest. Only external forces affect the motion of a system, according to Newton's first law. (The internal forces cancel each other out, as explained in the next section.) Therefore, we must define the boundaries of the system before we can determine which forces are external. Sometimes, the system is obvious, whereas at other times, identifying the boundaries of a system is more subtle. The concept of a system is fundamental to many areas of physics, as is the correct application of Newton's laws. This concept is revisited many times in the study of physics.

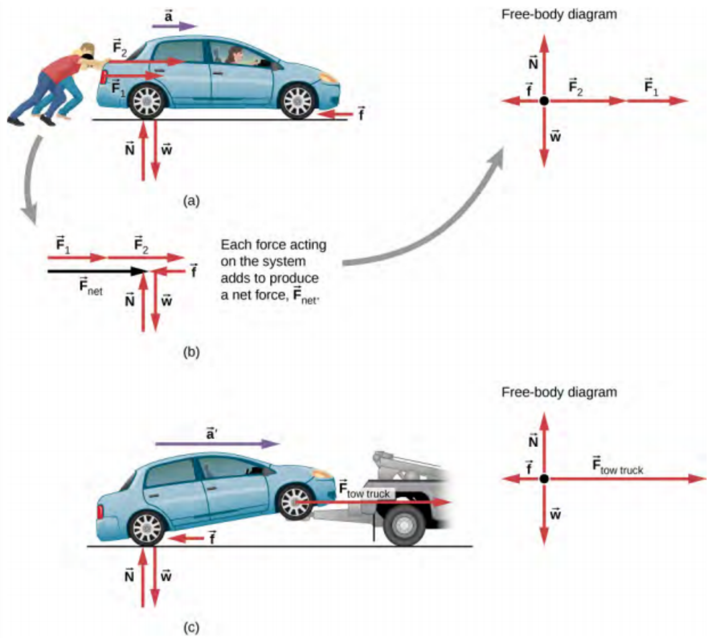


Figure 14.3.1: Different forces exerted on the same mass produce different accelerations. (a) Two students push a stalled car. All external forces acting on the car are shown. (b) The forces acting on the car are transferred to a coordinate plane (free-body diagram) for simpler analysis. (c) The tow truck can produce greater external force on the same mass, and thus greater acceleration.

From this example, you can see that different forces exerted on the same mass produce different accelerations. In Figure $14.3.1a$, the two students push a car with a driver in it. Arrows representing all external forces are shown. The system of interest is the car and its driver. The weight $\vec{w}$ of the system and the support of the ground $\vec{N}$ are also shown for completeness and are assumed to cancel (because there was no vertical motion and no imbalance of forces in the vertical direction to create a change in motion). The vector $\vec{f}$ represents the friction acting on the car, and it acts to the left, opposing the motion of the car. (We discuss friction in more detail in an upcoming chapter.) In Figure $14.3.1b$, all external forces acting on the system add together to produce the net force $\vec{F}_{net}$. The free-body diagram shows all of the forces acting on the system of interest. The dot represents the center of mass of the system. Each force vector extends from this dot. Because there are two forces acting to the right, the vectors are shown collinearly. Finally, in Figure $14.3.1c$, a larger net external force produces a larger acceleration ($\vec{a}' > \vec{a}$) when the tow truck pulls the car.

It seems reasonable that acceleration would be directly proportional to and in the same direction as the net external force acting on a system. This assumption has been verified experimentally and is illustrated in Figure $14.3.1$. To obtain an equation for Newton's second law, we first write the relationship of acceleration $\vec{a}$ and net external force $\vec{F}_{net}$ as the proportionality

$$\vec{a} \propto \vec{F}_{net} \tag{14.3.1}$$

where the symbol α means "proportional to." (Recall from the first section of this chapter that the net external force is the vector sum of all external forces and is sometimes indicated as $\sum \vec{F}$.) This proportionality shows what we have said in words—acceleration is directly proportional to net external force. Once the system of interest is chosen, identify the external forces and ignore the internal ones. It is a tremendous simplification to disregard the numerous internal forces acting between objects within the system, such as muscular forces within the students' bodies, let alone the myriad forces between the atoms in the objects. Still, this simplification helps us solve some complex problems.

It also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. As illustrated in Figure 14.3.2, the same net external force applied to a basketball produces a much smaller acceleration when it is applied to an SUV. The proportionality is written as

$$a \propto \frac{1}{m}, \tag{14.3.2}$$

where m is the mass of the system and a is the magnitude of the acceleration. Experiments have shown that acceleration is exactly inversely proportional to mass, just as it is directly proportional to net external force.

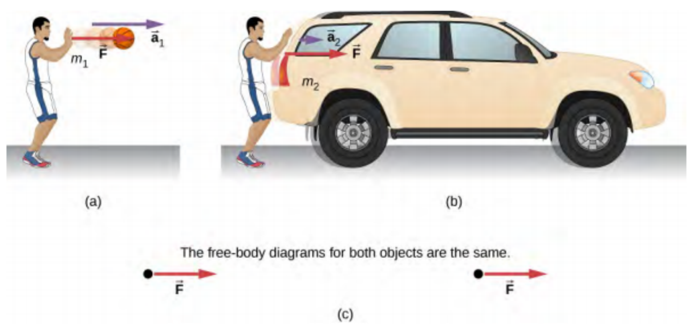


Figure 14.3.2: The same force exerted on systems of different masses produces different accelerations. (a) A basketball player pushes on a basketball to make a pass. (Ignore the effect of gravity on the ball.) (b) The same player exerts an identical force on a stalled SUV and produces far less acceleration. (c) The free-body diagrams are identical, permitting direct comparison of the two situations. A series of patterns for free-body diagrams will emerge as you do more problems and learn how to draw them in Drawing Free-Body Diagrams.

It has been found that the acceleration of an object depends only on the net external force and the mass of the object. Combining the two proportionalities just given yields **Newton's second law**.

Newton's Second Law of Motion

The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system and is inversely proportion to its mass. In equation form, Newton's second law is

$$\vec{a} = \frac{\vec{F}_{net}}{m}, \tag{14.3.3}$$

where $\vec{a}$ is the acceleration, $\vec{F}_{net}$ is the net force, and m is the mass. This is often written in the more familiar form

$$\vec{F}_{net} = \sum \vec{F} = m\vec{a}, \tag{14.3.4}$$

but the first equation gives more insight into what Newton's second law means.

The law is a cause-and-effect relationship among three quantities that is not simply based on their definitions. The validity of the second law is based on experimental verification. The free-body diagram, which you will learn to draw in Drawing Free-Body Diagrams, is the basis for writing Newton's second law.

Example 5.2: What Acceleration Can a Person Produce When Pushing a Lawn Mower?

Suppose that the net external force (push minus friction) exerted on a lawn mower is 51 N (about 11 lb.) parallel to the ground (Figure 14.3.3). The mass of the mower is 24 kg. What is its acceleration?

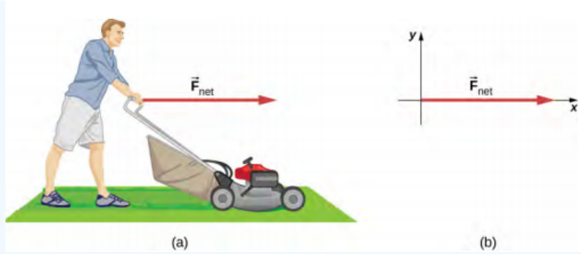


Figure 14.3.3: (a) The net force on a lawn mower is 51 N to the right. At what rate does the lawn mower accelerate to the right? (b) The free-body diagram for this problem is shown.

Strategy

This problem involves only motion in the horizontal direction; we are also given the net force, indicated by the single vector, but we can suppress the vector nature and concentrate on applying Newton's second law. Since F_{net} and m are given, the acceleration can be calculated directly from Newton's second law as F_{net} = ma.

Solution

The magnitude of the acceleration a is a = $\frac{F_{net}}{m}$. Entering known values gives

$$a = \frac{51\ N}{24\ kg}. \tag{14.3.5}$$

Substituting the unit of kilograms times meters per square second for newtons yields

$$a = \frac{51\ kg \cdot m/s^2}{24\ kg} = 2.1\ m/s^2. \tag{14.3.6}$$

Significance

The direction of the acceleration is the same direction as that of the net force, which is parallel to the ground. This is a result of the vector relationship expressed in Newton's second law, that is, the vector representing net force is the scalar multiple of the acceleration vector. There is no information given in this example about the individual external forces acting on the system, but we can say something about their relative magnitudes. For example, the force exerted by the person pushing the mower must be greater than the friction opposing the motion (since we know the mower moved forward), and the vertical forces must cancel because no acceleration occurs in the vertical direction (the mower is moving only horizontally). The acceleration found is small enough to be reasonable for a person pushing a mower. Such an effort would not last too long, because the person's top speed would soon be reached.

? Exercise 5.3

At the time of its launch, the HMS Titanic was the most massive mobile object ever built, with a mass of 6.0 x 10^7 kg. If a force of 6 MN (6 x 10^6 N) was applied to the ship, what acceleration would it experience?

In the preceding example, we dealt with net force only for simplicity. However, several forces act on the lawn mower. The weight pulls down on the mower, toward the center of Earth; this produces a contact force on the ground. The ground must exert an upward force on the lawn mower, known as the normal force $\vec{N}$, which we define in Motion from Forces. These forces are balanced and therefore do not produce vertical acceleration. In the next example, we show both of these forces. As you continue to solve problems using Newton's second law, be sure to show multiple forces.

✓ Example 5.3: Which Force Is Bigger?

a. The car shown in Figure 14.3.4 is moving at a constant speed. Which force is bigger, $\vec{F}_{engine}$ or $\vec{F}_{friction}$? Explain.
b. The same car is now accelerating to the right. Which force is bigger, $\vec{F}_{engine}$ or $\vec{F}_{friction}$? Explain.

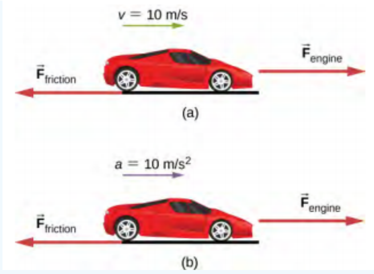


Figure 14.3.4: A car is shown (a) moving at constant speed and (b) accelerating. How do the forces acting on the car compare in each case? (a) What does the knowledge that the car is moving at constant velocity tell us about the net horizontal force on the car compared to the friction force? (b) What does the knowledge that the car is accelerating tell us about the horizontal force on the car compared to the friction force?

Strategy

We must consider Newton's first and second laws to analyze the situation. We need to decide which law applies; this, in turn, will tell us about the relationship between the forces.

Solution

a. The forces are equal. According to Newton's first law, if the net force is zero, the velocity is constant.
b. In this case, $\vec{F}_{engine}$ must be larger than $\vec{F}_{friction}$. According to Newton's second law, a net force is required to cause acceleration.

Significance

These questions may seem trivial, but they are commonly answered incorrectly. For a car or any other object to move, it must be accelerated from rest to the desired speed; this requires that the engine force be greater than the friction force. Once the car is moving at constant velocity, the net force must be zero; otherwise, the car will accelerate (gain speed). To solve problems involving Newton's laws, we must understand whether to apply Newton's first law (where $\sum \vec{F} = \vec{0}$) or Newton's second law (where $\sum \vec{F}$ is not zero). This will be apparent as you see more examples and attempt to solve problems on your own.

✔ Example 5.4: What Rocket Thrust Accelerates This Sled?

Before manned space flights, rocket sleds were used to test aircraft, missile equipment, and physiological effects on human subjects at high speeds. They consisted of a platform that was mounted on one or two rails and propelled by several rockets.

Calculate the magnitude of force exerted by each rocket, called its thrust T, for the four-rocket propulsion system shown in Figure 14.3.5. The sled's initial acceleration is 49 m/s^2, the mass of the system is 2100 kg, and the force of friction opposing the motion is 650 N.

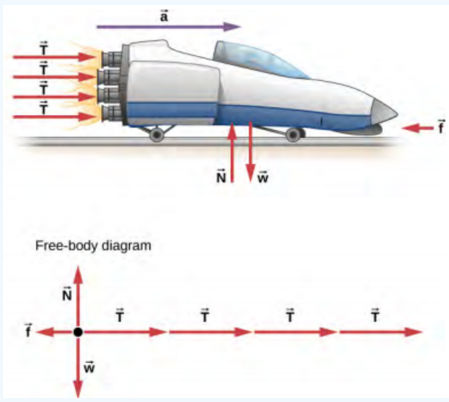


Figure 14.3.5: A sled experiences a rocket thrust that accelerates it to the right. Each rocket creates an identical thrust T. The system here is the sled, its rockets, and its rider, so none of the forces between these objects are considered. The arrow representing friction ($\vec{f}$) is drawn larger than scale.

Strategy

Although forces are acting both vertically and horizontally, we assume the vertical forces cancel because there is no vertical acceleration. This leaves us with only horizontal forces and a simpler one-dimensional problem. Directions are indicated with plus or minus signs, with right taken as the positive direction. See the free-body diagram in Figure 14.3.5.

Solution

Since acceleration, mass, and the force of friction are given, we start with Newton's second law and look for ways to find the thrust of the engines. We have defined the direction of the force and acceleration as acting "to the right," so we need to consider only the magnitudes of these quantities in the calculations. Hence we begin with

$$F_{net} = ma \tag{14.3.7}$$

where F_{net} is the net force along the horizontal direction. We can see from the figure that the engine thrusts add, whereas friction opposes the thrust. In equation form, the net external force is

$$F_{net} = 4T - f. \tag{14.3.8}$$

Substituting this into Newton's second law gives us

$$F_{net} = ma = 4T - f. \tag{14.3.9}$$

Using a little algebra, we solve for the total thrust 4T:

$$4T = ma + f. \tag{14.3.10}$$

Substituting known values yields

$$4T = ma + f = (2100\ kg)(49\ m/s^2) + 650\ N. \tag{14.3.11}$$

Therefore, the total thrust is

$$4T = 1.0 \times 10^5\ N. \tag{14.3.12}$$

Significance

The numbers are quite large, so the result might surprise you. Experiments such as this were performed in the early 1960s to test the limits of human endurance, and the setup was designed to protect human subjects in jet fighter emergency ejections. Speeds of 1000 km/h were obtained, with accelerations of 45 g's. (Recall that g, acceleration due to gravity, is 9.80 m/s^2. When we say that acceleration is 45 g's, it is 45 x 9.8 m/s^2, which is approximately 440 m/s^2.) Although living subjects are not used anymore, land speeds of 10,000 km/h have been obtained with a rocket sled.

In this example, as in the preceding one, the system of interest is obvious. We see in later examples that choosing the system of interest is crucial—and the choice is not always obvious.

Newton's second law is more than a definition; it is a relationship among acceleration, force, and mass. It can help us make predictions. Each of those physical quantities can be defined independently, so the second law tells us something basic and universal about nature.

? Exercise 5.4

A 550-kg sports car collides with a 2200-kg truck, and during the collision, the net force on each vehicle is the force exerted by the other. If the magnitude of the truck's acceleration is 10 m/s^2, what is the magnitude of the sports car's acceleration?

Component Form of Newton's Second Law

We have developed Newton's second law and presented it as a vector equation in Equation 14.3.4. This vector equation can be written as three component equations:

$$\sum \vec{F}_x = m\vec{a}_x, \sum \vec{F}_y = m\vec{a}_y, \sum \vec{F}_z = m\vec{a}_z. \tag{14.3.13}$$

The second law is a description of how a body responds mechanically to its environment. The influence of the environment is the net force $\vec{F}_{net}$, the body's response is the acceleration $\vec{a}$, and the strength of the response is inversely proportional to the mass m. The larger the mass of an object, the smaller its response (its acceleration) to the influence of the environment (a given net force). Therefore, a body's mass is a measure of its inertia, as we explained in Newton's First Law.

✓ Example 5.5: Force on a Soccer Ball

A 0.400-kg soccer ball is kicked across the field by a player; it undergoes acceleration given by $\vec{a}$ = 3.00 $\hat{i}$ + 7.00 $\hat{j}$ m/s^2. Find (a) the resultant force acting on the ball and (b) the magnitude and direction of the resultant force.

Strategy

The vectors in $\hat{i}$ and $\hat{j}$ format, which indicate force direction along the x-axis and the y-axis, respectively, are involved, so we apply Newton's second law in vector form.

Solution

a. We apply Newton's second law: $\vec{F}_{net} = m\vec{a} = (0.400\ kg)(3.00\ \hat{i} + 7.00\ \hat{j}\ m/s^2) = 1.20\ \hat{i} + 2.80\ \hat{j}\ N.$

b. . Magnitude and direction are found using the components of $\vec{F}_{net}$:

$$F_{net} = \sqrt{(1.20\ N)^2 + (2.80\ N)^2} = 3.05\ N\ and\ \theta = \tan^{-1}\left(\frac{2.80}{1.20}\right) = 66.8^o.$$

Significance

We must remember that Newton's second law is a vector equation. In (a), we are multiplying a vector by a scalar to determine the net force in vector form. While the vector form gives a compact representation of the force vector, it does not tell us how "big" it is, or where it goes, in intuitive terms. In (b), we are determining the actual size (magnitude) of this force and the direction in which it travels.

Example 5.6: Mass of a Car

Find the mass of a car if a net force of −600.0 $\hat{j}$ N produces an acceleration of −0.2 $\hat{j}$ m/s^2.

Strategy

Vector division is not defined, so $m = \frac{\vec{F}_{net}}{\vec{a}}$ cannot be performed. However, mass m is a scalar, so we can use the scalar form of Newton's second law, $m = \frac{F_{net}}{a}$.

Solution

We use m = $\frac{F_{net}}{a}$ and substitute the magnitudes of the two vectors: F_{net} = 600.0 N and a = 0.2 m/s^2. Therefore,

$$m = \frac{F_{net}}{a} = \frac{600.0\ N}{0.2\ m/s^2} = 3000\ kg.$$

Significance

Force and acceleration were given in the $\hat{i}$ and $\hat{j}$ format, but the answer, mass m, is a scalar and thus is not given in $\hat{i}$ and $\hat{j}$ form.

Example 5.7

Several Forces on a Particle A particle of mass m = 4.0 kg is acted upon by four forces of magnitudes. F_1 = 10.0 N, F_2 = 40.0 N, F_3 = 5.0 N, and F_4 = 2.0 N, with the directions as shown in the free-body diagram in Figure 14.3.6. What is the acceleration of the particle?

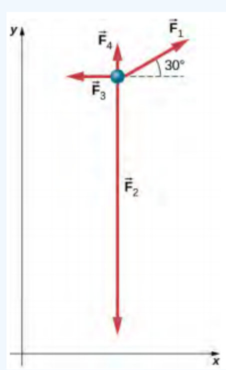


Figure 14.3.6: Four forces in the xy-plane are applied to a 4.0-kg particle.

Strategy

Because this is a two-dimensional problem, we must use a free-body diagram. First, $\vec{F}_1$ must be resolved into x- and y-components. We can then apply the second law in each direction.

Solution

We draw a free-body diagram as shown in Figure 14.3.6. Now we apply Newton's second law. We consider all vectors resolved into x- and y-components:

$$\sum F_x = ma_x \quad (14.3.14)$$
$$F_{1x} - F_{3x} = ma_x \quad (14.3.15)$$
$$F_1 \cos 30^o - F_{3x} = ma_x \quad (14.3.16)$$
$$(10.0\ N)(\cos 30^o) - 5.0\ N = (4.0\ kg)a_x \quad (14.3.17)$$
$$a_x = 0.92\ m/s^2. \quad (14.3.18)$$

$$\sum F_y = ma_y \quad (14.3.19)$$
$$F_{1y} + F_{4y} - F_{2y} = ma_y \quad (14.3.20)$$
$$F_1 \sin 30^o + F_{4y} - F_{2y} = ma_y \quad (14.3.21)$$
$$(10.0\ N)(\sin 30^o) + 2.0\ N - 40.0\ N = (4.0\ kg)a_y \quad (14.3.$$
$$a_y = -8.3\ m/s^2. \quad (14.3.23)$$

Thus, the net acceleration is

$$\vec{a} = (0.92\ \hat{i} - 8.3\ \hat{j})m/s^2, \quad (14.3.24)$$

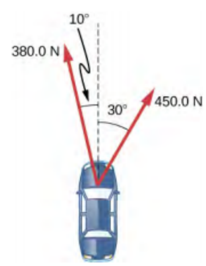
which is a vector of magnitude 8.4 m/s^2 directed at 276° to the positive x-axis.

Significance

Numerous examples in everyday life can be found that involve three or more forces acting on a single object, such as cables running from the Golden Gate Bridge or a football player being tackled by three defenders. We can see that the solution of this example is just an extension of what we have already done.

? Exercise 5.5

A car has forces acting on it, as shown below. The mass of the car is 1000.0 kg. The road is slick, so friction can be ignored. (a) What is the net force on the car? (b) What is the acceleration of the car?



Simulation

Explore the forces at work when pulling a cart or pushing a refrigerator, crate, or person. Create an applied force and see how it makes objects move. Put an object on a ramp and see how it affects its motion.

14.4: Details on Newton’s Third Law

We have thus far considered force as a push or a pull; however, if you think about it, you realize that no push or pull ever occurs by itself. When you push on a wall, the wall pushes back on you. This brings us to Newton’s third law.

Newton’s Third Law of Motion

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts. Mathematically, if a body A exerts a force $\vec{F}$ on body B, then B simultaneously exerts a force $-\vec{F}$ on A, or in vector equation form,

$$\vec{F}_{AB} = -\vec{F}_{BA}. \tag{14.4.1}$$

Newton’s third law represents a certain symmetry in nature: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as “action-reaction,” where the force exerted is the action and the force experienced as a consequence is the reaction. Newton’s third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system.

We can readily see Newton’s third law at work by taking a look at how people move about. Consider a swimmer pushing off the side of a pool (Figure 14.4.1). She pushes against the wall of the pool with her feet and accelerates in the direction opposite that of her push. The wall has exerted an equal and opposite force on the swimmer. You might think that two equal and opposite forces would cancel, but they do not **because they act on different systems**. In this case, there are two systems that we could investigate: the swimmer and the wall. If we select the swimmer to be the system of interest, as in the figure, then $F_{\text{wall on feet}}$ is an external force on this system and affects its motion. The swimmer moves in the direction of this force. In contrast, the force $F_{\text{feet on wall}}$ acts on the wall, not on our system of interest. Thus, $F_{\text{feet on wall}}$ does not directly affect the motion of the system and does not cancel $F_{\text{wall on feet}}$. The swimmer pushes in the direction opposite that in which she wishes to move. The reaction to her push is thus in the desired direction. In a free-body diagram, such as the one shown in Figure 14.4.1, we never include both forces of an action-reaction pair; in this case, we only use $F_{\text{wall on feet}}$, not $F_{\text{feet on wall}}$.

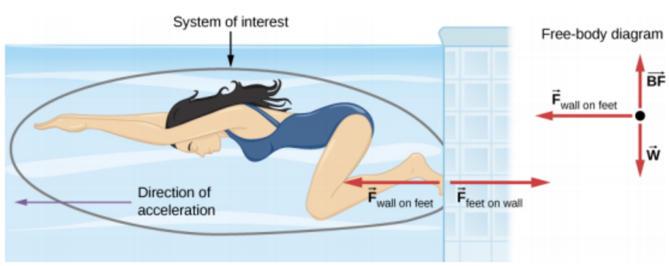


Figure 14.4.1: When the swimmer exerts a force on the wall, she accelerates in the opposite direction; in other words, the net external force on her is in the direction opposite of $F_{\text{feet on wall}}$. This opposition occurs because, in accordance with Newton’s third law, the wall exerts a force $F_{\text{wall on feet}}$ on the swimmer that is equal in magnitude but in the direction opposite to the one she exerts on it. The line around the swimmer indicates the system of interest. Thus, the free-body diagram shows only $F_{\text{wall on feet}}$, w (the gravitational force), and BF, which is the buoyant force of the water supporting the swimmer’s weight. The vertical forces w and BF cancel because there is no vertical acceleration.

Other examples of Newton’s third law are easy to find:

- As a professor paces in front of a whiteboard, he exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes him to accelerate forward.
- A car accelerates forward because the ground pushes forward on the drive wheels, in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw the rocks backward.
- Rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber; therefore, the gas exerts a large reaction force forward on the rocket. This reaction force, which pushes a body forward in response to a backward force, is called **thrust**. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases.
- Helicopters create lift by pushing air down, thereby experiencing an upward reaction force.

- Birds and airplanes also fly by exerting force on the air in a direction opposite that of whatever force they need. For example, the wings of a bird force air downward and backward to get lift and move forward.
- An octopus propels itself in the water by ejecting water through a funnel from its body, similar to a jet ski.
- When a person pulls down on a vertical rope, the rope pulls up on the person (Figure $14.4.2$).

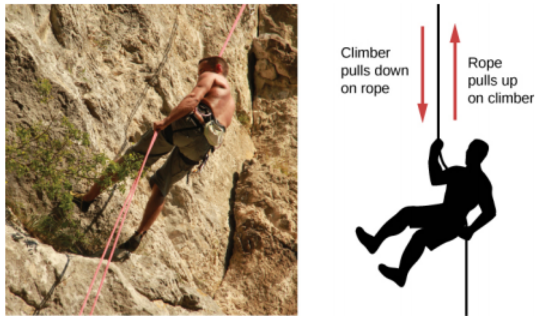


Figure $14.4.2$: When the mountain climber pulls down on the rope, the rope pulls up on the mountain climber.

There are two important features of Newton's third law. First, the forces exerted (the action and reaction) are always equal in magnitude but opposite in direction. Second, these forces are acting on different bodies or systems: A's force acts on B and B's force acts on A. In other words, the two forces are distinct forces that do not act on the same body. Thus, they do not cancel each other.

A person who is walking or running applies Newton's third law instinctively. For example, the runner in Figure $14.4.3$ pushes backward on the ground so that it pushes him forward.

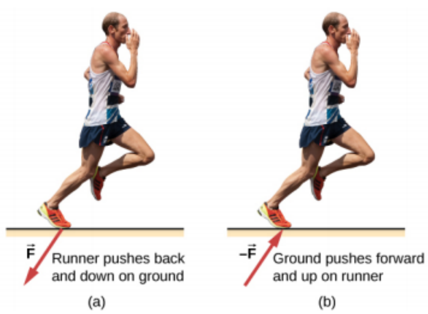


Figure $14.4.3$: The runner experiences Newton's third law. (a) A force is exerted by the runner on the ground. (b) The reaction force of the ground on the runner pushes him forward.

Example 5.9: Forces on a Stationary Object

The package in Figure $14.4.4$is sitting on a scale. The forces on the package are $\vec{S}$, which is due to the scale, and $-\vec{w}$, which is due to Earth's gravitational field. The reaction forces that the package exerts are $-\vec{S}$ on the scale and $\vec{w}$ on Earth. Because the package is not accelerating, application of the second law yields

$$\vec{S}-\vec{w}=m\vec{a}=\vec{0}, \tag{14.4.2}$$

so

$$\vec{S}=\vec{w}. \tag{14.4.3}$$

Thus, the scale reading gives the magnitude of the package's weight. However, the scale does not measure the weight of the package; it measures the force $-\vec{S}$ on its surface. If the system is accelerating, $\vec{S}$ and $-\vec{w}$ would not be equal, as explained in Applications of Newton's Laws.

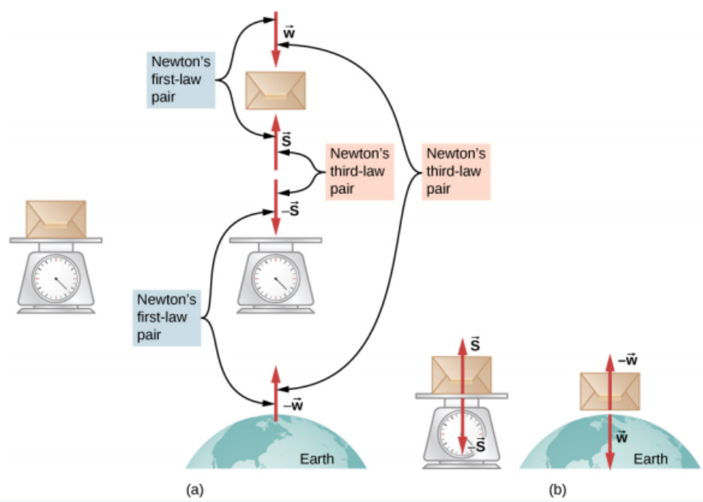


Figure 14.4.4: (a) The forces on a package sitting on a scale, along with their reaction forces. The force $\vec{w}$ is the weight of the package (the force due to Earth's gravity) and $\vec{S}$ is the force of the scale on the package. (b) Isolation of the package-scale system and the package-Earth system makes the action and reaction pairs clear.

Example 5.10: Getting Up to Speed: Choosing the Correct System

A physics professor pushes a cart of demonstration equipment to a lecture hall (Figure 14.4.5). Her mass is 65.0 kg, the cart's mass is 12.0 kg, and the equipment's mass is 7.0 kg. Calculate the acceleration produced when the professor exerts a backward force of 150 N on the floor. All forces opposing the motion, such as friction on the cart's wheels and air resistance, total 24.0 N.

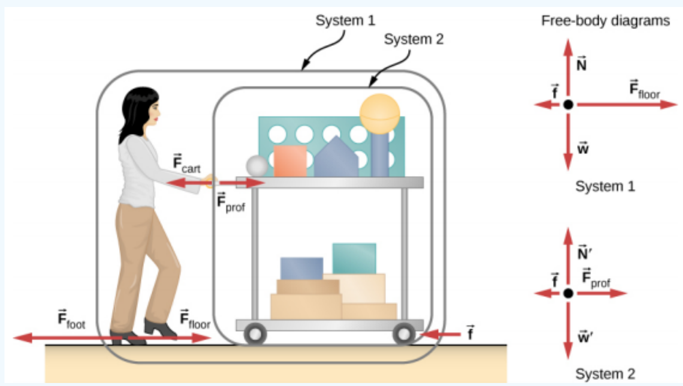


Figure 14.4.5: A professor pushes the cart with her demonstration equipment. The lengths of the arrows are proportional to the magnitudes of the forces (except for $\vec{f}$, because it is too small to drawn to scale). System 1 is appropriate for this example, because it asks for the acceleration of the entire group of objects. Only $\vec{F}_{floor}$ and $\vec{f}$ are external forces acting on System 1 along the line of motion. All other forces either cancel or act on the outside world. System 2 is chosen for the next example so that $\vec{F}_{prof}$ is an external force and enters into Newton's second law. The free-body diagrams, which serve as the basis for Newton's second law, vary with the system chosen.

Strategy

Since they accelerate as a unit, we define the system to be the professor, cart, and equipment. This is System 1 in Figure 14.4.5. The professor pushes backward with a force F_{foot} of 150 N. According to Newton's third law, the floor exerts a forward reaction force F_{floor} of 150 N on System 1. Because all motion is horizontal, we can assume there is no net force in the vertical direction. Therefore, the problem is one-dimensional along the horizontal direction. As noted, friction f opposes the motion and is thus in the opposite direction of F_{floor}. We do not include the forces F_{prof} or F_{cart} because these are internal forces, and we do not include F_{foot} because it acts on the floor, not on the system. There are no other significant forces acting on System 1. If the net external force can be found from all this information, we can use Newton's second law to find the acceleration as requested. See the free-body diagram in the figure.

Solution

Newton's second law is given by

$$a = \frac{F_{net}}{m}. \tag{14.4.4}$$

The net external force on System 1 is deduced from Figure 14.4.5 and the preceding discussion to be

$$F_{net} = F_{floor} - f = 150\ N - 24.0\ N = 126\ N. \tag{14.4.5}$$

The mass of System 1 is

$$m = (65.0 + 12.0 + 7.0)\ kg = 84\ kg. \tag{14.4.6}$$

These values of F_{net} and m produce an acceleration of

$$a = \frac{F_{net}}{m} = \frac{126\ N}{84\ kg} = 1.5\ m/s^2. \tag{14.4.7}$$

Significance

None of the forces between components of System 1, such as between the professor's hands and the cart, contribute to the net external force because they are internal to System 1. Another way to look at this is that forces between components of a system cancel because they are equal in magnitude and opposite in direction. For example, the force exerted by the professor on the cart results in an equal and opposite force back on the professor. In this case, both forces act on the same system and therefore cancel. Thus, internal forces (between components of a system) cancel. Choosing System 1 was crucial to solving this problem.

Example 5.11: Force on the Cart: Choosing a New System

Calculate the force the professor exerts on the cart in Figure 14.4.5, using data from the previous example if needed.

Strategy

If we define the system of interest as the cart plus the equipment (System 2 in Figure 14.4.5), then the net external force on System 2 is the force the professor exerts on the cart minus friction. The force she exerts on the cart, F_{prof}, is an external force acting on System 2. F_{prof} was internal to System 1, but it is external to System 2 and thus enters Newton's second law for this system.

Solution

Newton's second law can be used to find F_{prof}. We start with

$$a = \frac{F_{net}}{m}. \tag{14.4.8}$$

The magnitude of the net external force on System 2 is

$$F_{net} = F_{prof} - f. \tag{14.4.9}$$

We solve for F_{prof}, the desired quantity:

$$F_{prof} = F_{net} + f. \tag{14.4.10}$$

The value of f is given, so we must calculate net F_{net}. That can be done because both the acceleration and the mass of System 2 are known. Using Newton's second law, we see that

$$F_{net} = ma, \tag{14.4.11}$$

where the mass of System 2 is 19.0 kg (m = 12.0 kg + 7.0 kg) and its acceleration was found to be a = 1.5 m/s^2 in the previous example. Thus,

$$F_{net} = ma = (19.0\ kg)(1.5\ m/s^2) = 29\ N. \tag{14.4.12}$$

Now we can find the desired force:

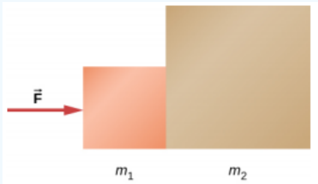
$$F_{prof} = F_{net} + f = 29\ N + 24.0\ N = 53\ N. \tag{14.4.13}$$

Significance

This force is significantly less than the 150-N force the professor exerted backward on the floor. Not all of that 150-N force is transmitted to the cart; some of it accelerates the professor. The choice of a system is an important analytical step both in solving problems and in thoroughly understanding the physics of the situation (which are not necessarily the same things).

? Exercise 5.7

Two blocks are at rest and in contact on a frictionless surface as shown below, with $m_1 = 2.0$ kg, $m_2 = 6.0$ kg, and applied force 24 N. (a) Find the acceleration of the system of blocks. (b) Suppose that the blocks are later separated. What force will give the second block, with the mass of 6.0 kg, the same acceleration as the system of blocks?



Note

View this video to watch examples of action and reaction. View this video to watch examples of Newton's laws and internal and external forces.

14.5: Free-Body Diagrams

Trying to draw every single force acting on every single object can very quickly become pretty messy. And anyway, this is not usually what we need: what we need is to separate cleanly all the forces acting on any given object, one object at a time, so we can apply Newton's second law, $F_{net} = ma$, to each object individually.

In order to accomplish this, we use what are known as *free-body diagrams*. In a free-body diagram, a potentially very complicated object is replaced symbolically by a dot or a small circle, and all the forces acting on the object are drawn (approximately to scale and properly labeled) as acting on the dot. Regardless of whether a force is a pulling or pushing force, the convention is to always draw it *as a vector that originates at the dot*. If the system is accelerating, it is also a good idea to indicate the acceleration's direction also somewhere on the diagram.

The figure below shows, as an example, a free-body diagram for a block, in the presence of both a nonzero acceleration and a friction force. The diagram includes all the forces, even gravity and the normal force.

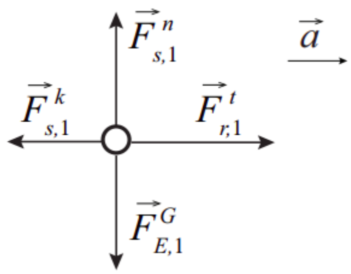


Figure 14.5.1, with the friction force adjusted so as to be compatible with a nonzero acceleration to the right.

Note that I have drawn F^n and the force of gravity $F^G_{E,1}$ as having the same magnitude, since there is no vertical acceleration for that block. If I know the value of the friction force, I should also try to draw F^k approximately to scale with the other two forces. Then, since I know that there is an acceleration to the right, I need to draw F^t greater than F^k, since the net force on the block must be to the right as well.

14.6: Vector Calculus

As we have seen, the study of physics is all about creating a mathematical abstraction of the world, and what kinds of mathematics are required depends on what we want to describe about the world. A basic feature of how the Universe works is "smoothly" - objects move gradually from one point to the next, without stopping. (The alternative to this might be some kind of pixilated version of the universe, in which objects can only exist on a grid, and they move by jumping from point to point. Like a video game or something?) That means that the mathematics we use to describe the universe must similarly be "smooth" - and the mathematics of gradual change mathematics is **calculus**. In addition to gradual change, we've also seen that some quantities in physics can be **vectors** - that is, have both magnitudes and directions. So, clearly, we are going to have to combine these two ideas if we are going to more fully understand how to describe the physical world, into a field of mathematics called **vector calculus**.

In truth, vector calculus can be enormously complicated (as well as enourmously rich and interesting!), but fortunately for mechanics we only need to know the basics of how calculus and vectors interact with each other. We will simply need to know how to take derivatives and integrals of vectors, *e.g.*

$$\vec{a}(t)=\frac{d}{dt}\vec{v}(t), \qquad \vec{v}(t)=\int \vec{r}(t')dt'. \tag{14.6.1}$$

(The two specific examples here are the acceleration as a derivative of velocity, and the velocity as an integral of positive.) At first glance, it might not be obvious how to proceed, but with a little reflection we can see the answer: rewrite the vectors using unit vectors,

$$\vec{v}(t)=v_x(t)\hat{x}+v_y(t)\hat{y}, \qquad \vec{r}(t)=x(t)\hat{x}+y(t)\hat{y}. \tag{14.6.2}$$

Now, if we just replace these two quantities in the expressions above, we can use the additive nature of integrals and derivatives to rewrite them in expressions we understand from usual calculus:

$$\vec{a}(t)=\frac{d}{dt}\vec{v}(t)=\frac{d}{dt}(v_x(t)\hat{x}+v_y(t)\hat{y})=\frac{dv_x(t)}{dt}\hat{x}+\frac{dv_y(t)}{dt}\hat{y} \tag{14.6.3}$$

$$\vec{v}(t)=\int \vec{r}(t')dt'=\int(x(t')\hat{x}+y(t')\hat{y})dt'=\left(\int x(t')dt'\right)\hat{x}+\left(\int y(t')dt'\right)\hat{y} \tag{14.6.4}$$

Notice the important thing that happened - *the unit vector is constant* (in time, in this case). In more detail, the derivative could be written as a chain rule,

$$\frac{d}{dt}(v_x(t)\hat{x})=\frac{dv_x(t)}{dt}\hat{x}+v_x(t)\frac{d\hat{x}}{dt}, \tag{14.6.5}$$

but since $\hat{x}$ is constant this second term is just zero. So, **the derivatives and integrals just pass right through the vector onto the components individually**, and we can do all our usual calculus operations on them without changing what anything means.

There is one slight complication that is probably worth mentioning - what happens to vector products, like dot products and cross products? For example, the definition of work is

$$W=\int \vec{F}(\vec{r})\cdot d\vec{r}. \tag{14.6.6}$$

The basic trick to understanding this expression is the same - use unit vectors. In cartesian coordinates, the force will simply be

$$\vec{F}=F_x(\vec{r})\hat{x}+F_y(\vec{r})\hat{y}, \tag{14.6.7}$$

while the infinitesimal element will be

$$d\vec{r}=dx\hat{x}+dy\hat{y} \tag{14.6.8}$$

(this expression looks a little strange, but it's simply dx in the x-direction and dy in the y-direction). Now we can take the dot product and follow the additive rules for integrals that we followed above:

$$\int \vec{F}(\vec{r})\cdot d\vec{r}=\int(F_x(\vec{r})dx+F_y(\vec{r})dy)=\int F_x(\vec{r})dx+\int F_y(\vec{r})dy. \tag{14.6.9}$$

Thus, the integrals break into an integral of dx and an integral of dy, which you can perform as you normally would. Now we are still not quite done - the force could be some complicated function of either the vector itself, like $\vec{F}(\vec{r}) = r^2\hat{r}$, or the coordinates like $\vec{F}(\vec{r}) = xy\hat{x} + y^2\hat{y}$. In this case we would have to have a relationship between x and y to perform the integrals (this is generally called a line integral, and you can find more information about those at Wikipedia or Khan Academy). We will have to treat these particular cases with a little bit of care!

14.7: Examples

Whiteboard Problem 14.7.1: A Pair of Vector Problems

Position to Velocity:

An object's position as a function of time is given by

$$\vec{r}(t)=\frac{c}{bt+1}\hat{x}+cbt\hat{y}+at^2\hat{z}, \tag{14.7.1}$$

where a, b, and c are constants.

1. What are the SI units of a, b, and c?
2. Find an expression of the object's *speed* as a function of time.

Velocity to Acceleration:

An object's velocity as a function of time has components

$$v_x(t)=bt^2+c \tag{14.7.2}$$

$$v_y(t)=qt \tag{14.7.3}$$

$$v_z(t)=0, \tag{14.7.4}$$

where $b=10$ m/s^3, $c=5$ m/s, and $q=-2.0$ m/s^2.

1. What is the magnitude of the object's acceleration at $t=0$?
2. What about at $t=3.0$ s?

CHAPTER OVERVIEW

15: N2) 1 Dimensional Kinematics

If the last chapter, on Newton's laws, can be considered the study of "why" objects move (because of the forces that act on them), we can consider this chapter to focus on the question of "how" objects move. This is the study of **kinematics**. In particular, we are going to focus on the motion of points (rather then extended objects), moving in 1 dimension, and moving under constant acceleration. To start with let's recall the definitions of velocity and acceleration, both the average and instantaneous:

$$v_{x,ave} = \frac{\Delta x}{\Delta t}, \quad v_x = \frac{dx}{dt}, \quad a_{x,ave} = \frac{\Delta v}{\Delta t}, \quad a_x = \frac{dv_x}{dt}. \tag{15.1}$$

Now, under the condition of constant acceleration in 1 dimension, there are two independent equations of motion - one of which describes the position as a function of time, and one of which describes the velocity as a function of time. We'll present them here, and derive them in this chapter:

$$x(t) = \frac{1}{2}a_x t^2 + v_{0x}t + x_0, \tag{15.2}$$

$$v_x(t) = a_x t + v_{0x}. \tag{15.3}$$

These are quite a jumble of symbols, so let's discuss them carefully. The left hand side of the first equation, $x(t)$, is the position as a function of time along the x-direction. That's relatively simple to understand, but keep in mind the functional notation here - for each value of t, we get a position $x(t)$ by evaluating the right hand side of this equation. On the right hand side, we have the acceleration in the x-direction a_x, the initial velocity in the x-direction v_{0x}, and the initial position in the x-direction x_0. Notice the "physicist" convention of labeling these initial values with a zero, 0 - they are pronounced "-naught", like "x-naught". The last variable on this side is the time t.

The second equation is actually simplier after you know what's going on in the first. It's the velocity in the x-direction (again in the functional notation), with the acceleration, time and initial position on the right hand side. The first two things to notice about these two equations:

* They share the exact same set of symbols, with on their the left hand side being unique to each. They are also tied together through the time parameter t.

* The second one is the derivative, with respect to time, of the first. This comes from the calculus behind how these equations work, but we can check it pretty easily:

$$\frac{d}{dt}x(t) = v_x(t), \tag{15.4}$$

$$\frac{d}{dt}\left(\frac{1}{2}a_x t^2 + v_{0x}t + x_0\right) = \frac{d}{dt}\left(\frac{1}{2}a_x t^2\right) + \frac{d}{dt}(v_{0x}t) + \frac{d}{dt}(x_0) = a_x t + v_0. \tag{15.5}$$

The first is just the definition of velocity, whereas the second is carrying out a few pretty easy differentiations.

Using these equations is pretty straightforward, and is usually just a matter of identifying the "knowns and unknowns" in your problem. For example, let's do a runner, who starts at a position of 5 m (from some origin), and accelerates from rest at 2 m/s^2. We can find the position and velocity of this runner at *any* point in the future, so let's say after 5 seconds. The first job is to identify all the variables in the two equations above: x_0=5 m, a_x=2 m/s^2, and v_{0x}=0 (since we said "at rest"). Since we were asked for $t = 5$ seconds, we can just plug in our values to find our two unknowns:

$$x(5\ s) = \frac{1}{2}(2\ m/s^2)(5\ s)^2 + (0)(5\ s) + (5\ m) = 30\ m, \tag{15.6}$$

$$v_x(5\ s) = (2\ m/s^2)(5\ s) + (0) = 10\ m/s. \tag{15.7}$$

It's important to realize that we did not label these positions "initial" and "final" - there is a reason for that. When we studied the conservation laws (momentum and energy), we were frequently concerned with very specific points - the final height of the ball, after a collision, etc. In the kinematic equations, we can consider *any value of time we like*, not only initial and final but any time in-between...not only 5 s, but also 5.1 s, and 4.9 s, or 4.99 s - you get the point. That is what is going on with the left side of these equations: $x(t)$ is "the position at a time t" - you have to know what time t you are specifically referring to in order to use these equations. Another way to think about that is the variable t is something you plug into these expressions - an input that needs to be specified to evaluate them. In this way, you might write these equations with that input more explicit:

$$x(\square) = \frac{1}{2}a_x(\square)^2 + v_{0x}(\square) + x_0, \tag{15.8}$$

$$v_x(\square) = a_x(\square) + v_{0x}. \tag{15.9}$$

Plug something into the boxes, and you are good to go!

15.1: Position, Displacement, Velocity

Kinematics is the part of mechanics that deals with the mathematical description of motion, leaving aside the question of what causes an object to move in a certain way. Kinematics, therefore, does not include such things as forces or energy, which fall instead under the heading of dynamics. It may be said, then, that kinematics by itself is not true physics, but only applied mathematics; yet it is still an essential part of classical mechanics, and its most natural starting point. This chapter (and parts of the next one) will introduce the basic concepts and methods of kinematics in one dimension.

Position

As stated in the previous section, we are initially interested only in describing the motion of a "particle," which can be thought of as a mathematical point in space. A point in three dimensions can be located by giving three numbers, known as its *Cartesian coordinates* (or, more simply, its *coordinates*). In two dimensions, this works as shown in Figure $15.1.1$ below. As you can see, the coordinates of a point just tell us how to find it by first moving a certain distance x, from a previously-agreed origin, along a horizontal (or x) axis, and then a certain distance y along a vertical (or y) axis. (Or, of course, you could equally well first move vertically and then horizontally.)

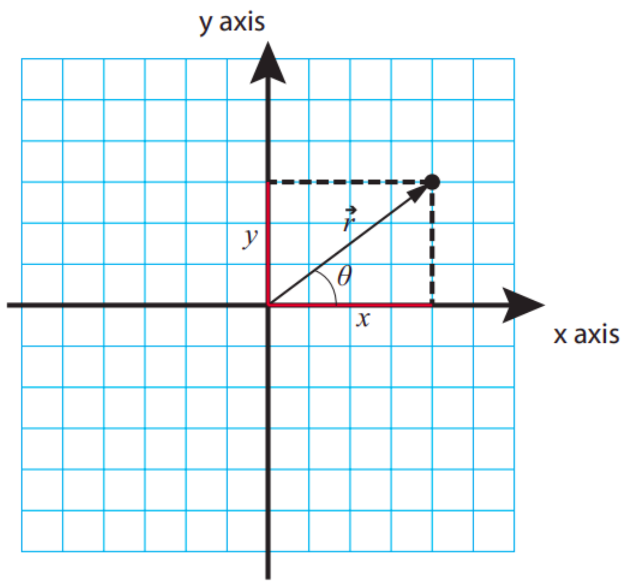


Figure 15.1.1: The position vector, $\vec{r}$, of a point, and its x and y components (the point's coordinates).

The quantities x and y are taken to be positive or negative depending on what side of the origin the point is on. Typically, we will always start by choosing a *positive direction* for each axis, as the direction along which the algebraic value of the corresponding coordinate increases. This is often chosen to be to the right for the horizontal axis, and upwards for the vertical axis, but there is nothing that says we cannot choose a different convention if it turns out to be more convenient. In Figure $15.1.1$, the arrows on the axes denote the positive direction for each. Going by the grid, the coordinates of the point shown are x = 4 units, y = 3 units.

In two or three dimensions (and even, in a sense, in one dimension), the coordinates of a point can be interpreted as the *components of a vector* that we call the point's **position vector**, and denote by $\vec{r}$ (sometimes boldface letters are used for vectors, instead of an arrow on top; in that case, the position vector would be denoted by **r**). A **vector** is a mathematical object, with specific geometric and algebraic properties, that physicists use to represent a quantity that has both a magnitude and a direction. The *magnitude* of the position vector in Figure $15.1.1$ is just the length of the arrow, which is to say, 5 length units (by the Pythagorean theorem, the length of $\vec{r}$, which we will often write using absolute value bars as $|\vec{r}|$, is equal to $\sqrt{x^2+y^2}$); this is just the straight-line distance of the point to the origin. The *direction* of $\vec{r}$, on the other hand, can be specified in a number of ways; a common convention is to give the value of the angle that it makes with the positive x axis, which I have denoted in the figure as θ (in this case, you can verify that $\theta=\tan^{-1}(y/x)=36.9^{\circ}$). In three dimensions, two angles would be needed to completely specify the direction of $\vec{r}$.

As you can see, giving the magnitude and direction of $\vec{r}$ is a way to locate the point that is completely equivalent to giving its coordinates x and y. By the same token, the coordinates x and y are a way to specify the vector $\vec{r}$ that is completely equivalent to giving its magnitude and direction. As I stated above, we call x and y the components (or sometimes, to be more specific, the Cartesian components) of the vector $\vec{r}$. All vectors can be described this way, so once you know how to deal with one vector, you can deal with them all.

For the first few chapters in this topic, we are going to be primarily concerned with motion in one dimension (that is to say, along a straight line, backwards or forwards), in which case all we need to locate a point is one number, its x (or y, or z) coordinate; we do not then need to worry particularly about vector algebra. Alternatively, we can simply say that a vector in one dimension is essentially the same as its only component, which is just a positive or negative number (the magnitude of the number being the magnitude of the vector, and its sign indicating its direction), and has the algebraic properties that follow naturally from that.

The description of the motion that we are aiming for is to find a *function of time*, which we denote by $x(t)$, that gives us the point's position (that is to say, the value of x) for any value of the time parameter, t. (Look ahead to Equation (15.1.10), for an example.) Remember that x stands for a number that can be positive or negative (depending on the side of the origin the point is on), and has dimensions of length, so when giving a numerical value for it you must always include the appropriate units (meters, centimeters, miles...). Similarly, t stands for the time elapsed since some more or less arbitrary "origin of time," or time zero. Normally t should always be positive, but in special cases it may make sense to consider negative times ("10 min before $t=0$" would be $t=-10$ min!\). Anyway, t also is a number with dimensions, and must be reported with its appropriate units: seconds, minutes, hours, etc.

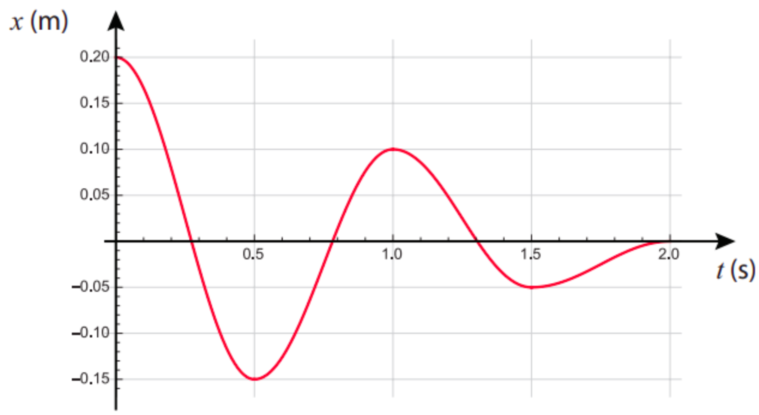


Figure 15.1.2: A possible position vs. time graph for an object moving in one dimension

We will be often interested in plotting the position of an object as a function of time—that is to say, the graph of the function $x(t)$. This may, in principle, have any shape, as you can see in Figure 15.1.2 above. In the lab, you will have a chance to use a position sensor that will automatically generate graphs like that for you on the computer, for any moving object that you aim the position sensor at. It is, therefore, important that you learn how to "read" such graphs. For example, Figure 15.1.2 shows an object that starts, at the time t = 0, a distance 0.2 m away and to the right of the origin (so $x(0)$ = 0.2 m), then moves in the negative direction to x = −0.15 m, which it reaches at t = 0.5 s; then turns back and moves in the opposite direction until it reaches the point x = 0.1 m, turns again, and so on. Physically, this could be tracking the oscillations of a system such as an object attached to a spring and sliding over a surface that exerts a friction force on it.

Displacement

In one dimension, the **displacement** of an object over a given time interval is a quantity that we denote as Δx, and equals the difference between the object's initial and final positions (in one dimension, we will often call the "position coordinate" simply the "position," for short):

$$\Delta x=x_f-x_i \tag{15.1.1}$$

Here the subscript i denotes the object's position at the beginning of the time interval considered, and the subscript f its position at the end of the interval. As we have previously discussed, the symbol Δ will consistently be used throughout this book to denote a *change* in the quantity following the symbol, meaning the difference between its initial value and its final value. The time interval itself will be written as Δt and can be expressed as

$$\Delta t=t_f-t_i \tag{15.1.2}$$

where again t_i and t_f are the initial and final values of the time parameter (imagine, for instance, that you are reading time in seconds on a digital clock, and you are interested in the change in the object's position between second 130 and second 132: then t_i = 130 s, t_2 = 132 s, and Δt = 2 s).

You can practice reading off displacements from Figure $15.1.2$ The displacement between t_i = 0.5 s and t_f = 1 s, for instance, is 0.25 m (x_i = −0.15 m, x_f = 0.1 m). On the other hand, between t_i = 1 s and t_f = 1.3 s, the displacement is Δx = 0 − 0.1 = −0.1 m

Notice two important things about the displacement. First, it can be positive or negative. Positive means the object moved, overall, in the positive direction; negative means it moved, overall, in the negative direction. Second, even when it is positive, the displacement does not always equal the distance traveled by the object (distance, of course, is always defined as a positive quantity), because if the object "doubles back" on its tracks for some distance, that distance does not count towards the overall displacement. For instance, looking again at Figure $15.1.2$, in between the times t_i = 0.5 s and t_f = 1.5 s the object moved first 0.25 m in the positive direction, and then 0.15 m in the negative direction, for a total distance traveled of 0.4 m; however, the total displacement was just 0.1 m.

In spite of these quirks, the total displacement is, mathematically, a useful quantity, because often we will have a way (that is to say, an equation) to calculate Δx for a given interval, and then we can rewrite Equation ($15.1.1$) so that it reads

$$x_f = x_i + \Delta x. \tag{15.1.3}$$

That is to say, if we know where the object started, and we have a way to calculate Δx, we can easily figure out where it ended up. You will see examples of this sort of calculation in the examples later on.

Extension to Two Dimensions

In two dimensions, we write the displacement as the vector

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_i. \tag{15.1.4}$$

The components of this vector are just the differences in the position coordinates of the two points involved; that is, $(\Delta \vec{r})_x$ (a subscript x, y, etc., is a standard way to represent the x, y. . . component of a vector) is equal to $x_f - x_i$, and similarly $(\Delta \vec{r})_y = y_f - y_i$.

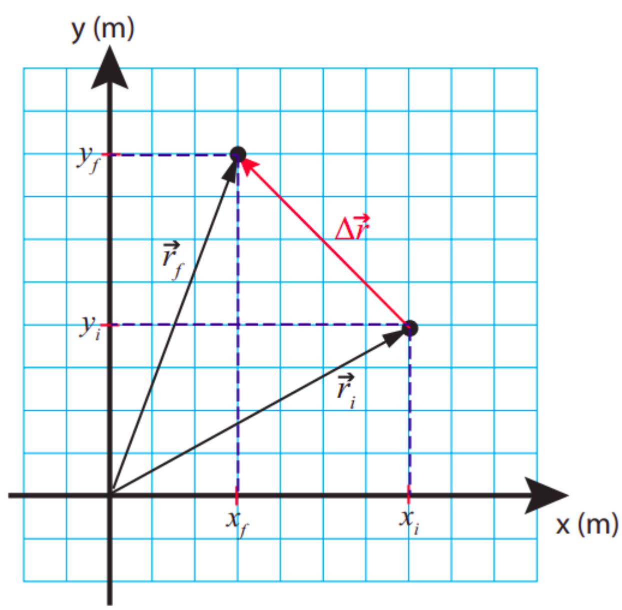


Figure $15.1.3$: The displacement vector for a particle that was initially at a point with position vector $\vec{r}_i$ and ended up at a point with position vector $\vec{r}_f$ is the *difference* of the position vectors.

Figure $15.1.3$ shows how this makes sense. The x component of $\Delta \vec{r}$ in the figure is Δx = 3 − 7 = −4 m; the y component is Δy = 8 − 4 = 4 m. This basically shows you how to subtract (and, by extension, add, since $\vec{r}_f = \vec{r}_i + \Delta \vec{r}$) vectors: you just subtract (or add) the corresponding components. Note how, by the Pythagorean theorem, the length (or magnitude) of the displacement vector, $|\Delta \vec{r}| = \sqrt{(x_f - x_i)^2 + (y_f - y_i)^2}$, equals the straight-line distance between the initial point and the final point, just as in one dimension; of course, the particle could have actually followed a very different path from the initial to the final point, and therefore traveled a different distance.

Velocity

Average Velocity

If you drive from Fayetteville to Fort Smith in 50 minutes, your average speed for the trip is calculated by dividing the distance of 59.2 mi by the time interval:

$$\text{average speed } = \frac{\text{distance}}{\Delta t} = \frac{59.2 \text{ mi}}{50 \text{ min}} = \frac{59.2 \text{ mi}}{50 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ hr}} = 71.0 \text{ mph} \tag{15.1.5}$$

The way we define *average velocity* is similar to average speed, but with one important difference: we use the *displacement*, instead of the distance. So, the average velocity v_{av} of an object, moving along a straight line, over a time interval Δt is

$$v_{av} = \frac{\Delta x}{\Delta t}. \tag{15.1.6}$$

This definition has all the advantages and the quirks of the displacement itself. On the one hand, it automatically comes with a sign (the same sign as the displacement, since Δt will always be positive), which tells us in what direction we have been traveling. On the other hand, it may not be an accurate estimate of our average *speed*, if we doubled back at all. In the most extreme case, for a roundtrip (leave Fayetteville and return to Fayetteville), the average velocity would be zero, since $x_f = x_i$ and therefore Δx = 0.

It is clear that this concept is not going to be very useful in general, if the object we are tracking has a chance to double back in the time interval Δt. A way to prevent this from happening, and also getting a more meaningful estimate of the object's speed at any instant, is to make the time interval very small. This leads to a new concept, that of *instantaneous velocity*.

Instantaneous Velocity

We define the instantaneous velocity of an object (a "particle"), at the time $t = t_i$, as the mathematical limit

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}. \tag{15.1.7}$$

The meaning of this is the following. Suppose we compute the ratio $\Delta x / \Delta t$ over successively smaller time intervals Δt (all of them starting at the same time t_i). For instance, we can start by making $t_f = t_i + 1\ s$, then try $t_f = t_i + 0.5\ s$, then $t_f = t_i + 0.1\ s$, and so on. Naturally, as the time interval becomes smaller, the corresponding displacement will also become smaller—the particle has less and less time to move away from its initial position, x_i. The hope is that the successive ratios $\Delta x / \Delta t$ will *converge* to a definite value: that is to say, that at some point we will start getting very similar values, and that beyond a certain point making Δt any smaller will not change any of the significant digits of the result that we care about. This limit value is the *instantaneous velocity* of the object at the time t_i.

When you think about it, there is something almost a bit self-contradictory about the concept of instantaneous velocity. You cannot (in practice) determine the velocity of an object if all you are given is a literal instant. You cannot even tell if the object is moving, if all you have is one instant! Motion requires more than one instant, the passage of time. In fact, all the "instantaneous" velocities that we can measure, with any instrument, are always really average velocities, only the average is taken over very short time intervals. Nevertheless, the fact is that for any reasonably well-behaved position function $x(t)$, the limit in Equation (15.1.7) is *mathematically* well-defined, and it equals what we call, in calculus, the *derivative* of the function $x(t)$:

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \tag{15.1.8}$$

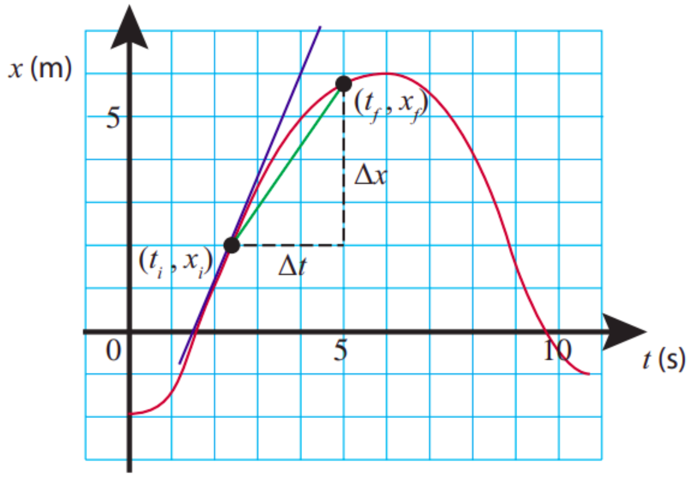


Figure 15.1.4: The slope of the green segment is the average velocity for the time interval Δt shown. As Δt becomes smaller, this approaches the slope of the tangent at the point (t_i, x_i)

In fact, there is a nice geometric interpretation for this quantity: namely, it is the slope of a line tangent to the x-vs-t curve at the point , (t_i, x_i). As Figure 15.1.4 shows, the average velocity $\Delta x/\Delta t$ is the slope (rise over run) of a line segment drawn from the point (t_i, x_i) to the point (t_f, x_f) (the green line in the figure). As we make the time interval smaller, by bringing t_f closer to t_i (and hence, also, x_f closer to x_i), the slope of this segment will approach the slope of the tangent line at (t_i, x_i) (the blue line), and this will be, by the definition (15.1.7), the instantaneous velocity at that point.

This geometric interpretation makes it easy to get a qualitative feeling, from the position-vs-time graph, for when the particle is moving more or less fast. A large slope means a steep rise or fall, and that is when the velocity will be largest—in magnitude. A steep rise means a large positive velocity, whereas a steep drop means a large negative velocity, by which I mean a velocity that is given by a negative number which is large in absolute value. In the future, to simplify sentences like this one, I will just use the word "speed" to refer to the magnitude (that is to say, the absolute value) of the instantaneous velocity. Thus, speed (like distance) is always a positive number, by definition, whereas velocity can be positive or negative; and a steep slope (positive or negative) means the speed is large there.

Conversely, looking at the sample x-vs-t graphs in this chapter, you may notice that there are times when the tangent is horizontal, meaning it has zero slope, and so the instantaneous velocity at those times is zero (for instance, at the time t = 1.0 s in Figure 15.1.2). This makes sense when you think of what the particle is actually doing at those special times: it is just changing direction, so its velocity is going, for instance, from positive to negative. The way this happens is, it slows down, down... the velocity gets smaller and smaller, and then, for just an instant (literally, a mathematical point in time), it becomes zero before, the next instant, going negative.

We will be coming back to this "reading of graphs" in the lab and the homework, as well as in the next section, when we introduce the concept of acceleration.

Motion With Constant Velocity

If the instantaneous velocity of an object never changes, it means that it is always moving in the same direction with the same speed. In that case, the instantaneous velocity and the average velocity coincide, and that means we can write $v = \Delta x/\Delta t$ (where the size of the interval Δt could now be anything), and rewrite this equation in the form

$$\Delta x = v\Delta t \tag{15.1.9}$$

which is the same as

$$x_f - x_i = v(t_f - t_i)$$

Now suppose we keep t_i constant (that is, we fix the initial instant) but allow the time t_f to change, so we will just write t for an arbitrary value of t_f, and x for the corresponding value of x_f. We end up with the equation

$$x - x_i = v(t - t_i)$$

which we can also write as

$$x(t)=x_i+v\left(t-t_i\right) \tag{15.1.10}$$

after some rearranging, and where the notation $x(t)$ has been introduced to emphasize that we want to think of x as a function of t. This is, not surprisingly, the equation of a straight line—a “curve” which is its own tangent and always has the same slope.

Caution

Please make sure that you are not confused by the notation in Equation (15.1.10). The parentheses around the t on the left-hand side mean that we are considering the position x as a function of t. On the other hand, the parentheses around the quantity $t-t_i$ on the right-hand side mean that we are multiplying this quantity by v, which is a constant here. This distinction will be particularly important when we introduce the function $v(t)$ next.

Either one of equations (15.1.9) or (15.1.11) can be used to solve problems involving motion with constant velocity.

Motion With Changing Velocity

If the velocity changes with time, obtaining an expression for the position of the object as a function of time may be a nontrivial task. In the next chapter we will study an important special case, namely, when the velocity changes at a constant rate (constant acceleration).

For the most general case, a graphical method that is sometimes useful is the following. Suppose that we know the function $v(t)$, and we graph it, as in Figure 15.1.5 below. Then the area under the curve in between any two instants, say t_i and t_f, is equal to the total displacement of the object over that time interval.

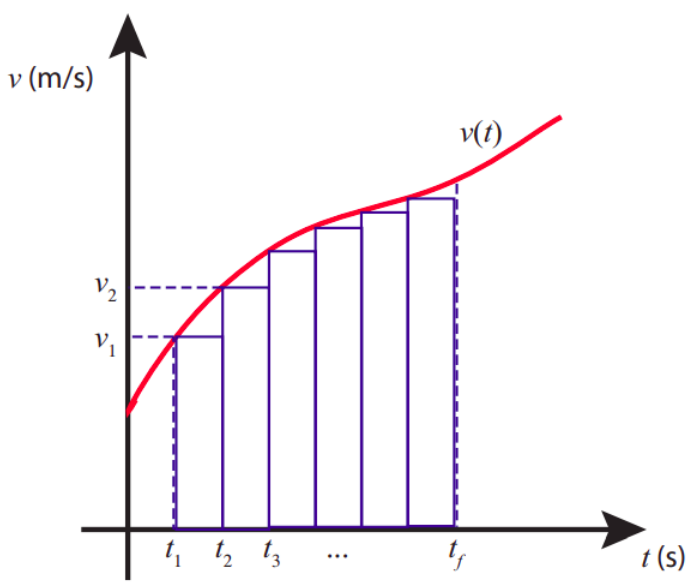


Figure 15.1.5: How to get the displacement from the area under the v-vs-t curve.

The idea involved is known in calculus as *integration*, and it goes as follows. Suppose that I break down the interval from t_i to t_f into equally spaced subintervals, beginning at the time t_i (which I am, equivalently, going to call t_1, that is, $t_1 \equiv t_i$, so I have now t_1, t_2, t_3,...t_f). Now suppose I treat the object’s motion over each subinterval as if it were motion with constant velocity, the velocity being that at the beginning of the subinterval. This, of course, is only an approximation, since the velocity is constantly changing; but, if you look at Figure 15.1.5, you can convince yourself that it will become a better and better approximation as I increase the number of intermediate points and the rectangles shown in the figure become narrower and narrower. In this approximation, the displacement during the first subinterval would be

$$\Delta x_1=v_1\left(t_2-t_1\right) \tag{15.1.11}$$

where $v_1=v(t_1)$; similarly, $\Delta x_2=v_2(t_3-t_2)$, and so on.

However, Equation (15.1.11) is just the area of the first rectangle shown under the curve in Figure 15.1.5 (the base of the rectangle has "length" $t_2 - t_1$, and its height is v_1). Similarly for the second rectangle, and so on. So the sum $\Delta x_1 + \Delta x_2 + \ldots$ is both an approximation to the area under the v-vs-t curve, and an approximation to the total displacement Δt. As the subdivision becomes finer and finer, and the rectangles narrower and narrower (and more numerous), both approximations become more and more accurate. In the limit of "infinitely many," infinitely narrow rectangles, you get both the total displacement and the area under the curve exactly, and they are both equal to each other. Mathematically, we would write

$$\Delta x = \int_{t_i}^{t_f} v(t) dt \tag{15.1.12}$$

where the stylized "S" (for "sum") on the right-hand side is the symbol of the operation known as *integration* in calculus. This is essentially the inverse of the process know as differentiation, by which we got the velocity function from the position function, back in Equation (15.1.8).

This graphical method to obtain the displacement from the velocity function is sometimes useful, if you can estimate the area under the v-vs-t graph reliably. An important point to keep in mind is that rectangles under the horizontal axis (corresponding to negative velocities) have to be added as having negative area (since the corresponding displacement is negative); see example 15.5.4 at the end of this chapter.

Extension to Two Dimensions

In two (or more) dimensions, you define the average velocity vector as a vector $\vec{v}_{av}$ whose components are $v_{av,x} = \Delta x / \Delta t$, $v_{av,y} = \Delta y / \Delta t$, and so on (where Δx, Δy,... are the corresponding components of the displacement vector $\Delta \vec{r}$). This can be written equivalently as the single vector equation

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}. \tag{15.1.13}$$

This tells you how to multiply (or divide) a vector by an ordinary number: you just multiply (or divide) each component by that number. Note that, if the number in question is positive, this operation does not change the direction of the vector at all, it just *scales* it up or down (which is why ordinary numbers, in this context, are called *scalars*). If the scalar is negative, the vector's direction is flipped as a result of the multiplication. Since Δt in the definition of velocity is always positive, it follows that the average velocity vector always points in the same direction as the displacement, which makes sense.

To get the instantaneous velocity, you just take the limit of the expression (15.1.13) as $\Delta t \to 0$, for each component separately. The resulting vector $\vec{v}$ has components $v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$, etc., which can also be written as $v_x = dx/dt, v_y = dy/dt, \ldots$.

All the results derived above hold for each spatial dimension and its corresponding velocity component. For instance, the graphical method shown in Figure 15.1.5 can always be used to get Δx if the function $v_x(t)$ is known, or equivalently to get Δy if you know $v_y(t)$, and so on.

Introducing the velocity vector at this point does cause a little bit of a notational difficulty. For quantities like x and Δx, it is pretty obvious that they are the x components of the vectors $\vec{r}$ and $\Delta \vec{r}$ respectively; however, the quantity that we have so far been calling simply v should more properly be denoted as v_x (or v_y if the motion is along the y axis). In fact, there is a convention that if you use the symbol for a vector without the arrow on top or any $x, y, \ldots$ subscripts, you must mean the *magnitude* of the vector. In this book, however, I have decided *not* to follow that convention, at least not until we get to Chapter 8 (and even then I will use it only for forces). This is because we will spend most of our time dealing with motion in only one dimension, and it makes the notation unnecessarily cumbersome to keep having to write the x or y subscripts on every component of every vector, when you really only have one dimension to worry about in the first place. So v will, throughout, refer to the relevant component of the velocity vector, to be inferred from the context, until we get to Chapter 8 and actually need to deal with both a v_x and a v_y explicitly.

Finally, notice that the magnitude of the velocity vector, $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$, is equal to the *instantaneous speed*, since, as $\Delta t \to 0$, the magnitude of the displacement vector, $|\Delta vecr|$, becomes the actual distance traveled by the object in the time interval Δt.

- **1.2: Position, Displacement, Velocity** by Julio Gea-Banacloche is licensed CC BY-SA 4.0. Original source: https://scholarworks.uark.edu/oer/3.

15.2: Acceleration

Average and Instantaneous Acceleration

Just as we defined average velocity in the previous chapter, using the concept of displacement (or change in position) over a time interval Δt, we define *average acceleration* over the time Δt using the change in velocity:

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}. \tag{15.2.1}$$

Here, v_i and v_f are the initial and final velocities, respectively, that is to say, the velocities at the beginning and the end of the time interval Δt. As was the case with the average velocity, though, the average acceleration is a concept of somewhat limited usefulness, so we might as well proceed straight away to the definition of the *instantaneous acceleration* (or just "the" acceleration, without modifiers), through the same sort of limiting process by which we defined the instantaneous velocity:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}. \tag{15.2.2}$$

Everything that we said in the previous chapter about the relationship between velocity and position can now be said about the relationship between acceleration and velocity. For instance (if you know calculus), the acceleration as a function of time is the derivative of the velocity as a function of time, which makes it the second derivative of the position function:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{15.2.3}$$

Similarly, we can "read off" the instantaneous acceleration from a *velocity* versus time graph, by looking at the slope of the line tangent to the curve at any point. However, if what we are given is a *position* versus time graph, the connection to the acceleration is more indirect. Figure $15.2.1$ provides you with such an example. See if you can guess at what points along this curve the acceleration is positive, negative, or zero.

The way to do this "from scratch," as it were, is to try to figure out what the velocity is doing, first, and infer the acceleration from that. Here is how that would go:

Starting at t = 0, and keeping an eye on the slope of the x-vs-t curve, we can see that the velocity starts at zero or near zero and increases steadily for a while, until t is a little bit more than 2 s (let us say, t = 2.2 s for definiteness). That would correspond to a period of positive acceleration, since Δv would be positive for every Δt in that range.

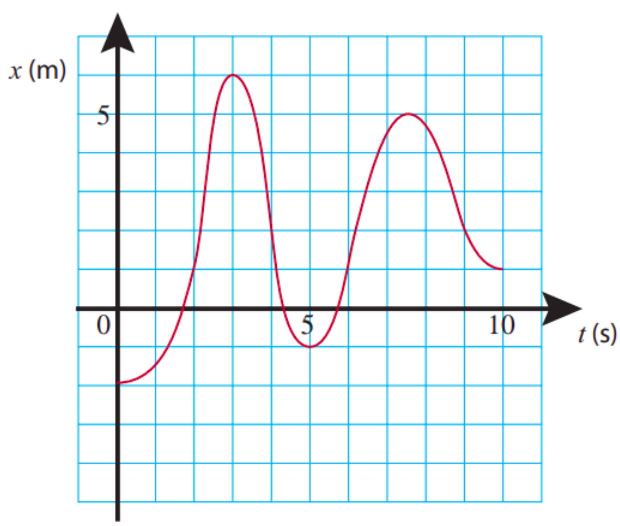


Figure 15.2.1: A possible position vs. time graph for an object whose acceleration changes with time.

Between t = 2.2 s and t = 2.5 s, as the object moves from x = 2 m to x = 4 m, the velocity does not appear to change very much, and the acceleration would correspondingly be zero or near zero. Then, around t = 2.5 s, the velocity starts to decrease noticeably, becoming (instantaneously) zero at t =3 s (x = 6 m). That would correspond to a negative acceleration. Note, however, that the velocity afterwards continues to decrease, becoming more and more negative until around t = 4 s. This also corresponds to a negative acceleration: even though the object is speeding up, it is speeding up in the negative direction, so Δv, and hence a, is negative for every time interval there. We conclude that $a < 0$ for all times between t = 2.5 s and t = 4 s.

Next, as we just look past t = 4 s, something else interesting happens: the object is still going in the negative direction (negative velocity), but now it is slowing down. Mathematically, that corresponds to a *positive* acceleration, since the algebraic value of the velocity is in fact increasing (a number like −3 is larger than a number like −5). Another way to think about it is that, if we have less and less of a negative thing, our overall trend is positive. So the acceleration is positive all the way from t = 4 s through t = 5 s (where the velocity is instantaneously zero as the object's direction of motion reverses), and beyond, until about t = 6 s, since between t = 5 s and t = 6 s the velocity is positive and growing.

You can probably figure out on your own now what happens after t = 6 s, reasoning as I did above, but you may also have noticed a pattern that makes this kind of analysis a lot easier. The acceleration, being proportional to the second derivative of the function $x(t)$ with respect to t, is directly related to the *curvature* of the x-vs-t graph. As figure $15.2.2$ below shows, if the graph is *concave* (sometimes called "concave upwards"), the acceleration is positive, whereas it is negative whenever the graph is *convex* (or "concave downwards"). It is (instantly) zero at those points where the curvature changes (which you may know as *inflection points*), as well as over stretches of time when the x-vs-t graph is a straight line (motion with constant velocity).

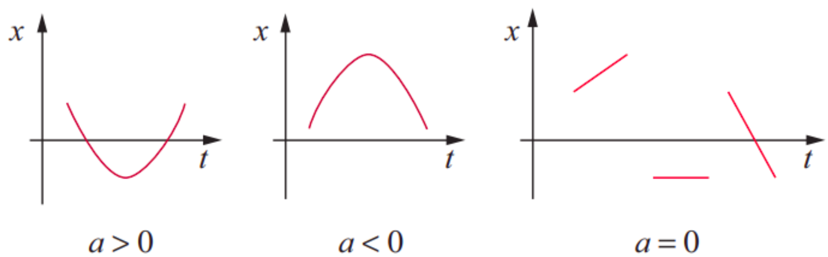


Figure 15.2.2: What the x-vs-t curves look like for the different possible signs of the acceleration.

Figure $15.2.3$ shows position, velocity, and acceleration versus time for a hypothetical motion case. Please study it carefully until every feature of every graph makes sense, relative to the other two! You will see many other examples of this in the homework and the lab.

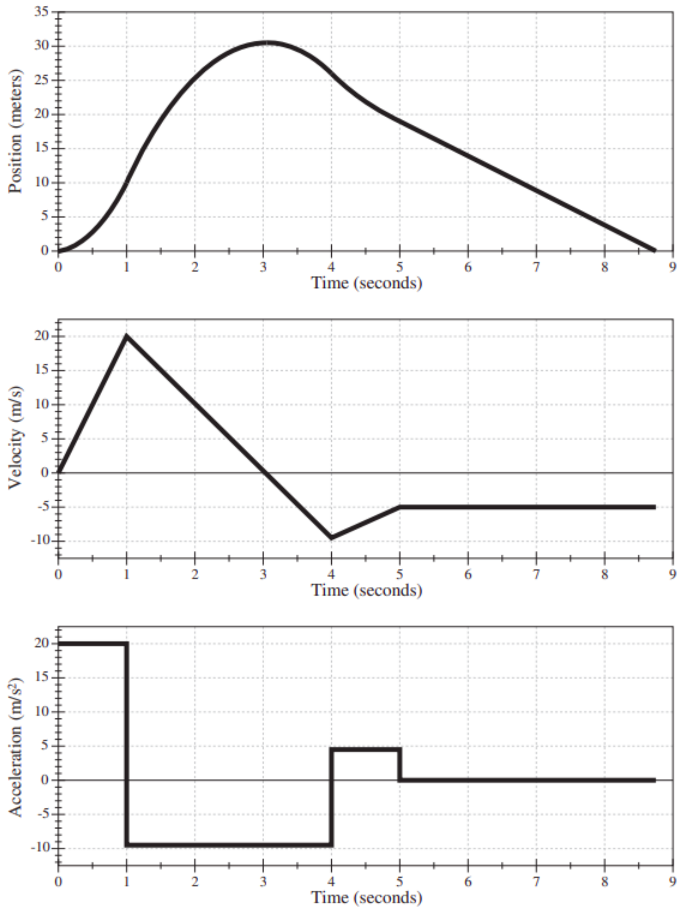


Figure $15.2.3$: Sample position, velocity and acceleration vs. time graphs for motion with piecewise-constant acceleration.

Notice that, in all these figures, the sign of x or v at any given time has nothing to do with the sign of a at that same time. It is true that, for instance, a negative a, if sustained for a sufficiently long time, will eventually result in a negative v (as happens, for instance, in Figure $15.2.3$ over the interval from t = 1 to t = 4 s) but this may take a long time, depending on the size of a and the initial value of v. The graphical clues to follow, instead, are: the acceleration is given by the slope of the tangent to the v-vs-t curve, or the curvature of the x-vs-t curve, as explained in Figure $15.2.2$; and the velocity is given by the slope of the tangent to the x-vs-t curve.

(Note: To make the interpretation of Figure $15.2.3$ simpler, I have chosen the acceleration to be "piecewise constant," that is to say, constant over extended time intervals and changing in value discontinuously from one interval to the next. This is physically unrealistic: in any real-life situation, the acceleration would be expected to change more or less smoothly from instant to instant. We will see examples of that later on, when we start looking at realistic models of collisions.)

Motion With Constant Acceleration

A particular kind of motion that is both relatively simple and very important in practice is motion with constant acceleration (see Figure $15.2.3$ again for examples). If a is constant, it means that the velocity changes with time at a constant rate, by a fixed number of m/s each second. (These are, incidentally, the units of acceleration: meters per second per second, or m/s^2.) The change in velocity over a time interval Δt is then given by

$$\Delta v = a\Delta t \tag{15.2.4}$$

which can also be written

$$v = v_i + a\left(t - t_i\right). \tag{15.2.5}$$

Equation ($15.2.5$) is the form of the velocity function (v as a function of t) for motion with constant acceleration. This, in turn, has to be the derivative with respect to time of the corresponding position function. If you know simple derivatives, then, you can verify that the appropriate form of the position function must be

$$x=x_i+v_i\left(t-t_i\right)+\frac{1}{2}a\left(t-t_i\right)^2 \tag{15.2.6}$$

or in terms of intervals,

$$\Delta x=v_i\Delta t+\frac{1}{2}a(\Delta t)^2. \tag{15.2.7}$$

Sometimes Equation (15.2.6) is written with the implicit assumption that the initial value of t is zero:

$$x=x_i+v_it+\frac{1}{2}at^2. \tag{15.2.8}$$

This is simpler, but not as general as Equation (15.2.6). Always make sure that you know what conditions apply for any equation you decide to use!

As you can see from Equation (15.2.5), for intervals during which the acceleration is constant, the velocity vs. time curve should be a straight line. Figure 15.2.3 illustrates this. Equation (15.2.6), on the other hand, shows that for those same intervals the position vs. time curve should be a (portion of a) parabola, and again this can be seen in Figure 15.2.3 (sometimes, if the acceleration is small, the curvature of the graph may be hard to see; this happens in Figure 15.2.3 for the interval between t = 4 s and t = 5 s).

The observation that v-vs-t is a straight line when the acceleration is constant provides us with a simple way to derive Equation (15.2.7), when combined with the result (from the end of the previous chapter) that the displacement over a time interval Δt equals the area under the v-vs-t curve for that time interval. Indeed, consider the situation shown in Figure 15.2.4. The total area under the segment shown is equal to the area of a rectangle of base Δt and height v_i, plus the area of a triangle of base Δt and height v_f-v_i. Since $v_f-v_i=a\Delta t$, simple geometry immediately yields Equation (15.2.7), or its equivalent (15.2.6).

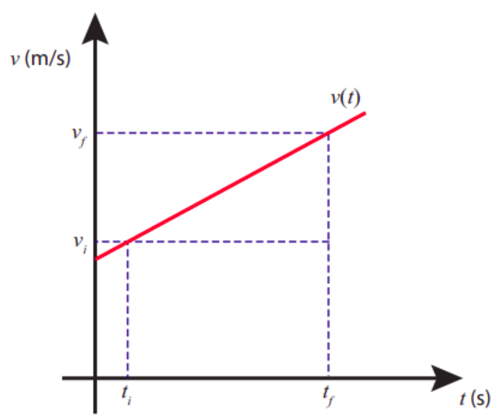


Figure 15.2.4: Graphical way to find the displacement for motion with constant acceleration.

Lastly, consider what happens if we solve Equation (15.2.4) for Δt and substitute the result in (15.2.7). We get

$$\Delta x=\frac{v_i\Delta v}{a}+\frac{(\Delta v)^2}{2a}. \tag{15.2.9}$$

Letting $\Delta v=v_f-v_i$, a little algebra yields

$$v_f^2-v_i^2=2a\Delta x. \tag{15.2.10}$$

This is a handy little result that can also be seen to follow, more directly, from the work-energy theorems to be introduced in Chapter 7[1].

[1] In fact, equation (15.2.10) turns out to be so handy that you will probably find yourself using it over and over this semester, and you may even be tempted to use it for problems involving motion in two dimensions. However, unless you really know what you are doing, you should resist the temptation, since it is very easy to use Equation (15.2.10) incorrectly when the acceleration and the displacement do not lie along the same line. You should use the appropriate form of a work-energy theorem instead.

Acceleration as a Vector

In two (or more) dimensions we introduce the average acceleration vector

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{1}{\Delta t}(\vec{v}_f - \vec{v}_i) \tag{15.2.11}$$

whose components are $a_{av,x} = \Delta v_x / \Delta t$, etc.. The instantaneous acceleration is then the vector given by the limit of Equation (15.2.11) as $\Delta t \to 0$, and its components are, therefore, $a_x = dv_x/dt, a_y = dv_y/dt$, . . .

Note that, since $\vec{v}_i$ and $\vec{v}_f$ in Equation (15.2.11) are vectors, and have to be subtracted as such, the acceleration vector will be nonzero whenever $\vec{v}_i$ and $\vec{v}_f$ are different, even if, for instance, their magnitudes (which are equal to the object's speed) are the same. In other words, you have accelerated motion whenever the *direction* of motion changes, even if the speed does not.

As long as we are working in one dimension, I will follow the same convention for the acceleration as the one I introduced for the velocity in Chapter 1: namely, I will use the symbol a, without a subscript, to refer to the relevant component of the acceleration ($a_x, a_y, \ldots$), and *not* to the magnitude of the vector $\vec{a}$.

Acceleration in Different Reference Frames

In Chapter 1 you saw that the following relation holds between the velocities of a particle P measured in two different reference frames, A and B:

$$\vec{v}_{AP} = \vec{v}_{AB} + \vec{v}_{BP}. \tag{15.2.12}$$

What about the acceleration? An equation like (15.2.12) will hold for the initial and final velocities, and subtracting them we will get

$$\Delta\vec{v}_{AP} = \Delta\vec{v}_{AB} + \Delta\vec{v}_{BP}. \tag{15.2.13}$$

Now suppose that reference frame B moves with *constant velocity* relative to frame A. In that case, $\vec{v}_{AB,f} = \vec{v}_{AB,i}$, so $\Delta\vec{v}_{AB}$ = 0, and then, dividing Equation (15.2.13) by Δt, and taking the limit $\Delta t \to 0$, we get

$$\vec{a}_{AP} = \vec{a}_{BP} \quad (\text{for constant } \vec{v}_{AB})\,. \tag{15.2.14}$$

So, if two reference frames are moving at constant velocity relative to each other, observers in both frames measure the *same* acceleration for any object they might both be tracking.

The result Equation (15.2.14) means, in particular, that if we have an inertial frame then any frame moving at constant velocity relative to it will be inertial too, since the respective observers' measurements will agree that an object's velocity does not change (otherwise put, its acceleration is zero) when no forces act on it. Conversely, an accelerated frame will *not* be an inertial frame, because Equation (15.2.14) will not hold. This is consistent with the examples I mentioned in Section 2.1 (the bouncing plane, the car coming to a stop). Another example of a non-inertial frame would be a car going around a curve, even if it is going at constant speed, since, as I just pointed out above, this is also an accelerated system. This is confirmed by the fact that objects in such a car tend to move—relative to the car—towards the outside of the curve, even though no actual force is acting on them.

15.3: Free Fall

An important example of motion with (approximately) constant acceleration is provided by *free fall* near the surface of the Earth. We say that an object is in "free fall" when the only force acting on it is the force of gravity (the word "fall" here may be a bit misleading, since the object could actually be moving upwards some of the time, if it has been thrown straight up, for instance). The space station is in free fall, but because it is nowhere near the surface of the earth its direction of motion (and hence its acceleration, regarded as a two-dimensional vector) is constantly changing. Right next to the surface of the earth, on the other hand, the planet's curvature is pretty much negligible and gravity provides an approximately constant, vertical acceleration, which, in the absence of other forces, turns out to be *the same for every object*, regardless of its size, shape, or weight.

The above result—that, in the absence of other forces, all objects should fall to the earth at the same rate, regardless of how big or heavy they are—is so contrary to our common experience that it took many centuries to discover it. The key, of course, as with the law of inertia, is to realize that, under normal circumstances, frictional forces are, in fact, acting all the time, so an object falling through the atmosphere is never *really* in "free" fall: there is always, at a minimum, and in addition to the force of gravity, an air drag force that opposes its motion. The magnitude of this force does depend on the object's size and shape (basically, on how "aerodynamic" the object is); and thus a golf ball, for instance, falls much faster than a flat sheet of paper. Yet, if you crumple up the sheet of paper till it has the same size and shape as the golf ball, you can see for yourself that they do fall at approximately the same rate! The equality can never be exact, however, unless you get rid completely of air drag, either by doing the experiment in an evacuated tube, or (in a somewhat extreme way), by doing it on the surface of the moon, as the Apollo 15 astronauts did with a hammer and a feather back in 1971[2].

This still leaves us with something of a mystery, however: the force of gravity is the only force known to have the property that it imparts all objects the *same* acceleration, regardless of their mass or constitution. A way to put this technically is that the force of gravity on an object is proportional to that object's *inertial mass*, a quantity that we will introduce properly in the next chapter. For the time being, we will simply record here that this acceleration, near the surface of the earth, has a magnitude of approximately 9.8 m/s^2, a quantity that is denoted by the symbol g. Thus, if we take the upwards direction as positive (as is usually done), we get for the acceleration of an object in free fall $a = -g$, and the equations of motion become

$$\Delta v = -g\Delta t \tag{15.3.1}$$

$$\Delta y = v_i\Delta t - \frac{1}{2}g(\Delta t)^2 \tag{15.3.2}$$

where I have used y instead of x for the position coordinate, since that is a more common choice for a vertical axis. Note that we could as well have chosen the downward direction as positive, and that may be a more natural choice in some problems. Regardless, the quantity g is always defined to be positive: g = 9.8 m/s^2. The acceleration, then, is g or $-g$, depending on which direction we take to be positive

In practice, the value of g changes a little from place to place around the earth, for various reasons (it is somewhat sensitive to the density of the ground below you, and it decreases as you climb higher away from the center of the earth). In a later chapter we will see how to calculate the value of g from the mass and radius of the earth, and also how to calculate the equivalent quantity for other planets.

In the meantime, we can use equations like (15.3.1) and (15.3.2) to answer a number of interesting questions about objects thrown or dropped straight up or down (always, of course, assuming that air drag is negligible). For instance, back at the beginning of this chapter I mentioned that if I dropped an object it might take about half a second to hit the ground. If you use Equation (15.3.2) with v_i = 0 (since I am dropping the object, not throwing it down, its initial velocity is zero), and substitute Δt = 0.5 s, you get Δy = 1.23 m (about 4 feet). This is a reasonable height from which to drop something.

On the other hand, you may note that half a second is not a very long time in which to make accurate observations (especially if you do not have modern electronic equipment), and as a result of that there was considerable confusion for many centuries as to the precise way in which objects fell. Some people believed that the speed did increase in some way as the object fell, while others appear to have believed that an object dropped would "instantaneously" (that is, at soon as it left your hand) acquire some speed and keep that unchanged all the way down. In reality, in the presence of air drag, what happens is a combination of both: initially the speed increases at an approximately constant rate (free, or nearly free fall), but the drag force increases with the speed as well, until eventually it balances out the force of gravity, and from that point on the speed does not increase anymore: we say that the object has reached "terminal velocity." Some objects reach terminal velocity almost instantly, whereas others (the more “aerodynamic” ones) may take a long time to do so. This accounts for the confusion that prevailed before Galileo’s experiments in the early 1600’s.

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[2]The video of this is available online: https://www.youtube.com/watch?v=oYEgdZ3iEKA. It is, however, pretty low resolution and hard to see. A very impressive modern-day demonstration involving feathers and a bowling ball in a completely evacuated (airless) room is available here: https://www.youtube.com/watch?v=E43-CfukEgs.

15.4: The Connection Between Displacement, Velocity, and Acceleration

Learning Objectives

- Derive the kinematic equations for constant acceleration using integral calculus.
- Use the integral formulation of the kinematic equations in analyzing motion.
- Find the functional form of velocity versus time given the acceleration function.
- Find the functional form of position versus time given the velocity function.

This section assumes you have enough background in calculus to be familiar with integration. In Instantaneous Velocity and Speed and Average and Instantaneous Acceleration we introduced the kinematic functions of velocity and acceleration using the derivative. By taking the derivative of the position function we found the velocity function, and likewise by taking the derivative of the velocity function we found the acceleration function. Using integral calculus, we can work backward and calculate the velocity function from the acceleration function, and the position function from the velocity function.

Kinematic Equations from Integral Calculus

Let's begin with a particle with an acceleration a(t) is a known function of time. Since the time derivative of the velocity function is acceleration,

$$\frac{d}{dt}v(t) = a(t), \tag{15.4.1}$$

we can take the indefinite integral of both sides, finding

$$\int \frac{d}{dt}v(t)dt = \int a(t)dt + C_1, \tag{15.4.2}$$

where C_1 is a constant of integration. Since $\int \frac{d}{dt}v(t)dt = v(t)$, the velocity is given by

$$v(t) = \int a(t)dt + C_1. \tag{15.4.3}$$

Similarly, the time derivative of the position function is the velocity function,

$$\frac{d}{dt}x(t) = v(t). \tag{15.4.4}$$

Thus, we can use the same mathematical manipulations we just used and find

$$x(t) = \int v(t)dt + C_2, \tag{15.4.5}$$

where C_2 is a second constant of integration.

We can derive the kinematic equations for a constant acceleration using these integrals. With a(t) = a, a constant, and doing the integration in Equation 15.4.3, we find

$$v(t) = \int a dt + C_1 = at + C_1. \tag{15.4.6}$$

If the initial velocity is $v(0) = v_0$, then

$$v_0 = 0 + C_1. \tag{15.4.7}$$

Then, $C_1 = v_0$ and

$$v(t) = v_0 + at, \tag{15.4.8}$$

which is Equation 3.5.12. Substituting this expression into Equation 15.4.5 gives

$$x(t) = \int (v_0 + at)dt + C_2. \tag{15.4.9}$$

Doing the integration, we find

$$x(t) = v_0 t + \frac{1}{2}at^2 + C_2. \tag{15.4.10}$$

If x(0) = x_0, we have

$$x_0 = 0 + 0 + C_2. \tag{15.4.11}$$

so, C_2 = x_0. Substituting back into the equation for x(t), we finally have

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2. \tag{15.4.12}$$

which is Equation 3.5.17.

Example 3.17: Motion of a Motorboat

A motorboat is traveling at a constant velocity of 5.0 m/s when it starts to decelerate to arrive at the dock. Its acceleration is a(t) = $-\frac{1}{4}$ t m/s^2. (a) What is the velocity function of the motorboat? (b) At what time does the velocity reach zero? (c) What is the position function of the motorboat? (d) What is the displacement of the motorboat from the time it begins to decelerate to when the velocity is zero? (e) Graph the velocity and position functions.

Strategy

(a) To get the velocity function we must integrate and use initial conditions to find the constant of integration. (b) We set the velocity function equal to zero and solve for t. (c) Similarly, we must integrate to find the position function and use initial conditions to find the constant of integration. (d) Since the initial position is taken to be zero, we only have to evaluate the position function at t = 0 .

Solution

We take t = 0 to be the time when the boat starts to decelerate.

a. From the functional form of the acceleration we can solve Equation 15.4.3 to get v(t):

$$v(t) = \int a(t)dt + C_1 = \int -\frac{1}{4}tdt + C_1 = -\frac{1}{8}t^2 + C_1. \tag{15.4.13}$$

At t = 0 we have v(0) = 5.0 m/s = 0 + C_1, so C_1 = 5.0 m/s or v(t) = 5.0 m/s − $\frac{1}{8}$ t^2.

b. v(t) = 0 = 5.0 m/s − $\frac{1}{8}$ t^2 (\Rightarrow\) t = 6.3 s

c. Solve Equation 15.4.5:

$$x(t) = \int v(t)dt + C_2 = \int (5.0 - \frac{1}{8}t^2)dt + C_2 = 5.0t - \frac{1}{24}t^3 + C_2. \tag{15.4.14}$$

At t = 0, we set x(0) = 0 = x_0, since we are only interested in the displacement from when the boat starts to decelerate. We have

$$x(0) = 0 = C_2. \tag{15.4.15}$$

Therefore, the equation for the position is

$$x(t) = 5.0t - \frac{1}{24}t^3. \tag{15.4.16}$$

d. Since the initial position is taken to be zero, we only have to evaluate x(t) when the velocity is zero. This occurs at t = 6.3 s. Therefore, the displacement is

$$x(6.3) = 5.0(6.3) - \frac{1}{24}(6.3)^3 = 21.1\ m. \tag{15.4.17}$$

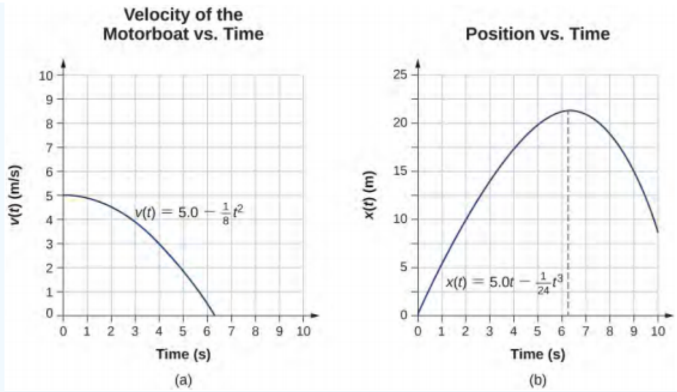


Figure 15.4.1: (a) Velocity of the motorboat as a function of time. The motorboat decreases its velocity to zero in 6.3 s. At times greater than this, velocity becomes negative—meaning, the boat is reversing direction. (b) Position of the motorboat as a function of time. At t = 6.3 s, the velocity is zero and the boat has stopped. At times greater than this, the velocity becomes negative—meaning, if the boat continues to move with the same acceleration, it reverses direction and heads back toward where it originated.

Significance

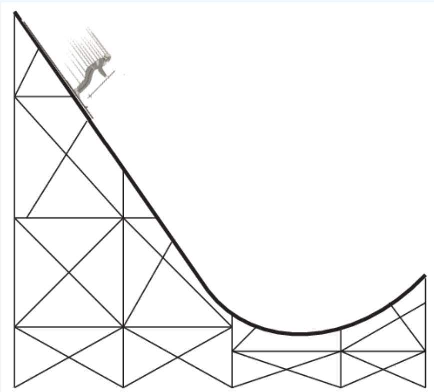
The acceleration function is linear in time so the integration involves simple polynomials. In Figure $15.4.1$, we see that if we extend the solution beyond the point when the velocity is zero, the velocity becomes negative and the boat reverses direction. This tells us that solutions can give us information outside our immediate interest and we should be careful when interpreting them.

? Exercise 3.8

A particle starts from rest and has an acceleration function $a(t) = \left(5 - \left(10\frac{1}{s}\right)t\right)\frac{m}{s^2}$. (a) What is the velocity function? (b) What is the position function? (c) When is the velocity zero?

15.5: Examples

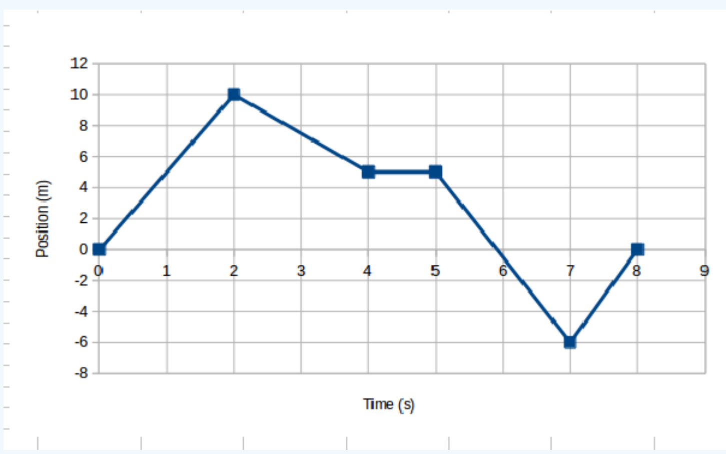
? Whiteboard Problem 15.5.1: Ski Jump!



Consider the ski jumper in the above figure. The mass of the skiier is 75 kg, and the slope down which he is traveling is 37° with respect to the horizontal (which is a typical ``in-run" to an olympic ski jump).

1. What is the normal force on this skiier, while they are traveling down the slope as shown in the figure?
2. Assuming the snow is perfectly frictionless, what is the acceleration of the skier?
3. How long will it take the skiier to go from the top of the slope to the bottom, if the slope is 115 m long? You may assume they started from rest.
4. How fast will the skiier be moving at the end of this slope?

? Whiteboard Problem 15.5.2: Average and Instantaneous Velocity



Consider the graph of position vs time in the figure.

1. Find the average velocity during the time intervals 0 s to 2 s, 2 s to 5 s, and 5 s to 7 s.
2. Find the instantaneous velocity at 1 s, 4.5 s, and 7.5 s.

? Whiteboard Problem 15.5.3: The Tortoise and the Hare, Redux!

The tortoise demands a rematch! They decide on the same distance (d = 1.0 km), and the tortoise starts off at his same speed (v_t = 0.20 m/s). This time, the hare is so convinced he will win that he waits at the start for the tortoise to get to d_1 = 0.80 km before he starts to run (hop?), starting from rest at a constant acceleration.

1. What acceleration will the hare need to have to finish the race at the same time as the tortoise?
2. The tortoise expects this kind of grandstanding from the hare, so he begins to accelerate at 0.01 m/s^2 when he sees the hare start the race. What acceleration must the hare have to still beat the tortoise?

Example 15.5.4: Motion with piecewise constant acceleration

Construct the position vs. time, velocity vs. time, and acceleration vs. time graphs for the motion described below. For each of the intervals (a)–(d) you'll need to figure out the position (height) and velocity of the rocket at the beginning and the end of the interval, and the acceleration for the interval. In addition, for interval (b) you need to figure out the maximum height reached by the rocket and the time at which it occurs. For interval (d) you need to figure out its duration, that is to say, the time at which the rocket hits the ground.

a. A rocket is shot upwards, accelerating from rest to a final velocity of 20 m/s in 1 s as it burns its fuel. (Treat the acceleration as constant during this interval.)
b. From t = 1 s to t = 4 s, with the fuel exhausted, the rocket flies under the influence of gravity alone. At some point during this time interval (you need to figure out when!) it stops climbing and starts falling.
c. At t = 4 s a parachute opens, suddenly causing an upwards acceleration (again, treat it as constant) lasting 1 s; at the end of this interval, the rocket's velocity is 5 m/s downwards.
d. The last part of the motion, with the parachute deployed, is with constant velocity of 5 m/s downwards until the rocket hits the ground.

Solution

(a) For this first interval (for which I will use a subscript "1" throughout) we have

$$\Delta y_1 = \frac{1}{2} a_1 (\Delta t_1)^2 \tag{15.5.1}$$

using Equation (15.2.7) for motion with constant acceleration with zero initial velocity (I am using the variable y, instead of x, for the vertical coordinate; this is more or less customary, but, of course, I could have used x just as well).

Since the acceleration is constant, it is equal to its average value:

$$a_1 = \frac{\Delta v}{\Delta t} = 20\ \frac{\text{m}}{\text{s}^2}.$$

Substituting this into (15.5.1) we get the height at t = 1 s is 10 m. The velocity at that time, of course, is v_{f1} = 20 m/s, as we were told in the statement of the problem.

(b) This part is free fall with initial velocity v_{i2} = 20 m/s. To find how high the rocket climbs, use Equation (15.3.1) in the form $v_{top} - v_{i2} = -g(t_{top} - t_{i2})$, with $v_{top} = 0$ (as the rocket climbs, its velocity decreases, and it stops climbing when its velocity is zero). This gives us t_{top} = 3.04 s as the time at which the rocket reaches the top of its trajectory, and then starts coming down. The corresponding displacement is, by Equation (15.3.2),

$$\Delta y_{top} = v_{i2}\left(t_{top} - t_{i2}\right) - \frac{1}{2} g \left(t_{top} - t_{i2}\right)^2 = 20.4\ \text{m}$$

so the maximum height it reaches is 30.4 m.

At the end of the full 3-second interval, the rocket's displacement is

$$\Delta y_2 = v_{i2} \Delta t_2 - \frac{1}{2} g (\Delta t_2)^2 = 15.9\ \text{m}$$

(so its height is 25.9 m above the ground), and the final velocity is

$$v_{f2} = v_{i2} - g \Delta t_2 = -9.43\ \frac{\text{m}}{\text{s}}.$$

(c) The acceleration for this part is $(v_{f3} - v_{i3})/\Delta t_3$ = (−5 + 9.43)/1=4.43 m/s^2. Note the positive sign. The displacement is

$$\Delta y_3 = -9.43 \times 1 + \frac{1}{2} \times 4.43 \times 1^2 = -7.22\ \text{m}$$

so the final height is 25.9 − 7.21 = 18.7 m.

(d) This is just motion with constant speed to cover 18.7 m at 5 m/s. The time it takes is 3.74 s. The graphs for this motion are shown earlier in the chapter, in Figure 15.2.3.

CHAPTER OVERVIEW

16: N3) 2 Dimensional Kinematics and Projectile Motion

In this chapter we are going to extend our work on kinematics in 1-dimension to include 2-dimensional motion. It turns out that the short answer is "2D is just like two copies of 1D", and we don't need to work hard to see that. In this course, there is really only one key example of 2D motion, and that is **projectile motion**.

Projectile motion is the motion of an object that is launched into the air and moves under the influence of gravity alone. During the entire motion, the object is subject to a constant acceleration of 9.8l meters per second squared, directed downwards (which we often take to be the negative direction). The path of the object is called a trajectory, and it can be predicted using a few basic principles.

The horizontal and vertical motions of the object are independent of each other. This means that the object will continue moving forward at a constant speed unless acted upon by external forces, while at the same time, it will be pulled downwards by gravity. The initial velocity and launch angle of the object determine its trajectory. The initial velocity is the speed at which the object is launched, while the launch angle is the angle at which it is launched relative to the horizontal. Together, these two factors determine the initial velocity vector, which can be broken down into its horizontal and vertical components.

The motion of the object can be analyzed using basic kinematic equations that describe the motion of an object under constant acceleration, which we presented in the last chapter. By using these equations, you can predict the maximum height reached by the object, the time it takes to reach the maximum height, the total time of flight, the range of the object, and the final velocity of the object when it hits the ground.

It is important to note that air resistance can affect the motion of a projectile, especially at high velocities or long distances. However, in many cases, air resistance can be neglected, and the motion of the object can be described using the basic principles of projectile motion. During our work in this course, we will nearly always be ignoring air resistance, which we will generally refer to as **freely falling motion**.

Overall, understanding projectile motion is essential for a wide range of applications, from sports to engineering to astronomy. By mastering the basic principles of projectile motion, you can make accurate predictions about the motion of objects in the real world and develop more sophisticated models and simulations.

16.1: Dealing with Forces in Two Dimensions

We have been able to get a lot of physics from our study of (mostly) one-dimensional motion only, but it goes without saying that the real world is a lot richer than that, and there are a number of new and interesting phenomena that appear when one considers motion in two or three dimensions. The purpose of this chapter is to introduce you to some of the simplest two-dimensional situations of physical interest.

A common feature to all these problems is that the forces acting on the objects under consideration will typically not line up with the displacements. This means, in practice, that we need to pay more attention to the vector nature of these quantities than we have done so far. This section will present a brief reminder of some basic properties of vectors, and introduce a couple of simple principles for the analysis of the systems that will follow.

To begin with, recall that a vector is a quantity that has both a magnitude and a direction. The magnitude of the vector just tells us how big it is: the magnitude of the velocity vector, for instance, is the speed, that is, just how fast something is moving. When working with vectors in one dimension, we have typically assumed that the entire vector (whether it was a velocity, an acceleration or a force) lay along the line of motion of the system, and all we had to do to indicate the direction was to give the vector's magnitude an appropriate sign. For the problems that follow, however, it will become essential to break up the vectors into their *components* along an appropriate set of axes. This involves very simple geometry, and follows the example of the position vector $\vec{r}$, whose components are just the Cartesian coordinates of the point it locates in space (as shown in Figure $16.1.1$). For a generic vector, for instance, a force, like the one shown in Figure $16.1.1$ below, the components F_x and F_y may be obtained from a right triangle, as indicated there:

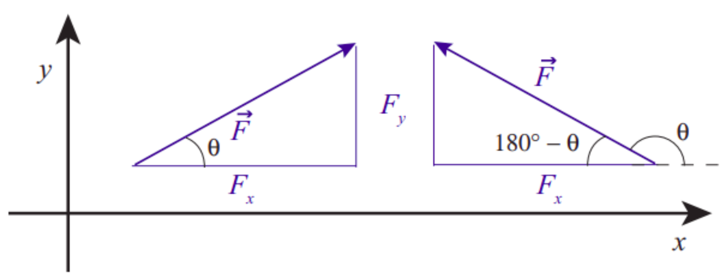


Figure 16.1.1: : The components of a vector that makes an angle θ with the positive x axis. Two examples are shown, for $\theta < 90^\circ$ (in which case $F_x > 0$) and for $90^\circ < \theta < 180^\circ$ (in which case $F_x < 0$). In both cases, $F_y > 0$.

The triangle will always have the vector's magnitude ($|\vec{F}|$ in this case) as the hypothenuse. The two other sides should be parallel to the coordinate axes. Their lengths are the corresponding components, except for a sign that depends on the orientation of the vector. If we happen to know the angle θ that the vector makes with the positive x axis, the following relations will always hold:

$$\begin{aligned} F_x &= |\vec{F}|\cos\theta \\ F_y &= |\vec{F}|\sin\theta \\ |\vec{F}| &= \sqrt{F_x^2 + F_y^2} \\ \theta &= \tan^{-1}\frac{F_y}{F_x}. \end{aligned} \tag{16.1.1}$$

Note, however, that in general this angle θ may not be one of the interior angles of the triangle (as shown on the right diagram in Figure $16.1.1$), and that in that case it may just be simpler to calculate the magnitude of the components using trigonometry and an interior angle (such as $180^\circ - \theta$ in the example), and give them the appropriate signs "by hand." In the example on the right, the length of the horizontal side of the triangle is equal to $|\vec{F}|\cos(180^\circ - \theta)$, which is a positive quantity; the correct value for F_x, however, is the negative number $|\vec{F}|\cos\theta = -|\vec{F}|\cos(180^\circ - \theta)$.

In any case, it is important not to get fixated on the notion that "the x component will always be proportional to the cosine of θ." The symbol θ is just a convenient one to use for a generic angle. There are four sections in this chapter, and in every one there is a θ used with a different meaning. When in doubt, just draw the appropriate right triangle and remember from your trigonometry classes which side goes with the sine, and which with the cosine (remember SOHCAHTOA!).

For the problems that we are going to study in this chapter, the idea is to break up all the forces involved into components along properly-chosen coordinate axes, then add all the components along any given direction, and apply $F_{net} = ma$ along that direction: that is to say, we will write (and eventually solve) the equations

$$\begin{aligned} F_{net,x} &= ma_x \\ F_{net,y} &= ma_y. \end{aligned} \tag{16.1.2}$$

We can show that Eqs. (16.1.2) must hold for any choice of orthogonal x and y axes, based on the fact that we know $\vec{F}_{net} = m\vec{a}$ holds along one particular direction, namely, the direction common to $\vec{F}_{net}$ and $\vec{a}$, and the fact that we have defined the projection procedure to be the same for any kind of vector. Figure 16.1.2 shows how this works. Along the dashed line you just have the situation that is by now familiar to us from one-dimensional problems, where $\vec{a}$ lies along $\vec{F}$ (assumed here to be the net force), and $|\vec{F}| = m|\vec{a}|$. However, in the figure I have chosen the axes to make an angle θ with this direction. Then, if you look at the projections of $\vec{F}$ and $\vec{a}$ along the x axis, you will find

$$\begin{aligned} a_x &= |\vec{a}| \cos\theta \\ F_x &= |\vec{F}| \cos\theta = m|\vec{a}| \cos\theta = ma_x \end{aligned} \tag{16.1.3}$$

and similarly, $F_y = ma_y$. In words, *each component of the force vector is responsible for only the corresponding component of the acceleration*. A force in the x direction does *not* cause any acceleration in the y direction, and vice-versa.

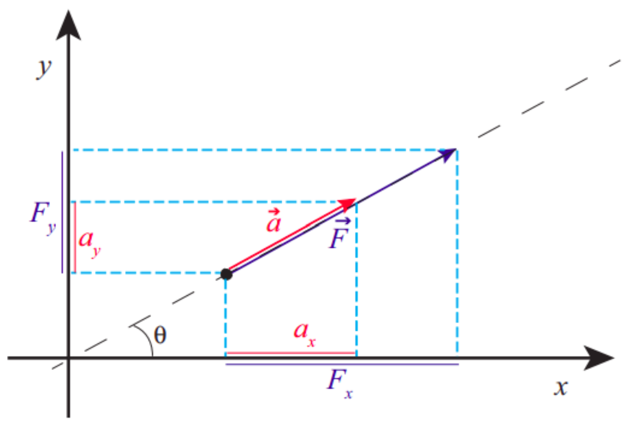


Figure 16.1.2: If you take the familiar, one-dimensional (see the black dashed line) form of $\vec{F} = m\vec{a}$, and project it onto orthogonal, rotated axes, you get the general two-dimensional case, showing that each orthogonal component of the acceleration is proportional, via the mass m, to only the corresponding component of the force (Eqs. (16.1.2)).

In the rest of the chapter we shall see how to use Eqs. (16.1.2) in a number of examples. One thing I can anticipate is that, in general, we will try to choose our axes (unlike in Figure 16.1.2 above) so that one of them does coincide with the direction of the acceleration, so the motion along the other direction is either nonexistent (v = 0) or trivial (constant velocity).

16.2: Motion in Two Dimensions and Projectile Motion

Motion in Two Dimensions and Projectile Motion

So far, we have studied motion under constant acceleration in one-dimension only. In this case, an object is restricted to move in a line (*i.e.* only along the x or y-directions), and the kinematic equations describe how the object moves. In this section, we will start looking at objects moving in two dimensions. A primary application of this topic is the study of objects moving in the gravitational field, which is called *projectile motion*.

Motion in Two Dimensions

How do we know when an object is moving in one dimension or two dimensions? The first answer to this is that it might depend on the coordinate system we choose. For instance, consider a car traveling at a constant speed in the north-west direction. In the coordinate system with x along east and y along north, the car is moving in two dimensions. However, if we pick a coordinate system which is tilted 45° with respect to the earth-north one, we can describe this car as only moving in one dimension.

However, we don't *always* have this choice. A better answer is to consider the vectors which describe the object's motion, $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$. If an object has either velocity or acceleration in more then one direction, then the object will move in more than one direction. Notice we have to consider both velocity *and* acceleration. For example, if we have an object with initial velocity in the x-direction, but acceleration in the y:

$$\vec{v}(t)=v_0\hat{x}, \qquad \vec{a}(t)=a_y\hat{y}, \tag{16.2.1}$$

then although the motion is initially just along the x-axis, the object will start to accelerate along the y-axis almost immediately. At some point later, the velocity will be

$$\vec{v}(t)=v_0\hat{x}+v_y(t)\hat{y}, \tag{16.2.2}$$

where $v_y(t)$ is the velocity after undergoing the acceleration a_y for some time interval.

Constant Acceleration in Two Directions

If you recall, in order to derive the kinematic equations in one-dimension, we started with the basic definition

$$a_x=\frac{dv_x}{dt}, \tag{16.2.3}$$

and integrated twice with respect to time (review those equations now if you don't remember them!). That required us to add in the initial values x_0 for the position and v_{0x} for the velocity. We didn't include anything about the y-direction because we assumed the acceleration in the y-direction was zero. But if that's not true, we would simply have

$$a_y=\frac{dv_y}{dt}, \tag{16.2.4}$$

and we could repeat the same analysis that we performed in the y-direction. In the end, we would end up with two copies of the kinematic equations, one in the x-direction and one in the y-direction,

$$x(t)=\frac{1}{2}a_xt^2+v_{0x}t+x_0, \quad v_x(t)=a_xt+v_{0x}, \quad y(t)=\frac{1}{2}a_yt^2+v_{0y}t+y_0, \quad v_y(t)=a_yt+v_{0y} \tag{16.2.5}$$

A key feature here is that the two directions are *connected by the time and nothing else*. Each direction has its own set of initial and final position, velocity, and acceleration, but they are parametrized by time. This means that finding the time interval under consideration is often the best way to solve any particular problem.

Projectile Motion

A particular case of two-dimensional motion under constant acceleration is projectile motion. In this case, we want to study the motion of an object which has only a single force acting on it, that of gravity. Often, this is an object which we throw, or propel into the air in some way. Since we usually don't know anything how *how* it is thrown (by hand? by canon? by gun?), we start our analysis right after it is in motion, when only gravity is acting on it.

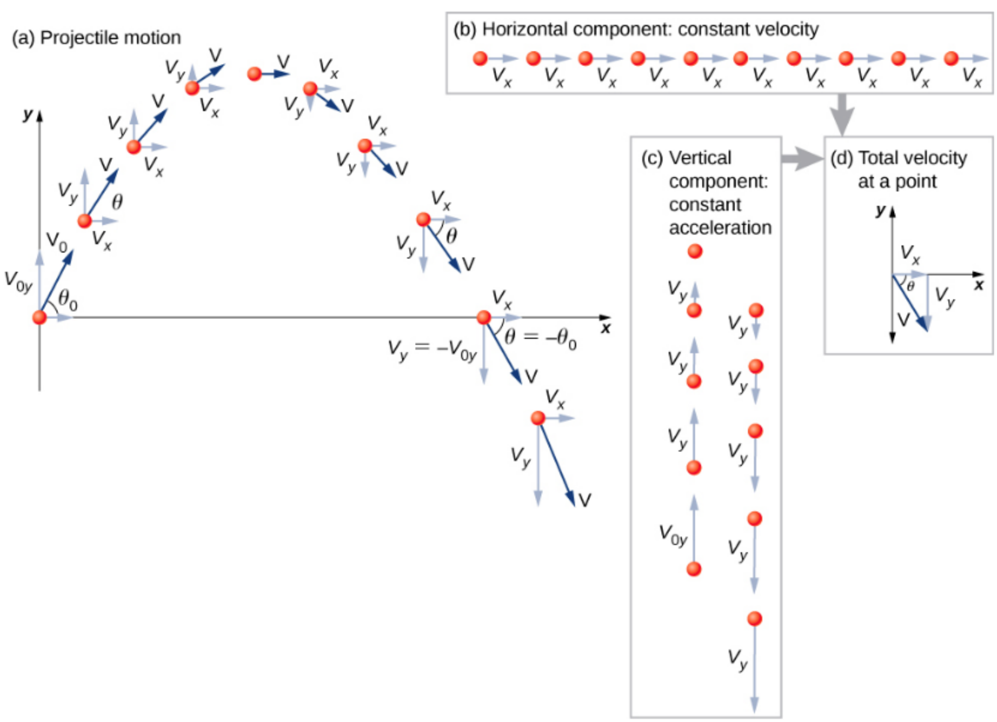
If gravity is the only force acting on the object, we can say right away what the magnitude of the acceleration is, $a = g = 9.81$ m/s ^{2}. It is also most convienient to set the coordinate system so that the y-direction is downwards, so that

$$\vec{a} = -g\hat{y}. \tag{16.2.6}$$

In other words, the acceleration in the x-direction is zero, $a_x = 0$. This will greatly simplify the kinematic equations, but we have to be a little careful - just because there is no acceleration in the x-direction does not mean there is no *motion* in the x-direction. If you throw an object into the air at an arbitrary angle, it will move in both the x- and y-directions, even though it will only accelerate downwards.

? Problem-Solving Strategy: Projectile Motion

1. Resolve the motion into horizontal and vertical components along the x- and y-axes. The magnitudes of the components of displacement $\vec{s}$ along these axes are x and y. The magnitudes of the components of velocity $\vec{v}$ are $v_x = v\cos\theta$ and $v_y = v\sin\theta$, where v is the magnitude of the velocity and θ is its direction relative to the horizontal, as shown in Figure 16.2.2
2. Treat the motion as two independent one-dimensional motions: one horizontal and the other vertical. Use the kinematic equations for horizontal and vertical motion presented earlier.
3. Solve for the unknowns in the two separate motions: one horizontal and one vertical. Note that the only common variable between the motions is time t. The problem-solving procedures here are the same as those for one-dimensional kinematics and are illustrated in the following solved examples.
4. Recombine quantities in the horizontal and vertical directions to find the total displacement $\vec{s}$ and velocity $\vec{v}$. Solve for the magnitude and direction of the displacement and velocity using
$s = \sqrt{x^2 + y^2}$. $\phi = \tan^{-1}\left(\frac{y}{x}\right)$, $v = \sqrt{v_x^2 + v_y^2}$. where ϕ is the direction of the displacement $\vec{s}$.



16.3: Inclined Planes

Another simple example of 2D motion is a block sliding down a plane. Now in this case, the block is not *actually* moving in 2D motion, since it's just moving in a straight line down the plane, but the forces are acting in 2 dimensions, which is what we have to understand to understand the motion. So here we will consider an inclined plane making an angle θ with the horizontal was $g \sin \theta$. This problem is going to introduce two kinds of friction as well, **kinetic friction $\boldsymbol{F_k}$**, which you get when an object is in motion, and **static friction $\boldsymbol{F_s}$**, which you get when an object is stuck in place. We will study friction more in Chapter 18 - for now just use your intuition about friction "slowing objects down", and "preventing them from moving".

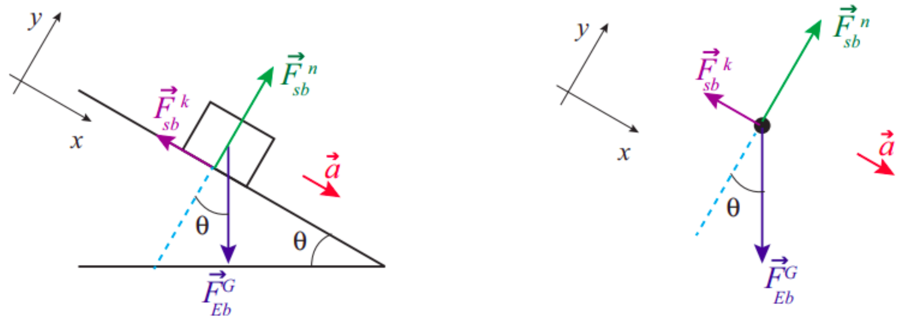


Figure 16.3.1: A block sliding down an inclined plane. The corresponding free-body diagram is shown on the right.

Figure 16.3.1 above shows, on the left, a block sliding down an inclined plane and all the forces acting on it. These are more clearly seen on the free-body diagram on the right. I have labeled all the forces using the $\vec{F}^{type}_{by,on}$ convention introduced back in 14.1 (so, for instance, $\vec{F}^k_{sb}$ is the force of kinetic friction exerted by the surface on the block); however, later on, for algebraic manipulations, and especially where x and y components need to be taken, I will drop the "by, on" subscripts, and just let the "type" superscript identify the force in question.

The diagrams also show the coordinate axes I have chosen: the x axis is along the plane, and the y is perpendicular to it. The advantage of this choice is obvious: the motion is entirely along one of the axes, and two of the forces (the normal force and the friction) already lie along the axes. The only force that does not is the block's weight (that is, the force of gravity), so we need to decompose it into its x and y components. For this, we can make use of the fact, which follows from basic geometry, that the angle of the incline, θ, is also the angle between the vector $\vec{F}^g$ and the *negative* y axis. This means we have

$$\begin{aligned} F^g_x &= F^g \sin\theta \\ F^g_y &= -F^g \cos\theta. \end{aligned} \tag{16.3.1}$$

Equations (16.3.1) also show another convention I will adopt from now, namely, that whenever the symbol for a vector is shown *without* an arrow on top *or* an x or y subscript, it will be understood to refer to the *magnitude* of the vector, which is always a positive number by definition.

Newton's second law, as given by equations (16.1.2) applied to this system, then reads:

$$F^g_x + F^k_x = ma_x = F^g \sin\theta - F^k \tag{16.3.2}$$

for the motion along the plane, and

$$F^g_y + F^n_y = ma_y = -F^g \cos\theta + F^n \tag{16.3.3}$$

for the direction perpendicular to the plane. Of course, since there is no motion in this direction, a_y is zero. This gives us immediately the value of the normal force:

$$F^n = F^g \cos\theta = mg\cos\theta \tag{16.3.4}$$

since $F^g = mg$. Now including our force of kinetic friction F_k, along with $F^G = mg$ in Equation (16.3.2), we get

$$ma_x = mg\sin\theta - F_k. \tag{16.3.5}$$

We can eliminate the mass to obtain finally

$$a_x = g\left(\sin\theta - \frac{F_k}{m}\right) \tag{16.3.6}$$

which is the desired result. In the absence of friction (μ_k = 0) this gives $a = g\sin\theta$, a result you might have seen already.

Of course, we know from experience that what happens when θ is very small is that the block does *not* slide: it is held in place by the force of static friction. The diagram for such a situation looks the same as Figure 16.3.1, except that $\vec{a}$ = 0, the force of friction is F^s instead of F^k, and of course its magnitude must match that of the x component of gravity. Equation (16.3.2) then becomes

$$ma_x = 0 = F^g \sin\theta - F^s. \tag{16.3.7}$$

It turns out that the force of static friction F_s does not have a fixed value - *it just has a maximum value*, over which the object starts to move. In this case we can find this maximum value by looking at the previous equation; as long as the static friction obeys

$$F_s \leq F^g \sin\theta, \tag{16.3.8}$$

the block will remain stationary. Again, we will look at this result more carefully in Chapter 18.

What if we send the block sliding *up* the plane instead? The acceleration would still be pointing down (since the object would be slowing down all the while), but now the force of kinetic friction would point in the direction *opposite* that indicated in Figure 16.3.1, since it always must oppose the motion. If you go through the same analysis I carried out above, you will get that $a_x = g(\sin\theta + F_k)$ in that case, since now friction and gravity are working together to slow the motion down.

16.4: Examples

? Whiteboard Problem 16.4.1: The Range Formula

The range of a projectile which starts and ends at the same altitude can be found with

$$R=\frac{2v_0^2\sin\theta\cos\theta}{g}, \tag{16.4.1}$$

where v_0 is the initial speed and θ is the launch angle.

1. Derive this formula from the kinematic equations for projectile motion.
2. Find the angle at which this range will be maximized. *Hint:* Use the trig identify

$$\sin 2\theta = 2\sin\theta\cos\theta. \tag{16.4.2}$$

✔ Example 16.4.2: A Fireworks Projectile Explodes high and away

During a fireworks display, a shell is shot into the air with an initial speed of 70.0 m/s at an angle of 75.0° above the horizontal, as illustrated in Figure 16.4.3. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. (a) Calculate the height at which the shell explodes. (b) How much time passes between the launch of the shell and the explosion? (c) What is the horizontal displacement of the shell when it explodes? (d) What is the total displacement from the point of launch to the highest point?

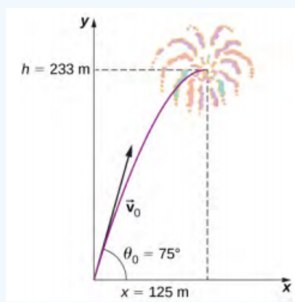


Figure 16.4.3: The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, which is found to be at a height of 233 m and 125 m away horizontally.

Strategy

The motion can be broken into horizontal and vertical motions in which $a_x = 0$ and $a_y = -g$. We can then define x_0 and y_0 to be zero and solve for the desired quantities.

Solution

a. By "height" we mean the altitude or vertical position y above the starting point. The highest point in any trajectory, called the apex, is reached when $v_y = 0$. Since we know the initial and final velocities, as well as the initial position, we use the following equation to find y: $v_y^2 = v_{0y}^2 - 2g(y - y_0)$. Because y_0 and v_y are both zero, the equation simplifies to $0 = v_{0y}^2 - 2gy$. Solving for y gives $y = \frac{v_{0y}^2}{2g}$. Now we must find v_{0y}, the component of the initial velocity in the y direction. It is given by $v_{0y} = v_0 \sin\theta_0$, where v_0 is the initial velocity of 70.0 m/s and $\theta_0 = 75°$ is the initial angle. Thus $v_{0y} = v_0\sin\theta = (70.0\ m/s)\sin 75^o = 67.6\ m/s$ and y is $y = \frac{(67.6\ m/s)^2}{2(9.80\ m/s^2)}$. Thus, we have $y = 233\ m$. Note that because up is positive, the initial vertical velocity is positive, as is the maximum height, but the acceleration resulting from gravity is negative. Note also that the maximum height depends only on the vertical component of the initial velocity, so that any projectile with a 67.6-m/s initial vertical component of velocity reaches a maximum height of 233 m (neglecting air resistance). The numbers in this example are reasonable for large fireworks displays, the shells of which do reach such heights before exploding. In practice, air resistance is not completely negligible, so the initial velocity would have to be somewhat larger than that given to reach the same height.

b. As in many physics problems, there is more than one way to solve for the time the projectile reaches its highest point. In this case, the easiest method is to use $v_y = v_{0y} - gt$. Because $v_y = 0$ at the apex, this equation reduces $0 = v_{0y} - gt$ or $t = \frac{v_{0y}}{g} = \frac{67.6\ m/s}{9.80\ m/s^2} = 6.90\ s$. This time is also reasonable for large fireworks. If you are able to see the launch of fireworks, notice that several seconds pass before the shell explodes. Another way of finding the time is by using $y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$. This is left for you as an exercise to complete.

c. Because air resistance is negligible, $a_x = 0$ and the horizontal velocity is constant, as discussed earlier. The horizontal displacement is the horizontal velocity multiplied by time as given by $x = x_0 + v_x t$, where x_0 is equal to zero. Thus, $x = v_x t,$ where v_x is the x-component of the velocity, which is given by $v_x = v_0 \cos\theta = (70.0\ m/s)\cos 75^o = 18.1\ m/s.$ Time t for both motions is the same, so x is $x = (18.1\ m/s)(6.90\ s) = 125\ m.$ Horizontal motion is a constant velocity in the absence of air resistance. The horizontal displacement found here could be useful in keeping the fireworks fragments from falling on spectators. When the shell explodes, air resistance has a major effect, and many fragments land directly below.

d. The horizontal and vertical components of the displacement were just calculated, so all that is needed here is to find the magnitude and direction of the displacement at the highest point: $\vec{s} = 125\hat{i} + 233\hat{j} |\vec{s}| = \sqrt{125^2 + 233^2} = 264\ m$ $\theta = \tan^{-1}\left(\frac{233}{125}\right) = 61.8^o$. Note that the angle for the displacement vector is less than the initial angle of launch. To see why this is, review Figure 16.4.1, which shows the curvature of the trajectory toward the ground level. When solving Example 4.7(a), the expression we found for y is valid for any projectile motion when air resistance is negligible. Call the maximum height y = h. Then, $h = \frac{v_{0y}^2}{2g}$. This equation defines the **maximum height of a projectile above its launch position** and it depends only on the vertical component of the initial velocity.

Exercise 16.4.3

A rock is thrown horizontally off a cliff 100.0 m high with a velocity of 15.0 m/s. (a) Define the origin of the coordinate system. (b) Which equation describes the horizontal motion? (c) Which equations describe the vertical motion? (d) What is the rock's velocity at the point of impact?

Example 16.4.4: Calculating projectile motion- Tennis Player

A tennis player wins a match at Arthur Ashe stadium and hits a ball into the stands at 30 m/s and at an angle 45° above the horizontal (Figure 16.4.4). On its way down, the ball is caught by a spectator 10 m above the point where the ball was hit. (a) Calculate the time it takes the tennis ball to reach the spectator. (b) What are the magnitude and direction of the ball's velocity at impact?

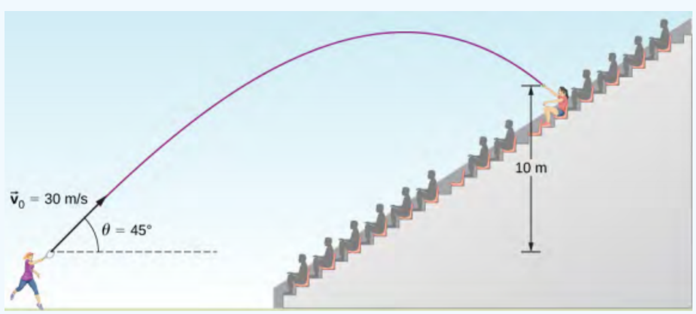


Figure 16.4.4: The trajectory of a tennis ball hit into the stands.

Strategy

Again, resolving this two-dimensional motion into two independent one-dimensional motions allows us to solve for the desired quantities. The time a projectile is in the air is governed by its vertical motion alone. Thus, we solve for t first. While the ball is rising and falling vertically, the horizontal motion continues at a constant velocity. This example asks for the final velocity. Thus, we recombine the vertical and horizontal results to obtain $\vec{v}$ at final time t, determined in the first part of the example.

Solution

a. While the ball is in the air, it rises and then falls to a final position 10.0 m higher than its starting altitude. We can find the time for this by using the third equation in 16.2.5: $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$. If we take the initial position y_0 to be zero, then the final position is y = 10 m. The initial vertical velocity is the vertical component of the initial velocity: $v_{0y} = v_0 \sin\theta_0 = (30.0\ m/s)\sin 45^o = 21.2\ m/s.$. Substituting into our kinematic equation for y gives us $10.0\ m = (21.2\ m/s)t - (4.90\ m/s^2)t^2$ Rearranging terms gives a quadratic equation in t: $(4.90\ m/s^2)t^2 - (21.2\ m/s)t + 10.0\ m = 0$.Use of the quadratic formula yields t = 3.79 s and t = 0.54 s. Since the ball is at a height of 10 m at two times during its trajectory—once on the way up and once on the way down—we take the longer solution for the time it takes the ball to reach the spectator: $t = 3.79\ s$.The time for projectile motion is determined completely by the vertical motion. Thus, any projectile that has an initial vertical velocity of 21.2 m/s and lands 10.0 m above its starting altitude spends 3.79 s in the air.

b. We can find the final horizontal and vertical velocities v_x and v_y with the use of the result from (a). Then, we can combine them to find the magnitude of the total velocity vector $\vec{v}$ and the angle θ it makes with the horizontal. Since v_x is constant, we can solve for it at any horizontal location. We choose the starting point because we know both the initial velocity and the initial angle. Therefore, $v_x = v_0 \cos\theta_0 = (30\ m/s)\cos 45^o = 21.2\ m/s.$. The final vertical velocity is given by the last equation in 16.2.5: $v_y = v_{0y} - gt.$ Since v_{0y} was found in part (a) to be 21.2 m/s, we have $v_y = 21.2\ m/s - (9.8\ m/s^2)(3.79s) = -15.9\ m/s$.The magnitude of the final velocity $\vec{v}$ is $v = \sqrt{v_x^2 + v_y^2} = \sqrt{(21.2\ m/s)^2 + (-15.9\ m/s)^2} = 26.5\ m/s$. The direction θ_v is found using the inverse tangent: $\theta_v = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{21.2}{-15.9}\right) = -53.1^o.$

Significance

a. As mentioned earlier, the time for projectile motion is determined completely by the vertical motion. Thus, any projectile that has an initial vertical velocity of 21.2 m/s and lands 10.0 m above its starting altitude spends 3.79 s in the air.

b. The negative angle means the velocity is 53.1° below the horizontal at the point of impact. This result is consistent with the fact that the ball is impacting at a point on the other side of the apex of the trajectory and therefore has a negative y component of the velocity. The magnitude of the velocity is less than the magnitude of the initial velocity we expect since it is impacting 10.0 m above the launch elevation.

✔ Example 16.4.5: Comparing golf shots

A golfer finds himself in two different situations on different holes. On the second hole he is 120 m from the green and wants to hit the ball 90 m and let it run onto the green. He angles the shot low to the ground at 30° to the horizontal to let the ball roll after impact. On the fourth hole he is 90 m from the green and wants to let the ball drop with a minimum amount of rolling after impact. Here, he angles the shot at 70° to the horizontal to minimize rolling after impact. Both shots are hit and impacted on a level surface. (a) What is the initial speed of the ball at the second hole? (b) What is the initial speed of the ball at the fourth hole? (c) Write the trajectory equation for both cases. (d) Graph the trajectories.

Strategy

We see that the range equation (see example problem 16.4.1) has the initial speed and angle, so we can solve for the initial speed for both (a) and (b). When we have the initial speed, we can use this value to write the trajectory equation.

Solution

a. $R = \frac{v_0^2 \sin 2\theta_0}{g} \Rightarrow v_0 = \sqrt{\frac{Rg}{\sin 2\theta_0}} = \sqrt{\frac{(90.0\ m)(9.8\ m/s^2)}{\sin(2(30^o))}} = 31.9\ m/s$

b. $R = \frac{v_0^2 \sin 2\theta_0}{g} \Rightarrow v_0 = \sqrt{\frac{Rg}{\sin 2\theta_0}} = \sqrt{\frac{(90.0\ m)(9.8\ m/s^2)}{\sin(2(70^o))}} = 37.0\ m/s$

c. $y = x\left[\tan\theta_0 - \frac{g}{2(v_0\cos\theta_0)^2}x\right]$ Second hole: $y = x\left[\tan 30^o - \frac{9.8\ m/s^2}{2[(31.9\ m/s)(\cos 30^o)]^2}x\right] = 0.58x - 0.0064x^2$ Fourth hole: $y = x\left[\tan 70^o - \frac{9.8\ m/s^2}{2[(37.0\ m/s)(\cos 70^o)]^2}x\right] = 2.75x - 0.0306x^2$

d. Using a graphing utility, we can compare the two trajectories, which are shown in Figure 16.4.6.

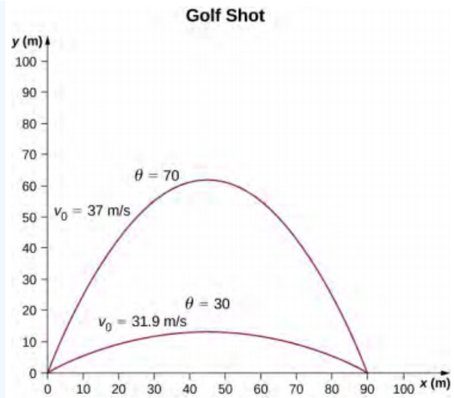


Figure 16.4.6: Two trajectories of a golf ball with a range of 90 m. The impact points of both are at the same level as the launch point.

Significance

The initial speed for the shot at 70° is greater than the initial speed of the shot at 30°. Note from Figure 16.4.6 that two projectiles launched at the same speed but at different angles have the same range if the launch angles add to 90°. The launch angles in this example add to give a number greater than 90°. Thus, the shot at 70° has to have a greater launch speed to reach 90 m, otherwise it would land at a shorter distance.

Exercise 16.4.6

If the two golf shots in Example 4.9 were launched at the same speed, which shot would have the greatest range?

Simulation

At PhET Explorations: Projectile Motion, learn about projectile motion in terms of the launch angle and initial velocity.

CHAPTER OVERVIEW

17: N4) Motion from Forces

In this chapter, we are going to talk about how to determine the motion of an object from the forces that act on that object. This is a combination of dynamics (forces), and kinematics (acceleration, velocity, and position). This process is generally the primary thing we are interested in as physicists - both *why* things move, and *how* they move.

The first, and most important thing, to understand is the two sides of Newton's second law, $\Sigma\vec{F} = m\vec{a}$. The left hand side is the addition of all the forces, and the right hand side is simply the mass times the acceleration. You need both sides in order to determine the motion of the object - typically the best strategy for solving these problems is to carefully write down everything you know about the forces (which forces, which directions), and everything you know about the accelerations (do you know anything?). Then it is a matter of solving the set of equations in front of you, acknowledging that they generally *will be a set* - that is, multiple equations and multiple unknowns.

There a few things to watch out for, and a few new ideas we will have to develop:

The Normal Force: This is a contact force between surfaces, like a block and the floor. The word normal here does not mean "usual", but means normal in the mathematical sense - that is, perpendicular. It's the force that prevents blocks from going through floors, or you hand from going through the wall when you push on it. This is not a *fundamental force*, like gravity or electric, but just an effective force that actually arises from the electromagnetic interactions between the atoms in the solid.

Tension in Ropes: We will be dealing a lot with ropes in this part of the class. At a basic level, ropes are just a way to transmit pulling forces (since you cannot push things with ropes!). The force that ropes transmit is called "tension", but even that is a more general word that can refer to other kinds of forces as well. The key aspect of ropes in this class is that they will be ideal - massless and inelastic. Among other things, this means that the tension the rope delivers is constant over it's length. So if a rope is apply a 50 N force to an object, that tension in that rope is 50 N.

Pulleys: Pulleys are nice objects to have when you have ropes around, because they allow you to change the direction of the forces in the ropes. In this class, we were generally only deal with ideal pulleys - massless and frictionless. Specifically, what that means is that pulleys can only change the direction of the tension in ropes, they cannot change the magnitude of the tension.

17.1: Solving Problems with Newton's Laws (Part 1)

Success in problem solving is necessary to understand and apply physical principles. We developed a pattern of analyzing and setting up the solutions to problems involving Newton's laws in Newton's Laws of Motion; in this chapter, we continue to discuss these strategies and apply a step-by-step process.

Problem-Solving Strategies

We follow here the basics of problem solving presented earlier in this text, but we emphasize specific strategies that are useful in applying Newton's laws of motion. Once you identify the physical principles involved in the problem and determine that they include Newton's laws of motion, you can apply these steps to find a solution. These techniques also reinforce concepts that are useful in many other areas of physics. Many problem-solving strategies are stated outright in the worked examples, so the following techniques should reinforce skills you have already begun to develop.

Problem-Solving Strategy: Applying Newton's Laws of Motion

1. Identify the physical principles involved by listing the givens and the quantities to be calculated.
2. Sketch the situation, using arrows to represent all forces.
3. Determine the system of interest, and draw a free body diagram.
4. Apply Newton's second law to solve the problem. If necessary, apply appropriate kinematic equations from the chapter on motion along a straight line.
5. Check the solution to see whether it is reasonable.

Let's apply this problem-solving strategy to the challenge of lifting a grand piano into a second-story apartment. Once we have determined that Newton's laws of motion are involved (if the problem involves forces), it is particularly important to draw a careful sketch of the situation. Such a sketch is shown in Figure $17.1.1a$. Then, as in Figure $17.1.1b$, we can represent all forces with arrows. Whenever sufficient information exists, it is best to label these arrows carefully and make the length and direction of each correspond to the represented force.

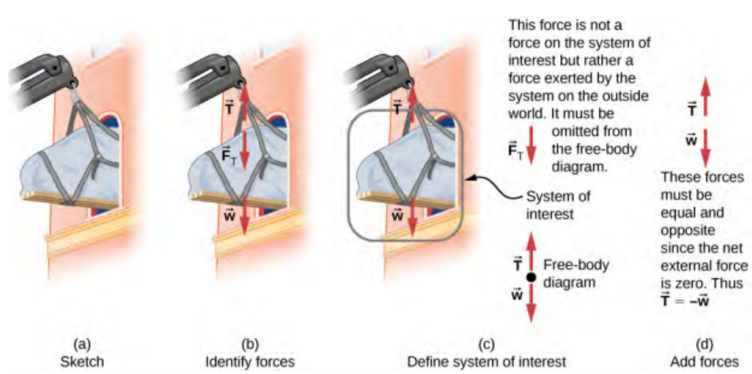


Figure $17.1.1$: (a) A grand piano is being lifted to a second-story apartment. (b) Arrows are used to represent all forces: $\vec{T}$ is the tension in the rope above the piano, $\vec{F}_T$ is the force that the piano exerts on the rope, and $\vec{w}$ is the weight of the piano. All other forces, such as the nudge of a breeze, are assumed to be negligible. (c) Suppose we are given the piano's mass and asked to find the tension in the rope. We then define the system of interest as shown and draw a free-body diagram. Now $\vec{F}_T$ is no longer shown, because it is not a force acting on the system of interest; rather, $\vec{F}_T$ acts on the outside world. (d) Showing only the arrows, the head-to-tail method of addition is used. It is apparent that if the piano is stationary, $\vec{T} = -\vec{w}$.

As with most problems, we next need to identify what needs to be determined and what is known or can be inferred from the problem as stated, that is, make a list of knowns and unknowns. It is particularly crucial to identify the system of interest, since Newton's second law involves only external forces. We can then determine which forces are external and which are internal, a necessary step to employ Newton's second law. (See Figure $17.1.1c$) Newton's third law may be used to identify whether forces are exerted between components of a system (internal) or between the system and something outside (external). As illustrated in Newton's Laws of Motion, the system of interest depends on the question we need to answer. Only forces are shown in free-body diagrams, not acceleration or velocity. We have drawn several free-body diagrams in previous worked examples. Figure $17.1.1c$ shows a free-body diagram for the system of interest. Note that no internal forces are shown in a free-body diagram.

Once a free-body diagram is drawn, we apply Newton's second law. This is done in Figure $17.1.1d$ for a particular situation. In general, once external forces are clearly identified in free-body diagrams, it should be a straightforward task to put them into equation form and solve for the unknown, as done in all previous examples. If the problem is one-dimensional—that is, if all forces are parallel—then the forces can be handled algebraically. If the problem is two-dimensional, then it must be broken down into a pair of one-dimensional problems. We do this by projecting the force vectors onto a set of axes chosen for convenience. As seen in previous examples, the choice of axes can simplify the problem. For example, when an incline is involved, a set of axes with one axis parallel to the incline and one perpendicular to it is most convenient. It is almost always convenient to make one axis parallel to the direction of motion, if this is known. Generally, just write Newton's second law in components along the different directions. Then, you have the following equations:

$$\sum F_x = ma_x, \quad \sum F_y = ma_y. \tag{17.1.1}$$

(If, for example, the system is accelerating horizontally, then you can then set ay = 0.) We need this information to determine unknown forces acting on a system.

As always, we must check the solution. In some cases, it is easy to tell whether the solution is reasonable. For example, it is reasonable to find that friction causes an object to slide down an incline more slowly than when no friction exists. In practice, intuition develops gradually through problem solving; with experience, it becomes progressively easier to judge whether an answer is reasonable. Another way to check a solution is to check the units. If we are solving for force and end up with units of millimeters per second, then we have made a mistake.

There are many interesting applications of Newton's laws of motion, a few more of which are presented in this section. These serve also to illustrate some further subtleties of physics and to help build problem-solving skills. We look first at problems involving particle equilibrium, which make use of Newton's first law, and then consider particle acceleration, which involves Newton's second law.

Particle Equilibrium

Recall that a particle in equilibrium is one for which the external forces are balanced. Static equilibrium involves objects at rest, and dynamic equilibrium involves objects in motion without acceleration, but it is important to remember that these conditions are relative. For example, an object may be at rest when viewed from our frame of reference, but the same object would appear to be in motion when viewed by someone moving at a constant velocity. We now make use of the knowledge attained in Newton's Laws of Motion, regarding the different types of forces and the use of free-body diagrams, to solve additional problems in particle equilibrium.

✔ Example 17.1.1: Different Tensions at Different Angles

Consider the traffic light (mass of 15.0 kg) suspended from two wires as shown in Figure $17.1.2$ Find the tension in each wire, neglecting the masses of the wires.

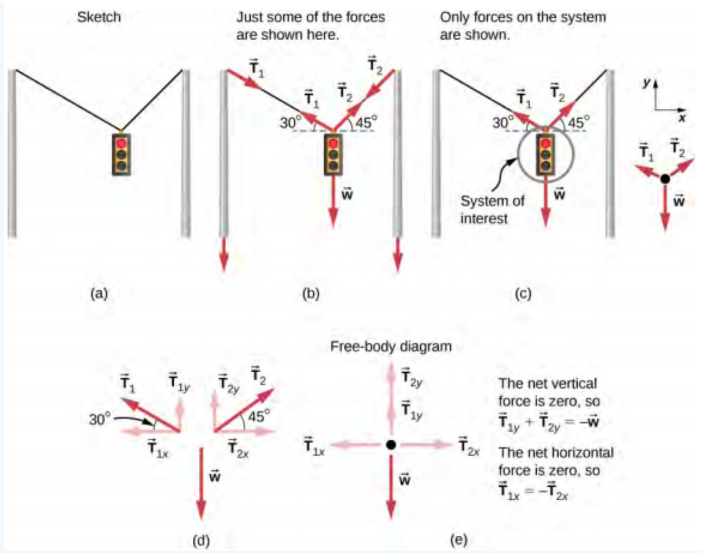


Figure 17.1.2: A traffic light is suspended from two wires. (b) Some of the forces involved. (c) Only forces acting on the system are shown here. The free-body diagram for the traffic light is also shown. (d) The forces projected onto vertical (y) and horizontal (x) axes. The horizontal components of the tensions must cancel, and the sum of the vertical components of the tensions must equal the weight of the traffic light. (e) The free-body diagram shows the vertical and horizontal forces acting on the traffic light.

Strategy

The system of interest is the traffic light, and its free-body diagram is shown in Figure 17.1.2c The three forces involved are not parallel, and so they must be projected onto a coordinate system. The most convenient coordinate system has one axis vertical and one horizontal, and the vector projections on it are shown in Figure 17.1.2d. There are two unknowns in this problem (T_1 and T_2), so two equations are needed to find them. These two equations come from applying Newton's second law along the vertical and horizontal axes, noting that the net external force is zero along each axis because acceleration is zero.

Solution

First consider the horizontal or x-axis:

$$F_{netx} = T_{2x} - T_{1x} = 0. \tag{17.1.2}$$

Thus, as you might expect,

$$T_{1x} = T_{2x}. \tag{17.1.3}$$

This give us the following relationship:

$$T_1 \cos 30^o = T_2 \cos 45^o. \tag{17.1.4}$$

Thus,

$$T_2 = 1.225 T_1. \tag{17.1.5}$$

Note that T_1 and T_2 are not equal in this case because the angles on either side are not equal. It is reasonable that T_2 ends up being greater than T_1 because it is exerted more vertically than T_1.

Now consider the force components along the vertical or y-axis:

$$F_{nety} = T_{1y} + T_{1x} - w = 0. \tag{17.1.6}$$

This implies

$$T_{1y} + T_{2y} = w. \tag{17.1.7}$$

Substituting the expressions for the vertical components gives

$$T_1 \sin 30^o + T_2 \sin 45^o = w. \tag{17.1.8}$$

There are two unknowns in this equation, but substituting the expression for T_2 in terms of T_1 reduces this to one equation with one unknown:

$$T_1(0.500) + (1.225T_1)(0.707) = w = mg, \tag{17.1.9}$$

which yields

$$1.366T_1 = (15.0\ kg)(9.80\ m/s^2). \tag{17.1.10}$$

Solving this last equation gives the magnitude of T_1 to be

$$T_1 = 108\ N. \tag{17.1.11}$$

Finally, we find the magnitude of T_2 by using the relationship between them, T_2 = 1.225 T_1, found above. Thus we obtain

$$T_2 = 132\ N. \tag{17.1.12}$$

Significance

Both tensions would be larger if both wires were more horizontal, and they will be equal if and only if the angles on either side are the same (as they were in the earlier example of a tightrope walker in Newton's Laws of Motion.)

✔ Example 17.1.2: Drag Force on a Barge

Two tugboats push on a barge at different angles (Figure 17.1.3). The first tugboat exerts a force of 2.7 x 10^5 N in the x-direction, and the second tugboat exerts a force of 3.6 x 10^5 N in the y-direction. The mass of the barge is 5.0 × 10^6 kg and its acceleration is observed to be 7.5 x 10^{-2} m/s^2 in the direction shown. What is the drag force of the water on the barge resisting the motion? (**Note**: Drag force is a frictional force exerted by fluids, such as air or water. The drag force opposes the motion of the object. Since the barge is flat bottomed, we can assume that the drag force is in the direction opposite of motion of the barge.)

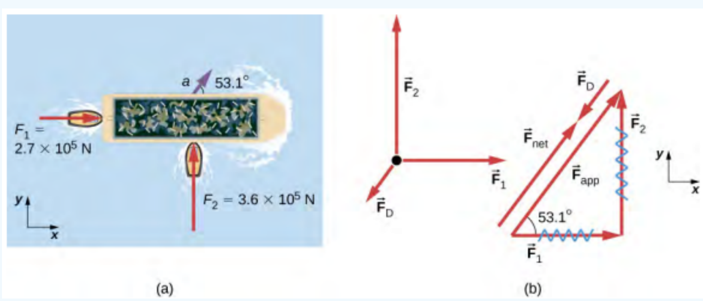


Figure 17.1.3: (a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Note that $\vec{F}_{app}$ is the total applied force of the tugboats.

Strategy

The directions and magnitudes of acceleration and the applied forces are given in Figure 17.1.3a. We define the total force of the tugboats on the barge as $\vec{F}_{app}$ so that

$$\vec{F}_{app} = \vec{F}_1 + \vec{F}_2. \tag{17.1.13}$$

The drag of the water $\vec{F}_D$ is in the direction opposite to the direction of motion of the boat; this force thus works against $\vec{F}_{app}$, as shown in the free-body diagram in Figure 17.1.3b The system of interest here is the barge, since the forces on it are given as well as its acceleration. Because the applied forces are perpendicular, the x- and y-axes are in the same direction as $\vec{F}_1$ and $\vec{F}_2$. The problem quickly becomes a one-dimensional problem along the direction of $\vec{F}_{app}$, since friction is in the direction opposite to $\vec{F}_{app}$. Our strategy is to find the magnitude and direction of the net applied force $\vec{F}_{app}$ and then apply Newton's second law to solve for the drag force $\vec{F}_D$.

Solution

Since F_x and F_y are perpendicular, we can find the magnitude and direction of $\vec{F}_{app}$ directly. First, the resultant magnitude is given by the Pythagorean theorem:

$$\vec{F}_{app} = \sqrt{F_1^2 + F_2^2} = \sqrt{(2.7 \times 10^5\ N)^2 + (3.6 \times 10^5\ N)^2} = 4.5 \times 10^5\ N. \tag{17.1.14}$$

The angle is given by

$$\theta = \tan^{-1}\left(\frac{F_2}{F_1}\right) = \tan^{-1}\left(\frac{3.6 \times 10^5\ N}{2.7 \times 10^5\ N}\right) = 53.1^o. \tag{17.1.15}$$

From Newton's first law, we know this is the same direction as the acceleration. We also know that $\vec{F}_D$ is in the opposite direction of $\vec{F}_{app}$, since it acts to slow down the acceleration. Therefore, the net external force is in the same direction as $\vec{F}_{app}$, but its magnitude is slightly less than $\vec{F}_{app}$. The problem is now one-dimensional. From the free-body diagram, we can see that

$$F_{net} = F_{app} - F_D. \tag{17.1.16}$$

However, Newton's second law states that

$$F_{net} = ma. \tag{17.1.17}$$

Thus,

$$F_{app} - F_D = ma. \tag{17.1.18}$$

This can be solved for the magnitude of the drag force of the water F_D in terms of known quantities:

$$F_D = F_{app} - ma. \tag{17.1.19}$$

Substituting known values gives

$$F_D = (4.5 \times 10^5\ N) - (5.0 \times 10^6\ kg)(7.5 \times 10^{-2}\ m/s^2) = 7.5 \times 10^4\ N. \tag{17.1.20}$$

The direction of $\vec{F}_D$ has already been determined to be in the direction opposite to $\vec{F}_{app}$, or at an angle of 53° south of west.

Significance

The numbers used in this example are reasonable for a moderately large barge. It is certainly difficult to obtain larger accelerations with tugboats, and small speeds are desirable to avoid running the barge into the docks. Drag is relatively small for a well-designed hull at low speeds, consistent with the answer to this example, where F_D is less than 1/600th of the weight of the ship.

In Newton's Laws of Motion, we discussed the normal force, which is a contact force that acts normal to the surface so that an object does not have an acceleration perpendicular to the surface. The bathroom scale is an excellent example of a normal force acting on a body. It provides a quantitative reading of how much it must push upward to support the weight of an object. But can you predict what you would see on the dial of a bathroom scale if you stood on it during an elevator ride?

Will you see a value greater than your weight when the elevator starts up? What about when the elevator moves upward at a constant speed? Take a guess before reading the next example.

✓ Example 17.1.3: What does the Bathroom Scale Read in an Elevator?

Figure 17.1.4 shows a 75.0-kg man (weight of about 165 lb.) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate of 1.20 m/s^2, and (b) if the elevator moves upward at a constant speed of 1 m/s.

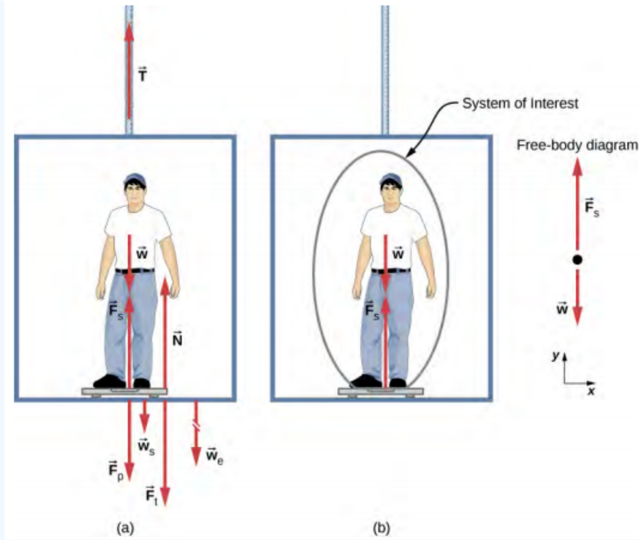


Figure 17.1.4: (a) The various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale. $\vec{T}$ is the tension in the supporting cable, $\vec{w}$ is the weight of the person, $\vec{w}_s$ is the weight of the scale, $\vec{w}_e$ is the weight of the elevator, $\vec{F}_s$ is the force of the scale on the person, $\vec{F}_p$ is the force of the person on the scale, $\vec{F}_t$ is the force of the scale on the floor of the elevator, and $\vec{N}$ is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person—and is the diagram we use for the solution of the problem.

Strategy

If the scale at rest is accurate, its reading equals $\vec{F}_p$, the magnitude of the force the person exerts downward on it. Figure $17.1.4a$ shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn, as in Figure $17.1.4b$ Analysis of the free-body diagram using Newton's laws can produce answers to both Figure $17.1.4a$ and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight $\vec{w}$ and the upward force of the scale $\vec{F}_s$. According to Newton's third law, $\vec{F}_p$ and $\vec{F}_s$ are equal in magnitude and opposite in direction, so that we need to find F_s in order to find what the scale reads. We can do this, as usual, by applying Newton's second law,

$$\vec{F}_{net} = m\vec{a}. \tag{17.1.21}$$

From the free-body diagram, we see that $\vec{F}_{net} = \vec{F}_s - \vec{w}$, so we have

$$F_s - w = ma. \tag{17.1.22}$$

Solving for F_s gives us an equation with only one unknown:

$$F_s = ma + w, \tag{17.1.23}$$

or, because w = mg, simply

$$F_s = ma + mg. \tag{17.1.24}$$

No assumptions were made about the acceleration, so this solution should be valid for a variety of accelerations in addition to those in this situation. (**Note**: We are considering the case when the elevator is accelerating upward. If the elevator is accelerating downward, Newton's second law becomes $F_s - w = -ma$.)

Solution

a. We have a = 1.20 m/s^2, so that $F_{s} = (75.0\; kg)(9.80\; m/s^{2}) + (75.0\; kg)(1.20\; m/s^{2})$yielding $F_{s} = 825\; N \ldotp$

b. Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight?

For any constant velocity—up, down, or stationary—acceleration is zero because $a = \frac{\Delta v}{\Delta t}$ and $\Delta v = 0$. Thus, $F_{s} = ma + mg = 0 + mg$or $F_{s} = (75.0\; kg)(9.80\; m/s^{2}),$which gives $F_{s} = 735\; N \ldotp$

Significance

The scale reading in Figure $17.1.4a$ is about 185 lb. What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:

$$F_{net} = ma = 0 = F_s - w \tag{17.1.25}$$

$$F_s = w = mg \tag{17.1.26}$$

$$F_s = (75.0\ kg)(9.80\ m/s^2) = 735\ N. \tag{17.1.27}$$

Thus, the scale reading in the elevator is greater than his 735-N (165-lb.) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward.

Clearly, the greater the acceleration of the elevator, the greater the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators. In Figure $17.1.4b$, the scale reading is 735 N, which equals the person's weight. This is the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

Exercise 17.1.4

Now calculate the scale reading when the elevator accelerates downward at a rate of 1.20 m/s^2.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward, a is negative, and the scale reading is **less** than the weight of the person. If a constant downward velocity is reached, the scale reading again becomes equal to the person's weight. If the elevator is in free fall and accelerating downward at g, then the scale reading is zero and the person appears to be weightless.

Example 17.1.5: Two Attached Blocks

Figure $17.1.5$ shows a block of mass m_1 on a frictionless, horizontal surface. It is pulled by a light string that passes over a frictionless and massless pulley. The other end of the string is connected to a block of mass m_2. Find the acceleration of the blocks and the tension in the string in terms of m_1, m_2, and g.

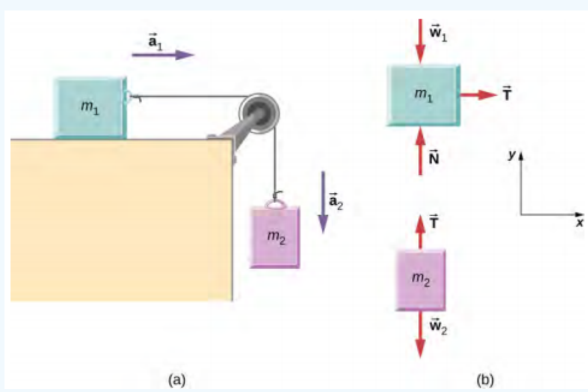


Figure $17.1.5$: (a) Block 1 is connected by a light string to block 2. (b) The free-body diagrams of the blocks.

Strategy

We draw a free-body diagram for each mass separately, as shown in Figure $17.1.5$. Then we analyze each one to find the required unknowns. The forces on block 1 are the gravitational force, the contact force of the surface, and the tension in the string. Block 2 is subjected to the gravitational force and the string tension. Newton's second law applies to each, so we write two vector equations:

For block 1: $\vec{T} + \vec{w}_1 + \vec{N} = m_1\vec{a}_1$

For block 2: $\vec{T}+\vec{w}_2=m_2\vec{a}_2$.

Notice that $\vec{T}$ is the same for both blocks. Since the string and the pulley have negligible mass, and since there is no friction in the pulley, the tension is the same throughout the string. We can now write component equations for each block. All forces are either horizontal or vertical, so we can use the same horizontal/vertical coordinate system for both objects.

Solution

The component equations follow from the vector equations above. We see that block 1 has the vertical forces balanced, so we ignore them and write an equation relating the x-components. There are no horizontal forces on block 2, so only the y-equation is written. We obtain these results:

Block 1	Block 2
$$\sum F_x=ma_x \tag{17.1.28}$$ $$T_x=m_1a_{1x} \tag{17.1.29}$$	$$\sum F_y=ma_y \tag{17.1.30}$$ $$T_y-m_2g=m_2a_{2y} \tag{17.1.31}$$

When block 1 moves to the right, block 2 travels an equal distance downward; thus, $a_{1x} = -a_{2y}$. Writing the common acceleration of the blocks as $a = a_{1x} = -a_{2y}$, we now have

$$T=m_1a \tag{17.1.32}$$

and

$$T-m_2g=-m_2a. \tag{17.1.33}$$

From these two equations, we can express a and T in terms of the masses m_1 and m_2, and g:

$$a=\frac{m_2}{m_1+m_2}g \tag{17.1.34}$$

and

$$T=\frac{m_1m_2}{m_1+m_2}g. \tag{17.1.35}$$

Significance

Notice that the tension in the string is less than the weight of the block hanging from the end of it. A common error in problems like this is to set $T = m_2g$. You can see from the free-body diagram of block 2 that cannot be correct if the block is accelerating.

? Check Your Understanding 17.1.6

Calculate the acceleration of the system, and the tension in the string, when the masses are m_1 = 5.00 kg and m_2 = 3.00 kg.

✓ Example 17.1.7: Atwood Machine

A classic problem in physics, similar to the one we just solved, is that of the Atwood machine, which consists of a rope running over a pulley, with two objects of different mass attached. It is particularly useful in understanding the connection between force and motion. In Figure 17.1.6, m_1 = 2.00 kg and m_2 = 4.00 kg. Consider the pulley to be frictionless. (a) If m_2 is released, what will its acceleration be? (b) What is the tension in the string?

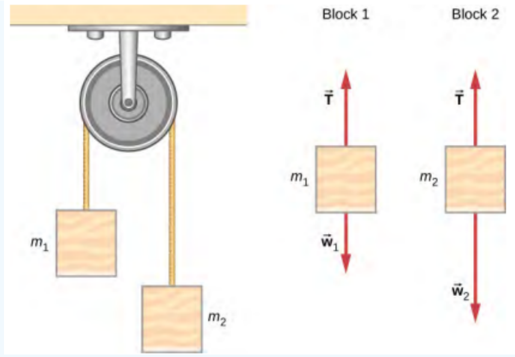


Figure 17.1.6: An Atwood machine and free-body diagrams for each of the two blocks.

Strategy

We draw a free-body diagram for each mass separately, as shown in the figure. Then we analyze each diagram to find the required unknowns. This may involve the solution of simultaneous equations. It is also important to note the similarity with the previous example. As block 2 accelerates with acceleration a_2 in the downward direction, block 1 accelerates upward with acceleration a_1. Thus, $a = a_1 = -a_2$.

Solution

a. We have $For\; m_{1}, \sum F_{y} = T - m_{1}g = m_{1}a \ldotp \quad For\; m_{2}, \sum F_{y} = T - m_{2}g = -m_{2}a \ldotp$(The negative sign in front of m_2 a indicates that m_2 accelerates downward; both blocks accelerate at the same rate, but in opposite directions.) Solve the two equations simultaneously (subtract them) and the result is $(m_{2} - m_{1})g = (m_{1} + m_{2})a \ldotp$Solving for a: $a = \frac{m_{2} - m_{1}}{m_{1} + m_{2}}g = \frac{4\; kg - 2\; kg}{4\; kg + 2\; kg} (9.8\; m/s^{2}) = 3.27\; m/s^{2} \ldotp$
b. Observing the first block, we see that $T - m_{1}g = m_{1}a$ $T = m_{1}(g + a) = (2\; kg)(9.8\; m/s^{2} + 3.27\; m/s^{2}) = 26.1\; N \ldotp$

Significance

The result for the acceleration given in the solution can be interpreted as the ratio of the unbalanced force on the system, $(m_2 - m_1)g$, to the total mass of the system, $m_1 + m_2$. We can also use the Atwood machine to measure local gravitational field strength.

? Exercise 6.3

Determine a general formula in terms of m_1, m_2 and g for calculating the tension in the string for the Atwood machine shown above.

17.2: Solving Problems with Newton's Laws (Part 2)

Newton’s Laws of Motion and Kinematics

Physics is most interesting and most powerful when applied to general situations that involve more than a narrow set of physical principles. Newton’s laws of motion can also be integrated with other concepts that have been discussed previously in this text to solve problems of motion. For example, forces produce accelerations, a topic of kinematics, and hence the relevance of earlier chapters.

When approaching problems that involve various types of forces, acceleration, velocity, and/or position, listing the givens and the quantities to be calculated will allow you to identify the principles involved. Then, you can refer to the chapters that deal with a particular topic and solve the problem using strategies outlined in the text. The following worked example illustrates how the problem-solving strategy given earlier in this chapter, as well as strategies presented in other chapters, is applied to an integrated concept problem.

Example 17.2.1: What Force Must a Soccer Player Exert to Reach Top Speed?

A soccer player starts at rest and accelerates forward, reaching a velocity of 8.00 m/s in 2.50 s. (a) What is her average acceleration? (b) What average force does the ground exert forward on the runner so that she achieves this acceleration? The player’s mass is 70.0 kg, and air resistance is negligible.

Strategy

To find the answers to this problem, we use the problem-solving strategy given earlier in this chapter. The solutions to each part of the example illustrate how to apply specific problem-solving steps. In this case, we do not need to use all of the steps. We simply identify the physical principles, and thus the knowns and unknowns; apply Newton’s second law; and check to see whether the answer is reasonable.

Solution

a. We are given the initial and final velocities (zero and 8.00 m/s forward); thus, the change in velocity is $\Delta v = 8.00$ m/s . We are given the elapsed time, so $\Delta t = 2.50$ s. The unknown is acceleration, which can be found from its definition: $a = \frac{\Delta v}{\Delta t} \ldotp$Substituting the known values yields $a = \frac{8.00\; m/s}{2.50\; s} = 3.20\; m/s^{2} \ldotp$
b. Here we are asked to find the average force the ground exerts on the runner to produce this acceleration. (Remember that we are dealing with the force or forces acting on the object of interest.) This is the reaction force to that exerted by the player backward against the ground, by Newton’s third law. Neglecting air resistance, this would be equal in magnitude to the net external force on the player, since this force causes her acceleration. Since we now know the player’s acceleration and are given her mass, we can use Newton’s second law to find the force exerted. That is, $F_{net} = ma \ldotp$Substituting the known values of m and a gives $F_{net} = (70.0\; kg)(3.20\; m/s^{2}) = 224\; N \ldotp$

This is a reasonable result: The acceleration is attainable for an athlete in good condition. The force is about 50 pounds, a reasonable average force.

Significance

This example illustrates how to apply problem-solving strategies to situations that include topics from different chapters. The first step is to identify the physical principles, the knowns, and the unknowns involved in the problem. The second step is to solve for the unknown, in this case using Newton’s second law. Finally, we check our answer to ensure it is reasonable. These techniques for integrated concept problems will be useful in applications of physics outside of a physics course, such as in your profession, in other science disciplines, and in everyday life.

Exercise 17.2.2

The soccer player stops after completing the play described above, but now notices that the ball is in position to be stolen. If she now experiences a force of 126 N to attempt to steal the ball, which is 2.00 m away from her, how long will it take her to get to the ball?

✓ Example 17.2.3: What Force Acts on a Model Helicopter?

A 1.50-kg model helicopter has a velocity of 5.00 $\hat{j}$ m/s at t = 0. It is accelerated at a constant rate for two seconds (2.00 s) after which it has a velocity of (6.00 $\hat{i}$ + 12.00 $\hat{j}$) m/s. What is the magnitude of the resultant force acting on the helicopter during this time interval?

Strategy

We can easily set up a coordinate system in which the x-axis ($\hat{i}$ direction) is horizontal, and the y-axis ($\hat{j}$ direction) is vertical. We know that Δt = 2.00s and Δv = (6.00 $\hat{i}$ + 12.00 $\hat{j}$ m/s) − (5.00 $\hat{j}$ m/s). From this, we can calculate the acceleration by the definition; we can then apply Newton's second law.

Solution

We have

$$a = \frac{\Delta v}{\Delta t} = \frac{(6.00\hat{i} + 12.00\hat{j}\ m/s) - (5.00\hat{j}\ m/s)}{2.00\ s} = 3.00\hat{i} + 3.50\hat{j}\ m/s^2\$\$\sum\vec{F} = m\vec{a} \\ = (1.50\ kg)(3.00\hat{i} + 3.50\hat{j}\ m/s^2) = 4.50\hat{i} + 5.25\hat{j}\ N. \tag{17.2.1}$$

The magnitude of the force is now easily found:

$$F = \sqrt{(4.50\ N)^2 + (5.25\ N)^2} = 6.91\ N. \tag{17.2.2}$$

Significance

The original problem was stated in terms of $\hat{i} - \hat{j}$ vector components, so we used vector methods. Compare this example with the previous example.

? Exercise 17.2.4

Find the direction of the resultant for the 1.50-kg model helicopter.

✓ Example $PageIndex{5}$: Baggage Tractor

Figure 17.2.7(a) shows a baggage tractor pulling luggage carts from an airplane. The tractor has mass 650.0 kg, while cart A has mass 250.0 kg and cart B has mass 150.0 kg. The driving force acting for a brief period of time accelerates the system from rest and acts for 3.00 s. (a) If this driving force is given by F = (820.0t) N, find the speed after 3.00 seconds. (b) What is the horizontal force acting on the connecting cable between the tractor and cart A at this instant?

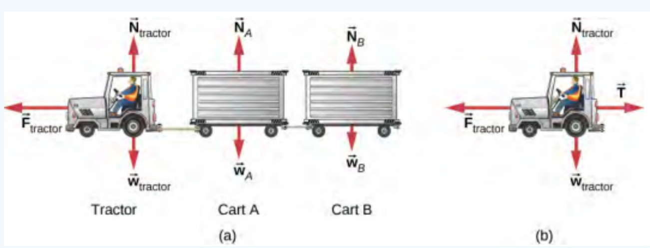


Figure 17.2.7: (a) A free-body diagram is shown, which indicates all the external forces on the system consisting of the tractor and baggage carts for carrying airline luggage. (b) A free-body diagram of the tractor only is shown isolated in order to calculate the tension in the cable to the carts.

Strategy

A free-body diagram shows the driving force of the tractor, which gives the system its acceleration. We only need to consider motion in the horizontal direction. The vertical forces balance each other and it is not necessary to consider them. For part b, we make use of a free-body diagram of the tractor alone to determine the force between it and cart A. This exposes the coupling force $\vec{T}$, which is our objective.

Solution

a. $\sum F_{x} = m_{system} a_{x}\; and\; \sum F_{x} = 820.0t,$so $820.0t = (650.0 + 250.0 + 150.0)a$ $a = 0.7809t \ldotp$Since acceleration is a function of time, we can determine the velocity of the tractor by using a = $\frac{dv}{dt}$ with the initial condition that v_0 = 0 at t = 0. We integrate from t = 0 to t = 3: $

$$\begin{aligned} dv &= adt \\ \int_0^3 dv = \int_0^{3.00} adt &= \int_0^{3.00} 0.7809tdt \\ v = 0.3905t^2\big]_0^{3.00} &= 3.51\ m/s. \end{aligned}$$

$

b. Refer to the free-body diagram in Figure 17.2.7(b) $

$$\begin{aligned} \sum F_x &= m_{tractor} a_x \\ 820.0t - T &= m_{tractor}(0.7805)t \\ (820.0)(3.00) - T &= (650.0)(0.7805)(3.00) \\ T &= 938\ N. \end{aligned}$$

$

Significance

Since the force varies with time, we must use calculus to solve this problem. Notice how the total mass of the system was important in solving Figure 17.2.7(a), whereas only the mass of the truck (since it supplied the force) was of use in Figure 17.2.7(b).

Recall that v = $\frac{ds}{dt}$ and a = $\frac{dv}{dt}$. If acceleration is a function of time, we can use the calculus forms developed in 1 Dimension Kinematics, as shown in this example. However, sometimes acceleration is a function of displacement. In this case, we can derive an important result from these calculus relations. Solving for dt in each, we have dt = $\frac{ds}{v}$ and dt = $\frac{dv}{a}$. Now, equating these expressions, we have $\frac{ds}{v}$ = $\frac{dv}{a}$. We can rearrange this to obtain a ds = v dv.

✔ Example 17.2.6: Motion of a Projectile Fired Vertically

A 10.0-kg mortar shell is fired vertically upward from the ground, with an initial velocity of 50.0 m/s (see Figure 17.2.8). Determine the maximum height it will travel if atmospheric resistance is measured as $F_D = (0.0100\ v^2)$ N, where v is the speed at any instant.

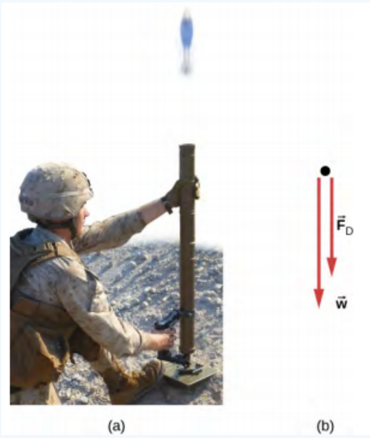


Figure 17.2.8: (a) The mortar fires a shell straight up; we consider the friction force provided by the air. (b) A free-body diagram is shown which indicates all the forces on the mortar shell.

Strategy

The known force on the mortar shell can be related to its acceleration using the equations of motion. Kinematics can then be used to relate the mortar shell's acceleration to its position.

Solution

Initially, $y_0 = 0$ and $v_0 = 50.0$ m/s. At the maximum height y = h, v = 0. The free-body diagram shows F_D to act downward, because it slows the upward motion of the mortar shell. Thus, we can write

$$\begin{aligned}\sum F_y &= ma_y \\ -F_D - w &= ma_y \\ -0.0100v^2 - 98.0 &= 10.0a \\ a &= -0.00100v^2 - 9.80.\end{aligned}$$

The acceleration depends on v and is therefore variable. Since a = f(v), we can relate a to v using the rearrangement described above,

$$ads = vdv. \tag{17.2.3}$$

We replace ds with dy because we are dealing with the vertical direction,

$$\begin{aligned}ady &= vdv \\ (-0.00100v^2 - 9.80)dy &= vdv.\end{aligned}$$

We now separate the variables (v's and dv's on one side; dy on the other):

$$\begin{aligned}\int_0^h dy &= \int_{50.0}^0 \frac{vdv}{(-0.00100v^2 - 9.80)} \\ &= -\int_{50.0}^0 \frac{vdv}{(-0.00100v^2 + 9.80)} \\ &= (-5 \times 10^3)\ln(0.00100v^2 + 9.80)\Big|_{50.0}^0.\end{aligned}$$

Thus, h = 114 m.

Significance

Notice the need to apply calculus since the force is not constant, which also means that acceleration is not constant. To make matters worse, the force depends on v (not t), and so we must use the trick explained prior to the example. The answer for the height indicates a lower elevation if there were air resistance.

Exercise 17.2.7

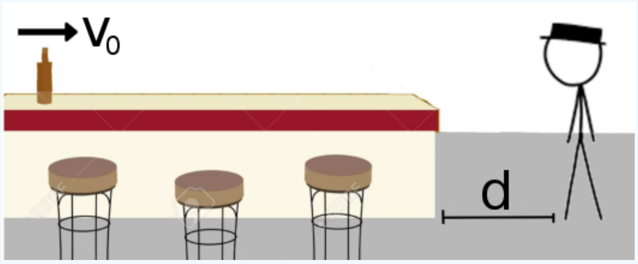
If atmospheric resistance is neglected, find the maximum height for the mortar shell. Is calculus required for this solution?

Simulation

Explore the forces at work in this simulation when you try to push a filing cabinet. Create an applied force and see the resulting frictional force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. View a free-body diagram of all the forces (including gravitational and normal forces).

17.3: Examples

? Whiteboard Problem 17.3.1: A Projectile Pint



A bartender slides a bottle down his bar to a customer. Unfortunately, he pushes it too hard and the bottle flies off the end of the bar!

1. The force of friction between the bar and the bottle is 0.150 N, and the bottle has a mass of 0.520 kg. What is the acceleration of the bottle as it travels down the bar?
2. If it started with a speed of 2.97 m/s, what is the speed of the bottle as it leaves the bar, 2.80 m away from where the bartender pushed it?
3. Black Hat from the webcomic XKCD is standing nearby, a distance 1.25 m from the end of the bar (which is 1.15 m tall). Does the bottle hit him after it slides off the end of the bar?

? Whiteboard Problem 17.3.2: Two Blocks in a Row

Consider two blocks which are in contact with each other on a horizontal frictionless surface. The masses of the blocks are 5 kg and 10 kg. The 5 kg block is being pushed with an unknown force F_p, which then pushes on the 10 kg block, making them both accelerate to the right at 2.00 m/s^2.

1. How big is the force pushing on the 10 kg block from the 5 kg block?
2. What is the magnitude of the pushing force F_p?

✓ Whiteboard Problem 17.3.3: Three Blocks in a Row

Consider three blocks which are in contact with each other on a horizontal frictionless surface. From left to right, the masses of the blocks are 5 kg, 10 kg, and 25 kg. The leftmost block is being pushed to the right with an unknown force F_p, and the blocks are accelerating to the right at 2.00 m/s^2.

1. What is the magnitude and direction of the forces the blocks are exerting on each other?
2. What is the magnitude of the pushing force F_p?

? Whiteboard Problem 17.3.4: Crates on a Lift

A 40.0 kg crate is sitting on top of a 60.0 kg crate on the floor of an elevator. The elevator floor is exerting an upwards force of 1050 N to the 60.0 kg crate.

1. Determine the magnitude and direction of the acceleration of the crates in the elevator.
2. Determine the magnitude and direction of the contact force that the 40.0 kg crate exerts on the 60.0 kg crate.
3. Determine the magnitude and direction of the contact force that the 60.0 kg crate exerts on the 40.0 kg crate.

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CHAPTER OVERVIEW

18: N5) Friction

18.1: Friction (Part 1)

Learning Objectives

- Describe the general characteristics of friction
- List the various types of friction
- Calculate the magnitude of static and kinetic friction, and use these in problems involving Newton's laws of motion

When a body is in motion, it has resistance because the body interacts with its surroundings. This resistance is a force of friction. Friction opposes relative motion between systems in contact but also allows us to move, a concept that becomes obvious if you try to walk on ice. Friction is a common yet complex force, and its behavior still not completely understood. Still, it is possible to understand the circumstances in which it behaves.

Static and Kinetic Friction

The basic definition of friction is relatively simple to state.

Friction

Friction is a force that opposes relative motion between systems in contact.

There are several forms of friction. One of the simpler characteristics of sliding friction is that it is parallel to the contact surfaces between systems and is always in a direction that opposes motion or attempted motion of the systems relative to each other. If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction. For example, friction slows a hockey puck sliding on ice. When objects are stationary, static friction can act between them; the static friction is usually greater than the kinetic friction between two objects.

Static and Kinetic Friction

If two systems are in contact and stationary relative to one another, then the friction between them is called **static friction**. If two systems are in contact and moving relative to one another, then the friction between them is called **kinetic friction**.

Imagine, for example, trying to slide a heavy crate across a concrete floor—you might push very hard on the crate and not move it at all. This means that the static friction responds to what you do—it increases to be equal to and in the opposite direction of your push. If you finally push hard enough, the crate seems to slip suddenly and starts to move. Now static friction gives way to kinetic friction. Once in motion, it is easier to keep it in motion than it was to get it started, indicating that the kinetic frictional force is less than the static frictional force. If you add mass to the crate, say by placing a box on top of it, you need to push even harder to get it started and also to keep it moving. Furthermore, if you oiled the concrete you would find it easier to get the crate started and keep it going (as you might expect).

Figure $18.1.1$ is a crude pictorial representation of how friction occurs at the interface between two objects. Close-up inspection of these surfaces shows them to be rough. Thus, when you push to get an object moving (in this case, a crate), you must raise the object until it can skip along with just the tips of the surface hitting, breaking off the points, or both. A considerable force can be resisted by friction with no apparent motion. The harder the surfaces are pushed together (such as if another box is placed on the crate), the more force is needed to move them. Part of the friction is due to adhesive forces between the surface molecules of the two objects, which explains the dependence of friction on the nature of the substances. For example, rubber-soled shoes slip less than those with leather soles. Adhesion varies with substances in contact and is a complicated aspect of surface physics. Once an object is moving, there are fewer points of contact (fewer molecules adhering), so less force is required to keep the object moving. At small but nonzero speeds, friction is nearly independent of speed.

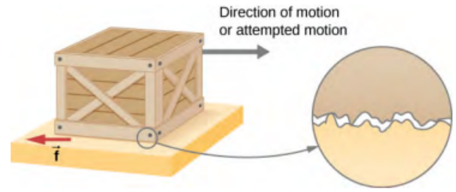


Figure 18.1.1: Frictional forces, such as $\vec{f}$, always oppose motion or attempted motion between objects in contact. Friction arises in part because of the roughness of the surfaces in contact, as seen in the expanded view. For the object to move, it must rise to where the peaks of the top surface can skip along the bottom surface. Thus, a force is required just to set the object in motion. Some of the peaks will be broken off, also requiring a force to maintain motion. Much of the friction is actually due to attractive forces between molecules making up the two objects, so that even perfectly smooth surfaces are not friction-free. (In fact, perfectly smooth, clean surfaces of similar materials would adhere, forming a bond called a "cold weld.")

The magnitude of the frictional force has two forms: one for static situations (static friction), the other for situations involving motion (kinetic friction). What follows is an approximate empirical (experimentally determined) model only. These equations for static and kinetic friction are not vector equations.

Magnitude of Static Friction

The magnitude of static friction f_s is

$$f_s \leq \mu_s N, \tag{18.1.1}$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force.

The symbol ≤ means **less than or equal to**, implying that static friction can have a maximum value of μ_sN. Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds f_s (max), the object moves. Thus,

$$f_s(max) = \mu_s N. \tag{18.1.2}$$

Magnitude of Kinetic Friction

The magnitude of kinetic friction f_k is given by

$$f_k = \mu_k N, \tag{18.1.3}$$

where μ_k is the coefficient of kinetic friction.

Unlike the static friction, kinetic friction only ever takes on a single value, the coefficient of kinetic friction times the normal force. It is worth noting that since these laws are experimentally determined, they are only approximate instead of universally true laws of the Universe (like Newton's laws or Universal Gravity). For more information on this, the Wikipedia article on friction gives some good information.The transition from static friction to kinetic friction is illustrated in Figure 18.1.2

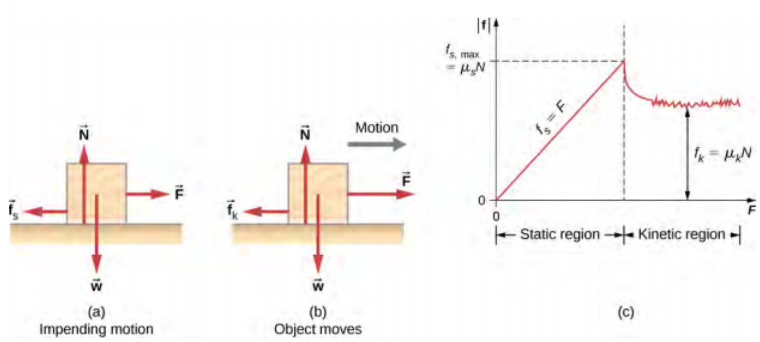


Figure 18.1.2: (a) The force of friction $\vec{f}$ between the block and the rough surface opposes the direction of the applied force $\vec{F}$. The magnitude of the static friction balances that of the applied force. This is shown in the left side of the graph in (c). (b) At some point, the magnitude of the applied force is greater than the force of kinetic friction, and the block moves to the right. This is shown in the right side of the graph. (c) The graph of the frictional force versus the applied force; note that fs (max) > f_k. This means that $\mu_s > \mu_k$

As you can see in Table 6.1, the coefficients of kinetic friction are less than their static counterparts. The approximate values of μ are stated to only one or two digits to indicate the approximate description of friction given by the preceding two equations.

Table 6.1 - Approximate Coefficients of Static and Kinetic Friction

System	Static Friction μ_s	Kinetic Friction μ_k
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.5-0.7	0.3-0.5
Wood on wood	0.5	0.3
Waxed wood on wet snow	0.14	0.1
Metal on wood	0.5	0.3
Steel on steel (dry)	0.6	0.3
Steel on steel (oiled)	0.05	0.03
Teflon on steel	0.04	0.04
Bone lubricated by synovial fluid	0.016	0.015
Shoes on wood	0.9	0.7
Shoes on ice	0.1	0.05
Ice on ice	0.1	0.03
Steel on ice	0.4	0.02

Equation 18.1.1 and Equation 18.1.3 include the dependence of friction on materials and the normal force. The direction of friction is always opposite that of motion, parallel to the surface between objects, and perpendicular to the normal force. For example, if the crate you try to push (with a force parallel to the floor) has a mass of 100 kg, then the normal force is equal to its weight,

$$w = F_g = mg = (100\ kg)(9.80\ m/s^2) = 980\ N, \tag{18.1.4}$$

perpendicular to the floor. If the coefficient of static friction is 0.45, you would have to exert a force parallel to the floor greater than

$$f_s(max) = \mu_s N = (0.45)(980\ N) = 440\ N \tag{18.1.5}$$

to move the crate. Once there is motion, friction is less and the coefficient of kinetic friction might be 0.30, so that a force of only

$$f_k = \mu_k N = (0.30)(980\ N) = 290\ N \tag{18.1.6}$$

keeps it moving at a constant speed. If the floor is lubricated, both coefficients are considerably less than they would be without lubrication. Coefficient of friction is a unitless quantity with a magnitude usually between 0 and 1.0. The actual value depends on the two surfaces that are in contact.

Many people have experienced the slipperiness of walking on ice. However, many parts of the body, especially the joints, have much smaller coefficients of friction—often three or four times less than ice. A joint is formed by the ends of two bones, which are connected by thick tissues. The knee joint is formed by the lower leg bone (the tibia) and the thighbone (the femur). The hip is a ball (at the end of the femur) and socket (part of the pelvis) joint. The ends of the bones in the joint are covered by cartilage, which provides a smooth, almost-glassy surface. The joints also produce a fluid (synovial fluid) that reduces friction and wear. A damaged or arthritic joint can be replaced by an artificial joint (Figure 18.1.3). These replacements can be made of metals (stainless steel or titanium) or plastic (polyethylene), also with very small coefficients of friction.

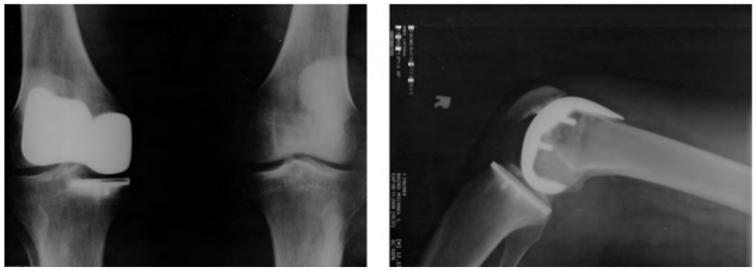


Figure 18.1.3: Artificial knee replacement is a procedure that has been performed for more than 20 years. These post-operative X-rays show a right knee joint replacement. (credit: Mike Baird)

Natural lubricants include saliva produced in our mouths to aid in the swallowing process, and the slippery mucus found between organs in the body, allowing them to move freely past each other during heartbeats, during breathing, and when a person moves. Hospitals and doctor's clinics commonly use artificial lubricants, such as gels, to reduce friction.

The equations given for static and kinetic friction are empirical laws that describe the behavior of the forces of friction. While these formulas are very useful for practical purposes, they do not have the status of mathematical statements that represent general principles (e.g., Newton's second law). In fact, there are cases for which these equations are not even good approximations. For instance, neither formula is accurate for lubricated surfaces or for two surfaces siding across each other at high speeds. Unless specified, we will not be concerned with these exceptions.

Example 6.10: Static and Kinetic Friction

A 20.0-kg crate is at rest on a floor as shown in Figure $18.1.4$ The coefficient of static friction between the crate and floor is 0.700 and the coefficient of kinetic friction is 0.600. A horizontal force $\vec{P}$ is applied to the crate. Find the force of friction if (a) $\vec{P}$ = 20.0 N, (b) $\vec{P}$ = 30.0 N, (c) $\vec{P}$ = 120.0 N, and (d) $\vec{P}$ = 180.0 N.

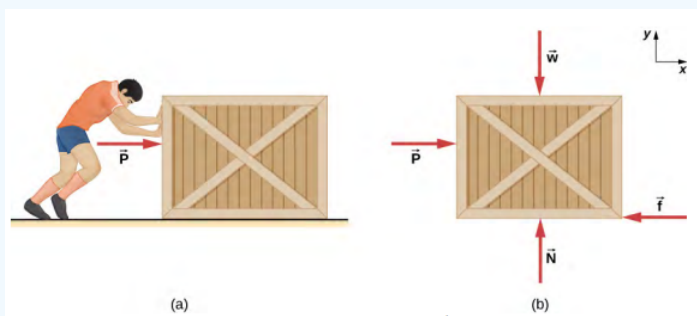


Figure 18.1.4: (a) A crate on a horizontal surface is pushed with a force $\vec{P}$. (b) The forces on the crate. Here, $\vec{f}$ may represent either the static or the kinetic frictional force.

Strategy

The free-body diagram of the crate is shown in Figure $18.1.4b$ We apply Newton's second law in the horizontal and vertical directions, including the friction force in opposition to the direction of motion of the box.

Solution

Newton's second law gives

$$\sum F_x = ma_x \tag{18.1.7}$$

$$P - f = ma_x \tag{18.1.8}$$

$$\sum F_y = ma_y \tag{18.1.9}$$

$$N - w = 0. \tag{18.1.10}$$

Here we are using the symbol f to represent the frictional force since we have not yet determined whether the crate is subject to station friction or kinetic friction. We do this whenever we are unsure what type of friction is acting. Now the weight of the crate is

$$w = (20.0\ kg)(9.80\ m/s^2) = 196\ N, \tag{18.1.11}$$

which is also equal to N. The maximum force of static friction is therefore (0.700)(196 N) = 137 N. As long as $\vec{P}$ is less than 137 N, the force of static friction keeps the crate stationary and $f_s = \vec{P}$. Thus, (a) f_s = 20.0 N, (b) f_s = 30.0 N, and (c) f_s = 120.0 N. (d) If $\vec{P}$ = 180.0 N, the applied force is greater than the maximum force of static friction (137 N), so the crate can no longer remain at rest. Once the crate is in motion, kinetic friction acts. Then

$$f_k = \mu_k N = (0.600)(196\ N) = 118\ N, \tag{18.1.12}$$

and the acceleration is

$$a_x = \frac{\vec{P} - f_k}{m} = \frac{180.0\ N - 118\ N}{20.0\ kg} = 3.10\ m/s^2. \tag{18.1.13}$$

Significance

This example illustrates how we consider friction in a dynamics problem. Notice that static friction has a value that matches the applied force, until we reach the maximum value of static friction. Also, no motion can occur until the applied force equals the force of static friction, but the force of kinetic friction will then become smaller.

Exercise 6.7

A block of mass 1.0 kg rests on a horizontal surface. The frictional coefficients for the block and surface are μ_s = 0.50 and μ_k = 0.40. (a) What is the minimum horizontal force required to move the block? (b) What is the block’s acceleration when this force is applied?

18.2: Friction (Part 2)

Friction and the Inclined Plane

One situation where friction plays an obvious role is that of an object on a slope. It might be a crate being pushed up a ramp to a loading dock or a skateboarder coasting down a mountain, but the basic physics is the same. We usually generalize the sloping surface and call it an inclined plane but then pretend that the surface is flat. Let's look at an example of analyzing motion on an inclined plane with friction.

Example 18.2.1: Downhill Skier

A skier with a mass of 62 kg is sliding down a snowy slope at a constant velocity. Find the coefficient of kinetic friction for the skier if friction is known to be 45.0 N.

Strategy

The magnitude of kinetic friction is given as 45.0 N. Kinetic friction is related to the normal force N by $f_k = \mu_k N$; thus, we can find the coefficient of kinetic friction if we can find the normal force on the skier. The normal force is always perpendicular to the surface, and since there is no motion perpendicular to the surface, the normal force should equal the component of the skier's weight perpendicular to the slope. (See Figure 18.2.1, which repeats a figure from the chapter on Newton's laws of motion.)

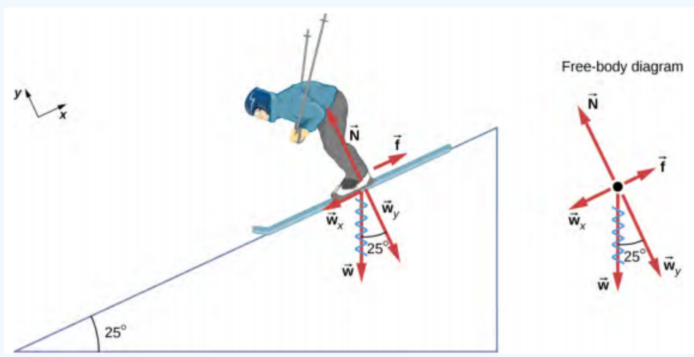


Figure 18.2.1: The motion of the skier and friction are parallel to the slope, so it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier). The normal force $\vec{N}$ is perpendicular to the slope, and friction $\vec{f}$ is parallel to the slope, but the skier's weight $\vec{w}$ has components along both axes, namely $\vec{w}_y$ and $\vec{w}_x$. The normal force $\vec{N}$ is equal in magnitude to $\vec{w}_y$, so there is no motion perpendicular to the slope. However, $\vec{f}$ is less than $\vec{w}_x$ in magnitude, so there is acceleration down the slope (along the x-axis).

We have

$$N = w_y = w\cos 25^o = mg\cos 25^o. \tag{18.2.1}$$

Substituting this into our expression for kinetic friction, we obtain

$$f_k = \mu_k mg\cos 25^o, \tag{18.2.2}$$

which can now be solved for the coefficient of kinetic friction μ_k.

Solution

Solving for μ_k gives

$$\mu_k = \frac{f_k}{N} = \frac{f_k}{w\cos 25^o} = \frac{f_k}{mg\cos 25^o}. \tag{18.2.3}$$

Substituting known values on the right-hand side of the equation,

$$\mu_k = \frac{45.0\ N}{(62\ kg)(9.80\ m/s^2)(0.906)} = 0.082. \tag{18.2.4}$$

Significance

This result is a little smaller than the coefficient listed in Table 6.1 for waxed wood on snow, but it is still reasonable since values of the coefficients of friction can vary greatly. In situations like this, where an object of mass m slides down a slope that makes an angle θ with the horizontal, friction is given by $f_k = \mu_k$ mg cos θ. All objects slide down a slope with constant acceleration under these circumstances.

We have discussed that when an object rests on a horizontal surface, the normal force supporting it is equal in magnitude to its weight. Furthermore, simple friction is always proportional to the normal force. When an object is not on a horizontal surface, as with the inclined plane, we must find the force acting on the object that is directed perpendicular to the surface; it is a component of the weight.

We now derive a useful relationship for calculating coefficient of friction on an inclined plane. Notice that the result applies only for situations in which the object slides at constant speed down the ramp.

An object slides down an inclined plane at a constant velocity if the net force on the object is zero. We can use this fact to measure the coefficient of kinetic friction between two objects. As shown in Example 18.2.1, the kinetic friction on a slope is $f_k = \mu_k$ mg cos θ. The component of the weight down the slope is equal to mg sin θ (see the free-body diagram in Figure 18.2.1). These forces act in opposite directions, so when they have equal magnitude, the acceleration is zero. Writing these out,

$$\mu_k mg\cos\theta = mg\sin\theta. \tag{18.2.5}$$

Solving for μ_k, we find that

$$\mu_k = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta. \tag{18.2.6}$$

Put a coin on a book and tilt it until the coin slides at a constant velocity down the book. You might need to tap the book lightly to get the coin to move. Measure the angle of tilt relative to the horizontal and find μ_k. Note that the coin does not start to slide at all until an angle greater than θ is attained, since the coefficient of static friction is larger than the coefficient of kinetic friction. Think about how this may affect the value for μ_k and its uncertainty.

Atomic-Scale Explanations of Friction

The simpler aspects of friction dealt with so far are its macroscopic (large-scale) characteristics. Great strides have been made in the atomic-scale explanation of friction during the past several decades. Researchers are finding that the atomic nature of friction seems to have several fundamental characteristics. These characteristics not only explain some of the simpler aspects of friction—they also hold the potential for the development of nearly friction-free environments that could save hundreds of billions of dollars in energy which is currently being converted (unnecessarily) into heat.

Figure 18.2.2 illustrates one macroscopic characteristic of friction that is explained by microscopic (small-scale) research. We have noted that friction is proportional to the normal force, but not to the amount of area in contact, a somewhat counterintuitive notion. When two rough surfaces are in contact, the actual contact area is a tiny fraction of the total area because only high spots touch. When a greater normal force is exerted, the actual contact area increases, and we find that the friction is proportional to this area.

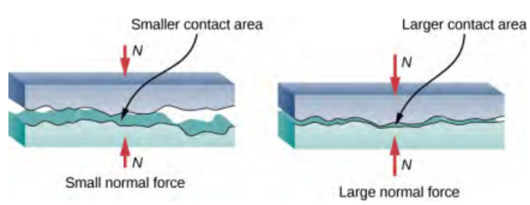


Figure 18.2.2: Two rough surfaces in contact have a much smaller area of actual contact than their total area. When the normal force is larger as a result of a larger applied force, the area of actual contact increases, as does friction.

However, the atomic-scale view promises to explain far more than the simpler features of friction. The mechanism for how heat is generated is now being determined. In other words, why do surfaces get warmer when rubbed? Essentially, atoms are linked with one another to form lattices. When surfaces rub, the surface atoms adhere and cause atomic lattices to vibrate—essentially creating sound waves that penetrate the material. The sound waves diminish with distance, and their energy is converted into heat. Chemical reactions that are related to frictional wear can also occur between atoms and molecules on the surfaces. Figure 18.2.3 shows how the tip of a probe drawn across another material is deformed by atomic-scale friction. The force needed to drag the tip can be measured and is found to be related to shear stress, which is nicely described in a Wikipedia article about that.

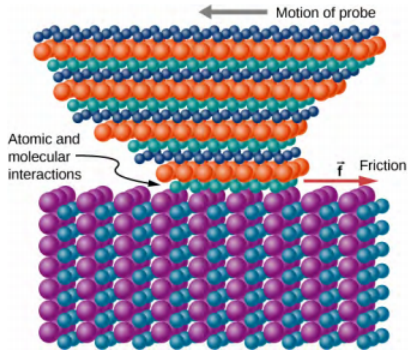


Figure 18.2.3: The tip of a probe is deformed sideways by frictional force as the probe is dragged across a surface. Measurements of how the force varies for different materials are yielding fundamental insights into the atomic nature of friction.

Simulation

Describe a model for friction on a molecular level. Describe matter in terms of molecular motion. The description should include diagrams to support the description; how the temperature affects the image; what are the differences and similarities between solid, liquid, and gas particle motion; and how the size and speed of gas molecules relate to everyday objects.

Example 18.2.2: Sliding Blocks

The two blocks of Figure 18.2.4 are attached to each other by a massless string that is wrapped around a frictionless pulley. When the bottom 4.00-kg block is pulled to the left by the constant force $\vec{P}$, the top 2.00-kg block slides across it to the right. Find the magnitude of the force necessary to move the blocks at constant speed. Assume that the coefficient of kinetic friction between all surfaces is 0.400.

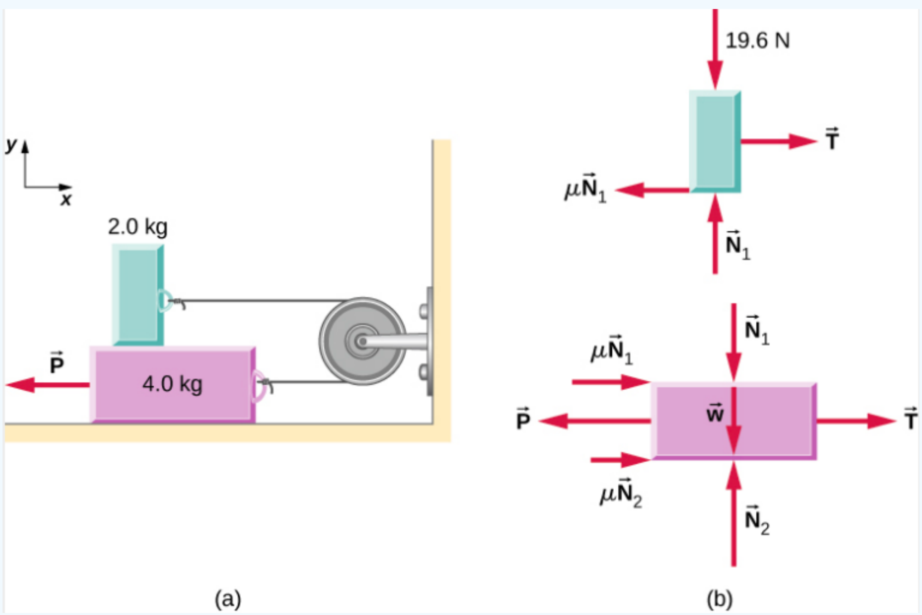


Figure 18.2.4: (a) Each block moves at constant velocity. (b) Free-body diagrams for the blocks.

Strategy

We analyze the motions of the two blocks separately. The top block is subjected to a contact force exerted by the bottom block. The components of this force are the normal force N_1 and the frictional force $-0.400\ N_1$. Other forces on the top block are the tension T in the string and the weight of the top block itself, 19.6 N. The bottom block is subjected to contact forces due to the top block and due to the floor. The first contact force has components $-N_1$ and $0.400\ N_1$, which are simply reaction forces to the contact forces that the bottom block exerts on the top block. The components of the contact force of the floor are N_2 and $0.400\ N_2$. Other forces on this block are $-P$, the tension T, and the weight –39.2 N. Solution Since the top block is moving horizontally to the right at constant velocity, its acceleration is zero in both the horizontal and the vertical directions. From Newton's second law,

$$\sum F_x = m_2 a_x \tag{18.2.7}$$

$$T - 0.400\ N_1 = 0 \tag{18.2.8}$$

$$\sum F_y = m_1 a_y \tag{18.2.9}$$

$$N_1 - 19.6\ N = 0. \tag{18.2.10}$$

Solving for the two unknowns, we obtain N_1 = 19.6 N and T = 0.40 N_1 = 7.84 N. The bottom block is also not accelerating, so the application of Newton's second law to this block gives

$$\sum F_x = m_2 a_x \tag{18.2.11}$$

$$T - P + 0.400\ N_1 + 0.400\ N_2 = 0 \tag{18.2.12}$$

$$\sum F_y = m_1 a_y \tag{18.2.13}$$

$$N_2 - 39.2\ N - N_1 = 0. \tag{18.2.14}$$

The values of N_1 and T were found with the first set of equations. When these values are substituted into the second set of equations, we can determine N_2 and P. They are

$$N_2 = 58.8\ N\ and\ P = 39.2\ N. \tag{18.2.15}$$

Significance

Understanding what direction in which to draw the friction force is often troublesome. Notice that each friction force labeled in Figure 18.2.4 acts in the direction opposite the motion of its corresponding block.

Example 18.2.3: A Crate on an Accelerating Truck

A 50.0-kg crate rests on the bed of a truck as shown in Figure 18.2.5. The coefficients of friction between the surfaces are μ_k = 0.300 and μ_s = 0.400. Find the frictional force on the crate when the truck is accelerating forward relative to the ground at (a) 2.00 m/s^2, and (b) 5.00 m/s^2.

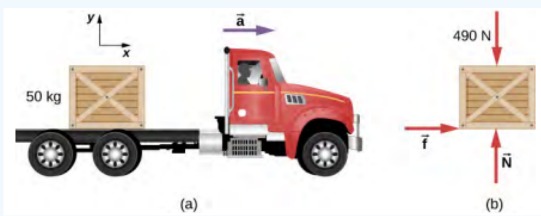


Figure 18.2.5: (a) A crate rests on the bed of the truck that is accelerating forward. (b) The free-body diagram of the crate.

Strategy

The forces on the crate are its weight and the normal and frictional forces due to contact with the truck bed. We start by assuming that the crate is not slipping. In this case, the static frictional force fs acts on the crate. Furthermore, the accelerations of the crate and the truck are equal.

Solution

a. Application of Newton's second law to the crate, using the reference frame attached to the ground, yields

$$\begin{aligned}\sum F_x &= ma_x \\ f_s &= (50.0\ kg)(2.00\ m/s^2) \\ &= 1.00 \times 10^2\ N\end{aligned}$$

$$\begin{aligned}\sum F_y &= ma_y \\ N - 4.90 \times 10^2\ N &= (50.0\ kg)(0) \\ N &= 4.90 \times 10^2\ N.\end{aligned}$$

We can now check the validity of our no-slip assumption. The maximum value of the force of static friction is $\mu_{s} N = (0.400)(4.90 \times 10^{2}\; N) = 196\; N,$whereas the **actual** force of static friction that acts when the truck accelerates forward at 2.00 m/s^2 is only 1.00 x 10^2 N. Thus, the assumption of no slipping is valid.

b. If the crate is to move with the truck when it accelerates at 5.0 m/s^2, the force of static friction must be $f_{s} = ma_{x} = (50.0\; kg)(5.00\; m/s^{2}) = 250\; N \ldotp$Since this exceeds the maximum of 196 N, the crate must slip. The frictional force is therefore kinetic and is $f_{k} = \mu_{k} N = (0.300)(4.90 \times 10^{2}\; N) = 147\; N \ldotp$The horizontal acceleration of the crate relative to the ground is now found from

$$\begin{aligned} \sum F_x &= ma_x \\ 147\ N &= (50.0\ kg)a_x, \\ so\ a_x &= 2.94\ m/s^2. \end{aligned}$$

Significance

Relative to the ground, the truck is accelerating forward at 5.0 m/s^2 and the crate is accelerating forward at 2.94 m/s^2. Hence the crate is sliding backward relative to the bed of the truck with an acceleration 2.94 m/s^2 − 5.00 m/s^2 = −2.06 m/s^2.

Example 18.2.4: Snowboarding

Earlier, we analyzed the situation of a downhill skier moving at constant velocity to determine the coefficient of kinetic friction. Now let's do a similar analysis to determine acceleration. The snowboarder of Figure 18.2.6 glides down a slope that is inclined at θ = 13° to the horizontal. The coefficient of kinetic friction between the board and the snow is μ_k = 0.20. What is the acceleration of the snowboarder?

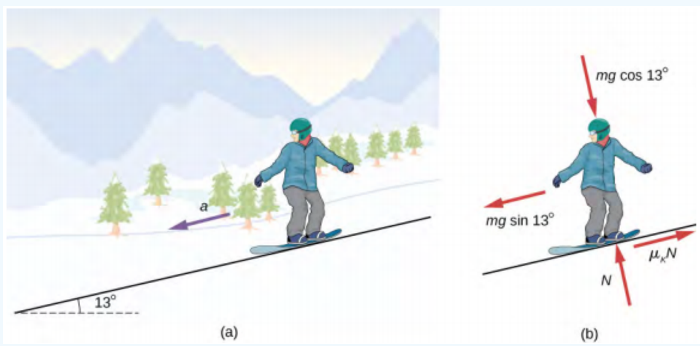


Figure 18.2.6: (a) A snowboarder glides down a slope inclined at 13° to the horizontal. (b) The free-body diagram of the snowboarder.

Strategy

The forces acting on the snowboarder are her weight and the contact force of the slope, which has a component normal to the incline and a component along the incline (force of kinetic friction). Because she moves along the slope, the most convenient reference frame for analyzing her motion is one with the x-axis along and the y-axis perpendicular to the incline. In this frame, both the normal and the frictional forces lie along coordinate axes, the components of the weight are mg sin θ along the slope and mg cos θ at right angles into the slope , and the only acceleration is along the x-axis (a_y = 0).

Solution

We can now apply Newton's second law to the snowboarder:

$$\begin{aligned} \sum F_x &= ma_x \\ mg\sin\theta - \mu_k N &= ma_x \end{aligned}$$	$$\begin{aligned} \sum F_y &= ma_y \\ N - mg\cos\theta &= m(0). \end{aligned}$$

From the second equation, N = mg cos θ. Upon substituting this into the first equation, we find

$$\begin{aligned} a_x &= g(\sin\theta - \mu_k \cos\theta) \\ &= g(\sin 13^o - 0.520 \cos 13^o) = 0.29\ m/s^2. \end{aligned}$$

Significance

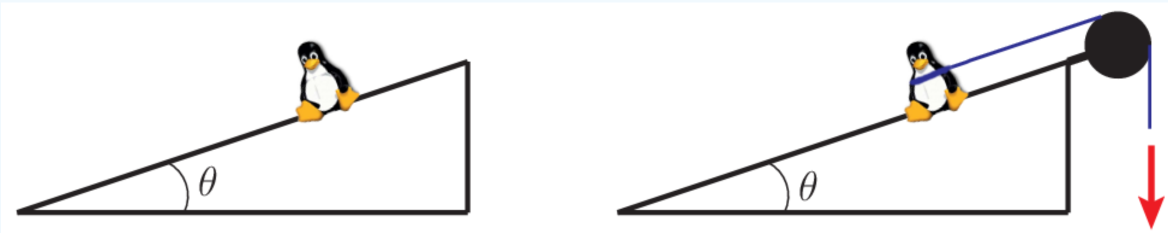
Notice from this equation that if θ is small enough or μ_k is large enough, a_x is negative, that is, the snowboarder slows down.

Exercise 18.2.4

The snowboarder is now moving down a hill with incline 10.0°. What is the skier’s acceleration?

18.3: More Examples

? Whiteboard Problem 18.3.1: Sliding Penguin



Consider a penguin sitting on a ramp, as shown in the figure on the left. The ramp makes an angle of 15° with respect to the floor, the mass of the penguin is 45 kg, and the coefficient of static friction between the penguin and the ramp is 0.20.

1. In the figure on the left, will the penguin slide down the ramp or not?
2. Now I tie a rope to the penguin, as shown in the figure on the right. This rope goes over a frictionless, massless pulley. How hard must I pull on the rope before the penguin just starts to move?

? Whiteboard Problem 18.3.2: Kinematics Review

Two children are going to race their sleds across a frozen pond. They run towards the pond and jump onto the sleds, and race for a point 10 m away from their starting point. The first child has a mass of 40 kg, and his sled has a coefficient of friction of 0.023. The second child is less massive (35 kg), but her sled has a larger coefficient of friction, 0.035.

1. Which child will win the race, assuming the only forces acting on them are gravity and the contact force between the ice and sleds?
2. When whoever wins reaches the finish line, how far behind them is the other?
3. What is each of their speeds at the end of the race?

✓ Example 18.3.6: Speeding up and slowing down

a. A 1400-kg car, starting from rest, accelerates to a speed of 30 mph in 10 s. What is the force on the car (assumed constant) over this period of time?
b. Where does this force comes from? That is, what is the (external) object that exerts this force on the car, and what is the nature of this force?
c. Draw a free-body diagram for the car. Indicate the direction of motion, and the direction of the acceleration.
d. Now assume that the driver, traveling at 30 mph, sees a red light ahead and pushes on the brake pedal. Assume that the coefficient of static friction between the tires and the road is μ_s = 0.7, and that the wheels don't "lock": that is to say, they continue rolling without slipping on the road as they slow down. What is the car's minimum stopping distance?
e. Draw a free-body diagram of the car for the situation in (d). Again indicate the direction of motion, and the direction of the acceleration.
f. Now assume that the driver again wants to stop as in part (c), but he presses on the brakes too hard, so the wheels lock, and, moreover, the road is wet, and the coefficient of kinetic friction is only μ_k = 0.2. What is the distance the car travels now before coming to a stop?

Solution

(a) First, let us convert 30 mph to meters per second. There are 1, 609 meters to a mile, and 3, 600 seconds to an hour, so 30 mph = 10 × 1609/3600 m/s = 13.4 m/s.

Next, for constant acceleration, we can use Equation (2.2.4): $\Delta v = a\Delta t$. Solving for a,

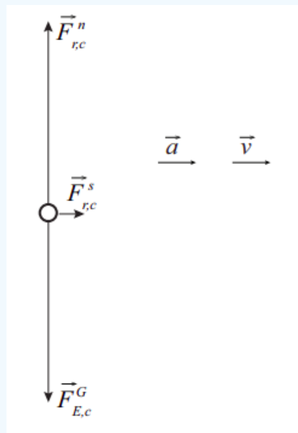
$$a = \frac{\Delta v}{\Delta t} = \frac{13.4 \text{ m/s}}{10 \text{ s}} = 1.34 \ \frac{\text{m}}{\text{s}^2}.$$

Finally, since $F = ma$, we have

$$F = ma = 1400 \text{ kg} \times 1.34 \ \frac{\text{m}}{\text{s}^2} = 1880 \text{ N}.$$

(b) The force must be provided by the road, which is the only thing external to the car that is in contact with it. The force is, in fact, the force of *static* friction between the car and the tires. As explained in the chapter, this is a reaction force (the tires push on the road, and the road pushes back). It is static friction because the tires are not slipping relative to the road. In fact, we will see in Chapter 9 that the point of the tire in contact with the road has an instantaneous velocity of zero (see Figure 9.6.1).

(c) This is the free-body diagram. Note the force of static friction pointing *forward*, in the direction of the acceleration. The forces have been drawn to scale.



(d) This is the opposite of part (a): the driver now relies on the force of static friction to *slow down* the car. The shortest stopping distance will correspond to the largest (in magnitude) acceleration, as per our old friend, Equation (2.2.10):

$$v_f^2 - v_i^2 = 2a\Delta x. \tag{18.3.1}$$

In turn, the largest acceleration will correspond to the largest force. As explained in the chapter, the static friction force cannot exceed $\mu_s F^n$ (Equation (6.3.8)). So, we have

$$F^s_{\text{max}} = \mu_s F^n = \mu_s mg$$

since, in this case, we expect the normal force to be equal to the force of gravity. Then

$$|a_{\text{max}}| = \frac{F^s_{\text{max}}}{m} = \frac{\mu_s mg}{m} = \mu_s g.$$

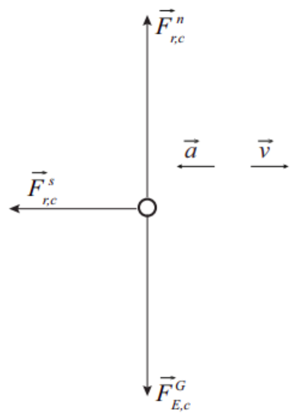
We can substitute this into Equation (18.3.1) with a negative sign, since the acceleration acts in the opposite direction to the motion (and we are implicitly taking the direction of motion to be positive). Also note that the final velocity we want is zero, v_f = 0. We get

$$-v_i^2 = 2a\Delta x = -2\mu_s g\Delta x.$$

From here, we can solve for Δx:

$$\Delta x = \frac{v_i^2}{2\mu_s g} = \frac{(13.4 \text{ m/s})^2}{2 \times 0.7 \times 9.81 \text{ m/s}^2} = 13.1 \text{ m}.$$

(e) Here is the free-body diagram. The interesting feature is that the force of static friction has reversed direction relative to parts (a)–(c). It is also much larger than before. (The forces are again to scale.)



(f) The math for this part is basically identical to that in part (d). The difference, physically, is that now you are dealing with the force of *kinetic* (or "sliding") friction, and that is always given by $F^k = \mu_k F^n$ (this is not an upper limit, it's just what F^k is). So we have $a = -F^k/m = -mu_k g$, and, just as before (but with μ_k replacing μ_s),

$$\Delta x = \frac{v_i^2}{2\mu_k g} = \frac{(13.4\ \mathrm{m/s})^2}{2 \times 0.2 \times 9.81\ \mathrm{m/s}^2} = 45.8\ \mathrm{m}.$$

This is a huge distance, close to half a football field! If these numbers are accurate, you can see that locking your brakes in the rain can have some pretty bad consequences.

CHAPTER OVERVIEW

19: N6) Statics and Springs

19.1: Conditions for Static Equilibrium

Learning Objectives

- Identify the physical conditions of static equilibrium.
- Draw a free-body diagram for a rigid body acted on by forces.
- Explain how the conditions for equilibrium allow us to solve statics problems.

We say that a rigid body is in **equilibrium** when both its linear and angular acceleration are zero relative to an inertial frame of reference. This means that a body in equilibrium can be moving, but if so, its linear and angular velocities must be constant. We say that a rigid body is in **static equilibrium** when it is at rest **in our selected frame of reference**. Notice that the distinction between the state of rest and a state of uniform motion is artificial—that is, an object may be at rest in our selected frame of reference, yet to an observer moving at constant velocity relative to our frame, the same object appears to be in uniform motion with constant velocity. Because the motion is **relative**, what is in static equilibrium to us is in dynamic equilibrium to the moving observer, and vice versa. Since the laws of physics are identical for all inertial reference frames, in an inertial frame of reference, there is no distinction between static equilibrium and equilibrium.

According to Newton’s second law of motion, the linear acceleration of a rigid body is caused by a net force acting on it, or

$$\sum_k \vec{F}_k = m\vec{a}_{CM}. \tag{19.1.1}$$

Here, the sum is of all external forces acting on the body, where m is its mass and $\vec{a}_{CM}$ is the linear acceleration of its center of mass (a concept we discussed in Linear Momentum and Collisions on linear momentum and collisions). In equilibrium, the linear acceleration is zero. If we set the acceleration to zero in Equation 19.1.1, we obtain the following equation:

First Equilibrium Condition

The first equilibrium condition for the static equilibrium of a rigid body expresses **translational** equilibrium:

$$\sum_k \vec{F}_k = \vec{0}. \tag{19.1.2}$$

The first equilibrium condition, Equation 19.1.2, is the equilibrium condition for forces, which we encountered when studying applications of Newton’s laws.

This vector equation is equivalent to the following three scalar equations for the components of the net force:

$$\sum_k F_{kx} = 0,\ \sum_k F_{ky} = 0,\ \sum_k F_{kz} = 0. \tag{19.1.3}$$

Although we have not talked about this yet, there is an analogous formula to Equation 19.1.1 for rotational motion, which looks like:

$$\sum_k \vec{\tau}_k = I\vec{\alpha}. \tag{19.1.4}$$

That quantity $\vec{\alpha}$ is the angular acceleration, so this equation says that the sum of all the torques on a body is equal to the moment of inertia times the angular acceleration. We will pick up on that in the next chapter, right now we just need to know it's like the acceleration, but for circular motion. In equilibrium, this rotational acceleration is zero. By setting to zero the right-hand side of Equation 19.1.4, we obtain the second equilibrium condition:

Second Equilibrium Condition

The second equilibrium condition for the static equilibrium of a rigid body expresses **rotational** equilibrium:

$$\sum_k \vec{\tau}_k = \vec{0}. \tag{19.1.5}$$

The second equilibrium condition, Equation 19.1.5, is the equilibrium condition for torques that we encountered when we studied rotational dynamics. It is worth noting that this equation for equilibrium is generally valid for rotational equilibrium about any axis of rotation (fixed or otherwise). Again, this vector equation is equivalent to three scalar equations for the vector components of the net torque:

$$\sum_k \tau_{kx} = 0,\ \sum_k \tau_{ky} = 0,\ \sum_k \tau_{kz} = 0. \tag{19.1.6}$$

The second equilibrium condition means that in equilibrium, there is no net external torque to cause rotation about any axis. The first and second equilibrium conditions are stated in a particular reference frame. The first condition involves only forces and is therefore independent of the origin of the reference frame. However, the second condition involves torque, which is defined as a cross product, $\vec{\tau}_k = \vec{r}_k \times \vec{F}_k$, where the position vector $\vec{r}_k$ with respect to the axis of rotation of the point where the force is applied enters the equation. Therefore, torque depends on the location of the axis in the reference frame. However, when rotational and translational equilibrium conditions hold simultaneously in one frame of reference, then they also hold in any other inertial frame of reference, so that the net torque about any axis of rotation is still zero. The explanation for this is fairly straightforward.

Suppose vector $\vec{R}$ is the position of the origin of a new inertial frame of reference S′ in the old inertial frame of reference S. From our study of relative motion, we know that in the new frame of reference S′, the position vector $\vec{r}\,'_k$ of the point where the force $\vec{F}_k$ is applied is related to $\vec{r}_k$ via the equation

$$\vec{r}\,'_k = \vec{r}_k - \vec{R}. \tag{19.1.7}$$

Now, we can sum all torques $\vec{\tau}\,'_k = \vec{r}\,'_k \times \vec{F}_k$ of all external forces in a new reference frame, S':

$$\sum_k \vec{\tau}\,'_k = \sum_k \vec{r}\,'_k \times \vec{F}_k = \sum_k (\vec{r}_k - \vec{R}) \times \vec{F}_k = \sum_k \vec{r}_k \times \vec{F}_k - \sum_k \vec{R} \times \vec{F}_k = \sum \vec{\tau}_k - \vec{R} \times \sum_k \vec{F}_k = \vec{0}. \tag{19.1.8}$$

In the final step in this chain of reasoning, we used the fact that in equilibrium in the old frame of reference, S, the first term vanishes because of Equation 19.1.5 and the second term vanishes because of Equation 19.1.2. Hence, we see that the net torque in any inertial frame of reference S′ is zero, provided that both conditions for equilibrium hold in an inertial frame of reference S.

The practical implication of this is that when applying equilibrium conditions for a rigid body, we are free to choose any point as the origin of the reference frame. Our choice of reference frame is dictated by the physical specifics of the problem we are solving. In one frame of reference, the mathematical form of the equilibrium conditions may be quite complicated, whereas in another frame, the same conditions may have a simpler mathematical form that is easy to solve. The origin of a selected frame of reference is called the pivot point.

In the most general case, equilibrium conditions are expressed by the six scalar equations (Equations 19.1.3 and 19.1.6). For planar equilibrium problems with rotation about a fixed axis, which we consider in this chapter, we can reduce the number of equations to three. The standard procedure is to adopt a frame of reference where the z-axis is the axis of rotation. With this choice of axis, the net torque has only a z-component, all forces that have non-zero torques lie in the xy-plane, and therefore contributions to the net torque come from only the x- and y-components of external forces. Thus, for planar problems with the axis of rotation perpendicular to the xy-plane, we have the following three equilibrium conditions for forces and torques:

$$F_{1x} + F_{2x} + \cdots + F_{Nx} = 0 \tag{19.1.9}$$

$$F_{1y} + F_{2y} + \cdots + F_{Ny} = 0 \tag{19.1.10}$$

$$\tau_1 + \tau_2 + \cdots + \tau_N = 0 \tag{19.1.11}$$

where the summation is over all N external forces acting on the body and over their torques. In Equation 19.1.11, we simplified the notation by dropping the subscript z, but we understand here that the summation is over all contributions along the z-axis, which is the axis of rotation. In Equation 19.1.11, the z-component of torque $\vec{\tau}_k$ from the force $\vec{F}_k$ is

$$\tau_k = r_k F_k \sin\theta \tag{19.1.12}$$

where r_k is the length of the lever arm of the force and F_k is the magnitude of the force (as you saw when we studied Angular Momentum). The angle θ is the angle between vectors $\vec{r}_k$ and $\vec{F}_k$, measuring **from vector** $\vec{r}_k$ **to vector** $\vec{F}_k$ in the **counterclockwise** direction (Figure 19.1.1). When using Equation 19.1.12, we often compute the magnitude of torque and assign its sense as either positive (+) or negative (−), depending on the direction of rotation caused by this torque alone. In Equation

19.1.11, net torque is the sum of terms, with each term computed from Equation 19.1.12, and each term must have the correct **sense**. Similarly, in Equation 19.1.9, we assign the + sign to force components in the + x-direction and the − sign to components in the − x-direction. The same rule must be consistently followed in Equation 19.1.10, when computing force components along the y-axis.

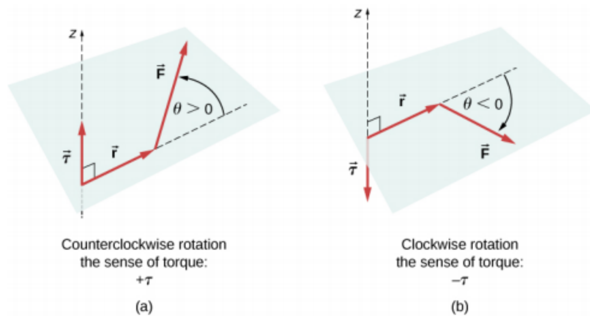


Figure 19.1.1: Torque of a force: (a) When the torque of a force causes counterclockwise rotation about the axis of rotation, we say that its sense is positive, which means the torque vector is parallel to the axis of rotation. (b) When torque of a force causes clockwise rotation about the axis, we say that its sense is negative, which means the torque vector is antiparallel to the axis of rotation.

Note

View this demonstration to see two forces act on a rigid square in two dimensions. At all times, the static equilibrium conditions given by Equation 19.1.9 through Equation 19.1.11 are satisfied. You can vary magnitudes of the forces and their lever arms and observe the effect these changes have on the square.

In many equilibrium situations, one of the forces acting on the body is its weight. In free-body diagrams, the weight vector is attached to the **center of gravity** of the body. For all practical purposes, the center of gravity is identical to the Center of Mass. Recall that the CM has a special physical meaning: When an external force is applied to a body at exactly its CM, the body as a whole undergoes translational motion and such a force does not cause rotation.

When the CM is located off the axis of rotation, a net **gravitational torque** occurs on an object. Gravitational torque is the torque caused by weight. This gravitational torque may rotate the object if there is no support present to balance it. The magnitude of the gravitational torque depends on how far away from the pivot the CM is located. For example, in the case of a tipping truck (Figure 19.1.2), the pivot is located on the line where the tires make contact with the road's surface. If the CM is located high above the road's surface, the gravitational torque may be large enough to turn the truck over. Passenger cars with a low-lying CM, close to the pavement, are more resistant to tipping over than are trucks.

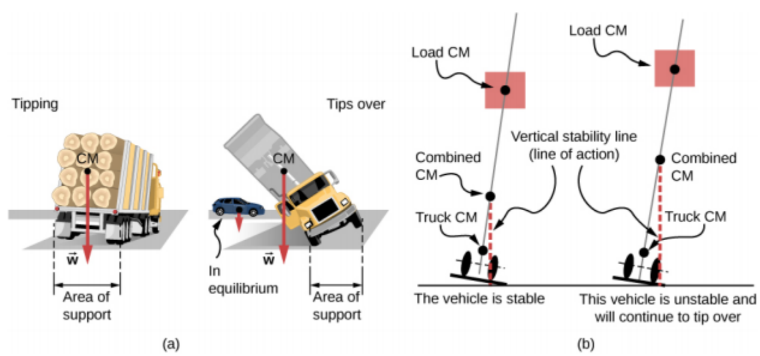


Figure 19.1.2: The distribution of mass affects the position of the center of mass (CM), where the weight vector $\vec{w}$ is attached. If the center of gravity is within the area of support, the truck returns to its initial position after tipping [see the left panel in (b)]. But if the center of gravity lies outside the area of support, the truck turns over [see the right panel in (b)]. Both vehicles in (b) are out of equilibrium. Notice that the car in (a) is in equilibrium: The low location of its center of gravity makes it hard to tip over.

> **Note**
>
> If you tilt a box so that one edge remains in contact with the table beneath it, then one edge of the base of support becomes a pivot. As long as the center of gravity of the box remains over the base of support, gravitational torque rotates the box back toward its original position of stable equilibrium. When the center of gravity moves outside of the base of support, gravitational torque rotates the box in the opposite direction, and the box rolls over. View this demonstration to experiment with stable and unstable positions of a box.

Example 12.1: Center of Gravity of a Car

A passenger car with a 2.5-m wheelbase has 52% of its weight on the front wheels on level ground, as illustrated in Figure 12.4. Where is the CM of this car located with respect to the rear axle?

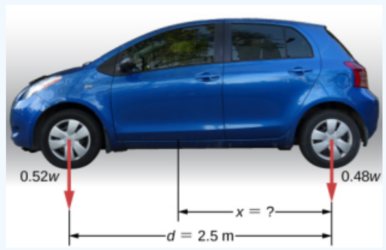


Figure 19.1.3: The weight distribution between the axles of a car. Where is the center of gravity located?

Strategy

We do not know the weight w of the car. All we know is that when the car rests on a level surface, 0.52w pushes down on the surface at contact points of the front wheels and 0.48w pushes down on the surface at contact points of the rear wheels. Also, the contact points are separated from each other by the distance d = 2.5 m. At these contact points, the car experiences normal reaction forces with magnitudes F_F = 0.52w and F_R = 0.48w on the front and rear axles, respectively. We also know that the car is an example of a rigid body in equilibrium whose entire weight w acts at its CM. The CM is located somewhere between the points where the normal reaction forces act, somewhere at a distance x from the point where F_R acts. Our task is to find x. Thus, we identify three forces acting on the body (the car), and we can draw a free-body diagram for the extended rigid body, as shown in Figure 19.1.4

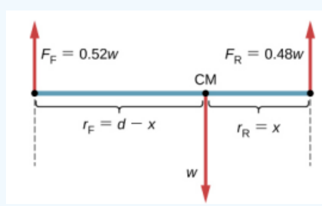


Figure 19.1.4: The free-body diagram for the car clearly indicates force vectors acting on the car and distances to the center of mass (CM). When CM is selected as the pivot point, these distances are lever arms of normal reaction forces. Notice that vector magnitudes and lever arms do not need to be drawn to scale, but all quantities of relevance must be clearly labeled.

We are almost ready to write down equilibrium conditions Equation 19.1.9 through Equation 19.1.11 for the car, but first we must decide on the reference frame. Suppose we choose the x-axis along the length of the car, the yaxis vertical, and the z-axis perpendicular to this xy-plane. With this choice we only need to write Equation 19.1.9 and Equation 19.1.11 because all the y-components are identically zero. Now we need to decide on the location of the pivot point. We can choose any point as the location of the axis of rotation (z-axis). Suppose we place the axis of rotation at CM, as indicated in the free-body diagram for the car. At this point, we are ready to write the equilibrium conditions for the car.

Solution

Each equilibrium condition contains only three terms because there are N = 3 forces acting on the car. The first equilibrium condition, Equation 19.1.9, reads

$$+F_F - w + F_R = 0. \tag{19.1.13}$$

This condition is trivially satisfied because when we substitute the data, Equation 19.1.13 becomes +0.52w − w + 0.48w = 0. The second equilibrium condition, Equation 19.1.11, reads

$$\tau_F + \tau_w + \tau_R = 0 \tag{19.1.14}$$

where τ_F is the torque of force F_F, τ_w is the gravitational torque of force w, and τ_R is the torque of force F_R. When the pivot is located at CM, the gravitational torque is identically zero because the lever arm of the weight with respect to an axis that passes through CM is zero. The lines of action of both normal reaction forces are perpendicular to their lever arms, so in Equation 19.1.12, we have $|\sin\theta| = 1$ for both forces. From the free-body diagram, we read that torque τ_F causes clockwise rotation about the pivot at CM, so its sense is negative; and torque τ_R causes counterclockwise rotation about the pivot at CM, so its sense is positive. With this information, we write the second equilibrium condition as

$$-r_F F_F + r_R F_R = 0. \tag{19.1.15}$$

With the help of the free-body diagram, we identify the force magnitudes F_R = 0.48w and F_F = 0.52w, and their corresponding lever arms r_R = x and r_F = d − x. We can now write the second equilibrium condition, Equation 19.1.15, explicitly in terms of the unknown distance x:

$$-0.52(d-x)w + 0.48xw = 0. \tag{19.1.16}$$

Here the weight w cancels and we can solve the equation for the unknown position x of the CM. The answer is x = 0.52d = 0.52(2.5 m) = 1.3 m. Solution Choosing the pivot at the position of the front axle does not change the result. The free-body diagram for this pivot location is presented in Figure 12.6. For this choice of pivot point, the second equilibrium condition is

$$-r_w w + r_R F_R = 0. \tag{19.1.17}$$

When we substitute the quantities indicated in the diagram, we obtain

$$-(d-x)w + 0.48dw = 0. \tag{19.1.18}$$

The answer obtained by solving Equation 19.1.15is, again, x = 0.52d = 1.3 m.

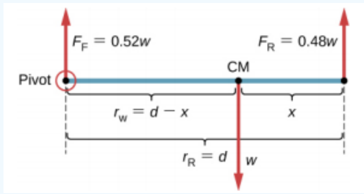


Figure 19.1.5: The equivalent free-body diagram for the car; the pivot is clearly indicated.

Significance

This example shows that when solving static equilibrium problems, we are free to choose the pivot location. For different choices of the pivot point we have different sets of equilibrium conditions to solve. However, all choices lead to the same solution to the problem.

? Exercise 12.1

Solve Example 12.1 by choosing the pivot at the location of the rear axle.

? Exercise 12.2

Explain which one of the following situations satisfies both equilibrium conditions: (a) a tennis ball that does not spin as it travels in the air; (b) a pelican that is gliding in the air at a constant velocity at one altitude; or (c) a crankshaft in the engine of a parked car.

A special case of static equilibrium occurs when all external forces on an object act at or along the axis of rotation or when the spatial extension of the object can be disregarded. In such a case, the object can be effectively treated like a point mass. In this special case, we need not worry about the second equilibrium condition, Equation 19.1.11, because all torques are identically zero and the first equilibrium condition (for forces) is the only condition to be satisfied. The free-body diagram and problem-solving strategy for this special case were outlined in Newton's Laws of Motion. You will see a typical equilibrium situation involving only the first equilibrium condition in the next example.

View this demonstration to see three weights that are connected by strings over pulleys and tied together in a knot. You can experiment with the weights to see how they affect the equilibrium position of the knot and, at the same time, see the vector-diagram representation of the first equilibrium condition at work.

Example 12.2: A Breaking Tension

A small pan of mass 42.0 g is supported by two strings, as shown in Figure 12.7. The maximum tension that the string can support is 2.80 N. Mass is added gradually to the pan until one of the strings snaps. Which string is it? How much mass must be added for this to occur?

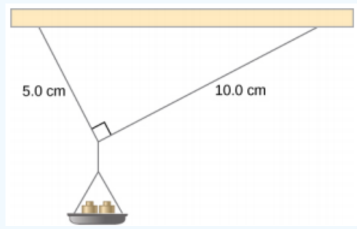


Figure 19.1.6: Mass is added gradually to the pan until one of the strings snaps.

Strategy

This mechanical system consisting of strings, masses, and the pan is in static equilibrium. Specifically, the knot that ties the strings to the pan is in static equilibrium. The knot can be treated as a point; therefore, we need only the first equilibrium condition. The three forces pulling at the knot are the tension $\vec{T}_1$ in the 5.0-cm string, the tension $\vec{T}_2$ in the 10.0-cm string, and the weight $\vec{w}$ of the pan holding the masses. We adopt a rectangular coordinate system with the y-axis pointing opposite to the direction of gravity and draw the free-body diagram for the knot (see Figure 12.8). To find the tension components, we must identify the direction angles α_1 and α_2 that the strings make with the horizontal direction that is the x-axis. As you can see in Figure 12.7, the strings make two sides of a right triangle. We can use the Pythagorean theorem to solve this triangle, shown in Figure 12.8, and find the sine and cosine of the angles α_1 and α_2. Then we can resolve the tensions into their rectangular components, substitute in the first condition for equilibrium (Equation 19.1.9and Equation 19.1.10), and solve for the tensions in the strings. The string with a greater tension will break first.

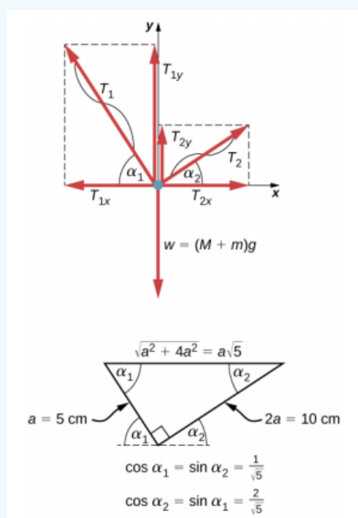


Figure 19.1.7: Free-body diagram for the knot in Example 12.2.

Solution

The weight w pulling on the knot is due to the mass M of the pan and mass m added to the pan, or w = (M + m)g. With the help of the free-body diagram in Figure 12.8, we can set up the equilibrium conditions for the knot:

in the x-direction,

$$-T_{1x} + T_{2x} = 0 \tag{19.1.19}$$

n the y-direction,

$$+T_{1y}+T_{2y}-w=0. \tag{19.1.20}$$

From the free-body diagram, the magnitudes of components in these equations are

$$\begin{aligned} T_{1x} &= T_1\cos\alpha_1 = \frac{T_1}{\sqrt{5}}, \quad T_{1y}=T_1\sin\alpha_1=\frac{2T_1}{\sqrt{5}} \\ T_{2x} &= T_2\cos\alpha_2 = \frac{2T_2}{\sqrt{5}}, \quad T_{2y}=T_2\sin\alpha_2=\frac{T_2}{\sqrt{5}}. \end{aligned}$$

We substitute these components into the equilibrium conditions and simplify. We then obtain two equilibrium equations for the tensions:

in x-direction,

$$T_1=2T_2 \tag{19.1.21}$$

in y-direction,

$$\frac{2T_1}{\sqrt{5}}+\frac{T_2}{\sqrt{5}}=(M+m)g. \tag{19.1.22}$$

The equilibrium equation for the x-direction tells us that the tension T_1 in the 5.0-cm string is twice the tension T_2 in the 10.0-cm string. Therefore, the shorter string will snap. When we use the first equation to eliminate T_2 from the second equation, we obtain the relation between the mass m on the pan and the tension T_1 in the shorter string:

$$\frac{2.5T_1}{\sqrt{5}}=(M+m)g. \tag{19.1.23}$$

The string breaks when the tension reaches the critical value of T_1 = 2.80 N. The preceding equation can be solved for the critical mass m that breaks the string:

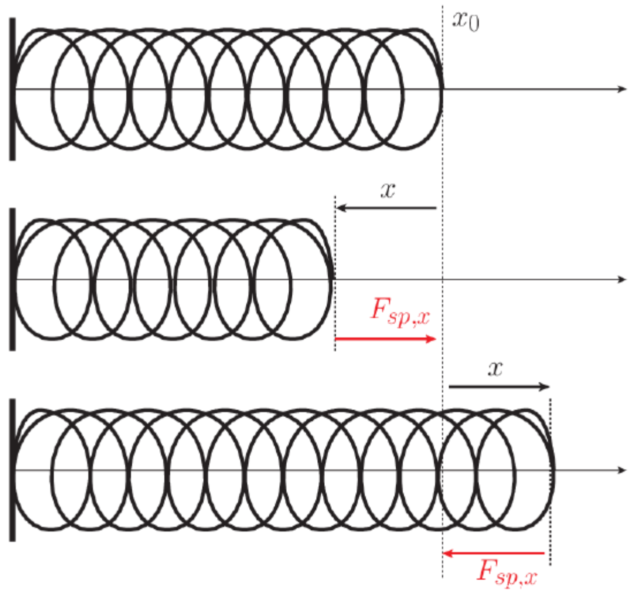
$$m=\frac{2.5}{\sqrt{5}}\frac{T_1}{g}-M=\frac{2.5}{\sqrt{5}}\frac{2.80\ N}{9.8\ m/s^2}-0.042\ kg=0.277\ kg=277.0\ g. \tag{19.1.24}$$

Significance

Suppose that the mechanical system considered in this example is attached to a ceiling inside an elevator going up. As long as the elevator moves up at a constant speed, the result stays the same because the weight w does not change. If the elevator moves up with acceleration, the critical mass is smaller because the weight of M + m becomes larger by an apparent weight due to the acceleration of the elevator. Still, in all cases the shorter string breaks first.

19.2: Springs

From Wikipedia: A spring is an elastic object that stores mechanical energy. Although real springs can be rather complicated, in physics we normally deal with *idealized springs*, which can be modeled relatively easily. In addition, this model is widely applicable for many other physical interactions, like between molecules or other fundamental particles. In fact, nearly every interaction in the universe looks like a spring if you consider just small displacements from equilibrium positions. In this section we will carefully outline how to model the interaction of a spring using forces - the chapter on potential energy has already discussed modeling the spring as an energy storage device.



A spring has an equilibrium position (see the figure above), at which it will stay until forces are applied to it. If we use a coordinate system with the origin at the fixed end of the spring, we will call this equilbrium position x_0. When the end of the spring is moved to a different location x, the response force from the spring is linear in the displacement $x - x_0|$, with a constant of proportionality k (**the spring constant**). This constant of proportionality is different for every spring, but remains constant during all interactions. The basic observation of linear proportionality is the content of Hooke's Law.

There are several equivalent ways of representing the force mathematically. Using the quantities defined above, we can write the magnitude as simply

$$|F_{sp}| = k|x - x_0|. \tag{19.2.1}$$

Of course, this doesn't take into account the direction of the force, only the size. Since the direction of the force is opposite of the displacement (compressed springs push against the compression, whereas stretched springs pull back - this is why the spring force is often called **a restoring force**), we need to be a little careful to get that right. One way to do this is to write the component of the force along the displacement as

$$F_{sp,x} = -k(x - x_0). \tag{19.2.2}$$

Notice the signs of the various quantities (the overall negative sign in particular) in comparison to the figure above. When the spring is stretched, we have $x > x_0$ so $x - x_0$ is positive, and the overall force is in the negative-x direction, as expected. When the spring is compressed, $x < x_0$ so $x - x_0$ is negative, but the overall negative sign ensures the force is now in the positive direction, as expected.

Generally, this is the most useful way to write the spring force. However, it is sometimes advantageous to set the equilibrium position of the spring as the origin of the coordinate system, $x_0 = 0$. In this case, the spring force is simply

$$F_{sp,x} = -kx. \tag{19.2.3}$$

You can check yourself that the signs work out the same way as they did in the other expression.

It's also possible to write this force as a vector, but we have to be a bit more careful. In cases where the idealized spring model is expected to be valid, one can write

$$\vec{F}_{sp} = -k(\vec{r} - \vec{r}_0), \tag{19.2.4}$$

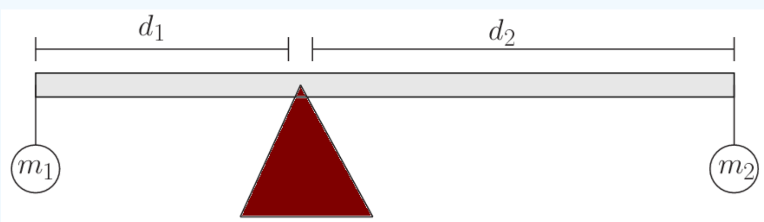
where $\vec{r}_0$ is the equilibrium position of the end of the spring, and $\vec{r}$ is the current position. However, this might start falling apart when considering various aspects of "real" springs. For example, if the end of the spring is allowed to rotate, the spring could find other equlibrium positions at distances $|r_0|$ from the origin, but at a different location than the original position. These can all be dealt with, so long as we understand how to apply the basic model of Hooke's law.

19.3: Examples

Problem-Solving Strategy: Static Equilibrium

1. Identify the object to be analyzed. For some systems in equilibrium, it may be necessary to consider more than one object. Identify all forces acting on the object. Identify the questions you need to answer. Identify the information given in the problem. In realistic problems, some key information may be implicit in the situation rather than provided explicitly.
2. Set up a free-body diagram for the object. (a) Choose the xy-reference frame for the problem. Draw a free-body diagram for the object, including only the forces that act on it. When suitable, represent the forces in terms of their components in the chosen reference frame. As you do this for each force, cross out the original force so that you do not erroneously include the same force twice in equations. Label all forces—you will need this for correct computations of net forces in the x- and y-directions. For an unknown force, the direction must be assigned arbitrarily; think of it as a 'working direction' or 'suspected direction.' The correct direction is determined by the sign that you obtain in the final solution. A plus sign (+) means that the working direction is the actual direction. A minus sign (−) means that the actual direction is opposite to the assumed working direction. (b) Choose the location of the rotation axis; in other words, choose the pivot point with respect to which you will compute torques of acting forces. On the free-body diagram, indicate the location of the pivot and the lever arms of acting forces—you will need this for correct computations of torques. In the selection of the pivot, keep in mind that the pivot can be placed anywhere you wish, but the guiding principle is that the best choice will simplify as much as possible the calculation of the net torque along the rotation axis.
3. Set up the equations of equilibrium for the object. (a) Use the free-body diagram to write a correct equilibrium condition Equation 12.2.9 for force components in the x-direction. (b) Use the free-body diagram to write a correct equilibrium condition Equation 12.2.13 for force components in the y-direction. (c) Use the free-body diagram to write a correct equilibrium condition Equation 12.2.11 for torques along the axis of rotation. Use Equation 12.2.12 to evaluate torque magnitudes and senses.
4. Simplify and solve the system of equations for equilibrium to obtain unknown quantities. At this point, your work involves algebra only. Keep in mind that the number of equations must be the same as the number of unknowns. If the number of unknowns is larger than the number of equations, the problem cannot be solved.
5. Evaluate the expressions for the unknown quantities that you obtained in your solution. Your final answers should have correct numerical values and correct physical units. If they do not, then use the previous steps to track back a mistake to its origin and correct it. Also, you may independently check for your numerical answers by shifting the pivot to a different location and solving the problem again, which is what we did in Example 12.1.

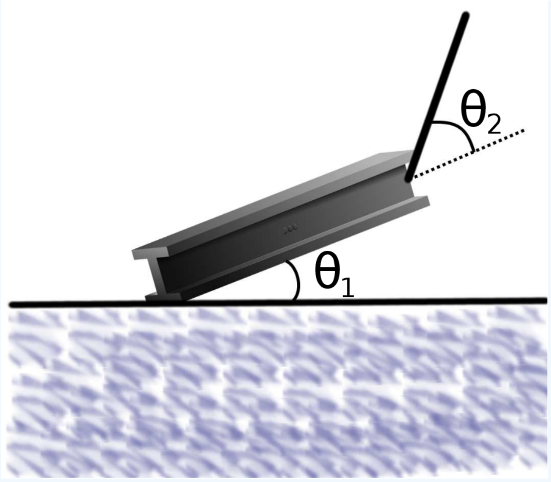
Whiteboard Problem 19.3.1: Seesaw



Consider the see-saw in the above figure - two masses attached to a massless board, balanced on a point between them.

1. If d_1 = 37.5 cm, d_2 = 113 cm, and m_1 = 15 kg, what should m_2 be so that this board is balanced?
2. How much force is the balance point acting on the board with?

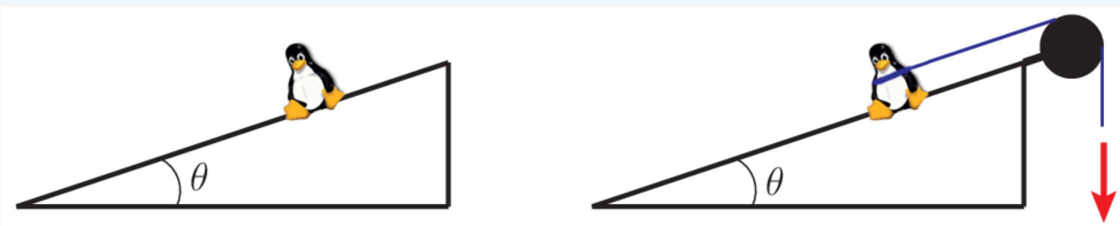
Whiteboard Problem 19.3.2: Steel Beam



Consider the steel beam shown in the figure, with a mass of 2450 kg, being held in place by a crane. The angle between the horizontal and the beam is 15°, and the angle between the axis of the beam and the cable is 63°.

1. What is the tension in the cable, if the length of the beam is 6.5 m?
2. How much force, and in which direction, is the ground acting on the beam with?

? Whiteboard Problem $19.3.3$: Sliding Penguin



Consider a penguin sitting on a ramp, as shown in the figure on the left. The ramp makes an angle of 15° with respect to the floor, the mass of the penguin is 45 kg, and the coefficient of static friction between the penguin and the ramp is 0.30.

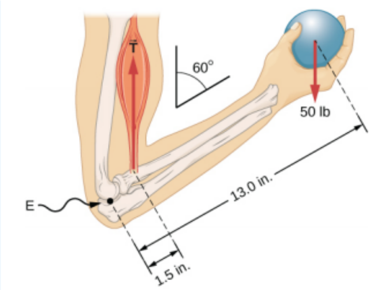
1. If the penguin is not moving, how large is the frictional force acting on it?
2. Now I tie a rope to the penguin, as shown in the figure on the right. This rope goes over a frictionless, massless pulley. How hard must I pull on the rope before the penguin just starts to move?

? Whiteboard Problem $19.3.4$: Sliding Penguin Reduex

Consider a penguin sitting on a ramp as shown in the lefthand figure for Whiteboard Problem $19.3.3$ (without the rope). This is an Emperor Penguin, so naturally it has a mass of 45 kg.

1. If the coefficient of static friction between the ramp and the penguin is 0.40, what is the maximum angle the ramp can have if the penguin is going to remain stationary?
2. If I increase the angle a little bit from part (a) then penguin will start to slide. Say I increase this angle by 10\%, and the coefficient of kinetic friction between the penguin and the ramp is 0.30, what will the acceleration of the penguin be?

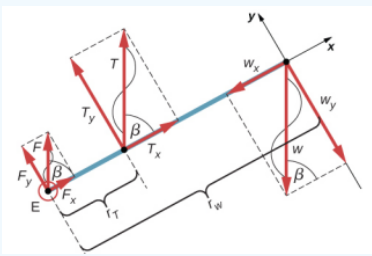
? Whiteboard Problem $19.3.5$: Curling for Torque!



A weightlifter is holding a 50.0-lb weight (equivalent to 222.4 N) with his forearm, as shown in the Figure. His forearm is positioned at $\theta = 60^\circ$ with respect to his upper arm, and supported by the biceps muscle, which causes a torque around the elbow (labeled ``E"). You can assume the tension T on the bicep muscle is directed straight up, opposite the direction of gravity, and you can ignore the weight of the arm.

1. What tension force is in the bicep muscle? (That is, ``find T"!)
2. What is the magnitude of the force at the elbow joint?
3. In what direction (describe or find an angle!) is the force at the elbow joint acting?

Note: this problem came from the Open Stax textbook University Physics Volume 1, and they solve it there using the following free body diagram. You are welcome to do that as well - but do you think that coordinate system is the best choice?



Example 19.3.6: The Torque Balance

Three masses are attached to a uniform meter stick, as shown in Figure 19.3.1. The mass of the meter stick is 150.0 g and the masses to the left of the fulcrum are m_1 = 50.0 g and m_2 = 75.0 g. Find the mass m3 that balances the system when it is attached at the right end of the stick, and the normal reaction force at the fulcrum when the system is balanced.

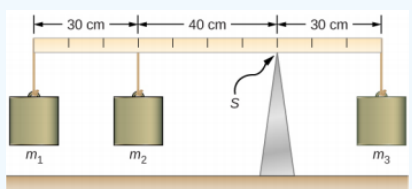


Figure 19.3.1: In a torque balance, a horizontal beam is supported at a fulcrum (indicated by S) and masses are attached to both sides of the fulcrum. The system is in static equilibrium when the beam does not rotate. It is balanced when the beam remains level.

Strategy

For the arrangement shown in the figure, we identify the following five forces acting on the meter stick:

1. $w_1 = m_1 g$ is the weight of mass m_1;
2. $w_2 = m_2 g$ is the weight of mass m_2;
3. w = mg is the weight of the entire meter stick;
4. $w_3 = m_3 g$ is the weight of unknown mass m_3;
5. F_S is the normal reaction force at the support point S.

We choose a frame of reference where the direction of the y-axis is the direction of gravity, the direction of the xaxis is along the meter stick, and the axis of rotation (the z-axis) is perpendicular to the x-axis and passes through the support point S. In other words, we choose the pivot at the point where the meter stick touches the support. This is a natural choice for the pivot because this point does not move as the stick rotates. Now we are ready to set up the free-body diagram for the meter stick. We indicate the pivot and attach five vectors representing the five forces along the line representing the meter stick, locating the forces with respect to the pivot

Figure 19.3.2. At this stage, we can identify the lever arms of the five forces given the information provided in the problem. For the three hanging masses, the problem is explicit about their locations along the stick, but the information about the location of the weight w is given implicitly. The key word here is "uniform." We know from our previous studies that the CM of a uniform stick is located at its midpoint, so this is where we attach the weight w, at the 50-cm mark.

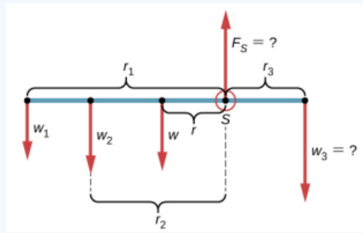


Figure 19.3.2: Free-body diagram for the meter stick. The pivot is chosen at the support point S.

Solution

With Figure 19.3.1 and Figure 19.3.2 for reference, we begin by finding the lever arms of the five forces acting on the stick:

$$\begin{aligned} r_1 &= 30.0\ cm + 40.0\ cm = 70.0\ cm \\ r_2 &= 40.0\ cm \\ r &= 50.0\ cm - 30.0\ cm = 20.0\ cm \\ r_S &= 0.0\ cm\ (because\ F_S\ is\ attached\ at\ the\ pivot) \\ r_3 &= 30.0\ cm. \end{aligned}$$

Now we can find the five torques with respect to the chosen pivot:

$$\begin{aligned} \tau_1 &= +r_1 w_1 \sin 90^o = +r_1 m_1 g \quad &(counterclockwise\ rotation,\ positive\ sense) \\ \tau_2 &= +r_2 w_2 \sin 90^o = +r_2 m_2 g \quad &(counterclockwise\ rotation,\ positive\ sense) \\ \tau &= +rw \sin 90^o = +rmg \quad &(gravitational\ torque) \\ \tau_S &= r_S F_S \sin\theta_S = 0 \quad &(because\ r_S = 0\ cm) \\ \tau_3 &= -r_3 w_3 \sin 90^o = -r_3 m_3 g \quad &(counterclockwise\ rotation,\ negative\ sense) \end{aligned}$$

The second equilibrium condition (equation for the torques) for the meter stick is

$$\tau_1 + \tau_2 + \tau + \tau_S + \tau_3 = 0. \tag{19.3.1}$$

When substituting torque values into this equation, we can omit the torques giving zero contributions. In this way the second equilibrium condition is

$$+r_1 m_1 g + r_2 m_2 g + rmg - r_3 m_3 g = 0. \tag{19.3.2}$$

Selecting the +y-direction to be parallel to $\vec{F}_S$, the first equilibrium condition for the stick is

$$-w_1 - w_2 - w + F_S - w_3 = 0. \tag{19.3.3}$$

Substituting the forces, the first equilibrium condition becomes

$$-m_1 g - m_2 g - mg + F_S - m_3 g = 0. \tag{19.3.4}$$

We solve these equations simultaneously for the unknown values m_3 and F_S. In Equation 19.3.2, we cancel the g factor and rearrange the terms to obtain

$$r_3 m_3 = r_1 m_1 + r_2 m_2 + rm. \tag{19.3.5}$$

To obtain m_3 we divide both sides by r_3, so we have

$$\begin{aligned} m_3 &= \frac{r_1}{r_3} m_1 + \frac{r_2}{r_3} m_2 + \frac{r}{r_3} m \\ &= \frac{70}{30}(50.0\ g) + \frac{40}{30}(75.0\ g) + \frac{20}{30}(150.0\ g) = 315.0 \left(\frac{2}{3}\right)\ g \simeq 317\ g. \end{aligned}$$

To find the normal reaction force, we rearrange the terms in Equation 19.3.4, converting grams to kilograms:

$$\begin{aligned} F_S &= (m_1 + m_2 + m + m_3) g \\ &= (50.0 + 75.0 + 150.0 + 316.7) \times (10^{-3}\ kg) \times (9.8\ m/s^2) = 5.8\ N. \end{aligned}$$

Significance

Notice that Equation 19.3.2 is independent of the value of g. The torque balance may therefore be used to measure mass, since variations in g-values on Earth's surface do not affect these measurements. This is not the case for a spring balance because it measures the force.

? Exercise 19.3.7

Repeat Example 12.3 using the left end of the meter stick to calculate the torques; that is, by placing the pivot at the left end of the meter stick.

In the next example, we show how to use the first equilibrium condition (equation for forces) in the vector form given by Equation 12.2.9 and Equation 12.2.10. We present this solution to illustrate the importance of a suitable choice of reference frame. Although all inertial reference frames are equivalent and numerical solutions obtained in one frame are the same as in any other, an unsuitable choice of reference frame can make the solution quite lengthy and convoluted, whereas a wise choice of reference frame makes the solution straightforward. We show this in the equivalent solution to the same problem. This particular example illustrates an application of static equilibrium to biomechanics.

? Exercise 19.3.8

Repeat Example 12.4 assuming that the forearm is an object of uniform density that weighs 8.896 N.

✓ Example19.3.9: A Ladder Resting Against a Wall

A uniform ladder is L = 5.0 m long and weighs 400.0 N. The ladder rests against a slippery vertical wall, as shown in Figure 19.3.6. The inclination angle between the ladder and the rough floor is β = 53°. Find the reaction forces from the floor and from the wall on the ladder and the coefficient of static friction μ_s at the interface of the ladder with the floor that prevents the ladder from slipping.

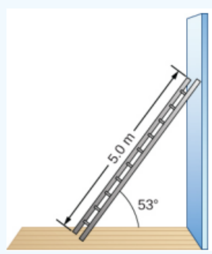


Figure 19.3.6 A 5.0-m-long ladder rests against a frictionless wall.

Strategy

We can identify four forces acting on the ladder. The first force is the normal reaction force N from the floor in the upward vertical direction. The second force is the static friction force f = μ_sN directed horizontally along the floor toward the wall—this force prevents the ladder from slipping. These two forces act on the ladder at its contact point with the floor. The third force is the weight w of the ladder, attached at its CM located midway between its ends. The fourth force is the normal reaction force F from the wall in the horizontal direction away from the wall, attached at the contact point with the wall. There are no other forces because the wall is slippery, which means there is no friction between the wall and the ladder. Based on this analysis, we adopt the frame of reference with the y-axis in the vertical direction (parallel to the wall) and the x-axis in the horizontal direction (parallel to the floor). In this frame, each force has either a horizontal component or a vertical component but not both, which simplifies the solution. We select the pivot at the contact point with the floor. In the free-body diagram for the ladder, we indicate the pivot, all four forces and their lever arms, and the angles between lever arms and the forces, as shown in Figure 19.3.7. With our choice of the pivot location, there is no torque either from the normal reaction force N or from the static friction f because they both act at the pivot.

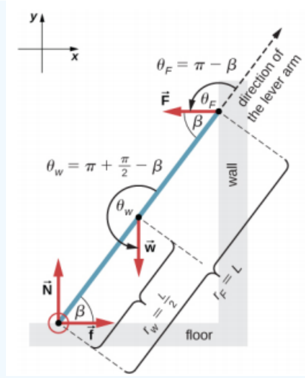


Figure 19.3.7: Free-body diagram for a ladder resting against a frictionless wall.

Solution

From the free-body diagram, the net force in the x-direction is

$$+f - F = 0 \tag{19.3.6}$$

the net force in the y-direction is

$$+N - w = 0 \tag{19.3.7}$$

and the net torque along the rotation axis at the pivot point is

$$\tau_w + \tau_F = 0. \tag{19.3.8}$$

where τ_w is the torque of the weight w and τ_F is the torque of the reaction F. From the free-body diagram, we identify that the lever arm of the reaction at the wall is r_F = L = 5.0 m and the lever arm of the weight is $r_w = \frac{L}{2}$ = 2.5 m. With the help of the free-body diagram, we identify the angles to be used in Equation 12.2.12 for torques: θ_F = 180° − β for the torque from the reaction force with the wall, and θ_w = 180° + (90° − β) for the torque due to the weight. Now we are ready to use Equation 12.2.12 to compute torques:

$$\tau_w = r_w w \sin\theta_w = r_w w \sin(180^o + 90^o - \beta) = -\frac{L}{2} w \sin(90^o - \beta) = -\frac{L}{2} w \cos\beta \tag{19.3.9}$$

$$\tau_F = r_F F \sin\theta_F = r_F F \sin(180^o - \beta) = LF\sin\beta. \tag{19.3.10}$$

We substitute the torques into Equation 19.3.8 and solve for F :

$$-\frac{L}{2} w \cos\beta + LF\sin\beta = 0 \tag{19.3.11}$$

$$F = \frac{w}{2}\cot\beta = \frac{400.0\ N}{2}\cot 53^o = 150.7\ N \tag{19.3.12}$$

We obtain the normal reaction force with the floor by solving Equation 19.3.7: N = w = 400.0 N. The magnitude of friction is obtained by solving Equation 19.3.6: f = F = 150.7 N. The coefficient of static friction is $\mu_s = \frac{f}{N} = \frac{150.7}{400.0}$ = 0.377.

The net force on the ladder at the contact point with the floor is the vector sum of the normal reaction from the floor and the static friction forces:

$$\vec{F}_{floor} = \vec{f} + \vec{N} = (150.7\ N)(-\hat{i}) + (400.0\ N)(+\hat{j}) = (-150.7\ \hat{i} + 400.0\ \hat{j})N. \tag{19.3.13}$$

Its magnitude is

$$F_{floor} = \sqrt{f^2 + N^2} = \sqrt{150.7^2 + 400.0^2}\ N = 427.4\ N \tag{19.3.14}$$

and its direction is

$$\varphi = \tan^{-1}\left(\frac{N}{f}\right) = \tan^{-1}\left(\frac{400.0}{150.7}\right) = 69.3^o\ above\ the\ floor. \tag{19.3.15}$$

We should emphasize here two general observations of practical use. First, notice that when we choose a pivot point, there is no expectation that the system will actually pivot around the chosen point. The ladder in this example is not rotating at all but firmly stands on the floor; nonetheless, its contact point with the floor is a good choice for the pivot. Second, notice when we use Equation

12.2.12 for the computation of individual torques, we do not need to resolve the forces into their normal and parallel components with respect to the direction of the lever arm, and we do not need to consider a sense of the torque. As long as the angle in Equation 12.2.12 is correctly identified—with the help of a free-body diagram—as the angle measured counterclockwise from the direction of the lever arm to the direction of the force vector, Equation 12.2.12 gives both the magnitude and the sense of the torque. This is because torque is the vector product of the lever-arm vector crossed with the force vector, and Equation 12.2.12 expresses the rectangular component of this vector product along the axis of rotation.

Significance

This result is independent of the length of the ladder because L is canceled in the second equilibrium condition, Equation 19.3.11. No matter how long or short the ladder is, as long as its weight is 400 N and the angle with the floor is 53°, our results hold. But the ladder will slip if the net torque becomes negative in Equation 19.3.11. This happens for some angles when the coefficient of static friction is not great enough to prevent the ladder from slipping.

Exercise 19.3.10

For the situation described in Example 12.5, determine the values of the coefficient μ_s of static friction for which the ladder starts slipping, given that β is the angle that the ladder makes with the floor.

Example 19.3.11: Forces on Door Hinges

A swinging door that weighs w = 400.0 N is supported by hinges A and B so that the door can swing about a vertical axis passing through the hinges Figure 19.3.8. The door has a width of b = 1.00 m, and the door slab has a uniform mass density. The hinges are placed symmetrically at the door's edge in such a way that the door's weight is evenly distributed between them. The hinges are separated by distance a = 2.00 m. Find the forces on the hinges when the door rests half-open.

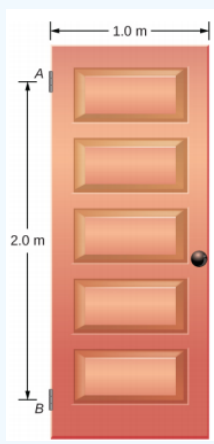


Figure 19.3.8: A 400-N swinging vertical door is supported by two hinges attached at points A and B.

Strategy

The forces that the door exerts on its hinges can be found by simply reversing the directions of the forces that the hinges exert on the door. Hence, our task is to find the forces from the hinges on the door. Three forces act on the door slab: an unknown force $\vec{A}$ from hinge A, an unknown force $\vec{B}$ from hinge B, and the known weight $\vec{w}$ attached at the center of mass of the door slab. The CM is located at the geometrical center of the door because the slab has a uniform mass density. We adopt a rectangular frame of reference with the y-axis along the direction of gravity and the x-axis in the plane of the slab, as shown in panel (a) of Figure 19.3.9, and resolve all forces into their rectangular components. In this way, we have four unknown component forces: two components of force $\vec{A}$ (A_x and A_y), and two components of force $\vec{B}$ (B_x and B_y). In the free-body diagram, we represent the two forces at the hinges by their vector components, whose assumed orientations are arbitrary. Because there are four unknowns (A_x, B_x, A_y, and B_y), we must set up four independent equations. One equation is the equilibrium condition for forces in the x-direction. The second equation is the equilibrium condition for forces in the y-direction. The third equation is the equilibrium condition for torques in rotation about a hinge. Because the weight is evenly distributed between the hinges, we have the fourth equation, $A_y = B_y$. To set up the equilibrium conditions, we draw a free-body diagram and choose the pivot point at the upper hinge, as shown in panel (b) of Figure 19.3.9. Finally, we solve the equations for the unknown force components and find the forces.

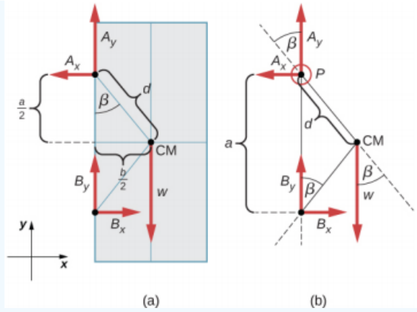


Figure 19.3.9: (a) Geometry and (b) free-body diagram for the door.

Solution

From the free-body diagram for the door we have the first equilibrium condition for forces:

in the x-direction, $-A_x + B_x = 0 \Rightarrow A_x + B_x$ in y-direction, $+A_y + B_y - w = 0 \Rightarrow A_y = B_y = \frac{w}{2} = \frac{400.0\ N}{2} = 200.0\ N$.

We select the pivot at point P (upper hinge, per the free-body diagram) and write the second equilibrium condition for torques in rotation about point P:

pivot at P:

$$\tau_w + \tau_{Bx} + \tau_{By} = 0. \tag{19.3.16}$$

We use the free-body diagram to find all the terms in this equation:

$$\begin{aligned} \tau_w &= dw \sin(-\beta) = -dw \sin\beta = -dw \frac{\frac{b}{2}}{d} = -w\frac{b}{2} \\ \tau_{Bx} &= aB_x \sin 90^o = +aB_x \\ \tau_{By} &= aB_y \sin 180^o = 0. \end{aligned}$$

In evaluating sin β, we use the geometry of the triangle shown in part (a) of the figure. Now we substitute these torques into Equation 19.3.16and compute B_x:

pivot at P: $-w\frac{b}{2} + aB_x = 0 \Rightarrow B_x = w\frac{b}{2a} = (400.0\ N)\frac{1}{2\cdot 2} = 100.0\ N$.

Therefore the magnitudes of the horizontal component forces are $A_x = B_x = 100.0$ N. The forces on the door are

at the upper hinge: $\vec{F}_{A\ on\ door} = -100.0\ N\ \hat{i} + 200.0\ N\ \hat{j}$at the lower hinge: $\vec{F}_{B\ on\ door} = +100.0\ N\ \hat{i} + 200.0\ N\ \hat{j}$.

The forces on the hinges are found from Newton's third law as

on the upper hinge: $\vec{F}_{door\ on\ A} = 100.0\ N\ \hat{i} - 200.0\ N\ \hat{j}$on the lower hinge: $\vec{F}_{door\ on\ B} = -100.0\ N\ \hat{i} - 200.0\ N\ \hat{j}$.

Significance

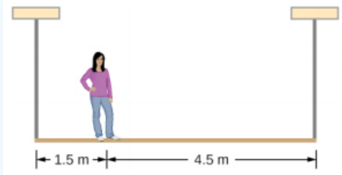
Note that if the problem were formulated without the assumption of the weight being equally distributed between the two hinges, we wouldn't be able to solve it because the number of the unknowns would be greater than the number of equations expressing equilibrium conditions.

? Exercise 19.3.11

Solve the problem in Example 12.6 by taking the pivot position at the center of mass.

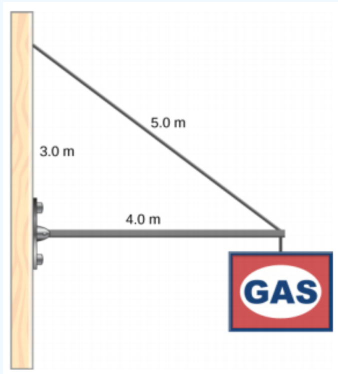
? Exercise 19.3.12

A 50-kg person stands 1.5 m away from one end of a uniform 6.0-m-long scaffold of mass 70.0 kg. Find the tensions in the two vertical ropes supporting the scaffold.



? Exercise 19.3.13

A 400.0-N sign hangs from the end of a uniform strut. The strut is 4.0 m long and weighs 600.0 N. The strut is supported by a hinge at the wall and by a cable whose other end is tied to the wall at a point 3.0 m above the left end of the strut. Find the tension in the supporting cable and the force of the hinge on the strut.



✔ Example 19.3.14: A Simple Spring Problem

Consider a spring of unknown spring constant. You first want to find out what the spring constant actually is, and then use the spring to determine the mass of an unknown object. To do this, first you measure the equilibrium length of the spring to be 10 cm. Then, you put a mass of 5 kg on the end, hang it vertically, and observe that the spring stretches to a total length of 12 cm. What is the spring constant?

Now that you know the spring constant, you put the unknown mass on the spring and notice that it stretches to a length of 17 cm. What is the mass of this object?

Solution

1. **Translate**: We will use the following variables:

$$y_0 = 10 \text{ cm}, \quad y_1 = 12 \text{ cm}, \quad y_2 = 17 \text{ cm}, m_1 = 5 \text{ kg}, \quad m_2 = ?. \tag{19.3.17}$$

   Notice that we are not specifying the coordinate system quite yet - since the spring is hanging vertically, the y-coordinate might end up being negative. These are just the lengths of the various quantities we measured.
2. **Model**: Since the only thing we know about this system are lengths and masses, we are clearly going to have to use Hooke's law, $F_{sp,y} = -k(y - y_0)$ (in the y-direction). Since this spring is hanging vertically, it makes sense to use Newton's 2nd law to model the equilibrium situation.
3. **Solve**: First, we write the condition for equilibrium when the known mass is hung on the spring. Here, we are going to take the vertical direction to be the y-coordinate, with positive upwards.

$$\Sigma F_y = 0 \rightarrow F_{sp,y} + F_{g,y} = 0 \rightarrow -k((-y_1) - (-y_0)) + m_1 g(-y_1) = 0 \rightarrow k = \frac{m_1 g y_1}{y_1 - y_0} \simeq 294 \text{ Nm} \tag{19.3.18}$$

   Notice carefully what we did with the coordinates - we made all of them negative, with the top of the spring being the origin. Also take care that you do the conversions from centimeters to meters in the final calculation!
   Now that we have the spring constant, we can do the same thing for the unknown mass. This equation is going to look very similar, but switching around our known and unknown variables:

$$\Sigma F_y = 0 \rightarrow F_{sp,y} + F_{g,y} = 0 \rightarrow -k((-y_2)-(-y_0)) + m_2 g(-y_2) = 0 \rightarrow m_2 = \frac{k(y_2-y_0)}{gy_2} \simeq 12.4\text{kg}. \tag{19.3.19}$$

Again, take careful note of what happens algebraically with the signs.

4. **Check**: This is consistent with our intuition - the spring stretched more for a heavier mass. Since the amount of stretching was 5 cm as compared to 2 cm, we would expect that the mass is also more then twice as big, which it is!

Notice that although the stretching length was exactly 3.5 times bigger (7 cm / 2 cm = 3.5), the unknown mass was not 3.5 times bigger: 3.5*5 kg = 17.5 kg. We can see why this happens by using the last formula we wrote down, but plugging in the equation for the spring constant we found in the first part:

$$m_2 = \frac{k(y_2-y_0)}{gy_2} = \left(\frac{m_1 g y_1}{y_1-y_0}\right)\left(\frac{(y_2-y_0)}{gy_2}\right) = \left(\frac{y_1}{y_2}\right)\left(\frac{y_2-y_0}{y_1-y_0}\right) m_1. \tag{19.3.20}$$

So, the ratio between m_1 and m_2 is not simply the ratio of the displacements $(y_2-y_0)/(y_1-y_0)$, but is also scaled by the ratio of the stretch, y_1/y_2.

✓ Example 19.3.15: Dropping and object on a weighing scale.

(Short version) Suppose you drop a 5-kg object on a spring scale from a height of 1 m. If the spring constant is k = 20, 000 N/m, what will the scale read?

(Long version) OK, let's break that up into parts. Suppose that a spring scale is just a platform (of negligible mass) sitting on top of a spring. If you put an object of mass m on top of it, the spring compresses so that (in equilibrium) it exerts an upwards force that matches that of gravity.

a. If the spring constant is k and the object's mass is m and the whole system is at rest, what distance is the spring compressed?
b. If you drop the object from a height h, what is the (instantaneous) *maximum* compression of the spring as the object is brought to a momentary rest? (This part is an *energy* problem! Assume that h is much greater than the actual compression of the spring, so you can neglect that when calculating the change in gravitational potential energy.)
c. What mass would give you that same compression, if you were to place it gently on the scale, and wait until all the oscillations died down?
d. OK, now answer the question at the top!

Solution

(a) The forces acting on the object sitting at rest on the platform are the force of gravity, $F^G_{E,o} = -mg$, and the normal force due to the platform, $F^n_{p,o}$. This last force is equal, in magnitude, to the force exerted on the platform by the spring (it has to be, because the platform itself is being pushed down by a force $F^n_{o,p} = -F^n_{p,o}$, and this has to be balanced by the spring force). This means we can, for practical purposes, pretend the platform is not there and just set the upwards force on the object equal to the spring force, $F^{spr}_{s,p} = -k(x-x_0)$. So, Newton's second law gives

$$F_{net} = F^G_{E,o} + F^{spr}_{s,p} = ma = 0. \tag{19.3.21}$$

For a compressed spring, $x-x_0$ is negative, and we can just let $d = x_0 - x$ be the distance the spring is compressed. Then Equation (19.3.21) gives

$$-mg + kd = 0$$

so

$$d = mg/k \tag{19.3.22}$$

when you just set an object on the scale and let it come to rest.

(b) This part, as the problem says, is a conservation of energy situation. The system formed by the spring, the object and the earth starts out with some gravitational potential energy, and ends up, with the object momentarily at rest, with only spring potential energy:

$$\begin{aligned} U_i^G + U_i^{spr} &= U_f^G + U_f^{spr} \\ mgy_i + 0 &= mgy_f + \frac{1}{2}kd_{\max}^2 \end{aligned} \tag{19.3.23}$$

where I have used the subscript "max" on the compression distance to distinguish it from what I calculated in part (a) (this kind of makes sense also because the scale is going to swing up and down, and we want only the maximum compression, which will give us the largest reading). The problem said to ignore the compression when calculating the change in U^G, meaning that, if we measure height from the top of the scale, $y_i = h$ and $y_f = 0$. Then, solving Equation (19.3.23) for d_{max}, we get

$$d_{\max} = \sqrt{\frac{2mgh}{k}}. \tag{19.3.24}$$

(c) For this part, let us rewrite Equation (19.3.22) as $m_{eq} = kd_{max}/g$, where m_{eq} is the "equivalent" mass that you would have to place on the scale (gently) to get the same reading as in part (b). Using then Equation (19.3.24),

$$m_{eq} = \frac{k}{g}\sqrt{\frac{2mgh}{k}} = \sqrt{\frac{2mkh}{g}}. \tag{19.3.25}$$

(d) Now we can substitute the values given: m = 5 kg, h = 1 m, k = 20, 000 N/m. The result is m_{eq} = 143 kg.

(Note: if you found the purely algebraic treatment above confusing, try substituting numerical values in Eqs. (19.3.22) and (19.3.24). The first equation tells you that if you just place the 5-kg mass on the scale it will compress a distance d = 2.45 mm. The second tells you that if you drop it it will compress the spring a distance d_{max} = 70 mm, about 28.6 times more, which corresponds to an "equivalent mass" 28.6 times greater than 5 kg, which is to say, 143 kg. Note also that 143 kg is an equivalent weight of 309 pounds, so if you want to try this on a bathroom scale I'd advise you to use smaller weights and drop them from a much smaller height!)

CHAPTER OVERVIEW

20: N7) Circular Motion

20.1: Motion on a Circle (Or Part of a Circle)

The last example of motion in two dimensions that I will consider in this chapter is motion on a circle. There are many examples of circular (or near-circular) motion in nature, particularly in astronomy (as we shall see in a later chapter, the orbits of most planets and many satellites are very nearly circular). There are also many devices that we use all the time that involve rotating or spinning objects (wheels, gears, turntables, turbines...). All of these can be mathematically described as collections of particles moving in circles.

In this section, I will first introduce the concept of *centripetal force*, which is the force needed to bend an object's trajectory into a circle (or an arc of a circle), and then I will also introduce a number of quantities that are useful for the description of circular motion in general, such as angular velocity and angular acceleration. The dynamics of rotational motion (questions having to do with rotational energy, and a new important quantity, angular momentum) will be the subject of Chapter 23.

Centripetal Acceleration and Centripetal Force

As you know by now, the law of inertia states that, in the absence of external forces, an object will move with constant speed on a straight line. A circle is not a straight line, so an object will not naturally follow a circular path unless there is a force acting on it.

Another way to see this is to go back to the definition of acceleration. If an object has a velocity vector $\vec{v}(t)$ at the time t, and a different velocity vector $\vec{v}(t+\Delta t)$ at the later time $t+\Delta t$, then its average acceleration over the time interval Δt is the quantity $\vec{v}_{av} = (\vec{v}(t+\Delta t) - \vec{v}(t))/\Delta t$. This is nonzero even if the *speed* does not change (that is, even if the two velocity vectors have the same magnitude), as long as they have different directions, as you can see from Figure $20.1.1$ below. Thus, motion on a circle (or an arc of a circle), even at constant speed, is *accelerated motion*, and, by Newton's second law, accelerated motion requires a force to make it happen.

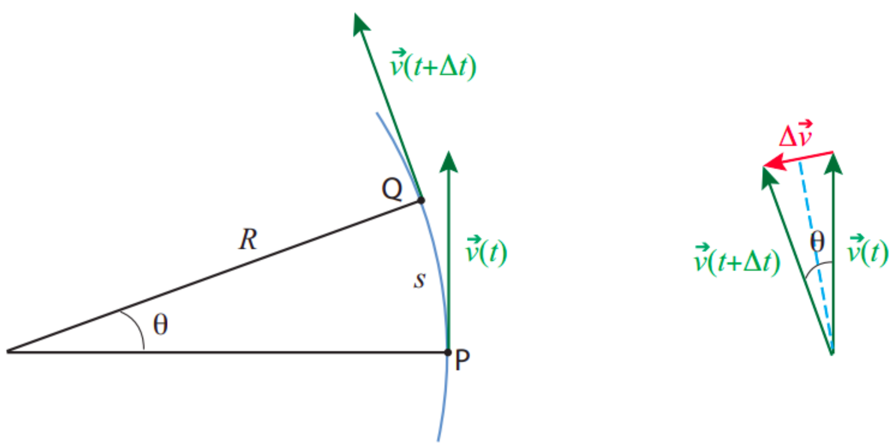


Figure 20.1.1: A particle moving along an arc of a circle of radius R. The positions and velocities at the times t and $t+\Delta t$ are shown. The diagram on the right shows the velocity difference, $\Delta\vec{v} = \vec{v}(t+\Delta t) - \vec{v}(t)$.

We can find out how large this acceleration, and the associated force, have to be, by applying a little geometry and trigonometry to the situation depicted in Figure $20.1.1$ Here a particle is moving along an arc of a circle of radius R, so that at the time t it is at point P and at the later time $t+\Delta t$ it is at point Q. The length of the arc between P and Q (the distance it has traveled) is $s = R\theta$, where the angle θ is understood to be in radians. I have assumed the speed to be constant, so the magnitude of the velocity vector, v, is just equal to the ratio of the distance traveled (along the circle), to the time elapsed: $v = s/\Delta t$. Combining these two expressions, we have a relationship for the angle in the figure:

$$\theta = \frac{s}{R} = \frac{v\Delta t}{R}. \tag{20.1.1}$$

Now, consider the second picture in the figure above. It shows the change of the velocity vector from $\vec{v}(t)$ to $\vec{v}(t+\Delta t)$, and it should be pretty easy to convince yourself that the angle θ between these two vectors is the same as in the left figure. Now, unlike the "definition of the angle" $\theta = s/r$ relationship we used above, this is a real triangle ($\Delta\vec{v}$ is a straight line), but we can say approximately the same thing is true,

$$\theta = \frac{\Delta v}{v(t)}. \tag{20.1.2}$$

Since these two are relationships for the angle (which should be the same), we can set them equal to each other:

$$\frac{v\Delta t}{R} = \frac{\Delta v}{v(t)} \rightarrow \frac{\Delta v}{\Delta t} = \frac{v^2}{R}. \tag{20.1.3}$$

In the last step we have solved for the change in velocity over the change in time, which is simply the acceleration,

$$a_c = \frac{v^2}{R}. \tag{20.1.4}$$

This acceleration is called the *centripetal acceleration*, which is why I have denoted it by the symbol a_c. The reason for that name is that it is *always pointing towards the center of the circle*. You can kind of see this from Figure 20.1.1: if you take the vector $\Delta\vec{v}$ shown there, and move it (without changing its direction, so it stays 'parallel to itself") to the midpoint of the arc, halfway between points P and Q, you will see that it does point almost straight to the center of the circle. (A more mathematically rigorous proof of this fact, using calculus, will be presented later in this section.)

The force $\vec{F}_c$ needed to provide this acceleration is called the *centripetal force*, and by Newton's second law it has to satisfy $\vec{F}_c = m\vec{a}_c$. Thus, the centripetal force has magnitude

$$F_c = ma_c = \frac{mv^2}{R} \tag{20.1.5}$$

and, like the acceleration $\vec{a}_c$, is always directed towards the center of the circle.

Physically, the centripetal force F_c, as given by Equation (20.1.5), is what it takes to bend the trajectory so as to keep it precisely on an arc of a circle of radius R and with constant speed v. Note that, since $\vec{F}_c$ is always perpendicular to the displacement (which, over any short time interval, is essentially tangent to the circle), it does *no work* on the object, and therefore (by Equation (10.2.7)) its kinetic energy does not change, so v does indeed stay constant when the centripetal force equals the net force. Note also that "centripetal" is just a job description: it is *not* a new type of force. In any given situation, the role of the centripetal force will be played by one of the forces we are already familiar with, such as the tension on a rope (or an appropriate component thereof) when you are swinging an object in a horizontal circle, or gravity in the case of the moon or any other satellite.

At this point, if you have never heard about the centripetal force before, you may be feeling a little confused, since you almost certainly have heard, instead, about a so-called *centrifugal* force that tends to push spinning things away from the center of rotation. In fact, however, this "centrifugal force" does not really exist: the "force" that you may feel pushing you towards the outside of a curve when you ride in a vehicle that makes a sharp turn is really nothing but your own inertia—your body "wants" to keep moving on a straight line, but the car, by bending its trajectory, is preventing it from doing so. The impression that you get that you would fly radially out, as opposed to along a tangent, is also entirely due to the fact that the reference frame you are in (the car) is continuously changing its direction of motion. Example 20.2.3 illustrates this in some detail.

On the other hand, getting a car to safely negotiate a turn is actually an important example of a situation that requires a definite *centripetal* force. On a flat surface (see the next section for a treatment of a banked curve!), you rely entirely on the force of static friction to keep you on the track, which can typically be modeled as an arc of a circle with some radius R. So, if you are traveling at a speed v, you need $F^s = mv^2/R$. Recalling that the force of static friction cannot exceed $\mu_s F^n$, and that on a flat surface you would just have $F^n = F^g = mg$, you see you need to keep mv^2/R smaller than $\mu_s mg$; or, canceling the mass,

$$\frac{v^2}{R} < \mu_s g. \tag{20.1.6}$$

This is the condition that has to hold in order to be able to make the turn safely. The maximum speed is then $v_{\max} = \sqrt{\mu_s g R}$, which, as you can see, will depend on the state of the road (for instance, if the road is wet the coefficient μ_s will be smaller). The posted, recommended speed will typically take this into consideration and will be as low as it has to be to keep you safe. Notice that the left-hand side of Equation (20.1.6) increases as the *square* of the speed, so doubling your speed makes that term four times larger! Do not even think of taking a turn at 60 mph if the recommended speed is 30, and do not exceed the recommended speed *at all* if the road is wet or your tires are worn.

Kinematic Angular Variables

Consider a particle moving on a circle, as in Figure $20.1.2$ below. Of course, we can just use the regular, cartesian coordinates, x and y, to describe its motion. But, in a way, this is carrying around more information than we typically need, and it is also not very transparent: a value of x and y does not immediately tell us how far the object has traveled along the circle itself.

Instead, the most convenient way to describe the motion of the particle, if we know the radius of the circle, is to give the *angle* θ that the position vector makes with some reference axis at any given time, as shown in Figure $20.1.2$. If we choose the x axis as the reference, then the conversion from a description based on the radius R and the angle θ to a description in terms of x and y is simply

$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \end{aligned} \tag{20.1.7}$$

so knowing the function $\theta(t)$ we can immediately get $x(t)$ and $y(t)$, if we need them. (Note: in this section we are using an uppercase R for the magnitude of the position vector, to emphasize that it is a constant, equal to the radius of the circle.)

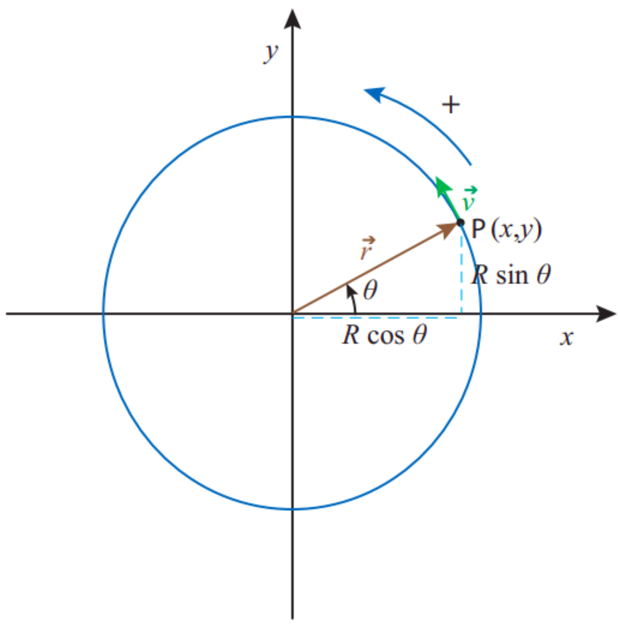


Figure $20.1.2$: A particle moving on a circle. The position vector has length R, so the x and y coordinates are $R\cos\theta$ and $R\sin\theta$, respectively. The conventional positive direction of motion is indicated. The velocity vector is always, as usual, tangent to the trajectory.

Although the angle θ itself is not a vector quantity, nor a component of a vector, it is convenient to allow for the possibility that it might be negative. The standard convention is that θ grows in the counterclockwise direction from the reference axis, and decreases in the clockwise direction. Of course, you can always get to any angle by coming from either direction, so the angle by itself does not tell you how the particle got there. Information on the direction of motion at any given time is best captured by the concept of the *angular velocity*, which we represent by the symbol ω and define in a manner analogous to the way we defined the ordinary velocity: if $\Delta\theta = \theta(t+\Delta t) - \theta(t)$ is the angular displacement over a time Δt, then

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}. \tag{20.1.8}$$

The standard convention is also to use radians as an angle measure in this context, so that the units of ω will be radians per second, or rad/s. Note that the radian is a *dimensionless* unit, so it “disappears” from a calculation when the final result does not call for it (as in Equation (20.1.12) below).

For motion with constant angular velocity, we clearly will have

$$\theta(t)=\theta_i+\omega\left(t-t_i\right) \quad \text { or } \quad \Delta \theta=\omega \Delta t \quad(\text { constant } \omega) \tag{20.1.9}$$

where ω is positive for counterclockwise motion, and negative for clockwise. (Recall that the direction of the vector $\vec{\omega}$ can be specified with the right hand rule, from section 7.1)

When ω changes with time, we can introduce an *angular acceleration* α, defined, again, in the obvious way:

$$\alpha=\lim _{\Delta t \rightarrow 0} \frac{\Delta \omega}{\Delta t}=\frac{d \omega}{d t}. \tag{20.1.10}$$

Then for motion with constant angular acceleration we have the formulas

$$\begin{aligned}
& \omega(t)=\omega_i+\alpha\left(t-t_i\right) \quad \text { or } \quad \Delta \omega=\alpha \Delta t \quad(\text { constant } \alpha) \\
& \theta(t)=\theta_i+\omega_i\left(t-t_i\right)+\frac{1}{2} \alpha\left(t-t_i\right)^2 \quad \text { or } \quad \Delta \theta=\omega_i \Delta t+\frac{1}{2} \alpha(\Delta t)^2 \quad(\text { constant } \alpha).
\end{aligned} \tag{20.1.11}$$

Equation (20.1.11) completely parallel the corresponding equations for motion in one dimension that we saw in Chapter 1. In fact, of course, a circle is just a line that has been bent in a uniform way, so the distance traveled along the circle itself is simply proportional to the angle swept by the position vector $\vec{r}$. As already pointed out in connection with Figure $20.1.1$, if we expressed θ in radians then the length of the arc corresponding to an angular displacement $\Delta\theta$ would be

$$s=R|\Delta \theta| \tag{20.1.12}$$

so multiplying Eqs. (20.1.9) or (20.1.11) by R directly gives the distance traveled along the circle in each case.

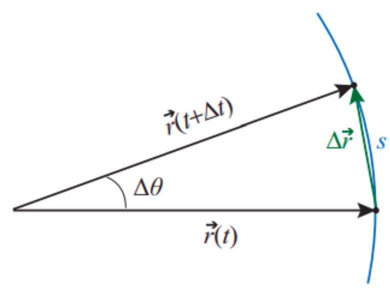


Figure 20.1.3: A small angular displacement. The distance traveled along the circle, $s=R\Delta\theta$, is almost identical to the straight-line distance $|\Delta\vec{r}|$ between the initial and final positions; the two quantities become the same in the limit $\Delta t \rightarrow 0$.

Figure $20.1.3$ shows that, for very small angular displacements, it does not matter whether the distance traveled is measured along the circle itself or on a straight line; that is, $s \simeq|\Delta \vec{r}|$. Dividing by Δt, using Equation (20.1.12) and taking the $\Delta t \rightarrow 0$ limit we get the following useful relationship between the angular velocity and the instantaneous speed v (defined in the ordinary way as the distance traveled per unit time, or the magnitude of the velocity vector):

$$|\vec{v}|=R|\omega|. \tag{20.1.13}$$

As we shall see later, the product $R\alpha$ is also a useful quantity. It is *not*, however, equal to the magnitude of the acceleration vector, but only one of its two components, the *tangential acceleration*:

$$a_t=R \alpha. \tag{20.1.14}$$

The sign convention here is that a positive a_t represents a vector that is tangent to the circle and points in the direction of increasing θ (that is, counterclockwise); the full acceleration vector is equal to the sum of this vector and the *centripetal acceleration* vector, introduced in the previous subsection, which always points towards the center of the circle and has magnitude

$$a_c=\frac{v^2}{R}=R \omega^2 \tag{20.1.15}$$

(making use of Eqs. (20.1.6) and (20.1.13)). These results will be formally established Chapter 22, after we introduce the vector product, although you could also verify them right now—if you are familiar enough with derivatives at this point—by using the chain rule to take the derivatives with respect to time of the components of the position vector, as given in Equation (20.1.7) (with $\theta=\theta(t)$, an arbitrary function of time).

The main thing to remember about the radial and tangential components of the acceleration is that the radial component (the centripetal acceleration) is *always* there for circular motion, whether the angular velocity is constant or not, whereas the tangential acceleration is only nonzero if the angular velocity is changing, that is to say, if the particle is slowing down or speeding up as it turns.

20.2: Banking

Example 20.2.1: Going around a banked curve

Roadway engineers often bank a curve, especially if it is a very tight turn, so the cars will not have to rely on friction alone to provide the required centripetal force. The picture shows a car going around such a curve, which we can model as an arc of a circle of radius r. In terms of r, the bank angle θ, and the coefficient of static friction, find the maximum safe speed around the curve.

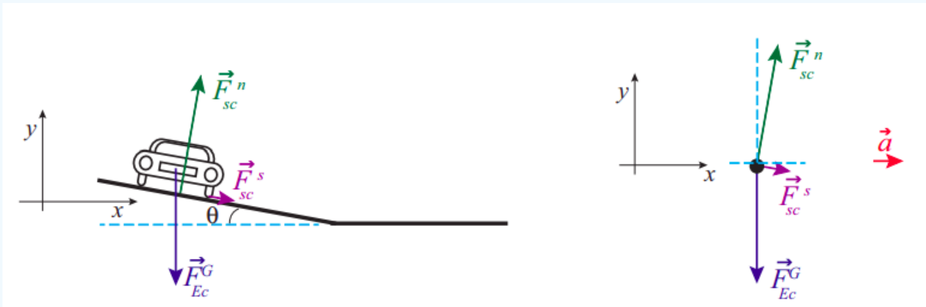


Figure 20.2.2: A car going around a banked curve (sketch and free-body diagram). The center of the circle is towards the right.

The figure shows the appropriate choice of axes for this problem. The criterion is, again, to choose the axes so that one of them will coincide with the direction of the acceleration. In this case, the acceleration is all centripetal, that is to say, pointing, horizontally, towards the center of the circle on which the car is traveling.

It may seem strange to see the force of static friction pointing *down* the slope, but recall that for a car turning on a flat surface it would have been pointing inwards (towards the center of the circle), so this is the natural extension of that. In general, you should always try to imagine which way the object would slide if friction disappeared altogether: $\vec{F}^{\,s}$ must point in the direction *opposite* that. Thus, for a car traveling at a reasonable speed, the direction in which it would skid is up the slope, and that means $\vec{F}^{\,s}$ must point down the slope. But, for a car just sitting still on the tilted road, $\vec{F}^{\,s}$ must point upwards, and we shall see in a moment that in general there is a minimum velocity required for the force of static friction to point in the direction we have chosen.

Apart from this, the main difference with the flat surface case is that now the normal force has a component along the direction of the acceleration, so it helps to keep the car moving in a circle. On the other hand, note that we now lose (for centripetal purposes) a little bit of the friction force, since it is pointing slightly downwards. This, however, is more than compensated for by the fact that the normal force is greater now than it would be for a flat surface, since the car is now, so to speak, “driving into” the road somewhat.

The dashed blue lines in the free-body diagram are meant to indicate that the angle θ of the bank is also the angle between the normal force and the positive y axis, as well as the angle that $\vec{F}^{\,s}$ makes *below* the positive x axis. It follows that the components of these two forces along the axes shown are:

$$\begin{aligned} F^n_x &= F^n \sin\theta \\ F^n_y &= F^n \cos\theta \end{aligned} \tag{20.2.1}$$

and

$$\begin{aligned} F^s_x &= F^s \cos\theta \\ F^n_y &= -F^s \sin\theta \end{aligned} \tag{20.2.2}$$

The vertical force equation is then:

$$0 = ma_y = F^n_y + F^s_y - F^G = F^n \cos\theta - F^s \sin\theta - mg. \tag{20.2.3}$$

This shows that $F^n = (mg + F^s \sin\theta)/\cos\theta$ is indeed greater than just mg for this problem, and must increase as the angle θ increases (since $\cos\theta$ decreases with increasing θ). The horizontal equation is:

$$ma_x = F_x^n + F_x^s = F^n \sin\theta + F^s \cos\theta = \frac{mv^2}{r} \tag{20.2.4}$$

where I have already substituted the value of the centripetal acceleration for a_x. Equations (20.2.3) and (20.2.4) form a system that needs to be solved for the two unknowns F^n and F^s. The result is:

$$\begin{aligned} F^n &= mg\cos\theta + \frac{mv^2}{r}\sin\theta \\ F^s &= -mg\sin\theta + \frac{mv^2}{r}\cos\theta. \end{aligned} \tag{20.2.5}$$

Note that the second equation would have F^s becoming negative if $v^2 < gr\tan\theta$. This means that below that speed, the force of static friction must actually point up the slope, as discussed above. We can call this particular speed, for which F^s becomes zero, $v_{no\ friction}$:

$$v_{\text{no friction}} = \sqrt{gr\tan\theta}. \tag{20.2.6}$$

What this means is that it is possible to arrange the banking angle so that a car going at a specific speed would not have to rely on friction at all in order to make the curve: the normal force would be just right to provide the required centripetal acceleration. A car going at that speed would not feel either pulled down or pushed up the slope. However, a car going faster than that would tend to "fly off", and the static friction force would be required to pull it in and keep it on the curve, whereas a car moving more slowly would tend to slide down and would have to be pushed up by the friction force. Friction, therefore, provides a range of safe speeds to drive in this case, just as it did in the flat surface case.

We can calculate the maximum safe speed as we did before, recalling that we must always have $F^s \leq \mu_s F^n$. Substituting Eqs. (20.2.5) in this expression, and solving for v, we get the condition

$$v_{\max} = \sqrt{gr}\sqrt{\frac{\mu_s + \tan\theta}{1 - \mu_s\tan\theta}}. \tag{20.2.7}$$

This reproduces our result (8.4.5) for θ = 0 (a flat road), as it should.

To put some numbers into this, suppose the curve has a radius of 20 m, and the coefficient of static friction between the tires and the road is μ_s = 0.7. Then, for a flat surface, we get v_{max} = 11.7 m/s, or about 26 mph, whereas for a bank angle of θ = 10 ° (the angle chosen for the figure above) we get v_{max} = 14 m/s, or about 31 mph.

Equation (20.2.7) actually indicates that the maximum velocity would "become infinite" for a finite bank angle, namely, if $1 - \mu_s\tan\theta = 0$, or $\tan\theta = 1/\mu_s$ (if μ_s = 0.7, this corresponds to θ = 55°). This is mathematically correct, but of course we cannot take it literally: it assumes that there is no limit to how large a normal force the roadway may exert without sustaining damage, and also that F^s can become arbitrarily large as long as it stays below the bound $F^s \leq \mu_s F^n$. Neither of these assumptions would hold in real life for very large speeds. Also, the angle $\theta = \tan^{-1}(1/\mu_s)$ is much too steep: recall that, according to Equation (8.3.11), the force of friction will only be able to keep an object (initially at rest) from sliding down the slope if $\tan\theta \leq \mu_s$, which for μ_s = 0.7 means $\theta \leq 35^\circ$. So, with a bank angle of 55° you *might* drive on the curve, provided you were going fast enough, but you could not park on it—the car would slide down! Bottom line, use Equation (20.2.7) only for moderate values of θ... and do not exceed $\theta = \tan^{-1}\mu_s$ if you want a car to be able to drive around the curve slowly without sliding down into the ditch.

Example 20.2.2: Staying on track

(This example studies a situation that you could easily setup experimentally at home (you can use a whole sphere instead of a half-sphere!), although to get the numbers to work out you really need to make sure that the friction between the surface and the object you choose is truly negligible. Essentially the same mathematical approach could be used to study the problem of a skier going over a mogul, or a car losing contact with the road if it is going too fast over a hill.)

A small object is placed at the top of a smooth (frictionless) dome shaped like a half-sphere of radius R, and given a small push so it starts sliding down the dome, initially moving very slowly $(v_i \simeq 0)$, but picking up speed as it goes, until at some point it flies off the surface.

a. At that point, when the object loses contact with the surface, what is the angle that its position vector (with origin at the center of the sphere) makes with the vertical?
b. How far away from the sphere does the object land?

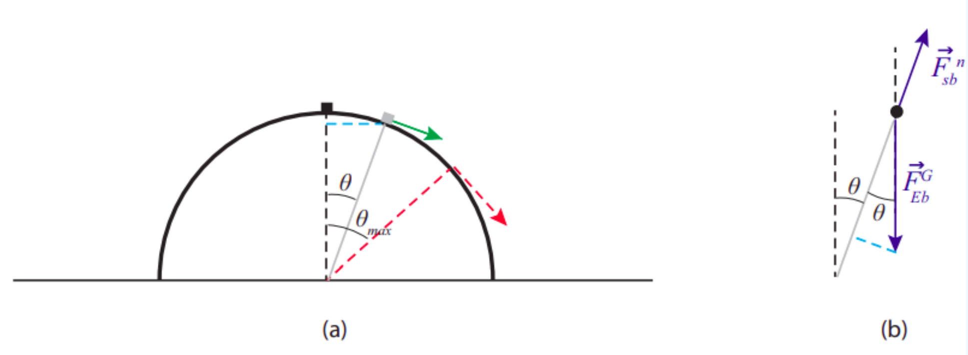


Figure 20.2.1: An object (small block) sliding on a hemispherical dome. The drawing (a) shows the angle θ_{max} at which the object flies off (red dashed line), and a smaller, generic angle θ. The drawing (b) shows the free-body diagram corresponding to the angle θ.

Solution

(a) As we saw in Section 8.4, in order to get an object to move along an arc of a circle, a centripetal force of magnitude mv^2/r is required. As long as our object is in contact with the surface, the forces acting on it are the normal force (which points along the radial direction, so it makes a *negative* contribution to the centrifugal force) and gravity, which has a component $mg\cos\theta$ along the radius, towards the center of the circle (see Figure $20.2.1$(b), the dashed light blue line). So, the centripetal force equation reads

$$\frac{mv^2}{R} = mg\cos\theta - F^n. \tag{20.2.8}$$

The next thing we need to do is find the value of the speed v for a given angle θ. If we treat the object as a particle, its only energy is kinetic energy, and $\Delta K = W_{net}$ (Equation (7.2.8)), where W_{net} is the work done on the particle by the net force acting on it. The normal force is always perpendicular to the displacement, so it does no work, whereas gravity is always vertical and does work $W_{grav} = -mg\Delta y$ (taking upwards as positive, so Δy is negative). In fact, from Figure $20.2.1$(a) (follow the dashed blue line) you can see that for a given angle θ, the height of the object above the ground is $R\cos\theta$, so the vertical displacement from its initial position is

$$\Delta y = -(R - R\cos\theta) \tag{20.2.9}$$

Hence we have, for the change in kinetic energy,

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_i^2 = mgR - mgR\cos\theta. \tag{20.2.10}$$

Assuming, as we are told in the text of the problem, that $v_i \simeq 0$, we get $v^2 \simeq 2gR - 2gR\cos\theta$, and using this in Equation (20.2.8)

$$2mg - 2mg\cos\theta = mg\cos\theta - F^n \tag{20.2.11}$$

or $F^n = 3mg\cos\theta - 2mg$. This shows that F^n starts out (when θ = 0) having its usual value of mg, and then it becomes progressively smaller as the object slides down. The point where the object loses contact with the surface is when F^n = 0, and that happens for

$$3\cos\theta_{max} = 2 \tag{20.2.12}$$

or $\theta_{\max} = \cos^{-1}(2/3) = 48.2^{\circ}$.

Recalling that $\Delta y = -(R - R\cos\theta)$, we see that when $\cos\theta$ = 2/3, the object has fallen a distance R/3; put otherwise, its height above the ground at the time it flies off is 2R/3, or 2/3 of the initial height.

(b) This is just a projectile problem now. We just have to find the values of the initial conditions (x_i, y_i, $v_{x,i}$ and $v_{y,i}$) and substitute in the equations (16.2.5). By inspecting the figure, you can see that, at the time the object flies off,

$$
\begin{aligned}
x_i &= R\sin\theta_{\max} = 0.745R \\
y_i &= R\cos\theta_{\max} = 0.667R.
\end{aligned} \tag{20.2.13}
$$

Also, we found above that $v^2 \simeq 2gR - 2gR\cos\theta$, and when $\theta = \theta_{max}$ this gives $v^2 = 0.667gR$, or $v = 0.816\sqrt{gR}$. The projection angle in this case is $-\theta_{max}$; that is, the initial velocity of the projectile (dashed red arrow in Figure $20.2.1$(a)) is at an angle 48.2° *below* the positive x axis, so we have:

$$
\begin{aligned}
v_{x,i} &= v_i\cos\theta_{\max} = 0.544\sqrt{gR} \\
v_{y,i} &= -v_i\sin\theta_{\max} = -0.609\sqrt{gR}
\end{aligned} \tag{20.2.14}
$$

Now we just use these results in Eqs. (16.2.5). Specifically, we want to know how long it takes for the object to reach the ground, so we use the last equation (16.2.5) with y = 0 and solve for t:

$$
0 = y_i + v_{y,i}t - \frac{1}{2}gt^2 \tag{20.2.15}
$$

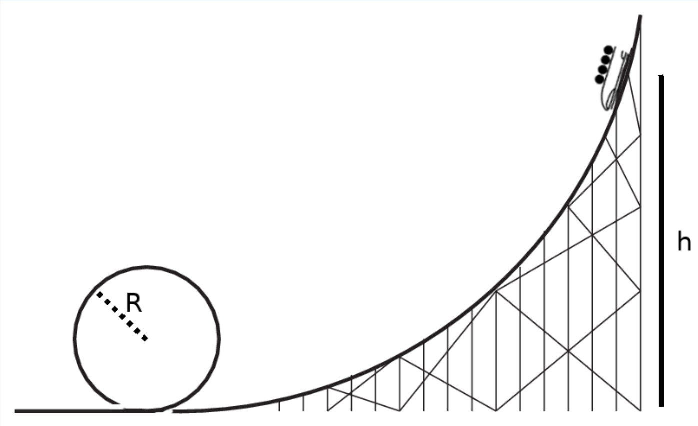
The result is $t = 0.697\sqrt{R/g}$. (You do not need to carry the "g" throughout; it would be OK to substitute 9.8 m/s^2 for it. I have just kept it in symbolic form so far to make it clear that the quantities we derive will have the right units.) Substituting this in the equation for x, we get

$$
x = x_i + v_{x,i}t = 0.745R + 0.544\sqrt{gR} \times 0.697\sqrt{R/g} = 1.125R \tag{20.2.16}
$$

(Note how the g cancels, so we would get the same result on any planet!) Since the sphere has a radius R, the object falls a distance $0.125R$away from the sphere.

20.3: Examples

? Whiteboard Problem 20.3.1: Loop the Loop!



Consider a rollarcoaster with a loop, shown in the figure above. The car starts at rest a height of $h = 35$ m above the loop, flies down the track and around the loop, which has a radius of $R = 7$ m. The car has a mass of 550 kg.

1. (energy review!) What speed is the car traveling at the top of the loop, if the track is completely frictionless?
2. What is the acceleration of the car at the top of the loop?
3. What is the normal force on the car at the top of the loop?

? Whiteboard Problem 20.3.2: This is better than a class trip, part 1

Consider the rollarcoaster loop in the figure, which has a teardrop shape. The cars ride on the inside of the loop and travel fast enough to ensure that the car stays on the track. The largest of these loops is 40 m high (*but not a circle!*) and the cars travel at a speed of 13 m/s at the top of the loop. At the top, the riders experience a centripetal acceleration of $2g$.

1. What is the radius of the arc of the track at the top?
2. If the mass of the car plus the riders is 1150 kg, what force does the track exert on the car at the top of the track?
3. If the designers had made this a circular loop of radius 20 m (so the same total height as the teardrop), what would the centripetal acceleration of the riders at the top be, assuming the car had the same speed as in the teardrop shape?
4. Which is more fun, a teardrop or a loop?

Example 20.3.3: The penny on the turntable

Suppose that you have a penny sitting on a turntable, a distance d = 10 cm from the axis of rotation. Assume the turntable starts moving, steadily spinning up from rest, in such a way that after 1.3 seconds it has reached its final rotation rate of 33.3 rpm (revolutions per minute). Answer the following questions:

a. What was the turntable's angular acceleration over the time interval from t = 0 to t = 1.3 s?
b. How many turns (complete and fractional) did the turntable make before reaching its final velocity?
c. Assuming the penny has not slipped, what is its centripetal acceleration once the turntable reaches its final velocity?
d. How large does the static friction coefficient between the penny and the turntable have to be for the penny not to slip throughout this process?

Solution

(a) We are told that the turntable spins up "steadily" from t = 0 to t = 1.3 s. The word "steadily" here is a keyword that means the (angular) acceleration is constant (that is, the angular velocity increases at a constant rate).

What is this rate? For constant α, we have, from Equation (20.1.10), $\alpha = \Delta\omega/\Delta t$. Here, the time interval Δt = 1.3 , so we just need to find $\Delta\omega$. By definition, $\Delta\omega = \omega_f - \omega_i$, and since we start from rest, ω_i = 0. So we just need ω_f. We are told that "the final rotation rate" is 33.3 rpm (revolutions per minute). What does this tell us about the angular velocity?

The angular velocity is the number of radians an object moving in a circle (such as the penny in this example) travels per second. A complete turn around the circle, or *revolution*, corresponds to 180° , or equivalently 2π radians. So, 33.3 revolutions, or turns, per minute means $33.3 \times 2\pi$ radians per 60 s, that is,

$$\omega_f = \frac{33.3 \times 2\pi \text{ rad}}{60 \text{ s}} = 3.49 \frac{\text{rad}}{\text{s}}. \tag{20.3.1}$$

The angular acceleration, therefore, is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t} = \frac{3.49 \text{ rad/s}}{1.3 \text{ s}} = 2.68 \frac{\text{rad}}{\text{s}^2}. \tag{20.3.2}$$

(b) The way to answer this question is to find out the total angular displacement, $\Delta\theta$, of the penny over the time interval considered (from t = 0 to t = 1.3 s), and then convert this to a number of turns, using the relationship 2π rad = 1 turn. To get $\Delta\theta$, we should use the equation (20.1.11) for motion with constant angular acceleration:

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha(\Delta t)^2. \tag{20.3.3}$$

We start from rest, so ω_i = 0, We know Δt = 1.3 s, and we just calculated α = 2.68 rad/s^2, so we have

$$\Delta\theta = \frac{1}{2} \times 2.68 \frac{\text{rad}}{\text{s}^2} \times (1.3 \text{ s})^2 = 2.26 \text{ rad}. \tag{20.3.4}$$

This is less than 2π radians, so it takes the turntable less than one complete revolution to reach its final angular velocity. To be precise, since 2π radians is one turn, 2.26 rad will be 2.26/(2π) turns, which is to say, 0.36 turns—a little more than 1/3 of a turn.

(c) For the questions above, the penny just served as a marker to keep track of the revolutions of the turntable. Now, we turn to the dynamics of the motion of the penny itself. First, to get its angular acceleration, we can just use Equation (20.1.15), in the form

$$a_c = R\omega^2 = 0.1 \text{ m} \times \left(3.49 \frac{\text{rad}}{\text{s}}\right)^2 = 1.22 \frac{\text{m}}{\text{s}^2} \tag{20.3.5}$$

noticing that R, the radius of the circle on which the penny moves, is just the distance d to the axis of rotation that we were given at the beginning of the problem, and ω, its angular velocity, is just the final angular velocity of the turntable (assuming, as we are told, that the penny has not slipped relative to the turntable).

(d) Finally, how about the force needed to keep the penny from slipping—that is to say, to keep it moving with the turntable? This is just the centripetal force needed "bend" the trajectory of the penny into a circle of radius R, so $F_c = ma_c$, where m is the mass of the penny and a_c is the centripetal acceleration we just calculated. Physically, we know that this force has to be provided by the *static* (as long as the penny does not slip!) friction force between the penny and the turntable. We know that $F^s \leq \mu_s F^n$, and we have for the normal force, in this simple situation, just $F^n = mg$. Therefore, setting $F^s = ma_c$ we have:

$$ma_c = F^s \leq \mu_s F^n = \mu_s mg. \tag{20.3.6}$$

This is equivalent to the single inequality $ma_c \leq \mu_s mg$, where we can cancel out the mass of the penny to conclude that we must have $a_c \leq \mu_s g$, and therefore

$$\mu_s \geq \frac{a_c}{g} = \frac{1.22 \text{ m/s}^2}{9.8 \text{ m/s}^2} = 0.124 \tag{20.3.7}$$

Example 20.3.4: Rotating frames of reference- centrifugal force and coriolis force

Imagine you are inside a rotating cylindrical room of radius R. There is a metal puck on the floor, a distance r from the axis of rotation, held in place with an electromagnet. At some time you switch off the electromagnet and the puck is free to slide without friction. Find where the puck strikes the wall, and show that, if it was not too far away from the wall to begin with, it appears as if it had moved straight for the wall as soon as it was released.

Solution

The picture looks as shown below, to an observer in an *inertial* frame, looking down. The puck starts at point A, with instantaneous velocity ωr pointing straight to the left at the moment it is released, so it just moves straight (in the inertial frame) until it hits the wall at point B. From the cyan-colored triangle shown, we can see that it travels a distance $\sqrt{R^2 - r^2}$, which takes a time

$$\Delta t = \frac{\sqrt{R^2 - r^2}}{\omega r}. \tag{20.3.8}$$

In this time, the room rotates counterclockwise through an angle $\Delta\theta_{room} = \omega \Delta t$:

$$\Delta\theta_{\text{room}} = \frac{\sqrt{R^2 - r^2}}{r}. \tag{20.3.9}$$

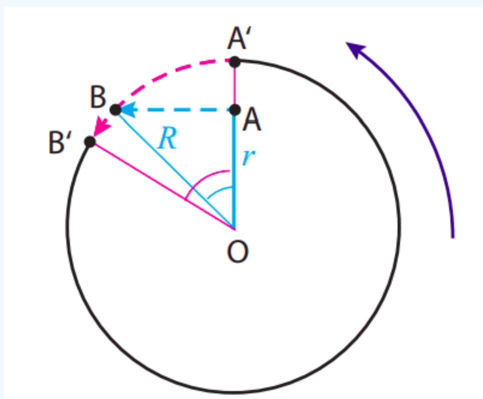


Figure 20.3.3: The motion of the puck (cyan) and the wall (magenta) as seen by an inertial observer.

This is the angle shown in magenta in the figure. As a result of this rotation, the point A′ that was initially on the wall straight across from the puck has moved (following the magenta dashed line) to the position B′, so to an observer in the rotating room, looking at things from the point O, the puck appears to head for the wall and drift a little to the right while doing so.

The cyan angle in the picture, which we could call $\Delta\theta_{part}$, has tangent equal to $\sqrt{R^2 - r^2}/r$, so we have

$$\Delta\theta_{\text{room}} = \tan(\Delta\theta_{\text{part}}). \tag{20.3.10}$$

This tells us the two angles are going to be pretty close if they are small enough, which is what happens if the puck starts close enough to the wall in the first place. The picture shows, for clarity, the case when $r = 0.7R$, which gives $\Delta\theta_{room}$ = 1.02 rad, and $\Delta\theta_{part} = \tan^{-1}(1.02)$ = 0.8 rad. For $r = 0.9R$, on the other hand, one finds $\Delta\theta_{room}$ = 0.48 rad, and $\Delta\theta_{part} = \tan^{-1}$(0.48) = 0.45 rad.

In terms of pseudoforces (forces that do not, physically, exist, but may be introduced to describe mathematically the motion of objects in non-inertial frames of reference), the non-inertial observer would say that the puck heads towards the wall because of a *centrifugal force* (that is, a force pointing away from the center of rotation), and while doing so it drifts to the right because of the so-called *Coriolis force*.

CHAPTER OVERVIEW

21: N8) Forces, Energy, and Work

21.1: Forces and Potential Energy

Consider the special case of two interacting objects, in which a lighter object is interacting with a much more massive one, so that the massive one essentially does not move at all as a result of the interaction. Note that this does not contradict Newton's 3rd law, Equation (6.1.5): the forces the two objects exert on each other are the same in magnitude, but the acceleration of each object is inversely proportional to its mass, so $F_{12} = -F_{21}$ implies

$$m_2 a_2 = -m_1 a_1 \tag{21.1.1}$$

and so if, for instance, $m_2 \gg m_1$, we get $|a_2| = |a_1| m_1/m_2 \ll |a_1|$. In words, the more massive object is less responsive than the less massive one to a force of the *same* magnitude. This is just how we came up with the concept of inertial mass in the first place!

Anyway, in this situation we could just write the potential energy function of the whole system as a function of only the lighter object's coordinate, $U(x)$. Let's use this simplified setup to demonstrate a very interesting relationship between potential energies and forces. Suppose this is a closed system in which no dissipation of energy is taking place. Then the total mechanical energy is a constant:

$$E_{\text{mech}} = \frac{1}{2}mv^2 + U(x) = \text{ constant} \tag{21.1.2}$$

(Here, m is the mass of the lighter object, and v its velocity; the more massive object does not contribute to the total kinetic energy, since it does not move!)

As the lighter object moves, both x and v in Equation (21.1.2) change with time. So I can take the derivative of Equation (21.1.2) with respect to time, using the chain rule, and noting that, since the whole thing is a constant, the total value of the derivative must be zero:

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{1}{2}m(v(t))^2 + U(x(t))\right) \\ &= mv(t)\frac{dv}{dt} + \frac{dU}{dx}\frac{dx}{dt}. \end{aligned} \tag{21.1.3}$$

But note that dx/dt is just the same as $v(t)$. So I can cancel that on both terms, and then I am left with

$$m\frac{dv}{dt} = -\frac{dU}{dx}. \tag{21.1.4}$$

But dv/dt is just the acceleration a, and $F = ma$. So this tells me that

$$F = -\frac{dU}{dx} \tag{21.1.5}$$

and this is how you can always get the force from a potential energy function.

Let us check it right away for the force of gravity: we know that $U^G = mgy$, so

$$F^G = -\frac{dU^G}{dy} = -\frac{d}{dy}(mgy) = -mg. \tag{21.1.6}$$

Is this right? It seems to be! Recall all objects fall with the same acceleration, $-g$ (assuming the upwards direction to be positive), so if $F = ma$, we must have $F^G = -mg$. So the gravitational force exerted by the earth on any object (which I would denote in full by $F^G_{E,o}$) is proportional to the inertial mass of the object—in fact, it is what we call the object's *weight*—but since to get the acceleration you have to divide the force by the inertial mass, that cancels out, and a ends up being the same for all objects, regardless of how heavy they are.

Now that we have this result under our belt, we can move on to the slightly more challenging case of two objects of comparable masses interacting through a potential energy function that must be, a function of just the relative coordinate $x_{12} = x_2 - x_1$.

I claim that in that case you can again get the force on object 1, F_{21}, by taking the derivative of $U(x_2 - x_1)$ with respect to x_1 (leaving x_2 alone), and reciprocally, you get F_{12} by taking the derivative of $U(x_2 - x_1)$ with respect to x_2. Here is how it works, again using the chain rule:

$$F_{21}=-\frac{d}{dx_1}U\left(x_{12}\right)=-\frac{dU}{dx_{12}}\frac{d}{dx_1}(x_2-x_1)=\frac{dU}{dx_{12}}$$
$$F_{12}=-\frac{d}{dx_2}U\left(x_{12}\right)=-\frac{dU}{dx_{12}}\frac{d}{dx_2}(x_2-x_1)=-\frac{dU}{dx_{12}} \tag{21.1.7}$$

and you can see that this automatically ensures that $F_{21}=-F_{12}$. In fact, it was in order to ensure this that I required that U should depend only on the *difference* of x_1 and x_2, rather than on each one separately. Since we got the condition $F_{21}=-F_{12}$ originally from conservation of momentum, you can see now how the two things are related[1].

The only example we have seen so far of this kind of potential energy function was in Chapter 9, for two carts interacting through an "ideal" spring. I told you there that the potential energy of the system could be written as $\frac{1}{2}k(x_2-x_1-x_0)^2$, where k was the "spring constant" and x_0 the relaxed length of the spring. If you apply Eqs. (21.1.7) to this function, you will find that the force exerted (through the spring) by cart 2 on cart 1 is

$$F_{21}=k\left(x_2-x_1-x_0\right). \tag{21.1.8}$$

Note that this force will be negative under the assumptions we made last chapter, namely, that cart 2 is on the right, cart 1 on the left, and the spring is compressed, so that $x_2-x_1<x_0$. Similarly,

$$F_{12}=-k\left(x_2-x_1-x_0\right) \tag{21.1.9}$$

and this one, as it should, is positive.

The results (21.1.8) and (21.1.9) basically tell you what we mean by an "ideal spring" in physics: it is a spring that pulls (if stretched) or pushes (if compressed) with a force that is proportional to the change from its equilibrium length. Thus, if you fasten one end of the spring at x = 0, and stretch it or compress it so that the other end is at x, the spring will respond by exerting a force

$$F^{spr}=-k\left(x-x_0\right). \tag{21.1.10}$$

As you can see, this is negative if $x>x_0>0$ (spring stretched, pulling force) and positive if $x<x_0$ (spring compressed, pushing force). In fact, the spring exerts an equal (in magnitude) and opposite (in direction) force at the other end (the one attached to the wall), so Equation (21.1.10) only gives the correct sign of the force at the end that is denoted by the coordinate value x. Equations (21.1.8) and (21.1.9) are a bit clearer in this respect: Equation (21.1.8) gives the correct sign of the force at point x_1, and Equation (21.1.9) the correct sign at point x_2.

Figure 21.1.1 shows, in black, all the forces exerted by a spring with one fixed end, according as to whether it is relaxed, compressed, or stretched. I have assumed that it is pushed or pulled by a hand (not shown) at the "free" end, hence the subscript "h", whereas the subscript "w" stands for "wall." Note that the wall and the hand, in turn, exert equal and opposite forces on the spring, shown in red in the figure.

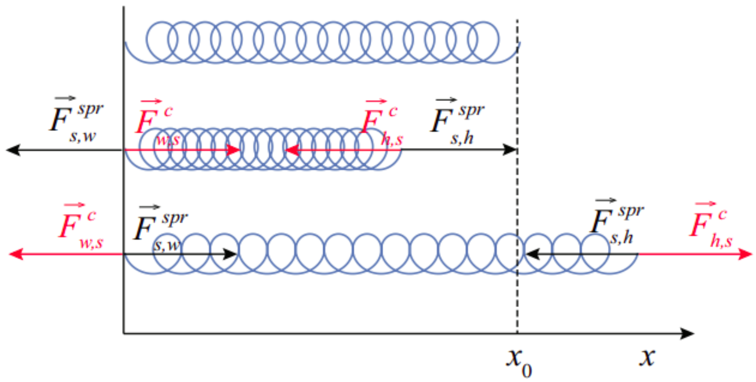


Figure 21.1.1: Forces (in black) exerted *by* a spring with one end attached to a wall and the other pushed or pulled by a hand (not shown). In every case the force is proportional to the change in the length of the spring from its equilibrium, or relaxed, value, shown here as x_0. For this figure I have set the proportionality constant k = 1. The forces exerted *on* the spring, by the wall and by the hand, are shown in red.

Equation (21.1.10) is generally referred to as *Hooke's law*, after the British scientist Robert Hooke (a contemporary of Newton). Of course, it is not a "law" at all, merely a useful approximation to the way most springs behave as long as you do not stretch them or compress them too much[2].

A note on the way the forces have been labeled in Figure 21.1.1. I have used the generic symbol “c”, which stands for “contact,” to indicate the type of force exerted by the wall and the hand on the spring. In fact, since each pair of forces (by the hand on the spring and by the spring on the hand, for instance), at the point of contact, arises from one and the same interaction, I should have used the same “type” notation for both, but it is widespread practice to use a superscript like “spr” to denote a force whose origin is, ultimately, a spring’s elasticity. This does not change the fact that the spring force, at the point where it is applied, is indeed a contact force.

So, next, a word on “contact” forces. Basically, what we mean by that is forces that arise where objects “touch,” and we mean this by opposition to what are called instead “field” forces (such as gravity, or magnetic or electrostatic forces) which “act at a distance.” The distinction is actually only meaningful at the macroscopic level, since at the microscopic level objects never *really* touch, and all forces are field forces, it is just that some are “long range” and some are “short range.” For our purposes, really, the word “contact” will just be a convenient, catch-all sort of moniker that we will use to label the force vectors when nothing more specific will do.

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[1]The result (21.1.7) generalizes to more dimensions, but to do it properly you need vectors and partial derivative notation, and I’m already bending the notational rules a little bit here..

[2]Assuming that you *can* compress them! Some springs, such as slinkies, actually cannot be compressed because their coils are already in contact when they are relaxed. Nevertheless, Equation (21.1.10) will still apply approximately to such a spring when it is stretched, that is, when $x > x_0$.

21.2: Work Done on a System By All the External Forces

Consider the most general possible system, one that might contain any number of particles, with possibly many forces, both internal and external, acting on each of them. We will again, for simplicity, start by considering what happens over a time interval so short that all the forces are approximately constant (the final result will hold for arbitrarily long time intervals, just by adding, or integrating, over many such short intervals). We will also work explicitly only the one-dimensional case, although again that turns out to not be a real restriction.

Let then $W_{all,1}$ be the work done on particle 1 by all the forces acting on it, $W_{all,2}$ the work done on particle 2, and so on. The total work is the sum $W_{all,sys} = W_{all,1} + W_{all,2} + \cdots$. However, by the results of Chapter 10, we have $W_{all,1} = \Delta K_1$ (the change in kinetic energy of particle 1), $W_{all,2} = \Delta K_2$, and so on, so adding all these up we get

$$W_{all,sys} = \Delta K_{sys} \tag{21.2.1}$$

where ΔK_{sys} is the change in kinetic energy of the whole system.

So far, of course, this is nothing new. To learn something else we need to look next at the work done by the internal forces. It is helpful here to start by considering the "no-dissipation case" in which all the internal forces can be derived from a potential energy[2]. We will consider the case where dissipative processes happen inside the system after we have gained a full understanding of the result we will obtain for this simpler case.

[2]Or else they do no work at all: the magnetic force between moving charges is an example of the latter.

The No-Dissipation Case

The internal forces are, by definition, forces that arise from the interactions between pairs of particles that are both inside the system. Because of Newton's 3rd law, the force F_{12} (we will omit the "type" superscript for now) exerted by particle 1 on particle 2 must be the negative of F_{21}, the force exerted by particle 2 on particle 1. Hence, the work associated with this interaction for this pair of particles can be written

$$W(1,2) = F_{12}\Delta x_2 + F_{21}\Delta x_1 = F_{12}\left(\Delta x_2 - \Delta x_1\right). \tag{21.2.2}$$

Notice that $\Delta x_2 - \Delta x_1$ can be rewritten as $x_{2,f} - x_{2,i} - x_{1,f} + x_{1,i} = x_{12,f} - x_{12,i} = \Delta x_{12}$, where $x_{12} = x_2 - x_1$ is the relative position coordinate of the two particles. Therefore,

$$W(1,2) = F_{12}\Delta x_{12}. \tag{21.2.3}$$

Now, if the interaction in question is associated with a potential energy, as I showed in section 21.1, $F_{12} = -dU/dx_{12}$. Assume the displacement Δx_{12} is so small that we can replace the derivative by just the ratio $\Delta U/\Delta x_{12}$ (which is consistent with our assumption that the force is approximately constant over the time interval considered); the result will then be

$$W(1,2) = F_{12}\Delta x_{12} \simeq -\frac{\Delta U}{\Delta x_{12}}\Delta x_{12} = -\Delta U. \tag{21.2.4}$$

Adding up very many such "infinitesimal" displacements will lead to the same final result, where ΔU will be the change in the potential energy over the whole process. This can also be proved using calculus, without any approximations:

$$W(1,2) = \int_{x_{12,i}}^{x_{12,f}} F_{12}dx_{12} = -\int_{x_{12,i}}^{x_{12,f}} \frac{dU}{dx_{12}}dx_{12} = -\Delta U. \tag{21.2.5}$$

We can apply this to every pair of particles and every internal interaction, and then add up all the results. On one side, we will get the total work done on the system by all the internal forces; on the other side, we will get the negative of the change in the system's total internal energy:

$$W_{int,sys} = -\Delta U_{sys}. \tag{21.2.6}$$

In words, *the work done by all the* (conservative) *internal forces is equal to the change in the system's potential energy*.

Let us now put Eqs. (21.2.1) and (21.2.6) together: the difference between the work done by all the forces and the work done by the internal forces is, of course, the work done by the *external* forces, but according to Eqs. (21.2.1) and (21.2.6), this is equal to

$$W_{ext,sys} = W_{all,sys} - W_{int,sys} = \Delta K_{sys} + \Delta U_{sys} \tag{21.2.7}$$

which is the change in the total *mechanical* (kinetic plus potential) energy of the system. If we further assume that the system, in the absence of the external forces, is closed, then there are no other processes (such as the absorption of heat) by which the total energy of the system might change, and we get the simple result that *the work done by the external forces equals the change in the system's total energy*:

Theorem 21.2.1: Generalized Work-Energy Theorem

$$W_{ext,sys} = \Delta E_{sys}. \tag{21.2.8}$$

As a first application of the result (21.2.8), Imagine you throw a ball of mass m upwards (see Figure 21.2.1), and it reaches a maximum height h above the point where your hand started to move. Let us define the system to be the ball and the earth, so the force exerted by your hand is an external force. Then you do work on the system during the throw, which in the figure is the interval, from A to B, during which your hand is on contact with the ball. The bar diagram on the side shows that some of this work goes into increasing the system's (gravitational) potential energy (because the ball goes up a little while in contact with your hand), and the rest, which is typically most of it, goes into increasing the system's kinetic energy (in this case, just the ball's; the earth's kinetic energy does not change in any measurable way!).

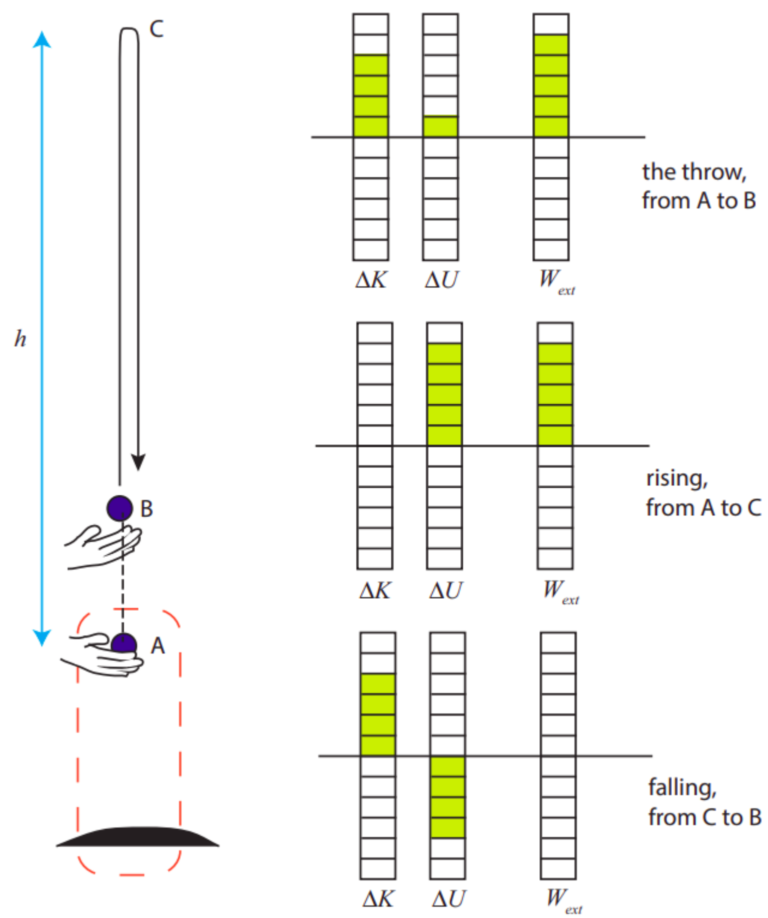

Figure 21.2.1: Tossing a ball into the air. We consider the system formed by the ball and the earth. The force exerted by the hand (which is in contact with the ball from point A to point B) is therefore an external force. The diagrams show the system's energy balance over three different intervals.

So how much work did you actually do? If we knew the distance from A to B, and the magnitude of the force you exerted, and if we could assume that your force was constant throughout, we could calculate W from the definition (10.2.5). But in this case, and many others like it, it is actually easier to find out how much total energy the system gained and just use Equation (21.2.8). To find ΔE in practice, all we have to do is see how high the ball rises. At the ball's maximum height (point C), as the second diagram shows, all the energy in the system is gravitational potential energy, and (as long as the system stays closed), all that energy is still equal to the work you did initially, so if the distance from A to C is h you must have done an amount of work

$$W_{you} = \Delta U^G = mgh. \tag{21.2.9}$$

The third diagram in Figure 21.2.1 shows the work-energy balance for another time interval, during which the ball falls from C to B. Over this time, no *external* forces act on the ball (recall we have taken the system to be the ball and the earth, so gravity is an *internal* force). Then, the work done by the external forces is zero, and the change in the total energy of the system is also zero. The diagram just shows an increase in kinetic energy at the expense of an equal decrease in potential energy.

What about the work done by the *internal* forces? Equation (21.2.6) tells us that this work is equal to the negative of the change in potential energy. In this case, the internal force is gravity, and the corresponding energy is gravitational potential energy. This change in potential energy is clearly visible in all the diagrams; however, when you add to it the change in kinetic energy, the result is always equal to the work done by the external force *only*. Put otherwise, the internal forces do not change the system's total energy, they just "redistribute" it among different kinds—as in, for instance, the last diagram in Figure 21.2.1, where you can clearly see that gravity is causing the kinetic energy of the system to increase at the expense of the potential energy.

We will use diagrams like the ones in Figure 21.2.1 to look at the work-energy balance for different systems. The idea is that the sum of all the columns on the left (the change in the system's total energy) has to equal the result on the far-right column (the work done by the net external force): that is the content of the theorem (21.2.8). Note that, unlike the energy diagrams we used in Chapter 5, these columns represent *changes* in the energy, so they could be positive or negative.

Just as for the earlier energy diagrams, the picture we get will be different, even for the same physical situation, depending on the choice of system. This is illustrated in Figure 21.2.2 below, where I have taken the same throw shown in Figure 21.2.1, but now the system I'm looking at is the ball only. This means gravity is now an external force, as is the force of the hand, and the ball only has kinetic energy. Normally one would show the sum of the work done by the two external forces on a single column, but here I have chosen to break it up into two columns for clarity.

As you can see, during the throw the hand does positive work, whereas gravity does a comparatively small amount of negative work, and the change in kinetic energy is the sum of the two. For the longer interval from A to C (second diagram), gravity continues to do negative work until all the kinetic energy of the ball is gone. For the interval from C to B, the only external force is gravity, which now does positive work, equal to the increase in the ball's kinetic energy.

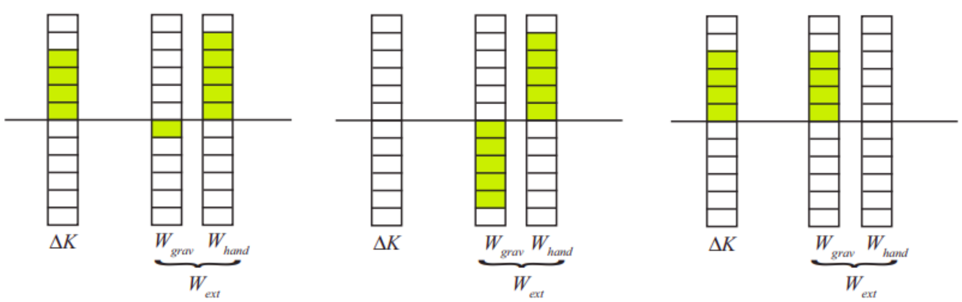

Figure 21.2.2: Work-energy balance diagrams for the same toss illustrated in Figure 21.2.1, but now the system is taken to be the ball only.

Of course, the numerical value of the actual work done by any particular force does not depend on our choice of system: in each case, gravity does the same amount of work in the processes illustrated in Figure 21.2.2 as in those illustrated in Figure 21.2.1. The difference, however, is that for the system in Figure 21.2.2, gravity is an external force, and now the work it does actually changes the system's total energy, because the gravitational potential energy is now *not* included in that total.

Formally, it works like this: in the case shown in Figure 21.2.1, where the system is the ball and the earth, we have $\Delta K + \Delta U^G = W_{hand}$. By the result (21.2.6), however, we have $\Delta U^G = -W_{grav}$, and so this equation can be rearranged to read $\Delta K = W_{grav} + W_{hand}$, which is just the result (21.2.8) when the system is the ball alone.

Ultimately, the reason we emphasize the importance of the choice of system is to prevent double counting: if you want to count the work done by gravity as contributing to the change in the system's total energy, it means that you are, implicitly, treating gravity as an external force, and therefore your system must be something that does not have, by itself, gravitational potential energy (the case of the ball in Figure $21.2.2$); conversely, if you insist on counting gravitational potential energy as contributing to the system's total energy, then you must treat gravity as an internal force, and leave it out of the calculation of the work done on the system by the external forces, which are the only ones that can change the system's total energy.

The General Case- Systems With Dissipation

We are now ready to consider what happens when some of the internal interactions in a system are not conservative. There are two key observations to keep in mind: first, of course, that energy will always be conserved in a closed system, regardless of whether the internal forces are "conservative" or not: if they are not, it merely means that they will convert some of the "organized," mechanical energy, into disorganized (primarily thermal) energy.

The second observation is that the work done by an external force on a system does not depend on where the force comes from—that is to say, what physical arrangement we use to produce the force. Only the value of the force at each step and the displacement of the point of application are involved in the definition (10.2.5). This means, in particular, that we can use a conservative interaction to do the work for us. It turns out, then, that the generalization of the result (21.2.8) to apply to all sorts of interactions becomes straightforward.

To see the idea, consider, for example, the situation in Figure $21.2.3$ below. Here I have broken it up into two systems. System A, outlined in blue, consists of block 1 and the surface on which it slides, and includes a dissipative interaction—namely, kinetic friction—between the block and the surface. The force doing work on this system is the tension force from the rope, $\vec{F}^{t}_{r,1}$.

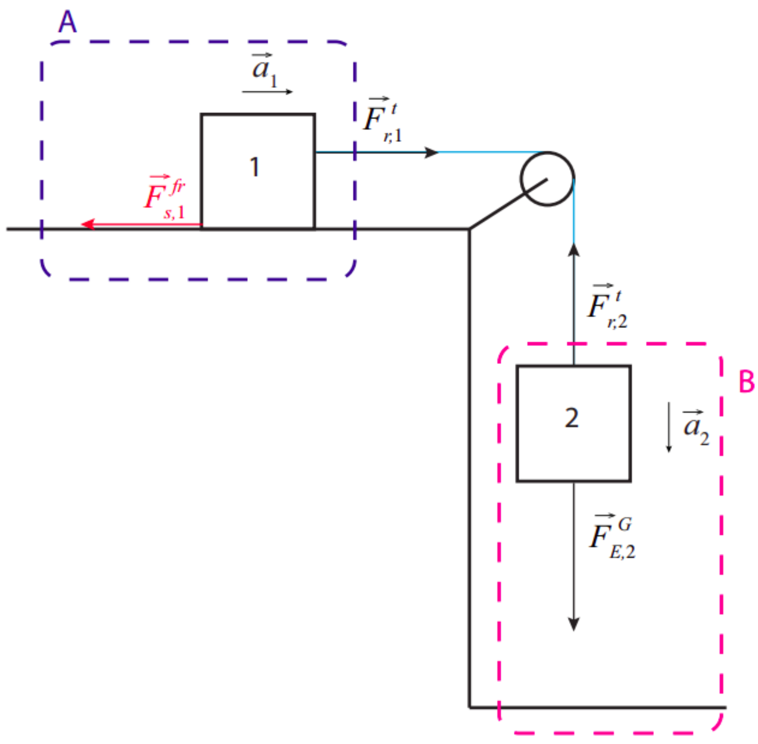

Figure $21.2.3$: Block sliding on a surface, with friction, being pulled by a rope attached to a block falling under the action of gravity. The motion of this system was solved for in Section 6.3.

Because the rope is assumed to have negligible mass, this force is the same in magnitude as the force $\vec{F}^{t}_{r,2}$ that is doing *negative* work on system B. System B, outlined in magenta, consists of block 2 and the earth and thus it includes only one internal interaction, namely gravity, which is conservative. This means that we can immediately apply the theorem (21.2.8) to it, and conclude that the work done on B by $\vec{F}^{t}_{r,2}$ is equal to the change in system B's total energy:

$$W_{r,B} = \Delta E_B. \tag{21.2.10}$$

However, since the rope is inextensible, the two blocks move the same distance in the same time, and the force exerted on each by the rope is the same in magnitude, so the work done by the rope on system A is equal in magnitude but opposite in sign to the work it does on system B:

$$W_{r,A} = -W_{r,B} = -\Delta E_B. \tag{21.2.11}$$

Now consider the *total* system formed by A+B. Assuming it is a closed system, its total energy must be constant, and so any change in the total energy of B must be equal and opposite the corresponding change in the total energy of A: $\Delta E_B = -\Delta E_A$. Therefore,

$$W_{r,A} = -\Delta E_B = \Delta E_A. \tag{21.2.12}$$

So we conclude that the work done by the external force on system A must be equal to the total change in system A's energy. In other words, Equation (21.2.8) applies to system A as well, as it does to system B, even though the interaction between the parts that make up system A is dissipative.

Although I have shown this to be true just for one specific example, the argument is quite general: if I use a conservative system B to do some work on another system A, two things happen: first, by virtue of (21.2.8), the work done by B comes at the expense of its total energy, so $W_{ext,A} = -\Delta E_B$. Second, if A and B together form a closed system, the change in A's energy must be equal and opposite the change in B's energy, so $\Delta E_A = -\Delta E_B = W_{ext,A}$. So the result (21.2.8) holds for A, regardless of whether its internal interactions are conservative or not.

What is essential in the above reasoning is that A and B together should form a closed system, that is, one that does not exchange energy with its environment. It is very important, therefore, if we want to apply the theorem (21.2.8) to a general system—that is, one that includes dissipative interactions—that we draw the boundary of the system in such a way as to ensure that *no dissipation is happening at the boundary*. For example, in the situation illustrated in Figure 21.2.3, if we want the result (21.2.12) to apply we must take system A to include both block 1 *and the surface on which it slides*. The reason for this is that the energy "dissipated" by kinetic friction when two objects rub together goes into both objects. So, as the block slides, kinetic friction is converting some of its kinetic energy into thermal energy, but not all this thermal energy stays inside block 1. Put otherwise, in the presence of friction, block 1 by itself is not a closed system: it is "leaking" energy to the surface. On the other hand, when you include (enough of) the surface in the system, you can be sure to have "caught" all the dissipated energy, and the result (21.2.8) applies.

Energy Dissipated by Kinetic Friction

In the situation illustrated in Figure 21.2.3, we might calculate the energy dissipated by kinetic friction by indirect means. For instance, we can use the fact that the energy of system A is of two kinds, kinetic and "dissipated," and therefore, by theorem (21.2.8), we have

$$\Delta K + \Delta E_{diss} = F^t_{r,1} \Delta x_1. \tag{21.2.13}$$

Back in section 6.3, we used Newton's laws to solve for the acceleration of this system and the tension in the rope; using those results, we can calculate the displacement Δx_1 over any time interval, and the corresponding change in K, and then we can solve Equation (21.2.13) for ΔE_{diss}.

If we do this, we will find out that, in fact, the following result holds,

$$\Delta E_{diss} = -F^k_{s,1} \Delta x_1 \tag{21.2.14}$$

where $F^k_{s,1}$ is the force of kinetic friction exerted by the surface on block 1, and must be understood to be negative in this equation (so that ΔE_{diss} will come out positive, as it must be).

It is tempting to think of the product $F^k_{s,1}\Delta x$ as the work done by the force of kinetic friction on the block, and most of the time there is nothing wrong with that, but it is important to realize that the "point of application" of the friction force is not a single point: rather, the force is "distributed," that is to say, spread over the whole contact area between the block and the surface. As a consequence of this, a more general expression for the energy dissipated by kinetic friction between an object o and a surface s should be

$$\Delta E_{\text{diss}} = |F^k_{s,o}||\Delta x_{so}| \tag{21.2.15}$$

where we are using the subscript notation x_{AB} to refer to "the position of B in the frame of A" (or "relative to A"); in other words, Δx_{so} is the change in the position of the object relative to the surface or, more simply, *the distance that the object and the surface slip past each other* (while rubbing against each other, and hence dissipating energy). If the surfaces is at rest (relative to the Earth), Δx_{so} reduces to Δx_{Eo}, the displacement of the object in the Earth reference frame, and we can remove the subscript E, as we typically do, for simplicity; however, in the rare cases when both the surface and the objet are moving what matters is how far they move *relative to each other*. In that case we have $|\Delta x_{so}| = |\Delta x_o - \Delta x_s|$ (with both Δx_o and Δx_s measured in the Earth reference frame).

21.3: Forces Not Derived From a Potential Energy

As we have seen in the previous section, for interactions that are associated with a potential energy, we are always able to determine the forces from the potential energy by simple differentiation. This means that we do not have to rely exclusively on an equation of the type $F = ma$, to *infer* the value of a force from the observed acceleration; rather, we can work in reverse, and *predict* the value of the acceleration (and from it all the subsequent motion) from our knowledge of the force.

I have said before that, on a microscopic level, all the interactions can be derived from potential energies, yet at the macroscopic level this is not generally true: we have many kinds of interactions for which the associated "stored" or converted energy cannot, in general, be written as a function of the macroscopic position variables for the objects making up the system (by which I mean, typically, the positions of their centers of mass). So what do we do in those cases?

The forces of this type with which we shall deal this semester actually fall into two different categories: the ones that do not dissipate energy, and that we *could*, in fact, associate with a potential energy if we wanted to[3], and the ones that definitely dissipate energy and need special handling. The former category includes the normal force, tension, and the static friction force; the second category includes the force of kinetic (or sliding) friction, and air resistance. A brief description of all these forces, and the methods to deal with them, follows.

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[3]If we wanted to complicate our life, that is...

Tensions

Tension is the force exerted by a stretched spring, and, similarly, by objects such as cables, ropes, and strings in response to a stretching force (or load) applied to them. It is ultimately an elastic force, so, as I said above, we could in principle describe it by a potential energy, but in practice cables, strings and the like are so stiff that it is often all right to neglect their change in length altogether and assume that *no* potential energy is, in fact, stored in them. The price we pay for this simplification (and it *is* a simplification) is that we are left without an independent way to determine the value of the tension in any specific case; we just have to infer it from the acceleration of the object on which it acts (since it is a reaction force, it can assume any value as required to adjust to any circumstance—up to the point where the rope snaps, anyway).

Thus, for instance, in the picture below, which shows two blocks connected by a rope over a pulley, the tension force exerted by the rope on block 1 must equal $m_1 a_1$, where a_1 is the acceleration of that block, provided there are no other horizontal forces (such as friction) acting on it. For the hanging block, on the other hand, the net force is the sum of the tension on the other end of the rope (pulling up) and gravity, pulling down. If we choose the upward direction as positive, we can write Newton's second law for the second block as

$$F^t_{r,2} - m_2 g = m_2 a_2. \tag{21.3.1}$$

Two things need to be realized now. First, if the rope is inextensible, both blocks travel the same distance in the same time, so their speeds are always the same, and hence the *magnitude* of their accelerations will always be the same as well; only the sign may be different depending on which direction we choose as positive. If we take to the right to be positive for the horizontal motion, we will have $a_2 = -a_1$. I'm just going to call $a_1 = a$, so then $a_2 = -a$.

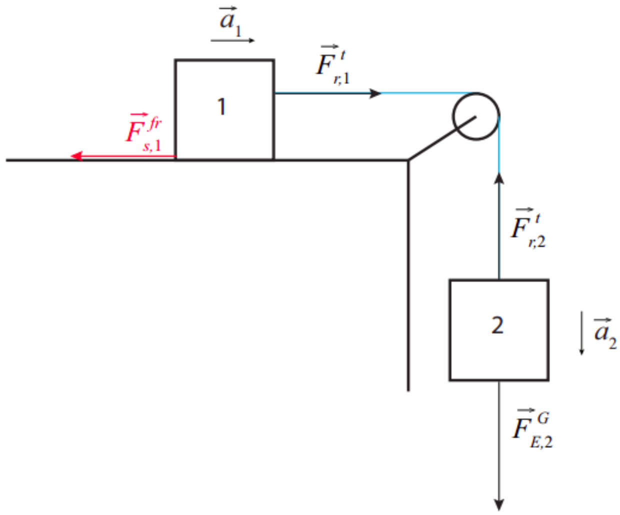


Figure 21.3.1: Two blocks joined by a massless, inextensible strength threaded over a massless pulley. An optional friction force (in red, where fr could be either s or k) is shown for use later, in the discussion below in subsection "Static and Kinetic Friction Forces." In this subsection, however, it is assumed to be zero.

The second thing to note is that, if the rope's mass is negligible, it will, like an ideal spring, pull with a force with the same *magnitude* on both ends. With our specific choices (up and to the right is positive), we then have $F^t_{r,2} = F^t_{r,1}$, and I'm just going to call this quantity F^t . All this yields, then, the following two equations:

$$\begin{aligned} F^t &= m_1 a \\ F^t - m_2 g &= -m_2 a. \end{aligned} \tag{21.3.2}$$

The system (21.3.2) can be easily solved to get

$$\begin{aligned} a &= \frac{m_2 g}{m_1 + m_2} \\ F^t &= \frac{m_1 m_2 g}{m_1 + m_2}. \end{aligned} \tag{21.3.3}$$

Normal Forces

Normal force is the reaction force with which a surface pushes back when it is being pushed on. Again, this works very much like an extremely stiff spring, this time under compression instead of tension. And, again, we will eschew the potential energy treatment by assuming that the surface's actual displacement is entirely negligible, and we will just calculate the value of F^n as whatever is needed in order to make Newton's second law work. Note that this force will always be perpendicular to the surface, by definition (the word "normal" means "perpendicular" here); the task of dealing with a sideways push on the surface will be delegated to the static friction force, to be covered next.

If I am just standing on the floor and not falling through it, the net vertical force acting on me must be zero. The force of gravity on me is mg downwards, and so the upwards normal force must match this value, so for this situation $F^n = mg$. But don't get too attached to the notion that the normal force must always be equal to mg, since this will often not be the case. Imagine, for instance, a person standing inside an elevator at the time it is accelerating upwards. With the upwards direction as positive, Newton's second law for the person reads

$$F^n - mg = ma \tag{21.3.4}$$

and therefore for this situation

$$F^n = mg + ma. \tag{21.3.5}$$

If you were weighing yourself on a bathroom scale in the elevator, this is the upwards force that the bathroom scale would have to exert on you, and it would do that by compressing a spring inside, and it would record the "extra" compression (beyond that required by your actual weight, mg) as extra weight. Conversely, if the elevator were accelerating downward, the scale would record you as being lighter. In the extreme case in which the cable of the elevator broke and you, the elevator and the scale ended up (briefly, before the emergency brake caught on) in free fall, you would all be falling with the same acceleration, you would not be pushing down on the scale at all, and its normal force as well as your recorded weight would be zero. This is ultimately the reason for the apparent weightlessness experienced by the astronauts in the space station, where the force of gravity is, in fact, not very much smaller than on the surface of the earth. (We will return to this effect after we have a good grip on two-dimensional, and in particular circular, motion.)

Static and Kinetic Friction Forces

The *static friction* force is a force that prevents two surfaces in contact from slipping relative to each other. It is an extremely useful force, since we would not be able to drive a car, or ride a bicycle, or even walk, without it—as we know from experience, if we have ever tried to do any of those things on a low-friction surface (such as a sheet of ice).

The science behind friction (known technically as *tribology*) is actually not very simple at all, and it is of great current interest for many reasons—whether the ultimate goal is to develop ways to reduce friction or to increase it. On an elementary level, we are all aware of the fact that even a surface that looks smooth on a macroscopic scale will actually exhibit irregularities, such as ridges and valleys, under a microscope. It makes sense, then, that when two such surfaces are pressed together, the bumps on one of them will hit, and be held in place by, the bumps on the other one, and that will prevent sliding until and unless a sufficient force is applied to temporarily "flatten" the bumps enough to allow the thing to move[4].

As long as this does not happen, that is, as long as the surfaces do *not* slide relative to each other, we say we are dealing with the *static* friction force, which is, at least approximately, an elastic force that does not dissipate energy: the small distortion of the "bumps" on the surfaces that takes place when you push on them typically happens slowly enough, and is small enough, to be reversible, so that when you stop pushing the two surfaces just go back to their initial state. This is no longer the case once the surfaces start sliding relative to each other. At that point the character of the friction force changes, and we have to deal with the *sliding*, or *kinetic* friction force, as I will explain below.

The static friction force is also, like tension and the normal force, a reaction force that will adjust itself, within limits, to take any value required to prevent slippage in a given circumstance. Hence, its actual value in a particular situation cannot really be ascertained until the other relevant forces— the other forces pushing or pulling on the object—are known.

For instance, for the system in Figure $21.3.1$, imagine there is a force of static friction between block 1 and the surface on which it rests, sufficiently large to keep it from sliding altogether. How large does this have to be? If there is no acceleration (a = 0), the equivalent of system (21.3.2) will be

$$\begin{aligned} &F_{s,1}^{s}+F^{t}=0 \\ &F^{t}-m_{2}g=0 \end{aligned} \tag{21.3.6}$$

where $F^s_{s,1}$ is the force of static friction exerted by the surface on block 1, and we are going to let the math tell us what sign it is supposed to have. Solving the system (21.3.6) we just get the condition

$$F_{s,1}^{s}=-m_{2}g \tag{21.3.7}$$

so this is how large $F^s_{s,1}$ has to be in order to keep the whole system from moving in this case.

There is an empirical formula that tells us approximately how large the force of static friction *can* get in a given situation. The idea behind it is that, microscopically, the surfaces are in contact only near the top of their respective ridges. If you press them together harder, some of the ridges get flattened and the effective contact area increases; this in turn makes the surfaces more resistant to slippage. A direct measure of how strongly the two surfaces press against each other is, actually, just the normal force they exert on each other. So, in general, we expect the maximum force that static friction will be able to exert to be proportional to the *normal* force between the surfaces:

$$\left|F_{s1,s2}^{s}\right|_{\max}=\mu_{s}\left|F_{s1,s2}^{n}\right| \tag{21.3.8}$$

where s_1 and s_2 just mean "surface 1" and "surface 2," respectively, and the number μ_s is known as the *coefficient of static friction*: it is a tabulated quantity that is determined experimentally, by testing the slippage of different surfaces against each other under different loads.

In our example, the normal force exerted by the surface on block 1 has to be equal to m_1g, since there is no vertical acceleration for that block, and so the maximum value that F^s may have in this case is $\mu_s m_1 g$, whatever μ_s might happen to be. In fact, this setup would give us a way to determine μ_s for these two surfaces: start with a small value of m_2, and gradually increase it until the system starts moving. At that point we will know that $m_2 g$ has just exceeded the maximum possible value of $|F^s_{12}|$, namely, $\mu_s m_1 g$, and so $\mu_s = (m_2)_{max}/m_1$, where $(m_2)_{max}$ is the largest mass we can hang before the system starts moving.

By contrast with all of the above, the *kinetic friction* force, which always acts so as to oppose the relative motion of the two surfaces when they are actually slipping, is not elastic, it is definitely dissipative, and, most interestingly, it is also *not* much of a reactive force, meaning that its value can be approximately predicted for any given circumstance, and does not depend much on things such as how fast the surfaces are actually moving relative to each other. It *does* depend on how hard the surfaces are pressing against each other, as quantified by the normal force, and on another tabulated quantity known as the *coefficient of kinetic friction*:

$$\left|F^k_{s1,s2}\right| = \mu_k \left|F^n_{s1,s2}\right| \tag{21.3.9}$$

Note that, unlike for static friction, this is *not* the maximum possible value of $|F^k|$, but its *actual* value; so if we know F^n (and μ_k) we know F^k without having to solve any other equations (its sign does depend on the direction of motion, of course). The coefficient μ_k is typically a little smaller than μ_s, reflecting the fact that once you get something you have been pushing on to move, keeping it in motion with constant velocity usually does not require the same amount of force.

To finish off with our example in Figure $21.3.1$, suppose the system *is* moving, and there is a kinetic friction force $F^k_{s,1}$ between block 1 and the surface. The equations (21.3.2) then have to be changed to

$$\begin{aligned} F^t - \mu_k m_1 g &= m_1 a \\ F^t - m_2 g &= -m_2 a \end{aligned} \tag{21.3.10}$$

and the solution now is

$$\begin{aligned} a &= \frac{m_2 - \mu_k m_1}{m_1 + m_2} g \\ F^t &= \frac{m_1 m_2 \left(1 + \mu_k\right)}{m_1 + m_2} g \end{aligned} \tag{21.3.11}$$

You may ask, why does kinetic friction dissipate energy? A qualitative answer is that, as the surfaces slide past each other, their small (sometimes microscopic) ridges are constantly “bumping” into each other; so you have lots of microscopic collisions happening all the time, and they cannot all be perfectly elastic. So mechanical energy is being “lost.” In fact, it is primarily being converted to thermal energy, as you can verify experimentally: this is why you rub your hands together to get warm, for instance. More dramatically, this is how some people (those who really know what they are doing!) can actually start a fire by rubbing sticks together.

[4]This picture based, essentially, on classical physics, leaves out an atomic-scale effect that may be important in some cases, which is the formation of weak bonds between the atoms of both surfaces, resulting in an actual “adhesive” force. This is, for instance, how geckos can run up vertical walls. For our purposes, however, the classical picture (of small ridges and valleys bumping into each other) will suffice to qualitatively understand all the examples we will cover this semester.

Air Resistance

Air resistance is an instance of fluid resistance or *drag*, a force that opposes the motion of an object through a fluid. Microscopically, you can think of it as being due to the constant collisions of the object with the air molecules, as it cleaves its way through the air. As a result of these collisions, some of its momentum is transferred to the air, as well as some of its kinetic energy, which ends up as thermal energy (as in the case of kinetic friction discussed above). The very high temperatures that air resistance can generate can be seen, in a particularly dramatic way, on the re-entry of spacecraft into the atmosphere.

Unlike kinetic friction between solid surfaces, the fluid drag force does depend on the velocity of the object (relative to the fluid), as well as on a number of other factors having to do with the object’s shape and the fluid’s density and viscosity. Very roughly speaking, for low velocities the drag force is proportional to the object’s speed, whereas for high velocities it is proportional to the square of the speed.

In principle, one can use the appropriate drag formula together with Newton’s second law to calculate the effect of air resistance on a simple object thrown or dropped; in practice, this requires a somewhat more advanced math than we will be using this course, and the formulas themselves are complicated, so I will not introduce them here.

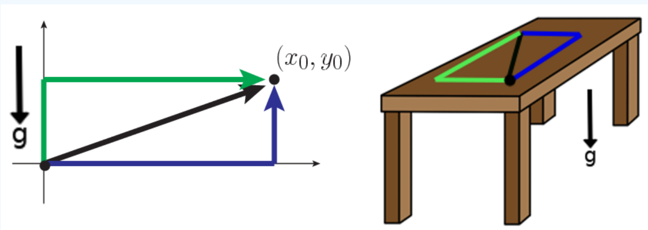
One aspect of air resistance that deserves to be mentioned is what is known as “terminal velocity”. Since air resistance increases with speed, if you drop an object from a sufficiently great height, the upwards drag force on it will increase as it accelerates, until at some point it will become as large as the downward force of gravity. At that point, the net force on the object is zero, so it stops accelerating, and from that point on it continues to fall with constant velocity. When the Greek philosopher Aristotle was trying to figure out the motion of falling bodies, he reasoned that, since air was just another fluid, he could slow down the fall (in order to study it better) without changing the physics by dropping objects in liquids instead of air. The problem with this approach, though, is that terminal velocity is reached much faster in a liquid than in air, so Aristotle missed entirely the early stage of approximately constant acceleration, and concluded (wrongly) that the natural way all objects fell was with constant velocity. It took almost two thousand years until Galileo disproved that notion by coming up with a better method to slow down the falling motion—namely, by using inclined planes.

21.4: Examples

Problem-Solving Strategy: Work-Energy Theorem

1. Draw a free-body diagram for each force on the object.
2. Determine whether or not each force does work over the displacement in the diagram. Be sure to keep any positive or negative signs in the work done.
3. Add up the total amount of work done by each force.
4. Set this total work equal to the change in kinetic energy and solve for any unknown parameter.
5. Check your answers. If the object is traveling at a constant speed or zero acceleration, the total work done should be zero and match the change in kinetic energy. If the total work is positive, the object must have sped up or increased kinetic energy. If the total work is negative, the object must have slowed down or decreased kinetic energy

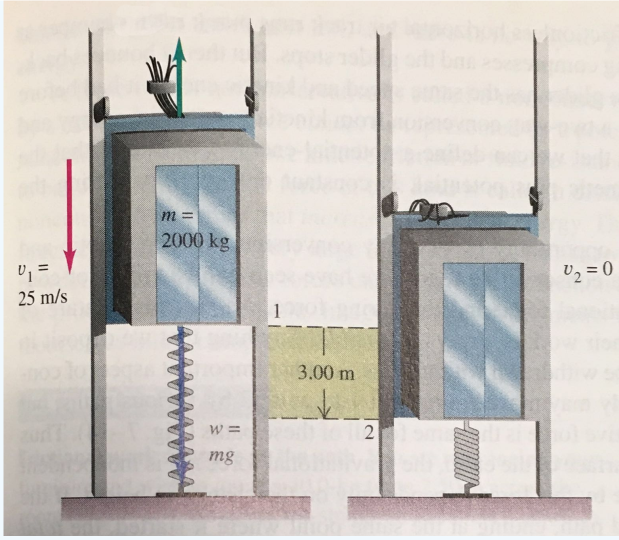
Whiteboard Problem 21.4.1: Conservative vs Nonconservative



Consider moving an object, of mass m = 8.75 kg, from the origin to point $(x_0, y_0) = (7.2, 3.5)$ m, along three different paths as shown in the figure.

1. First assume the only force acting on the object is gravity, which is directed downwards (so $\vec{g} = -g\hat{y}$). Find the work done by the gravitational force on the object for each of the three paths in the figure on the left.
2. Now consider dragging this object around the surface of a table, as shown in the figure on the right. If the coefficient of friction between the table and the particle is 0.15, find the work done by friction for each path.

Whiteboard Problem 21.4.2: Terror on an Elevator



You have been tasked with designing a ``worst-case scenario" safety system for an elevator. If this 2000-kg elevator's cables break at the top of it's tower, it will be moving with a speed of 25 m/s when it reaches the bottom. Your job is to determine the parameters required if this elevator will be stopped by brakes or a safety spring, over a distance of 3.00 m.

1. Using brakes that apply a constant force, how much force would the brakes have to apply to stop the elevator?
2. Using a spring, what would the spring constant need to be to stop this elevator?
3. What if you used both? If the brakes apply a force of 17 kN, what would the spring constant need to be?

✔ Example 21.4.3: Loop-the-Loop

The frictionless track for a toy car includes a loop-the-loop of radius R. How high, measured from the bottom of the loop, must the car be placed to start from rest on the approaching section of track and go all the way around the loop?

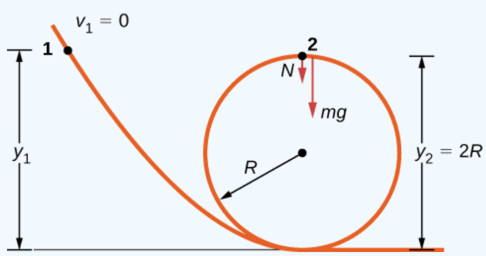


Figure 21.4.2: A frictionless track for a toy car has a loop-the-loop in it. How high must the car start so that it can go around the loop without falling off?

Strategy

The free-body diagram at the final position of the object is drawn in Figure $21.4.2$ The gravitational work is the only work done over the displacement that is not zero. Since the weight points in the same direction as the net vertical displacement, the total work done by the gravitational force is positive. From the work-energy theorem, the starting height determines the speed of the car at the top of the loop,

$$mg(y_2 - y_1) = \frac{1}{2}mv_2^2,$$

where the notation is shown in the accompanying figure. At the top of the loop, the normal force and gravity are both down and the acceleration is centripetal, so

$$a_{top} = \frac{F}{m} = \frac{N + mg}{m} = \frac{v_2^2}{R}.$$

The condition for maintaining contact with the track is that there must be some normal force, however slight; that is, $N > 0$. Substituting for v_2^2 and N, we can find the condition for y_1.

Solution

Implement the steps in the strategy to arrive at the desired result:

$$N = -mg + \frac{mv_2^2}{R} = \frac{-mgR + 2mg(y_1 - 2R)}{R} > 0 \ or \ y_1 > \frac{5R}{2}.$$

Significance

On the surface of the loop, the normal component of gravity and the normal contact force must provide the centripetal acceleration of the car going around the loop. The tangential component of gravity slows down or speeds up the car. A child would find out how high to start the car by trial and error, but now that you know the work-energy theorem, you can predict the minimum height (as well as other more useful results) from physical principles. By using the work-energy theorem, you did not have to solve a differential equation to determine the height.

Exercise 21.4.4

Suppose the radius of the loop-the-loop in Example 21.4.1 is 15 cm and the toy car starts from rest at a height of 45 cm above the bottom. What is its speed at the top of the loop?

In situations where the motion of an object is known, but the values of one or more of the forces acting on it are not known, you may be able to use the work-energy theorem to get some information about the forces. Work depends on the force and the distance over which it acts, so the information is provided via their product.

Example 21.4.5: Determining a Stopping Force

A bullet has a mass of 40 grains (2.60 g) and a muzzle velocity of 1100 ft./s (335 m/s). It can penetrate eight 1-inch pine boards, each with thickness 0.75 inches. What is the average stopping force exerted by the wood, as shown in Figure 21.4.3?

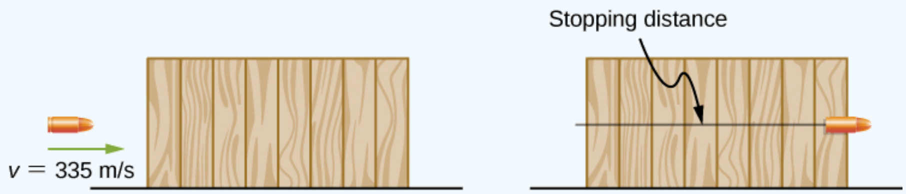


Figure 21.4.3: The boards exert a force to stop the bullet. As a result, the boards do work and the bullet loses kinetic energy

Strategy

We can assume that under the general conditions stated, the bullet loses all its kinetic energy penetrating the boards, so the work-energy theorem says its initial kinetic energy is equal to the average stopping force times the distance penetrated. The change in the bullet's kinetic energy and the net work done stopping it are both negative, so when you write out the work-energy theorem, with the net work equal to the average force times the stopping distance, that's what you get. The total thickness of eight 1-inch pine boards that the bullet penetrates is 8 x $\frac{3}{4}$ in. = 6 in. = 15.2 cm.

Solution

Applying the work-energy theorem, we get

$$W_{net} = -F_{ave}\Delta s_{stop} = -K_{initial},$$

so

$$F_{ave} = \frac{\frac{1}{2}mv^2}{\Delta s_{stop}} = \frac{\frac{1}{2}(2.66 \times 10^{-3}\ kg)(335\ m/s)^2}{0.152\ m} = 960\ N.$$

Significance

We could have used Newton's second law and kinematics in this example, but the work-energy theorem also supplies an answer to less simple situations. The penetration of a bullet, fired vertically upward into a block of wood, is discussed in one section of Asif Shakur's recent article ["Bullet-Block Science Video Puzzle." **The Physics Teacher** (January 2015) 53(1): 15-16]. If the bullet is fired dead center into the block, it loses all its kinetic energy and penetrates slightly farther than if fired off-center. The reason is that if the bullet hits off-center, it has a little kinetic energy after it stops penetrating, because the block rotates. The work-energy theorem implies that a smaller change in kinetic energy results in a smaller penetration.

Learn more about work and energy in this PhET simulation (https://phet.colorado.edu/en/simulation/the-ramp) called "the ramp." Try changing the force pushing the box and the frictional force along the incline. The work and energy plots can be examined to note the total work done and change in kinetic energy of the box.

Example 21.4.6: Work, energy and the choice of system- dissipative case

Consider again the situation shown in Figure 21.3.1. Let m_1 = 1 kg, m_2 = 2 kg, and μ_k = 0.3. Use the solutions provided in section 21.3 to calculate the work done by all the forces, and the changes in all energies, when the system undergoes a displacement of 0.5 m, and represent the changes graphically using bar diagrams like the ones in Figure 21.2.2 (for system A and B separately)

Solution

From Equation (21.3.11), we have

$$\begin{aligned} a &= \frac{m_2 - \mu_k m_1}{m_1 + m_2} g = 5.55 \ \frac{\mathrm{m}}{\mathrm{s}^2} \\ F^t &= \frac{m_1 m_2 \left(1 + \mu_k\right)}{m_1 + m_2} g = 8.49 \ \mathrm{N} \end{aligned} \tag{21.4.1}$$

We can use the acceleration to calculate the change in kinetic energy, since we have Equation (15.2.10) for motion with constant acceleration:

$$v_f^2 - v_i^2 = 2a\Delta x = 2 \times \left(5.55 \ \frac{\mathrm{m}}{\mathrm{s}^2}\right) \times 0.5 \ \mathrm{m} = 5.55 \ \frac{\mathrm{m}^2}{\mathrm{s}^2} \tag{21.4.2}$$

so the change in kinetic energy of the two blocks is

$$\begin{aligned} \Delta K_1 &= \frac{1}{2} m_1 \left(v_f^2 - v_i^2\right) = 2.78 \ \mathrm{J} \\ \Delta K_2 &= \frac{1}{2} m_2 \left(v_f^2 - v_i^2\right) = 5.55 \ \mathrm{J}. \end{aligned} \tag{21.4.3}$$

We can also use the tension to calculate the work done by the external force on each system:

$$\begin{aligned} W_{ext,A} &= F_{r,1}^t \Delta x = (8.49 \ \mathrm{N}) \times (0.5 \ \mathrm{m}) = 4.25 \ \mathrm{J} \\ W_{ext,B} &= F_{r,2}^t \Delta y = (8.49 \ \mathrm{N}) \times (-0.5 \ \mathrm{m}) = -4.25 \ \mathrm{J}. \end{aligned} \tag{21.4.4}$$

Lastly, we need the change in the gravitational potential energy of system B:

$$\Delta U_B^G = m_2 g \Delta y = (2 \ \mathrm{kg}) \times \left(9.8 \ \frac{\mathrm{m}}{\mathrm{s}^2}\right) \times (-0.5 \ \mathrm{m}) = -9.8 \ \mathrm{J} \tag{21.4.5}$$

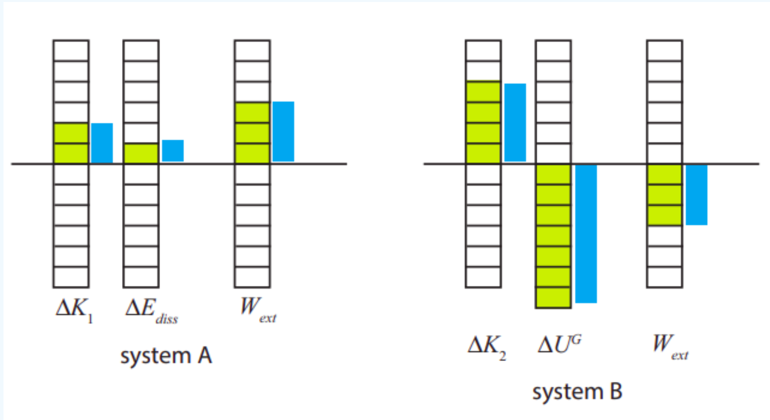
and the increase in dissipated energy in system A, which we can get from Equation (21.2.16):

$$\Delta E_{\text{diss}} = -F_{s,1}^k \Delta x = \mu_k F_{s,1}^n \Delta x = \mu_k m_1 g \Delta x = 0.3 \times (1 \ \mathrm{kg}) \times \left(9.8 \ \frac{\mathrm{m}}{\mathrm{s}^2}\right) \times (0.5 \ \mathrm{m}) = 1.47 \ \mathrm{J}. \tag{21.4.6}$$

We can now put all this together to show that Equation (21.2.8) indeed holds:

$$\begin{aligned} W_{ext,A} &= \Delta E_A = \Delta K_1 + \Delta E_{diss} = 2.78 \ \mathrm{J} + 1.47 \ \mathrm{J} = 4.25 \ \mathrm{J} \\ W_{ext,B} &= \Delta E_B = \Delta K_2 + \Delta U_B^G = 5.55 \ \mathrm{J} - 9.8 \ \mathrm{J} = -4.25 \ \mathrm{J}. \end{aligned} \tag{21.4.7}$$

To plot all this as energy bars, if you do not have access to a very precise drawing program, you typically have to make some approximations. In this case, we see that $\Delta K_2 = 2\Delta K_1$ (exactly), whereas $\Delta K_1 \simeq 2\Delta E_{diss}$, so we can use one box to represent E_{diss}, two boxes for ΔK_1, three for $W_{ext,A}$, four for ΔK_2, and so on. The result is shown in green in the picture below; the blue bars have been drawn more exactly to scale, and are shown for your information only.



Example 21.4.7: Work, energy and the choice of system- non-dissipative case

Suppose you hang a spring from the ceiling, then attach a block to the end of the spring and let go. The block starts swinging up and down on the spring. Consider the initial time just before you let go, and the final time when the block momentarily stops at the bottom of the swing. For each of the choices of a system listed below, find the net energy change of the system in this process, and relate it explicitly to the work done on the system by an external force (or forces)

a. System is the block and the spring.
b. System is the block alone.
c. System is the block and the earth.

Solution

(a) The block alone has kinetic energy, and the spring alone has (elastic) potential energy, so the total energy of this system is the sum of these two. For the interval considered, the change in kinetic energy is zero, because the block starts and ends (momentarily) at rest, so only the spring energy changes. This has to be equal to the work done by gravity, which is the only external force.

So, if the spring stretches a distance d, its potential energy goes from zero to $\frac{1}{2}kd^2$, and the block falls the same distance, so gravity does an amount of work equal to mgd, and we have

$$W_{grav} = mgd = \Delta E_{sys} = \Delta K + \Delta U^{spr} = 0 + \frac{1}{2}kd^2. \tag{21.4.8}$$

(b) If the system is the block alone, the only energy it has is kinetic energy, which, as stated above, does not see a net change in this process. This means the net work done on the block by the external forces must be zero. The external forces in this case are the spring force and gravity, so we have

$$W_{spr} + W_{grav} = \Delta K = 0. \tag{21.4.9}$$

We have calculated W_{grav} above, so from this we get that the work done by the spring on the block, as it stretches, is $-mgd$, or (by Equation (21.4.8)) $-\frac{1}{2}kd^2$. Note that the force exerted by the spring is *not* constant as it stretches (or compresses) so we cannot just use Equation (9.2.1) to calculate it; rather, we need to calculate it as an integral, as in Equation (21.2.5), or derive it in some indirect way as we have just done here.

(c) If the system is the block and the earth, it has kinetic energy and gravitational potential energy. The force exerted by the spring is an external force now, so we have:

$$W_{spr} = \Delta E_{sys} = \Delta K + \Delta U^G = 0 - mgd \tag{21.4.10}$$

so we end up again with the result that $W_{spr} = -mgd = -\frac{1}{2}kd^2$. Note that both the work done by the spring and the work done by gravity are equal to the negative of the changes in their respective potential energies, as they should be.

Example 21.4.8: Jumping

For a standing jump, you start standing straight (A) so your body's center of mass is at a height h_1 above the ground. You then bend your knees so your center of mass is now at a (lower) height h_2 (B). Finally, you straighten your legs, pushing hard on the ground, and take off, so your center of mass ends up achieving a maximum height, h_3, above the ground (C). Answer the following questions in as much detail as you can.

a. Consider the system to be your body only. In going from (A) to (B), which external forces are acting on it? How do their magnitudes compare, as a function of time?
b. In going from (A) to (B), does any of the forces you identified in part (a) do work on your body? If so, which one, and by how much? Does your body's energy increase or decrease as a result of this? Into what kind of energy do you think this work is primarily converted?
c. In going from (B) to (C), which external forces are acting on you? (Not all of them need to be acting all the time.) How do their magnitudes compare, as a function of time?
d. In going from (B) to (C), does any of the forces you identified do work on your body? If so, which one, and by how much? Does your body's kinetic energy see a net change from (B) to (C)? What other energy change needs to take place in order for Equation (21.2.8) (always with your body as the system) to be valid for this process?

Solution

(a) The external forces on your body are gravity, pointing down, and the normal force from the floor, pointing up. Initially, as you start lowering your center of mass, the normal force has to be slightly smaller than gravity, since your center of mass acquires a small downward acceleration. However, eventually F^n would have to exceed F^G in order to stop the downward motion.

(b) The normal force does no work, because its point of application (the soles of your feet) does not move, so Δx in the expression $W = F\Delta x$ (Equation (9.2.1)) is zero.

Gravity, on the other hand, does positive work, since you may always treat the center of mass as the point of application of gravity. We have $F^G_y = -mg$, and $\Delta y = h_2 - h_1$, so

$$W_{grav} = F^G_y \Delta y = -mg\left(h_2 - h_1\right) = mg\left(h_1 - h_2\right).$$

Since this is the net work done by all the external forces on my body, and it is positive, the total energy in my body must have increased (by the theorem ((21.2.8)): $W_{ext,sys} = \Delta E_{sys}$). In this case, it is clear that the main change has to be an increase in my body's *elastic potential energy*, as my muscles tense for the jump. (An increase in thermal energy is always possible too.)

(c) During the jump, the external forces acting on me are again gravity and the normal force, which together determine the acceleration of my center of mass. At the beginning of the jump, the normal force has to be much stronger than gravity, to give me a large upwards acceleration. Since the normal force is a reaction force, I accomplish this by pushing very hard with my feet on the ground, as I extend my leg's muscles: by Newton's third law, the ground responds with an equal and opposite force upwards.

As my legs continue to stretch, and move upwards, the force they exert on the ground decreases, and so does F^n, which eventually becomes less than F^G. At that point (probably even before my feet leave the ground) the acceleration of my center of mass becomes negative (that is, pointing down). This ultimately causes my upwards motion to stop, and my body to come down.

(d) The only force that does work on my body during the process described in (c) is gravity, since, again, the point of application of F^n is the point of contact between my feet and the ground, and that point does not move up or down—it is always level with the ground. So $W_{ext,sys} = W_{grav}$, which in this case is actually *negative*: $W_{grav} = -mg(h_3 - h_2)$.

In going from (B) to (C), there is no change in your kinetic energy, since you start at rest and end (momentarily) with zero velocity at the top of the jump. So the fact that there is a net negative work done on you means that the energy inside your body must have gone down. Clearly, some of this is just a decrease in elastic potential energy. However, since h_3 (the final height of your center of mass) is greater than h_1 (its initial height at (A), before crouching), there is a *net* loss of energy in your body as a result of the whole process. The most obvious place to look for this loss is in chemical energy: you "burned" some calories in the process, primarily when pushing hard against the ground.

Example 21.4.9: analyzing a car crash

At a stoplight, a large truck (3000 kg) collides with a motionless small car (1200 kg). The truck comes to an instantaneous stop; the car slides straight ahead, coming to a stop after sliding 10 meters. The measured coefficient of friction between the car's tires and the road was 0.62. How fast was the truck moving at the moment of impact?

Strategy

At first it may seem we don't have enough information to solve this problem. Although we know the initial speed of the car, we don't know the speed of the truck (indeed, that's what we're asked to find), so we don't know the initial momentum of the system. Similarly, we know the final speed of the truck, but not the speed of the car immediately after impact. The fact that the car eventually slid to a speed of zero doesn't help with the final momentum, since an external friction force caused that. Nor can we calculate an impulse, since we don't know the collision time, or the amount of time the car slid before stopping. A useful strategy is to impose a restriction on the analysis.

Suppose we define a system consisting of just the truck and the car. The momentum of this system isn't conserved, because of the friction between the car and the road. But if we could find the speed of the car the instant after impact—before friction had any measurable effect on the car—then we could consider the momentum of the system to be conserved, with that restriction.

Can we find the final speed of the car? Yes; we invoke the work-kinetic energy theorem.

Solution

First, define some variables. Let:

- M_c and M_T be the masses of the car and truck, respectively
- $v_{T,i}$ and $v_{T,f}$ be the velocities of the truck before and after the collision, respectively
- $v_{c,i}$ and $v_{c,f}$ be the velocities of the car before and after the collision, respectively
- K_i and K_f be the kinetic energies of the car immediately after the collision, and after the car has stopped sliding (so $K_f = 0$).
- d be the distance the car slides after the collision before eventually coming to a stop.

Since we actually want the initial speed of the truck, and since the truck is not part of the work-energy calculation, let's start with conservation of momentum. For the car + truck system, conservation of momentum reads

$$\begin{aligned} p_i &= p_f \\ M_c v_{c,i} + M_T v_{T,i} &= M_c v_{c,f} + M_T v_{T,f}. \end{aligned}$$

Since the car's initial velocity was zero, as was the truck's final velocity, this simplifies to

$$v_{T,i} = \frac{M_c}{M_T} v_{c,f}. \tag{21.4.11}$$

So now we need the car's speed immediately after impact. Recall that

$$W = \Delta K \tag{21.4.12}$$

where

$$\begin{aligned} \Delta K &= K_f - K_i \\ &= 0 - \frac{1}{2} M_c v_{c,f}^2. \end{aligned}$$

Also,

$$W = \vec{F} \cdot \vec{d} = F d \cos\theta. \tag{21.4.13}$$

The work is done over the distance the car slides, which we've called d. Equating:

$$F d \cos\theta = -\frac{1}{2} M_c v_{c,f}^2. \tag{21.4.14}$$

Friction is the force on the car that does the work to stop the sliding. With a level road, the friction force is

$$F = \mu_k M_c g. \tag{21.4.15}$$

Since the angle between the directions of the friction force vector and the displacement d is 180°, and cos(180°) = –1, we have

$$-(\mu_k M_c g)d = -\frac{1}{2} M_c v_{c,f}^2 \tag{21.4.16}$$

(Notice that the car's mass divides out; evidently the mass of the car doesn't matter.)

Solving for the car's speed immediately after the collision gives

$$v_{c,f} = \sqrt{2\mu_k g d}. \tag{21.4.17}$$

Substituting the given numbers:

$$\begin{aligned} v_{c,f} &= \sqrt{2(0.62)(9.81\ m/s^2)(10\ m)} \\ &= 11.0\ m/s. \end{aligned}$$

Now we can calculate the initial speed of the truck:

$$v_{T,i} = \left(\frac{1200\ kg}{3000\ kg}\right)(11.0\ m/s) = 4.4\ m/s. \tag{21.4.18}$$

Significance

This is an example of the type of analysis done by investigators of major car accidents. A great deal of legal and financial consequences depend on an accurate analysis and calculation of momentum and energy.

CHAPTER OVERVIEW

22: N9) Rotational Motion

22.1: Rotational Variables

We have already talked about angular velocity in Chapter 6, so here we will review it, provide a few more details that we will need, and then move on to the discuss the notion of angular acceleration.

Angular Velocity

Uniform circular motion (discussed previously in Circular Motion) is motion in a circle at constant speed. Although this is the simplest case of rotational motion, it is very useful for many situations, and we use it here to introduce rotational variables.

In Figure 22.1.1, we show a particle moving in a circle. The coordinate system is fixed and serves as a frame of reference to define the particle's position. Its position vector from the origin of the circle to the particle sweeps out the angle θ, which increases in the counterclockwise direction as the particle moves along its circular path. The angle θ is called the **angular position** of the particle. As the particle moves in its circular path, it also traces an arc length s.

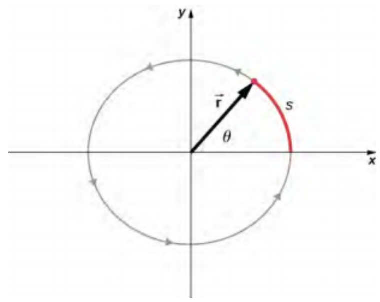


Figure 22.1.1: A particle follows a circular path. As it moves counterclockwise, it sweeps out a positive angle θ with respect to the x-axis and traces out an arc length s.

The angle is related to the radius of the circle and the arc length by

$$\theta=\frac{s}{r}. \tag{22.1.1}$$

We can assign vectors to the quantities in Equation 22.1.1. The angle $\vec{\theta}$ is a vector out of the page in Figure 22.1.1. The angular position vector $\vec{r}$ and the arc length $\vec{s}$ both lie in the plane of the page. These three vectors are related to each other by

$$\vec{s}=\vec{\theta}\times\vec{r}. \tag{22.1.2}$$

That is, the arc length is the cross product of the angle vector and the position vector, as shown in Figure 22.1.2.

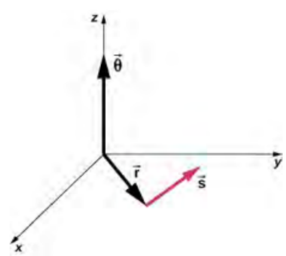


Figure 22.1.2: The angle vector points along the z-axis and the position vector and arc length vector both lie in the xy-plane. We see that $\vec{s}=\vec{\theta}\times\vec{r}$. All three vectors are perpendicular to each other.

The magnitude of the angular velocity, denoted by ω, is the time rate of change of the angle θ as the particle moves in its circular path. The instantaneous angular velocity is defined as the limit in which $\Delta t \rightarrow 0$ in the average angular velocity $\bar{\omega}=\frac{\Delta\theta}{\Delta t}$:

$$\omega=\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}=\frac{d\theta}{dt}, \tag{22.1.3}$$

where θ is the angle of rotation (Figure 22.1.2).

We can see how angular velocity is related to the tangential speed of the particle by differentiating Equation 22.1.1 with respect to time. We rewrite Equation 22.1.1 as

$$s=r\theta. \tag{22.1.4}$$

Taking the derivative with respect to time and noting that the radius r is a constant, we have

$$\frac{ds}{dt}=\frac{d}{dt}(r\theta)=\theta\frac{dr}{dt}+r\frac{d\theta}{dt}=r\frac{d\theta}{dt} \tag{22.1.5}$$

where $\theta\frac{dr}{dt}$ = 0. Here, $\frac{ds}{dt}$ is just the tangential speed v_t of the particle in Figure 22.1.1. Thus, by using Equation 22.1.3, we arrive at an expression we first encounted in Section 6.1:

$$v_t=r\omega. \tag{22.1.6}$$

That is, the tangential speed of the particle is its angular velocity times the radius of the circle. From Equation 22.1.6, we see that the tangential speed of the particle increases with its distance from the axis of rotation for a constant angular velocity. This effect is shown in Figure 22.1.3. Two particles are placed at different radii on a rotating disk with a constant angular velocity. As the disk rotates, the tangential speed increases linearly with the radius from the axis of rotation. In Figure 22.1.3, we see that $v_1 = r_1\omega_1$ and $v_2 = r_2\omega_2$. But the disk has a constant angular velocity, so $\omega_1=\omega_2$. This means $\frac{v_1}{r_1}=\frac{v_2}{r_2}$ or $v_2 = \left(\frac{r_2}{r_1}\right)v_1$. Thus, since $r_2 > r_1$, $v_2 > v_1$.

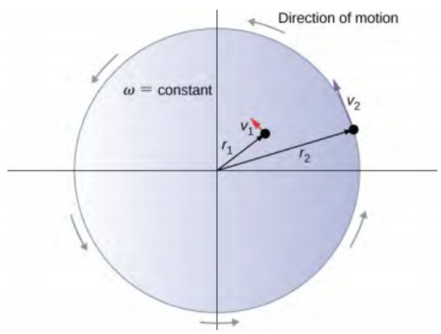


Figure 22.1.3: Two particles on a rotating disk have different tangential speeds, depending on their distance to the axis of rotation.

Up until now, we have discussed the magnitude of the angular velocity $\omega=\frac{d\theta}{dt}$, which is a scalar quantity—the change in angular position with respect to time. The vector $\vec{\omega}$ is the vector associated with the angular velocity and points along the axis of rotation. This is useful because when a rigid body is rotating, we want to know both the axis of rotation and the direction that the body is rotating about the axis, clockwise or counterclockwise. The angular velocity $\vec{\omega}$ gives us this information. The angular velocity $\vec{\omega}$ has a direction determined by what is called the right-hand rule. The right-hand rule is such that if the fingers of your right hand wrap counterclockwise from the x-axis (the direction in which θ increases) toward the y-axis, your thumb points in the direction of the positive z-axis (Figure 22.1.4). An angular velocity $\vec{\omega}$ that points along the positive z-axis therefore corresponds to a counterclockwise rotation, whereas an angular velocity $\vec{\omega}$ that points along the negative z-axis corresponds to a clockwise rotation.

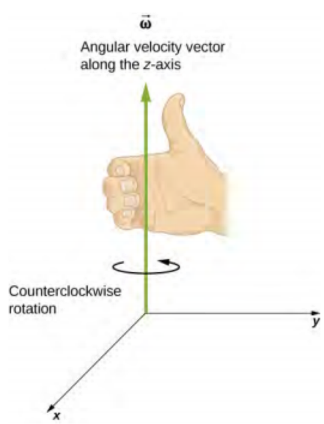


Figure 22.1.4: For counterclockwise rotation in the coordinate system shown, the angular velocity points in the positive z-direction by the right-hand-rule.

We can verify the right-hand-rule using the vector expression for the arc length $\vec{s}=\vec{\theta}\times\vec{r}$, Equation 22.1.2. If we differentiate this equation with respect to time, we find

$$\frac{d\vec{s}}{dt} = \frac{d}{dt}(\vec{\theta} \times \vec{r}) = \left(\frac{d\theta}{dt} \times \vec{r}\right) + \left(\vec{\theta} \times \frac{d\vec{r}}{dt}\right) = \frac{d\theta}{dt} \times \vec{r}. \tag{22.1.7}$$

Since $\vec{r}$ is constant, the term $\vec{\theta} \times \frac{d\vec{r}}{dt} = 0$. Since $\vec{v} = \frac{d\vec{s}}{dt}$ is the tangential velocity and $\omega = \frac{d\vec{\theta}}{dt}$ is the angular velocity, we have

$$\vec{v} = \vec{\omega} \times \vec{r}. \tag{22.1.8}$$

That is, the tangential velocity is the cross product of the angular velocity and the position vector, as shown in Figure 22.1.5. From part (a) of this figure, we see that with the angular velocity in the positive z-direction, the rotation in the xy-plane is counterclockwise. In part (b), the angular velocity is in the negative z-direction, giving a clockwise rotation in the xy-plane.

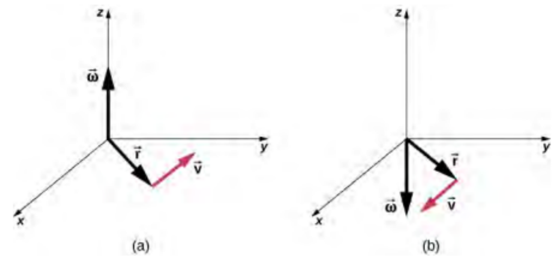


Figure 22.1.5: The vectors shown are the angular velocity, position, and tangential velocity. (a) The angular velocity points in the positive z-direction, giving a counterclockwise rotation in the xy-plane. (b) The angular velocity points in the negative z-direction, giving a clockwise rotation.

Example 22.1.1: Rotation of a Flywheel

A flywheel rotates such that it sweeps out an angle at the rate of θ = ωt = (45.0 rad/s)t radians. The wheel rotates counterclockwise when viewed in the plane of the page. (a) What is the angular velocity of the flywheel? (b) What direction is the angular velocity? (c) How many radians does the flywheel rotate through in 30 s? (d) What is the tangential speed of a point on the flywheel 10 cm from the axis of rotation?

Strategy

The functional form of the angular position of the flywheel is given in the problem as θ(t) = ωt, so by taking the derivative with respect to time, we can find the angular velocity. We use the right-hand rule to find the angular velocity. To find the angular displacement of the flywheel during 30 s, we seek the angular displacement $\Delta\theta$, where the change in angular position is between 0 and 30 s. To find the tangential speed of a point at a distance from the axis of rotation, we multiply its distance times the angular velocity of the flywheel.

Solution

a. $\omega = \frac{d\theta}{dt}$ = 45 rad/s. We see that the angular velocity is a constant.
b. By the right-hand rule, we curl the fingers in the direction of rotation, which is counterclockwise in the plane of the page, and the thumb points in the direction of the angular velocity, which is out of the page.
c. $\Delta\theta$ = θ(30 s) − θ(0 s) = 45.0(30.0 s) − 45.0(0 s) = 1350.0 rad.
d. v_t = rω = (0.1 m)(45.0 rad/s) = 4.5 m/s.

Significance

In 30 s, the flywheel has rotated through quite a number of revolutions, about 215 if we divide the angular displacement by 2π. A massive flywheel can be used to store energy in this way, if the losses due to friction are minimal. Recent research has considered superconducting bearings on which the flywheel rests, with zero energy loss due to friction.

Angular Acceleration

We have just discussed angular velocity for uniform circular motion, but not all motion is uniform. Envision an ice skater spinning with his arms outstretched—when he pulls his arms inward, his angular velocity increases. Or think about a computer's hard disk slowing to a halt as the angular velocity decreases. We will explore these situations later, but we can already see a need to define an **angular acceleration** for describing situations where ω changes. The faster the change in ω, the greater the angular acceleration. We define the **instantaneous angular acceleration** α as the derivative of angular velocity with respect to time:

$$\alpha=\lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t}=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2},\tag{22.1.9}$$

where we have taken the limit of the average angular acceleration, $\bar{\alpha}=\frac{\Delta\omega}{\Delta t}$ as $\Delta t\to 0$. The units of angular acceleration are (rad/s)/s, or rad/s^2.

In the same way as we defined the vector associated with angular velocity $\vec{\omega}$, we can define $\vec{\alpha}$, the vector associated with angular acceleration (Figure $22.1.6$). If the angular velocity is along the positive z-axis, as in Figure $22.1.4$, and $\frac{d\omega}{dt}$ is positive, then the angular acceleration $\vec{\alpha}$ is positive and points along the +z- axis. Similarly, if the angular velocity $\vec{\omega}$ is along the positive z-axis and $\frac{d\omega}{dt}$ is negative, then the angular acceleration is negative and points along the +z-axis.

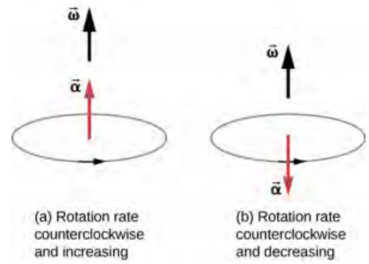


Figure $22.1.6$: The rotation is counterclockwise in both (a) and (b) with the angular velocity in the same direction. (a) The angular acceleration is in the same direction as the angular velocity, which increases the rotation rate. (b) The angular acceleration is in the opposite direction to the angular velocity, which decreases the rotation rate.

We can express the tangential acceleration vector as a cross product of the angular acceleration and the position vector. This expression can be found by taking the time derivative of $\vec{v}=\vec{\omega}\times\vec{r}$; the result is:

$$\vec{a}=\vec{\alpha}\times\vec{r}.\tag{22.1.10}$$

The vector relationships for the angular acceleration and tangential acceleration are shown in Figure $22.1.7$.

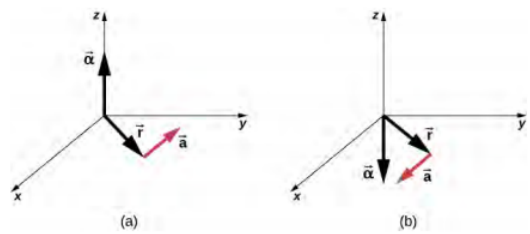


Figure $22.1.7$: (a) The angular acceleration is the positive z-direction and produces a tangential acceleration in a counterclockwise sense. (b) The angular acceleration is in the negative z-direction and produces a tangential acceleration in the clockwise sense.

We can relate the tangential acceleration of a point on a rotating body at a distance from the axis of rotation in the same way that we related the tangential speed to the angular velocity. If we differentiate Equation 22.1.6 with respect to time, noting that the radius r is constant, we obtain

$$a_t=r\alpha.\tag{22.1.11}$$

Thus, the tangential acceleration a_t is the radius times the angular acceleration. Equations 22.1.6 and 22.1.11 are important for the discussion of rolling motion.

Let's apply these ideas to the analysis of a few simple fixed-axis rotation scenarios. Before doing so, we present a problem-solving strategy that can be applied to rotational kinematics: the description of rotational motion.

? Problem-Solving Strategy: Rotational Kinematics

1. Examine the situation to determine that rotational kinematics (rotational motion) is involved.
2. Identify exactly what needs to be determined in the problem (identify the unknowns). A sketch of the situation is useful.
3. Make a complete list of what is given or can be inferred from the problem as stated (identify the knowns).
4. Solve the appropriate equation or equations for the quantity to be determined (the unknown). It can be useful to think in terms of a translational analog, because by now you are familiar with the equations of translational motion.

5. Substitute the known values along with their units into the appropriate equation and obtain numerical solutions complete with units. Be sure to use units of radians for angles.
6. Check your answer to see if it is reasonable: Does your answer make sense?

Now let's apply this problem-solving strategy to a few specific examples.

Example 22.1.2: A Spinning Bicycle Wheel

A bicycle mechanic mounts a bicycle on the repair stand and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s. (a) Calculate the average angular acceleration in rad/s^2. (b) If she now hits the brakes, causing an angular acceleration of −87.3 rad/s^2, how long does it take the wheel to stop?

Strategy

The average angular acceleration can be found directly from its definition $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$ because the final angular velocity and time are given. We see that $\Delta\omega = \omega_{final} - \omega_{initial}$ = 250 rev/min and Δt is 5.00 s. For part (b), we know the angular acceleration and the initial angular velocity. We can find the stopping time by using the definition of average angular acceleration and solving for Δt, yielding

$$\Delta t = \frac{\Delta\omega}{\alpha}. \tag{22.1.12}$$

Solution

a. Entering known information into the definition of angular acceleration, we get $\bar{\alpha} = \frac{\Delta \omega}{\Delta t} = \frac{250\; rpm}{5.00\; s} \ldotp$Because $\Delta\omega$ is in revolutions per minute (rpm) and we want the standard units of rad/s^2 for angular acceleration, we need to convert from rpm to rad/s: $\Delta \omega = 250 \frac{rev}{min}\; \cdotp \frac{2 \pi\; rad}{rev}\; \cdotp \frac{1\; min}{60\; s} = 26.2\; rad/s \ldotp$Entering this quantity into the expression for α, we get $\bar{\alpha} = \frac{\Delta \omega}{\Delta t} = \frac{26.2\; rpm}{5.00\; s} = 5.24\; rad/s^{2} \ldotp$
b. Here the angular velocity decreases from 26.2 rad/s (250 rpm) to zero, so that $\Delta\omega$ is −26.2 rad/s, and α is given to be –87.3 rad/s^2. Thus $\Delta t = \frac{-26.2\; rad/s}{-87.3\; rad/s^{2}} = 0.300\; s \ldotp$

Significance

Note that the angular acceleration as the mechanic spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero.

Exercise 22.1.1

The fan blades on a turbofan jet engine (shown below) accelerate from rest up to a rotation rate of 40.0 rev/s in 20 s. The increase in angular velocity of the fan is constant in time. (The GE90-110B1 turbofan engine mounted on a Boeing 777, as shown, is currently the largest turbofan engine in the world, capable of thrusts of 330–510 kN.) (a) What is the average angular acceleration? (b) What is the instantaneous angular acceleration at any time during the first 20 s?

Example 22.1.3: Wind Turbine

A wind turbine (Figure 22.1.9) in a wind farm is being shut down for maintenance. It takes 30 s for the turbine to go from its operating angular velocity to a complete stop in which the angular velocity function is ω(t) = $\left[\frac{(ts^{-1}-30.0)^2}{100.0}\right]$ rad/s. If the turbine is rotating counterclockwise looking into the page, (a) what are the directions of the angular velocity and acceleration vectors? (b) What is the average angular acceleration? (c) What is the instantaneous angular acceleration at t = 0.0, 15.0, 30.0 s?

Figure 22.1.9: A wind turbine that is rotating counterclockwise, as seen head on.

Strategy

a. We are given the rotational sense of the turbine, which is counterclockwise in the plane of the page. Using the right hand rule (Figure 10.5), we can establish the directions of the angular velocity and acceleration vectors.
b. We calculate the initial and final angular velocities to get the average angular acceleration. We establish the sign of the angular acceleration from the results in (a).
c. We are given the functional form of the angular velocity, so we can find the functional form of the angular acceleration function by taking its derivative with respect to time.

Solution

a. Since the turbine is rotating counterclockwise, angular velocity $\vec{\omega}$ points out of the page. But since the angular velocity is decreasing, the angular acceleration $\vec{\alpha}$ points into the page, in the opposite sense to the angular velocity.
b. The initial angular velocity of the turbine, setting t = 0, is ω = 9.0 rad/s. The final angular velocity is zero, so the average angular acceleration is \$\bar{\alpha} \frac{\Delta \omega}{\Delta t} = \frac{\omega - \omega_{0}}{t - t_{0}} = \frac{0 - 9.0\; rad/s}{30.0 - 0\; s} = -0.3\; rad/s^{2} \ldotp\$
c. Taking the derivative of the angular velocity with respect to time gives $\alpha = \frac{d\omega}{dt} = \frac{(t-30.0)}{50.0}$ rad/s^2 \$\alpha (0.0; s) = -0.6\; rad/s^{2}, \alpha (15.0\; s) = -0.3\; rad/s^{2}, and\; \alpha (30.0\; s) = 0\; rad/s \ldotp\$

Significance

We found from the calculations in (a) and (b) that the angular acceleration α and the average angular acceleration $\bar{\alpha}$ are negative. The turbine has an angular acceleration in the opposite sense to its angular velocity.

We now have a basic vocabulary for discussing fixed-axis rotational kinematics and relationships between rotational variables. We discuss more definitions and connections in the next section.

22.2: Rotation with Constant Angular Acceleration

Learning Objectives

- Derive the kinematic equations for rotational motion with constant angular acceleration
- Select from the kinematic equations for rotational motion with constant angular acceleration the appropriate equations to solve for unknowns in the analysis of systems undergoing fixed-axis rotation
- Use solutions found with the kinematic equations to verify the graphical analysis of fixed-axis rotation with constant angular acceleration

In the preceding section, we defined the rotational variables of angular displacement, angular velocity, and angular acceleration. In this section, we work with these definitions to derive relationships among these variables and use these relationships to analyze rotational motion for a rigid body about a fixed axis under a constant angular acceleration. This analysis forms the basis for rotational kinematics. If the angular acceleration is constant, the equations of rotational kinematics simplify, similar to the equations of linear kinematics discussed in Motion along a Straight Line and Motion in Two and Three Dimensions. We can then use this simplified set of equations to describe many applications in physics and engineering where the angular acceleration of the system is constant. Rotational kinematics is also a prerequisite to the discussion of rotational dynamics later in this chapter.

Kinematics of Rotational Motion

Using our intuition, we can begin to see how the rotational quantities θ, ω, α, and t are related to one another. For example, we saw in the preceding section that if a flywheel has an angular acceleration in the same direction as its angular velocity vector, its angular velocity increases with time and its angular displacement also increases. On the contrary, if the angular acceleration is opposite to the angular velocity vector, its angular velocity decreases with time. We can describe these physical situations and many others with a consistent set of rotational kinematic equations under a constant angular acceleration. The method to investigate rotational motion in this way is called **kinematics of rotational motion**.

To begin, we note that if the system is rotating under a constant acceleration, then the average angular velocity follows a simple relation because the angular velocity is increasing linearly with time. The average angular velocity is just half the sum of the initial and final values:

$$\bar{\omega}=\frac{\omega_0+\omega_f}{2}. \tag{22.2.1}$$

From the definition of the average angular velocity, we can find an equation that relates the angular position, average angular velocity, and time:

$$\bar{\omega}=\frac{\Delta\theta}{\Delta t}. \tag{22.2.2}$$

Solving for θ, we have

$$\theta_f=\theta_0+\bar{\omega}t, \tag{22.2.3}$$

where we have set t_0 = 0. This equation can be very useful if we know the average angular velocity of the system. Then we could find the angular displacement over a given time period. Next, we find an equation relating ω, α, and t. To determine this equation, we start with the definition of angular acceleration:

$$\alpha=\frac{d\omega}{dt}. \tag{22.2.4}$$

We rearrange this to get αdt = dω and then we integrate both sides of this equation from initial values to final values, that is, from t_0 to t and ω_0 to ω_f. In uniform rotational motion, the angular acceleration is constant so it can be pulled out of the integral, yielding two definite integrals:

$$\alpha\int_{t_0}^{t}dt'=\int_{\omega_0}^{\omega_f}d\omega. \tag{22.2.5}$$

Setting t_0 = 0, we have

$$\alpha t=\omega_f-\omega_0. \tag{22.2.6}$$

We rearrange this to obtain

$$\omega_f=\omega_0+\alpha t, \tag{22.2.7}$$

where ω_0 is the initial angular velocity. Equation 22.2.7 is the rotational counterpart to the linear kinematics equation $v_f = v_0 + at$. With Equation 22.2.7, we can find the angular velocity of an object at any specified time t given the initial angular velocity and the angular acceleration.

Let's now do a similar treatment starting with the equation $\omega=\frac{d\theta}{dt}$. We rearrange it to obtain $\omega dt = d\theta$ and integrate both sides from initial to final values again, noting that the angular acceleration is constant and does not have a time dependence. However, this time, the angular velocity is not constant (in general), so we substitute in what we derived above:

$$\begin{aligned}\int_{t_0}^{t_f}(\omega_0+\alpha t')dt' &=\int_{\theta_0}^{\theta_f}d\theta;\\ \int_{t_0}^{t}\omega_0 dt+\int_{t_0}^{t}\alpha t dt &=\int_{\theta_0}^{\theta_f}d\theta=\left[\omega_0 t'+\alpha\left(\frac{(t')^2}{2}\right)^2\right]_{t_0}^{t}=\omega_0 t+\alpha\left(\frac{t^2}{2}\right)=\theta_f-\theta_0.\end{aligned}$$

where we have set $t_0 = 0$. Now we rearrange to obtain

$$\theta_f=\theta_0+\omega_0 t+\frac{1}{2}\alpha t^2. \tag{22.2.8}$$

Equation 22.2.8 is the rotational counterpart to the linear kinematics equation found in Motion Along a Straight Line for position as a function of time. This equation gives us the angular position of a rotating rigid body at any time t given the initial conditions (initial angular position and initial angular velocity) and the angular acceleration.

We can find an equation that is independent of time by solving for t in Equation 22.2.7 and substituting into Equation 22.2.8. Equation 22.2.8 becomes

$$\begin{aligned}\theta_f &=\theta_0+\omega_0\left(\frac{\omega_f-\omega_0}{\alpha}\right)+\frac{1}{2}\alpha\left(\frac{\omega_f-\omega_0}{\alpha}\right)^2\\ &=\theta_0+\frac{\omega_0\omega_f}{\alpha}-\frac{\omega_0^2}{\alpha}+\frac{1}{2}\frac{\omega_f^2}{\alpha}-\frac{\omega_0\omega_f}{\alpha}+\frac{1}{2}\frac{\omega_0^2}{\alpha}\\ &=\theta_0+\frac{1}{2}\frac{\omega_f^2}{\alpha}-\frac{1}{2}\frac{\omega_0^2}{\alpha},\\ \theta_f-\theta_0 &=\frac{\omega_f^2-\omega_0^2}{2\alpha}\end{aligned}$$

or

$$\omega_f^2=\omega_0^2+2\alpha(\Delta\theta). \tag{22.2.9}$$

Equation 22.2.3 through Equation 22.2.9 describe fixed-axis rotation for constant acceleration and are summarized in Table 10.1.

Table 10.1 - Kinematic Equations

Angular displacement from average angular velocity	$\theta_f=\theta_0+\bar{\omega}t$
Angular velocity from angular acceleration	$\omega_f=\omega_0+\alpha t$
Angular displacement from angular velocity and angular acceleration	$\theta_f=\theta_0+\omega_0 t+\frac{1}{2}\alpha t^2$

Angular velocity from angular displacement and angular acceleration	$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$

Applying the Equations for Rotational Motion

Now we can apply the key kinematic relations for rotational motion to some simple examples to get a feel for how the equations can be applied to everyday situations.

Example 10.4: Calculating the Acceleration of a Fishing Reel

A deep-sea fisherman hooks a big fish that swims away from the boat, pulling the fishing line from his fishing reel. The whole system is initially at rest, and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of 110 rad/s^2 for 2.00 s (Figure $22.2.1$).

a. What is the final angular velocity of the reel after 2 s?
b. How many revolutions does the reel make?

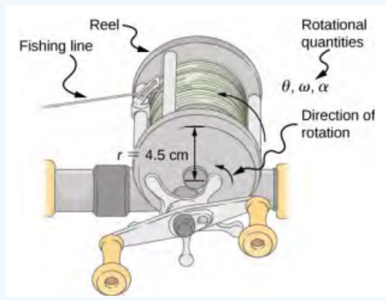


Figure $22.2.1$: Fishing line coming off a rotating reel moves linearly

Strategy

Identify the knowns and compare with the kinematic equations for constant acceleration. Look for the appropriate equation that can be solved for the unknown, using the knowns given in the problem description.

Solution

a. We are given α and t and want to determine ω. The most straightforward equation to use is $\omega_f = \omega_0 + \alpha t$, since all terms are known besides the unknown variable we are looking for. We are given that ω_0 = 0 (it starts from rest), so $\omega_{f} = 0 + (110\; rad/s^{2})(2.00\; s) = 220\; rad/s \ldotp$
b. We are asked to find the number of revolutions. Because 1 rev = 2π rad, we can find the number of revolutions by finding θ in radians. We are given α and t, and we know ω_0 is zero, so we can obtain θ by using $

$$\begin{aligned} \theta_f &= \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ &= 0 + 0 + (0.500)(110\ rad/s^2)(2.00\ s)^2 = 220\ rad. \end{aligned}$$

$Converting radians to revolutions gives $Number\; of\; rev = (220\; rad) \left(\dfrac{1\; rev}{2 \pi\; rad}\right) = 35.0\; rev \ldotp$

Significance

This example illustrates that relationships among rotational quantities are highly analogous to those among linear quantities. The answers to the questions are realistic. After unwinding for two seconds, the reel is found to spin at 220 rad/s, which is 2100 rpm. (No wonder reels sometimes make high-pitched sounds.)

In the preceding example, we considered a fishing reel with a positive angular acceleration. Now let us consider what happens with a negative angular acceleration.

Example 10.5: Calculating the Duration When the Fishing Reel Slows Down and Stops

Now the fisherman applies a brake to the spinning reel, achieving an angular acceleration of −300 rad/s^2. How long does it take the reel to come to a stop?

Strategy

We are asked to find the time t for the reel to come to a stop. The initial and final conditions are different from those in the previous problem, which involved the same fishing reel. Now we see that the initial angular velocity is ω_0 = 220 rad/s and the final angular velocity ω is zero. The angular acceleration is given as α = −300 rad/s^2. Examining the available equations, we see all quantities but t are known in $\omega_f = \omega_0 + \alpha t$, making it easiest to use this equation.

Solution

The equation states

$$\omega_f = \omega_0 + \alpha t. \tag{22.2.10}$$

We solve the equation algebraically for t and then substitute the known values as usual, yielding

$$t = \frac{\omega_f - \omega_0}{\alpha} = \frac{0 - 220.0\ rad/s}{-300.0\ rad/s^2} = 0.733\ s. \tag{22.2.11}$$

Significance

Note that care must be taken with the signs that indicate the directions of various quantities. Also, note that the time to stop the reel is fairly small because the acceleration is rather large. Fishing lines sometimes snap because of the accelerations involved, and fishermen often let the fish swim for a while before applying brakes on the reel. A tired fish is slower, requiring a smaller acceleration.

Exercise 10.2

A centrifuge used in DNA extraction spins at a maximum rate of 7000 rpm, producing a “g-force” on the sample that is 6000 times the force of gravity. If the centrifuge takes 10 seconds to come to rest from the maximum spin rate: (a) What is the angular acceleration of the centrifuge? (b) What is the angular displacement of the centrifuge during this time?

Example 10.6: Angular Acceleration of a Propeller

Figure 22.2.2 shows a graph of the angular velocity of a propeller on an aircraft as a function of time. Its angular velocity starts at 30 rad/s and drops linearly to 0 rad/s over the course of 5 seconds. (a) Find the angular acceleration of the object and verify the result using the kinematic equations. (b) Find the angle through which the propeller rotates during these 5 seconds and verify your result using the kinematic equations.

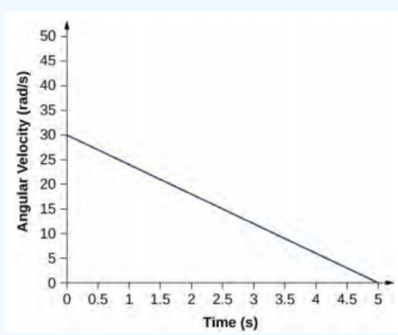


Figure 22.2.2: A graph of the angular velocity of a propeller versus time.

Strategy

a. Since the angular velocity varies linearly with time, we know that the angular acceleration is constant and does not depend on the time variable. The angular acceleration is the slope of the angular velocity vs. time graph, $\alpha = \frac{d\omega}{dt}$. To calculate the slope, we read directly from Figure 22.2.2, and see that ω_0 = 30 rad/s at t = 0 s and ω_f = 0 rad/s at t = 5 s. Then, we can verify the result using $\omega = \omega_0 + \alpha t$.

b. We use the equation $\omega = \frac{d\theta}{dt}$; since the time derivative of the angle is the angular velocity, we can find the angular displacement by integrating the angular velocity, which from the figure means taking the area under the angular velocity graph. In other words: $\int_{\theta_0}^{\theta_f} d\theta = \theta_f - \theta_0 = \int_{t_0}^{t_f} \omega(t)dt.$ Then we use the kinematic equations for constant acceleration to verify the result.

Solution

a. Calculating the slope, we get $\alpha = \frac{\omega - \omega_0}{t - t_0} = \frac{(0-30.0)\ rad/s}{(5.0-0)\ s} = -6.0\ rad/s^2.$ We see that this is exactly Equation 22.2.7 with a little rearranging of terms.

b. We can find the area under the curve by calculating the area of the right triangle, as shown in Figure 22.2.3.

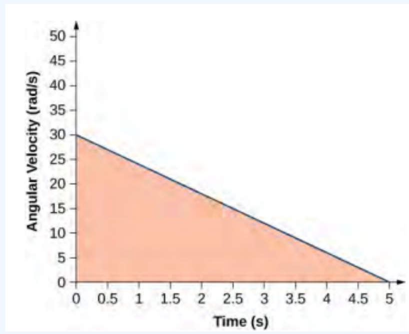


Figure 22.2.3: The area under the curve is the area of the right triangle.

$$\Delta\theta = area(triangle) = \frac{1}{2}(30\ rad/s)(5\ s) = 75\ rad. \tag{22.2.12}$$

We verify the solution using Equation 22.2.8:

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2. \tag{22.2.13}$$

Setting θ_0 = 0, we have

$$\theta_0 = (30.0\ rad/s)(5.0\ s) + \frac{1}{2}(-6.0\ rad/s^2)(5.0\ s)^2 = 150.0 - 75.0 = 75.0\ rad. \tag{22.2.14}$$

This verifies the solution found from finding the area under the curve.

Significance

We see from part (b) that there are alternative approaches to analyzing fixed-axis rotation with constant acceleration. We started with a graphical approach and verified the solution using the rotational kinematic equations. Since $\alpha = \frac{d\omega}{dt}$, we could do the same graphical analysis on an angular acceleration-vs.-time curve. The area under an α-vs.-t curve gives us the change in angular velocity. Since the angular acceleration is constant in this section, this is a straightforward exercise.

22.3: Relating Angular and Translational Quantities

In this section, we relate each of the rotational variables to the translational variables defined in 1 Dimensional Kinematics and 2 Dimensional Kinematics. This will complete our ability to describe rigid-body rotations.

Angular vs. Linear Variables

In Rotational Variables, we introduced angular variables. If we compare the rotational definitions with the definitions of linear kinematic variables, we find that there is a mapping of the linear variables to the rotational ones. Linear position, velocity, and acceleration have their rotational counterparts, as we can see when we write them side by side:

	Linear	Rotational
Position	x	θ
Velocity	$v=\frac{dx}{dt}$	$\omega=\frac{d\theta}{dt}$
Acceleration	$a=\frac{dv}{dt}$	$a=\frac{d\omega}{dt}$
Mass[1]	m	I

Let's compare the linear and rotational variables individually. The linear variable of position has physical units of meters, whereas the angular position variable has dimensionless units of radians, as can be seen from the definition of $\theta=\frac{s}{r}$, which is the ratio of two lengths. The linear velocity has units of m/s, and its counterpart, the angular velocity, has units of rad/s. In Rotational Variables, we saw in the case of circular motion that the linear tangential speed of a particle at a radius r from the axis of rotation is related to the angular velocity by the relation $v_t = r\omega$. This could also apply to points on a rigid body rotating about a fixed axis. Here, we consider only circular motion. In circular motion, both uniform and nonuniform, there exists a centripetal acceleration (Motion in Two and Three Dimensions). The centripetal acceleration vector points inward from the particle executing circular motion toward the axis of rotation. The derivation of the magnitude of the centripetal acceleration is given in Motion in Two and Three Dimensions. From that derivation, the magnitude of the centripetal acceleration was found to be

$$a_c=\frac{v_t^2}{r}, \tag{22.3.1}$$

where r is the radius of the circle.

Thus, in uniform circular motion when the angular velocity is constant and the angular acceleration is zero, we have a linear acceleration—that is, centripetal acceleration—since the tangential speed in Equation 22.3.1 is a constant. If nonuniform circular motion is present, the rotating system has an angular acceleration, and we have both a linear centripetal acceleration that is changing (because v_t is changing) as well as a linear tangential acceleration. These relationships are shown in Figure 22.3.1, where we show the centripetal and tangential accelerations for uniform and nonuniform circular motion.

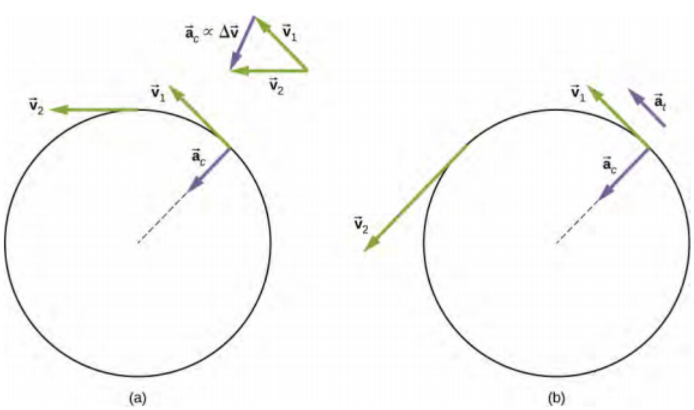


Figure 22.3.1: (a) Uniform circular motion: The centripetal acceleration a_c has its vector inward toward the axis of rotation. There is no tangential acceleration. (b) Nonuniform circular motion: An angular acceleration produces an inward centripetal acceleration that is changing in magnitude, plus a tangential acceleration a_t.

The centripetal acceleration is due to the change in the direction of tangential velocity, whereas the tangential acceleration is due to any change in the magnitude of the tangential velocity. The tangential and centripetal acceleration vectors $\vec{a}_t$ and $\vec{a}_c$ are always perpendicular to each other, as seen in Figure $22.3.1$. To complete this description, we can assign a **total linear acceleration** vector to a point on a rotating rigid body or a particle executing circular motion at a radius r from a fixed axis. The total linear acceleration vector $\vec{a}$ is the vector sum of the centripetal and tangential accelerations,

$$\vec{a} = \vec{a}_c + \vec{a}_t. \tag{22.3.2}$$

The total linear acceleration vector in the case of nonuniform circular motion points at an angle between the centripetal and tangential acceleration vectors, as shown in Figure $22.3.2$. Since $\vec{a}_c \perp \vec{a}_t$, the magnitude of the total linear acceleration is

$$|\vec{a}| = \sqrt{a_c^2 + a_t^2}. \tag{22.3.3}$$

Note that if the angular acceleration is zero, the total linear acceleration is equal to the centripetal acceleration.

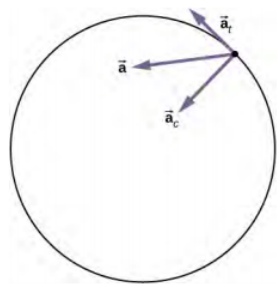


Figure $22.3.2$: A particle is executing circular motion and has an angular acceleration. The total linear acceleration of the particle is the vector sum of the centripetal acceleration and tangential acceleration vectors. The total linear acceleration vector is at an angle in between the centripetal and tangential accelerations.

Relationships between Rotational and Translational Motion

We can look at two relationships between rotational and translational motion.

1. Generally speaking, the linear kinematic equations have their rotational counterparts. Table 10.2 lists the four linear kinematic equations and the corresponding rotational counterpart. The two sets of equations look similar to each other, but describe two different physical situations, that is, rotation and translation.

Table 10.2 - Rotational and Translational Kinematic Equations

Rotational	Translational
$\theta_f = \theta_0 + \bar{\omega}t$	$x = x_0 + \bar{v}t$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}at^2$	$x_f = x_0 + v_0 t + \frac{1}{2}\omega t^2$
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	$v_f^2 = v_0^2 + 2a(\Delta x)$

2. The second correspondence has to do with relating linear and rotational variables in the special case of circular motion. This is shown in Table 10.3, where in the third column, we have listed the connecting equation that relates the linear variable to the rotational variable. The rotational variables of angular velocity and acceleration have subscripts that indicate their definition in circular motion.

Table 10.3 - Rotational and Translational Quantities: Circular Motion

Rotational	Translational	Relationship (r = radius
θ	s	$\theta = \frac{s}{r}$
ω	v_t	$\omega = \frac{v_t}{r}$
α	a_t	$\alpha = \frac{a_t}{r}$

Rotational	Translational	Relationship (r = radius
	a_c	$a_c = \frac{v_t^2}{r}$

Example 10.7: Linear Acceleration of a Centrifuge

A centrifuge has a radius of 20 cm and accelerates from a maximum rotation rate of 10,000 rpm to rest in 30 seconds under a constant angular acceleration. It is rotating counterclockwise. What is the magnitude of the total acceleration of a point at the tip of the centrifuge at t = 29.0s? What is the direction of the total acceleration vector?

Strategy

With the information given, we can calculate the angular acceleration, which then will allow us to find the tangential acceleration. We can find the centripetal acceleration at t = 0 by calculating the tangential speed at this time. With the magnitudes of the accelerations, we can calculate the total linear acceleration. From the description of the rotation in the problem, we can sketch the direction of the total acceleration vector.

Solution

The angular acceleration is

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - (1.0 \times 10^4)\left(\frac{2\pi\ rad}{60.0\ s}\right)}{30.0\ s} = -34.9\ rad/s^2. \tag{22.3.4}$$

Therefore, the tangential acceleration is

$$a_t = r\alpha = (0.2\ m)(-34.9\ rad/s^2) = -7.0\ m/s^2. \tag{22.3.5}$$

The angular velocity at t = 29.0 s is

$$\begin{aligned} \omega &= \omega_0 + \alpha t = (1.0 \times 10^4)\left(\frac{2\pi\ rad}{60.0\ s}\right) + (-39.49\ rad/s^2)(29.0\ s) \\ &= 1047.2\ rad/s - 1012.71\ rad/s = 35.1\ rad/s. \end{aligned}$$

Thus, the tangential speed at t = 29.0 s is

$$v_t = r\omega = (0.2\ m)(35.1\ rad/s) = 7.0\ m/s. \tag{22.3.6}$$

We can now calculate the centripetal acceleration at t = 29.0 s:

$$a_c = \frac{v^2}{r} = \frac{(7.0\ m/s)^2}{0.2\ m} = 245.0\ m/s^2. \tag{22.3.7}$$

Since the two acceleration vectors are perpendicular to each other, the magnitude of the total linear acceleration is

$$|\vec{a}| = \sqrt{a_c^2 + a_t^2} = \sqrt{(245.0)^2 + (-7.0)^2} = 245.1\ m/s^2. \tag{22.3.8}$$

Since the centrifuge has a negative angular acceleration, it is slowing down. The total acceleration vector is as shown in Figure 22.3.3. The angle with respect to the centripetal acceleration vector is

$$\theta = \tan^{-1}\left(\frac{-7.0}{245.0}\right) = -1.6^o. \tag{22.3.9}$$

The negative sign means that the total acceleration vector is angled toward the clockwise direction.

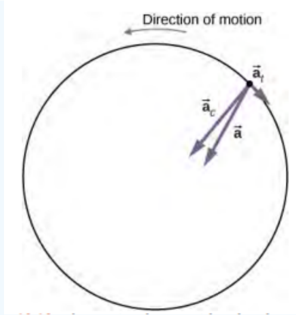


Figure $22.3.3$: The centripetal, tangential, and total acceleration vectors. The centrifuge is slowing down, so the tangential acceleration is clockwise, opposite the direction of rotation (counterclockwise).

Significance

From Figure $22.3.3$, we see that the tangential acceleration vector is opposite the direction of rotation. The magnitude of the tangential acceleration is much smaller than the centripetal acceleration, so the total linear acceleration vector will make a very small angle with respect to the centripetal acceleration vector.

? Exercise 10.3

A boy jumps on a merry-go-round with a radius of 5 m that is at rest. It starts accelerating at a constant rate up to an angular velocity of 5 rad/s in 20 seconds. What is the distance travelled by the boy?

Simulation

Check out this PhET simulation to change the parameters of a rotating disk (the initial angle, angular velocity, and angular acceleration), and place bugs at different radial distances from the axis. The simulation then lets you explore how circular motion relates to the bugs' xy-position, velocity, and acceleration using vectors or graphs.

[1]It is a little bit strange to put mass in this table of quantities that are purely related to the motion of the object; however, it is worth pointing out that the moment of inertia (I) is very much the rotational equivalent of mass (m)

22.4: Newton's Second Law for Rotation

In this subsection, we put together all the pieces learned so far in this chapter to analyze the dynamics of rotating rigid bodies. We have analyzed motion with kinematics and rotational kinetic energy but have not yet connected these ideas with force and/or torque. In this subsection, we introduce the rotational equivalent to Newton's second law of motion and apply it to rigid bodies with fixed-axis rotation.

Newton's Second Law for Rotation

We have thus far found many counterparts to the translational terms used throughout this text, most recently, torque, the rotational analog to force. This raises the question: Is there an analogous equation to Newton's second law, $\sum \vec{F}$ = m$\vec{a}$, which involves torque and rotational motion? To investigate this, we start with Newton's second law for a single particle rotating around an axis and executing circular motion. Let's exert a force $\vec{F}$ on a point mass m that is at a distance r from a pivot point (Figure $22.4.1$). The particle is constrained to move in a circular path with fixed radius and the force is tangent to the circle. We apply Newton's second law to determine the magnitude of the acceleration a = $\frac{F}{m}$ in the direction of $\vec{F}$. Recall that the magnitude of the tangential acceleration is proportional to the magnitude of the angular acceleration by a = rα. Substituting this expression into Newton's second law, we obtain

$$F = mr\alpha. \tag{22.4.1}$$

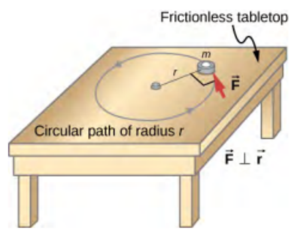


Figure 22.4.1: An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force $\vec{F}$ is applied to the object perpendicular to the radius r, causing it to accelerate about the pivot point. The force is perpendicular to r.

Multiply both sides of this equation by r,

$$rF = mr^2\alpha. \tag{22.4.2}$$

Note that the left side of this equation is the torque about the axis of rotation, where r is the lever arm and F is the force, perpendicular to r. Recall that the moment of inertia for a point particle is I = mr^2. The torque applied perpendicularly to the point mass in Figure $22.4.1$is therefore

$$\tau = I\alpha. \tag{22.4.3}$$

The torque on the particle is equal to the moment of inertia about the rotation axis times the angular acceleration. We can generalize this equation to a rigid body rotating about a fixed axis.

Newton's Second Law for Rotation

If more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration:

$$\sum_i \tau_i = I\alpha. \tag{22.4.4}$$

The term Iα is a scalar quantity and can be positive or negative (counterclockwise or clockwise) depending upon the sign of the net torque. Remember the convention that counterclockwise angular acceleration is positive. Thus, if a rigid body is rotating clockwise and experiences a positive torque (counterclockwise), the angular acceleration is positive.

Equation $22.4.4$ is **Newton's second law for rotation** and tells us how to relate torque, moment of inertia, and rotational kinematics. This is called the equation for **rotational dynamics**. With this equation, we can solve a whole class of problems involving force and rotation. It makes sense that the relationship for how much force it takes to rotate a body would include the moment of inertia, since that is the quantity that tells us how easy or hard it is to change the rotational motion of an object.

Deriving Newton's Second Law for Rotation in Vector Form

As before, when we found the angular acceleration, we may also find the torque vector. The second law $\sum \vec{F} = m\vec{a}$ tells us the relationship between net force and how to change the translational motion of an object. We have a vector rotational equivalent of this equation, which can be found by using Equation 10.2.10 and Figure 10.2.7. Equation 10.2.10 relates the angular acceleration to the position and tangential acceleration vectors:

$$\vec{a} = \vec{\alpha} \times \vec{r}. \tag{22.4.5}$$

We form the cross product of this equation with $\vec{r}$ and use a cross product identity (note that $\vec{r} \cdot \vec{\alpha} = 0$):

$$\vec{r} \times \vec{a} = \vec{r} \times (\vec{\alpha} \times \vec{r}) = \vec{\alpha}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot \vec{\alpha}) = \vec{\alpha}(\vec{r} \cdot \vec{r}) = \vec{\alpha} r^2. \tag{22.4.6}$$

We now form the cross product of Newton's second law with the position vector $\vec{r}$,

$$\sum (\vec{r} \times \vec{F}) = \vec{r} \times (m\vec{a}) = m\vec{r} \times \vec{a} = mr^2 \vec{\alpha}. \tag{22.4.7}$$

Identifying the first term on the left as the sum of the torques, and mr^2 as the moment of inertia, we arrive at Newton's second law of rotation in vector form:

$$\sum \tau = I\alpha. \tag{22.4.8}$$

This equation is exactly Equation 22.4.4 but with the torque and angular acceleration as vectors. An important point is that the torque vector is in the same direction as the angular acceleration.

Applying the Rotational Dynamics Equation

Before we apply the rotational dynamics equation to some everyday situations, let's review a general problem-solving strategy for use with this category of problems.

Problem-Solving Strategy: Rotational Dynamics

1. Examine the situation to determine that torque and mass are involved in the rotation. Draw a careful sketch of the situation.
2. Determine the system of interest.
3. Draw a free-body diagram. That is, draw and label all external forces acting on the system of interest.
4. Identify the pivot point. If the object is in equilibrium, it must be in equilibrium for all possible pivot points—chose the one that simplifies your work the most.
5. Apply $\sum_i \tau_i = I\alpha$, the rotational equivalent of Newton's second law, to solve the problem. Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.
6. As always, check the solution to see if it is reasonable.

Example 10.16: Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in Figure 22.4.2 He exerts a force of 250 N at the edge of the 200.0-kg merry-go-round, which has a 1.50-m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m away from the center. Consider the merry-go-round itself to be a uniform disk with negligible friction.

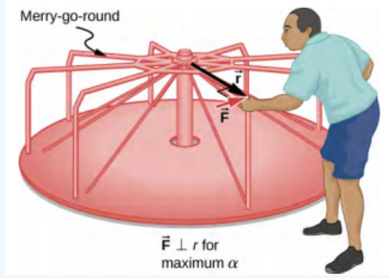


Figure 22.4.2: A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

Strategy

The net torque is given directly by the expression $\sum_i \tau_i = I\alpha$, To solve for α, we must first calculate the net torque τ (which is the same in both cases) and moment of inertia I (which is greater in the second case).

Solution

a. The moment of inertia of a solid disk about this axis is given in Figure 10.5.4 to be $\frac{1}{2}MR^2$.. We have M = 50.0 kg and R = 1.50 m, so

$$I = (0.500)(50.0\ kg)(1.50\ m)^2 = 56.25\ kg \cdot m^2. \tag{22.4.9}$$

To find the net torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

$$\tau = rF\sin\theta = (1.50\ m)(250.0\ N) - 375.0\ N \cdot m. \tag{22.4.10}$$

Now, after we substitute the known values, we find the angular acceleration to be
\[\alpha = \frac{\tau}{I} = \frac{375.0\; N\; \cdotp m}{56.25\; kg\; \cdotp m^{2}} = 6.67\; rad/s^{2} \ldotp

b. We expect the angular acceleration for the system to be less in this part because the moment of inertia is greater when the child is on the merry-go-round. To find the total moment of inertia I, we first find the child's moment of inertia I_c by approximating the child as a point mass at a distance of 1.25 m from the axis. Then

$$I_c = mR^2 = (18.0\ kg)(1.25\ m)^2 = 28.13\ kg \cdot m^2. \tag{22.4.11}$$

The total moment of inertia is the sum of the moments of inertia of the merry-go-round and the child (about the same axis):

$$I = (28.13\ kg \cdot m^2) + (56.25\ kg \\ ]cdotpm^2) = 84.38\ kg \cdot m^2. \tag{22.4.12}$$

Substituting known values into the equation for α gives

$$\alpha = \frac{\tau}{I} = \frac{375.0\ N \cdot m}{84.38\ kg \cdot m^2} = 4.44\ rad/s. \tag{22.4.13}$$

Significance

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-goround an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case.

> **? Exercise 10.7**
>
> The fan blades on a jet engine have a moment of inertia 30.0 kg • m^2. In 10 s, they rotate counterclockwise from rest up to a rotation rate of 20 rev/s. (a) What torque must be applied to the blades to achieve this angular acceleration? (b) What is the torque required to bring the fan blades rotating at 20 rev/s to a rest in 20 s?

We do not actually need the force of static friction to keep an object rolling on a flat surface (as I mentioned above, the motion could in principle go on "unforced" forever), but things are different on an inclined plane. Figure 22.4.2 shows an object rolling down an inclined plane, and the corresponding extended free-body diagram.

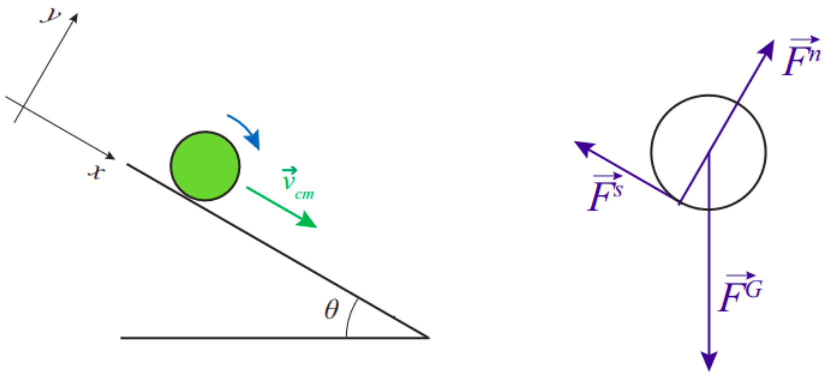


Figure 22.4.2: An object rolling down an inclined plane, and the extended free-body diagram. Note that neither gravity (applied at the CM) nor the normal force (whose line of action passes through the CM) exert a torque around the center of mass; only the force of static friction, $\vec{F}^s$, does.

The basic equations we use to solve for the object's motion are the sum of forces equation:

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \tag{22.4.14}$$

the net torque equation, with torques taken around the center of mass:

$$\sum \vec{\tau}_{ext} = I\vec{\alpha} \tag{22.4.15}$$

and the extension of the condition of rolling without slipping, (11.2.1), to the accelerations:

$$|a_{cm}| = R|\alpha|. \tag{22.4.16}$$

For the situation shown in Figure 22.4.2, if we take down the plane as the positive direction for linear motion, and clockwise torques as negative, we have to write $a_{cm} = -R\alpha$. In the direction perpendicular to the plane, we conclude from (22.4.14) that $F^n = Mg\cos\theta$, an equation we will not actually need; in the direction along the plane, we have

$$Ma_{cm} = Mg\sin\theta - F^s \tag{22.4.17}$$

and the torque equation just gives $-F^sR = I\alpha$, which with $a_{cm} = -R\alpha$ becomes

$$F^sR = I\frac{a_{cm}}{R}. \tag{22.4.18}$$

We can eliminate F^s in between these two equations and solve for a_{cm}:

$$a_{cm} = \frac{g\sin\theta}{1 + I/(MR^2)}. \tag{22.4.19}$$

Now you can see why, earlier in the semester, we were always careful to assume that all the objects we sent down inclined planes were *sliding*, not rolling! The acceleration for a rolling object is *never* equal to simply $g\sin\theta$. Most remarkably, the correction factor depends only on the shape of the rolling object, and not on its mass or size, since the ratio of I to MR^2 is independent of m and R for any given geometry. Thus, for instance, for a disk, $I = \frac{1}{2}MR^2$, so $a_{cm} = \frac{2}{3}g\sin\theta$, whereas for a hoop, $I = MR^2$, so

$a_{cm} = \frac{1}{2} g \sin\theta$. So any disk or solid cylinder will always roll down the incline faster than *any* hoop or hollow cylinder, regardless of mass or size.

This rather surprising result may be better understood in terms of energy. First, let's show (a result that is somewhat overdue) that for a rigid object that is rotating around an axis passing through its center of mass with angular velocity ω we can write the total kinetic energy as

$$K = K_{cm} + K_{rot} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I \omega^2. \tag{22.4.20}$$

This is because for every particle the velocity can be written as $\vec{v} = \vec{v}_{cm} + \vec{v}'$, where $\vec{v}'$ is the velocity relative to the center of mass (that is, in the CM frame). Since in this frame the motion is a simple rotation, we have $|v'| = \omega r$, where r is the particle's distance to the axis. Therefore, the kinetic energy of that particle will be

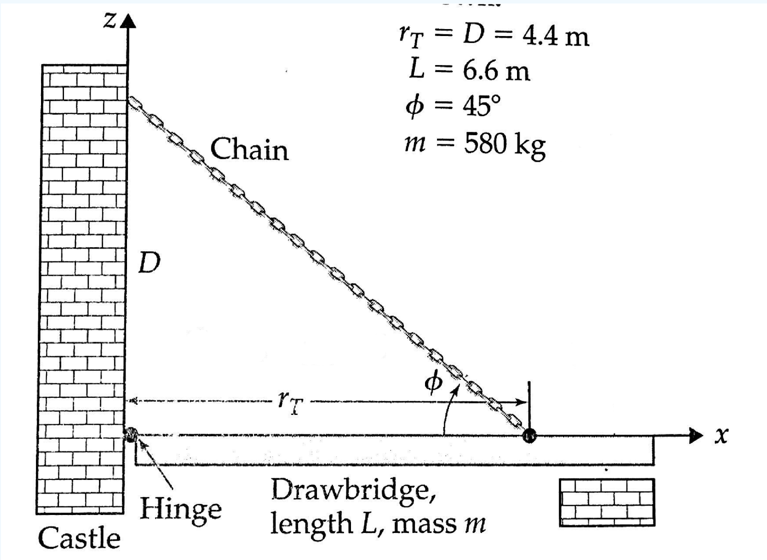
$$\begin{aligned} \frac{1}{2} m v^2 = \frac{1}{2} \vec{v} \cdot \vec{v} \ &= \frac{1}{2} m \left(\vec{v}_{cm} + \vec{v}'\right) \cdot \left(\vec{v}_{cm} + \vec{v}'\right) \\ &= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} m v'^2 + m \vec{v}_{cm} \cdot \vec{v}' \\ &= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} m r^2 \omega^2 + \vec{v}_{cm} \cdot \vec{p}' \end{aligned} \tag{22.4.21}$$

(Note how I have made use of the *dot product* to calculate the magnitude squared of a vector.) On the last line, the quantity $\vec{p}'$ is the momentum of that particle in the CM frame. Adding those momenta for all the particles should give zero, since, as we saw in an earlier chapter, the center of mass frame *is* the zero momentum frame. Then, adding the contributions of all particles to the first and second terms in 22.4.21gives Equation (22.4.20).

22.5: Examples

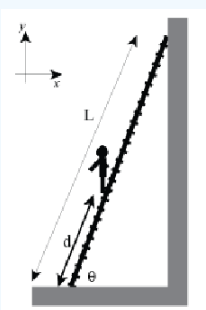
After a couple of whiteboard problems, we will present a few additional examples in this section show you have to set up and solve the equations of motion for somewhat more complicated systems, and you should study them carefully.

❓ Whiteboard Problem 22.5.1



Consider the drawbridge in the figure above, which is in static equilibrium. Determine the magnitude and direction of all the forces on the bridge - tension, gravity, and the hinge.

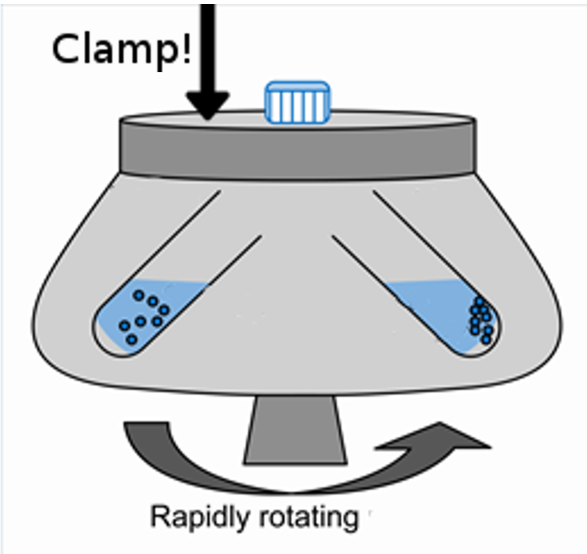
❓ Whiteboard Problem 22.5.2



Consider a very light ladder, of length 5.0 m, placed against a wall as shown at an angle of 60°. The wall is smooth (no friction), but the floor is rough and the coefficient of static friction between the ladder and the floor is 0.5. A painter of mass 65 kg is standing on the ladder at a distance d from the bottom. *(Hint: Draw a free body diagram, and make sure the object appears to be in rotational equilibrium before deciding you've identified all the forces!)*

1. What is the normal force exerted by the floor on the ladder?
2. How far up the ladder d can the painter go before the ladder starts to slide?

❓ Whiteboard Problem 22.5.3

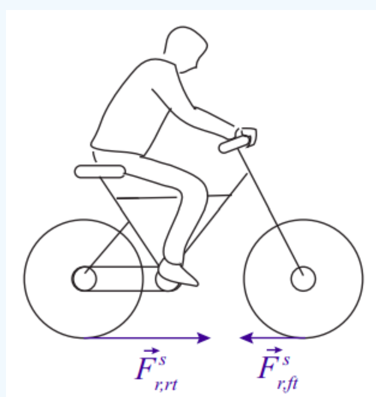


A specific centrifuge can spin at a rate of 1000 rpm.

1. What is the angular speed of this centrifuge, in radians per second?
2. The centrifuge is slowed down by applying a clamp at a distance 15 cm from the center of rotation. This clamp applies a force of 32 N to the centrifuge. What is the largest amount of torque this clamp could be exerting?
3. If the clamp exerts the largest amount of torque (which you just found), what will the angular acceleration of the centrifuge be? Take the centrifuge to be a uniform disk with radius 27 cm and total mass 4.5 kg.
4. How long will it take for this centrifuge to come to a stop?

Example $22.5.4$: Torques and forces on the wheels of an accelerating bicycle

Consider an accelerating bicycle. The rider exerts a torque on the pedals, which is transmitted to the rear wheel by the chain (possibly amplified by the gears, etc). How does this "drive" torque on the rear wheel (call it τ_d) relate to the final acceleration of the center of mass of the bicycle?



Solution

We need first to figure out how many external forces, at a minimum, we have to deal with. As the bicycle accelerates, two things happen: the wheels (both wheels) turn faster, so there must be a net torque (clockwise in the picture, if the bicycle is accelerating to the right) on *each* wheel; and the center of mass of the system accelerates, so there must be a net external force on the whole system. The system is only in contact with the road, and so, as long as no slippage happens, the only external source of torques or forces on the wheels has to be the force of static friction between the tires and the road.

For the front wheel, this is in fact the only external force, and the only force of any sort that exerts a torque on that wheel (there are forces acting at the axle, but they exert no torque around the axle). Since the torque has to be clockwise, then, the force of static friction on the front wheel, applied as it is at the point of contact with the road, must point *backwards*, that is, opposite the direction of motion. We get then one equation of motion (of the type (22.4.13)) for that wheel:

$$-F^s_{r,ft}R = I\alpha \tag{22.5.1}$$

where the subscript "ft" stands for "front tire", and the wheel is supposed to have a radius R and moment of inertia I.

For the rear wheel, we have the "drive torque" τ_d, exerted by the chain, and another torque exerted by the force of static friction, $\vec{F}^s_{r,rt}$, between that tire and the road. However, now the force $\vec{F}^s_{r,rt}$ needs to point *forward*. This is because the net external force on the whole bicycle-rider system is $\vec{F}^s_{r,rt} + \vec{F}^s_{r,ft}$, and that has to point forward, or the center of mass could never accelerate in that direction. Since we have established that $F^s_{r,ft}$ has to point backwards, it follows that $F^s_{r,rt}$ needs to be larger, and in the forward direction. This means we get, for the center of mass acceleration, the equation ($F_{net} = Ma_{cm}$)

$$F^s_{r,rt} - F^s_{r,ft} = Ma_{cm} \tag{22.5.2}$$

and for the rear wheel, the torque equation

$$F^s_{r,rt}R - \tau_d = I\alpha. \tag{22.5.3}$$

I am following the convention that clockwise torques are negative, and also that a force symbol without an arrow on top represents the magnitude of the force. If a clockwise angular acceleration is likewise negative, the condition of rolling without slipping [Equation (22.4.11)] needs to be written as

$$a_{cm} = -R\alpha. \tag{22.5.4}$$

These are all the equations we need to relate the acceleration to τ_d. We can start by solving (22.5.1) for $F^s_{r,ft}$ and substituting in (22.5.2), then likewise solving (22.5.3) for $F^s_{r,rt}$ and substituting in (22.5.2). The result is

$$\frac{I\alpha + \tau_d}{R} + \frac{I\alpha}{R} = Ma_{cm} \tag{22.5.5}$$

then use Eq (22.5.4) to write $\alpha = -a_{cm}/R$, and solve for a_{cm}:

$$a_{cm} = \frac{\tau_d}{MR + 2I/R} \tag{22.5.6}$$

Example 22.5.5: Blocks connected by rope over a pulley with non-zero mass

Consider again the setup illustrated in the figure below, but now assume that the pulley has a mass M and radius R. For simplicity, leave the friction force out. What is now the acceleration of the system?

Solution

The figure below shows the setup, plus free-body diagrams for the two blocks (the vertical forces on block 1 have been left out to avoid cluttering the figure, since they are not relevant here), and an extended free-body diagram for the pulley. (You can see from the pulley diagram that there has to be another force acting on it, to balance the two forces shown. This would be a contact force at the axle, directed upwards and to the left. If this was a statics problem, I would have to include it, but since it does not exert a torque around the axis of rotation, it does not contribute to the dynamics of the system, so I have left it out as well.)

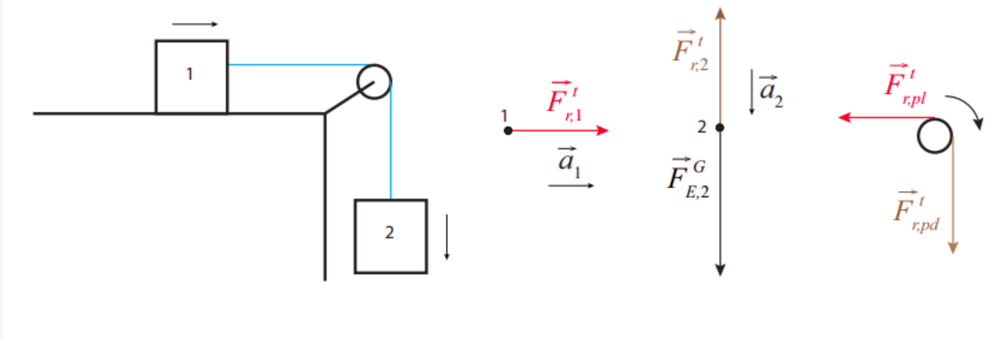


Figure \(\PageIndex{1}\)

The key new feature of this problem is that the tension on the string has to have different values on either side of the pulley, because there has to be a net torque on the pulley. Hence, the leftward force on the pulley ($F^t_{r,pl}$) has to be smaller than the downward force ($F^t_{r,pd}$).

On the other hand, as long as the mass of the rope is negligible, it will still be the case that the horizontal part of the rope will pull with equal strength on block 1 and on the pulley, and similarly the vertical part of the rope will pull with equal strength on the pulley and on block 2. (To make this point clearer, I have “color-coded” these matching forces in the figure.) This means that we can write $F^t_{r,pl} = F^t_{r,1}$ and $F^t_{r,pd} = F^t_{r,2}$, and write the torque equation (22.4.10) for the pulley as

$$F^t_{r,1} R - F^t_{r,2} R = I\alpha. \tag{22.5.7}$$

We also have $F = ma$ for each block:

$$F^t_{r,1} = m_1 a \tag{22.5.8}$$

$$F^t_{r,2} - m_2 g = -m_2 a \tag{22.5.9}$$

where I have taken a to be $a = |\vec{a}_1| = |\vec{a}_2|$. The condition of rolling without slipping, Equation (22.4.11), applied to the pulley, gives then

$$-R\alpha = a \tag{22.5.10}$$

since, in the situation shown, α will be negative, and a has been defined as positive. Substituting Eqs. (22.5.8), (22.5.9), and (22.5.10) into (22.5.7), we get

$$m_1 a R - (m_2 g - m_2 a) R = -\frac{Ia}{R} \tag{22.5.11}$$

which is easily solved for a:

$$a = \frac{m_2 g}{m_1 + m_2 + I/R^2}. \tag{22.5.12}$$

If you look at the structure of this equation, it all makes sense. The numerator is the force of gravity on block 2, which is, ultimately, the force responsible for setting the whole thing in motion. The denominator is, essentially, the inertia of the system: ordinary inertia for the blocks, and rotational inertia for the pulley. Note further that, if we treat the pulley as a flat, homogeneous disk of mass M, then $I = \frac{1}{2}MR^2$, and the denominator of (22.5.12) becomes just $m_1 + m_2 + M/2$.

CHAPTER OVERVIEW

23: N10) Simple Harmonic Motion

23.1: Introduction- The Physics of Oscillations

It is probably not an exaggeration to suggest that we are all introduced to oscillatory motion from our first moments of life. Babies, it seems, are constantly rocked to sleep, in many cases using devices, such as cradles and rocking chairs, that exemplify the kind of mechanical oscillator with which this chapter is concerned. And then, of course, there are swings, which function essentially like the pendulum depicted below.

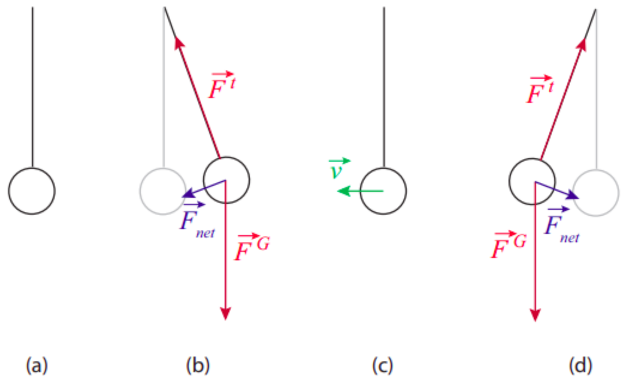


Figure 23.1.1: A simple pendulum. In (a), the equilibrium position, the tension and gravity forces balance out. In (b), they combine to produce a restoring force (in blue) pointing back towards equilibrium. In (c), the bob is passing through equilibrium and the net force on it at that instant is again zero, but its momentum keeps it going. At (d) we have the mirror image of (b).

In fact, oscillatory motion is extremely common, both in natural systems and in human-made structures. It essentially requires only two things: a stable equilibrium configuration, where the stability is ensured by what we call a *restoring force*; and inertia, which, of course, every physical system has.

The pendulum in Figure 23.1.1 illustrates how these things combine to produce an oscillation. As the pendulum bob is displaced from its equilibrium position, a net force on it appears (a combination of gravity and the tension in the string), pointing back towards the vertical. When the bob is released, it accelerates under the influence of this force, with the result that when it reaches back the equilibrium position, its inertia (or, if you prefer, its momentum) causes it to overshoot it. Once this happens, the restoring force changes direction, always trying to bring the mass back to equilibrium; as a result, the bob slows down, and eventually reverses course, accelerates again towards the vertical, overshoots it again... the process will repeat itself, until all the energy we initially put in the system (gravitational potential energy, in this case) is dissipated away (or *damped*), mostly through friction at the pivot point, though air resistance plays a small part as well.

That the motion, in the absence of dissipation, must be symmetric around the equilibrium position follows from conservation of energy: the speed of the bob at any given height must be the same on either side, in order for the sum of its potential and kinetic energies to be the same. In particular, if released from rest from some height, it will stop when it reaches the same height on the other side. In the presence of dissipation, the motion is neither exactly symmetric, nor exactly periodic (that is to say, it does not repeat itself exactly—the maximum height gets lower every time, the speed as it passes through the equilibrium position gets also smaller and smaller), but when the dissipation is not very large one can always define an approximate *period* (which we will denote with the letter T) as the time it takes to complete one full swing.

The inverse of the period is the *frequency*, f, which tells us how many full swings the pendulum completes per second. These two quantities, T and f, can be defined for any type of periodic (or approximately periodic) motion, and will always satisfy the relationship

$$f=\frac{1}{T}. \tag{23.1.1}$$

The units of frequency are, of course, inverse seconds (s^{-1}). In this context, however, this unit is called a ”hertz,” and abbreviated Hz.

- **11.1: Introduction- The Physics of Oscillations** by Julio Gea-Banacloche is licensed CC BY-SA 4.0. Original source: https://scholarworks.uark.edu/oer/3.

23.2: Simple Harmonic Motion

A particularly important kind of oscillatory motion is called *simple harmonic motion*. This is what happens when the restoring force is linear in the displacement from the equilibrium position: that is to say, in one dimension, if x_0 is the equilibrium position, the restoring force has the form

$$F=-k\left(x-x_{0}\right). \tag{23.2.1}$$

We are familiar with this from Hooke's "law" for an ideal spring (see Chapter 19). So, an object attached to an ideal, massless spring, as in the figure below, should perform simple harmonic motion. This kind of oscillation is distinguished by the following characteristics:

- The position as a function of time, $x(t)$, is a sinusoidal function.
- The period of the oscillations does not depend on their amplitude (by "amplitude" we mean the maximum displacement from the equilibrium position).

What this second property means is that, for instance, with reference to Figure $23.2.1$, you can displace the mass a distance A, or $A/2$, or $A/3$, or whatever you choose, and the period (and frequency) of the resulting oscillations will be the same regardless. (This means, actually, that if you displace it farther it has to end up moving faster, to cover the larger distance in the same time.)

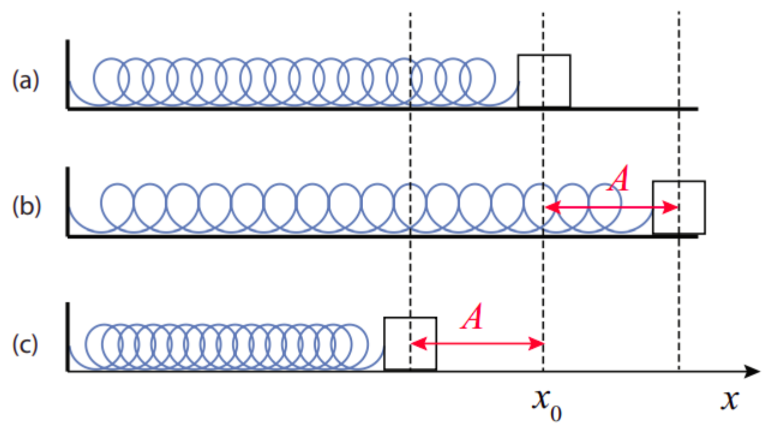


Figure $23.2.1$ (a) shows the spring in its relaxed state (the "equilibrium" position for the mass, at coordinate x_0). If displaced from equilibrium a distance A and released (b), the mass will perform simple harmonic oscillations with amplitude A.

Since we know that "Hooke's law" is actually just an approximation, valid only provided that the spring is not compressed or stretched too much, we expect that in real life the "ideal" simple harmonic motion properties I have listed above will only hold approximately, as well; so, in fact, if you stretch a spring too much you will get a different period, eventually, than if you stay in the "linear," Hooke's law regime. This is a general characteristic of most physical systems: simple harmonic motion only happens for relatively small oscillations, but "relatively small" can still be fairly large sometimes, and even as an approximation it is often an extremely valuable one.

The other distinctive characteristic of simple harmonic motion is that the position function is sinusoidal, by which I mean a sine or a cosine. Thus, for example, if the mass in Figure $23.2.1$ is *released from rest* at t = 0, and the position x is measured from the equilibrium position x_0 (that is, the point $x=x_0$ is taken as the origin of coordinates), the function $x(t)$ will be

$$x(t)=A\cos(\omega t) \tag{23.2.2}$$

where the quantity ω, known as the oscillator's *angular frequency*, is given by

$$\omega=\sqrt{\frac{k}{m}}. \tag{23.2.3}$$

Here, k is the spring constant, and m the mass of the object (remember the spring is assumed to be massless). I will prove that Equation (23.2.2), together with (23.2.3), satisfy Newton's second law of motion for this system in a moment; first, however, I need to say a couple of things about ω. You'll recall that we have used this symbol before, in Chapter 6, to represent the *angular velocity* of a particle moving in a circle (or, more generally, of any rotating object). Why bring it up again now for an apparently completely different purpose?

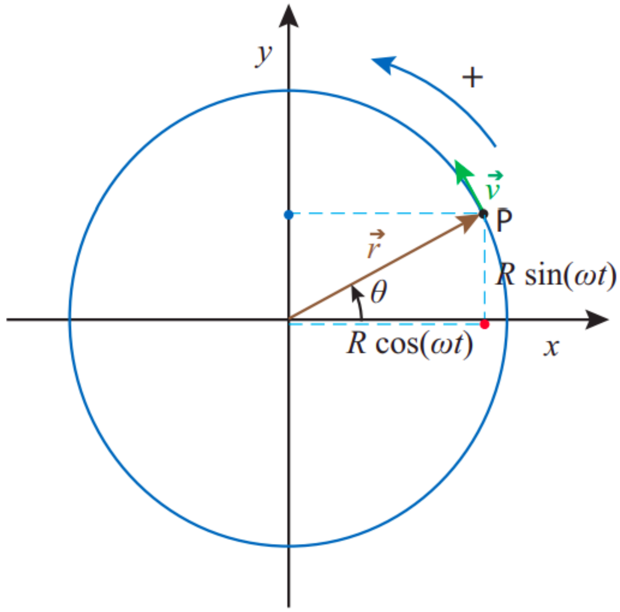


Figure $23.2.2$: A particle moving on a circle with constant angular velocity ω. Assuming $\theta = 0$ at $t = 0$, we have $\theta = \omega t$, and therefore the particle's x coordinate is given by the function $x(t) = R\cos(\omega t)$. This means the corresponding point on the x axis (the red dot) performs simple harmonic motion with angular frequency ω as the particle rotates.

The answer is that there is a very close relationship between simple harmonic motion and circular motion with constant speed, as Figure $23.2.2$ illustrates: as the point P rotates with constant angular velocity ω, its projection onto the x axis (the red dot in the figure) performs simple harmonic motion with angular frequency ω (and amplitude R). (Of course, there is nothing special about the x axis; the projection on *any* other axis will also perform simple harmonic motion with the same angular frequency; for example, the blue dot on the figure.)

If the angular velocity of the particle in Figure $23.2.2$ is constant, then its "orbital period" (the time needed to complete one revolution) will be $T = 2\pi/\omega$, and this will also be the period of the associated harmonic motion (the time it takes for the motion to repeat itself). You can see this directly from Equation (23.2.2): if you increase the time t by $2\pi/\omega$, you get the same value of x:

$$x\left(t + \frac{2\pi}{\omega}\right) = A\cos\left[\omega\left(t + \frac{2\pi}{\omega}\right)\right] = A\cos(\omega t + 2\pi) = A\cos(\omega t) = x(t). \tag{23.2.4}$$

Since the frequency f of an oscillator is equal to $1/T$, this gives us the following relationship between f and ω:

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{23.2.5}$$

One way to tell whether one is talking about an oscillator's frequency (f) or its angular frequency (ω)—apart from the different symbols, of course—is to pay attention to the units. The frequency f is usually given in hertz, whereas the angular frequency ω is always given in radians per second. Apart from the factor of 2π, they are, of course, completely equivalent; sometimes one is just more convenient than the other. On the other hand, the only way to tell whether ω is a harmonic oscillator's angular frequency or the angular velocity of something moving in a circle is from the context. (In this chapter, of course, it will always be the former).

Let us go back now to Equation 23.2.2 for our block-on-a-spring system. The derivative with respect to time will give us the block's velocity. This is a simple application of the chain rule of calculus:

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t). \tag{23.2.6}$$

Another derivative will then give us the acceleration:

$$a(t) = \frac{dv}{dt} = -\omega^2 A\cos(\omega t). \tag{23.2.7}$$

Note that the acceleration is always proportional to the position, only with the opposite sign. The proportionality constant is ω^2. Since the force exerted by the spring on the block is $F = -kx$ (because we are measuring the position from the equilibrium position x_0), Newton's second law, $F = ma$, gives us

$$ma = -kx \tag{23.2.8}$$

and you can check for yourself that this will be satisfied if x is given by Equation (23.2.2), a is given by Equation (23.2.7), and ω is given by Equation (23.2.3).

The expression (23.2.3) for ω is typical of what we find for many different kinds of oscillators: the restoring force (here represented by the spring constant k) and the object's inertia (m) together determine the frequency of the motion, acting in opposite directions: a larger restoring force means a higher frequency (faster oscillations) whereas a larger inertia means a lower frequency (slower oscillations—a more "sluggish" response).

The position, velocity and acceleration graphs for the motion (23.2.2) are shown in Figure 23.2.3 below. You may want to pay attention to some of their main features. For instance, the position and the velocity are what we call "90° out of phase": one is maximum (or minimum) when the other one is zero. The acceleration, on the other hand, is "180° out of phase" (that is to say, in complete opposition) with the position. As a result of that, all combinations of signs for a and v are possible: the object may be moving to the right with positive or negative acceleration (depending on which side of the origin it's on), and likewise when it is moving to the left.

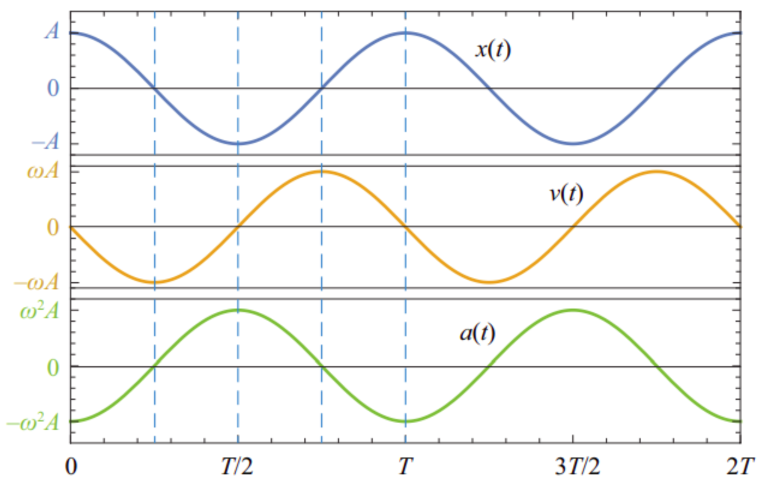


Figure 23.2.3: Position, velocity and acceleration as a function of time for an object performing simple harmonic motion according to Equation (23.2.2).

Since the time we choose as $t = 0$ is arbitrary, the function in Equation (23.2.2) (which assumes that $t = 0$ is when the object's displacement is maximum and positive) is clearly not the most general formula for simple harmonic motion. Another way to see this is to realize that we could have started the motion differently. For instance, we could have hit the block with a sharp, "impulsive" force, lasting only a very short time, so it would have acquired a substantial velocity before it could have moved very far from its initial (equilibrium) position. In such a case, the motion would be better described by a sine function, such as $x(t) = A\sin(\omega t)$, which is zero at $t = 0$ but whose derivative (the object's velocity) is maximum at that time.

If we stick to using cosines, for definiteness, then the most general equation for the position of a simple harmonic oscillator is as follows:

$$x(t) = A\cos(\omega t + \phi) \tag{23.2.9}$$

where ϕ is what we call a "phase angle," that allows us to match the function to the initial conditions—by which I mean, the object's initial position and velocity. Specifically, you can see, by setting $t = 0$ in Equation (23.2.9) and its derivative, that the initial position and velocity of the motion described by Equation (23.2.9) are

$$\begin{aligned} x_i &= A\cos\phi \\ v_i &= -\omega A\sin\phi. \end{aligned} \tag{23.2.10}$$

Conversely, if you are given x_i and v_i, you can use Eqs. (23.2.10) to determine A and ϕ, which is what you need to know in order to use Equation (23.2.9) (note that the angular frequency, ω, does *not* depend on the initial conditions—it is always the same regardless of how you choose to start the motion). Specifically, you can verify that Eqs. (23.2.10) imply the following:

$$A^2 = x_i^2 + \frac{v_i^2}{\omega^2} \tag{23.2.11}$$

and then, once you know A, you can get ϕ from either $x_i = A\cos\phi$ or $v_i = -\omega A\sin\phi$ (in fact, since the inverse sine and inverse cosine are both multivalued functions, you should use *both* equations, to make sure you get the correct sign for ϕ).

Energy in Simple Harmonic Motion

Equation (23.2.10) above actually follows from the conservation of energy principle for a harmonic oscillator. Consider again the mass on the spring in Figure 23.2.2 Its kinetic energy is clearly $K = \frac{1}{2}mv^2$, whereas the potential energy in the spring is $\frac{1}{2}kx^2$. Using Equation (23.2.9) and its derivative, we have

$$\begin{aligned} U^{spr} &= \frac{1}{2}kA^2\cos^2(\omega t+\phi) \\ K &= \frac{1}{2}m\omega^2A^2\sin^2(\omega t+\phi). \end{aligned} \tag{23.2.12}$$

Recalling Equation (23.2.3), note that $\omega^2 = k/m$, so if you substitute this in the second equation above, you can see that the amplitude of both the potential and the kinetic energy is the same, namely, $\frac{1}{2}kA^2$. Since, for any angle θ, it is always true that $\cos^2\theta + \sin^2\theta = 1$, we find

$$E_{sys} = U^{spr} + K = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2 \tag{23.2.13}$$

so the total energy of the system is constant (independent of time), at it should be, in the absence of dissipation. Figure 23.2.4 shows how the potential and kinetic energies oscillate in opposition, so one is maximum whenever the other is minimum. It also shows that they oscillate twice as fast as the oscillator itself: for example, the potential energy is maximum both when the displacement is maximum (spring maximally stretched) and when it is minimum (spring maximally compressed). Similarly, the kinetic energy is maximum when the oscillator passes through the equilibrium position, regardless of whether it is moving to the left or to the right.

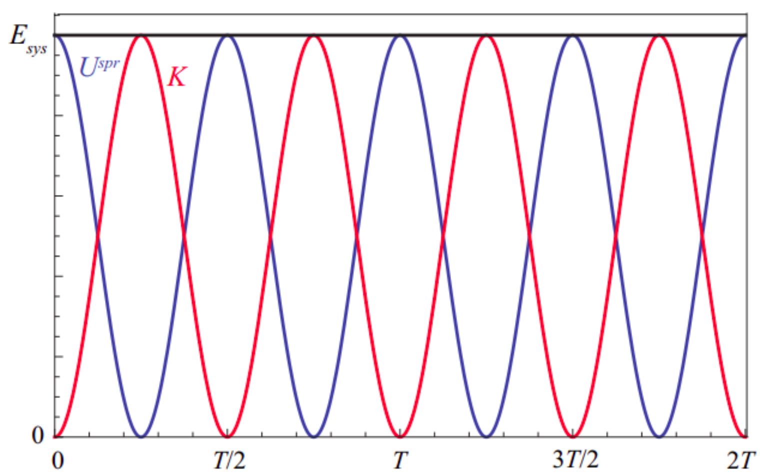


Figure 23.2.4: Kinetic (red), potential (blue) and total (black) energy for the oscillator shown in Figure 23.2.3.

Harmonic Oscillator Subject to an External, Constant Force

Consider a mass hanging from an ideal spring suspended from the ceiling, as in Figure 23.2.5 below (next page). Supposed the relaxed length of the spring is l, such that, in the absence of gravity, the object's equilibrium position would be at the height y_0 shown in figure 23.2.5(a). In the presence of gravity, of course, the spring needs to stretch, to balance the object's weight, and so the actual equilibrium position for the system will be y_0', as shown in figure 23.2.5(b). The upwards force from the spring at that point will be $-k\left(y_0' - y_0\right)$, and to balance gravity we must have

$$-k\left(y_0' - y_0\right) - mg = 0. \tag{23.2.14}$$

Suppose that we now stretch the spring beyond this new equilibrium position, so the mass is now at a height y (figure $23.2.5$(c)). What happens then? The net upwards force will be $-k\left(y-y_{0}\right)-m g$, but using Equation (23.2.14) this can be rewritten as

$$F_{net}=-k\left(y-y_{0}\right)-m g=-k(y-\left[\frac{m g}{k}+y_{0}^{\prime}\right])-m g=-k y+m g+k y_{0}^{\prime}-m g=-k\left(y-y_{0}^{\prime}\right) . \tag{23.2.15}$$

This is a remarkable result, because the force of gravity has disappeared completely from the final expression. Basically, the system behaves as if it consisted of just a spring of constant k with equilibrium length $l^{\prime}=l+y_{0}-y_{0}^{\prime}$, and *no gravity*. In other words, the only thing gravity does is to change the equilibrium position, so that if you now displace the mass, it will oscillate around $y_{0}^{\prime}$ instead of around y_{0}. The oscillation's period and frequency are the same as if the spring was horizontal.

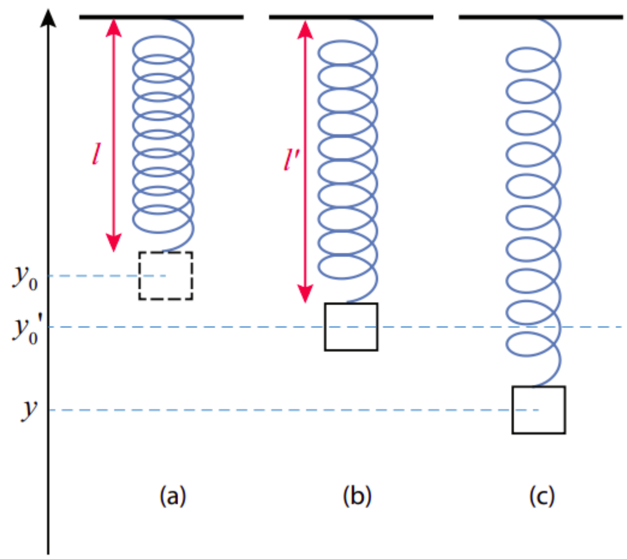


Figure $23.2.5$: (a) An ideal (massless) spring hanging from the ceiling, in its relaxed position. (b) With a mass m hanging from its end, the spring stretches to a new length $l^{\prime}$, so that $k\left(l^{\prime}-l\right)=m g$. (c) If the mass is now displaced from this equilibrium position (labeled $y_{0}^{\prime}$ in the figure) it will perform harmonic oscillations symmetrically around the point $y_{0}^{\prime}$, with the same frequency as if the spring was horizontal.

Although I have established this here for the specific case where the oscillator involves a spring, and the external force is gravity, this is a completely general result, valid for any simple harmonic oscillator, since for such a system the restoring force will always be a linear function of the displacement (which is all that is required for the math to work). As long as the external force is constant, the frequency of the oscillations will not be affected, and only the equilibrium position will change. In an example at the end of the chapter (under "Advanced Topics") I will show you how you can make use of this to calculate the effect of friction on the horizontal mass-spring combination in Figure $23.2.1$.

One thing you need to keep in mind, however, is that when the oscillator is subjected to an external force, as was the case here, its energy will *not*, in general, remain constant (unlike what we saw in the previous subsection "Energy in Simple Harmonic Motion"), since the external force will be doing work on the system as it oscillates. If the external force is constant, and does not change direction, this work will be positive half the time, and negative half the time. If it is kinetic friction, then of course it will change direction every half cycle, and the work will be negative all the time.

In the case shown in Figure $23.2.5$, the external force is gravity, which we know to be a conservative force, so the energy that will be conserved will be the total energy of the system that includes the oscillation and the Earth, and hence also the gravitational potential energy (for which we can use here the familiar form $U^{G}=m g y$):

$$E_{\text{osc+earth}}=U^{spr}+K+U^{G}=\frac{1}{2}\left(y-y_{0}\right)^{2}+\frac{1}{2} m v^{2}+m g y= \text{ const } . \tag{23.2.16}$$

The reason it is no longer possible to combine the terms $U^{spr}+K$ into the constant $\frac{1}{2} k A^{2}$, as in Equation (23.2.13), is that now we have

$$\begin{aligned} &y(t)=y_{0}^{\prime}+A \cos (\omega t+\phi) \\ &v(t)=-\omega A \sin (\omega t+\phi) \end{aligned} \tag{23.2.17}$$

so the oscillations are centered around the new equilibrium position y_0', but the spring energy is not zero at that point: it is zero at $y = y_0$ instead. You can check for yourself, however, that if you substitute Eqs. (23.2.17) into Equation (23.2.16), and make use of the fact that $k\left(y_0' - y_0\right) = -mg$ (Equation (23.2.14)), you do indeed get a constant, as you should.

23.3: Pendulums

The Simple Pendulum

Besides masses on springs, pendulums are another example of a system that will exhibit simple harmonic motion, at least approximately, as long as the amplitude of the oscillations is small. The simple pendulum is just a mass (or "bob"), approximated here as a point particle, suspended from a massless, inextensible string, as in Figure $23.3.1$.

We could analyze the motion of the bob by using the general methods introduced in Chapter 8 to deal with motion in two dimensions—breaking down all the forces into components and applying $\vec{F}_{net} = m\vec{a}$ along two orthogonal directions—but this turns out to be complicated by the fact that both the direction of motion and the direction of the acceleration are constantly changing. Although, under the assumption of small oscillations, it turns out that simply using the vertical and horizontal directions is good enough, this is not immediately obvious, and arguably it is not the most pedagogical way to proceed.

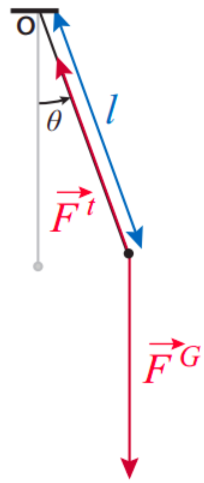


Figure $23.3.1$: A simple pendulum. The mass of the bob is m, the length of the string is l, and torques are calculated around the point of suspension O. The counterclockwise direction is taken as positive.

Instead, I will take advantage of the obvious fact that the bob moves on an arc of a circle, and that we have developed already in Chapter 22 a whole set of tools to deal with that kind of motion. Let us, therefore, describe the position of the pendulum by the angle it makes with the vertical, θ, and let $\alpha = d^2\theta/dt^2$ be the angular acceleration; we can then write the equation of motion in the form $\tau_{net} = I\alpha$, with the torques taken around the center of rotation—which is to say, the point from which the pendulum is suspended. Then the torque due to the tension in the string is zero (since its line of action goes through the center of rotation), and τ_{net} is just the torque due to gravity, which can be written

$$\tau_{net} = -mgl\sin\theta. \tag{23.3.1}$$

The minus sign is there to enforce a consistent sign convention for θ and τ: if, for instance, I choose counterclockwise as positive for both, then I note that when θ is positive (pendulum to the right of the vertical), τ is clockwise, and hence negative, and vice-versa. This is characteristic of a *restoring torque*, that is to say, one that will always try to push the system back to its equilibrium position (the vertical in this case).

As for the moment of inertia of the bob, it is just $I = ml^2$ (if we treat it as just a point particle), so the equation $\tau_{net} = I\alpha$ takes the form

$$ml^2\frac{d^2\theta}{dt^2} = -mgl\sin\theta. \tag{23.3.2}$$

The mass and one factor of l cancel, and we get

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta. \tag{23.3.3}$$

Equation (23.3.3) is an example of what is known as a *differential equation*. The problem is to find a function of time, $\theta(t)$, that satisfies this equation; that is to say, when you take its second derivative the result is equal to $-(g/l)\sin[\theta(t)]$. Such functions exist and are called *elliptic functions*; they are included in many modern mathematical packages, but they are still not easy to use. More importantly, the oscillations they describe, in general, are not of the simple harmonic type.

On the other hand, if the amplitude of the oscillations is small, so that the angle θ, expressed in radians, is a small number, we can make an approximation that greatly simplifies the problem, namely,

$$\sin\theta \simeq \theta. \tag{23.3.4}$$

This is known as the *small angle approximation*, and requires θ to be in radians. As an example, if θ = 0.2 rad (which corresponds to about 11.5°), we find $\sin\theta$ = 0.199, to three-figure accuracy.

With this approximation, the equation to solve become much simpler:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta. \tag{23.3.5}$$

We have, in fact, already solved an equation completely equivalent to this one in the previous section: that was equation (23.2.8) for the mass-on-a-spring system, which can be rewritten as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{23.3.6}$$

since $a = d^2x/dt^2$. Just like the solutions to (23.3.6) could be written in the form $x(t) = A\cos(\omega t+\phi)$, with $\omega = \sqrt{k/l}$, the solutions to (23.3.5) can be written as

$$\begin{aligned} \theta(t) &= A\cos(\omega t+\phi) \\ \omega &= \sqrt{\frac{g}{l}}. \end{aligned} \tag{23.3.7}$$

This tells us that if a pendulum is not pulled too far away from the vertical (say, about 10° or less) it will perform approximate simple harmonic oscillations, with a period of

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}. \tag{23.3.8}$$

This depends only on the length of the pendulum, and remains constant even as the oscillations wind down, which is why it became the basis for time-keeping devices, beginning with the invention of the *pendulum clock* by Christiaan Huygens in 1656. In particular, a pendulum of length l = 1 m will have a period of almost exactly 2 s, which is what gives you the familiar "tick-tock" rhythm of a "grandfather's clock," once per second (that is to say, once every half period).

The "Physical Pendulum"

By a "physical pendulum" one means typically any pendulum-like device for which the moment of inertia is not given by the simple expression $I = ml^2$. This means that the mass is not concentrated into a single point-like particle a distance l away from the point of suspension; rather, for example, the bob could have a size that is not negligible compared to l (as in Figure 23.3.2a), or the "string" could have a substantial mass of its own—it could, for instance, be a chain, like in a playground swing, or a metal rod, as in most pendulum clocks.

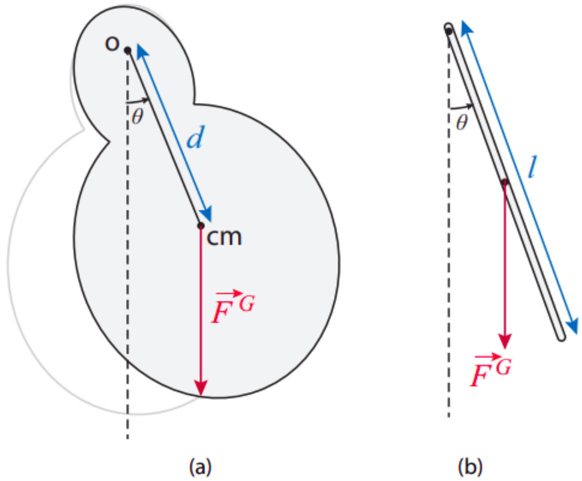


Figure 23.3.2 (b) shows the special case of a thin rod of length l pivoted at one end (the distance $d = l/2$ in this case). In both cases, there is an additional force (not shown) acting at the pivot point, to balance gravity.

Regardless of the reason, having to deal with a distributed mass means also that one needs to use the center of mass of the system as the point of application of the force of gravity. When this is done, the motion of the pendulum can again be described by the angle between the vertical and a line connecting the point of suspension and the center of mass. If the distance between these two points is d, then the torque due to gravity is $-mgd\sin\theta$, and the only other force on the system, the force at the pivot point, exerts no torque around that point, so we can write the equation of motion in the form

$$I\frac{d^2\theta}{dt^2} = -mgd\sin\theta. \tag{23.3.9}$$

Under the small-angle approximation, this will again lead to simple harmonic motion, only now with an angular frequency given by

$$\omega = \sqrt{\frac{mgd}{I}}. \tag{23.3.10}$$

As an example, consider the oscillations of a uniform, thin rod of length l and mass m pivoted at one end. We then have $I = ml^2/3$, and $d = l/2$, so Equation (23.3.10) gives

$$\omega = \sqrt{\frac{3g}{2l}}. \tag{23.3.11}$$

This is about 22% larger than the result (23.3.7) for a simple pendulum of the same length, implying a correspondingly shorter period.

23.4: Advanced Topics

Mass on a Spring Damped By Friction with a Surface

Consider the system depicted in Figure 23.2.1 in the presence of friction between the block and the surface. Let the coefficient of kinetic friction be μ_k and the coefficient of static friction be μ_s. As usual, we will assume that $\mu_s \geq \mu_k$.

As the mass oscillates, it will experience a kinetic friction force of magnitude $F^k = \mu_k mg$, in the direction opposite the direction of motion; that is to say, a force that changes direction every half period. As shown in section 23.2, this force does not change the frequency of the motion, but it displaces the equilibrium position in the direction of the force. Let's study this process in more detail.

First, think about that spring moving to the right - as the friction force acts on it, it shifts the position of the equilibrium, like in equation (23.2.14). We can determine the new equilibrium position using Newton's second law; we get

$$-kx_0' - \mu_k mg = 0 \rightarrow x_0' = -\frac{\mu_k mg}{k}. \tag{23.4.1}$$

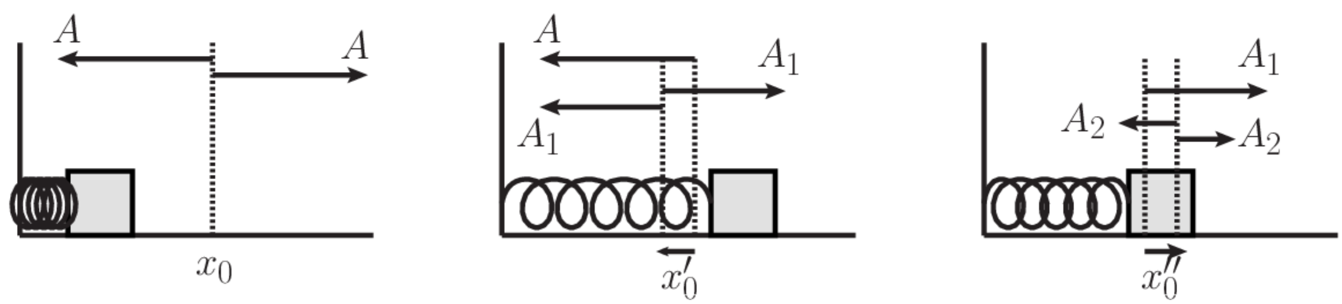
(Notice we have to be careful with the signs here - when moving to the right, the friction acts the other way!) Since the equilibrium moves to the right, the actual amplitude of this motion is (see the figure) $A_1 = A - |x_0'| = A - \frac{\mu_k mg}{k}$. Now when the block turns around, at the other maximum, during the leftward moving period the equlibrium changes again. We can find this again:

$$-kx_0'' + \mu_k mg = 0 \rightarrow x_0'' = \frac{\mu_k mg}{k}. \tag{23.4.2}$$

and the new amplitude is

$$A_2 = A_1 - |x_0''| = A - \frac{\mu_k mg}{k} - \frac{\mu_k mg}{k} = A - 2\frac{\mu_k mg}{k}. \tag{23.4.3}$$

In other words, the amplitude after n "half swings" is $A_n = A - n\frac{\mu_k mg}{k}$. The amplitude gets smaller and smaller each time, and in fact it vanishes for some number of these half-swings.



The figure below shows an example of how this would go, for the following choice of parameters: period T = 1 s, μ_k = 0.1, and A = 0.18 m. Note that, since x_0' depends only on the ratio $m/k = 1/\omega^2$, there is no need to specify m and k separately. We can determine the number of half-swings by just asking when the amplitude vanishes,

$$A_n = A - n\frac{\mu_k g}{\omega^2} = 0 \rightarrow n = \frac{A\omega^2}{\mu_k g} = 4.13. \tag{23.4.4}$$

So that means the motion goes for 4 half-swings before stopping.

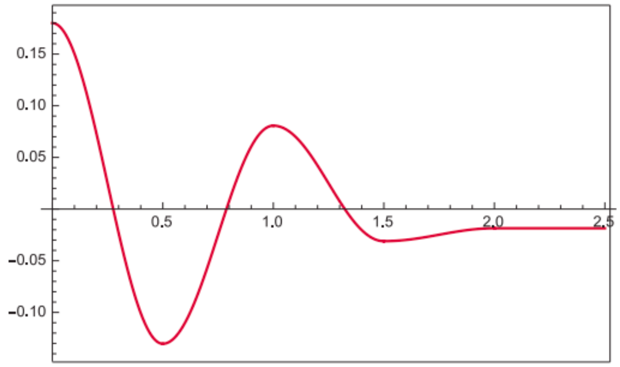


Figure 23.4.1: Damped oscillations.

Just for the record, this is *not* the way dissipation in simple harmonic motion is usually handled. The conventional thing is to assume a damping force that is proportional to the oscillator's velocity. You will almost certainly see this more standard approach (which leads to a relatively simple differential equation) in some later course.

We can study this same example in a little more detail using energy. Consider the total mechanical energy of the system, $E=\frac{1}{2}mv^2+\frac{1}{2}kx^2$. This energy does not include the friction, so we don't anticipate $\Delta E=0$; but we can still take a time derivative of this formula to see what happens:

$$\frac{dE}{dt}=mv\frac{dv}{dt}+kx\frac{dx}{dt}=mva+kxv. \tag{23.4.5}$$

The first equality is taking the derivative, being careful to use the chain run on the velocity and position. The second equality is just recognizing that the derivative of velocity and position is the accleration and velocity, respectively. The resulting expression looks a bit strange, but notice that the velocity is a common term, so we can write

$$\frac{dE}{dt}=v(ma+kx). \tag{23.4.6}$$

The expression inside the parenthesis looks interesting, because it looks something like Newton's second law for this situation - being sure of the signs (assuming the block is moving in the positive direction) we have

$$-kx-\mu_k mg=ma\rightarrow ma+kx=-\mu_k mg. \tag{23.4.7}$$

Plugging this in above we finally find

$$\frac{dE}{dt}=-v\mu_k mg. \tag{23.4.8}$$

So the rate of energy loss is negative - exactly what we would have expected. Using a little calculus we can take this analysis further - recall that $v=dx/dt$, so we can intergrate

$$\int_{E_i}^{E_f}dE=-\mu_k mg\int_{v_i}^{v_f}vdt=-mu_k mg\int_{x_i}^{x_f}dx\rightarrow\Delta E=-\mu mg\Delta x. \tag{23.4.9}$$

Of course, this is the work-energy theorem! (What else could we possibly have gotten by calculating E_f-E_i ??) If we pick that half-swing from above, $\Delta x=2\tilde{A}$ and we can determine that our system looses $\Delta E=-2\mu mg\tilde{A}$ each half cycle - although notice that the amplitude $\tilde{A}$ is actually getting smaller during this process, so we can't easily compare our two approaches.

The Cavendish Experiment- How to Measure G with a Torsion Balance

Suppose that you want to try and duplicate Cavendish's experiment to measure directly the gravitational force between two masses (and hence, indirectly, the value of G). You take two relatively small, identical objects, each of mass m, and attach them to the ends of a rod of length l (let us say the mass of the rod is negligible, for simplicity), making a sort of dumbbell; then you suspend this from the ceiling, by the midpoint, using a nylon line.

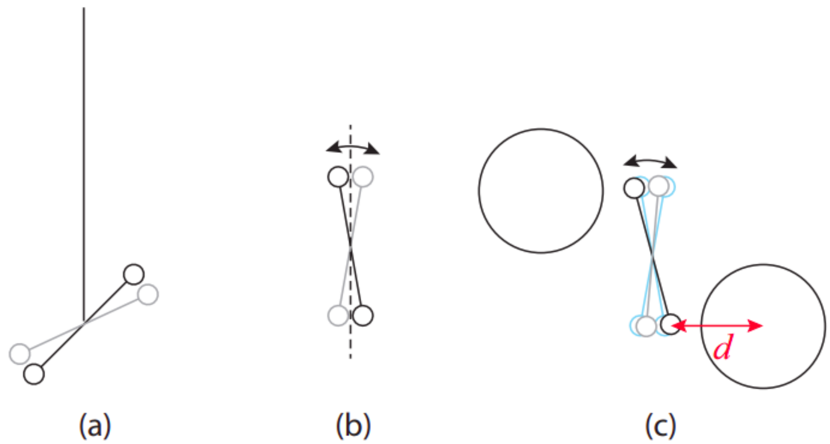


Figure 23.4.2: : (a) Torsion balance. The extremes of the oscillation are drawn in black and gray, respectively. (b) The view from the top. The dashed line indicates the equilibrium position. (c) In the presence of two nearby large masses, the equilibrium position is tilted very slightly; the light blue lines in the background show the oscillation in the absence of the masses, for reference.

You have now made a *torsion balance* similar to the one Cavendish used. You will probably find out that it it is very hard to keep it motionless: the slightest displacement causes it to oscillate around an equilibrium position. The way it works is that an angular displacement θ from equilibrium puts a small twist on the line, which results in a restoring torque $\tau=-\kappa\theta$, where κ is the *torsion constant* for your setup. If your dumbbell has moment of inertia I, then the equation of motion $\tau=I\alpha$ gives you

$$I\frac{d^2\theta}{dt^2}=-\kappa\theta. \tag{23.4.10}$$

If you compare this to Equation (23.3.3), and follow the derivation there, you can see that the period of oscillation is

$$T=2\pi\sqrt{\frac{I}{\kappa}} \tag{23.4.11}$$

so if you measure T you can get κ, since $I=2m(l/2)^2=ml^2/2$ for the dumbbell.

Now suppose you bring two large masses, a distance d each from each end of the dumbbell, perpendicular to the dumbbell axis, and one on either side, as in the figure. The gravitational force $F^G=GmM/d^2$ between the large and small mass results in a net “external” torque of magnitude

$$\tau_{ext}=2F^G\times\frac{l}{2}=F^Gl. \tag{23.4.12}$$

This torque will cause a very small displacement, so small that the change in d will be practically negligible, so you can treat F^G, and hence τ_{ext}, as a constant. Then the situation is analogous to that of an oscillator subjected to a constant external force (section 23.2): the frequency of the oscillations will not change, but the equilibrium position will. In Equation (23.2.14) we found that $y_0'-y_0=F_{ext}/k$ for a spring of spring constant k, where y_0 was the old and y_0' the new equilibrium position (the force was equal to $-mg$; the displacement of the equilibrium position will be in the direction of the force). For the torsion balance, the equivalent result is

$$\theta_0'-\theta_0=\frac{\tau_{ext}}{\kappa}=\frac{F^Gl}{\kappa}. \tag{23.4.13}$$

So, if you measure the angular displacement of the equilibrium position, you can get F^G. This displacement is going to be very small, but you can try to monitor the position of the dumbbell by, for instance, reflecting a laser from it (or, one or both of your small masses could be a small laser). Tracking the oscillations of the point of laser light on the wall, you might be able to detect the very small shift predicted by Equation (23.4.13).

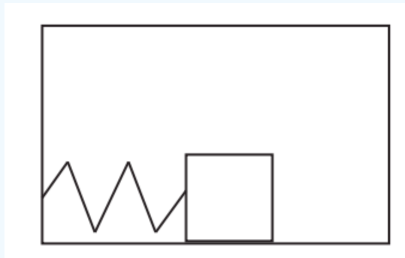
Historically, this experimental set up was used in the first precision calculation of the gravitational constant G, from the force F^G. This experiment was carried out in the late 1790s by Henry Cavendish, but several others were involved in the several decades beforehand (you can check the story out on Wikipedia).

23.5: Examples

Example 23.5.1: Oscillator in a box (a basic accelerometer!)

Consider a block-spring system inside a box, as shown in the figure. The block is attached to the spring, which is attached to the inside wall of the box. The mass of the block is 0.2 kg. For parts (a) through (f), assume that the box does not move.

Suppose you pull the block 10 cm to the right and release it. The angular frequency of the oscillations is 30 rad/s. Neglect friction between the block and the bottom of the box.



a. What is the spring constant?
b. What will be the amplitude of the oscillations?
c. Taking to the right to be positive, at what point in the oscillation is the velocity minimum and what is its minimum value?
d. At what point in the oscillation is the acceleration minimum, and what is its minimum value?
e. What is the total energy of the spring-block system?
f. If you take $t = 0$ to be the instant when you release the block, write an equation of motion for the oscillation, $x(t) = ?$, identifying the values of all constants that you use.
g. Imagine now that the box, with the spring and block in it, starts moving to the left with an acceleration $a = -4$ m/s^2. By how much does the equilibrium position of the block shift (relative to the box), and in what direction?

Solution

Most of this is really pretty straightforward, since it is just a matter of using the equations introduced in this chapter properly:

(a) Since we know that for this kind of situation, the angular frequency, the mass and the spring constant are related by

$$\omega=\sqrt{\frac{k}{m}}$$

we con solve this for k:

$$k=m\omega^2=0.2\text{ kg}\times\left(30\ \frac{\text{rad}}{\text{s}}\right)^2=180\ \frac{\text{N}}{\text{m}}$$

(b) The amplitude will be 10 cm, since it is released at that point with no kinetic energy.

(c) The velocity is minimum (largest in magnitude, but with a negative sign) as the object passes through the equilibrium position moving to the left.

$$v_{\text{min}}=-\omega A=-\left(30\ \frac{\text{rad}}{\text{s}}\right)\times 0.1\text{ m}=-3\ \frac{\text{m}}{\text{s}}$$

(d) The acceleration is minimum (again, largest in magnitude, but with a negative sign) when the spring is maximally stretched (block is farthest to the right), since this gives you the maximal force in the negative direction:

$$a_{\text{min}}=-\omega^2A=-\left(30\ \frac{\text{rad}}{\text{s}}\right)^2\times 0.1\ m=-90\ \frac{\text{m}}{\text{s}^2}$$

(e) The total energy is given by the formula (either one is acceptable)

$$E=\frac{1}{2}m\omega^2A^2=\frac{1}{2}kA^2=\frac{1}{2}(180\text{ N/m})\times(0.1\text{ m})^2=0.9\text{ J}$$

(You could also use $E = \frac{1}{2} m v_{max}^2$.)

(f) The result is

$$x(t) = A\cos(\omega t) = A\sin\left(\omega t + \frac{\pi}{2}\right)$$

with A = 0.1 m and ω = 30 rad/s. You could also just write the numbers directly in the formula, but in that case you need to include the units implicitly or explicitly. What I mean by "implicitly" is to say something like: "$x(t) = 0.1\cos(30t)$, with x in meters and t in seconds."

(g) The equilibrium position is where the block could sit at rest relative to the box. In that case, relative to the ground outside the box, it would be moving with an acceleration a = −4 m/s^2, and the spring force (which is the only actual force acting on the block) would have to provide this acceleration:

$$F_x^{spr} = -k\Delta x = ma$$

so

$$\Delta x = -\frac{ma}{k} = \frac{0.2\ kg \times 4\ m/s^2}{180\ N/m} = 0.00444\ m$$

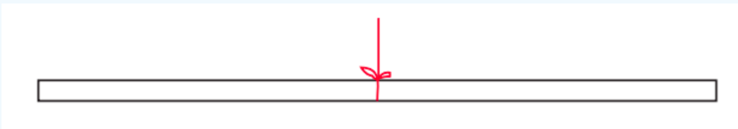
or 4.44 mm. This is positive, so the spring stretches—the equilibrium position for the block is shifted to the right, relative to the box's walls.

Another way to see this is the following. As we saw in example 20.3.4, an accelerated reference system, with acceleration a, appears "from the inside" as an inertial reference system subject to a gravitational interaction that pulls any object with mass m with a force equal to ma in the direction opposite the acceleration. Therefore, inside the box, which is accelerating towards the left, the block behaves as if there was a force of gravity of magnitude ma, pulling it to the right. In other words, we have a situation like the one illustrated in Figure 11.2.5, only sideways. As in that case, we find the equilibrium position is shifted just enough for the force of the stretched spring to match the "force of gravity," and in this way we get again the equation $F_x^{spr} = ma$.

To get an accelerometer, we provide the box with some readout mechanism that can tell us the change in the oscillator's equilibrium position. This basic principle is one of the ways accelerometers— and so-called "inertial navigation systems"— work.

Example 23.5.2: Meter stick as a physical pendulum

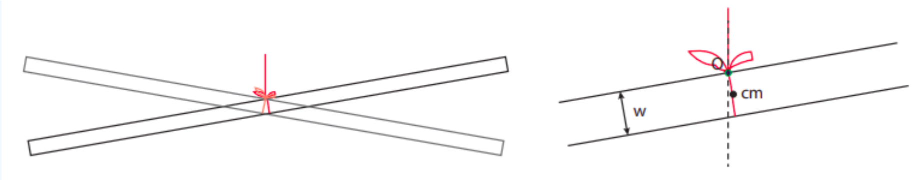
While working on the lab on torques, you notice that a meter stick suspended from the middle behaves a little like a pendulum, in that it performs very slow oscillations when you tilt it slightly. Intrigued, you notice that it is suspended by a simple loop of string tied in a knot at the top (see figure). You measure the period of the oscillations to be about 5 s, and the width of the stick to be about 2.5 cm.



a. What does this tell you about the quantity I/M, where M is the mass of the stick, and I its moment of inertia around a certain point?
b. What is the "certain point" mentioned in (a)?

Solution

As the picture below shows, the stick will behave like a physical pendulum, oscillating around the point of suspension O, which in this case is just next to the stick, where the knot is. As seen in the blown-up detail, if the width of the stick is w, the center of mass of the stick is located a distance $d = w/2$ away from the point of suspension:



As shown in Section 23.3, we have then

$$\omega=\sqrt{\frac{Mgw}{2I}}. \tag{23.5.1}$$

Squaring this, and solving for I/M,

$$\frac{I}{M}=\frac{gw}{2\omega^2}=\frac{9.8\text{ m/s}^2\times 0.025\text{ m}}{2\times(2\pi/5\text{ s})^2}=0.0776\text{ m}^2. \tag{23.5.2}$$

The moment of inertia is to be calculated around the point O, that is to say, the point of suspension (where the knot is in the figure). For reference, the moment of inertia of a thin rod of length l around its midpoint is $Ml^2/12=0.083l^2$. The length of the meter stick is, of course, 1 m, so the result $I/M\sim$ 0.08 m^2 obtained above seems reasonable.

Exercise 23.5.3

A block of mass m is sliding on a frictionless, horizontal surface, with a velocity v_i. It hits an ideal spring, of spring constant k, which is attached to the wall. The spring compresses until the block momentarily stops, and then starts expanding again, so the block ultimately bounces off.

a. Write down an equation of motion (a function $x(t)$) for the block, which is valid for as long as it is in contact with the spring. For simplicity, assume the block is initially moving to the right, take the time when it first makes contact with the spring to be t = 0, and let the position of the block at that time to be x = 0. Make sure that you express any constants in your equation (such as A or ω) in terms of the given data, namely, m, v_i, and k.
b. Sketch the function $x(t)$ for the relevant time interval.

Exercise 23.5.4

For this problem, imagine that you are on a ship that is oscillating up and down on a rough sea. Assume for simplicity that this is simple harmonic motion (in the vertical direction) with amplitude 5 cm and frequency 2 Hz. There is a box on the floor with mass m = 1 kg.

a. Assuming the box remains in contact with the floor throughout, find the maximum and minimum values of the normal force exerted on it by the floor over an oscillation cycle.
b. How large would the amplitude of the oscillations have to become for the box to lose contact with the floor, assuming the frequency remains constant? (Hint: what is the value of the normal force at the moment the box loses contact with the floor?)

Exercise 23.5.5

Imagine a simple pendulum swinging in an elevator. If the cable holding the elevator up was to snap, allowing the elevator to go into free fall, what would happen to the frequency of oscillation of the pendulum? Justify your answer.

Exercise 23.5.6

Consider a block of mass m attached to *two* springs, one on the left with spring constant k_1 and one on the right with spring constant k_2. Each spring is attached on the other side to a wall, and the block slides without friction on a horizontal surface. When the block is sitting at x = 0, both springs are relaxed.

Write Newton's second law, $F=ma$, as a differential equation for an arbitrary position x of the block. What is the period of oscillation of this system?

Exercise 23.5.7

Consider the block hanging from a spring shown in Figure 23.2.5. Suppose the mass of the block is 1.5 kg and the system is at rest when the spring has been stretched 2 cm from its original length (that is, with reference to the figure, $y_0 - y_0'$ = 0.02 m).

a. What is the value of the spring constant k?
b. If you stretch the spring by an additional 2 cm downward from this equilibrium position, and release it, what will be the frequency of the oscillations?
c. Now consider the system formed by the spring, the block, and the earth. Take the “zero” of gravitational potential energy to be at the height y_0' (the equilibrium point; you may also use this as the origin for the vertical coordinate!), and calculate all the energies in the system (kinetic, spring/elastic, and gravitational) at the highest point in the oscillation, the equilibrium point, and the lowest point. Verify that the sum is indeed constant.

CHAPTER OVERVIEW

24: Waves in One Dimension

24.1: Traveling Waves

In our study of mechanics we have so far dealt with particle-like objects (objects that have only translational energy), and extended, rigid objects, which may also have rotational energy. We have, however, implicitly assumed that all the objects we studied had some internal structure, or were to some extent deformable, whenever we allowed for the possibility of their storing other forms of energy, such as chemical or thermal.

This chapter deals with a very common type of organized (as opposed to incoherent) motion exhibited by extended elastic objects, namely, *wave motion*. (Often, the "object" in which the wave motion takes place is called a "medium.") Waves can be "traveling" or "standing," and we will start with the traveling kind, since they are the ones that most clearly exhibit the characteristics typically associated with wave motion.

A *traveling wave* in a medium is a *disturbance* of the medium that propagates through it, in a definite direction and with a definite velocity. By a "disturbance" we typically mean a displacement of the parts that make up the medium, away from their rest or equilibrium position. The idea here is to regard each part of an elastic medium as, potentially, an oscillator, which couples to the neighboring parts by pushing or pulling on them (for an example of how to model this mathematically, see Advanced Topic 12.6 at the end of this chapter). When the traveling wave reaches a particular location in the medium, it sets that part of the medium in motion, by giving it some energy and momentum, which it then passes on to a neighboring part, and so on down the line.

You can see an example of how this works in a slinky. Start by stretching the slinky somewhat, then grab a few coils, bunch them up at one end, and release them. You should see a "compression pulse" traveling down the slinky, with very little distortion; you may even be able to see it being reflected at the other end, and coming back, before all its energy is dissipated away.

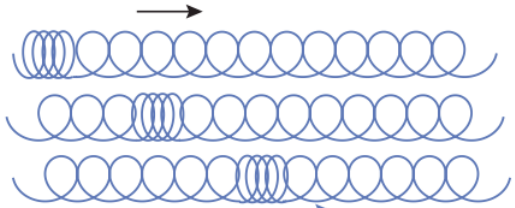


Figure 24.1.1: A longitudinal (compression) wave pulse traveling down a slinky.

The compression pulse in the slinky in Figure $24.1.1$ is an example of what is called a *longitudinal wave*, because the displacement of the parts that make up the medium (the rings, in this case) takes place along the same spatial dimension along which the wave travels (the horizontal direction, in the figure). The most important examples of longitudinal waves are *sound waves*, which work a bit like the longitudinal waves on the slinky: a region of air (or some other medium) is compressed, and as it expands it pushes on a neighboring region, causing it to compress, and passing the disturbance along. In the process, regions of rarefaction (where the density drops below its average value) are typically produced, alongside the regions of compression (increased density).

The opposite of a longitudinal wave is a *transverse wave*, in which the displacement of the medium's parts takes place in a direction *perpendicular* to the wave's direction of travel. It is actually also relatively easy to produce a transverse wave on a slinky: again, just stretch it somewhat and give one end a vigorous shake up and down. It is, however, a little hard to draw the resulting pulse on a long spring with all the coils, so in Figure $24.1.2$ below I have instead drawn a transverse wave pulse on a *string*, which you can produce in the same way. (Strings have other advantages: they are also easier to describe mathematically, and they are very relevant, particularly to the production of musical sounds.)

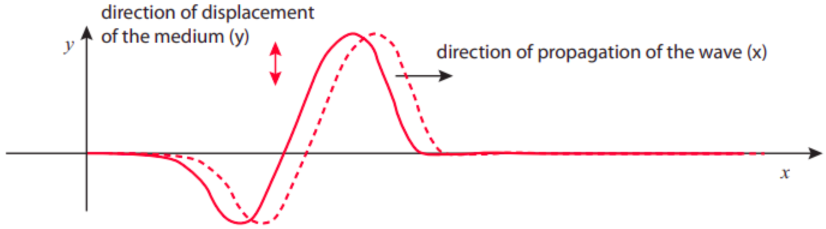


Figure 24.1.2: A transverse wave pulse traveling down a string. This pulse can be generated by giving an end of the string a strong shake, while holding the string taut. (You can do this on a slinky, too.)

Perhaps the most important (and remarkable) property of wave motion is that it can carry energy and momentum over relatively long distances *without an equivalent transport of matter*. Again, think of the slinky: the “pulse” can travel through the slinky’s entire length, carrying momentum and energy with it, but each individual ring does not move very far away from its equilibrium position. Ideally, after the pulse has passed through a particular location in the medium, the corresponding part of the medium returns to its equilibrium position and does not move any more: all the energy and momentum it momentarily acquired is passed forward. The same is (ideally) true for the transverse wave on the string in Figure $24.1.2$

Since this is meant to be a very elementary introduction to waves, I will consider only this case of “ideal” (technically known as “linear and dispersion-free”) wave propagation, in which the speed of the wave does not depend on the shape or size of the disturbance. In that case, the disturbance retains its “shape” as it travels, as I have tried to illustrate in figures $24.1.1$ and $24.1.2$

The "Wave Shape" Function- Displacement and Velocity of the Medium

In a slinky, what I have been calling the “parts” of the medium are very clearly seen (they are, naturally, the individual rings); in a “homogeneous” medium (one with no visible parts), the way to describe the wave is to break up the medium, in your mind, into infinitely many small parts or “particles” (as we have been doing for extended systems all semester), and write down equations that tell us how each part moves. Physically, you should think of each of these “particles” as being large enough to contain many molecules, but small enough that its position in the medium may be represented by a mathematical point.

The standard way to label each “particle” of the medium is by the position vector of its equilibrium position (the place where the particle sits at rest in the absence of a wave). In the presence of the wave, the particle that was initially at rest at the point $\vec{r}$ will undergo a displacement that I am going to represent by the vector $\vec{\xi}$ (where ξ is the Greek letter “xi”). This displacement will in general be a function of time, and it may also be different for different particles, so it will also be a function of $\vec{r}$, the equilibrium position of the particle we are considering. The particle’s position under the influence of the wave becomes then

$$\vec{r}+\vec{\xi}(\vec{r}, t). \tag{24.1.1}$$

This is very general, and it can be given a simpler form for simple cases. For instance, for a transverse wave on a string, we can label each part of the string at rest by its x coordinate, and then take the displacement to lie along the y axis; the position vector, then, could be written in component form as $(x, \xi(x, t), 0)$. Similarly, we can consider a “plane” sound wave as a longitudinal wave traveling in the x direction, where the density of the medium is independent of y and z (that is, it is constant on planes perpendicular to the direction of propagation). In that case, the equilibrium coordinate x can be used to refer to a whole “slice” of the medium, and the position of that slice, along the x axis, at the time t will be given by $x+\xi(x, t)$. In both of these cases, the displacement vector ξ reduces to a single nonzero component (along the y or x axis, respectively), which can, of course, be positive or negative. I will restrict myself implicitly to these simple cases and treat ξ as a scalar from this point on.

Under these conditions, the function $\xi(x, t)$ (which is often called the *wave function*) gives us the shape of the “displacement wave,” that is to say, the displacement of every part of the medium, labeled by its equilibrium x-coordinate, at any instant in time. Accordingly, taking the derivative of ξ gives us the velocity of the corresponding part of the medium:

$$v_{\text{med}}=\frac{d \xi}{d t}. \tag{24.1.2}$$

This is also, in general, a vector (along the direction of motion of the wave, if the wave is longitudinal, or perpendicular to it if the wave is transverse). It is also a function of time, and in general *will be different from the speed of the wave itself*, which we have taken to be constant, and which I will denote by c instead.

Harmonic Waves

An important class of waves are those for which the wave function is sinusoidal. This means that the different parts of the medium execute simple harmonic motion, all with the same frequency, but each (in general) with a different phase. Specifically, for a sinusoidal wave we have

$$\xi(x, t)=\xi_0 \sin \left[\frac{2 \pi x}{\lambda}-2 \pi f t\right]. \tag{24.1.3}$$

In Equation (24.1.3), f stands for the frequency, and plays the same role it did in the previous chapter: it tells us how often (that is, how many times per second) the corresponding part of the medium oscillates around its equilibrium position. The constant ξ_0 is just the amplitude of the oscillation (what we used to call A in the previous chapter). The constant λ, on the other hand, is sometimes known as the "spatial period," or, most often, the *wavelength* of the wave: it tells you how far you have to travel along the x axis, from a given point x, to find another one that is performing the same oscillation with the same amplitude and phase.

A couple of snapshots of a harmonic wave are shown in Figure $24.1.3$. The figure shows the displacement ξ, at two different times, and as a function of the coordinate x used to label the parts into which we have broken up the medium (as explained in the previous subsection). As such, the wave it represents could equally well be longitudinal or transverse. If it is transverse, like a wave on a string, then you can think of ξ as being essentially just y, and then the displacement curve (the blue line) just gives you the shape of the string. If the wave is longitudinal, however, then it is a bit harder to visualize what is going on just from the plot of $\xi(x,t)$. This is what I have tried to do with the density plots at the bottom of the figure.

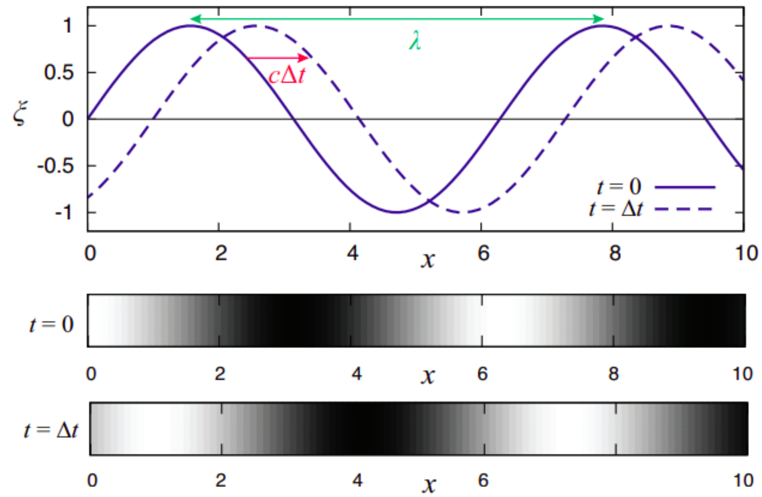


Figure $24.1.3$: Top: two snapshots of a traveling harmonic wave at t = 0 (solid) and at $t = \Delta t$ (dashed). The quantity ξ is the displacement of a typical particle of the medium at each point x (the wave is traveling in the positive x direction). Units for both x and ξ are arbitrary. Bottom: The corresponding densities, for the case of a longitudinal wave.

Imagine the wave is longitudinal, and consider the $x = \pi$ point on the t = 0 curve (the first zero, not counting the origin). A particle of the medium immediately to the left of that point has a positive displacement, that is, it is pushed towards $x = \pi$, whereas a slice on the right has a negative displacement—which means it is *also* pushed towards $x = \pi$. We therefore expect the density of the medium to be highest around that point, whereas around $x = 2\pi$ the opposite occurs: particles to the left are pushed to the left and those to the right are pushed to the right, resulting in a low-density region. The density plot labeled t = 0 attempts to show this using a grayscale where darker and lighter correspond to regions of higher and smaller density, respectively. At the later time $t = \Delta t$ the high and low density regions have moved a distance $c\Delta t$ to the right, as shown in the second density plot.

Regardless of whether the wave is longitudinal or transverse, if it is harmonic, the spatial pattern will repeat itself every wavelength; you can think of the wavelength λ as the distance between two consecutive crests (or two consecutive troughs) of the displacement function, as shown in the figure. If the wave is traveling with a speed c, an observer sitting at a fixed point x would see the disturbance pass through that point, the particles move up and down (or back and forth), and the motion repeat itself after the wave has traveled a distance λ, that is, after a time λ/c. This means the period of the oscillation at every point is $T = \lambda/c$, and the corresponding frequency $f = 1/T = c/\lambda$:

$$f = \frac{c}{\lambda}. \tag{24.1.4}$$

This is the most basic equation for harmonic waves. Making use of it, Equation (24.1.3) can be rewritten as

$$\xi(x,t) = \xi_0 \sin\left[\frac{2\pi}{\lambda}(x - ct)\right]. \tag{24.1.5}$$

This suggests that if we want to have a wave moving to the left instead, all we have to do is change the sign of the term proportional to c, which is indeed the case.

In contrast to the wave speed, which is a constant, the speed of any part of the medium, with equilibrium position x, at the time t, can be calculated from Eqs. (24.1.2) and (24.1.3) to be

$$v_{med}(x,t)=2\pi f\xi_0\cos\left[\frac{2\pi x}{\lambda}-2\pi ft\right]=\omega\xi_0\cos\left[\frac{2\pi x}{\lambda}-2\pi ft\right] \tag{24.1.6}$$

(where I have introduced the angular frequency $\omega=2\pi f$). Again, this is a familiar result from the theory of simple harmonic motion: the velocity is “90 degrees out of phase” with the displacement, so it is maximum or minimum where the displacement is zero (that is, when the particle is passing through its equilibrium position in one direction or the other).

Note that the result (24.1.6) implies that, for a longitudinal wave, the *“velocity wave” is in phase with the “density wave”*: that is, the medium velocity is large and positive where the density is largest, and large and negative where the density is smallest (compare the density plots in Figure 24.1.3). If we think of the momentum of a volume element in the medium as being proportional to the product of the instantaneous density and velocity, we see that for this wave, which is traveling in the positive x direction, there is more “positive momentum” than “negative momentum” in the medium at any given time (of course, if the wave had been traveling in the opposite direction, the sign of v_{med} in Equation (24.1.6) would have been negative, and we would have found the opposite result). This confirms our expectation that the wave carries a net amount of momentum in the direction of propagation. A detailed calculation (which is beyond the scope of this book) shows that the time-average of the “momentum density” (momentum per unit volume) can be written as

$$\frac{p}{V}=\frac{1}{2c}\rho_0\omega^2\xi_0^2 \tag{24.1.7}$$

where ρ_0 is the medium’s average mass density (mass per unit volume). Interestingly, this result applies also to a transverse wave!

As mentioned in the introduction, the wave also carries energy. Equation (24.1.6) could be used to calculate the kinetic energy of a small region of the medium (with volume V and density ρ_0, and therefore $m=\rho_0 V$), and its time average. This turns out to be equal to the time average of the elastic potential energy of the same part of the medium (recall that we had the same result for harmonic oscillators in the previous chapter). In the end, the total time-averaged energy density (energy per unit volume) in the region of the medium occupied by the wave is given by

$$\frac{E}{V}=\frac{1}{2}\rho_0\omega^2\xi_0^2. \tag{24.1.8}$$

Comparing (24.1.7) and (24.1.8), you can see that

$$\frac{E}{V}=\frac{cp}{V}. \tag{24.1.9}$$

This relationship between the energy and momentum densities (one is just c times the other) is an extremely general result that applies to all sorts of waves, including electromagnetic waves!

The Wave Velocity

You may ask, what determines the speed of a wave in a material medium? The answer, qualitatively speaking, is that c always ends up being something of the form

$$c\sim\sqrt{\frac{\text{stiffness}}{\text{inertia}}} \tag{24.1.10}$$

where “stiffness” is some measure of how rigid the material is (how hard it is to compress it or, in the case of a transverse wave, shear it), whereas “inertia” means some sort of mass density.

For a transverse wave on a string, for instance, we find

$$c=\sqrt{\frac{Ft}{\mu}} \tag{24.1.11}$$

where F^t is the tension in the string and μ is not the “reduced mass” of anything (sorry about the confusion!), but a common way to write the “mass per unit length” of the string. We could also just write $\mu=M/L$, where M is the total mass of the string and L its length. Note that the tension is a measure of the stiffness of the string, so this is, indeed, of the general form (24.1.10). For two strings under the same tension, but with different densities, the wave will travel more slowly on the denser one.

For a sound wave in a fluid (liquid or gas), the speed of sound is usually written

$$c=\sqrt{\frac{B}{\rho_0}} \tag{24.1.12}$$

where ρ_0 is the regular density (mass per unit volume), and B is the so-called *bulk modulus*, which gives the fluid's resistance to a change in volume when a pressure P is applied to it: $B=P/(\Delta V/V)$. So, once again, we get something of the form (24.1.10). In this case, however, we find that for many fluids the density and the stiffness are linked, so they increase together, which means we cannot simply assume that the speed of sound will be automatically smaller in a denser medium. For gases, this does work well: the speed of sound in a lighter gas, like helium, is greater than in air, whereas in a denser gas like sulfur hexafluoride the speed of sound is less than in air[1]. However, if you compare the speed of sound in water to the speed of sound in air, you find it is much greater in water, since water is much harder to compress than air: in this case, the increase in stiffness more than makes up for the increase in density.

The same thing happens if you go from a liquid like water to a solid, where the speed of sound is given by

$$c=\sqrt{\frac{Y}{\rho_0}} \tag{24.1.13}$$

where Y is, again, a measure of the stiffness of the material, called the *Young modulus*. Since a solid is typically even harder to compress than a liquid, the speed of sound in solids such as metals is much greater than in water, despite their being also denser. For reference, the speed of sound in steel would be about c = 5,000 m/s; in water, about 1,500 m/s; and in air, "only" about 340 m/s.

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[1]This effect can be used to produce "funny voices," because of the relationship $f=c/\lambda$ (Equation (24.1.4)), which will be discussed in greater detail in the section on standing waves.

Reflection and Transmission of Waves at a Medium Boundary

Suppose that you have two different elastic media, joined in some way at a common boundary, and you have a wave in the first medium traveling towards the boundary. Examples of media connected this way could be two different strings tied together, or two springs with different spring constants joined at the ends; or, for sound waves, it could just be something like water with air above it: a compression wave in air traveling towards the water surface will push on the surface and set up a sound wave there, and vice-versa.

The first thing to notice is that, if the incident wave has a frequency f, it will cause the medium boundary, when it arrives there, to oscillate at that frequency. As a result of that, the wave that is set up in the second medium—which we call the *transmitted wave*—will also have the same frequency f. Again, think of the two strings tied together, so the first string "drives" the second one at the frequency f; or the sound at the air-water boundary, driving (pushing) the water surface at the frequency f.

So, the incident and transmitted waves will have the same frequency, but it is clear that, if the wave speeds in the two media are different, they cannot have the same wavelength: since the relation (24.1.4) has to hold, we will have $\lambda_1=c_1/f$, and $\lambda_2=c_2/f$. Thus, if a periodic wave goes from a slower to a faster medium, its wavelength will increase, and if it goes from a faster to a slower one, the wavelength will decrease.

It is easy to see physically why this happens, and how it has to be the case even for non-periodic waves, that is, wave pulses: a pulse going into a faster medium will widen in length (stretch), whereas a pulse going into a slower medium will become narrower (squeezed). Imagine, for example, several people walking in line, separated by the same distance d, all at the same pace, until they reach a line beyond which they are supposed to start running. When the first person reaches the line, he starts running, but the second one is still walking, so by the time the second one reaches the line the first one has increased his distance from the second. The same thing will happen between the second and the third, and so on: the original "bunch" will become spread out. (If you watch car races, chances are you have seen this kind of thing happen already!)

Besides setting up a transmitted wave, with the properties I have just discussed, the incident wave will almost always cause a *reflected wave* to start traveling in the first medium, moving backwards from the boundary. The reflected wave also has the same frequency as the incident one, and since it is traveling in the same medium, it will also have the same wavelength. A non-periodic pulse, when reflected, will therefore not be stretched or squeezed, but it will be "turned around" back-to-front, since the first part to reach the boundary also has to be the first to leave. See Figure $24.1.4$(the top part) for an example.

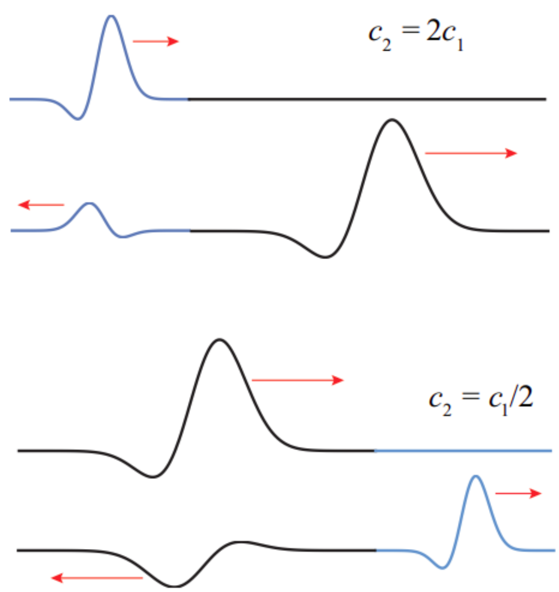

Figure 24.1.4: Reflection and transmission of a pulse at the boundary where two strings of different densities are joined. ("Before" and "after" situations are shown for each case.) Top figure: the string on the right is less dense, so the pulse travels faster (the tension on both strings is supposed to be the same). The reflected pulse is upright but reversed left-to-right. Bottom figure: the string on the right is more dense, so the transmitted pulse travels more slowly. The reflected pulse is reversed left-to-right and flipped upside down.

What is the physical reason for the reflected wave? Ultimately, it has to do with the energy carried by the incident wave, and whether it is possible for the transmitted wave alone to handle the incoming energy flux or not. As we saw earlier (Equation (24.1.8)), the energy per unit volume in a harmonic wave of angular frequency ω and amplitude ξ_0 is $E/V = \frac{1}{2}\rho_0\omega^2\xi_0^2$. If the wave is traveling at a speed c, then the energy *flux* (energy transported per unit time per unit area) is equal to $(E/V)c$, which is to say

$$I = \frac{1}{2}c\rho_0\omega^2\xi_0^2. \tag{24.1.14}$$

This is often called the *intensity* of the wave. It can be written as $I = \frac{1}{2}Z\omega^2\xi_0^2$, where I have defined the medium's *mechanical impedance* (or simply the *impedance*) as

$$Z = c\rho_0 \tag{24.1.15}$$

(for a string, the mass per unit length μ instead of the mass per unit volume ρ_0 should be used). You can see that if the two media have the same impedance, then the energy flux in medium 2 will exactly match that in medium 1, provided the incident and transmitted waves have the same amplitudes. In that case, there will be *no* reflected wave: even if the two media have different densities and wave velocities, as long as they have the same impedance, the wave will be completely transmitted.

On the other hand, if the media have different impedances, then it will in general be impossible to match the energy flux with only a transmitted wave, and reflection will occur. This is not immediately obvious, since it looks like all you have to do, to compensate for the different impedances in Equation (24.1.14), is to give the transmitted wave an amplitude that is different from that of the incident wave. But the point is precisely that, mathematically, you cannot do that without introducing a reflected wave. This is because the actual amplitude of the oscillation at the boundary has to be the same on *both* sides, since the two media are connected there, and oscillating together; so, if $\xi_{0,\text{ inc}}$ is going to be different from $\xi_{0,\text{ trans}}$, you need to have another wave in medium 1, the reflected wave, to insure that $\xi_{0,\text{ inc}} + \xi_{0,\text{ refl}} = \xi_{0,\text{ trans}}$.

Another way to see this is to dig in a little deeper into the physical meaning of the impedance. This is a worthwhile detour, because impedance in various forms recurs in a number of physics and engineering problems. For a sound wave in a solid, for instance, we can see from Eqs. (24.1.13) and (24.1.15) that $Z = c\rho_0 = \sqrt{Y\rho_0}$; so a medium can have a large impedance either by being very stiff (large Y) or very dense (large ρ_0) or both; either way, one would have to work harder to set up a wave in such a medium than in one with a smaller impedance. On the other hand, once the wave is set up, all that work gets stored as energy of the wave, so a wave in a medium with larger Z will also carry a larger amount of energy (as is also clear from Equation (24.1.14))[2] for a given displacement ξ_0.

So, when a wave is trying to go from a low impedance to a large impedance medium, it will find it hard to set up a transmitted wave: the transmitted wave amplitude will be small (compared to that of the incident wave), and the only way to satisfy the condition $\xi_{0,\text{ inc }}+\xi_{0,\text{ refl }}=\xi_{0,\text{ trans }}$ will be to set up a reflected wave with a *negative* amplitude[3]—in effect, to flip the reflected wave upside down, in addition to left-to-right. This is the case illustrated in the bottom drawing in Figure $24.1.4$

Conversely, you might think that a wave trying to go from a high impedance to a low impedance medium would have no trouble setting up a transmitted wave there, and that is true—but because of its low impedance, the transmitted wave will still not be able to carry all the energy flux by itself. In this case, $\xi_{0,trans}$ will be greater than $\xi_{0,inc}$, and this will also call for a reflected wave in the first medium, only now it will be "upright," that is, $\xi_{0,\text{ refl }}=\xi_{0,\text{ trans }}-\xi_{0,\text{ inc }}>0$.

To finish up the subject of impedance, note that the observation we just made, that impedance will typically go as the square root of the product of the medium's "stiffness" times its density, is quite general. Hence, a medium's density will typically be a good proxy for its impedance, at least as long as the "stiffness" factor is independent of the density (as for strings, where it is just equal to the tension) or, even better, increases with it (as is typically the case for sound waves in most materials). Thus, you will often hear that a reflected wave is inverted (flipped upside down) when it is reflected from a denser medium, without any reference to the impedance—it is just understood that "denser" also means "larger impedance" in this case. Also note, along these lines, that a "fixed end," such as the end of a string that is tied down (or, for sound waves, the closed end of an organ pipe), is essentially equivalent to a medium with "infinite" impedance, in which case there is no transmitted wave at that end, and all the energy is reflected.

Finally, the expression $\xi_{0,inc}+\xi_{0,refl}$ that I wrote earlier, for the amplitude of the wave in the first medium, implicitly assumes a very important property of waves, which is the phenomenon known as *interference*, or equivalently, the "linear superposition principle." According to this principle, when two waves overlap in the same region of space, the total displacement is just equal to the algebraic sum of the displacements produced by each wave separately. Since the displacements are added with their signs, one may get destructive interference if the signs are different, or constructive interference if the signs are the same. This will play an important role in a moment, when we start the study of standing waves.

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[2]In this respect, it may help you to think of the impedance of an extended medium as being somewhat analog to the inertia (mass) of a single particle. The larger the mass, the harder it is to accelerate a particle, but once you have given it a speed v, the larger mass also carries more energy.

[3]A better way to put this would be to say that the amplitude is positive as always, but the reflected wave is 180° out of phase with the incident wave, so the amplitude of the total wave on the medium 1 side of the boundary is $\xi_{0,inc}-\xi_{0,refl}$.

24.2: Standing Waves and Resonance

Imagine you have a sinusoidal traveling wave of the form (12.1.5), only traveling to the left, incident from the right on a "fixed end" at x = 0. The incident wave will go as $\xi_0 \sin[2\pi(x+ct)/\lambda]$; the reflected wave should be flipped left to right and upside down, so change x to $-x$ and put an overall minus sign on the displacement, to get $-\xi_0 \sin[2\pi(-x+ct)/\lambda]$. The sum of the two waves in the region $x > 0$ is then

$$\xi(x,t) = \xi_0 \sin\left[\frac{2\pi}{\lambda}(x+ct)\right] - \xi_0 \sin\left[\frac{2\pi}{\lambda}(-x+ct)\right] = 2\xi_0 \sin\left(\frac{2\pi x}{\lambda}\right)\cos(2\pi f t) \tag{24.2.1}$$

using a trigonometrical identity for $\sin(a+b)$, and $f = c/\lambda$.

The result on the right-hand side of Equation (24.2.1) is called a *standing wave*. It does not travel anywhere, it just oscillates "in place": every point x behaves like a separate oscillator with an amplitude $2\xi_0 \sin(2\pi x/\lambda)$. This amplitude is zero at special points, where $2x/\lambda$ is equal to an integer. These points are called *nodes*.

We could think of "confining" a wave of this sort to a string fixed at both ends, if we make the string have an end at x = 0 and the other one at one of these points where the amplitude is zero; this means we want the length L of the string to satisfy

$$2L = n\lambda \tag{24.2.2}$$

where n = 1, 2,.... Alternatively, we can think of L as being fixed and Equation (24.2.2) as giving us the possible values of λ that will give us standing waves: $\lambda = 2L/n$. Since $f = c/\lambda$, we see that all these possible standing waves, for fixed L and c, have different frequencies that we can write as

$$f_n = \frac{nc}{2L}, \quad n = 1,2,3,\ldots. \tag{24.2.3}$$

Note that these are all multiples of the frequency $f_1 = c/2L$. We call this the *fundamental frequency* of oscillation of a string fixed at both ends. The period corresponding to this fundamental frequency is the roundtrip time of a wave pulse around the string, $2L/c$.

The first three standing waves are plotted in Figure $24.2.1$. Their wave functions are given by the right-hand side of Equation (24.2.1), for 0 ≤ x ≤ L, with $\lambda = 2L/n$ (n = 1, 2, 3), and $f = f_n = nc/2L$. The amplitude is arbitrary; in the figure I have set it equal to 1 for convenience. Calling the corresponding function $u_n(x,t)$ is more or less common practice in other contexts:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos(2\pi f_n t). \tag{24.2.4}$$

These functions are called the *normal modes of vibration* of the string. In Figure $24.2.1$ I have shown, for each of them, the displacement at the initial time, t = 0, as a solid line, and then half a period later as a dashed line. In addition to this, notice that the wave function vanishes identically (the string is flat) at the quarter-period intervals, $t = 1/4f_n$ and $t = 3/4f_n$. At those times, the wave has no elastic potential energy (since the string is unstretched): as with a simple oscillator passing through the equilibrium position, all its energy is kinetic. For $n > 1$, there are also nodes (places where the oscillation amplitude is always zero) at points other than the ends. Including the endpoints, the n-th normal mode has $n+1$ nodes. The places where the oscillation amplitude is largest are called *antinodes*.

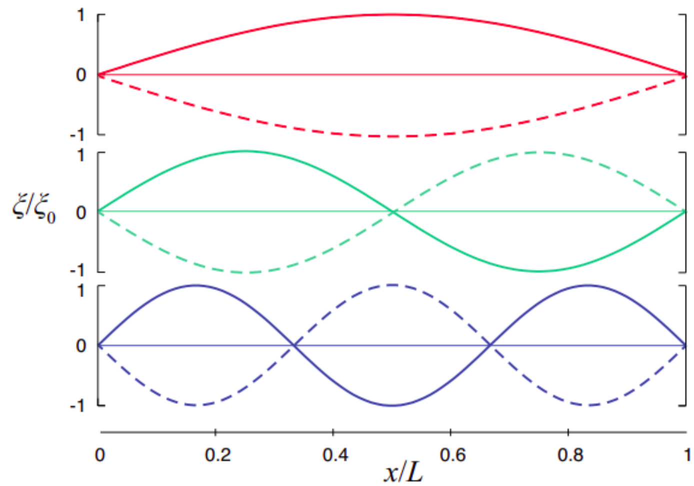

Figure 24.2.1: The three lowest-frequency normal modes of vibration of a string held down at both ends, corresponding to, from top to bottom, n = 1, 2, 3

Animations of these standing waves can be found in many places; one I particularly like is here: http://newt.phys.unsw.edu.au/jw/strings.html#standing. It also shows graphically how the standing wave can be considered as a superposition of two oppositely-directed traveling waves, as in Equation (24.2.1).

If we initially bent the string into one of the shapes shown in Figure 24.2.1, and then released it, it would oscillate at the corresponding frequency f_n, keeping the same shape, only scaling it up and down by a factor $\cos(2\pi f_n t)$ as time elapsed. So, another way to think of standing waves is as the *natural modes of vibration of an extended system*—the string, in this case, although standing waves can be produced in any medium that can carry a traveling wave.

What I mean by a "natural mode of vibration" is the following: a single oscillator, say, a pendulum, has a single "natural" frequency; if you displace it or hit it, it just oscillates at that frequency with a constant amplitude. An extended system, like the string, can be viewed as a collection of coupled oscillators, which may in general oscillate in many different and complicated ways; yet, there is a specific set of frequencies—for the string with two ends fixed, the sequence f_n of Equation (24.2.3)—and associated shapes that will result in all the parts of the string performing simple harmonic motion, in synchrony, all at the same frequency.

Of course, to produce just one of these specific modes of oscillation requires some care ("driving" the string at the right frequency is probably the easiest way; see next paragraph); however, if you simply hit or pluck the string in any random way, a remarkable thing happens: the resulting motion will be, mathematically, described as a sum of sinusoidal standing waves, each with one of the frequencies f_n, and each with a different amplitude A_n. In a musical instrument, this will eventually generate a superposition of sound waves with frequencies $f_1 = c/2L$, $f_2 = 2f_1$, $f_3 = 3f_1$ ... (called, in this context, the *fundamental*, f_1, and its *overtones*, $f_n = nf_1$). Each one of these frequencies corresponds to a different pitch, or musical note, and the result will sound a little like a chord, although not nearly as pronounced—we will mostly hear only the fundamental, which corresponds to the root note of the chord, but all the notes in a major triad are in fact present in the vibration of a single guitar or piano string[4].

But wait, there's more! Suppose that you try to get the string to oscillate by "driving" it: that is to say, grabbing a hold of one end and shaking it at some frequency, only with a very small amplitude, so the displacement at that end remains always close to zero. In that case, you will typically get only very small amplitude oscillations, until the driving frequency hits one of the special frequencies f_n, at which point you will get a large oscillation with the shape of the corresponding standing wave. This is a phenomenon known as *resonance*, and the f_n are the *resonant frequencies* of this system.

Note that the effect I just described is essentially the same as you experience when you are "pumping," or simply pushing, a swing. Unless you do it at the right frequency, you do not get very far; but if you do it at the right frequency (which is the swing's natural frequency, the one at which it will swing on its own), you can get huge amplitude oscillations. So, the frequencies (24.2.3) may be said to be the string's natural oscillation frequencies in the same two ways: they are the ones at which it will oscillate if you just pluck it, and they are the ones at which you have to drive it if you want to get large oscillations.

Pretty much everything I have just shown you above for standing waves on a string applies to sound waves inside a tube or pipe open at both ends. In that case, however, it is not the displacement, but the pressure (or density) wave that must have zeros at the ends (since the ends are open, the pressure there must be just the average atmospheric pressure; note that the pressure or density waves in a sound wave do not give the absolute pressure or density, but the deviation, positive or negative, from the average). The math, however, is identical, and one finds the same set of normal modes and resonance frequencies as above. These are then the frequencies that would be produced when blowing in a flute or an organ pipe open at both ends. So, both from pipes and strings we get the same "harmonic series" of frequencies (24.2.3) that has been the foundation of Western music since at least the time of Pythagoras.

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[4]It works like this: say f_1 corresponds to a C, then f_2 is the C above that, f_3 the G above that, f_4 the C above that, and f_5 the E above that.

24.3: Conclusion, and Further Resources

This chapter on one-dimensional waves has barely scratched the surface of the extremely rich world of wave phenomena. I have only given you a passing glance at interference, and I have not said anything at all about diffraction, the Doppler effect, polarization, refraction.... Many of these things you will learn about in later courses, most likely when you encounter electromagnetic waves (which are non-mechanical, but described by the same mathematical equation).

Waves are such an intrinsically kinetic phenomenon that they are best appreciated by watching them in action, or, as a second-best alternative, through animations. A wonderful repository of such movies and animations is PHYSCLIPS at the University of New South Wales:

http://www.animations.physics.unsw.edu.au/waves-sound/oscillations/index.html

They also have a set of pages on the "physics of music" that I have already mentioned a couple of times. If you are interested in this topic, you should go spend some time there!

http://newt.phys.unsw.edu.au/music

Finally, closer to home, the fellows at PhET (University of Colorado), have this great interactive app to explore waves on a string:

https://phet.colorado.edu/en/simulation/wave-on-a-string

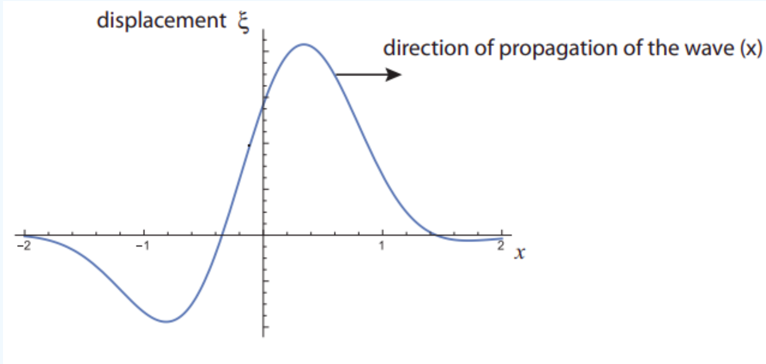
24.4: In Summary

1. A traveling wave in an elastic medium is a collective disturbance of the particles in the medium (a displacement, or change in pressure or density) that carries energy and momentum from one point of the medium to another, over a distance that is typically much larger than the displacement of the individual particles making up the wave.
2. In a longitudinal wave, the displacement of the particles is along the line of motion of the wave; in a transverse wave, it is perpendicular to the wave's motion.
3. An important kind of waves are periodic waves, in which the disturbance repeats itself at each point in the medium with a period T. Sinusoidal, periodic waves are called harmonic waves. Their spatial period is called the wavelength λ. If the speed of the wave is c, one has $c = f\lambda$, where $f = 1/T$ is the wave frequency.
4. The time-averaged energy density in a harmonic wave (sum of kinetic and elastic potential energy per unit volume) is $E/V = \rho_0\omega^2\xi_0^2/2$, where ρ_0 is the medium's density, and ξ_0 the amplitude of the displacement oscillations. The time average momentum density is E/cV. The *intensity* of the wave (energy carried per unit time per unit area) is cE/V.
5. Sound is a longitudinal compression-and-rarefaction wave in an elastic medium. It can be described in terms of displacement, pressure or density. The pressure or density disturbance is maximal where the displacement is zero, and vice-versa.
6. The speed of sound in a solid with Young modulus Y is $c = \sqrt{Y/\rho_0}$; in a fluid with bulk modulus B, it is $c = \sqrt{B/\rho_0}$. In an ideal gas, this depends only on the ratio of specific heats, the molar mass, and the temperature.
7. Transverse waves on a string with mass per unit length μ and under a tension F^t travel with a speed $c = \sqrt{F^t/\mu}$.
8. When a wave reaches the boundary between two media, it is typically partly reflected and partly transmitted. The incident, reflected and transmitted waves all have the same frequency. The transmitted wave has a wavelength c_2/f, where c_2 is the wave speed in the second medium.
9. The quantity that determines how much of the energy is reflected or transmitted is the *mechanical impedance*, defined for each medium as $Z = c\rho_0$. If $Z_1 = Z_2$ there is no reflected wave. If $Z_1 < Z_2$, the reflected wave is inverted (flipped upside-down) relative to the incident wave. If $Z_1 > Z_2$, it is upright.
10. Standing waves arise in a medium that is confined to a region of space, and are the normal (or "natural") modes of vibration of the system. In a standing wave, each particle of the medium oscillates with an amplitude that is a fixed function of the particle's position (a sinusoidal function in one dimension). This amplitude is zero at points called *nodes*.
11. In one dimension, all the standing wave frequencies are multiples of a fundamental frequency $f_1 = c/2L$, where L is the length of the medium (as long as the boundary conditions at both ends of the medium are identical). These are the *resonant* frequencies of the system: if disturbed, it will naturally oscillate in a superposition of these frequencies, and if driven at one of these frequencies, one will obtain a large response.

24.5: Examples

Example 24.5.1: Displacement and Density/pressure in a longitudinal wave

The picture below shows the displacement of a medium (let's say air) as a sound pulse travels through it. (Don't worry about the units on the axes right now! We are only interested in qualitative results here.)



a. Sketch the corresponding pressure (or density) pulse. Note: pressure and density are in phase, so one is large where the other is large. In either case what is always plotted is the *difference* between the actual pressure or density and the average pressure (for air, atmospheric pressure) or density of the medium.
b. If this sound pulse is incident on water, sketch the reflected pulse, both in a displacement and in a pressure/density plot.

Solution

(a) The purpose of this example is to help refine the intuition you may have gotten from Figure 12.1.3 regarding the relationship between the displacement and the pressure/density in a longitudinal wave. When discussing Figure 12.1.3, I argued that the density should be high at a point like $x = \pi$ in that figure, because the particles to the left of that point were being pushed to the right, and those to the right were being pushed to the left. However, a similar argument can be made to show that the density should be higher than its equilibrium value whenever the displacement curve has a *negative* slope, in general.

For instance, consider point x = 1 in the figure above. Particles both to the left and the right of that point are being pushed to the right (positive displacement), but the displacement is larger for the ones on the left, which will result in a bunching at x = 1.

Conversely, if you look at a point with positive slope, such as x = 0, you see that the particles on the right are pushed farther to the right than the particles on the left, which means the density around x = 0 will drop.

From this you may conclude that the density versus position graph will look somewhat like the negative of the derivative of the ξ-vs.-x graph: positive when ξ falls, negative when it rises, and zero at the "turning points" (maxima or minima of $\xi(x)$). This is, in fact, mathematically true, and is illustrated in the figure below.

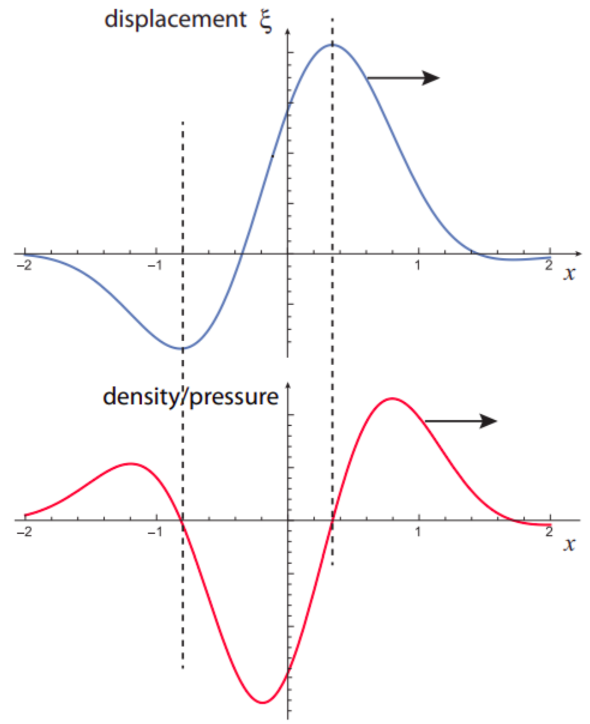


Figure 24.5.1: Illustrating the relationship between displacement (blue curve) and pressure/density (red) in a longitudinal wave. The dashed lines separate the regions where the pressure (or density) is positive (higher than in the absence of the wave) from those where it is negative.

(b) If this sound wave is incident from air into water, it means it is going from a low impedance to a high impedance medium (both the density and the speed of sound are much greater in water than in air, giving a much larger $Z = c\rho_0$; see Equation (12.1.15) for the definition of impedance). This means the reflected displacement pulse will be flipped upside down, as well as left to right (see the figure on the next page). This is just (except for the scale, which here is arbitrary) like the bottom part of Figure 12.1.4.

However, if you now try to figure out the shape of the density/pressure wave based on the displacement wave, as we did in part (a), you'll see that it is only reversed left to right, but *not* flipped upside down! This is a general property of longitudinal waves: the reflected pressure/density wave behaves in exactly the opposite way as the displacement wave, as far as the upside-down "flip" is concerned: it gets flipped when going from high impedance to low impedance, and not when going from low to high.

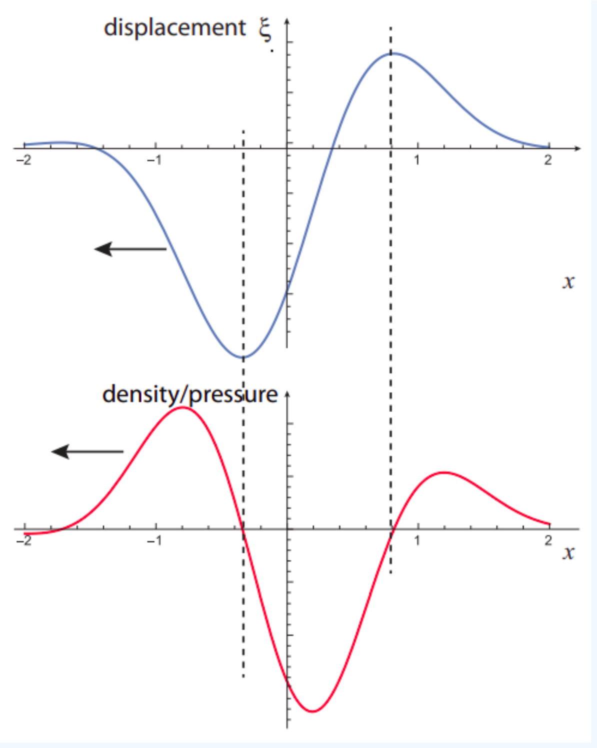


Figure 24.5.2: What the wave pulse in the previous figure would look like if reflected from a high-impedance medium. The displacement wave is reversed left to right and flipped upside down. The pressure/density wave is only reversed left to right.

If you are curious to see how this happens mathematically, the idea is that the density wave is proportional to $-d\xi/dx$, and the reflected displacement wave goes like $\xi_{refl} = -\xi_{inc}(-x)$, where the first minus sign gives the vertical flip and the second the horizontal one. Taking the derivative of this last expression with respect to x then removes the minus sign in front.

Example 24.5.1: violin sounds

The "sounding length" of a violin string, from the bridge to the nut at the upper end of the fingerboard, is about 32 cm.

a. If the string is tuned so that its fundamental frequency corresponds to a concert A (440 Hz), what is the speed of a wave on that string?
b. If the string's density is 0.66 g/m (note: the "g" stands for "grams"!), what is the tension on the string?
c. When the string is played, its vibration is transmitted through the bridge to the violin plates. At what frequency will the plates vibrate?
d. The vibration of the plates then sets up a sound wave in air. What is the wavelength of this wave?

Solution

(a) In Section 12.2 we saw that the fundamental frequency of a string fixed at both ends is $f_1 = c/2L$ (corresponding to Equation (12.2.3) with n = 1). Setting this equal to 440 Hz, and solving for c,

$$c = 2Lf_1 = 2 \times (0.32\text{ m}) \times 440\text{ s}^{-1} = 282\ \frac{\text{m}}{\text{s}}$$

(b) From Section 12.1, we have that the speed of a wave on a string is $c = \sqrt{F^t/\mu}$, where F^t is the tension and μ the mass per unit length (Equation (12.1.11). Solving for F^t,

$$F^t = c^2\mu = \left(282\ \frac{\text{m}}{\text{s}}\right)^2 \times 6.6 \times 10^{-4}\ \frac{\text{kg}}{\text{m}} = 52.5\text{ N}$$

(c) The plates will vibrate at the same frequency as the string, 440 Hz, since they are driven by the motion of the string.

(d) The basic relationship to use here is Equation (12.1.4), $f = c/\lambda$, which we can solve for λ if we know c, the speed of sound in air. In Section 12.1 it was stated that the speed of sound in air is about 340 m/s, so we have

$$\lambda = \frac{c}{f} = \frac{340\ \mathrm{m/s}}{440\ \mathrm{s}^{-1}} = 0.77\ \mathrm{m}$$

24.6: Advanced Topics

Chain of Masses Coupled with Spring- Dispersion, and Long-Wavelength Limit

Consider a model of an extended elastic medium in which, for simplicity, we separate the two main medium properties, inertia and elasticity, by describing it as a chain of point-like masses (particles) connected by massless springs, as in Figure 24.6.1 below. I will show you here how one can get “ideal” wave behavior in this system, provided we work in the “long-wavelength” limit, that is to say, we consider only waves whose wavelength is much greater than the average distance between neighboring masses.

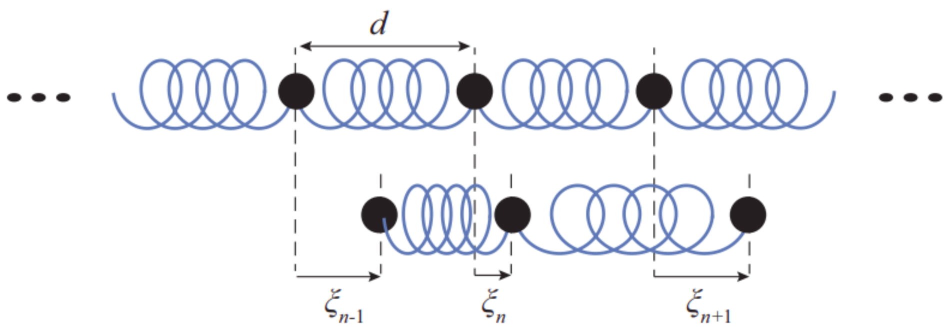


Figure 24.6.1: A chain of masses connected by massless springs. The top picture shows the equilibrium positions with the springs relaxed, the bottom one the situation where each mass has undergone a displacement ξ.

In the figure above I have explicitly shown the n-th mass and the two springs that push and/or pull on it, both in equilibrium (top drawing) and when the chain is in motion (bottom). In the latter case, the length of the springs depends on the relative displacements of all three masses shown. Specifically, the length of the spring on the left is $d+\xi_n-\xi_{n-1}$, where d is the distance between the masses in equilibrium, and the length of the spring on the right is $d+\xi_{n+1}-\xi_n$. If the left spring is stretched (length greater than d) it will pull to the left on the n-th mass, and, conversely, if the right spring is stretched (length greater than d) it will pull to the right on the n-th mass. So, if all the springs have the same constant k, the force equation $F=ma$ for mass n is

$$ma_n=-k\left(\xi_n-\xi_{n-1}\right)+k\left(\xi_{n+1}-\xi_n\right) \tag{24.6.1}$$

which we can rewrite as

$$m\frac{d^2\xi_n}{dt^2}=k\xi_{n-1}-2k\xi_n+k\xi_{n+1}. \tag{24.6.2}$$

Now let us try to see if we can get a sinusoidal solution to this system of differential equations. By analogy with Equation (12.1.3) let

$$\xi_n(t)=A\sin\left[2\pi\left(\frac{x_n}{\lambda}-ft\right)\right]$$

where $x_n=nd$ is the equilibrium position of the n-th mass. Then for each of the three masses considered, we have

$$\begin{aligned}\xi_{n-1}(t) &=A\sin[2\pi((n-1)d/\lambda-ft)]=A\sin[2\pi(nd/\lambda-ft)-2\pi d/\lambda] \\ \xi_n(t) &=A\sin[2\pi(nd/\lambda-ft)] \\ \xi_{n+1}(t) &=A\sin[2\pi((n+1)d/\lambda-ft)]=A\sin[2\pi(nd/\lambda-ft)+2\pi d/\lambda]\end{aligned} \tag{24.6.3}$$

We want to substitute all this in Equation (24.6.2). We can use the trigonometric identity $\sin(a-b)+\sin(a+b)=2\sin a\cos b$ to simplify $\xi_{n-1}+\xi_{n+1}$:

$$\xi_{n-1}+\xi_{n+1}=2A\sin\left[2\pi\left(\frac{nd}{\lambda}-ft\right)\right]\cos\left(\frac{2\pi d}{\lambda}\right) \tag{24.6.4}$$

then use $1-\cos x=2\sin^2(x/2)$ to yield

$$k\xi_{n-1}-2k\xi_n+k\xi_{n+1}=-4kA\sin^2\left(\frac{\pi d}{\lambda}\right)\sin\left[2\pi\left(\frac{nd}{\lambda}-ft\right)\right]=-4k\sin^2\left(\frac{\pi d}{\lambda}\right)\xi_n \tag{24.6.5}$$

It is clear now that Equation (24.6.2) will be satisfied provided the following condition holds:

$$m(2\pi f)^2 = 4k\sin^2\left(\frac{\pi d}{\lambda}\right). \tag{24.6.6}$$

Or, taking the square root and simplifying,

$$f = \frac{1}{\pi}\sqrt{\frac{k}{m}}\sin\left(\frac{\pi d}{\lambda}\right). \tag{24.6.7}$$

This is clearly a more complicated relation between f and λ than just Equation (12.1.4). However, since we can argue that Equation (12.1.4) must always hold for a sinusoidal wave, what we have actually found is that the chain of masses and springs in Figure $24.6.1$ will support a sinusoidal wave provided the *wave velocity depends on the wavelength* as required by Eqs. (12.1.4) and (24.6.7):

$$c = \lambda f = \sqrt{\frac{k}{m}}\frac{\lambda}{\pi}\sin\left(\frac{\pi d}{\lambda}\right). \tag{24.6.8}$$

This is an instance of the phenomenon called *dispersion*: sinusoidal waves of different frequencies (or wavelengths) have different velocities. One thing that happens in the presence of dispersion is that, although a single (infinite), sinusoidal wave can travel without changing its shape (provided f and λ satisfy Equation (24.6.7)), a general pulse will be distorted as it propagates through the medium, often severely so.

In the long wavelength limit, however, the dispersion in this model disappears. We can see this as follows. In that limit, $\lambda \gg d$ (the wavelength is much greater than the distance between the masses), and therefore $\pi d/\lambda \ll 1$; we can then make the small-angle approximation in Equation (24.6.8), $\sin(\pi d/\lambda) \simeq \pi d/\lambda$, and end up with

$$c \simeq d\sqrt{\frac{k}{m}}. \tag{24.6.9}$$

This is of the general form $\sqrt{\text{stiffness}/\text{inertia}}$ (as per Equation (12.1.10)). Basically, in the long-wavelength limit, the medium appears homogeneous to the wave—it cannot “tell” that it is a chain of discrete particles. When you consider that everything that looks homogeneous on a macroscopic scale is actually made of discrete atoms or molecules at the microscopic level, you can see that this model is perhaps not as artificial as it might seem, and that in general you should, in fact, expect some kind of dispersion to occur in any medium, at sufficiently small wavelengths.

24.7: Exercises

Exercise 24.7.1

When plucked, the D string on a guitar vibrates with a frequency of 147 Hz.

a. What would happen to this frequency if you were to *increase* the tension in the string?
b. The vibration of the string eventually produces a sound wave of the same frequency, traveling through the air. If the speed of sound in air is 340 m/s, what is the wavelength of this wave?

Exercise 24.7.2

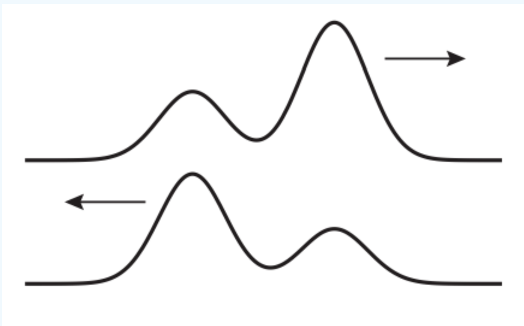
Think of a flute as basically a cylindrical tube of length 0.6 m, open to the atmosphere at both ends. If the speed of sound in air is 340 m/s

a. What is the fundamental (lowest) frequency of a sound wave in a flute?
b. Is this a transverse or a longitudinal wave?
c. The speed of sound in helium is about 3 times that in air. How would the flute's resonance frequencies change if you filled it with helium instead of air?

Justify each of your answers briefly.

Exercise 24.7.3

The top picture shows a wave pulse on a string (string 1) traveling to the right, where the string is attached to another one (string 2, not shown). The bottom picture shows the reflected wave some time later.



If the tension on both strings is the same,

a. Is string 2 more or less dense than string 1?
b. In which string will the wave travel faster?
c. Sketch what the reflected wave would look like if the strings' densities were the opposite of what you answered in part (a).

Explain each of your answers briefly.

CHAPTER OVERVIEW

25: Thermodynamics

25.1: Introduction

The last two lectures this semester are about thermodynamics, an extremely important branch of physics that developed throughout the 19th century, motivated in part by the development of the steam engines that brought about the Industrial Revolution. Physics majors will study thermodynamics at much greater length in University Physics III and subsequent courses, whereas Engineering and Chemistry majors will encounter it also in specialized courses in their own disciplines.

There is really no escaping thermodynamics, but you may wonder why bring it up here (in this course, at this time) at all? The answer is twofold:

- From the point of view of the study of energy and its transformations, which has been one of the major themes of this course, thermodynamics provides us with the last missing pieces: it is here that we find out what thermal energy really is, and how it is different from other forms of energy (so much so, that we say that energy has been “dissipated” or “lost” when it becomes thermal energy). It is also here that we deal with the **other** way that energy can be transferred from a system to another (other, that is, than by doing work): this is the “direct transfer of thermal energy,” or what is normally called an *exchange of heat*.
- From the point of view of the study of motion, which has been also another running theme, thermodynamics also represents the next logical step beyond what we have learned so far. Recall that we started looking at the motion of extended objects as if they were simple point particles, moving as a whole along with their center of mass, and slowly introduced tools to deal with more complex kinds of motion: first rigid body rotations, then elastic deformations (waves) in which the constituent parts of an object move relative to each other in a way that looks “organized,” or synchronized, from a macroscopic perspective. What is needed next is to account for the random motion, on a microscopic scale, of the smallest parts (atoms or molecules) that make up an extended object. This motion is constantly happening, and it is a key ingredient of the concepts of thermal energy and temperature.

Conceptually, thermodynamics involves the introduction of two new physical quantities, *temperature* and *entropy*. Temperature will be introduced in this lecture, and entropy in the next one. It is interesting to note from the start, however, that these are very different from all the quantities we have introduced so far this semester, in a fundamental way. In classical physics, at least, there is no difficulty in extending all those other quantities to the study of the smallest parts making up an object: we can perfectly well talk about the position, velocity or energy of a molecule. But temperature and entropy are *statistical* quantities, which are only properly defined, from a fundamental point of view, for a large collection of (small) subsystems: it makes no sense to speak about the temperature or the entropy of a single molecule. This shows that there was really a profound change in perspective and methodology in classical physics when *statistical mechanics* (the part of physics that provides a microscopic foundation for thermodynamics) was developed.

25.2: Introducing Temperature

Temperature and Heat Capacity

The change in perspective that I just mentioned also means that it is not easy to even *define* temperature, beyond our natural intuition of "hot" and "cold," or the somewhat circular notion that temperature is simply "what thermometers measure." Roughly speaking, though, temperature is a measure of the amount (or, to be somewhat more precise, the *concentration*) of thermal energy in an object. When we directly put an amount of thermal energy, ΔE_{th} (what we will be calling *heat* in a moment), in an object, we typically observe its temperature to increase in a way that is approximately proportional to ΔE_{th}, at least as long as ΔE_{th} is not too large:

$$\Delta T=\frac{\Delta E_{th}}{C}. \tag{25.2.1}$$

The proportionality constant C is called the *heat capacity* of the object: according to Equation (25.2.1), a system with a large heat capacity could absorb (or give off—the equation is supposed to apply in either case) a large amount of thermal energy without experiencing a large change in temperature. If the system does not do any work in the process (recall Equation (7.4.8)!), then its internal energy will increase (or decrease) by exactly the same amount of thermal energy it has taken in (or given off)[1], and we can use the heat capacity[2] to, ultimately, relate the system's temperature to its energy content in a one-to-one-way.

What is found experimentally is that the heat capacity of a homogeneous object (that is, one made of just one substance) is, in general, proportional to its mass. This is why, instead of tables of heat capacities, what we compile are tables of *specific heats*, which are heat capacities per kilogram (or sometimes per mole, or per cubic centimeter... but all these things are ultimately proportional to the object's mass). In terms of a specific heat $c=C/m$, and again assuming no work done or by the system, we can rewrite Equation (25.2.1) to read

$$\Delta E_{sys}=mc\Delta T \tag{25.2.2}$$

or, again,

$$\Delta T=\frac{\Delta E_{sys}}{mc} \tag{25.2.3}$$

which shows what I said above, that temperature really measures, not the *total* energy content of an object, but its *concentration*—the thermal energy "per unit mass," or, if you prefer (and somewhat more fundamentally) "per molecule." An object can have a great deal of thermal energy just by virtue of being huge, and yet still be pretty cold (water in the ocean is a good example).

In fact, we can also rewrite Eqs. (25.2.1–25.2.3) in the (somewhat contrived-looking) form

$$C=mc=m\frac{\Delta E_{sys}/m}{\Delta T} \tag{25.2.4}$$

which tells you that an object can have a large heat capacity in two ways: one is simply to have a lot of mass, and the other is to have a large specific heat. The first of these ways is kind of boring (but potentially useful, as I will discuss below); the second is interesting, because it means that a relatively large change in the internal energy per molecule (roughly speaking, the numerator of (25.2.4)) will only show as a relatively small change in temperature (the denominator of (25.2.4); a large numerator and a small denominator make for a large fraction!).

Put differently, and somewhat fancifully, *substances with a large specific heat are very good at hiding their thermal energy from thermometers* (see Figure 25.2.1 for an example). This, as I said, is an interesting observation, but it also means that measuring heat capacities—or, for that matter, measuring temperature itself—may not be an easy matter. How do we get at the object's internal energy if not through its temperature? Where does one start?

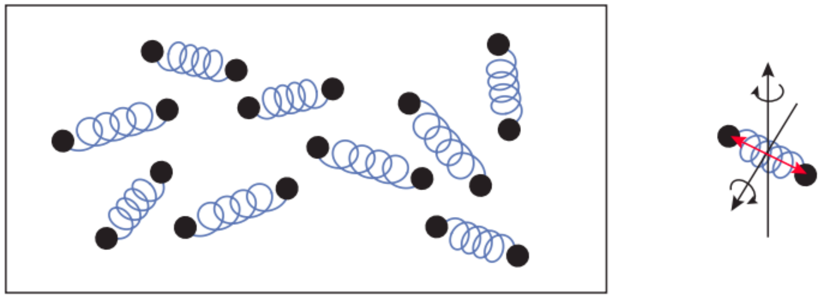


Figure 25.2.1: In this simple model of a gas of diatomic molecules, each molecule can store "vibrational" potential energy (both potential and kinetic, through the oscillations of the "spring" that models the interaction between the atoms), plus as at least two kinds of rotational kinetic energy (corresponding to rotations around the axes shown), in addition to just the translational kinetic energy of its center of mass. The latter is the only one directly measured by a gas thermometer, so a diatomic gas has many more ways of "hiding" its thermal energy (and hence, a larger specific heat) than a monoatomic gas.

[1]If the system *does* do some work (or has work done on it), then Equation (13.3.1) applies.

[2]Which, it must be noted, may also be a function of the system's temperature—another complication we will cheerfully ignore here.

The Gas Thermometer

A good start, at least conceptually, is provided by looking at a system that has no place to hide its thermal energy—it has to show it all, have it, as it were, in full view all the time. Such a system is what has come to be known as *an ideal gas*—which we model, microscopically, as a collection of molecules (or, more properly, atoms) with no dimension and no structure: just pointlike things whizzing about and continually banging into each other and against the walls of their container. For such a system *the only possible kind of internal energy is the sum of the molecules' translational kinetic energy*. We may expect this to be easily detected by a thermometer (or any other energy-sensitive probe), because as the gas molecules bang against the thermometer, they will indirectly reveal the energy they carry, both by how often and how hard they collide.

As it turns out, we can be a lot more precise than that. We can analyze the theoretical model of an ideal gas that we have just described fairly easily, using nothing but the concepts we have introduced earlier in the semester (plus a few simple statistical ideas) and obtain the following result for the gas' pressure and volume:

$$PV = \frac{2}{3}N\left\langle K_{\text{trans}}\right\rangle \tag{25.2.5}$$

where N is the total number of molecules, and $\langle K_{\text{trans}}\rangle$ is the average translational kinetic energy per molecule. Now, you are very likely to have seen, in high-school chemistry, the *experimentally* derived "ideal gas law,"

$$PV = nRT \tag{25.2.6}$$

where n is the number of moles, and R the "ideal gas constant." Comparing Equation (25.2.5) (a theoretical prediction for a mathematical model) and Equation (25.2.6) (an empirical result approximately valid for many real-world gases under a wide range of pressure and temperature, where "temperature" literally means simply "what any good thermometer would measure") immediately tells us what temperature *is*, at least for this extremely simple system: it is just a measure of the average (translational) kinetic energy per molecule.

It would be tempting to leave it at that, and immediately generalize the result to all kinds of other systems. After all, presumably, a thermometer inserted in a liquid is fundamentally responding to the same thing as a thermometer inserted in an ideal gas: namely, to how often, and how hard, the liquid's molecules bang against the thermometer's wall. So we can assume that, in fact, it must be measuring the same thing in both cases—and that would be the average translational kinetic energy per molecule. Indeed, there is a result in classical statistical mechanics that states that for *any* system (liquid, solid, or gas) in "thermal equilibrium" (a state that I will define more precisely later), the average translational kinetic energy per molecule must be

$$\left\langle K_{\text{trans}}\right\rangle = \frac{3}{2}k_BT \tag{25.2.7}$$

where k_B is a constant called *Boltzmann's constant* (k_B = 1.38×10^{-23} J/K), and T, as in Equation (25.2.6) is measured in degrees Kelvin.

There is nothing wrong with this way to think about temperature, except that it is too selflimiting. To simply identify temperature with the translational kinetic energy per molecule leaves out a lot of other possible kinds of energy that a complex system might have (a sufficiently complex molecule may also rotate and vibrate, for instance, as shown in Figure 25.2.1; these are some of the ways the molecule can "hide" its energy from the thermometer, as I suggested above). Typically, *all* those other forms of internal energy also go up as the temperature increases, so it would be at least a bit misleading to think of the temperature as having to do with only K_{trans}, Equation (25.2.7) notwithstanding. Ultimately, in fact, it is the *total* internal energy of the system that we want to relate to the temperature, which means having to deal with those pesky specific heats I introduced in the previous section. (As an aside, the calculation of specific heats was one of the great challenges to the theoretical physicists of the late 19th and early 20th century, and eventually led to the introduction of quantum mechanics—but that is another story!)

In any case, the ideal gas not only provides us with an insight into the microscopic picture behind the concept of temperature, it may also serve as a thermometer itself. Equation (25.2.6) shows that the volume of an ideal gas held at constant pressure will increase in a way that's directly proportional to the temperature. This is just how a conventional, old-fashioned mercury thermometer worked—as the temperature rose, the volume of the liquid in the tube went up. The ideal gas thermometer is a bit more cumbersome (a relatively small temperature change may cause a pretty large change in volume), but, as I stated earlier, typically works well over a very large temperature range.

By using an ideal (or nearly ideal) gas as a thermometer, based on Equation (25.2.6), we are, in fact, implicitly defining a specific temperature scale, the Kelvin scale (indeed, you may recall that for Equation (25.2.6) to work, the temperature must be measured in degrees Kelvin). The zero point of that scale (what we call absolute zero) is the theoretical point at which an ideal gas would shrink to precisely zero volume. Of course, no gas stays ideal (or even gaseous!) at such low temperatures, but the point can easily be found by extrapolation: for instance, imagine plotting experimental values of V vs T, at constant pressure, for a nearly ideal gas, using any kind of thermometer scale to measure T, over a wide range of temperatures. Then, connect the points by a straight line, and extend the line to where it crosses the T axis (so V = 0); that point gives you the value of absolute zero in the scale you were using, such as −273.15 Celsius, for instance, or −459.67 Fahrenheit.

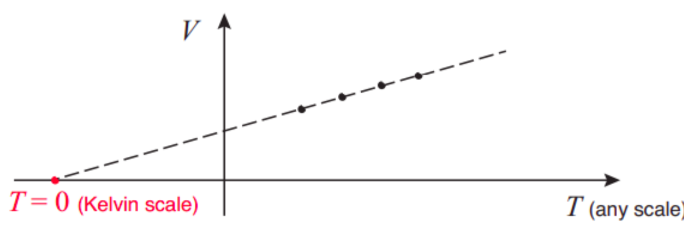


Figure 25.2.2: Illustrating how a gas thermometer can be used to define the Kelvin, or absolute, temperature scale.

The connection between Kelvin (or *absolute*) temperature and microscopic motion expressed by equations like (25.2.5) through (25.2.7) immediately tells us that as you lower the temperature the atoms in your system will move more and more slowly, until, when you reach absolute zero, all microscopic motion would cease. This does not quite happen, because of quantum mechanics, and we also believe that it is impossible to really reach absolute zero for other reasons, but it is true to a very good approximation, and experimentalists have recently become very good at cooling small ensembles of atoms to temperatures extremely close to absolute zero, where the atoms move, literally, slower than snails (instead of whizzing by at close to the speed of sound, as the air molecules do at room temperature).

The Zero-th Law

Historically, thermometers became useful because they gave us a way to quantify our natural perception of cold and hot, but the quantity they measure, temperature, would have been pretty useless if it had not exhibited an important property, which we naturally take for granted, but which is, in fact, surprisingly not trivial. This property, which often goes by the name of the *zero-th law of thermodynamics*, can be stated as follows:

Suppose you place two systems A and B in contact, so they can directly exchange thermal energy (more about this in the next section), while isolating them from the rest of the world (so their joint thermal energy has no other place to go). Then, eventually, they will reach a state, called *thermal equilibrium*, in which they will both have the same temperature.

This is important for many reasons, not the least of which being that that is what allows us to measure temperature with a thermometer in the first place: the thermometer tells us the temperature of the object with which we place it in contact, by first adopting itself that temperature! Of course, a good thermometer has to be designed so that it will do that while changing the temperature of the system being measured as little as possible; that is to say, the thermometer has to have a much smaller heat capacity than the system it is measuring, so that it only needs to give or take a very small amount of thermal energy in order to match its temperature. But the main point here is that the match actually happens, and when it does, the temperature measured by the thermometer will be the same for any other systems that are, in turn, in thermal equilibrium with—that is, at the same temperature as—the first one.

The zero-th law only assures us that thermal equilibrium will eventually happen, that is, the two systems will eventually reach one and the same temperature; it does not tell us how long this may take, nor even, by itself, what that final temperature will be. The latter point, however, can be easily determined if you make use of conservation of energy (the first law, coming up!) and the concept of heat capacity introduced above (think about it for a minute).

Still, as I said above, this result is far from trivial. Just imagine, for instance, two different ideal gases, whose molecules have different masses, that you bring to a state of joint thermal equilibrium. Equations ($25.2.5$) through ($25.2.7$) tell us that in this final state the average translational kinetic energy of the "A" molecules and the "B" molecules will be the same. This means, in particular, that the more massive molecules will end up moving more slowly, on average, so $m_a v_{a,av}^2 = m_b v_{b,av}^2$. But why is that? Why should it be the kinetic energies that end up matching, on average, and not, say, the momenta, or the molecular speeds themselves? The result, though undoubtedly true, defied a rigorous mathematical proof for decades, if not centuries; I am not sure that a rigorous proof exists, even now.

25.3: Heat and the First Law

"Direct Exchange of Thermal Energy" and Early Theories of Heat

In the previous section I have considered the possibility of "direct exchange of thermal energy" between two objects. This is a phenomenon with which we are all familiar: when a colder object is placed in contact with a warmer one, the warmer one cools off and the colder one warms up. This "warmth" that seems to flow out of one object and into the other is conventionally called "heat."

Naturally, this observation was made long before the concept of "energy" was even developed, and so heat was thought of, for a time, as an "invisible fluid" (called, at one point, "caloric fluid"), a sort of indestructible "substance" that literally passed from one body to another. By "indestructible" I mean that they had a notion of this caloric fluid being conserved: it was not created or destroyed, only exchanged from one body to another. This makes sense, in a way: if it was really something material, how could it be created or destroyed? Conservation of matter was pretty much accepted scientific "dogma" already by the end of the 18th century.

This idea of conservation of the caloric fluid led to the whole field of "calorimetry," as essentially a way to try to quantify (that is, measure) the amount of "caloric" that materials would take in or give off. The connection with temperature led directly to the definition of heat capacities and specific heats, just as I have introduced them above (in section 2.1); only instead of "change in energy" you would use "change in caloric content." This would be measured in units, called calories, defined by the amount of caloric that led to a given temperature change in a reference substance, such as water.

To be precise, let 1 calorie be the amount of "caloric" needed to raise the temperature of one gram of water by one degree Celsius at a pressure of one atmosphere. This makes the specific heat of water, by definition, 1 calorie/$^\circ$C·gram. Now imagine you place a hot object in a container of water, insulated from the rest of the world, and wait until thermal equilibrium is reached. Then you can calculate the "amount of caloric that flowed into the water," from the change in its temperature, and if you assume that all this came from the hot object then you can calculate its heat capacity (in calories/$^\circ$C) from the change in *its* temperature. By proceeding in this fashion, scientists developed tables of specific heats that do not need any change today—only the recognition that "caloric" is not really a fluid at all, but a form of energy, and can, therefore, be measured in energy units.

Clearly, conservation of caloric was a very good idea in its own way, since much of what was established back then still works if you simply replace the word "caloric" or "heat" by "thermal energy." It was, however, ultimately unsatisfactory precisely because it restricted itself to what we would today recognize as just one kind of energy, and so it failed to recognize thermal energy as something that could be converted into, or from, other kinds of energy

In hindsight, it is a bit surprising that the belief in the conservation of caloric could have held for so long. What today appear to us like obvious instances of the transformation of (macroscopic) mechanical energy into thermal energy, such as the warmth generated when you rub two objects together, were explained away as instances of mechanically "squeezing" caloric fluid out of the objects. Around the turn of the 19th century, an American expatriate, Count Rumford, observed one of the most egregious instances of this in the enormous amount of "heat" that was generated in the boring of cannons (which involved, basically, a huge metal tool drilling a hole in a large metal cylinder). He noticed that the total mass of the metal, including all the shavings, did not appear to change in the process, and concluded that caloric had to be virtually massless, since enormous quantities of it could be "squeezed" out without an appreciable mass loss. He speculated that caloric was not a fluid at all, but rather "a form of motion," since only something like that could be made to increase without any apparent limit.

Rumford's theory was not generally accepted at the time, but later in the 19th century the direct conversion of mechanical energy into thermal energy was established beyond a doubt by James Prescott Joule in a series of painstaking experiments in which he used a system of weights to turn some vanes, or paddles, that stirred water in a container and eventually caused its temperature to rise. By measuring the mechanical energy deficit (gravitational plus kinetic) of his system of weights and paddles, he could tell how much energy the water must have gained, and by measuring the water's change in temperature he could then establish the equivalent "amount of caloric" that had gone into it. He thus established what was called "the mechanical equivalent of heat," which we would express today by saying that a calorie does not measure the amount of some (nonexistent) caloric fluid, but simply an amount of energy equal to 4.18 joules (and yes, the Joule is named after him!).

The First Law of Thermodynamics

The upshot of all this experimentation was the full development of the concept of energy as a conserved quantity that manifested itself in different ways and could be "converted" among different kinds. To the observation, already familiar from macroscopic mechanics, that the energy of a system could be changed by doing work on it (or letting it do work on its environment) was added the observation, coming from thermal physics, that *thermal energy* could also be directly exchanged between two objects merely by placing them in contact, without any macroscopic work being involved. The two things taken together led to the principle of conservation of energy in its most general (pre-relativistic) form:

$$\Delta E=W+Q \tag{25.3.1}$$

which says simply that a change in the total energy of a system may result from work (W) or from "heat exchange" (Q). "Heat," in physics usage today, is simply what we call the thermal energy that is directly transferred from one object to another, typically by contact; the convention used for this term is the same as for the work term, that is, Q is positive if thermal energy flows into the system and negative if thermal energy leaves the system.

Equation (25.3.1) is the *first law of thermodynamics*. Note that, in terms of Q, the precise definition of a system's heat capacity is $C=Q/\Delta T$, and so this will only be equal to $\Delta E/\Delta t$ when the system does no work, which is why I was careful to include that condition in the derivation of Equation (13.2.2).

25.4: The Second Law and Entropy

The second law of thermodynamics is really little more than a formal statement of the observation that heat always flows spontaneously from a warmer to a colder object, and never in reverse.

More precisely, consider two systems, at different temperatures, that can exchange heat with each other but are otherwise isolated from the rest of the world. The second law states that under those conditions the heat will only flow from the warmer to the colder one.

The closure of the system—its isolation from any sources of energy—is important in the above statement. It is certainly possible to build devices that will remove heat from a relatively cold place (like the inside of your house on a hot summer day) and exhaust it to a warmer environment. These devices are called refrigerators or heat pumps, and the main thing about them is that they need to be plugged in to operate: that is, they require an external energy source.

If you have an energy source, then, you *can* move heat from a colder to a warmer object. To avoid unnecessary complications and loopholes (what if the energy source is a battery that is physically inside your "closed" system?) an alternative formulation of the basic principle, due to Clausius, goes as follows:

No process is possible whose *sole result* is the transfer of heat from a cooler to a hotter body.

The words "sole result" are meant to imply that in order to accomplish this "unnatural" transfer of heat you must draw energy from some source, and so you must be, in some way, *depleting* that source (the battery, for instance). On the other hand, for the reverse, "spontaneous" process—the flow from hotter to cooler—no such energy source is necessary.

A mathematical way to formulate the second law would be as follows. Consider two systems, in thermal equilibrium at temperatures T_1 and T_2, that you place in contact so they can exchange heat. For simplicity, assume that exchange of heat is all that happens; no work is done either by the systems or on them, and no heat is transferred to or from the outside world either. Then, if Q_1 and Q_2 are the amounts of heat gained by each system, we must have, by the conservation of energy, $Q_2 = -Q_1$, so one of these is positive and the other one is negative, and, by the second law, the system with the positive Q (the one that gains thermal energy) must be the colder one. This is ensured by the following inequality:

$$Q_1 (T_2 - T_1) \geq 0. \tag{25.4.1}$$

So, if $T_2 > T_1$, Q_1 must be positive, and if $T_1 > T_2$, Q_1 must be negative. (The equal sign is there to allow for the case in which $T_1 = T_2$, in which case the two systems are initially in thermal equilibrium already, and no heat transfer takes place.)

Equation (25.4.1) is valid regardless of the temperature scale. If we use the Kelvin scale, in which all the temperatures are positive[3], we can rewrite it by dividing both sides by the product $T_1 T_2$, and using $Q_2 = -Q_1$, as

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} \geq 0. \tag{25.4.2}$$

This more symmetric statement of the second law is a good starting point from which to introduce the concept of *entropy*, which I will proceed to do next.

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[3]For a while in the 1970's some people were very excited by the concept of *negative* absolute temperatures, but that is mostly an artificial contrivance used to describe systems that are not really in thermal equilibrium anyway.

Entropy

In Equations (25.4.1) and (25.4.2), we have taken T_1 and T_2 to be the initial temperatures of the two systems, but in general, of course, these temperatures will change during the heat transfer process. It is useful to consider an "infinitesimal" heat transfer, dQ, so small that it leads to a negligible temperature change, and then define the *change in the system's entropy* by

$$dS = \frac{dQ}{T}. \tag{25.4.3}$$

Here, S denotes a new system variable, the entropy, which is implicitly defined by Equation (25.4.3). That is to say, suppose you take a system from one initial state to another by adding or removing a series of infinitesimal amounts of heat. We take the change in entropy over the whole process to be

$$\Delta S=S_f-S_i=\int_i^f \frac{dQ}{T}. \tag{25.4.4}$$

Starting from an arbitrary state, we could use this to find the entropy for any other state, at least up to a (probably) unimportant constant (a little like what happens with the energy: the absolute value of the energy does not typically matter, it is only the energy differences that are meaningful). This may be easier said than done, though; there is no *a priori* guarantee that any two arbitrary states of a system could be connected by a process for which (25.4.4) could be calculated, and conversely, it might also happen that two states could be connected by several possible processes, and the integral in (25.4.4) would have different values for all those. In other words, there is no guarantee that the entropy thus defined will be a true *state function*—something that is uniquely determined by the other variables that characterize a system's state in thermal equilibrium.

Nevertheless, it turns out that it is possible to show that the integral (25.4.4) is indeed independent of the "path" connecting the initial and final states, at least as long as the physical processes considered are "reversible" (a constraint that basically amounts to the requirement that heat be exchanged, and work done, only in small increments at a time, so that the system never departs too far from a state of thermal equilibrium). I will not attempt the proof here, but merely note that this provides the following, alternative formulation of the second law of thermodynamics:

For every system in thermal equilibrium, there is a state function, the entropy, with the property that it can never decrease for a closed system.

You can see how this covers the case considered in the previous section, of two objects, 1 and 2, in thermal contact with each other but isolated from the rest of the world. If object 1 absorbs some heat dQ_1 while at temperature T_1 its change in entropy will be $dS_1=dQ_1/T_1$, and similarly for object 2. The total change in the entropy of the closed system formed by the two objects will then be

$$dS_{\text{total}}=dS_1+dS_2=\frac{dQ_1}{T_1}+\frac{dQ_2}{T_2} \tag{25.4.5}$$

and the requirement that this cannot be negative (that is, S_{total} must not decrease) is just the same as Equation (25.4.2), in differential form.

Once again, this simply means that the hotter object gives off the heat and the colder one absorbs it, but when you look at it in terms of entropy it is a bit more interesting than that. You can see that the entropy of the hotter object decreases (negative dQ), and that of the colder one increases (positive dQ), but *by a different amount*: in fact, it increases so much that it makes the total change in entropy for the system positive. This shows that entropy is rather different from energy (which is simply conserved in the process). You can always make it increase just by letting a process "take its normal course"—in this case, just letting the heat flow from the warmer to the colder object until they reach thermal equilibrium with each other (at which point, of course, the entropy will stop increasing, since it is a function of the state and the state will no longer change).

Although not immediately obvious from the above, the absolute (or Kelvin) temperature scale plays an essential role in the definition of the entropy, in the sense that only in such a scale (or another scale *linearly proportional* to it) is the entropy, as defined by Equation (25.4.4), a state variable; that is, only when using such a temperature scale is the integral (25.4.4) path-independent. The proof of this (which is much too complicated to even sketch here) relies essentially on the Carnot principle, to be discussed next.

The Efficiency of Heat Engines

By the beginning of the 19th century, an industrial revolution was underway in England, due primarily to the improvements in the efficiency of steam engines that had taken place a few decades earlier. It was natural to ask how much this efficiency could ultimately be increased, and in 1824, a French engineer, Nicolas Sadi Carnot, wrote a monograph that provided an answer to this question.

Carnot modeled a "heat engine" as an abstract machine that worked in a cycle. In the course of each cycle, the engine would take in an amount of heat Q_h from a "hot reservoir," give off (or "exhaust") an amount of heat $|Q_c|$ to a "cold reservoir," and produce an amount of work $|W|$. (I am using absolute value bars here because, from the point of view of the engine, Q_c and W must be negative quantities.) At the end of the cycle, the engine should be back to its initial state, so $\Delta E_{engine}=0$. The hot and cold reservoirs were supposed to be systems with very large heat capacities, so that the change in their temperatures as they took in or gave off the heat from or to the engine would be negligible.

If $\Delta E_{engine} = 0$, we must have

$$\Delta E_{\text{engine}} = Q_h + Q_c + W = Q_h - |Q_c| - |W| = 0 \tag{25.4.6}$$

that is, the work produced by the engine must be

$$|W| = Q_h - |Q_c|. \tag{25.4.7}$$

The energy input to the engine is Q_h, so it is natural to define the efficiency as $\epsilon = |W|/Q_h$; that is to say, the Joules of work done per Joule of heat taken in. A value of $\epsilon = 1$ would mean an efficiency of 100%, that is, the complete conversion of thermal energy into macroscopic work. By Equation (25.4.7), we have

$$\epsilon = \frac{|W|}{Q_h} = \frac{Q_h - |Q_c|}{Q_h} = 1 - \frac{|Q_c|}{Q_h} \tag{25.4.8}$$

which shows that ϵ will always be less than 1 as long as the heat exhausted to the cold reservoir, Q_c, is nonzero. This is always necessarily the case for steam engines: the steam needs to be cooled off at the end of the cycle, so a new cycle can start again.

Carnot considered a hypothetical "reversible" engine (sometimes called a *Carnot machine*), which could be run *backwards*, while interacting with the same two reservoirs. In backwards mode, the machine would work as a refrigerator or heat pump. It would take in an amount of work W per cycle (from some external source) and use that to absorb the amount of heat $|Q_c|$ from the cold reservoir and dump the amount Q_h to the hot reservoir. Carnot argued that *no heat engine could have a greater efficiency than a reversible one working between the same heat reservoirs*, and, consequently, that *all reversible engines, regardless of their composition, would have the same efficiency when working in between the same temperatures*. His argument was based on the observation that a hypothetical engine with a greater efficiency than the reversible one could be used to drive a reversible one in refrigerator mode, to produce as the sole result the transfer of some net amount of heat from the cold to the hot reservoir[4], something that we argued in Section 1 should be impossible.

What makes this result more than a theoretical curiosity is the fact that an ideal gas would, in fact, provide a suitable working substance for a Carnot machine, if put through the following cycle (the so-called "Carnot cycle"): an isothermal expansion, followed by an adiabatic expansion, then an isothermal compression, and finally an adiabatic compression. What makes this ideally reversible is the fact that the heat is exchanged with each reservoir only when the gas is at (nearly) the same temperature as the reservoir itself, so by just "nudging" the temperature up or down a little bit you can get the exchange to go either way. When the ideal gas laws are used to calculate the efficiency of such a machine, the result (the *Carnot efficiency*) is

$$\epsilon_C = 1 - \frac{T_c}{T_h} \tag{25.4.9}$$

where the temperatures must be measured in degrees Kelvin, the natural temperature scale for an ideal gas.

It is actually easy to see the connection between this result and the entropic formulation of the second law presented above. Suppose for a moment that Carnot's principle does not hold, that is to say, that we can build an engine with $\epsilon > \epsilon_C = 1 - T_c/T_h$. Since (25.4.8) must hold in any case (because of conservation of energy), we find that this would imply

$$1 - \frac{|Q_c|}{Q_h} > 1 - \frac{T_c}{T_h} \tag{25.4.10}$$

and then some very simple algebra shows that

$$-\frac{Q_h}{T_h} + \frac{|Q_c|}{T_c} < 0. \tag{25.4.11}$$

But now consider the total entropy of the system formed by the engine and the two reservoirs. The engine's entropy does not change (because it works in a cycle); the entropy of the hot reservoir goes *down* by an amount $-Q_h/T_h$; and the entropy of the cold reservoir goes *up* by an amount $|Q_c|/T_c$. So the left-hand side of Equation (25.4.11) actually equals the total change in entropy, and Equation (25.4.11) is telling us that this change is *negative* (the total entropy goes *down*) during the operation of this hypothetical heat engine whose efficiency is greater than the Carnot limit (25.4.9). Since this is impossible (the total entropy of a closed system can never decrease), we conclude that the Carnot limit must always hold.

As you can see, the seemingly trivial observation with which I started this section (namely, that heat always flows spontaneously from a hotter object to a colder object, and never in reverse) turns out to have profound consequences. In particular, it means that *the complete conversion of thermal energy into macroscopic work is essentially impossible*[5], which is why we treat mechanical energy as “lost” once it is converted to thermal energy. By Carnot’s theorem, to convert some of that thermal energy back to work we would need to introduce a colder reservoir (and take advantage, so to speak, of the natural flow of heat from hotter to colder), and then we would only get a relatively small conversion efficiency, unless the cold reservoir is really at a very low Kelvin temperature (and to create such a cold reservoir would typically require refrigeration, which again consumes energy). It is easy to see that Carnot efficiencies for reservoirs close to room temperature are rather pitiful. For instance, if T_h = 300 K and T_c = 273 K, the best conversion efficiency you could get would be 0.09, or 9%.

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[4]The greater efficiency engine could produce the same amount of work as the reversible one while absorbing less heat from the hot reservoir and dumping less heat to the cold one. If all the work output of this engine were used to drive the reversible one in refrigerator mode, the result would be, therefore, a net flow of heat out of the cold one and a net flow of heat into the hot one.

[5]At least it is impossible to do using a device that runs in a cycle. For a one-use-only, you might do something like pump heat into a gas and allow it to expand, doing work as it does so, but eventually you will run out of room to do your expanding into...

But What IS Entropy, Anyway?

The existence of this quantity, the entropy, which can be measured or computed (up to an arbitrary reference constant) for any system in thermal equilibrium, is one of the great discoveries of 19th century physics. There are tables of entropies that can be put to many uses (for instance, in chemistry, to figure out which reactions will happen spontaneously and which ones will not), and one could certainly take the point of view that those tables, plus the basic insight that the total entropy can never decrease for a closed system, are all one needs to know about it. From this perspective, entropy is just a convenient number that we can assign to any equilibrium state of any system, which gives us some idea of which way it is likely to go if the equilibrium is perturbed.

Nonetheless, it is natural for a physicist to ask to what, exactly, does this number correspond? What property of the equilibrium state is actually captured by this quantity? Especially, in the context of a microscopic description, since that is, by and large, how physicists have always been trying to explain things, by breaking them up into little pieces, and figuring out what the pieces were doing. What are the molecules or atoms of a system doing in a state of high entropy that is different from a state of low entropy?

The answer to this question is provided by the branch of physics known as Statistical Mechanics, which today is mostly quantum-based (since you need quantum mechanics to describe most of what atoms or molecules do, anyway), but which started in the context of pure classical mechanics in the mid-to-late 1800’s and, despite this handicap, was actually able to make surprising headway for a while.

From this microscopic, but still classical, perspective (which applies, for instance, moderately well to an ideal gas), the entropy can be seen as a measure of the *spread* in the velocities and positions of the molecules that make up the system. If you think of a probability distribution, it has a mean value and a standard deviation. In statistical mechanics, the molecules making up the system are described statistically, by giving the probability that they might have a certain velocity or be at some point or another. These probability distributions may be very narrow (small standard deviation), if you are pretty certain of the positions or the velocities, or very broad, if you are not very certain at all, or rather expect the actual velocities and positions to be spread over a considerable range of values. A state of large entropy corresponds to a broad distribution, and a state of small entropy to a narrow one.

For an ideal gas, the temperature determines both the average molecular speed and the spread of the velocity distribution. This is because the average velocity is zero (since it is just as likely to be positive or negative), so the only way to make the average speed (or root-mean-square speed) large is to have a broad velocity distribution, which makes large speeds comparatively more likely. Then, as the temperature increases, so does the range of velocities available to the molecules, and correspondingly the entropy. Similarly (but more simply), for a given temperature, a gas that occupies a smaller volume will have a smaller entropy, since the range of positions available to the molecules will be smaller.

These considerations may help us understand an important property of entropy, which is that it increases in all irreversible processes. To begin with, note that this makes sense, since, by definition, these are processes that do not “reverse” spontaneously. If a process involves an increase in the total entropy of a closed system, then the reverse process will not happen, because it would require a spontaneous decrease in entropy, which the second law forbids. But, moreover, we can see the increase in entropy directly in many of the irreversible processes we have considered this semester, such as the ones involving friction. As I just pointed out above, in general, we may expect that increasing the temperature of an object will increase its entropy (other things being equal), regardless of how the increase in temperature comes about. Now, when mechanical energy is lost due to friction, the temperature of *both* of the objects (surfaces) involved increases, so the total entropy will increase as well. That marks the process as irreversible.

Another example of an irreversible process might be the mixing of two gases (or of two liquids, like cream and coffee). Start with all the "brown" molecules to the left of a partition, and all the "white" molecules to the right. After you remove the partition, the system will reach an equilibrium state in which the range of positions available to both the brown and white molecules has increased substantially—and this is, according to our microscopic picture, a state of higher entropy (other things, such as the average molecular speeds, being equal[6]).

For quantum mechanical systems, where the position and velocity are not simultaneously well defined variables, one uses the more abstract concept of "state" to describe what each molecule is doing. The entropy of a system in thermal equilibrium is then defined as a measure of the total number of states available to its microscopic components, compatible with the constraints that determine the macroscopic state (such as, again, total energy, number of particles, and volume).

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[6]In the case of cream and coffee, the average molecular speeds will not be equal—the cream will be cold and the coffee hot—but the resulting exchange of heat is just the kind of process I described at the beginning of the chapter, and we have seen that it, too, results in an increase in the total entropy.

25.5: In Summary

1. *Temperature* is a statistical quantity that provides a (typically indirect) measure of the *concentration* of thermal energy in a system. For a system that is (approximately) well described by classical mechanics, the temperature, as measured by a conventional thermometer, is directly proportional to the average translational kinetic energy per molecule.
2. In a process in which a system does no work, a change in the system's temperature is related to a change in its total internal energy (which typically includes more than just translational kinetic energy contributions) by $\Delta E = C\Delta T$, where C is the system's *heat capacity* for the process.
3. The transfer of thermal energy between two systems without either one doing macroscopic work on each other is generally possible. Thermal energy transferred in this way is called *heat*, and denoted by the symbol Q.
4. The actual definition of a system's heat capacity is $C = Q/\Delta T$. For a homogeneous system (made of just one substance), $C = mc$, where m is the system's mass and c the substance's specific heat. Specific heats typically depend on temperature in nontrivial ways.
5. Two systems isolated from the rest of the world but allowed to exchange thermal energy with each other will eventually reach a state of *thermal equilibrium* in which their temperatures will be the same (zero-th law of thermodynamics).
6. The work done on (or by) a system by (or on) its environment, plus the heat given to (or taken from) the system by its environment, always equals the net change in the system's total energy (conservation of energy, or first law of thermodynamics; Equation (13.3.1)).
7. For any system in thermal equilibrium, there exists a state variable, called *entropy*, with the property that it can never decrease for a closed system. When a system at temperature T takes in a small amount of heat dQ, its change in entropy is given by $dS = dQ/T$.
8. This principle of never-decreasing entropy is equivalent to the statement that "No process is possible whose *sole result* is the transfer of heat from a cooler to a hotter body."
9. The principle 7. is also equivalent to Carnot's theorem, which states that "it is impossible for an engine that operates in a cycle, taking in heat from a hot reservoir at temperature T_h and exhausting heat to a cold reservoir at temperature T_c, to do work with an efficiency greater than $1 - T_c/T_h$."
10. Either one of 7., 8., or 9., above, may be regarded as an equivalent statement of the *second law of thermodynamics*.
11. Carnot's theorem shows the limitations inherent in the conversion of thermal energy into macroscopic work, which is the reason why one usually regards mechanical energy that is converted into thermal energy as "lost."
12. Microscopically, the entropy of a system is a measure of the range of distinct states available to its microscopic components (atoms or molecules) that are compatible with the set of macroscopic constraints that determine its thermal equilibrium state. More entropy means a greater range of possible "microstates."
13. Entropy always increases in irreversible processes.

25.6: Examples

Example 25.6.1: Calorimetry

The specific heat of aluminum is 900 J/kg·K, and that of water is 4186 J·K. Suppose you drop a block of aluminum of mass 1 kg at a temperature of 80°C in a liter of water (which also has a mass of 1 kg) at a temperature of 20°C. What is the final temperature of the system, assuming no exchange of heat with the environment takes place? How much energy does the aluminum lose/the water gain?

Solution

Let us call T_{Al} the initial temperature of the aluminum, T_{water} the initial temperature of the water, and T_f their final common temperature. The thermal energy given off by the aluminum equals $\Delta E_{Al} = C_{Al}(T_f - T_{Al})$ (this follows from the definition (13.2.1) of heat capacity; we could equally well call this quantity "the heat given off by the aluminum"). In the same way, the thermal energy change of the water (heat absorbed by the water) equals $\Delta E_{water} = C_{water}(T_f - T_{water})$. If the total system is closed, the sum of these two quantities, each with its appropriate sign, must be zero:

$$0 = \Delta E_{\text{Al}} + \Delta E_{\text{water}} = C_{\text{Al}}\left(T_f - T_{\text{Al}}\right) + C_{\text{water}}\left(T_f - T_{\text{water}}\right). \tag{25.6.1}$$

This equation for T_f has the solution

$$T_f = \frac{C_{\text{Al}}T_{\text{Al}} + C_{\text{water}}T_{\text{water}}}{C_{\text{Al}} + C_{\text{water}}}. \tag{25.6.2}$$

As you can see, the result is a weighted average of the two starting temperatures, with the corresponding heat capacities as the weighting factors.

The heat capacities C are equal to the given specific heats multiplied by the respective masses. In this case, the mass of aluminum and the mass of the water are the same, so they will cancel in the final result. Also, we can use the temperatures in degrees Celsius, instead of Kelvin. This is not immediately obvious from the final expression (25.6.2), but if you look at (25.6.1) you'll see it involves only temperature differences, and those have the same value in the Kelvin and Celsius scales.

Substituting the given values in (25.6.2), then, we get

$$T_f = \frac{900 \times 80 + 4186 \times 20}{900 + 4186} = 30.6^{\circ}\text{C}. \tag{25.6.3}$$

This is much closer to the initial temperature of the water, as expected, since it has the greater heat capacity. The amount of heat exchanged is

$$C_{\text{water}}\left(T_f - T_{\text{water}}\right) = 4186 \times (30.6 - 20) = 44,440 \text{ J} = 44.4 \text{ kJ}. \tag{25.6.4}$$

So, 1 kg of aluminum gives off 44.4 kJ of thermal energy and its temperature drops almost 50°C, from 80°C to 30.6°C, whereas 1 kg of water takes in the same amount of thermal energy and its temperature only rises about 10.6°C.

Example 25.6.2: Equipartition of energy

Estimate the speed of an oxygen molecule in air at room temperature (about 300 K).

Solution

Recall that in Section 13.2 I mentioned that the average translational kinetic energy of a molecule in a system at a temperature T is $\frac{3}{2}k_BT$ (Equation (13.2.7), where k_B, Boltzmann's constant, is equal to 1.38 × 10^{-23} J/K. So, at T = 300 K, a molecule of oxygen (or of anything else, for that matter) should have, on average, a kinetic energy of

$$\langle K_{\text{trans}}\rangle = \frac{3}{2}k_BT = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 \text{ J} = 6.21 \times 10^{-21} \text{ J}. \tag{25.6.5}$$

Since $K = \frac{1}{2}mv^2$, we can figure out the average value of v^2 if we know the mass of an oxygen molecule. This is something you can look up, or derive like this: One mole of oxygen atoms has a mass of 16 grams (16 is the atomic mass number of oxygen) and contains Avogadro's number of atoms, 6.02×10^{23}. So a single atom has a mass of 0.016 kg/6.02×10^{23} = 2.66×10^{-26} kg. A molecule of oxygen contains two atoms, so it has twice the mass, m = 5.32×10^{-26} kg. Then,

$$\left\langle v^2 \right\rangle = \frac{2\left\langle K_{\text{trans}} \right\rangle}{m} = \frac{2 \times 6.21 \times 10^{-21}\ \mathrm{J}}{5.32 \times 10^{-26}\ \mathrm{kg}} = 2.33 \times 10^5\ \frac{\mathrm{m}^2}{\mathrm{s}^2}. \tag{25.6.6}$$

The square root of this will give us what is called the "root mean square" velocity, or v_{rms}:

$$v_{rms} = \sqrt{2.33 \times 10^5\ \frac{\mathrm{m}^2}{\mathrm{s}^2}} = 483\ \frac{\mathrm{m}}{\mathrm{s}}. \tag{25.6.7}$$

This is of the same order of magnitude as (but larger than) the speed of sound in air at room temperature (about 340 m/s, as you may recall from Chapter 12).

25.7: Exercises

Exercise 25.7.1

Consider a system of two objects in contact, one initially hotter than the other, so they may directly exchange thermal energy, in isolation from the rest of the world. According to the laws of thermodynamics, what must happen to the system's total energy and entropy? (Do they change, increase, decrease, stay constant...?)

Exercise 25.7.2

Consider the same two objects in Problem 1 and suppose the heat capacity of the colder object is much greater than the heat capacity of the hotter one. When the system reaches thermal equilibrium, will its final temperature will be closer to the initial temperature of the hot object, the colder object, or exactly halfway between the two initial temperatures? Why?

Exercise 25.7.3

Which of the following is *not* a valid formulation of the second law of thermodynamics?

a. For any system in thermal equilibrium, there exists a state variable, called entropy, with the property that it can never decrease for a closed system.
b. No process is possible whose sole result is the transfer of heat from a cooler to a hotter body.
c. It is impossible for an engine that operates in a cycle, taking in heat from a hot reservoir at temperature T_h and exhausting heat to a cold reservoir at temperature T_c, to do work with an efficiency greater than $1-T_c/T_h$.
d. The entropy of any system goes to zero as T (the absolute, or Kelvin) temperature goes to zero.

Exercise 25.7.4

Which of the following statements is true?

a. Once the entropy of a system increases, it is impossible to bring it back down.
b. Once some amount of mechanical energy is converted to thermal energy, it is impossible to turn any of it back into mechanical energy.
c. It is always possible to reduce the entropy of a system, for instance, by cooling it.
d. All of the above statements are true.
e. None of the above statements are true.

Other Questions

- Can you tell the temperature of a gas by measuring the translational kinetic energy of a single molecule?
- Does a shuffled deck of cards have more or less entropy (in the thermodynamic sense) than an identical, ordered set of cards? Assume they are at the same temperature.
- A diatomic gas molecule, such as O_2, can store kinetic energy in the form of vibrations and rotations, in addition to just translation of the center of mass. By contrast, a monoatomic gas molecule such as C has virtually no kinetic energy (at normal temperatures) other than translational kinetic energy. Which kind of gas do you expect to have a larger molar heat capacity (heat capacity per molecule)?

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