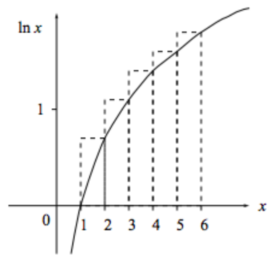
10.5: E- Stirling's Approximation

The Stirling formula is an approximation for $n!$ that is good at large values of n.

$$n!=1\cdot 2\cdot 3\cdots(n-1)\cdot n \tag{10.5.1}$$

$$\ln(n!)=\underbrace{\ln 1}_{0}+\ln 2+\ln 3+\cdots+\ln(n-1)+\ln(n) \tag{10.5.2}$$



Note that the function $\ln x$ is nearly flat for large values of x. For example, $\ln 10^{23}$ is about equal to 23.

From the figure

$$\ln(6!)=\text{ area under the staircase } >\int_1^6 \ln x dx \tag{10.5.3}$$

and in general

$$\ln(n!)>\int_1^n \ln x dx=[x\ln x-x]_1^n=n\ln n-n+1. \tag{10.5.4}$$

For large values of n, where the $\ln n$ function is nearly flat, the two expressions above become quite close. Also, the 1 becomes negligible. We conclude that

$$\ln(n!)\approx n\ln n-n \quad \text{ for } n\gg 1. \tag{10.5.5}$$

This is Stirling's formula. For corrections to the formula, see M. Boas, *Mathematical Methods in the Physical Sciences*, sections 9-10 and 9-11. You know that

$$A^n \tag{10.5.6}$$

increases rapidly with n for positive A, but

$$n!\approx\left(\frac{n}{e}\right)^n \tag{10.5.7}$$

increases a bit more rapidly still.

E.1 Problem: An upper bound for the factorial function

Stirling's approximation gives a rigorous lower bound for $n!$.

a. Use the general ideas presented in the derivation of that lower bound to show that

$$\int_1^n \ln(x+1)dx>\ln n!. \tag{10.5.8}$$

b. Conclude that

$$(n+1)\ln(n+1)-n+1-2\ln 2>\ln n!>n\ln n-n+1. \tag{10.5.9}$$