10.3: C- Clinic on the Gamma Function

The gamma function $\Gamma(s)$ is defined, for $s > 0$, by

$$\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,dx. \tag{10.3.1}$$

Upon seeing any integral, your first thought is to evaluate it. Stay calm. . . first make sure that the integral exists. A quick check shows that the integral above converges when $s > 0$.

There is no simple formula for the gamma function for arbitrary s. But for $s = 1$,

$$\Gamma(1)=\int_0^\infty e^{-x}\,dx=-e^{-x}\Big|_0^\infty=1. \tag{10.3.2}$$

For $s > 1$ we may integrate by parts,

$$\int_0^\infty x^{s-1}e^{-x}\,dx=-x^{s-1}e^{-x}\Big|_0^\infty+(s-1)\int_0^\infty x^{s-2}e^{-x}\,dx, \tag{10.3.3}$$

giving

$$\Gamma(s)=(s-1)\Gamma(s-1)\quad \text{for } s>1. \tag{10.3.4}$$

Apply equation (C.4) repeatedly for n a positive integer,

$$\Gamma(n)=(n-1)\Gamma(n-1)=(n-1)(n-2)\Gamma(n-2)=(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1)=(n-1)!, \tag{10.3.5}$$

to find a relation between the gamma function and the factorial function. Thus the gamma function generalizes the factorial function to non-integer values, and can be used to define the factorial function through

$$x!=\Gamma(x+1)\quad \text{for any } x>-1. \tag{10.3.6}$$

In particular,

$$0!=\Gamma(1)=1. \tag{10.3.7}$$

(It is a deep and non-obvious result that the gamma function is in fact the simplest generalization of the factorial function.)

The gamma function can be simplified for half-integral arguments. For example

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty x^{-1/2}e^{-x}\,dx=\int_0^\infty y^{-1}e^{-y^2}(2y\,dy)=\int_{-\infty}^\infty e^{-y^2}\,dy=\sqrt{\pi} \tag{10.3.8}$$

where we used the substitution $y=\sqrt{x}$. Thus

$$\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}=\left(\frac{1}{2}\right)!, \tag{10.3.9}$$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{4}\sqrt{\pi}, \tag{10.3.10}$$

and so forth.