6.2: The Interchange Rule

Before turning to statistical mechanics, let us review the quantal “interchange rule”. The wavefunction for a system of three particles is a function of three variables: $\psi(x_A, x_B, x_C)$. [The symbol x represents whatever is needed to specify the state: For a spinless particle in one dimension x represents the coordinate x or, if you are working in momentum space, the coordinate *p*. For a particle with spin moving in three dimensions, x represents (x, y, z, m_z), or perhaps (p_x, p_y, p_z, m_x).] If the three particles are identical, then the wavefunction must be either symmetric under the interchange of any pair of variables

$$\psi\left(\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C\right)=+\psi\left(\mathbf{x}_B, \mathbf{x}_A, \mathbf{x}_C\right)=+\psi\left(\mathbf{x}_B, \mathbf{x}_C, \mathbf{x}_A\right), \quad \text{etc. ,} \tag{6.2.1}$$

or else antisymmetric under the interchange of any pair of variables,

$$\psi\left(\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C\right)=-\psi\left(\mathbf{x}_B, \mathbf{x}_A, \mathbf{x}_C\right)=+\psi\left(\mathbf{x}_B, \mathbf{x}_C, \mathbf{x}_A\right), \quad \text{etc.} \tag{6.2.2}$$

The foregoing assertion is an empirical rule that cannot be derived from any of the other principles of quantum mechanics. (Indeed there is currently considerable interest in *anyons*, hypothetical particles that obey all the principles of quantum mechanics except the interchange rule.) The rule holds for all quantal states, not just energy eigenstates. It holds for interacting as well as for non-interacting particles. And the rule has a number of surprising consequences, both within the domain of quantum mechanics and atomic physics, and, as well shall soon see in detail, within the domain of statistical mechanics.

The sign of the interchange symmetry, either + or −, is governed only by the type of particle involved: for pions it is always +, for electrons it is always −. Particles for which the sign is always + are called bosons, and those for which it is always − are called *fermions*. It is an experimental fact that particles with integral spin *s* are bosons and those with half integral spin *s* are fermions.

6.2.1 Problems

6.1 The interchange rule in another representation

Full information about the state of three identical spinless particles is contained not only in the configurational wave function $\psi(\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C)$ but also in the momentum space wave function

$$\tilde{\psi}\left(\mathbf{p}_A, \mathbf{p}_B, \mathbf{p}_C\right)=\frac{1}{(2 \pi \hbar)^{9 / 2}} \int d^3 x_A \int d^3 x_C \psi\left(\mathbf{x}_A, \mathbf{x}_B, \mathbf{x}_C\right) e^{-i\left(\mathbf{p}_A \cdot \mathbf{x}_A+\mathbf{p}_B \cdot \mathbf{x}_B+\mathbf{p}_C \cdot \mathbf{x}_C\right) / \hbar}. \tag{6.2.3}$$

Show that if one representation is symmetric (or antisymmetric), then the other one is as well.

6.2 Symmetrization

Show that for any function $f(x, y, z)$, the function

$$f_S(x, y, z)=f(x, y, z)+f(x, z, y)+f(z, x, y)+f(z, y, x)+f(y, z, x)+f(y, x, z) \tag{6.2.4}$$

is symmetric under the interchange of any pair of variables. How many such pairs are there? Is there a similar algorithm for building up an antisymmetric function from any garden-variety function?

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