2.4: Statistical Definition of Entropy

Logarithms and dimensions

You cannot take the logarithm of a number with dimensions. Perhaps you have heard this rule phrased as "you can't take the logarithm of 3.5 meters" or "you can't take the logarithm of five oranges". Why not? A simple argument is "Well, what would be the units of $\ln(3.5\text{meters})$?" A more elaborate argument follows. The logarithm function is the inverse of the exponential function:

$$y=\ln(x) \quad \text{means the same as} \quad x=e^{y}.$$

But remember that

$$x=e^{y}=1+y+\frac{1}{2!}y^{2}+\frac{1}{3!}y^{3}+\cdots.$$

If y had the dimensions of length, then the expression above would be a meaningless sum of 1 plus a length plus an area plus a volume plus so forth.

You cannot exponentiate a number with dimensions, and you cannot take the logarithm of a number with dimensions.

Additivity (see problem 2.10) . . . don't physically mix.

$$S(E,\Delta E,V,N)=k_{B}\ln\Omega(E,\Delta E,V,N) \tag{2.4.1}$$

The constant k_B in this definition is called the "Boltzmann constant". It is clear from the argument that the Boltzmann constant could have been chosen to have any value: 1, π, whatever. For historical reasons, it was chosen to have the value

$$k_{B}=1.38\times 10^{-23}\text{ joule / kelvin.} \tag{2.4.2}$$

There is no particular physical significance to this number: its logical role is analogous to 2.54 cm/inch or 4.186 joule/calorie. In other words, the origin of this number is not to be sought in nature, but in the history of the definition of the joule and the kelvin. It is a conversion factor.

Problems

2.8 (E) **Accessible regions of phase space**

Suppose that N non-interacting particles, each of mass m, move freely in a one-dimensional box (i.e. an infinite square well). Denote the position coordinates by $x_1, x_2, \ldots, x_N$ and the momentum coordinates by $p_1, p_2, \ldots, p_N$. The box restricts all the positions to fall between $x_i=0$ and $x_i=L$. The energy of the system lies between E and $E+\Delta E$.

a. If only one particle is present, draw the system's phase space and shade the regions of phase space that are accessible.

b. If two particles are present then phase space is four dimensional, which makes it difficult to draw. Draw separately the part of phase space involving positions and the part involving momenta. Shade the accessible regions of phase space.

c. Suppose two particles are present, and consider the slice of phase space for which $x_1=(2/3)L$ and p_2 equals some constant called $\bar{p}_2$. Draw a (carefully labeled) sketch of this slice with the accessible regions shaded.

d. Describe the accessible regions of phase space if N particles are present.

2.9 **Accessible configurations of a spin system**

Consider an isolated system of N spin-$\frac{1}{2}$ atoms in a magnetic field H. The atoms are fixed at their lattice sites and the spins do not interact. Each atom has a magnetic moment m that can point either "up" (parallel to the field H) or "down" (antiparallel to H). A microstate (or configuration) of this system is specified by giving the direction of every spin. An up spin has energy $-mH$, a down spin has energy $+mH$, so a configuration with $n_\uparrow$ up spins and $n_\downarrow$ down spins has energy

$$E=-(n_{\uparrow}-n_{\downarrow})mH. \tag{2.4.3}$$

This system is called the "ideal paramagnet".

a. Not every energy is possible for this model. What is the maximum possible energy? The minimum? What is the minimum possible non-zero energy difference between configurations?

b. Suppose we know that the system has $n_\uparrow$ up spins and $n_\downarrow$ down spins, but we do not know how these spins are arranged. How many microstates are consistent with this knowledge?

c. The variables $n_\uparrow$ and $n_\downarrow$ cannot be determined directly from macroscopic measurements. Find expressions for $n_\uparrow$ and $n_\downarrow$ in terms of N, E, and H. (Hand a paramagnet sample to an experimentalist and ask her to find the number of up spins. She will just look at you quizzically. But ask her to find the number, the energy, and the magnetic field and she'll be happy to.)

d. Consider the energy range from E to $E+\Delta E$ where ΔE is small compared to NmH but large compared to mH. What is the approximate number of states $\Omega(E, \Delta E, H, N)$ lying in this energy range? Express your answer in a form that does not include the quantities $n_\uparrow$ or $n_\downarrow$.

2.10 **Microstates for a combined system**

System #1 is in a macrostate with three corresponding microstates, labeled A, B, and C. System #2 is in a macrostate with four corresponding microstates, labeled α, β, γ, and δ. How many microstates are accessible to the combined system consisting of system #1 and system #2? List all such microstates.

2.11 (E) **The logarithm**

Suppose that a differentiable function satisfies

$$f(xy) = f(x) + f(y) \tag{2.4.4}$$

for all positive x and y. Show that

$$f(x) = k\ln(x). \tag{2.4.5}$$

[Clues: 1) Take derivative with respect to x, then set $x = 1.2$) Set $y = 1$ in Equation 2.4.4.]
