6.9: Additional Problems

6.27 What if there were no interchange rule?

Suppose that the interchange rule did not apply, so that the basis on page 141 were the correct one for three identical particles. (Alternatively, consider a gas of N non-identical particles.) Find and sketch the heat capacity as a function of temperature.

6.28 Pressure and energy density

(This problem was inspired by Reif problem 9.5.) Any non-relativistic monatomic ideal gas, whether classical or quantal, satisfies

$$p=\frac{2}{3}\frac{E}{V}. \tag{6.9.1}$$

This remarkable fact can be demonstrated most easily in the canonical ensemble. (Throughout this problem the particle number is fixed so the N dependence of functions is never mentioned explicitly.)

a. Use $E=\frac{3}{2}Nk_BT$ to demonstrate the relation for the classical monatomic ideal gas.

b. From thermodynamics, show that

$$E(T,V)=-\frac{\partial \ln Z}{\partial \beta})_V \quad \text{and} \quad p(T,V)=\frac{1}{\beta}\frac{\partial \ln Z}{\partial V})_T. \tag{6.9.2}$$

c. Argue that temperature and energy eigenvalues enter into the canonical partition function as quotients:

$$\ln Z(T,V)=\mathcal{F}\left(\beta\epsilon_1,\beta\epsilon_2,\ldots,\beta\epsilon_M\right) \tag{6.9.3}$$

d. Show that in non-relativistic quantum mechanics the free particle energy eigenvalues depend on box volume through

$$\frac{\partial \epsilon_r}{\partial V}=-\frac{2}{3}\frac{\epsilon_r}{V}. \tag{6.9.4}$$

e. Use the last three items together to prove the pressure-energy density relation.

f. How are the pressure and energy density related for blackbody radiation? At what stage does the above proof break down in this case?

6.29 Pressure comparison

(This problem is modified from one in a GRE Physics test.)

Consider three systems of non-interacting identical particles, each with the same *T*, *V*, and *N*. In one system the particles are fermions, in another they are bosons, and in a third they behave classically. Which system has the greatest pressure? Which has the smallest?

6.30 Challenge

For many years I suspected that the chemical potential μ of an ideal gas would have to decrease or at least remain constant when the temperature increased (with V and N constant). I tried proving this in a number of ways, and in my attempts I came across several interesting facts (such as the results of problem 3.36 and part (f.) of problem 6.11) but I was never been able to prove the desired result. That's because the result is false! The chemical potential increases with temperature for ideal fermions in one dimension. Can you show this?