1.5: Additional Problems

1.2 (I*) Compressibility, expansion coefficient

The "isothermal compressibility" of a substance is defined as

$$\kappa_T(p,T) = -\frac{1}{V(p,T)}\frac{\partial V(p,T)}{\partial p}, \tag{1.5.1}$$

where the volume $V(p, T)$ is treated as a function of pressure and temperature.

a. Justify the name "compressibility". If a substance has a large κ_T is it hard or soft? Since "squeeze" is a synonym for "compress", is "squeezeabilty" a synonym for "compressibility"? Why were the negative sign and the factor of 1/V included in the definition?

b. The "expansion coefficient" is

$$\beta(p,T) = \frac{1}{V(p,T)}\frac{\partial V(p,T)}{\partial T} \tag{1.5.2}$$

In most situations β is positive, but it can be negative. Give one such circumstance.

c. What are κ_T and β in a region of two-phase coexistence (for example, a liquid in equilibrium with its vapor)?

d. Find κ_T and β for an ideal gas, $V(p, T) = Nk_BT/p$.

e. Show that

$$\frac{\partial \kappa_T(p,T)}{\partial T} = -\frac{\partial \beta(p,T)}{\partial p} \tag{1.5.3}$$

for all substances.

f. Verify this relation for the ideal gas. (Clue: The two expressions are *not* both equal to $-Nk_B/p^2V$.)

1.3 (I*) Heat capacity as a susceptibility

Later in this book we will find that the "heat capacity" of a fluid[3] is

$$C_V(T,V,N) = \frac{\partial E(T,V,N)}{\partial T}, \tag{1.5.4}$$

where the energy $E(T, V, N)$ is considered as a function of temperature, volume, and number of particles. The heat capacity is easy to measure experimentally and is often the first quantity observed when new regimes of temperature or pressure are explored. (For example, the first sign of superfluid He^3 was an anomalous dip in the measured heat capacity of that substance.)

a. Explain how to measure the heat capacity of a gas given a strong, insulated bottle, a thermometer, a resistor, a voltmeter, an ammeter, and a clock.

b. Near the superfluid transition temperature T_c, the heat capacity of Helium is given by

$$C_V(T) = -A\ln(|T-T_c|/T_c). \tag{1.5.5}$$

Sketch the heat capacity and the energy as a function of temperature in this region.

c. The heat capacity is one member of a class of thermodynamic quantities called "susceptibilities". Why does it have that name? (Clues: A change in temperature causes a change in energy, but how much of a change? If the heat capacity is relatively high, is the system relatively sensitive or insensitive (i.e. susceptible or insusceptible) to such temperature changes?)

d. Interpret the isothermal compressibility (1.6) as a susceptibility. (Clue: A change in pressure causes a change in volume.)

1.4 (I) The meaning of "never"

(This problem is modified from Kittel and Kroemer, *Thermal Physics*, second edition, page 53.) It has been said[4] that "six monkeys, set to strum unintelligently on typewriters for millions of years, would be bound in time to write all the books in the British Museum". This assertion gives a misleading impression concerning very large numbers.[5] Consider the following situation:

- The quote considers six monkeys. Let's be generous and allow 10^{10} monkeys to work. (About twice the present human population.)
- The quote vaguely mentions "millions of years". The age of the universe is about 14 billion years, so let's allow all of that time, about 10^{18} seconds.
- The quote wants to write out all the books in a very large library. Let's be modest and demand only the production of Shakespeare's Hamlet, a work of about 105 characters.
- Finally, assume that a monkey can type ten characters per second, and for definiteness, assume a keyboard of 29 characters (letters, comma, period, space. . . ignore caPitALIzaTion).

a. Show that the probability that a given sequence of 10^5 characters comes out through a random striking of 10^5 keys is

$$\frac{1}{29^{100000}} \approx 10^{-146240}. \tag{1.5.6}$$

How did you perform the arithmetic?

b. Show that the probability of producing Hamlet through the "unintelligent strumming" of 10^{10} monkeys over 10^{18} seconds is about $10^{-146\,211}$, which is small enough to be considered zero for most purposes.

1.5 Human genetics

There are about 21,000 genes in the human genome. Suppose that each gene could have any of three possible states (called "alleles"). (For example, if there were a single gene for hair color, the three alleles might be black, brown, and blond.) Then how many genetically distinct possible people would there be? Compare to the current worldwide human population. Estimate how long it would take for every possible genetically distinct individual to be realized. Compare to the age of the universe.

[3]Technically, the "heat capacity at constant volume".

[4]J. Jeans, *Mysterious Universe* (Cambridge University Press, Cambridge, 1930) p. 4.

[5]An insightful discussion of the "monkeys at typewriters" problem, and its implications for biological evolution, is given by Richard Dawkins in his book *The Blind Watchmaker* (Norton, New York, 1987) pp. 43–49.