3.12: Additional Problems

3.43 Cool mountain air

Model the earth's atmosphere as an ideal gas (nitrogen) in a uniform gravitational field. Ignore all winds. Let m denote the mass of a gas molecule, g the acceleration of gravity, and z the height above sea level.

a. Use ideas from Newtonian mechanics to show that the change of atmospheric pressure p with height z is

$$\frac{dp}{dz} = -\frac{mg}{k_B T(z)} p(z). \tag{3.12.1}$$

b. If the atmosphere is a poor conductor of heat, then the decrease in pressure with height is due to an adiabatic expansion. (Clue: In other words: At the bottom of the mountain, fill an insulated balloon with air at the local density, pressure, and temperature. Transport that balloon to the top of the mountain. During the journey the balloon will expand, so the density, pressure, and temperature will change. According to our assumption, during this journey the balloon's density, pressure, and temperature will match those of the atmosphere outside.) Show that under this assumption

$$\frac{dp}{dT} = \frac{\gamma}{\gamma - 1} \frac{p(T)}{T} \tag{3.12.2}$$

and hence that

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma} \frac{mg}{k_B}. \tag{3.12.3}$$

Evaluate this expression in kelvin per kilometer for nitrogen, which has $\gamma = 1.4$.

c. In contrast, if the atmosphere were a good conductor of heat, then temperature would be uniform. Find $p(z)$ under such circumstances. Denote the sea-level pressure and temperature by p_0 and T_0.

d. Similarly find $p(z)$ for an adiabatic atmosphere. 3.44 The speed of sound When a sound wave passes through a fluid (liquid or gas), the period of vibration is short compared to the time necessary for significant heat flow, so the compressions may be considered adiabatic. Analyze the compressions and rarefactions of fluid in a tube. The equilibrium mass density is ρ_0. Apply $F = ma$ to a slug of fluid of thickness Δx, and show that if the variations in pressure $p(x, t)$ are small then pressure satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \tag{3.12.4}$$

where c, the speed of sound, is given by

$$c = \frac{1}{\sqrt{\rho_0 \kappa_S}}. \tag{3.12.5}$$

Optional: Use the results of problems 1.2 and 3.35 to show that, for an ideal gas,

$$c = \sqrt{\gamma \frac{k_B T}{m}}. \tag{3.12.6}$$

3.45 Thermodynamics of a plastic rod

(This problem is based on Reif problem 5.14.)

For a restricted range of lengths L and temperatures T, the tension force in a stretched plastic rod is

$$F(T, L) = aT^2 (L - L_0), \tag{3.12.7}$$

where a is a positive constant and L_0 is the relaxed (unstretched) length of the rod. When $L = L_0$, the heat capacity C_L of the rod (measured at constant length) is given by $C_L(T, L_0) = bT$, where b is independent of temperature.

a. Write down the fundamental thermodynamic relation for this rod, expressing dE in terms of dS and dL.

b. Compute $(\partial S/\partial L)T$. (Clue: Derive an appropriate Maxwell relation for the assembly with variables T and L.)

c. Knowing $S(T_0, L_0)$, integrate along an appropriate path to find $S(T, L)$ at any temperature and length within the range of applicability of the equation for $F(T, L)$.

d. If you start at $T = T_i$ and $L = L_i$ and then quasi-statically stretch a thermally insulated rod until it reaches length L_f, what is the final temperature T_f? Show than when $L_0 \leq L_i < L_f$, the rod is cooled by this process.

e. Find the heat capacity $C_L(L, T)$ of the rod when its length is not necessarily L_0.

f. Find $(\partial T/\partial L)S$ for arbitrary T and L. Can insulated stretches warm as well as cool the rod?

3.46 Magnetic cooling

At low temperatures, paramagnetic salts obey the Curie law

$$M = c\frac{H}{T}, \tag{3.12.8}$$

where c is a positive constant (see equation (3.100)). Assume that the heat capacity C_H is a constant independent of temperature and field. Suppose a sample at magnetic field H_i and temperature T_i is wrapped in insulation, and then the magnetic field is slowly reduced to zero. Find the final temperature, and show that it is less than T_i. This technique, known as "adiabatic demagnetization" is the refrigeration method used to produce the temperatures from about 1 kelvin to 1 microkelvin.

3.47 Thermodynamics of an electrochemical cell

Reif 5.16.

3.48 Thermodynamics and evolution

Read the essay "Thermodynamics and Evolution" by John W. Patterson, in *Scientists Confront Creationism*, Laurie R. Godfrey, ed. (Norton, New York, 1983), pages 99–116, on reserve in the science library.

a. When a snowflake forms, its surroundings increase in entropy ("become more disordered"). What is the name of the heat flow associated with this entropy change?

b. Patterson argues that $\Delta S < 0$ on Earth, due to biological evolution, and that $\Delta S > 0$ somewhere else in the universe in order to make up for it. Where is that entropy increase taking place?

c. Patterson feels the need to invoke "self-organization" and Prigogine (pages 110–111) to explain *how* his ram pumps could be made. Is this necessary? List two or more situations from nature in which water *does* flow uphill.

3.49 Entropy and evolution

Creationists sometimes claim that the second law of thermodynamics prohibits biological evolution.

a. The surface of the Sun (mean temperature 5778 K) heats the surface of the Earth (mean temperature 288 K) through visible and near-infrared radiation. The solar energy absorbed by the Earth each second is 1.732×10^{17} J. What is the entropy change per second (due to this process) of the Sun? The Earth? Does the entropy of "Sun plus Earth" increase or decrease?

b. Yet the mean temperature of the Earth changes slowly, if at all. This is because almost all of the solar energy absorbed by the Earth is then emitted through far-infrared radiation which in turn heats "outer space" — the cosmic microwave background (CMB; temperature 2.728 K). What is the entropy change per second (due to this process) of the Earth? The CMB? Does the entropy of "Earth plus CMB" increase or decrease?

c. Now refine the model by supposing that, due to evolution, the entropy of the Earth is not exactly constant, but is decreasing. (In this case the entropy of the CMB would have to be increasing faster than rate predicted in part (b).) Suppose that, due to evolution, each individual organism is 1000 times "more improbable" than the corresponding individual was 100 years ago. In other words, if Ω_i is the number of microstates consistent with the specification of an organism 100 years ago, and if Ωf is the number of microstates consistent with the specification of today's "improved and less probable" organism, then $\Omega_f = 10-3\Omega_i$. What is the corresponding change in entropy per organism?

d. The population of Earth is about 1018 eukaryotic individuals and 1032 prokaryotic individuals. If the estimate of part (c) holds for each one of them, what is the change in entropy due to evolution each second?

e. How accurately would you have to measure the entropy flux of part (b) in order to notice the diversion of entropy flux calculated in part (d)? Has any scientific quantity ever been measured to this accuracy?

f. It is generally agreed that the greatest rate of evolution fell during the Cambrian period, from 542 million years ago to 488 million years ago. During this so-called “Cambrian explosion” multicellular organisms first formed and then radiated into remarkable variety. Suppose that during the Cambrian period entropy was diverted into the evolution of living things at the rate calculated in part (d). And suppose that at the end of the Cambrian there were 10^{18} multicellular individuals. How much “improved and less probable” would each organism be, relative to its single-celled ancestor at the beginning of the Cambrian period?

The moral of the story? There’s plenty of entropy to go around.