6.6: Fermi-Dirac Statistics

In three dimensions, the chemical potential μ decreases with temperature. Why? It is clear that at $T = 0$, $\mu = \mathcal{E}_F > 0$. But as the temperature rises the gas approaches the classical limit, for which $\mu < 0$ (see problem 2.23). This is not proof, but it makes sense that μ should decrease with increasing temperature. A proof is available but surprisingly difficult.

6.6.1 Problems

6.17 Qualitative origin of $\mathcal{E}_F$

(This problem is stolen from a GRE Physics test.)

The mean kinetic energy of electrons in metals at room temperature is usually many times the thermal energy k_BT. Which of the following can best be used to explain this fact?

a. The time-energy uncertainty relation.

b. The Pauli exclusion principle.

c. The degeneracy of the energy levels.

d. The Born approximation.

e. The wave-particle duality.

6.18 Fermion gas in two dimensions

(This problem is based on one in Ashcroft and Mermin, page 53.)

Consider a gas of free, independent, spin-$\frac{1}{2}$ fermions in two dimensions. The gas is contained within an area (or two dimensional volume) of A.

a. What is the density of one-particle levels in k-space?

b. How does the Fermi energy $\mathcal{E}_F$ depend upon the density N/A?

c. Use $\sum_r \langle n_r \rangle = N$ to show that

$$\mu + k_BT \ln\left(1 + e^{-\mu/kBT}\right) = \mathcal{E}_F. \tag{6.6.1}$$

Notice that the chemical potential μ decreases with temperature.

6.19 Dependence of chemical potential on temperature

Show that for independent (not necessarily free) fermions, the $\mu(T)$ curve has slope (when N and V are constant)

$$\frac{d\mu}{dT} = -\frac{1}{T}\frac{\int_0^\infty G(\mathcal{E})\,\mathrm{sech}^2(\beta(\mathcal{E}-\mu)/2)(\mathcal{E}-\mu)d\mathcal{E}}{\int_0^\infty G(\mathcal{E})\,\mathrm{sech}^2(\beta(\mathcal{E}-\mu)/2)d\mathcal{E}}. \tag{6.6.2}$$

Can you use this result to show that the chemical potential must decrease with temperature? I can't.

6.20 Thermodynamics of the fermion gas

Consider a collection of free and independent spin-$\frac{1}{2}$ fermions. Do not assume that the temperature vanishes.

a. Use the fundamental grand canonical result $\Pi = -k_BT \ln \Xi$ to show that

$$p(T,\mu)V = k_BT \int_0^\infty G(\mathcal{E}) \ln\left(1 + e^{\beta(\mu-\mathcal{E})}\right) d\mathcal{E}. \tag{6.6.3}$$

b. Use the expression for $G(\mathcal{E})$ and the change of variable $x = \beta\mathcal{E}$ to find

$$p(T,\mu) = (k_BT)^{5/2}\left[\frac{\sqrt{2m^3}}{\pi^2\hbar^3}\right] \int_0^\infty \sqrt{x} \ln\left(1 + e^{\beta\mu}e^{-x}\right) dx. \tag{6.6.4}$$

c. Integrate by parts to obtain

$$p(T, \mu)=\frac{2}{3}\left(k_{B} T\right)^{5/2}\left[\frac{\sqrt{2 m^{3}}}{\pi^{2} \hbar^{3}}\right] \int_{0}^{\infty} \frac{x^{3/2}}{e^{x} e^{-\beta \mu}+1} d x \tag{6.6.5}$$

(Do not attempt to evaluate the integral that remains.)

d. Meanwhile, show that the total energy

$$E(T, V, \mu)=\int_{0}^{\infty} G(\mathcal{E}) f(\mathcal{E}) \mathcal{E} d \mathcal{E} \tag{6.6.6}$$

is given by

$$E(T, V, \mu)=V\left(k_{B} T\right)^{5/2}\left[\frac{\sqrt{2 m^{3}}}{\pi^{2} \hbar^{3}}\right] \int_{0}^{\infty} \frac{x^{3/2}}{e^{x} e^{-\beta \mu}+1} d x \tag{6.6.7}$$

Thus the pressure and energy are related by

$$p V=\frac{2}{3} E, \tag{6.6.8}$$

which is exactly the same relationship they have in the classical monatomic ideal gas! (See problem 6.28.)

6.21 Pressure at zero temperature

Use the result of the previous problem to show that at absolute zero the pressure of a gas of free and independent spin-$\frac{1}{2}$ fermions does not vanish (as it does for a classical ideal gas). Instead, the zero-temperature pressure is \(\frac{2}{5} \rho \mathcal{E}_{F}).

6.22 Mass-radius relation for white dwarf stars

(This problem is modified from Kittel and Kroemer, *Thermal Physics*, second edition, page 216.)

A white dwarf star (see Kittel and Kroemer, pp. 196–198) consists of highly compressed hydrogen in which the atoms have ionized into independent protons and electrons. In most cases it is a good model to assume that the electrons are free and independent non-relativistic fermions at zero temperature. Consider a white dwarf of mass M and radius R, containing N electrons.

a. Show that to a very good approximation, $N = M/m_p$, where m_p is the mass of a proton.

b. Show that the gravitational potential energy of a uniform sphere of mass M and radius R is $-cGM^2/R$, where G is the gravitational constant and c is a dimensionless constant. (In fact $c=\frac{3}{5}$ but this is tedious to show.)

c. Show that the kinetic energy of the electrons is

$$\mathrm{KE}=\frac{9}{20}\left(\frac{3 \pi^{2}}{2}\right)^{1/3} \frac{\hbar^{2}}{m_{e} m_{p}^{5/3}} \frac{M^{5/3}}{R^{2}} \tag{6.6.9}$$

where m_e is the mass of an electron.

d. If the potential and kinetic energies satisfy

$$\mathrm{KE}=-\frac{1}{2} \mathrm{PE}, \tag{6.6.10}$$

as required by the virial theorem of mechanics, show that

$$R M^{1/3}= \text{ a constant of approximate value } 10^{17} \mathrm{mkg}^{1/3}. \tag{6.6.11}$$

Evaluate the constant assuming that $c=\frac{3}{5}$. Note that the radius decreases as the mass increases.

e. If the white dwarf has the mass of the sun (2×10^{30} kg), what is its radius (in km)? Compare this to the radius of our sun.

f. Neutron stars (observed as pulsars) are also zero temperature fermion gases, but in this case the fermions are neutrons rather than electrons. Derive the mass-radius relation for a neutron star, and use it to find the radius of a neutron star with the mass of the sun.