10.4: D- Volume of a Sphere in d Dimensions

I will call the volume of a d-dimensional sphere, as a function of radius, $V_d(r)$. You know, of course, that

$$V_2(r) = \pi r^2 \tag{10.4.1}$$

(two-dimensional volume is commonly called “area”) and that

$$V_3(r) = \frac{4}{3}\pi r^3. \tag{10.4.2}$$

But what is the formula for arbitrary d? There are a number of ways to find it. I will use induction on dimensionality d. That is, I will use the formula for $d = 2$ to find the formula for $d = 3$, the formula for $d = 3$ to find the formula for $d = 4$, and in general use the formula for d to find the formula for $d + 1$. This is not the most rigorous formal method to derive the formula, but it is very appealing and has much to recommend it.

To illustrate the process, I will begin with a well-known and easily visualized stage, namely deriving $V_3(r)$ from $V_2(r)$. Think of a 3-dimensional sphere (of radius r) as a stack of pancakes of various radii, but each with infinitesimal thickness dz. The pancake on the very bottom of the stack ($z = -r$) has zero radius. The one above it is slightly broader. They get broader and broader until we get to the middle of the stack ($z = 0$), where the pancake has radius r. The pancakes stacked still higher become smaller and smaller, until they vanish again at the top of the stack ($z = +r$). Because the equation for the sphere is

$$x^2 + y^2 + z^2 = r^2, \tag{10.4.3}$$

the radius of the pancake at height z_0 is

$$\sqrt{r^2 - z_0^2}. \tag{10.4.4}$$

This whole process shows that

$$V_3(r) = \int_{-r}^{+r} dz V_2\left(\sqrt{r^2 - z^2}\right). \tag{10.4.5}$$

It is easy to check this integral against the known result for $V_3(r)$:

$$V_3(r) = \int_{-r}^{+r} dz\pi\left(r^2 - z^2\right) \tag{10.4.6}$$

$$= \pi\left[r^2 z - \frac{1}{3}z^3\right]_{-r}^{+r} \tag{10.4.7}$$

$$= \pi\left[2r^3 - \frac{2}{3}r^3\right] \tag{10.4.8}$$

$$= \frac{4}{3}\pi r^3. \tag{10.4.9}$$

So we haven’t gone wrong yet.

Now, how to derive $V_4(r)$ from $V_3(r)$? This requires a more vivid imagination. Last time we started with a two-dimensional disk of radius r_0 in (x, y) space and thickened it a bit into the third dimension (z) to form a pancake of three-dimensional volume $dz\, V_2(r_0)$. Stacking an infinite number of such pancakes in the z direction, from $z = -r$ to $z = +r$, gave us a three-dimensional sphere. Now we begin with a three-dimensional sphere of radius r_0 in (w, x, y) space and thicken it a bit into the fourth dimension (z) to form a thin four-dimensional pancake of four-dimensional volume $dz\, V_3(r_0)$. Stacking an infinite number of such pancakes in the z direction, from $z = -r$ to $z = +r$, gives a four-dimensional sphere. Because the equation for the four-sphere is

$$w^2 + x^2 + y^2 + z^2 = r^2, \tag{10.4.10}$$

the radius of the three-dimensional sphere at height z_0 is

$$\sqrt{r^2 - z_0^2}, \tag{10.4.11}$$

and the volume of the four-sphere is

$$V_4(r)=\int_{-r}^{+r} dz V_3\left(\sqrt{r^2-z^2}\right). \tag{10.4.12}$$

In general, the volume of a $(d+1)$-sphere is

$$V_{d+1}(r)=\int_{-r}^{+r} dz V_d\left(\sqrt{r^2-z^2}\right). \tag{10.4.13}$$

If we guess that the formula for $V_d(r)$ takes the form

$$V_d(r)=C_d r^d \tag{10.4.14}$$

(which is certainly true for two and three dimensions, and which is reasonable from dimensional analysis), then

$$V_{d+1}(r)=\int_{-r}^{+r} dz C_d\left(r^2-z^2\right)^{d/2} \tag{10.4.15}$$

$$=\int_{-1}^{+1} r du C_d\left(r^2-r^2u^2\right)^{d/2} \tag{10.4.16}$$

$$=r^{d+1}C_d\int_{-1}^{+1} du\left(1-u^2\right)^{d/2}. \tag{10.4.17}$$

This proves our guess and gives us a recursive formula for C_d:

$$C_{d+1}=C_d\int_{-1}^{+1} du\left(1-u^2\right)^{d/2}. \tag{10.4.18}$$

The problem below shows how to build this recursive chain up from $C_2=\pi$ to

$$C_d=\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}=\frac{\pi^{d/2}}{(d/2)!}. \tag{10.4.19}$$

Thus the volume of a d-dimensional sphere of radius r is

$$V_d(r)=\frac{\pi^{d/2}}{(d/2)!}r^d. \tag{10.4.20}$$

D.1 (I) Problem: Volume of a d-dimensional sphere

Before attempting this problem, you should read the material concerning beta functions in an applied mathematics textbook, such as George Arfken's *Mathematical Methods for Physicists* or Mary Boas's *Mathematical Methods in the Physical Sciences*. (Or in the Digital Library of Mathematical Functions.)

a. Show that

$$\int_{-1}^{+1}\left(1-u^2\right)^{d/2}du=B\left(\frac{1}{2},\frac{d}{2}+1\right). \tag{10.4.21}$$

b. Use

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{10.4.22}$$

and $C_2=\pi$ to conclude that

$$C_d=\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}. \tag{10.4.23}$$

D.2 (I) Problem: Volume of a d-dimensional ellipse

Show that the volume of the d-dimensional ellipse described by the equation

$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 + \cdots + \left(\frac{x_d}{a_d}\right)^2 = 1 \tag{10.4.24}$$

is

$$V_d(r) = \frac{\pi^{d/2}}{(d/2)!} a_1 a_2 a_3 \cdots a_d. \tag{10.4.25}$$