# 10.2: B- Evaluating the Gaussian Integral

The integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx \tag{10.2.1}$$

called the Gaussian integral, does not fall to any of the methods of attack that you learned in elementary calculus. But it can be evaluated quite simply using the following trick.

Define the value of the integral to be $A$. Then

$$A^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy e^{-(x^2+y^2)}. \tag{10.2.2}$$

At the last step we have written $A^2$ as a two-variable integral over the entire plane. This seems perverse, because most of the times we work hard to reduce two-dimensional integrals to one-dimensional integrals, whereas here we are going in reverse. But look at the integrand again. When regarded as an integral on the plane, it is clear that we can regard $x^2 + y^2$ as just $r^2$, and this suggests we should convert the integral from Cartesian $(x, y)$ to polar $(r, \theta)$ coordinates:

$$A^2 = \int_0^{\infty} dr \int_0^{2\pi} r d\theta e^{-r^2} = 2\pi \int_0^{\infty} r e^{-r^2} dr \tag{10.2.3}$$

The last integral immediately suggests the substitution $u = r^2$, giving

$$A^2 = \pi \int_0^{\infty} e^{-u} du = -\pi e^{-u} \Big|_0^{\infty} = \pi. \tag{10.2.4}$$

We conclude that

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{10.2.5}$$

*B.1 (I) Problem: Another integral*

Show that

$$\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx = \sqrt{\pi}. \tag{10.2.6}$$

(Clue: Use the substitution $y = \sqrt{x}$.

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