2.10: Derivation of Wien's and Stefan's Laws

Wien's and Stefan's Laws are found, respectively, by differentiation and integration of Planck's equation. Neither of these is particularly easy, and they are not found in every textbook. Therefore, I derive them here.

Wien's Law

Planck's equation for the exitance per unit wavelength interval (equation 2.6.1) is

$$\frac{M}{C}=\frac{1}{\lambda^{5}\left(e^{K/\lambda T}-1\right)},\tag{2.11.1}$$

in which I have omitted some subscripts. Differentiation gives

$$\frac{1}{C}\frac{dM}{d\lambda}=-\frac{1}{\left(e^{K/\lambda T}-1\right)^{2}}\cdot\left[5\lambda^{4}\cdot\left(e^{K/\lambda T}-1\right)+\lambda^{5}\cdot\left(-\frac{K}{\lambda^{2}T}\right)e^{K/\lambda T}\right].\tag{2.11.2}$$

M is greatest when this is zero; that is, when

$$x=5\left(1-e^{-x}\right),\tag{2.11.3}$$

where

$$x=\frac{K}{\lambda T}.\tag{2.11.4}$$

Hence, with equation 2.6.9, the wavelength at which M is a maximum, is given by

$$\lambda=\frac{hc}{kxT}.\tag{2.11.5}$$

The maximum value of M is found be substituting this vale of λ back into Planck's equation, to arrive at equation 2.7.16. The corresponding versions of Wien's Law appropriate to the other version's of Planck's equation are found similarly.

Stefan's Law

Integration of Planck's equation to arrive at Stefan's law is a bit more tricky.

It should be clear that $\int_0^\infty M_\lambda d\lambda=\int_0^\infty M_\nu d\nu$, and therefore I choose to integrate the easier of the functions, namely M_ν. To integrate M_λ, the first thing we would do anyway would be to make the substitution $\nu=c/\lambda$.

Planck's equation for the blackbody exitance per unit frequency interval is

$$M_\nu=C_3\int_0^\infty\frac{\nu^3d\nu}{e^{K_2\nu/T}-1}.\tag{2.11.6}$$

Let $x=K_2\nu/T$; then

$$M_\nu=\frac{2\pi k^4T^4}{c^2h^3}\int_0^\infty\frac{x^3dx}{e^x-1},\tag{2.11.7}$$

And, except for the numerical value of the integral, we already have Stefan's law. The integral can be evaluated numerically, but not without difficulty, and there is an analytical solution for it.

Consider the indefinite integral and integrate it by parts:

$$\int\frac{x^3dx}{e^x-1}=x^3\ln\left(1-e^{-x}\right)-3\int x^2\ln\left(1-e^{-x}\right)dx+\text{const.}\tag{2.10.1}$$

Now put the limits in:

$$\int_0^\infty\frac{x^3dx}{e^x-1}=-3\int_0^\infty x^2\ln\left(1-e^{-x}\right)dx.\tag{2.10.2}$$

Write down the Maclaurin expansion of the integrand:

$$\int_0^\infty \frac{x^3dx}{e^x-1} = 3\int_0^\infty x^2\left(e^{-x}+\frac{1}{2}e^{-2x}+\frac{1}{3}e^{-3x}+\dots\right)dx \tag{2.10.3}$$

and integrate term by term to obtain

$$\int_0^\infty \frac{x^3dx}{e^x-1} = 6\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right). \tag{2.11.8}$$

We must now evaluate $1+\frac{1}{2^4}+\frac{1}{3^4}+\dots$

The series $\sum_1^\infty \frac{1}{n^m}$ is the Riemann ζ-function. For $m=1$, it diverges. For $m=3,5,7$, etc., it has to be evaluated numerically. For $m=2,4,6$, etc., the sums can be written explicitly in terms of π. For example:

$$\zeta(2) = \frac{\pi^2}{6}, \tag{2.10.4}$$

$$\zeta(4) = \frac{\pi^4}{90}, \tag{2.10.5}$$

$$\zeta(6) = \frac{\pi^6}{945}. \tag{2.10.6}$$

One of the stages necessary in evaluating the ζ-function is to derive the infinite product

$$\frac{\sin\alpha\pi}{\alpha\pi} = [1-\alpha^2]\left[1-\left(\frac{1}{2}\alpha\right)^2\right]\left[1-\left(\frac{1}{3}\alpha\right)^2\right]\dots \tag{2.11.9}$$

If we can do that, we are more than halfway there.

Let's start by considering the Fourier expansion of $\cos\theta x$:

$$\cos\theta x = \sum_0^\infty a_n\cos nx \tag{2.11.10}$$

In Equation 2.11.10 n is an integer, θ not necessarily so; we shall suppose that θ is some number between 0 and 1. There is no need to consider any sine terms, because $\cos\theta x$ is an even function of x. We work out what the Fourier coefficients are in the usual way, to get

$$a_n = (-1)^n\frac{2\theta\sin\theta\pi}{\theta^2-n^2}, \quad n=1,2,3,\dots \tag{2.11.11}$$

As usual, and for the usual reason, a_0 is an exception:

$$a_0 = \frac{\sin\theta\pi}{\theta\pi}. \tag{2.11.12}$$

We have therefore arrived at the Fourier expansion of $\cos\theta x$:

$$\cos\theta x = \frac{2\theta\sin\theta\pi}{\pi}\left(\frac{1}{2\theta^2}-\frac{\cos x}{\theta^2-1^2}+\frac{\cos 2x}{\theta^2-2^2}-\frac{\cos 3x}{\theta^2-3^2}+\dots\right). \tag{2.11.13}$$

Put $x=\pi$ and rearrange slightly:

$$\pi\cot\theta\pi-\frac{1}{\theta} = 2\theta\left(\frac{1}{\theta^2-1^2}+\frac{1}{\theta^2-2^2}+\dots\right). \tag{2.11.14}$$

Since we are assuming that θ is some number between 0 and 1, we shall re-write this so that the denominators are all positive:

$$\pi\cot\theta\pi-\frac{1}{\theta} = -\frac{2\theta}{1^2-\theta^2}-\frac{2\theta}{2^2-\theta^2}-\dots \tag{2.11.15}$$

Now multiply both sides by $d\theta$ and integrate from $\theta=0$ to $\theta=\alpha$. The integration must be done with care. The indefinite integral of the left hand side is $\ln\sin\theta\pi-\ln\theta+\text{constant}$, i.e. $\ln\left(\frac{\sin\theta\pi}{\theta}\right)+\text{constant}$. The definite integral between 0 and α is $\ln\left(\frac{\sin\alpha\pi}{\alpha}\right)-\lim_{\theta\to0}\ln\left(\frac{\sin\theta\pi}{\theta}\right)$.

The limit of the second term is $\ln\pi$, so the definite integral is $\ln\left(\frac{\sin\alpha\pi}{\alpha\pi}\right)$. Integrating the right hand side is a bit easier, so we arrive at

$$\ln\left(\frac{\sin\alpha\pi}{\alpha\pi}\right)=\ln\left(\frac{1^2-\alpha^2}{1^2}\right)+\ln\left(\frac{2^2-\alpha^2}{2^2}\right)+\ldots \tag{2.11.16}$$

On taking the antilogarithm, we arrive at the required infinite product:

$$\frac{\sin\alpha\pi}{\alpha\pi}=\left[1-\alpha^2\right]\left[1-\left(\frac{1}{2}\alpha\right)^2\right]\left[1-\left(\frac{1}{3}\alpha\right)^2\right]\ldots \tag{2.11.17}$$

Now expand this as a power series in α^2:

$$\frac{\sin\alpha\pi}{\alpha\pi}=1+()\alpha^2+()\alpha^4+()\alpha^6+\ldots \tag{2.11.18}$$

The first one is easy, but subsequent ones rapidly get more difficult, but you do have to get at least as far as α^4.

Now compare this expansion with the ordinary Maclaurin expansion:

$$\frac{\sin\alpha\pi}{\alpha\pi}=1-\frac{\pi^2}{3!}\alpha^2+\frac{\pi^4}{5!}\alpha^4-\ldots \tag{2.11.19}$$

and we arrive at the correct expressions for the Riemann ζ-functions. We then get for Stefan's law:

$$M=\frac{2\pi^5k^4}{15h^3c^2}T^4=\sigma T^4, \tag{2.11.20}$$

where $\sigma=5.6705\times10^{-8}\ \mathrm{W\ m^{-2}K^{-4}}$.

Questions

Finally, now that you have struggled through Riemann's zeta-function, let's just make sure that you have understood the really simple stuff, so here are a couple of easy questions – and you won't have to bother with zeta-functions.

1. By what factor should the temperature of a black body be increased so that

   a) The integrated radiance (over all frequencies) is doubled?

   b) The frequency at which its radiance is greatest is doubled?

   c) The spectral radiance per unit wavelength interval at its wavelength of maximum spectral radiance is doubled?

2. A block of shiny silver (absorptance = 0.23) has a bubble inside it of radius $2.2\mathrm{cm}$, and it is held at a temperature of $1200\mathrm{K}$.

A block of dull black carbon (absorptance = 0.86) has a bubble inside it of radius $4.3\mathrm{cm}$, and it is held at a temperature of $2300\mathrm{K}$,

Calculate the ratio

$$\frac{\text{Integrated radiation energy density inside the carbon bubble}}{\text{Integrated radiation energy density insdie the silver bubble}}. \tag{2.10.7}$$

Answers. 1. a) 1.189 b) 2.000 c) 1.149

2. 13.5