10.10: APPENDIX B- Radiation Damping as Functions of Angular Frequency, Frequency and Wavelength

It occurred to me while preparing this Chapter as well as the preceding and following ones, that sometimes I have been using angular frequency as argument, sometimes frequency, and sometimes wavelength. In this Appendix, I bring together the salient formulas for radiation damping in terms of $\Delta\omega = \omega - \omega_0$, $\Delta\nu = \nu - \nu_0$ and $\Delta\lambda = \lambda - \lambda_0$. I reproduce equation 10.2.11 for the absorption coefficient for a set of forced, damped oscillators, except that I replace n, the number per unit volume of oscillators with $n_1 f_{12}$, the effective number of atoms per unit volume in the lower level of a line, and I replace the classical damping constant γ with Γ, which may include a pressure broadening component.

$$\alpha = \frac{n_1 f_{12} \Gamma e^2 \omega^2}{m\varepsilon_0 c[(\omega^2 - \omega_0^2)^2 + \Gamma^2\omega^2]} \quad \mathrm{m}^{-1}. \tag{10.B.1}$$

You should check that the dimensions of this expression are L^{-1}, which is appropriate for linear absorption coefficient. You may note that $[e^2/\varepsilon_0] \equiv \mathrm{ML^3T^{-2}}$ and $[\Gamma] \equiv \mathrm{T}^{-1}$. Indeed check the dimensions of all expressions that follow, at each stage.

We can write $\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) = \Delta\omega(2\omega_0 + \Delta\omega)$, and the equation becomes

$$\alpha = \frac{n_1 f_{12} \Gamma e^2 (\omega_0 + \Delta\omega)^2}{m\varepsilon_0 c[(\Delta\omega)^2(2\omega_0 + \Delta\omega)^2 + \Gamma^2(\omega_0 + \Delta\omega)^2]} \quad \mathrm{m}^{-1}. \tag{10.B.2}$$

Now I think it will be owned that the width of a spectrum line is very, very much smaller than its actual wavelength, except perhaps for extremely Stark-broadened hydrogen lines, so that, in the immediate vicinity of a line, $\Delta\omega$ can be neglected compared with ω_0; and a very long way from the line, where this might not be so, the expression is close to zero anyway. (Note that you can neglect $\Delta\omega$ *only with respect to* ω; you cannot just put $\Delta\omega = 0$ where it lies alone in the denominator!) In any case, I have no compunction at all in making the approximation

$$\alpha(\Delta\omega) = \frac{n_1 f_{12} \Gamma e^2}{4m\varepsilon_0 c[(\Delta\omega)^2 + (\frac{1}{2}\Gamma)^2]} \quad \mathrm{m}^{-1}. \tag{10.B.3}$$

The maximum of the $\alpha(\Delta\omega)$ curve is

$$\alpha(0) = \frac{e^2 n_1 f_{12}}{m\varepsilon_0 c\Gamma} \quad \mathrm{m}^{-1}. \tag{10.B.4}$$

The optical thickness at the line centre (whether or not the line is optically thin) is

$$\tau(0) = \frac{e^2 \mathcal{N}_1 f_{12}}{m\varepsilon_0 c\Gamma}. \tag{10.B.5}$$

$\mathcal{N}_1$ is the number of atoms in level 1 per unit area in the line of sight, whereas n_1 is the number per unit volume.

The HWHM of $\alpha(\Delta\omega)$ curve is

$$\mathrm{HWHM} = \frac{1}{2}\Gamma \quad \mathrm{rad\,s^{-1}}. \tag{10.B.6}$$

The area under the $\alpha(\Delta\omega)$ curve is

$$\mathrm{Area} = \frac{\pi e^2 n_1 f_{12}}{2m\varepsilon_0 c} \quad \mathrm{m^{-1} rad\,s^{-1}}. \tag{10.B.7}$$

As expected, the area does not depend upon Γ.

To express the absorption coefficient as a function of *frequency*, we note that $\omega = 2\pi\nu$, and we obtain

$$\alpha(\Delta\nu) = \frac{n_1 f_{12} \Gamma e^2}{16\pi^2 m\varepsilon_0 c[(\Delta\nu)^2 + \left(\frac{\Gamma}{4\pi}\right)^2]} \quad \mathrm{m}^{-1}. \tag{10.B.8}$$

The maximum of this is (of course) the same as equation 10.B.4.

The HWHM of the $\alpha(\Delta\nu)$ curve is

$$\mathrm{HWHM} = \Gamma/(4\pi) \quad \mathrm{s}^{-1}. \tag{10.B.9}$$

The area under the $\alpha(\Delta\nu)$ curve is

$$\mathrm{Area} = \frac{e^2 n_1 f_{12}}{4m\varepsilon_0 c} \quad \mathrm{m}^{-1}\mathrm{s}^{-1}. \tag{10.B.10}$$

$$\alpha = \frac{n_1 f_{12}\Gamma e^2}{m\varepsilon_0 c} \cdot \left(\frac{\lambda^2\lambda_0^4}{4\pi^2 c^2 \left(\lambda_0^2 - \lambda^2\right)^2 + \lambda^2\lambda_0^4\Gamma^2} \right) \quad \mathrm{m}^{-1}. \tag{10.B.11}$$

In a manner similar to our procedure following equation 10.B.12, we write $\lambda_0^2 - \lambda^2 = (\lambda_0 - \lambda)(\lambda_0 + \lambda)$, and , $\lambda = \lambda_0 + \Delta\lambda$, and neglect $\Delta\lambda$ with respect to λ_0, and we obtain:

$$\alpha(\Delta\lambda) = \frac{n_1 f_{12}\Gamma e^2}{16\pi^2 m\varepsilon_0 c^3} \cdot \left(\frac{\lambda_0^4}{(\Delta\lambda)^2 + \frac{\lambda_0^4\Gamma^2}{16\pi^2 c^2}} \right) \quad \mathrm{m}^{-1}. \tag{10.B.12}$$

The maximum of this is (of course) the same as equation 10.B.4. (Verifying this will serve as a check on the algebra.)

The HWHM of the $\alpha(\Delta\lambda)$ curve is

$$\mathrm{HWHM} = \frac{\lambda_0^2\Gamma}{4\pi c} \quad \mathrm{m}. \tag{10.B.13}$$

The area under the $\alpha(\Delta\lambda)$ curve is

$$\mathrm{Area} = \frac{\lambda_0^2 e^2 n_1 f_{12}}{4m\varepsilon_0 c^2}. \tag{10.B.14}$$

Did I forget to write down the units after this equation?

These results for α might be useful in tabular form. For τ, replace n_1 by N_1 .

$$\begin{array}{cccc} & \Delta\omega & \Delta\nu & \Delta\lambda \\ & \frac{\Gamma e^2 n_1 f_{12}}{4m\varepsilon_0 c[(\Delta\omega)^2 + (\frac{1}{2}\Gamma)^2]} & \frac{\Gamma e^2 n_1 f_{12}}{16\pi^2 m\varepsilon_0 c[(\Delta\nu)^2 + \left(\frac{\Gamma}{4\pi}\right)^2]} & \frac{\Gamma e^2 \lambda_0^4 n_1 f_{12}}{16\pi^2 m\varepsilon_0 c^3\left[(\Delta\lambda)^2 + \frac{\lambda_0^4\Gamma^2}{16\pi^2 c^2}\right]} \\ \text{Height} & \frac{e^2 n_1 f_{12}}{m\varepsilon_0 c\Gamma} & \frac{e^2 n_1 f_{12}}{m\varepsilon_0 c\Gamma} & \frac{e^2 n_1 f_{12}}{m\varepsilon_0 c\Gamma} \\ \text{Area} & \frac{\pi e^2 n_1 f_{12}}{2m\varepsilon_0 c} & \frac{e^2 n_1 f_{12}}{4m\varepsilon_0 c} & \frac{\lambda_0^2 e^2 n_1 f_{12}}{4m\varepsilon_0 c^2} \\ \text{HWMH} & \frac{1}{2}\Gamma & \Gamma/(4\pi) & \frac{\lambda_0^2\Gamma}{4\pi c} \end{array} \tag{10.10.1}$$

It is to be noted that if the radiation damping profile is thermally broadened, the height of the absorption coefficient curve diminishes, while the area is unaltered provided that the line is optically thin. The optically thick situation is dealt with in the following chapter. It might also be useful to note that a gaussian profile of the form

$$\alpha(\Delta\lambda) = \alpha(0)\exp\left(-\frac{c^2(\Delta\lambda)}{V_\mathrm{m}^2\lambda_0^2}\right) \tag{10.B.15}$$

has an area of $\frac{\lambda_0^2 e^2 n_1 f_{12}}{4m\varepsilon_0 c^2}$ if

$$\alpha(0) = \frac{\lambda_0 e^2 n_1 f_{12}}{4\sqrt{\pi}m\varepsilon_0 c V_\mathrm{m}}. \tag{10.B.16}$$