

9.4: Einstein B Coefficient

In section 9.2 on oscillator strengths, we first defined what we meant by absorption oscillator strength f_{12} . We then showed that the equivalent width of a line is proportional to $\varpi_1 f_{12}$. We followed this by defining an emission oscillator strength f_{21} by the equation $\varpi_2 f_{21} = \varpi_1 f_{12}$. Thereafter we defined a weighted oscillator strength ϖf to be used more or less as a single symbol equal to either $\varpi_2 f_{21}$ or $\varpi_1 f_{12}$. Can we do a similar sort of thing with Einstein coefficient? That is, we have defined A_{21} , the Einstein coefficient for spontaneous emission (i.e. downward transition) without any difficulty, and we have shown that the intensity or radiance of an emission line is proportional to $\varpi_2 A_{21}$. Can we somehow define an Einstein absorption coefficient A_{12} ? But this would hardly make any sense, because atoms do not make spontaneous upward transitions! An upward transition requires either absorption of a photon or collision with another atom.

For absorption lines (upwards transitions) we can define an *Einstein B coefficient* such that the rate of upward transitions from level 1 to level 2 is proportional to the product of two things, namely the number of atoms N_1 currently in the initial (lower) level and the amount of radiation that is available to excite these upward transitions. The proportionality constant is the Einstein coefficient for the transition, B_{12} . There is a real difficulty in that by “amount of radiation” different authors mean different things. It could mean, for example, any of the four things:

u_λ the energy density per unit wavelength interval at the wavelength of the line, expressed in $\text{J m}^{-3} \text{m}^{-1}$;

u_ν the energy density per unit frequency interval at the frequency of the line, expressed in $\text{J m}^{-3} \text{Hz}^{-1}$;

L_λ radiance (unorthodoxly called “specific intensity” or even merely “intensity” and given the symbol I by many astronomers) per unit wavelength interval at the wavelength of the line, expressed in $\text{W m}^{-2} \text{sr}^{-1} \text{m}^{-1}$;

L_ν radiance per unit frequency interval at the frequency of the line, expressed in $\text{W m}^{-2} \text{sr}^{-1} \text{Hz}^{-1}$.

Thus there are at least four possible definitions of the Einstein B coefficient and it is rarely clear which definition is intended by a given author. It is essential in all one’s writings to make this clear and always, in numerical work, to state the units. If we use the symbols B_{12}^a , B_{12}^b , B_{12}^c , B_{12}^d for these four possible definitions of the Einstein B coefficient, the SI units and dimensions for each are

$$B_{12}^a : \quad \text{s}^{-1} (\text{J m}^{-3} \text{m}^{-1})^{-1} \quad \text{M}^{-1} \text{L}^2 \text{T}$$

$$B_{12}^b : \quad \text{s}^{-1} (\text{J m}^{-3} \text{Hz}^{-1})^{-1} \quad \text{M}^{-1} \text{L}$$

$$B_{12}^c : \quad \text{s}^{-1} (\text{W m}^{-2} \text{sr}^{-1} \text{m}^{-1})^{-1} \quad \text{M}^{-1} \text{LT}^2$$

$$B_{12}^d : \quad \text{s}^{-1} (\text{W m}^{-2} \text{sr}^{-1} \text{Hz}^{-1})^{-1} \quad \text{M}^{-1} \text{L}$$

You can, of course, find equivalent ways of expressing these units (for example, you could express B_{12}^b in metres per kilogram if you thought that that was helpful!), but the ones given make crystal clear the meanings of the coefficients.

The relations between them are (omitting the subscripts 12):

$$B^a = \frac{\lambda^2}{c} B^b = \frac{c}{4\pi} B^c = \frac{\lambda^2}{4\pi} B^d; \quad (9.4.1)$$

$$B^b = \frac{\nu^2}{4\pi} B^c = \frac{c}{4\pi} B^d = \frac{\nu^2}{c} B^a; \quad (9.4.2)$$

$$B^c = \frac{\lambda^2}{c} B^d = \frac{4\pi}{c} B^a = \frac{4\pi\lambda^2}{c^2} B^b; \quad (9.4.3)$$

$$B^d = \frac{4\pi\nu^2}{c^2} B^a = \frac{4\pi}{c} B^b = \frac{\nu^2}{c} B^c; \quad (9.4.4)$$

For the derivation of these, you will need to refer to equations 1.3.1, 1.15.3 and 1.17.12,

From this point henceforth, unless stated otherwise, I shall use the first definition without a superscript, so that the Einstein coefficient, when written B_{12} , will be understood to mean B_{12}^a . Thus the rate of radiation-induced upward transitions from level 1 to level 2 will be taken to be B_{12} times N_1 times u_λ .

Induced downward transitions.

The Einstein B_{12} coefficient and the oscillator strength f_{12} (which are closely related to each other in a manner that will be shown later this section) are concerned with the forced upward transition of an atom from a level 1 to a higher level 2 by radiation of a wavelength that corresponds to the energy difference between the two levels. The Einstein A_{21} coefficient is concerned with the spontaneous downward decay of an atom from a level 2 to a lower level 1.

There is another process. Light of the wavelength that corresponds to the energy difference between levels 2 and 1 may *induce a downward transition* from an atom, initially in level 2, to the lower level 1. When it does so, the light is not absorbed; rather, the atom emits another photon of that wavelength. Of course the light that is irradiating the atoms induces upward transitions from level 1 to level 2, as well as inducing downward transitions from level 2 to level 1, and since, for any finite positive temperature, there are more atoms in level 1 than in level 2, there is a net absorption of light. (The astute reader will note that there may be more atoms in level 2 than in level 1 if it has a larger statistical weight, and that the previous statement should refer to states rather than levels.) If, however, the atoms are not in thermodynamic equilibrium and there are more atoms in the higher levels than in the lower (the atom is “top heavy”, corresponding to a negative excitation temperature), there will be *Light Amplification by Stimulated Emission of Radiation* (LASER). In this section, however, we shall assume a Boltzmann distribution of atoms among their energy levels and a finite positive excitation temperature. The number of induced downward transitions per unit time from level 2 to level 1 is given by $B_{21}N_2u_\lambda$. Here B_{21} is the Einstein coefficient for induced downward transition.

Let m denote a particular atomic level. Let n denote any level lower than m and let n' denote any level higher than m . Let N_m be the number of atoms in level m at some time. The rate at which N_m decreases with time as a result of these processes is

$$-\dot{N}_m = N_m \sum_n A_{mn} + N_m \sum_n B_{mn} u_{\lambda_{mn}} + N_m \sum_{n'} B_{mn'} u_{\lambda_{mn'}}. \quad (9.4.5)$$

This equation describes only the rate at which N_m is depleted by the three radiative processes. It does not describe the rate of replenishment of level m by transitions from other levels, nor with its depletion or replenishment by collisional processes. Equation 9.4.5 when integrated results in

$$N_m(t) = N_m(0)e^{-\Gamma_m t}. \quad (9.4.6)$$

Here

$$\Gamma_m = \sum_n A_{mn} + \sum_n B_{mn} u_{\lambda_{mn}} + \sum_{n'} B_{mn'} u_{\lambda_{mn'}} \quad (9.4.7)$$

(Compare equation 9.3.3, which dealt with a two-level atom in the absence of stimulating radiation.)

The reciprocal of Γ_m is the mean lifetime of the atom in level m .

Consider now just two levels – a level 2 and a level below it, 1. The rate of spontaneous and induced downward transitions from m to n is equal to the rate of forced upward transitions from n to m :

$$A_{21}N_2 + B_{21}N_2u_\lambda = B_{12}N_1u_\lambda. \quad (9.4.8)$$

I have omitted the subscripts 21 to λ , since there is only one wavelength involved, namely the wavelength corresponding to the energy difference between the levels 2 and 1. Let us assume that the gas and the radiation field are in thermodynamic equilibrium. In that case the level populations are governed by Boltzmann’s equation (equation 8.4.19), so that equation 9.4.8 becomes

$$(A_{21} + B_{21}u_\lambda)N_0 \frac{\varpi_2}{\varpi_0} e^{-E_2/(kT)} = B_{12}u_\lambda N_0 \frac{\varpi_1}{\varpi_0} e^{-E_1/(kT)}, \quad (9.4.9)$$

from which

$$u_\lambda = \frac{A_{21}\varpi_2}{B_{12}\varpi_1 e^{hc/(\lambda kT)} - B_{21}\varpi_2}, \quad (9.4.10)$$

where I have made use of

$$E_2 - E_1 = hc/\lambda. \quad (9.4.11)$$

Now, still assuming that the gas and photons are in thermodynamic equilibrium, the radiation distribution is governed by Planck’s equation (equations 2.6.4, 2.6.5, 2.6.9; see also equation 2.4.1):

$$u_{\lambda} = \frac{8\pi hc}{\lambda^5 (e^{hc/(\lambda kT)} - 1)}. \quad (9.4.12)$$

On comparing equations 9.4.10 and 9.4.12, we obtain

$$\varpi_1 B_{12} = \varpi_2 B_{21} = \frac{\lambda^5}{8\pi hc} \varpi_2 A_{21}. \quad (9.4.13)$$

A reminder here may be appropriate that the B here is B^a as defined near the beginning of this section. Also, in principle there would be no objection to defining an ϖB such that $\varpi B = \varpi_1 B_{12} = \varpi_2 B_{21}$, just as was done for oscillator strength, although I have never seen this done.

Einstein B_{12} coefficient and Equivalent width.

Imagine a continuous radiant source of radiance per unit wavelength interval L_{λ} , and in front of it an optically thin layer of gas containing N_1 atoms per unit area in the line of sight in level 1. The number of upward transitions per unit area per unit time to level 2 is $B_{12}^c \mathcal{N}_1 L_{\lambda_{12}}$, and each of these absorbs an amount hc/λ_{12} of energy. The rate of absorption of energy per unit area per unit solid angle is therefore $\frac{1}{4\pi} \times B_{12}^c \mathcal{N}_1 L_{\lambda_{12}} \times \frac{hc}{\lambda_{12}}$. This, by definition of equivalent width (in wavelength units), is equal to $WL_{\lambda_{12}}$.

Therefore

$$W = \frac{B_{12}^c \mathcal{N}_1 hc}{4\pi \lambda_{12}} = \frac{h}{\lambda_{12}} \mathcal{N}_1 B_{12}^a. \quad (9.4.14)$$

If we compare this with equation 9.2.4 we obtain the following relation between a B_{12} and f_{12} :

$$B_{12}^a = \frac{e^2 \lambda^3}{4\epsilon_0 h m c^2} f_{12}. \quad (9.4.15)$$

It also follows from equations 9.4.13 and 9.4.15 that

$$\varpi_2 A_{21} = \frac{2\pi e^2}{\epsilon_0 m c \lambda^2} \varpi_1 f_{12}. \quad (9.4.16)$$

I shall summarize the various relations between oscillator strength, Einstein coefficient and line strength in section 9.9.

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