

## 7.5: One-dimensional Waves in a Stretched String

The last sentence in the previous section may strike you as rather odd, because one sometimes has the impression that quantum mechanics is replete with *ad hoc* assumptions, peppered as it is with various quantum numbers which can assume only integral values for some mysterious reason that no one can understand. Therefore, before moving on to wave mechanics as applied to atomic spectra, it will be useful to remind ourselves of some aspects of the behaviour of waves in a taut, stretched string. By doing this, we may be able to take some of the "mystery" out of quantum mechanics, and to see that many of its assumptions are by no means *ad hoc*, and arise rather naturally from the elementary theory of waves.

We'll start by imagining a long, taut string, which is constrained so that it can vibrate only in a single plane. Let us suppose that it suffers a brief disturbance, such as by being struck or plucked. In that case a wave (not necessarily periodic) of the form

$$\Psi_1 = f_1(x - ct) \quad (7.5.1)$$

will travel down the string to the right (positive  $x$ -direction) with speed  $c$  (this is not intended to mean the speed of light - just the speed at which waves are propagated in the string, which depends on its mass per unit length and its tension), and another wave

$$\Psi_2 = f_2(x + ct) \quad (7.5.2)$$

will move to the left. The symbol  $\Psi$  is just the transverse displacement of the string. It can easily be verified by direct substitution that either of these satisfies a differential Equation of the form

$$c^2 \frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 \Psi}{\partial t^2}. \quad (7.5.3)$$

Indeed it is also easy to verify that any linear combination such as

$$\Psi = A f_1(x - ct) + B f_2(x + ct) \quad (7.5.4)$$

also satisfies the differential Equation. Since Equation 7.5.3 is a second-order differential equation, there are just two arbitrary constants of integration, and, since there are two arbitrary constants in Equation 7.5.4, the latter is the most general solution of the differential Equation. The differential Equation is the general differential Equation for a wave in one dimension. Two points are worth noting. One, there is no minus sign in it. Two, if your memory fails you, you can easily determine which side of the Equation the  $c^2$  is on by considering the dimensions.

If the function is periodic, it can be represented as the sum of a number (perhaps an infinite number) of sine and cosine terms with frequencies related to each other by rational fractions. Even if the function is not periodic, it can still be represented by sine and cosine functions, except that in this case there is a continuous distribution of component frequencies. The distribution of component frequencies is then the Fourier transform of the wave profile. (If you are shaky with Fourier transforms, do not worry - I promise not to mention them again in this chapter.) Because of this, I shall assume all waves to be sinusoidal functions.

Let us now assume that the string is fixed at both ends. (Until this point I had not mentioned whether the ends of the string were fixed or not, although I had said that the string was taut. I suppose it would be possible for those of mathematical bent to imagine a taut string of infinite length, though my imagination begins to falter after the first few parsecs.) If a sine wave travels down the string, when it reaches the end it reverses its direction, and it does so again at the other end. What happens is that you end up with two waves travelling in opposite directions:

$$\begin{aligned} \Psi &= a \sin k(x - ct) + a \sin k(x + ct) \\ &= a \sin(kx - \omega t) + a \sin(kx + \omega t). \end{aligned} \quad (7.5.5)$$

Here  $k$  is the propagation constant  $2\pi/\lambda$  and  $\omega$  is the angular frequency  $2\pi\nu$ .

By a trigonometric identity this can be written:

$$\Psi = 2a \cos \omega t \sin kx. \quad (7.5.6)$$

This is a stationary sin wave ( $\sin kx$ ) whose amplitude ( $2a \cos \omega t$ ) varies periodically with time. In other words, it is a stationary or standing wave system. Notice particularly that a stationary or standing wave system is represented by the *product of a function of space and a function of time*:

$$\Psi(x, t) = \Psi(x) \cdot \chi(t). \quad (7.5.7)$$

Because the string is fixed at both ends (these are *fixed boundary conditions*) the only possible wavelengths are such that there is a node at each fixed end, and consequently there can only be an integral number of antinodes or half wavelengths along the length of the string. If the length of the string is  $l$ , the fundamental mode of vibration has a wavelength  $2l$  and a fundamental frequency of  $c/(2l)$ . Other modes (the higher harmonics) are equal to an integral number of times this fundamental frequency; that is  $nc/(2l)$ , where  $n$  is an integer. Note that the introduction of this number  $n$ , which is restricted to integral values (a "quantum number", if you will) is a consequence of the *fixed boundary conditions*.

Which modes are excited and with what relative amplitudes depends upon the *initial conditions* – that is, on whether the string is plucked (initially  $\Psi \neq 0$ ,  $\dot{\Psi} = 0$ ) or struck (initially  $\Psi = 0$ ,  $\dot{\Psi} \neq 0$ ), and where it was plucked or struck. It would require some skill and practice (ask any musician) to excite only one vibrational mode, unless you managed to get the initial conditions exactly right, and the general motion is a *linear superposition of the normal modes*.

I'll mention just one more thing here, which you should recall if you have studied waves at all, namely that the *energy* of a wave of a given frequency is proportional to the square of its amplitude.

Volumes could be written about the vibrations of a stretched string. I would ask the reader to take notice especially of these four points.

1. A stationary solution is the product of a function of space and a function of time.
2. Restriction to a discrete set of frequencies involving an integral number is a consequence of fixed boundary conditions.
3. The general motion is a linear combination of the normal modes.
4. The energy of a wave is proportional to the square of its amplitude.

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