2.9: Dimensionless forms of Planck's equation

The Planck functions (of wavelength or frequency and temperature) can be collapsed on to dimensionless functions of a single variable if we express the exitance in units of the maximum exitance, and the wavelength or frequency in units of the wavelength or frequency at which the maximum occurs. Equations 2.7.1-4 and 16-20 will be needed to achieve this, and the reader might enjoy doing it as a challenge. (I said "might".) The results are

$$M_\lambda = \frac{b_1}{\lambda^5\left(e^{x_1/\lambda}-1\right)} \tag{2.10.1}$$

$$N_\lambda = \frac{b_2}{\lambda^4\left(e^{x_2/\lambda}-1\right)} \tag{2.10.2}$$

$$M_\nu = \frac{b_3\nu^3}{e^{x_3\nu}-1} \tag{2.10.3}$$

$$N_\nu = \frac{b_4\nu^2}{e^{x_4\nu}-1} \tag{2.10.4}$$

where

$$b_n = e^{x_n}-1 \quad (n=1,2,3,4) \tag{2.10.5}$$

The numerical values of x_n are given in equations 2.7.8-11, and the values of b_n are

$$b_1 = 142.32492 \tag{2.10.6}$$

$$b_2 = 49.435253 \tag{2.10.7}$$

$$b_3 = 15.801016 \tag{2.10.8}$$

$$b_4 = 3.9215536 \tag{2.10.9}$$

The numbers x_n y_n, b_n are independent of the values of any physical constants such as h, c or k, and will not change as our knowledge of these values improves. These functions, which are independent of temperature, are drawn in figures II.1,2,3,4 shown at the end of this chapter.

Example

Here is an example to show the use of the dimensionless functions to calculate the blackbody radiance quickly. What is the radiance per unit wavelength of a $5000\ \mathrm{K}$ black body at $400\ \mathrm{nm}$? You may prefer to calculate this directly from equation 2.6.1, but let's try it using the dimensionless form. From equation 2.7.1 we find that the wavelength at which maximum exitance per unit wavelength occurs is $579.56\ \mathrm{nm}$, and therefore our dimensionless wavelength to be inserted into equation 2.10.1 is 0.6902. This gives a radiance per unit wavelength interval (in units of the maximum) of 0.6832. But equation 2.7.16 gives the maximum radiance per unit wavelength interval as $4.0178\times10^{13}\ \mathrm{W\ m^{-2}m^{-1}}$ and therefore the radiance at $400\ \mathrm{nm}$ is $2.745\times10^{13}\ \mathrm{W\ m^{-2}m^{-1}}$.

I don't think there is much point in integrating the dimensionless functions 2.10.1-4 over all wavelengths and frequencies, but, for the record:

$$\int_0^\infty M_\lambda d\lambda = \frac{\pi^4 x_1}{15y_1} = 1.52080 \tag{2.10.10}$$

$$\int_0^\infty N_\lambda d\lambda = \frac{2\zeta(3)x_2}{y_2} = 1.97199 \tag{2.10.11}$$

$$\int_0^\infty M_\nu d\nu = \frac{\pi^4}{15x_3y_3} = 1.61924 \tag{2.10.12}$$

$$\int_0^\infty N_\nu d\nu = \frac{2\zeta(3)}{x_4y_4} = 2.32946 \tag{2.10.13}$$

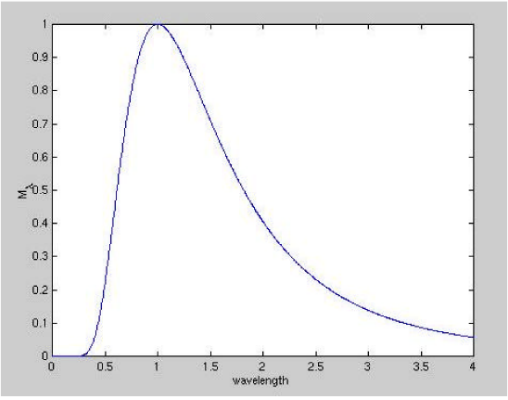

Figure II.1. Blackbody exitance per unit wavelength interval. The equation is equation 2.10.1, with constants given by equations 2.10.6 and 2.7.8. The maximum value is given by equations 2.7.16 and 2.7.27. It occurs at a wavelength given by equations 2.7.1 and 2.7.12.

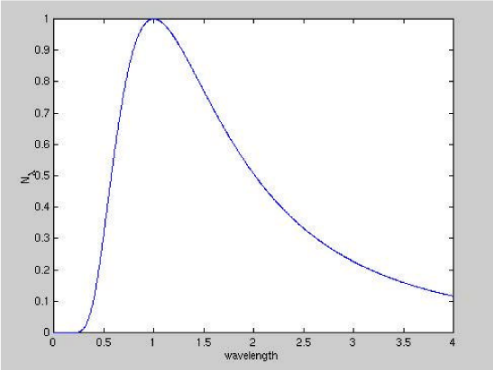

Figure II.2. Blackbody photon exitance per unit wavelength interval. The equation is equation 2.10.2 with constants given by equations 2.10.7 and 2.7.9. The maximum value is given by equations 2.7.17 and 2.7.28. It occurs at a wavelength given by equations 2.7.2 and 2.7.13.

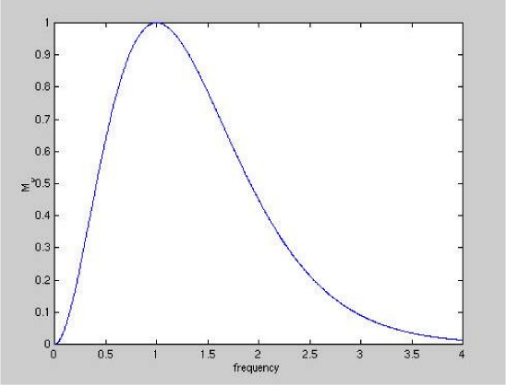

Figure II.3. Blackbody radiance per unit frequency interval. The equation is equation 2.10.3 with constants given by equations 2.10.8 and 2.7.10. The maximum value is given by equations 2.7.18 and 2.7.29. It occurs at a frequency given by equations 2.7.3 and 2.7.14.

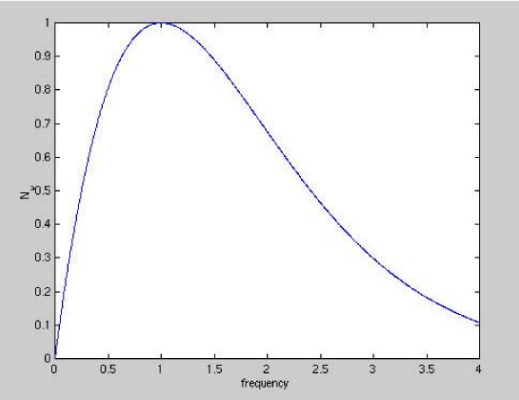

Figure II.4. Blackbody photon radiance per unit frequency interval. The equation is equation 2.10.4 with constants given by equations 2.10.9 and 2.7.11. The maximum value is given by equations 2.7.19 and 2.7 30. It occurs at a frequency given by equations 2.7.4 and 2.7.15.
