7.6: Vibrations of a Uniform Sphere

This is a three-dimensional problem and the wave equation is

$$c^2\nabla^2\Psi = \ddot{\Psi}. \tag{7.6.1}$$

The wave-function here, Ψ, which is a function of the coordinates and the time, can be thought of as the density of the sphere; it describes how the density of the sphere is varying in space and with time.

In problems of spherical symmetry it is convenient to write this in spherical coordinates. The expression for ∇^2 in spherical coordinates is fairly lengthy. You have probably not memorized it, but you will have seen it or know where to look it up. Stationary solutions are of the form

$$\Psi(r,\theta,\phi;t) = \psi(r,\theta,\phi).\,\chi(t). \tag{7.6.2}$$

Is it necessary to know the mathematical details of these functions? Probably not if it is not your intention to have a career in theoretical spectroscopy. If you *are* going to have a career in theoretical spectroscopy, it probably wouldn't hurt to do the detailed calculation - but in practice a great deal of calculation can be and is done without reference to the detailed algebra; all that is necessary to know and become familiar with are the properties of the functions and how they react to the several operators encountered in quantum mechanics. At this stage, we are just looking at some general principles, and need not worry about the details of the algebra, which can be fairly involved.

The spherical coordinates r, θ, ϕ are independent variables, and consequently the timeindependent part of the wave function can be written as the product of three functions:

$$\psi(r,\theta,\phi) = R(r).\,\Theta(\theta).\,\Phi(\phi). \tag{7.6.3}$$

Again, it is not immediately necessary to know the detailed forms of these functions - you will find them in various books on physics or chemistry or spectroscopy. The simplest is Φ − it is a simple sinusoidal function, usually written as $e^{-im\phi}$. The function Θ is a bit more complicated and it involves things called Legendre polynomials. The function R involves somewhat less-familiar polynomials called Laguerre polynomials. But there are *boundary conditions*. Thus ϕ goes only from 0 to 2π, and in that interval there can only be an integral number of half waves. Likewise, θ goes only from 0 to π, and r goes only from 0 to a, where a is the radius of the sphere. All three of these functions have integral "quantum numbers" associated with them. There is nothing mysterious about this, nor is it necessary to see the detailed algebra to see why this must be so; it is one of the inevitable constraints of fixed boundary conditions. The function R, the radial part of the wavefunction, has associated with it an integer n, which can take only integral values $1,\ 2.\ 3, \ldots$. The function Θ, the meridional wavefunction, has associated with it an integer l, which, for a given radial function (i.e. a given value of n) can have only the n different integral values $0,\ 1,\ 2,\ \ldots\ \ldots n-1$. Finally, the function Φ, the azimuthal wavefunction, has associated with it an integer m, which, for a given meridional function (i.e. a given value of l) can have only the $2l+1$ different integral values $-l,\ -l+1,\ \ldots\ 0,\ \ldots l-1,\ l$. Thus for a given n, the number of possible wavefunctions is

$$\sum_{0}^{n-1} 2l+1.$$

You will have to remember how to sum arithmetic series in order to evaluate this, so please do so. The answer is n^2.

When I first came across these quantum numbers, it was in connection with the wave mechanics of the hydrogen atom, and I thought there was something very mysterious about atomic physics. I was reassured only rather later - as I hope you will be reassured now - that the introduction of these quantum numbers is nothing to do with some special mysterious properties of atoms, but comes quite naturally out of the classical theory of vibrating spheres. Indeed, if you come to think about it, it would be very difficult indeed to believe that the wavefunctions of vibrating spheres did not involve numbers that had to be integers with restrictions on them.

The time-independent part of the wavefunction can be written:

$$\psi_{lmn}(r,\theta,\phi) = R_{nl}(r).\,\Theta_{lm}(\theta).\,\Phi_m(\phi). \tag{7.6.4}$$

Often the angular parts of the wavefunction are combined into a single function:

$$Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta).\,\Phi_m(\phi). \tag{7.6.5}$$

The functions Y_{lm} are known as *spherical harmonics*.

When performing various manipulations on the wavefunctions, very often all that happens is that you end up with a similar function but with different values of the integers ("quantum numbers"). Anyone who does a lot of such calculations soon gets used to this, and consequently, rather than write out the rather lengthy functions in full, what is done is merely to list the quantum number of a function between two special symbols known as a "ket". (That's one half of a bracket.) Thus a wavefunction may be written merely as $|lmn\rangle$. This is done frequently in the quantum mechanics of atoms. I have never seen it done in other, more "classical" branches of physics, but I see no reason why it should not be done, and I dare say that it is in some circles.

The problem of a vibrating sphere of uniform density is fairly straightforward. Similar problems face geophysicists or planetary scientists when discussing the interiors of the Earth or the Moon or other planets, or astrophysicists studying "helioseismology" or "asteroseismology" - except that you have to remember there that you are not dealing with *uniform* spheres.
