7.20: Orbiting and Spinning Charges

In the next section, section 7.21, we are going to look at the Zeeman effect, which is the symmetric splitting of spectrum lines in a magnetic field. Before we get to that, however, it will be useful to review some elementary principles of classical mechanics and electromagnetism in relation to orbiting and spinning electric charges. In particular, an orbiting or a spinning electric charge has *angular momentum* and also a *magnetic moment*. In this section we shall calculate both of these quantities for an orbiting charge and for a spinning charge. In particular we shall calculate for each the ratio of the magnetic moment to the angular momentum for each. This quantity should be called the *magnetogyric ratio*, though for some reason some people refer to it, entirely illogically, as the "gyromagnetic" ratio.

We shall start by considering a point electric charge of mass m and charge e moving with speed v in a circle of radius a. It is elementary classical mechanics that its *angular momentum* is mva. This may be seen either by thinking of angular momentum as synonymous with *moment of momentum* (in which case the moment of momentum is mv times a) or thinking of angular momentum as $I\omega$ (in which case it is ma^2 times v/a). The dimensions of angular momentum are $\mathrm{ML^2T^{-1}}$, and the SI units are conveniently expressed as J s (joule seconds). You should check that the dimensions of J s are correct.

Before calculating the magnetic moment, it is worth while to consider what is *meant* by magnetic moment. This is because there are several different ways in which magnetic moment could be defined (I can think of twelve plausible definitions!) and it by no means clear what different authors think that they mean by the term. I shall use the following concept (which is standard SI). If a magnet is placed in an external magnetic field $\mathbf{B}$ (of which the SI unit is tesla), the magnet will in general experience a *torque*. (I say "in general" because if the magnetic moment is parallel or antiparallel to the field, there will be no torque.) The magnitude of the torque depends on the orientation of the magnet with respect to the torque. There are two particular directions (opposite to each other - i.e. differing by 180°) in which the torque is a maximum. Definition: *The magnetic dipole moment is the maximum torque experienced in unit magnetic field* $\mathbf{B}$. It is a vector quantity, and I propose to use the symbol $\boldsymbol{\mu}$ for it, or μ for its magnitude, unless contexts arise in which μ might be confused with something else (such as permeability, for example). I shall warn if I need to use an alternative symbol. With that definition, the expression for the torque $\boldsymbol{\tau}$ on a magnet of moment $\boldsymbol{\mu}$ in a external field $\mathbf{B}$ is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} \tag{7.20.1}$$

This equation alone will not define $\boldsymbol{\mu}$ uniquely, for (if you have experienced solutions of vector equations before) the equation cannot be solved uniquely for $\boldsymbol{\mu}$. However, in concert with the definition given in the italicized sentence above, the magnetic moment is defined uniquely and without ambiguity. The SI unit is $\mathrm{N\,m\,T^{-1}}$. ($\mathrm{J\,T^{-1}}$ would also do in principle, except that joule does not suggest the concept of *torque* as well as $\mathrm{N\,m}$ does.) An example is usually given in introductory electricity courses for the torque on a plane, current-carrying coil, in which it is shown that the magnetic moment in that case is equal to the current times the area of the coil (times the number of turns if there are more than one). Thus equivalent and fully acceptable SI units are $\mathrm{A\,m^2}$. The dimensions of magnetic moment are $\mathrm{IL^2}$. (It will be noted that magnetic moment is often quoted in other units such as $\mathrm{T\,m^3}$ or $\mathrm{G\,cm^3}$. If not actually wrong, these must describe dimensionally different definitions of what is meant by magnetic moment. $\mathrm{T\,m^3}$, for example, is not dimensionally the same as $\mathrm{N\,m\,T^{-1}}$. I do not pursue this aspect here; I have given the standard SI treatment.)

With that introduction, we can now move to calculate the magnetic moment of a point electric charge e moving with speed v in a circle of radius a. The effective current is the charge divided by the period of the orbit, which is $e \div (2\pi a/v) = ev/(2\pi a)$. The area of the circle is πa^2, so the magnetic moment is $eav/2$.

The ratio of the magnetic moment to the angular momentum, i.e. the *magnetogyric ratio*, is $\frac{e}{2m}$. The SI unit is $\mathrm{C\,kg^{-1}}$. For an electron moving in a circle (of whatever radius and at whatever speed) its magnitude is $8.794 \times 10^{10}\mathrm{C\,kg^{-1}}$.

What about a spinning sphere of finite radius? Well, if the charge and the mass are both uniformly distributed throughout the sphere, or if they are not uniformly distributed but charge and mass are *equally* distributed throughout the sphere, then the magnetogyric ratio is again just $\frac{e}{2m}$, because every little element of the spinning sphere can be considered to be a point charge moving in a circle.

But what if the charge and mass densities are distributed differently? For example, if the sphere were a charged metal sphere, in which the mass is distributed uniformly through the sphere, but the charge is confined to the surface? In this case you would expect the magnetogyric ratio to be a good deal greater than $\frac{1}{2} \cdot \frac{e}{m}$. You will probably find it easy to calculate the *angular momentum* of a uniform sphere of mass m and radius a spinning at angular speed ω. It will be slightly more difficult, but a very good exercise, to work out the *magnetic moment*, assuming that all the charge is on the surface. You should find that the magnetogyric ratio is $\frac{5}{6} \cdot \frac{e}{m}$, which, as expected, is a good deal larger than $\frac{1}{2} \cdot \frac{e}{m}$. From this it might be anticipated that, if we could somehow measure (perhaps from the Zeeman effect) the magnetogyric ratio of an electron (which, indeed, we can, and with quite extraordinary precision) we can get some information about how mass and charge are distributed throughout the electron. In fact it turns out that the magnetogyric ratio of a spinning electron is very close to e/m. This suggests not only that the charge is distributed near to the surface but that the mass is concentrated near to the centre. Is this true? It is true that we can measure the magnetogyric ratio spectroscopically, and it is true that it turns out to be very close to e/m. Whether deducing from this that an electron is a sphere in which the charge is held near the surface and the mass is centrally concentrated is a bit more problematical. It is not necessarily "wrong", but it may not be a very useful model to describe other properties of the electron beyond its magnetogyric ratio. Particle physicists generally regard an electron as a "lepton" without any discernible internal structure. Perhaps physics cannot say what an electron (or anything else) really "is"; we can describe its observable properties and use whatever models appear best to describe these and to predict its behaviour.

I move now to the question of what is the *potential energy* of a magnet of moment $\boldsymbol{\mu}$ when it is situated in a magnetic field $\mathbf{B}$. When the angle between $\boldsymbol{\mu}$ and $\mathbf{B}$ is θ, there is a torque on the magnet of magnitude $\mu B \sin\theta$ (see equation 7.20.1)). The work needed to increase θ by $d\theta$ is $\mu B \sin\theta d\theta$, and the work needed to go from θ_1 to θ_2 is therefore $\mu B(\cos\theta_1 - \cos\theta_2)$, which is the difference in potential energy between the two positions. We may choose the zero level for potential energy where we will. If, for example, we choose the potential energy to be zero when $\boldsymbol{\mu}$ and $\mathbf{B}$ are parallel, the potential energy at angle θ is $\mu B(1-\cos\theta)$. If, on the other hand, we choose the potential energy to be zero when $\boldsymbol{\mu}$ and $\mathbf{B}$ are at right angles to each other (as is very often done), the potential energy at angle θ is $-\mu B \cos\theta$, or $-\boldsymbol{\mu} \cdot \mathbf{B}$. With this convention, the potential energy is $-\mu B$, 0 or $+\mu B$ for $\boldsymbol{\mu}$ and $\mathbf{B}$ parallel, perpendicular or antiparallel.

One last topic before proceeding to the Zeeman effect. Those who have studied classical mechanics will appreciate that if a spinning body, of angular momentum $\mathbf{L}$ is subjected to an external torque $\boldsymbol{\tau}$, it will *precess* with an angular speed $\boldsymbol{\Omega}$, and the three vectors are related by

$$\boldsymbol{\tau} = \boldsymbol{\Omega} \times \mathbf{L}. \tag{7.20.2}$$

Again, those familiar with vector equations will note that this equation cannot be solved uniquely for $\boldsymbol{\Omega}$, because we don't know the angle between $\boldsymbol{\tau}$ and $\mathbf{L}$. If, however, we are dealing with a nonvertical top spinning on a table, the torque is at right angles to the angular momentum vector and the rate of precession is mgl/L, where l is the distance between the bottom of the top (sorry for the choice of words) and the centre of mass. (Those who are *very* familiar with the theory of the top, will recognize that we are here referring to the rate of true or pseudo regular precession, after nutational motion has been damped out - but there is no need for such niceties here.)

Now consider a spinning magnet whose magnetic moment is $\boldsymbol{\mu}$ and whose angular momentum is $\mathbf{L}$. Suppose it is placed in a magnetic field $\mathbf{B}$. It now experiences a torque $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$. The result of this is that, regardless of the angle between $\mathbf{L}$ and $\mathbf{B}$, $\mathbf{L}$ will precess around $\mathbf{B}$ at an angular speed $\mu B/L$. That is to say, at an angular speed equal to B times the magnetogyric ratio. In the case of a particle of charge e and mass m moving in a circle the angular speed of precession is $\frac{eB}{2m}$. This is called the *Larmor angular speed*, and $\frac{eB}{4\pi m}$ is the Larmor frequency. If e, B and m are expressed in C, T and kg respectively, the Larmor frequency will be in Hz.