5.9: More on the Equation of Transfer

Refer to equation 5.5.1. We see from what had been subsequently discussed that $[\alpha(\nu)+\sigma(\nu)]dx = d\tau(\nu)$ and that $j_\nu dx = d\tau(\nu)$. Therefore

$$\frac{dI_v}{d\tau(\nu)} = S_\nu - I_\nu, \tag{5.9.1}$$

and this is another form of the *equation of transfer*.

Now consider a spherical star with a shallow atmosphere ("plane parallel atmosphere"). In figure V.6, radial distance r is measured radially outwards from the centre of the star. Optical depth is measured from outside towards the centre of the star. The thickness of the layer is dr. The coordinate z is measured from the centre of the star towards the observer, and the path length through the atmosphere in that direction at angle θ is $dz = dr\sec\theta$. The equation of transfer can be written

$$dI_\nu(\theta) = -[\kappa(\nu)I_\nu(\theta) - j_\nu]dz. \tag{5.9.2}$$

Now $\kappa(\nu)dz = -\sec\theta d\tau(\nu)$ and $j_\nu = \kappa(\nu)S_\nu$. Therefore

$$\cos\theta\frac{dI_\nu(\theta)}{d\tau(\nu)} = I_\nu(\theta) - S_\nu \tag{5.9.3}$$

This is yet another form of the equation of transfer. The quantity $\cos\theta$ is often written μ, so that equation 5.9.3 is often written

$$\mu\frac{dI_v(\theta)}{d\tau(\nu)} = I_\nu(\theta) - S_\nu \tag{5.9.4}$$

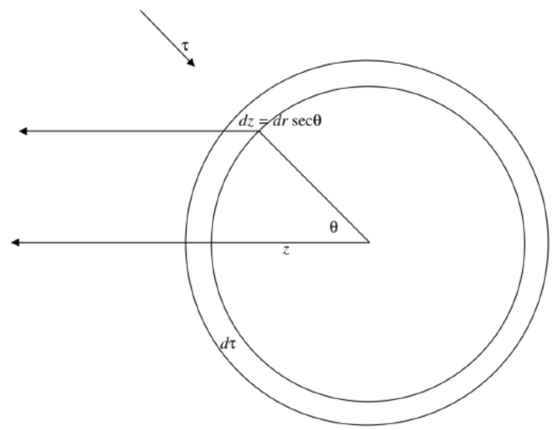

FIGURE V.6

Let us do $\frac{1}{4\pi}\oint d\omega$ to each term in equation 5.9.4. By $\oint$ I mean integrate over 4π steradians. In spherical coordinates $d\omega = \sin\theta d\theta d\phi$. We obtain

$$\frac{1}{4\pi}\oint\frac{dI_\nu(\theta)}{d\tau(\nu)}\cos\theta d\omega = \frac{1}{4\pi}\oint I_\nu d\omega - \frac{1}{4\pi}\oint S_\nu d\omega \tag{5.9.5}$$

The left hand side is $dH_\nu/d\tau(\nu)$ and the first term on the right hand side is J_ν. (See the definitions - equations 4.5.2 and 4.7.1.) In the case of isotropic scattering, the source function is isotropic so that, in this case

$$\frac{dH_\nu}{d\tau(\nu)} = J_\nu - S_\nu, \tag{5.9.6}$$

and this is another form of the equation of transfer.

On the other hand, if we do $\frac{1}{4\pi}\oint\cos\theta d\omega$ to each term in equation 5.15, we obtain

$$\frac{1}{4\pi}\oint\frac{dI_\nu(\theta)}{d\tau(\nu)}\cos^2\theta d\omega=\frac{1}{4\pi}\oint I_\nu\cos\theta d\omega-\frac{1}{4\pi}\oint S_\nu\cos\theta d\omega \tag{5.9.7}$$

In the case of isotropic scattering the last integral is zero, so that

$$\frac{dK_\nu}{d\tau(\nu)}=H_\nu, \tag{5.9.8}$$

and this is yet another form of the equation of transfer.

Now H_ν is independent of optical depth (why? - in a plane parallel atmosphere, this just expresses the fact that the flux (watts per square metre) is conserved), so we can integrate equation 5.9.8 to obtain

$$K_\nu=H_\nu\tau(\nu)+\text{constant} \tag{5.9.9}$$

Note also that $H_\nu=F_\nu/(4\pi)$, and, if the radiation is isotropic, $K_\nu=J_\nu/3$ so that,

$$J_\nu=\frac{3F_\nu\tau(\nu)}{4\pi}+J_\nu(0) \tag{5.9.10}$$

where $J_\nu(0)$ is the mean specific intensity (radiance) at the surface, which is half the specific intensity at the surface (since the radiance of the sky above the surface is zero). Thus

$$J_\nu(0)=\frac{1}{2}I_\nu(0)=F_\nu/(2\pi) \tag{5.9.11}$$

Therefore

$$J_\nu=\frac{F_\nu}{2\pi}\left(1+\frac{3}{2}\tau(\nu)\right) \tag{5.9.12}$$

This shows, to this degree of approximation (which includes the approximation that the radiation in the atmosphere is isotropic - which can be the case exactly only at the centre of the star) how the mean specific intensity increases with optical depth.

Let T be the temperature at optical depth τ.

Let T_0 be the surface temperature.

Let T_{eff} be the effective temperature, defined by $F(0)=\sigma T_{\text{eff}}^4$,

We also have $\pi J=\sigma T^4$ and $\pi J(0)=\sigma T_0^4=\frac{1}{2}F$.

From these we find the following relations between these temperatures:

$$T^4=\left(1+\frac{3}{2}\tau\right)T_0^4=\frac{1}{2}\left(1+\frac{3}{2}\tau\right)T_{\text{eff}}^4 \tag{5.9.13}$$

$$T_0^4=\frac{2}{2+3\pi}T^4=\frac{1}{2}T_{\text{eff}}^4 \tag{5.9.14}$$

$$T_{\text{eff}}^4=\frac{4}{2+3\tau}T^4=2T_0^4 \tag{5.9.15}$$

Note also that $T=T_{\text{eff}}$ at $\tau=2/3$, and $T=T_0$ at $\tau=0$.