4.7: The Cone

The Equation

$$x^2+y^2=a^2z^2 \tag{4.7.1}$$

represents a circular cone whose vertex in at the origin and whose axis coincides with the z-axis. The semi-vertical angle α of the cone is given by

$$\alpha=\tan^{-1} a. \tag{4.7.2}$$

In this context, the word "vertical" has nothing to do with "upright"; it merely means "of or at the vertex". A little knowledge of Latin is useful even today!

The Equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2 \tag{4.7.3}$$

is a cone with elliptical cross-section.

If the vertex of the cone remains at the origin, but the axis is in some arbitrary direction (described, for example, by the direction cosines, or by spherical angles θ and ϕ) the Equation can be derived by a rotation of coordinate axes. This will introduce terms in yz, zx and xy, but it will not produce any terms in x, y or z, nor will it introduce a constant. Therefore the Equation to such a cone will have only second degree terms in it. The Equation will be of the form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \tag{4.7.4}$$

With one proviso, the converse is usually true, namely that Equation 4.7.4 represents a cone with vertex at the origin. The one proviso is that if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0 \tag{4.7.5}$$

Equation 4.7.4 will factorize into two linear expressions, and will represent two planes, which intersect in a line that contains the origin - and this could be regarded as a special case of a cone, with zero vertical angle. If, however, the two linear factors are identical, the two planes are coincident.

The Equation to a cone of semi vertical angle α whose vertex is at the origin and whose axis has direction cosines (l,m,n) can be found as follows. The Equation

$$lx+my+nz+h=0 \tag{4.7.6}$$

represents a plane that is perpendicular to the axis of the cone and is at a distance h from the origin (i.e. from the vertex of the cone). Let P (x,y,z) be a point in the plane and also on the surface of the cone, at a distance r from the origin. The semi vertical angle of the cone is then given by

$$\cos\alpha=h/r. \tag{4.7.7}$$

But

$$r^2=x^2+y^2+z^2 \tag{4.7.8}$$

and, from Equation 4.7.6,

$$h^2=(lx+my+nz)^2 \tag{4.7.9}$$

Thus

$$(x^2+y^2+z^2)\cos^2\alpha=(lx+my+nz)^2 \tag{4.7.10}$$

is the required Equation to the cone.

It might be thought that the Equation to an inclined cone is unlikely to find much application in astronomy. Here is an application.

A fireball (i.e. a bright meteor, potentially capable of depositing a meteorite) moves down through the atmosphere with speed V along a straight line trajectory with direction cosines (l, m, n) referred to some coordinate system whose xy-plane is on the surface of Earth (assumed flat). If v is the speed of sound, and $V > v$, the meteoroid will generate a conical shock front of semi vertical angle α given by

$$\sin\alpha = v/V. \tag{4.7.11}$$

At a time t before impact, the coordinates of the vertex will be (lVt, mVt, nVt), and the Equation to the conical shock front will then be

$$\left[(x+lVt)^2+(y+mVt)^2+(z+nVt)^2\right]\cos^2\alpha = \left[l(x+lVt)+m(y+mVt)+n(z+nVt)\right]^2 \tag{4.7.12}$$

Part of this shock front (at time t before impact) has already reached ground level, and it intersects the ground in a conic section given by putting $z = 0$ in Equation 4.7.12:

$$(\cos^2\alpha - l^2)x^2 + (\cos^2\alpha - m^2)y^2 - 2lmxy - 2Vt\sin^2\alpha(lx+my) - V^2t^2\sin^2\alpha = 0 \tag{4.7.13}$$

and witnesses on the ground on any point on this conic section will hear the shock front at the same time. Further details on this can be found in Tatum, J.B., *Meteoritics and Planetary Science*, **34**, 571 (1999) and Tatum, J.B., Parker, L. C. and Stumpf, L. L., *Planetary and Space Science* **48**, 921 (2000).
