9.7: Position in a Hyperbolic Orbit

If an interstellar object were to encounter the solar system from interstellar space, it would pursue a hyperbolic orbit around the Sun. The first known such object with an original hyperbolic orbit was detected in 2017, and was given the name Oumuamua. However, a comet with a near-parabolic orbit from the Oort belt may approach Jupiter on its way in to the inner solar system, and its orbit may be perturbed into a hyperbolic orbit. This will result in its ultimate loss from the solar system. Several examples of such cometary orbits are known. There is evidence, from radar studies of meteors, of meteoroidal dust encountering Earth at speeds that are hyperbolic with respect to the Sun, although whether these are on orbits that are originally hyperbolic (and are therefore from interstellar space) or whether they are of solar system origin and have been perturbed by Jupiter into hyperbolic orbits is not known.

I must admit to not having actually carried out a calculation for a hyperbolic orbit, but I think we can just proceed in a manner similar to an ellipse or a parabola. Thus we can start with the angular momentum per unit mass:

$$h = r^2\dot{v} = \sqrt{G\mathbf{M}l}, \tag{9.8.1}$$

where

$$r = \frac{l}{1+e\cos v} \tag{9.8.2}$$

and

$$l = a(e^2-1). \tag{9.8.3}$$

If we use astronomical units for distance and mass, we obtain

$$\int_0^v \frac{dv}{(1+e\cos v)^2} = \frac{2\pi}{a^{3/2}(e^2-1)^{3/2}}\int_T^t dt. \tag{9.8.4}$$

Here I am using astronomical units of distance and mass and have therefore substituted $4\pi^2$ for $G\mathbf{M}$.

I'm going to write this as

$$\int_0^v \frac{dv}{(1+e\cos v)^2} = \frac{2\pi(t-T)}{a^{3/2}(e^2-1)^{3/2}} = \frac{Q}{(e^2-1)^{3/2}} \tag{9.8.5}$$

where $Q = \frac{2\pi(t-T)}{a^{3/2}}$. Now we have to integrate this.

Method 1

Guided by the elliptical case, but bearing in mind that we are now dealing with a hyperbola, I'm going to try the substitution

$$\cos v = \frac{e-\cosh E}{e\cosh E - 1} \tag{9.8.6}$$

If you try this, I think you'll end up with

$$e\sinh E - E = Q. \tag{9.8.7}$$

This is just the analogy of Kepler's Equation.

The procedure, then, would be to calculate Q from Equation 9.8.5. Then calculate E from Equation 9.8.7. This could be done, for example, by a Newton-Raphson iteration in quite the same way as was done for Kepler's Equation in the elliptic case, the iteration now taking the form

$$E = \frac{Q + e(E\cosh E - \sinh E)}{e\cosh E - 1}. \tag{9.8.8}$$

Then v is found from Equation 9.8.6, and the heliocentric distance is found from the polar Equation to a hyperbola:

$$r = \frac{a(e^2-1)}{1+e\cos v}. \tag{9.8.9}$$

Method 2

Method 1 should work all right, but it has the disadvantage that you may not be as familiar with sinh and cosh as you are with sin and cos, or there may not be a sinh or cosh button your calculator. I believe there are SINH and COSH functions in $\mathrm{FORTRAN}$, and there may well be in other computing languages. Try it and see. But maybe we'd like to try to avoid hyperbolic functions, so let's try the brilliant substitution

$$\cos v = -\frac{u(u-2e)+1}{u(eu-2)+e}. \tag{9.8.10}$$

You may have noticed, when you were learning calculus, that often the professor would make a brilliant substitution, and you could see that it worked, but you could never understand what made the professor think of the substitution. I don't want to tell you what made me think of this substitution, because, when I do, you'll see that it isn't really very brilliant at all. I remembered that

$$\cosh E = \frac{1}{2}(\mathrm{e}^{E} + \mathrm{e}^{E}) \tag{9.8.11}$$

and then I let $\mathrm{e}^{E} = u$, so

$$\cosh E = \frac{1}{2}(u + 1/u), \tag{9.8.12}$$

and I just substituted this into Equation 9.8.6 and I got Equation 9.8.10. Now if you put the expression 9.8.10 for $\cos v$ into Equation 9.8.5, you eventually, after a few lines, get something that you can integrate. Please do work through it. In the end, on integration of Equation 9.8.5, you should get

$$\frac{1}{2}e(u - \frac{1}{u}) - \ln u = Q. \tag{9.8.13}$$

You already know from Chapter 1 how to solve the Equation $f(x) = 0$, so there is no difficulty in solving Equation 9.8.13 for u. Newton-Raphson iteration results in

$$u = \frac{2u[e - u(1 - Q\ln u)]}{u(eu-2)+2}, \tag{9.8.14}$$

and this should converge in the usual rapid fashion.

So the procedure in method 2 is to calculate Q from Equation 9.8.5, then calculate u from Equation 9.8.14, and finally v from Equation 9.8.10 – all very straightforward.

Exercise 9.7.1

Set yourself a problem to make sure that you can carry through the calculation. Then write a computer program that will generate v and r as a function of t.