4.8: The General Second Degree Equation in Three Dimensions

The general second degree Equation in three dimensions is

$$ax^2 + by^2 + cz + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \tag{4.8.1}$$

This may represent a plane or pair of planes (which, if not parallel, define a straight line), or an ellipsoid, paraboloid, hyperboloid, cylinder or cone. The Equation could, if convenient, be divided through by d (or any of the other constants), and there are in reality only nine independent constants. Therefore nine points in space are sufficient to determine the second degree surface on this they lie.

If d is zero, the surface contains the origin. If u, v and w are all zero, and the surface is an ellipsoid, hyperbolic paraboloid or a hyperboloid, the origin is at the centre of the figure. If the figure is an elliptic paraboloid, the origin is at the vertex. If u, v, w and d are all zero, the surface is a cone with the proviso mentioned in section 4.7. If a, b, c, f, g, h are all zero, the surface is a plane.

Let us consider a particular example:

$$3x^2 - 4y^2 + 6z^2 + 8yz - 2zx + 4xy + 14x - 10y - 4z + 5 = 0 \tag{4.8.2}$$

What sort of a surface is this?

We need to do two things. First we need to rotate the coordinate axes so that they are parallel to the figure axes. The Equation referred to the figure axes will have no terms in yz, zx or xy. Then we need to translate the axes so that the origin is at the centre of the figure (or at the vertex, if it is an elliptical paraboloid).

Mathematically, we need to find the eigenvectors of the matrix

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -4 & 4 \\ -1 & 4 & 6 \end{vmatrix} \tag{4.8.3}$$

Some readers will readily know how to do this. Others may not, and may not even be quite certain what an eigenvector is. Section 4.9 may be of interest to either group of readers. In any case, the eigenvectors are found to be

$$\begin{pmatrix} l_{11} \\ l_{12} \\ l_{13} \end{pmatrix} = \begin{pmatrix} -0.069\,5481 \\ +0.318\,8310 \\ +0.945\,2565 \end{pmatrix} \quad \begin{pmatrix} l_{12} \\ l_{22} \\ l_{32} \end{pmatrix} = \begin{pmatrix} -0.240\,6405 \\ +0.914\,2071 \\ -0.326\,0635 \end{pmatrix} \quad \begin{pmatrix} l_{13} \\ l_{23} \\ l_{33} \end{pmatrix} = \begin{pmatrix} -0.968\,1194 \\ -0.2501441 \\ +0.013\,1423 \end{pmatrix} \tag{4.8.1}$$

with corresponding eigenvalues $7.422\,7590,\ 5.953\,0969,\ 3.530\,3380$

The elements of the eigenvectors are the direction cosines of the present coordinate axis with respect to the figure axes. To express the Equation to the surface relative to coordinate axes that are parallel to the figure axes, we replace

$$x \text{ by } l_{11}x + l_{12}y + l_{13}z \tag{4.8.2}$$

$$y \text{ by } l_{21}x + l_{22}y + l_{23}z \tag{4.8.3}$$

$$z \text{ by } l_{31}x + l_{32}y + l_{33}z \tag{4.8.4}$$

This will make the terms in yz, zx and xy vanish; this should be checked numerically, particularly as it is easy to rotate the axes in the wrong sense. When the substitutions are made, the Equation is found to be

$$7.422\,7590x^2 - 5.9530969y^2 + 3.530\,3380z^2 - 7.9430994x - 11.2067840y - 11.1047998z + 5 = 0. \tag{4.8.4}$$

Notice that there are now no terms in yz, zx or xy.

Now we need to translate the origin of coordinates to the centre of the figure (or to the vertex if it is an elliptic paraboloid). It will readily be seen that this can be done by substituting

$$x - \alpha \quad \text{for } x \tag{4.8.5}$$

$$y - \beta \quad \text{for } y \tag{4.8.6}$$

$$z - \gamma \quad \text{for } z \tag{4.8.7}$$

where

$$\alpha=(\text{coefficient of } x)/(\text{twice the coefficient of } x^2)=-0.535\ 050\ 336 \tag{4.8.8}$$

$$\beta=(\text{coefficient of } y)/(\text{twice the coefficient of } y^2)=+0.941\ 256\ 643 \tag{4.8.9}$$

$$\gamma=(\text{coefficient of } z)/(\text{twice the coefficient of } z^2)=-1.572\ 767\ 224 \tag{4.8.10}$$

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The Equation then becomes

$$7.422\ 7590x^2-5.953\ 0969y^2+3.530\ 3380z^2-0.583\ 3816=0 \tag{4.8.11}$$

or

$$\frac{x^2}{(0.280\ 346)^2}-\frac{y^2}{(0.313\ 044)^2)}+\frac{z^2}{(0.406\ 507)^2}=1. \tag{4.8.12}$$

The surface is a hyperboloid of one sheet, elliptical in any $y=$ constant cross-section.

The surfaces described by second-degree Equations in three dimensions - ellipsoids, paraboloids, hyperboloids, cones and cylinders - are generally called quadric surfaces. The surface described by the Equation

$$\left[(x^2+y^2)^{\frac{1}{2}}-a\right]^2+z^2=b^2, \quad b<a \tag{4.8.13}$$

is not one of the quadric surfaces. If the square root is isolated and squared, the resulting Equation will contain terms of degree four. The surface is a fairly familiar one, and the reader should try to imagine what it is. Failing that, if your computer skills are up to it, you might try to draw the surface in three-dimensional space. The only hint I give is to suggest that you put $y=0$ in Equation ??? to see what the section is in the z-x plane.