13.10: Higher-order Approximation

The reason that we made the approximation to order τ^3 was that, in evaluating the expressions for F_1, G_1, F_3 and G_3, we did not know the radial velocity $\dot{r}_2$. Perhaps we can now evaluate it.

Exercise. Show that the radial velocity of a particle in orbit around the Sun, when it is at a distance r from the Sun, is

$$\text{Ellipse}: \quad \dot{r}=\mp\sqrt{\frac{GM}{a_0}}\left(\frac{a^2e^2-(a-r)^2}{ar^2}\right)^{1/2}, \tag{13.10.1}$$

$$\text{Parabola}: \quad \dot{r}=\mp\sqrt{\frac{GM(r-q)}{a_0}}, \tag{13.10.2}$$

$$\text{Hyperbola}: \quad \dot{r}=\mp\sqrt{\frac{GM}{a_0}}\left(\frac{(a+r)^2-a^2e^2}{ar^2}\right)^{1/2}. \tag{13.10.3}$$

Show that the radial velocity is greatest at the ends of a latus rectum.

Here a_0 is the astronomical unit, a is the semi major axis of the elliptic orbit or the semi transverse axis of the hyperbolic orbit, q is the perihelion distance of the parabolic orbit, and e is the orbital eccentricity. The $-$ sign is for pre-perihelion, and the $+$ sign is for post-perihelion.

Unfortunately, while this is a nice exercise in orbit theory, we do not know the eccentricity, so these formulas at present are of no use to us.

However, we can calculate the heliocentric distances at the times of the first and third observations by exactly the same method as we used for the second observation. Here are the results for our numerical example, after one iteration. The units, of course, are au. Also indicated are the instants of the observations, taking $t_2=0$ and expressing the other instants in units of $1/k$ (see section 13.8).

$$t_1=-\tau_3=-0.086\,010\,494\,75 \quad r_1=3.419\,52$$

$$t_2=0 \quad r_2=3.416\,73$$

$$t_3=+\tau_1=+0.172\,020\,989\,5 \quad r_3=3.410\,82$$

We can fit a quadratic expression to this, of the form:

$$r=c_0+c_1t+c_2t^2 \tag{13.10.4}$$

With our choice of time origin $t_2=0$, c_0 is obviously just equal to r_2, so we have just two constants, c_1 and c_2 to solve for. We can then calculate the radial velocity at the time of the second observation from

$$\dot{r}_2=c_1+2c_2t_2. \tag{13.10.5}$$

We can calculate A_1, A_2 and A_3 in the same manner as before, up to τ^4 rather than just τ^3. The algebra is slightly long and tedious, but straightforward. Likewise, the results look long and unwieldy, but there is no difficulty in programming them for a computer, and the actual calculation is, with a modern computer, virtually instantaneous. The results of the algebra that I give below are taken from the book *Determination of Orbits* by A.D. Dubyago (which has been the basis of much of this chapter). I haven't checked the algebra myself, but the conscientious reader will probably want to do so himself or herself.

$$A_1=\frac{1}{2}\sqrt{l}\tau_1\left(1-\frac{\tau_1^2}{6r_2^3}+\frac{\tau_1^3}{4r_2^4}\dot{r}_2\right), \tag{13.10.6}$$

$$A_2=\frac{1}{2}\sqrt{l}\tau_2\left(1-\frac{\tau_2^2}{6r_2^3}+\frac{\tau_2^2(\tau_1-\tau_3)}{4r_2^4}\dot{r}_2\right), \tag{13.10.7}$$

$$A_3=\frac{1}{2}\sqrt{l}\tau_3\left(1-\frac{\tau_3^2}{6r_2^3}-\frac{\tau_3^3}{4r_2^4}\dot{r}_2\right). \tag{13.10.8}$$

And from these,

$$a_1 = \frac{\tau_1}{\tau_2}\left(1 + \frac{\tau_3(\tau_2+\tau_1)}{6r_2^3} + \frac{\tau_3(\tau_3(\tau_1+\tau_3)-\tau_1^2)}{4r_2^4}\dot{r}_2\right). \tag{13.10.9}$$

and

$$a_3 = \frac{\tau_3}{\tau_2}\left(1 + \frac{\tau_1(\tau_2+\tau_3)}{6r_2^3} - \frac{\tau_1(\tau_1(\tau_1+\tau_3)-\tau_3^2}{4r_2^4}\dot{r}_2\right). \tag{13.10.10}$$

This might result in slightly better values for a_1 and a_3. I have not calculated this for our numerical example here, for reasons given in Section 13.9. We can move on to the next section, using our current vales of a_1 and a_3, namely

$$a_1 = 0.666\,770 \quad \text{and} \quad a_3 = 0.333\,416.$$