13.15: Calculating the Elementss

We can now immediately calculate the semi latus rectum from Equation 13.12.6a (recalling that $2f_3 = v_2 - v_1$, so that everything except l in the Equation is already known.) In fact we have three opportunities for calculating the semi latus rectum by using each of Equations 13.12.6a,b,c, and this serves as a check on the arithmetic. For our numerical example, I obtain

$$l = 2.61779$$

identically (at least to eleven significant figures) for each of the three permutations.

Now, on referring to Equation 2.3.37, we recall that the polar Equation to an ellipse is

$$r = \frac{l}{1 + e\cos v}. \tag{13.15.1}$$

We therefore have, for the first and third observations,

$$e\cos v_1 = l/r_1 - 1 \tag{13.15.2}$$

and, admitting that $v_3 = v_1 + 2f_2$,

$$e\cos(v_1 + 2f_2) = l/r_3 - 1. \tag{13.15.3}$$

We observe that, in Equations 13.15.2and 13.15.3, the only quantities we do not already know are v_1 and e – so we are just about to find our first orbital element, the eccentricity!

A hint for solving Equations 13.15.2and 3: Expand $cos(v_1 + 2f_2)$. Take $e\sin v$ to the left hand side, and Equation 13.15.3will become

$$e\sin v_1 = \frac{(l/r_1 - 1).\cos 2f_2 - (l/r_3 - 1)}{\sin 2f_2}. \tag{13.15.4}$$

After this, it is easy to solve Equations 13.15.2and 13.15.4for e and for v_1. The other true anomalies are given by $v_2 = v_1 + 2f_3$ and $v_3 = v_1 + 2f_2$. A check on the arithmetic may (and should) be performed by carrying out the same calculation for the first and second observations and for the second and third observations. For all three, I obtained

$$\underline{e = 0.23875}$$

We have our first orbital element!

(The MPC value for the eccentricity for this epoch is 0.22994 – but this is based on all available observations, and we cannot expect to get the MPC value from just three hypothetical “observations”.)

The true anomalies at the times of the three observations are

$$v_1 = 191^\circ.99814 \quad v_2 = 192^\circ.68221 \quad v_3 = 194^\circ.05377$$

After that, the semi major axis is easy from Equation 2.3.10, $l = a(1 - e^2)$, for the semi latus rectum of an ellipse. We find

$$\underline{a = 2.77602 \text{ au}}$$

The period in sidereal years is given by $P^2 = a^3$, and is therefore 4.62524 sidereal years. This is not one of the six independent elements, since it is always related to the semi major axis by Kepler’s third law, so it doesn’t merit the extra dignity of being underlined. However, it is certainly worth converting it to mean solar days by multiplying by 365.25636 We find that $P = 1689.39944$days.

The next element to yield will be the time of perihelion passage. We find the eccentric anomalies for each of the three observations from any of Equations 2.3.16, 17a, 17b or 17c. For example:

$$\cos E = \frac{e + \cos v}{1 + e\cos v}. \tag{13.15.5}$$

Then the time of perihelion passage will come from Equations 9.6.4 and 9.6.5:

$$T = t - \frac{P}{2\pi}(E - e\sin E) + nP. \tag{13.15.6}$$

With $n=1$ I make this $\underline{T=t_1+756^{\mathrm{d}}.1319}$

The next step is to calculate the Ps and Qs. These are defined in Equation 10.9.40. They are the direction cosines relating the heliocentric plane-of-orbit basis set to the heliocentric equatorial basis set.

Exercise. Apply Equation 10.9.50 to the first and third observations to show that

$$P_x=\frac{\xi_1 r_3\sin v_3-\xi_3 r_1\sin v_1}{r_1 r_3\sin 2f_2} \tag{13.15.7}$$

and

$$Q_x=\frac{\xi_3 r_1\cos v-\xi_1 r_3\cos v_3}{r_1 r_2\sin 2f_2} \tag{13.15.8}$$

From Equations 10.9.51 and 52, find similar Equations for P_y, Q_y, P_z, Q_z.

The numerical work can and should be checked by calculating these direction cosines also from the first and second, and from the second and third, observations. Check also that $P_x^2+P_y^2+P_z^2=Q_x^2+Q_y^2+Q_z^2=1$. I get

$P_x=-0.48044 \quad P_y=+0.86568 \quad P_z=-0.14059$

$Q_x=-0.87392 \quad Q_y=-0.45907 \quad Q_z=+0.15978$

(Remember that my computer is carrying all significant figures to double precision, though I print out here only a limited number of significant figures. You will not get exactly my numbers unless you, too, carry all significant figures and do not prematurely round off.)

The direction cosines are related to the Eulerian angles, of course, by Equations 10.9.41- 46 (how could you possibly forget?!). All (!) you have to do, then, is to solve these six Equations for the Eulerian angles. (You need six Equations to remove quadrant ambiguity from the angles. Remember the $\mathrm{ATAN2}$ function on your computer – it's an enormous help with quadrants.)

Exercise. Show that (or verify at any rate) that:

$$\sin\omega\sin i=P_z\cos\varepsilon-P_y\sin\varepsilon \tag{13.15.9}$$

and

$$\cos\omega\sin i=Q_z\cos\varepsilon-Q_y\sin\varepsilon. \tag{13.15.10}$$

You can now solve this for the argument of perihelion ω. Don't yet try to solve it for the inclination. (Why not?!) Using $\varepsilon=23^{\circ}.438\,960$ for the obliquity of the ecliptic of date (calculated from page $\mathrm{B}18$ of the 2002 *Astronomical Almanac*), I get

$$\underline{\omega=304^{\circ}.81849}$$

Exercise. Show that (or verify at any rate) that:

$$\sin\Omega=(P_y\cos\omega-Q_y\sin\omega)\sec\varepsilon \tag{13.15.11}$$

and

$$\cos\Omega=P_x\cos\omega-Q_x\sin\omega. \tag{13.15.12}$$

From these, I find:

$$\underline{\Omega=172^{\circ}.64776}$$

One more to go!

Exercise. Show that (or verify at any rate) that:

$$\cos i=-(P_x\sin\omega+Q_x\cos\omega)\csc\Omega. \tag{13.15.13}$$

You can now solve this with Equation 13.15.9or 13.15.10(or both, as a check on the arithmetic) for the inclination. I get

$$\underline{i=35^{\circ}.20872}$$

Here they are, all together:

$$\begin{aligned} a &= 2.77602 \text{ AU} & i &= 35^\circ.20872 \\ e &= 0.23875 & \Omega &= 172^\circ.64776 \\ T &= t_1 + 756^{\text{d}}.1319 & \omega &= 304^\circ.81849 \end{aligned} \tag{13.15.1}$$

Have we made any mistakes? Well, presumably after you read chapter 10 you wrote a program to generate an ephemeris. So now, use these elements to see whether they will reproduce the original observations! Incidentally, to construct an ephemeris, there is no need actually to use the elements – you can use the Ps and Qs instead.