9.10: Mean Distance in an Elliptic Orbit

It is sometimes said that "a" in an elliptic orbit is the "mean distance" of a planet from the Sun. In fact a is the semi major axis of the orbit. Whether and it what sense it might also be the "mean distance" is worth a moment of thought.

It was the late Professor C. E. M Joad whose familiar answer to the weighty questions of the day was "It all depends what you mean by..." And the "mean distance" depends on whether you mean the distance averaged over the true anomaly v or over the time. The mean distance averaged over the true anomaly is $\frac{1}{\pi}\int_0^{\pi} rdv$, where $r = l/(1+ecosv)$. If you are looking for some nice substitution to help you to integrate this, Equation 2.13.6 does very nicely, and you soon find the unexpected result that the mean distance, averaged over the mean anomaly, is b, the semi *minor* axis.

On the other hand, the mean distance averaged over the time is $\frac{1}{2}P\int_0^{\frac{1}{2}P} rdt$. This one is slightly more tricky, but, following the hint for evaluating $\frac{1}{\pi}\int_0^{\pi} rdv$, you could try expressing r and v in terms of the eccentric anomaly. It will take you a moment or so, but you should eventually find that the mean distance averaged over the time is $a(1+\frac{1}{2}e^2)$.

It is often pointed out that, because of Kepler's second law, a planet spends more time far from the Sun that it does near to the Sun, which is why we have longer summers than winters in the northern hemisphere. An easy exercise would be to ask you what fraction of its orbital period does a planet spend on the sunny side of a latus rectum. A slightly more difficult exercise would be to ask: What fraction of its orbital period does a planet spend closer to the Sun than its mean (time-averaged) distance? You'd first have to ask, what is the true anomaly when $r = a(1+\frac{1}{2}e^2)$? Then you need to calculate the fraction of the area of the orbit. Area in polar coordinates is $\frac{1}{2}\int r^2 dv$. I haven't tried this, but, if it proves difficult, I'd try and write r and v in terms of the eccentric anomaly E and see if that helped.