13.7: Geocentric and Heliocentric Distances - First Attempt

Let us write down the three heliocentric equatorial components of Equation 13.2.1:

$$\xi_2 = a_1\xi_1 + a_3\xi_3, \tag{13.7.1}$$

$$\eta_2 = a_1\eta_1 + a_3\eta_3, \tag{13.7.2}$$

$$\zeta_2 = a_1\zeta_1 + a_3\zeta_3. \tag{13.7.3}$$

Now write $l\,\Delta - \mathfrak{x}_o$ for ξ, etc., from Equations 13.5.1,2,3 and rearrange to take the solar coordinates to the right hand side:

$$l_1 a_1\,\Delta_1 - l_2\,\Delta_2 + l_3 a_3\,\Delta_3 = a_1\mathfrak{x}_{o1} - \mathfrak{x}_{o2} + a_3\mathfrak{x}_{o3}, \tag{13.7.4}$$

$$m_1 a_1\,\Delta_1 - m_2\,\Delta_2 + m_3 a_3\,\Delta_3 = a_1\mathfrak{y}_{o1} - \mathfrak{y}_{o2} + a_3\mathfrak{y}_{o3}, \tag{13.7.5}$$

$$n_1 a_1\,\Delta_1 - n_2\,\Delta_2 + n_3 a_3\,\Delta_3 = a_1\mathfrak{z}_{o1} - \mathfrak{z}_{o2} + a_3\mathfrak{z}_{o3}. \tag{13.7.6}$$

As a very first, crude, approximation, we can let $a_1 = b_1$ and $a_3 = b_3$, for we know b_1 and b_3 (in our numerical example, $b_1 = \frac{2}{3}$, $b_3 = \frac{1}{3}$), so we can solve Equations 13.7.4,5,6 for the three geocentric distances. However, we shall eventually need to find the correct values of a_1 and a_3.

When we have solved these Equations for the geocentric distances, we can then find the heliocentric distances from Equations 13.5.1,2 and 3. For example,

$$\xi_1 = l_1\,\Delta_1 - \mathfrak{x}_{o1} \tag{13.7.7}$$

and of course

$$r_1^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2. \tag{13.7.8}$$

In our numerical example, we have

$$\begin{aligned} l_1 &= +0.722\,980\,907 \\ l_2 &= +0.715\,380\,933 \\ l_3 &= +0.698\,125\,992 \end{aligned}$$

$$\begin{aligned} m_1 &= -0.631\,808\,343 \\ m_2 &= -0.641\,649\,261 \\ m_3 &= -0.664\,816\,398 \end{aligned}$$

$$\begin{aligned} n_1 &= +0.279\,493\,876 \\ n_2 &= +0.276\,615\,882 \\ n_3 &= +0.265\,780\,465 \end{aligned}$$

As a check on the arithmetic, the reader can - and should - verify that

$$l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 = 1$$

This does not verify the signs of the direction cosines, for which care should be taken.

From *The Astronomical Almanac* for 2002, we find that

$$\begin{array}{llll} \mathfrak{x}_{o1} = -306\,728\,3 & \mathfrak{y}_{o1} = +0.889\,290\,0 & \mathfrak{z}_{o1} = +0.385\,549\,5 & \text{AU} \\ \mathfrak{x}_{o2} = -386\,194\,4 & \mathfrak{y}_{o2} = +0.862\,645\,7 & \mathfrak{z}_{o2} = +0.373\,999\,6 & \\ \mathfrak{x}_{o3} = -536\,330\,8 & \mathfrak{y}_{o3} = +0.791\,387\,2 & \mathfrak{z}_{o2} = +0.343\,100\,4 & \end{array}$$

(For a fraction of a day, which will usually be the case, these coordinates can be obtained by nonlinear interpolation – see chapter 1, section 1.10.)

Equations 13.7.4,5,6 become

$$+0.481\,987\,271\,\Delta_1 - 0.\,715\,380\,933\,\Delta_2 + 0.\,232\,708\,664\,\Delta_3 = 0.002\,931\,933$$

$$-0.421\,205\,562\,\Delta_1 + 0.641\,649\,261\,\Delta_2 - 0.221\,605\,466\,\Delta_3 = -0.005\,989\,967$$

$$+0.186\,329\,251\,\Delta_1 - 0.276\,615\,882\,\Delta_2 - 0.088\,593\,488\,\Delta_3 = -0.002\,599\,800$$

I give below the solutions to these Equations, which are our first crude approximations to the geocentric distances in AU, together with the corresponding heliocentric distances. I also give, for comparison, the correct values, from the published MPC ephemeris

$$\begin{array}{llll} \multicolumn{2}{c}{\text{First crude estimates}} & \multicolumn{2}{c}{\text{MPC}} \\ \Delta_1 = 2.725\ 71 & r_1 = 3.485\ 32 & \Delta_1 = 2.644 & r_1 = 3.406 \\ \Delta_2 = 2.681\ 60 & r_2 = 3.481\ 33 & \Delta_2 = 2.603 & r_2 = 3.404 \\ \Delta_3 = 2.610\ 73 & r_3 = 3.474\ 71 & \Delta_3 = 2.536 & r_3 = 3.401 \end{array} \tag{13.7.1}$$

This must justifiably give cause for some satisfaction, because we now have some idea of the geocentric distances of the planet at the instants of the three observations, though it is a little early to open the champagne bottles. We still have a little way to go, for we must refine our values of a_1 and a_3. Our first guesses, $a_1 = b_1$ and $a_3 = b_3$, are not quite good enough.

The key to finding the geocentric and heliocentric distances is to be able to determine the triangle ratios $a_1 = A_1/A_2$, $a_3 = A_3/A_2$ and the triangle/sector ratios a/b. The sector ratios are found easily from Kepler's second law. We have made our first very crude attempt to find the geocentric and heliocentric distances by assuming that the triangle ratios are equal to the sector ratios. It is now time to improve on that assumption, and to obtain better triangle ratios. After what may seem like a considerable amount of work, we shall obtain approximate formulas, Equations 13.8.35a,b, for improved triangle ratios. The reader who does not wish to burden himself with the details of the derivation of these Equations may proceed directly to them, near the end of Section 13.7