13.8: Improved Triangle Ratios

The Equation of motion of the orbiting body is

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}. \tag{13.8.1}$$

If we recall Equation 13.4.2, this can be written

$$\ddot{\mathbf{r}} = -k^2\left(\frac{a^3}{r^3}\right)\mathbf{r}. \tag{13.8.2}$$

If we now agree to express r in units of a (i.e. in Astronomical Units of length (au)) and time in units of $1/k$ ($1/k = 58.132\,440\,87$ mean solar days), this becomes merely

$$\ddot{\mathbf{r}} = -\frac{1}{r^3}\mathbf{r}. \tag{13.8.3}$$

In these units, GM has the value 1.

Now write the x- and y- components of this Equation, where $(x,\ y)$ are heliocentric coordinates in the plane of the orbit (see sections 13.5 or 10.7).

$$\ddot{x} = -\frac{x}{r^3} \tag{13.8.4}$$

and

$$\ddot{y} = -\frac{y}{r^3}, \tag{13.8.5}$$

where

$$x^2 + y^2 = r^2. \tag{13.8.6}$$

The areal speed is $\frac{1}{2}h = \frac{1}{2}\sqrt{GMl}$ or, in these units, $\frac{1}{2}\sqrt{l}$ where l is the semi latus rectum of the orbit in au

Let the planet be at $(x,\ y)$ at time t. Then at time $t+\delta t$ it will be at $(x+\delta x, y+\delta y)$, where

$$\delta x = \dot{x}\delta t + \frac{1}{2!}\ddot{x}(\delta t)^2 + \frac{1}{3!}\dddot{x}(\delta t)^3 + \frac{1}{4!}\ddddot{x}(\delta t)^4 + \ldots \tag{13.8.7}$$

and similarly for y.

Now, starting from Equation 13.8.4 we obtain

$$\dddot{x} = \frac{3x\dot{r}}{r^4} - \frac{\dot{x}}{r^3} \tag{13.8.8}$$

and

$$\ddddot{x} = 3\left(\frac{\dot{x}\dot{r}}{r^4} + \frac{x\ddot{r}}{r^4} - \frac{4x\dot{r}^2}{r^5}\right) - \frac{r^3\ddot{x} - 3r^2\dot{x}\dot{r}}{r^6}. \tag{13.8.9}$$

(The comment in the paragraph preceding Equation 3.4.16 may be of help here, in case this is heavy-going.)

Now $\ddot{x}$ and x are related by Equation 13.8.4, so that we can write Equation 13.8.9 with no time derivatives of x higher than the first, and indeed it is not difficult, because Equation 13.8.4 is just $r^3\ddot{x} = -x$. We obtain

and

$$\ddddot{x} = x\left(\frac{1}{r^6} - \frac{12\dot{r}^2}{r^5} + \frac{3\ddot{r}}{r^4}\right) + \frac{6\dot{x}\dot{r}}{r^4}. \tag{13.8.10}$$

In a similar fashion, because of the relation 13.8.4, all higher time derivatives of x can be written with no derivatives of x higher than the first, and a similar argument holds for y.

Thus we can write Equation 13.8.7 as

$$x+\delta x=Fx+G\dot{x} \tag{13.8.11}$$

and similarly for y:

$$y+\delta y=Fy+G\dot{y}, \tag{13.8.12}$$

where

$$F=1-\frac{1}{2r^3}(\delta t)^2+\frac{\dot{r}}{2r^4}(\delta t)^3+\frac{1}{24}\left(\frac{1}{r^6}-\frac{12\dot{r}^2}{r^5}+\frac{3\ddot{r}}{r^4}\right)(\delta t)^4+\ldots \tag{13.8.13}$$

and

$$G=\delta t-\frac{1}{6r^3}(\delta t)^3+\frac{\dot{r}}{4r^4}(\delta t)^4+\ldots \tag{13.8.14}$$

Now we are going to look at the triangle and sector areas. From figure XIII.1 we can see that

$$\mathbf{A}_1=\frac{1}{2}\mathbf{r}_2\times\mathbf{r}_3,\ \mathbf{A}_2=\frac{1}{2}\mathbf{r}_1\times\mathbf{r}_3,\ \mathbf{A}_3=\frac{1}{2}\mathbf{r}_1\times\mathbf{r}_2. \tag{13.8.15a,b,c}$$

Also, angular momentum per unit mass is $\mathbf{r}\times\mathbf{v}$ and Kepler's second law tells us that areal speed is half the angular momentum per unit mass and that it is constant and equal to $\frac{1}{2}\sqrt{l}$ (in the units that we are using), so that

$$\dot{\mathbf{B}}_1=\frac{1}{2}\mathbf{r}_1\times\dot{\mathbf{r}}_1=\frac{1}{2}\mathbf{r}_2\times\dot{\mathbf{r}}_2=\frac{1}{2}\mathbf{r}_3\times\dot{\mathbf{r}}_3. \tag{13.8.16a,b,c}$$

All four of these vectors are parallel and perpendicular to the plane of the orbit, to that their magnitudes are just equal to their z-components. From the usual formulas for the components of a vector product we have, then,

$$A_1=\frac{1}{2}(x_2y_3-y_2x_3),\quad A_2=\frac{1}{2}(x_1y_3-y_1x_3),\quad A_3=\frac{1}{2}(x_1y_2-y_1x_2) \tag{13.8.17a,b,c}$$

and

$$\frac{1}{2}\sqrt{l}=\frac{1}{2}(x_1\dot{y}_1-y_1\dot{x}_1)=\frac{1}{2}(x_2\dot{y}_2-y_2\dot{x}_2)=\frac{1}{2}(x_3\dot{y}_3-y_3\dot{x}_3). \tag{13.8.18a,b,c}$$

Now, start from the second observation (x_2,y_2) at instant t_2. We shall try to predict the third observation, using Equations 13.8.11-14, in which $x+\delta x$ is x_3 and δt is t_3-t_2, which we are calling (see section 13.3) τ_1. I shall make the subscripts for F and G the same as the subscripts for τ. Thus the F and G that connect observations 2 and 3 will have subscript 1, just as we are using the notation τ_1 for t_3-t_2.

Thus we have

$$x_2=F_1x_2+G_1\dot{x}_2 \tag{13.8.19}$$

and

$$y_3=F_1y_2+G_1\dot{y}_2, \tag{13.8.20}$$

where

$$F_1=1-\frac{1}{2r_2^3}\tau_1^2+\frac{\dot{r}_2}{2r_2^4}\tau_1^3+\frac{1}{24}\left(\frac{1}{r_2^6}-\frac{12\dot{r}_2^2}{r_2^5}+\frac{3\ddot{r}_2}{r_2^4}\right)\tau_1^4+\ldots \tag{13.8.21}$$

and

$$G_1=\tau_1-\frac{1}{6r_2^3}\tau_1^3+\frac{\dot{r}_2}{4r_2^3}\tau_1^4+\ldots\quad . \tag{13.8.22}$$

Similarly, the first observation is given by

$$x_1=F_3x_2+G_3\dot{x}_2 \tag{13.8.23}$$

and

$$y_1 = F_3 y_2 + G_3 \dot{y}_2, \tag{13.8.24}$$

where, by substitution of $-\tau_3$ for δt,

$$F_3 = 1 - \frac{1}{2r_2^3}\tau_3^2 - \frac{\dot{r}_2}{2r_2^4}\tau_3^3 + \frac{1}{24}\left(\frac{1}{r_2^6} - \frac{12\dot{r}_2^2}{r_2^5} + \frac{3\ddot{r}_2}{r_2^4}\right)\tau_3^4 + \dots \tag{13.8.25}$$

and

$$G_3 = -\tau_3 + \frac{1}{6r_2^3}\tau_3^3 + \frac{\dot{r}_2}{4r_2^4}\tau_3^4 + \dots \quad . \tag{13.8.26}$$

From Equations 13.8.17,18,19,20,23,24, we soon find that

$$A_1 = \frac{1}{2}G_1\sqrt{l}, \quad A_2 = \frac{1}{2}(F_3G_1 - F_1G_3)\sqrt{l}, \quad A_3 = -\frac{1}{2}G_3\sqrt{l}. \tag{13.8.27a,b,c}$$

Now we do not yet know $\dot{r}$ or $\ddot{r}$, but we can take the expansions of F and G as far as τ^2. We then obtain, correct to τ^3 :

$$A_1 = \frac{1}{2}\sqrt{l}\tau_1\left(1 - \frac{\tau_1^2}{6_2^3}\right), \tag{13.8.28}$$

$$A_2 = \frac{1}{2}\sqrt{l}\tau_2\left(1 - \frac{\tau_1^2}{6r_2^3}\right), \tag{13,8.29}$$

and

$$A_3 = \frac{1}{2}\sqrt{l}\tau_3\left(1 - \frac{\tau_3^2}{6r_2^3}\right). \tag{13.8.30}$$

Thus the triangle ratio $a_1 = A_1/A_2$ is

$$a_1 = \frac{\tau_1}{\tau_2}\left(1 - \frac{\tau_1^2}{6r_2^3}\right)\left(1 - \frac{\tau_2^2}{6r_2^3}\right)^{-1}, \tag{13.8.31}$$

which, to order τ^3, is

$$a_1 = \frac{\tau_1}{\tau_2}\left(1 + \frac{(\tau_2^2 - \tau_1^2)}{6r_2^3}\right), \tag{13.8.32}$$

or, with $\tau_2 - \tau_1 = \tau_3$,

$$a_1 = \frac{\tau_1}{\tau_2}\left(1 + \frac{\tau_3(\tau_2 + \tau_1)}{6r_2^3}\right). \tag{13.8.33}$$

Similarly,

$$a_3 = \frac{\tau_3}{\tau_2}\left(1 + \frac{\tau_1(\tau_2 + \tau_3)}{6r_2^3}\right). \tag{13.8.34}$$

Further, with $\tau_1/\tau_2 = b_1$ and $\tau_3/\tau_2 = b_3$,

$$a_1 = b_1 + \frac{\tau_1\tau_3}{6r_2^3}(1 + b_1) \quad \text{and} \quad a_3 = b_3 + \frac{\tau_1\tau_3}{6r_2^3}(1 + b_3). \tag{13.8.35a,b}$$

These will serve as better approximations for the triangle ratios. Be aware, however, that Equations 13.8.35a,b are only approximations, and do not give the *exact* values for a_1 and a_3.