5.13: Pressure at the Centre of a Uniform Sphere

What is the pressure at the centre of a sphere of radius a and of uniform density ρ?

(Preliminary thought: Show by dimensional analysis that it must be something times $G\rho^2 a^2$.)

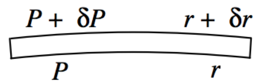

FIGURE V.27

Consider a portion of the sphere between radii r and $r+\delta r$ and cross-sectional area A. Its volume is $A\delta r$ and its mass is $\rho A\delta r$. (Were the density not uniform throughout the sphere, we would here have to write $\rho(r)A\delta r$.) Its weight is $\rho gA\delta r$, where $g=GM_r/r^2=\frac{4}{3}\pi G\rho r$. We suppose that the pressure at radius r is P and the pressure at radius $r+\delta r$ is $P+\delta P$. (δP is negative.) Equating the downward forces to the upward force, we have

$$A(P+\delta P)+\frac{4}{3}\pi AG\rho^2 r\delta r=AP. \tag{5.13.1}$$

That is:

$$\delta P=-\frac{4}{3}\pi G\rho^2 r\delta r. \tag{5.13.2}$$

Integrate from the centre to the surface:

$$\int_{P_0}^{0} dP=-\frac{4}{3}\pi G\rho^2\int_0^a rdr. \tag{5.13.3}$$

Thus:

$$P=\frac{2}{3}\pi G\rho^2 a^2. \tag{5.13.4}$$