5.8.6: Rods

Refer to figure V.5. The potential at P due to the element δx is $-\frac{G\lambda\delta x}{r} = -G\lambda\sec\theta\delta\theta$. The total potential at P is therefore

$$\psi = -G\lambda\int_\alpha^\beta \sec\theta d\theta = -G\lambda\ln\left[\frac{\sec\beta+\tan\beta}{\sec\alpha+\tan\alpha}\right]. \tag{5.8.15}$$

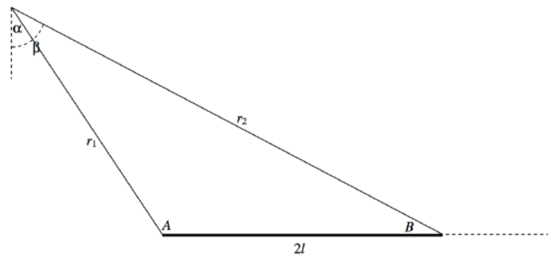

FIGURE V.24

Refer now to figure V.24, in which $A = 90^\circ + \alpha$ and $B = 90^\circ - \beta$.

$$\frac{\sec\beta+\tan\beta}{\sec\alpha+\tan\alpha} = \frac{\cos\alpha(1+\sin\beta)}{\cos\beta(1+\sin\alpha)} = \frac{\sin A(1+\cos B)}{\sin B(1-\cos A)} = \frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A.\,2\cos^2\frac{1}{2}B}{2\sin\frac{1}{2}B\cos\frac{1}{2}B.\,2\sin^2\frac{1}{2}A} = \cot\frac{1}{2}A\cot\frac{1}{2}B \tag{5.8.6.1}$$

$$= \sqrt{\frac{s(s-r_2)}{(s-r_1)(s-2l)}}\cdot\sqrt{\frac{s(s-r_1)}{(s-2l)(s-r_2)}},$$

where $s = \frac{1}{2}(r_1+r_2+2l)$. (You may want to refer here to the formulas on pp. 37 and 38 of Chapter 2.)

Hence

$$\psi = -G\lambda\ln\left[\frac{r_1+r_2+2l}{r_1+r_2-2l}\right]. \tag{5.8.16}$$

If r_1 and r_2 are very large compared with l, they are nearly equal, so let's put $r_1+r_2 = 2r$ and write Equation 5.8.17 as

$$\psi = -\frac{Gm}{2l}\ln\left[\frac{2r\left(1+\frac{2l}{2r}\right)}{2r\left(1-\frac{2l}{2r}\right)}\right] = -\frac{Gm}{2l}\left[\ln\left(1+\frac{l}{r}\right)-\ln\left(1-\frac{l}{r}\right)\right]. \tag{5.8.6.2}$$

Maclaurin expand the logarithms, and you will see that, at large distances from the rod, the potential is, expected, $-Gm/r$.

Let us return to the near vicinity of the rod and to Equation 5.8.16. We see that if we move around the rod in such a manner that we keep r_1+r_2 constant and equal to $2a$, say − that is to say if we move around the rod in an *ellipse* (see our definition of an ellipse in Chapter 2, Section 2.3) − the potential is constant. In other words the equipotentials are confocal ellipses, with the foci at the ends of the rod. Equation 5.8.16 can be written

$$\psi = -G\lambda\ln\left(\frac{a+l}{a-l}\right). \tag{5.8.17}$$

For a given potential ψ, the equipotential is an ellipse of major axis

$$2a = 2l\left(\frac{e^{\psi/(G\lambda)}+1}{e^{\psi/(G\lambda)}-1}\right), \tag{5.8.20}$$

where $2l$ is the length of the rod. This knowledge is useful if you are exploring space and you encounter an alien spacecraft or an asteroid in the form of a uniform rod of length $2l$.