5.10: Nabla, Gradient and Divergence

We are going to meet, in this section, the symbol ∇. In North America it is generally pronounced "del", although in the United Kingdom and elsewhere one sometimes hears the alternative pronunciation "nabla", called after an ancient Assyrian harp-like instrument of approximately that shape.

In section 5.7, particularly Equation 5.7.1, we introduced the idea that the gravitational field g is minus the gradient of the potential, and we wrote $g = -d\psi/dx$. This Equation refers to an essentially one-dimensional situation. In real life, the gravitational potential is a three dimensional scalar function $\psi(x,y,z)$, which varies from point to point, and its *gradient* is

$$\mathbf{grad}\psi = \mathbf{i}\frac{\partial\psi}{\partial x} + \mathbf{j}\frac{\partial\psi}{\partial y} + \mathbf{k}\frac{\partial\psi}{\partial x}, \tag{5.10.1}$$

which is a vector field whose magnitude and direction vary from point to point. The gravitational field, then, is given by

$$\mathbf{g} = -\mathbf{grad}\psi. \tag{5.10.2}$$

Here, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the x-, y- and z-directions.

The operator ∇ is $\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial x}$, so that Equation 5.10.2 can be written

$$\mathbf{g} = -\nabla\psi. \tag{5.10.3}$$

I suppose one could write a long book about ∇, but I am going to try to restrict myself in this section to some bare essentials.

Let us suppose that we have some vector field, which we might as well suppose to be a gravitational field, so I'll call it $\mathbf{g}$. (If you don't want to be restricted to a gravitational field, just call the field $\mathbf{A}$ as some sort of undefined or general vector field.) We can calculate the quantity

$$\nabla\cdot\mathbf{g} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial x} + \mathbf{k}\frac{\partial}{\partial x}\right)\cdot\left(\mathbf{i}g_x + \mathbf{j}g_y + \mathbf{k}g_z\right). \tag{5.10.4}$$

When this is multiplied out, we obtain a *scalar* field called the *divergence* of $\mathbf{g}$:

$$\nabla\cdot\mathbf{g} = \mathrm{div}\mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}. \tag{5.10.5}$$

Is this of any use?

Here's an example of a possible useful application. Let us imagine that we have some field $\mathbf{g}$ which varies in magnitude and direction through some volume of space. Each of the components, g_x, g_y, g_z can be written as functions of the coordinates. Now suppose that we want to calculate the surface integral of g through the closed boundary of the volume of space in question. Can you just imagine what a headache that might be? For example, suppose that $g = x^2\mathbf{i} - xy\mathbf{j} - xz\mathbf{k}$, and I were to ask you to calculate the surface integral over the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. It would be hard to know where to begin.

Well, there is a theorem, which I am not going to derive here, but which can be found in many books on mathematical physics, and is not particularly difficult, which says:

The surface integral of a vector field over a closed surface is equal to the volume integral of its divergence.

In symbols:

$$\int\int \mathbf{g}\cdot d\mathbf{A} = \int\int\int div\mathbf{g}dV. \tag{5.10.6}$$

If we know g_x, g_y and g_z as functions of the coordinates, then it is often very simple and straightforward to calculate the divergence of $\mathbf{g}$, which is a scalar function, and it is then often equally straightforward to calculate the volume integral. The example I gave in the previous paragraph is trivially simple (it is a rather artificial example, designed to be ridiculously simple) and you will readily find that $\mathbf{div}\ \mathbf{g}$ is everywhere zero, and so the surface integral over the ellipsoid is zero.

If we combine this very general theorem with Gauss's theorem (which applies to an inverse square field), which is that the surface integral of the field over a closed volume is equal to $-4\pi G$ times the enclosed mass (Equation 5.5.1) we understand immediately that the divergence of $\mathbf{g}$ at any point is related to the density at that point and indeed that

$$\text{div}\ \mathbf{g} = \nabla \cdot \mathbf{g} = -4\pi G\rho. \tag{5.10.7}$$

This may help to give a bit more physical meaning to the divergence. At a point in space where the local density is zero, div $\mathbf{g}$, of course, is also zero.

Now Equation 5.10.2 tells us that $\mathbf{g} = -\nabla\psi$, so that we also have

$$\nabla \cdot (-\nabla\psi) = -\nabla \cdot (\nabla\psi) = -4\pi G\rho. \tag{5.10.8}$$

If you write out the expressions for ∇ and for $\nabla\psi$ in full and calculate the dot product, you will find that $\nabla \cdot (\nabla\psi)$, which is also written $\nabla^2\psi$, is $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{dy^2} + \frac{\partial^2\psi}{\partial z^2}$. Thus we obtain

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 4\pi G\rho. \tag{5.10.9}$$

This is *Poisson's Equation*. At any point in space where the local density is zero, it becomes

$$\nabla^2\psi = 0 \tag{5.10.10}$$

which is *Laplace's Equation*. Thus, no matter how complicated the distribution of mass, the potential as a function of the coordinates must satisfy these Equations.

We leave this topic here. Further details are to be found in books on mathematical physics; our aim here was just to obtain some feeling for the physical meaning. I add just a few small comments. One is, yes, it is certainly possible to operate on a vector field with the operator $\nabla\times$. Thus, if $\mathbf{A}$ is a vector field, $\nabla \times \mathbf{A}$ is called the $\mathbf{curl}$ of $\mathbf{A}$. The $\mathbf{curl}$ of a gravitational field is zero, and so there is no need for much discussion of it in a chapter on gravitational fields. If, however, you have occasion to study fluid dynamics or electromagnetism, you will need to become very familiar with it. I particularly draw your attention to a theorem that says

*The line integral of a vector field around a closed plane circuit is equal to the surface integral of its **curl**.*

This will enable you easily to calculate two-dimensional line integrals in a similar manner to that in which the divergence theorem enables you to calculate threedimensional surface integrals.

Another comment is that very often calculations are done in spherical rather than rectangular coordinates. The formulas for $\mathbf{grad}$, div, $\mathbf{curl}$ and ∇^2 are then rather more complicated than their simple forms in rectangular coordinates.

Finally, there are dozens and dozens of formulas relating to nabla in the books, such as "$\mathbf{curl\,curl} = \mathbf{grad}$ div minus nabla-squared". While they should certainly never be memorized, they are certainly worth becoming familiar with, even if we do not need them immediately here.
