13.12: Sector-Triangle Ratio

We recall that it is easy to determine the ratio of adjacent sectors swept out by the radius vector. By Kepler's second law, it is just the ratio of the two time intervals. What we really need, however, are the triangle ratios, which are related to the heliocentric distance by Equation 13.2.1. Oh, wouldn't it just be so nice if someone were to tell us the ratio of a sector area to the corresponding triangle area! We shall try in this section to do just that.

$$\textit{Notation}: \text{ Triangle ratios}: \quad a_1 = A_1/A_2, \quad a_3 = A_3/A_2. \tag{13.12.1a,b}$$

$$\text{Sector ratios}: \quad b_1 = B_1/B_2, \quad b_3 = B_3/B_2. \tag{13.12.2a,b}$$

$$\text{Sector-triangle ratios}: \quad R_1 = \frac{B_1}{A_1}, \quad R_2\frac{B_2}{A_2}, \quad R_3 = \frac{B_3}{A_3}, \tag{13.12.3a,b,c}$$

from which it follows that

$$a_1 = \frac{R_2}{R_1}b_1, \quad a_3 = \frac{R_2}{R_3}b_3. \tag{13.12.4a,b}$$

We also recall that subscript 1 for areas refers to observations 2 and 3; subscript 2 to observations 3 and 1; and subscript 3 to observations 1 and 2. Let us see, then, whether we can determine R_3 from the first and second observations.

Readers who wish to avoid the heavy algebra may proceed direct to Equations 13.12.25 and 13.12.26, which will enable the calculation of the sector-triangle ratios.

Let (r_1, v_1) and (r_2, v_2) be the polar coordinates (i.e. heliocentric distance and true anomaly) in the plane of the orbit of the planet at the instant of the first two observations. In concert with our convention for subscripts involving two observations, let

$$2f_3 = v_2 - v_1. \tag{13.12.5}$$

We have $R_3 = B_3/A_3$. From Equation 13.4.1, which is Kepler's second law, we have, in the units that we are using, in which $GM = 1$, $\dot{B} = \frac{1}{2}\sqrt{l}$ and therefore $B_3 = \frac{1}{2}\sqrt{l}\tau_3$. Also, from the z-component of Equation 13.8.15c, we have $A_3 = \frac{1}{2}r_1 r_2 \sin(v_2 - v_1)$.

Therefore

$$R_3 = \frac{\sqrt{l}\tau_3}{r_1 r_2 \sin(v_2 - v_1)} = \frac{\sqrt{l}\tau_3}{r_1 r_2 \sin 2f_3}. \tag{13.12.6a}$$

In a similar manner, we have

$$R_1 = \frac{\sqrt{l}\tau_1}{r_2 r_3 \sin(v_3 - v_2)} = \frac{\sqrt{l}\tau_1}{r_2 r_3 \sin 2f_1} \tag{13.12.6b}$$

$$R_2 = \frac{\sqrt{l}\tau_2}{r_3 r_1 \sin(v_3 - v_1)} = \frac{\sqrt{l}\tau_2}{r_3 r_1 \sin 2f_2}. \tag{13.12.6c}$$

I would like to eliminate l from here.

I now want to recall some geometrical properties of an ellipse and a property of an elliptic orbit. By glancing at figure II.11, or by multiplying Equations 2.3.15 and 2.3.16, we immediately see that $r\cos v = a(\cos E - e)$, and hence by making use of a trigonometric identity we find

$$r\cos^2\frac{1}{2}v = a(1-e)\cos^2\frac{1}{2}E, \tag{13.12.7}$$

and in a similar manner it is easy to show that

$$r\sin^2\frac{1}{2}v = a(1+e)\sin^2\frac{1}{2}E. \tag{13.12.8}$$

Here E is the eccentric anomaly.

Also, the mean anomaly at time t is defined as $\frac{2\pi}{P}(t-T)$ and is also equal (via Kepler's Equation) to $E-e\sin E$. The period of the orbit is related to the semi major axis of its orbit by Kepler's third law: $P^2=\frac{4\pi^2}{GM}a^3$. (This material is covered on Chapter 10.) Hence we have (in the units that we are using, in which $GM=1$):

$$E-e\sin E=\frac{t-T}{a^{3/2}}, \tag{13.12.9}$$

where T is the instant of perihelion passage.

Now introduce

$$2f_3=v_2-v_1, \tag{13.12.10}$$

$$2F_3=v_2+v_1, \tag{13.12.11}$$

$$2g_3=E_2-E_1, \tag{13.12.12}$$

$$2G_3=E_2+E_1. \tag{13.12.13}$$

From Equation 13.12.7 I can write

$$\sqrt{r_1r_2}\cos\frac{1}{2}v_1\cos\frac{1}{2}v_2=a(1-e)\cos\frac{1}{2}E_1\cos\frac{1}{2}E_2 \tag{13.12.14}$$

and from Equation 13.12.8 I can write

$$\sqrt{r_1r_2}\sin\frac{1}{2}v_1\sin\frac{1}{2}v_2=a(1+e)\sin\frac{1}{2}E_1\sin\frac{1}{2}E_2. \tag{13.12.15}$$

I now make use of the sum of the sum-and-difference formulas from page 38 of chapter 3, namely $\cos A\cos B=\frac{1}{2}(\cos S+\cos D)$ and $\sin A\sin B=\frac{1}{2}(\cos D-\cos S)$, to obtain

$$\sqrt{r_1r_2}(\cos F_3+\cos f_3)=a(1-e)(\cos G_3+\cos g_3) \tag{13.12.16}$$

and

$$\sqrt{r_1r_2}(\cos f_3-\cos F_3)=a(1+e)(\cos g_3-\cos G_3). \tag{13.12.17}$$

On adding these, we obtain

$$\sqrt{r_1r_2}\cos f_3=a(\cos g_3-e\cos G_3). \tag{13.12.18}$$

I leave it to the reader to derive in a similar manner (also making use of the formula for the semi latus rectum $l=a(1-e^2)$)

$$\sqrt{r_1r_2}\sin f_3=\sqrt{a}\sqrt{l}\sin g \tag{13.12.19}$$

and

$$r_1+r_2=2a(1-e\cos g_3\cos G_3). \tag{13.12.20}$$

We can eliminate $e\cos G$ from Equations 13.12.18 and 13.12.20:

$$r_1+r_2-2\sqrt{r_1r_2}\cos f_3\cos g_3=2a\sin^2 g_3 \tag{13.12.21}$$

Also, if we write Equation 13.12.9 for the first and second observations and take the difference, and then use the formula on page 35 of chapter 3 for the difference between two sines, we obtain

$$2(g_3-e\sin g_3\cos G_3)=\frac{\tau_3}{a^{3/2}}. \tag{13.12.22}$$

Eliminate $e\cos G_3$ from Equations 13.12.18 and 13.12.22:

$$2g_3-\sin 2g_3+\frac{2\sqrt{r_1r_2}}{a}\sin g_3\cos f_3=\frac{\tau_3}{a^{3/2}}. \tag{13.12.23}$$

Also, eliminate l from Equations 13.12.6 and 13.12.19:

$$R_3 = \frac{\tau_3}{2\sqrt{a}\sqrt{r_1 r_2}\cos f_3 \sin g_3}. \tag{13.12.24}$$

We have now eliminated F_3, G_3 and e, and we are left with Equations 13.12.21, 23 and 24, the first two of which I now repeat for easy reference:

$$r_1 + r_2 - 2\sqrt{r_1 r_2}\cos f_3 \cos g_3 = 2a\sin^2 g_3 \tag{13.12.21}$$

$$2g_3 - \sin 2g_3 + \frac{2\sqrt{r_1 r_2}}{a}\sin g_3 \cos f_3 = \frac{\tau_3}{a^{3/2}}. \tag{13.12.23}$$

In these Equations we already know an approximate value for f_3 (we'll see how when we resume our numerical example); the unknowns in these Equations are R_3, a and g_3, and it is R_3 that we are trying to find. Therefore we need to eliminate a and g_3. We can easily obtain a from Equation 13.12.24, and, on substitution in Equations 13.12.21 and 23 we obtain, after some algebra:

$$R_3^2 = \frac{M_3^2}{N_3 - \cos g_3} \tag{13.12.25}$$

and

$$R_3^3 - R_3^2 = \frac{M_3^2 (g_3 - \sin g_3 \cos g_3)}{\sin^3 g_3}, \tag{13.12.26}$$

where

$$M_3 = \frac{\tau_3}{2\left(\sqrt{r_1 r_2}\cos f_3\right)^{3/2}} \tag{13.12.27}$$

and

$$N_3 = \frac{r_1 + r_2}{2\sqrt{r_1 r_2}\cos f_3}. \tag{13.12.28}$$

Similar Equations for R_1 and R_2 can be obtained by cyclic permutation of the subscripts. Equations 13.12.25 and 26 are two simultaneous Equations in R_3 and g_3. Their solution is given as an example in section 1.9 of chapter 1, so we can now assume that we can calculate the sector-triangle ratios.

We can then calculate better triangle ratios from Equations 13.12.4 and return to Equations 13.7.4, 5 and 6 to get better geocentric distances. From Equations 13.7.8 and 9 calculate the heliocentric distances. Make the light-time corrections. (I am not doing this in our numerical example because our original positions were not actual observations, but rather were ephemeris positions.) Then go straight to this section (13.12) again, until you get to here again. Repeat until the geocentric distances do not change.