5.12: Gravitational Potential of any Massive Body

You might just want to look at **Chapter 2** of Classical Mechanics (Moments of Inertia) before proceeding further with this chapter.

In figure VIII.26 I draw a massive body whose centre of mass is C, and an external point P at a distance R from C. I draw a set of $\text{C}xyz$ axes, such that P is on the z-axis, the coordinates of P being $(0, 0, z)$. I indicate an element δm of mass, distant r from C and l from P. I'll suppose that the density at δm is ρ and the volume of the mass element is $\delta\tau$, so that $\delta m = \rho\delta\tau$.

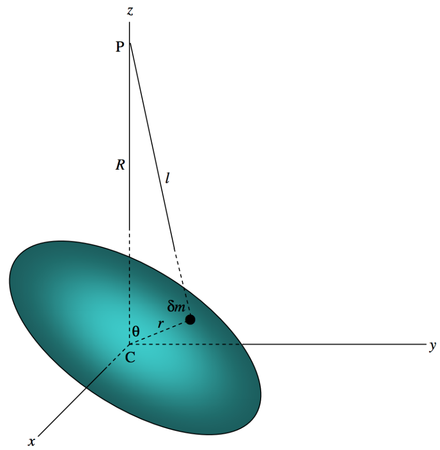

FIGURE V.26

The potential at P is

$$\psi = -G\int\frac{dm}{l} = -G\int\frac{\rho d\tau}{l}. \tag{5.12.1}$$

But $l^2 = R^2 + r^2 - 2Rr\cos 2\theta$,

so

$$\psi = -G\left[\frac{1}{R}\int\rho d\tau + \frac{1}{R^2}\int\rho r\cos\theta d\tau + \frac{1}{R^3}\int\rho r^2 P_2(\cos\theta)d\tau + \frac{1}{R^4}\int\rho r^3 P_3(\cos\theta)d\tau\ldots\right]. \tag{5.12.2}$$

The integral is to be taken over the entire body, so that $\int\rho d\tau = M$, where M is the mass of the body. Also $\int\rho r\cos\theta d\tau = \int zdm$, which is zero, since C is the centre of mass. The third term is

$$\frac{1}{2R^3}\int\rho r^2(3\cos^2\theta - 1)d\tau = \frac{1}{2R^3}\int\rho r^2(2 - 3\sin^2\theta)d\tau. \tag{5.12.3}$$

Now

$$\int 2\rho r^2 d\tau = \int 2r^2 dm = \int\left[(y^2+z^2)+(z^2+x^2)+(x^2+y^2)\right]dm = A+B+C \tag{5.12.1}$$

where A, B and C are the second moments of inertia with respect to the axes $\text{C}x$, $\text{C}y$, $\text{C}z$ respectively. But $A+B+C$ is invariant with respect to rotation of axes, so it is also equal to $A_0 + B_0 + C_0$, where A_0, B_0, C_0 are the *principal moments of inertia*.

Lastly, $\int\rho r^2\sin^2\theta d\tau$ is equal to C, the moment of inertia with respect to the axis $\text{C}z$.

Thus, if R is sufficiently larger than r so that we can neglect terms of order $(r/R)^3$ and higher, we obtain

$$\psi = -\frac{GM(2MR^2 + A_0 + B_0 + C_0 - 3C)}{2R^3}. \tag{5.12.4}$$

In the special case of an *oblate symmetric top*, in which $A_0 = B_0 < C_0$, and the line CP makes an angle γ with the principal axis, we have

$$C = A_0 + (C_0 - A_0)\cos^2\gamma = A_0 + (C_0 - A_0)Z^2/R^2, \tag{5.12.5}$$

so that

$$\psi = -\frac{G}{R}\left[M + \frac{C_0 - A_0}{2R^2}\left(1 - \frac{3Z^2}{R^2}\right)\right]. \tag{5.12.6}$$

Now consider a uniform oblate spheroid of polar and equatorial diameters $2c$ and $2a$ respectively. It is easy to show that

$$C_0 = \frac{2}{5}Ma^2. \tag{5.12.7}$$

? Exercise 5.12.1

Confirm Equation 5.12.7.

It is slightly less easy to show (*Exercise*: Show it.) that

$$A_0 = \frac{1}{5}M\left(a^2 + c^2\right). \tag{5.12.8}$$

For a symmetric top, the integrals of the odd polynomials of Equation 5.12.2 are zero, and the potential is generally written in the form

$$\psi = -\frac{GM}{R}\left[1 + \left(\frac{a}{R}\right)^2 J_2 P_2(\cos\gamma) + \left(\frac{a}{R}\right) J_4 P_4(\cos\gamma)\ldots\right] \tag{5.12.9}$$

Here γ is the angle between CP and the principal axis. For a uniform oblate spheroid, $J_2 = \frac{C_0 - A_0}{Mc^2}$. This result will be useful in a later chapter when we discuss precession.