1.14: Legendre Polynomials

Consider the expression

$$(1-2rx+r^2)^{-1/2}, \tag{1.14.1}$$

in which $|x|$ and $|r|$ are both less than or equal to one. Expressions similar to this occur quite often in theoretical physics - for example in calculating the gravitational or electrostatic potentials of bodies of arbitrary shape. See, for example, Chapter 5, Sections 5.11 and 5.12.

Expand the expression 1.14.1 by the binomial theorem as a series of powers of r. This is straightforward, though not particularly easy, and you might expect to spend several minutes in obtaining the coefficients of the first few powers of r. You will find that the coefficient of r^l is a polynomial expression in x of degree l. Indeed the expansion takes the form

$$(1-2rx+r^2)^{-1/2} = P_0(x) + P_1(x)r + P_2(x)r^2 + P_3(x)r^3 \ldots \tag{1.14.2}$$

The coefficients of the successive power of r are the *Legendre polynomials*; the coefficient of r^l, which is $P_l(x)$, is the Legendre polynomial of order l, and it is a polynomial in x including terms as high as x^l. We introduce these polynomials in this section because we shall need them in Section 1.15 on the derivation of Gaussian Quadrature.

If you have conscientiously tried to expand expression 1.14.1, you will have found that

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \tag{1.14.3}$$

though you probably gave up with exhaustion after that and didn't take it any further. If you look carefully at how you derived the first few polynomials, you may have discovered for yourself that you can obtain the next polynomial as a function of two earlier polynomials. You may even have discovered for yourself the following *recursion relation*:

$$P_{l+1} = \frac{(2l+1)xP_l - lP_{l-1}}{l+1}. \tag{1.14.4}$$

This enables us very rapidly to obtain higher order Legendre polynomials, whether numerically or in algebraic form. For example, put $l=1$ and show that Equation 1.12.4 results in $P_2 = \frac{1}{2}(3x^2-1)$. You will then want to calculate P_3, and then P_4, and more and more and more. Another way to generate them is form the Equation

$$P_{l+1} = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l. \tag{1.14.5}$$

Here are the first eleven Legendre polynomials:

$$\begin{aligned}
&P_0 = 1\\
&P_1 = x\\
&P_2 = \tfrac{1}{2}(3x^2-1)\\
&P_3 = \tfrac{1}{2}(5x^3-3x)\\
&P_4 = \tfrac{1}{8}(35x^4-30x^2+3)\\
&P_5 = \tfrac{1}{16}(63x^5-70x^3+15x)\\
&P_6 = \tfrac{1}{16}(231x^6-315x^4+105x^2-5)\\
&P_7 = \tfrac{1}{16}(429x^7-693x^5+315x^3-35x)\\
&P_8 = \tfrac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35\\
&P_9 = \tfrac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)\\
&P_{10} = \tfrac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)
\end{aligned} \tag{1.14.6}$$

The polynomials with argument $\cos\theta$ are given in Section 5.11 of Chapter 5.

In what follows in the next section, we shall also want to know the roots of the Equations $P_l = 0$ for $l > 1$. Inspection of the forms of these polynomials will quickly show that all odd polynomials have a root of zero, and all nonzero roots occur in positive/negative pairs. Having read Sections 1.4 and 1.5, we shall have no difficulty in finding the roots of these Equations. The roots $x_{l,i}$ are in the following table, which also lists certain coefficients $c_{l,i}$, that will be explained in Section 1.15.

Roots of $P_l = 0$

l	$x_{l,i}$	$c_{l,i}$
2	±0.577 350 269 190	1.000 000 000 00
3	±0.774 596 669 241	0.555 555 555 56
	0.000 000 000 000	0.888 888 888 89
4	±0.861 136 311 594	0.347 854 845 14
	±0.339 981 043 585	0.652 145 154 86
5	±0.906 179 845 939	0.236 926 885 06
	±0.538 469 310 106	0.478 628 670 50
	0.000 000 000 000	0.568 888 888 89
6	±0.932 469 514 203	0.171 324 492 38
	±0.661 209 386 466	0.360 761 573 05
	±0.238 619 186 083	0.467 913 934 57
7	±0.949 107 912 343	0.129 484 966 17
	±0.741 531 185 599	0.279 705 391 49
	±0.405 845 151 377	0.381 830 050 50
	0.000 000 000 000	0.417 959 183 68
8	±0.960 289 856 498	0.101 228 536 29
	±0.796 666 477 414	0.222 381 034 45
	±0.525 532 409 916	0.313 706 645 88
	±0.183 434 642 496	0.362 683 783 38
9	±0.968 160 239 508	0.081 274 388 36
	±0.836 031 107 327	0.180 648 160 69
	±0.613 371 432 701	0.260 610 696 40
	±0.324 253 423 404	0.312 347 077 04
	0.000 000 000 000	0.330 239 355 00
10	±0.973 906 528 517	0.066 671 343 99
	±0.865 063 366 689	0.149 451 349 64
	±0.679 409 568 299	0.219 086 362 24
	±0.433 395 394 129	0.269 266 719 47
	±0.148 874 338 982	0.295 524 224 66
11	±0.978 228 658 146	0.055 668 567 12
	±0.887 062 599 768	0.125 580 369 46
	±0.730 152 005 574	0.186 290 210 93
	±0.519 096 129 207	0.233 193 764 59
	±0.269 543 155 952	0.262 804 544 51

	0.000 000 000 000	0.272 925 086 78
12	±0.981 560 634 247	0.047 175 336 39
	±0.904 117 256 370	0.106 939 325 99
	±0.769 902 674 194	0.160 078 328 54
	±0.587 317 954 287	0.203 167 426 72
	±0.367 831 498 998	0.233 492 536 54
	±0.125 233 408 511	0.249 147 045 81
13	±0.984 183 054 719	0.040 484 004 77
	±0.917 598 399 223	0.092 121 499 84
	±0.801 578 090 733	0.138 873 510 22
	±0.642 349 339 440	0.178 145 980 76
	±0.448 492 751 036	0.207 816 047 54
	±0.230 458 315 955	0.226 283 180 26
	0.000 000 000 000	0.232 551 553 23
14	±0.986 283 808 697	0.035 119 460 33
	±0.928 434 883 664	0.080 158 087 16
	±0.827 201 315 070	0.121 518 570 69
	±0.687 292 904 812	0.157 203 167 16
	±0.515 248 636 358	0.185 538 397 48
	±0.319 112 368 928	0.205 198 463 72
	±0.108 054 948 707	0.215 263 853 46
15	±0.987 992 518 020	0.030 753 242 00
	±0.937 273 392 401	0.070 366 047 49
	±0.848 206 583 410	0.107 159 220 47
	±0.724 417 731 360	0.139 570 677 93
	±0.570 972 172 609	0.166 269 205 82
	±0.394 151 347 078	0.186 161 000 02
	±0.201 194 093 997	0.198 431 485 33
	0.000 000 000 000	0.202 578 241 92
16	±0.989 400 934 992	0.027 152 459 41
	±0.944 575 023 073	0.062 253 523 94
	±0.865 631 202 388	0.095 158 511 68
	±0.755 404 408 355	0.124 628 971 26
	±0.617 876 244 403	0.149 595 988 82
	±0.458 016 777 657	0.169 156 519 39
	±0.281 603 550 779	0.182 603 415 04
	±0.095 012 509 838	0.189 450 610 46
17	±0.990 575 475 315	0.024 148 302 87
	±0.950 675 521 769	0.055 459 529 38
	±0.880 239 153 727	0.085 036 148 32
	±0.781 514 003 897	0.111 883 847 19
	±0.657 671 159 217	0.135 136 368 47

(1.14.1)

$\pm 0.512\,690\,537\,086$	$0.154\,045\,761\,08$
$\pm 0.351\,231\,763\,454$	$0.168\,004\,102\,16$
$\pm 0.178\,484\,181\,496$	$0.176\,562\,705\,37$
$0.000\,000\,000\,000$	$0.179\,446\,470\,35$

For interest, I draw graphs of the Legendre polynomials in figures I.7 and I.8.

Figure I.7. Legendre polynomials for even I

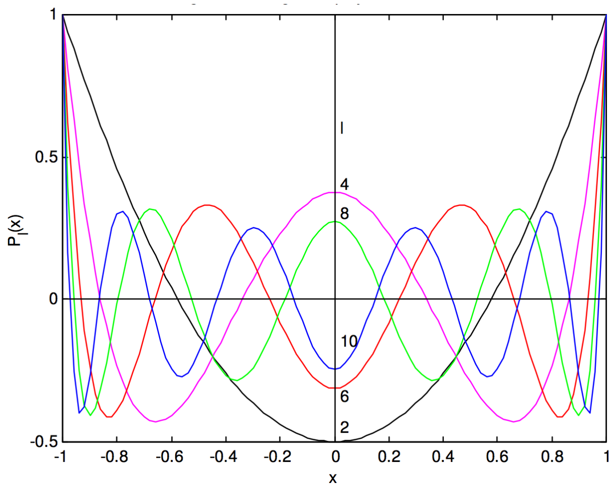
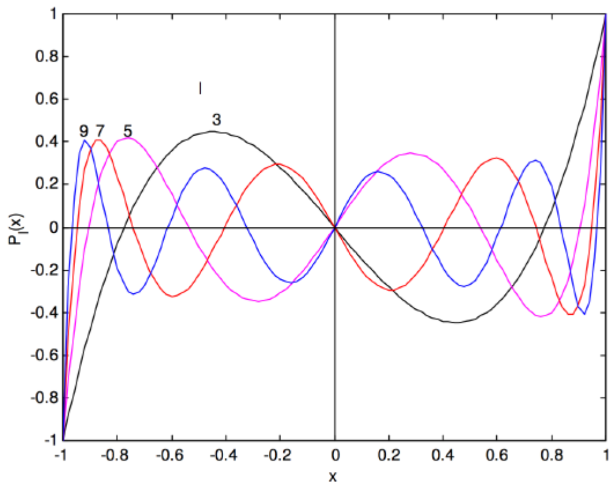


Figure I.8. Legendre polynomials for odd I



For further interest, it should be easy to verify, by substitution, that the Legendre polynomials are solutions of the differential Equation

$$(1-x^2)y''-2xy'+l(l+1)y=0. \tag{1.14.7}$$

The Legendre polynomials are solutions of this and related Equations that appear in the study of the vibrations of a solid sphere (spherical harmonics) and in the solution of the Schrödinger Equation for hydrogen-like atoms, and they play a large role in quantum mechanics.