# 5.6: Calculating Surface Integrals

While the concept of a surface integral sounds easy enough, how do we actually calculate one in practice? In this section I do two examples.

**Example 5.6.1**

In Figure V.19 I show a small mass $m$, and I have surrounded it with a cylinder of radius $a$ and height $2h$. The problem is to calculate the surface integral $\int \mathbf{g} \cdot d\mathbf{A}$ through the entire surface of the cylinder. Of course we already know, from Gauss's theorem, that the answer is $= -4\pi Gm$, but we would like to see a surface integral actually carried out.

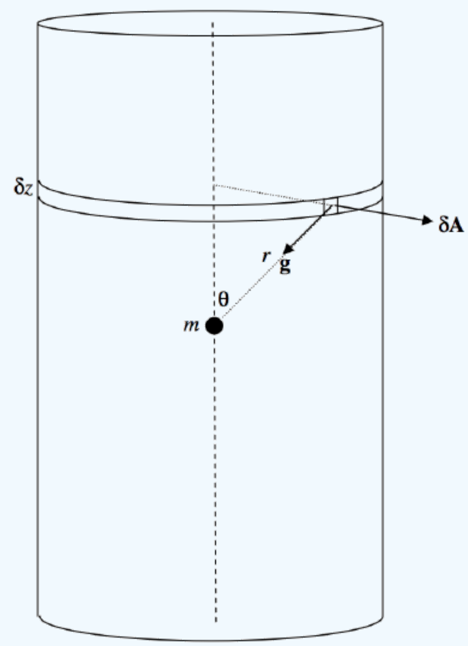


FIGURE V.19

I have drawn a small element of the surface. Its area $\delta A$ is $dz$ times $a\delta\phi$, where $\phi$ is the usual azimuthal angle of cylindrical coordinates. That is, $\delta A = a\,\delta z\,\delta\phi$ . The magnitude $g$ of the field there is $Gm/r^2$, and the angle between $\mathbf{g}$ and $d\mathbf{A}$ is $90^\circ + \theta$ . The outward flux through the small element is

$$\mathbf{g} \cdot \delta\mathbf{A} = \frac{Gma\cos(\theta + 90^\circ)\delta z\delta\phi}{r^2}. \tag{5.6.1}$$

(This is negative – i.e. it is actually an inward flux – because $\cos(\theta + 90^\circ) = -\sin\theta$ .) When integrated around the elemental strip $\delta z$, this is $-\frac{2\pi Gma\sin\theta\delta z}{r^2}$. To find the flux over the total curved surface, let's integrate this from $z = 0$ to $h$ and double it, or, easier, from $\theta = \pi/2$ to $\alpha$ and double it, where $\tan\alpha = a/h$. We'll need to express $z$ and $r$ in terms of $\theta$ (that's easy:- $z = a\cot\theta$ and $r = a\csc\theta$ ),and the integral becomes

$$4\pi Gm \int_{\pi/2}^{\alpha} \sin\theta\, d\theta = -4\pi Gm\cos\alpha \tag{5.6.1}$$

Let us now find the flux through one of the flat ends of the cylinder.

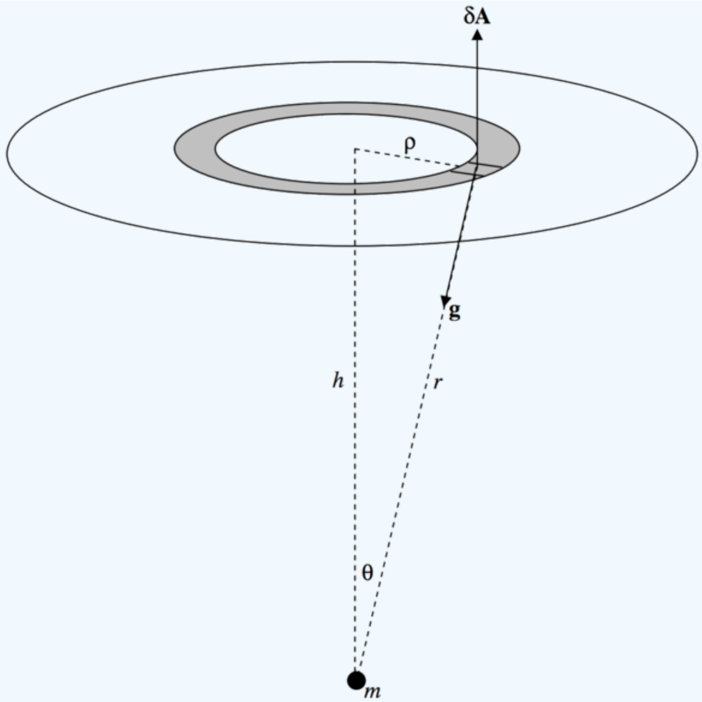


FIGURE V.20

This time, $\delta A = \rho\,\delta\rho\,\delta\phi$, $g = Gm/r^2$ and the angle between $\mathbf{g}$ and $\delta\mathbf{A}$ is $180^\circ - \theta$. The outwards flux through the small element is $\frac{Gm\rho\cos(180^\circ-\theta)\delta\rho\delta\phi}{r^2}$ and when integrated around the annulus this becomes $-\frac{2\pi Gm\cos\theta\,\rho\delta\rho}{r^2}$. We now have to integrate this from $\rho = 0$ to $a$, or, better, from $\theta = 0$ to $\alpha$. We have $r = h\sec\theta$ and $\rho = h\tan\theta$, and the integral becomes

$$-2\pi Gm\int_0^\alpha \sin\theta d\theta = -2\pi Gm(1-\cos\alpha). \tag{5.6.2}$$

There are two ends, so the total flux through the entire cylinder is twice this plus Equation 5.6.1 to give

$$\phi = -4\pi Gm, \tag{5.6.3}$$

as expected from Gauss's theorem.

Example 5.6.2

In figure V.21 I have drawn (part of) an infinite rod whose mass per unit length is $\lambda$. I have drawn around it a sphere of radius $a$. The problem will be to determine the total normal flux through the sphere. From Gauss's theorem, we know that the answer must be $-8\pi G\alpha\lambda$.

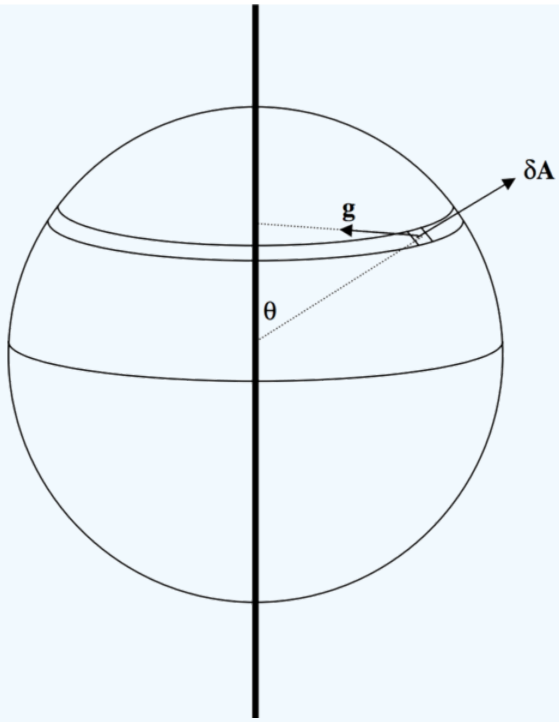


FIGURE V.21

The vector $\delta\mathbf{A}$ representing the element of area is directed away from the centre of the sphere, and the vector $\mathbf{g}$ is directed towards the nearest point of the rod. The angle between them is $\theta+90^{\circ}$ . The magnitude of $\delta\mathbf{A}$ in spherical coordinates is $a^2\sin\theta\delta\theta\delta\phi$ , and the magnitude of $\mathbf{g}$ is (see Equation 5.4.15) $\frac{2G\lambda}{a\sin\theta}$. The dot product $\mathbf{g}\cdot\delta\mathbf{A}$ is

$$\frac{2G\lambda}{a\sin\theta}\cdot a^2\sin\theta\delta\theta\delta\phi\cdot\cos(\theta+90^{\circ})=-2G\lambda a\sin\theta\delta\theta\delta\phi. \tag{5.6.4}$$

To find the total flux, this must be integrated from $\phi=0$ to $2\pi$ and from $\theta=0$ to $\pi$. The result, as expected, is $-8\pi G\alpha\lambda$.