14.5: Motion Around an Oblate Symmetric Top

In Section 5.12, we developed an expression (Equation 5.12.6) for the gravitational potential of an oblate symmetric top (e.g. an oblate spheroid). With a slight change of notation to conform to the present context, we obtain for the perturbing function

$$R=\frac{Gm(C-A)}{2r^3}\left(1-\frac{3z^2}{r^2}\right). \tag{14.5.1}$$

This is the negative of the additional potential *energy* of a mass m at a point whose cylindrical coordinates are $(r,\ z)$ in the vicinity of a symmetric top (which I'll henceforth call an oblate spheroid) whose principal second moments of inertia are C (polar) and A (equatorial). This is correct to order r/a, where a is the equatorial radius of the spheroid.

Let us imagine a particle of mass m in orbit around an oblate spheroid – e.g. an artificial satellite in orbit around Earth. Suppose the orbit is inclined at an angle i to the equator, and the argument of perigee is ω. At some instant, when the cylindrical coordinates of the satellite are $(r,\ z)$, its true anomaly is v.

Exercise 14.5.1: Geometry

Show that $z/r=\sin\ i\ \sin(\omega+v)$.

Having done that, we see that the perturbing function can be written

$$R=\frac{Gm(C-A)}{2r^3}\left(1-3\sin^2 i\sin^2(\omega+v)\right). \tag{14.5.2}$$

Here, r and v vary with time, or what amounts to the same thing, with the mean anomaly $\mathbf{M}$. With a (nontrivial) effort, this can be expanded as a series, including a constant (time independent) term plus periodic terms of the form $\cos\mathbf{M}$, $\cos 2\mathbf{M}$, $\cos 3\mathbf{M}$, etc. If the spirit moves me, I may post the details at a later date, but for the present I give the result that, if the expansion is taken as far as e^2 (i.e. we are assuming that the orbit of the satellite is not strongly eccentric), the constant (time-independent) part of the perturbing function is

$$R=\frac{Gm(C-A)}{2a^3}(1+\frac{3}{2}e^2)(1-\frac{3}{2}\sin^2 i). \tag{14.5.3}$$

Now look at Lagrange's Equations, and you see that the secular parts of $\dot{a}, \dot{e}$ and $\dot{i}$ are all zero. That is, although there may be periodic variations (which we have not examined) in these elements, to this order of approximation (e^2) there is no secular change in these elements.

On the other hand, application of Equation 14.4.5 gives for the secular rate of change of the longitude of the nodes

$$\dot{\Omega}=-\frac{3\sqrt{GM}}{2}\frac{(C-A)}{M}\frac{1}{a^{7/2}}(1+2e^2)\cos i. \tag{14.5.4}$$

The reader will no doubt be relieved to note that this expression does not contain m, the mass of the orbiting satellite; M is the mass of the Earth. The reader may also note the minus sign, indicating that the nodes regress. To obtain the factor (1 + 2e^2), readers will have to do a little bit of work, and to expand, by the binomial theorem, whatever expression in e they get, as far as e^2.

Let a be the equatorial radius of Earth. Multiply top and bottom of Equation 14.5.4 by $\mathrm{a}^{7/2}$, and the Equation becomes

$$\dot{\Omega}=-\frac{3}{2}\sqrt{\frac{GM}{\mathrm{a}^3}}\frac{(C-A)}{M\mathrm{a}^2}\left(\frac{\mathrm{a}}{a}\right)^{7/2}(1+2e^2)\cos i. \tag{14.5.5}$$

Here M is the mass of Earth (not of the orbiting satellite), a is the semi major axis of the satellite's orbit, and a is the equatorial radius of Earth.

[If we assume Earth is an oblate spheroid of uniform density, then, according to example 1.iii of Section 2.20 of Chapter 2 of our notes on Classical Mechanics, . $C=\frac{2}{5}M\mathrm{a}^2$. In that case, Equation 14.5.5 becomes $\dot{\Omega}=-\frac{3}{5}\sqrt{\frac{GM}{\mathrm{a}^3}}\frac{(C-A)}{C}\left(\frac{\mathrm{a}}{a}\right)^{7/2}(1+2e^2)\cos i$. But the density of Earth is not uniform, so we'll leave Equation 14.5.5 as it is.] For a nearly circular orbit, Equation 14.5.5 becomes just

$$\dot{\Omega} = -\frac{3}{2}\sqrt{\frac{GM}{\mathrm{a}^3}}\frac{(C-A)}{Ma^2}\left(\frac{\mathrm{a}}{a}\right)^{7/2}\cos i. \tag{14.5.6}$$

This tells us that the line of nodes of a satellite in orbit around an oblate planet (i.e $C > A$) *regresses*. From the rate of regression of the line of nodes, we can deduce the difference, $C - A$ between the principal moments of inertia, though we cannot deduce either moment separately. (If we *could* determine the moment of inertia from the rate of regression of the nodes – which we cannot – how well can we determine the density distribution inside Earth? See Problem 14 in Chapter A of our Classical Mechanics notes to determine the answer to this. It will be found that knowledge of the moment of inertia places only weak constraints on the core size and density.)

Numerically it is known for Earth that the quantity $\frac{3}{2}\sqrt{\frac{GM}{\mathrm{a}^3}}\frac{(C-A)}{M\mathrm{a}^2}$ is about $2.04\ \mathrm{rad\,s}^{-1}$, or about 10.1 degrees per day. Thus the rate of regression of the nodes of a satellite in orbit around Earth in a near-circular orbit is about

$$\dot{\Omega} = -10.1\left(\frac{\mathrm{a}}{a}\right)^{7/2}\cos i \quad \text{degrees per day}.$$

We can refer to Equations 14.4.4 and 14.5.3 to determine the rate of motion of the line of apsides, $\dot{\omega}$. After some algebra, and neglect of terms of order e^2 and higher, we find

$$\dot{\omega} = \frac{3(C-A)}{4a^{7/2}}\sqrt{\frac{G}{M}}(5\cos^2 i - 1). \tag{14.5.7}$$

or, if we multiply top and bottom by $\mathrm{a}^{7/2}$,

$$\dot{\omega} = \frac{3(C-A)}{4M\mathrm{a}^2}\sqrt{\frac{GM}{\mathrm{a}^3}}\left(\frac{\mathrm{a}}{a}\right)^{7/2}(5\cos^2 i - 1). \tag{14.5.8}$$

Thus we find that the line of apsides advances if the inclination of the orbit to the equator is less than 63° and it regresses if the inclination is greater than this.

In this section, I have demanded a fair amount of work from the reader – in particular for the expansion of Equation 14.5.2. While the work requires some patience and persistence, it is straightforward, and the resolute reader will be able to work out the expansion in terms of the mean anomaly and the time, and hence, by making use of Lagrange's planetary Equations, will be able to predict the periodic variations in a, e and i. For the time being, I am not going to do this, since no new principles are involved, the aim of the chapter being to give the reader a start on how to start to calculate the changes in the orbital elements if one can express the perturbing function analytically.

For the effect of the perturbation of a planetary orbit by the presence of other planets, we have to solve the problem numerically by the techniques of *special perturbations*, which, I hope, some time in the future, may be the subject of an additional chapter.
