9.1: Kepler's Laws

Kepler's law of planetary motion (the first two announced in 1609, the third in 1619) are as follows:

1. Every planet moves around the Sun in an orbit that is an ellipse with the Sun at a focus.
2. The radius vector from Sun to planet sweeps out equal areas in equal times.
3. The squares of the periods of the planets are proportional to the cubes of their semi major axes.

The first law is a consequence of the inverse square law of gravitation. An inverse square law of attraction will actually result in a path that is a *conic section* – that is, an ellipse, a parabola or a hyperbola, although only an ellipse, of course, is a closed orbit. An inverse square law of repulsion (for example, α-particles being deflected by gold nuclei in the famous Geiger-Marsden experiment) will result in a hyperbolic path. An attractive force that is directly proportional to the first power of the distance also results in an elliptical path (a Lissajous ellipse) - for example a mass whirled at the end of a Hooke's law elastic spring - but in that case the centre of attraction is at the centre of the ellipse, rather than at a focus.

We shall derive, in Section 9.5, Kepler's first and third laws from an assumed inverse square law of attraction. The problem facing Newton was the opposite: Starting from Kepler's laws, what is the law of attraction governing the motions of the planets? To start with, he had to invent the differential and integral calculus. This is a far cry from the popular notion that he "discovered" gravity by seeing an apple fall from a tree.

The second law is a consequence of conservation of angular momentum, and would be valid for any law of attraction (or repulsion) as long as the force was entirely radial with no transverse component. We derive it in Section 9.3.

Although a full treatment of the first and third laws awaits Section 9.5, the third law is trivially easy to derive in the case of a *circular* orbit. For example, if we suppose that a planet of mass m is in a circular orbit of radius a around a Sun of mass M, M being supposed to be so much larger than m that the Sun can be regarded as stationary, we can just equate the product of mass and centripetal acceleration of the planet, $ma\omega^2$, to the gravitational force between planet and Sun, GMm/a^2 ; and, with the period being given by $P = 2\pi/\omega$, we immediately obtain the third law:

$$P^2 = \frac{4\pi^2}{GM}a^3. \tag{9.2.1}$$

The reader might like to show that, if the mass of the Sun is not so high that the Sun's motion can be neglected, and that planet and Sun move in circular orbits around their mutual centre of mass, the period is

$$P^2 = \frac{4\pi^2}{G(M+m)}a^3. \tag{9.2.2}$$

Here a is the distance between Sun and planet.

Exercise 9.1.1

Express the period in terms of a_1, the radius of the planet's circular orbit around the centre of mass.