5.11: Legendre Polynomials

In this section we cover just enough about Legendre polynomials to be useful in the following section. Before starting, I want you to expand the following expression, by the binomial theorem, for $|x|<1$, up to x^4 :

$$\frac{1}{(1-2x\cos\theta+x^2)^{1/2}}. \tag{5.11.1}$$

Please do go ahead and do it. Well, you probably won't, so I'd better do it myself:

I'll start with

$$(1-X)^{-1/2}=1+\frac{1}{2}X+\frac{3}{8}X^2+\frac{5}{16}X^3+\frac{35}{128}X^4\ldots \tag{5.11.2}$$

and therefore

$$[1-x(2\cos\theta-x)]^{-1/2}=1+\frac{1}{2}x(2\cos\theta-x)+\frac{3}{8}x^2(2\cos\theta-x)^2+\frac{5}{16}x^3(2\cos\theta-x)^3+\frac{35}{128}x^4(2\cos\theta-x)^4\ldots \tag{5.11.3}$$

$$\begin{aligned}&=1+x\cos\theta-\frac{1}{2}x^2+\frac{3}{8}x^2(4\cos^2\theta-4x\cos\theta+x^2)+\frac{5}{16}x^3(8\cos^3\theta-12x\cos^2\theta+6x^2\cos\theta-x^3)\\&\quad+\frac{35}{128}x(16\cos^4\theta-32x\cos^3\theta+24x^2\cos^2\theta-8x^3\cos\theta+x^4)\ldots\end{aligned} \tag{5.11.4}$$

$$=1+x\cos\theta+x^2\left(-\frac{1}{2}+\frac{3}{2}\cos^2\theta\right)+x^3\left(-\frac{3}{2}\cos\theta+\frac{5}{2}\cos^3\theta\right)+x^4\left(\frac{3}{8}-\frac{15}{4}\cos^2\theta+\frac{35}{8}\cos^4\theta\right)\ldots \tag{5.11.5}$$

The coefficients of the powers of x are the *Legendre polynomials* $P_l(\cos\theta)$, so that

$$\frac{1}{(1-2x\cos\theta+x^2)^{1/2}}=1+xP_1(\cos\theta)+x^2P_2(\cos\theta)+x^3P_3(\cos\theta)+x^4P_4(\cos\theta)+\ldots \tag{5.11.6}$$

The Legendre polynomials with argument $\cos\theta$ can be written as series of terms in powers of $\cos\theta$ by substitution of $\cos\theta$ for x in Equations 1.12.5 in Section 1.12 of Chapter 1. Note that x in Section 1 is not the same as x in the present section. Alternatively they can be written as series of cosines of multiples of θ as follows.

$$\begin{aligned}P_0&=1\\P_1&=\cos\theta\\P_2&=\frac{1}{4}(3\cos2\theta+1)\\P_3&=\frac{1}{8}(5\cos3\theta+3\cos\theta)\\P_4&=\frac{1}{64}(35\cos4\theta+20\cos2\theta+9)\\P_5&=\frac{1}{128}(63\cos5\theta+35\cos3\theta+30\cos\theta)\\P_6&=\frac{1}{512}(231\cos6\theta+126\cos4\theta+105\cos2\theta+50)\\P_7&=\frac{1}{1024}(429\cos7\theta+231\cos5\theta+189\cos3\theta+175\cos\theta)\\P_8&=(6435\cos8\theta+3432\cos6\theta+2772\cos4\theta+2520\cos2\theta+1225)/2^{14}\end{aligned} \tag{5.11.7}$$

For example, $P_6(\cos\theta)$ can be written either as given by Equation 5.11.7, or as given by Equation 1, namely

$$P_6=\frac{1}{16}(231c^6-315c^4+105c^2-5),\ \text{where}\ c=\cos\theta. \tag{5.11.8}$$

The former may look neater, and the latter may look "awkward" because of all the powers. However, the latter is far faster to compute, particularly when written as nested parentheses:

$$P_6=(-5+C(105+C(-315+231C)))/16,\ \text{where}\ C=\cos^2\theta. \tag{5.11.9}$$