14.3: The Poisson Brackets for the Orbital Elements

A worked example is in order. From Equations 14.2.7 and 14.2.8, we see that the Poisson brackets are defined by

$$\{A_i, A_k\}_{\alpha_j,\beta_j} = \sum_j \left(\frac{\partial A_i}{\partial \alpha_j}\frac{\partial A_k}{\partial \beta_j} - \frac{\partial A_i}{\partial \beta_j}\frac{\partial A_k}{\partial \alpha_j} \right). \tag{14.3.1}$$

The A_i are the orbital elements.

For our example, we shall calculate $\{\Omega, i\}$ and we write out the sum in full:

$$\{\Omega, i\} = \sum_j \left(\frac{\partial \Omega}{\partial \alpha_j}\frac{\partial i}{\partial \beta_j} - \frac{\partial \Omega}{\partial \beta_j}\frac{\partial i}{\partial \alpha_j} \right) \tag{14.3.2}$$

$$= \frac{\partial \Omega}{\partial \alpha_1}\frac{\partial i}{\partial \beta_1} + \frac{\partial \Omega}{\partial \alpha_2}\frac{\partial i}{\partial \beta_2} + \frac{\partial \Omega}{\partial \alpha_3}\frac{\partial i}{\partial \beta_3} - \frac{\partial \Omega}{\partial \beta_1}\frac{\partial i}{\partial \alpha_1} - \frac{\partial \Omega}{\partial \beta_2}\frac{\partial i}{\partial \alpha_2} - \frac{\partial \Omega}{\partial \beta_3}\frac{\partial i}{\partial \alpha_3}. \tag{14.3.3}$$

Refer now to Equations 10.11.27 and 29, and we find

$$\{\Omega, i\} = 0 + 0 + 0 - 0 + \frac{1}{\alpha_3\sqrt{1-\alpha_2^2/\alpha_3^2}} - 0. \tag{14.3.4}$$

Finally, referring to Equations 10.11.20 and 21, we obtain

$$\{\Omega, i\} = \frac{1}{\sqrt{GMm^2a(1-e^2).\sin i}}. \tag{14.3.5}$$

Proceeding in a similar manner for the others, we obtain

$$\{a, T\} = -\frac{2a^2}{GMm}, \tag{14.3.6}$$

$$\{e, T\} = -\frac{a(1-e^2)}{GMme}, \tag{14.3.7}$$

$$\{i, \omega\} = \frac{1}{\sqrt{GMm^2a(1-e^2).\tan i}}. \tag{14.3.8}$$

$$\{e, \omega\} = -\frac{\sqrt{1-e^2}}{em\sqrt{GMa}}, \tag{14.3.9}$$

In addition, we have, of course,

$$\{i, \Omega\} = -\{\Omega, i\}, \{T, a\} = -\{a, T\}, \{T, e\} = -\{e, T\} \text{ and } \{\omega, i\} = -\{i, \omega\}. \tag{14.3.10}$$

All other pairs are zero.