14.2: Contact Transformations and General Perturbation Theory

(Before reading this section, it may be well to re-read section 10.11 of Chapter 10.)

Suppose that we have a simple problem in which we know the hamiltonian H_0 and that the Hamilton-Jacobi Equation has been solved:

$$H_0\left(q_1, \frac{\partial S}{\partial q_1}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{14.2.1}$$

Now suppose we have a similar problem, but that the hamiltonian, instead of being just H_0 is $H = H_0 - R$, and $K = H + \frac{\partial S}{\partial t}$.

Let us make a contact transformation from $(p_i,\, q_i)$ to $(P_i,\, Q_i)$, where $\dot{Q}_i = \frac{\partial K}{\partial P_i}$ and $\dot{P}_i = -\frac{\partial K}{\partial Q_i}$. In the orbital context, following Section 10.11, we identify Q_i with α_i and P_i with $-\beta_i$, which are functions (given in Section 10.11) of the orbital elements and which can serve in place of the orbital elements. The parameters are constants with respect to the unperturbed problem, but are variables with respect to the perturbing function. They are given, as functions of time, by the solution of Hamilton's Equations of motion, which retain their form under a contact transformation.

$$\dot{\alpha}_i = \frac{\partial R}{\partial \beta_i} \text{ and } \dot{\beta}_i = -\frac{\partial R}{\partial \alpha_i}. \tag{14.2.2a,b}$$

Perturbation theory will show , then, how the α_i and β_i will vary with a given perturbation. The conventional elements $a,\, e,\, i,\, \Omega,\, \omega,\, T$ are functions of $\alpha_i,\, \beta_i$, and our aim is to find how the conventional elements vary with time under the perturbation R.

We can do that as follows. Let A_i be an orbital element, given by

$$A_i = A_i(\alpha_i, \beta_i). \tag{14.2.3}$$

Then

$$\dot{A}_i = \sum_j \frac{\partial A_i}{\partial \alpha_j}\dot{\alpha}_j + \sum_j \frac{\partial A_i}{\partial \beta_j}\dot{\beta}_j. \tag{14.2.4}$$

By Equations 14.2.2a,b, this becomes

$$\dot{A}_i = \sum_j \frac{\partial A_i}{\partial \alpha_j}\frac{\partial R}{\partial \beta_j} - \sum_j \frac{\partial A_i}{\partial \beta_j}\frac{\partial R}{\partial \beta_j}. \tag{14.2.5}$$

But

$$\frac{\partial R}{\partial \alpha_j} = \sum_k \frac{\partial R}{\partial A_k}\frac{\partial A_k}{\partial \alpha_j} \quad \text{and} \quad \frac{\partial R}{\partial \beta_j} = \sum_k \frac{\partial R}{\partial A_k}\frac{\partial A_k}{\partial \beta_j}. \tag{14.2.6a,b}$$

$$\therefore \quad \dot{A}_i = \sum_j \sum_k \frac{\partial R}{\partial A_k}\left(\frac{\partial A_i}{\partial a_j}\frac{\partial A_k}{\partial \beta_j} - \frac{\partial A_i}{\partial \beta_j}\frac{\partial A_k}{\partial \alpha_j}\right) \tag{14.2.6}$$

That is

$$\dot{A}_i = \sum_k \frac{\partial R}{\partial A_k}\sum_j\left(\frac{\partial A_i}{\partial a_j}\frac{\partial A_k}{\partial \beta_j} - \frac{\partial A_i}{\partial \beta_j}\frac{\partial A_k}{\partial \alpha_j}\right) \tag{14.2.7}$$

This can be written, in shorthand:

$$\dot{A}_i = \sum_k \frac{\partial R}{\partial A_k}\{A_i, A_k\}_{\alpha_j,\beta_j}. \tag{14.2.8}$$

Here the symbol $\{A_i, A_k\}_{\alpha_i,\beta_i}$ is called the *Poisson bracket* of $A_i,\ A_k$ with respect to α_j, β_j. (In the language of the typographer, the symbols (), [] and {} are, respectively, parentheses, brackets and braces; you may refer to Poisson braces if you wish, but the usual term, in spite of the symbols, is Poisson bracket.)

Note the property $\{A_i, A_k\}_{\alpha_j,\beta_j} = -\{A_k, A_i\}_{\alpha_j,\beta_j}$.