9.6: Position in a Parabolic Orbit

When a "long-period" comet comes in from the Oort belt, it typically comes in on a highly eccentric orbit, of which we can observe only a very short arc. Consequently, it is often impossible to determine the period or semi major axis with any degree of reliability or to distinguish the orbit from a parabola. There is therefore frequent occasion to have to understand the dynamics of a parabolic orbit.

We have no mean or eccentric anomalies. We must try to get v directly as a function of t without going through these intermediaries.

The angular momentum per unit mass is given by Equation 9.5.28a:

$$h = r^2\dot{v} = \sqrt{2G\mathbf{M}q}, \tag{9.7.1}$$

where v is the true anomaly and q is the perihelion distance.

But the Equation to the parabola (see Equation 2.4.16) is

$$r = \frac{2q}{1+\cos v}, \tag{9.7.2}$$

or (see section 3.8 of Chapter 3), by making use of the identity

$$\cos v = \frac{1-u^2}{1+u^2}, \quad \text{where} \quad u = \tan\frac{1}{2}v, \tag{9.7.3a,b}$$

the Equation to the parabola can be written

$$r = q\sec^2\frac{1}{2}v. \tag{9.7.4}$$

Thus, by substitution of Equation 9.7.4 into 9.7.1 and integrating, we obtain

$$q^2\int_0^v \sec^4(\frac{1}{2}v)dv = \sqrt{2G\mathbf{M}q}\int_T^t dt. \tag{9.7.5}$$

Upon integration (drop me an email if you get stuck!) this becomes

$$u + \frac{1}{3}u^3 = \frac{\sqrt{\frac{1}{2}G\mathbf{M}}}{q^{3/2}}(t-T). \tag{9.7.6}$$

This Equation, when solved for u (which, remember, is $\tan\frac{1}{2}v$), gives us v as a function of t. As explained at the end of section 9.5, if q is in astronomical units and $t-T$ is in sidereal years, and if the mass of the comet is negligible compared with the mass of the Sun, this becomes

$$u + \frac{1}{3}u^3 = \frac{\pi\sqrt{2}(t-T)}{q^{3/2}} \tag{9.7.7}$$

or

$$3u + u^3 - C = 0, \quad \text{where} \quad C = \frac{\pi\sqrt{18}(t-T)}{q^{3/2}}. \tag{9.7.8a,b}$$

There is a choice of methods available for solving Equation 9.7.8a,b, so it might be that the only difficulty is to decide which of the several methods you want to use! The constant $\frac{1}{3}C$ is sometimes called the "parabolic mean anomaly".

Method 1: Just solve it by Newton-Raphson iteration. Thus $f = 3u + u^3 - C = 0$ and $f' = 3(1+u^2)$, so that the Newton-Raphson $u = u - f/f'$ becomes

$$u = \frac{2u^3 + C}{3(1+u^2)}, \tag{9.7.9}$$

which should converge quickly. For economy, calculate u^2 only once per iteration.

Method 2:

Let

$$u = x - 1/x \quad \text{and} \quad C = c - 1/c. \tag{9.7.10a,b}$$

Then Equation 9.7.8a becomes

$$x = c^{1/3}. \tag{9.7.11}$$

Thus, as soon as c is found, x, u and v can be calculated from Equations 9.7.11, 10a, and 3a or b, and the problem is finished – as soon as c is found!

So, how do we find c? We have to solve Equation 9.7.10b.

Method 2a:

Equation 9.7.10b can be written as a quadratic Equation:

$$c^2 - Cc - 1 = 0. \tag{9.7.12}$$

Just be careful that you choose the correct root; you should end with v having the same sign as $t - T$.

Method 2b:

Let

$$C = 2\cot 2\phi \tag{9.7.13}$$

and calculate ϕ. But by a trigonometric identity,

$$2\cot 2\phi = \cot\phi - 1/\cot\phi \tag{9.7.14}$$

so that, by comparison with Equation 9.7.10b, we see that

$$c = \cot\phi. \tag{9.7.15}$$

Again, just make sure that you choose the right quadrant in calculating ϕ from Equation 9.7.13, so as to be sure that you end with v having the same sign as $t - T$.

Method 3.

I am told that Equation 9.7.8 has the exact analytic solution

$$u = \frac{1}{2} w^{\frac{1}{3}} - 2w^{-\frac{1}{3}}, \tag{9.7.16}$$

where

$$w = 4C + \sqrt{64 + 16C^2}. \tag{9.7.17}$$

I haven't verified this for myself, so you might like to have a go.

Example: Solve the Equation $3u + u^3 = 1.6$ by all four methods. (Methods 1, 2a, 2b and 3.)

Example: A comet is moving in a parabolic orbit with perihelion distance 0.9 AU. Calculate the true anomaly and heliocentric distance 20 days after perihelion passage. (A sidereal year is 365.25636 days.)

Exercise: Write a computer program that will return the true anomaly as a function of time, given the perihelion distance of a parabolic orbit. Test it with your answer for the previous example.

This page titled 9.6: Position in a Parabolic Orbit is shared under a CC BY-NC 4.0 license and was authored, remixed, and/or curated by Jeremy Tatum via source content that was edited to the style and standards of the LibreTexts platform.