4.5: The Hyperboloid

The Equation

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \tag{4.5.1}$$

is a hyperbola, and a is the semi transverse axis. (As described in Chapter 2, c is the semi transverse axis of the conjugate hyperbola.)

If this figure is rotated about the z-axis through 360°, the surface swept out is a *circular hyperboloid* (or *hyperboloid of revolution*) *of one sheet*. Its Equation is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1. \tag{4.5.2}$$

Imagine two horizontal rings, one underneath the other. The upper one is fixed. The lower one is suspended from the upper one by a large number of vertical strings attached to points equally spaced around the circumference of each ring. Now twist the lower one through a few degrees about a vertical axis, so that the strings are no longer quite vertical, and the lower ring rises slightly. These strings are generators of a circular hyperboloid of one sheet.

If the figure is rotated about the x-axis through 360°, the surface swept out is a *circular hyperboloid* (or *hyperboloid of revolution*) *of two sheets*. Its Equation is

$$\frac{x^2}{a^2} - \frac{y^2}{c^2} - \frac{z^2}{c^2} = 1. \tag{4.5.3}$$

The Equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{4.5.4}$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{4.5.5}$$

represent hyperbolas of one and two sheets respectively, but are not hyperbolas of revolution, since their cross sections in the planes $z =$ constant and $x =$ constant $> a$ respectively are ellipses rather than circles. The reader should imagine what the cross-sections of all four hyperboloids are like in the planes $x = 0$, $y = 0$ and $z = 0$.