5.4.7: Solid Cylinder

Page ID
8137

We do this not because it has any particular relevance to celestial mechanics, but because it is easy to do. We imagine a solid cylinder, density ρ, radius a, length l. We seek to calculate the field at a point P on the axis, at a distance h from one end of the cylinder (figure V.8).

FIGURE V.8

The field at P from an elemental disc of thickness δz a distance z below P is (from Equation 5.4.9)

$$\delta g = G\rho\delta z\omega. \tag{5.4.19}$$

Here ω is the solid angle subtended at P by the disc, which is $2\pi\left[1-\frac{z}{(z^2+a^2)^{1/2}}\right]$. Thus the field at P from the entire cylinder is

$$g = 2\pi G\rho\int_h^{l+h}\left[1-\frac{z}{(z^2+a^2)^{1/2}}\right]dz, \tag{5.4.20}$$

or

$$g = 2\pi G\rho\left(l-\sqrt{(l+h)^2+a^2}+\sqrt{h^2+a^2}\right), \tag{5.4.21}$$

or

$$g = 2\pi G\rho(l-r_2+r_1). \tag{5.4.22}$$

It might also be of interest to express g in terms of the height $y(=\frac{1}{2}l+h)$ of the point P above the mid-point of the cylinder. Instead of Equation 5.4.21, we then have

$$g = 2\pi G\rho\left(l-\sqrt{(y+\frac{1}{2}l)^2+a^2}+\sqrt{(y-\frac{1}{2}l)^2+a^2}\right). \tag{5.4.23}$$

If the point P is *inside* the cylinder,at a distance h below the upper end of the cylinder, the limits of integration in Equation 5.4.20 are h and $l-h$, and the distance y is $\frac{1}{2}l-h$. In terms of y the gravitational field at P is then

$$g = 2\pi G\rho\left(2y-\sqrt{(y+\frac{1}{2}l)^2+a^2}+\sqrt{(y-\frac{1}{2}l)^2+a^2}\right). \tag{5.4.24}$$

In the graph below I have assumed, by way of example, that l and a are both 1, and I have plotted g in units of $2\pi G\rho$ (counting g as positive when it is directed downwards) from $y=-1$ to $y=+1$. The portion inside the cylinder $(-\frac{1}{2}\leq y\leq\frac{1}{2}l)$, represented by Equation 5.4.24, is almost, but not quite, linear. The field at the centre of the cylinder is, of course, zero.

Below, I draw the same graph, but for a thin disc, with $a = 1$ and $l = 0.1$. We see how it is that the field reaches a maximum immediately above or below the disc, but is zero at the centre of the disc.