

## 10.8: Rolling Motion

### Learning Objectives

- Describe the physics of rolling motion without slipping
- Explain how linear variables are related to angular variables for the case of rolling motion without slipping
- Find the linear and angular accelerations in rolling motion with and without slipping
- Calculate the static friction force associated with rolling motion without slipping
- Use energy conservation to analyze rolling motion

Rolling motion is that common combination of rotational and translational motion that we see everywhere, every day. Think about the different situations of wheels moving on a car along a highway, or wheels on a plane landing on a runway, or wheels on a robotic explorer on another planet. Understanding the forces and torques involved in **rolling motion** is a crucial factor in many different types of situations.

For analyzing rolling motion in this chapter, refer to [Figure 10.5.4 in Fixed-Axis Rotation](#) to find moments of inertia of some common geometrical objects. You may also find it useful in other calculations involving rotation.

### 10.8.1 Rolling Motion without Slipping

People have observed rolling motion without slipping ever since the invention of the wheel. For example, we can look at the interaction of a car's tires and the surface of the road. If the driver depresses the accelerator to the floor, such that the tires spin without the car moving forward, there must be kinetic friction between the wheels and the surface of the road. If the driver depresses the accelerator slowly, causing the car to move forward, then the tires roll without slipping. It is surprising to most people that, in fact, the bottom of the wheel is at rest with respect to the ground, indicating there must be static friction between the tires and the road surface. In Figure 10.8.1, the bicycle is in motion with the rider staying upright. The tires have contact with the road surface, and, even though they are rolling, the bottoms of the tires deform slightly, do not slip, and are at rest with respect to the road surface for a measurable amount of time. There must be static friction between the tire and the road surface for this to be so.



Figure 10.8.1: (a) The bicycle moves forward, and its tires do not slip. The bottom of the slightly deformed tire is at rest with respect to the road surface for a measurable amount of time. (b) This image shows that the top of a rolling wheel appears blurred by its motion, but the bottom of the wheel is instantaneously at rest. (credit a: modification of work by Nelson Lourenço; credit b: modification of work by Colin Rose)

To analyze rolling without slipping, we first derive the linear variables of velocity and acceleration of the center of mass of the wheel in terms of the angular variables that describe the wheel's motion. The situation is shown in Figure 10.8.2

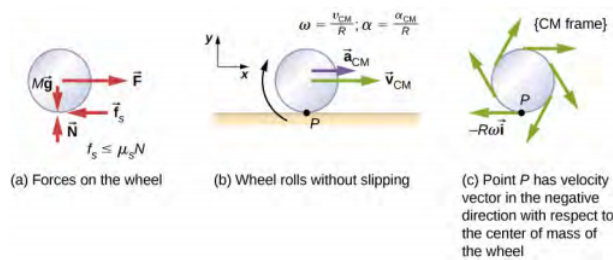


Figure 10.8.2: (a) A wheel is pulled across a horizontal surface by a force  $\vec{F}$ . The force of static friction  $\vec{f}_s$ ,  $|\vec{f}_s| = \mu_s N$  is large enough to keep it from slipping. (b) The linear velocity and acceleration vectors of the center of mass and the relevant expressions for  $\omega$  and  $\alpha$ . Point P is at rest relative to the surface. (c) Relative to the center of mass (CM) frame, point P has linear velocity  $-R\omega\hat{i}$ .

From Figure 10.8.2(a), we see the force vectors involved in preventing the wheel from slipping. In (b), point P that touches the surface is at rest relative to the surface. Relative to the center of mass, point P has velocity  $-R\omega\hat{i}$ , where R is the radius of the wheel and  $\omega$  is the wheel's angular velocity about its axis. Since the wheel is rolling, the velocity of P with respect to the surface is its velocity with respect to the center of mass plus the velocity of the center of mass with respect to the surface:

$$\vec{v}_P = -R\omega\hat{i} + v_{CM}\hat{i}. \quad (10.8.1)$$

Since the velocity of P relative to the surface is zero,  $v_P = 0$ , this says that

$$v_{CM} = R\omega. \quad (10.8.2)$$

Thus, the velocity of the wheel's center of mass is its radius times the angular velocity about its axis. We show the correspondence of the linear variable on the left side of the equation with the angular variable on the right side of the equation. This is done below for the linear acceleration.

If we differentiate Equation 10.8.2 on the left side of the equation, we obtain an expression for the linear acceleration of the center of mass. On the right side of the equation, R is a constant and since  $\alpha = \frac{d\omega}{dt}$ , we have

$$a_{CM} = R\alpha. \quad (10.8.3)$$

Furthermore, we can find the distance the wheel travels in terms of angular variables by referring to Figure 10.8.3. As the wheel rolls from point A to point B, its outer surface maps onto the ground by exactly the distance traveled, which is  $d_{CM}$ .

We see from Figure 10.8.3 that the length of the outer surface that maps onto the ground is the arc length  $R\theta$ . Equating the two distances, we obtain

$$d_{CM} = R\theta. \quad (10.8.4)$$

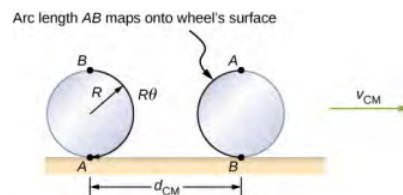


Figure 10.8.3: As the wheel rolls on the surface, the arc length  $R\theta$  from A to B maps onto the surface, corresponding to the distance  $d_{CM}$  that the center of mass has moved.

### ✓ Example 10.8.1: Rolling Down an Inclined Plane

A solid cylinder rolls down an inclined plane without slipping, starting from rest. It has mass m and radius r. (a) What is its acceleration? (b) What condition must the coefficient of static friction  $\mu_s$  satisfy so the cylinder does not slip?

#### Strategy

Draw a sketch and free-body diagram, and choose a coordinate system. We put x in the direction down the plane and y upward perpendicular to the plane. Identify the forces involved. These are the normal force, the force of gravity, and the force due to friction. Write down Newton's laws in the x- and y-directions, and Newton's law for rotation, and then solve for the acceleration and force due to friction.

### Solution

- a. The free-body diagram and sketch are shown in Figure 10.8.4 including the normal force, components of the weight, and the static friction force. There is barely enough friction to keep the cylinder rolling without slipping. Since there is no slipping, the magnitude of the friction force is less than or equal to  $\mu_s N$ . Writing down Newton's laws in the x- and y-directions, we have

$$\sum F_x = ma_x; \sum F_y = ma_y. \quad (10.8.5)$$

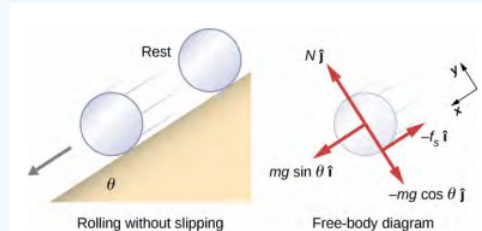


Figure 10.8.4: A solid cylinder rolls down an inclined plane without slipping from rest. The coordinate system has x in the direction down the inclined plane and y perpendicular to the plane. The free-body diagram is shown with the normal force, the static friction force, and the components of the weight  $m\vec{g}$ . Friction makes the cylinder roll down the plane rather than slip.

Substituting in from the free-body diagram

$$\begin{aligned} mg \sin \theta - f_s &= m(a_{CM})_x, \\ N - mg \cos \theta &= 0 \end{aligned}$$

we can then solve for the linear acceleration of the center of mass from these equations:

$$a_{CM} = g \sin \theta - \frac{f_s}{m}. \quad (10.8.6)$$

However, it is useful to express the linear acceleration in terms of the moment of inertia. For this, we write down Newton's second law for rotation,

$$\sum \tau_{CM} = I_{CM} \alpha. \quad (10.8.7)$$

The torques are calculated about the axis through the center of mass of the cylinder. The only nonzero torque is provided by the friction force. We have

$$f_s r = I_{CM} \alpha. \quad (10.8.8)$$

Finally, the linear acceleration is related to the angular acceleration by

$$(a_{CM})_x = r \alpha. \quad (10.8.9)$$

These equations can be used to solve for  $a_{CM}$ ,  $\alpha$ , and  $f_s$  in terms of the moment of inertia, where we have dropped the x-subscript. We write  $a_{CM}$  in terms of the vertical component of gravity and the friction force, and make the following substitutions.

$$f_s = \frac{I_{CM} \alpha}{r} = \frac{I_{CM} a_{CM}}{r^2} \quad (10.8.10)$$

From this we obtain

$$\begin{aligned} a_{CM} &= g \sin \theta - \frac{I_{CM} a_{CM}}{mr^2}, \\ &= \frac{mg \sin \theta}{m + \left( \frac{I_{CM}}{r^2} \right)}. \end{aligned}$$

Note that this result is independent of the coefficient of static friction,  $\mu_s$ .

Since we have a solid cylinder, from Figure 10.5.4, we have  $I_{CM} = \frac{mr^2}{2}$  and

$$a_{CM} = \frac{mg \sin \theta}{m + \left(\frac{mr^2}{2r^2}\right)} = \frac{2}{3}g \sin \theta. \quad (10.8.11)$$

Therefore, we have

$$\alpha = \frac{a_{CM}}{r} = \frac{2}{3r}g \sin \theta. \quad (10.8.12)$$

b. Because slipping does not occur,  $f_s = \mu_s N$ . Solving for the friction force,

$$f_s = I_{CM} \frac{\alpha}{r} = I_{CM} \frac{(a_{CM})}{r^2} = \left(\frac{I_{CM}}{r^2}\right) \left(\frac{mg \sin \theta}{m + \left(\frac{I_{CM}}{r^2}\right)}\right) = \frac{mg I_{CM} \sin \theta}{mr^2 + I_{CM}}. \quad \text{Substituting this expression into the}$$

condition for no slipping, and noting that  $N = mg \cos \theta$ , we have  $\frac{mg I_{CM} \sin \theta}{mr^2 + I_{CM}} \leq \mu_s mg \cos \theta$  or  $\mu_s \geq \frac{\tan \theta}{1 + \left(\frac{mr^2}{I_{CM}}\right)}.$

For the solid cylinder, this becomes  $\mu_s \geq \frac{\tan \theta}{1 + \left(\frac{2mr^2}{mr^2}\right)} = \frac{1}{3} \tan \theta.$

### Significance

- The linear acceleration is linearly proportional to  $\sin \theta$ . Thus, the greater the angle of the incline, the greater the linear acceleration, as would be expected. The angular acceleration, however, is linearly proportional to  $\sin \theta$  and inversely proportional to the radius of the cylinder. Thus, the larger the radius, the smaller the angular acceleration.
- For no slipping to occur, the coefficient of static friction must be greater than or equal to  $\frac{1}{3} \tan \theta$ . Thus, the greater the angle of incline, the greater the coefficient of static friction must be to prevent the cylinder from slipping.

### ? Exercise 10.8.2

A hollow cylinder is on an incline at an angle of  $60^\circ$ . The coefficient of static friction on the surface is  $\mu_s = 0.6$ . (a) Does the cylinder roll without slipping? (b) Will a solid cylinder roll without slipping?

It is worthwhile to repeat the equation derived in this example for the acceleration of an object rolling without slipping:

$$a_{CM} = \frac{mg \sin \theta}{m + \left(\frac{I_{CM}}{r^2}\right)}. \quad (10.8.13)$$

This is a very useful equation for solving problems involving rolling without slipping. Note that the acceleration is less than that for an object sliding down a frictionless plane with no rotation. The acceleration will also be different for two rotating cylinders with different rotational inertias.

## 10.8.2 Rolling Motion with Slipping

In the case of rolling motion with slipping, we must use the coefficient of kinetic friction, which gives rise to the kinetic friction force since static friction is not present. The situation is shown in Figure 10.8.5. In the case of slipping,  $v_{CM} - R\omega \neq 0$ , because point P on the wheel is not at rest on the surface, and  $v_P \neq 0$ . Thus,  $\omega \neq \frac{v_{CM}}{R}$ ,  $\alpha \neq \frac{a_{CM}}{R}$ .

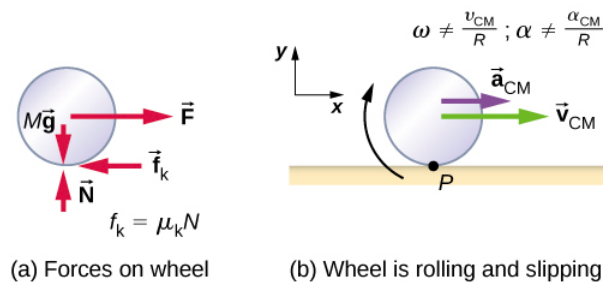


Figure 10.8.5: (a) Kinetic friction arises between the wheel and the surface because the wheel is slipping. (b) The simple relationships between the linear and angular variables are no longer valid.

### ✓ Example 10.8.2: Rolling Down an Inclined Plane with Slipping

A solid cylinder rolls down an inclined plane from rest and undergoes slipping (Figure 10.8.6). It has mass  $m$  and radius  $r$ . (a) What is its linear acceleration? (b) What is its angular acceleration about an axis through the center of mass?

#### Strategy

Draw a sketch and free-body diagram showing the forces involved. The free-body diagram is similar to the no-slipping case except for the friction force, which is kinetic instead of static. Use Newton's second law to solve for the acceleration in the  $x$ -direction. Use Newton's second law of rotation to solve for the angular acceleration.

#### Solution

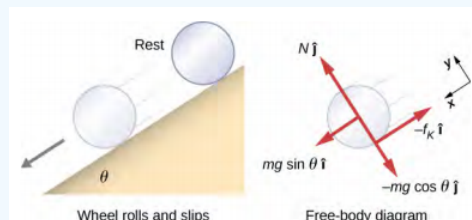


Figure 10.8.6: A solid cylinder rolls down an inclined plane from rest and undergoes slipping. The coordinate system has  $x$  in the direction down the inclined plane and  $y$  upward perpendicular to the plane. The free-body diagram shows the normal force, kinetic friction force, and the components of the weight  $m\vec{g}$ .

The sum of the forces in the  $y$ -direction is zero, so the friction force is now  $f_k = \mu_k N = \mu_k mg \cos \theta$ . Newton's second law in the  $x$ -direction becomes

$$\sum F_x = ma_x,$$

$$mg \sin \theta - \mu_k mg \cos \theta = m(a_{CM})_x,$$

or

$$(a_{CM})_x = g(\sin \theta - \mu_k \cos \theta).$$

The friction force provides the only torque about the axis through the center of mass, so Newton's second law of rotation becomes

$$\sum \tau_{CM} = I_{CM} \alpha,$$

$$f_k r = I_{CM} \alpha = \frac{1}{2} m r^2 \alpha.$$

Solving for  $\alpha$ , we have

$$\alpha = \frac{2f_k}{mr} = \frac{2\mu_k g \cos \theta}{r}.$$

#### Significance

We write the linear and angular accelerations in terms of the coefficient of kinetic friction. The linear acceleration is the same as that found for an object sliding down an inclined plane with kinetic friction. The angular acceleration about the axis of rotation is linearly proportional to the normal force, which depends on the cosine of the angle of inclination. As  $\theta \rightarrow 90^\circ$ , this force goes to zero, and, thus, the angular acceleration goes to zero.

### 10.8.3 Conservation of Mechanical Energy in Rolling Motion

In the preceding chapter, we introduced rotational kinetic energy. Any rolling object carries rotational kinetic energy, as well as translational kinetic energy and potential energy if the system requires. Including the gravitational potential energy, the total mechanical energy of an object rolling is

$$E_T = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 + mgh. \quad (10.8.14)$$

In the absence of any nonconservative forces that would take energy out of the system in the form of heat, the total energy of a rolling object without slipping is conserved and is constant throughout the motion. Examples where energy is not conserved are a rolling object that is slipping, production of heat as a result of kinetic friction, and a rolling object encountering air resistance.

You may ask why a rolling object that is not slipping conserves energy, since the static friction force is nonconservative. The answer can be found by referring back to Figure 10.8.2. Point P in contact with the surface is at rest with respect to the surface. Therefore, its infinitesimal displacement  $d\vec{r}$  with respect to the surface is zero, and the incremental work done by the static friction force is zero. We can apply energy conservation to our study of rolling motion to bring out some interesting results.

#### ✓ Example 10.8.3: Curiosity Rover

The **Curiosity** rover, shown in Figure 10.8.7, was deployed on Mars on August 6, 2012. The wheels of the rover have a radius of 25 cm. Suppose astronauts arrive on Mars in the year 2050 and find the now-inoperative **Curiosity** on the side of a basin. While they are dismantling the rover, an astronaut accidentally loses a grip on one of the wheels, which rolls without slipping down into the bottom of the basin 25 meters below. If the wheel has a mass of 5 kg, what is its velocity at the bottom of the basin?



Figure 10.8.7: The NASA Mars Science Laboratory rover Curiosity during testing on June 3, 2011. The location is inside the Spacecraft Assembly Facility at NASA's Jet Propulsion Laboratory in Pasadena, California. (credit: NASA/JPL-Caltech)

#### Strategy

We use mechanical energy conservation to analyze the problem. At the top of the hill, the wheel is at rest and has only potential energy. At the bottom of the basin, the wheel has rotational and translational kinetic energy, which must be equal to the initial potential energy by energy conservation. Since the wheel is rolling without slipping, we use the relation  $v_{CM} = r\omega$  to relate the translational variables to the rotational variables in the energy conservation equation. We then solve for the velocity. From Figure 10.8.7 we see that a hollow cylinder is a good approximation for the wheel, so we can use this moment of inertia to simplify the calculation.

### Solution

Energy at the top of the basin equals energy at the bottom:

$$mgh = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2.$$

The known quantities are  $I_{CM} = mr^2$ ,  $r = 0.25$  m, and  $h = 25.0$  m.

We rewrite the energy conservation equation eliminating  $\omega$  by using  $\omega = v_{CM}/r$ . We have

$$mgh = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}mr^2 \frac{v_{CM}^2}{r^2}$$

or

$$gh = \frac{1}{2}v_{CM}^2 + \frac{1}{2}v_{CM}^2 \Rightarrow v_{CM} = \sqrt{gh}.$$

On Mars, the acceleration of gravity is  $3.71$  m/s<sup>2</sup>, which gives the magnitude of the velocity at the bottom of the basin as

$$v_{CM} = \sqrt{(3.71 \text{ m/s}^2)(25.0 \text{ m})} = 9.63 \text{ m/s}.$$

### Significance

This is a fairly accurate result considering that Mars has very little atmosphere, and the loss of energy due to air resistance would be minimal. The result also assumes that the terrain is smooth, such that the wheel wouldn't encounter rocks and bumps along the way.

Also, in this example, the kinetic energy, or energy of motion, is equally shared between linear and rotational motion. If we look at the moments of inertia in [Figure 10.5.4](#), we see that the hollow cylinder has the largest moment of inertia for a given radius and mass. If the wheels of the rover were solid and approximated by solid cylinders, for example, there would be more kinetic energy in linear motion than in rotational motion. This would give the wheel a larger linear velocity than the hollow cylinder approximation. Thus, the solid cylinder would reach the bottom of the basin faster than the hollow cylinder.

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