7.4: Center of Mass

Learning Objectives

- Explain the meaning and usefulness of the concept of center of mass
- Calculate the center of mass of a given system
- Apply the center of mass concept in two and three dimensions

7.4.1 Definition

Our task is to determine what part of the extended object, if any, is obeying Equation ???.

It's tempting to take the next step; does the following equation mean anything?

$$\vec{F} = M\vec{a} \tag{7.4.1}$$

If it **does** mean something (acceleration of what, exactly?), then we could write

$$M\vec{a} = \frac{d\vec{p}_{CM}}{dt} \tag{7.4.2}$$

Looking at this calculation, notice that (inside the parentheses) we are calculating the product of each particle's mass with its position, adding all N of these up, and dividing this sum by the total mass of particles we summed. This is reminiscent of an average; inspired by this, we'll (loosely) interpret it to be the weighted average position of the mass of the extended object. It's actually called the **center of mass** of the object. Notice that the position of the center of mass has units of meters; that suggests a definition:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{j=1}^{N} m_j\vec{r}_j. \tag{7.4.3}$$

So, the point that obeys Equation ??? (and therefore Equation 7.4.1 as well) is the center of mass of the object, which is located at the position vector $\vec{r}_{CM}$.

It may surprise you to learn that there does not have to be any actual mass at the center of mass of an object. For example, a hollow steel sphere with a vacuum inside it is spherically symmetrical (meaning its mass is uniformly distributed about the center of the sphere); all of the sphere's mass is out on its surface, with no mass inside. But it can be shown that the center of mass of the sphere is at its geometric center, which seems reasonable. Thus, there is no mass at the position of the center of mass of the sphere. (Another example is a doughnut.) The procedure to find the center of mass is illustrated in Figure $7.4.2$

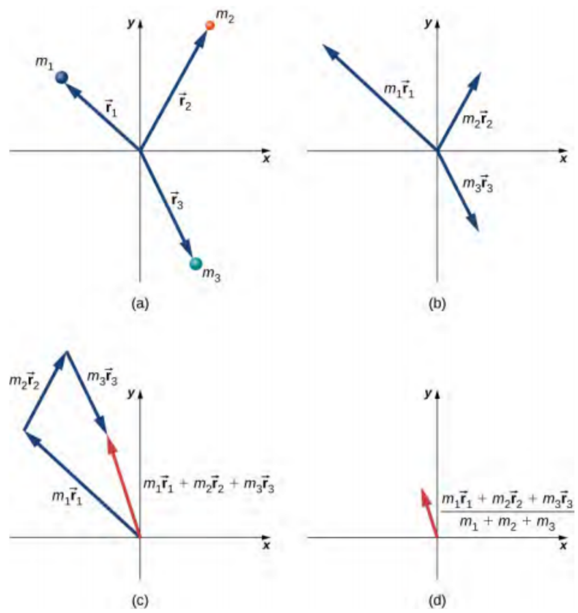


Figure 7.4.2: Finding the center of mass of a system of three different particles. (a) Position vectors are created for each object. (b) The position vectors are multiplied by the mass of the corresponding object. (c) The scaled vectors from part (b) are added together. (d) The final vector is divided by the total mass. This vector points to the center of mass of the system. Note that no mass is actually present at the center of mass of this system.

Since $\vec{r}_j = x_j\hat{i} + y_j\hat{j} + z_j\hat{k}$, it follows that:

$$r_{CM,x} = \frac{1}{m}\sum_{j=1}^{N} m_j x_j \tag{7.4.4}$$

$$r_{CM,y} = \frac{1}{m}\sum_{j=1}^{N} m_j y_j \tag{7.4.5}$$

$$r_{CM,z} = \frac{1}{m}\sum_{j=1}^{N} m_j z_j \tag{7.4.6}$$

and thus

$$\vec{r}_{CM} = r_{CM,x}\hat{i} + r_{CM,y}\hat{j} + r_{CM,z}\hat{k} \tag{7.4.7}$$

$$r_{CM} = |\vec{r}_{CM}| = (r^2_{CM,x} + r^2_{CM,y} + r^2_{CM,z})^{1/2}. \tag{7.4.8}$$

Therefore, you can calculate the components of the center of mass vector individually.

Finally, to complete the kinematics, the instantaneous velocity of the center of mass is calculated exactly as you might suspect:

$$\vec{v}_{CM} = \frac{d}{dt}\left(\frac{1}{M}\sum_{j=1}^{N} m_j\vec{r}_j\right) = \frac{1}{M}\sum_{j=1}^{N} m_j\vec{v}_j \tag{7.4.9}$$

and this, like the position, has x-, y-, and z-components.

7.4.2 Finding the Center of Mass

To calculate the center of mass in actual situations, we recommend the following procedure:

? Problem-Solving Strategy: Calculating the Center of Mass

The center of mass of an object is a position vector. Thus, to calculate it, do these steps:

1. Define your coordinate system. Typically, the origin is placed at the location of one of the particles. This is not required, however.
2. Determine the x, y, z-coordinates of each particle that makes up the object.
3. Determine the mass of each particle, and sum them to obtain the total mass of the object. Note that the mass of the object at the origin must be included in the total mass.
4. Calculate the x-, y-, and z-components of the center of mass vector, using Equation 7.4.4, Equation 7.4.5, and Equation 7.4.6.
5. If required, use the Pythagorean theorem to determine its magnitude.

7.4.3 Examples

Here are a few examples that will give you a feel for what the center of mass is.

Example 7.4.1

Find the center of mass of a sphere of mass M and radius R and a cylinder of mass m, radius r, and height h arranged as shown in Figure 7.4.3. Express your answers in a coordinate system that has the origin at the center of the cylinder

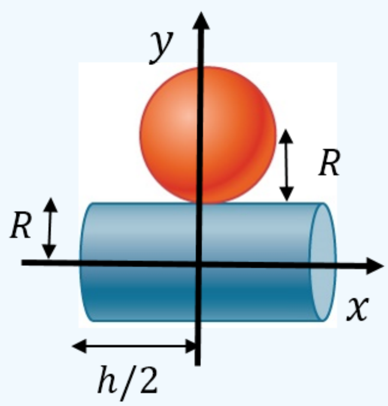


Figure 7.4.3.

Solution

The center of mass can be found by:

$$x_{CM} = \frac{1}{M} \sum_{j=1}^{N} m_j x_j.$$

and

$$y_{CM} = \frac{1}{M} \sum_{j=1}^{N} m_j y_j.$$

In this case:

$$x_{CM} = \frac{1}{M_{sp} + M_{cy}} (M_{sp} x_{sp} + M_{cy} x_{cy}) = \frac{1}{M+m} (M(0) + m(0)) = 0.$$

$$y_{CM} = \frac{1}{M_{sp} + M_{cy}} (M_{sp} y_{sp} + M_{cy} y_{cy}) = \frac{1}{M+m} (M(2R) + m(0)) = \frac{2RM}{M+m}.$$

Example 7.4.2: Center of Mass of the Earth-Moon System

Using data from text appendix, determine how far the center of mass of the Earth-moon system is from the center of Earth. Compare this distance to the radius of Earth, and comment on the result. Ignore the other objects in the solar system.

Strategy

We get the masses and separation distance of the Earth and moon, impose a coordinate system, and use Equation 7.4.3with just N = 2 objects. We use a subscript “e” to refer to Earth, and subscript “m” to refer to the moon.

Solution

Define the origin of the coordinate system as the center of Earth. Then, with just two objects, Equation 7.4.3becomes

$$R = \frac{m_c r_c + m_m r_m}{m_c + m_m}. \tag{7.4.10}$$

From Appendix D,

$$m_c = 5.97 \times 10^{24}\ kg \tag{7.4.11}$$

$$m_m = 7.36 \times 10^{22}\ kg \tag{7.4.12}$$

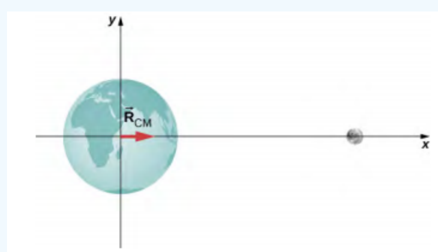
$$r_m = 3.82 \times 10^{5}\ m. \tag{7.4.13}$$

We defined the center of Earth as the origin, so r_e = 0 m. Inserting these into the equation for R gives

$$\begin{aligned} R &= \frac{(5.97 \times 10^{24}\ kg)(0\ m) + (7.36 \times 10^{22}\ kg)(3.82 \times 10^{8}\ m)}{(5.98 \times 10^{24}\ kg) + (7.36 \times 10^{22}\ kg)} \\ &= 4.64 \times 10^{6}\ m. \end{aligned}$$

Significance

The radius of Earth is 6.37 x 10^6 m, so the center of mass of the Earth-moon system is (6.37 − 4.64) x 10^6 m = 1.73 x 10^6 m = 1730 km (roughly 1080 miles) **below** the surface of Earth. The location of the center of mass is shown (not to scale).



Example 7.4.3: Center of Mass of a Salt Crystal

Figure 7.4.5 shows a single crystal of sodium chloride—ordinary table salt. The sodium and chloride ions form a single unit, NaCl. When multiple NaCl units group together, they form a cubic lattice. The smallest possible cube (called the unit cell) consists of four sodium ions and four chloride ions, alternating. The length of one edge of this cube (i.e., the bond length) is 2.36 x 10^{-10} m. Find the location of the center of mass of the unit cell. Specify it either by its coordinates ($r_{CM,x}$, $r_{CM,y}$, $r_{CM,z}$), or by r_{CM} and two angles.

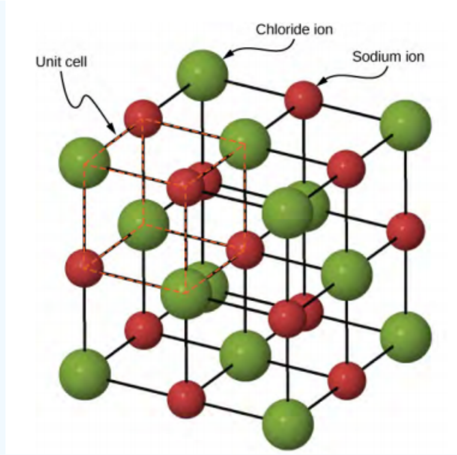


Figure 7.4.5: A drawing of a sodium chloride (NaCl) crystal.

Strategy

We can look up all the ion masses. If we impose a coordinate system on the unit cell, this will give us the positions of the ions. We can then apply Equation 7.4.4, Equation 7.4.5, and Equation 7.4.6(along with the Pythagorean theorem).

Solution

Define the origin to be at the location of the chloride ion at the bottom left of the unit cell. Figure 7.4.6 shows the coordinate system.

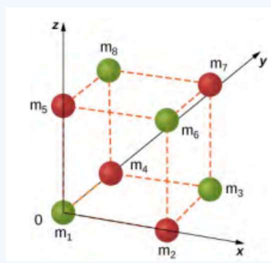


Figure 7.4.6: A single unit cell of a NaCl crystal.

There are eight ions in this crystal, so N = 8:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{j=1}^{8} m_j \vec{r}_j. \tag{7.4.14}$$

The mass of each of the chloride ions is

$$35.453u \times \frac{1.660 \times 10^{-27}\ kg}{u} = 5.885 \times 10^{-26}\ kg \tag{7.4.15}$$

so we have

$$m_1 = m_3 = m_6 = m_8 = 5.885 \times 10^{-26}\ kg. \tag{7.4.16}$$

For the sodium ions,

$$m_2 = m_4 = m_5 = m_7 = 3.816 \times 10^{-26}\ kg. \tag{7.4.17}$$

The total mass of the unit cell is therefore

$$M = (4)(5.885 \times 10^{-26}\ kg) + (4)(3.816 \times 10^{-26}\ kg) = 3.880 \times 10^{-25}\ kg. \tag{7.4.18}$$

From the geometry, the locations are

$$\begin{aligned}
\vec{r}_1 &= 0 \\
\vec{r}_2 &= (2.36 \times 10^{-10}\ m)\hat{i} \\
\vec{r}_3 &= r_{3x}\hat{i} + r_{3y}\hat{j} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{j} \\
\vec{r}_4 &= (2.36 \times 10^{-10}\ m)\hat{j} \\
\vec{r}_5 &= (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_6 &= r_{6x}\hat{i} + r_{6z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_7 &= r_{7x}\hat{i} + r_{7y}\hat{j} + r_{7z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{j} + (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_8 &= r_{8y}\hat{j} + r_{8z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{j} + (2.36 \times 10^{-10}\ m)\hat{k}.
\end{aligned}$$

Substituting:

$$\begin{aligned}
|\vec{r}_{CM,x}| &= \sqrt{r_{CM,x}^2 + r_{CM,y}^2 + r_{CM,z}^2} \\
&= \frac{1}{M}\sum_{j=1}^{8} m_j (r_x)_j \\
&= \frac{1}{M}(m_1 r_{1x} + m_2 r_{2x} + m_3 r_{3x} + m_4 r_{4x} + m_5 r_{5x} + m_6 r_{6x} + m_7 r_{7x} + m_8 r_{8x}) \\
&= \frac{1}{3.8804 \times 10^{-25}\ kg}\Big[(5.885 \times 10^{-26}\ kg)(0\ m) + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) \\
&\quad + (5.885 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + 0 + 0 \\
&\quad + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + 0\Big] \\
&= 1.18 \times 10^{-10}\ m.
\end{aligned}$$

Similar calculations give $r_{CM,y} = r_{CM,z} = 1.18 \times 10^{-10}$ m (you could argue that this must be true, by symmetry, but it's a good idea to check).

Significance

As it turns out, it was not really necessary to convert the mass from atomic mass units (u) to kilograms, since the units divide out when calculating r_{CM} anyway.

To express r_{CM} in terms of magnitude and direction, first apply the three-dimensional Pythagorean theorem to the vector components:

$$\begin{aligned}
r_{CM} &= \sqrt{r_{CM,x}^2 + r_{CM,y}^2 + r_{CM,z}^2} \\
&= (1.18 \times 10^{-10}\ m)\sqrt{3} \\
&= 2.044 \times 10^{-10}\ m.
\end{aligned}$$

Since this is a three-dimensional problem, it takes two angles to specify the direction of $\vec{r}_{CM}$. Let ϕ be the angle in the x,y-plane, measured from the +x-axis, counterclockwise as viewed from above; then:

$$\phi = \tan^{-1}\left(\frac{r_{CM,y}}{r_{CM,x}}\right) = 45^o. \tag{7.4.19}$$

Let θ be the angle in the y,z-plane, measured downward from the +z-axis; this is (not surprisingly):

$$\theta = \tan^{-1}\left(\frac{R_z}{R_y}\right) = 45^o. \tag{7.4.20}$$

Thus, the center of mass is at the geometric center of the unit cell. Again, you could argue this on the basis of symmetry

Two crucial concepts come out of these examples:

1. 1. As with all problems, you must define your coordinate system and origin. For center-of-mass calculations, it often makes sense to choose your origin to be located at one of the masses of your system. That choice automatically defines its distance in Equation 7.4.3 to be zero. However, you must still include the mass of the object at your origin in your calculation of M, the total mass Equation ???. In the Earth-moon system example, this means including the mass of Earth. If you hadn't, you'd have ended up with the center of mass of the system being at the center of the moon, which is clearly wrong.
2. In the second example (the salt crystal), notice that there is no mass at all at the location of the center of mass. This is an example of what we stated above, that there does not have to be any actual mass at the center of mass of an object.
