3.4: Graphical Analysis of Motion

Interpreting Graphs

Motion graphs represent what is happening to the various dependent variables (x, v, and a) over time. There are three goals here:

1. To interpret a graph in terms of the physical motion of the object it represents.
2. To sketch a graph that represents the physical motion of an object, given a description of that motion.
3. To sketch a graph of one or two dependent variables based on the graph of another dependent variable.

> **Alert**
>
> *These are not always easy tasks to perform, for two main reasons: First, our first instinct when we see a graph is to interpret it as a picture, rather than a plot of a quantity vs. time. The second problem (and this persists throughout the study of physics) is the tendency to confuse the change of a quantity for the value of that quantity. More precisely, we tend to lose sight of the fact that a variable's value at an instant and its rate of change are quite independent of each other.*

For task #1, here are some of the questions we should be able to answer:

Q1: Where is the object (which side of the origin is it on)?

This would seem to be quite trivial (and it is): The position at any given time is the value on the vertical axis for the x vs. t graph. Where we run into trouble is thinking that we might have some idea of how to answer this question for the velocity and acceleration graphs. Velocity and acceleration graphs only give us information about the object's changing position and changing speed, respectively, not where the object is at any given time. If we are separately given where the object is at some point in time (say at $t=0$), then we can determine its position at other times. One way to think of this is that the velocity graph gives us the *shape* of the position graph, but that shape could be located anywhere up-and-down the vertical axis.

Q2: Is the object at rest, or is it moving?

Another seemingly obvious question to answer, but again there are things to keep in mind. Although this is a property of velocity we *can* answer it using the position graph (we only get unknown constants when we integrate, not when we take derivatives). Mathematically, we know that the velocity is the slope of the position graph, so since "at rest" means zero velocity, the object is at rest when the tangent line to the x vs. t graph has zero slope. But we should strive to look at this *physically* as well. Obviously an object that is moving is one whose position is changing, so if the x value is changing, the object is moving. If we are given a v vs. t graph, we have to be careful not to use the same criterion as we did for the x vs. t graph. Instead, whether the object is moving or not is a simple matter of whether or not the value of v is zero. If we have the acceleration graph, then integrating it to get the velocity graph leaves an unknown constant (v_o). We know the shape of the v vs. t graph, but not where it is located up-and-down the vertical axis. This means that with just the acceleration graph we cannot know where the velocity graph crosses the horizontal axis, and therefore have no idea where the object is coming to rest.

Q3: Which way is the object moving?

The direction of motion of the object can also be obtained from both the position and velocity graphs.

- From the position graph, we know that the sign of the slope is the sign of the velocity (which is the direction of motion).
- On the velocity graph, we simply need to determine if the value of the velocity is positive or negative (i.e. is the graph below or above the horizontal axis). A common mistake is to confuse these two things. For example, the position graph being below the horizontal axis does not mean the object is moving in the $-x$ direction, and a positive slope of the velocity graph does not mean that the object is moving in the $+x$ direction.
- The acceleration graph does not – by itself – provide information about the direction of the object's motion, because the question of above-or-below the horizontal axis for the velocity graph cannot be answered when the acceleration graph only gives the v vs. t graph's shape.

Q4: Is the object speeding-up or slowing down?

This is probably the trickiest question of all, because it doesn't have a direct correlation to the value or slope of any of the graphs. To make this determination, you actually need *two* pieces of information – the directions of both the velocity and the acceleration. This is because if the object is accelerating in the same direction that it is moving, then it is speeding up, and if it is accelerating in the opposite direction as the direction of motion, then it is slowing down. We therefore cannot determine the answer to this question from the acceleration graph alone, because that graph by itself does not provide the direction of motion (the function $v(t)$ associated with this acceleration could be above or below the horizontal axis anywhere).

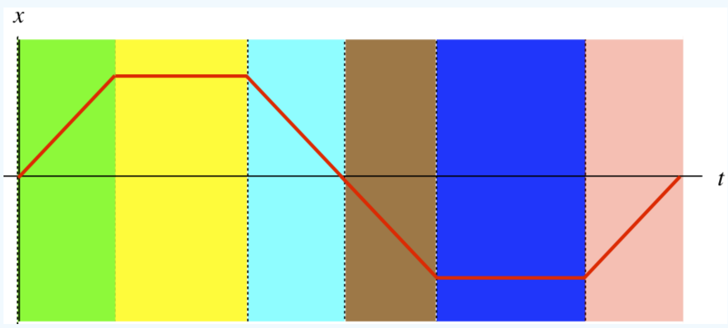
We can determine speeding-up/slowing-down from the v vs. t graph, by comparing the slope of the graph with the value of the graph at the same point. If they have the same sign, then the acceleration is in the same direction as the velocity, and it is speeding up. If they are opposite, then it is slowing down. But there is a simpler, physical way to make this determination: If the v vs. t graph at the point in question is heading closer to the horizontal axis, then its velocity is heading toward zero, and it is slowing down, while if it is heading away, it is speeding up. Naturally horizontal parts of the the v vs. t graph represent motion in which the object is neither speeding up nor slowing down.

Making this determination from the x vs. t graph is even more challenging. Clearly if a section of the x vs. t graph is a straight line, then the velocity is constant, and the object is neither speeding up nor slowing down. So what about when $x(t)$ is curved? The trick to use here is to determine if continuing this curve will eventually cause the graph to go horizontal (i.e. reach a max or a min), or vertical. If it is the former, then the object is slowing (a horizontal slope is stationary), and the latter is speeding up. Note that both of these can occur for either concave or convex curves, for positive or negative slopes, and above or below the horizontal axis.

Examples

Example 3.4.1

For the position vs. time graph of an object moving in one dimension given below answer each of the four questions given above for every segment of time indicated by the different colors.



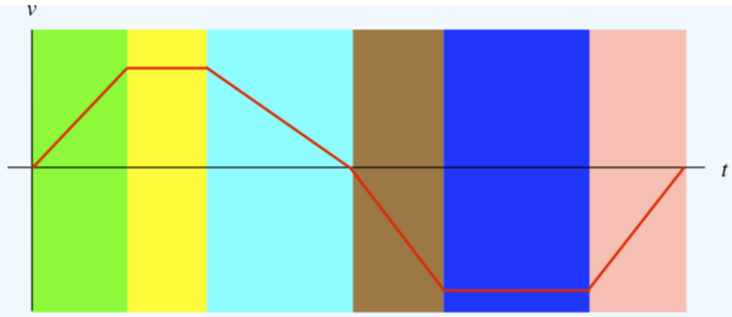
Solution

Where is the object	At rest or moving?	Moving which way?	Speeding up or slowing down?
+ side of origin	**moving**	**+x direction**	**neither - constant speed**
+ side of origin	**at rest**	**not moving**	**neither - not moving**
+ side of origin	**moving**	**-x direction**	**neither - constant speed**
- side of origin	**moving**	**-x direction**	**neither - constant speed**
- side of origin	**at rest**	**not moving**	**neither - not moving**
- side of origin	**moving**	**+ x direction**	**neither - constant speed**

Example 3.4.2

For the velocity vs. time graph of an object moving in one dimension given below:

a. *Answer each of the four questions given above for every segment of time indicated by the different colors.*
b. *Sketch the position vs. time and acceleration vs. time graphs associated with this same motion. Assume that the object was at the origin at time $t=0$.*

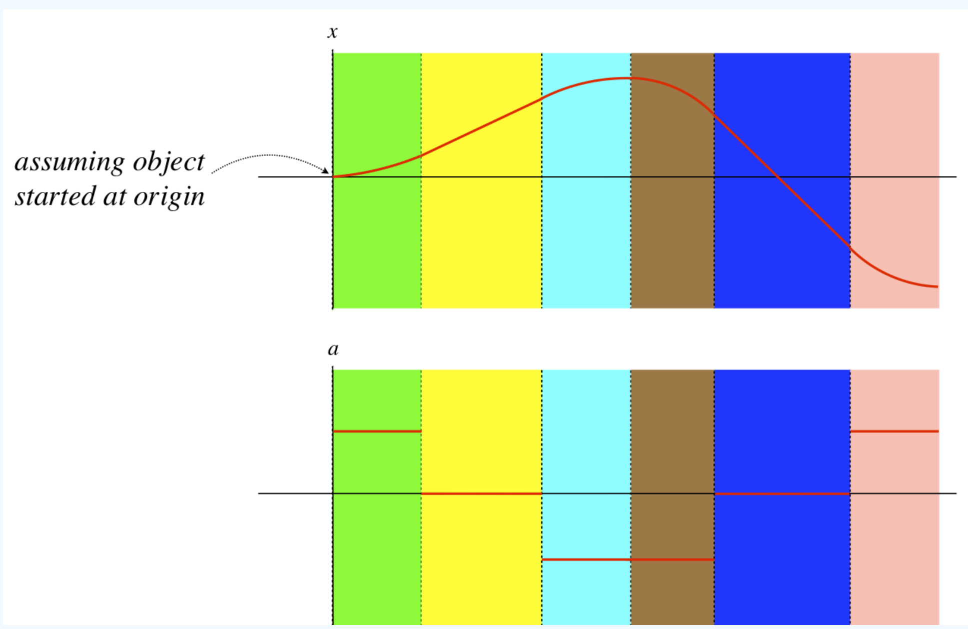


Solution

a.

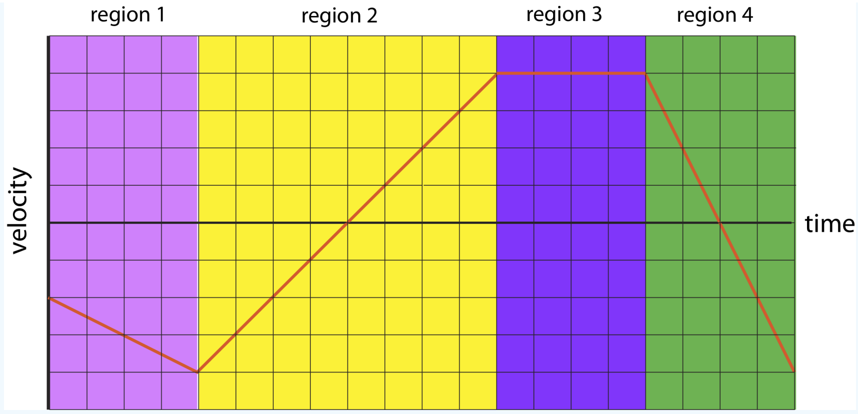
Where is the object	At rest or moving?	Moving which way?	Speeding up or slowing down?
don't know	**moving**	**+x direction**	**speeding up**
don't know	**moving**	**+x direction**	**neither - constant velocity**
don't know	**moving**	**+x direction**	**slowing down**
don't know	**moving**	**-x direction**	**speeding up**
don't know	**moving**	**-x direction**	**neither - constant speed**
don't know	**moving**	**-x direction**	**slowing down**

b.



Example 3.4.3

Below is a velocity versus time plot.



a) For each marked region specify the direction of motion (right, left, or turning around) and describe the speed (zero, speeding up, slowing down, or constant speed).

b) Make an acceleration versus time plot for the entire time shown.

Solution

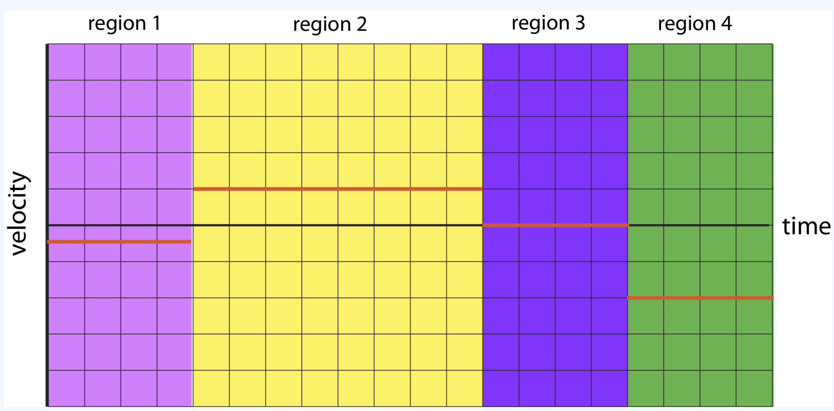
a) Region 1: object starts with non-zero velocity, is moving to the left since velocity is negative and speeding up since the magnitude of velocity is increasing.

Region 2: object is moving to the left and slowing down. After 4 units of time in this region, the object turns around, after which it's moving to the right and speeding up.

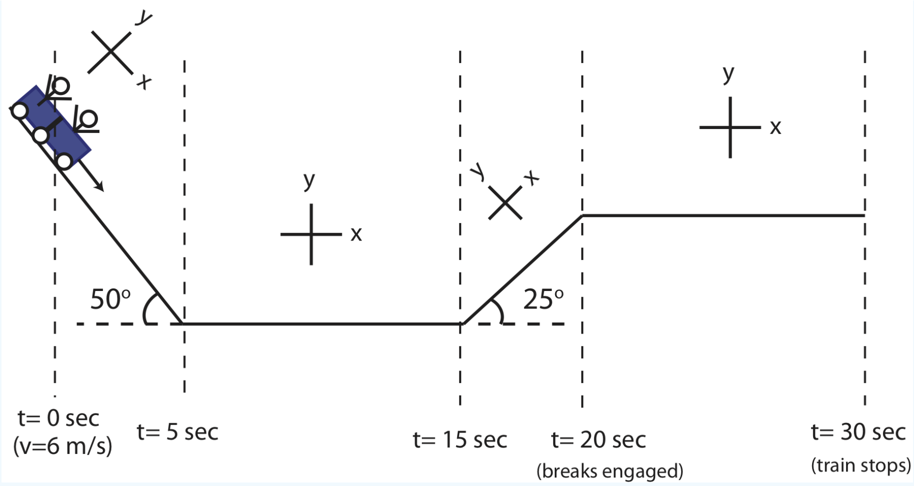
Region 3: object is moving to the right with constant speed.

Region 4: object is moving to the right and slowing down, after 2 units of time it turns around, then moving to the left and speeds up.

b) Below is the acceleration versus time plot. The plots in each region are obtained from the slope of velocity versus time plot for each marked region: region 1 the slope is -1/2, in region 2 the slope is 1, in region 3 the slope is zero, and in region 4 the slope is-2.



Example 3.4.4



Shown above is a rollercoaster ride. At the start of a drop, defined as t=0 sec, the train is moving with a speed of 6 m/s. For the first 5 seconds, the acceleration is due to the x-component of gravity, $a=\frac{F_{gx}}{m}=g\sin\theta=(9.8m/s^2)(\sin 50^{\circ})=7.51m/s^2$. The velocity is related to acceleration as:

$$a=\frac{\Delta v}{\Delta t}=\frac{v_f-v_i}{t_f-t_i}$$

The initial time is 0 seconds. Solving for v_f at the bottom of the ramp, when $t_f=5sec$:

$$v_f=v_i+at_f=(6m/s)+(7.51m/s^2)(5sec)=43.5m/s$$

Between 5 and 15 seconds acceleration is zero since there is no net force, so velocity remains constant at 43.5 m/s. Between 15 and 20 seconds the net forces points in the negative x-direction, again due to the x-component of gravity, $a=\frac{F_{gx}}{m}=-g\sin\theta=-(9.8m/s^2)(\sin 25^{\circ})=-4.14m/s^2$. Solving for v_f at 20 seconds at the top of the ramp:

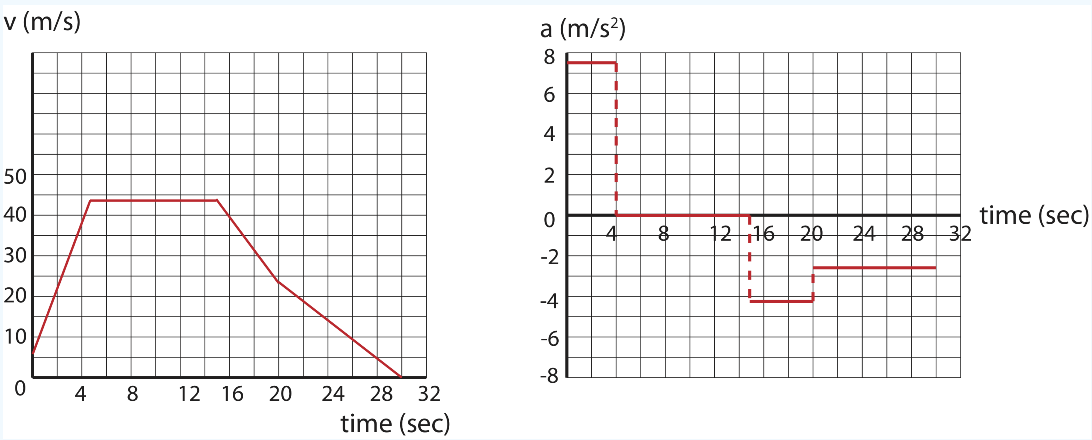
$$v_f=v_i+a(t_f-t_i)=(43.5m/s)+(-4.14m/s^2)(20sec-15sec)=22.8m/s$$

Between 20 and 30 seconds, we don't know the magnitude of the force, but we know that the train stops at 30 seconds:

$$a=\frac{v_f-v_i}{t_f-t_i}=\frac{0-22.8m/s}{30sec-20sec}=-2.28m/s^2$$

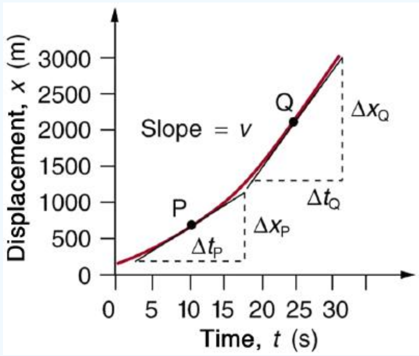
Make a plot for component of velocity and acceleration horizontal the track (x-direction as shown in the figure) as a function of time from t=0 to t=30 sec.

Solution



Example 3.4.5:

Calculate the velocity of the jet car at a time of 25 s by finding the slope of the \(\displaystyle vs. \(\displaystyle graph in the graph below



t (s)	x (m)
0	200
5	338
10	600
15	988
20	1500
25	2138
30	2900

Figure $3.4.5$: The slope of an \(vs. \(*graph is velocity. This is shown at two points. Instantaneous velocity at any point is the slope of the tangent at that point.*

Strategy

The slope of a curve at a point is equal to the slope of a straight line tangent to the curve at that point. This principle is illustrated in Figure, where Q is the point at $t = 25s$.

Solution

1. Find the tangent line to the curve at $t = 25s$.
2. Determine the endpoints of the tangent. These correspond to a position of 1300 m at time 19 s and a position of 3120 m at time 32 s.
3. Plug these endpoints into the equation to solve for the slope, .

$$slope = v_Q = \frac{\Delta x_Q}{\Delta t_Q} = \frac{(3120m - 1300m)}{(32s - 19s)}$$

Thus,

$$v_Q = \frac{1820m}{13s} = 140m/s.$$

Discussion

This is the value given in this figure's table for v at $t = 25s$. The value of 140 m/s for v_Q is plotted in Figure. The entire graph of v vs. t can be obtained in this fashion.

Carrying this one step further, we note that the slope of a velocity versus time graph is acceleration. Slope is rise divided by run; on a v vs. t graph, rise = change in velocity Δv and run = change in time Δt.

THE SLOPE OF V VS. T

The slope of a graph of velocity v vs. time t is acceleration a.

$$slope = \frac{\Delta v}{\Delta t} = a$$

Example 3.4.6: Calculating Acceleration from a Graph of Velocity versus Time

Now consider the motion of the jet car as it goes from 165 m/s to its top velocity of 250 m/s, graphed in Figure $3.4.6$. Time starts at zero, and the initial position and velocity are 2900 m and 165 m/s, respectively. (These were the final position and velocity of the car in the motion graphed in Figure $3.4.4$) Acceleration gradually decreases from $5.0m/s^2$ to zero when the car hits 250 m/s. The slope of the x vs. t graph increases until $t = 55s$, after which time the slope is constant. Similarly, velocity increases until 55 s and then becomes constant, since acceleration decreases to zero at 55 s and remains zero afterward.

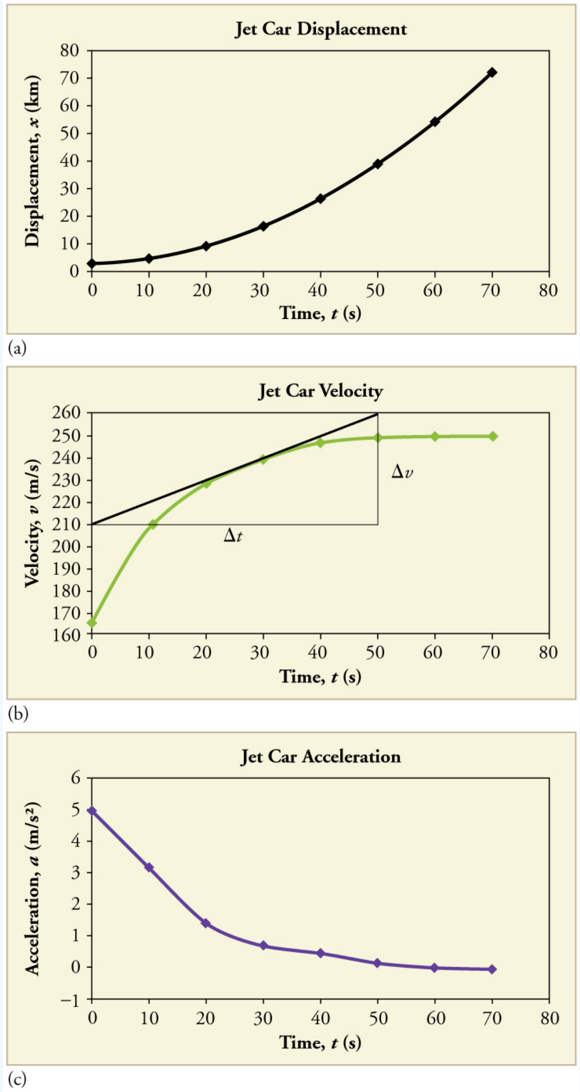


Figure $3.4.3$ ends. (a) The slope of this graph is velocity; it is plotted in the next graph. (b) The velocity gradually approaches its top value. The slope of this graph is acceleration; it is plotted in the final graph. (c) Acceleration gradually declines to zero when velocity becomes constant.

Calculate the acceleration of the jet car at a time of 25 s by finding the slope of the v vs. t graph in Figure $3.4.6b$.

Strategy

The slope of the curve at $t = 25s$ is equal to the slope of the line tangent at that point, as illustrated in Figure $3.4.6b$

Solution

Determine endpoints of the tangent line from the figure, and then plug them into the equation to solve for slope, .

$$slope = \frac{\Delta v}{\Delta t} = \frac{(260m/s - 210m/s)}{(51s - 1.0s)}$$

$$a = \frac{50m/s}{50s} = 1.0m/s^2.$$

Discussion

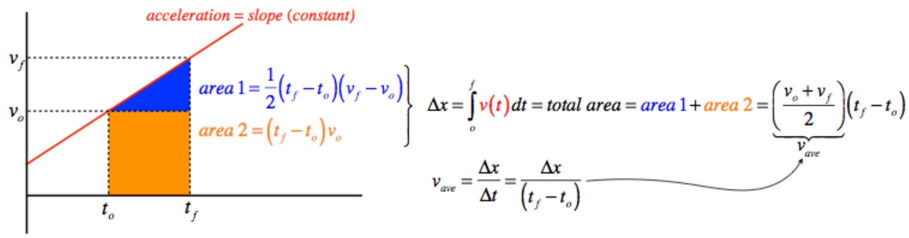
Note that this value for a is consistent with the value plotted in Figure(c) at $t = 25s$.

A graph of displacement versus time can be used to generate a graph of velocity versus time, and a graph of velocity versus time can be used to generate a graph of acceleration versus time. We do this by finding the slope of the graphs at every point. If the graph is linear (i.e., a line with a constant slope), it is easy to find the slope at any point and you have the slope for every point. Graphical analysis of motion can be used to describe both specific and general characteristics of kinematics. Graphs can also be used for other topics in physics. An important aspect of exploring physical relationships is to graph them and look for underlying relationships.

Integrating Using Graphs

We have already seen that we can derive equations of motion for $v(t)$ and $x(t)$ by integrating their derivatives, and we know that integrals of functions equal the areas under the curves those functions represent, so we can use this knowledge to tie together these two facts. If we are given the graph of a motion, we can compute the area under the curve between the starting and ending points to get a definite integral, and therefore the change between the starting and ending values. So for example, if we again assume constant acceleration, a velocity-vs-time graph is a straight line whose slope is the acceleration. The area under this line from the starting time to the ending time will be the displacement between these two times (note: we still don't know the positions for these times, only the change in positions). This actually demonstrates the average velocity relation we found earlier:

Figure 1.5.1 – Area Under Velocity Curve Is Displacement



Notice that it is vital that the acceleration is constant for this formula for average velocity to come out, because the area under the curve involves the area of a triangle that requires a straight line on top. Of course, the average velocity could *accidentally* come out to equal the arithmetic average of the starting and ending velocities when the acceleration is not constant (if the area under the curved graph happens to equal the area under the straight line graph between the same two points), but we cannot rely on such coincidences when solving problems. Moreover, this means that we cannot assume the converse – if the arithmetic mean of a starting and ending velocity equals the average velocity, we cannot conclude that the acceleration was constant over that time interval.

Key Points

As you work with analysis of motion for different physical situations, here are a list of a few key points to keep in mind when making acceleration, velocity and position plots:

General:

- Acceleration, velocity, and position are vectors which can have positive, negative, or zero values depending on their direction and location of origin.
- If you know one of the graphs, you can obtain the other two as long as you know initial conditions.

Acceleration:

- Acceleration points in the same direction as the net force.
- Acceleration is zero when the net force is zero.
- Acceleration plot alone does not contain any information whether the object is speeding up or slowing down, since it does not tell you which way the object is moving, only in which direction its velocity is changing.

Velocity:

- The slope of the velocity plot is the acceleration.
- Initial velocity has to be known to make the velocity versus time plot.
- When acceleration and velocity have the same sign, the system is speeding up.
- When acceleration and velocity have opposite signs, the system is slowing down.
- When velocity plot crosses the x-axis (changes sign), the system is turning around.
- When the slope of velocity plot is zero, acceleration is zero, implying zero net force.

Position:

- The slope of the position plot is the velocity.
- Initial position has to be known to make the position versus time plot.
- Positive slope means the object is moving to the right.
- Negative slope means the object is moving to the left.
- Increasing magnitude of slope (either negative or positive) means the object is speeding up.
- Decreasing magnitude of slope (either negative or positive) means the object is slowing down.
- Zero slope means the object is stationary.
- When the sign of the slope of the position plot changes, this means the object has changed direction of motion.
- The sign of the position plot does not tell us about the direction of motion, but an indication of whether the object is located on the positive or negative side of the origin.