5.11: Forces on Rotating Bodies

Learning Objectives

- Describe how the magnitude of a torque depends on the magnitude of the lever arm and the angle the force vector makes with the lever arm
- Determine the sign (positive or negative) of a torque using the right-hand rule
- Calculate individual torques about a common axis and sum them to find the net torque

An important quantity for describing the dynamics of a rotating rigid body is torque. We see the application of torque in many ways in our world. We all have an intuition about torque, as when we use a large wrench to unscrew a stubborn bolt. Torque is at work in unseen ways, as when we press on the accelerator in a car, causing the engine to put additional torque on the drive train. Or every time we move our bodies from a standing position, we apply a torque to our limbs. In this section, we define torque and make an argument for the equation for calculating torque for a rigid body with fixed-axis rotation.

5.11.1 Defining Torque

So far we have defined many variables that are rotational equivalents to their translational counterparts. Let's consider what the counterpart to force must be. Since forces change the translational motion of objects, the rotational counterpart must be related to changing the rotational motion of an object about an axis. We call this rotational counterpart **torque**.

In everyday life, we rotate objects about an axis all the time, so intuitively we already know much about torque. Consider, for example, how we rotate a door to open it. First, we know that a door opens slowly if we push too close to its hinges; it is more efficient to rotate a door open if we push far from the hinges. Second, we know that we should push perpendicular to the plane of the door; if we push parallel to the plane of the door, we are not able to rotate it. Third, the larger the force, the more effective it is in opening the door; the harder you push, the more rapidly the door opens. The first point implies that the farther the force is applied from the axis of rotation, the greater the angular acceleration; the second implies that the effectiveness depends on the angle at which the force is applied; the third implies that the magnitude of the force must also be part of the equation. Note that for rotation in a plane, torque has two possible directions. Torque is either clockwise or counterclockwise relative to the chosen pivot point. Figure $5.11.1$ shows counterclockwise rotations.

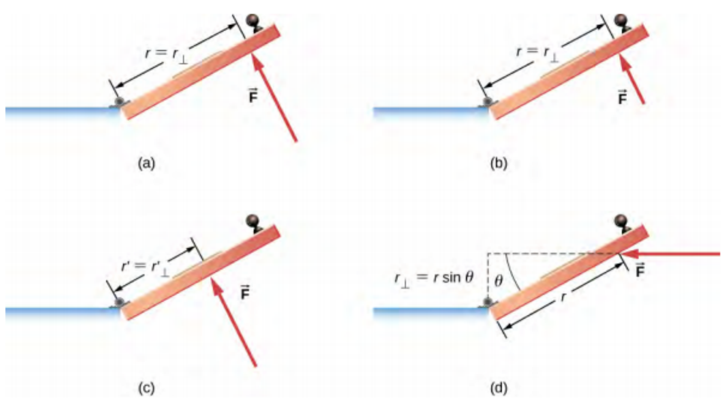

Figure 5.11.1: Torque is the turning or twisting effectiveness of a force, illustrated here for door rotation on its hinges (as viewed from overhead). Torque has both magnitude and direction. (a) A counterclockwise torque is produced by a force $\vec{F}$ acting at a distance r from the hinges (the pivot point). (b) A smaller counterclockwise torque is produced when a smaller force $\vec{F}'$ acts at the same distance r from the hinges. (c) The same force as in (a) produces a smaller counterclockwise torque when applied at a smaller distance from the hinges. (d) A smaller counterclockwise torque is produced by the same magnitude force as (a) acting at the same distance as (a) but at an angle θ that is less than 90°.

Now let's consider how to define torques in the general three-dimensional case.

Torque

When a force $\vec{F}$ is applied to a point P whose position is $\vec{r}$ relative to O (Figure $5.11.2$), the torque $\vec{\tau}$ around O is

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{5.11.1}$$

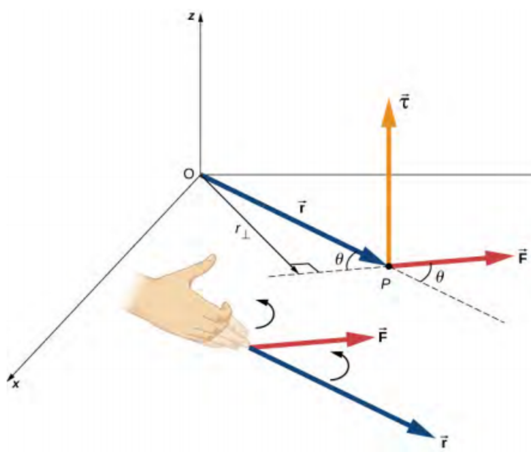

Figure $5.11.2$: The torque is perpendicular to the plane defined by $\vec{r}$ and $\vec{F}$ and its direction is determined by the right-hand rule.

From the definition of the cross product, the torque $\vec{\tau}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ and has magnitude

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = rF\sin\theta, \tag{5.11.2}$$

where θ is the angle between the vectors $\vec{r}$ and $\vec{F}$. The SI unit of torque is newtons times meters, usually written as N • m. The quantity $r_\perp = r\sin\theta$ is the perpendicular distance from O to the line determined by the vector $\vec{F}$ and is called the **lever arm**. Note that the greater the lever arm, the greater the magnitude of the torque. In terms of the lever arm, the magnitude of the torque is

$$|\vec{\tau}| = r_\perp F. \tag{5.11.3}$$

The cross product $\vec{r} \times \vec{F}$ also tells us the sign of the torque. In Figure $5.11.2$, the cross product $\vec{r} \times \vec{F}$ is along the positive z-axis, which by convention is a positive torque. If $\vec{r} \times \vec{F}$ is along the negative z-axis, this produces a negative torque.

If we consider a disk that is free to rotate about an axis through the center, as shown in Figure $5.11.3$, we can see how the angle between the radius $\vec{r}$ and the force $\vec{F}$ affects the magnitude of the torque. If the angle is zero, the torque is zero; if the angle is 90°, the torque is maximum. The torque in Figure $5.11.3$ is positive because the direction of the torque by the right-hand rule is out of the page along the positive z-axis. The disk rotates counterclockwise due to the torque, in the same direction as a positive angular acceleration.

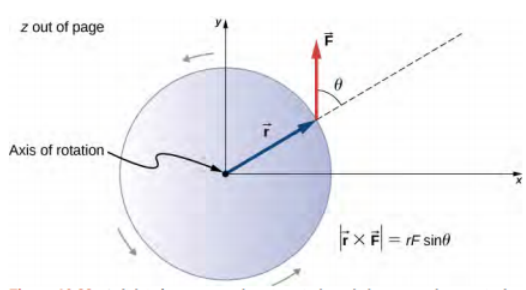

Figure $5.11.3$: A disk is free to rotate about its axis through the center. The magnitude of the torque on the disk is rFsin θ.When θ = 0°, the torque is zero and the disk does not rotate. When θ = 90°, the torque is maximum and the disk rotates with maximum angular acceleration.

Any number of torques can be calculated about a given axis. The individual torques add to produce a net torque about the axis. When the appropriate sign (positive or negative) is assigned to the magnitudes of individual torques about a specified axis, the net torque about the axis is the sum of the individual torques:

$$|\vec{\tau}_{net}| = \sum_i |\vec{\tau}_i|. \tag{5.11.4}$$

5.11.2 Calculating Net Torque for Rigid Bodies on a Fixed Axis

In the following examples, we calculate the torque both abstractly and as applied to a rigid body. We first introduce a problem-solving strategy.

Problem-Solving Strategy: Finding Net Torque

1. Choose a coordinate system with the pivot point or axis of rotation as the origin of the selected coordinate system.
2. Determine the angle between the lever arm $\vec{r}$ and the force vector.
3. Take the cross product of $\vec{r}$ and $\vec{F}$ to determine if the torque is positive or negative about the pivot point or axis.
4. Evaluate the magnitude of the torque using $r_\perp F$.
5. Assign the appropriate sign, positive or negative, to the magnitude.
6. Sum the torques to find the net torque.

Example 10.14: Calculating Torque

Four forces are shown in Figure 5.11.4 at particular locations and orientations with respect to a given xy-coordinate system. Find the torque due to each force about the origin, then use your results to find the net torque about the origin.

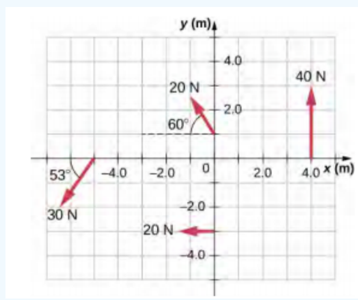

Figure 5.11.4: Four forces producing torques.

Strategy

This problem requires calculating torque. All known quantities—forces with directions and lever arms—are given in the figure. The goal is to find each individual torque and the net torque by summing the individual torques. Be careful to assign the correct sign to each torque by using the cross product of $\vec{r}$ and the force vector $\vec{F}$.

Solution

Use $|\vec{\tau}| = r_\perp F = rF\sin\theta$ to find the magnitude and $\vec{\tau} = \vec{r} \times \vec{F}$ to determine the sign of the torque.

The torque from force 40 N in the first quadrant is given by (4)(40)sin 90° = 160 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page, positive.

The torque from force 20 N in the third quadrant is given by −(3)(20)sin 90° = − 60 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is into the page, so it is negative.

The torque from force 30 N in the third quadrant is given by (5)(30)sin 53° = 120 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page, positive.

The torque from force 20 N in the second quadrant is given by (1)(20)sin 30° = 10 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page.

The net torque is therefore $\tau_{net} = \sum_i |\tau_i|$ = 160 − 60 + 120 + 10 = 230 N • m.

Significance

Note that each force that acts in the counterclockwise direction has a positive torque, whereas each force that acts in the clockwise direction has a negative torque. The torque is greater when the distance, force, or perpendicular components are greater.

✓ Example 10.15: Calculating Torque on a rigid body

Figure 5.11.5 shows several forces acting at different locations and angles on a flywheel. We have $|\vec{F}_1|$ = 20 N, $|\vec{F}_2|$ = 30 N, $|\vec{F}_3|$ = 30 N, and r = 0.5 m. Find the net torque on the flywheel about an axis through the center.

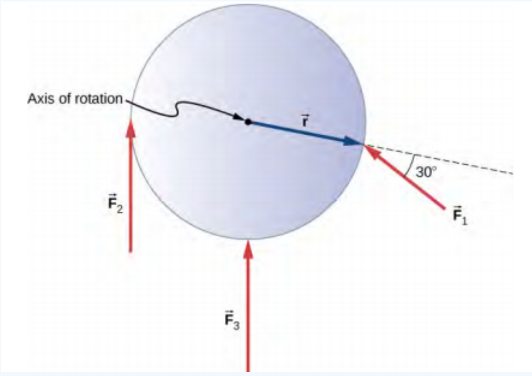

Figure 5.11.5: Three forces acting on a flywheel.

Solution

We calculate each torque individually, using the cross product, and determine the sign of the torque. Then we sum the torques to find the net torque. Solution We start with $\vec{F}_1$. If we look at Figure 5.11.5 we see that $\vec{F}_1$ makes an angle of 90° + 60° with the radius vector $\vec{r}$. Taking the cross product, we see that it is out of the page and so is positive. We also see this from calculating its magnitude:

$$|\vec{\tau}_1| = rF_1 \sin 150^o = (0.5\ m)(20\ N)(0.5) = 5.0\ N \cdot m. \tag{5.11.5}$$

Next we look at $\vec{F}_2$. The angle between $\vec{F}_2$ and $\vec{r}$ is 90° and the cross product is into the page so the torque is negative. Its value is

$$|\vec{\tau}_2| = -rF_2 \sin 90^o = (-0.5\ m)(30\ N) = -15.0\ N \cdot m. \tag{5.11.6}$$

When we evaluate the torque due to $\vec{F}_3$, we see that the angle it makes with $\vec{r}$ is zero so $\vec{r} \times \vec{F}_3$ = 0. Therefore, $\vec{F}_3$ does not produce any torque on the flywheel.

We evaluate the sum of the torques:

$$\tau_{net} = \sum_i |\tau_i| = 5 - 15 = -10\ N \cdot m. \tag{5.11.7}$$

Significance

The axis of rotation is at the center of mass of the flywheel. Since the flywheel is on a fixed axis, it is not free to translate. If it were on a frictionless surface and not fixed in place, $\vec{F}_3$ would cause the flywheel to translate, as well as $\vec{F}_1$. Its motion would be a combination of translation and rotation.

5.11.3 Rotational dynamics for a single particle

Suppose that a single force, $\vec{F}$, is acting on a particle of mass m. Newton's Second Law for the particle is then given by:

$$\vec{F} = m\vec{a}$$

We can define a point of rotation such that $\vec{r}$ is the position of the particle relative to that point. We can take the cross-product of $\vec{r}$ with both sides of the equation in Newton's Second Law:

$$\vec{r} \times \vec{F} = m\vec{r} \times \vec{a}$$

The left hand-side of the equation is called “the torque of $\vec F$ relative to the point of rotation”, and is usually denoted by $\vec\tau$:

$$\vec\tau = \vec r \times \vec F \tag{5.11.8}$$

The right-hand side of the equation is related to the angular acceleration vector, $\vec\alpha$, about that point of rotation:

$$m\vec r \times \vec a = mr^2\vec\alpha$$

Putting this altogether, we get:

$$\vec\tau = mr^2\vec\alpha$$

If more than one force is exerted on the particle, it is easy to show that the **net torque** from the net force on the particle **is equal to the sum of the torques on the particle**:

$$\begin{aligned} \vec r \times (\vec F_1 + \vec F_2 + \vec F_3 + \ldots) &= (\vec r \times \vec F_1 + \vec r \times \vec F_2 + \vec r \times \vec F_3 + \ldots) \\ \therefore \vec r \times \sum \vec F &= \sum \vec\tau = \vec\tau^{net} \end{aligned}$$

We can write “Newton’s Second Law for the rotational dynamics of a particle”:

$$\sum \vec\tau = \vec\tau^{net} = mr^2\vec\alpha \tag{5.11.9}$$

This equation provides us an alternate formulation to Newton’s Second Law that is useful for describing the motion of a particle that is rotating. The left-hand side of the equation corresponds to the “causes of motion” (much like the sum of the forces in Newton’s Second Law), and the right-hand side of the equation to the inertia and the kinematics. A few things to note when comparing to Newton’s Second Law:

1. The rotational quantities, torque and angular acceleration, **are only defined with respect to a point or axis of rotation** (as this determines the vector $\vec r$). If one chooses a different point of rotation, then the torque and angular acceleration will be different.
2. The angular acceleration of a particle is proportional to the net torque exerted on it, much like the linear acceleration is proportional to the net force exerted on the particle.
3. Torque about a center of rotation can be thought of as the equivalent of a force that causes things rotate about an axis that goes through the point of rotation and that is parallel to the torque/angular acceleration vectors.
4. Instead of mass, it is mass times r^2 that plays the role of inertia and determines how large of an angular acceleration a particle will experience for a given net torque.

Example 5.11.1

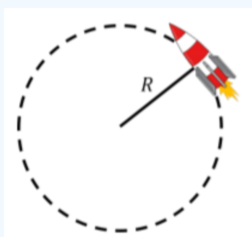

Figure 5.11.1: A toy rocket accelerating around a circle of radius R, as seen from above.

A toy rocket is attached to a string on a horizontal frictionless table, as shown in Figure 5.11.1. The rocket has a mass m and produces a constant force of thrust with a magnitude F that accelerates the rocket along a circle of radius R (the length of the string). If the rocket starts at rest, what distance along the circumference of the circle will the rocket have traveled after a time, t?

Solution

We can model the rocket as a point particle of mass m with the following forces exerted on it:

1. $\vec F$, the thrust of the rocket, always acting tangent to the circle.
2. $\vec T$, the force of tension in the string, always acting towards the center of the circle.
3. $\vec F_g$, the rocket’s weight, acting into the page, with magnitude mg.
4. $\vec N$, a normal force exerted by the table, out of the page, with magnitude mg.

Because the normal force and the weight are equal in magnitude and opposite in direction, the net force will be the sum of the force of thrust and the force of tension, which are always perpendicular to each other. Thinking about this with Newton's Second Law, we could model the force of thrust as increasing the speed of the particle, while the force of tension keeps the rocket moving in a circle (it can do no work to increase the speed, since it is always perpendicular to the motion).

We can also think about this in terms of torques and angular acceleration about the center of the circle. The thrust will result in a net torque about the center of rotation, which will lead to the rocket having an angular acceleration. By determining the angular acceleration, we can then model the displacement at some time, t, using kinematics. The force of tension will create no torque about the center of the circle because the force of tension is always co-linear with the position vector, $\vec{r}$ (the cross-product of co-linear vectors is always zero).

We introduce a coordinate system whose origin coincides with the center of the circle, as shown in Figure $5.11.2$ so that $\vec{r}$ corresponds to the position of the rocket relative to the origin. The force of thrust and the tension are also shown in the diagram. We choose the direction of the x axis such that the rocket was located at the intersection of the x axis and the circle at time, $t=0$.

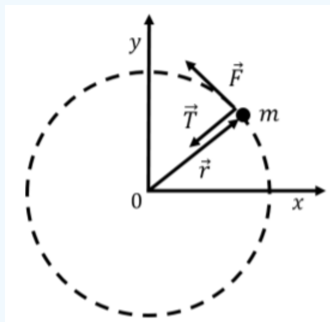

Figure $5.11.2$: Coordinate system to describe the motion of the rocket.

The net torque on the rocket about the point of rotation is given by the cross-product between the thrust force, $\vec{F}$, and the position vector, $\vec{r}$:

$$\vec{\tau}^{net} = \vec{r} \times \vec{F}$$

and will point in the positive z direction (as given by the right hand rule). $\vec{r}$ and $\vec{F}$ are perpendicular, so the magnitude of the net torque is given by:

$$\tau^{net} = rF\sin(90^{\circ}) = RF$$

where R is the magnitude of $\vec{r}$. The net torque vector is thus:

$$\vec{\tau}^{net} = RF\hat{z}$$

Applying the rotational version of Newton's Second Law allows us to determine the angular acceleration:

$$\begin{aligned}\vec{\tau}^{net} &= mr^2\vec{\alpha} \\ RF\hat{z} &= mR^2\vec{\alpha} \\ \therefore \vec{\alpha} &= \frac{F}{mR}\hat{z}\end{aligned}$$

The angular acceleration vector points in the positive z direction (as does the net torque), and indicates that the rocket is accelerating in the counter-clockwise direction about the z axis.

After a period of time t, the rocket will have covered an angular displacement, $\Delta\theta$, given by:

$$\begin{aligned}\Delta\theta &= \theta(t) - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= \frac{1}{2}\frac{F}{mR}t^2\end{aligned}$$

The linear displacement, Δs, that corresponds to this angular displacement is:

$$\Delta s=R\Delta\theta=\frac{1}{2}\frac{F}{m}t^2$$

Discussion:

The formula that we found for the total linear displacement is the same that we would have found if the particle were moving in a straight line with a net force F applied to it (as the particle would have a constant acceleration given by F/m).