3.3: Acceleration

Learning Objectives

- Calculate the average acceleration between two points in time.
- Calculate the instantaneous acceleration given the functional form of velocity.
- Calculate the acceleration vector given the velocity function in unit vector notation.
- Describe the motion of a particle with a constant acceleration in three dimensions.
- Use the one-dimensional motion equations along perpendicular axes to solve a problem in two or three dimensions with a constant acceleration.
- Find instantaneous acceleration at a specified time on a graph of velocity versus time.

Mathematically, acceleration and velocity are similar. Acceleration is related to velocity the same way velocity is related to position.

3.3.1 Average Acceleration

The **average acceleration** $\vec{a}_{avg}$ is defined as the change in velocity between two times divided by the time taken for the change.

Average Velocity

If $\vec{v}_1$ and $\vec{v}_2$ are the velocities of an object at times t_1 and t_2, respectively, then

$$\overrightarrow{Average\ acceleration} = \vec{a}_{avg} = \frac{\overrightarrow{change\ in\ velocity\ between\ the\ two\ times}}{Elapsed\ time} \tag{3.3.1}$$

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}. \tag{3.3.2}$$

Because acceleration is velocity in meters divided by time in seconds, the SI units for acceleration are often abbreviated m/s^2—that is, meters per second squared or meters per second per second. This literally means by how many meters per second the velocity changes every second. Recall that velocity is a vector—it has both magnitude and direction—which means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in direction. For example, if a runner traveling at constant speed of 10 km/h on a circular track, her velocity has changed as a result of the change in direction, although the **magnitude** of the velocity has not changed. Thus, acceleration occurs when velocity changes in magnitude (an increase or decrease in speed) or in direction, or both.

Acceleration as a Vector

Acceleration is a vector in the same direction as the **change** in velocity, Δv. Since velocity is a vector, it can change in magnitude or in direction, or both. Acceleration is, therefore, a change in speed or direction, or both.

Keep in mind that acceleration is not always in the direction of motion. When an object slows down, its acceleration is opposite to the direction of its motion. When only the direction of the velocity changes, the acceleration is perpendicular to the direction of motion.

Alert

1. *In common day language acceleration means speeding up. In physics acceleration occurs when we speed up, slow down, or change direction. There acceleration whenever the velocity changes.*
2. *Acceleration is in the direction of motion when the object speeds up, is opposite to the direction of motion when the object slows down and perpendicular to the direction of motion when the object changes direction.*
3. *There is a term that is often used when talking of an object slowing down, deceleration. The term is not necessary to describe motion since acceleration already accounts for both speeding up and slowing down. It will not be used in this course.*

3.3.2 Instantaneous Acceleration

In addition to obtaining the displacement and velocity vectors of an object in motion, we often want to know its **acceleration vector** at any point in time along its trajectory. This acceleration vector is the instantaneous acceleration and it can be obtained from the derivative with respect to time of the velocity function, as we have seen in a previous chapter. The only difference in two or three dimensions is that these are now vector quantities. Taking the derivative with respect to time $\vec{v}$(t), we find

$$\vec{a}(t)=\lim_{t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}=\frac{d\vec{v}(t)}{dt}. \tag{3.3.3}$$

Since $\vec{v}(t)=\frac{d\vec{r}(t)}{dt}$, we can also write:

$$\vec{a}(t)=\frac{d\vec{v}(t)}{dt}=\frac{d^2\vec{r}(t)}{dt^2}. \tag{3.3.4}$$

The acceleration in terms of components is

$$\vec{a}(t)=\frac{dv_x(t)}{dt}\ \hat{i}+\frac{dv_y(t)}{dt}\ \hat{j}+\frac{dv_z(t)}{dt}\ \hat{k}. \tag{3.3.5}$$

Also, since the velocity is the derivative of the position function, we can write the acceleration in terms of the second derivative of the position function:

$$\vec{a}(t)=\frac{d^2x(t)}{dt^2}\ \hat{i}+\frac{d^2y(t)}{dt^2}\ \hat{j}+\frac{d^2z(t)}{dt^2}\ \hat{k}. \tag{3.3.6}$$

3.3.3 Another Look at Kinematic Quantities

Let’s begin with a particle with an acceleration $\vec{a}(t)$ is a known function of time. Since the time derivative of the velocity function is acceleration,

$$\frac{d}{dt}\vec{v}(t)=\vec{a}(t), \tag{3.3.7}$$

we can take the indefinite integral of both sides, finding

$$\int\frac{d}{dt}\vec{v}(t)dt=\int\vec{a}(t)dt+\overrightarrow{C_1}, \tag{3.3.8}$$

where $\overrightarrow{C_1}$ is a constant of integration. Since $\int\frac{d}{dt}\vec{v}(t)dt=\vec{v}(t)$, the velocity is given by

$$\vec{v}(t)=\int\vec{a}(t)dt+\overrightarrow{C_1}. \tag{3.3.9}$$

Similarly, the time derivative of the position function is the velocity function,

$$\frac{d}{dt}\vec{r}(t)=\vec{v}(t). \tag{3.3.10}$$

Thus, we can use the same mathematical manipulations we just used and find

$$x(t)=\int\vec{v}(t)dt+\overrightarrow{C_2}, \tag{3.3.11}$$

where $\overrightarrow{C_2}$ is a second constant of integration.

3.3.4 Constant Acceleration

We can derive the kinematic equations for a constant acceleration using these integrals. With $\vec{a}(t)=\vec{a}$, $\vec{a}$ constant, and doing the integration in Equation 3.3.9, we find

$$\vec{v}(t)=\int\vec{a}dt+\overrightarrow{C_1}=\vec{a}t+\overrightarrow{C_1}. \tag{3.3.12}$$

If the initial velocity is $\vec{v}(0)=\vec{v}_0$, then

$$\vec{v}_0=0+\overrightarrow{C_1}. \tag{3.3.13}$$

Then, $\overrightarrow{C_1}=\vec{v}_0$ and

$$\vec{v}(t)=\vec{v}_0+\vec{a}t, \tag{3.3.14}$$

Substituting this expression into Equation 3.3.11 gives

$$\vec{r}(t)=\int(\vec{v}_0+\vec{a}t)dt+\overrightarrow{C_2}. \tag{3.3.15}$$

Doing the integration, we find

$$\vec{r}(t)=\vec{v}_0t+\frac{1}{2}\vec{a}t^2+\overrightarrow{C_2}. \tag{3.3.16}$$

If $\vec{r}(0)=\vec{r}_0$, we have

$$\vec{r}_0=0+0+\overrightarrow{C_2}. \tag{3.3.17}$$

so, $C_2=\vec{r}_0$ Substituting back into the equation for $\vec{r}(t)$, we finally have

$$\vec{r}(t)=\vec{r}_0+\vec{v}_0t+\frac{1}{2}\vec{a}t^2. \tag{3.3.18}$$

Like all motion, multidimensional motion with constant acceleration is equivalent to three "independent" one-dimensional motions, each along an axis perpendicular to the others. To develop the relevant equations in each direction, let’s consider the two-dimensional problem of a particle moving in the xy plane with constant acceleration, ignoring the z-component for the moment. The acceleration vector is

$$\vec{a}=a_{0x}\ \hat{i}+a_{0y}\ \hat{j}. \tag{3.3.19}$$

Each component of the motion has a separate set of equations similar to Equation 3.10–Equation 3.14 of the previous chapter on one-dimensional motion. We show only the equations for position and velocity in the x- and y-directions. A similar set of kinematic equations could be written for motion in the z-direction:

$$x(t)=x_0+(v_x)_{avg}t \tag{3.3.20}$$

$$v_x(t)=v_{0x}+a_xt \tag{3.3.21}$$

$$x(t)=x_0+v_{0x}t+\frac{1}{2}a_xt^2 \tag{3.3.22}$$

$$v_x^2(t)=v_{0x}^2+2a_x(x-x_0) \tag{3.3.23}$$

$$y(t)=y_0+(v_y)_{avg}t \tag{3.3.24}$$

$$v_y(t)=v_{0y}+a_yt \tag{3.3.25}$$

$$y(t)=y_0+v_{0y}t+\frac{1}{2}a_yt^2 \tag{3.3.26}$$

$$v_y^2(t)=v_{0y}^2+2a_y(y-y_0). \tag{3.3.27}$$

Here the subscript 0 denotes the initial position or velocity. Equation 3.3.20 to 3.3.27 can be substituted into Equation 4.2 and Equation 4.5 without the z-component to obtain the position vector and velocity vector as a function of time in two dimensions:

$$\vec{r}(t)=x(t)\ \hat{i}+y(t)\ \hat{j} \tag{3.3.28}$$

and

$$\vec{v}(t)=v_x(t)\ \hat{i}+v_y(t)\ \hat{j}. \tag{3.3.29}$$

With Equations 3.3.3-3.3.6 we have completed the set of expressions for the position, velocity, and acceleration of an object moving in two or three dimensions. If the trajectories of the objects look something like the “Red Arrows” in the opening picture for the chapter, then the expressions for the position, velocity, and acceleration can be quite complicated. In the sections to follow we examine two special cases of motion in two and three dimensions by looking at projectile motion and circular motion.

3.3.5 The Independence of Perpendicular Motions

When we look at the three-dimensional equations for position and velocity written in unit vector notation, Equation ??? and Equation ???, we see the components of these equations are separate and unique functions of time that do not depend on one another. We can write:

$$\vec{v}(t) = v_x(t)\ \hat{i} + v_y(t)\ \hat{j} + v_z(t)\ \hat{k} = \frac{dx(t)}{dt}\ \hat{i} + \frac{dy(t)}{dt}\ \hat{j} + \frac{dz(t)}{dt}\ \hat{k}. \tag{3.3.30}$$

and,

$$\vec{v}_{avg}(t) = v_{x_{avg}}\ \hat{i} + v_{y_{avg}}\ \hat{j} + v_{z_{avg}}\ \hat{k} = \frac{\Delta x}{\Delta t}\ \hat{i} + \frac{\Delta y}{\Delta t}\ \hat{j} + \frac{\Delta z}{\Delta t}\ \hat{k}. \tag{3.3.31}$$

Motion along the x direction has no part of its motion along the y and z directions, and similarly for the other two coordinate axes. Thus, the motion of an object in two or three dimensions can be divided into separate, independent motions along the perpendicular axes of the coordinate system in which the motion takes place.

To illustrate this concept with respect to displacement, consider a person walking from point A to point B in a city with square blocks. The person taking the path from A to B may walk east for so many blocks and then north (two perpendicular directions) for another set of blocks to arrive at B. How far the person walks east is affected only by the motion eastward. Similarly, how far the person walks north is affected only by the motion northward.

Independence of Motion

In the kinematic description of motion, we are able to treat the horizontal and vertical components of motion separately. In many cases, motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

Table 3.3.2 - Equations of Motion along the various directions

Quantity	General Expression	Along x	Along y
	$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$		
	$\vec{v}(t) = v_x(t)\ \hat{i} + v_y(t)\ \hat{j}$		
	$\vec{a} = a_x(t)\ \hat{i} + a_y(t)\ \hat{j}$		
Position	$\vec{r}(t)$	$x(t)$	$y(t)$
Average Velocity	$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$	$v_{x_{avg}} = \frac{\Delta x}{\Delta t}$	$v_{y_{avg}} = \frac{\Delta y}{\Delta t}$
Instantaneous Velocity	$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$	$v_x(t) = \frac{dx(t)}{dt}$	$v_y(t) = \frac{dy(t)}{dt}$
Average Acceleration	$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$	$a_{x_{avg}} = \frac{\Delta v_x}{\Delta t}$	$a_{y_{avg}} = \frac{\Delta v_y}{\Delta t}$
Instantaneous Acceleration	$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}$	$a_x(t) = \frac{dv_x(t)}{dt} = \frac{d^2x(t)}{dt^2}$	$a_y(t) = \frac{dv_y(t)}{dt} = \frac{d^2y(t)}{dt^2}$
Motion at constant acceleration			
	$\vec{r}(t) = \vec{r}_0 + (\vec{v})_{avg}t$	$x(t) = x_0 + (v_x)_{avg}t$	$y(t) = y_0 + (v_y)_{avg}t$
	$\vec{v}_y(t) = \vec{v}_{0y} + a_yt$	$v_x(t) = v_{0x} + a_xt$	$v_y(t) = v_{0y} + a_yt$
	$\vec{r}(t) = \vec{r}_0 + \vec{v}_0t + \frac{1}{2}a_yt^2$	$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_xt^2$	$y(t) = y_0 + v_{0y}t + \frac{1}{2}a_yt^2$
	$v_y^2(t) = \vec{v}_0^2 + 2a_y(\vec{r} - \vec{r}_0)$	$v_x^2(t) = v_{0x}^2 + 2a_x(x - x_0)$	$v_y^2(t) = v_{0y}^2 + 2a_y(y - y_0)$

the acceleration, the greater the change in velocity over a given time. Acceleration is widely seen in experimental physics. In linear particle accelerator experiments, for example, subatomic particles are accelerated to very high velocities in collision experiments, which tell us information about the structure of the subatomic world as well as the origin of the universe. In space, cosmic rays are subatomic particles that have been accelerated to very high energies in supernovas (exploding massive stars) and active galactic nuclei. It is important to understand the processes that accelerate cosmic rays because these rays contain highly penetrating radiation that can damage electronics flown on spacecraft, for example.

Instantaneous Acceleration

Instantaneous acceleration a, or **acceleration at a specific instant in time**, is obtained using the same process discussed for instantaneous velocity. That is, we calculate the average velocity between two points in time separated by Δt and let Δt approach zero. The result is the derivative of the velocity function v(t), which is **instantaneous acceleration** and is expressed mathematically as

$$a(t)=\frac{d}{dt}v(t). \tag{3.3.32}$$

Thus, similar to velocity being the derivative of the position function, instantaneous acceleration is the derivative of the velocity function. We can show this graphically in the same way as instantaneous velocity. In Figure $3.3.5$, instantaneous acceleration at time t_0 is the slope of the tangent line to the velocity-versus-time graph at time t_0. We see that average acceleration $\bar{a}=\frac{\Delta v}{\Delta t}$ approaches instantaneous acceleration as ?t approaches zero. Also in part (a) of the figure, we see that velocity has a maximum when its slope is zero. This time corresponds to the zero of the acceleration function. In part (b), instantaneous acceleration at the minimum velocity is shown, which is also zero, since the slope of the curve is zero there, too. Thus, for a given velocity function, the zeros of the acceleration function give either the minimum or the maximum velocity

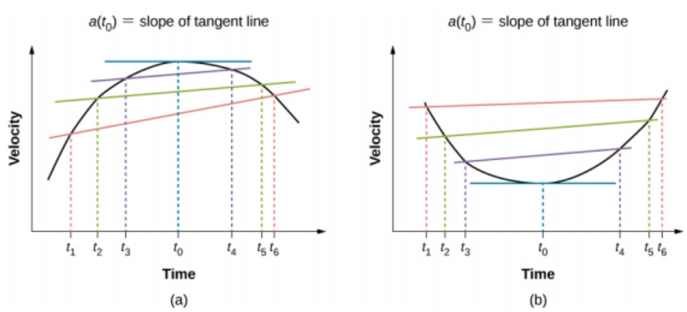

Figure $3.3.5$: In a graph of velocity versus time, instantaneous acceleration is the slope of the tangent line. (a) Shown is average acceleration $\bar{a}=\frac{\Delta v}{\Delta t}=\frac{v_f-v_0}{t_f-t_0}$ between times Δt = t_6 - t_1, Δt = t_5 - t_2, and Δt = t_4 - t_3. When Δt ? 0, the average acceleration approaches instantaneous acceleration at time t_0. In view (a), instantaneous acceleration is shown for the point on the velocity curve at maximum velocity. At this point, instantaneous acceleration is the slope of the tangent line, which is zero. At any other time, the slope of the tangent line—and thus instantaneous acceleration—would not be zero. (b) Same as (a) but shown for instantaneous acceleration at minimum velocity.

To illustrate this concept, let's look at two examples. First, a simple example is shown using Figure 3.3.4(b), the velocityversus-time graph of Example 3.3, to find acceleration graphically. This graph is depicted in Figure $3.3.6$(a), which is a straight line. The corresponding graph of acceleration versus time is found from the slope of velocity and is shown in Figure $3.3.6$(b). In this example, the velocity function is a straight line with a constant slope, thus acceleration is a constant. In the next example, the velocity function is has a more complicated functional dependence on time.

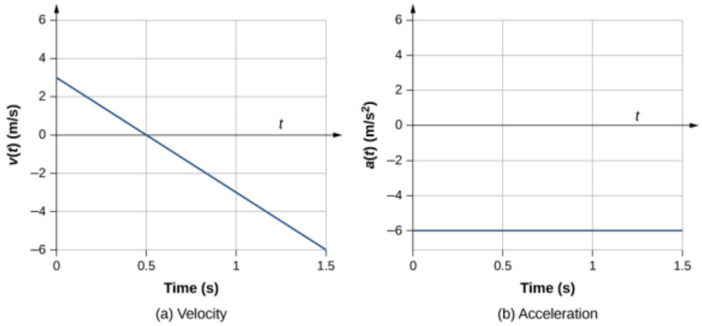

Figure 3.3.6: (a, b) The velocity-versus-time graph is linear and has a negative constant slope (a) that is equal to acceleration, shown in (b).

If we know the functional form of velocity, v(t), we can calculate instantaneous acceleration a(t) at any time point in the motion using Equation 3.3.32.

3.3.7 Getting a Feel for Acceleration

You are probably used to experiencing acceleration when you step into an elevator, or step on the gas pedal in your car. However, acceleration is happening to many other objects in our universe with which we don't have direct contact. Table 3.2 presents the acceleration of various objects. We can see the magnitudes of the accelerations extend over many orders of magnitude.

3.3.7.1 Table - Typical Values of Acceleration

(credit: Wikipedia: Orders of Magnitude (acceleration))

Acceleration	Value (m/s^2)
High-speed train	0.25
Elevator	2
Cheetah	5
Object in a free fall without air resistance near the surface of Earth	9.8
Space shuttle maximum during launch	29
Parachutist peak during normal opening of parachute	59
F16 aircraft pulling out of a dive	79
Explosive seat ejection from aircraft	147
Sprint missile	982
Fastest rocket sled peak acceleration	1540
Jumping flea	3200
Baseball struck by a bat	30,000
Closing jaws of a trap-jaw ant	1,000,000
Proton in the large Hadron collider	1.9×10^9

In this table, we see that typical accelerations vary widely with different objects and have nothing to do with object size or how massive it is. Acceleration can also vary widely with time during the motion of an object. A drag racer has a large acceleration just after its start, but then it tapers off as the vehicle reaches a constant velocity. Its average acceleration can be quite different from its instantaneous acceleration at a particular time during its motion. Figure 3.3.8 compares graphically average acceleration with instantaneous acceleration for two very different motions.

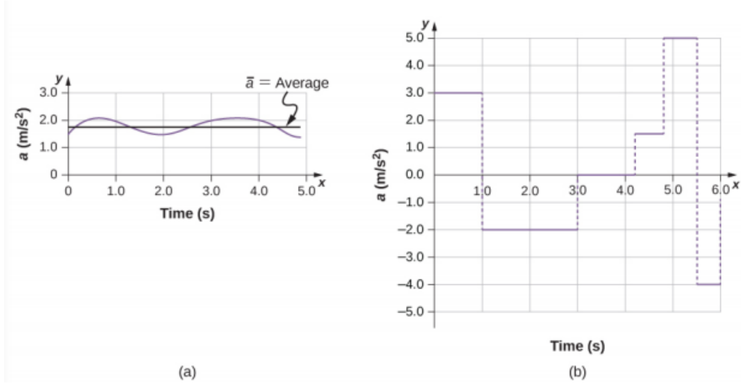

Figure 3.3.8: Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Acceleration varies only slightly and is always in the same direction, since it is positive. The average over the interval is nearly the same as the acceleration at any given time. (b) Acceleration varies greatly, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0–1.0 s) with constant or nearly constant acceleration in such a situation.

3.3.8 Examples

Example 3.3.1: Finding an Acceleration Vector

A particle has a velocity of $\vec{v}(t) = 5.0t\hat{i} + t^2\hat{j} - 2.0t^3\hat{k}\, m/s$.

a. What is the acceleration function?
b. What is the acceleration vector at t = 2.0 s? Find its magnitude and direction.

Solution

a. We take the first derivative with respect to time of the velocity function to find the acceleration. The derivative is taken component by component:

$$\vec{a}(t) = 5.0\ \hat{i} + 2.0t\ \hat{j} - 6.0t^2\ \hat{k}\ m/s^2.$$

b. Evaluating $\vec{a}(2.0\ s) = 5.0\hat{i} + 4.0\hat{j} - 24.0\hat{k}\, m/s^2$ gives us the direction in unit vector notation. The magnitude of the acceleration is

$$|\vec{a}(2.20\ s)| = \sqrt{5.0^2 + 4.0^2 + (-24.0)^2} = 24.8\ m/s^2.$$

Significance

In this example we find that acceleration has a time dependence and is changing throughout the motion. Let's consider a different velocity function for the particle.

Example 3.3.2: Finding a Particle Acceleration

A particle has a position function: $\vec{r}(t) = (10t - t^2)\hat{i} + 5t\hat{j} + 5t\hat{k}\, m$.

a. What is the velocity?
b. What is the acceleration?
c. Describe the motion from $t = 0\ s$.

Strategy

We can gain some insight into the problem by looking at the position function. It is linear in y and z, so we know the acceleration in these directions is zero when we take the second derivative. Also, note that the position in the x direction is zero for t = 0 s and t = 10 s.

Solution

a. Taking the derivative with respect to time of the position function, we find $\vec{v}(t) = (10 - 2t)\hat{i} + 5\hat{j} + 5\hat{k}\, m/s$. The velocity function is linear in time in the x direction and is constant in the y and z directions.
b. Taking the derivative of the velocity function, we find

$$\vec{a}(t) = -2\ \hat{i}\ m/s^2.$$

The acceleration vector is a constant in the negative x-direction.
c. The trajectory of the particle can be seen in Figure 3.3.1 Let's look in the y and z directions first. The particle's position increases steadily as a function of time with a constant velocity in these directions. In the x direction, however, the particle follows a path in positive x until t = 5 s, when it reverses direction. We know this from looking at the velocity function, which becomes zero at this time and negative thereafter. We also know this because the acceleration is negative and constant—meaning, the particle is decelerating, or accelerating in the negative direction. The particle's position reaches 25 m, where it then reverses direction and begins to accelerate in the negative x direction. The position reaches zero at t = 10 s.

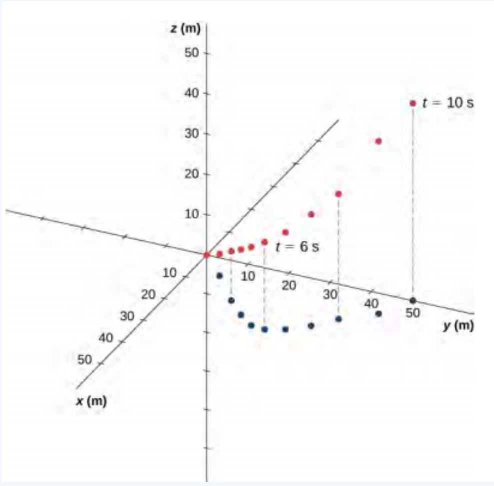

Figure 3.3.1: The particle starts at point (x, y, z) = (0, 0, 0) with position vector $\vec{r}$ = 0. The projection of the trajectory onto the xy-plane is shown. The values of y and z increase linearly as a function of time, whereas x has a turning point at t = 5 s and 25 m, when it reverses direction. At this point, the x component of the velocity becomes negative. At t = 10 s, the particle is back to 0 m in the x direction.

The following example illustrates a practical use of the kinematic equations in two dimensions.

Example 3.3.3: A Skier

Figure 3.3.2 shows a skier moving with an acceleration of 2.1 m/s^2 down a slope of 15° at t = 0. With the origin of the coordinate system at the front of the lodge, her initial position and velocity are

$$\vec{r}(0) = (7.50\ \hat{i} - 50.0\ \hat{j})m$$

and

$$\vec{v}(0) = (4.1\ \hat{i} - 1.1\ \hat{j})m/s$$

a. What are the x- and y-components of the skier's position and velocity as functions of time?
b. What are her position and velocity at t = 10.0 s?

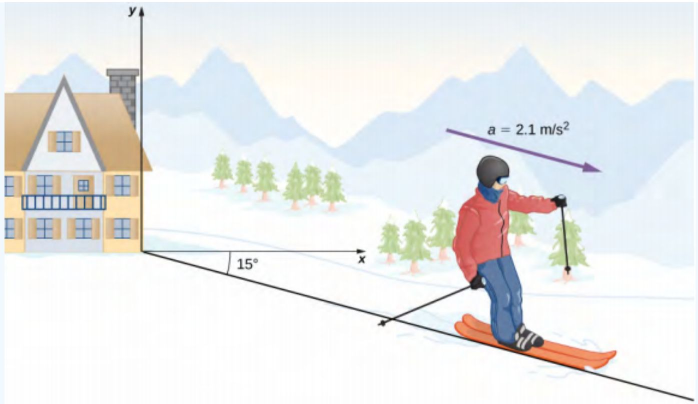

Figure 3.3.2: A skier has an acceleration of 2.1 m/s^2 down a slope of 15°. The origin of the coordinate system is at the ski lodge.

Strategy

Since we are evaluating the components of the motion equations in the x and y directions, we need to find the components of the acceleration and put them into the kinematic equations. The components of the acceleration are found by referring to the coordinate system in Figure 3.3.2 Then, by inserting the components of the initial position and velocity into the motion equations, we can solve for her position and velocity at a later time t.

Solution

a. The origin of the coordinate system is at the top of the hill with y-axis vertically upward and the x-axis horizontal. By looking at the trajectory of the skier, the x-component of the acceleration is positive and the y-component is negative. Since the angle is 15° down the slope, we find
$a_x = (2.1\ m/s^2)\cos(15^o) = 2.0\ m/s^2\ a_y = (-2.1\ m/s^2)\sin(15^o) = -0.54\ m/s^2.$
Inserting the initial position and velocity into Equations 3.3.21and 3.3.22for x, we have
$x(t) = 75.0\ m + (4.1\ m/s)t + \frac{1}{2}(2.0\ m/s^2)t^2 v_x(t) = 4.1\ m/s + (2.0\ m/s^2)t.$
For y, we have
$y(t) = -50.0.0\ m + (-1.1\ m/s)t + \frac{1}{2}(-0.54\ m/s^2)t^2\ v_y(t) = -1.1\ m/s + (-0.54\ m/s^2)t.$

b. Now that we have the equations of motion for x and y as functions of time, we can evaluate them at t = 10.0 s:
$x(10.0\ s) = 75.0\ m + (4.1\ m/s)(10.0\ s) + \frac{1}{2}(2.0\ m/s^2)(10.0\ s)^2 = 216.0\ m$
$v_x(10.0\ s) = 4.1\ m/s + (2.0\ m/s^2)(10.0\ s) = 24.1\ m/s$
$y(10.0) = -50.0.0\ m + (-1.1\ m/s)(10.0\ s) + \frac{1}{2}(-0.54\ m/s^2)(10.0\ s)^2$
$v_y(10.0\ s) = -1.1\ m/s + (-0.54\ m/s^2)(10.0\ s).$
The position and velocity at t = 10.0 s are, finally
$\vec{r}(10.0\ s) = (216.0\ \hat{i} - 88.0\ \hat{j})m\vec{v}(10.0\ s) = (24.1\ \hat{i} - 6.5\ \hat{j})m/s.$
The magnitude of the velocity of the skier at 10.0 s is 25 m/s, which is 60 mi/h.

Significance

It is useful to know that, given the initial conditions of position, velocity, and acceleration of an object, we can find the position, velocity, and acceleration at any later time.

✓ Example 3.3.4: Calculating Average Acceleration: A Racehorse Leaves the Gate

A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?

Figure 3.3.3: Racehorses accelerating out of the gate. (credit: Jon Sullivan)

Strategy

First we draw a sketch and assign a coordinate system to the problem Figure 3.3.4 This is a simple problem, but it always helps to visualize it. Notice that we assign east as positive and west as negative. Thus, in this case, we have negative velocity.

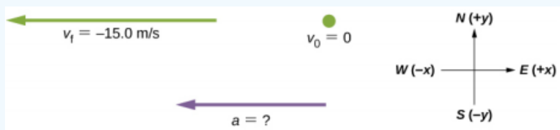

Figure 3.3.4: Identify the coordinate system, the given information, and what you want to determine.

We can solve this problem by identifying Δv and Δt from the given information, and then calculating the average acceleration directly from the equation $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$.

Solution

First, identify the knowns: v_0 = 0, v_f = -15.0 m/s (the negative sign indicates direction toward the west), Δt = 1.80 s. Second, find the change in velocity. Since the horse is going from zero to –15.0 m/s, its change in velocity equals its final velocity:

$$\Delta v = v_f - v_0 = v_f = -15.0\ m/s. \tag{3.3.33}$$

Last, substitute the known values (Δv and Δt) and solve for the unknown $\bar{a}$:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-15.0\ m/s}{1.80\ s} = -8.33\ m/s^2. \tag{3.3.34}$$

Significance

The negative sign for acceleration indicates that acceleration is toward the west. An acceleration of 8.33 m/s^2 due west means the horse increases its velocity by 8.33 m/s due west each second; that is, 8.33 meters per second per second, which we write as 8.33 m/s^2. This is truly an average acceleration, because the ride is not smooth. We see later that an acceleration of this magnitude would require the rider to hang on with a force nearly equal to his weight.

✔ Example 3.3.5: Calculating Instantaneous Acceleration

A particle is in motion and is accelerating. The functional form of the velocity is $v(t) = 20t - 5t^2$ m/s.

a. Find the functional form of the acceleration.
b. Find the instantaneous velocity at t = 1, 2, 3, and 5 s.
c. Find the instantaneous acceleration at t = 1, 2, 3, and 5 s.
d. Interpret the results of (c) in terms of the directions of the acceleration and velocity vectors.

Strategy

We find the functional form of acceleration by taking the derivative of the velocity function. Then, we calculate the values of instantaneous velocity and acceleration from the given functions for each. For part (d), we need to compare the directions of velocity and acceleration at each time.

Solution

a. a(t) = $\frac{dv(t)}{dt}$dv(t) dt = 20 - 10t m/s_2
b. v(1 s) = 15 m/s, v(2 s) = 20 m/s, v(3 s) = 15 m/s, v(5 s) = -25 m/s
c. a(1 s) = 10m/s^2, a(2 s) = 0m/s^2, a(3 s) = -10m/s^2, a(5 s) = -30m/s^2
d. At t = 1 s, velocity v(1 s) = 15 m/s is positive and acceleration is positive, so both velocity and acceleration are in the same direction. The particle is moving faster.

At t = 2 s, velocity has increased to v(2 s) = 20 m/s , where it is maximum, which corresponds to the time when the acceleration is zero. We see that the maximum velocity occurs when the slope of the velocity function is zero, which is just the zero of the acceleration function.

At t = 3 s, velocity is v(3 s) = 15 m/s and acceleration is negative. The particle has reduced its velocity and the acceleration vector is negative. The particle is slowing down.

At t = 5 s, velocity is v(5 s) = -25 m/s and acceleration is increasingly negative. Between the times t = 3 s and t = 5 s the particle has decreased its velocity to zero and then become negative, thus reversing its direction. The particle is now speeding up again, but in the opposite direction.

We can see these results graphically in Figure $3.3.7$.

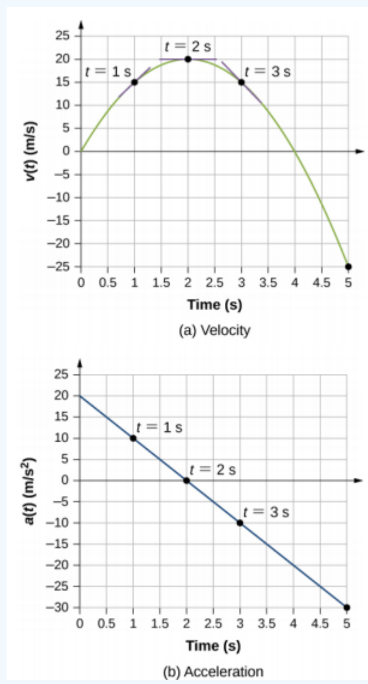

Figure $3.3.7$: (a) Velocity versus time. Tangent lines are indicated at times 1, 2, and 3 s. The slopes of the tangents lines are the accelerations. At t = 3 s, velocity is positive. At t = 5 s, velocity is negative, indicating the particle has reversed direction. (b) Acceleration versus time. Comparing the values of accelerations given by the black dots with the corresponding slopes of the tangent lines (slopes of lines through black dots) in (a), we see they are identical.

Significance

By doing both a numerical and graphical analysis of velocity and acceleration of the particle, we can learn much about its motion. The numerical analysis complements the graphical analysis in giving a total view of the motion. The zero of the acceleration function corresponds to the maximum of the velocity in this example. Also in this example, when acceleration is positive and in the same direction as velocity, velocity increases. As acceleration tends toward zero, eventually becoming negative, the velocity reaches a maximum, after which it starts decreasing. If we wait long enough, velocity also becomes negative, indicating a reversal of direction. A real-world example of this type of motion is a car with a velocity that is increasing to a maximum, after which it starts slowing down, comes to a stop, then reverses direction.

Example 3.3.6: Motion of a Motorboat

A motorboat is traveling at a constant velocity of 5.0 m/s when it starts to decelerate to arrive at the dock. Its acceleration is a(t) = $-\frac{1}{4}$ t m/s^2. (a) What is the velocity function of the motorboat? (b) At what time does the velocity reach zero? (c) What is the position function of the motorboat? (d) What is the displacement of the motorboat from the time it begins to decelerate to when the velocity is zero? (e) Graph the velocity and position functions.

Strategy

(a) To get the velocity function we must integrate and use initial conditions to find the constant of integration. (b) We set the velocity function equal to zero and solve for t. (c) Similarly, we must integrate to find the position function and use initial conditions to find the constant of integration. (d) Since the initial position is taken to be zero, we only have to evaluate the position function at t = 0 .

Solution

We take t = 0 to be the time when the boat starts to decelerate.

a. From the functional form of the acceleration we can solve Equation 3.3.9to get v(t):
$v(t) = \int a(t)dt + C_1 = \int -\frac{1}{4}tdt + C_1 = -\frac{1}{8}t^2 + C_1.$ At t = 0 we have v(0) = 5.0 m/s = 0 + C_1, so C_1 = 5.0 m/s or v(t) = 5.0 m/s - $\frac{1}{8}$ t^2.
b. v(t) = 0 = 5.0 m/s - $\frac{1}{8}$ t^2 (\Rightarrow\) t = 6.3 s
c. Solve Equation 3.3.11: $x(t) = \int v(t)dt + C_2 = \int (5.0 - \frac{1}{8}t^2)dt + C_2 = 5.0t - \frac{1}{24}t^3 + C_2.$ At t = 0, we set x(0) = 0 = x_0, since we are only interested in the displacement from when the boat starts to decelerate. We have $x(0) = 0 = C_2.$ Therefore, the equation for the position is $x(t) = 5.0t - \frac{1}{24}t^3.$
d. Since the initial position is taken to be zero, we only have to evaluate x(t) when the velocity is zero. This occurs at t = 6.3 s. Therefore, the displacement is $x(6.3) = 5.0(6.3) - \frac{1}{24}(6.3)^3 = 21.1\ m.$

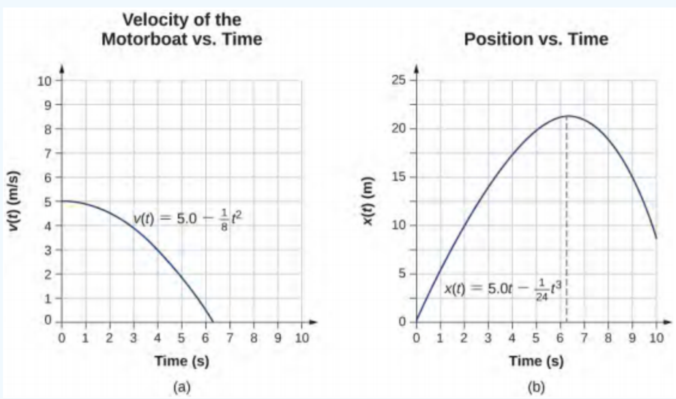

Figure 3.3.1: (a) Velocity of the motorboat as a function of time. The motorboat decreases its velocity to zero in 6.3 s. At times greater than this, velocity becomes negative—meaning, the boat is reversing direction. (b) Position of the motorboat as a function of time. At t = 6.3 s, the velocity is zero and the boat has stopped. At times greater than this, the velocity becomes negative—meaning, if the boat continues to move with the same acceleration, it reverses direction and heads back toward where it originated.

Significance

The acceleration function is linear in time so the integration involves simple polynomials. In Figure 3.3.1, we see that if we extend the solution beyond the point when the velocity is zero, the velocity becomes negative and the boat reverses direction. This tells us that solutions can give us information outside our immediate interest and we should be careful when interpreting them.

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