9.4: Rolling motion

In this section, we examine how to model the motion of an object that is rolling along a surface, such as the motion of a bicycle wheel. Consider the motion of a wheel of radius, R, rotating with angular velocity, $\vec\omega$, about an axis perpendicular to the wheel and through its center of mass, **as observed in the center of mass frame**. This is illustrated in Figure $9.4.1$.

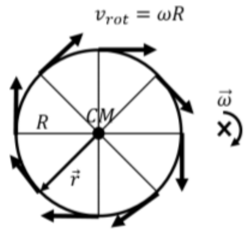

Figure $9.4.1$: A wheel rotating with angular velocity $\vec\omega$ about an axis through its center of mass.

In the frame of reference of the center of mass, each point on the edge of the wheel has a velocity, $\vec v_{rot}$, due to rotation given by:

$$\vec v_{rot} = \vec\omega \times \vec r$$

where $\vec r$ is a vector (of magnitude R) from the center of mass to the corresponding point on the edge of the wheel (shown in Figure $9.4.1$ for a point on the lower left of the wheel). The vector $\vec r$ is always perpendicular to $\vec\omega$, so that the speed of all points on the edge, as measured in the frame of reference of the center of mass, is the same:

$$v_{rot} = \omega R \tag{9.4.1}$$

as illustrated in Figure $9.4.1$.

Now, suppose that the whole wheel is moving, as it rolls on the ground, such that the center of mass of the wheel moves with a velocity, $\vec v_{CM}$, as illustrated in Figure $9.4.2$.

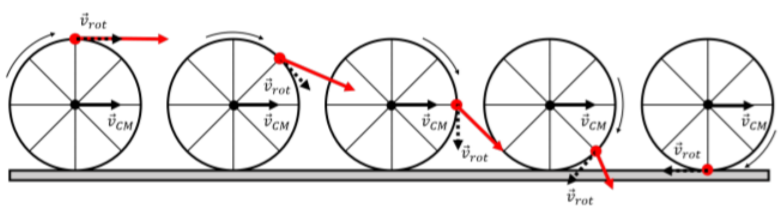

Figure $9.4.2$: A wheel rolling without slipping on the ground, with the center of mass moving with velocity $\vec v_{CM}$. The wheel is shown at different instants in time, as the point shown in red moves around the center of mass.

In the frame of reference of the ground, each point on the edge of the wheel will have a velocity $\vec v$ given by:

$$\vec v = \vec v_{rot} + \vec v_{CM}$$

That is, in the frame reference of the ground, each point will have a velocity obtained by (vectorially) adding its velocity relative to the center of mass, $\vec v_{rot}$, and the velocity of the center of mass relative to the ground, $\vec v_{CM}$. This is illustrated in Figure $9.4.2$ for one specific point, shown in red. The red vector corresponds to the velocity of the red point as the wheel rotates, and is obtained by adding the velocity of the center of mass, $\vec v_{CM}$, and the velocity, $\vec v_{rot}$, relative to the center of mass (shown as the dashed vector, tangent to the edge of the wheel).

Consider, specifically, the instant in time when the red point is at the bottom of the wheel, where the wheel makes contact with the ground. **If the wheel is not slipping with respect to the ground**, then the point is, at that instant, at rest relative to the ground. We call this type of motion "rolling without slipping"; the point on the rotating object that is in contact with the ground is instantaneously at rest relative to the ground. This is the scenario illustrated in Figure $9.4.2$.

For the point in contact with the ground, the vectors $\vec v_{rot}$ and $\vec v_{CM}$ are anti-parallel, horizontal, and must sum to zero. Writing out the horizontal component of the velocity of that point (choosing the positive direction to be in the direction of the velocity of the center of mass):

$$v = -v_{rot} + v_{CM} = 0$$
$$\therefore v_{rot} = v_{CM}$$

and we find that, for rolling without slipping, the speed due to rotation about the center of mass has to be equal to the speed of the center of mass. The speed due to rotation about the center of mass can be expressed using the angular velocity of the wheel about the center of mass (*Equation 12.2.1*). For rolling without slipping, we thus have the following relationship between angular velocity and the speed of the center of mass:

$$\omega R = v_{CM} \qquad \text{(rolling without slipping)} \tag{9.4.2}$$

It makes sense for the angular velocity to be related to the speed of the center of mass. The faster the wheel rotates, the faster the center of mass will move. If the wheel is slipping with respect to the ground, then the point of contact is no longer stationary relative to the ground, and there is no relation between the angular velocity and the speed of the center of mass. For rolling with slipping, imagine the motion of your bicycle wheel as you try to ride your bike on a slick sheet of ice.

For rolling without slipping, the magnitude of the linear acceleration of the center of mass, a_{CM}, is similarly related to the magnitude of the angular acceleration of the wheel, α, about the center of mass:

$$a_{CM} = \frac{dv_{CM}}{dt} = \frac{d}{dt}\omega R = R\frac{d\omega}{dt}$$
$$\therefore a_{CM} = R\alpha$$

Exercise 9.4.1

For rolling without slipping (Figure 9.4.2), the speed of the point on the wheel that is in contact with the ground is 0. What is the speed of the point at the top of the wheel?

A. 0.
B. v_{CM}.
C. $2v_{CM}$.
D. None of the above.

Answer

Example 9.4.1

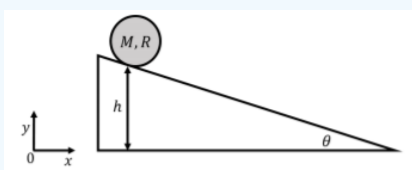

Figure 9.4.3: A disk rolling without slipping down an incline.

Solution

A disk of mass M and radius R is placed on an incline at a height h above the ground. The incline makes an angle θ with respect to the horizontal, as shown in Figure 9.4.3 If the disk starts at rest and rolls without slipping down the incline, what speed will the center of mass have when the disk reaches the bottom of the incline? We can use the conservation of mechanical energy to determine the speed of the center of mass at the bottom of the incline, as there are no non-conservative forces doing work on the disk. If we choose to define gravitational potential energy such that it is zero at the bottom of the incline, we can write the total mechanical energy of the disk at the top of the incline as:

$$E = K + U = (0) + Mgh$$

where the kinetic energy is zero, since the disk starts at rest[1]. At the bottom of the incline, the disk will have only kinetic energy, since the potential energy at the bottom is defined to be zero. The kinetic energy of the disk will have a component from the rotation of the disk about the center of mass, with angular speed ω, and a component from the translation of the center of mass with speed v_{CM}. The mechanical energy at the bottom of the incline is thus:

$$E' = K' + U = K'_{rot} + K'_{trans} + (0) = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{cm}^2$$

Since the disk is rolling without slipping, its angular speed is related to the speed of center of mass:

$$\omega = \frac{v_{CM}}{R}$$

The moment of inertia of the disk about its center of mass is given by:

$$I_{CM} = \frac{1}{2}MR^2$$

We can thus write the mechanical energy at the bottom of the incline as:

$$\begin{aligned} E' &= \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{cm}^2 \\ &= \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2}Mv_{cm}^2 \\ &= \frac{3}{4}Mv_{cm}^2 \end{aligned}$$

Applying conservation of energy allows us to determine the speed of the center of mass at the bottom of the incline:

$$\begin{aligned} E &= E' \\ Mgh &= \frac{3}{4}Mv_{cm}^2 \\ \therefore v_{CM} &= \sqrt{\frac{4}{3}gh} \end{aligned}$$

Discussion:

This example showed how we can use the conservation of energy to model the motion of an object that is rolling without slipping. The constraint of rolling without slipping allowed for the angular speed of the object to be related to the speed of its center of mass.

? Exercise 9.4.2

A hoop, a disk, and a sphere roll without slipping down an incline. If they are all released at the same time, in what order will they arrive at the bottom?

A. Hoop, disk, sphere.
B. Sphere, disk, hoop.
C. Disk, sphere, hoop.
D. Disk, hoop, sphere.

Answer

The instantaneous axis of rotation

When an object is rolling without slipping, we can model its motion as the superposition of rotation about the center of mass and translational motion of the center of mass, as in the previous section. However, because the point of contact between the rolling object and the ground is stationary, we can also model the motion as if the object were instantaneously rotating with angular velocity, $\vec{\omega}$, about a stationary axis through the point of contact. That is, we can model the motion as rotation only, with no translation, if we choose an axis of rotation through the point of contact between the ground and the wheel.

We call the axis through the point of contact the "instantaneous axis of rotation", since, instantaneously, it appears as if the whole wheel is rotating about that point. This is illustrated in Figure 9.4.4, which shows, in red, the velocity vector for each point on the edge of the wheel, relative to the instantaneous axis of rotation. Because the axis of rotation is fixed to the ground, the velocity of each point about that axis of rotation corresponds to the same velocity relative to the ground that is depicted in Figure 9.4.2.

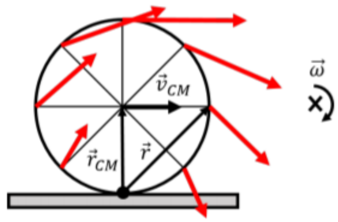

Figure 9.4.4: A wheel that is rolling without slipping, as viewed if rotating about the instantaneous axis of rotation that passes through the point of contact with the ground.

In particular, the angular velocity, $\vec{\omega}$, about the instantaneous axis of rotation is the same as when we model the motion as translation plus rotation about the center of mass ,as in the previous section. Indeed, relative to the instantaneous axis of rotation, the center of mass must still have a velocity $\vec{v}_{CM}$, which is given by:

$$\begin{aligned} \vec{v}_{CM} &= \vec{\omega} \times \vec{r}_{CM} \\ \therefore v_{CM} &= \omega R \end{aligned}$$

where $\vec{r}_{CM}$ is the vector from the axis of rotation to the center of mass. This is the same condition for rolling without slipping that we found before. Similarly, the velocity of any point on the wheel, relative to the ground, is given by:

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where $\vec{r}$ is the vector from the axis of rotation to the point of interest (shown in Figure $9.4.4$ for the point on the right side of the wheel). In particular, the velocity vector (in red) for any point is always perpendicular to the vector $\vec{r}$ for that point, which was not necessarily obvious when modeling the motion as rotation plus translation, as in Figure $9.4.2$.

Example $9.4.2$

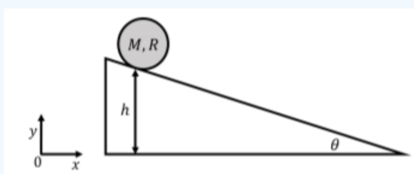

Figure $9.4.5$: A disk rolling without slipping down an incline.

A disk of mass M and radius R is placed on an incline at a height h above the ground. The incline makes an angle θ with respect to the horizontal, as shown in Figure $9.4.5$. What is the angular acceleration of the disk, about an axis through its center of mass, as it rolls without slipping down the slope?

Solution

In order to determine the angular acceleration of the disk about the center of mass, we need to model the forces that are exerted on the disk. The forces exerted on the disk are:

1. $\vec{F}_g$, the weight of the disk, exerted downwards at the center of mass, with magnitude Mg.
2. $\vec{N}$, a normal force perpendicular to the incline, exerted by the incline at the point of contact with the disk.
3. $\vec{f}_s$, a force of static friction parallel to the incline, exerted by the incline at the point of contact with the disk. Without this force, the disk would simply slide down the incline without rotating.

These forces are illustrated in Figure $9.4.6$ along with the acceleration of the center of mass, and our choice of coordinate system (we choose the x axis parallel to the acceleration of the center of mass, to facilitate applying Newton's Second Law).

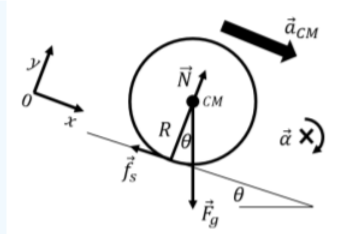

Figure 9.4.6: The forces on the disk rolling without slipping down an incline.

The angular acceleration of the disk about the center of mass, $\vec{\alpha}$ is given by Newton's Second Law for rotational dynamics:

$$\vec{\tau}^{ext} = I_{CM}\vec{\alpha}$$

where $\vec{\tau}^{ext}$ is the net external torque on the disk about the center of mass (which will be in the negative z direction).

The only force that can exert a torque about the center of mass is the force of static friction. Gravity has a lever arm of zero and the normal force is anti-parallel to the vector that goes from the center of mass to the point where the force is exerted. The net torque about the center of mass is thus:

$$\vec{\tau}^{ext} = \vec{\tau}_{f_s} = \vec{r}_{f_s} \times \vec{f}_s = -Rf_s\hat{z}$$

The angular acceleration will thus be in the negative z direction, and the magnitude is given by:

$$\alpha = \frac{\tau^{ext}}{I_{CM}} = \frac{Rf_s}{\frac{1}{2}MR^2} = \frac{2f_s}{MR}$$

However, we do not know the magnitude of the force of static friction. We can use the x and y components of Newton's Second Law to determine it (with acceleration of the center of mass in the x direction):

$$\begin{aligned} \sum F_x &= F_g\sin\theta - f_s = Ma_{CM} \\ \sum F_y &= N - F_g\cos\theta = 0 \end{aligned}$$

Because the disk is rolling without slipping, the acceleration of the center of mass is related to the angular acceleration of the disk:

$$a_{cm} = \alpha R$$

The x component of Newton's Second Law can thus be used to determine the magnitude of the force of static friction in terms of the angular acceleration:

$$\begin{aligned} Mg\sin\theta - f_s &= M\alpha R \\ \therefore f_s &= Mg\sin\theta - M\alpha R \end{aligned}$$

We can then substitute out the force of friction from our previous formula for the angular acceleration:

$$\begin{aligned} \alpha &= \frac{2f_s}{MR} \\ &= \frac{2Mg\sin\theta - 2M\alpha R}{MR} = \frac{2g\sin\theta}{R} - 2\alpha \\ \therefore \alpha &= \frac{2g\sin\theta}{3R} \end{aligned}$$

Instead of modeling the motion of the disk as rotation about the center of mass and translation of the center of mass, we can also model it about the instantaneous axis of rotation.

The angular acceleration about the instantaneous axis of rotation will be the same as the angular acceleration about the center of mass. About the instantaneous axis of rotation, only the force of gravity can exert a torque, since the normal force and the force of friction both have a lever arm of zero. The torque from the force of gravity, about the instantaneous axis of rotation is:

$$\vec{\tau}_g = -F_gR\sin\theta\hat{z} = -MgR\sin\theta\hat{z}$$

The torque from the force of gravity is equal to the moment of inertia of the disk about the instantaneous axis of rotation, I, multiplied by its angular acceleration:

$$\begin{aligned}\tau^{ext} &= \tau_g = I\alpha \\ \therefore \alpha &= \frac{\tau_g}{I} = \frac{MgR\sin\theta}{I}\end{aligned}$$

The moment of inertia about the instantaneous axis of rotation is easily found using the parallel axis theorem:

$$I = I_{CM} + MR^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

This allows us to find the angular acceleration of the disk:

$$\begin{aligned}\alpha &= \frac{MgR\sin\theta}{I} = \frac{MgR\sin\theta}{\frac{3}{2}MR^2} \\ &= \frac{2g\sin\theta}{3R}\end{aligned}$$

as we found previously, but in this case, we did not need to use Newton's Second Law to determine the force of friction.

Discussion:

We saw that we can model the dynamics of the rolling body using either an axis through the center of mass, or an axis through the instantaneous axis of rotation. The latter was easier in this case, because it did not require using Newton's Second Law.

By using an axis through the center of mass to model the motion of the disk, it was clear that the force of static friction is required in order for the disk to rotate. Without the force of static friction, the disk would slide along the surface of the incline. The disk could still rotate if there is a force of kinetic friction that causes a torque that rotates the disk. If the surface were completely frictionless, the disk would simply slide down the incline, and we could model it as a sliding block. If the incline is too steep the force of static friction is no longer sufficient to provide the necessary torque required for the angular acceleration to be that which corresponds to rolling without slipping, and the disk would slip.

Footnotes

1. Technically, the potential energy should be taken for the height of the center of mass, which is a distance $h_{CM} = h + R\cos\theta$ from the ground at the top of the incline, and a height $h'_{CM} = R$ at the bottom of the incline. The net difference in height for the center of mass is thus $h_{CM} - h'_{CM} = h + R(1-\cos\theta)$. If we assume that h is much bigger than R, then this is negligible, otherwise, that is what we should use instead of h for the potential energy.

More about Rolling Motion

As a step up from a statics problem, we may consider a situation in which the sum of the external forces is zero, as well as the sum of the external torques, yet the system is moving. We call this "unforced motion." The first condition, $\sum \vec{F}_{ext} = 0$, means that the center of mass of the system must be moving with constant velocity; the second condition means that the total angular momentum must be constant. For a rigid body, this means that the most general kind of unforced motion can be described as a translation of the center of mass with constant velocity, accompanied by a rotation with constant angular velocity around the center of mass. For an extended, deformable system, on the other hand, the presence of internal forces can make the general motion a lot more complicated. Just think, for instance, of the solar system: although everything is, loosely speaking, revolving around the sun, the motions of individual planets and (especially) moons can be fairly complicated.

A simple example of (for practical purposes) unforced motion is provided by a symmetric, rigid object (such as a ball, or a wheel) rolling on a flat surface. The normal and gravity forces cancel each other, and since they lie along the same line their torques cancel too, so both $\vec{v}_{cm}$ and $\vec{L}$ remain constant. In principle, you could imagine removing the ground and gravity and nothing would change: the same motion (in the absence of air resistance) would just continue forever.

In practice, there is energy dissipation associated with rolling motion, primarily because, if the rolling object is not perfectly rigid[6], then, as it rolls, different parts of it get compressed under the combined pressure of gravity and the normal force, expand again, get compressed again... This kind of constant "squishing" ends up converting macroscopic kinetic energy into thermal energy: you may have noticed that the tires on a car get warm as you drive around, and you may also be familiar with the fact that you get a better gas mileage (less energy dissipation) when your tires are inflated to the right pressure than when they are low (because they are more "rigid," less deformable, in the first case).

This conversion of mechanical energy into thermal energy can be formally described by introducing another "friction" force that we call the force of *rolling friction*. Eventually, rolling friction alone would bring any rolling object to a stop, even in the absence of air resistance. It is, however, usually much weaker than sliding friction, so we will continue to ignore it from now on. You may have noticed already that typically an object can roll on a surface much farther than it can slide without rolling on the same surface. In fact, what happens often is that, if you try to send the object (for instance, a billiard ball) sliding, it will lose kinetic energy rapidly to the force of kinetic friction, but it will also start spinning under the influence of the same force, until a critical point is reached when the *condition for rolling without slipping* is satisfied:

$$|\vec{v}_{cm}| = R|\omega|. \tag{9.4.3}$$

At this point, the object will start rolling without slipping, and losing speed at a much slower rate.

The origin of the condition (9.4.3) is fairly straightforward. You can imagine an object that is rolling without slipping as "measuring the surface" as it rolls (or vice-versa, the surface measuring the circumference of the object as its different points are pressed against it in succession). So, after it has completed a revolution (2π radians), it should have literally "covered" a distance on the surface equal to $2\pi R$, that is, advanced a distance $2\pi R$. But the same has to be true, proportionately, for any rotation angle $\Delta\theta$ other that 2π: since the length of the corresponding arc is $s = R|\Delta\theta|$, in a rotation over an angle $|\Delta\theta|$ the center of mass of the object must have advanced a distance $|\Delta x_{cm}| = s = R|\Delta\theta|$. Dividing by Δt as $\Delta t \to 0$ then yields Equation (9.4.3).

Figure 9.4.1: Left: illustrating the rolling without slipping condition. The cyan line on the surface has the same length as the cyan-colored arc, and will be the distance traveled by the disk when it has turned through an angle θ. Right: velocities for four points on the edge of the disk. The pink arrows are the velocities in the center of mass frame. In the Earth reference frame, the velocity of the center of mass, $\vec{v}_{cm}$, in green, has to be added to each of them. The resultant is shown in blue for two of them.

Note that, unlike Equation (8.4.12), which it very much resembles, Equation (9.4.3) is *not* a "vector identity in disguise": there is nothing like Equation (9.3.6) that we could substitute for it in order to make the signs automatically come out right. You should just treat it as a relationship between the magnitudes of $\vec{v}_{cm}$ and $\vec{\omega}$ and just pick the signs appropriately for each circumstance, based on your convention for positive directions of translation and rotation.

In fact, we could use Equation (9.3.6) to find the velocity of any point on the circle, if we go to a reference frame where the center is at rest—which is to say, the center of mass (CM) reference frame; then, to go back to the Earth frame, we just have to add $\vec{v}_{cm}$ (as a vector) to the vector we obtained in the CM frame. Figure 9.4.1 shows the result. Note, particularly, that the point at the very bottom of the circle has a velocity $-R|\omega|$ in the CM frame, but when we go back to the Earth frame, its velocity is $-R|\omega| + v_{cm} = -R|\omega| + R|\omega| = 0$ (by the condition (9.4.3)). Thus, as long as the condition for rolling without slipping holds, the point (or points) on the rolling object that are momentarily in contact with the surface have zero instantaneous velocity. This means that, even if there was a force acting on the object at that point (such as the force of static friction), it would do no work, since the instantaneous power Fv for a force applied there would always be equal to zero.

We do not actually need the force of static friction to keep an object rolling on a flat surface (as I mentioned above, the motion could in principle go on "unforced" forever), but things are different on an inclined plane. Figure 9.4.2 shows an object rolling down an inclined plane, and the corresponding extended free-body diagram.

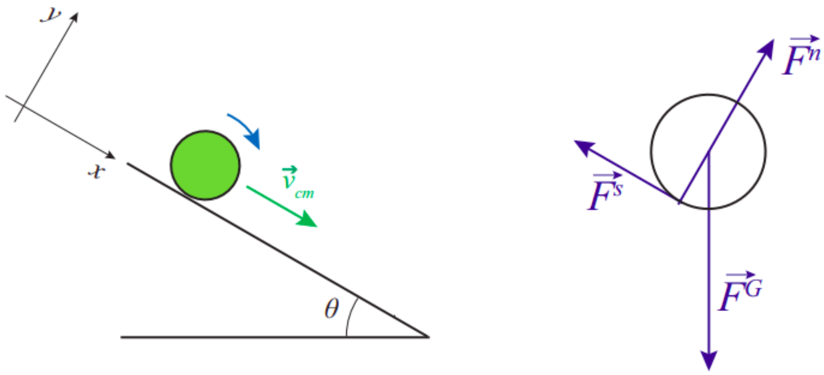

Figure 9.4.2: An object rolling down an inclined plane, and the extended free-body diagram. Note that neither gravity (applied at the CM) nor the normal force (whose line of action passes through the CM) exert a torque around the center of mass; only the force of static friction, $\vec{F}^s$, does.

The basic equations we use to solve for the object's motion are the sum of forces equation:

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \tag{9.4.4}$$

the net torque equation, with torques taken around the center of mass[7]

$$\sum \vec{\tau}_{ext} = I\vec{\alpha} \tag{9.4.5}$$

and the extension of the condition of rolling without slipping, (9.4.3), to the accelerations:

$$|a_{cm}| = R|\alpha|. \tag{9.4.6}$$

For the situation shown in Figure 9.4.2, if we take down the plane as the positive direction for linear motion, and clockwise torques as negative, we have to write $a_{cm} = -R\alpha$. In the direction perpendicular to the plane, we conclude from (9.4.4) that $F^n = Mg\cos\theta$, an equation we will not actually need[8]; in the direction along the plane, we have

$$Ma_{cm} = Mg\sin\theta - F^s \tag{9.4.7}$$

and the torque equation just gives $-F^sR = I\alpha$, which with $a_{cm} = -R\alpha$ becomes

$$F^sR = I\frac{a_{cm}}{R}. \tag{9.4.8}$$

We can eliminate F^s in between these two equations and solve for a_{cm}:

$$a_{cm} = \frac{g\sin\theta}{1 + I/(MR^2)}. \tag{9.4.9}$$

Now you can see why, earlier in the semester, we were always careful to assume that all the objects we sent down inclined planes were *sliding*, not rolling! The acceleration for a rolling object is *never* equal to simply $g\sin\theta$. Most remarkably, the correction factor depends only on the shape of the rolling object, and not on its mass or size, since the ratio of I to MR^2 is independent of m and R for any given geometry. Thus, for instance, for a disk, $I = \frac{1}{2}MR^2$, so $a_{cm} = \frac{2}{3}g\sin\theta$, whereas for a hoop, $I = MR^2$, so $a_{cm} = \frac{1}{2}g\sin\theta$. So any disk or solid cylinder will always roll down the incline faster than *any* hoop or hollow cylinder, regardless of mass or size.

This rather surprising result may be better understood in terms of energy. First, let me show (a result that is somewhat overdue) that for a rigid object that is rotating around an axis passing through its center of mass with angular velocity ω we can write the total kinetic energy as

$$K = K_{cm} + K_{rot} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2. \tag{9.4.10}$$

This is because for every particle the velocity can be written as $\vec{v} = \vec{v}_{cm} + \vec{v}'$, where $\vec{v}'$ is the velocity relative to the center of mass (that is, in the CM frame). Since in this frame the motion is a simple rotation, we have $|v'| = \omega r$, where r is the particle's distance to the axis. Therefore, the kinetic energy of that particle will be

$$\begin{aligned}\frac{1}{2}mv^2=\frac{1}{2}\vec{v}\cdot\vec{v}&=\frac{1}{2}m\left(\vec{v}_{cm}+\vec{v}'\right)\cdot\left(\vec{v}_{cm}+\vec{v}'\right)\\&=\frac{1}{2}mv_{cm}^2+\frac{1}{2}mv'^2+m\vec{v}_{cm}\cdot\vec{v}'\\&=\frac{1}{2}mv_{cm}^2+\frac{1}{2}mr^2\omega^2+\vec{v}_{cm}\cdot\vec{p}'\end{aligned}\tag{9.4.11}$$

(Note how I have made use of the *dot product* to calculate the magnitude squared of a vector.) On the last line, the quantity $\vec{p}'$ is the momentum of that particle in the CM frame. Adding those momenta for all the particles should give zero, since, as we saw in an earlier chapter, the center of mass frame *is* the zero momentum frame. Then, adding the contributions of all particles to the first and second terms in 9.4.11 gives Equation (9.4.10).

Coming back to our rolling body, using Equation (9.4.10) and the condition of rolling without slipping (9.4.3), we see that the ratio of the translational to the rotational kinetic energy is

$$\frac{K_{cm}}{K_{rot}}=\frac{mv_{cm}^2}{I\omega^2}=\frac{mR^2}{I}.\tag{9.4.12}$$

The amount of energy available to accelerate the object initially is just the gravitational potential energy of the object-earth system, and that has to be split between translational and rotational in the proportion (9.4.12). An object with a proportionately larger I is one that, for a given angular velocity, needs more rotational kinetic energy, because more of its mass is away from the rotation axis. This leaves less energy available for its translational motion.

Resources

Unfortunately, we will not really have enough time this semester to explore further the many interesting effects that follow from the vector nature of Equation (9.4.2), but you are at least subconsciously familiar with some of them if you have ever learned to ride a bicycle! A few interesting Internet references (some of which could perhaps inspire a good Honors project!) are the following:

- Walter Lewin's lecture on gyroscopic motion (and rolling motion):
  https://www.youtube.com/watch?v=N92FYHHT1qM
- A "Veritasium" video on "antigravity":
  https://www.youtube.com/watch?v=GeyDf4ooPdo
  https://www.youtube.com/watch?v=tLMpdBjA2SU
- And the old trick of putting a gyroscope (flywheel) in a suitcase:
  https://www.youtube.com/watch?v=zdN6zhZSJKw
  If any of the above links are dead, try googling them. (You may want to let me know, too!)

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[6]There is a simple argument, based on Einstein's theory of relativity, that shows an infinitely rigid object cannot exist: if it did, you could send a signal instantly from one end of it to the other, just by pushing or pulling on your end. In practice, such motion cannot reach the other end faster than the speed of a sound wave (that is to say, a compression wave) in the material. We will study such waves (which imply the medium is not infinitely rigid) in Chapter 12.

[7]You may feel a little uneasy about the fact that the CM frame is now an accelerated, and therefore *non-inertial*, frame. How do we know Equation (9.4.5) even applies there? This is, indeed, a non-trivial point. However, as we shall see in a later chapter, being in a uniformly-accelerated reference frame is equivalent to being in a uniform gravitational field, and we have just shown that, for all torque-related purposes, one may treat such a field as a single force applied to the center of mass of an object. Such a force (or "pseudoforce" in this case) clearly does not contribute to the torque about the center of mass, and so Equation (9.4.5) applies in the CM frame.

[8]Unless we were trying to answer a question such as "how steep does the plane have to be for rolling without slipping to become impossible?"
