2.2: Right Angle Triangle Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°(\frac{\pi}{6})$,$45°(\frac{\pi}{4})$,and $60°(\frac{\pi}{3})$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We can defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, however, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

We can used a unit circle to define the *trigonometric functions*. In this section, we use definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure $2.2.1$ shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x,y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

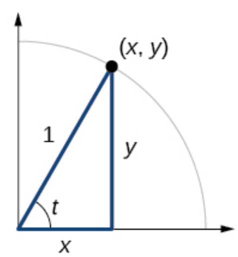

Figure $2.2.1$: Graph of quarter circle with radius of 1. Inscribed triangle with an angle of t. Point of (x,y) is at intersection of terminal side of angle and edge of circle.

We know

$$\cos t = \frac{x}{1} = x \tag{2.2.1}$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \tag{2.2.2}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x,y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means “next to.”) Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $2.2.2$.

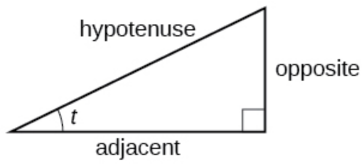

Figure 2.2.2: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{2.2.3}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{2.2.4}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{2.2.5}$$

A common mnemonic for remembering these relationships is SohCahToa[1], formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 2.2.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure $2.2.3$, find the value of $\cos \alpha$.

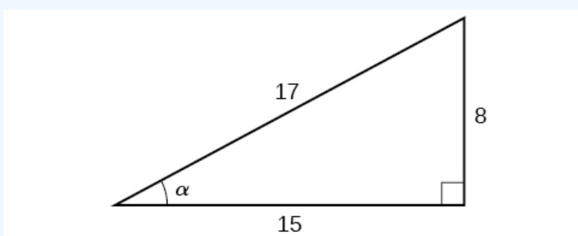

Figure $2.2.3$: A right triangle with side lengths of 8, 15, and 17. Angle alpha also labeled which is opposite to the side labeled 8.

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 2.2.4:

$$\begin{aligned} \cos(\alpha) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{15}{17} \end{aligned}$$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure $2.2.5$. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

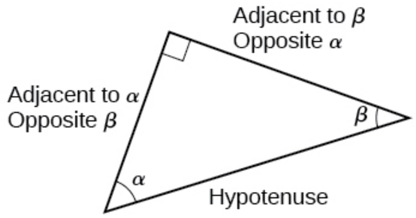

Figure 2.2.5: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 2.2.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 2.2.6, evaluate $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

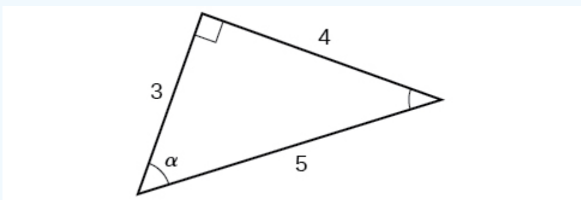

Figure 2.2.6: Right triangle with sides of 3, 4, and 5. Angle alpha is also labeled which is opposite the side labeled 4.

Solution

$$\begin{aligned}
\sin\alpha &= \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5} \\
\cos\alpha &= \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5} \\
\tan\alpha &= \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3} \\
\sec\alpha &= \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3} \\
\csc\alpha &= \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4} \\
\cot\alpha &= \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}
\end{aligned}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

Here we use unit circle relationships to evaluate triangles that contain **special angles**. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of 30°, 60°, and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a $30^\circ, 60^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45^\circ, 45^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure $2.2.8$.

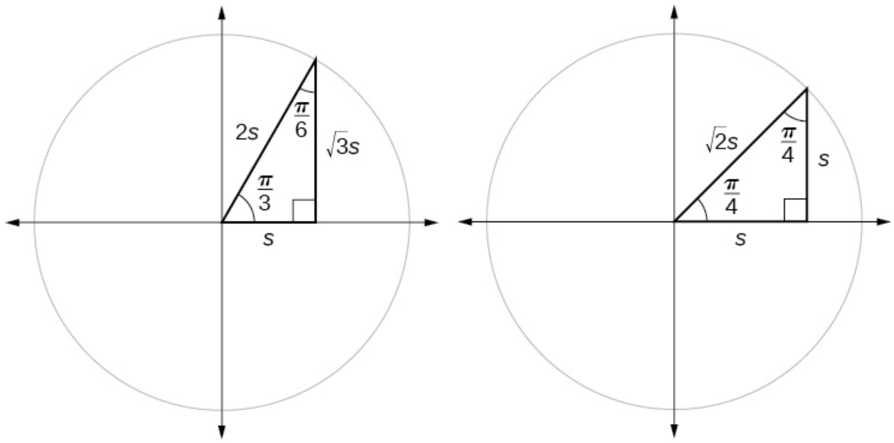

Figure $2.2.8$: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure $2.2.8$ for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 2.2.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Using Equal Co-function of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure $2.2.9$).

$$\begin{aligned} \sin\frac{\pi}{3} &= \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2} \\ \sin\frac{\pi}{6} &= \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2} \end{aligned}$$

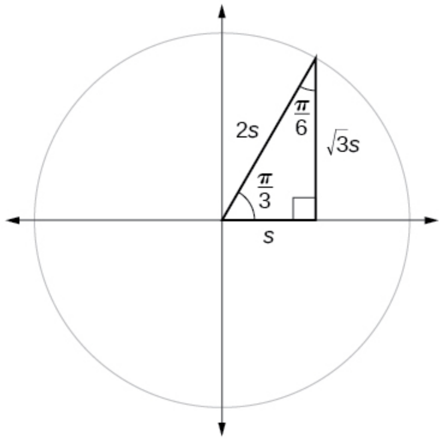

Figure 2.2.9: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure $2.2.9$, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, π,and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure $2.2.10$.

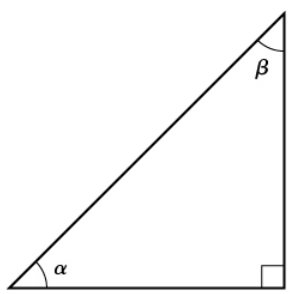

Figure 2.2.10: Co-function identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

CO-FUNCTION IDENTITIES

The **co-function identities** in radians are listed in Table $2.2.1$.

Table $2.2.1$

$\cos t=\sin(\frac{\pi}{2}-t)$	$\sin t=\cos(\frac{\pi}{2}-t)$
$\tan t=\cot(\frac{\pi}{2}-t)$	$\cot t=\tan(\frac{\pi}{2}-t)$
$\sec t=\csc(\frac{\pi}{2}-t)$	$\csc t=\sec(\frac{\pi}{2}-t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 2.2.4: Using Cofunction Identities

If $\sin t=\frac{5}{12}$, find $(\cos\frac{\pi}{2}-t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t=\cos(\frac{\pi}{2}-t).$$

So

$$\cos(\frac{\pi}{2}-t)=\frac{5}{12}.$$

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 2.2.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 2.2.11.

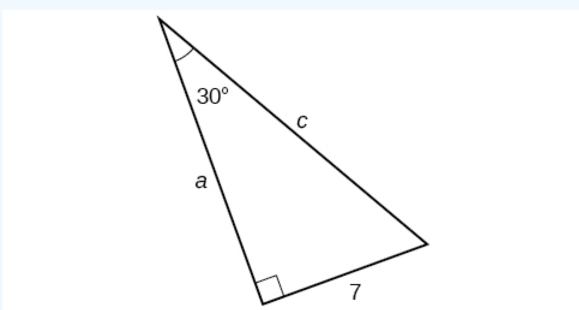

Figure 2.2.11: A right triangle with sides a, c, and 7. Angle of 30 degrees is also labeled which is opposite the side labeled 7.

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30^\circ)} \tag{2.2.6}$$
$$= 12.1 \tag{2.2.7}$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30^\circ)} = 14$$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure $2.2.12$

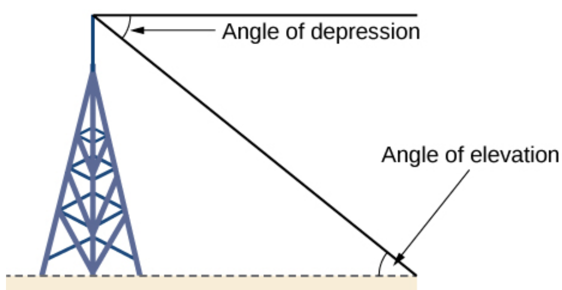

Figure $2.2.12$: Diagram of a radio tower with line segments extending from the top and base of the tower to a point on the ground some distance away. The two lines and the tower form a right triangle. The angle near the top of the tower is the angle of depression. The angle on the ground at a distance from the tower is the angle of elevation.

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 2.2.6: Measuring a Distance Indirectly[2]

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure 2.2.13. Find the height of the tree.

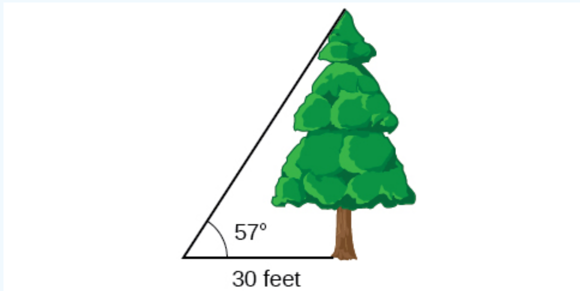

Figure 2.2.13: A tree with angle of 57 degrees from vantage point. Vantage point is 30 feet from tree.

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{2.2.8}$$

The tree is approximately 46 feet tall.

Key Equations

Co-function Identities

$$\begin{aligned} \cos t &= \sin(\frac{\pi}{2} - t) \\ \sin t &= \cos(\frac{\pi}{2} - t) \\ \tan t &= \cot(\frac{\pi}{2} - t) \\ \cot t &= \tan(\frac{\pi}{2} - t) \\ \sec t &= \csc(\frac{\pi}{2} - t) \\ \csc t &= \sec(\frac{\pi}{2} - t) \end{aligned}$$

Glossary

adjacent side

in a right triangle, the side between a given angle and the right angle

angle of depression

the angle between the horizontal and the line from the object to the observer’s eye, assuming the object is positioned lower than the observer

angle of elevation

the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side

in a right triangle, the side most distant from a given angle

hypotenuse

the side of a right triangle opposite the right angle

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[1]Other ways are "Oscar Had A Heap Of Apples" though this way you need to know sine, cosine, and tangent are in that order, or "Studying Our Homework Can Always Help To Obtain Achievement." Wolfram Math has some more alternatives to SOHCAHTOA which might be preferred by some.

[2]A famous example of measuring indirectly does not require trigonometry but is nevertheless an interesting tale. That tale is about Thales and his technique to measure the pyramid with the shadow of a stick (or himself). There is some question if it really happened or not, though the technique was actually correct regardless.
