2.7.15: Exponential and Logarithmic Functions

Learning Objectives

- Identify the form of an exponential function.
- Explain the difference between the graphs of x^b and b^x.
- Recognize the significance of the number e.
- Identify the form of a logarithmic function.
- Explain the relationship between exponential and logarithmic functions.
- Describe how to calculate a logarithm to a different base.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e.

Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y, and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number, then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = \left(\sqrt{9}\right)^3 = 27$.

Exercise 2.7.15.1

Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Answer

$f(4) = 900$

$f(10) = 24,300$

For any base $b > 0$, $b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x, we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \to \infty$ as $x \to \infty$, whereas $b^x \to 0$ as $x \to -\infty$. On the other hand, if $0 < b < 1$, $f(x) = b^x$ is decreasing on $(-\infty, \infty)$ and $b^x \to 0$ as $x \to \infty$ whereas $b^x \to \infty$ as $x \to -\infty$ (Figure 2.7.15.2).

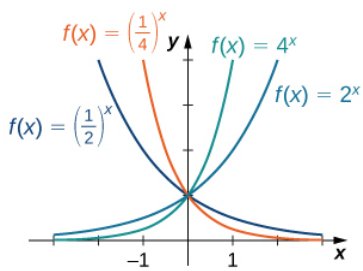


Figure 2.7.15.2: If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If $0 < b < 1$, then b^x is decreasing on $(-\infty, \infty)$.

Note that exponential functions satisfy the general laws of exponents. To remind you of these laws, we state them as rules.

Laws of Exponents

For any constants $a > 0$, $b > 0$, and for all x and y,

1. $$b^x \cdot b^y = b^{x+y}$$
2. $$\frac{b^x}{b^y} = b^{x-y}$$
3. $$(b^x)^y = b^{xy}$$
4. $$(ab)^x = a^x b^x$$

5. $$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$

Example 2.7.15.2: Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

a. $\dfrac{(2x^{2/3})^3}{(4x^{-1/3})^2}$

b. $\dfrac{(x^3y^{-1})^2}{(xy^2)^{-2}}$

Soution

a. We can simplify as follows:

$$\frac{(2x^{2/3})^3}{(4x^{-1/3})^2} = \frac{2^3(x^{2/3})^3}{4^2(x^{-1/3})^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{(x^3y^{-1})^2}{(xy^2)^{-2}} = \frac{(x^3)^2(y^{-1})^2}{x^{-2}(y^2)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

Exercise 2.7.15.2

Use the laws of exponents to simplify $\dfrac{6x^{-3}y^2}{12x^{-4}y^5}$.

Hint

$x^a/x^b = x^{a-b}$

Answer

$x/(2y^3)$

The Number e

A special type of exponential function appears frequently in real-world applications involves the Euler number e:

$$e \approx 2.718282.$$

Leonhard Euler

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. Since $e > 1$, we know $f(x) = e^x$ is increasing on $(-\infty, \infty)$. In Figure 2.7.15.3 we show a graph of $f(x) = e^x$ along with a tangent line to the graph of f at $x = 0$. The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1. As we see later in the text, having this property makes the natural exponential function the most simple exponential function to use in many instances.

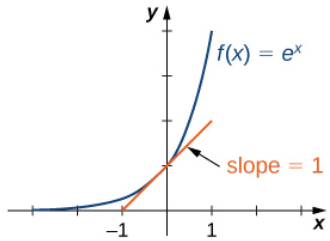


Figure 2.7.15.3: The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$.

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as the pH scale in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is *one-to-one*, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the logarithmic function with base b. For any $b > 0,\ b \neq 1$, the logarithmic function with base b, denoted $\log_b$, has domain $(0, \infty)$ and range $(-\infty, \infty)$,and satisfies

$$\log_b(x) = y$$

if and only if $b^y = x$.

For example,

$$\log_2(8) = 3$$

since $2^3 = 8$,

$$\log_{10}\left(\frac{1}{100}\right) = -2$$

since $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$,

$$\log_b(1) = 0$$

since $b^0 = 1$ for any base $b > 0$.

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

$$\log_b(b^x) = x$$

and

$$b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

$$\begin{aligned} \ln(e) &= \log_e(e) = 1 \\ \ln(e^3) &= \log_e(e^3) = 3 \\ \ln(1) &= \log_e(1) = 0. \end{aligned}$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ (Figure 2.7.15.4).

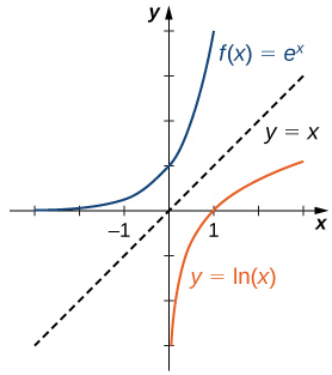


Figure 2.7.15.4: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

In general, for any base $b > 0$, $b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ (Figure 2.7.15.5).

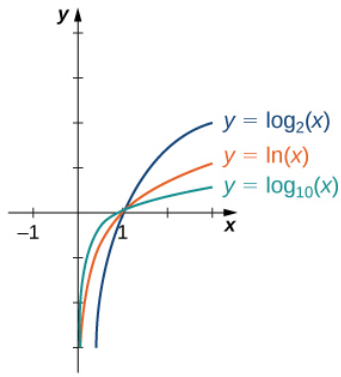


Figure 2.7.15.5: Graphs of $y = \log_b(x)$ are depicted for $b = 2,\ e,\ 10$.

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Properties of Logarithms

If $a,\ b,\ c > 0,\ b \neq 1$, and r is any real number, then

- **Product property**

$$\log_b(ac) = \log_b(a) + \log_b(c) \tag{2.7.15.1}$$

- **Quotient property**

$$\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c) \tag{2.7.15.2}$$

- **Power property**

$$\log_b(a^r) = r\log_b(a) \tag{2.7.15.3}$$

Example 2.7.15.4: Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$
b. $e^x + 6e^{-x} = 5$

Solution

a. Applying the natural logarithm function to both sides of the equation, we have

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

$$x \ln 5 = \ln 2.$$

Therefore,

$$x = \frac{\ln 2}{\ln 5}.$$

b. Multiplying both sides of the equation by e^x,we arrive at the equation

$$e^{2x} + 6 = 5e^x \ .$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0 \ ,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Exercise 2.7.15.4

Solve

$$e^{2x}/(3 + e^{2x}) = 1/2.$$

Hint

First solve the equation for e^{2x}

Answer

$x = \dfrac{\ln 3}{2}$.

Example 2.7.15.5: Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\dfrac{1}{x}\right) = 4$
b. $\log_{10} \sqrt{x} + \log_{10} x = 2$
c. $\ln(2x) - 3\ln(x^2) = 0$

Solution

a. By the definition of the natural logarithm function,

$$\ln\left(\frac{1}{x}\right) = 4$$

- if and only if $e^4 = \dfrac{1}{x}$.

Therefore, the solution is $x = 1/e^4$.

b. Using the product (Equation 2.7.15.1) and power (Equation 2.7.15.3) properties of logarithmic functions, rewrite the left-hand side of the equation as

$$\begin{aligned}\log_{10}\sqrt{x}+\log_{10}x &= \log_{10}x\sqrt{x}\\ &= \log_{10}x^{3/2}\\ &= \frac{3}{2}\log_{10}x.\end{aligned}$$

Therefore, the equation can be rewritten as

$$\frac{3}{2}\log_{10}x = 2$$

or

$$\log_{10}x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

c. Using the power property (Equation 2.7.15.3) of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln(x^6) = 0$. Using the quotient property (Equation 2.7.15.2), this becomes

$$\ln\left(\frac{2}{x^5}\right) = 0$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

Exercise 2.7.15.5

Solve $\ln(x^3) - 4\ln(x) = 1$.

Hint

First use the power property, then use the product property of logarithms.

Answer

$$x = \frac{1}{e}$$

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are $\log_{10}$ or log, called the **common logarithm**, or ln, which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base b. If you need to use a calculator to evaluate an expression with a different base, you can apply the change-of-base formulas first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Rule: Change-of-Base Formulas

Let $a > 0$, $b > 0$, and $a \neq 1$, $b \neq 1$.

1. $a^x = b^{x\log_b a}$ for any real number x.

   If $b = e$, this equation reduces to $a^x = e^{x\log_e a} = e^{x\ln a}$.

2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0$.

   If $b = e$, this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}$.

Example 2.7.15.6: Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earlier.

Solution

Use the second equation with $a = 3$ and $b = e$: $\log_3 7 = \dfrac{\ln 7}{\ln 3} \approx 1.77124$.

Exercise 2.7.15.6

Use the change-of-base formula and a calculating utility to evaluate $\log_4 6$.

Hint

Use the change of base to rewrite this expression in terms of expressions involving the natural logarithm function.

Answer

$$\log_4 6 = \frac{\ln 6}{\ln 4} \approx 1.29248$$

Example 2.7.15.7: The Richter Scale for Earthquakes

In 1935, Charles Richter developed a scale (now known as the Richter scale) to measure the magnitude of an **earthquake**. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake?

Figure 2.7.15.6: (credit: modification of work by Robb Hannawacker, NPS)

A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

$$R_1 - R_2 = \log_{10}\left(\frac{A1}{A2}\right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

$$8 - 7 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore,

$$\log_{10}\left(\frac{A1}{A2}\right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2, we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

$$\log_{10}\left(\frac{A1}{A2}\right) = 8 - 6 = 2.$$

Therefore, $A_1 = 100A_2$.That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10}\left(\frac{A1}{A2}\right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

Exercise 2.7.15.7

Compare the relative severity of a magnitude 8.4 earthquake with a magnitude 7.4 earthquake.

Hint

$R_1 - R_2 = \log_{10}(A1/A2)$.

Answer

The magnitude 8.4 earthquake is roughly 10 times as severe as the magnitude 7.4 earthquake.

Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0 = 1$, we can convert this to logarithmic form and get $\log_a 1 = 0$. Also, since $a^1 = a$, we get $\log_a a = 1$.

Definition 2.7.15.1

Properties of Logarithms

$$\log_a 1 = 0 \quad \log_a a = 1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

Example 2.7.15.1

Evaluate using the properties of logarithms:

a. $\log_8 1$
b. $\log_6 6$

Solution:

a.

$$\log_8 1$$

Use the property, $\log_a 1 = 0$.

$$0 \quad \log_8 1 = 0$$

b.

$$\log_6 6$$

Use the property, $\log_a a = 1$.

$$1 \quad \log_6 6 = 1$$

Exercise 2.7.15.1

Evaluate using the properties of logarithms:

a. $\log_{13} 1$
b. $\log_9 9$

Answer

a. 0
b. 1

Exercise 2.7.15.2

Evaluate using the properties of logarithms:

a. $\log_5 1$
b. $\log_7 7$

Answer

a. 0
b. 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x$, which is a true statement for positive values for x only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x$, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power. These two properties show the **composition** of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

Definition 2.7.15.2

Inverse Properties of Logarithms

For $a > 0, x > 0$ and $a \neq 1$,

$$a^{\log_a x} = x \quad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

Example 2.7.15.2

Evaluate using the properties of logarithms:

a. $4^{\log_4 9}$
b. $\log_3 3^5$

Solution:

a.

$$4^{\log_4 9}$$

Use the property, $a^{\log_a x} = x$.

$$9 \quad 4^{\log_4 9} = 9$$

b.

$$\log_3 3^5$$

Use the property, $a^{\log_a x} = x$.

$$5 \quad \log_3 3^5 = 5$$

Exercise 2.7.15.3

Evaluate using the properties of logarithms:

a. $5^{\log_5 15}$
b. $\log_7 7^4$

Answer

a. 15
b. 4

Exercise 2.7.15.4

Evaluate using the properties of logarithms:

a. $2^{\log_2 8}$
b. $\log_2 2^{15}$

Answer

a. 8
b. 15

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic function are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m \cdot a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_a M \cdot N = \log_a M + \log_a N$ tells us to take the log of a product, we add the log of the factors.

Definition 2.7.15.3

Product Property of Logarithms

If $M > 0, N > 0, \text{a} > 0$ and $\text{a} \neq 1$, then

$$\log_a (M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

Example 2.7.15.3

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3 7x$
b. $\log_4 64xy$

Solution:

a.

$$\log_3 7x$$

Use the Product Property, $\log_a (M \cdot N) = \log_a M + \log_a N$.

$$\log_3 7 + \log_3 x$$
$$\log_3 7x = \log_3 7 + \log_3 x$$

b.

$$\log_4 64xy$$

Use the Product Property, $\log_a(M \cdot N) = \log_a M + \log_a N$.

$$\log_4 64 + \log_4 x + \log_4 y$$

Simplify be evaluating, $\log_4 64$.

$$3 + \log_4 x + \log_4 y$$
$$\log_4 64xy = 3 + \log_4 x + \log_4 y$$

Exercise 2.7.15.5

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3 3x$
b. $\log_2 8xy$

Answer

a. $1 + \log_3 x$
b. $3 + \log_2 x + \log_2 y$

Exercise 2.7.15.6

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_9 9x$
b. $\log_3 27xy$

Answer

a. $1 + \log_9 x$
b. $3 + \log_3 x + \log_3 y$

Similarly, in the Quotient Property of Exponents, $\frac{a^m}{a^n} = a^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_a \frac{M}{N} = \log_a M - \log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Definition 2.7.15.4

Quotient Property of Logarithms

If $M > 0, N > 0, \mathrm{a} > 0$ and $\mathrm{a} \neq 1$, then

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M = \log_a N \neq \log_a(M - N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

Example 2.7.15.4

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_5 \frac{5}{7}$
b. $\log \frac{x}{100}$

Solution:

a.

$$\log_5 \frac{5}{7}$$

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.

$$\log_5 5 - \log_5 7$$

Simplify.

$$1 - \log_5 7$$

$$\log_5 \frac{5}{7} = 1 - \log_5 7$$

b.

$$\log \frac{x}{100}$$

Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.

$$\log x - \log 100$$

Simplify.

$$\log x - 2$$

$$\log \frac{x}{100} = \log x - 2$$

Exercise 2.7.15.7

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_4 \frac{3}{4}$
b. $\log \frac{x}{1000}$

Answer

a. $\log_4 3 - 1$
b. $\log x - 3$

Exercise 2.7.15.8

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_2 \frac{5}{4}$
b. $\log \frac{10}{y}$

Answer

a. $\log_2 5 - 2$
b. $1 - \log y$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Definition 2.7.15.5

Power Property of Logarithms

If $M > 0, \mathrm{a} > 0, \mathrm{a} \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

Example 2.7.15.5

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_5 4^3$
b. $\log x^{10}$

Solution:

a.

$$\log_5 4^3$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$3 \log_5 4$$

$$\log_5 4^3 = 3 \log_5 4$$

b.

$$\log x^{10}$$

Use the Power Property, $\log_a M^p = p \log_a M$.

$$10 \log x$$

$$\log x^{10} = 10 \log x$$

Exercise 2.7.15.9

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_7 5^4$
b. $\log x^{100}$

Answer

a. $4 \log_7 5$
b. $100 \cdot \log x$

Exercise 2.7.15.10

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_2 3^7$
b. $\log x^{20}$

Answer

a. $7 \log_2 3$
b. $20 \cdot \log x$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Properties of Logarithms

If $M > 0, \mathrm{a} > 0, \mathrm{a} \neq 1$ and p is any real number then,

Table 10.4.1

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

Now that we have the properties we can use them to "expand" a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

Example 2.7.15.6

Use the Properties of Logarithms to expand the logarithm $\log_4 \left(2x^3y^2\right)$. Simplify, if possible.

Solution:

Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$.

Use the Power Property, $\log_a M^p = p \log_a M$, on the last two terms. Simplify.

Exercise 2.7.15.11

Use the Properties of Logarithms to expand the logarithm $\log_2 \left(5x^4y^2\right)$. Simplify, if possible.

Answer

$\log_2 5 + 4 \log_2 x + 2 \log_2 y$

Exercise 2.7.15.12

Use the Properties of Logarithms to expand the logarithm $\log_3 \left(7x^5y^3\right)$. Simplify, if possible.

Answer

$\log_3 7 + 5 \log_3 x + 3 \log_3 y$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

Example 2.7.15.7

Use the Properties of Logarithms to expand the logarithm $\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$. Simplify, if possible.

Solution

$$\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$$

Rewrite the radical with a rational exponent.

$$\log_2\left(\frac{x^3}{3y^2z}\right)^{\frac{1}{4}}$$

Use the Power Property, $\log_a M^p = p\log_a M$.

$$\frac{1}{4}\log_2\left(\frac{x^3}{3y^2z}\right)$$

Use the Quotient Property, $\log_a M \cdot N = \log_a M - \log_a N$.

$$\frac{1}{4}\left(\log_2\left(x^3\right) - \log_2\left(3y^2z\right)\right)$$

Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$, in the second term.

$$\frac{1}{4}\left(\log_2\left(x^3\right) - \left(\log_2 3 + \log_2 y^2 + \log_2 z\right)\right)$$

Use the Power Property, $\log_a M^p = p\log_a M$, inside the parentheses.

$$\frac{1}{4}(3\log_2 x - (\log_2 3 + 2\log_2 y + \log_2 z))$$

Simplify by distributing.

$$\frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$$

$$\log_2\sqrt[4]{\frac{x^3}{3y^2z}} = \frac{1}{4}(3\log_2 x - \log_2 3 - 2\log_2 y - \log_2 z)$$

Exercise 2.7.15.13

Use the Properties of Logarithms to expand the logarithm $\log_4\sqrt[5]{\frac{x^4}{2y^3z^2}}$. Simplify, if possible.

Answer

$\frac{1}{5}\left(4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z\right)$

Exercise 2.7.15.14

Use the Properties of Logarithms to expand the logarithm $\log_3\sqrt[3]{\frac{x^2}{5yz}}$. Simplify, if possible.

Answer

$\frac{1}{3}(2\log_3 x - \log_3 5 - \log_3 y - \log_3 z)$

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

Example 2.7.15.8

Use the Properties of Logarithms to condense the logarithm $\log_4 3 + \log_4 x - \log_4 y$. Simplify, if possible.

Solution:

The log expressions all have the same base, 4.

The first two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M : N$.

Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$.

Exercise 2.7.15.15

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

Answer

$\log_2 \frac{5x}{y}$

Exercise 2.7.15.16

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

Answer

$\log_3 \frac{6}{xy}$

Example 2.7.15.9

Use the Properties of Logarithms to condense the logarithm $2\log_3 x + 4\log_3(x+1)$. Simplify, if possible.

Solution:

The log expressions have the same base, 3.

$$2\log_3 x + 4\log_3(x+1)$$

Use the Power Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$\log_3 x^2 + \log_3(x+1)^4$$

The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.

$$\log_3 x^2(x+1)^4$$
$$2\log_3 x + 4\log_3(x+1) = \log_3 x^2(x+1)^4$$

Exercise 2.7.15.17

Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2(x-1)$. Simplify, if possible.

Answer

$\log_2 x^3(x-1)^2$

Exercise 2.7.15.18

Use the Properties of Logarithms to condense the logarithm $2\log x + 2\log(x+1)$. Simplify, if possible.

Answer

$\log x^2(x+1)^2$

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the **Change-of-Base Formula**. We will show how this is derived.

Suppose we want to evaluate $\log_a M$	$\log_a M$
Let $y = \log_a M$.	$y = \log_a M$
Rewrite the epression in exponential form.	$a^y = M$
Take the $\log_b$ of each side.	$\log_b a^y = \log_b M$
Use the Power Property.	$y \log_b a = \log_b M$
Solve for y.	$y = \frac{\log_b M}{\log_b a}$
Substiture $y = \log_a M$.	$\log_a M = \frac{\log_b M}{\log_b a}$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b > 0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Definition 2.7.15.6

Change-of-Base Formula

For any logarithmic bases a, b and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$

new base b new base 10 new base e

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol ($\approx$).

Example 2.7.15.10

Rounding to three decimal places, approximate $\log_4 35$.

Solution:

Table 10.4.2

	$\log_4 35$
Use the Change-of-Base Formula.	$\log_a M = \frac{\log_b M}{\log_b a}$
Identify a and M. Choose 10 for b.	$\log_4 35 = \frac{\log 35}{\log 4}$
Enter the expression $\frac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.	$\log_4 35 \approx 2.565$

Exercise 2.7.15.19

Rounding to three decimal places, approximate $\log_3 42$.

Answer

3.402

Exercise 2.7.15.20

Rounding to three decimal places, approximate $\log_5 46$.

Answer

2.379

In the previous section, we derived two important properties of logarithms, which allowed us to solve some basic exponential and logarithmic equations.

properties of logs

Inverse Properties:

$$\log_b(b^x) = x \tag{2.7.15.4}$$

$$b^{\log_b x} = x \tag{2.7.15.5}$$

Exponential Property:

$$\log_b(A^r) = r\log_b(A) \tag{2.7.15.6}$$

Change of Base:

$$\log_b(A) = \frac{\log_c(A)}{\log_c(b)} \tag{2.7.15.7}$$

While these properties allow us to solve a large number of problems, they are not sufficient to solve all problems involving exponential and logarithmic equations.

properties of logs

Sum of Logs Property:

$$\log_b(A) + \log_b(C) = \log_b(AC) \tag{2.7.15.8}$$

Difference of Logs Property:

$$\log_b(A) - \log_b(C) = \log_b\left(\frac{A}{C}\right) \tag{2.7.15.9}$$

It's just as important to know what properties logarithms do not satisfy as to memorize the valid properties listed above. In particular, the logarithm is not a linear function, which means that it does not distribute:

$$\log A + B \neq \log A + \log B. \tag{2.7.15.10}$$

To help in this process we offer a proof of Equation 2.7.15.10 to help solidify our new rules and show how they follow from properties you've already seen.

Proof

Let $a = \log_b(A)$ and $c = \log_b(C)$.

By definition of the logarithm, $b^a = A$ and $b^c = C$.

Using these expressions,

$$AC = b^a b^c$$

Using exponent rules on the right,

$$AC = b^{a+c}$$

Taking the log of both sides, and utilizing the inverse property of logs,

$$\log_b(AC) = \log_b\left(b^{a+c}\right) = a + c$$

Replacing a and c with their definition establishes the result

$$\log_b(AC) = \log_b A + \log_b C$$

The proof for the difference property is very similar.

With these properties, we can rewrite expressions involving multiple logs as a single log, or break an expression involving a single log into expressions involving multiple logs.

Example 2.7.15.1

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution

Using the sum of logs property on the first two terms,

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to

$$\log_3(40)-\log_3(2)$$

Then using the difference of logs property,

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20)$$

Example 2.7.15.2

Evaluate $2\log(5)+\log(4)$ without a calculator by first rewriting as a single logarithm.

Solution

On the first term, we can use the exponent property of logs to write

$$2\log(5)=\log\left(5^2\right)=\log(25)$$

With the expression reduced to a sum of two logs, $\log(25)+\log(4)$, we can utilize the sum of logs property

$$\log(25)+\log(4)=\log(4\cdot 25)=\log(100)$$

Since $100=10^2$, we can evaluate this log without a calculator:

$$\log(100)=\log\left(10^2\right)=2$$

Exercise 2.7.15.1

Without a calculator evaluate by first rewriting as a single logarithm:

$$\log_2(8)+\log_2(4)$$

Answer

$$\log_2(8\cdot 4)=\log_2(32)=\log_2\left(2^5\right)=5$$

Example 2.7.15.3

Rewrite $\ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs

Solution

First, noticing we have a quotient of two expressions, we can utilize the difference property of logs to write

$$\ln\left(\frac{x^4y}{7}\right)=\ln\left(x^4y\right)-\ln(7)$$

Then seeing the product in the first term, we use the sum property

$$\ln\left(x^4 y\right) - \ln(7) = \ln\left(x^4\right) + \ln(y) - \ln(7)$$

Finally, we could use the exponent property on the first term

$$\ln\left(x^4\right) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

Log properties in Solving Equations

The logarithm properties often arise when solving problems involving logarithms. First, we'll look at a simpler log equation.

Example 2.7.15.5

Solve $\log(2x-6) = 3$.

Solution

To solve for x, we need to get it out from inside the log function. There are two ways we can approach this.

Method 1: Rewrite as an exponential.

Recall that since the common log is base 10, $\log(A) = B$ can be rewritten as the exponential $10^B = A$. Likewise, $\log(2x-6) = 3$ can be rewritten in exponential form as

$$10^3 = 2x - 6$$

Method 2: Exponentiate both sides.

If $A = B$, then $10^A = 10^B$. Using this idea, since $\log(2x-6) = 3$, then $10^{\log(2x-6)} = 10^3$. Use the inverse property of logs to rewrite the left side and get $2x - 6 = 10^3$.

Using either method, we now need to solve $2x - 6 = 10^3$. Evaluate 10^3 to get

$$2x - 6 = 1000$$

Add 6 to both sides

$$2x = 1006$$

Divide both sides by 2

$$x = 503$$

Occasionally the solving process will result in extraneous solutions – answers that are outside the domain of the original equation. In this case, our answer looks fine.

Example 2.7.15.6

Solve $\log(50x+25) - \log(x) = 2$.

Solution

In order to rewrite in exponential form, we need a single logarithmic expression on the left side of the equation. Using the difference property of logs, we can rewrite the left side:

$$\log\left(\frac{50x+25}{x}\right) = 2$$

Rewriting in exponential form reduces this to an algebraic equation:

$$\frac{50x+25}{x} = 10^2 = 100$$

Multiply both sides by x

$$50x+25=100x$$

Combine like terms

$$25=50x$$

Divide by 50

$$x=\frac{25}{50}=\frac{1}{2}$$

Checking this answer in the original equation, we can verify there are no domain issues, and this answer is correct.

Exercise 2.7.15.2

Solve $\log(x^2-4)=1+\log(x+2)$.

Answer

$$\log(x^2-4)=1+\log(x+2)$$

Move both logs to one side

$$\log\left(x^2-4\right)-\log(x+2)=1$$

Use the difference property of logs

$$\log\left(\frac{x^2-4}{x+2}\right)=1$$

Factor

$$\log\left(\frac{(x+2)(x-2)}{x+2}\right)=1$$

Simplify

$$\log(x-2)=1$$

Rewrite as an exponential

$$10^1=x-2$$

Add 2 to both sides

$$x=12$$

Example 2.7.15.7

Solve $\ln(x+2)+\ln(x+1)=\ln(4x+14)$.

Solution

$$\ln(x+2)+\ln(x+1)=\ln(4x+14)$$

Use the sum of logs property on the right

$$\ln((x+2)(x+1))=\ln(4x+14)$$

Expand

$$\ln\left(x^2+3x+2\right)=\ln(4x+14)$$

We have a log on both side of the equation this time. Rewriting in exponential form would be tricky, so instead we can exponentiate both sides.

$$e^{\ln(x^2+3x+2)} = e^{\ln(4x+13)}$$

Use the inverse property of logs

$$x^2+3x+2=4x+14$$

Move terms to one side

$$x^2-x-12=0$$

Factor

$$(x+4)(x-3)=0$$

$$x=-4 \text{ or } x=3$$

Checking our answers, notice that evaluating the original equation at $x=-4$ would result in us evaluating $\ln(-2)$, which is undefined. That answer is outside the domain of the original equation, so it is an extraneous solution and we discard it. There is one solution: $x=3$.

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

Example $2.7.15.8a$

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year(World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 24, 2010). If these trends continue, when will the population of Kenya match that of Sudan?

Solution

We start by writing an equation for each population in terms of t, the number of years after 2008.

$$Kenya(t)=38.8(1+0.0264)^t$$
$$Sudan(t)=41.3(1+0.0224)^t$$

To find when the populations will be equal, we can set the equations equal

$$38.8(1.0264)^t=41.3(1.0224)^t$$

For our first approach, we take the log of both sides of the equation.

$$\log\left(38.8(1.0264)^t\right)=\log\left(41.3(1.0224)^t\right)$$

Utilizing the sum property of logs, we can rewrite each side,

$$\log(38.8)+\log\left(1.0264^t\right)=\log(41.3)+\log\left(1.0224^t\right)$$

Then utilizing the exponent property, we can pull the variables out of the exponent

$$\log(38.8)+t\log(1.0264)=\log(41.3)+t\log(1.0224)$$

Moving all the terms involving t to one side of the equation and the rest of the terms to the other side,

$$t\log(1.0264)-t\log(1.0224)=\log(41.3)-\log(38.8)$$

Factoring out the t on the left,

$$t\left(\log(1.0264)-\log(1.0224)\right)=\log(41.3)-\log(38.8)$$

Dividing to solve for t

$$t = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)} \approx 15.991$$

It will be 15.991 years until the populations will be equal.

Example 2.7.15.8b

Solve the problem above by rewriting before taking the log.

Solution

Starting at the equation

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

Divide to move the exponential terms to one side of the equation and the constants to the other side

$$\frac{1.0264^t}{1.0224^t} = \frac{41.3}{38.8}$$

Using exponent rules to group on the left,

$$\left(\frac{1.0264}{1.0224}\right)^t = \frac{41.3}{38.8}$$

Taking the log of both sides

$$\log\left(\left(\frac{1.0264}{1.0224}\right)^t\right) = \log\left(\frac{41.3}{38.8}\right)$$

Utilizing the exponent property on the left,

$$t\log\left(\frac{1.0264}{1.0224}\right) = \log\left(\frac{41.3}{38.8}\right)$$

Dividing gives

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} \approx 15.991 \text{ years}$$

While the answer does not immediately appear identical to that produced using the previous method, note that by using the difference property of logs, the answer could be rewritten:

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)}$$

While both methods work equally well, it often requires fewer steps to utilize algebra before taking logs, rather than relying solely on log properties.

Exercise 2.7.15.3

Tank A contains 10 liters of water, and 35% of the water evaporates each week. Tank B contains 30 liters of water, and 50% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

Answer

Tank A: $A(t) = 10(1-0.35)^t$. Tank B: $B(t) = 30(1-0.50)^t$

Solving A(t) = B(t),

$$10(0.65)^t = 30(0.5)^t$$

Using the method from Example 8b

$$\frac{(0.65)^t}{(0.5)^t} = \frac{30}{10}$$

Regroup

$$\left(\frac{0.65}{0.5}\right)^t = 3$$

Simplify

$$(1.3)^t = 3$$

Take the log of both sides

$$\log\left((1.3)^t\right) = \log(3)$$

Use the exponent property of logs

$$t\log(1.3) = \log(3)$$

Divide and evaluate

$$t = \frac{\log(3)}{\log(1.3)} \approx 4.1874 \text{ weeks}$$

Applications

While we have explored some basic applications of exponential and logarithmic functions, in this section we explore some important applications in more depth.

Radioactive Decay

In Nuclear Physics we discuss radioactive decay – the idea that radioactive isotopes change over time. One of the common terms associated with radioactive decay is half-life.

Definition: Half Life

The **half-life** of a radioactive isotope is the time it takes for half the substance to decay.

Given the basic exponential growth/decay equation $h(t) = ab^t$, half-life can be found by solving for when half the original amount remains; by solving $\frac{1}{2}a = a(b)^t$, or more simply $\frac{1}{2} = b^t$. Notice how the initial amount is irrelevant when solving for half-life.

Example 2.7.15.1

Bismuth-210 is an isotope that decays by about 13% each day. What is the half-life of Bismuth-210?

Solution

We were not given a starting quantity, so we could either make up a value or use an unknown constant to represent the starting amount. To show that starting quantity does not affect the result, let us denote the initial quantity by the constant a. Then the decay of Bismuth-210 can be described by the equation $Q(d) = a(0.87)^d$.

To find the half-life, we want to determine when the remaining quantity is half the original: $\frac{1}{2}a$. Solving,

$$\frac{1}{2}a = a(0.87)^d$$

Divide by a,

$$\frac{1}{2} = 0.87^d$$

Take the log of both sides

$$\log\left(\frac{1}{2}\right) = \log\left(0.87^d\right)$$

Use the exponent property of logs

$$\log\left(\frac{1}{2}\right) = d\log(0.87)$$

Divide to solve for d

$$d = \frac{\log\left(\frac{1}{2}\right)}{\log(0.87)} \approx 4.977 \text{ days}$$

This tells us that the half-life of Bismuth-210 is approximately 5 days.

Example 2.7.15.2

Cesium-137 has a half-life of about 30 years. If you begin with 200 mg of cesium-137, how much will remain after 30 years? 60 years? 90 years?

Solution

Since the half-life is 30 years, after 30 years, half the original amount, 100 mg, will remain.

After 60 years, another 30 years have passed, so during that second 30 years, another half of the substance will decay, leaving 50 mg.

After 90 years, another 30 years have passed, so another half of the substance will decay, leaving 25 mg.

Example 2.7.15.3

Cesium-137 has a half-life of about 30 years. Find the annual decay rate.

Solution

Since we are looking for an annual decay rate, we will use an equation of the form $Q(t) = a(1+r)^t$. We know that after 30 years, half the original amount will remain. Using this information

$$\frac{1}{2}a = a(1+r)^{30}$$

Dividing by a

$$\frac{1}{2} = (1+r)^{30}$$

Taking the 30th root of both sides

$$\sqrt[30]{\frac{1}{2}} = 1+r$$

Subtracting one from both sides,

$$r = \sqrt[30]{\frac{1}{2}} - 1 \approx -0.02284$$

This tells us cesium-137 is decaying at an annual rate of 2.284% per year.

Exercise 2.7.15.1

Chlorine-36 is eliminated from the body with a biological half-life of 10 days (www.ead.anl.gov/pub/doc/chlorine.pdf). Find the daily decay rate.

Answer

$r = \sqrt[10]{\frac{1}{2}} - 1 \approx -0.067$ or 6.7% is the daily rate of decay.

Example 2.7.15.4

Carbon-14 is a radioactive isotope that is present in organic materials, and is commonly used for dating historical artifacts. Carbon-14 has a half-life of 5730 years. If a bone fragment is found that contains 20% of its original carbon-14, how old is the bone?

Solution

To find how old the bone is, we first will need to find an equation for the decay of the carbon-14. We could either use a continuous or annual decay formula, but opt to use the continuous decay formula since it is more common in scientific texts. The half life tells us that after 5730 years, half the original substance remains. Solving for the rate,

$$\frac{1}{2}a = ae^{r5730}$$

Dividing by a

$$\frac{1}{2} = e^{r5730}$$

Taking the natural log of both sides

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{r5730}\right)$$

Use the inverse property of logs on the right side

$$\ln\left(\frac{1}{2}\right) = 5730r$$

Divide by 5730

$$r = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -0.000121$$

Now we know the decay will follow the equation $Q(t) = ae^{-0.000121t}$. To find how old the bone fragment is that contains 20% of the original amount, we solve for t so that $Q(t) = 0.20a$.

$$0.20a = ae^{-0.000121t}$$

$$0.20 = e^{-0.000121t}$$

$$\ln(0.20) = \ln\left(e^{-0.000121t}\right)$$

$$\ln(0.20) = -0.000121t$$

$$t = \frac{\ln(0.20)}{-0.000121} \approx 13301 \text{ years}$$

The bone fragment is about 13,300 years old.

Exercise 2.7.15.2

In Example 2, we learned that Cesium-137 has a half-life of about 30 years. If you begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Answer

Less than 230 years, 229.3157 to be exact.

Doubling Time

For decaying quantities, we asked how long it takes for half the substance to decay. For growing quantities we might ask how long it takes for the quantity to double.

Definition: Doubling Time

The **doubling time** of a growing quantity is the time it takes for the quantity to double.

Given the basic exponential growth equation $h(t) = ab^t$, doubling time can be found by solving for when the original quantity has doubled; by solving $2a = a(b)^x$, or more simply $2 = b^x$. Like with decay, the initial amount is irrelevant when solving for doubling time.

Example 2.7.15.5

Cancer cells sometimes increase exponentially. If a cancerous growth contained 300 cells last month and 360 cells this month, how long will it take for the number of cancer cells to double?

Solution

Defining t to be time in months, with $t = 0$ corresponding to this month, we are given two pieces of data: this month, (0, 360), and last month, (-1, 300).

From this data, we can find an equation for the growth. Using the form $C(t) = ab^t$, we know immediately a = 360, giving $C(t) = 360b^t$. Substituting in (-1, 300),

$$300 = 360b^{-1}$$
$$300 = \frac{360}{b}$$
$$b = \frac{360}{300} = 1.2$$

This gives us the equation $C(t) = 360(1.2)^t$

To find the doubling time, we look for the time when we will have twice the original amount, so when $C(t) = 2a$.

$$2a = a(1.2)^t$$

$$2 = (1.2)^t$$

$$\log(2) = \log\left(1.2^t\right)$$

$$\log(2) = t\log(1.2)$$

$$t = \frac{\log(2)}{\log(1.2)} \approx 3.802$$

months for the number of cancer cells to double.

Example 2.7.15.6

Use of a new social networking website has been growing exponentially, with the number of new members doubling every 5 months. If the site currently has 120,000 users and this trend continues, how many users will the site have in 1 year?

Solution

We can use the doubling time to find a function that models the number of site users, and then use that equation to answer the question. While we could use an arbitrary *a* as we have before for the initial amount, in this case, we know the initial amount was 120,000.

If we use a continuous growth equation, it would look like $N(t) = 120e^{rt}$, measured in thousands of users after *t* months. Based on the doubling time, there would be 240 thousand users after 5 months. This allows us to solve for the continuous growth rate:

$$240 = 120e^{r5}$$

$$2 = e^{r5}$$

$$\ln 2 = 5r$$

$$r = \frac{\ln 2}{5} \approx 0.1386$$

Now that we have an equation, $N(t) = 120e^{0.1386t}$, we can predict the number of users after 12 months:

$$N(12) = 120e^{0.1386(12)} = 633.140 \text{ thousand users}$$

.

So after 1 year, we would expect the site to have around 633,140 users.

Exercise 2.7.15.3

If tuition at a college is increasing by 6.6% each year, how many years will it take for tuition to double?

Answer

Solving $a(1+0.066)^t = 2a$, it will take $t = \frac{log(2)}{log(1.066)} \approx 10.845$ years, or approximately 11 years, for tuition to double.

Newton's Law of Cooling

When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off towards the surrounding air temperature. This "leveling off" will correspond to a horizontal asymptote in the graph of

the temperature function. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function.

Definition: Newton's Law of Cooling

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = ae^{kt} + T_s \tag{2.7.15.11}$$

Where

- t is time
- a is a constant determined by the initial temperature of the object
- k is a constant, the continuous rate of cooling of the object

While an equation of the form $T(t) = ab^t + T_s$ could be used, the continuous growth form is more common.

Example 2.7.15.7

A cheesecake is taken out of the oven with an ideal internal temperature of 165 degrees Fahrenheit, and is placed into a 35 degree refrigerator. After 10 minutes, the cheesecake has cooled to 150 degrees. If you must wait until the cheesecake has cooled to 70 degrees before you eat it, how long will you have to wait?

Solution

Since the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially towards 35, following the equation

$$T(t) = ae^{kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$. Substituting in these values,

$$\begin{aligned} 165 &= ae^{k0} + 35 \\ 165 &= a + 35 \\ a &= 130 \end{aligned}$$

We were given another pair of data, $T(10) = 150$, which we can use to solve for k

$$150 = 130e^{k10} + 35$$

$$\begin{aligned} 115 &= 130e^{k10} \\ \frac{115}{130} &= e^{10k} \\ \ln\left(\frac{115}{130}\right) &= 10k \\ k &= \frac{\ln\left(\frac{115}{130}\right)}{10} = -0.0123 \end{aligned}$$

Together this gives us the equation for cooling:

$$T(t) = 130e^{-0.0123t} + 35$$

Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$70 = 130e^{-0.0123t} + 35$$

$$35 = 130e^{-0.0123t}$$

$$\frac{35}{130} = e^{-0.0123t}$$

$$\ln\left(\frac{35}{130}\right) = -0.0123t$$

$$t = \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \approx 106.68$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool. Of course, if you like your cheesecake served chilled, you'd have to wait a bit longer.

Exercise 2.7.15.4

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Answer

$T(t) = ae^{kt} + 70$. Substituting (0, 40), we find $a = -30$. Substituting (1, 45), we solve

$$45 = -30e^{k(1)} + 70$$

to get

$$k = ln(\frac{25}{30}) = -0.1823$$

Solving $60 = -30e^{-0.1823t} + 70$ gives

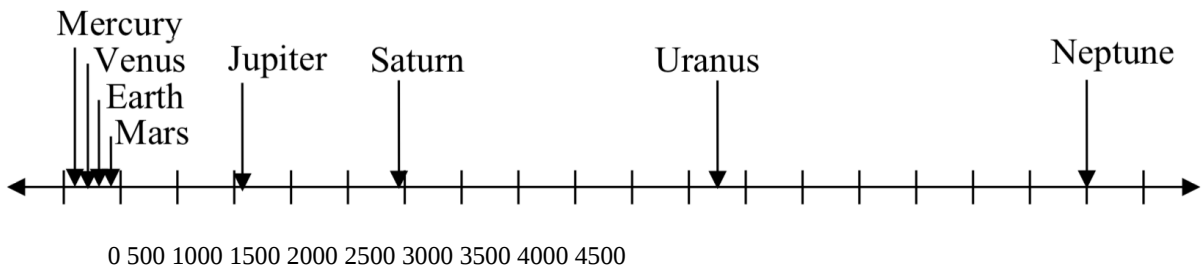
$$t = \frac{ln(1/3)}{-0.1823} = 6.026 \text{ hours}$$

Logarithmic Scales

For quantities that vary greatly in magnitude, a standard scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the average distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

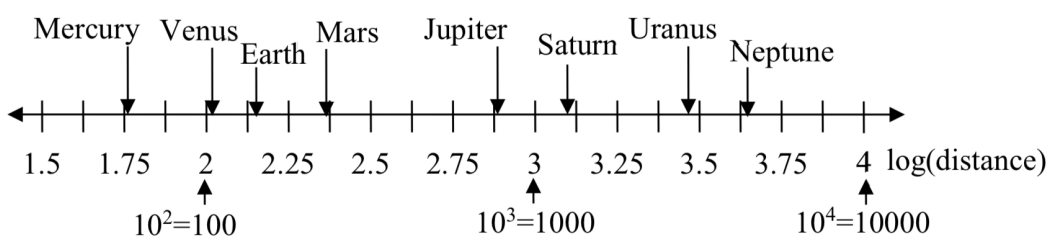
Planet	Distance (millions of km)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

Placed on a linear scale – one with equally spaced values – these values get bunched up.

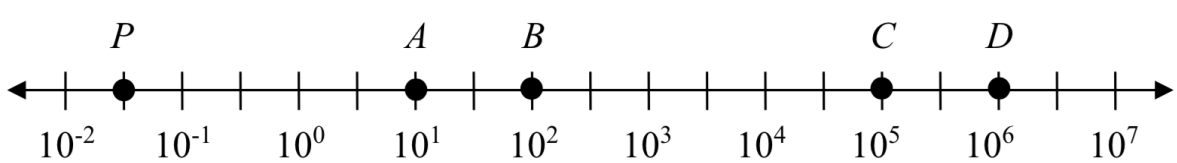


However, computing the logarithm of each value and plotting these new values on a number line results in a more manageable graph, and makes the relative distances more apparent.(It is interesting to note the large gap between Mars and Jupiter on the log number line. The asteroid belt is located there, which scientists believe is a planet that never formed because of the effects of the gravity of Jupiter.)

Planet	Distance (millions of km)	log(distance)
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65



Sometimes, as shown above, the scale on a logarithmic number line will show the log values, but more commonly the original values are listed as powers of 10, as shown below.



Example 2.7.15.8

Estimate the value of point P on the log scale above

The point P appears to be half way between -2 and -1 in log value, so if V is the value of this point,

$$\log(V) \approx -1.5$$

Rewriting in exponential form,

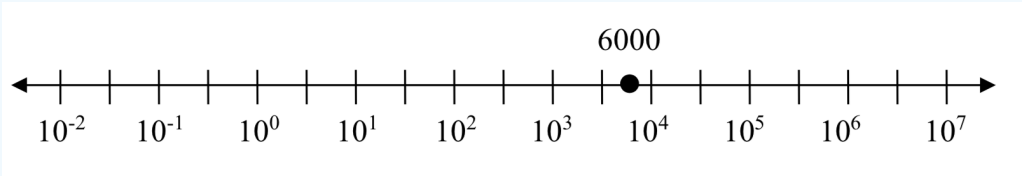
$$V \approx 10^{-1.5} = 0.0316$$

Example 2.7.15.9

Place the number 6000 on a logarithmic scale.

Solution

Since $\log(6000) \approx 3.8$, this point would belong on the log scale about here:

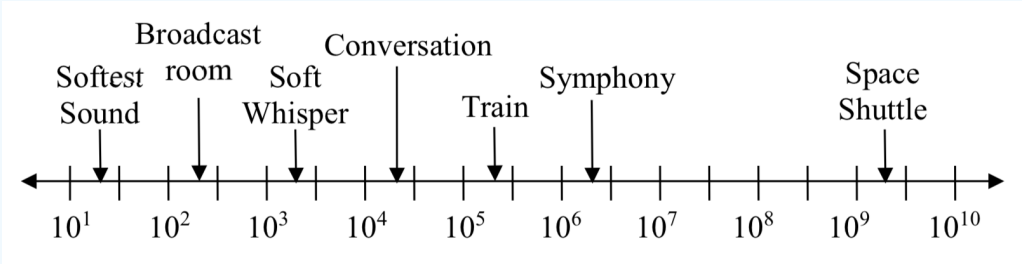


Exercise 2.7.15.5

Plot the data in the table below on a logarithmic scale (From http://www.epd.gov.hk/epd/noise_educ...1/intro_5.html, retrieved Oct 2, 2010).

Source of Sound/Noise	Approximate Sound Pressure in μ Pa (micro Pascals)
Launching of the Space Shuttle	2000,000,000
Full Symphony Orchestra	2000,000
Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000
Unoccupied Broadcast Studio	200
Softest Sound a human can hear	20

Answer



Notice that on the log scale above Example 8, the visual distance on the scale between points A and B and between C and D is the same. When looking at the values these points correspond to, notice B is ten times the value of A, and D is ten times the value of C. A visual *linear* difference between points corresponds to a relative (ratio) change between the corresponding values.

Logarithms are useful for showing these relative changes. For example, comparing \$1,000,000 to \$10,000, the first is 100 times larger than the second.

$$\frac{1,000,000}{10,000} = 100 = 10^2$$

Likewise, comparing \$1000 to \$10, the first is 100 times larger than the second.

$$\frac{1,000}{10} = 100 = 10^2$$

When one quantity is roughly ten times larger than another, we say it is one **order of magnitude** larger. In both cases described above, the first number was two orders of magnitude larger than the second.

Notice that the order of magnitude can be found as the common logarithm of the ratio of the quantities. On the log scale above, B is one order of magnitude larger than A, and D is one order of magnitude larger than C.

Definition: Orders of magnitude

Given two values A and B, to determine how many **orders of magnitude** A is greater than B,

Difference in orders of magnitude = $\log(\frac{A}{B})$

Example 2.7.15.10

On the log scale above Example 8, how many orders of magnitude larger is C than B?

Solution

The value B corresponds to $10^2 = 100$

The value C corresponds to $10^5 = 100,000$

The relative change is $\frac{100,000}{100} = 1000 = \frac{10^5}{10^2} = 10^3$. The log of this value is 3.

C is three orders of magnitude greater than B, which can be seen on the log scale by the visual difference between the points on the scale.

Exercise 2.7.15.6

Using the table from Try it Now #5, what is the difference of order of magnitude between the softest sound a human can hear and the launching of the space shuttle?

Answer

$\frac{2 \times 10^9}{2 \times 10^1} = 10^8$. The sound pressure in μPa created by launching the space shuttle is 8 orders of magnitude greater than the sound pressure in μPa created by the softest sound a human ear can hear.

Earthquakes

An example of a logarithmic scale is the Moment Magnitude Scale (MMS) used for earthquakes. This scale is commonly and mistakenly called the Richter Scale, which was a very similar scale succeeded by the MMS.

Moment Magnitude Scale

For an earthquake with seismic moment S, a measurement of earth movement, the MMS value, or magnitude of the earthquake, is

$$M = \frac{2}{3} log(\frac{S}{S_0}) \tag{2.7.15.12}$$

Where $S_0 = 10^{16}$ is a baseline measure for the seismic moment.

Example 2.7.15.11

If one earthquake has a MMS magnitude of 6.0, and another has a magnitude of 8.0, how much more powerful (in terms of earth movement) is the second earthquake?

Solution

Since the first earthquake has magnitude 6.0, we can find the amount of earth movement for that quake, which we'll denote S_1. The value of S_0 is not particularity relevant, so we will not replace it with its value.

$$6.0 = \frac{2}{3} log(\frac{S_1}{S_0})$$

$$6.0(\frac{3}{2} = log(\frac{S_1}{S_0})$$

$$9 = log(\frac{S_1}{S_0})$$

$$\frac{S_1}{S_0} = 10^9$$

$$S_1 = 10^9 S_0$$

This tells us the first earthquake has about 10^9 times more earth movement than the baseline measure.

Doing the same with the second earthquake, S_2, with a magnitude of 8.0,

$$8.0 = \frac{2}{3} log(\frac{S_2}{S_0})$$

$$S_2 = 10^{12} S_0$$

Comparing the earth movement of the second earthquake to the first,

$$\frac{S_2}{S_1} = \frac{10^{12} S_0}{10^9 S_0} = 10^3 = 1000$$

The second value's earth movement is 1000 times as large as the first earthquake.

Example 2.7.15.12

One earthquake has magnitude of 3.0. If a second earthquake has twice as much earth movement as the first earthquake, find the magnitude of the second quake.

Solution

Since the first quake has magnitude 3.0,

$$3.0 = \frac{2}{3} log(\frac{S}{S_0})$$

Solving for S,

$$3.0 \frac{3}{2} = log(\frac{S}{S_0})$$

$$4.5 = log(\frac{S}{S_0})$$

$$10^{4.5} = \frac{S}{S_0}$$

$$S = 10^{4.5} S_0$$

Since the second earthquake has twice as much earth movement, for the second quake,

$$S = 2 \cdot 10^{4.5} S_0$$

Finding the magnitude,

$$M = \frac{2}{3} log(\frac{2 \cdot 10^{4.5} S_0}{S_0})$$

$$M = \frac{2}{3} log(2 \cdot 10^{4.5}) \approx 3.201$$

The second earthquake with twice as much earth movement will have a magnitude of about 3.2.

In fact, using log properties, we could show that whenever the earth movement doubles, the magnitude will increase by about 0.201:

$$M = \frac{2}{3} log(\frac{2S}{S_0}) = \frac{2}{3} log(2 \cdot \frac{S}{S_0})$$

$$M = \frac{2}{3} (log(2) + log(\frac{S}{S_0}))$$

$$M = \frac{2}{3} log(2) + \frac{2}{3} log(\frac{S}{S_0})$$

$$M = 0.201 + \frac{2}{3} log(\frac{S}{S_0})$$

This illustrates the most important feature of a log scale: that $multiplying$the quantity being considered will add to the scale value, and vice versa.

Key Concepts

- $\log_a 1 = 0 \quad \log_a a = 1$
- **Inverse Properties of Logarithms**
  - For $a > 0, x > 0$ and $a \neq 1$

$$a^{\log_a x} = x \quad \log_a a^x = x$$

- **Product Property of Logarithms**
  - If $M > 0, N > 0, a > 0$ and $a \neq 1$, then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

    The logarithm of a product is the sum of the logarithms.
- **Quotient Property of Logarithms**
  - If $M > 0, N > 0, \mathrm{a} > 0$ and $a \neq 1$, then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**
  - If $M > 0, a > 0, a \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

- **Properties of Logarithms Summary**
  If $M > 0, a > 0, a \neq 1$ and p is any real number then,

Table 10.4.1

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**
  For any logarithmic bases a and b, and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \quad \log_a M = \frac{\log M}{\log a} \quad \log_a M = \frac{\ln M}{\ln a}$$

$$\text{new base } b \qquad \text{new base } 10 \qquad \text{new base } e$$

Key Concepts

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a, we can make a change of base to convert this function to any base $b > 0$, $b \neq 1$. We typically convert to base e.

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