8.1: Newton's Second Law for Rotation

Learning Objectives

- Calculate the torques on rotating systems about a fixed axis to find the angular acceleration
- Explain how changes in the moment of inertia of a rotating system affect angular acceleration with a fixed applied torque

In this subsection, we put together all the pieces learned so far in this chapter to analyze the dynamics of rotating rigid bodies. We have analyzed motion with kinematics and rotational kinetic energy but have not yet connected these ideas with force and/or torque. In this subsection, we introduce the rotational equivalent to Newton's second law of motion and apply it to rigid bodies with fixed-axis rotation.

Newton's Second Law for Rotation

We have thus far found many counterparts to the translational terms used throughout this text, most recently, torque, the rotational analog to force. This raises the question: Is there an analogous equation to Newton's second law, $\sum \vec{F} = m\vec{a}$, which involves torque and rotational motion? To investigate this, we start with Newton's second law for a single particle rotating around an axis and executing circular motion. Let's exert a force $\vec{F}$ on a point mass m that is at a distance r from a pivot point (Figure 8.1.1). The particle is constrained to move in a circular path with fixed radius and the force is tangent to the circle. We apply Newton's second law to determine the magnitude of the acceleration $a = \frac{F}{m}$ in the direction of $\vec{F}$. Recall that the magnitude of the tangential acceleration is proportional to the magnitude of the angular acceleration by $a = r\alpha$. Substituting this expression into Newton's second law, we obtain

$$F = mr\alpha. \tag{8.1.1}$$

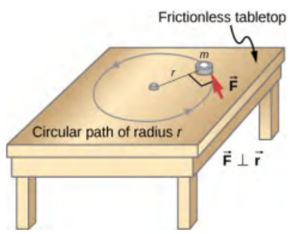

Figure 8.1.1: An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force $\vec{F}$ is applied to the object perpendicular to the radius r, causing it to accelerate about the pivot point. The force is perpendicular to r.

Multiply both sides of this equation by r,

$$rF = mr^2\alpha. \tag{8.1.2}$$

Note that the left side of this equation is the torque about the axis of rotation, where r is the lever arm and F is the force, perpendicular to r. Recall that the moment of inertia for a point particle is $I = mr^2$. The torque applied perpendicularly to the point mass in Figure 8.1.1 is therefore

$$\tau = I\alpha. \tag{8.1.3}$$

The torque on the particle is equal to the moment of inertia about the rotation axis times the angular acceleration. We can generalize this equation to a rigid body rotating about a fixed axis.

Newton's Second Law for Rotation

If more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration:

$$\sum_i \vec{\tau}_i = I\vec{\alpha}. \tag{8.1.4}$$

The term Iα is a scalar quantity and can be positive or negative (counterclockwise or clockwise) depending upon the sign of the net torque. Remember the convention that counterclockwise angular acceleration is positive. Thus, if a rigid body is rotating clockwise and experiences a positive torque (counterclockwise), the angular acceleration is positive.

Equation 8.1.4 is **Newton's second law for rotation** and tells us how to relate torque, moment of inertia, and rotational kinematics. This is called the equation for **rotational dynamics**. With this equation, we can solve a whole class of problems involving force and rotation. It makes sense that the relationship for how much force it takes to rotate a body would include the moment of inertia, since that is the quantity that tells us how easy or hard it is to change the rotational motion of an object.

Deriving Newton's Second Law for Rotation in Vector Form

We have seen before that we can express the tangential acceleration vector as a cross product of the angular acceleration and the position vector. This expression can be found by taking the time derivative of $\vec{v}=\vec{\omega}\times\vec{r}$ and is left as an exercise:

$$\vec{a}=\vec{\alpha}\times\vec{r}. \tag{8.1.5}$$

The vector relationships for the angular acceleration and tangential acceleration are shown in Figure 8.1.2.

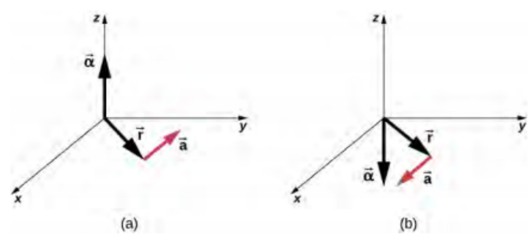

Figure 8.1.2: (a) The angular acceleration is the positive z-direction and produces a tangential acceleration in a counterclockwise sense. (b) The angular acceleration is in the negative z-direction and produces a tangential acceleration in the clockwise sense.

Similarly, we may find the torque vector. The second law $\sum\vec{F}$ = m$\vec{a}$ tells us the relationship between net force and how to change the translational motion of an object. We have a vector rotational equivalent of this equation, which can be found by using Equation 8.1.5 and (Figure 8.1.2) . Equation 8.1.5 relates the angular acceleration to the position and tangential acceleration vectors. We form the cross product of this equation with $\vec{r}$ and use a cross product identity (note that $\vec{r}\cdot\vec{\alpha}$ = 0):

$$\vec{r}\times\vec{a}=\vec{r}\times(\vec{\alpha}\times\vec{r})=\vec{\alpha}(\vec{r}\cdot\vec{r})-\vec{r}(\vec{r}\cdot\vec{\alpha})=\vec{\alpha}(\vec{r}\cdot\vec{r})=\vec{\alpha}r^2. \tag{8.1.6}$$

We now form the cross product of Newton's second law with the position vector $\vec{r}$,

$$\sum(\vec{r}\times\vec{F})=\vec{r}\times(m\vec{a})=m\vec{r}\times\vec{a}=mr^2\vec{\alpha}. \tag{8.1.7}$$

Identifying the first term on the left as the sum of the torques, and mr^2 as the moment of inertia, we arrive at Newton's second law of rotation:

$$\sum\vec{\tau}=I\vec{\alpha}. \tag{8.1.8}$$

This equation is exactly Equation 8.1.4. An important point is that the torque vector is in the same direction as the angular acceleration.

Applying the Rotational Dynamics Equation

Before we apply the rotational dynamics equation to some everyday situations, let's review a general problem-solving strategy for use with this category of problems.

Problem-Solving Strategy: Rotational Dynamics

1. Examine the situation to determine that torque and mass are involved in the rotation. Draw a careful sketch of the situation.
2. Determine the system of interest.
3. Draw a free-body diagram. That is, draw and label all external forces acting on the system of interest.
4. Identify the pivot point. If the object is in equilibrium, it must be in equilibrium for all possible pivot points—chose the one that simplifies your work the most.

5. Apply $\sum_i \tau_i = I\alpha$, the rotational equivalent of Newton's second law, to solve the problem. Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.
6. As always, check the solution to see if it is reasonable.

✔ Example 8.1.1: Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in Figure 8.1.2. He exerts a force of 250 N at the edge of the 200.0-kg merry-go-round, which has a 1.50-m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m away from the center. Consider the merry-go-round itself to be a uniform disk with negligible friction.

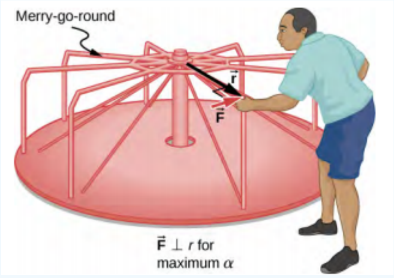

Figure 8.1.2: A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

Strategy

The net torque is given directly by the expression $\sum_i \tau_i = I\alpha$, To solve for α, we must first calculate the net torque τ (which is the same in both cases) and moment of inertia I (which is greater in the second case).

Solutiony

a. The moment of inertia of a solid disk about this axis is given in Figure 10.5.4 to be

$$\frac{1}{2}MR^2. \tag{8.1.9}$$

We have M = 50.0 kg and R = 1.50 m, so

$$I = (0.500)(50.0\ kg)(1.50\ m)^2 = 56.25\ kg \cdot m^2. \tag{8.1.10}$$

To find the net torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

$$\tau = rF\sin\theta = (1.50\ m)(250.0\ N) - 375.0\ N \cdot m. \tag{8.1.11}$$

Now, after we substitute the known values, we find the angular acceleration to be

$$\alpha = \frac{\tau}{I} = \frac{375.0\ N \cdot m}{56.25\ kg \cdot m^2} = 6.67\ rad/s^2. \tag{8.1.12}$$

b. We expect the angular acceleration for the system to be less in this part because the moment of inertia is greater when the child is on the merry-go-round. To find the total moment of inertia I, we first find the child's moment of inertia I_c by approximating the child as a point mass at a distance of 1.25 m from the axis. Then

$$I_c = mR^2 = (18.0\ kg)(1.25\ m)^2 = 28.13\ kg \cdot m^2. \tag{8.1.13}$$

The total moment of inertia is the sum of the moments of inertia of the merry-go-round and the child (about the same axis):

$$I = (28.13\ kg \cdot m^2) + (56.25\ kg \cdot m^2) = 84.38\ kg \cdot m^2. \tag{8.1.14}$$

Substituting known values into the equation for α gives

$$\alpha = \frac{\tau}{I} = \frac{375.0\ N \cdot m}{84.38\ kg \cdot m^2} = 4.44\ rad/s. \tag{8.1.15}$$

Significance

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-goround an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case.

Example $8.1.2$

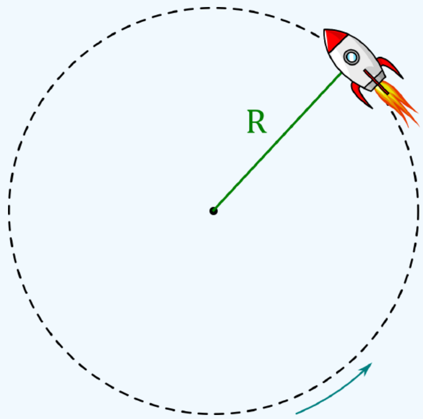

Figure $8.1.3$: A toy rocket accelerating around a circle of radius R, as seen from above.

A toy rocket is attached to a string on a horizontal frictionless table, as shown in Figure $8.1.3$. The rocket has a mass m and produces a constant force of thrust with a magnitude $\vec{F}$ that accelerates the rocket along a circle of radius R (the length of the string). If the rocket starts at rest, what distance along the circumference of the circle will the rocket have traveled after a time, t?

Solution

We can model the rocket as a point particle of mass m with the following forces exerted on it:

1. $\vec{F}$, the thrust of the rocket, always acting tangent to the circle.
2. $\vec{T}$, the force of tension in the string, always acting towards the center of the circle.
3. $\vec{F}_g$, the rocket's weight, acting into the page, with magnitude mg.
4. $\vec{F}_N$, a normal force exerted by the table, out of the page, with magnitude mg.

Because the normal force and the weight are equal in magnitude and opposite in direction, the net force will be the sum of the force of thrust and the force of tension, which are always perpendicular to each other. Thinking about this with Newton's Second Law, we could model the force of thrust as increasing the speed of the particle, while the force of tension keeps the rocket moving in a circle (it cannot increase the speed, since it is always perpendicular to the motion).

We introduce a coordinate system whose origin coincides with the center of the circle, as shown in Figure $8.1.4$, so that $\vec{r}$ corresponds to the position of the rocket relative to the origin. The force of thrust and the tension are also shown in the diagram. We choose the direction of the x axis such that the rocket was located at the intersection of the x axis and the circle at time, $t=0$.

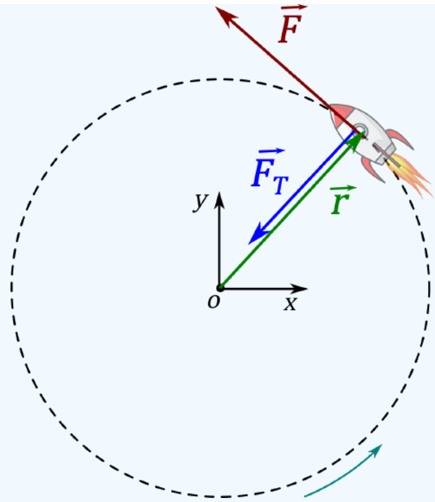

Figure 8.1.4: Coordinate system to describe the motion of the rocket.

The net torque on the rocket about the point of rotation is given by the cross-product between the thrust force, $\vec{F}$, and the position vector, $\vec{r}$:

$$\vec{\tau}^{net} = \vec{r} \times \vec{F}$$

and will point in the positive z direction (as given by the right hand rule). $\vec{r}$ and $\vec{F}$ are perpendicular, so the magnitude of the net torque is given by:

$$\tau^{net} = rF\sin(90^{\circ}) = RF$$

where R is the magnitude of $\vec{r}$. The net torque vector is thus:

$$\vec{\tau}^{net} = RF\hat{z}$$

Applying the rotational version of Newton's Second Law allows us to determine the angular acceleration:

$$\begin{aligned} \vec{\tau}^{net} &= mr^2\vec{\alpha} \\ RF\hat{z} &= mR^2\vec{\alpha} \\ \therefore \vec{\alpha} &= \frac{F}{mR}\hat{z} \end{aligned}$$

The angular acceleration vector points in the positive z direction (as does the net torque), and indicates that the rocket is accelerating in the counter-clockwise direction about the z axis.

After a period of time t, the rocket will have covered an angular displacement, $\Delta\theta$, given by:

$$\begin{aligned} \Delta\theta &= \theta(t) - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= \frac{1}{2}\frac{F}{mR}t^2 \end{aligned}$$

The linear displacement, Δs, that corresponds to this angular displacement is:

$$\Delta s = R\Delta\theta = \frac{1}{2}\frac{F}{m}t^2$$

Discussion

The formula that we found for the total linear displacement is the same that we would have found if the particle were moving in a straight line with a net force F applied to it (as the particle would have a constant acceleration given by F/m).

Exercise 10.7

The fan blades on a jet engine have a moment of inertia 30.0 kg • m^2. In 10 s, they rotate counterclockwise from rest up to a rotation rate of 20 rev/s. (a) What torque must be applied to the blades to achieve this angular acceleration? (b) What is the torque required to bring the fan blades rotating at 20 rev/s to a rest in 20 s?
