4.6: Relative Motion

Learning Objectives

- Explain the concept of reference frames.
- Write the position and velocity vector equations for relative motion.
- Draw the position and velocity vectors for relative motion.
- Analyze one-dimensional and two-dimensional relative motion problems using the position and velocity vector equations.

Motion does not happen in isolation. If you’re riding in a train moving at 10 m/s east, this velocity is measured relative to the ground on which you’re traveling. However, if another train passes you at 15 m/s east, your velocity relative to this other train is different from your velocity relative to the ground. Your velocity relative to the other train is 5 m/s west. To explore this idea further, we first need to establish some terminology.

4.6.1 Reference Frames

To discuss relative motion in one or more dimensions, we first introduce the concept of **reference frames**. When we say an object has a certain velocity, we must state it has a velocity with respect to a given reference frame. In most examples we have examined so far, this reference frame has been Earth. If you say a person is sitting in a train moving at 10 m/s east, then you imply the person on the train is moving relative to the surface of Earth at this velocity, and Earth is the reference frame. We can expand our view of the motion of the person on the train and say Earth is spinning in its orbit around the Sun, in which case the motion becomes more complicated. In this case, the solar system is the reference frame. In summary, all discussion of relative motion must define the reference frames involved. We now develop a method to refer to reference frames in relative motion.

4.6.2 Relative Motion in One Dimension

We introduce relative motion in one dimension first, because the velocity vectors simplify to having only two possible directions. Take the example of the person sitting in a train moving east. If we choose east as the positive direction and Earth as the reference frame, then we can write the velocity of the train with respect to the Earth as $\vec{v}_{TE}$ = 10 m/s $\hat{i}$ east, where the subscripts TE refer to train and Earth. Let’s now say the person gets up out of /her seat and walks toward the back of the train at 2 m/s. This tells us she has a velocity relative to the reference frame of the train. Since the person is walking west, in the negative direction, we write her velocity with respect to the train as $\vec{v}_{PT}$ = −2 m/s $\hat{i}$. We can add the two velocity vectors to find the velocity of the person with respect to Earth. This relative velocity is written as

$$\vec{v}_{PE} = \vec{v}_{PT} + \vec{v}_{TE}. \tag{4.6.1}$$

Note the ordering of the subscripts for the various reference frames in Equation 4.6.1. The subscripts for the coupling reference frame, which is the train, appear consecutively in the right-hand side of the equation. Figure 4.6.1 shows the correct order of subscripts when forming the vector equation.

$$\vec{v}_{\boldsymbol{PE}} = \vec{v}_{\boldsymbol{PT}} + \vec{v}_{\boldsymbol{TE}}. \tag{4.6.2}$$

Figure 4.6.1: When constructing the vector equation, the subscripts for the coupling reference frame appear consecutively on the inside. The subscripts on the left-hand side of the equation are the same as the two outside subscripts on the right-hand side of the equation.

Adding the vectors, we find $\vec{v}_{PE}$ = 8 m/s $\hat{i}$, so the person is moving 8 m/s east with respect to Earth. Graphically, this is shown in Figure 4.6.2.

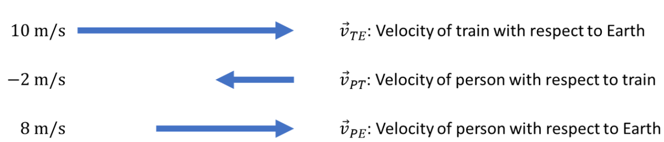


Figure 4.6.2: Velocity vectors of the train with respect to Earth, person with respect to the train, and person with respect to Earth.

Example 4.6.1: Dart Gun on a Bus

a) Imagine that you have a dart gun with a **muzzle velocity** of 45 mph. Further imagine that you are on a bus traveling along a straight highway at $55mph$ and that you point the gun so that the barrel is level and pointing directly forward, toward the front of the bus. Assuming no recoil, as it leaves the muzzle of the gun, how fast is the dart traveling relative to the road?

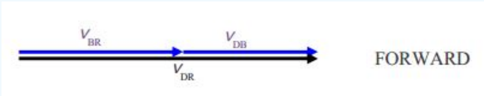
b) What if you fire a dart gun so that it has a muzzle velocity of 45 mph straight backward, (toward the back of the bus). Find the velocity of the dart, relative to the road, as it leaves the gun.

Solution

a) That's right! 100 mph The dart is already traveling forward at 55 mph relative to the road just because it is on a bus that is moving at 55 mph relative to the road. Add to that the velocity of 45 mph that it acquires as a result of the firing of the gun and you get the total velocity of the dart relative to the road. Let us discuss how to mathematically solve the problem.

Defining

- $\vec{V}_{BR}$ to be the velocity of the bus relative to the road,
- $\vec{V}_{DB}$ to be the velocity of the dart relative to the bus, and
- $\vec{V}_{DR}$ to be the velocity of the dart relative to the road; we have



Using the vector notation, with the forward direction represented by the unit vector $\hat{\imath}$,

$$\vec{V}_{BR} = (55 \text{ mph })\hat{\imath}$$
$$\vec{V}_{DB} = (45 \text{ mph })\hat{\imath}$$

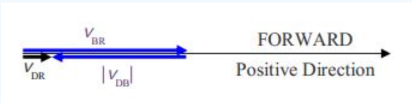
The vector addition problem this illustrates is

$$\vec{V}_{DR} = \vec{V}_{DB} + \vec{V}_{BR}$$
$$\vec{V}_{DR} = (45 \text{ mph })\hat{\imath} + (55 \text{ mph })\hat{\imath}$$
$$\vec{V}_{DR} = (100 \text{ mph })\hat{\imath}$$

$|\vec{V}_{DR}| = (100 \text{ mph})$ in the direction in which the bus is traveling

The results demonstrate that we are dealing with that very special case in which the magnitude of the resultant is just the sum of the magnitudes of the vectors we are adding.

b) In this case:



$$\vec{V}_{BR} = (55 \text{ mph })\hat{\imath}$$
$$\vec{V}_{DB} = (-45 \text{ mph })\hat{\imath}$$

The vector addition problem this illustrates is

$$\vec{V}_{DR} = \vec{V}_{DB} + \vec{V}_{BR}$$
$$\vec{V}_{DR} = (-45 \text{ mph })\hat{\imath} + (55 \text{ mph })\hat{\imath}$$
$$\vec{V}_{DR} = 10 \text{ mph } \hat{\imath}$$

$|\vec{V}_{DR}| = (10 \text{ mph})$ in the direction in which the bus is traveling

It would be odd looking at that dart from the side of the road. Relative to you it would still be moving in the direction that the bus is traveling, tail first, at (10 mph).

4.6.3 Relative Velocity in Two Dimensions

We can now apply these concepts to describing motion in two dimensions. Consider a particle P and reference frames S and S′, as shown in Figure $4.6.3$. The position of the origin of S′ as measured in S is $\vec{r}_{S'S}$, the position of P as measured in S′ is $\vec{r}_{PS'}$, and the position of P as measured in S is $\vec{r}_{PS}$.

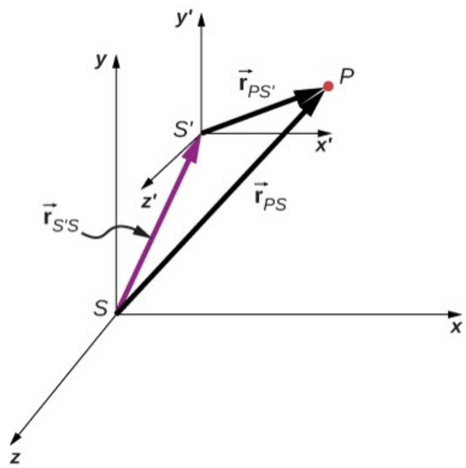


Figure $4.6.3$: The positions of particle P relative to frames S and S′ are $\vec{r}_{PS}$ and $\vec{r}_{PS'}$, respectively.

From Figure $4.6.3$ we see that

$$\vec{r}_{PS} = \vec{r}_{PS} + \vec{r}_{S'S}. \tag{4.6.3}$$

The relative velocities are the time derivatives of the position vectors. Therefore,

$$\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S}. \tag{4.6.4}$$

The velocity of a particle relative to S is equal to its velocity relative to S′ plus the velocity of S′ relative to S.

We can extend Equation $4.6.4$ to any number of reference frames. For particle P with velocities $\vec{v}_{PA}$, $\vec{v}_{PB}$, and $\vec{v}_{PC}$ in frames A, B, and C,

$$\vec{v}_{PC} = \vec{v}_{PA} + \vec{v}_{AB} + \vec{v}_{BC}. \tag{4.6.5}$$

We can also see how the accelerations are related as observed in two reference frames by differentiating Equation $4.6.4$:

$$\vec{a}_{PS} = \vec{a}_{PS'} + \vec{a}_{S'S}. \tag{4.6.6}$$

We see that if the velocity of S′ relative to S is a constant, then $\vec{a}_{S'S}$ = 0 and

$$\vec{a}_{PS} = \vec{a}_{PS'}. \tag{4.6.7}$$

This says the acceleration of a particle is the same as measured by two observers moving at a constant velocity relative to each other.

✓ Example $4.6.3$: Motion of a Car Relative to a Truck

A truck is traveling south at a speed of 70 km/h toward an intersection. A car is traveling east toward the intersection at a speed of 80 km/h (Figure $4.6.4$). What is the velocity of the car relative to the truck?

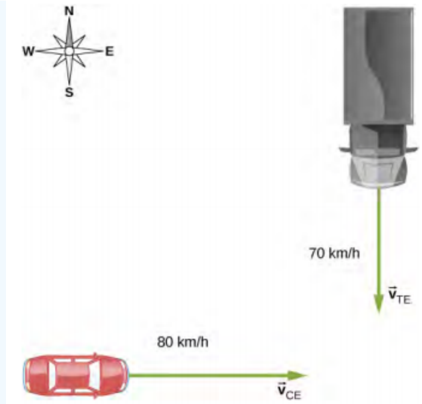


Figure 4.6.4: A car travels east toward an intersection while a truck travels south toward the same intersection.

Strategy

First, we must establish the reference frame common to both vehicles, which is Earth. Then, we write the velocities of each with respect to the reference frame of Earth, which enables us to form a vector equation that links the car, the truck, and Earth to solve for the velocity of the car with respect to the truck.

Solution

The velocity of the car with respect to Earth is $\vec{v}_{CE}$ = 80 km/h $\hat{i}$. The velocity of the truck with respect to Earth is $\vec{v}_{TE}$ = −70 km/h $\hat{j}$. Using the velocity addition rule, the relative motion equation we are seeking is

$$\vec{v}_{CT} = \vec{v}_{CE} + \vec{v}_{ET}. \tag{4.6.8}$$

Here, $\vec{v}_{CT}$ is the velocity of the car with respect to the truck, and Earth is the connecting reference frame. Since we have the velocity of the truck with respect to Earth, the negative of this vector is the velocity of Earth with respect to the truck: $\vec{v}_{ET} = -\vec{v}_{TE}$. The vector diagram of this equation is shown in Figure 4.6.5

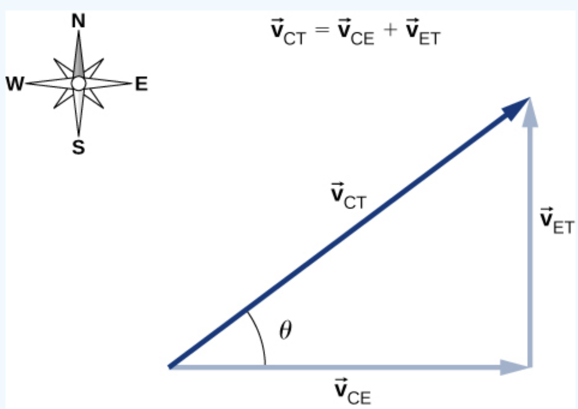


Figure 4.6.5: Vector diagram of the vector Equation 4.6.8.

We can now solve for the velocity of the car with respect to the truck:

$$|\vec{v}_{CT}| = \sqrt{(80.0\ km/h)^2 + (70.0\ km/h)^2} = 106.\ km/h$$

and

$$\theta = \tan^{-1}\left(\frac{70.0}{80.0}\right) = 41.2^o\ north\ of\ east.$$

Significance

Drawing a vector diagram showing the velocity vectors can help in understanding the relative velocity of the two objects.

Example 4.6.4: Flying a Plane in a Wind

A pilot must fly his plane due north to reach his destination. The plane can fly at 300 km/h in still air. A wind is blowing out of the northeast at 90 km/h. (a) What is the speed of the plane relative to the ground? (b) In what direction must the pilot head her plane to fly due north?

Strategy

The pilot must point her plane somewhat east of north to compensate for the wind velocity. We need to construct a vector equation that contains the velocity of the plane with respect to the ground, the velocity of the plane with respect to the air, and the velocity of the air with respect to the ground. Since these last two quantities are known, we can solve for the velocity of the plane with respect to the ground. We can graph the vectors and use this diagram to evaluate the magnitude of the plane's velocity with respect to the ground. The diagram will also tell us the angle the plane's velocity makes with north with respect to the air, which is the direction the pilot must head her plane.

Solution

The vector equation is $\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$, where P = plane, A = air, and G = ground. From the geometry in Figure $4.6.6$ we can solve easily for the magnitude of the velocity of the plane with respect to the ground and the angle of the plane's heading, θ.

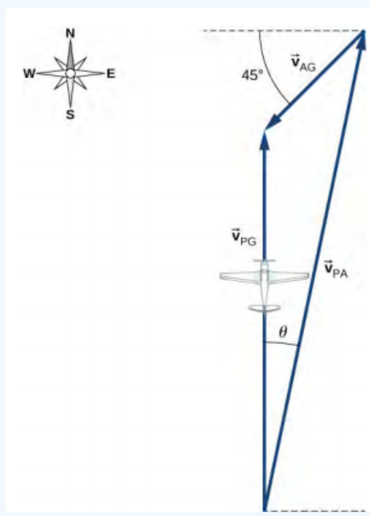


Figure 4.6.6: Vector diagram for Equation 4.6.3 showing the vectors $\vec{v}_{PA}$, $\vec{v}_{AG}$, and $\vec{v}_{PG}$.

a. Known quantities: $|\vec{v}_{PA}| = 300\ km/h$ $|\vec{v}_{AG}| = 90\ km/h$ Substituting into the equation of motion, we obtain $|\vec{v}_{PG}|$ = 230 km/h.

b. The angle $\theta = \tan^{-1}\left(\frac{63.64}{300}\right)$ = 12° east of north.

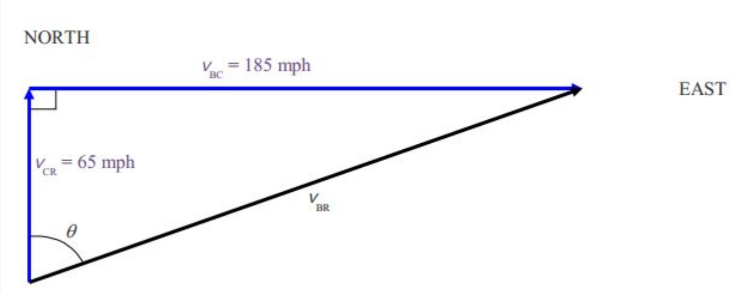
Example 4.6.5: BB Shot from a Noving Car

A boy sitting in a car that is traveling due north at $65 mph$ aims a BB gun (a gun which uses a compressed gas to fire a small metal or plastic ball called a BB), with a muzzle velocity of $185 mph$, due east, and pulls the trigger. Recoil (the backward movement of the gun resulting from the firing of the gun) is negligible. In what compass direction does the BB go?

Solution

Defining

- $\vec{V}_{CR}$ to be the velocity of the car relative to the road,
- $\vec{V}_{BC}$ to be the velocity of the BB relative to the car, and
- $\vec{V}_{BR}$ to be the velocity of the BB relative to the road; we have



$$tan\theta = \frac{V_{BC}}{V_{CR}}$$

$$\theta = tan^{-1}\frac{V_{BC}}{V_{CR}}$$

$$\theta = tan^{-1}\frac{185mph}{65mph}$$

$$\theta = 70.6^{\circ}$$

The BB travels in the direction for which the compass heading is 70.6°.

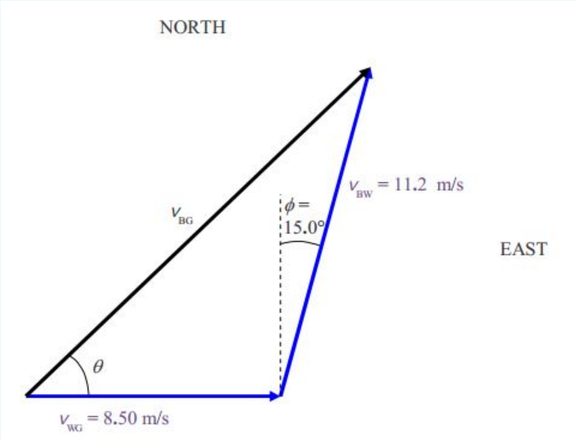
✔ Example 4.6.6: Boat Crossing aRiver

A boat is traveling across a river that flows due east at $8.50m/s$. The compass heading of the boat is 15.0°. Relative to the water, the boat is traveling straight forward (in the direction in which the boat is pointing) at $11.2m/s$. How fast and which way is the boat moving relative to the banks of the river?

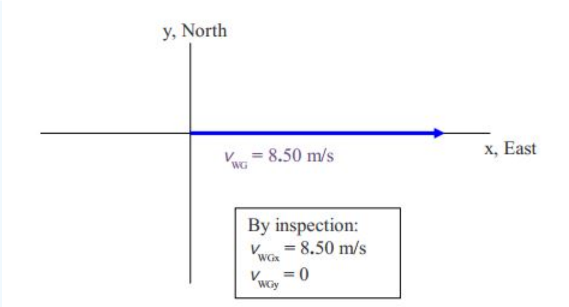
Solution

Okay, here we have a situation in which the boat is being carried downstream by the movement of the water at the same time that it is moving relative to the water. Note the given information means that if the water was dead still, the boat would be going 11.2 m/s at 15.0° East of North. The water, however, is not still. Defining

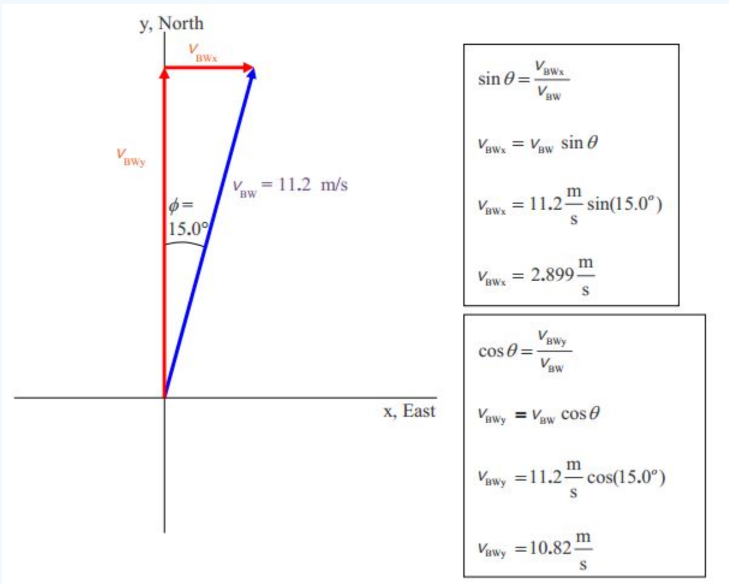
- $\vec{V}_{WG}$ to be the velocity of the water relative to the ground,
- $\vec{V}_{BW}$ to be the velocity of the boat relative to the water, and
- $\vec{V}_{BG}$ to be the velocity of the boat relative to the ground; we have



Solving this problem is just a matter of following the vector addition recipe. First we define $+x$ to be eastward and $+y$ to be northward. Then we draw the vector addition diagram for $\vec{V}_{WG}$. Breaking it up into components is trivial since it lies along the x-axis:



Breaking $\vec{V}_{BW}$ does involve a little bit of work:



Now we add the x components to get the x-component of the resultant

$$V_{BGx} = V_{WGx} + V_{BWx}$$

$$V_{BGx} = 8.50 \frac{m}{s} + 2.899 \frac{m}{s}$$

$$V_{BGx} = 11.299 \frac{m}{s}$$

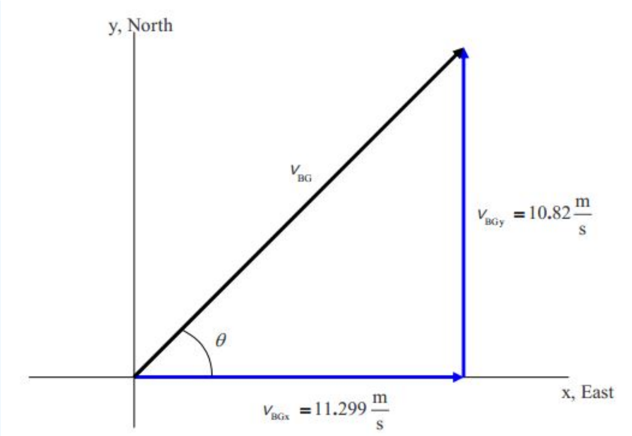
and we add the y components to get the y-component of the resultant:

$$V_{BGy} = V_{WGy} + V_{BWy}$$

$$V_{BGy} = 0 \frac{m}{s} + 10.82 \frac{m}{s}$$

$$V_{BGy} = 10.82 \frac{m}{s}$$

Now we have both components of the velocity of the boat relative to the ground. We need to draw the vector component diagram for $\vec{V}_{BG}$ to determine the direction and magnitude of the velocity of the boat relative to the ground.



We then use the Pythagorean Theorem to get the magnitude of the velocity of the boat relative to the ground,

$$\vec{V}_{BG} = \sqrt{V_{BGx}^2 + V_{BGy}^2}$$

$$\vec{V}_{BG} = \sqrt{(11.299m/s)^2 + (10.82m/s)^2}$$

$$\vec{V}_{BG} = 15.64m/s$$

and the definition of the tangent to determine the direction of $\vec{V}_{BG}$:

$$tan\theta = \frac{V_{BGy}}{V_{BGx}}$$

$$\theta = tan^{-1}\frac{V_{BGy}}{V_{BGx}}$$

$$\theta = tan^{-1}\frac{10.82m/s}{11.299m/s}$$

$$\theta = 43.8^\circ$$

Hence, $\vec{V}_{BG} = 15.64m/s$at 43.8° North of East.

4.6.4 Relative Velocity Vectors

We begin by introducing some language. When an observer – who we will call "A" – in a given reference frame measures the velocity vector of an object (or another frame) – which we will call "B" – we express this vector in words and symbols in this way:

$$\text{" velocity of } B \text{ relative to } A \text{ "} \quad \Longleftrightarrow \quad \overrightarrow{v}_{B\ rel\ A} \tag{4.6.9}$$

Let's see if we can put the above example into this language. There are three entities here. Two are frames and on is a moving object. The moving object is Ann, and she is being observed by Bob, in the reference frame of the train, and Chu, in the reference frame of the earth. In the example, we expressed three different relative velocity vectors:

$$\begin{array}{l} \text{" velocity of Bob relative to Chu "} \Longleftrightarrow \overrightarrow{v}_{b\ rel\ c} = (60mph)\,\widehat{north} \\ \text{" velocity of Ann relative to Bob "} \Longleftrightarrow \overrightarrow{v}_{a\ rel\ b} = (10mph)\,\widehat{south} \\ \text{" velocity of Ann relative to Chu "} \Longleftrightarrow \overrightarrow{v}_{a\ rel\ c} = (50mph)\,\widehat{north} \end{array} \tag{4.6.10}$$

Let's represent these three vectors as arrows beside each other in a diagram:

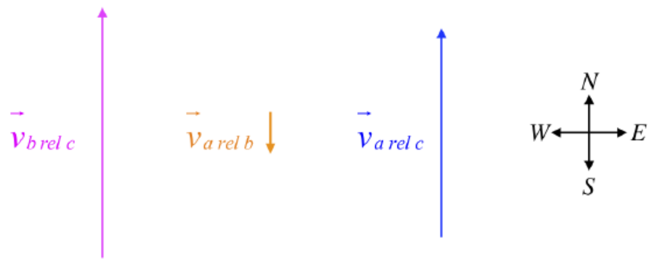


Figure 4.6.7: Relative Velocity Vectors.

The first thing we notice when we look closely at these is that our intuitive understanding of the original statement of the situation can be represented as a vector addition. Placing the the tail of the first vector at the head of the second vector, we find that the third vector can connect the open tail to the open head. In other words, we can express the result of the above example as a vector addition:

$$\vec{v}_{a\ rel\ b} + \vec{v}_{b\ rel\ c} = \vec{v}_{a\ rel\ c} \tag{4.6.11}$$

Note the ordering of the frames here is like a chain connecting Ann to Chu through Bob: Ann relative to Bob, then Bob relative to Chu, gives Ann relative to Chu. It turns out that this vector equation works not only when the velocities lie along a line, but also when they do not. For example, we can use the same vector equation if Ann were walking *across* the train (perpendicular to its motion.

There is one other feature of these relative velocity vectors that we will need, and that is reversing the perspective. In the case above, we have that Ann is moving 10 mph south relative to the Bob, but we can also talk about how Ann sees Bob moving *relative to her*. Bob starts off south of her, and as she runs by him, he ends up north of her. Therefore from Ann's perspective, Bob is moving north at 10 mph. So there is a simple way to alter a relative vector to reverse the perspective of reference frames: Switch the two frames in the subscript, and reverse the direction of the vector (i.e. multiply the original vector by –1). Here is a summary of these two rules:

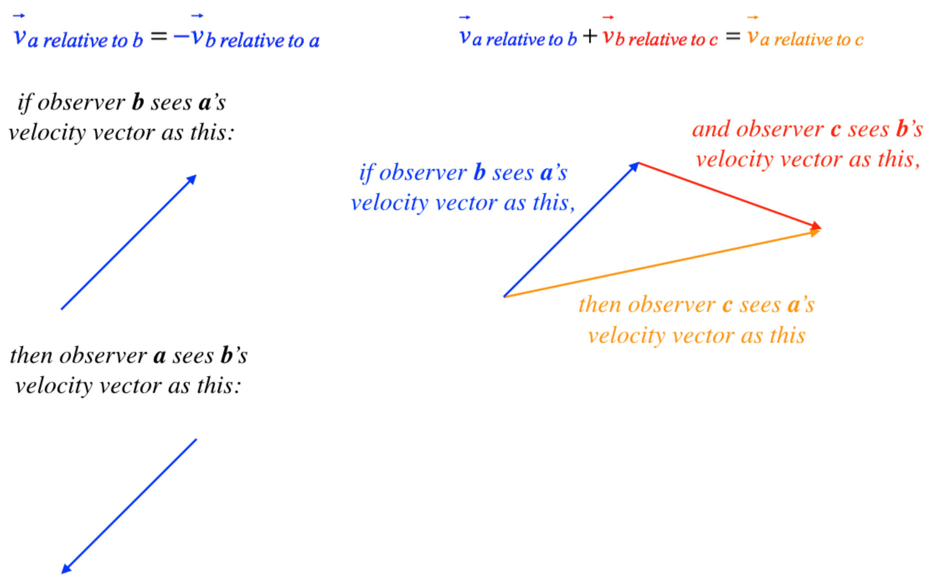


Figure 4.6.8: Summary of Relative Velocity Rules.

Example 4.6.7: Swimming Across a River

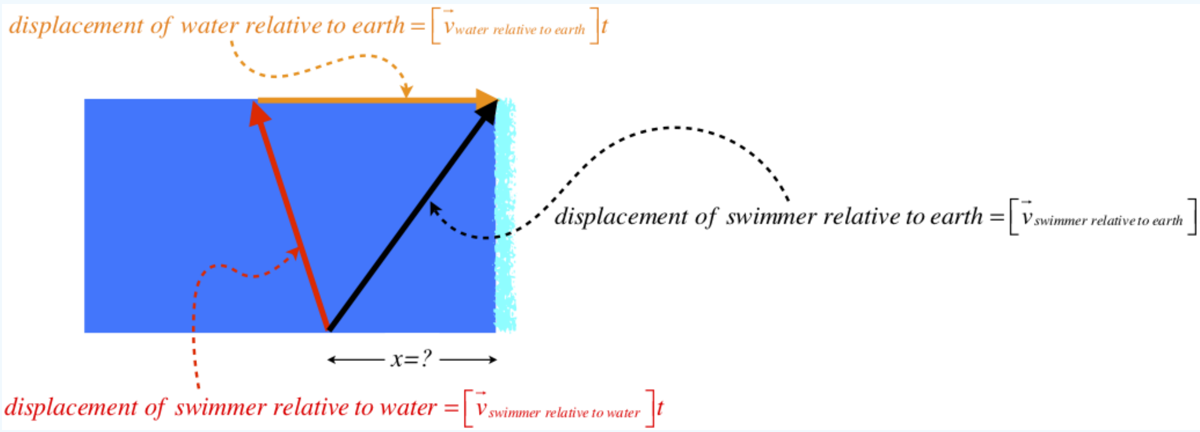
You stand on the bank of a river, contemplating swimming across, but the place where you hope to cross is close to a dangerous waterfall. When you look at the speed of the river, you estimate that it is about the same speed as you are able to swim. You realize that you can only swim so far in the cold water at this speed before your muscles shut down, and in still water you estimate that this distance is about $100m$*. The width of the river is about* $80m$*.*

a. *Find the minimum distance that you must start upstream of the waterfall in order to not be swept over it.*

b. *If the river flows west-to-east and you start on its south shore, compute the direction in which you must swim in order to get safely across if you leave from the starting point computed in part (a).*

Solution

a. Clearly to minimize the distance upstream that you need to start, you must swim with a component of your velocity relative to the river being upstream. The more you are able to turn yourself upstream, the less you will float downstream, and the closer you can start to the waterfall. But there is a limit to how far you can swim relative to the water, so your angle with the river must be such that when you reach your limit relative to the river, you reach the other side. The velocities are all constant and the time spans are all equal, so they are proportional to the displacements, which we can draw:



We are given that the speed of the river relative to the earth is the same as the speed of the swimmer relative to the water, so we'll call that quantity v, and the width of the river (which we know), we'll call w. From the pythagorean theorem we can get the distance swum upstream against the current:

$$distance\ swum\ against\ water = \sqrt{(vt)^2 - (w)^2}$$

The distance the water moves downstream relative to the earth is clearly vt, so the total distance the swimmer moves downstream is:

$$x = vt - \sqrt{(vt)^2 - (w)^2}$$

But we actually know the value of vt, because it is the maximum distance that the swimmer can go in the water. Plugging in all the values therefore gives our answer:

$$x = (100m) - \sqrt{(100m)^2 - (80m)^2} = \boxed{40m}$$

b. The angle is easy to determine, since we know the length of the displacement vector of the swimmer relative to the water and the width of the river:

$$\cos\theta = \frac{w}{vt} = \frac{80m}{100m} \quad \Rightarrow \quad \theta = \boxed{37^o west\ of\ north}$$

4.6.5 Galilean Transformation

Let's now consider two observers in difference reference frames that are moving at a constant speed relative to one another, which we will call v. We'll define the coordinate systems of these two observers such that their origins coincide at time $t = 0$, and both observers agree on this starting time. Since the frames are moving relative to each other, this common origin only lasts for that one instant in time. We'll also define the coordinate systems such that they have common x, y, and z axes when their origins coincide, and have their relative motion be along their common x-axis. We will label position coordinates and time measured by the frame moving in the $+x$-direction with a prime, to distinguish it from the other frame.

Suppose both observers record the motion of the same object. One observer gets equations of motion of this object for its three spatial coordinates (x, y, z) as a function of time t, while the other observer gets equations of motion of the object for (x', y', z') as a function of time t'. The question we want to answer is, "Given what we know about how these frames are related to each other, what are the relations between the primed and unprimed coordinates?"

Let's start by noting that when the primed observer's origin has moved a distance s relative to the unprimed observer's origin, the x-component of an object's position measured in the unprimed frame will be greater than the same component measured in the primed frame by that amount:

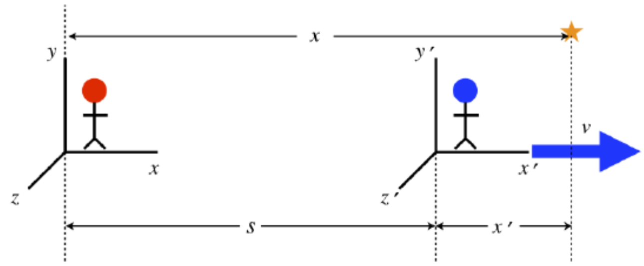


Figure 4.6.9: Relating Coordinates of Reference Frames.

We defined the frames so that their origins coincided when each of them measured the time to be zero, so the distance s is simply equal to vt. The only difference in the two frames is in the x-direction, and the clocks are synchronized, so we have a complete translation of the two frames:

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned} \tag{4.6.12}$$

These are referred to as the *Galilean transformation equations*. They translate the coordinates of one frame into another that is moving relative to the first, with the restrictions indicated above regarding coinciding origins and so on. While this may not seem particularly interesting, keep in mind that these coordinates (when combined with cartesian unit vectors) compose the position vector, whose first derivative with respect to time is the velocity vector, etc. That is, every element of 3-dimensional kinematics – all the equations of motion of observed objects – can be translated into what they would be in another frame of reference through this transformation.

Example 4.6.8:

Ann and Bob are observers from different reference frames in relative motion, with all of the conditions necessary for their coordinate systems to be related by the Galilean transformation given above (Bob is in the primed frame, moving in the x-direction relative to Ann at a speed v). Ann observes a toy rocket moving in the y-direction with a speed u. Show that the velocity vector of this same rocket as measured by Bob is the same as would be obtained using the method of relative velocity vectors described in the previous section.

Solution

Let's start by computing the velocity vector of the ball according to Bob using the Galilean transformation. Taking the derivative of the position components with respect to time gives the components of the velocity vector seen by Bob, so substituting for t' and x' in the derivative gives:

$$\left.\begin{aligned} \frac{dx'}{dt'} &= \frac{d\left(x-vt\right)}{dt} = \frac{dx}{dt} - v = -v \\ \frac{dy'}{dt'} &= \frac{dy}{dt} = u \\ \frac{dz'}{dt'} &= \frac{dz}{dt} = 0 \end{aligned}\right\} \quad \Rightarrow \quad \overrightarrow{u}' = -v\hat{i} + u\hat{j}$$

Now let's use the tail-to-head relative velocity vector method from the previous section. The velocity of the rocket relative to Ann is $u\hat{j}$, and the velocity of Bob relative to Ann is $+v\hat{i}$. To get the velocity of the rocket relative to Bob, we need to form *the "vector chain," which means we first need to get the velocity of Ann relative to Bob. Swapping the relative order requires only a minus sign, so doing this and putting together the vector chain gives:*

$$\left.\begin{array}{l}\text{velocity of rocket relative to Bob} = \vec{u}\,' \\ \text{velocity of rocket relative to Ann} = u\hat{j} \\ \text{velocity of Ann relative to Bob} = -v\hat{i}\end{array}\right\} \quad \Rightarrow \quad \vec{u}\,' = u\hat{j} - v\hat{i}$$

Vectors add like vectors, not like numbers. Except in that very special case in which the vectors you are adding lie along one and the same line, you can't just add the magnitudes of the vectors.

Everything up to this point assumes that we are using a fixed, previously agreed upon **reference frame**. Basically, this is just an origin and a set of axes along which to measure our coordinates, as shown in Figure $4.6.1$.

There are, however, a number of situations in physics that call for the use of different reference frames, and, more importantly, that require us to *convert* various physical quantities from one reference frame to another. For instance, imagine you are on a boat on a river, rowing downstream. You are moving with a certain velocity relative to the water around you, but the water itself is flowing with a different velocity relative to the shore, and your actual velocity relative to the shore is the sum of those two quantities. Ships generally have to do this kind of calculation all the time, as do airplanes: the "airspeed" is the speed of a plane relative to the air around it, but that air is usually moving at a substantial speed relative to the earth.

The way we deal with all these situations is by introducing two reference frames, which here I am going to call A and B. One of them, say A, is "at rest" relative to the earth, and the other one is "at rest" relative to something else—which means, really, moving along with that something else. (For instance, a reference frame at rest "relative to the river" would be a frame that's moving along with the river water, like a piece of driftwood that you could measure your progress relative to.)

In any case, graphically, this will look as in Figure $4.6.1$, which I have drawn for the two-dimensional case because I think it makes it easier to visualize what's going on:

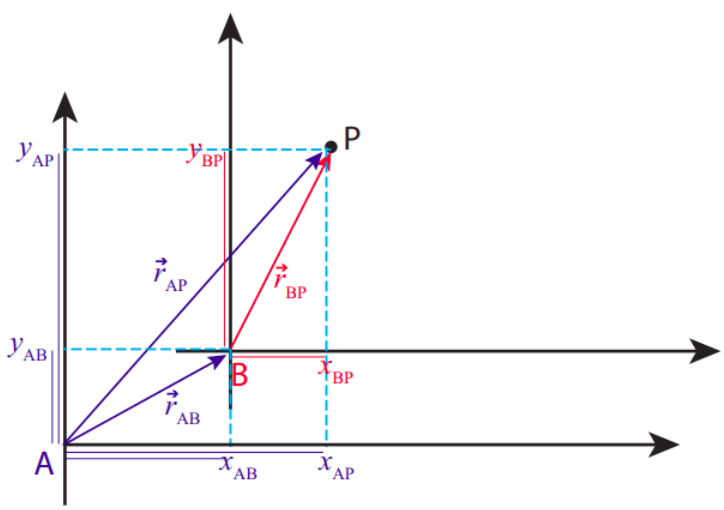


Figure $4.6.10$: Position vectors and coordinates of a point P in two different reference frames, A and B.

In the reference frame A, the point P has position coordinates (x_{AP}, y_{AP}). Likewise, in the reference frame B, its coordinates are (x_{BP}, y_{BP}). As you can see, the notation chosen is such that every coordinate in A will have an "A" as a first subscript, while the second subscript indicates the object to which it refers, and similarly for coordinates in B.

The coordinates (x_{AB}, y_{AB}) are special: they are the coordinates, in the reference frame A, of the origin of reference frame B. This is enough to fully locate the frame B in A, as long as the frames are not rotated relative to each other.

The thin colored lines I have drawn along the axes in Figure $4.6.1$ are intended to make it clear that the following equations hold:

$$\begin{aligned} x_{AP} &= x_{AB} + x_{BP} \\ x_{AP} &= x_{AB} + x_{BP} \end{aligned} \tag{4.6.13}$$

Although the figure is drawn for the easy case where all these quantities are positive, you should be able to convince yourself that Eqs. (4.6.13) hold also when one or more of the coordinates have negative values.

All these coordinates are also the components of the respective position vectors, shown in the figure and color-coded by reference frame (so, for instance, $\vec{r}_{AP}$ is the position vector of P in the frame A), so the equations (4.6.13) can be written more compactly as the single vector equation

$$\vec{r}_{AP} = \vec{r}_{AB} + \vec{r}_{BP}. \tag{4.6.14}$$

From all this you can see how to add vectors: algebraically, you just add their components separately, as in Eqs. (4.6.13); graphically, you draw them so the tip of one vector coincides with the tail of the other (we call this "tip-to-tail"), and then draw the sum vector from the tail of the first one to the tip of the other one. (In general, to get two arbitrary vectors tip-to-tail you may need to displace one of them; this is OK provided you do not change its orientation, that is, provided you only displace it, not rotate it. We'll see how this works in a moment with velocities, and later on with forces.)

Of course, I showed you already how to *subtract* vectors with Figure $4.6.3$: again, algebraically, you just subtract the corresponding coordinates, whereas graphically you draw them with a common origin, and then draw the vector from the tip of the vector you are subtracting to the tip of the other one. If you read the previous paragraph again, you can see that Figure $4.6.3$ can equally well be used to show that $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$, as to show that $\vec{r}_f = \vec{r}_i + \Delta\vec{r}$.

In a similar way, you can see graphically from Figure $4.6.1$ (or algebraically from Equation (4.6.14)) that the position vector of P in the frame B is given by $\vec{r}_{BP} = \vec{r}_{AP} - \vec{r}_{AB}$. The last term in this expression can be written in a different way, as follows. If I follow the convention I have introduced above, the quantity x_{BA} (with the order of the subscripts reversed) would be the x coordinate of the origin of frame A in frame B, and algebraically that would be equal to $-x_{AB}$, and similarly $y_{BA} = -y_{AB}$. Hence the vector equality $\vec{r}_{AB} = -\vec{r}_{BA}$ holds. Then,

$$\vec{r}_{BP} = \vec{r}_{AP} - \vec{r}_{AB} = \vec{r}_{AP} + \vec{r}_{BA}. \tag{4.6.15}$$

This is, in a way, the "inverse" of Equation (4.6.14): it tells us how to get the position of P in the frame B if we know its position in the frame A.

Let me show next you how all this extends to displacements and velocities. Suppose the point P indicates the position of a particle at the time t. Over a time interval Δt, both the position of the particle and the relative position of the two reference frames may change. We can add yet another subscript, i or f, (for initial and final) to the coordinates, and write, for example,

$$\begin{aligned} x_{AP,i} &= x_{AB,i} + x_{BP,i} \\ x_{AP,f} &= x_{AB,f} + x_{BP,f} \end{aligned} \tag{4.6.16}$$

Subtracting these equations gives us the corresponding displacements:

$$\Delta x_{AP} = \Delta x_{AB} + \Delta x_{BP}. \tag{4.6.17}$$

Dividing Equation (4.6.17) by Δt we get the average velocities[1], and then taking the limit $\Delta t \to 0$ we get the instantaneous velocities. This applies in the same way to the y coordinates, and the result is the vector equation

$$\vec{v}_{AP} = \vec{v}_{BP} + \vec{v}_{AB}. \tag{4.6.18}$$

I have rearranged the terms on the right-hand side to (hopefully) make it easier to visualize what's going on. In words: the velocity of the particle P relative to (or *measured in*) frame A is equal to the (vector) sum of the velocity of the particle as measured in frame B, plus the velocity of frame B relative to frame A.

The result (4.6.18) is just what we would have expected from the examples I mentioned at the beginning of this section, like rowing in a river or an airplane flying in the wind. For instance, for the airplane $\vec{v}_{BP}$ could be its "airspeed" (only it has to be a vector, so it would be more like its "airvelocity": that is, its velocity relative to the air around it), and $\vec{v}_{AB}$ would be the velocity of the air relative to the earth (the wind velocity, at the location of the airplane). In other words, A represents the earth frame of reference and B the air, or wind, frame of reference. Then, $\vec{v}_{AP}$ would be the "true" velocity of the airplane relative to the earth. You can see how it would be important to add these quantities as vectors, in general, by considering what happens when you fly in a cross wind, or try to row across a river, as in Figure $4.6.2$ below.

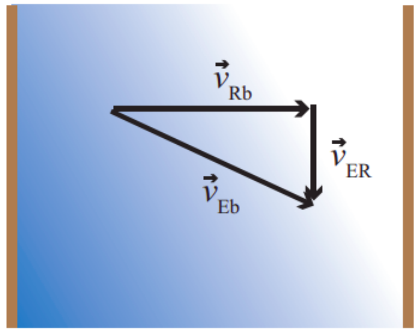


Figure 4.6.11: Rowing across a river. If you head "straight across" the river (with velocity vector $\vec{v}_{Rb}$ in the moving frame of the river, which is flowing with velocity $\vec{v}_{ER}$ in the frame of the earth), your actual velocity relative to the shore will be the vector $\vec{v}_{Eb}$. This is an instance of Equation (4.6.18), with frame A being E (the earth), frame B being R (the river), and "b" (for "boat") standing for the point P we are tracking.

As you can see from this couple of examples, Equation (4.6.18) is often useful as it is written, but sometimes the information we have is given to us in a different way: for instance, we could be given the velocity of the object in frame A $(\vec{v}_{AP})$, and the velocity of frame B as seen in frame A $(\vec{v}_{AB})$, and told to calculate the velocity of the object as seen in frame B. This can be easily accomplished if we note that the vector $\vec{v}_{AB}$ is equal to $-\vec{v}_{BA}$; that is to say, the velocity of frame B as seen from frame A is just the opposite of the velocity of frame A as seen from frame B. Hence, Equation (4.6.18) can be rewritten as

$$\vec{v}_{AP} = \vec{v}_{BP} - \vec{v}_{BA}. \tag{4.6.19}$$

For most of the next few chapters we are going to be considering only motion in one dimension, and so we will write Equation (4.6.18) (or (4.6.19)) without the vector symbols, and it will be understood that v refers to the component of the vector $\vec{v}$ along the coordinate axis of interest.

A quantity that will be particularly important later on is the *relative velocity* of two objects, which we could label 1 and 2. The velocity of object 2 relative to object 1 is, by definition, the velocity which an observer moving along with 1 would measure for object 2. So it is just a simple frame change: let the earth frame be frame E and the frame moving with object 1 be frame 1, then the velocity we want is v_{12} ("velocity of object 2 in frame 1"). If we make the change A $\rightarrow$ 1, B $\rightarrow$ E, and P $\rightarrow$ 2 in Equation (4.6.19), we get

$$v_{12} = v_{E2} - v_{E1}. \tag{4.6.20}$$

In other words, the velocity of 2 relative to 1 is just the velocity of 2 minus the velocity of 1. This is again a familiar effect: if you are driving down the highway at 50 miles per hour, and the car in front of you is driving at 55, then its velocity relative to you is 5 mph, which is the rate at which it is moving away from you (in the forward direction, assumed to be the positive one).

It is important to realize that all these velocities are *real* velocities, each in its own reference frame. Something may be said to be truly moving at some velocity in one reference frame, and just as truly moving with a different velocity in a different reference frame. I will have a lot more to say about this in the next chapter, but in the meantime you can reflect on the fact that, if a car moving at 55 mph collides with another one moving at 50 mph in the same direction, the damage will be basically the same as if the first car had been moving at 5 mph and the second one had been at rest. For practical purposes, where you are concerned, another car's velocity relative to yours is that car's "real" velocity.

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[1] We have made a very natural assumption, that the time interval Δt is the same for observers tracking the particle's motion in frames A and B, respectively (where each observer is understood to be moving along with his or her frame). This, however, turns out to be *not* true when any of the velocities involved is close to the speed of light, and so the simple addition of velocities formula (4.6.18) does not hold in Einstein's relativity theory. (This is actually the first bit of real physics I have told you about in this book, so far; unfortunately, you will have no use for it this semester!)

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