9.9: Conservative and Non-Conservative Forces

Learning Objectives

- Characterize a conservative force in several different ways
- Specify mathematical conditions that must be satisfied by a conservative force and its components
- Relate the conservative force between particles of a system to the potential energy of the system
- Calculate the components of a conservative force in various cases
- Define conservative force, potential energy, and mechanical energy.
- Explain the potential energy of a spring in terms of its compression when Hooke's law applies.
- Use the work-energy theorem to show how having only conservative forces implies conservation of mechanical energy.

9.9.1 Conservative and Non-Conservative Forces

In Potential Energy and Conservation of Energy, any transition between kinetic and potential energy conserved the total energy of the system. This was path independent, meaning that we can start and stop at any two points in the problem, and the total energy of the system—kinetic plus potential—at these points are equal to each other. This is characteristic of a **conservative force**. We dealt with conservative forces in the preceding section, such as the gravitational force and spring force. When comparing the motion of the football in Figure 8.2.1, the total energy of the system never changes, even though the gravitational potential energy of the football increases, as the ball rises relative to ground and falls back to the initial gravitational potential energy when the football player catches the ball. **Non-conservative forces** are dissipative forces such as friction or air resistance. These forces take energy away from the system as the system progresses, energy that you can't get back. These forces are path dependent; therefore it matters where the object starts and stops.

Definition: Conservative Force

The work done by a conservative force is independent of the path; in other words, the work done by a conservative force is the same for any path connecting two points:

$$W_{AB,\ path-1} = \int_{AB,\ path-1} \vec{F}_{cons} \cdot d\vec{r} = W_{AB,\ path-2} = \int_{AB,\ path-2} \vec{F}_{cons} \cdot d\vec{r}. \tag{9.9.1}$$

The work done by a non-conservative force depends on the path taken. Equivalently, a force is conservative if the work it does around any closed path is zero:

$$W_{closed\ path} = \oint \vec{E}_{cons} \cdot d\vec{r} = 0. \tag{9.9.2}$$

In Equation 9.9.2, we use the notation of a circle in the middle of the integral sign for a line integral over a closed path, a notation found in most physics and engineering texts.] Equations 9.9.1 and 9.9.2 are equivalent because any closed path is the sum of two paths: the first going from A to B, and the second going from B to A. The work done going along a path from B to A is the negative of the work done going along the same path from A to B, where A and B are any two points on the closed path:

$$\begin{aligned} 0 = \int \vec{F}_{cons} \cdot d\vec{r} &= \int_{AB,\ path-1} \vec{F}_{cons} \cdot d\vec{r} + \int_{BA,\ path-2} \vec{F}_{cons} \cdot d\vec{r} \\ &= \int_{AB,\ path-1} \vec{F}_{cons} \cdot d\vec{r} - \int_{AB,\ path-2} \vec{F}_{cons} \cdot d\vec{r} = 0. \end{aligned}$$

You might ask how we go about proving whether or not a force is conservative, since the definitions involve any and all paths from A to B, or any and all closed paths, but to do the integral for the work, you have to choose a particular path. One answer is that the work done is independent of path if the infinitesimal work $\vec{F} \cdot d\vec{r}$ is an **exact differential**, the way the infinitesimal net work was equal to the exact differential of the kinetic energy, $dW_{net} = m\vec{v} \cdot d\vec{v} = d\frac{1}{2}mv^2$, when we derived the work-energy theorem in Work-Energy Theorem. There are mathematical conditions that you can use to test whether the infinitesimal work done by a force is an exact differential, and the force is conservative. These conditions only involve differentiation and are thus relatively easy to apply. In two dimensions, the condition for $\vec{F} \cdot d\vec{r} = F_x dx + F_y dy$ to be an exact differential is

$$\frac{dF_x}{dy} = \frac{dF_y}{dx}. \tag{9.9.3}$$

You may recall that the work done by the force in Example 7.2.4 depended on the path. For that force,

$$F_x = (5\ N/m)y \ and\ F_y = (10\ N/m)x. \tag{9.9.4}$$

Therefore,

$$\left(\frac{dF_x}{dy}\right) = 5\ N/m \neq \left(\frac{dF_y}{dx}\right) = 10\ N/m, \tag{9.9.5}$$

which indicates it is a non-conservative force. Can you see what you could change to make it a conservative force?

Figure 9.9.1: A grinding wheel applies a non-conservative force, because the work done depends on how many rotations the wheel makes, so it is path-dependent.

Example 9.9.1: Conservative or Not?

Which of the following two-dimensional forces are conservative and which are not? Assume a and b are constants with appropriate units:

a. $axy^3\hat{i} + ayx^3\hat{j}$,
b. $a\left[\left(\frac{y^2}{x}\right)\hat{i} + 2y\ln\left(\frac{x}{b}\right)\hat{j}\right]$,
c. $\frac{ax\hat{i}+ay\hat{j}}{x^2+y^2}$

Strategy

Apply the condition stated in Equation 9.9.3, namely, using the derivatives of the components of each force indicated. If the derivative of the y-component of the force with respect to x is equal to the derivative of the x-component of the force with respect to y, the force is a conservative force, which means the path taken for potential energy or work calculations always yields the same results.

Solution

a:

$$\frac{dF_x}{dy} = \frac{d(axy^3)}{dy} = 3axy^2$$

and

$$\frac{dF_y}{dx} = \frac{d(ayx^3)}{dx} = 3ayx^2,$$

so this force is non-conservative.

b:

$$\frac{dF_x}{dy} = \frac{d\left(\frac{ay^2}{x}\right)}{dy} = \frac{2ay}{x}$$

and

$$\frac{dF_y}{dx} = \frac{d(2ay\ln\left(\frac{x}{b}\right))}{dx} = \frac{2ay}{x},$$

so this force is conservative.

c:

$$\frac{dF_x}{dy} = \frac{d\left(\frac{ax}{(x^2+y^2)}\right)}{dy} = -\frac{ax(2y)}{(x^2+y^2)^2} = \frac{dF_y}{dx} = \frac{d\left(\frac{ay}{(x^2+y^2)}\right)}{dx}, \tag{9.9.6}$$

again conservative.

Significance

The conditions in Equation 9.9.3 are derivatives as functions of a single variable; in three dimensions, similar conditions exist that involve more derivatives.

? Exercise 9.9.1

A two-dimensional, conservative force is zero on the x- and y-axes, and satisfies the condition $\left(\frac{dF_x}{dy}\right) = \left(\frac{dF_y}{dy}\right)$ = (4 N/m^3)xy. What is the magnitude of the force at the point $x = y = 1\,m$?

Before leaving this section, we note that non-conservative forces do not have potential energy associated with them because the energy is lost to the system and can't be turned into useful work later. So there is always a conservative force associated with every potential energy. We have seen that potential energy is defined in relation to the work done by conservative forces. That relation, Equation 8.2.1, involved an integral for the work; starting with the force and displacement, you integrated to get the work and the change in potential energy. However, integration is the inverse operation of differentiation; you could equally well have started with the potential energy and taken its derivative, with respect to displacement, to get the force. The infinitesimal increment of potential energy is the dot product of the force and the infinitesimal displacement,

$$dU = -\vec{F}\cdot d\vec{l} = -F_l dl. \tag{9.9.7}$$

Here, we chose to represent the displacement in an arbitrary direction by d$\vec{l}$, so as not to be restricted to any particular coordinate direction. We also expressed the dot product in terms of the magnitude of the infinitesimal displacement and the component of the force in its direction. Both these quantities are scalars, so you can divide by dl to get

$$F_l = -\frac{dU}{dl}. \tag{9.9.8}$$

This equation gives the relation between force and the potential energy associated with it. In words, the component of a conservative force, in a particular direction, equals the negative of the derivative of the corresponding potential energy, with respect to a displacement in that direction. For one-dimensional motion, say along the x-axis, Equation 9.9.8 give the entire vector force,

$$\bar{F} = F_x\hat{i} = -\frac{\partial U}{\partial x}\hat{i}. \tag{9.9.9}$$

In two dimensions,

$$\bar{F} = F_x\hat{i} + F_y\hat{j} \tag{9.9.10}$$

$$= -\left(\frac{\partial U}{\partial x}\right)\hat{i} - \left(\frac{\partial U}{\partial y}\right)\hat{j}. \tag{9.9.11}$$

From this equation, you can see why Equation 9.9.8 is the condition for the work to be an exact differential, in terms of the derivatives of the components of the force. In general, a partial derivative notation is used. If a function has many variables in it, the derivative is taken only of the variable the partial derivative specifies. The other variables are held constant. In three dimensions, you add another term for the z-component, and the result is that the force is the negative of the gradient of the potential energy. However, we won't be looking at three-dimensional examples just yet.

Example 9.9.2: Force due to a Quartic Potential Energy

The potential energy for a particle undergoing one-dimensional motion along the x-axis is

$$U(x) = \frac{1}{4}cx^4,$$

where c = 8 N/m^3. Its total energy at x = 0 is 2 J, and it is not subject to any non-conservative forces. Find (a) the positions where its kinetic energy is zero and (b) the forces at those positions.

Strategy

a. We can find the positions where K = 0, so the potential energy equals the total energy of the given system.
b. Using Equation 9.9.8, we can find the force evaluated at the positions found from the previous part, since the mechanical energy is conserved.

Solution

a. The total energy of the system of 2 J equals the quartic elastic energy as given in the problem $2\; J = \frac{1}{4} (8\; N/m^{3}) x_{f}^{4} \ldotp$Solving for x_f results in $x_f = \pm 1$ m.
b. From Equation 9.9.8, $F_{x} = - \frac{dU}{dx} = -cx^{3} \ldotp$ Thus, evaluating the force at ±1 m , we get $\vec{F} = -(8\; N/m^{3})(\pm 1\; m)^{3} \hat{i} = \pm 8\; N \hat{i} \ldotp$At both positions, the magnitude of the forces is 8 N and the directions are toward the origin, since this is the potential energy for a restoring force.

Significance

Finding the force from the potential energy is mathematically easier than finding the potential energy from the force, because differentiating a function is generally easier than integrating one.

Exercise 9.9.2

Find the forces on the particle in Example 9.9.2 when its kinetic energy is 1.0 J at $x = 0$.

9.9.2 Conservative Forces and Potential Energy

Work is done by a force, and some forces, such as weight, have special characteristics. A *conservative force* is one, like the gravitational force, for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken. We can define a *potential energy (PE)* for any conservative force, just as we did for the gravitational force. For example, when you wind up a toy, an egg timer, or an old-fashioned watch, you do work against its spring and store energy in it. (We treat these springs as ideal, in that we assume there is no friction and no production of thermal energy.) This stored energy is recoverable as work, and it is useful to think of it as potential energy contained in the spring. Indeed, the reason that the spring has this characteristic is that its force is *conservative*. That is, a conservative force results in stored or potential energy. Gravitational potential energy is one example, as is the energy stored in a spring. We will also see how conservative forces are related to the conservation of energy.

Potential Energy and Conservative Forces

Potential energy is the energy a system has due to position, shape, or configuration. It is stored energy that is completely recoverable.

A conservative force is one for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken.

We can define a potential energy (PE) for any conservative force. The work done against a conservative force to reach a final configuration depends on the configuration, not the path followed, and is the potential energy added.

9.9.3 Potential Energy of a Spring

First, let us obtain an expression for the potential energy stored in a spring $(PE_s$. We calculate the work done to stretch or compress a spring that obeys Hooke's law. (Hooke's law was examined in Elasticity: Stress and Strain, and states that the magnitude of force F on the spring and the resulting deformation ΔL are proportional, $F = k\Delta L$). (See Figure.) For our spring, we will replace ΔL (the amount of deformation produced by a force F) by the distance x that the spring is stretched or compressed along its length. So the force needed to stretch the spring has magnitude $F = kx$, where k is the spring's force constant. The force increases linearly from 0 at the start to kx in the fully stretched position. The average force is $kx/2$. Thus the work done in stretching or compressing the spring is

$$W_s = Fd = (\frac{kx}{2})x = \frac{1}{2}kx^2. \tag{9.9.12}$$

Alternatively, we noted in Kinetic Energy and the Work-Energy Theorem that the area under a graph of F vs. x is the work done by the force. In Figure (c) we see that this area is also $\frac{1}{2}kx^2$. We therefore define the **potential energy of a spring**, \(PE_s|) to be

$$PE_s = \frac{1}{2}kx^2, \tag{9.9.13}$$

where k is the spring's force constant and x is the displacement from its undeformed position. The potential energy represents the work done *on* the spring and the energy stored in it as a result of stretching or compressing it a distance x. The potential energy of the spring PE_s does not depend on the path taken; it depends only on the stretch or squeeze x in the final configuration.

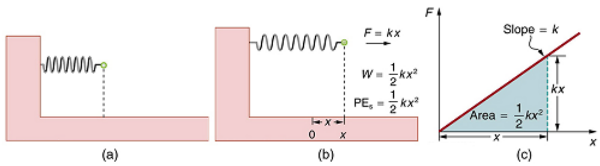

Figure 9.9.1: (a) An undeformed spring has no PE_s stored in it. (b) The force needed to stretch (or compress) the spring a distance x has a magnitude $F = kx$, and the work done to stretch (or compress) it is $\frac{1}{2}kx^2$. Because the force is conservative, this work is stored as potential energy (PE_s) in the spring, and it can be fully recovered. (c) A graph of F vs x has a slope of k, and the area under the graph is $\frac{1}{2}kx^2$. Thus the work done or potential energy stored is $\frac{1}{2}kx^2$.

The equation $PE_s = \frac{1}{2}kx^2$ has general validity beyond the special case for which it was derived. Potential energy can be stored in any elastic medium by deforming it. Indeed, the general definition of potential energy is energy due to position, shape, or configuration. For shape or position deformations, stored energy is $PE_s = \frac{1}{2}kx^2$, where k is the force constant of the particular system and x is its deformation. Another example is seen in Figure for a guitar string.

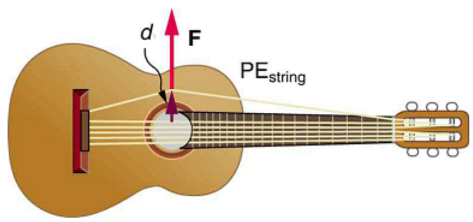

Figure 9.9.2: Work is done to deform the guitar string, giving it potential energy. When released, the potential energy is converted to kinetic energy and back to potential as the string oscillates back and forth. A very small fraction is dissipated as sound energy, slowly removing energy from the string.

9.9.4 Conservation of Mechanical Energy

Let us now consider what form the work-energy theorem takes when only conservative forces are involved. This will lead us to the conservation of energy principle. The work-energy theorem states that the net work done by all forces acting on a system equals its change in kinetic energy. In equation form, this is

$$W_{net} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \Delta KE. \tag{9.9.14}$$

If only conservative forces act, then

$$W_{net} = W_c \tag{9.9.15}$$

where W_c is the total work done by all conservative forces. Thus,

$$W_c = \Delta KE. \tag{9.9.16}$$

Now, if the conservative force, such as the gravitational force or a spring force, does work, the system loses potential energy. That is,

$$W_c = -\Delta PE \tag{9.9.17}$$

Therefore,

$$-\Delta PE = KE \tag{9.9.18}$$

or

$$\Delta PE + \Delta KE = 0. \tag{9.9.19}$$

This equation means that the total kinetic and potential energy is constant for any process involving only conservative forces. That is,

$$KE + PE = constant \tag{9.9.20}$$

or

$$KE_i + PE_i = KE_f + PE_f \tag{9.9.21}$$

$$(conservative\, forces\, only), \tag{9.9.22}$$

where i and f denote initial and final values. This equation is a form of the work-energy theorem for conservative forces; it is known as the **conservation of mechanical energy principle**. Remember that this applies to the extent that all the forces are conservative, so that friction is negligible. The total kinetic plus potential energy of a system is defined to be its **mechanical energy,** $(KE + PE)$. In a system that experiences only conservative forces, there is a potential energy associated with each force, and the energy only changes form between KE and the various types of PE, with the total energy remaining constant.

Example 9.9.1: Using Conservation of Mechanical Energy to Calculate the Speed of a Toy Car

A 0.100-kg toy car is propelled by a compressed spring, as shown in Figure. The car follows a track that rises 0.180 m above the starting point. The spring is compressed 4.00 cm and has a force constant of 250.0 N/m. Assuming work done by friction to be negligible, find (a) how fast the car is going before it starts up the slope and (b) how fast it is going at the top of the slope.

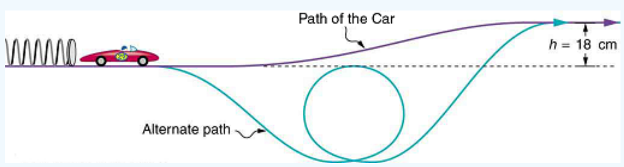

Figure 9.9.3: A toy car is pushed by a compressed spring and coasts up a slope. Assuming negligible friction, the potential energy in the spring is first completely converted to kinetic energy, and then to a combination of kinetic and gravitational potential energy as the car rises. The details of the path are unimportant because all forces are conservative—the car would have the same final speed if it took the alternate path shown.

Strategy

The spring force and the gravitational force are conservative forces, so conservation of mechanical energy can be used. Thus,

$$KE_i + PE_i = KE_f + PE_f \tag{9.9.23}$$

or

$$\frac{1}{2}mv_i^2 + mgh_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgh_f + \frac{1}{2}kx_f^2$$

where h is the height (vertical position) and x is the compression of the spring. This general statement looks complex but becomes much simpler when we start considering specific situations. First, we must identify the initial and final conditions in a problem; then, we enter them into the last equation to solve for an unknown.

Solution for (a)

This part of the problem is limited to conditions just before the car is released and just after it leaves the spring. Take the initial height to be zero, so that both h_i and h_f are zero. Furthermore, the initial speed v_i is zero and the final compression of the spring x_f is zero, and so several terms in the conservation of mechanical energy equation are zero and it simplifies to

$$\frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2.$$

In other words, the initial potential energy in the spring is converted completely to kinetic energy in the absence of friction. Solving for the final speed and entering known values yields

$$\begin{aligned} v_f &= \sqrt{\frac{k}{m}}x_i \\ &= \sqrt{\frac{250N/m}{0.100kg}}(0.0400\,m) \\ &= 2.00m/s \end{aligned}$$

Solution for (b)

One method of finding the speed at the top of the slope is to consider conditions just before the car is released and just after it reaches the top of the slope, completely ignoring everything in between. Doing the same type of analysis to find which terms are zero, the conservation of mechanical energy becomes

$$\frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgh_f.$$

This form of the equation means that the spring's initial potential energy is converted partly to gravitational potential energy and partly to kinetic energy. The final speed at the top of the slope will be less than at the bottom. Solving for v_f and substituting known values gives

$$v_f = \sqrt{\frac{kx^2}{m} - 2gh_f}$$

$$= \sqrt{\left(\frac{250N/m}{0.100kg}\right)(0.0400m)^2 - 2(9.80m/s^2)(0.180m)}$$

$$= 0.687\,m/s$$

Discussion

Another way to solve this problem is to realize that the car's kinetic energy before it goes up the slope is converted partly to potential energy—that is, to take the final conditions in part (a) to be the initial conditions in part (b).

Note that, for conservative forces, we do not directly calculate the work they do; rather, we consider their effects through their corresponding potential energies, just as we did in Example. Note also that we do not consider details of the path taken—only the starting and ending points are important (as long as the path is not impossible). This assumption is usually a tremendous simplification, because the path may be complicated and forces may vary along the way.

9.9.5 Summary

- A conservative force is one for which work depends only on the starting and ending points of a motion, not on the path taken.
- We can define potential energy $(PE$ for any conservative force, just as we defined PE_g for the gravitational force.
- The potential energy of a spring is $PE_s = \frac{1}{2}kx^2$, where k is the spring's force constant and |(x\) is the displacement from its undeformed position.
- Mechanical energy is defined to be $KE = PE$ for conservative force.
- When only conservative forces act on and within a system, the total mechanical energy is constant. In equation form,

$$KE + PE = constant \tag{9.9.24}$$

or

$$KE_i + PE_i = KE_f + PE_f \tag{9.9.25}$$

where i and f denote initial and final values. This is known as the conservation of mechanical energy.