# 3.2: Velocity

**LEARNING OBJECTIVES**

- Calculate the velocity vector given the position vector as a function of time.
- Calculate the average velocity in multiple dimensions.
- Explain the difference between average velocity and instantaneous velocity.
- Describe the difference between velocity and speed.
- Calculate the instantaneous velocity given the mathematical equation for the velocity.
- Calculate the speed given the instantaneous velocity.

To calculate the other physical quantities in kinematics we must introduce the time variable. The time variable allows us not only to state where the object is (its position) during its motion, but also how fast it is moving. How fast an object is moving is given by the rate at which the position changes with time.

## 3.2.1 Average Velocity

The **average velocity** $\vec{v}_{avg}$ is defined as the total displacement between two positions divided by the time taken to travel between them. The time taken to travel between two points is called the **elapsed time** $\Delta t$.

**Average Velocity**

If $\vec{r}_1$ and $\vec{r}_2$ are the positions of an object at times $t_1$ and $t_2$, respectively, then

$$\overrightarrow{Average\ velocity} = \vec{v}_{avg} = \frac{\overrightarrow{Displacement\ between\ two\ points}}{Elapsed\ time\ between\ two\ points} \tag{3.2.1}$$

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}. \tag{3.2.2}$$

**Alert**

*A common error is to write:* $\vec{v}_{avg} = \frac{\vec{v}_1 + \vec{v}_2}{2}$

*This equation is valid only when the velocity changes at a constant rate.*

## 3.2.2 Instantaneous Velocity

We can find the instantaneous velocity by calculating the derivative of the position function with respect to time. In taking the derivative, we use the same methods used for functions but in one, two and three dimensions, we use vectors. The instantaneous **velocity vector** is now

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}(t)}{dt}. \tag{3.2.3}$$

Let's look at the relative orientation of the position vector and velocity vector graphically. In Figure $3.2.1$ we show the vectors $\vec{r}$(t) and $\vec{r}$(t + $\Delta$t), which give the position of a particle moving along a path represented by the gray line. As $\Delta$t goes to zero, the velocity vector, given by Equation 3.2.4, becomes tangent to the path of the particle at time t.

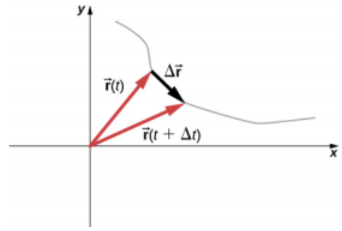


Figure 3.2.1: A particle moves along a path given by the gray line. In the limit as $\Delta$t approaches zero, the velocity vector becomes tangent to the path of the particle.

Equation 3.2.3 can also be written in terms of the components of $\vec{v}$(t). Since

$$\vec{r}(t)=x(t)\ \hat{i}+y(t)\ \hat{j}+z(t)\ \hat{k}, \tag{3.2.4}$$

we can write

$$\vec{v}(t)=v_x(t)\ \hat{i}+v_y(t)\ \hat{j}+v_z(t)\ \hat{k} \tag{3.2.5}$$

where

$$v_x(t)=\frac{dx(t)}{dt},\quad v_y(t)=\frac{dy(t)}{dt},\quad v_z(t)=\frac{dz(t)}{dt}. \tag{3.2.6}$$

If only the average velocity is of concern, we have the vector equivalent of the one-dimensional average velocity for two and three dimensions:

$$\vec{v}_{avg}=\frac{\vec{r}(t_2)-\vec{r}(t_1)}{t_2-t_1}. \tag{3.2.7}$$

Alert

*The direction of the velocity vector is always in the direction of motion.*

### 3.2.3 Speed

In everyday language, most people use the terms speed and velocity interchangeably. In physics, however, they do not have the same meaning and are distinct concepts. One major difference is that speed has no direction; that is, speed is a scalar.

We can calculate the **average speed** by finding the total distance traveled divided by the elapsed time:

$$Average\ speed=\bar{s}=\frac{Total\ distance}{Elapsed\ time}. \tag{3.2.8}$$

Average speed is not necessarily the same as the magnitude of the average velocity, which is found by dividing the magnitude of the total displacement by the elapsed time. For example, if a trip starts and ends at the same location, the total displacement is zero, and therefore the average velocity is zero. The average speed, however, is not zero, because the total distance traveled is greater than zero. If we take a road trip of 300 km and need to be at our destination at a certain time, then we would be interested in our average speed.

However, we can calculate the **instantaneous speed** from the magnitude of the instantaneous velocity:

$$Instantaneous\ speed=|v(t)|. \tag{3.2.9}$$

If a particle is moving along the x-axis at +7.0 m/s and another particle is moving along the same axis at −7.0 m/s, they have different velocities, but both have the same speed of 7.0 m/s. Some typical speeds are shown in the following table.

Table 3.2.1 - Speeds of Various Objects

| Speed | m/s | mi/h |
|---|---|---|
| Continental drift | $10^{-7}$ | $2 \times 10^{-7}$ |
| Brisk walk | 1.7 | 3.9 |
| Cyclist | 4.4 | 10 |

| Speed | m/s | mi/h |
|---|---|---|
| Sprint runner | 12.2 | 27 |
| Rural speed limit | 24.6 | 56 |
| Official land speed record | 341.1 | 763 |
| Speed of sound at sea level | 343 | 768 |
| Space shuttle on reetry | 7800 | 17,500 |
| Escape velocity of Earth* | 11,200 | 25,000 |
| Orbital speed of Earth around the Sun | 29,783 | 66,623 |
| Speed of light in a vacuum | 299,792,458 | 670,616,629 |
| *Escape velocity is the velocity at which an object must be launched so that it overcomes Earth's gravity and is not pulled back toward Earth. | | |

## 3.2.4 The Independence of Perpendicular Motions

When we look at the three-dimensional equations for position and velocity written in unit vector notation, Equation 3.2.4 and Equation 3.2.5, we see the components of these equations are separate and unique functions of time that do not depend on one another. We can write:

$$\vec{v}(t) = v_x(t)\ \hat{i} + v_y(t)\ \hat{j} + v_z(t)\ \hat{k} = \frac{dx(t)}{dt}\ \hat{i} + \frac{dy(t)}{dt}\ \hat{j} + \frac{dz(t)}{dt}\ \hat{k}. \tag{3.2.10}$$

and,

$$\vec{v}_{avg}(t) = v_{x_{avg}}\ \hat{i} + v_{y_{avg}}\ \hat{j} + v_{z_{avg}}\ \hat{k} = \frac{\Delta x}{\Delta t}\ \hat{i} + \frac{\Delta y}{\Delta t}\ \hat{j} + \frac{\Delta z}{\Delta t}\ \hat{k}. \tag{3.2.11}$$

Motion along the x direction has no part of its motion along the y and z directions, and similarly for the other two coordinate axes. Thus, the motion of an object in two or three dimensions can be divided into separate, independent motions along the perpendicular axes of the coordinate system in which the motion takes place.

To illustrate this concept with respect to displacement, consider a person walking from point A to point B in a city with square blocks. The person taking the path from A to B may walk east for so many blocks and then north (two perpendicular directions) for another set of blocks to arrive at B. How far the person walks east is affected only by the motion eastward. Similarly, how far the person walks north is affected only by the motion northward.

**Independence of Motion**

In the kinematic description of motion, we are able to treat the horizontal and vertical components of motion separately. In many cases, motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

Table 3.2.2 - Equations of Motion along the various directions

| Quantity | General Expression | Along $x$ | Along $y$ | Along $z$ |
|---|---|---|---|---|
| Position | $\vec{r}(t)$ | $x(t)$ | $y(t)$ | $z(t)$ |
| Average Velocity | $\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$ | $v_{x_{avg}} = \frac{\Delta x}{\Delta t}$ | $v_{y_{avg}} = \frac{\Delta y}{\Delta t}$ | $v_{z_{avg}} = \frac{\Delta z}{\Delta t}$ |
| Instantaneous Velocity | $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ | $v_x(t) = \frac{dx(t)}{dt}$ | $v_y(t) = \frac{dy(t)}{dt}$ | $v_z(t) = \frac{dz(t)}{dt}$ |

## 3.2.5 Examples

**Example 3.2.1: Calculating the Velocity Vector**

The position function of a particle is $\vec{r}$(t) = 2.0t$^2$ $\hat{i}$ + (2.0 + 3.0t) $\hat{j}$+ 5.0t $\hat{k}$ m. (a) What is the instantaneous velocity and speed at t = 2.0 s? (b) What is the average velocity between 1.0 s and 3.0 s?

**Solution**

Using Equation 3.2.5 and Equation 3.2.6, and taking the derivative of the position function with respect to time, we find

a.

$$v(t) = \frac{d\vec{r}(t)}{dt} = 4.0t\ \hat{i} + 3.0\ \hat{j} + 5.0\ \hat{k}\ m/s$$
$$\vec{v}(2.0\ s) = 8.0\ \hat{i} + 3.0\ \hat{j} + 5.0\ \hat{k}\ m/s$$
$$Speed\ |\vec{v}(2.0\ s)| = \sqrt{8^2 + 3^2 + 5^2} = 9.9\ m/s.$$

b.

From Equation 3.2.7,

$$\vec{v}_{avg} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} = \frac{\vec{r}(3.0\ s) - \vec{r}(1.0\ s)}{3.0\ s - 1.0\ s} = \frac{(18\ \hat{i} + 11\ \hat{j} + 15\ \hat{k})m - (2\ \hat{i} + 5\ \hat{j} + 5\ \hat{k})m}{2.0\ s}$$
$$= \frac{(16\ \hat{i} + 6\ \hat{j} + 10\ \hat{k})m}{2.0\ s} = 8.0\ \hat{i} + 3.0\ \hat{j} + 5.0\ \hat{k}\ m/s.$$

**Significance**

We see the average velocity is the same as the instantaneous velocity at t = 2.0 s, as a result of the velocity function being linear. This need not be the case in general. In fact, most of the time, instantaneous and average velocities are not the same.

### Example 3.2.2: Delivering Flyers

Jill sets out from her home to deliver flyers for her yard sale, traveling due east along her street lined with houses. At 0.5 km and 9 minutes later she runs out of flyers and has to retrace her steps back to her house to get more. This takes an additional 9 minutes. After picking up more flyers, she sets out again on the same path, continuing where she left off, and ends up 1.0 km from her house. This third leg of her trip takes 15 minutes. At this point she turns back toward her house, heading west. After 1.75 km and 25 minutes she stops to rest.

a. What is Jill's total displacement to the point where she stops to rest?
b. What is the magnitude of the final displacement?
c. What is the average velocity during her entire trip?
d. What is the total distance traveled?
e. Make a graph of position versus time. A sketch of Jill's movements is shown in Figure 3.2.3.

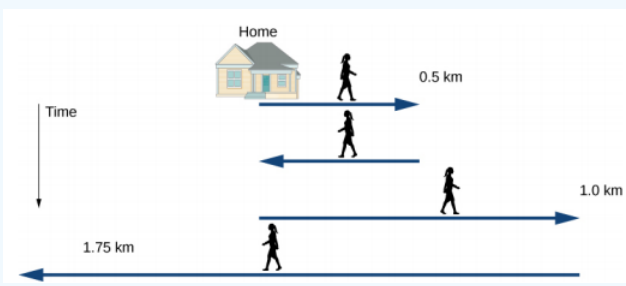


Figure 3.2.3: Timeline of Jill's movements.

**Strategy**

The problem contains data on the various legs of Jill's trip, so it would be useful to make a table of the physical quantities. We are given position and time in the wording of the problem so we can calculate the displacements and the elapsed time. We take east to be the positive direction. From this information we can find the total displacement and average velocity. Jill's home is the starting point $x_0$. The following table gives Jill's time and position in the first two columns, and the displacements are calculated in the third column.

| Time $t_i$ (min) | Position $x_i$ (km) | Displacement $\Delta\ x_i$ (km) |
|---|---|---|
| $t_0 = 0$ | $x_0 = 0$ | $\Delta\ x_0 = 0$ |
| $t_1 = 9$ | $x_1 = 0.5$ | $\Delta x_1 = x_1 - x_0 = 0.5$ |

| | | |
|---|---|---|
| $t_2 = 18$ | $x_2 = 0$ | $\Delta x2 = x_2 - x_1 = -0.5$ |
| $t_3 = 33$ | $x_3 = 1.0$ | $\Delta x_3 = x_3 - x_2 = 1.0$ |
| $t_4 = 58$ | $x_4 = -0.75$ | $\Delta x_4 = x_4 - x_3 = -1.75$ |

**Solution**

a. From the above table, the total displacement is $\sum \Delta x_i = 0.5 - 0.5 + 1.0 - 1.75\ km = -0.75\ km.$
b. The magnitude of the total displacement is |−0.75| km = 0.75 km.
c. $Average\ velocity = \frac{Total\ displacement}{Elapsed\ time} = \bar{v} = \frac{-0.75\ km}{58\ min} = -0.013\ km/min$
d. The total distance traveled (sum of magnitudes of individual displacements) is $x_{Total} = \sum |\Delta x_i| = 0.5 + 0.5 + 1.0 + 1.75\ km = 3.75\ km.$
e. We can graph Jill's position versus time as a useful aid to see the motion; the graph is shown in Figure $3.2.4$

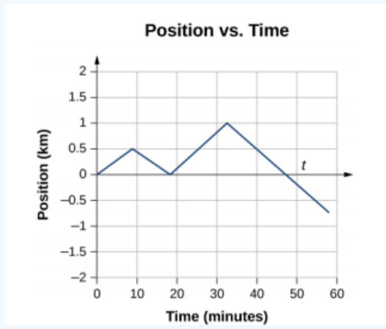


Figure $3.2.4$: This graph depicts Jill's position versus time. The average velocity is the slope of a line connecting the initial and final points.

**Significance**

Jill's total displacement is −0.75 km, which means at the end of her trip she ends up 0.75 km due west of her home. The average velocity means if someone was to walk due west at 0.013 km/min starting at the same time Jill left her home, they both would arrive at the final stopping point at the same time. Note that if Jill were to end her trip at her house, her total displacement would be zero, as well as her average velocity. The total distance traveled during the 58 minutes of elapsed time for her trip is 3.75 km.

## ✔ Example $3.2.3$: Finding Velocity from a Position-Versus-Time Graph

Given the position-versus-time graph of Figure $3.2.2$, find the velocity-versus-time graph.

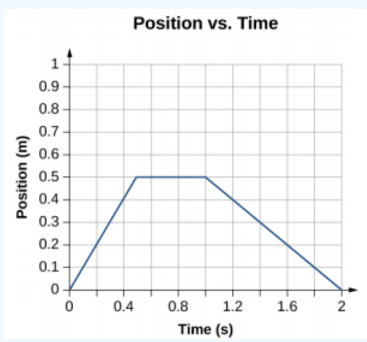


Figure $3.2.2$: The object starts out in the positive direction, stops for a short time, and then reverses direction, heading back toward the origin. Notice that the object comes to rest instantaneously, which would require an infinite force. Thus, the graph is an approximation of motion in the real world. (The concept of force is discussed in Newton's Laws of Motion.)

**Strategy**

The graph contains three straight lines during three time intervals. We find the velocity during each time interval by taking the slope of the line using the grid.

**Solution**

Time interval 0 s to 0.5 s: $\bar{v} = \frac{\Delta x}{\Delta t} = \frac{0.5\ m - 0.0\ m}{0.5\ s - 0.0\ s} = 1.0\ m/s$

Time interval 0.5 s to 1.0 s: $\bar{v} = \frac{\Delta x}{\Delta t} = \frac{0.0\ m - 0.0\ m}{1.0\ s - 0.5\ s} = 0.0\ m/s$

Time interval 1.0 s to 2.0 s: $\bar{v} = \frac{\Delta x}{\Delta t} = \frac{0.0\ m - 0.5\ m}{2.0\ s - 1.0\ s} = -0.5\ m/s$

The graph of these values of velocity versus time is shown in Figure 3.2.3

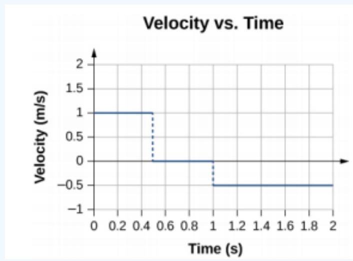


Figure 3.2.3: The velocity is positive for the first part of the trip, zero when the object is stopped, and negative when the object reverses direction.

**Significance**

During the time interval between 0 s and 0.5 s, the object's position is moving away from the origin and the position-versus-time curve has a positive slope. At any point along the curve during this time interval, we can find the instantaneous velocity by taking its slope, which is +1 m/s, as shown in Figure 3.2.3 In the subsequent time interval, between 0.5 s and 1.0 s, the position doesn't change and we see the slope is zero. From 1.0 s to 2.0 s, the object is moving back toward the origin and the slope is −0.5 m/s. The object has reversed direction and has a negative velocity.

### Example 3.2.4: Instantaneous Velocity Versus Average Velocity

The position of a particle is given by $x(t) = 3.0t + 0.5t^3$ m.

a. Using Equation 3.2.5 and Equation 3.2.6, find the instantaneous velocity at t = 2.0 s.
b. Calculate the average velocity between 1.0 s and 3.0 s.

**Strategy**

Equation 3.2.5give the instantaneous velocity of the particle as the derivative of the position function. Looking at the form of the position function given, we see that it is a polynomial in t. Therefore, we can use Equation 3.2.6, the power rule from calculus, to find the solution. We use Equation 3.2.7to calculate the average velocity of the particle.

**Solution**

a. $v(t) = \frac{dx(t)}{dt} = 3.0 + 1.5t^2$ m/s. Substituting t = 2.0 s into this equation gives $v(2.0\ s) = [3.0 + 1.5(2.0)^2]$ m/s = 9.0 m/s.
b. To determine the average velocity of the particle between 1.0 s and 3.0 s, we calculate the values of x(1.0 s) and x(3.0 s):

$$x(1.0s) = \left[(3.0)(1.0) + 0.5(1.0)^3\right] m = 3.5\ m \tag{3.2.12}$$

$$x(3.0s) = \left[(3.0)(3.0) + 0.5(3.0)^3\right] m = 22.5\ m \tag{3.2.13}$$

Then the average velocity is

$$\bar{v} = \frac{x(3.0\ s) - x(1.0\ s)}{t(3.0\ s) - t(1.0\ s)} = \frac{22.5 - 3.5\ m}{3.0 - 1.0\ s} = 9.5\ m/s. \tag{3.2.14}$$

**Significance**

In the limit that the time interval used to calculate $\bar{v}$ goes to zero, the value obtained for $\bar{v}$ converges to the value of v.

## Example 3.2.5: Instantaneous Velocity Versus Speed

Consider the motion of a particle in which the position is $x(t) = 3.0t - 3t^2$ m.

a. What is the instantaneous velocity at $t = 0.25$ s, $t = 0.50$ s, and $t = 1.0$ s?
b. What is the speed of the particle at these times?

**Strategy**

The instantaneous velocity is the derivative of the position function and the speed is the magnitude of the instantaneous velocity. We use Equation 3.2.5and Equation 3.2.6to solve for instantaneous velocity.

**Solution**

a. $v(t) = \frac{dx(t)}{dt} = 3.0 - 6.0t$ m/s
b. $v(0.25\ s) = 1.50$ m/s, $v(0.5\ s) = 0$ m/s, $v(1.0\ s) = -3.0$ m/s
c. Speed $= |v(t)| = 1.50$ m/s, 0.0 m/s, and 3.0 m/s

**Significance**

The velocity of the particle gives us direction information, indicating the particle is moving to the left (west) or right (east). The speed gives the magnitude of the velocity. By graphing the position, velocity, and speed as functions of time, we can understand these concepts visually Figure 3.2.4 In (a), the graph shows the particle moving in the positive direction until t = 0.5 s, when it reverses direction. The reversal of direction can also be seen in (b) at 0.5 s where the velocity is zero and then turns negative. At 1.0 s it is back at the origin where it started. The particle's velocity at 1.0 s in (b) is negative, because it is traveling in the negative direction. But in (c), however, its speed is positive and remains positive throughout the travel time. We can also interpret velocity as the slope of the position-versus-time graph. The slope of x(t) is decreasing toward zero, becoming zero at 0.5 s and increasingly negative thereafter. This analysis of comparing the graphs of position, velocity, and speed helps catch errors in calculations. The graphs must be consistent with each other and help interpret the calculations.

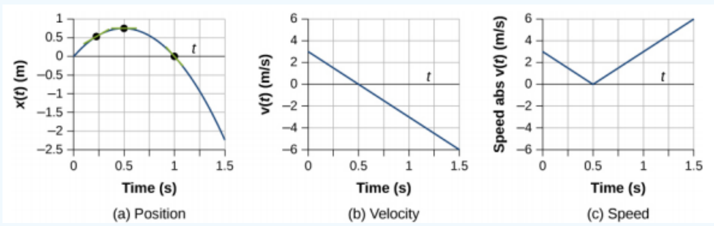


Figure 3.2.4: (a) Position: x(t) versus time. (b) Velocity: v(t) versus time. The slope of the position graph is the velocity. A rough comparison of the slopes of the tangent lines in (a) at 0.25 s, 0.5 s, and 1.0 s with the values for velocity at the corresponding times indicates they are the same values. (c) Speed: |v(t)| versus time. Speed is always a positive number.