19.1: Introduction to Cycloids

Let us set up a coordinate system Oxy, and a horizontal straight line $y = 2a$. We imagine a circle of diameter $2a$ between the x-axis and the line $y = 2a$, and initially the lowest point on the circle, P, coincides with the origin of coordinates O. We now allow the circle to roll counterclockwise without slipping on the line $y = 2a$, so that the centre of the circle moves to the right. As the circle rolls on the line, the point P describes a curve, which is known as a *cycloid*.

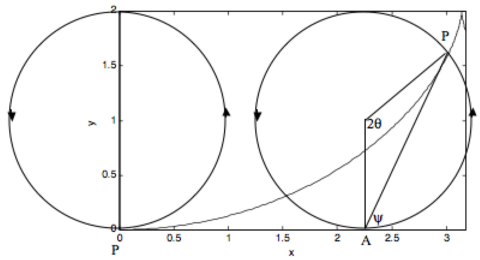

FIGURE XIX.1

When the circle has rolled through an angle 2θ, the centre of the circle has moved to the right by a horizontal distance $2a\theta$, while the horizontal distance of the point P from the centre of the circle is $a \sin 2\theta$ and the vertical distance of the point P below the centre of the circle is $a \cos 2\theta$. Thus the coordinates of the point P are

$$x = a(2\theta + \sin 2\theta) \tag{19.1.1}$$

and

$$y = a(1 - \cos 2\theta). \tag{19.1.2}$$

Equations 19.1.1 and 19.1.2 are the parametric equations of the cycloid. Using a simple trigonometric identity, Equation 19.1.2 can also be written

$$y = 2a \sin^2 \theta. \tag{19.1.3}$$

Example 19.1.1

When the x-coordinate of P is $2.500a$, what (to four significant figures) is its y-coordinate?

Solution

We have to find 2θ by solution of $2\theta + \sin 2\theta$. By Newton-Raphson iteration or otherwise, we find 2θ = 0.931 599 201 radians, and hence y = $0.9316a$.