2.17: Solid Body Rotation and the Inertia Tensor

It is intended that this chapter should be limited to the calculation of the moments of inertia of bodies of various shapes, and not with the huge subject of the rotational dynamics of solid bodies, which requires a chapter on its own. In this section I mention merely for interest two small topics involving the principal axes, and a third topic in a bit more detail as necessary before proceeding to Section 2.18.

Everyone knows that the relation between translational kinetic energy and linear momentum is $E = p^2/(2m)$. Similarly rotational kinetic energy is related to angular momentum L by $E = L^2/(2I)$, where I is the moment of inertia. If an isolated body (such as an asteroid) is rotating about a non-principal axis, it will be subject to internal stresses. If the body is nonrigid this will result in distortions (strains) which may cause the body to vibrate. If in addition the body is inelastic the vibrations will rapidly die out (if the damping is greater than critical damping, indeed, the body will not even vibrate). Energy that was originally rotational kinetic energy will be converted to heat (which will be radiated away.) The body loses rotational kinetic energy. In the absence of external torques, however, L remains constant. Therefore, while E diminishes, I increases. The body adjusts its rotation until it is rotating around its axis of maximum moment of inertia, at which time there are no further stresses, and the situation remains stable.

In general the rotational motion of a solid body whose momental ellipse is triaxial is quite complicated and chaotic, with the body tumbling over and over in apparently random fashion. However, if the body is nonrigid and inelastic (as all real bodies are in practice), it will eventually end up rotating about its axis of maximum moment of inertia. The time taken for a body, initially tumbling chaotically over and over, until it reaches its final blissful state of rotation about its axis of maximum moment of inertia, depends on how fast it is rotating. For most irregular small asteroids the time taken is comparable to or longer than the age of formation of the solar system, so that it is not surprising to find some asteroids with non-principal axis (NPA) rotation. However, a few rapidly-rotating NPA asteroids have been discovered, and, for rapid rotators, one would expect PA rotation to have been reached a long time ago. It is thought that something (such as a collision) must have happened to these rapidly-rotating NPA asteroids relatively recently in the history of the solar system.

Another interesting topic is that of the *stability* of a rigid rotator that is rotating about a principal axis, against small perturbations from its rotational state. Although I do not prove it here (the proof can be done either mathematically, or by a qualitative argument) rotation about either of the axes of maximum or of minimum moment of inertia is stable, whereas rotation about the intermediate axis is unstable. The reader can observe this for him- or herself. Find anything that is triaxial - such as a small block of wood shaped as a rectangular parallelepiped with unequal sides. Identify the axes of greatest, least and intermediate moment of inertia. Toss the body up in the air at the same time setting it rotating about one or the other of these axes, and you will be able to see for yourself that the rotation is stable in two cases but unstable in the third.

Inertia Tensor

I now deal with a third topic in rather more detail, namely the relation between angular momentum $\mathbf{L}$ and angular velocity $\boldsymbol{\omega}$. The reader will be familiar from elementary (and two- dimensional) mechanics with the relation $L = I\omega$. What we are going to find in the three- dimensional solid-body case is that the relation is $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$. Here $\mathbf{L}$ and $\boldsymbol{\omega}$ are, of course, vectors, but they *are not necessarily parallel to each other.* They are parallel only if the body is rotating about a principal axis of rotation. The quantity $\mathbb{I}$ is a tensor known as the *inertia tensor*. Readers will be familiar with the equation $\mathbf{F} = m\mathbf{a}$. Here the two vectors are in the same direction, and m is a scalar quantity that does not change the direction of the vector that it multiplies. A tensor usually (unless its matrix representation is *diagonal*) changes the direction as well as the magnitude of the vector that it multiplies. The reader might like to think of other examples of tensors in physics. There are several. One that comes to mind is the permittivity of an anisotropic crystal; in the equation $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$ and $\mathbf{E}$ are not parallel unless they are both directed along one of the crystallographic axes.

If there are no external torques acting on a body, $\mathbf{L}$ is constant in both magnitude and direction. The instantaneous angular velocity vector, however, is not fixed either in space or with respect to the body - unless the body is rotating about a principal axis and the inertia tensor is diagonal.

So much for a preview and a qualitative description. Now down to work.

I am going to have to assume familiarity with the equation for the components of the cross product of two vectors:

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_zB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_z)\hat{\mathbf{z}} \tag{2.17.1}$$

I am also going to assume that the reader knows that the angular momentum of a particle of mass m at position vector $\mathbf{r}$ (components $(x, y, z\ )$) and moving with velocity $\mathbf{v}$ (components ($\dot{x}, \dot{y}, \dot{z}$)) is $m\mathbf{r} \times \mathbf{v}$. For a collection of particles, (or an extended solid body, which, I'm told, consists of a collection of particles called atoms), the angular momentum is

$$\mathbf{L} = \sum m\mathbf{r} \times \mathbf{v} \tag{2.17.2}$$

$$= \sum [m(y\dot{z} - z\dot{y})\hat{\mathbf{x}} + m(z\dot{x} - x\dot{z})\hat{\mathbf{y}} + m(x\dot{y} - y\dot{x})\hat{\mathbf{z}}] \tag{2.17.3}$$

I also assume that the relation between linear velocity $\mathbf{v}(\ \dot{x}, \dot{y}, \dot{z}\ )$and angular velocity $\boldsymbol{\omega}\ (\omega_x, \omega_y, \omega_z)$ is understood to be $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, so that , for example $\dot{z} = \omega_x y - \omega_y x.$ then

$$\mathbf{L} \ = \sum [m(y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z)\hat{\mathbf{x}} + (etc.)\hat{\mathbf{y}} + (etc.)\hat{\mathbf{z}}] \tag{2.17.4}$$

$$= (\omega_x \sum my^2 - \omega_y \sum mxy - \omega_z \sum mzx + \omega_x \sum mz^2)\hat{\mathbf{x}} + etc. \tag{2.17.5}$$

$$= (A\omega_x - H\omega_y - G\omega_z)\hat{\mathbf{x}} + ()\hat{\mathbf{y}} + ()\hat{\mathbf{z}}. \tag{2.17.6}$$

Finally we obtain

$$\mathbf{L} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{2.17.7}$$

This is the equation $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$ referred to above. The inertia tensor is sometimes written in the form

$$\mathbb{I}I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix}$$

so that, for example, $I_{xy} = -H$. It is a symmetric matrix (but it is not an orthogonal matrix).