21.2: Motion Under a Central Force

I consider the two-dimensional motion of a particle of mass m under the influence of a conservative central force $F(r)$, which can be either attractive or repulsive, but depends only on the radial coordinate r. Recalling the formula $\ddot{r} - r\dot{\theta}^2$ \)for acceleration in polar coordinates (the second term being the centripetal acceleration), we see that the equation of motion is

$$m\ddot{r} - mr\dot{\theta}^2 = F(r). \tag{21.2.1}$$

This describes, in polar coordinates, two-dimensional motion in a plane. But since there are no transverse forces, the angular momentum $m^2\dot{\theta}^2$ is constant and equal to L, say. Thus we can write Equation 21.2.1 as

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3}. \tag{21.2.2}$$

This has reduced it to a one-dimensional equation; that is, we are describing, relative to a co-rotating frame, how the distance of the particle from the centre of attraction (or repulsion) varies with time. In this co-rotating frame it is as if the particle were subject not only to the force $F(r)$, but also to an additional force $\frac{L^2}{mr^3}$. In other words the total force on the particle (referred to the co-rotating frame) is

$$F'(r) = F(r) + \frac{L^2}{mr^3}. \tag{21.2.3}$$

Now $F(r)$, being a conservative force, can be written as minus the derivative of a potential energy function, $F = -\frac{dV}{dr}$. Likewise, $\frac{L^2}{m^3}$ is minus the derivative of $\frac{L^2}{2mr^2}$. Thus, in the co-rotating frame, the motion of the particle can be described as constrained by the potential energy function V', where

$$V' = V + \frac{L^2}{2mr^2}. \tag{21.2.4}$$

This is the *equivalent potential energy*. If we divide both sides by the mass m of the orbiting particle, this becomes

$$\Phi' = \Phi + \frac{h^2}{2r^2}. \tag{21.2.5}$$

Here h is the angular momentum per unit mass of the orbiting particle, Φ is the potential in the inertial frame, and $\Phi'$is the *equivalent potential* in the corotating frame.