

## 14.3: Hamilton's Equations of Motion

In classical mechanics we can describe the state of a system by specifying its Lagrangian as a function of the coordinates and their time rates of change:

$$L = L(q_i, \dot{q}_i) \quad (14.3.1)$$

If the coordinates and the velocities increase, the corresponding increment in the Lagrangian is

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i. \quad (14.3.2)$$

### Definition: generalized momenta

The *generalized momentum*  $p_i$  associated with the generalized coordinate  $q_i$  is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \quad (14.3.3)$$

[You have seen this before, in Section 13.4. Remember “ignorable coordinate”?]

It follows from the Lagrangian equation of motion (Equation 13.4.14)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

that

$$\dot{p}_i = \frac{\partial L}{\partial q_i}. \quad (14.3.4)$$

Thus

$$dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i. \quad (14.3.5)$$

(I am deliberately numbering this Equation 14.3.5 to maintain an analogy between this section and Section 14.2.)

However, it is sometimes convenient to change the basis of the description of the state of a system from  $q_i$  and  $\dot{q}_i$  to  $q_i$  and  $p_i$  by defining a quantity called the hamiltonian  $H$  defined by

$$H = \sum_i p_i \dot{q}_i - L. \quad (14.3.6)$$

### Definition: hamiltonian

In that case, if the state of the system changes, then

$$\begin{aligned} dH &= \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - dL \\ &= \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \sum_i p_i d\dot{q}_i \end{aligned}$$

That is

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i. \quad (14.3.7)$$

We are regarding the hamiltonian as a function of the generalized coordinates and generalized momenta:

$$H = H(q_i, p_i) \quad (14.3.8)$$

so that

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i, \quad (14.3.9)$$

from which we see that

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \quad (14.3.10)$$

and

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (14.3.11)$$

In summary, then, Equations [14.3.3](#), [14.3.4](#), [14.3.10](#) and [14.3.11](#):

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (14.3.12)$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \quad (14.3.13)$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \quad (14.3.14)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (14.3.15)$$

which I personally find impossible to commit accurately to memory (although note that there is one dot in each equation) except when using them frequently, may be regarded as Hamilton's equations of motion. I'll refer to these equations as **A**, **B**, **C** and **D**.

Note that, in Equation [14.3.13](#) if the Lagrangian is independent of the coordinate  $q_i$  the coordinate  $q_i$  is referred to as an "ignorable coordinate". I suppose it is called "ignorable" because you can ignore it when calculating the lagrangian, but in fact a so-called "ignorable" coordinate is usually a very interesting coordinate indeed, because it means (look at the second equation) that the corresponding generalized momentum is conserved.

Now the kinetic energy of a system is given by  $T = \frac{1}{2} \sum_i p_i \dot{q}_i$  (for example,  $\frac{1}{2} m v v$ ), and the hamiltonian (Equation [14.3.6](#)) is defined as  $H = \sum_i p_i \dot{q}_i - L$ . For a *conservative system*,  $L = T - V$ , and hence, for a conservative system,  $H = T + V$ . If you are asked in an examination to explain what is meant by the hamiltonian, by all means say it is  $T + V$ . That's fine for a conservative system, and you'll probably get half marks. That's 50% - a D grade, and you've passed. If you want an A+, however, I recommend Equation [14.3.6](#)

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