3.12: Torque, Angular Momentum and a Moving Point

In Figure III.7 I draw the particle m_i, which is just one of n particles, $n-1$ of which I haven't drawn and are scattered around in 3-space. I draw an arbitrary origin O, the centre of mass C of the system, and another point Q, which may (or may not) be moving with respect to O. The question I am going to ask is: Does the equation $\dot{\mathbf{L}} = \boldsymbol{\tau}$ apply to the point Q? It obviously does if Q is stationary, just as it applies to O. But what if Q is moving? If it does not apply, just what *is* the appropriate relation?

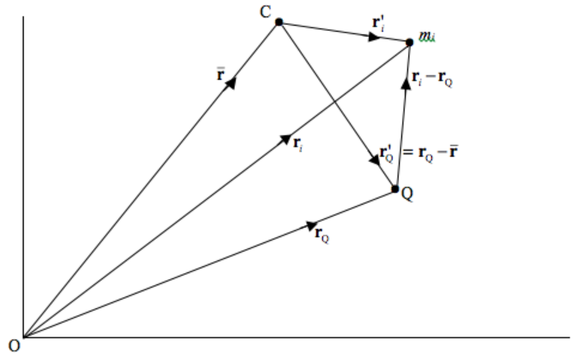

FIGURE III.7

The theorem that we shall prove – and interpret - is

$$\dot{\mathbf{L}}_Q = \boldsymbol{\tau}_Q + M\mathbf{r}'_Q \times \ddot{\mathbf{r}}_Q. \tag{3.12.1}$$

We start:

$$\mathbf{L}_Q = \sum (\mathbf{r}_i - \mathbf{r}_Q) \times [m_i(\mathbf{v}_i - \mathbf{v}_Q)] \tag{3.12.2}$$

$$\therefore \quad \dot{\mathbf{L}}_Q = \sum (\mathbf{r}_i - \mathbf{r}_Q) \times m_i(\dot{\mathbf{v}}_i - \dot{\mathbf{v}}_Q) + \sum (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_Q) \times m_i(\mathbf{v}_i - \mathbf{v}_Q). \tag{3.12.3}$$

The second term is zero, because $\dot{\mathbf{r}} = \mathbf{v}$

Continue:

$$\dot{\mathbf{L}} = \sum (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \dot{\mathbf{v}}_i - \sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_Q + \sum m_i \mathbf{r}_Q \times \dot{\mathbf{v}}_Q \tag{3.12.4}$$

Now $m_i \dot{\mathbf{v}}_i = \mathbf{F}_i$, so that the first term is just $\boldsymbol{\tau}_Q$

Continue:

$$\begin{aligned} \dot{\mathbf{L}} &= \boldsymbol{\tau}_Q - \sum m_i \mathbf{r}_i \times dot \mathbf{v}_Q + \sum M_i \mathbf{r}_Q \times dot \mathbf{v}_Q \\ &= \boldsymbol{\tau}_Q - M\bar{\mathbf{r}} \times \ddot{\mathbf{r}}_Q + M\mathbf{r}_Q \times \ddot{\mathbf{r}}_Q \\ &= \boldsymbol{\tau}_Q + M(\mathbf{r}_Q - \bar{\mathbf{r}}) \times \ddot{\mathbf{r}}_Q \\ \therefore \dot{\mathbf{L}}_Q &= \boldsymbol{\tau}_Q + M\mathbf{r}'_Q \times \ddot{r}_Q \qquad Q.E.D \end{aligned} \tag{3.12.5}$$

Thus in general, $\dot{\mathbf{L}}_Q \neq \boldsymbol{\tau}_Q$, but $\dot{\mathbf{L}}_Q = \boldsymbol{\tau}_Q$ under any of the following three circumstances:

i. $\mathbf{r}'_Q = 0$ - that is, Q coincides with C.
ii. $\ddot{\mathbf{r}}_Q = 0$ - that is, Q is not accelerating.
iii. $\ddot{\mathbf{r}}_Q$ and $\mathbf{r}'_Q$ are parallel, which would happen, for example, if O were a centre of attraction or repulsion and Q were accelerating towards or away from O.