1.1: Introduction and Some Definitions

This chapter deals with the calculation of the positions of the centres of mass of various bodies. We start with a brief explanation of the meaning of centre of mass, centre of gravity and centroid, and a very few brief sentences on their physical significance. Many students will have seen the use of calculus in calculating the positions of centres of mass, and we do this for

Plane areas

i for which the equation is given in $x-y$ coordinates;

ii for which the equation is given in polar coordinates.

Plane curves

i for which the equation is given in $x-y$ coordinates;

ii for which the equation is given in polar coordinates.

Three-dimensional figures such as solid and hollow hemispheres and cones.

There are some figures for which interesting geometric derivations can be done without calculus; for example, triangular laminas, and solid tetrahedra, pyramids and cones. And the theorems of Pappus allows you to find the centres of mass of semicircular laminas and arcs in your head with no calculus.

First, some definitions.

Consider several point masses in the $x-y$ plane:

$$m_1 \text{ at } (x_1, y_1)$$

$$m_2 \text{ at } (x_2, y_2)$$

etc.

The centre of mass is a point $(\bar{x}, \bar{y})$ whose coordinates are defined by

$$\bar{x} = \frac{\sum m_i x_i}{M} \qquad \bar{y} = \frac{\sum m_i y_i}{M} \tag{1.1.1}$$

where M is the total mass $\sum m_i$. The sum $\sum m_i x_i$ is the first moment of mass with respect to the y axis. The sum $\sum m_i y_i$ is the first moment of mass with respect to the x axis.

If the masses are distributed in three-dimensional space, with m_1 at $(x_1, y_1, z_1\ )$, etc,. the centre of mass is a point ($\bar{x}, \bar{y}, \bar{z}$) such that

$$\bar{x} = \frac{\sum m_i x_i}{M} \qquad \bar{y} = \frac{\sum m_i y_i}{M} \qquad \bar{y} = \frac{\sum m_i z_i}{M} \tag{1.1.2}$$

In this case, $\sum m_i x_i$, $\sum m_i y_i$, $\sum m_i z_i$ are the first moments of mass with respect to the $y-z, z-x$ and $x-y$ planes respectively.

In either case we can use vector notation and suppose that $\mathbf{r_1, r_2, r_3}$ are the position vectors of m_1, m_2, m_3 with respect to the origin, and the centre of mass is a point whose position vector $\bar{\mathbf{r}}$ is defined by

$$\bar{r} = \frac{\sum m_i \mathbf{r_i}}{M} \tag{1.1.3}$$

In this case the sum is a vector sum and $\sum m_i \mathbf{r_i}$ a vector quantity, is the first moment of mass with respect to the origin. Its scalar components in the two-dimensional case are the moments with respect to the axes; in the three dimensional case they are the moments with respect to the planes.

Many early books, and some contemporary ones, use the term "centre of gravity". Strictly the centre of gravity is a point whose position is defined by the ratio of the first moment of weight to the total weight. This will be identical to the centre of mass provided that the strength of the gravitational field g (or gravitational acceleration) is the same throughout the space in which the masses are situated. This is usually the case, though it need not necessarily be so in some contexts.

For a plane geometrical figure, the centroid or centre of area, is a point whose position is defined as the ratio of the first moment of area to the total area. This will be the same as the position of the centre of mass of a plane lamina of the same size and shape provided that the lamina is of uniform surface density.

Calculating the position of the centre of mass of various figures could be considered as merely a make-work mathematical exercise. However, the centres of gravity, mass and area have important applications in the study of mechanics.

For example, most students at one time or another have done problems in static equilibrium, such as a ladder leaning against a wall. They will have dutifully drawn vectors indicating the forces on the ladder at the ground and at the wall, and a vector indicating the weight of the ladder. They will have drawn this as a single arrow at the centre of gravity of the ladder as if the entire weight of the ladder could be "considered to act" at the centre of gravity. In what sense can we take this liberty and "consider all the weight as if it were concentrated at the centre of gravity"? In fact the ladder consists of many point masses (atoms) all along its length. One of the equilibrium conditions is that there is no net torque on the ladder. The definition of the centre of gravity is such that the sum of the moments of the weights of all the atoms about the base of the ladder is equal to the total weight times the horizontal distance to the centre of gravity, and it is in that sense that all the weight "can be considered to act" there. Incidentally, in this example, "centre of gravity" is the correct term to use. The distinction would be important if the ladder were in a nonuniform gravitational field.

In dynamics, the total linear momentum of a system of particles is equal to the total mass times the velocity of the centre of mass. This may be "obvious", but it requires formal proof, albeit one that follows very quickly from the definition of the centre of mass.

Likewise the kinetic energy of a rigid body in two dimensions equals $\frac{1}{2}MV^2 + \frac{1}{2}I\omega^2$ where M is the total mass, V the speed of the centre of mass, I the rotational inertia and ω the angular speed, both around the centre of mass. Again it requires formal proof, but in any case it furnishes us with another example to show that the calculation of the positions of centres of mass is more than merely a make-work mathematical exercise and that it has some physical significance.

If a vertical surface is immersed under water (e.g. a dam wall) it can be shown that the total hydrostatic force on the vertical surface is equal to the area times the pressure at the centroid. This requires proof (readily deduced from the definition of the centroid and elementary hydrostatic principles), but it is another example of a physical application of knowing the position of the centroid.
