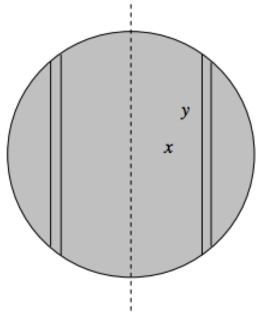
2.6: Three-dimensional Solid Figures. Spheres, Cylinders, Cones.

Sphere, mass m, radius a.



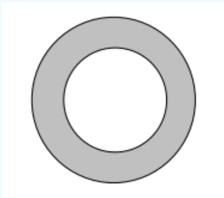
The volume of an elemental cylinder of radii x, $x+\delta x$, height $2y$ is $4\pi yx\delta x = 4\pi(a^2-x^2)^{1/2}x\delta x$. Its mass is $m\times\frac{4\pi(a^2-x^2)^{\frac{1}{2}}x\delta x}{\frac{4}{3}\pi a^3} = \frac{3m}{a^3}\times(a^2-x^2)^{\frac{1}{2}}x\delta x$. It' second moment of intertia is $=\frac{3m}{a^3}\times(a^2-x^2)^{\frac{1}{2}}x^3\delta x$. The second moment of inertia of the entire sphere is

$$=\frac{3m}{a^3}\times\int_0^a(a^2-x^2)^{\frac{1}{2}}x^3\delta x=\frac{2}{5}ma^2.$$

The moment of inertia of a uniform solid hemisphere of mass m and radius a about a diameter of its base is also , $\frac{2}{5}ma^2$, because the distribution of mass around the axis is the same as for a complete sphere.

Exercise 2.6.1

A hollow sphere is of mass M, external radius a and internal radius xa. Its rotational inertia is $0.5Ma^2$. Show that x is given by the solution of

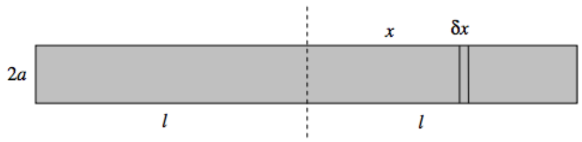


$$1-5x^3+4x^5=0$$

and calculate x to four significant figures.

(Answer = 0.6836.)

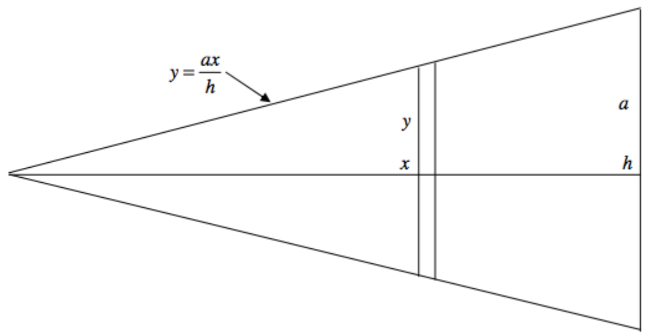
Solid cylinder, mass m, radius a, length $2l$



The mass of an elemental disc of thickness δx is $\frac{m\delta x}{2l}$. Its moment of inertia about its diameter is $\frac{1}{4}\frac{m\delta x}{2l}a^2=\frac{ma^2\delta x}{8l}$. Its moment of inertia about the dashed axis through the centre of the cylinder is $\frac{ma^2\delta x}{8l}+\frac{m\delta x}{2l}x^2=\frac{m(a^2+4x^2)\delta x}{8l}$. The moment of inertia of the entire cylinder about the dashed axis is $2\int_0^1\frac{m(a^2+4x^2)\delta x}{8l}=m(\frac{1}{4}a^2+\frac{1}{3}l^2)$.

In a similar manner it can be shown that the moment of inertia of a uniform solid triangular prism of mass m, length $2l$, cross section an equilateral triangle of side $2a$about an axis through its centre and perpendicular to its length is $m(\frac{1}{6}a^2+\frac{1}{3}l^2)$.

Solid cone, mass m, height h, base radius a.



The mass of elemental disc of thickness δx is

$$m \times \frac{\pi y^2 \delta x}{\frac{1}{3}\pi a^2 h} = \frac{3my^2\delta x}{a^2 h} \ .$$

Its second moment of inertia about the axis of the cone is

$$\frac{1}{2} \times \frac{3my^2\delta x}{a^2 h} \times y^2 = \frac{3my^4\delta x}{2a^2 h} \ .$$

But y and x are related through $y = \frac{ax}{h}$, so the moment of inertia of the elemental disk is

$$\frac{3ma^2x^4\delta x}{2h^5}.$$

The moment of inertia of the entire cone is

$$\frac{3ma^2}{2h^5}\int_0^h x^4 dx = \frac{3ma^2}{10} .$$

The following, for a solid cone of mass m, height h, base radius a, are left as an exercise: