18.3: Equation of the Catenary in Rectangular Coordinates, and Other Simple Relations

The slope at some point is $y' = \frac{dy}{dx} = \tan\psi = \frac{s}{a}$ from which $\frac{ds}{dx} = a\frac{d^2y}{dx^2}$. But, from the usual pythagorean relation between intrinsic and rectangular coordinates $ds = (1+y'^2)^{\frac{1}{2}}dx$, this becomes

$$(1+y'^2)^{\frac{1}{2}} = a\frac{dy'}{dx}. \tag{18.3.1}$$

On integration, with the condition that $y' = 0$ where $x = 0$, this becomes

$$y' = \sinh\left(\frac{x}{a}\right), \tag{18.3.2}$$

and, on further integration,

$$y = a\cosh\left(\frac{x}{a}\right) + C \tag{18.3.3}$$

If we fix the origin of coordinates so that the lowest point of the catenary is at a height a above the x-axis, this becomes

$$y = a\cosh\left(\frac{x}{a}\right) \tag{18.3.4}$$

This, then, is the x, y Equation to the catenary. The x-axis is the *directrix* of this catenary.

The following additional simple relations are easily derived and are left to the reader:

$$s = a\sinh\left(\frac{x}{a}\right) \tag{18.3.5}$$

$$y^2 = a^2 + s^2 , \tag{18.3.6}$$

$$y = a\sec\psi. \tag{18.3.7}$$

$$x = a\ln\left(\sec\psi + \tan\psi\right) \tag{18.3.8}$$

$$T = \mu g y \tag{18.3.9}$$

Equations 18.3.7 and 18.3.8 may be regarded as parametric Equations to the catenary.

If one end of the chain is fixed, and the other is looped over a smooth peg, Equation 18.3.9 shows that the loosely hanging vertical portion of the chain just reaches the directrix of the catenary, and the tension at the peg is equal to the weight of the vertical portion.

Exercise 18.3.1

By expanding Equation 18.3.4 as far as x^2, show that, near the bottom of the catenary, or for a tightly stretched catenary with a small sag, the curve is approximately a parabola. Actually, it does not matter what Equation 18.3.4 is – if you expand it as far as x^2, provided the x^2 term is not zero, you'll get a parabola – so, in order not to let you off so lightly, show that the semi latus rectum of the parabola is a.

Exercise 18.3.2

Expand Equation 18.3.5 as far as x^3.

Now: let $2s$ = total length of chain, $2k$ = total span, and d = sag. Show that for a shallow catenary $s - k = k^3/(6a^3)$ and $k^2 = 2ad$ hence that length − span = $\frac{8}{3}$ sag^2/span.

Example 18.3.1

A cord is stretched between points on the same horizontal level. How large a force must be applied so that the cord is no longer a catenary, but is accurately a straight line?

Answer:

> There is no force however great
>
> Can stretch a string however fine
>
> Into a horizontal line
>
> That shall be accurately straight.

I am indebted to Hamilton Carter of Texas A & M University for drawing my attention to a note by C. A. Chant in J. Roy. Astron. Soc. Canada 33, 72, (1939), where this doggerel is attributed to the early nineteenth century Cambridge mathematician William Whewell.

Exercise 18.3.3

And here's something for engineers. We, the general public, expect engineers to built safe bridges for us. The suspension chain of a suspension bridge, though scarcely shallow, is closer to a parabola than to a catenary. There is a reason for this. Discuss.
