2.19: Moment of Inertia with Respect to a Point

By “moment of inertia” we have hitherto meant the second moment of mass *with respect to an axis.* We were easily able to identify it with the *rotational inertia* with respect to the axis, namely the ratio of an applied torque to the resulting angular acceleration.

I am now going to define the (second) moment of inertia with respect to a point, which I shall take unless otherwise specified to mean the origin of coordinates. If we have a collection of mass points m_i at distances r_i from the origin, I define

$$\iota=\sum_i m_i r_i^2=\sum_i m_i(x_i^2+y_i^2+z_i^2) \tag{2.19.1}$$

as the (second) moment of inertia with *respect to the origin,* also sometimes called the “geometric moment of inertia”. I cannot relate it in an obvious way to a simple dynamical concept in the same way that I related moment of inertia with respect to an axis to rotational inertia, but we shall see that it is by no means merely a tedious exercise in arithmetic, and it does have its uses. The symbol *I* has probably been used rather a lot in this chapter; so to describe the geometric moment of inertia I am going to use the symbol ι.

The moment of inertia with respect to the origin is clearly something that does not depend on the orientation of any particular basis set of orthogonal axes, since it depends only on the distances of the particles from the origin.

If you recall the definitions of A, B and C from Section 2.15, you will easily see that

$$\iota=\frac{1}{2}(A+B+C) \tag{2.19.2}$$

and we already noted (see Equation 2.16.2) that $A+B+C$ is invariant under rotation of axes. In Section 2.18 we expressed it slightly more generally by saying "the trace of a symmetric matrix is invariant under an orthogonal transformation". By now it probably seems slightly less mysterious.

> *The trace of a* ***symmetric*** *matrix is invariant under an* ***orthogonal*** *transformation*

Let us now calculate the geometric moment of inertia of a uniform solid sphere of radius a, mass m, density ρ, with respect to the center of the sphere. It is

$$\iota=\int_{sphere} r^2 dm. \tag{2.19.3}$$

The element of mass, dm, here is the mass of a shell of radii $r, r+dr;$ that is $4\pi\rho r2dr$. Thus

$$\iota=4\pi\rho\int_0^a r^4 dr=\frac{4}{5}\pi\rho a^5. \tag{2.19.4}$$

With $m=\frac{4}{3}\pi a^3\rho$, this becomes

$$\iota=\frac{3}{5}ma^2. \tag{2.19.5}$$

Indeed, for any *spherically symmetric distribution of matter,* since $A=B=C$, it will be clear from Equation 2.19.2, that *the moment of inertia with respect to the center is 3/2 times the moment of inertia with respect to an axis through the center.* For example, it is obvious from the definition of moment of inertia with respect to the center that for a hollow spherical shell it is just Ma^2, and therefore the moment of inertia with respect to an axis through the center is $\frac{2}{3}ma^2$. In other words, you can work out that the moment of inertia of a hollow spherical shell with respect to an axis through its center is $\frac{2}{3}ma^2$ in your head without any of the integration that we did in Section 2.7!

By way of illustration, consider three spheres, each of radius a and mass M, but the density between center and surface varies as

$$\rho=\rho_0(1-\frac{kr}{a}),\quad \rho=\rho_0(1-\frac{kr^2}{a^2}),\quad \rho=\rho_0\sqrt{1-\frac{kr^2}{a^2}}$$

for the three spheres.

Example 2.19.1

Calculate for each the moment of inertia about an axis through the center of the sphere. Express the answer in the form $\frac{2}{5}Ma^2 \times f(k)$.

Solution

The mass of a sphere is

$$M = 4\pi \int_0^a \rho(r) r^2 dr$$

and so

$$\frac{2}{5}Ma^2 = \frac{8\pi a^2}{5} \int_0^a \rho(r) r^2 dr$$

The moment of inertia about the center is

$$\iota = 4\pi \int_0^a \rho(r) r^4 dr.$$

and so the moment of inertia about an axis through the center is

$$I = \frac{8}{3} \int_0^a \rho(r) r^4 dr.$$

Therefore

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{5\int_o^a \rho(r) r^4 dr}{3a^2 \int_o^a \rho(r) r^2 dr}$$

For the first two spheres the integrations are straightforward. I make it

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{12 - 10k}{12 - 9k}$$

for the first sphere, and

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{35 - 25k}{35 - 21k}$$

for the second sphere. The integrations for the third sphere need a little more patience, but I make the answer

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{5(12\alpha - 3\sin 2\alpha - 3\sin 4\alpha + \sin 6\alpha)}{18\sin^2\alpha(4\alpha - \sin 4\alpha)}$$

where $\sin\alpha = \sqrt{k}$.

Example 2.19.1 should be enough to convince that the concept of ι is useful – but it is not its only use. We shall meet it again in Chapter 3 on the dynamics of systems of particles; in particular, it will play a role in what we shall become familiar with as the virial theorem.