13.4: The Lagrangian Equations of Motion

This section might be tough – but do not be put off by it. I promise that, after we have got over this section, things will be easy. But in this section I do not like all these summations and subscripts any more than you do.

Suppose that we have a system of N particles, and that the force on the ith particle ($i = 1$ to N) is $\mathbf{F_i}$. If the ith particle undergoes a displacement $\delta \mathbf{r_i}$, the total work done on the system is $\sum_i \mathbf{F_i} \cdot \partial \mathbf{r_i}$. The position vector $\mathbf{r}$ of a particle can be written as a function of its generalized coordinates; and a change in $\mathbf{r}$ can be expressed in terms of the changes in the generalized coordinates. Thus the total work done on the system is

$$\sum_i \mathbf{F_i} \cdot \sum_\mathbf{j} \frac{\partial \mathbf{r_i}}{\partial \mathbf{q_j}} \delta \mathbf{q_j} \tag{13.4.1}$$

which can be written

$$\sum_j \sum_i \mathbf{F_i} \cdot \frac{\partial \mathbf{r_i}}{\partial \mathbf{q_j}} \delta \mathbf{q_j}. \tag{13.4.2}$$

But by definition of the generalized force, the work done on the system is also

$$\sum_j P_j \cdot \delta q_j. \tag{13.4.3}$$

Thus the generalized force P_j associated with generalized coordinate q_j is given by

$$P_j = \sum_i \mathbf{F_i} \cdot \frac{\partial \mathbf{r_i}}{\partial \mathbf{q_j}}. \tag{13.4.4}$$

Now $\mathbf{F_i} = \mathbf{m_i} \ddot{\mathbf{r}}_\mathbf{i}$, so that

$$P_j = \sum_i m_i \ddot{r}_i \cdot \frac{\partial \mathbf{r_i}}{\partial q_j}. \tag{13.4.5}$$

Also

$$\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r_i}}{\partial q_j}\right) = \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r_i}}{\partial q_j} + \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r_i}}{\partial q_j}\right). \tag{13.4.6}$$

Substitute for $\ddot{\mathbf{r}}_i \cdot \dfrac{\partial \mathbf{r_i}}{\partial q_j}$ from Equation 13.4.6 into Equation 13.4.5 to obtain

$$P_j = \sum_i m_i \left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r_i}}{\partial q_j}\right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r_i}}{\partial q_j}\right)\right]. \tag{13.4.7}$$

Now

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} \tag{13.4.8}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}. \tag{13.4.9}$$

Therefore

$$P_j = \sum_i m_i \left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}\right) - \dot{\mathbf{r}}_i \cdot \left(\frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}\right)\right] \tag{13.4.10}$$

> You may not be immediately comfortable with the assertions in Equations 13.4.8 and 13.4.9 so I'll interrupt the flow briefly here with an example to try to justify these assertions and to understand what they mean.

Consider the relation between the coordinate x and the spherical coordinates r, θ, ϕ:

$$x = r \sin\theta \cos\phi \tag{A1}$$

In this example, x would correspond to one of the components of $\mathbf{r_i}$, and r, θ, ϕ are the q_1, q_2, q_3.

From Equation A1, we easily derive

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \tag{A2.1}$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \tag{A2.2}$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \tag{A2.3}$$

and differentiating Equation A1 with respect to time, we obtain

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\dot{\theta}\cos\phi - r\sin\theta\sin\phi\dot{\phi} \tag{A3}$$

And from this we see that

$$\frac{\partial \dot{x}}{\partial \dot{r}} = \sin\theta\cos\phi \tag{A4.1}$$

$$\frac{\partial \dot{x}}{\partial \dot{\theta}} = r\cos\theta\cos\phi \tag{A4.2}$$

$$\frac{\partial \dot{x}}{\partial \dot{\phi}} = -r\sin\theta\sin\phi \tag{A4.3}$$

Thus the first assertion is justified in this example, and I think you'll see that it will always be true no matter what the functional dependence of $\mathbf{r_i}$ on the q_j.

For the second assertion, consider

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \tag{A5}$$

and hence

$$\frac{d}{dt}\frac{\partial x}{\partial r} = \cos\theta\dot{\theta}\cos\phi - \sin\theta\sin\phi\dot{\phi}. \tag{A6}$$

From Equation A3 we find that

$$\frac{\partial \dot{x}}{\partial r} = \cos\theta\dot{\theta}\cos\phi - \sin\theta\sin\phi\dot{\phi}, \tag{A7}$$

and the second assertion is justified. Again, I think you'll see that it will always be true no matter what the functional dependence of $\mathbf{r_i}$ on the q_j.

The kinetic energy T is

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \tag{13.4.11}$$

Therefore

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \tag{13.4.12}$$

and

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}. \tag{13.4.13}$$

On substituting these in Equation 13.4.10we obtain

$$P_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}. \tag{13.4.14}$$

This is one form of Lagrange's equation of motion, and it often helps us to answer the question posed in the last sentence of Section 13.2 – namely to determine the generalized force associated with a given generalized coordinate.

Conservative Forces

If the various forces in a particular problem are **conservative** (gravity, springs and stretched strings, including valence bonds in a molecule) then the generalized force can be obtained by the negative of the gradient of a potential energy function – i.e. $P_j = -\frac{\partial V}{\partial q_j}$. In that case, Lagrange's equation takes the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}. \tag{13.4.15}$$

In my experience, this is the most useful and most often encountered version of Lagrange's equation.

The quantity $L = T - V$ is known as the **lagrangian** for the system, and Lagrange's equation can then be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0. \tag{13.4.16}$$

This form of the equation is seen more often in theoretical discussions than in the practical solution of problems. It does enable us to see one important result. If, for one of the generalized coordinates, $\frac{\partial L}{\partial q_j} = 0$ (this could happen if neither T nor V depends on q_j – but of course it could also happen if $\frac{\partial T}{\partial q_j}$ and $\frac{\partial V}{\partial q_j}$ were nonzero but equal and opposite in sign), then that generalized coordinate is called an *ignorable coordinate* – presumably because one can ignore it in setting up the lagrangian. However, it does not really mean that it should be ignored altogether, because it *immediately reveals a constant* of the motion. In particular, if $\frac{\partial L}{\partial q_j}$ = 0, then $\frac{\partial L}{\partial \dot{q}_j}$ *is constant*. It will be seen that if q_j has the dimensions of length, $\frac{\partial L}{\partial \dot{q}_j}$ has the dimensions of linear momentum. And if q_i is an angle, $\frac{\partial L}{\partial \dot{q}_j}$ has the dimensions of angular momentum. The derivative $\frac{\partial L}{\partial \dot{q}_j}$ is usually given the symbol p_j and is called *the generalized momentum conjugate to the generalized coordinate* q_j. If q_j is an "ignorable coordinate", then p_j is a constant of the motion.

In each of Equations 13.4.14, 13.4.15and 13.4.16one of the qs has a dot over it. You can see which one it is by thinking about the *dimensions* of the various terms. Dot has dimension T^{-1}.

So, we have now derived Lagrange's equation of motion. It was a hard struggle, and in the end we obtained three versions of an equation which at present look quite useless. But from this point, things become easier and we rapidly see how to use the equations and find that they are indeed very useful.