1.4: Plane Curves

Plane Curves Expressed in $x-y$ coordinates

FIGURE I.7

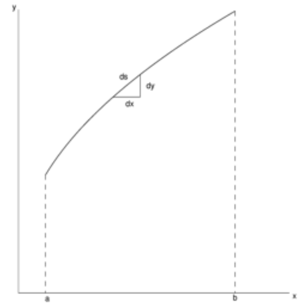

Figure I.7 shows how an elemental length δs is related to the corresponding increments in x and y:

$$\delta s=\sqrt{\delta x^{2}+\delta y^{2}}=\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,\delta x=\sqrt{\left(\frac{dx}{dy}\right)^{2}+1}\,dy \tag{1.4.1}$$

Consider a wire of mass per unit length (linear density) λ bent into the shape $y=y(x)$ between $x=a$ and $x=b$. The mass of an element ds is $\lambda\delta s$, so the total mass is

$$\int\lambda\,ds=\int_{a}^{b}\lambda\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx \tag{1.4.2}$$

The first moments of mass about the y - and x -axes are respectively

$$\int_{a}^{b}\lambda x\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx \tag{1.4.3}$$

and

$$\int_{a}^{b}\lambda y\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx \tag{1.4.4}$$

If the wire is uniform and λ is therefore not a function of x or y, λ can come outside the integral signs in Equations 1.4.2 - 1.4.4, and we hence obtain

$$\bar{x}=\frac{\int_{a}^{b}x\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx}{\int_{a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx} \tag{1.4.5}$$

and

$$\bar{y}=\frac{\int_{a}^{b}y\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx}{\int_{a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx} \tag{1.4.6}$$

the denominator in each of these expressions merely being the total length of the wire.

Example 1.4.1

Consider a uniform wire bent into the shape of the semicircle $x^2 + y^2 = a^2$, $x > 0$.

First, it might be noted that one would expect $\overline{x} > 0.4244a$ (the value for a plane semicircular lamina).

The length (i.e. the denominators in Equations 1.4.5 and 1.4.6) is just πa. Since there are, between x and $x + \delta x$, two elemental lengths to account for, one above and one below the x axis, the numerator of Equation 1.4.5 must be

$$2\int_0^a x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

In this case

$$y = \sqrt{a^2 - x^2}$$

and

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$$

The first moment of length of the entire semicircle is

$$\overline{x} = 2\int_0^a x\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx = 2a\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}}$$

From this point the student is left to his or her own devices to solve this integral and derive $\overline{x} = \frac{2a}{\pi} = 0.6366a$.

Plane Curves Expressed in Polar Coordinates

FIGURE I.8

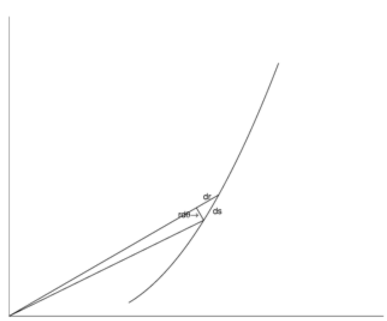

Figure I.8 shows how an elemental length δs is related to the corresponding increments in r and θ:

$$\delta s = \sqrt{(\delta r)^2 + (r\delta\theta)^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,\delta\theta = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\,\delta r. \tag{1.4.7}$$

The mass of the curve (between $\theta = a$ and $\theta = b$) is

$$\int_\alpha^\beta \lambda\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta.$$

The first moments about the y - and x -axes are (recalling that $x = r\cos\theta$ and $y = r\sin\theta$)

$$\int_\alpha^\beta \lambda r\cos\theta\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta$$

and

$$\int_{\alpha}^{\beta} \lambda r \sin\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta.$$

If λ is not a function of r or θ , we obtain

$$\bar{x} = \frac{1}{L} \int_{\alpha}^{\beta} r \cos\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \tag{1.4.8}$$

and

$$\bar{y} = \frac{1}{L} \int_{\alpha}^{\beta} r \sin\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \tag{1.4.9}$$

where L is the length of the wire.

Example 1.4.2

Again consider the uniform wire of Figure I.8 bent into the shape of a semicircle. The equation in polar coordinates is simply $r = a$, and the integration limits are $\theta = \frac{-\pi}{2}$ to $\theta = \frac{+\pi}{2}$ and the length is πa.

Thus

$$\bar{x} = \frac{1}{\pi a} \int_{-\pi/2}^{+\pi/2} acos\theta [0 - a^2]^{\frac{1}{2}}\, d\theta = \frac{2a}{\pi}.$$

The reader should now find the position of the center of mass of a wire bent into the arc of a circle of angle 2α. The expression obtained should go to $\frac{2a}{\pi}$ as α goes to $\frac{\pi}{2}$, and to a as α goes to zero.