19.5: Motion on a Cycloid, Cusps Up

We shall imagine either a particle sliding down the inside of a smooth cycloidal bowl, or a bead sliding down a smooth cycloidal wire, Figure XIX.6.

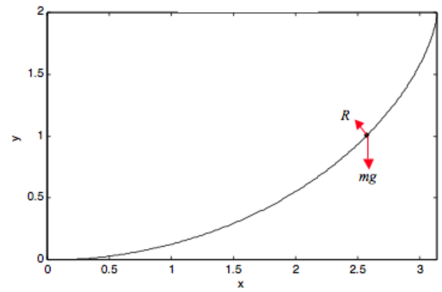

FIGURE XIX.6

We shall work in intrinsic coordinates to obtain the tangential and normal Equations of motion. These Equations are, respectively:

$$\ddot{s} = -g\sin\psi \tag{19.5.1}$$

and

$$\frac{mv^2}{\rho} = R - mg\cos\psi. \tag{19.5.2}$$

Here R is the normal (and only) reaction of the bowl or wire on the particle and ρ is the radius of curvature. The radius of curvature is $ds/d\psi$, which, from Equation 19.3.1, (or Equations 19.4.3 and 19.4.5) is

$$\rho = 4a\cos\psi \tag{19.5.3}$$

From Equations 19.3.1 and 19.5.1 we see that the tangential Equation of motion can be written, without approximation:

$$\ddot{s} = -\frac{g}{4a}s. \tag{19.5.4}$$

This is simple harmonic motion of period $4\pi\sqrt{a/g}$, independent of the amplitude of the motion. This is the *isochronous* property of the cycloid. Likewise, if the particle is released from rest, it will reach the bottom of the cycloid in a time $\pi\sqrt{a/g}$ whatever the starting position.

Let us see if we can find the value of R where the generating angle is ψ. Let us suppose that the particle is released from rest at a height y_0 above the x-axis (generating angle = ψ_0); what is its speed v when it has reached a height y (generating angle ψ)? Clearly this is given by

$$\frac{1}{2}mv^2 = mg(y_0 - y), \tag{19.5.5}$$

and, following Equation 19.3.2, and recalling that $\theta = \psi$, this is

$$v^2 = 2ga(cos2\psi - cos2\psi_0). \tag{19.5.6}$$

On substituting this and Equation 19.5.3 into Equation 19.5.2, we find for R:

$$R = \frac{mg}{2\cos\psi}(1 + 2\cos 2\psi - \cos 2\psi_0) \tag{19.5.7}$$