11.5ii: Heavy damping- γ>2 ω 0 γ>2ω0

The motion is given by Equations 11.5.4 and 11.5.6 where, this time, k_1 and k_2 are each real and negative. For convenience, I am going to write $\lambda_1 = -k_1$ and $\lambda_2 = -k_2$. λ_1 are λ_2 both real and positive, with $\lambda_2 > \lambda_1$ given by

$$\lambda_1 = \frac{1}{2}\gamma - \sqrt{(\frac{1}{2}\gamma)^2 - \omega_0^2}, \quad \lambda_2 = \frac{1}{2}\gamma + \sqrt{(\frac{1}{2}\gamma)^2 - \omega_0^2} \tag{11.5.19}$$

The general solution for the displacement as a function of time is

$$x = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t}. \tag{11.5.20}$$

The speed is given by

$$\dot{x} = -A\lambda_1 e^{-\lambda_1 t} - B\lambda_2 e^{-\lambda_2 t}. \tag{11.5.21}$$

The constants A and B depend on the initial conditions. Thus:

$$x_0 = A + B \tag{11.5.22}$$

and

$$(\dot{x})_0 = -(A\lambda_1 + B\lambda_2). \tag{11.5.23}$$

From these, we obtain

$$A = \frac{(\dot{x})_0 + \lambda_2 x_0}{\lambda_2 - \lambda_1}, \qquad B = -[\frac{(\dot{x})_0 + \lambda_1 x_0}{\lambda_2 - \lambda_1}]. \tag{11.5.24}$$

Example $11.5ii.1$

$x_0 \neq 0, \quad (\dot{x})_0 = 0.$

$$x = \frac{x_0}{\lambda_2 - \lambda_1}(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}). \tag{11.5.25}$$

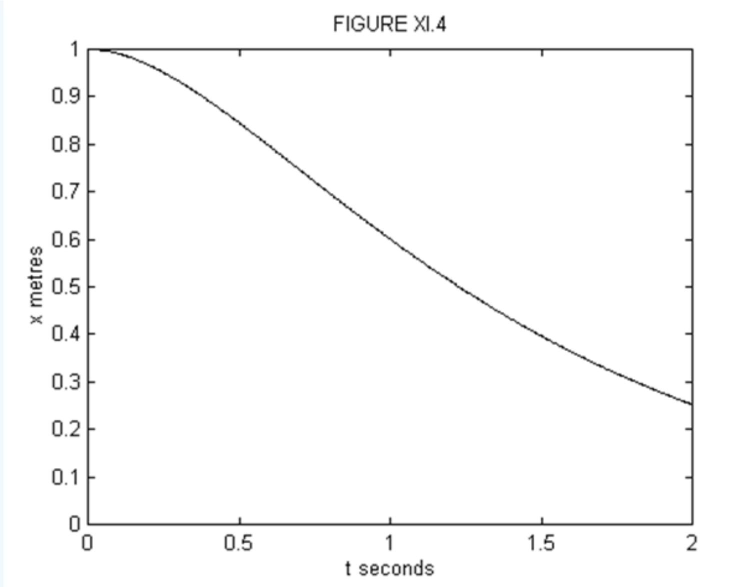
Figure XI.4 shows $x \quad : \quad t$ for x_0 = 1 m, λ_1 = 1 s^{-1}, λ_2 = 2 s^{-1}.

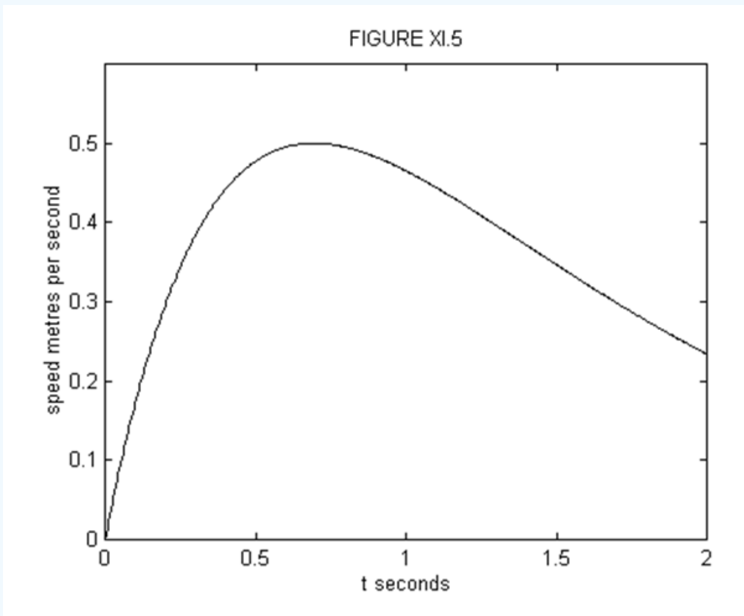
The displacement will fall to half of its initial value at a time given by putting $\frac{x}{x_0} = \frac{1}{2}$ in Equation 11.5.25. This will in general require a numerical solution. In our example, however, the equation reduces to $\frac{1}{2} = 2e^{-t} - e^{-2t}$ and if we let $u = e^{-t}$, this becomes $u^2 - 2u + \frac{1}{2} = 0$. The two solutions of this are $u = 1.707107$ or 0.292893 The first of these gives a negative t, so we want the second solution, which corresponds to $t = 1.228$ seconds.

The velocity as a function of time is given by

$$\dot{x} = -\frac{\lambda_1\lambda_2 x_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{\lambda_2 t}). \tag{11.5.26a}$$

This is always negative. In figure XI.5, is shown the speed, which is $|\dot{x}|$ as a function of time, for our numerical example. Those who enjoy differentiating can show that the maximum speed is reached in a time $-\dot{x}$ and that the maximum speed is $\frac{\lambda_1\lambda_2 x_0}{\lambda_2-\lambda_1}[(\frac{\lambda_1}{\lambda_2})^{\frac{\lambda_2}{\lambda_2-\lambda_1}} - (\frac{\lambda_1}{\lambda_2})^{\frac{\lambda_1}{\lambda_2-\lambda_1}}]$. (Are these dimensionally correct?) In our example, the maximum speed, reached at $t = \ln 2 = 0.6931$ seconds, is 0.5 m s^{-1}.





Example $11.5ii.2$

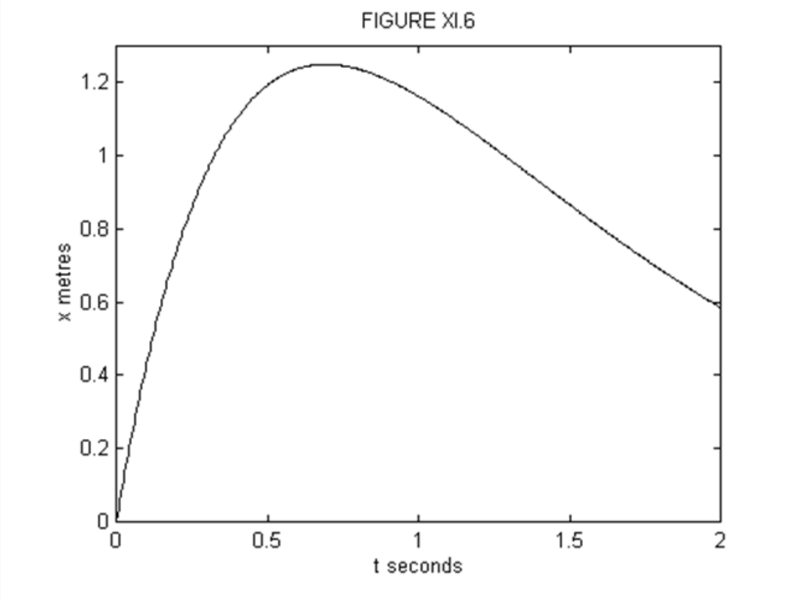
$x_0 = 0, \quad (\dot{x})_0 = V(>0)$.

In this case it is easy to show that

$$x = \frac{V}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}). \tag{11.5.26b}$$

It is left as an exercise to show that x reaches a maximum value of $\frac{V}{\lambda_2}(\frac{\lambda_1}{\lambda_2})^{\frac{\lambda_1}{\lambda_2-\lambda_1}}$ when $t = \frac{\ln(\frac{\lambda_2}{\lambda_1})}{\lambda_2-\lambda_1}$. Figure XI.6 illustrates Equation 11.5.26a for λ_1 = 1 s-1, λ_2 = 2 s-1, V = 5 m s-1. The maximum displacement of 1.25 m is reached when $t = \ln 2 = 0.6831$ s. It is also left as an exercise to show that equation 11.5.26acan be written

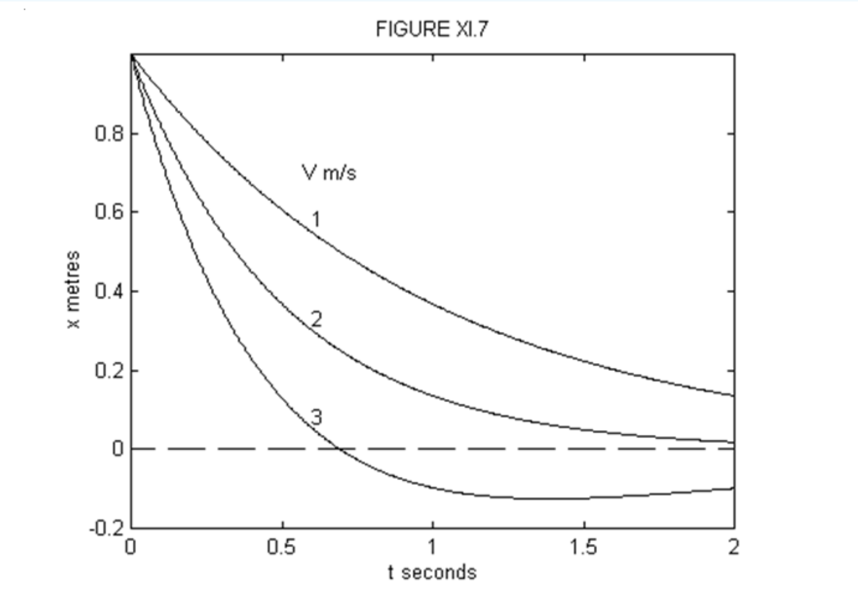
$$x = \frac{2Ve^{-\frac{1}{2}\lambda t}}{\lambda_2 - \lambda_1}\sinh(\frac{1}{4}\gamma^2 - \omega_0^2). \tag{11.5.27}$$



Example 11.5ii.3

$x_0 \neq 0, \quad (\dot{x})_0 = -V$.

This is the really exciting example, because the suspense-filled question is whether the particle will shoot past the origin at some finite time and then fall back to the origin; or whether it will merely tamely fall down asymptotically to the origin without ever crossing it. The tension will be almost unbearable as we find out. In fact, I cannot wait; I am going to plot x versus t in figure XI.7 for λ_1 = 1 s^{-1}, λ_2 = 2 s $^{-1}$, x_0 = 1 m, and three different values of V, namely 1, 2 and 3 m s^{-1}.



We see that if V = 3 m s^{-1} the particle overshoots the origin after about 0.7 seconds. If V = 1 m s^{-1}, it does not look as though it will ever reach the origin. And if V = 2 m s^{-1}, I'm not sure. Let's see what we can do. We can find out when it crosses the origin by putting x= 0 in Equation 11.5.20, where A and B are found from Equations 11.5.24with $(\dot{x})_0 = -V$. This gives, for the time when it crosses the origin,

$$t = \frac{1}{\lambda_2 - \lambda_1} \ln(\frac{V - \lambda_1 x_0}{V - \lambda_2 x_0}). \tag{11.5.28}$$

Since $\lambda_2 > \lambda_1$, this implies that the particle will overshoot the origin if $V > \lambda_2 x_0$, and this in turn implies that, for a given V, it will overshoot only if

$$\gamma < \frac{\frac{V^2}{x_0^2} + \omega_0^2}{\frac{V}{x_0}}. \tag{11.5.29}$$

For our example, $\lambda_2 x_0$= 2 m s^{-1}, so that it just fails to overshoot the origin if V= 2 m s^{-1}. For V = 3 m s^{-1}, it crosses the origin at $t = \ln 2 = 0.6931$ s. In order to find out how far past the origin it goes, and when, we can do this just as in

I make it that it reaches its maximum negative displacement of -0.125 m at $t = \ln 4 = 1.386$ s.