4.2: Angular Velocity and Eulerian Angles

Let $Oxyz$ be a set of space-fixed axis, and let $Ox_0y_0z_0$ be the body-fixed principal axes of a rigid body.

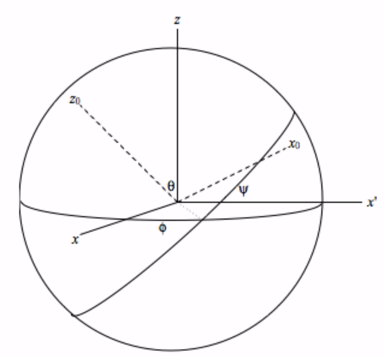

FIGURE IV.1a

The orientation of the body-fixed principal axes $Ox_0y_0z_0$ with respect to the space-fixed axes $Oxyz$ can be described by the three Euler angles: θ, ϕ, and ψ. These are illustrated in Figure IV.1a. Those who are not familiar with Euler angles or who would like a reminder can refer to their detailed description in Chapter 3 of my notes on Celestial Mechanics.

We are going to examine the motion of a body that is rotating about a non-principal axis. If the body is freely rotating in space with no external torques acting upon it, its angular momentum $\mathbf{L}$ will be constant in magnitude and direction. The angular velocity vector ω, however, will not be constant, but will wander with respect to both the space-fixed and body-fixed axes, and we shall be examining this motion. I am going to call the instantaneous components of ω relative to the body-fixed axes $\omega_1, \omega_2, \omega_3$, and its magnitude ω. As the body tumbles over and over, its Euler angles will be changing continuously. We are going to establish a geometrical relation between the instantaneous rates of change of the Euler angles and the instantaneous components of ω. That is, we are going to find how ω_1, ω_2 and ω_3 are related to $\dot{\theta}, \dot{\phi}$ and $\dot{\psi}$.

I have indicated, in Figure VI.2a, the angular velocities $\dot{\boldsymbol{\theta}}, \dot{\boldsymbol{\phi}}$ and $\dot{\boldsymbol{\psi}}$ as vectors in what I hope will be agreed are the appropriate directions.

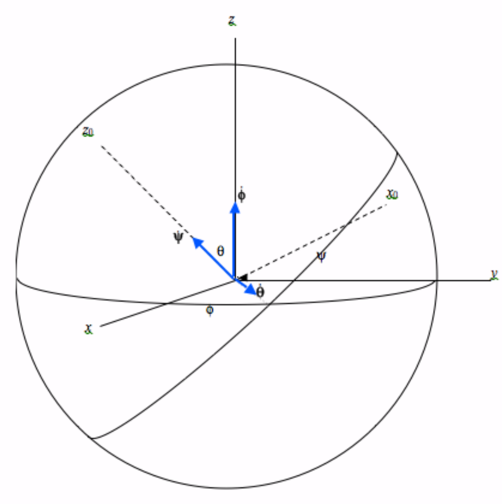

FIGURE IV.2a

It should be clear that ω_1 is equal to the x_0-component of $\dot{\boldsymbol{\phi}}$ plus the x_0-component of $\dot{\boldsymbol{\theta}}$ and that ω_2 is equal to the y_0-component of $\dot{\boldsymbol{\phi}}$ plus the y_0-component of $\dot{\boldsymbol{\theta}}$ and that ω_3 is equal to the z_0-component of $\dot{\boldsymbol{\phi}}$ plus $\dot{\boldsymbol{\psi}}$

Let us look at Figure IV.2b

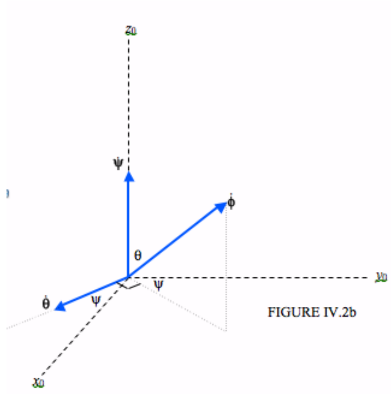

FIGURE IV.2b

We see that the x_0 and y_0 components of $\dot{\boldsymbol{\theta}}$ are $\dot{\theta}\cos\psi$ and $-\dot{\theta}\sin\psi$ respectively. The x_0, y_0 and z_0 components of $\dot{\boldsymbol{\phi}}$ are, respectively:

- $\dot{\phi}\sin\theta\sin\psi$,
- $\dot{\phi}\sin\theta\sin\psi$, and
- $\dot{\phi}\cos\theta$.

Hence we arrive at

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi. \tag{4.2.1}$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi. \tag{4.2.2}$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \tag{4.2.3}$$