11.1: Simple Harmonic Motion

I am assuming that this is by no means the first occasion on which the reader has met simple harmonic motion, and hence in this section I merely summarize the familiar formulas without spending time on numerous elementary examples

Simple harmonic motion can be defined as follows: It a point P moves in a circle of radius a at constant angular speed ω (and hence period $\frac{2\pi}{\omega}$) its projection Q on a diameter moves with simple harmonic motion. This is illustrated in Figure XI.1, in which the velocity and acceleration of P and of Q are shown as coloured arrows. The velocity of P is just $a\omega$ and its acceleration is the centripetal acceleration $a\omega^2$. As in Chapter 8 and elsewhere, I use blue arrows for velocity vectors and green for acceleration.

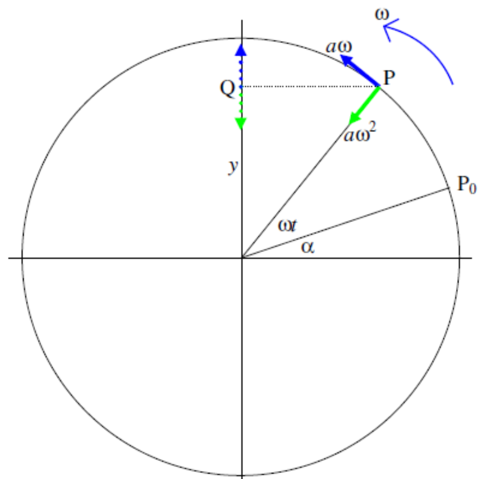

FIGURE XI.1

P_0 is the initial position of P - i.e. the position of P at time $t=0$ - and a is the *initial phase angle.* At time t later, the phase angle is $\omega t+\alpha$. The projection of P upon a diameter is Q. The displacement of Q from the origin, and its velocity and acceleration, are

$$y=a\sin(\omega t+\alpha) \tag{11.1.1}$$

$$v=\dot{y}=a\omega\cos(\omega t+\alpha) \tag{11.1.2}$$

$$\ddot{y}=-a\omega^2\sin(\omega t+\alpha). \tag{11.1.3}$$

Equations 11.1.2 and 11.1.3 can be obtained immediately either by inspection of Figure XI.1 or by differentiation of Equation 11.1.1. Elimination of the time from Equations 11.1.1 and 11.1.2 and from Equations 11.1.1 and 11.1.3 leads to

$$v=\dot{y}=\omega(a^2-y^2)^{\frac{1}{2}} \tag{11.1.4}$$

and

$$\ddot{y}=-\omega^2 y \tag{11.1.5}$$

An alternative definition of simple harmonic motion is to define as simple harmonic motion any motion that obeys the differential Equation 11.1.5. We then have the problem of solving this differential Equation. We can make no progress with this unless we remember to write $\ddot{y}$ as $v\frac{dv}{dy}$ (recall that we did this often in Chapter 6.) Equation 11.1.5 then immediately integrates to

$$v^2=\omega^2(a^2-y^2)$$

A further integration, with $v=\frac{dy}{dt}$, leads to

$$y=a\sin(\omega t+\alpha)$$

provided we remember to use the appropriate initial conditions. Differentiation with respect to time then leads to Equation 11.1.2, and all the other Equations follow.

❓ Exercise 11.1.1

Important Problem.

Show that $y = a\sin(\omega t + \alpha)$ can be written

$$y = A\sin\omega t + B\cos\omega t \tag{11.1.6}$$

where $A = a\cos\alpha$ and $B = a\sin\alpha$. The converse of these are $a = \sqrt{A^2 + B^2}$, $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2+B^2}}$. It is important to note that, if A and B are known, in order to calculate a without ambiguity of quadrant it is entirely necessary to calculate both $\cos\alpha$ and $\sin\alpha$. It will not do, for example, to calculate α solely from $\alpha = \tan^{-1}(\frac{y}{x})$ because this will give two possible solutions for a differing by 180$^{\text{o}}$.

Show also that Equation 11.1.6 can also be written

$$y = Me^{i\omega t} + Ne^{-i\omega t}, \tag{11.1.7}$$

where $M = \frac{1}{2}(B - iA)$ and $N = \frac{1}{2}(B + iA)$ show that the right hand side of Equation 11.1.7 is real.

The four large satellites of Jupiter furnish a beautiful demonstration of simple harmonic motion. Earth is almost in the plane of their orbits, so we see the motion of satellites projected on a diameter. They move to and fro in simple harmonic motion, each with different amplitude (radius of the orbit), period (and hence angular speed ω) and initial phase angle α.