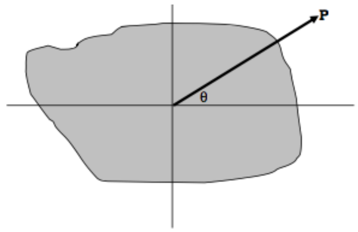
2.13: Momental Ellipse

Consider a plane lamina such that its radius of gyration about some axis through the centre of mass is k. Let $\mathbf{P}$ be a vector in the direction of that axis, originating at the centre of mass, given by

$$\mathbf{P}=\frac{a^2}{k}\hat{\mathbf{r}} \tag{2.13.1}$$



Here $\hat{\mathbf{r}}$ is a unit vector in the direction of interest; k is the radius of gyration, and a is an arbitrary length introduced so that the dimensions of $\mathbf{P}$ are those of length, and the length of the vector$\mathbf{P}$ is inversely proportional to the radius of gyration. The moment of inertia is $Mk^2=\frac{Ma^4}{P^2}$. That is to say

$$\frac{Ma^4}{P^2}=A\cos^2\theta-2H\sin\theta\cos\theta+B\sin^2\theta, \tag{2.13.2}$$

where A, H and B are the moments with respect to the x- and y-axes. Let (x,y) be the coordinates of the tip of the vector $\mathbf{P}$, so that $x=P\cos\theta$ and $y=P\sin\theta$. Then

$$Ma^4=Ax^2-2Hxy+By^2. \tag{2.13.2}$$

Thus, no matter what the shape of the lamina, however irregular and asymmetric, the tip of the vector $\mathbf{P}$ traces out an ellipse, whose axes are inclined at angles $\frac{1}{2}\tan^{-1}(\frac{2H}{B-A})$ to the x - axis.

This is the *momental ellipse*, and the axes of the momental ellipse are the principal axes of the lamina.

Example 2.13.1

Consider a regular n-gon. By symmetry the moment of inertia is the same about any two axes in the plane inclined at $2\pi/n$ to each other. This is possible only if the momental ellipse is a circle. It follows that the moment of inertia of a uniform polygonal plane lamina is the same about any axis in its plane and passing through its centroid.

Exercise 2.13.1

Show that the moment of inertia of a uniform plane n - gon of side $2a$ about any axis in its plane and passing through its centroid is $\frac{1}{12}ma^2(1+3\cot^2(\pi/n))$.

What is this for a square? For an equilateral triangle?