19.7: The Brachystochrone Property of the Cycloid

A small point. The word is sometimes spelled brachistochrone, and I have no recommendation one way or the other. For what it's worth, the only dictionary within easy reach of my desk has brachiopod and brachycephalic. In any case, the word is derived from Greek, and means shortest time.

The famous brachystochrone problem problem is this: A smooth wire, which can be of any desired length, is to connect two points O and P; P is at a lower level than O, but is not vertically below O. The wire is to be bent to a shape, and cut to a length, such that the time taken for a bead to slide down the wire from O to P is least.

It is not easy to prove that the required curve is a cusps-up cycloid; but it is quite reasonable to speculate or to guess that this might be so. And, having speculated that it might be a cycloid, it is easy to verify that the required curve is indeed a cusps-up cycloid, the bead starting from rest at a cusp of the cycloid.

A speculation might go something like this. Generally one would expect that the further P is from O, the longer it will take for the bead to slide from O to P. But, if O and P are connected with a cycloidal wire, the time taken to go from O to P does not increase with distance. (See the isochronous property of the cycloid discussed in Section 19.5.) Thus, as you increase the distance between O and P, the time taken to travel by any route other than the cycloidal one must take longer than the cycloidal route. This argument may not sound like a rigorous proof, though it is enough to arouse our suspicions and to test whether it is correct.

Since I am going to deal with a bead sliding downwards under gravity, I am going to find it convenient to set up our coordinate axes such that x increases to the right, and y increases *downwards*. In that case, the parametric equations to a cusps-up cycloid, with the origin at a cusp, are

$$x=a(2\theta-\sin 2\theta) \tag{19.7.1}$$

and

$$y=2a\sin^2\theta \tag{19.7.2}$$

− and these are the equations that we shall be testing.

The time taken for the bead to travel a distance ds along the wire, while it is moving at speed v is ds/v. In (x,y) coordinates, ds is $\sqrt{1+y'^2}\,dx$ where $y'=dy/dx$. Also, the speed reached is related (by equating the gain in kinetic energy to the loss of potential energy) to the vertical distance y dropped by $v=\sqrt{2gy}$.Thus the time taken to go from O to P is

$$\frac{1}{\sqrt{2g}}=\int_0^P \frac{\sqrt{1y'^2}}{\sqrt{y}}dx=\frac{1}{2g}\int_O^P f(y,y')dx. \tag{19.7.3}$$

This is least (see Chapter 18 for a discussion of this theorem from the calculus of variations) for a function $y(x)$ that satisfies

$$\frac{d}{dx}\frac{\partial f}{\partial y'}=\frac{\partial f}{\partial y}. \tag{19.7.4}$$

We have:

$$f=\frac{1+y'^2}{\sqrt{y}}, \tag{19.7.5}$$

$$\frac{\partial f}{\partial y}=-\frac{(1+y'^2)^{1/2}}{2y^{3/2}} \tag{19.7.6}$$

$$\frac{\partial f}{\partial y}=-\frac{y'}{y^{1/2}(1+y'^2)^{1/2}} \tag{19.7.7}$$

It is left for the reader to see whether equations 19.7.1 and 19.7.2 satisfy equation 19.7.4. You should find that both sides of the equation are equal to Thus our speculation is confirmed, and a cusps-up cycloid is indeed the curve that offers passage from O to P in the shortest time.