2.16: Rotation of Axes - Three Dimensions

Let Oxyz be one set of mutually orthogonal axes, and let O$x_1y_1z_1$ be another set of axes inclined to the first. The coordinates (x_1, y_1, z_1) of a point with respect to the second basis set are related to the coordinates (x, y, z) with respect to the first by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.16.1}$$

Here the c_{ij} are the cosines of the angles between the axes of one basis set with respect to the axes of the other. For example, c_{12} is the cosine of the angle between Ox_1 and Oy, and c_{23} is the cosine of the angles between Oy_1 and Oz.

Some readers may know how to express these cosines in terms of complicated expressions involving the *Eulerian angles*. While these are important, they are not essential for following the present development, so we shall not make use of the Eulerian angles just here.

The matrix of direction cosines is *orthogonal*. Among the several properties of an orthogonal matrix is the fact that its reciprocal (inverse) is equal to its transpose - i.e. the reciprocal of an orthogonal matrix is found merely my interchanging the rows and columns. This enables us easily to find (x, y, z) in terms of (x_1, y_1, z_1).

A number of other properties of an orthogonal matrix are useful in detecting, locating and even correcting arithmetic mistakes in computing the elements. These properties are

1. The sum of the squares of the elements in any row or column is unity. This merely expresses the fact that the magnitude of a unit vector along any of the six axes is indeed unity.
2. The sum of the products of corresponding elements of any two rows or of any two columns is zero. This merely expresses the fact that the scalar product of any two orthogonal vectors is zero. It will be noted that checking for property 1 will not detect any mistakes in sign of the elements, whereas checking for property 2 will do so.
3. Every element is equal to ± its own cofactor. This expresses the fact that the cross product of two unit orthogonal vectors is equal to the third.
4. The determinant of the matrix is ± 1. If the sign is negative, it means that the chiralities (handedness) of the two basis sets of axes are opposite; i.e. one of them is a right-handed set and the other is a left-handed set. It is usually convenient to choose both sets as right- handed.

If it is possible to find a set of axes with respect to which the product moments F, G and H are all zero, these axes are called the principal axes of the body, and the moments of inertia with respect to these axes are the principal moments of inertia, for which we shall use the notation A_0, B_0, C_0, with the convention $A_0 \leq B_0 \leq C_0$. We shall see shortly that it is indeed possible, and we shall show how to do it. A vector whose length is inversely proportional to the radius of gyration traces out in space an ellipsoid, known as the *momental ellipsoid.*

In the study of solid body rotation (whether by astronomers studying the rotation of asteroids or by chemists studying the rotation of molecules) bodies are classified as follows.

1. $A_0 \neq B_0 \neq C_0$ The ellipsoid is a triaxial ellipsoid, and the body is an *asymmetric top.*
2. $A_0 < B_0 = C_0$ The ellipsoid is a prolate spheroid and the body is a *prolate symmetric top.*
3. $A_0 = B_0 < C_0$ The ellipsoid is an oblate spheroid and the body is an *oblate symmetric top.*
4. $A_0 = B_0 = C_0$ The ellipsoid is a sphere and the body is a *spherical top.*
5. One moment is zero. The ellipsoid is an infinite elliptical cylinder, and the body is a *linear top.*

Example 2.16.1

We know from Section 2.5 that the moment of inertia of a plane square lamina of side $2a$ about an axis through its centroid and perpendicular to its area is $\frac{2}{3}ma^2$, and it will hence be obvious that the moment of inertia of a uniform solid cube of side 2a about an axis passing through the mid-points of opposite sides is also $\frac{2}{3}ma^2$. It will clearly be the same about an axis passing through the mid-points of *any* pairs of opposite sides. Therefore the cube is a spherical top and the momental ellipsoid is a sphere. Therefore the moment of inertia of a uniform solid cube about any axis through its centre (including, for example, a diagonal) is also $\frac{2}{3}ma^2$.

> **Exercise 2.16.1**
>
> What is the ratio of the length to the diameter of a uniform solid cylinder such that it is a spherical top?
>
> **Answer:**
>
> $\sqrt{3}/2 = 0.866.$

Let us note in passing that

$$A+B+C=2\sum m(x^2+y^2+z^2)=2\sum mr^2, \tag{2.16.2}$$

which is independent of the orientation of the basis axes In other words, regardless of how A, B and C may depend on the orientation of the axes with respect to the body, the sum $A+B+C$ is invariant under a rotation of axes.

We shall deal with the determination of the principal axes in Section 2.18 - but don't skip Section 2.17.
