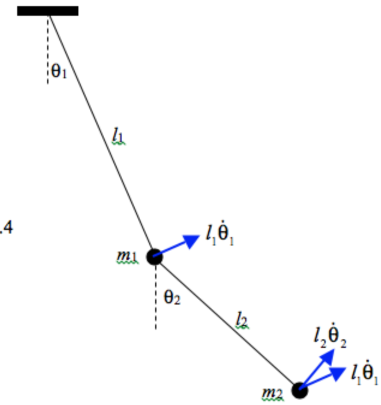
17.5: Double Pendulum

This is another similar problem, though, instead of assuming Hooke's law, we shall assume that angles are small ($\sin\theta \approx \theta, \cos\theta \approx 1 - \frac{1}{2}\theta^2$). For clarity of drawing, however, I have drawn large angles in Figure XVIII.4.

FIGURE XVII.4



Because I am going to use the lagrangian equations of motion, I have not marked in the forces and accelerations; rather, I have marked in the velocities. I hope that the two components of the velocity of m_2 that I have marked are self-explanatory; the speed of m_2 is given by $v_2^2 = l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_2-\theta_1)$. The kinetic and potential energies are

$$T = \frac{1}{2}m_1l_1^2\dot\theta_1^2 + \frac{1}{2}m_1[l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_2-\theta_1)], \tag{17.5.1}$$

$$V = constant - m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2). \tag{17.5.2}$$

If we now make the small angle approximation, these become

$$T = \frac{1}{2}m_1l_1^2\dot\theta_1^2 + \frac{1}{2}m_2(l_1\dot\theta_1 + l_2\dot\theta_2)^2 \tag{17.5.3}$$

and

$$V = contant + \frac{1}{2}m_1gl_1\theta_1^2 + \frac{1}{2}m_1g(l_1\theta_1^2 + l_2\theta_2^2) - m_1gl_1 - m_2gl_2. \tag{17.5.4}$$

Apply the lagrangian equation in turn to θ_1 and θ_2:

$$(m_1+m_2)l_1^2\ddot\theta_1 + m_2l_1l_2\ddot\theta = -(m_1+m_2)gl_1\theta_1 \tag{17.5.5}$$

and

$$m_2l_1l_2\ddot\theta_1 + m_2l_2^2\ddot\theta_2 = -m_2gl_2\theta_2. \tag{17.5.6}$$

Seek solutions in the form of $\dot\theta_1 = -\omega\theta_1$ and $\dot\theta_2 = -\omega^2\theta_2$.

Then

$$(m_1+m_2)(l_2\omega^2 - g)\theta_1 + m_2l_1l_2\omega^2\theta_2 = 0 \tag{17.5.7}$$

and

$$l_1\omega^2\theta_1 + (l_2\omega^2 - g)\theta_2 = 0. \tag{17.5.8}$$

Either of these gives the displacement ratio θ_2/θ_1. Equating the two expressions for the ratio θ_2/θ_1, or putting the determinant of the coefficients to zero, gives the following equation for the frequencies of the normal modes:

$$m_1l_1l_2\omega^4 - (m_1+m_2)g(l_1+l_2)\omega^2 + (m_1+m_2)g^2 = 0. \tag{17.5.9}$$

As in the previous examples, there is a slow in-phase mode, and fast out-of-phase mode.

For example, suppose m_1 = 0.01 kg, m_2 = 0.02 kg, l_1 = 0.3 m, l_2 = 0.6 m, g = 9.8 m s^{-2}.

Then $0.0018\omega^4 - 0.02446\omega^2 = 0$. The slow solution is ω = 3.441 rad s^{-1} (P = 1.826 s), and the fast solution is ω = 11.626 rad s^{-1} (P =0.540 s). If we put the first of these (the slow solution) in either of equations 17.5.7 or 8 (or both, as a check against mistakes) we obtain the displacement ratio θ_2/θ_1 = 1.319, which is an in-phase mode. If we put the second (the fast solution) in either equation, we obtain θ_2/θ_1 = −0.5689 , which is an out-of-phase mode. If you were to start withθ_2/θ_1 = 1.319 and let go, the pendulum would swing in the slow in-phase mode. If you were to start with θ_2/θ_1 = −0.5689 and let go, the pendulum would swing in the fast out-of-phase mode. Otherwise the motion would be a linear combination of the normal modes, with the fraction of each determined by the initial conditions, as in the example in Section 17.3.