13.5: Acceleration Components

In Section 3.4 of the Celestial Mechanics "book", I derived the radial and transverse components of velocity and acceleration in two-dimensional coordinates. The radial and transverse velocity components are fairly obvious and scarcely need derivation; they are just $\dot{\rho}$ and $\rho\dot{\phi}$. For the acceleration components I reproduce here an extract from that chapter:

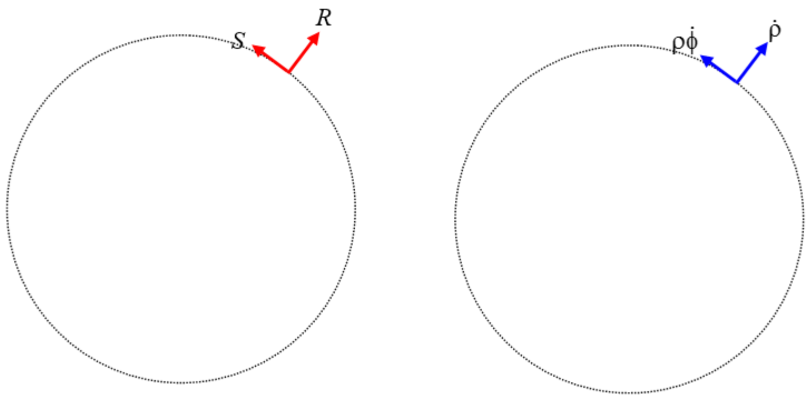
"The radial and transverse components of acceleration are therefore $(\ddot{\rho}-\rho\dot{\phi}^2)$ and $(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi})$ respectively."

I also derived the radial, meridional and azimuthal components of velocity and acceleration in three-dimensional spherical coordinates. Again the velocity components are rather obvious; they are $\dot{r}$, $r\dot{\theta}$ and $r\sin\theta\dot{\phi}$ while for the acceleration components I reproduce here the relevant extract from that chapter.

"On gathering together the coefficients of $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$ we find that the components of acceleration are:

- Radial: $\ddot{r}-r\dot{\theta}^2-r\sin^2\theta\dot{\phi}^2$
- Meridional: $r\ddot{\theta}+2\dot{r}\dot{\theta}-r\sin\theta\cos\theta\dot{\phi}^2$
- Azimuthal: $2\dot{r}\dot{\phi}\sin\theta+2r\dot{\theta}\dot{\phi}\cos\theta+r\sin\theta\ddot{\phi}$. "

You might like to look back at these derivations now. However, I am now going to derive them by a different method, using Lagrange's equation of motion. You can decide for yourself which you prefer.



We'll start in two dimensions. Let R and S be the radial and transverse components of a force acting on a particle. ("Radial" means in the direction of increasing ρ; "transverse" means in the direction of increasing ϕ.) If the radial coordinate were to increase by $\delta\rho$, the work done by the force would be just $R\delta\rho$. Thus the generalized force associated with the coordinate ρ is just $P_\rho=R$. If the azimuthal angle were to increase by $\delta\phi$, the work done by the force would be $S\rho\delta\phi$. Thus the generalized force associated with the coordinate ϕ is $P_\phi=S\rho$. Now we do not have to think about how to start; in Lagrangian mechanics, the first line is always "T= ...", and I hope you'll agree that

$$T=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\phi}^2). \tag{13.5.1}$$

If you now apply Equation 13.4.12 in turn to the coordinates ρ and ϕ, you obtain

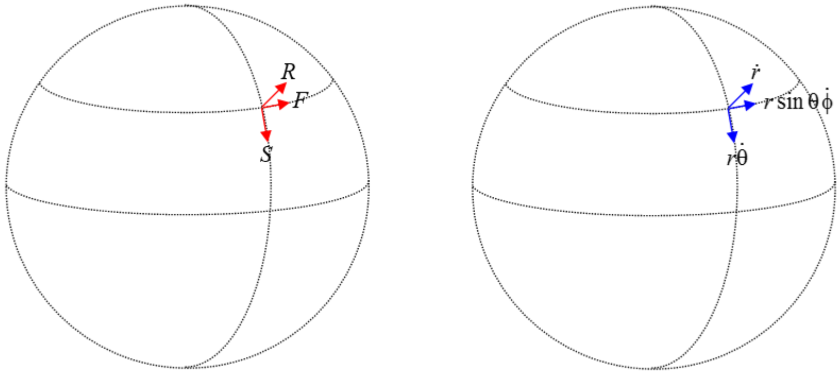
$$P_\rho=m(\ddot{\rho}-\rho\dot{\phi}^2) \quad and \quad P_\phi=m\rho(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi}), \tag{13.5.2a,b}$$

and so

$$R=m(\ddot{\rho}-\rho\dot{\phi}^2) \quad and \quad S=m(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi}). \tag{13.5.3a,b}$$

Therefore the radial and transverse components of the acceleration are $(\ddot{\rho}-\rho\dot{\phi}^2)$ and $(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi})$ respectively.

We can do exactly the same thing to find the acceleration components in three-dimensional spherical coordinates. Let R, S and F be the radial, meridional and azimuthal (i.e. in direction of increasing r, θ and ϕ) components of a force on a particle.



- If r increases by δr, the work on the particle done is $R\delta r$.
- If θ increases by $\delta\theta$, the work done on the particle is $Sr\delta\theta$.
- If ϕ increases by $\delta\phi$, the work done on the particle is $Fr\sin\theta\delta\phi$.

Therefore $P_r = R, \quad P_\theta = Sr$ and $P_\phi = Fr\sin\theta$.

Start:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \tag{13.5.4}$$

If you now apply Equation 13.4.12 in turn to the coordinates r, θ and ϕ, you obtain

$$P_r = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2), \tag{13.5.5}$$

$$P_\theta = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2) \tag{13.5.6}$$

and

$$P_\phi = m(r^2\sin^2\theta\ddot{\phi} + 2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta + 2r\dot{r}\dot{\phi}\sin^2\theta). \tag{13.5.7}$$

Therefore

$$R = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2), \tag{13.5.8}$$

$$S = m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2) \tag{13.5.9}$$

and

$$F = m(r\sin\theta\ddot{\phi} + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta). \tag{13.5.10}$$

Thus the acceleration components are

- Radial: $\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2$
- Meridional: $r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2$
- Azimuthal: $2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}$.

Be sure to check the dimensions. Since dot has dimension T^{-1}, and these expressions must have the dimensions of acceleration, there must be an r and two dots in each term.
