2.18: Determination of the Principal Axes

We now need to address ourselves to the determination of the principal axes. Unlike the two- dimensional case, we do not have a nice, simple explicit expression similar to Equation 2.12.12 to calculate the orientations of the principal axes. The determination is best done through a numerical example.

Example 2.18.1

Consider four masses whose positions and coordinates are as follows:

M	x	y	z
1	3	1	4
2	1	5	9
3	2	6	5
4	3	5	9

Relative to the first particle, the coordinates are

1	0	0	0
2	-2	4	5
3	-1	5	1
4	0	4	5

From this, it is easily found that the coordinates of the centre of mass relative to the first particle are (-0.7, 3.9, 3.3), and the moments of inertia with respect to axes through the first particle are

- $A=324$
- $B=164$
- $C=182$
- $F=135$
- $G=-23$
- $H=-31$

From the parallel axes theorems we can find the moments of inertia with respect to axes passing through the centre of mass:

- $A=63.0$
- $B=50.2$
- $C=25.0$
- $F=6.3$
- $G=0.1$
- $H=-3.7$

The inertia tensor is therefore

$$\begin{pmatrix} 63.0 & 3.7 & -0.1 \\ 3.7 & 50.2 & -6.3 \\ -0.1 & -6.3 & 25.0 \end{pmatrix}$$

We understand from what has been written previously that if $\boldsymbol{\omega}$, the instantaneous angular velocity vector, is along any of the principal axes, then $\mathbf{I}\boldsymbol{\omega}$ will be in the same direction as $\boldsymbol{\omega}$. In other words, if (l, m, n) are the direction cosines of a principal axis, then

$$\begin{pmatrix} A & -H & G \\ -H & B & -F \\ -G & -F & C \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix},$$

where λ is a scalar quantity. In other words, a vector with components l, m, n(direction cosines of a principal axis) is an eigenvector of the inertia tensor, and λ is the corresponding principal moment of inertia. There will be three eigenvectors (at right angles to each other) and three corresponding eigenvalues, which we'll initially call $\lambda_1, \lambda_2, \lambda_3$, though, as soon as we know which is the largest and which the smallest, we'll call A_0, B_0, C_0, according to our convention $A_0 \leq B_0 \leq C_0$.

The Characteristic Equation is

$$\begin{bmatrix} a-\lambda & -H & -G \\ -H & B-\lambda & -F \\ -G & -F & C-\lambda \end{bmatrix} = 0.$$

In this case, this results in the cubic equation

$$a_0 + a_1\lambda + a_2\lambda^2 - \lambda^3 = 0,$$

where

- $a_0 = 76226.44$
- $a_1 = -5939.21$
- $a_2 = 138.20$

The three solutions for λ, which we shall call A_0, B_0, C_0 in order of increasing size are

- $A_0 = 23.498256$
- $B_0 = 50.627521$
- $C_0 = 64.074223$

and these are the principal moments of inertia. From the theory of equations, we note that the sum of the roots is exactly equal to a_2, and we also note that it is equal to $A+B+C$, consistent with what we wrote in Section 2.16 (Equation 2.16.2). The sum of the diagonal elements of a matrix is known as the *trace* of the matrix. Mathematically we say that "the trace of a symmetric matrix is invariant under an orthogonal transformation".

Two other relations from the theory of equations may be used as a check on the correctness of the arithmetic. The product of the solutions equals a_0, which is also equal to the determinant of the inertia tensor, and the sum of the products taken two at a time equals $-a_1$.

We have now found the magnitudes of the principal moments of inertia; we have yet to find the direction cosines of the three principal axes. Let's start with the axis of least moment of inertia, for which the moment of inertia is $A_0 = 23.498256$. Let the direction cosines of this axis be (l_1, m_1, n_1). Since this is an eigenvector with eigenvalue 23.498 256 we must have

$$\begin{pmatrix} 63.0 & 3.7 & -0.1 \\ 3.7 & 50.2 & -6.3 \\ -0.1 & -6.3 & 25.0 \end{pmatrix} \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = 23.498256 \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$$

These are three linear equations in $l_1 m_1, n_1$, with no constant term. Because of the lack of a constant term, the theory of equations tells us that the third equation, if it is consistent with the other two, must be a linear combination of the first two. We have, in effect, only two independent equations, and we are going to need a third, independent equation if we are to solve for the three direction cosines. If we let $l' = l/n$ and $m' = m/n$, then the first two equations become

$$39.501744l' + 3.7m' - 0.1 = 0$$

$$3.7l' + 26.701744m' - 6.3 = 0.$$

The solutions are

- $l' = -0.019825485$
- $m' = +0.238686617.$

The correctness of the arithmetic can and should be checked by verifying that these solutions also satisfy the third equation.

The additional equation that we need is provided by Pythagoras's theorem, which gives for the relation between three direction cosines

$$l_1^2+m_1^2+n_1^2=1,$$

or

$$n_1^2=\frac{1}{l'^2+m'^2+1}$$

whence

$$n_1 \pm 0.972495608.$$

Thus we have, for the direction cosines of the axis corresponding to the moment of inertia A_0,

- $l_1=\mp 0.019280197$
- $m_1=\pm 0.232121881$
- $n_1=\pm 0.972495608$

(Check that $l_1^2+m_1^2+n_1^2=1.$)

It does not matter which sign you choose - after all, the principal axis goes both ways.

Similar calculations for B_0 yield

- $l_2=\pm 0.280652440$
- $m_2=\mp 0.932312706$
- $n_2=\pm 0.228094774$

and for C_0

- $l_3=\pm 0.959615796$
- $m_3=\pm 0.277330987$
- $n_3=\mp 0.047170415$

For the first two axes, it does not matter whether you choose the upper or the lower sign. For the third axes, however, in order to ensure that the principal axes form a right-handed set, choose the sign such that the determinant of the matrix of direction cosines is +1.

We have just seen that, if we know the moments and products of inertia A, B, C, F, G, H with respect to some axes (i.e. if we know the elements of the inertia tensor) we can find the principal moments of inertia $A_0, B_0, C_0$by diagonalizing the inertia tensor, or finding its eigenvalues. If, on the other hand, we know the principal moments of inertia of a system of particles (or of a solid body, which is a collection of particles), how can we find the moment of inertia I about an axis whose direction cosines with respect to the principal axes are (l, m, n) ?

First, some geometry.

Let Oxyz be a coordinate system, and let P(x, y, z) be a point whose position vector is

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$$

Let L be a straight line passing through the origin, and let the direction cosines of this line be

(l, m, n). A unit vector $\mathbf{e}$ directed along L is represented by

$$\mathbf{e}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$$

The angle θ between $\mathbf{r}$ and $\mathbf{e}$ is found from the scalar product $\mathbf{r}\cdot\mathbf{e}$, given by

$$r\cos\theta=\mathbf{r}\cdot\mathbf{e}.$$

I.e.

$$(x^2+y^2+z^2)^{\frac{1}{2}}\cos\theta = lx+my+nz$$

The perpendicular distance p from P to L is

$$p = r\sin\theta = (x^2+y^2+z^2)^{\frac{1}{2}}\sin\theta.$$

If we write $\sin\theta = (1-\cos^2\theta)^{\frac{1}{2}}$, we soon obtain

$$p^2 = x^2+y^2+z^2-(lx+my+nz)^2.$$

Noting that $l^2 = 1-m^2-n^2, m^2 = 1-n^2-l^2, n^2 = 1-l^2-m^2,$ we find, after further manipulation:

$$p^2 = l^2(y^2+z^2)+m^2(z^2+x^2)+n^2(x^2+y^2)-2(mnyz+nlzx+lmxy).$$

Now return to our collection of particles, and let Oxyz be the principal axes of the system. The moment of inertia of the system with respect to the line L is

$$I = \sum Mp^2.$$

where I have omitted a subscript i on each symbol. Making use of the expression for p and noting that the product moments of the system with respect to Oxyz are all zero, we obtain

$$I = l^2A_0+m^2B_0+n^2C_0. \tag{2.18.1}$$

Also, let A, B, C, F, G, H be the moments and products of inertia with respect to a set of nonprincipal orthogonal axes; then the moment of inertia about some other axis with direction cosines l, m, n with respect to these nonprincipal axes is

$$I = l^2A+m^2B+n^2C-2mnF-2nlG-2lmH. \tag{2.18.2}$$

Example 2.18.2: Consider a brick

We saw in Section 2.16 that the moment of inertia of a uniform solid cube of mass M and side $2a$ about a body diagonal is $\frac{2}{3}Ma^2$, and we saw how very easy this was. At that time the problem of finding the moment of inertia of a uniform solid rectangular parallelepiped of sides $2a, 2b, 2c$ must have seemed intractable, but by now it is not at all hard.

$$A_0 = \frac{1}{3}M(b^2+c^2)$$
$$B_0 = \frac{1}{3}M(c^2+a^2)$$
$$C_0 = \frac{1}{3}M(a^2+b^2)$$

Thus we have:

$$l = \frac{a}{(a^2+b^2+c^2)^{\frac{1}{2}}}$$
$$m = \frac{b}{(a^2+b^2+c^2)^{\frac{1}{2}}}$$
$$n = \frac{c}{(a^2+b^2+c^2)^{\frac{1}{2}}}$$

We obtain:

$$I = \frac{2M(b^2c^2+c^2a^2+a^2b^2)}{3(a^2+b^2+c^2)}$$

We note:

i. This is dimensionally correct;
ii. It is symmetric in a, b, c;
iii. If $a = b = c$, it reduces to $\frac{2}{3}Ma^2$.
