14.5: Poisson Brackets

Let f and g be functions of the generalized coordinates and momenta. Think first of all of one coordinate, say q_i, and its conjugate momentum p_i(defined, you may remember, as $\frac{\partial L}{\partial \dot{q}_i}$). I now ask the question: Is $\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$ the same thing as $\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$?

After thinking about it you will probably say something like: Well, I dare say that you *might be able to find* two functions such that that is so, but I do not see why it should be so for *any* two arbitrary functions. If that is what you thought, you thought right. Pairs of functions such that these two expressions are equal are of special significance. And pairs of functions such that these two expressions are *not* equal are also of special significance

The *Poisson bracket* of two functions of the coordinates and momenta is defined as

$$[f, g] \quad = \quad \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \tag{14.5.1}$$

(Poisson brackets are sometimes written as *braces* - i.e. {}. I'm not sure whether braces {} or brackets [] are the commoner. I have chosen brackets here, so that I do not have to call them Poisson braces.)

Poisson brackets have important applications in celestial mechanics and in quantum mechanics. In celestial mechanics, they are used in the developments of *Lagrange's planetary equations*, which are used to calculate the perturbations of the elements of the planetary orbits under small deviations from ideal two-body point-source orbits. See, for example, Chapter 14 of the Celestial Mechanics set of these notes. Readers who have had an introductory course in quantum mechanics may have come across the *commutator* of two operators, and will (or should!) understand the significance of two operators that commute. (It means that a function can be found that is simultaneously an eigenfunction of both operators.) You may not have thought of the commutator as being a *Poisson bracket*, but you soon will.

Let's suppose (because it does not make any essential difference) that there is just a single generalized coordinate and its conjugate generalized momentum, so that the Poisson bracket is just

$$[f, g] \quad = \quad \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{14.5.2}$$

Example 14.5.1

Now let's suppose that f is just q, the coordinate, and that g is the Hamiltonian, H, which is defined, you will recall, as $p\dot{q} - L$, and is a function of the coordinate and the momentum. What, then is the Poisson bracket $[q, H]$?

Solution

$$[q, H] \quad = \quad \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial H}{\partial q}. \tag{14.5.3}$$

The coordinate and the momentum are independent variables, so that $\frac{\partial q}{\partial p}$ is zero, so the second term on the right hand side of Equation 14.5.3is zero. In the first term on the right hand side, $\frac{\partial q}{\partial q}$ is of course 1, and $\frac{\partial H}{\partial p}$, by Hamilton's equations of motion, is $\dot{q}$. Thus, the answer is

$$[q, H] = \dot{q}. \tag{14.5.4}$$

In a similar vein, you will find (DO IT!!) that

$$[p, H] = \dot{p} \tag{14.5.5}$$

Thus neither the generalized coordinate nor the generalized momentum commutes with the Hamiltonian.

Now go a little further, and suppose that there are more than one coordinate and more than one momentum. Two will do, so that

$$[f, g] = \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} + \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2} \tag{14.5.6}$$

Exercise 14.5.1

Can you show that:

$$[p_1, p_2] = [q_1, q_2] = [p_1, q_2] = [q_1, p_2] = 0; \quad [q_1, p_1] = 1.? \tag{14.5.7}$$

I shan't go any further than that here, because it would take us too far into quantum mechanics. However, those readers who have done some introductory quantum mechanics may recall that there are various pairs of operators that do or do not commute, and may now begin to appreciate the relation between the Poisson brackets of certain pairs of observable quantities and the commutator of the operators representing these quantities. For example, consider the last of these. It shows that a coordinate such as x does not commute with its corresponding momentum p_x. There is nothing more certain that this. So certain is it that it ought to be called Heisenberg's Certainty Principle. But for some reason people often seem to present quantum mechanics as something uncertain or mysterious, whereas in reality there is nothing uncertain or mysterious about it at all.
