15.9: The FitzGerald-Lorentz Contraction

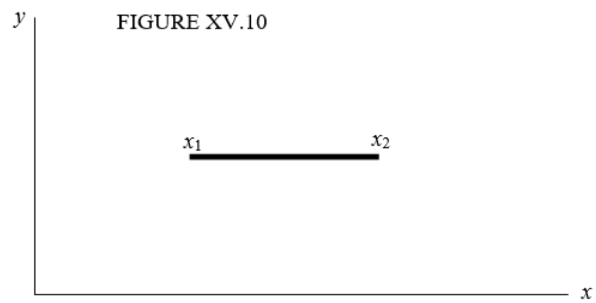
This is sometimes described in words something like the following:

If a measuring-rod is moving with respect to a "stationary" observer, it "appears" to be shorter than it "really" is.

This is not a very precise statement, and the words that I have placed in inverted commas call for some clarification.

We have seen that, while the interval between two events is invariant between reference frames, the distance between two points (and hence the length of a rod) depends on the coordinate frame to which the points are referred. Let us now define what we mean by the *length* of a rod. Figure XV.10 shows a reference frame, and a rod lying parallel to the x-axis. For the moment I am not specifying whether the rod is moving with respect to the reference frame, or whether it is stationary.

FIGURE XV.10



Let us suppose that the *x*-coordinate of the left-hand end of the rod is x_1, and that, *at the same time referred to this reference frame*, the x-coordinate of the right-hand end is x_2. The length l of the rod is defined as $l = x_2 - x_1$. That could scarcely be a simpler statement – but note the little phrase "at the same time referred to this reference frame". That simple phrase is important.

Now let's look at the FitzGerald-Lorentz contraction. See Figure XV.11.

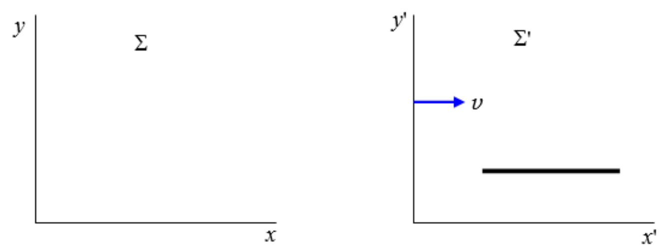


FIGURE XV.11

The are two reference frames, $\sum$ and sum'. The frame $\sum'$ is moving to the right with respect to $\sum$ with speed ν. A rod *is at rest with respect to the frame* $\sum'$, and is therefore moving to the right with respect to $\sum$ at speed ν.

In my younger days I often used to travel by train, and I still like to think of railway trains whenever I discuss relativity. Modern students usually like to think of spacecraft, presumably because they are more accustomed to this mode of travel. In the very early days of railways, it was customary for the stationmaster to wear top hat and tails. Those days are long gone, but, when thinking about the FitzGerald-Lorentz contraction, I like to think of $\sum$ as being a railway station in which there resides a stationmaster in top hat and tails, while $\sum'$ is a railway train.

The length of the rod, referred to the frame $\sum'$, is $l' = x'_2 - x'_1$, in what I hope is obvious notation, and of course these two coordinates are determined at the same time referred to $\sum'$.

The length of the rod *referred to a frame in which it is at rest* is called its *proper length.* Thus l' is the proper length of the rod.

Now it should be noted that, according to the way in which we have defined distance and time by means of the Lorentz transformation, although x'_2 and x'_1 are measured simultaneously with respect to $\sum'$, these two events (the determination of the coordinates of the two ends of the rod) are not simultaneous when referred to the frame $\sum$ (a point to which we shall return in a later section dealing with simultaneity). The length of the rod referred to the frame $\sum$ is given by $l = x_2 - x_1$, where these two coordinates are to be determined at the same time when referred to $\sum$. Now Equation 15.5.16 tells us that $x_2 = \frac{x_2'}{\gamma} + \nu t$ and $x_1 = \frac{x_1'}{\gamma} + \nu t$. (Readers should note this derivation very carefully, for it is easy to go wrong. In particular, be very clear what is meant in these two equations by the symbol t. It is the single instant of time, referred to $\sum$, when the coordinates of the two ends are determined simultaneously with respect to $\sum$.) From these we reach the result:

$$l = \frac{l'}{\gamma}. \tag{15.9.1}$$

This is the FitzGerald-Lorentz contraction.

It is sometimes described thus: A railway train of proper length 100 yards is moving past a railway station at 95% of the speed of light (γ = 3.2026.) To the stationmaster the train "appears" to be of length 31.22 yards; or the stationmaster "thinks" the length of the train is 31.22 yards; or, "according to" the stationmaster the length of the train is 31.22 yards. This gives a false impression, as though the stationmaster is under some sort of misapprehension concerning the length of the train, or as if he is labouring under some sort of illusion, and it introduces some sort of unnecessary "mystery" into what is nothing more than simple algebra. In fact what the stationmaster "thinks" or "asserts" is entirely irrelevant. Two correct statements are: 1. The length of the train, referred to a reference frame in which it is at rest – i.e. the proper length of the train – is 100 yards. 2. The length of the train when referred to a frame with respect to which it is moving at a speed of 0.95c is 31.22 yards. And that is all there is to it. Any phrase such as "this observer thinks that" or "according to this observer" should always be interpreted in this manner. It is not a matter of what an observer "thinks". It is a matter of which frame a measurement is referred to. Nothing more, nothing less.

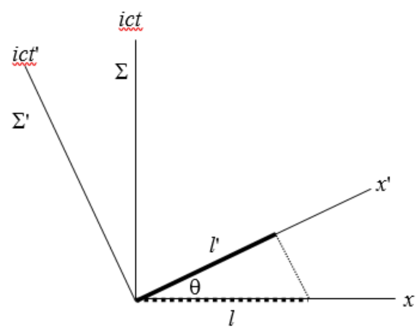


FIGURE XV.12

It is possible to describe the Lorentz-FitzGerald contraction by interpreting the Lorentz transformations as a rotation in 4-space. Whether it is helpful to do so only you can decide. Thus Figure XV.12 shows $\sum$ and $\sum'$ related by a rotation in the manner described in Section 15.7. The thick continuous line shows a rod oriented so that its two ends are drawn at the same time with respect to $\sum'$. Its length is, referred to sum', l', and this is its proper length. The thick dotted line shows the two ends at the same time with respect to $\sum$. Its length referred to $\sum$ is $l = \frac{l'}{\cos\theta}$. And, since $\cos\theta = \gamma$, which is greater than 1, this means that, in spite of appearances in the Figure, $l < l'$. The Figure is deceptive because, as discussed in Section 15.7, θ is imaginary. As I say, only you can decide whether this way of looking at the contraction is helpful or merely confusing. It is, however, at least worth looking at, because I shall be using this concept of rotation in a forthcoming section on simultaneity and order of events. Illustrating the Lorentz transformations as a rotation like this is called a *Minkowski diagram.*