9.1: Introduction

In Chapter 7 we dealt with forces on a particle that depend on the speed of the particle. In Chapter 8 we dealt with forces that depend on the time. In this chapter, we deal with forces that depend only on the position of a particle. Such forces are called *conservative forces*. While only conservative forces act, the sum of potential and kinetic energies is conserved.

Conservative forces have a number of properties. One is that the work done by a conservative force (or, what amounts to the same thing, the *line integral* of a conservative force) as it moves from one point to another is route-independent. The work done depends only on the coordinates of the beginning and end points, and not on the path taken to get from one to the other. It follows from this that the work done by a conservative force, or its line integral, round a closed path is zero. (If you are reminded here of the properties of a *function of state* in thermodynamics, all to the good.) Another property of a conservative force is that it can be derived from a potential energy function. Thus for any conservative force, there exists a scalar function $V(x, y, z)$ such that the force is equal to -**grad** V, or $-\nabla V$. In a one-dimensional situation, a sufficient condition for a force to be conservative is that it is a function of its position alone. In two- and three-dimensional situations, this is a necessary condition, but it is not a sufficient one. That a conservative force must be derivable from the gradient of a potential energy function and that its line integral around a closed path must be zero implies that the **curl** of a conservative force must be zero, and indeed a zero **curl** is a necessary and a sufficient condition for a force to be conservative.

This is all very well, but suppose you are stuck in the middle of an exam and your mind goes blank and you can't think what a line integral or a **grad** or a **curl** are, or you never did understand them in the first place, how can you tell if a force is conservative or not? Here is a rule of thumb that will almost never fail you: If the force is the tension in a stretched elastic string or spring, or the thrust in a compressed spring, or if the force is gravity or if it is an electrostatic force, the force is conservative. If it is not one of these, it is not conservative.

Example $9.1.1$

A man lifts up a basket of groceries from a table. Is the force that he exerts a conservative force?

Solution

No, it is not. The force is not the tension in a string or a spring, nor is it electrostatic. And, although he may be fighting against gravity, the force that he exerts with his muscles is not a gravitational force. Therefore it is not a conservative force. You see, he may be accelerating as he moves the basket up, in which case the force that he is exerting is greater than the weight of the groceries. If he is moving at constant speed, the force he exerts is equal to the weight of the groceries. Thus the force he exerts depends on whether he is accelerating or not; the force does not depend only on the position.

Example $9.1.2$

But you are not in an exam now, and you have ample time to remind yourself what a **curl** is. Each of the following two forces are functions of position only - a necessary condition for them to be conservative. But it is not a sufficient condition. In fact one of them is conservative and the other isn't. You will have to find out by evaluating the **curl** of each. The one that has zero **curl** is the conservative one. When you have identified it, work out the potential energy function from which it can be derived. In other words, find $V(x, y, z)$ such that $\mathbf{F}=-\mathbf{\nabla V}$.

i. $\mathbf{F}=\mathbf{(3x^2z-3y^2z)i-6xyzj+(x^3-xy^2)k}$

ii. $\mathbf{F}=\mathbf{ax^2yzi-bxy^2zj+cxyz^2k}$

When you have identified which of these forces is irrotational (i.e. has zero **curl**), you can find the potential function by calculating the work done when the force moves from the origin to (x,y,z) along any route you choose. Indeed, you might try more than one route to convince yourself that the line integral is route-independent.

One could devise many exercises in determining whether various force functions are conservative, and, if so, what the corresponding potential energy functions are, but I am going to restrict this chapter to just one more topic, namely