20.3: Shear Modulus and Torsion Constant

Imagine that we have a rectangular block of solid material, as shown on the left hand side of Figure XX.6. We now apply a couple of tangential forces F as shown on the right hand side. (I have not decided to go all chatty and informal by saying “a couple” of forces; far from it – I am using the word “couple” in its formal sense in mechanics.) The material will undergo an angular deformation, and the ratio of the tangential force per unit area to the resulting angular deformation is called the *shear modulus* or the *rigidity modulus*. Its SI unit is N m^{-2} rad^{-1} and its dimensions are $ML^{-1}T^{-2}\theta^{-1}$. (I'd advise against using “pascals” per radian. The unit “pascal” is best restricted to pressure, which is normal force per unit area, and is not quite the same thing as the tangential force per unit area that we are discussing here.) You should convince yourself that the definition must specify the force F, not the torque provided by the couple. If the block were twice as thick, and the forces were the same, you'd still get the same angular deformation.

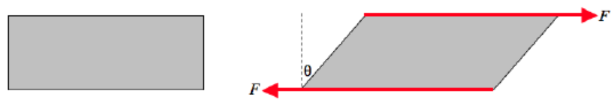

FIGURE XX.6

If you hold one end of a wire or rod fixed and apply a torque to the other end, this end will twist through an angle, and the ratio of the applied torque to the angle through which the wire twists is the *torsion constant*, *c*, of the wire. You can see how the torsion constant depends on the shear modulus η of the metal and the radius a and length l of the wire by the method of dimensions. You can start by supposing that

$$c \propto \eta^{\alpha} a^{\beta} l^{\gamma},$$

but you will soon find yourself in difficulty because a and l are each of dimension L. However, you will probably have no difficulty with making the assumption that $\gamma = -1$ (the longer the wire, the easier it is to twist), and dimensional analysis will soon show that $\alpha = 1$ and $\beta = 4$- which, being interpreted, means that it is much more difficult to twist a thick wire than a thin wire. But can we do better and get an expression other than a mere proportionality for the torsion constant? Can we find the proportionality constant? Let's try some simpler problems first, and see how things go.

Let us consider a long, thin strip or ribbon of metal. By long and thin I mean that its length is much greater than its width, and its width is much greater than its thickness. I can use any symbol I like to represent any quantity I like, so I could, if I wished, use Ξ for the length, m_{α} for the width, and G_2 for the thickness. Instead, the symbols that I shall choose to represent the length, width and thickness of the strip are going to be, respectively, $l, 2\pi r$ and δr. This seems silly at the moment, but in the end you'll be glad that I made this choice. The strip is shown at the left hand side of Figure XX.7.

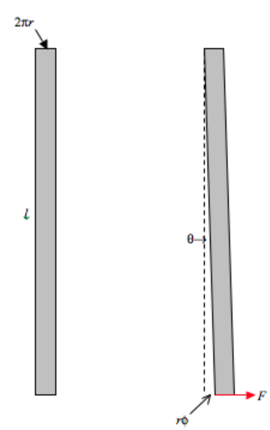

FIGURE XX.7

I am now going to fix the upper end of the strip and apply a force F to the lower end, as shown in the right hand side of Figure XX.7, and I can use any symbol I like to represent the displacement of the lower end, and I choose the symbol $r\phi$. This means that the angular displacement θ is equal to $r\phi/l$.The tangential force per unit area is $F/(2\pi r\delta r)$, and therefore

$$\eta = \frac{Fl}{2\pi\phi r^2 \delta r}, \tag{20.3.1}$$

or

$$F = \frac{2\pi\eta\phi r^2 \delta r}{l}. \tag{20.3.2}$$

Now I'm going to restore the strip to its original shape, and then I'm going to roll it into a hollow cylindrical tube, so that it now looks like a metal drinking straw. The circumference of the straw is $2\pi r$, its radius is r and its thickness is δr (Figure XX.8). (Now my notation is beginning to make some sense!)

FIGURE XX.8

I shall hold the upper end of the tube fixed and I shall apply a torque $\tau = Fr$ to the lower end. The tube will evidently twist through an azimuthal angle ϕ given by

$$\tau = \frac{2\pi\eta^3 \delta r}{l}\phi. \tag{20.3.3}$$

The torsion constant of the hollow tube is therefore

$$c = \frac{2\pi\eta r^3 \delta r}{l}. \tag{20.3.4}$$

The torsion constant of a long solid cylinder (a wire) of radius a is the integral of this from 0 to a, which is

$$c = \frac{\pi\eta a^4}{2l} \tag{20.3.5}$$