15.7: The Lorentz Transformation as a Rotation

The Lorentz transformation can be written

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{15.7.1}$$

where $x_1 = x,\ x_2 = y,\ x_3 = z$ and $x_4 = -ict$, and similarly for primed quantities. Please do not just take my word for this; multiply the matrices, and verify that this Equation does indeed represent the Lorentz transformation. You could, if you wish, also write this for short:

$$\mathbf{x}' = \lambda \mathbf{x}. \tag{15.7.2}$$

Another way of writing the Lorentz transformation is

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_0' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix}, \tag{15.7.3}$$

where $x_1 = x,\ x_2 = y,\ x_3 = z$ and $x_0 = ct$, and similarly for primed quantities.

Some people prefer one version; others prefer the other. In any case, a set of four quantities that transforms like this is called a 4-*vector*. Those who dislike version 15.7.1 dislike it because of the introduction of imaginary quantities. Those who like version 15.7.1 point out that the expression

$$\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + (\Delta x_4)^2}$$

(the "interval" between two events) is invariant in four-space – that is, it has the same value in all uniformly-moving reference frames, just as the distance between two points in three-space, $[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{\frac{1}{2}}$, is independent of the position or orientation of any reference frame. In version 15.7.3, the invariant interval is

$$\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + (\Delta x_0)^2}\ .$$

Those who prefer version 15.7.1 dislike the minus sign in the expression for the interval. Those who prefer version 15.7.3 dislike the imaginary quantities of version 15.7.1.

For the time being, I am going to omit y and z, so that I can concentrate my attention on the relations between x and t. Thus I am going to write 15.7.1 as

$$\begin{pmatrix} x_1' \\ x_4' \end{pmatrix} = \begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ict \end{pmatrix} \tag{15.7.4}$$

and Equation 15.7.3 as

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{15.7.5}$$

Readers may notice how closely Equation 15.7.4 resembles the Equation for the transformation of coordinates between two reference frames that are inclined to each other at an angle. (See Celestial Mechanics Section 3.6.) Indeed, if we let $\cos\theta = \gamma$ and $\sin\theta = i\beta\gamma$, Equation 15.7.4 becomes

$$\begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{15.7.6}$$

The matrices in Equations 15.7.1, 15.7.4 and 15.7.6 are orthogonal matrices and they satisfy each of the criteria for orthogonality described, for example, in Celestial Mechanics Section 3.7. We can obtain the converse relations (i.e. we can express x and t in terms of x' and t') by interchanging the primed and unprimed quantities and either reversing the sign of β or of θ or by interchanging the rows and columns of the matrix.

There is a difficulty in making the analogy between the Lorentz transformation as expressed by Equation 15.7.4 and rotation of axes as expressed by Equation 15.7.6 in that, since $\gamma > 1$, θ is an imaginary angle. (At this point you may want to reach for your ancient, brittle, yellowed notes on complex numbers and hyperbolic functions.) Thus

$\theta = \cos^{-1}\gamma$ and for $\gamma > 1$, this means that $\theta = i\cosh^{-1}\gamma = i\ln\left(\gamma + \sqrt{\gamma^2 - 1}\right)$. And $\theta = \sin^{-1}(i\beta\gamma) = i\sinh^{-1}(\beta\gamma) = i\ln\left(\beta\gamma + \sqrt{\beta^2\gamma^2 + 1}\right)$. Either of these expressions reduces to $\theta = i\ln[\gamma(1+\beta)]$. Perhaps a yet more convenient way of expressing this is

$$\theta = i\tanh^{-1}\beta = \frac{1}{2}i\ln\left(\frac{1+\beta}{1-\beta}\right). \tag{15.7.7}$$

For example, if β = 0.8, θ = 1.0986i, which might be written (not necessarily particularly usefully) as $i62^{\circ}\ 57'$.

At this stage, you are probably thinking that you much prefer the version of Equation 15.7.5, in which all quantities are real, and the expression for the interval between two events is $[(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - (\Delta x_0)^2]^{\frac{1}{2}}$. The minus sign in the expression is a small price to pay for the realness of all quantities. Equation 15.7.5 can be written

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{15.7.8}$$

where $\cosh\phi = \gamma,\ \sinh\phi = \beta\gamma,\ \tanh\phi = \beta$. On the face of it, this looks much simpler.

No messing around with imaginary angles. Yet this formulation is not without its own set of difficulties. For example, neither the matrix of Equation 15.7.5 nor the matrix of Equation 15.7.8 is orthogonal. You cannot invert the Equation to find x and t in terms of x' and t' merely by interchanging the primed and unprimed symbols and interchanging the rows and columns. The converse of Equation 15.7.8 is in fact

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{15.7.9}$$

which can also (understandably!) be written

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh(-\phi) & \sinh(-\phi) \\ \sinh(-\phi) & \cosh(-\phi) \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{15.7.10}$$

which demands as much skill in handling hyperbolic functions as the other formulation did in handling complex numbers. A further problem is that the formulation 15.7.5 does not allow the analogy between the Lorenz transformation and the rotation of axes. You take your choice.

It may be noticed that the determinants of the matrices of Equations 15.7.5 and 15.7.8 are each unity, and it may therefore be thought that each matrix is orthogonal and that its reciprocal is its transpose. But this is not the case, for the condition that the determinant is inity is not a sufficient condition for a matrix to be orthogonal. The necessary tests are summarized in Celestial Mechanics, Section 3.7, and it will be found that several of the conditions are not satisfied.