3.4: Notation

In this section I am going to suppose that we n particles scattered through three-dimensional space. We shall be deriving some general properties and theorems – and, to the extent that a solid body can be considered to be made up of a system of particles, these properties and theorems will apply equally to a solid body.

In the Figure III.5, I have drawn just two of the particles, (the rest of them are left to your imagination) and the centre of mass C of the system.

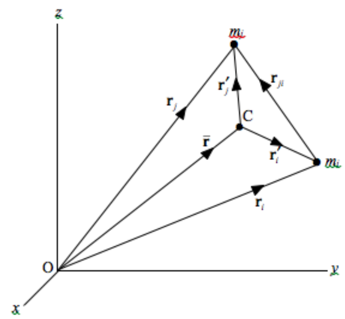

FIGURE III.5

A given particle may have an *external force* $\mathbf{F_i}$ acting upon it. (It may, of course, have *several* external forces acting on it, but I mean by $\mathbf{F_i}$ the vector sum of all the external forces acting on the *i* th particle.) It may also interact with the other particles in the system, and consequently it may have *internal forces* $\mathbf{F_{ij}}$ acting upon it, where j goes from 1 to n except for i. I define the vector sum $\mathbf{F} = \sum \mathbf{F_i}$ as the total external force acting upon the *system*.

I am going to establish the following notation for the purposes of this chapter.

- Mass of the i th particle = m_i
- Total mass of the system $M = \sum m_i$
- Position vector of the i th particle referred to a fixed point O: $\mathbf{r}_i = x_i\hat{\mathbf{x}} + y_i\hat{\mathbf{y}} + z_i\hat{\mathbf{z}}$
- Velocity of the i th particle referred to a fixed point O: $\mathbf{r}_i$ or $\mathbf{v}_i$ (Speed = v_i)
- Linear momentum of the i th particle referred to a fixed point O: $\mathbf{p}_i = m_i\mathbf{v}_i$
- Linear momentum of the *system*: $\mathbf{P} = \sum \mathbf{P}_i = \sum m_i\mathbf{v}_i$
- External force on the i th particle: $\mathbf{F}_i$
- Total external force on the system: $\mathbf{F} = \sum \mathbf{F}_i$
- Angular momentum (moment of momentum) of the i th particle referred to a fixed point O:

$$\mathbf{l}_i = \mathbf{r}_i \times \mathbf{p}_i$$

- Angular momentum of the system: $\mathbf{L} = \sum \mathbf{l}_i = \sum \mathbf{r}_i \times \mathbf{p}_i$
- Torque on the *i* th particle referred to a fixed point O: $\boldsymbol{\tau}_i = \mathbf{r}_i \times \mathbf{F}_i$
- Total external torque on the system with respect to the origin:

$$\boldsymbol{\tau} = \sum \boldsymbol{\tau}_i = \sum \mathbf{r}_i \times \mathbf{F}_i$$

Kinetic energy of the system: (We are dealing with a system of *particles* – so we are dealing only with *translational* kinetic energy – no rotation or vibration):

$$T = \sum \tfrac{1}{2} m_i v_i^2$$

Position vector of the *centre of mass* referred to a fixed point O: $\bar{\mathbf{r}}_i = \bar{x}\hat{\mathbf{x}} + \bar{y}\hat{\mathbf{y}} + \bar{z}\hat{\mathbf{z}}$

The centre of mass is defined by $M\bar{\mathbf{r}} = \sum m_i\mathbf{r}_i$

Velocity of the centre of mass referred to a fixed point O: $\bar{\mathbf{r}} = \bar{\mathbf{v}}$ (Speed = $\bar{v}$)

For position vectors, unprimed single-subscript symbols will refer to O. Primed single-subscript symbols will refer to C. This will be clear, I hope, from Figure III.5.

Position vector of the i th particle *referred to the centre of mass* C: $\mathbf{r}_i' = \mathbf{r}_i - \overline{\mathbf{r}}_i$

Position vector of particle j with respect to particle i: $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$

(Internal) force exerted on particle i by particle j: $\mathbf{F}_{ij}$

(Internal) force exerted on particle j by particle i: $\mathbf{F}_{ji}$

If the force between two particles is *repulsive* (e.g. between electrically-charged particles of the same sign), then $\mathbf{F}_{ji}$ and $\mathbf{r}_{ji}$ are in the same direction. But if the force is an *attractive* force, $\mathbf{F}_{ji}$ and $\mathbf{r}_{ji}$ are in opposite directions.

According to Newton's Third Law of Motion (Lex III), $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$

Total angular momentum of system referred to the centre of mass C: $\mathbf{L}_C$

Total external torque on system referred to the centre of mass C: $\boldsymbol{\tau}_C$

For the velocity of the centre of mass I may use either $\dot{\overline{\mathbf{r}}}$ or $\overline{\mathbf{v}}$

O is an arbitrary origin of coordinates. C is the centre of mass.

Note that

$$\mathbf{r}_i = \overline{\mathbf{r}} + \mathbf{r}_i' \tag{3.4.1}$$

and therefore

$$\dot{\overline{\mathbf{r}}}_i = \dot{\overline{\mathbf{r}}} + \dot{\mathbf{r}}_i'; \tag{3.4.2}$$

that is to say

$$\mathbf{v}_i = \overline{\mathbf{v}} + \mathbf{v}_i' \tag{3.4.3}$$

Note also that

$$\sum m_i \mathbf{r}_i' = 0 \tag{3.4.4}$$

Note further that

$$\sum m_i \mathbf{v}_i' = \sum m_i (\mathbf{v}_i - \overline{\mathbf{v}}) = \sum m_i \mathbf{v}_i - \overline{\mathbf{v}} \sum m_i = M\overline{\mathbf{v}} - \overline{\mathbf{v}}M = 0 \tag{3.4.5}$$

That is, *the total linear momentum with respect to the centre of mass is zero.*

Having established our notation, we now move on to some theorems concerning systems of particles. It may be more useful for you to conjure up a physical picture in your mind what the following theorems mean than to memorize the details of the derivations.