18.4: Area of a Catenoid

A theorem from the branch of mathematics known as the calculus of variations is as follows. Let $y = y(x)$ with $y' = dy/dx$ and let $f(y, y', x)$ be some function of y, y' and x. Consider the line integral of f from A to B along the route $y = y(x)$.

$$\int_A^B f(y, y', x) dx \tag{18.4.1}$$

In general, and unless f is a function of x and y alone, and not of y', the value of this integral will depend on the *route* (i.e. $y = y(x)$) over which this line integral is calculated. The theorem states that the integral is an extremum for a route that satisfies

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y} \tag{18.4.2}$$

By "extremum" we mean either a minimum or a maximum, or an inflection, though in many – perhaps most – cases of physical interest, it is a minimum. It can be difficult for a newcomer to this theorem to try to grasp exactly what this theorem means, so perhaps the best way to convey its meaning is to start by giving a simple example. Following that, I give an example involving the catenary. There will be another example, involving a famous problem in dynamics, in Chapter 19, and in fact we have already encountered an application of it in Chapter 14 in the discussion of Hamilton's variational principle.

Let us consider, for example, the problem of calculating the distance, measured along some route $y(x)$ between two points; that is, we want to calculate the arc length $\int ds$ From the usual pythagorean relation between ds, dx and dy, this is $\int (1 + y')^{1/2} dx$. The variational principle says that this distance – measured along $y(x)$ – is least for a route $y(x)$ that satisfies Equation 18.4.2, in which in this case $f = (1 + y')^{1/2}$

For this case, we have $\frac{df}{dy} = 0$ and $\frac{df}{dy} = \frac{y'}{(1+y')^{1/2}}$. Thus integration of Equation 18.4.2 gives

$$y' = c(1 + y'^2)^{1/2}, \tag{18.4.3}$$

where c is the integration constant. If we solve this for y', we obtain which is just another constant, which I'll write as a, so that $y' = a$. Integrate this to find

$$y = ax + b \tag{18.4.4}$$

This probably seems rather a long way to prove that the shortest distance between two points is a straight line – but that wasn't the point of the exercise. The aim was merely to understand the meaning of the variational principle.

Let's try another example, in which the answer will not be so obvious.

Consider some curve $y = y(x)$, and let us rotate the curve through an angle ϕ (which need not necessarily be a full (\2 \pi \) radians) about the y-axis. An element ds of the curve can be written as $\sqrt{1 + y'^2 dx}$ and the distance moved by the element ds (which is at a distance x from the y-axis) during the rotation is ϕx. Thus the area swept out by the curve is

$$A = \phi \int x\sqrt{1 + y'^2}dx. \tag{18.4.5}$$

For what shape of curve, $y = y(x)$, is this area least? The answer is – a curve that satisfies Equation 18.4.2, where $f = x\sqrt{1 = y'^2}$. For this function, we have $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial y} = \frac{xy'}{\sqrt{1+y'^2}}$.

Therefore the required curve satisfies

$$\frac{xy'}{\sqrt{1 + y'^2}} = a. \tag{18.4.6}$$

That is,

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}. \tag{18.4.7}$$

On substitution of $x = a\cosh\theta$ and looking up everything we have forgotten about hyperbolic functions, and integrating, we obtain

$$y = a\cosh(x/a). \tag{18.4.8}$$

Thus the required curve is a catenary.

If a soap bubble is formed between two identical horizontal rings, one beneath the other, it will take up the shape of least area, namely a catenoid.