19.6: Motion on a Cycloid, Cusps Down

We imagine a particle sliding down the outside of an inverted smooth cycloidal bowl, or a bead sliding down a smooth cycloidal wire. We shall suppose that, at time $t = 0$, the particle was at the top of the cycloid and was projected forward with a horizontal velocity v_0. See Figure XIX.7.

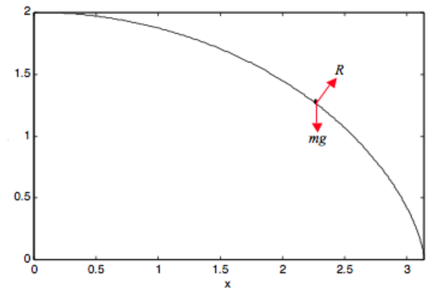

FIGURE XIX.7

This time, the equations of motion are

$$\ddot{s} = gsin\psi \tag{19.6.1}$$

and

$$\frac{mv^2}{\rho} = mg\cos\phi - R. \tag{19.6.2}$$

By arguments similar to those made in Section 19.5, we find that

$$\ddot{s} = \frac{gs}{4a} \tag{19.6.3}$$

The general solution to this is

$$s = Ae^{pt} + Be^{-pt}, \tag{19.6.4}$$

where

$$p = \sqrt{g/(2a)}. \tag{19.6.5}$$

With the initial condition given (at $t = 0$, $s = 0$, $\dot{s} = v_0$), we can find A and B and hence:

$$s = v_0\sqrt{\frac{a}{g}}(e^{pt} - e^{-pt}) \tag{19.6.6}$$

Again proceeding as in Section 19.5, we find for R:

$$R = \frac{m}{4\cos\psi}(4ga\cos 2\psi - v_0^2). \tag{19.6.7}$$

So – what happens?

If the constraint is *two-sided* (bead sliding on a wire) R becomes zero when $\cos 2\pi = v_0^2/(2/ga)$, and thereafter R is in the opposite direction.

If the constraint is *one-sided* (particle sliding down the outside of a smooth cycloidal bowl):

1. If $v_0^2 > 4ga$, the particle loses contact at the moment of projection.
2. If If $v_0^2 < 4ga$ the particle loses contact as soon as $\cos 2\pi = v_0^2/(2ga)$, is very small (i.e. very much smaller than $\sqrt{(}2ga)$), this will happen when $\psi = 45\circ$; for faster initial speeds, contact is lost sooner.

Example 19.6.1

A particle is projected horizontally with speed $v_0 = 1$ m s^{-1} from the vertex of the smooth cycloidal hill

$$x = a(2\theta + \sin 2\theta$$

$$y = 2a\cos^2\theta,$$

where $a = 2$ m. Assuming that g = 9.8 m s^{-2}, how long does it take to get halfway down the hill (i.e. to $y = a$)?

Solution

We have to use Equation 19.6.6. With the numerical data given, this is

$$s = 0.451754(e^{1.565248t} - e^{-1.565248t}).$$

We can find s from Equation 19.4.12, which gives us s = 2.828427 m. If we let we now have to solve 6.26099 = $\xi - 1/\xi$, or $\xi^2 - 6.26099\xi - 1 = 0$. From this, ξ = 6.41683 and hence t = 1.19 s.

I leave it to the reader to calculate R at this time – and indeed to see whether the particle loses contact with the hill before then. Perhaps the fact that I got a positive real root for ξ means that we are all right and the particle is still in contact – but I wouldn't be sure of that. I leave it to the reader to investigate further.