3.11: Torque and Rate of Change of Angular Momentum

Theorem:

The rate of change of the total angular momentum of a system of particles is equal to the sum of the external torques on the system.

Thus:

$$L = \sum_i \mathbf{r}_i \times p_i \tag{3.11.1}$$

$$\therefore \qquad \dot{\mathbf{L}} = \sum_i \dot{\mathbf{r}}_i \times \dot{\mathbf{p}}_i \tag{3.11.2}$$

But the first term is zero, because $\dot{\mathbf{r}}$ and $\mathbf{p}_i$ are parallel.

Also

$$\dot{\mathbf{r}}_i = \mathbf{F}_i + \sum \mathbf{F}_{ij} \tag{3.11.3}$$

$$\dot{\mathbf{L}}_i = \sum_i \mathbf{r}_i \times (\mathbf{r}_i + \sum_j \mathbf{F}_{ij}) = \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_i \mathbf{r}_i \times \sum_j \mathbf{F}_{ii}$$

$$\therefore \qquad \sum_i r_i \times F_i + \sum_i r_i \times \sum_j F_{ii}$$

But $\sum_i \sum_j \mathbf{F}_{ij} = 0$ by Newton's third law of motion, and so $\sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij} = 0$.

Also $\sum_i \mathbf{r}_i \times \mathbf{F}_i = \boldsymbol{\tau}$, and so we arrive at

$$\dot{\mathbf{L}} = \boldsymbol{\tau} \tag{3.11.4}$$

which was to be demonstrated.

Corollary: Law of Conservation of Angular Momentum

If the sum of the external torques on a system is zero, the angular momentum is constant.