15.25: Addition of Kinetic Energies

I want now to consider two particles moving at nonrelativistic speeds – by which I mean that the kinetic energy is given to a sufficient approximation by the expression $\frac{1}{2}mu^2$ and so that parallel velocities add linearly.

Consider the particles in figure XV.37, in which the velocities are shown relative to laboratory space.

FIGURE XV.37

Referred to laboratory space, the kinetic energy is $\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2$. However, the centre of mass is moving to the right with speed $V=\frac{(m_1u_1+m_2u_2)}{(m_1+m_2)}$, and, referred to centre of mass space, the kinetic energy is $\frac{1}{2}m_1(u_1-V)^2+\frac{1}{2}m(u_2+V)^2$. On the other hand, if we refer the situation to a frame in which m_1 is at rest, the kinetic energy is $\frac{1}{2}m_2(u_1+u_2)^2$, and, if we refer the situation to a frame in which m_2 is at rest, the kinetic energy is $\frac{1}{2}m_1(u_1+u_2)^2$.

All we are saying is that the kinetic energy depends on the frame to which speed are referred – and this is not something that crops up only for relativistic speeds.

Let us put some numbers in. Let us suppose, for example that

$$m_1=3\,\text{kg}\;u_1=4\,\text{m s}^{-1}$$
$$m_2=2\,\text{kg}\;u_3=4\,\text{m s}^{-1}$$

so that

$$V=1.2\,\text{m s}^{-1}.$$

In that case, the kinetic energy

referred to laboratory space is 33 J,

referred to centre of mass space is 29.4 J,

referred to m_1 is 49 J,

referred to m_2 is 73.5 J.

In this case the kinetic energy is least when referred to centre of mass space, and is greatest when referred to the lesser mass.

Exercise. Is this always so, whatever the values of m_1, m_2 , u_1 and u_2?

It may be worthwhile to look at the special case in which the two masses are equal (m) and the two speeds(whether in laboratory or centre of mass space) are equal (u).

In that case the kinetic energy in laboratory or centre of mass space is mu^2, while referred to either of the masses it is $2mu^2$.

We shall now look at the same problem for particles travelling at relativistic speeds, and we shall see that the kinetic energy referred to a frame in which one of the particles is at rest is very much greater than (not merely twice) the energy referred to a centre of mass frame.

If two particles are moving towards each other with "speeds" given by g_1 and g_2 in centre of mass space, the g of one relative to the other is given by equation 15.16.14, and, since K = g - 1, it follows that if the two particles have kinetic energies K_1 and K_2 in centre of mass space (in units of the m_0c^2 of each), then the kinetic energy of one relative to the other is

$$K=K_1\oplus K_2=K_1+K_2+K_1K_2+\sqrt{K_1K_2(K_1+2)(K_2+2)}. \tag{15.25.1}$$

If two identical particles, each of kinetic energy K_1 times m_0c^2, approach each other, the kinetic energy of one relative to the other is

$$K=2K_1(K_1+2). \tag{15.25.2}$$

For nonrelativistic speeds as $K_1\rightarrow 0$, this tends to $K=4K_1$, as expected.

Let us suppose that two protons are approaching each other at 99% of the speed of light in centre of mass space (K_1 = 6.08881). Referred to a frame in which one proton is at rest, the kinetic energy of the other will be K = 98.5025, the relative speeds being 0.99995 times the speed of light. Thus $K = 16K_1$ rather than merely $4K_1$ as in the nonrelativistic calculation. For more energetic particles, the ratio $\frac{K}{K_1}$ is even more. These calculations are greatly facilitated if you wrote, as suggested in Section 15.3, a program that instantly connects all the relativity factors given there.

Exercise 15.25.1

Two protons approach each other, each having a kinetic energy of 500 GeV in laboratory or centre of mass space. (Since the two rest masses are equal, these TWO spaces are identical.) What is the kinetic energy of one proton in a frame in which the other is at rest?

(Answer: I make it 535 TeV.)

The factor K (the kinetic energy in units of m_0c^2) is the last of several factors used in this chapter to describe the speed at which a particle is moving, and I take the opportunity here of summarising the formulas that have been derived in the chapter for combining these several measures of speed. These are

$$\beta_1 \oplus \beta_2 = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}. \tag{15.16.7}$$

$$\gamma_1 \oplus \gamma_2 = \gamma_1\gamma_2 + \sqrt{(\gamma_1^2-1)(\gamma_2^2-1)}. \tag{15.16.14}$$

$$k_1 \oplus k_2 = k_1k_2 \tag{15.18.11}$$

$$z_1 \oplus z_2 = z_1z_2 + z_1 + z_2. \tag{15.18.12}$$

$$K = K_1 \oplus K_2 = K_1 + K_2 + K_1K_2 + \sqrt{K_1K_2(K_1+2)(K_2+2)} \quad .$$

$$\frac{\phi_1}{\phi_2} = \phi_1 + \phi_2 \tag{15.16.11}$$

If the two speeds to be combined are equal, these become

$$\beta_1 \oplus \beta_1 = \frac{2\beta_1}{1+\beta_1^2}. \tag{15.25.3}$$

$$\gamma_1 \oplus \gamma_1 = 2\gamma_1^2 - 1 \tag{15.25.4}$$

$$\frac{k_1}{k_1} = k_1^2 \tag{15.25.5}$$

$$z_1 \oplus z_1 = z_1(z_1+2) \tag{15.25.6}$$

$$K_1 \oplus K_1 = 2K_1(K_1+2). \tag{15.25.7}$$

$$\phi_1 \oplus \phi_1 = 2\phi. \tag{15.25.8}$$

These formulas are useful, but for numerical examples, if you already have a program for interconverting between all of these factors, the easiest and quickest way of combinng two "speeds" is to convert them to ϕ. We have seen examples of how this works in Sections 15.16 and 15.18. We can do the same thing with our example from the present section when combining two kinetic energies. Thus we were combining two kinetic energies in laboratory space, each of magnitude K_1 = 6.08881 (ϕ_1 = 2.64665). From this, ϕ = 5.29330, which corresponds to K = 98.5025.