1.7: Uniform Solid Tetrahedron, Pyramid and Cone

Definition

A median of a tetrahedron is a line from a vertex to the centroid of the opposite face.

Theorem I.

The four medians of a tetrahedron are concurrent at a point 3/4 of the way from a vertex to the centroid of the opposite face.

Theorem II

The centre of mass of a uniform solid tetrahedron is at the meet of the medians.

Theorem I can be derived by a similar vector geometric argument used for the plane triangle. It is slightly more challenging than for the plane triangle, and it is left as an exercise for the reader. I draw two diagrams (Figure I.14). One shows the point $\mathrm{C_1}$ that is 3/4 of the way from the vertex A to the centroid of the opposite face. The other shows the point $\mathrm{C_2}$ that is 3/4 of the way from the vertex B to the centroid of its opposite face. You should be able to show that

$$\mathbf{C_1} = (\mathbf{A}+\mathbf{B}+\mathbf{D})/4$$

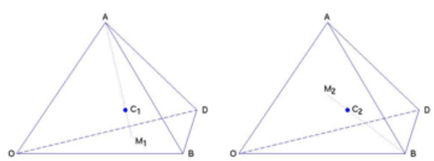

FIGURE I.14

In fact this suffices to prove Theorem I, because, from the symmetry between $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$, one is bound to arrive at the same expression for the three-quarter way mark on any of the four medians. But for reassurance you should try to show, from the second figure, that

\({\bf \text{C}_{2}={\bf\text{A+B+D}})/{4}\)

The argument for Theorem II is easy, and is similar to the corresponding argument for plane triangles.

Pyramid.

A right pyramid whose base is a regular polygon (for example, a square) can be considered to be made up of several tetrahedra stuck together. Therefore the centre of mass is 3/4 of the way from the vertex to the mid point of the base.

Cone.

A right circular cone is just a special case of a regular pyramid in which the base is a polygon with an infinite number of infinitesimal sides. Therefore the centre of mass of a uniform right circular cone is 3/4 of the way from the vertex to the centre of the base.

We can also find the position of the centre of mass of a solid right circular cone by calculus. We can find its volume by calculus, too, but we'll suppose that we already know, from the theorem of Pappus, that the volume is $\frac{1}{3}$ × base × height.

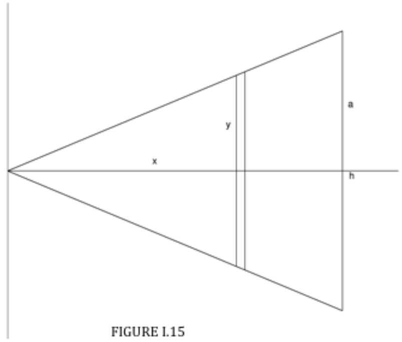

FIGURE I.15

Consider the cone in Figure I.15, generated by rotating the line $y = \frac{ax}{h}$ (between $x = 0$ and $x = h$) through 360º about the x axis. The radius of the elemental slice of thickness dx at x is $\frac{ax}{h}$. Its volume is $\frac{\pi a^2 x^2 \delta x}{h^2}$.

Since the volume of the entire cone is $\frac{\pi a^2 h}{3}$, the mass of the slice is

$$M \times \frac{\pi a^2 x^2 \delta x}{h^2} \div \frac{\pi a^2 h}{3} = \frac{3Mx^2 \delta x}{h^3}$$

where M is the total mass of the cone. The first moment of mass of the elemental slice with respect to the y axis is $\frac{3Mx^3 dx}{h^3}$.

The position of the centre of mass is therefore

$$\overline{x} = \frac{3}{h^3} \int_0^h x^3 dx = \frac{3}{4} h$$