15.29: Force

Force is defined as rate of change of momentum, and we wish to find the transformation between forces referred to frames in uniform relative motion such that this relation holds on all such frames.

Suppose that, in Σ', a mass has instantaneous mass m' and velocity whose instantaneous components are $u'_{x'}$ and $u'_{y'}$. If a force acts on it, then the velocity *and hence also the mass* are functions of time. The x-component of the force is given by

$$F'_{x'} = \frac{d}{dt}(m'u'_{x'}). \tag{15.29.1}$$

We want to express everything on the right hand side in terms of unprimed quantities. Thus from Equation 15.21.8 and the inverse of Equation 15.16.2, we obtain

$$m'u'_{x'} = m\gamma(u_x - v). \tag{15.29.2}$$

Also

$$\frac{d}{dt'} = \frac{dt}{dt'}\frac{d}{dt} \tag{15.29.3}$$

Let us first evaluate $\frac{d}{dt}(m\gamma u_x - m\gamma v)$. In this expression, v and γ are independent of time (the frame Σ' is moving at constant velocity relative to Σ), and $\frac{d}{dt}$ of mu_x is the x-component of the force in Σ, that is F_x. Thus

$$\frac{d}{dt}(m\gamma u_x - m\gamma v) = \gamma\left(F_x - v\frac{dm}{dt}\right). \tag{15.29.4}$$

Now we need to evaluate $\frac{d}{dt'}$ *in terms of unprimed quantities.* If we start with

$$dt' = \left(\frac{\partial t'}{\partial x}\right)_t dx + \left(\frac{\partial t'}{\partial t}\right)_x dt \tag{15.29.5}$$

and we'll evaluate $\frac{dt'}{dt}$ which, being a total derivative, is the reciprocal of $\frac{dt}{dt'}$. The partial derivatives are given by Equations 15.15.3j,k and l, while $\frac{dx}{dt} = u_x$. Hence we obtain

$$\frac{dt}{dt'} = \frac{1}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}. \tag{15.29.6}$$

Thus we arrive at

$$F'_{x'} = \frac{F_x - v\left(\frac{dm}{dt}\right)}{1 - \frac{u_x v}{c^2}} \tag{15.29.7}$$

The mass is not constant (i.e. $\frac{dm}{dt}$ is not zero) because there is a force acting on the body, and we have to relate the term $\frac{dm}{dt}$ to the force. At some instant when the force and velocity (in Σ) are $\mathbf{F}$ and $\mathbf{u}$, the rate at which $\mathbf{F}$ is doing work on the body is $\mathbf{F} * \mathbf{u} = F_x u_x + F_y u_y + F_z u_z$ and this is equal to the rate of increase of energy of the body, which is $\dot{m}c^2$. (In Section 15.24, in deriving the expression for kinetic energy, I wrote that the rate of doing work was equal to the rate of increase of *kinetic* energy. Now I have just written that it is equal to the rate of increase of (total) energy. Which is right?)

$$\frac{dm}{dt} = \frac{1}{c^2}(F_x u_x + F_y u_y + F_z u_z). \tag{15.29.8}$$

Substitute this into Equation 15.29.7and, after a very little more algebra, we finally obtain the transformation for $F'_{x'}$:

$$F'_{x'} = F_x - \frac{v}{c^2 - u_x v}(u_y F_y + u_z F_z). \tag{15.29.9}$$

The $y'-$ and $z'-$ components are a little easier, and I leave it as an exercise to show that

$$F'_{y'} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{u_x v}{c}} F_y \tag{15.29.10}$$

$$F'_{z'} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{u_x v}{c}} F_z. \tag{15.29.11}$$

As usual, the inverse transformations are found by interchanging the primed and unprimed quantities and changing the sign of v.

The force on a particle and its resultant acceleration are not in general in the same direction, because the mass is not constant. (Newton's second law is not $\mathbf{F} = \mathbf{ma}$; it is $\mathbf{F} = \dot{\mathbf{p}}$) Thus

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{u}) = m\mathbf{a} + \dot{m}\mathbf{u}. \tag{15.29.12}$$

Here

$$m = \frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{1}{2}}} \tag{15.29.13}$$

and so

$$\dot{m} = \frac{m_0 u a}{c^2\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}. \tag{15.29.14}$$

Thus

$$\mathbf{F} = \frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{1}{2}}}\left(\mathbf{a} + \frac{ua}{c^2-u^2}\mathbf{u}\right). \tag{15.29.15}$$