

3.13: The Virial Theorem

First, let me say that I am not sure how this theorem got its name, other than that my Latin dictionary tells me that *vis*, *viris* means *force*, and its plural form, *vires*, *virium* is generally translated as *strength*. The term was apparently introduced by Rudolph Clausius of thermodynamics fame. We do not use the word *strength* in any particular technical sense in classical mechanics, although we do talk about the *tensile strength* of a wire, which is the force that it can summon up before it snaps. We use the word *energy* to mean *the ability to do work*; perhaps we could use the word *strength* to mean *the ability to exert a force*. But enough of these idle speculations.

Before proceeding, I define the quantity

$$\iota = \sum_i m_i r_i^2 \quad (3.13.1)$$

as the second moment of mass of a system of particles with respect to the origin. As discussed in Chapter 2, mass is (apart from some niceties in general relativity) synonymous with inertia, and the second moment of mass is used so often that it is nearly always called simply “the” moment of inertia, as though there were only one moment, the second, worth considering. Note carefully, however, that you are probably much more used to thinking about the moment of inertia with respect to an *axis* rather than with respect to a *point*. This distinction is discussed in Section 2.19. Note also that, since the symbol I tends to be heavily used in any discussion of moments of inertia, for moment of inertia with respect to a point I am using the symbol ι .

I can also write Equation 3.13.1 as

$$\iota = \sum_i m_i (\mathbf{r}_i \times \mathbf{r}_i) \quad (3.13.2)$$

Differentiate twice with respect to time:

$$\dot{\iota} = 2 \sum_i m_i (\mathbf{r}_i \dot{\mathbf{r}}_i), \quad (3.13.3)$$

and

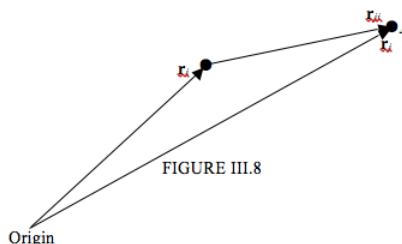
$$\ddot{\iota} = 2 \sum_i m_i (\dot{\mathbf{r}}_i^2 + \mathbf{r}_i \ddot{\mathbf{r}}_i), \quad (3.13.4)$$

or

$$\ddot{\iota} = 4T + 2 \sum_i \mathbf{r}_i m_i \ddot{\mathbf{r}}_i, \quad (3.13.5)$$

where T is the *kinetic energy* of the system of particles. The sums are understood to be over all particles - i.e. i from 1 to n .

$m_i \ddot{\mathbf{r}}_i$ is the force on the i th particle. I am now going to suppose that there are no *external* forces on any of the particles in the system, but the particles interact with each other with conservative forces, \mathbf{F}_{ij} being the force exerted on particle i by particle j . I am also going to introduce the notation $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$, which is a vector directed from particle i to particle j . The relation between these three vectors is shown in Figure III.8.



I have not drawn the force \mathbf{F}_{ij} , but it will be in the opposite direction to \mathbf{r}_{ji} if it is a repulsive force and in the same direction as \mathbf{r}_{ji} if it is an attractive force.

The total force on particle i is $\sum_{j \neq i} \mathbf{F}_{ij}$ and this is equal to $m_i \ddot{\mathbf{r}}_i$. Therefore, Equation 3.13.5 becomes

$$\ddot{\mathbf{r}} = 4T + 2 \sum_i \mathbf{r}_i \sum_{j \neq i} \mathbf{F}_{ij} \quad (3.13.6)$$

Now it is clear that

$$\sum_i \mathbf{r}_i \sum_{j \neq i} \mathbf{F}_{ij} = \sum_i \sum_{j < i} \mathbf{r}_{ij} \mathbf{F}_{ij} \quad (3.13.7)$$

However, in case, like me, you find double subscripts and summations confusing and you have really no idea what Equation 3.13.7 means, and it is by no means at all clear, I write it out in full in the case where there are five particles. Thus:

$$\begin{aligned} \sum_i \mathbf{r}_i \sum_{j \neq i} \mathbf{F}_{ij} &= \mathbf{r}_1(\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_{15}) \\ &+ \mathbf{r}_2(\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_{25}) \\ &+ \mathbf{r}_3(\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_{34} + \mathbf{F}_{35}) \\ &+ \mathbf{r}_4(\mathbf{F}_{41} + \mathbf{F}_{42} + \mathbf{F}_{43} + \mathbf{F}_{45}) \\ &+ \mathbf{r}_5(\mathbf{F}_{51} + \mathbf{F}_{52} + \mathbf{F}_{53} + \mathbf{F}_{54}) \end{aligned}$$

Now apply Newton's third law of motion:

$$\begin{aligned} \sum_i \mathbf{r}_i \sum_{j \neq i} \mathbf{F}_{ij} &= \mathbf{r}_1(-\mathbf{F}_{21} - \mathbf{F}_{31} - \mathbf{F}_{41} + \mathbf{F}_{51}) \\ &+ \mathbf{r}_2(\mathbf{F}_{21} - \mathbf{F}_{32} - \mathbf{F}_{42} - \mathbf{F}_{52}) \\ &+ \mathbf{r}_3(\mathbf{F}_{31} - \mathbf{F}_{32} - \mathbf{F}_{43} - \mathbf{F}_{53}) \\ &+ \mathbf{r}_4(\mathbf{F}_{41} - \mathbf{F}_{42} - \mathbf{F}_{43} - \mathbf{F}_{54}) \\ &+ \mathbf{r}_5(\mathbf{F}_{51} - \mathbf{F}_{52} - \mathbf{F}_{53} + \mathbf{F}_{54}) \end{aligned}$$

Now bear in mind that $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_{21}$, and we see that this becomes

$$\begin{aligned} \sum_i \mathbf{r}_i \sum_{j \neq i} \mathbf{F}_{ij} &= \mathbf{F}_{21} * \mathbf{r}_{21} + \mathbf{F}_{31} * \mathbf{r}_{31} + \mathbf{F}_{41} * \mathbf{r}_{41} + \mathbf{F}_{51} * \mathbf{r}_{51} \\ &+ \mathbf{F}_{32} * \mathbf{r}_{32} + \mathbf{F}_{42} * \mathbf{r}_{42} + \mathbf{F}_{52} * \mathbf{r}_{52} \\ &+ \mathbf{F}_{43} * \mathbf{r}_{43} + \mathbf{F}_{53} * \mathbf{r}_{53} \\ &+ \mathbf{F}_{54} * \mathbf{r}_{54} \end{aligned}$$

and we have arrived at Equation 3.13.7. Equation 3.13.6 then becomes

$$\ddot{\mathbf{r}} = 4T + 2 \sum_i \sum_{j < i} \mathbf{r}_{ij} \mathbf{F}_{ij} \quad (3.13.8)$$

This is the most general form of the virial Equation. It tells us whether the cluster is going to disperse ($\ddot{\mathbf{r}}$ positive) or collapse ($\ddot{\mathbf{r}}$ negative) – though this will evidently depend on the nature of the force law \mathbf{F}_{ij} .

Now suppose that the particles attract each other with a force that is inversely proportional to the n th power of their distance apart. For gravitating particles, of course, $n = 2$. The force between two particles can then be written in various forms, such as

$$\mathbf{F}_{ij} = -\mathbf{F}_{ij} \hat{\mathbf{r}}_{ij} = -\frac{k}{r_{ij}^n} \hat{\mathbf{r}}_{ij} = -\frac{k}{r_{ij}^{n+1}} \mathbf{r}_{ij} \quad (3.13.9)$$

and the mutual potential energy between two particles is minus the integral of $\mathbf{F}_{ij} d\mathbf{r}$, or

$$U_{ij} = -\frac{k}{(n-1)r_{ij}^{n-1}} \quad (3.13.10)$$

I now suppose that the forces between the particles are gravitational forces, such that

$$\mathbf{F}_{ij} = -\frac{Gm_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \quad (3.13.11)$$

Now return to the term $\mathbf{r}_{ij} \mathbf{F}_{ij}$ which occurs in Equation 3.13.8

$$\mathbf{r}_{ij} \mathbf{F}_{ij} = -\frac{k}{r_{ij}^{n+1}} \mathbf{r}_{ij} \mathbf{r}_{ij} = -\frac{k}{r_{ij}^{n-1}} = (n-1)U_{ij} \quad (3.13.12)$$

Thus Equation 3.13.8 becomes

$$\ddot{I} = 4T + 2(n-1)U, \quad (3.13.13)$$

where T and U are the kinetic and potential energies of the system. Note that for gravitational interaction (or any attractive) forces, the quantity U is *negative*. **Equation 3.13.13 is the virial theorem for a system of particles with an r^{-2} attractive force between them.** The system will disperse or collapse according to the sign of \ddot{I} . For a system of **gravitationally-interacting** particles, $n = 2$, and so the virial theorem takes the form

$$\ddot{I} = 4T + 2U \quad (3.13.14)$$

changing from moment to moment, but always in such a manner that Equation 3.13.13 is satisfied.

In a *stable, bound* system, by which I mean that, over a long period of time, there is no long-term change in the moment of inertia of the system, and the system is neither irreversibly dispersing or contracting, that is to say in a system in which the average value of \ddot{I} over a long period of time is zero (I'll define "long" soon), the virial theorem for a stable, bound system of r^n particles takes the form

$$2\langle t \rangle + \langle u \rangle = 0, \quad (3.13.15)$$

and for a **stable** system of **gravitationally-interacting** particles,

$$2\langle t \rangle + \langle u \rangle = 0, \quad (3.13.16)$$

Here the angular brackets are understood to mean the average values of the kinetic and potential energies over a long period of time. By a "long" period we mean, for example, long compared with the time that a particle takes to cross from one side of the system to the other, or long compared with the time that a particle takes to move in an orbit around the centre of mass of the system. (In the absence of external forces, of course, the centre of mass does not move, or it moves with a constant velocity.)

For example, if a bound cluster of stars occupies a spherical volume of uniform density, the potential energy is $\frac{3GM^2}{5a}$ (Equation 5.9.1 of Celestial Mechanics), so the virial theorem (Equation 3.13.16) will enable you to work out the mean kinetic energy and hence speed of the stars. A globular cluster has roughly spherical symmetry, but it is not of uniform density, being centrally condensed. If you assume some functional form for the density distribution, this will give a slightly different formula for the potential energy, and you can then still use the virial theorem to calculate the mean kinetic energy.

✓ Example 3.13.1

Consider a planet of mass m moving in a circular orbit of radius a around a Sun of mass M , such that $m \ll M$ and the Sun does not move.

The potential energy of the system is $U = -GM\frac{m}{a}$.

The speed of the planet is given by equating $\frac{mv^2}{a}$ to $\frac{GMm}{a^2}$, from which $T = GM\frac{m}{(2a)}$,

so we easily see in this case that $2T + U = 0$.

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