4.4: Lagrange's Equations of Motion

In Section 4.5 I want to derive Euler's equations of motion, which describe how the angular velocity components of a body change when a torque acts upon it. In deriving Euler's equations, I find it convenient to make use of Lagrange's equations of motion. This will cause no difficulty to anyone who is already familiar with Lagrangian mechanics. Those who are not familiar with Lagrangian mechanics may wish just to understand what it is that Euler's equations are dealing with and may wish to skip over their derivation at this stage. Later in this series, I hope to add a longer chapter on Lagrangian mechanics, when all will be made clear (maybe). In the meantime, for those who are not content just to accept Euler's equations but must also understand their derivation, this section gives a five-minute course in Lagrangian mechanics.

To begin with, I have to introduce the idea of *generalized coordinates* and *generalized forces.*

The geometrical description of a mechanical system at some instant of time can be given by specifying a number of *coordinates.* For example, if the system consists of just a single particle, you could specify its rectangular coordinates xyz or its cylindrical coordinates $\rho\phi z$, or its spherical coordinates $r\theta\phi$. Certain theorems to be developed will be equally applicable to any of these, so we can think of *generalized coordinates* $q_1 q_2 q_3$, which could mean any one of the rectangular, cylindrical of spherical set.

In a more complicated system, for example a polyatomic molecule, you might describe the geometry of the molecule at some instant by a set of interatomic distances plus a set of angles between bonds. A fairly large number of distances and angles may be necessary. These distances and angles can be called the *generalized coordinates.* Notice that generalized coordinates need not always be of dimension L. Some generalized coordinates, for example, may have the dimensions of angle.

[See Appendix of this Chapter for a brief discussion as to whether angle is a dimensioned or a dimensionless quantity.]

While the generalized coordinates at an instant of time describe the geometry of a system at an instant of time, they alone do not predict the future behaviour of the system.

I now introduce the idea of *generalized forces*. With each of the generalized coordinates there is associated a *generalized force.* With the generalized coordinate q_i there is associated a corresponding generalized force P_i. It is defined as follows. If, when the generalized coordinate q_i increases by δq_i, the work done on the system is $P_i \delta q_i$ then P_i is the generalized force associated with the generalized coordinate q_i. For example, in our simple example of a single particle, if one of the generalized coordinates is merely the x-coordinate, the generalized force associated with x is the x-component of the force acting on the particle.

Note, however, that often one of the generalized coordinates might be an *angle*. In that case the generalized force associated with it is a *torque* rather than a force. In other words, a generalized force need not necessarily have the dimensions MLT^{-2}.

Before going on to describe Lagrange's equations of motion, let us remind ourselves how we solve problems in mechanics using Newton's law of motion. We may have a ladder leaning against a smooth wall and smooth floor, or a cylinder rolling down a wedge, the hypotenuse of which is rough (so that the cylinder does not slip) and the smooth base of which is free to obey Newton's third law of motion on a smooth horizontal table, or any of a number of similar problems in mechanics that are visited upon us by our teachers. The way we solve these problems is as follows. We draw a large diagram using a pencil , ruler and compass. Then we mark in red all the *forces*, and we mark in green all the *accelerations*. If the problem is a two-dimensional problem, we write $F = ma$ in any two directions; if it is a three-dimensional problem, we write $F = ma$ in any three directions. Usually, this is easy and straightforward. Sometimes it does not seem to be as easy as it sounds, and we may prefer to solve the problem by Lagrangian methods.

To do this, as before, we draw a large diagram using a pencil , ruler and compass. But this time we mark in blue all the *velocities* (including angular velocities).

> *Lagrange, in the Introduction to his book La méchanique analytique (modern French spelling omits the h) pointed out that there were no diagrams at all in his book, since all of mechanics could be done analytically – hence the title of the book. Not all of us, however, are as mathematically gifted as Lagrange, and we cannot bypass the step of drawing a large, neat and clear diagram.*

Having drawn in the velocities (including angular velocities), we now calculate the *kinetic energy*, which in advanced texts is often given the symbol T, presumably because potential energy is traditionally written U or V. There would be no harm done if you prefer to write E_k, E_p and E for kinetic, potential and total energy. I shall stick to T, U or V, and E.

Now, instead of writing $F = ma$, we write, for each generalized coordinate, the Lagrangian equation (whose proof awaits a later chapter):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial \dot{q}_i} = P_i \tag{4.4.1}$$

The only further intellectual effort on our part is to determine what is the generalized force associated with that coordinate. Apart from that, the procedure goes quite automatically. We shall use it in use in the next section.

That ends our five-minute course on Lagrangian mechanics.