SECTION OVERVIEW

11.5: Damped Oscillatory Motion

As pointed out in Section 11.2, the equation of motion for a mass m vibrating on the end of a spring of force constant k, in the absence of any damping, is

$$m\ddot{x} = -kx. \tag{11.5.1}$$

Here, I am assuming that the displacement x is a function of time, and a dot denotes $\frac{d}{dt}$.

However, in most real situations, there is some damping, or loss of mechanical energy, which is dissipated as heat. In the case of our example of a mass oscillating on a horizontal table, damping may be caused by friction between the mass and the table. For a mass hanging vertically from a spring, we might imagine the mass to be immersed in a viscous fluid. These are obvious examples. Slightly less obvious, it may be that the constant bending and stretching of the spring produces heat, and the motion is damped from this cause. In any case, in this analysis we shall assume that, in addition to the restoring force kx, there is also a damping force that is proportional to the speed at which the particle is moving. I shall denote the damping force by $b\dot{x}$. The equation of motion is then

$$m\ddot{x} = -kx - b\dot{x}. \tag{11.5.2}$$

If I divide by m and write ω_0^2 for $\frac{k}{m}$ and γ for $\frac{b}{m}$, we obtain the equation of motion in its usual form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0. \tag{11.5.3}$$

Here γ is the *damping constant*, which we have already met in Chapter 10, and, from Section 11.4, we are ready to solve the differential Equation 11.5.3. Indeed, we know that the general solution is

$$x = Ae^{k_1 t} + Be^{k_2 t}, \tag{11.5.4}$$

where k_1 and k_2 are the solutions of the quadratic equation

$$k^2 + \gamma k + \omega_0^2 = 0. \tag{11.5.5}$$

An exception occurs if $k_1 = k_2$, and we shall deal with that exceptional case in due course (subsection 11.5iii). Otherwise, k_1 and k_2 are given by

$$k_1 = -\frac{1}{2}\gamma + \sqrt{\frac{1}{4}\gamma^2 - \omega_0^2}, \quad k_2 = -\frac{1}{2}\gamma - \sqrt{\frac{1}{4}\gamma - \omega_0^2}. \tag{11.5.6}$$

In Section 11.5.3, we pointed out that the nature of the solution depends on whether b^2 is less than, equal to or greater than $4ac$, or, in the present case, upon whether γ is less than, equal to or greater than. These cases are referred to, respectively, as lightly damped, critically damped and heavily damped. We shall start by considering light damping.

Topic hierarchy