15.5: The Lorentz Transformations

For the remainder of this chapter I am taking, as a fundamental postulate, that

> *It is impossible to determine the speed of motion of a uniformly-moving reference frame by any means whatever, whether by a mechanical or electrical or indeed any experiment performed entirely or partially within that frame, or even by reference to another frame*

and consequently I am choosing to believe that my speedometer will not work. If it is impossible by any electrical experiment to determine our speed, we must assume that all the electromagnetic equations that we know, not just the ones that we have quoted, but indeed Maxwell's equations, which embrace all electromagnetic phenomena, are the same in all uniformly-moving reference frames.

One of the many predictions of Maxwell's equations is that electromagnetic radiation (which includes light) travels at a speed

$$c=\frac{1}{\sqrt{\mu_0\epsilon_0}} \tag{15.5.1}$$

Presumably neither the permeability nor the permittivity of space changes merely because we believe that we are travelling through space – indeed it would defy common sense to suppose that they would. Consequently, our acceptance of the fundamental principle of special relativity is equivalent to accepting as a fundamental postulate that the speed of light *in vacuo* is the same for all observers in uniform relative motion. We shall take anything other than this to be an outrage against common sense – though acceptance of the principle will require a careful examination of our ideas concerning the relations between time and space.

Let us imagine two reference frames, $\sum$ and $\sum'$. $\sum'$is moving to the right (positive x-direction) at speed ν relative to $\sum$. (For brevity, I shall from time to time refer to S as the "stationary" frame, in the hope that this liberty will not lead to misunderstanding.) At time $t=t'=0$ the two frames coincide, and at that instant someone strikes a match at the common origin of the two frames. At a later time, which I shall call t if referred to the frame $\sum$, and t' if referred to $\sum'$, the light from the match forms a spherical wavefront travelling radially outward at speed c from the origin O of $\sum$, and the equation to this wavefront, when referred to the frame $\sum$, is

$$x^2+y^2+z^2-c^2t^2=0. \tag{15.5.2}$$

Referred to $\sum'$, it also travels outward at speed c from the origin O' of S', and the equation to this wavefront, when referred to the frame $\sum'$, is

$$x'^2+y'^2+z'^2-c^2t'^2=0. \tag{15.5.3}$$

Most readers will accept, I think, that $y=y'$ and $z=z'$. Some formal algebra may be needed for a rigorous proof, but that would distract from our main purpose of finding a transformation between the primed and unprimed coordinates such that

$$x'^2-c^2t'^2=x^2-c^2t^2. \tag{15.5.4}$$

It is easy to show that the "Galilean" transformation $x'=x-ct$, $t'=t$ does not satisfy this equality, so we shall have to try harder.

Let us seek linear transformations of the form

$$x'=Ax+Bt, \tag{15.5.5}$$

$$t'=Cx+Dt, \tag{15.5.6}$$

which satisfy Equation 15.5.4.

We have

$$\frac{x'}{t'}=\frac{Ax+Bt}{Cx+Dt}, \tag{15.5.7}$$

and, by inversion,

$$\frac{x}{t} = \frac{Dx' - Ct'}{Ax' - Ct'}. \tag{15.5.8}$$

Consider the motion of O' relative to $\sum$ and to $\sum'$. We have $\frac{x}{t} = \nu$ and $x' = 0$.

$$\nu = -\frac{B}{A}. \tag{15.5.9}$$

Consider the motion of O relative to $\sum'$ and to $\sum$. We have $\frac{x'}{t'} = -\nu$ and $x = 0$.

$$-\nu = \frac{B}{D} \tag{15.5.10}$$

From these we find that $D = A$ and $B = -A\nu$, so we arrive at

$$x' = A(x - \nu t) \tag{15.5.11}$$

and

$$t' = Cx + At. \tag{15.5.12}$$

On substitution of Equations 15.5.11and 15.5.12into Equation 15.5.4, we obtain

$$A^2(x - \nu t)^2 - c^2(Cx + At)^2 = x^2 - c^2t^2. \tag{15.5.13}$$

Equate powers of t^2 to obtain

$$A = \frac{1}{\sqrt{\frac{1-\nu^2}{c^2}}} = \gamma. \tag{15.5.14}$$

Equate powers of xt to obtain

$$C = -\frac{\nu\gamma}{c}. \tag{15.5.15}$$

Equating powers of x^2 produces no new information.

We have now determined A, B, C and D, and we can substitute them into Equations 15.5.5and 15.5.6, and hence we arrive at

$$x' = \gamma(x - \nu t) \tag{15.5.16}$$

and

$$t' = \gamma\left(\frac{t - \nu x}{c^2}\right). \tag{15.5.17}$$

These, together with $y = y'$ and $z = z'$, constitute the *Lorentz transformations*, which, by suitable choice of axes, guarantee the invariance of the speed of light in all reference frames moving at constant velocities relative to one another.

To express x and t in terms of x' and t', you may, if you are good at algebra, solve Equations 15.5.16and 15.5.17simultaneously for x' and t', or, if instead, you have good physical insight, you will merely reverse the sign of ν and interchange the primed and unprimed quantities. Either way, you should obtain

$$x = \gamma(x' + \nu t') \tag{15.5.18}$$

and

$$t = \gamma\left(\frac{t' + \nu x'}{c^2}\right) \tag{15.5.19}$$