13.1: Introduction to Lagrangian Mechanics

The usual way of using newtonian mechanics to solve a problem in dynamics is first of all to draw a large, clear diagram of the system, using a ruler and a compass. Then mark in the forces on the various parts of the system with red arrows and the accelerations of the various parts with green arrows. Then apply the equation $F = ma$ in two different directions if it is a two-dimensional problem or in three directions if it is a three-dimensional problem, or $\tau = I\ddot{\theta}$ if torques are involved. More correctly, if a mass or a moment of inertia is not constant, the equations are $F = \dot{p}$ and $\tau = \dot{L}$. In any case, we arrive at one or more *equations of motion*, which are differential equations which we integrate with respect to space or time to find the desired solution. Most of us will have done many, many problems of that sort.

Sometimes it is not all that easy to find the equations of motion as described above. There is an alternative approach known as lagrangian mechanics which enables us to find the equations of motion when the newtonian method is proving difficult. In lagrangian mechanics we start, as usual, by drawing a large, clear diagram of the system, using a ruler and a compass. But, rather than drawing the forces and accelerations with red and green arrows, we draw the *velocity* vectors (including angular velocities) with blue arrows, and, from these we write down the *kinetic energy* of the system. If the forces are *conservative* forces (gravity, springs and stretched strings), we write down also the *potential energy*. That done, the next step is to write down the *lagrangian equations of motion* for each coordinate. These equations involve the kinetic and potential energies, and are a little bit more involved than $F = ma$, though they do arrive at the same results.

I shall derive the lagrangian equations of motion, and while I am doing so, you will think that the going is very heavy, and you will be discouraged. At the end of the derivation you will see that the lagrangian equations of motion are indeed rather more involved than $F = ma$, and you will begin to despair – but do not do so! In a very short time after that you will be able to solve difficult problems in mechanics that you would not be able to start using the familiar newtonian methods, and the speed at which you do so will be limited solely by the speed at which you can write. Indeed, you scarcely have to stop and think. You know straight away what you have to do. Draw the diagram. Mark the velocity vectors. Write down expressions for the kinetic and potential energies, and apply the lagrangian equations. It is automatic, fast, and enjoyable.

Incidentally, when Lagrange first published his great work *La méchanique analytique* (the modern French spelling would be *mécanique*), he pointed out with some pride in his introduction that there were no drawings or diagrams in the book – because all of mechanics could be done *analytically* – i.e. with algebra and calculus. Not all of us, however, are as gifted as Lagrange, and we cannot omit the first and very important step of drawing a large and clear diagram with ruler and compass and marking all the velocity vectors.