# 21.6: A General Central Force

Let us suppose that we have a particle that is moving under the influence of a central force $F(r)$. The equations of motion are

Radial:

$$m(\ddot{r}-r\dot{\theta}^2)=F(r) \tag{21.6.2}$$

Transverse:

$$r\ddot{\theta}+2\dot{r}\dot{\theta}=0. \tag{21.6.3}$$

These can also be written

$$\ddot{r}-\dot{\theta}^2=a(r) \tag{21.6.4}$$

$$r^2\dot{\theta}=h. \tag{21.6.5}$$

Here $a$ is the radial force per unit mass (i.e. the radial acceleration) and $h$ is the (constant) angular momentum per unit mass. [If you are unsure of why Equations 21.6.3 and 21.6.5 are the same, differentiate equation 21.6.5 with respect to time.]

These are two simultaneous equations in $r, \theta, t$. In principle, if we could eliminate $t$ between them, we would obtain a relation between $r$ and $\theta$, which would tell us the shape of the path pursued by the particle. In Chapter 9 of my Celestial Mechanics notes we do this for the gravitational case, and we find that the path is an ellipse of the form $r=\frac{l}{1+e\cos\theta}$ . Or perhaps we could eliminate $r$ and hence find out how the angle $\theta$ changes with time. Or again we might be able to eliminate $\theta$ and hence get a relation telling us how $r$ varies with the time. Yet again we might be told the shape of the path $r(\theta)$, and asked to find the force law $F(r)$. Or again, rather than the force, we might be given the form of the potential energy $V(r)$, which is related to the force by $F=-dV/dr$. The potential $\Phi$ is the potential energy per unit mass, and $-d\Phi/dr$ is the radial force per unit mass - i.e. it is the radial acceleration $a(r)$ of the orbiting particle. The angular momentum of the particle, which is constant, is $L-mr^2\dot{\theta}$ , and the angular momentum per unit mass is $h=r^2\dot{\theta}$ , which is twice the rate at which the radius vector sweeps out area.

We might also remember that, if we are given the potential energy $V$ or the potential $\Phi$ in an inertial frame, we might also want to work in a co-rotating frame, making use of the equivalent potential energy $V'=V+\frac{L^2}{2mr^2}$ or the equivalent potential $\Omega'=\Omega+\frac{h^2}{2^2}$ .

One last thing to bear in mind before starting any problems of this class. It turns out that, very often, a change of variable $u=1/r$ turns out to be useful. Conservation of angular momentum then takes the form $\dot{\theta}/u^2=h$ Also

$$\dot{r}=\frac{dr}{du}\frac{du}{dt}=\frac{dr}{d\theta}\frac{dr}{dt}\frac{du}{d\theta}=-\frac{\dot{\theta}}{u^2}\frac{du}{d\theta}=-h\frac{du}{d\theta} \tag{21.6.6}$$

and

$$\ddot{r}=\frac{d}{dt}\left(-h\frac{du}{d\theta}\right)=-h\frac{d}{dt}\frac{du}{d\theta}=-h\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta}=-h.\,hu^2.\,\frac{d^2u}{d\theta^2}=-h^2u^2\frac{d^2u}{d\theta^2}. \tag{21.6.7}$$

Equations 21.5.4 and 21.6.5 now become

$$h^2u^2\frac{d^2u}{d\theta^2}+h^2u^3=-a(r). \tag{21.6.8}$$

and

$$\dot{\theta}=hu^2. \tag{21.6.9}$$

We can now easily eliminate the time which was one of our aims:

$$h^2u^2\frac{d^2u}{d\theta^2}+h^2u^3=-a(r). \tag{21.6.10}$$

[As ever, check the dimensions.] This equation, which does not contain the time, when integrated will give us the equation to the path.

With these remarks in mind, let us try a few problems. For example:

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