11.2: Mass Attached to an Elastic Spring

I am thinking of a mass m resting on a smooth horizontal table, rather than hanging downwards, because I want to avoid the unimportant distraction of the gravitational force (weight) acting on the mass. The mass is attached to one end of a spring of force constant k, the other end of the spring being fixed, and the motion is restricted to one dimension.

I suppose that the force required to stretch or compress the spring through a distance x is proportional to x and to no higher powers; that is, the spring obeys *Hooke's Law*. When the spring is stretched by an amount x there is a *tension* kx in the spring; when the spring is compressed by x there is a *thrust* kx in the spring. The constant k is the *force constant* of the spring.

When the spring is stretched by an distance x, its acceleration $\ddot{x}$ is given by

$$m\ddot{x} = -kx. \tag{11.2.1}$$

This is an Equation of the type 11.1.5, with $\omega^2 = \frac{k}{m}$, and the motion is therefore simple harmonic motion of period

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \tag{11.2.2}$$

At this stage you should ask yourself two things: Does this expression have dimensions T? Physically, would you expect the oscillations to be slow for a heavy mass and a weak spring? The reader might be interested to know (and this is literally true) that when I first typed Equation 11.2.2, I inadvertently typed $\sqrt{\frac{k}{m}}$ and I immediately spotted my mistake by automatically asking myself these two questions. The reader might also like to note that you can deduce that $P\infty\sqrt{\frac{m}{k}}$ by the method of dimensions, although you cannot deduce the proportionality constant 2π. Try it.

Energy Considerations. The work required to stretch (or compress) a Hooke's law spring by x is $\frac{1}{2}kx^2$, and this can be described as the potential energy or the elastic energy stored in the spring. I shall not pause to derive this result here. It is probably already known by the reader, or s/he can derive it by calculus. Failing that, just consider that, in stretching the spring, the force increases linearly from 0 to kx, so the average force used over the distance x is $\frac{1}{2}kx$ and so the work done is $\frac{1}{2}kx^2$.

If we assume that, while the mass is oscillating, no mechanical energy is dissipated as heat, the total energy of the system at any time is the sum of the elastic energy $\frac{1}{2}kx^2$ stored in the spring and the kinetic energy $\frac{1}{2}mv^2$ of the mass. (I am assuming that the mass of the spring is negligible compared with m.)

Thus

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \tag{11.2.3}$$

and there is a continual exchange of energy between elastic and kinetic. When the spring is fully extended, the kinetic energy is zero and the total energy is equal to the elastic energy then, $\frac{1}{2}ka^2$ when the spring is unstretched and uncompressed, the energy is entirely kinetic; the mass is then moving at its maximum speed $a\omega$ and the total energy is equal to the kinetic energy then, $\frac{1}{2}ma^2\omega^2$. Any of these expressions is equal to the total energy:

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}ka^2 = \frac{1}{2}ma^2\omega^2 \tag{11.2.4}$$