15.24: Kinetic Energy

If a force $\mathbf{F}$ acts on a particle moving with velocity $\mathbf{u}$, the rate of doing work – i.e. the rate of increase of kinetic energy T is $\dot{T} = \mathbf{F} \cdot \mathbf{u}$. But $\mathbf{F} = \dot{\mathbf{p}}$ where $\mathbf{p} = \mathbf{mu} = \gamma \mathbf{m_0 u}$.

(A point about notation may be in order here. I have been using the symbol $\mathbf{v}$ and v for the velocity and speed of a frame Σ' relative to a frame Σ, and my choice of axes without significant loss of generality has been such that $\mathbf{v}$ has been directed parallel to the x-axis. I have been using the symbol $\mathbf{u}$ for the velocity (speed = u) of a particle relative to the frame Σ. Usually the symbol γ has meant $\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$, but here I am using it to mean $\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$. I hope that this does not cause too much confusion and that the context will make it clear. I toyed with the idea of using a different symbol, but I thought that this might make matters worse. Just be on your guard, anyway.)

We have, then

$$\mathbf{F} = \mathbf{m_0}(\dot{\gamma}\mathbf{u} + \gamma\dot{\mathbf{u}}) \tag{15.24.1}$$

and therefore

$$\dot{T} = m_0(\dot{\gamma}u^2 + \gamma\dot{\mathbf{u}} \cdot \mathbf{u}). \tag{15.24.2}$$

Making use of Equations 15.23.5 and 15.23.6 we obtain

$$\dot{T} = \dot{\gamma} m_0 c^2 \tag{15.24.3}$$

Integrate with respect to time, with the condition that when γ = 1, T = 0, and we obtain the following expression for the kinetic energy:

$$T = (\gamma - 1) m_0 c^2. \tag{15.24.4}$$

Exercise. Expand γ by the binomial theorem as far as $\frac{u^2}{c^2}$, and show that, to this order, $T = \frac{1}{2}mu^2$.

I here introduce the dimensionless symbol

$$K = \frac{T}{m_0 c^2} = \gamma - 1 \tag{15.24.5}$$

to mean the kinetic energy in units of $m_0 c^2$. The second half of this was already given as Equation 15.3.5.