11.4: Ordinary Homogeneous Second-order Differential Equations

This is not a full mathematical course on differential Equations, but it may be useful as a reminder for those who have already studied differential Equations, and may even be just enough for our purposes for those who have not.

We suppose that $y = y(x)$ and y' denotes $\frac{dy}{dx}$. An ordinary homogenous second-order differential equation is an Equation of the form

$$ay'' + by' + cy = 0, \tag{11.4.1}$$

and we have to find a function $y(x)$ which satisfies this. It turns out that it is quite easy to do this, although the nature of the solutions depends on whether b^2 is less than, equal to or greater than $4ac$.

A first point to notice is that, if $y = f(x)$ is a solution, so is $Af(x)$ - just try substituting this in the Equation 11.4.1. If $y = g(x)$ is another solution, the same is true of g - i.e. $Bg(x)$ is also a solution. And you can also easily verify that any linear combination, such as

$$y = Af(x) + Bg(x), \tag{11.4.2}$$

is also a solution.

Now Equation 11.4.1 is a second-order Equation - i.e. the highest derivative is a second derivative - and therefore there can be only two arbitrary constants of integration in the solution - and we already have two in Equation 11.4.1, and consequently there are no further solutions. All we have to do, then, is to find two functions that satisfy the differential Equation.

It will not take long to discover that solutions of the form $y = e^{kx}$ satisfy the Equation, because then $y' = ky$ and $y'' = k^2y$, and, if you substitute these in Equation 11.4.1, you obtain

$$(ak^2 + bk + c)y = 0. \tag{11.4.3}$$

You can always find two values of k that satisfy this, although these may be complex, which is why the nature of the solutions depends on whether b^2 is less than or greater than $4ac$. Thus the general solution is

$$y = Ae^{k_1 x} + Be^{k_2 x} \tag{11.4.4}$$

where k_1 and k_2 are the solutions of the Equation

$$ax^2 + bx + c = 0. \tag{11.4.5}$$

There is one complication, however, if $b^2 = 4ac$ because then the two solutions of Equation 11.4.5 are each equal to $(\frac{-b}{(2a)}$. The solution of the differential Equation is then

$$y = (A + B)\exp\left[\frac{-bx}{(2a)}\right] \tag{11.4.6}$$

and the two constants can be combined into a single constant $C = A + B$ so that Equation 11.4.6 can be written

$$y = C\exp\left[\frac{-bx}{(2a)}\right]. \tag{11.4.7}$$

This solution has only one independent arbitrary constant, and so an additional solution must be possible. Let us try and see whether a function of the form

$$y = xe^{mx} \tag{11.4.8}$$

might be a solution of Equation 11.4.1. From Equation 11.4.8 we obtain $y' = (1 + mx)e^{mx}$ and $y'' = m(2 + mx)e^{mx}$. On substituting these into Equation 11.4.1, remembering that $c = \frac{b^2}{(4a)}$, we obtain for the left hand side of Equation 11.4.1, after some algebra,

$$\frac{e^{mx}}{4a}\left[(2am+b)^2x+4a(2am+b)\right]. \tag{11.4.9}$$

This is identically zero if $m=\frac{-b}{(2a)}$, and hence

$$y=x\exp\left[\frac{-bx}{(2a)}\right] \tag{11.4.10}$$

is a solution of Equation 11.4.1the general solution of Equation 11.4.1, if $b^2=4ac$, is therefore

$$y=(C=Dx)\exp\left[\frac{-bx}{(2a)}\right]. \tag{11.4.11}$$

We shall discover what these solutions actually look like in the next section.