4.1: Introduction to Rigid Body Rotation

No real solid body is perfectly rigid. A rotating nonrigid body will be distorted by centrifugal force* or by interactions with other bodies. Nevertheless most people will allow that in practice some solids are fairly rigid, are rotating at only a modest speed, and any distortion is small compared with the overall size of the body. No excuses, therefore, are needed or offered for analyzing, to begin with the rotation of a rigid body.

*I do not in this chapter delve deeply into whether there really is “such thing” as “centrifugal force”. Some would try to persuade us that there is no such thing. But is there “such thing” as a “gravitational force”? And is one any more or less “real” than the other? These are deep questions best left to the philosophers. In physics we use the concept of “force” – or indeed any other concept – according to whether it enables us to supply a description of how physical bodies behave. Many of us would, I think, be challenged if we were faced with an examination question: “Explain, without using the term *centrifugal force*, why Earth bulges at its equator.”

We have already discussed some aspects of solid body rotation in Chapter 2 on Moment of Inertia, and indeed the present Chapter 4 should not be plunged into without a good understanding of what is meant by “moment of inertia”. One of the things that we found was that, while the comfortable relation $L = I\omega$ we are familiar with from elementary physics is adequate for problems in two dimensions, in three dimensions the relation becomes $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, where $\mathbf{I}$ is the *inertia tensor*, whose properties were discussed at some length in Chapter 2. We also learned in Chapter 2 about the concepts of *principal moments of inertia*, and we introduced the notion that, unless a body is rotating about one of its principal axes, the equation $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ implies that the angular momentum and angular velocity vectors *are not in the same direction.* We shall discuss this in more detail in this chapter.

A full treatment of the rotation of an *asymmetric top* (whose three principal moments of inertia are unequal and which has as its momental ellipsoid a triaxial ellipsoid) is very lengthy, since there are so many cases to consider. I shall restrict consideration of the motion of an asymmetric top to a qualitative argument that shows that rotation about the principal axis of greatest moment of inertia or about the axis of least moment of inertia is stable, whereas rotation about the intermediate axis is unstable.

I shall treat in more detail the free rotation of a *symmetric top* (which has two equal principal moments of inertia) and we shall see how it is that the angular velocity vector precesses while the angular momentum vector (in the absence of external torques) remains fixed in magnitude and direction.

I shall also discuss the situation in which a symmetric top is subjected to an external torque (in which case $\mathbf{L}$ is certainly not fixed), such as the motion of a top. A similar situation, in which Earth is subject to external torques from the Sun and Moon, causes Earth’s axis to precess with a period of 26,000 years, and this will be dealt with in a chapter of the notes on Celestial Mechanics.

Before discussing these particular problems, there are a few preparatory topics, namely, angular velocity and Eulerian angles, kinetic energy, Lagrange’s equations of motion, and Euler’s equations of motion.