1.3: Plane Areas

Plane areas in which the equation is given in $x - y$ *coordinates*

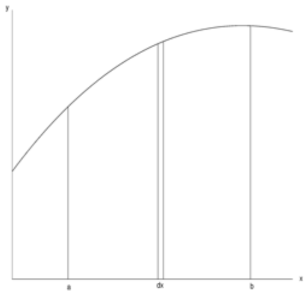


FIGURE I.3

We have a curve $y = y(x)$ (Figure I.3) and we wish to find the position of the centroid of the area under the curve between $x = a$ and $x = b$. We consider an elemental slice of width δx at a distance x from the y axis. Its area is $y\delta x$, and so the total area is

$$A = \int_a^b ydx \tag{1.3.1}$$

The first moment of area of the slice with respect to the y axis is $xy\delta x$, and so the first moment of the entire area is $\int_a^b xydx$.

Therefore

$$\bar{x} = \frac{\int_a^b xydyx}{\int_a^b ydyx} = \frac{\int_a^b xydyx}{A} \tag{1.3.2}$$

$$labeleq: 1.3.2$$

For $\bar{y}$ we notice that the distance of the centroid of the slice from the x axis is $\frac{1}{2}y$, and therefore the first moment of the area about the x axis is $\frac{1}{2}y.\,y\delta x$.

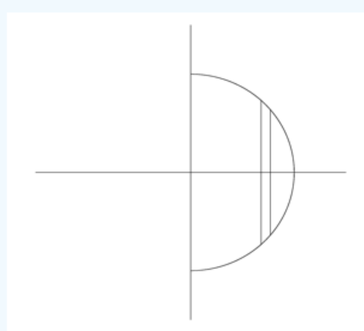
Therefore

$$\bar{y} = \frac{\int_a^b y^2dx}{2A} \tag{1.3.3}$$

Example 1.3.1

Consider a semicircular lamina, $x^2 + y^2 = a^2$, see Figure I.4:

FIGURE I.4

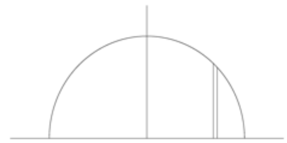


We are dealing with the parts both above and below the \(x \) axis, so the area of the semicircle is $2\int_0^a ydx$ and the first moment of area is $2\int_0^a xydx$.

You should find $\bar{x} = 4a/(3\pi) = 0.4244a$.

Now consider the lamina $x^2 + y^2 = a^2$, $y > 0$ (Figure I.5):

FIGURE I.5



The area of the elemental slice this time is $y\delta x$ (not $2y\delta x$), and the integration limits are from $-a$ to $+a$. To find $\overline{y}$, use Equation 1.3.3, and you should get $y = 0.4244a$.

Plane areas in which the equation is given in polar coordinates.

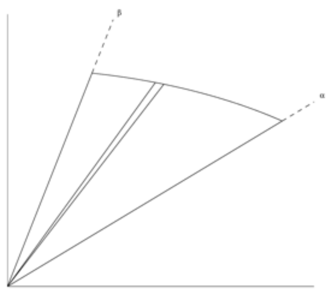


FIGURE I.6

We consider an elemental triangular sector (Figure I.6) between θ and $\theta + \delta\theta$. The "height" of the triangle is r and the "base" is $r\delta\theta$. The area of the triangle is $\frac{1}{2}r^2\delta\theta$.

Therefore the whole area =

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \tag{1.3.4}$$

The horizontal distance of the centroid of the elemental sector from the origin (more correctly, from the "pole" of the polar coordinate system) is $\frac{2}{3}r\cos\theta$. The first moment of area of the sector with respect to the y axis is

$$\tfrac{2}{3}r\cos\theta \times \tfrac{1}{2}r^2\delta\theta = \tfrac{1}{3}r^3\cos\theta\delta\theta$$

so the first moment of area of the entire figure between $\theta = \alpha$ and $\theta = \beta$ is

$$\tfrac{1}{3}\int_{\alpha}^{\beta} r^3\cos\theta d\theta$$

Therefore

$$\overline{x} = \frac{2\int_{\alpha}^{\beta} r^3\cos\theta d\theta}{3\int_{\alpha}^{\beta} r^2 d\theta} \tag{1.3.5}$$

Similarly

$$\overline{x} = \frac{2\int_{\alpha}^{\beta} r^3\sin\theta d\theta}{3\int_{\alpha}^{\beta} r^2 d\theta} \tag{1.3.6}$$

Example 1.3.2

Consider the semicircle $r = a$, $\theta = \frac{-\pi}{2}$ to $\frac{+\pi}{2}$

$$\overline{x} = \frac{2a\int_{-\pi/2}^{+\pi/2}\cos\theta d\theta}{3\int_{-\pi/2}^{+\pi/2} d\theta} = \frac{2a}{3\pi}\int_{-\pi/2}^{+\pi/2}\cos\theta d\theta = \frac{4a}{3\pi} \tag{1.3.7}$$

The reader should now try to find the position of the centroid of a circular sector (slice of pizza!) of angle 2α . The integration limits will be $-\alpha$ to $+\alpha$.

When you arrive at a formula (which you should keep in a notebook for future reference), check that it goes to $\frac{4\alpha}{3\pi}$ if $\alpha = \frac{\pi}{2}$, and to $\frac{2\pi}{3}$ if $\alpha = 0$.