15.16: Addition of Velocities

A railway train trundles towards the east at speed ν_1, and a passenger strolls towards the front at speed ν_2. What is the speed of the passenger relative to the railway station? We might at first be tempted to reply: “Why, $\nu_1 + \nu_2$ of course.” In this section we shall show that the answer as predicted from the Lorentz transformations is a little less than this, and we shall develop a formula for calculating it. We have already discussed (in Section 15.6) our answer to the objection that this defies common sense. We pointed out there that the answer (to the perfectly reasonable objection) that “at the speeds we are accustomed to we would hardly notice the difference” is not a satisfactory response. The reason that the resultant speed is a little less than $\nu_1 + \nu_2$ results from the way in which we have defined the Lorentz transformations between references frames and the way in which distances and time intervals are defined with reference to reference frames in uniform relative motion

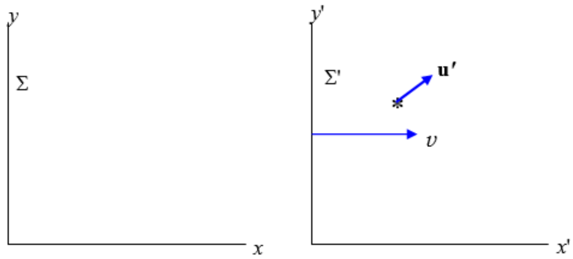


FIGURE XV.17

Figure XV.17 shows two references frames, Σ and Σ', the latter moving at speed ν with respect to the former. A particle is moving with velocity $\mathbf{u}'$ in Σ', with components $u'_{x'}$ and $u'_{y'}$. (“ in Σ' ” = “referred to the reference frame Σ'”.)

What is the velocity of the particle in Σ?

Let us start with the x-component.

We have:

$$u = \frac{dx}{dt} = \frac{\left(\frac{\partial x}{\partial x'}\right)_{t'} dx' + \left(\frac{\partial x}{\partial t'}\right)_{x'} dt'}{\left(\frac{\partial t}{\partial x'}\right)_{t'} dx' + \left(\frac{\partial t}{\partial t'}\right)_{x'} dt'} = \frac{\left(\frac{\partial x}{\partial x'}\right)_{t'} u' + \left(\frac{\partial x}{\partial t'}\right)_{x'}}{\left(\frac{\partial t}{\partial x'}\right)_{t'} u' + \left(\frac{\partial t}{\partial t'}\right)_{x'}} \tag{15.16.1}$$

We take the derivatives from Equations 15.15.3a,b,c,d, and, writing $\frac{\nu}{c}$ for β, we obtain

$$u_x = \frac{u'_x + \nu}{1 + u'_{x'} \frac{\nu}{c^2}}. \tag{15.16.2}$$

The inverse is obtained by interchanging the primed and unprimed symbols and reversing the sign of ν.

The y-component is found in an exactly similar manner, and I leave its derivation to the reader. The result is

$$u_y = \frac{u' + \nu}{1 + u' \frac{\nu}{c^2}} \tag{15.16.3}$$

Special cases:

I. If $u'_{x'} = u'$ and $u'_{y'} = 0$, then

$$u_x = \frac{u' + \nu}{1 + u' \frac{\nu}{c^2}} \tag{15.16.4a}$$

$$u_y = 0 \tag{15.16.4b}$$

II. If $u'_{x'}=0$ and $u'_{y'}=u'$ then

$$u_x=\nu \tag{15.16.5a}$$

$$u_y=\frac{u'}{\gamma} \tag{15.16.5b}$$

Equation 15.16.4a as written is not easy to commit to memory, though it is rather easier if we write $\beta_1=\frac{\nu}{c}$, $\beta_2=\frac{u'}{c}$ and $\beta=\frac{u_x}{c}$. Then the equation becomes

$$\beta=\frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \tag{15.16.4}$$

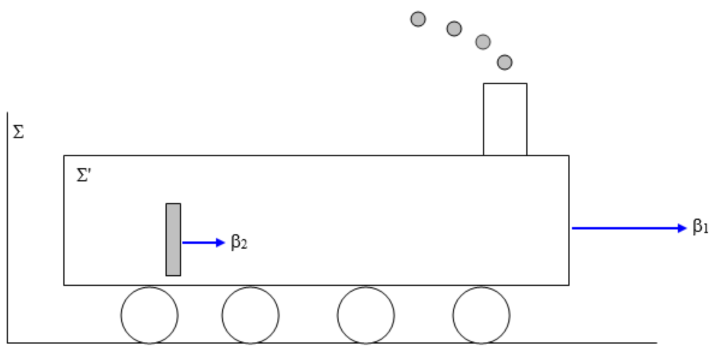
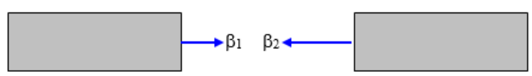


FIGURE XV.18

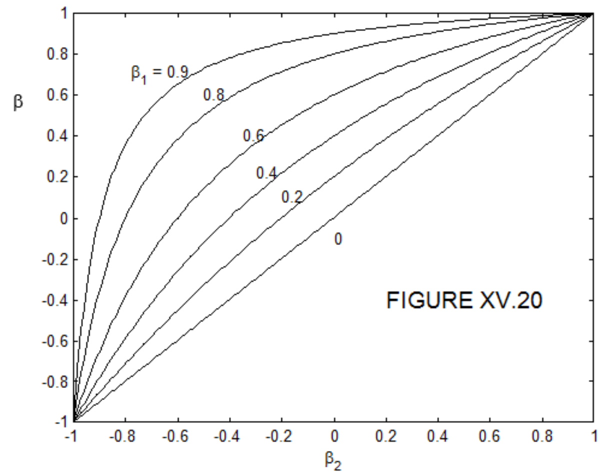


In Figure XV.18, a train Σ' is trundling with speed β_1 (times the speed of light) towards the right, and a passenger is strolling towards the front at speed β_2. The speed β of the passenger relative to the station Σ is then given by Equation 15.16.4. In Figure XV.19, two trains, one moving at speed β_1 and the other moving at speed β_2, are moving towards each other. (If you prefer to think of protons rather than trains, that is fine.) Again, the relative speed b of one train relative to the other is given by Equation 15.16.4.

Example 15.16.1

A train trundles to the right at 90% of the speed of light relative to Σ, and a passenger strolls to the right at 15% of the speed of light relative to Σ'. The speed of the passenger relative to Σ is 92.5% of the speed of light.

The relation between β_1, β_2 and β is shown graphically in Figure XV.20.



If I use the notation $\frac{\beta_1}{\beta_2}$ to mean "combining β_1 with β_2", I can write Equation 15.16.4as

$$\beta_1 \oplus \beta_2 = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{15.16.5}$$

You may notice the similarity of Equation 15.16.4$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_1}$ to the hyperbolic function identity

$$\tanh(\phi_1 + \phi_2) = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2} \tag{15.16.6}$$

Thus I can represent the speed of an object by giving the value of ϕ, where

$$\beta = \tanh \phi \tag{15.16.7}$$

or

$$\phi = \tanh^{-1} \beta = \frac{1}{2} \ln \left(\frac{1 + \beta}{1 - \beta} \right) \tag{15.16.8}$$

The factor ϕ combines simply as

$$\frac{\phi_2}{\phi_2} = \phi_1 + \phi_2 \tag{15.16.9}$$

If you did what I suggested in Section 15.3 and programmed your calculator or computer to convert instantly from one relativity factor to another, you now have a quick way of adding speeds.

Example 15.16.2

A train trundles to the right at 90% of the speed of light (ϕ_1 = 1.47222) relative to S, and a passenger strolls to the right at 15% of the speed of light (ϕ_2 = 0.15114) relative to Σ'. The speed of the passenger relative to Σ is ϕ = 1.62336, or 92.5% of the speed of light.

Example 15.16.3

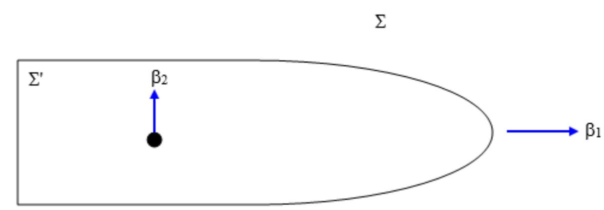


FIGURE XV.22

(Sorry – there is no Figure XV.21.)

An ocean liner Σ' sails serenely eastwards at a speed β_1 = 0.9c (g_1 = 2.29416) relative to the ocean Σ. A passenger ambles athwartships at a speed β_2 = 0.5c relative to the ship. What is the velocity of the passenger relative to the ocean?

The northerly component of her velocity is given by Equation 15.16.5b and is 0.21794c. Her easterly component is just 0.9c. Her velocity relative to the ocean is therefore 0.92601c in a direction 13° 37' north of east.

Exercise 15.16.1

Show that, if the speed of the ocean liner is β_1 and the athwartships speed of the passenger is β_2, the resultant speed β of the passenger relative to the ocean is given by

$$\beta^2 = \beta_1^2 + \beta_2^2 - \beta_1^2\beta_2^2 \tag{15.16.10}$$

and that her velocity makes and angle α with the velocity of the ship given by

$$\tan\alpha = \beta_2\sqrt{1 - \frac{\beta_1^2}{\beta_1}}. \tag{15.16.11}$$

Example 15.16.4

A railway train Σ' of proper length L_0 = 100 yards thunders past a railway station Σ at such a speed that the stationmaster thinks its length is only 40 yards. (Correction: It is not a matter of what he "thinks". What I should have said is that the length of the train, referred to a reference frame Σ in which the stationmaster is at rest, is 40 yards.) A dachshund waddles along the corridor towards the front of the train. (A dachshund, or badger hound, is a cylindrical dog whose proper length is normally several times its diameter.) The proper length l_0 of the dachshund is 24 inches, but to a seated passenger, it appears to be... no, sorry, I mean that its length, referred to the reference frame Σ', is 15 inches. What is the length of the dachshund referred to the reference frame Σ in which the stationmaster is at rest?

We are told, in effect, that the speed of the train relative to the station is given by γ_1 = 2.5, and that the speed of the dachshund relative to the train is given by γ_2 = 1.6. So how do these two gammas combine to make the factor γ for the dachshund relative to the station?

There are several ways in which you could do this problem. One is to develop a general algebraic method of combining two gamma factors. Thus:

Exercise 15.16.2

Show that two gamma factors combine according to

$$\gamma_1 \oplus \gamma_2 = \gamma_1\gamma_2 + \sqrt{(\gamma_1^2 - 1)(\gamma_2^2 - 1)}\,. \tag{15.16.12}$$

I'll leave you to try that. The other way is to take advantage of the programme you wrote when you read Section 15.3, by which you can instantaneously convert one relativity factor to another. Thus you instantly convert the gammas to phis.

Thus $\gamma_1 = 2.5 \ \Rightarrow\ \phi_1 = 1.56680$

and $\gamma_1 = 1.6 \ \Rightarrow\ \phi_1 = 1.04697$

$$\therefore \qquad \phi = 2.61377 \Rightarrow \gamma = 6.86182$$

Is this what Equation 15.16.12gets?

Therefore, referred to the railway station, the length of the dachshund is $\dfrac{24}{\gamma}$ = 3.5 inches.