9.2: The Time and Energy Equation

Consider a one-dimensional situation in which there is a force $F(x)$ that depends on the one coordinate only and is therefore a conservative force. If a particle moves under this force, its equation of motion is

$$m\ddot{x} = F(x)$$

and we can obtain the space integral in the usual fashion by writing $\ddot{x}$ as $v\frac{dv}{dx}$.

Thus

$$mv\frac{dv}{dx} = F(x)$$

Integration yields

$$\frac{1}{2}mv^2 = \int F(x)dx + T_0$$

Here $\frac{1}{2}mv^2$ is called the *kinetic energy* and the integration constant T_0 can be interpreted as the initial kinetic energy. Thus the gain in kinetic energy is

$$T - T_0 = \int F(x)dx \tag{9.2.4}$$

the right hand side merely being the work done by the force.

Since F is a function of x alone, we can find a V such that $F = -\frac{dV}{dx}$. [It is true that we could also find a function V such that $F = +\frac{dV}{dx}$, but we shall shortly find that the choice of the minus sign gives V a desirable property that we can make use of.] If we integrate this equation, we find

$$V = -\int F(x)dx + V_0 \tag{9.2.5}$$

Here V is the *potential energy* and V_0 is the initial potential energy. From Equations 9.2.4 and 9.2.5 we obtain

$$V + T = V_0 + T_0 \tag{9.2.6}$$

Thus the quantity $V+T$ is conserved under the action of a conservative force. (This would not have been the case if we had chosen the + sign in our definition of V.) We may call the sum of the two energies E, the total energy, and we have

$$T = E - V(x) \tag{9.2.7}$$

or

$$\frac{1}{2}mv^2 = E - V(x) \tag{9.2.8}$$

With $v = \frac{dx}{dt}$, we obtain, by integrating Equation 9.2.8,

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E-V(x)}} \tag{9.2.9a}$$

This may at first appear to be a very formal and laborious way of arriving at something very obvious and something we have known since we first studied physics, but we shall see that it can often be a quite useful equation. You might, by the way, check that this equation is dimensionally correct.

The choice of the sign in Equation 9.2.9a may require some care, as will be evident in the examples that follow in the next section. If the particle is moving away from the origin, then its speed is $v = \frac{dx}{dt}$, and we choose the positive sign. If the particle is moving towards from the origin, then its speed is $v = -\frac{dx}{dt}$, and we choose the negative sign. However, I believe the following to be true: If the particle is moving away from the origin, then the initial value of x is smaller than the final value. If the particle is moving toward the origin, then the initial value of x is larger than the final value. It would seem to be safe, then, always to use the positive sign, but then the lower limit of integration is the smaller value of x (not necessarily the initial value), and the upper limit of integration is the larger value of x (not necessarily the final value). It may therefore be easier to write the equation in the form

$$t=\pm\sqrt{\frac{m}{2}}\int_{x_{smaller}}^{x_{larger}}\frac{dx}{\sqrt{E-V(x)}} \tag{9.2.9b}$$

All that this means is that, for a conservative force, the time taken for a "return" journey is just equal to the time taken for the outbound journey, so one might as well always calculate the time for the outbound journey.

In some classes of problem such as pendulums, or rods falling over, the potential energy can be written as a function of an angle, and the kinetic energy is rotational kinetic energy written in the form $\frac{1}{2}I\omega^2$ where $\omega=\frac{d\theta}{dt}$. In that case, Equation 9.2.9b takes the form

$$t=\pm\sqrt{\frac{I}{2}}\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{E-V(\theta)}}. \tag{9.2.10}$$

You should check that this, too, is dimensionally correct.