15.15: Derivatives

We'll pause here and establish a few derivatives just for reference and in case we need them later.

We recall that the Lorentz relations are

$$x=\gamma(x'+\nu t') \tag{15.15.1}$$

and

$$t=\gamma\left(t'+\frac{\beta x'}{c}\right) \tag{15.15.2}$$

From these we immediately find that

$$\left(\frac{\partial x}{\partial x'}\right)_{t'}=\gamma;\quad \left(\frac{\partial x}{\partial t'}\right)_{x'}=\gamma\nu;\quad \left(\frac{\partial t}{\partial x'}\right)_{t'}=\frac{\beta\gamma}{c};\quad \left(\frac{\partial t}{\partial t'}\right)_{x'}=\gamma. \tag{15.15.3a,b,c,d}$$

We shall need these in future sections.

Caution

It is not impossible to make a mistake with some of these derivatives if one allows one's attention to wander. For example, one might suppose that, since $\frac{\partial x}{\partial x'}=\gamma$ then "obviously" $\frac{\partial x'}{\partial x}=\frac{1}{\gamma}$ - and indeed this is correct if t' is being held constant. However, we have to be sure that this is really what we want. The difficulty is likely to arise if, when writing a partial derivative, we neglect to specify what variables are being held constant, and no great harm would be done by insisting that these always be specified when writing a partial derivative. If you want the *inverses* rather than the *reciprocals* of Equations 15.15.3a,b,c,dthe rule, as ever, is: Interchange the primed and unprimed symbols and change the sign of ν or β. For example, the reciprocal of $\left(\frac{\partial x}{\partial x'}\right)_{t'}$ is $\left(\frac{\partial x'}{\partial x}\right)_{t'}$, while its inverse is $\left(\frac{\partial x'}{\partial x}\right)_{t}$. For completeness, and reference, then, I write down all the possibilities:

$$\left(\frac{\partial x'}{\partial x}\right)_{t'}=\frac{1}{\gamma};\quad \left(\frac{\partial t'}{\partial x}\right)_{x'}=\frac{1}{\gamma\nu};\quad \left(\frac{\partial x'}{\partial t}\right)_{t'}=\frac{c}{\beta\gamma};\quad \left(\frac{\partial t'}{\partial t}\right)_{x'}=\frac{1}{\gamma}. \tag{5.15.3e,f,g,h}$$

$$\left(\frac{\partial x'}{\partial x}\right)_{t}=\gamma;\quad \left(\frac{\partial x'}{\partial t}\right)_{x}=-\gamma\nu;\quad \left(\frac{\partial t'}{\partial x}\right)_{t}=-\frac{\beta\gamma}{c};\quad \left(\frac{\partial t'}{\partial t}\right)_{x}=\gamma. \tag{15.15.3i,j,k,l}$$

$$\left(\frac{\partial x}{\partial x'}\right)_{t}=\frac{1}{\gamma};\quad \left(\frac{\partial t}{\partial x'}\right)_{x}=-\frac{1}{\gamma\nu};\quad \left(\frac{\partial x}{\partial t'}\right)_{t}=-\frac{c}{\beta\gamma};\quad \left(\frac{\partial t}{\partial t'}\right)_{x}=\frac{1}{\gamma}. \tag{15.15.3m,n,o,p}$$

Now let's suppose that $\psi=\psi(x,t)$ where x and t are in turn functions (Equations 15.15.1and 15.15.2) of x' and t'. Then

$$\frac{\partial\psi}{\partial x'}=\frac{\partial x}{\partial x'}\frac{\partial\psi}{\partial t}+\frac{\partial t}{\partial x'}\frac{\partial\psi}{\partial t}=\gamma\frac{\partial\psi}{\partial x}+\frac{\beta\gamma}{c}\frac{\partial\psi}{\partial t} \tag{15.15.3}$$

and

$$\frac{\partial\psi}{\partial t'}=\frac{\partial x}{\partial t'}\frac{\partial\psi}{\partial x}+\frac{\partial t}{\partial t'}\frac{\partial\psi}{\partial t}=\gamma\nu\frac{\partial\psi}{\partial x}+\gamma\frac{\partial\psi}{\partial t}. \tag{15.15.4}$$

The reader will doubtless notice that I have here ignored my own advice and I have not indicated which variables are to be held constant. It would be worth spending a moment here thinking about this.

We can write Equations 15.15.3and 15.15.4as equivalent operators:

$$\frac{\partial}{\partial x'}=\gamma\left(\frac{\partial}{\partial x}+\frac{\beta}{c}\frac{\partial}{\partial t}\right) \tag{15.15.5}$$

and

$$\frac{\partial}{\partial t'}=\gamma\left(\nu\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right). \tag{15.15.6}$$

We can also, if we wish, find the second derivatives. Thus

$$\frac{\partial^2 \psi}{\partial x'^2} = \gamma^2 \left(\frac{\partial^2}{\partial x^2} + \frac{2\beta}{c} \frac{\partial^2}{\partial x \partial t} + \frac{\beta^2}{c^2} \frac{\partial^2}{\partial t^2} \right). \tag{15.15.7}$$

In a similar manner we obtain

$$\frac{\partial^2}{\partial x' \partial t'} = \gamma^2 \left(\nu \frac{\partial^2}{\partial x^2} + (1+\beta^2) \frac{\partial^2}{\partial x \partial t} + \frac{\beta}{c} \frac{\partial^2}{\partial t^2} \right) \tag{15.15.8}$$

and

$$\frac{\partial^2}{\partial t'^2} = \gamma^2 \left(\nu^2 \frac{\partial^2}{\partial x^2} + 2\nu \frac{\partial^2}{\partial x \partial t} + \frac{\partial^2}{\partial t^2} \right). \tag{15.15.9}$$

The inverses of all of these relations are to be found by interchanging the primed and unprimed coordinates and changing the signs of ν and β.
