University of Victoria

Classical Mechanics

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CHAPTER OVERVIEW

1: Centers of Mass

Topic hierarchy

1.1: Introduction and Some Definitions

This chapter deals with the calculation of the positions of the centres of mass of various bodies. We start with a brief explanation of the meaning of centre of mass, centre of gravity and centroid, and a very few brief sentences on their physical significance. Many students will have seen the use of calculus in calculating the positions of centres of mass, and we do this for

Plane areas

i for which the equation is given in $x-y$ coordinates;

ii for which the equation is given in polar coordinates.

Plane curves

i for which the equation is given in $x-y$ coordinates;

ii for which the equation is given in polar coordinates.

Three-dimensional figures such as solid and hollow hemispheres and cones.

There are some figures for which interesting geometric derivations can be done without calculus; for example, triangular laminas, and solid tetrahedra, pyramids and cones. And the theorems of Pappus allows you to find the centres of mass of semicircular laminas and arcs in your head with no calculus.

First, some definitions.

Consider several point masses in the $x-y$ plane:

$$m_1 \text{ at } (x_1, y_1)$$

$$m_2 \text{ at } (\ x_2, y_2\ )$$

etc.

The centre of mass is a point $(\bar{x}, \bar{y})$ whose coordinates are defined by

$$\bar{x} = \frac{\sum m_i x_i}{M} \qquad \bar{y} = \frac{\sum m_i y_i}{M} \tag{1.1.1}$$

where M is the total mass $\sum m_i$. The sum $\sum m_i x_i$ is the first moment of mass with respect to the y axis. The sum $\sum m_i y_i$ is the first moment of mass with respect to the x axis.

If the masses are distributed in three-dimensional space, with m_1 at $(x_1, y_1, z_1\ )$, etc,. the centre of mass is a point ($\bar{x}, \bar{y}, \bar{z}$) such that

$$\bar{x} = \frac{\sum m_i x_i}{M} \qquad \bar{y} = \frac{\sum m_i y_i}{M} \qquad \bar{y} = \frac{\sum m_i z_i}{M} \tag{1.1.2}$$

In this case, $\sum m_i x_i$, $\sum m_i y_i$, $\sum m_i z_i$ are the first moments of mass with respect to the $y-z, z-x$ and $x-y$ planes respectively.

In either case we can use vector notation and suppose that $\mathbf{r_1, r_2, r_3}$ are the position vectors of m_1, m_2, m_3 with respect to the origin, and the centre of mass is a point whose position vector $\bar{\mathbf{r}}$ is defined by

$$\bar{r} = \frac{\sum m_i \mathbf{r_i}}{M} \tag{1.1.3}$$

In this case the sum is a vector sum and $\sum m_i \mathbf{r_i}$ a vector quantity, is the first moment of mass with respect to the origin. Its scalar components in the two-dimensional case are the moments with respect to the axes; in the three dimensional case they are the moments with respect to the planes.

Many early books, and some contemporary ones, use the term "centre of gravity". Strictly the centre of gravity is a point whose position is defined by the ratio of the first moment of weight to the total weight. This will be identical to the centre of mass provided that the strength of the gravitational field g (or gravitational acceleration) is the same throughout the space in which the masses are situated. This is usually the case, though it need not necessarily be so in some contexts.

For a plane geometrical figure, the centroid or centre of area, is a point whose position is defined as the ratio of the first moment of area to the total area. This will be the same as the position of the centre of mass of a plane lamina of the same size and shape provided that the lamina is of uniform surface density.

Calculating the position of the centre of mass of various figures could be considered as merely a make-work mathematical exercise. However, the centres of gravity, mass and area have important applications in the study of mechanics.

For example, most students at one time or another have done problems in static equilibrium, such as a ladder leaning against a wall. They will have dutifully drawn vectors indicating the forces on the ladder at the ground and at the wall, and a vector indicating the weight of the ladder. They will have drawn this as a single arrow at the centre of gravity of the ladder as if the entire weight of the ladder could be "considered to act" at the centre of gravity. In what sense can we take this liberty and "consider all the weight as if it were concentrated at the centre of gravity"? In fact the ladder consists of many point masses (atoms) all along its length. One of the equilibrium conditions is that there is no net torque on the ladder. The definition of the centre of gravity is such that the sum of the moments of the weights of all the atoms about the base of the ladder is equal to the total weight times the horizontal distance to the centre of gravity, and it is in that sense that all the weight "can be considered to act" there. Incidentally, in this example, "centre of gravity" is the correct term to use. The distinction would be important if the ladder were in a nonuniform gravitational field.

In dynamics, the total linear momentum of a system of particles is equal to the total mass times the velocity of the centre of mass. This may be "obvious", but it requires formal proof, albeit one that follows very quickly from the definition of the centre of mass.

Likewise the kinetic energy of a rigid body in two dimensions equals $\frac{1}{2}MV^2 + \frac{1}{2}I\omega^2$ where M is the total mass, V the speed of the centre of mass, I the rotational inertia and ω the angular speed, both around the centre of mass. Again it requires formal proof, but in any case it furnishes us with another example to show that the calculation of the positions of centres of mass is more than merely a make-work mathematical exercise and that it has some physical significance.

If a vertical surface is immersed under water (e.g. a dam wall) it can be shown that the total hydrostatic force on the vertical surface is equal to the area times the pressure at the centroid. This requires proof (readily deduced from the definition of the centroid and elementary hydrostatic principles), but it is another example of a physical application of knowing the position of the centroid.

1.2: Plane Triangular Lamina

Definition: A median of a triangle is a line from a vertex to the midpoint of the opposite side.

Theorem I. The three medians of a triangle are concurrent (meet at a single, unique point) at a point that is two-thirds of the distance from a vertex to the midpoint of the opposite side.

Theorem II. The centre of mass of a uniform triangular lamina (or the centroid of a triangle) is at the meet of the medians.

The proof of I can be done with a nice vector argument (Figure I.1):

Let $\mathbf{A}$, $\mathbf{B}$ be the vectors OA , OB . Then $\mathbf{A+B}$ is the diagonal of the parallelogram of which OA and OB are two sides, and the position vector of the point C_1 is $\frac{1}{3}\mathbf{(A+B)}$.

To get C_2, we see that

$$\mathbf{C}_2 = \mathbf{A} + \frac{2}{3}(\mathrm{AM}_2) = \mathbf{A} + \frac{2}{3}(\mathbf{M_2} - \mathbf{A}) = \mathbf{A} + \frac{2}{3}(\frac{1}{2}\mathbf{B} - \mathbf{A}) = \frac{1}{3}(\mathbf{A} + \mathbf{B})$$

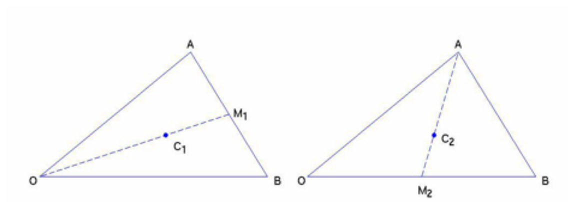


FIGURE I.1

Thus the points C_1 and C_2 are identical, and the same would be true for the third median, so Theorem I is proved.

Now consider an elemental slice as in Figure I.2. The centre of mass of the slice is at its mid-point. The same is true of any similar slices parallel to it. Therefore the centre of mass is on the locus of the mid-points - i.e. on a median. Similarly, it is on each of the other medians, and Theorem II is proved.

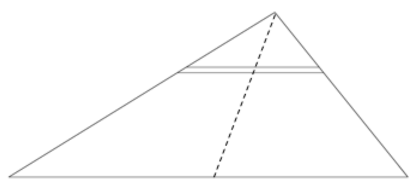


FIGURE 1.2

That needed only some vector geometry. We now move on to some calculus.

1.3: Plane Areas

Plane areas in which the equation is given in $x-y$ *coordinates*

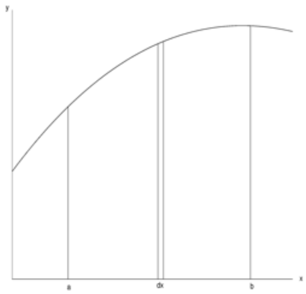


FIGURE I.3

We have a curve $y=y(x)$ (Figure I.3) and we wish to find the position of the centroid of the area under the curve between $x=a$ and $x=b$. We consider an elemental slice of width δx at a distance x from the y axis. Its area is $y\delta x$, and so the total area is

$$A=\int_a^b ydx \tag{1.3.1}$$

The first moment of area of the slice with respect to the y axis is $xy\delta x$, and so the first moment of the entire area is $\int_a^b xydx$.

Therefore

$$\bar{x}=\frac{\int_a^b xydyx}{\int_a^b ydyx}=\frac{\int_a^b xydyx}{A} \tag{1.3.2}$$

$$labeleq:1.3.2$$

For $\bar{y}$ we notice that the distance of the centroid of the slice from the x axis is $\frac{1}{2}y$, and therefore the first moment of the area about the x axis is $\frac{1}{2}y.\,y\delta x$.

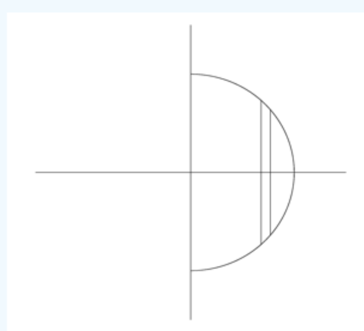
Therefore

$$\bar{y}=\frac{\int_a^b y^2dx}{2A} \tag{1.3.3}$$

Example 1.3.1

Consider a semicircular lamina, $x^2+y^2=a^2$, see Figure I.4:

FIGURE I.4

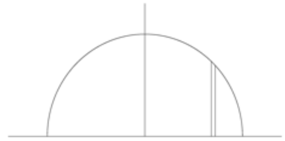


We are dealing with the parts both above and below the \(x \) axis, so the area of the semicircle is $2\int_0^a ydx$ and the first moment of area is $2\int_0^a xydx$.

You should find $\bar{x}=4a/(3\pi)=0.4244a$.

Now consider the lamina $x^2+y^2=a^2$, $y>0$ (Figure I.5):

FIGURE I.5



The area of the elemental slice this time is $y\delta x$ (not $2y\delta x$), and the integration limits are from $-a$ to $+a$. To find $\overline{y}$, use Equation 1.3.3, and you should get $y = 0.4244a$.

Plane areas in which the equation is given in polar coordinates.

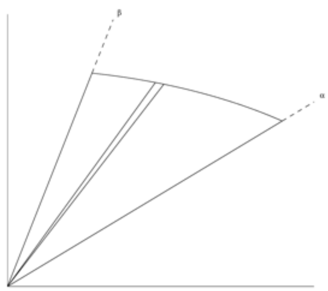


FIGURE I.6

We consider an elemental triangular sector (Figure I.6) between θ and $\theta + \delta\theta$. The "height" of the triangle is r and the "base" is $r\delta\theta$. The area of the triangle is $\frac{1}{2}r^2\delta\theta$.

Therefore the whole area =

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \tag{1.3.4}$$

The horizontal distance of the centroid of the elemental sector from the origin (more correctly, from the "pole" of the polar coordinate system) is $\frac{2}{3}r\cos\theta$. The first moment of area of the sector with respect to the y axis is

$$\tfrac{2}{3}r\cos\theta \times \tfrac{1}{2}r^2\delta\theta = \tfrac{1}{3}r^3\cos\theta\delta\theta$$

so the first moment of area of the entire figure between $\theta = \alpha$ and $\theta = \beta$ is

$$\tfrac{1}{3}\int_{\alpha}^{\beta} r^3\cos\theta d\theta$$

Therefore

$$\overline{x} = \frac{2\int_{\alpha}^{\beta} r^3\cos\theta d\theta}{3\int_{\alpha}^{\beta} r^2 d\theta} \tag{1.3.5}$$

Similarly

$$\overline{x} = \frac{2\int_{\alpha}^{\beta} r^3\sin\theta d\theta}{3\int_{\alpha}^{\beta} r^2 d\theta} \tag{1.3.6}$$

✔ Example 1.3.2

Consider the semicircle $r = a$, $\theta = \frac{-\pi}{2}$ to $\frac{+\pi}{2}$

$$\overline{x} = \frac{2a\int_{-\pi/2}^{+\pi/2}\cos\theta d\theta}{3\int_{-\pi/2}^{+\pi/2} d\theta} = \frac{2a}{3\pi}\int_{-\pi/2}^{+\pi/2}\cos\theta d\theta = \frac{4a}{3\pi} \tag{1.3.7}$$

The reader should now try to find the position of the centroid of a circular sector (slice of pizza!) of angle 2α. The integration limits will be $-\alpha$ to $+\alpha$.

When you arrive at a formula (which you should keep in a notebook for future reference), check that it goes to $\frac{4\alpha}{3\pi}$ if $\alpha = \frac{\pi}{2}$, and to $\frac{2\pi}{3}$ if $\alpha = 0$.

1.4: Plane Curves

Plane Curves Expressed in $x-y$ coordinates

FIGURE I.7

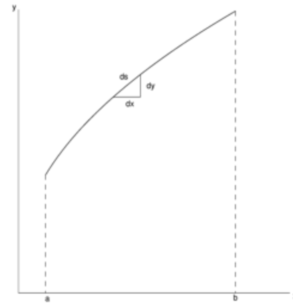


Figure I.7 shows how an elemental length δs is related to the corresponding increments in x and y:

$$\delta s=\sqrt{\delta x^2+\delta y^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\delta x=\sqrt{\left(\frac{dx}{dy}\right)^2+1}\,dy \tag{1.4.1}$$

Consider a wire of mass per unit length (linear density) λ bent into the shape $y=y(x)$ between $x=a$ and $x=b$. The mass of an element ds is $\lambda\delta s$, so the total mass is

$$\int\lambda\,ds=\int_a^b\lambda\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{1.4.2}$$

The first moments of mass about the y - and x -axes are respectively

$$\int_a^b\lambda x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{1.4.3}$$

and

$$\int_a^b\lambda y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{1.4.4}$$

If the wire is uniform and λ is therefore not a function of x or y, λ can come outside the integral signs in Equations 1.4.2 - 1.4.4, and we hence obtain

$$\bar{x}=\frac{\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx} \tag{1.4.5}$$

and

$$\bar{y}=\frac{\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx} \tag{1.4.6}$$

the denominator in each of these expressions merely being the total length of the wire.

> **Example 1.4.1**
>
> Consider a uniform wire bent into the shape of the semicircle $x^2 + y^2 = a^2$, $x > 0$.
>
> First, it might be noted that one would expect $\overline{x} > 0.4244a$ (the value for a plane semicircular lamina).
>
> The length (i.e. the denominators in Equations 1.4.5 and 1.4.6) is just πa. Since there are, between x and $x + \delta x$, two elemental lengths to account for, one above and one below the x axis, the numerator of Equation 1.4.5 must be
>
> $$2\int_0^a x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
>
> In this case
>
> $$y = \sqrt{a^2 - x^2}$$
>
> and
>
> $$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2-x^2}}$$
>
> The first moment of length of the entire semicircle is
>
> $$\overline{x} = 2\int_0^a x\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx = 2a\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}}$$
>
> From this point the student is left to his or her own devices to solve this integral and derive $\overline{x} = \dfrac{2a}{\pi} = 0.6366a$.

Plane Curves Expressed in Polar Coordinates

FIGURE I.8

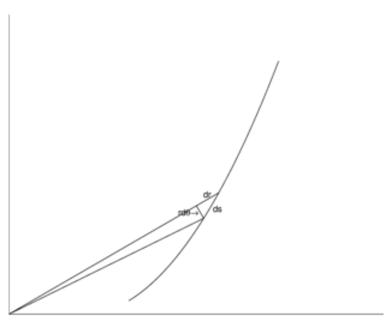


Figure I.8 shows how an elemental length δs is related to the corresponding increments in r and θ:

$$\delta s = \sqrt{(\delta r)^2 + (r\delta\theta)^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,\delta\theta = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\,\delta r. \tag{1.4.7}$$

The mass of the curve (between $\theta = a$ and $\theta = b$) is

$$\int_\alpha^\beta \lambda\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta.$$

The first moments about the y - and x -axes are (recalling that $x = r\cos\theta$ and $y = r\sin\theta$)

$$\int_\alpha^\beta \lambda r\cos\theta\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta$$

and

$$\int_{\alpha}^{\beta} \lambda r \sin\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta.$$

If λ is not a function of r or θ , we obtain

$$\bar{x} = \frac{1}{L} \int_{\alpha}^{\beta} r \cos\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \tag{1.4.8}$$

and

$$\bar{y} = \frac{1}{L} \int_{\alpha}^{\beta} r \sin\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \tag{1.4.9}$$

where L is the length of the wire.

Example 1.4.2

Again consider the uniform wire of Figure I.8 bent into the shape of a semicircle. The equation in polar coordinates is simply $r = a$, and the integration limits are $\theta = \frac{-\pi}{2}$ to $\theta = \frac{+\pi}{2}$ and the length is πa.

Thus

$$\bar{x} = \frac{1}{\pi a} \int_{-\pi/2}^{+\pi/2} a cos\theta [0 - a^2]^{\frac{1}{2}}\, d\theta = \frac{2a}{\pi}.$$

The reader should now find the position of the center of mass of a wire bent into the arc of a circle of angle 2α. The expression obtained should go to $\frac{2a}{\pi}$ as α goes to $\frac{\pi}{2}$, and to a as α goes to zero.

1.5: Summary of the Formulas for Plane Laminas and Curves

Uniform Plane Lamina

$y = y(x)$	$r = r(\theta)$
$\bar{x} = \frac{1}{A}\int_a^b xydx$ $\bar{y} = \frac{1}{2A}\int_a^b y^2 dx$	$\bar{x} = \frac{2\int_\alpha^\beta r^3 cos\theta d\theta}{3\int_\alpha^\beta r^2 d\theta}$ $\bar{y} = \frac{2\int_\alpha^\beta r^3 sin\theta d\theta}{3\int_\alpha^\beta r^2 d\theta}$

Uniform Plane Curve

$y = y(x)$	$r = r(\theta)$
$\bar{x} = \frac{1}{L}\int_a^b x[1+(\frac{dy}{dx})^2]^{\frac{1}{2}}$ $\bar{y} = \frac{1}{L}\int_a^b y[1+(\frac{dy}{dx})^2]^{\frac{1}{2}}$	$\bar{x} = \frac{1}{L}\int_\alpha^\beta rcos\theta[(\frac{dr}{d\theta})^2+r^2]^{\frac{1}{2}}$ $\bar{y} = \frac{1}{L}\int_\alpha^\beta rsin\theta[(\frac{dr}{d\theta})^2+r^2]^{\frac{1}{2}}$

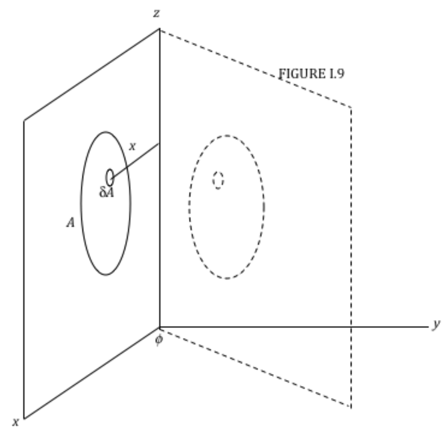
1.6: The Theorems of Pappus

(Pappus Alexandrinus, Greek mathematician, approximately 3rd or 4th century AD.)

I. If a plane area is rotated about an axis in its plane, but which does not cross the area, the volume swept out equals the area times the distance moved by the centroid.
II. If a plane curve is rotated about an axis in its plane, but which does not cross the curve, the area swept out equals the length times the distance moved by the centroid.

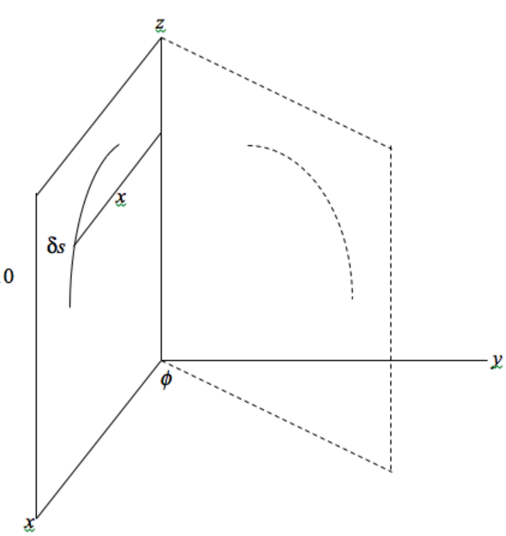
These theorems enable us to work out the volume of a solid of revolution if we know the position of the centroid of a plane area, or *vice versa*; or to work out the area of a surface of revolution if we know the position of the centroid of a plane curve or *vice versa*. It is not necessary that the plane or the curve be rotated through a full 360º.

We prove the theorems first. We then follow with some examples.



Consider an area A in the zx plane (Figure I.9), and an element δA within the area at a distance x from the z axis. Rotate the area through an angle ϕ about the z axis. The length of the arc traced by the element dA in moving through an angle ϕ is $x\phi$, so the volume swept out by δA is $x\phi\delta A$. The volume swept out by the entire area is $\phi\int xdA$. But the definition of the centroid of A is such that its distance from the z axis is given by $\overline{x}A = \int xdA$. Therefore the volume swept out by the area is $\phi\overline{x}A$. But $\phi\overline{x}$ is the distance moved by the centroid, so the first theorem of Pappus is proved.

FIGURE I.10



Consider a curve of length L in the zx plane (Figure I.10), and an element δs of the curve at a distance x from the z axis. Rotate the curve through an angle ϕ about the z axis. The length of the arc traced by the element δsin moving through an angle ϕ is $x\phi$, so the area swept out by δs is $x\phi\delta s$. The area swept out by the entire curve is $\phi \int xds$. But the definition of the centroid is such that its distance from the z axis is given by $\bar{x}L = \int xds$. Therefore the area swept out by the curve is $\phi\bar{x}L$. But $\phi\bar{x}$ is the distance moved by the centroid, so the second theorem of Pappus is proved.

Applications of the Theorems of Pappus

Rotate a plane semicircular figure of area $\frac{1}{2}\pi a^2$ through 360° about its diameter. The volume swept out is $\frac{4}{3}\pi a^3$, and the distance moved by the centroid is $2\pi\bar{x}$ Therefore by the theorem of Pappus, $\bar{x} = \frac{4a}{(3\pi)}$.

Rotate a plane semicircular arc of length π a through 360° about its diameter. Use a similar argument to show that $\bar{x} = \frac{2a}{\pi}$.

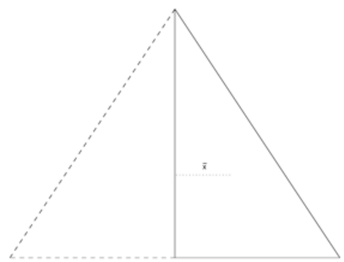
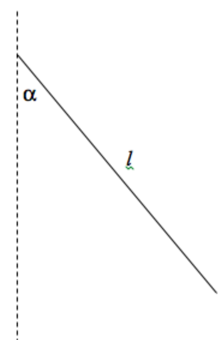


FIGURE I.11

Consider a right-angled triangle, height h, base a (Figure I.11). Its centroid is at a distance $\frac{a}{3}$ from the height h. The area of the triangle is $\frac{ah}{2}$. Rotate the triangle through 360° about h. The distance moved by the centroid is $\frac{2\pi a}{3}$. The volume of the cone swept out is $\frac{ah}{2}$ times $\frac{2\pi}{3}$, equals $\frac{\pi a^2 h}{3}$.

Now consider a line of length l inclined at an angle α to the y axis (Figure I.12). Its centroid is at a distance $\frac{1}{2} l \sin\alpha$ from the y axis. Rotate the line through 360° about the y axis. The distance moved by the centroid is $2\pi \times \frac{1}{2} l \sin\alpha = \pi l \sin\alpha$. The surface area of the cone swept out is $l \times \pi\, l \sin\alpha = \pi l^2 \sin\alpha$.

FIGURE I.12



The centre of a circle of radius b is at a distance a from the y axis. It is rotated through 360° about the y axis to form a torus (Figure I.13). Use the theorems of Pappus to show that the volume and surface area of the torus are, respectively, $2\pi^2 ab^2$ and $4\pi^2 ab$.

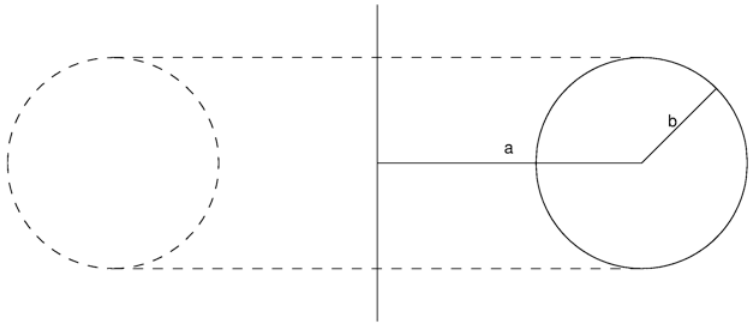


FIGURE I.13

1.7: Uniform Solid Tetrahedron, Pyramid and Cone

Definition

A median of a tetrahedron is a line from a vertex to the centroid of the opposite face.

Theorem I.

The four medians of a tetrahedron are concurrent at a point 3/4 of the way from a vertex to the centroid of the opposite face.

Theorem II

The centre of mass of a uniform solid tetrahedron is at the meet of the medians.

Theorem I can be derived by a similar vector geometric argument used for the plane triangle. It is slightly more challenging than for the plane triangle, and it is left as an exercise for the reader. I draw two diagrams (Figure I.14). One shows the point C_1 that is 3/4 of the way from the vertex A to the centroid of the opposite face. The other shows the point C_2 that is 3/4 of the way from the vertex B to the centroid of its opposite face. You should be able to show that

$$\mathbf{C_1} = (\mathbf{A}+\mathbf{B}+\mathbf{D})/4$$

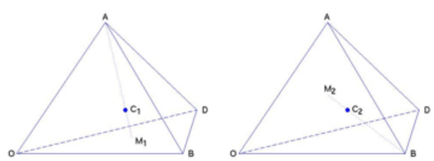


FIGURE I.14

In fact this suffices to prove Theorem I, because, from the symmetry between $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$, one is bound to arrive at the same expression for the three-quarter way mark on any of the four medians. But for reassurance you should try to show, from the second figure, that

$$\mathbf{C}_{2}=(\mathbf{A+B+D})/4$$

The argument for Theorem II is easy, and is similar to the corresponding argument for plane triangles.

Pyramid.

A right pyramid whose base is a regular polygon (for example, a square) can be considered to be made up of several tetrahedra stuck together. Therefore the centre of mass is 3/4 of the way from the vertex to the mid point of the base.

Cone.

A right circular cone is just a special case of a regular pyramid in which the base is a polygon with an infinite number of infinitesimal sides. Therefore the centre of mass of a uniform right circular cone is 3/4 of the way from the vertex to the centre of the base.

We can also find the position of the centre of mass of a solid right circular cone by calculus. We can find its volume by calculus, too, but we'll suppose that we already know, from the theorem of Pappus, that the volume is $\frac{1}{3}$ × base × height.

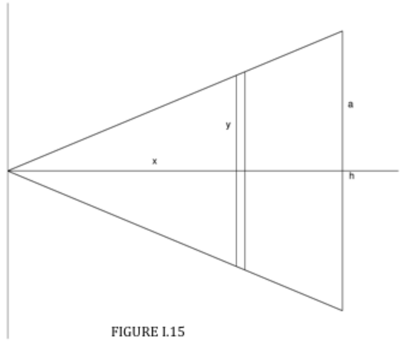


FIGURE I.15

Consider the cone in Figure I.15, generated by rotating the line $y=\frac{ax}{h}$ (between $x=0$ and $x=h$) through 360º about the x axis. The radius of the elemental slice of thickness dx at x is $\frac{ax}{h}$. Its volume is $\frac{\pi a^2x^2\delta x}{h^2}$.

Since the volume of the entire cone is $\frac{\pi a^2 h}{3}$, the mass of the slice is

$$M\times\frac{\pi a^2x^2\delta x}{h^2}\div\frac{\pi a^2h}{3}=\frac{3Mx^2\delta x}{h^3}$$

where M is the total mass of the cone. The first moment of mass of the elemental slice with respect to the y axis is $\frac{3Mx^3dx}{h^3}$.

The position of the centre of mass is therefore

$$\overline{x}=\frac{3}{h^3}\int_0^h x^3dx=\frac{3}{4}h$$

1.8: Hollow Cone

The surface of a hollow cone can be considered to be made up of an infinite number of infinitesimally slender isosceles triangles, and therefore the centre of mass of a hollow cone (without base) is 2/3 of the way from the vertex to the midpoint of the base.

1.9: Hemispheres

Uniform solid hemisphere

Figure I.4 will serve. The argument is exactly the same as for the cone. The volume of the elemental slice is $\pi y^2 \delta x = \pi(a^2 - x^2)\delta x$ and the volume of the hemisphere is $\frac{2\pi a^3}{3}$, so the mass of the slice is

$$M \times \pi(a^2 - x^2)\delta x \div (2\pi a/3) = \frac{3M(a^2 - x^2)\delta x}{2a^3}$$

where M is the mass of the hemisphere. The first moment of mass of the elemental slice is x times this, so the position of the centre of mass is

$$\overline{x} = \frac{3}{2a^3}\int_0^a x(a^2 - x^2)dx = \frac{3a}{8}$$

Hollow hemispherical shell.

We may note to begin with that we would expect the centre of mass to be further from the base than for a uniform solid hemisphere.

Again, Figure I.4 will serve. The area of the elemental annulus is $2\pi a\delta x$ (NOT $2\pi y\delta x$!) and the area of the hemisphere is $2\pi a^2$. Therefore the mass of the elemental annulus is

$$M \times 2\pi a\delta x \div (2\pi a^2) = M\delta x/a$$

The first moment of mass of the annulus is x times this, so the position of the centre of mass is

$$\overline{x} = \int_0^a \frac{xdx}{a} = \frac{a}{2}$$

1.S: Centers of Mass (Summary)

SUMMARY

Triangular lamina: 2/3 of way from vertex to midpoint of opposite side

Solid Tetrahedron, Pyramid, Cone: 3/4 of way from vertex to centroid of opposite face.

Hollow cone: 2/3 of way from vertex to midpoint of base.

Semicircular lamina: $\frac{4a}{3\pi}$

Lamina in form of a sector of a circle, angle $2\alpha:\frac{(2a\sin\alpha)}{(3\alpha)}$

Semicircular wire: $\frac{2a}{\pi}$

Wire in form of an arc of a circle, angle $2\alpha:\frac{(a\sin\alpha)}{\alpha}$

Solid hemisphere: $\frac{3a}{8}$

Hollow hemisphere: $\frac{a}{2}$

CHAPTER OVERVIEW

2: Moments of Inertia

In this chapter we shall consider how to calculate the (second) moment of inertia for different sizes and shapes of body, as well as certain associated theorems. But the question should be asked: "What is the purpose of calculating the squares of the distances of lots of particles from an axis, multiplying these squares by the mass of each, and adding them all together?

2.1: Definition of Moment of Inertia

Consider a straight line (the "axis") and a set of point masses m_1, m_2, m_3 ... such that the distance of the mass m_i from the axis is r_i. The quantity $m_i r_i^2$ is the second moment of the i th mass with respect to (or "about") the axis, and the sum $\sum m_i r_i^2$ is the second moment of mass of all the masses with respect to the axis.

Apart from some subtleties encountered in general relativity, the word "inertia" is synonymous with mass - the inertia of a body is merely the ratio of an applied force to the resulting acceleration. Thus $\sum m_i r_i^2$ can also be called the **second moment of inertia**. The second moment of inertia is discussed so much in mechanics that it is usually referred to as just "the" moment of inertia.

In this chapter we shall consider how to calculate the (second) moment of inertia for different sizes and shapes of body, as well as certain associated theorems. But the question should be asked: "What is the purpose of calculating the squares of the distances of lots of particles from an axis, multiplying these squares by the mass of each, and adding them all together? Is this merely a pointless make-work exercise in arithmetic? Might one just as well, for all the good it does, calculate the sum $\sum m_i r_i^2$? Does $\sum r_i m_i^2$ have any physical significance?"

2.2: Meaning of Rotational Inertia

If a force acts of a body, the body will accelerate. The ratio of the applied force to the resulting acceleration is the inertia (or mass) of the body.

If a torque acts on a body that can rotate freely about some axis, the body will undergo an angular acceleration. *The ratio of the applied torque to the resulting angular acceleration is the rotational inertia* of the body. It depends not only on the mass of the body, but also on how that mass is distributed with respect to the axis.

Consider the system shown in Figure II.1.

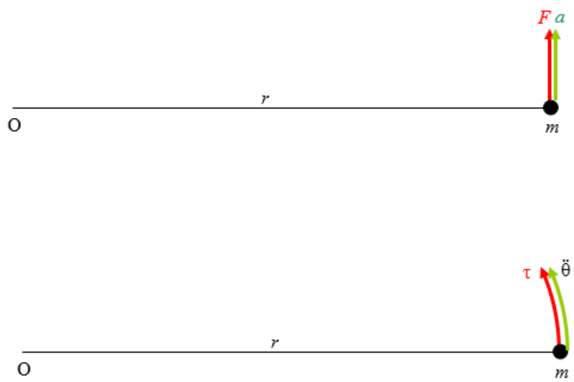


FIGURE II.1

A particle of mass m is attached by a light (i.e. zero or negligible mass) arm of length r to a point at O, about which it can freely rotate. A force F is applied, and the mass consequently undergoes a linear acceleration $a = \frac{F}{m}$. The angular acceleration is then $\ddot{\Theta} = F.mr$. Also, the torque is $\tau = Fr$. The ratio of the applied torque to the angular acceleration is therefore mr^2. Thus the rotational inertia is the second moment of inertia. Rotational inertia and (second) moment of inertia are one and the same thing, except that rotational inertia is a physical concept and moment of inertia is its mathematical representation.

2.3: Moments of Inertia of Some Simple Shapes

A student may well ask: "For how many different shapes of body must I commit to memory the formulas for their moments of inertia?" I would be tempted to say: "None". However, if any are to be committed to memory, I would suggest that the list to be memorized should be limited to those few bodies that are likely to be encountered very often (particularly if they can be used to determine quickly the moments of inertia of other bodies) and for which it is easier to remember the formulas than to derive them. With that in mind I would recommend learning no more than five. In the following, each body is supposed to be of mass m and rotational inertia I.

Formula 1.

A rod of length $2l$ about an axis through the middle, and at right angles to the rod:

$$I=\frac{1}{3}ml^2 \tag{2.3.1}$$

Formula 2.

A uniform circular disc of radius a about an axis through the center and perpendicular to the plane of the disc:

$$I=\frac{1}{2}ma^2 \tag{2.3.2}$$

Formula 3.

A uniform right-angled triangular lamina about one of its shorter sides - i.e. not the hypotenuse. The other not-hypotenuse side is of length a:

$$I=\frac{1}{6}ma^2 \tag{2.3.3}$$

Formula 4.

A uniform solid sphere of radius a about an axis through the center.

$$I=\frac{2}{5}ma^2 \tag{2.3.4}$$

Formula 5.

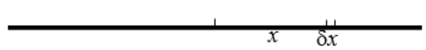
A uniform spherical shell of radius a about an axis through the center.

$$I=\frac{2}{3}ma^2 \tag{2.3.5}$$

I shall now derive the first three of these by calculus. The derivations for the spheres will be left until later.

1.Rod, length $2l$ (Figure II.2)

FIGURE II.2



The mass of an element δx at a distance x from the middle of the rod is $\frac{m\delta x}{2l}$.

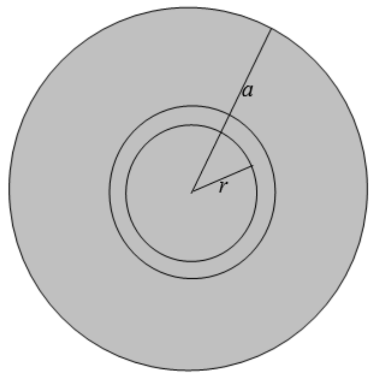
and its second moment of inertia is

$$\frac{mx^2\delta x}{2l}.$$

$$\frac{m}{2l}\int_{-l}^{l}x^2dx = \frac{m}{l}\int_0^l x^2dx = \frac{1}{3}ml^2.$$

2. Disc, radius a . (Figure II.3)

FIGURE II.3



The area of an elemental annulus, radii r is $r+\delta r$ is $2\pi r\delta r$.

The area of the entire disc is πa^2.

Therefore the mass of the annulus is

$$\frac{2\pi r\delta rm}{\pi a^2} = \frac{2mr\delta r}{a^2}.$$

and its second moment of inertia is

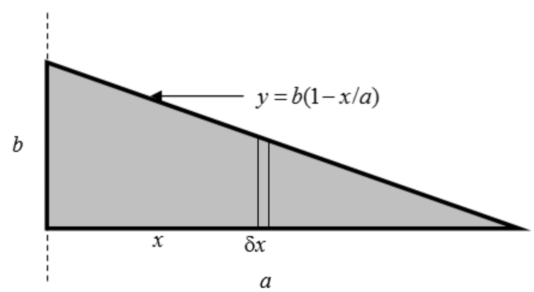
$$\frac{2mr^3\delta r}{a^2}.$$

The moment of inertia of the entire disc is

$$\frac{2m}{a^2}\int_0^a r^3dr = \frac{1}{2}ma^2.$$

3. Right-angled triangular lamina. (Figure II.4)

FIGURE II.4



The equation to the hypotenuse is $y = b(1-x/a)$.

The area of the elemental strip is $y\delta x = b(1-x/a)\delta x$ and the area of the entire triangle is $\frac{ab}{2}$.

Therefore the mass of the elemental strip is $\frac{2m(a-x)\delta x}{a^2}$.

and its second moment of inertia is $\frac{2mx^2(a-x)\delta x}{a^2}$.

The second moment of inertia of the entire triangle is the integral of this from $x = 0$ to $x = a$, which is $\frac{ma^2}{6}$.

Uniform circular lamina about a diameter.

For the sake of one more bit of integration practice, we shall now use the same argument to show that the moment of inertia of a uniform circular disc about a diameter is $\frac{ma^2}{4}$. However, we shall see later that it is not necessary to resort to integral calculus to arrive at this result, nor is it necessary to commit the result to memory. In a little while it will become immediately apparent and patently obvious, with no calculation, that the moment of inertia must be $\frac{ma^2}{4}$. However, for the time being, let us have some more calculus practice. See Figure II.5.

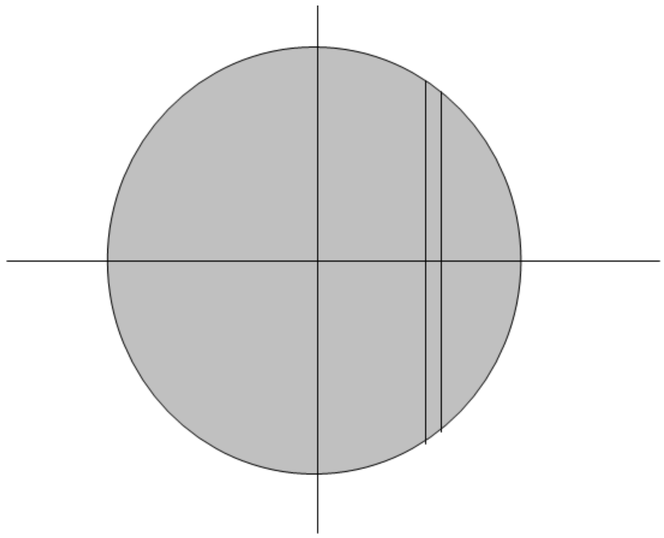


FIGURE II.5

The disc is of radius a, and the area of the elemental strip is $2y\delta x$. But y and x are related through the equation to the circle, which is $y = (a^2 - x^2)^{1/2}$. Therefore the area of the strip is $2(a^2 - x^x)^{1/2}\delta x$. The area of the whole disc is πa^2, so the mass of the strip is

$$m \times \frac{2(a^2 - x^2)^{1/2}\delta x}{\pi a^2} = \frac{2m}{\pi a^2} \times (a^2 - x^2)^{1/2}\delta x.$$

The second moment of inertia about the y-axis is

$$\frac{2m}{\pi a^2} \times x^2(a^2 - x^2)^{1/2}\delta x.$$

For the entire disc, we integrate from $x = -a$ to $x = +a$, or, if you prefer, from $x = 0$ to $x = a$ and then double it. The result $\frac{ma^2}{4}$ should follow. If you need a hint about how to do the integration, let $x = acos\theta$ (which it is, anyway), and be sure to get the limits of integration with respect to θ right.

The moment of inertia of a uniform *semi*circular lamina of mass m and radius a about its base, or diameter, is also $\frac{ma^2}{4}$, since the mass distribution with respect to rotation about the diameter is the same. $\frac{ma^2}{4}$, since the mass distribution with respect to rotation about the diameter is the same.

2.4: Radius of Gyration

The second moment of inertia of any body can be written in the form mk^2. Thus, for the rod, the disc (about an axis perpendicular to its plane), the triangle and the disc (about a diameter), k has the values

- $\dfrac{l}{\sqrt{3}} = 0.866l$,
- $\dfrac{a}{\sqrt{2}} = 0.707a$,
- $\dfrac{a}{\sqrt{6}} = 0.408a$, and
- $\dfrac{a}{2} = 0.500a$

respectively.

k is called the **radius of gyration**. If you were to concentrate all the mass of a body at its radius of gyration, its moment of inertia would remain the same.

2.5: Plane Laminas and Mass Points distributed in a Plane

In Figure II.6a, the two unbroken lines represent two fixed coordinate axes. I have drawn several point masses m_1, m_2, m_3 distributed in a plane.

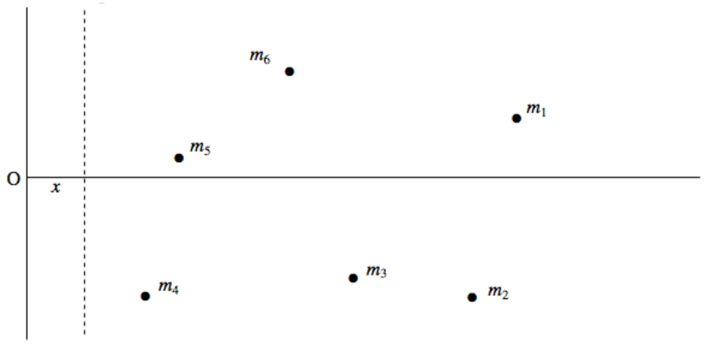


FIGURE II.6a

The x-coordinate of mass m_i is x_i. The dashed line is moveable, and it x-coordinate is x, so that the distance of m_i this line is $x_i - x$ The moment of inertia of the system of masses about the dashed line is

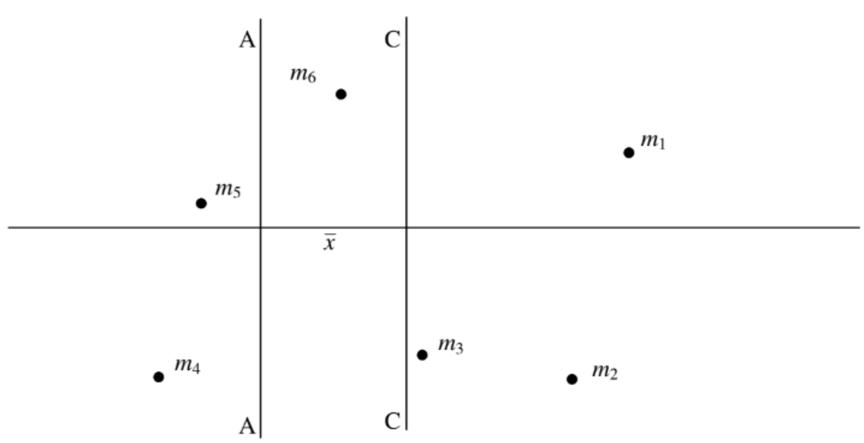
$$I = m_1(x_1 - x)^2 + m_2(x_2 - x)^2 + m_3(x_3 - x)^2 + \ldots. \tag{2.5.1}$$

Now imagine what happens if the dashed line is moved to the right. The moment of inertia decreases – and decreases - and decreases. But eventually the line finds itself to the right of m_4, and then of m_5, and then of m_6. After that is by no means obvious that the moment of inertia is going to continue to decrease. Indeed, by this time it is clear that at some point I is going to go through a minimum and then start to increase again as more and more of the masses find themselves to the left of the dashed line. Just where is the dashed line when the moment of inertia is a minimum? I'll leave you to differentiate Equation 2.5.1 with respect to x, and hence show that I is least when

$$x = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} \tag{2.5.2}$$

That is, the moment of inertia is least when $\bar{x} = x$. That is, the moment of inertia is least for an axis passing through the centre of mass.

In Figure II.6b, the line CC passes through the centre of mass; the moment of inertia is least about this line. The line AA is at a distance $\bar{x}$ from CC, and the moment of inertia is greater about AA than about CC. The *Parallel Axes Theorem* tells us by how much.



Let us measure distances from CC, so that the distance of m_i from CC is x_i and the distance of m_i from AA is $x_i + \bar{x}$.

It is clear that $I_{CC} = \sum m_i x_i^2$

and that

$$I_{AA} = \sum m_i (x_i + \overline{x})^2 = \sum m_i x_i^2 + 2\overline{x} \sum m_i x_i + \overline{x}^2 \sum m_i. \tag{2.5.3}$$

The first term on the right hand side is I_{CC}. The sum in the second term is the first moment of mass about the centre of mass, and is zero. The sum in the third term is the total mass. We therefore arrive at the *Parallel Axes Theorem.*

$$I_{AA} = I_{CC} + M\overline{x}^2. \tag{2.5.4}$$

In words, the moment of inertia about an arbitrary axis is equal to the moment of inertia about a parallel axis through the centre of mass plus the total mass times the square of the distance between the parallel axes. The theorem holds also for masses distributed in three-dimensional space.

The *Perpendicular Axes Theorem*, on the other hand, *holds only for masses distributed in a plane, or for plane laminas.*

FIGURE II.7

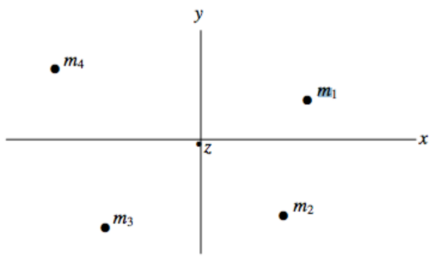


Figure II.7 shows some point masses distributed in the xy plane, the z axis being perpendicular to the plane of the paper. The moments of inertia about the x, y and z axes are denoted respectively by A, B and C. The distance of m_i from the z axis is $(x_i^2 + y_i^2)^{\frac{1}{2}}$ Therefore the moment of inertia of the masses about the z axis is

$$C = \sum m_i (x_i^2 + y_i^2) \tag{2.5.5}$$

That is to say:

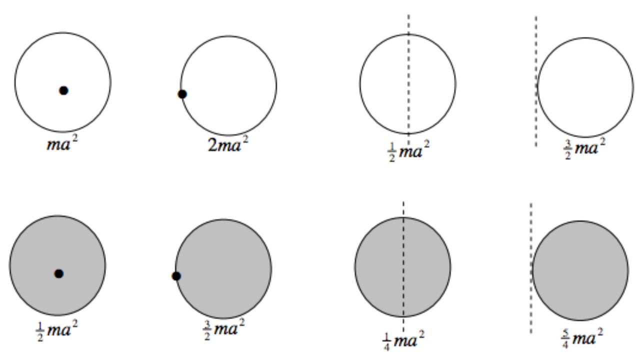
$$C = A + B \tag{2.5.6}$$

This is the *Perpendicular Axes Theorem*. Note again very carefully that, unlike the parallel axes theorem, this theorem applies only to plane laminas and to point masses distributed in a plane.

Examples of the Use of the Parallel and Perpendicular Axes Theorems.

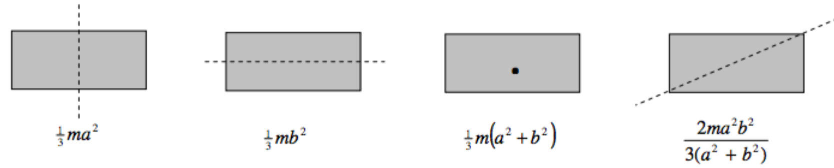
From Section 2.3 we know the moments of inertia of discs, rods and triangular laminas. We can make use of the parallel and perpendicular axes theorems to write down the moments of inertia of most of the following examples almost by sight, with no calculus.

Hoop and discs, radius a.

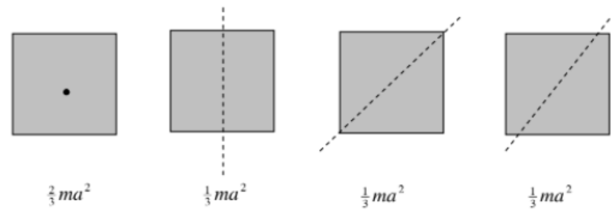


Rods, length $2l$.

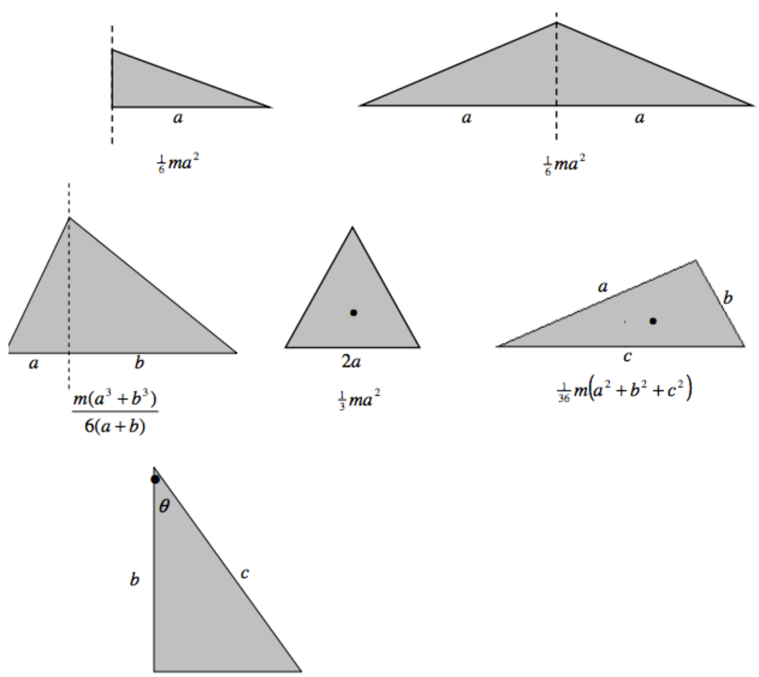
Rectangular laminas, sides $2a$ and $2b$; $a > b$.



Square laminas, side $2a$.



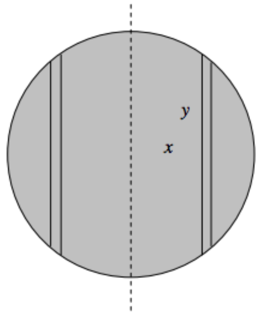
Triangular laminas.



$$I = \frac{1}{6}ma^2(1+3cot^2\theta) = \frac{1}{6}mb^2(3+tan^2\theta) = \frac{1}{6}mc^2(3-2sin^2\theta)$$
$$= \frac{1}{6}m(2b^2+c^2) = \frac{1}{6}m(3c^2-2a^2) = \frac{1}{6}m(a^3b^2)$$

2.6: Three-dimensional Solid Figures. Spheres, Cylinders, Cones.

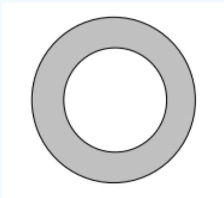
Sphere, mass m, radius a.



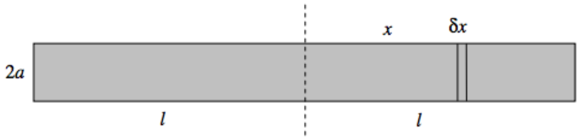
The volume of an elemental cylinder of radii x, $x+\delta x$, height $2y$ is $4\pi yx\delta x = 4\pi(a^2-x^2)^{1/2}x\delta x$. Its mass is $m \times \frac{4\pi(a^2-x^2)^{\frac{1}{2}}x\delta x}{\frac{4}{3}\pi a^3} = \frac{3m}{a^3}\times(a^2-x^2)^{\frac{1}{2}}x\delta x$. It' second moment of intertia is $= \frac{3m}{a^3}\times(a^2-x^2)^{\frac{1}{2}}x^3\delta x$. The second moment of inertia of the entire sphere is

$$= \frac{3m}{a^3}\times\int_0^a(a^2-x^2)^{\frac{1}{2}}x^3\delta x = \frac{2}{5}ma^2.$$

The moment of inertia of a uniform solid hemisphere of mass m and radius a about a diameter of its base is also , $\frac{2}{5}ma^2$, because the distribution of mass around the axis is the same as for a complete sphere.

> **? Exercise 2.6.1**
>
> A hollow sphere is of mass M, external radius a and internal radius xa. Its rotational inertia is $0.5Ma^2$. Show that x is given by the solution of
>
> 
>
> $$1-5x^3+4x^5=0$$
>
> and calculate x to four significant figures.
>
> (Answer = 0.6836.)

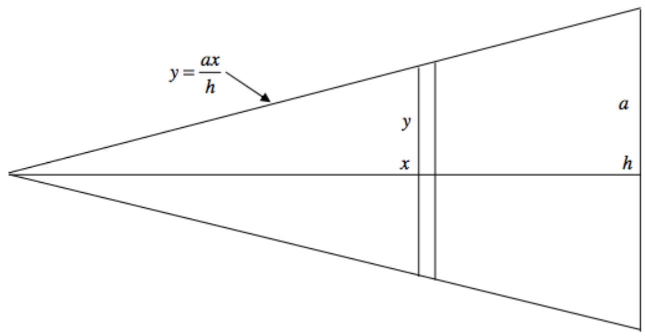
Solid cylinder, mass m, radius a, length $2l$



The mass of an elemental disc of thickness δx is $\frac{m\delta x}{2l}$. Its moment of inertia about its diameter is $\frac{1}{4}\frac{m\delta x}{2l}a^2 = \frac{ma^2\delta x}{8l}$. Its moment of inertia about the dashed axis through the centre of the cylinder is $\frac{ma^2\delta x}{8l}+\frac{m\delta x}{2l}x^2 = \frac{m(a^2+4x^2)\delta x}{8l}$. The moment of inertia of the entire cylinder about the dashed axis is $2\int_0^1\frac{m(a^2+4x^2)\delta x}{8l} = m\left(\frac{1}{4}a^2+\frac{1}{3}l^2\right)$.

In a similar manner it can be shown that the moment of inertia of a uniform solid triangular prism of mass m, length $2l$, cross section an equilateral triangle of side $2a$about an axis through its centre and perpendicular to its length is $m\left(\frac{1}{6}a^2+\frac{1}{3}l^2\right)$.

Solid cone, mass m, height h, base radius a.



The mass of elemental disc of thickness δx is

$$m \times \frac{\pi y^2 \delta x}{\frac{1}{3}\pi a^2 h} = \frac{3my^2 \delta x}{a^2 h} .$$

Its second moment of inertia about the axis of the cone is

$$\frac{1}{2} \times \frac{3my^2 \delta x}{a^2 h} \times y^2 = \frac{3my^4 \delta x}{2a^2 h} .$$

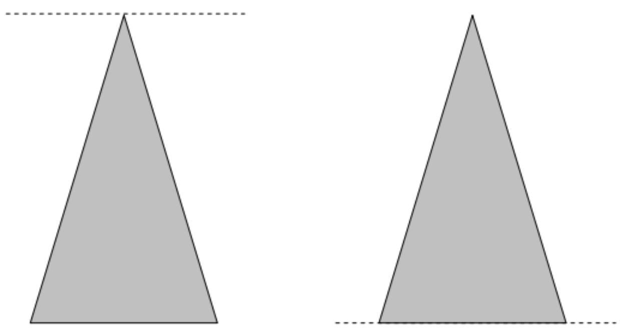
But y and x are related through $y = \frac{ax}{h}$, so the moment of inertia of the elemental disk is

$$\frac{3ma^2 x^4 \delta x}{2h^5}.$$

The moment of inertia of the entire cone is

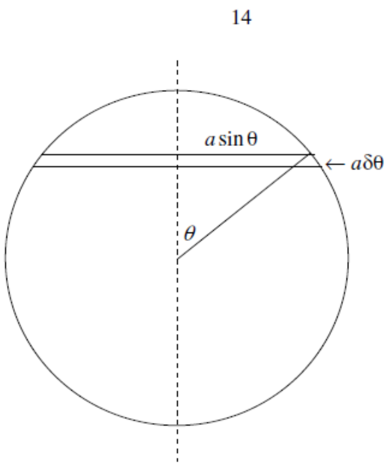
$$\frac{3ma^2}{2h^5} \int_0^h x^4 dx = \frac{3ma^2}{10} .$$

The following, for a solid cone of mass m, height h, base radius a, are left as an exercise:



2.7: Three-dimensional Hollow Figures. Spheres, Cylinders, Cones

Hollow*spherical shell, mass m, radius a*



The area of the elemental zone is $2\pi a^2 \sin\theta\delta\theta$. Its mass is

$$m \times \frac{2\pi a^2 \sin\theta\delta\theta}{4\pi a^2} = \frac{1}{2} m \sin\theta\delta\theta$$

Its moment of inertia is $\frac{1}{2} m \sin\theta\delta\theta \times a^2 \sin^2\theta = \frac{1}{2} ma^2 \sin^3\theta\delta\theta$

The moment of inertia of the entire spherical shell is

$$\frac{1}{2} ma^2 \int_0^\pi \sin^3\theta\delta\theta = \frac{2}{3} ma^2$$

This result can be used to calculate, by integration, the moment of inertia $\frac{2}{5}ma^2$ of a solid sphere. Or, if you start with $\frac{2}{5}ma^2$ for a solid sphere, you can differentiate to find the result $\frac{2}{3}ma^2$ for a hollow sphere. Write the moment of inertia for a solid sphere in terms of its density rather than its mass. Then add a layer da and calculate the increase dI in the moment of inertia. We can also use the moment of inertia for a hollow sphere ($\frac{2}{3}ma^2$) to calculate the moment of inertia of a nonuniform solid sphere in which the density varies as $\rho = \rho(r)$. For example, if $\rho = \rho_0 \sqrt{1 - (\frac{r}{a})^2}$, see if you can show that the mass of the sphere is $2.467\rho_0 a^3$ and that its moment of inertia is $\frac{1}{3}ma^2$. A much easier method will be found in Section 19.

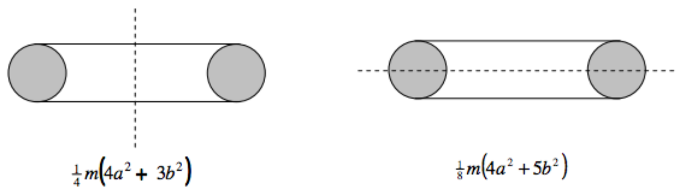
Using methods similar to that given for a solid cylinder, it is left as an exercise to show that the moment of inertia of an open hollow cylinder about an axis perpendicular to its length passing through its centre of mass is $m(\frac{1}{2}a^2 + \frac{1}{3}l^2)$, where a is the radius and $2l$ is the length.

The moment of inertia of a baseless hollow cone of mass m, base radius a, about the axis of the cone could be found by integration. However, those who have an understanding of the way in which the moment of inertia depends on the distribution of mass should readily see, without further ado, that the moment of inertia is $\frac{1}{2}ma^2$. (Look at the cone from above; it looks just like a disc, and indeed it has the same radial mass distribution as a uniform disc.)

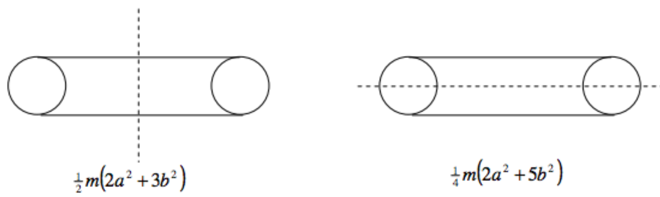
2.8: Torus

The rotational inertias of solid and hollow toruses (large radius a, small radius b) are given below for reference and without derivation. They can be derived by integral calculus, and their derivation is recommended as a challenge to the reader.

Solid torus:



Hollow torus:



2.9: Linear Triatomic Molecule

Here is an interesting problem. It should be straightforward to calculate the rotational inertia of the above molecule with respect to an axis perpendicular to the molecule and passing through the center of mass. In practice it is quite easy to *measure* the rotational inertia very precisely from the spacing between the lines in a molecular band in the infrared region of the spectrum.

If you know the three masses (which you do if you know the atoms that make up the molecule) can you calculate the two interatomic spacings x and y ? That would require determining two unknown quantities, x and y, from a single measurement of the rotational inertia, I. Evidently that cannot be done; a second measurement is required. Can you suggest what might be done? We shall answer that shortly. In the meantime, it is an exercise to show that the rotational inertia is given by

$$ax^2 + 2hxy + by^2 + c = 0, \tag{2.9.1}$$

where

$$a = m_1(m_2 + m_3)/M \tag{2.9.2}$$

$$h = m_1 m_3/M \tag{2.9.3}$$

$$b = m_3(m_1 + m_2)/M \tag{2.9.4}$$

$$M = m_1 + m_2 + m_3 \tag{2.9.4}$$

$$c = -I \tag{2.9.5}$$

Example 2.9.1: OCS

Suppose the molecule is the linear molecule OCS, and the three masses are 16, 12 and 32 respectively, and, from infrared spectroscopy, it is determined that the moment of inertia is 20. (For this hypothetical illustrative example, I am not concerning myself with units). In that case, equation 2.9.1 becomes

$$11.7\bar{3}x^2 + 17.0\bar{6}xy + 14.9\bar{3}y^2 - 20 = 0. \tag{2.9.6}$$

We need another equation to solve for x and y. What can be done chemically is to prepare an isotopically-substituted molecule (isotopomer) such as ^{18}OCS, and measure *its* moment of inertia from its spectrum, making the probably very justified assumption that the interatomic distances are unaffected by the isotopic substitution. This results in a second equation:

$$a'x^2 + 2h'xy + b'y^2 + c' = 0. \tag{2.9.7}$$

Let's suppose that the new moment of inertia is $I' = 21$, and I leave it to the reader to work out the numerical values of a', h' and b' with the stern caution to retain all the decimal places on your calculator. That is, do not round off the numbers until the very end of the calculation.

You now have two equations, 2.9.1 and 2.9.7, to solve for x and y. These are two simultaneous quadratic equations, and it may be that some guidance in solving them would be helpful. I have three suggestions.

1. Treat equation 2.9.1 as a quadratic equation in x and solve it for x in terms of y. Then substitute this in equation 2.9.7. I expect you will *very* soon become bored with this method and will want to try something a little less tedious.
2. You have two equations of the form $S(x, y) = 0$, $S'(x, y) = 0$. There are standard ways of solving these iteratively by an extension of the Newton-Raphson process. This is described, for example, in Section 1.9 of my **Celestial Mechanics** notes, and this general method for two or more nonlinear equations should be known by anyone who expects to engage in much numerical calculation.

For this particular case, the detailed procedure would be as follows. This is an iterative method, and it is first necessary to make a guess at the solutions for x and y. The guesses need not be particularly good. That done, compute the following six quantities:

$$S = x(ax + 2hy) + by^2 + c$$

$$S' = x(a'x + 2h'y) + b'y^2 + c'$$
$$S_x = 2(ax + hy)$$
$$S_y = 2(hx + by)$$
$$S'_x = 2(a'x + h'y)$$
$$S'_y = 2(h'x + b'y)$$

Here the subscripts denote the partial derivatives. Now if

$$x \text{ (true)} = x \text{ (guess)} + \epsilon$$

and

$$y \text{ (true)} = y \text{ (guess)} + \eta$$

the errors ϵ and η can be found from the solution of

$$S_x\epsilon + S_y\eta + S = 0$$

and

$$S'_x\epsilon + S'_y\eta + S' = 0$$

If we calculate

$$F = \frac{1}{S_yS'_x - S_xS'_y}$$

The solutions for the errors are

$$\epsilon = F(S'_yS - S_yS')$$
$$\eta = F(S_xS' - S'_xS)$$

This will enable a better guess to be made, and the procedure can be repeated until the errors are as small as desired. Generally only a very few iterations are required. If this is not the case, a programming mistake is indicated.

3. While method 2 can be used for any nonlinear simultaneous equations, in this particular case we have two simultaneous quadratic equations, and a little familiarity with conic sections provides a rather nice method.

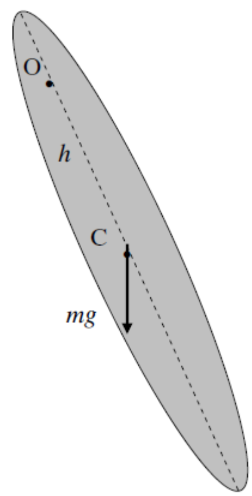
Thus, if $S = 0$ and $S' = 0$ are equations 2.9.1 and 2.9.7 respectively. Each of these equations represents a conic section, and they intersect at four points. We wish to find the point of intersection that lies in the all-positive quadrant - i.e. with x and y both positive. Since the two conic sections are very similar, in order to calculate where they intersect it is necessary to calculate with great accuracy. Therefore, do not round off the numbers until the very end of the calculation. Form the equation $c'S - cS' = 0$. This is also a quadratic equation representing a conic section passing through the four points. Furthermore, it has no constant term, and it therefore represents the two straight lines that pass through the four points. The equation can be factorized into two linear terms, $\alpha\beta - 0$, where $\alpha = 0$ and $\beta = 0$ are the two straight lines. Choose the one with positive slope and solve it with $S = 0$ or with $S' = 0$ (or with both, as a check against arithmetic mistakes) to find x and y. In this case, the solutions are $x = 0.2529, y = 1.000$

2.10: Pendulums

In Section 2.2, we discussed the physical meaning of the rotational inertia as being the ratio of the applied torque to the resulting angular acceleration. In linear motion, we are familiar with the equation $F = ma$. The corresponding Equation when dealing with torques and angular acceleration is $\tau = I\ddot{\theta}$. We are also familiar with the equation of motion for a mass vibrating at the end of a spring of force constant k : $m\ddot{x} = -kx$. This is simple harmonic motion of period $2\pi\sqrt{\frac{m}{k}}$. The mechanics of the *torsion pendulum* is similar.

The torsion constant c of a wire is the torque required to twist it through unit angle. If a mass is suspended from a torsion wire, and the wire is twisted through an angle θ, the restoring torque will be $c\theta$, and the Equation of motion is $I\ddot{\theta} = -c\theta$, which is simple harmonic motion of period $2\pi\sqrt{\frac{I}{c}}$. The torsion constant of a wire of circular cross-section, by the way, is proportional to its shear modulus, the fourth power of its radius, and inversely as its length. The derivation of this takes a little trouble, but it can be verified by dimensional analysis. Thus a thick wire is very much harder to twist than a thin one. A wire of narrow rectangular cross-section, such as a strip or a ribbon has a relatively small torsion constant.

Now let's look not at a torsion pendulum, but at a pendulum swinging about an axis under gravity.



We suppose the pendulum, of mass m, is swinging about a point O, which is at a distance h from the center of mass C. The rotational inertia about O is I. The line OC makes an angle θ with the vertical, so that the horizontal distance between O and C is $h \sin\theta$. The torque about O is $mgh \sin\theta$, so that the equation of motion is

$$I\ddot{\theta} = -mgh \sin\theta. \tag{2.10.1}$$

For small angles ($\sin\theta \approx \theta$), this is

$$I\ddot{\theta} = -mgh\theta. \tag{2.10.2}$$

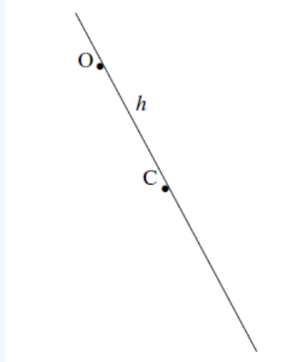
This is simple harmonic motion of period

$$P = 2\pi\sqrt{\frac{I}{mgh}}. \tag{2.10.3}$$

We'll look at two examples - a uniform rod, and an arc of a circle.

Example 2.10.1

First, a uniform rod.



The center of mass is C. The rotational inertia about C is $\frac{1}{3}ml^2$, so the rotational inertia about O is $I = \frac{1}{3}ml^2 + mh^2$. If we substitute this in equation 2.10.3, we find for the period of small oscillations

$$P = 2\pi\sqrt{\frac{l^2+3h^2}{3gh}}. \tag{2.10.4}$$

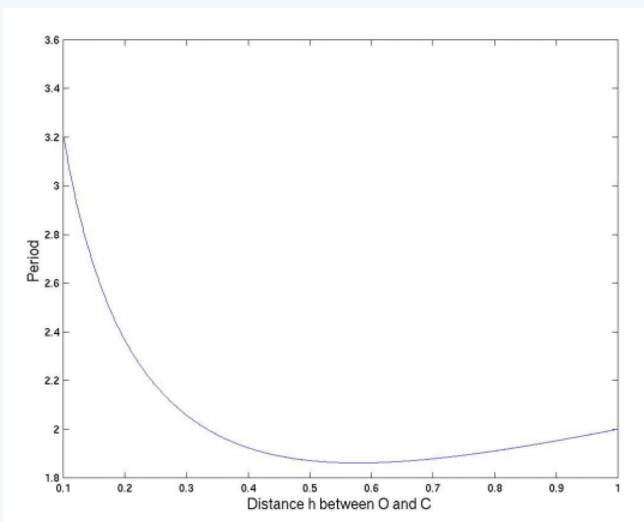
This can be written

$$P = 2\pi\sqrt{\frac{l}{3g}} \cdot \sqrt{\frac{l+3(\frac{h}{l})^2}{\frac{h}{l}}}. \tag{2.10.5}$$

or, if we write $\mathsf{P} = \frac{P}{2\pi\sqrt{\frac{l}{3g}}}$ and $\mathsf{h} = \frac{h}{l}$:

$$P = \sqrt{\frac{1+3\mathsf{h}^2}{\mathsf{h}}}. \tag{2.10.6}$$

The figure shows a graph of P versus h.



Equation 2.10.6 can be written

$$\mathsf{P}^2 = \frac{1}{\mathsf{h}} + 3\mathsf{h}. \tag{2.10.7}$$

and, by differentiation of P^2 with respect to h, we find that the period is least when $\mathsf{h} = \frac{1}{\sqrt{3}}$.

This least period is given by $\mathsf{P}^2 = \sqrt{12}$,or $\mathsf{P} = 1.861$.

Equation 2.10.7 can also be written

$$3h^2 - P^2h + 1 = 0 \tag{2.10.8}$$

This quadratic Equation shows that there are two positions of the support O that give rise to the same period of small oscillations. The period is least when the two solutions of Equation 2.10.8 are equal, and by the theory of quadratic Equations, then, the least period is given by $P^2 = \sqrt{12}$ as we also deduced by differentiation of Equation 2.10.7, and this occurs when $h = \frac{1}{\sqrt{3}}$.

For periods longer than this, there are two solutions for h. Let h_1 be the smaller of these, and let h_2 be the larger. By the theory of quadratic Equations, we have

$$h_1 + h_2 = \frac{1}{3}P^2 \tag{2.10.9}$$

and

$$h_1h_2 = \frac{1}{3} \tag{2.10.10}$$

Let $H = h_2 - h_1$ be the distance between two points O that give the same period of oscillation.

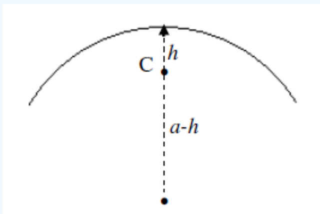
Then

$$H^2 = (h_2 - h_1)^2 = (h_2 + h_1)^2 - 4h_1h_2 = \frac{P^4 - 12}{9} \tag{2.10.11}$$

If we measure H for a given period P and recall the definition of P we see that this provides a method for determining g. Although this is a common undergraduate laboratory exercise, the graph shows that the minimum is very shallow and consequently H and hence g are very difficult to measure with any precision.

Example 2.10.2

For another example, let us look at a wire bent into the arc of a circle of radius a oscillating in a vertical plane about its mid-point. In the figure, C is the center of mass.



The rotational inertia about the centerof the circle is ma^2. By two applications of the parallel axes theorem, we see that the rotational inertia about the point of oscillation is $I = ma^2 - m(a-h)^2 + mh^2 = 2mah$ .Thus, from Equation 2.10.3 we find

$$P = 2\pi\sqrt{\frac{2a}{g}}, \tag{2.10.12}$$

and the period is independent of the length of the arc.

2.11: Plane Laminas. Product Moment. Translation of Axes (Parallel Axes Theorem)

We consider a set of point masses distributed in a plane, or a plane lamina. We have hitherto met three second moments of inertia:

$$A=\sum m_i y_i^2, \tag{2.11.1}$$

$$B=\sum m_i x_i^2, \tag{2.11.2}$$

$$C=\sum m_i (x_i^2+y_i^2), \tag{2.11.3}$$

These are respectively the moments of inertia about the x- and y-axes (assumed to be in the plane of the masses or the lamina) and the z-axis (normal to the plane). Clearly, $C=A+B$, which is the perpendicular axes theorem for a plane lamina.

We now introduce another quantity, H, called the *product moment of inertia* with respect to the x- and y-axes, defined by

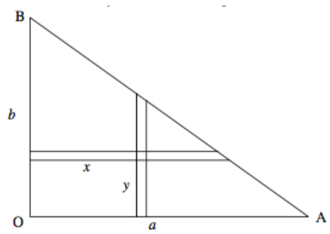
$$H=\sum m_i x_i y_i \tag{2.11.4}$$

We'll need sometime to ask ourselves whether this has any particular physical significance, or whether it is merely something to calculate for the sake of passing the time of day. In the meantime, the reader should recall the parallel axes theorems (Section 2.5) and, using arguments similar to those given in that section, should derive

$$H=H_C+M\bar{x}\bar{y} \tag{2.11.5}$$

It may also be noted that Equation 2.11.4 does not contain any squared terms and therefore the product moment of inertia, depending on the distribution of masses, is just as likely to be a negative quantity as a positive one.

We shall defer discussing the physical significance, if any, of the product moment until section 2.12. In the meantime let us try to calculate the product moment for a plane right triangular lamina:



The area of the triangle is $\frac{1}{2}ab$ and so the mass of the element $\delta x\delta x$ is $\frac{2M\delta x\delta y}{ab}$, where M is the mass of the complete triangle. The product moment of the element with respect to the sides OA, OB is $\frac{2Mxy\delta x\delta y}{ab}$ and so the product moment of the entire triangle is $\frac{2M}{ab}\int\int xydxdy$. We have to consider carefully the limits of integration. We'll integrate first with respect to x ; that is to say we integrate along the horizontal (y constant) strip from the side OB to the side AB. That is to say we integrate $x\delta x$ from where $x=0$ to where $x=a(1-\frac{y}{b})$. The product moment is therefore

$$\frac{2M}{ab}\int y\frac{1}{2}a^2(1-\frac{y}{b})^2dy.$$

We now have to add up all the horizontal strips from the side OA, where $y=0$, to B, where $y=b$.

Thus

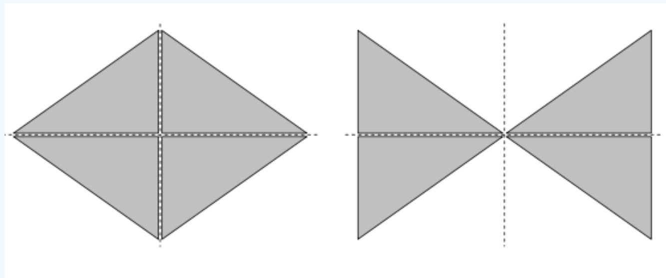
$$H=\frac{Ma}{b}\int_0^b y\left(1-\frac{y}{b}\right)^2dy,$$

which, after some algebra, comes to $H=\frac{1}{12}Mab$.

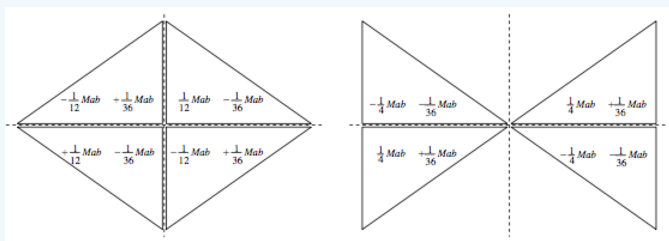
The coordinates of the centre of mass with respect to the sides OA, OB are $\frac{1}{3}a$, $\frac{1}{3}b$, so that, from Equation 2.11.5, we find that the product moment with respect to axes parallel to OA, OB and passing through the centre of mass is $-\frac{1}{36}Mab$.

? Exercise 2.11.1

Calculate the product moments of the following eight laminas, each of mass M, with respect to horizontal and vertical axes through the origin, and with respect to horizontal and vertical axes through the centroid of each. (We have just done the first of these, above.) The horizontal base of each is of length a, and the height of each is b. You are going to have to take great care with the signs, and with the limits of integration. If you get an answer right except for the sign, then you have got the answer wrong.

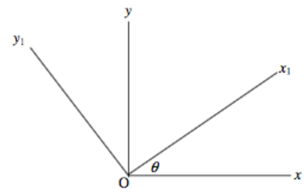


I make the answers as follows.



2.12: Rotation of Axes

We start by recalling a result from elementary geometry. Consider two sets of axes Oxy and Ox_1y_1, the latter being inclined at an angle θ to the former. Any point in the plane can be described by the coordinates (x, y) or by (x_1, y_1).



These coordinates are related by a rotation matrix:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.12.1}$$

The rotation matrix is orthogonal; one of the several properties of an orthogonal matrix is that its reciprocal is its transpose.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{2.12.2}$$

Now let us apply this to the moments of inertia of a plane lamina. Let us suppose that the axes are in the plane of the lamina and that O is the centre of mass of the lamina. A, B and H are the moments of inertia with respect to the axes Oxy, and A_1, B_1 and H_1 are the moments of inertia with respect to Ox_1y_1. Strictly speaking a lamina implies a continuous distribution of matter in a plane, but, since matter, we are told, is composed of discrete atoms, there is little difficulty in justifying treating a lamina as though it we a distribution of point masses in the plane. In any case the results that follow are valid either for a collection of point masses in a plane or for a genuine continuous lamina.

We have, by definition:

$$A_1 = \sum m y_1^2 \tag{2.12.3}$$

$$B_1 = \sum m x_1^2 \tag{2.12.4}$$

$$H_1 = \sum m x_1 y_1 \tag{2.12.5}$$

Now let us apply Equation 2.12.1to Equation 2.12.3:

$$\begin{aligned} A_1 &= \sum m(-x\sin\theta + y\cos\theta)^2 \\ &= \sin^2\theta \sum mx^2 - 2\sin\theta\cos\theta \sum mxy + \cos^2\theta \sum my^2. \end{aligned}$$

That is to say (writing the third term first, and the first term last)

$$A_1 = A\cos^2\theta - 2H\sin\theta\cos\theta + B\sin^2\theta. \tag{2.12.6}$$

In a similar fashion, we obtain for the other two moments

$$B_1 = A\sin^2\theta + 2H\sin\theta\cos\theta + B\cos^2\theta. \tag{2.12.7}$$

and

$$H_1 = A\sin\theta\cos\theta + H\sin(\cos^2\theta - \sin^2\theta) - B\sin\theta\cos\theta. \tag{2.12.8}$$

It is usually more convenient to make use of trigonometric identities to write these as

$$A_1 = \frac{1}{2}(B+A) - \frac{1}{2}(B-A)\cos 2\theta - H\sin 2\theta, \tag{2.12.9}$$

$$B_1 = \frac{1}{2}(B+A) + \frac{1}{2}(B-A)\cos 2\theta + H\sin 2\theta, \tag{2.12.10}$$

$$H_1 = H\cos 2\theta - \frac{1}{2}(B - A)\sin 2\theta \tag{2.12.11}$$

These equations enable us to calculate the moments of inertia with respect to the axes Ox_1y_1 if we know the moments with respect to the axes Oxy. Further, a matter of importance, we see, from Equation 2.12.11, that if

$$\tan 2\theta = \frac{2H}{B - A}, \tag{2.12.12}$$

the product moment H_1 with respect to the axes Oxy is zero. This gives some physical meaning to the product moment, namely: If we *can* find some axes (which we can, by means of Equation 2.12.12) with respect to which the product moment is zero, these axes are called the principal axes of the lamina, and the moments of inertia with respect to the principal axes are called the *principal moments of inertia*. I shall use the symbols A_0 and B_0 for the principal moments of inertia, and I shall adopt the convention that $A_0 \le B_0$.

Example 2.12.1

Consider three point masses at the coordinates given below:

Mass	Coordinates
5	(1 , 1)
3	(4 , 2)
2	(3 , 4)

The moments of inertia are $A = 49, B = 71, C = 53$. The coordinates of the centre of mass are (2.3 , 1.9). If we use the parallel axes theorem, we can find the moments of inertia with respect to axes parallel to the original ones but with origin at the centre of mass. With respect to these axes we find $A = 12.9, B = 18.1, H = +9.3$. The principal axes are therefore inclined at angles θ to the x -axis given (Equation 2.12.12) by $\tan 2\theta = 3.57669$; That is θ = 37°11' and 127° 11'. On using Equation 2.12.9 or 2.12.10 with these two angles, together with the convention that $A_0 \le B_0$, we obtain for the principal moments of inertia $A_0 = 5.84$ and $B_0 = 25.16$.

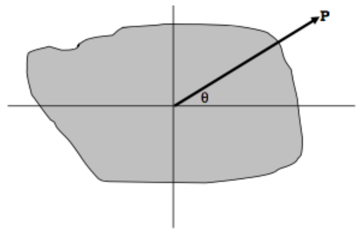
Example 2.12.2

Consider the right-angled triangular lamina of Section 11. The moments of inertia with respect to axes passing through the centre of mass and parallel to the orthogonal sides of the triangle are $A = \frac{1}{18}Mb^2, B = \frac{1}{18}Ma^2, H = -\frac{1}{36}Mab$. The angles that the principal axes make with the a- side are given by $\tan 2\theta = \frac{ab}{b^2 - a^2}$. The interested reader will be able to work out expressions, in terms of M, a, b, for the principal moments.

2.13: Momental Ellipse

Consider a plane lamina such that its radius of gyration about some axis through the centre of mass is k. Let $\mathbf{P}$ be a vector in the direction of that axis, originating at the centre of mass, given by

$$\mathbf{P} = \frac{a^2}{k}\hat{\mathbf{r}} \tag{2.13.1}$$



Here $\hat{\mathbf{r}}$ is a unit vector in the direction of interest; k is the radius of gyration, and a is an arbitrary length introduced so that the dimensions of $\mathbf{P}$ are those of length, and the length of the vector$\mathbf{P}$ is inversely proportional to the radius of gyration. The moment of inertia is $Mk^2 = \frac{Ma^4}{P^2}$. That is to say

$$\frac{Ma^4}{P^2} = A\cos^2\theta - 2H\sin\theta\cos\theta + B\sin^2\theta, \tag{2.13.2}$$

where A, H and B are the moments with respect to the x- and y-axes. Let (x, y) be the coordinates of the tip of the vector $\mathbf{P}$, so that $x = P\cos\theta$ and $y = P\sin\theta$. Then

$$Ma^4 = Ax^2 - 2Hxy + By^2. \tag{2.13.2}$$

Thus, no matter what the shape of the lamina, however irregular and asymmetric, the tip of the vector $\mathbf{P}$ traces out an ellipse, whose axes are inclined at angles $\frac{1}{2}\tan^{-1}(\frac{2H}{B-A})$ to the x - axis.

This is the *momental ellipse*, and the axes of the momental ellipse are the principal axes of the lamina.

Example 2.13.1

Consider a regular n-gon. By symmetry the moment of inertia is the same about any two axes in the plane inclined at $2\pi/n$ to each other. This is possible only if the momental ellipse is a circle. It follows that the moment of inertia of a uniform polygonal plane lamina is the same about any axis in its plane and passing through its centroid.

Exercise 2.13.1

Show that the moment of inertia of a uniform plane n - gon of side $2a$ about any axis in its plane and passing through its centroid is $\frac{1}{12}ma^2(1 + 3\cot^2(\pi/n))$.

What is this for a square? For an equilateral triangle?

2.14: Eigenvectors and Eigenvalues

In Sections 11-13, we have been considering some aspects of the moments of inertia of plane laminas, and we have discussed such matters as rotation of axes, and such concepts as product moments of inertia, principal axes, principal moments of inertia and the momental ellipse. We next need to develop the same concepts with respect to three-dimensional solid bodies. In doing so, we shall need to make use of the algebraic concepts of eigenvectors and eigenvalues. If you are already familiar with such matters, you may want to skip this section and move on to the next. If the ideas of eigenvalues and eigenvectors are new to you, or if you are a bit rusty with them, this section may be helpful. I do assume that the reader is at least familiar with the elementary rules of matrix multiplication.

Example 2.14.1

Consider what happens when you multiply a vector, for example the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

by a square matrix, for example the matrix $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ we obtain:

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The result of the operation is another vector that is in quite a different direction from the original one.

However, now let us multiply the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by the same matrix. The result is $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$. The result of the multiplication is merely to multiply the vector by 3 without changing its direction. The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a very special one, and it is called an *eigenvector* of the matrix, and the multiplier 3 is called the corresponding *eigenvalue*. "Eigen" is German for "own" in the sense of "my own book". There is one other eigenvector of the matrix; it is the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Try it; you should that the corresponding eigenvalue is 2.

In short, given a square matrix **A**, if you can find a vector **a** such that **Aa** = λ **a**, where λ is merely a scalar multiplier that does not change the direction of the vector **a**, then **a** is an eigenvector and λ is the corresponding eigenvalue.

In the above, I told you what the two eigenvectors were, and you were able to verify that they were indeed eigenvectors and you were able to find their eigenvalues by straightforward arithmetic. But, what if I hadn't told you the eigenvectors? How would you find them?

Let $\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector with corresponding eigenvalue λ. Then we must have

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$

That is,

$$(A_{11} - \lambda)x_1 + A_{12}x_2 = 0$$

and

$$A_{21}x_1 + (A_{22} - \lambda)x_2 = 0.$$

These two equations are consistent only if the determinant of the coefficients is zero. That is,

$$\begin{bmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{bmatrix} = 0$$

This equation is a quadratic equation in λ, known as the *characteristic equation*, and its two roots, the *characteristic* or *latent roots* are the eigenvalues of the matrix. Once the eigenvalues are found the ratio of x_1 to x_2 is easily found, and hence the eigenvectors.

Similarly, if **A** is a 3 × 3 matrix, the characteristic equation is

$$\begin{bmatrix} A_{11}-\lambda & A_{12} & A_{13} \\ A_{21} & A_{22}-\lambda & A_{23} \\ A_{31} & A_{32} & A_{33}-\lambda \end{bmatrix} = 0$$

This is a cubic equation in λ, the three roots being the eigenvalues. For each eigenvalue, the ratio $x_1 : x_2 : x_3$ can easily be found and hence the eigenvectors. The characteristic equation is a cubic equation, and is best solved numerically, not by algebraic formula. The cubic equation can be written in the form

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

and the solutions can be checked from the following results from the theory of equations:

$$\lambda_1 + \lambda_2 + \lambda_3 = -a_2,$$

$$\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = a_1,$$

$$\lambda_1\lambda_2\lambda_3 = -a_0.$$

2.15: Solid Body

The moments of inertia of a collection of point masses distributed in three-dimensional space (or of a solid three-dimensional body, which, after all, is a collection of point masses (atoms)) with respect to axes Oxyz are:

$$\begin{array}{ll} A=\sum m(y^2+z^2) & F=\sum myz \\ B=\sum m(z^2+x^2) & G=\sum mzx \\ C=\sum m(x^2+y^2) & H=\sum mxy \end{array}$$

Suppose that A, B, C, F, G, H, are the moments and products of inertia with respect to axes whose origin is at the centre of mass. The *parallel axes theorems* (which the reader should prove) are as follows: Let P be some point not at the centre of mass, such that the coordinates of the centre of mass with respect to axes parallel to the axes Oxyz but with origin at P are $(\overline{x}, \overline{y}, \overline{z})$.

Then the moments and products of inertia with respect to the axes through P are

$$\begin{array}{ll} A+M(\overline{y}^2+\overline{z}^2) & F+M\overline{yz} \\ B+M(\overline{z}^2+\overline{x}^2) & G+M\overline{zx} \\ C+M(\overline{x}^2+\overline{y}^2) & H+M\overline{yx} \end{array}$$

where M is the total mass.

Unless stated otherwise, in what follows we shall suppose that the moments and products of inertia under discussion are referred to a set of axes with the centre of mass as origin.

2.16: Rotation of Axes - Three Dimensions

Let Oxyz be one set of mutually orthogonal axes, and let O$x_1y_1z_1$ be another set of axes inclined to the first. The coordinates (x_1, y_1, z_1) of a point with respect to the second basis set are related to the coordinates (x, y, z) with respect to the first by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2.16.1}$$

Here the c_{ij} are the cosines of the angles between the axes of one basis set with respect to the axes of the other. For example, c_{12} is the cosine of the angle between Ox_1 and Oy, and c_{23} is the cosine of the angles between Oy_1 and Oz.

Some readers may know how to express these cosines in terms of complicated expressions involving the *Eulerian angles*. While these are important, they are not essential for following the present development, so we shall not make use of the Eulerian angles just here.

The matrix of direction cosines is *orthogonal*. Among the several properties of an orthogonal matrix is the fact that its reciprocal (inverse) is equal to its transpose - i.e. the reciprocal of an orthogonal matrix is found merely my interchanging the rows and columns. This enables us easily to find (x, y, z) in terms of (x_1, y_1, z_1).

A number of other properties of an orthogonal matrix are useful in detecting, locating and even correcting arithmetic mistakes in computing the elements. These properties are

1. The sum of the squares of the elements in any row or column is unity. This merely expresses the fact that the magnitude of a unit vector along any of the six axes is indeed unity.
2. The sum of the products of corresponding elements of any two rows or of any two columns is zero. This merely expresses the fact that the scalar product of any two orthogonal vectors is zero. It will be noted that checking for property 1 will not detect any mistakes in sign of the elements, whereas checking for property 2 will do so.
3. Every element is equal to ± its own cofactor. This expresses the fact that the cross product of two unit orthogonal vectors is equal to the third.
4. The determinant of the matrix is ± 1. If the sign is negative, it means that the chiralities (handedness) of the two basis sets of axes are opposite; i.e. one of them is a right-handed set and the other is a left-handed set. It is usually convenient to choose both sets as right- handed.

If it is possible to find a set of axes with respect to which the product moments F, G and H are all zero, these axes are called the principal axes of the body, and the moments of inertia with respect to these axes are the principal moments of inertia, for which we shall use the notation A_0, B_0, C_0, with the convention $A_0 \leq B_0 \leq C_0$. We shall see shortly that it is indeed possible, and we shall show how to do it. A vector whose length is inversely proportional to the radius of gyration traces out in space an ellipsoid, known as the *momental ellipsoid.*

In the study of solid body rotation (whether by astronomers studying the rotation of asteroids or by chemists studying the rotation of molecules) bodies are classified as follows.

1. $A_0 \neq B_0 \neq C_0$ The ellipsoid is a triaxial ellipsoid, and the body is an *asymmetric top.*
2. $A_0 < B_0 = C_0$ The ellipsoid is a prolate spheroid and the body is a *prolate symmetric top.*
3. $A_0 = B_0 < C_0$ The ellipsoid is an oblate spheroid and the body is an *oblate symmetric top.*
4. $A_0 = B_0 = C_0$ The ellipsoid is a sphere and the body is a *spherical top.*
5. One moment is zero. The ellipsoid is an infinite elliptical cylinder, and the body is a *linear top.*

Example 2.16.1

We know from Section 2.5 that the moment of inertia of a plane square lamina of side $2a$ about an axis through its centroid and perpendicular to its area is $\frac{2}{3}ma^2$, and it will hence be obvious that the moment of inertia of a uniform solid cube of side 2a about an axis passing through the mid-points of opposite sides is also $\frac{2}{3}ma^2$. It will clearly be the same about an axis passing through the mid-points of *any* pairs of opposite sides. Therefore the cube is a spherical top and the momental ellipsoid is a sphere. Therefore the moment of inertia of a uniform solid cube about any axis through its centre (including, for example, a diagonal) is also $\frac{2}{3}ma^2$.

Exercise 2.16.1

What is the ratio of the length to the diameter of a uniform solid cylinder such that it is a spherical top?

Answer:

$\sqrt{3}/2 = 0.866$.

Let us note in passing that

$$A+B+C=2\sum m(x^2+y^2+z^2)=2\sum mr^2, \tag{2.16.2}$$

which is independent of the orientation of the basis axes In other words, regardless of how A, B and C may depend on the orientation of the axes with respect to the body, the sum $A+B+C$ is invariant under a rotation of axes.

We shall deal with the determination of the principal axes in Section 2.18 - but don't skip Section 2.17.

2.17: Solid Body Rotation and the Inertia Tensor

It is intended that this chapter should be limited to the calculation of the moments of inertia of bodies of various shapes, and not with the huge subject of the rotational dynamics of solid bodies, which requires a chapter on its own. In this section I mention merely for interest two small topics involving the principal axes, and a third topic in a bit more detail as necessary before proceeding to Section 2.18.

Everyone knows that the relation between translational kinetic energy and linear momentum is $E = p^2/(2m)$. Similarly rotational kinetic energy is related to angular momentum L by $E = L^2/(2I)$, where I is the moment of inertia. If an isolated body (such as an asteroid) is rotating about a non-principal axis, it will be subject to internal stresses. If the body is nonrigid this will result in distortions (strains) which may cause the body to vibrate. If in addition the body is inelastic the vibrations will rapidly die out (if the damping is greater than critical damping, indeed, the body will not even vibrate). Energy that was originally rotational kinetic energy will be converted to heat (which will be radiated away.) The body loses rotational kinetic energy. In the absence of external torques, however, L remains constant. Therefore, while E diminishes, I increases. The body adjusts its rotation until it is rotating around its axis of maximum moment of inertia, at which time there are no further stresses, and the situation remains stable.

In general the rotational motion of a solid body whose momental ellipse is triaxial is quite complicated and chaotic, with the body tumbling over and over in apparently random fashion. However, if the body is nonrigid and inelastic (as all real bodies are in practice), it will eventually end up rotating about its axis of maximum moment of inertia. The time taken for a body, initially tumbling chaotically over and over, until it reaches its final blissful state of rotation about its axis of maximum moment of inertia, depends on how fast it is rotating. For most irregular small asteroids the time taken is comparable to or longer than the age of formation of the solar system, so that it is not surprising to find some asteroids with non-principal axis (NPA) rotation. However, a few rapidly-rotating NPA asteroids have been discovered, and, for rapid rotators, one would expect PA rotation to have been reached a long time ago. It is thought that something (such as a collision) must have happened to these rapidly-rotating NPA asteroids relatively recently in the history of the solar system.

Another interesting topic is that of the *stability* of a rigid rotator that is rotating about a principal axis, against small perturbations from its rotational state. Although I do not prove it here (the proof can be done either mathematically, or by a qualitative argument) rotation about either of the axes of maximum or of minimum moment of inertia is stable, whereas rotation about the intermediate axis is unstable. The reader can observe this for him- or herself. Find anything that is triaxial - such as a small block of wood shaped as a rectangular parallelepiped with unequal sides. Identify the axes of greatest, least and intermediate moment of inertia. Toss the body up in the air at the same time setting it rotating about one or the other of these axes, and you will be able to see for yourself that the rotation is stable in two cases but unstable in the third.

Inertia Tensor

I now deal with a third topic in rather more detail, namely the relation between angular momentum $\mathbf{L}$ and angular velocity $\boldsymbol{\omega}$. The reader will be familiar from elementary (and two- dimensional) mechanics with the relation $L = I\omega$. What we are going to find in the three- dimensional solid-body case is that the relation is $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$. Here $\mathbf{L}$ and $\boldsymbol{\omega}$ are, of course, vectors, but they *are not necessarily parallel to each other.* They are parallel only if the body is rotating about a principal axis of rotation. The quantity $\mathbb{I}$ is a tensor known as the *inertia tensor*. Readers will be familiar with the equation $\mathbf{F} = m\mathbf{a}$. Here the two vectors are in the same direction, and m is a scalar quantity that does not change the direction of the vector that it multiplies. A tensor usually (unless its matrix representation is *diagonal*) changes the direction as well as the magnitude of the vector that it multiplies. The reader might like to think of other examples of tensors in physics. There are several. One that comes to mind is the permittivity of an anisotropic crystal; in the equation $\mathbf{D} = \boldsymbol{\epsilon}\mathbf{E}$ and $\mathbf{E}$ are not parallel unless they are both directed along one of the crystallographic axes.

If there are no external torques acting on a body, $\mathbf{L}$ is constant in both magnitude and direction. The instantaneous angular velocity vector, however, is not fixed either in space or with respect to the body - unless the body is rotating about a principal axis and the inertia tensor is diagonal.

So much for a preview and a qualitative description. Now down to work.

I am going to have to assume familiarity with the equation for the components of the cross product of two vectors:

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_zB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_z)\hat{\mathbf{z}} \tag{2.17.1}$$

I am also going to assume that the reader knows that the angular momentum of a particle of mass m at position vector $\mathbf{r}$ (components $(x, y, z\ )$) and moving with velocity $\mathbf{v}$ (components ($\dot{x}, \dot{y}, \dot{z}$)) is $m\mathbf{r} \times \mathbf{v}$. For a collection of particles, (or an extended solid body, which, I'm told, consists of a collection of particles called atoms), the angular momentum is

$$\mathbf{L} = \sum m\mathbf{r} \times \mathbf{v} \tag{2.17.2}$$

$$= \sum [m(y\dot{z} - z\dot{y})\hat{\mathbf{x}} + m(z\dot{x} - x\dot{z})\hat{\mathbf{y}} + m(x\dot{y} - y\dot{x})\hat{\mathbf{z}}] \tag{2.17.3}$$

I also assume that the relation between linear velocity $\mathbf{v}(\ \dot{x}, \dot{y}, \dot{z}\ )$and angular velocity $\boldsymbol{\omega}\ (\omega_x, \omega_y, \omega_z)$ is understood to be $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, so that , for example $\dot{z} = \omega_x y - \omega_y x.$ then

$$\mathbf{L} \ = \sum [m(y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z)\hat{\mathbf{x}} + (etc.)\hat{\mathbf{y}} + (etc.)\hat{\mathbf{z}}] \tag{2.17.4}$$

$$= (\omega_x \sum my^2 - \omega_y \sum mxy - \omega_z \sum mzx + \omega_x \sum mz^2)\hat{\mathbf{x}} + etc. \tag{2.17.5}$$

$$= (A\omega_x - H\omega_y - G\omega_z)\hat{\mathbf{x}} + ()\hat{\mathbf{y}} + ()\hat{\mathbf{z}}. \tag{2.17.6}$$

Finally we obtain

$$\mathbf{L} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{2.17.7}$$

This is the equation $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$ referred to above. The inertia tensor is sometimes written in the form

$$\mathbb{I}I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix}$$

so that, for example, $I_{xy} = -H$. It is a symmetric matrix (but it is not an orthogonal matrix).

2.18: Determination of the Principal Axes

We now need to address ourselves to the determination of the principal axes. Unlike the two- dimensional case, we do not have a nice, simple explicit expression similar to Equation 2.12.12 to calculate the orientations of the principal axes. The determination is best done through a numerical example.

Example 2.18.1

Consider four masses whose positions and coordinates are as follows:

M	x	y	z
1	3	1	4
2	1	5	9
3	2	6	5
4	3	5	9

Relative to the first particle, the coordinates are

1	0	0	0
2	-2	4	5
3	-1	5	1
4	0	4	5

From this, it is easily found that the coordinates of the centre of mass relative to the first particle are (−0.7, 3.9, 3.3), and the moments of inertia with respect to axes through the first particle are

- $A = 324$
- $B = 164$
- $C = 182$
- $F = 135$
- $G = -23$
- $H = -31$

From the parallel axes theorems we can find the moments of inertia with respect to axes passing through the centre of mass:

- $A = 63.0$
- $B = 50.2$
- $C = 25.0$
- $F = 6.3$
- $G = 0.1$
- $H = -3.7$

The inertia tensor is therefore

$$\begin{pmatrix} 63.0 & 3.7 & -0.1 \\ 3.7 & 50.2 & -6.3 \\ -0.1 & -6.3 & 25.0 \end{pmatrix}$$

We understand from what has been written previously that if $\boldsymbol{\omega}$, the instantaneous angular velocity vector, is along any of the principal axes, then $\mathbf{I}\boldsymbol{\omega}$ will be in the same direction as $\boldsymbol{\omega}$. In other words, if (l, m, n) are the direction cosines of a principal axis, then

$$\begin{pmatrix} A & -H & G \\ -H & B & -F \\ -G & -F & C \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix},$$

where λ is a scalar quantity. In other words, a vector with components l, m, n(direction cosines of a principal axis) is an eigenvector of the inertia tensor, and λ is the corresponding principal moment of inertia. There will be three eigenvectors (at right angles to each other) and three corresponding eigenvalues, which we'll initially call $\lambda_1, \lambda_2, \lambda_3$, though, as soon as we know which is the largest and which the smallest, we'll call A_0, B_0, C_0, according to our convention $A_0 \leq B_0 \leq C_0$.

The Characteristic Equation is

$$\begin{bmatrix} a-\lambda & -H & -G \\ -H & B-\lambda & -F \\ -G & -F & C-\lambda \end{bmatrix} = 0.$$

In this case, this results in the cubic equation

$$a_0 + a_1\lambda + a_2\lambda^2 - \lambda^3 = 0,$$

where

- $a_0 = 76226.44$
- $a_1 = -5939.21$
- $a_2 = 138.20$

The three solutions for λ, which we shall call A_0, B_0, C_0 in order of increasing size are

- $A_0 = 23.498256$
- $B_0 = 50.627521$
- $C_0 = 64.074223$

and these are the principal moments of inertia. From the theory of equations, we note that the sum of the roots is exactly equal to a_2, and we also note that it is equal to $A+B+C$, consistent with what we wrote in Section 2.16 (Equation 2.16.2). The sum of the diagonal elements of a matrix is known as the *trace* of the matrix. Mathematically we say that "the trace of a symmetric matrix is invariant under an orthogonal transformation".

Two other relations from the theory of equations may be used as a check on the correctness of the arithmetic. The product of the solutions equals a_0, which is also equal to the determinant of the inertia tensor, and the sum of the products taken two at a time equals $-a_1$.

We have now found the magnitudes of the principal moments of inertia; we have yet to find the direction cosines of the three principal axes. Let's start with the axis of least moment of inertia, for which the moment of inertia is $A_0 = 23.498256$. Let the direction cosines of this axis be (l_1, m_1, n_1). Since this is an eigenvector with eigenvalue 23.498 256 we must have

$$\begin{pmatrix} 63.0 & 3.7 & -0.1 \\ 3.7 & 50.2 & -6.3 \\ -0.1 & -6.3 & 25.0 \end{pmatrix} \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = 23.498256 \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$$

These are three linear equations in $l_1 m_1, n_1$, with no constant term. Because of the lack of a constant term, the theory of equations tells us that the third equation, if it is consistent with the other two, must be a linear combination of the first two. We have, in effect, only two independent equations, and we are going to need a third, independent equation if we are to solve for the three direction cosines. If we let $l' = l/n$ and $m' = m/n$, then the first two equations become

$$39.501744l' + 3.7m' - 0.1 = 0$$

$$3.7l' + 26.701744m' - 6.3 = 0.$$

The solutions are

- $l' = -0.019825485$
- $m' = +0.238686617.$

The correctness of the arithmetic can and should be checked by verifying that these solutions also satisfy the third equation.

The additional equation that we need is provided by Pythagoras's theorem, which gives for the relation between three direction cosines

$$l_1^2 + m_1^2 + n_1^2 = 1,$$

or

$$n_1^2 = \frac{1}{l'^2 + m'^2 + 1}$$

whence

$$n_1 \pm 0.972495608.$$

Thus we have, for the direction cosines of the axis corresponding to the moment of inertia A_0,

- $l_1 = \mp 0.019280197$
- $m_1 = \pm 0.232121881$
- $n_1 = \pm 0.972495608$

(Check that $l_1^2 + m_1^2 + n_1^2 = 1.$)

It does not matter which sign you choose - after all, the principal axis goes both ways.

Similar calculations for B_0 yield

- $l_2 = \pm 0.280652440$
- $m_2 = \mp 0.932312706$
- $n_2 = \pm 0.228094774$

and for C_0

- $l_3 = \pm 0.959615796$
- $m_3 = \pm 0.277330987$
- $n_3 = \mp 0.047170415$

For the first two axes, it does not matter whether you choose the upper or the lower sign. For the third axes, however, in order to ensure that the principal axes form a right-handed set, choose the sign such that the determinant of the matrix of direction cosines is +1.

We have just seen that, if we know the moments and products of inertia A, B, C, F, G, H with respect to some axes (i.e. if we know the elements of the inertia tensor) we can find the principal moments of inertia $A_0, B_0, C_0$by diagonalizing the inertia tensor, or finding its eigenvalues. If, on the other hand, we know the principal moments of inertia of a system of particles (or of a solid body, which is a collection of particles), how can we find the moment of inertia I about an axis whose direction cosines with respect to the principal axes are (l, m, n) ?

First, some geometry.

Let Oxyz be a coordinate system, and let P(x, y, z) be a point whose position vector is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Let L be a straight line passing through the origin, and let the direction cosines of this line be

(l, m, n). A unit vector $\mathbf{e}$ directed along L is represented by

$$\mathbf{e} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$$

The angle θ between $\mathbf{r}$ and $\mathbf{e}$ is found from the scalar product $\mathbf{r} \cdot \mathbf{e}$, given by

$$r\cos\theta = \mathbf{r} \cdot \mathbf{e}.$$

I.e.

$$(x^2+y^2+z^2)^{\frac{1}{2}}\cos\theta = lx+my+nz$$

The perpendicular distance p from P to L is

$$p = r\sin\theta = (x^2+y^2+z^2)^{\frac{1}{2}}\sin\theta.$$

If we write $\sin\theta = (1-\cos^2\theta)^{\frac{1}{2}}$, we soon obtain

$$p^2 = x^2+y^2+z^2-(lx+my+nz)^2.$$

Noting that $l^2 = 1-m^2-n^2, m^2 = 1-n^2-l^2, n^2 = 1-l^2-m^2,$ we find, after further manipulation:

$$p^2 = l^2(y^2+z^2)+m^2(z^2+x^2)+n^2(x^2+y^2)-2(mnyz+nlzx+lmxy).$$

Now return to our collection of particles, and let Oxyz be the principal axes of the system. The moment of inertia of the system with respect to the line L is

$$I = \sum Mp^2.$$

where I have omitted a subscript i on each symbol. Making use of the expression for p and noting that the product moments of the system with respect to Oxyz are all zero, we obtain

$$I = l^2A_0+m^2B_0+n^2C_0. \tag{2.18.1}$$

Also, let A, B, C, F, G, H be the moments and products of inertia with respect to a set of nonprincipal orthogonal axes; then the moment of inertia about some other axis with direction cosines l, m, n with respect to these nonprincipal axes is

$$I = l^2A+m^2B+n^2C-2mnF-2nlG-2lmH. \tag{2.18.2}$$

Example 2.18.2: Consider a brick

We saw in Section 2.16 that the moment of inertia of a uniform solid cube of mass M and side $2a$ about a body diagonal is $\frac{2}{3}Ma^2$, and we saw how very easy this was. At that time the problem of finding the moment of inertia of a uniform solid rectangular parallelepiped of sides $2a, 2b, 2c$ must have seemed intractable, but by now it is not at all hard.

$$A_0 = \frac{1}{3}M(b^2+c^2)$$

$$B_0 = \frac{1}{3}M(c^2+a^2)$$

$$C_0 = \frac{1}{3}M(a^2+b^2)$$

Thus we have:

$$l = \frac{a}{(a^2+b^2+c^2)^{\frac{1}{2}}}$$

$$m = \frac{b}{(a^2+b^2+c^2)^{\frac{1}{2}}}$$

$$n = \frac{c}{(a^2+b^2+c^2)^{\frac{1}{2}}}$$

We obtain:

$$I = \frac{2M(b^2c^2+c^2a^2+a^2b^2)}{3(a^2+b^2+c^2)}$$

We note:

i. This is dimensionally correct;
ii. It is symmetric in a, b, c;
iii. If $a = b = c,$ it reduces to $\frac{2}{3}Ma^2$.

2.19: Moment of Inertia with Respect to a Point

By "moment of inertia" we have hitherto meant the second moment of mass *with respect to an axis.* We were easily able to identify it with the *rotational inertia* with respect to the axis, namely the ratio of an applied torque to the resulting angular acceleration.

I am now going to define the (second) moment of inertia with respect to a point, which I shall take unless otherwise specified to mean the origin of coordinates. If we have a collection of mass points m_i at distances r_i from the origin, I define

$$\iota = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2 + z_i^2) \tag{2.19.1}$$

as the (second) moment of inertia with *respect to the origin,* also sometimes called the "geometric moment of inertia". I cannot relate it in an obvious way to a simple dynamical concept in the same way that I related moment of inertia with respect to an axis to rotational inertia, but we shall see that it is by no means merely a tedious exercise in arithmetic, and it does have its uses. The symbol I has probably been used rather a lot in this chapter; so to describe the geometric moment of inertia I am going to use the symbol ι.

The moment of inertia with respect to the origin is clearly something that does not depend on the orientation of any particular basis set of orthogonal axes, since it depends only on the distances of the particles from the origin.

If you recall the definitions of A, B and C from Section 2.15, you will easily see that

$$\iota = \frac{1}{2}(A+B+C) \tag{2.19.2}$$

and we already noted (see Equation 2.16.2) that $A+B+C$ is invariant under rotation of axes. In Section 2.18 we expressed it slightly more generally by saying "the trace of a symmetric matrix is invariant under an orthogonal transformation". By now it probably seems slightly less mysterious.

> *The trace of a* ***symmetric*** *matrix is invariant under an* ***orthogonal*** *transformation*

Let us now calculate the geometric moment of inertia of a uniform solid sphere of radius a, mass m, density ρ, with respect to the center of the sphere. It is

$$\iota = \int_{sphere} r^2 dm. \tag{2.19.3}$$

The element of mass, dm, here is the mass of a shell of radii $r, r+dr$; that is $4\pi\rho r 2dr$. Thus

$$\iota = 4\pi\rho \int_0^a r^4 dr = \frac{4}{5}\pi\rho a^5. \tag{2.19.4}$$

With $m = \frac{4}{3}\pi a^3 \rho$, this becomes

$$\iota = \frac{3}{5} m a^2. \tag{2.19.5}$$

Indeed, for any *spherically symmetric distribution of matter,* since $A = B = C$, it will be clear from Equation 2.19.2, that *the moment of inertia with respect to the center is 3/2 times the moment of inertia with respect to an axis through the center.* For example, it is obvious from the definition of moment of inertia with respect to the center that for a hollow spherical shell it is just Ma^2, and therefore the moment of inertia with respect to an axis through the center is $\frac{2}{3}ma^2$. In other words, you can work out that the moment of inertia of a hollow spherical shell with respect to an axis through its center is $\frac{2}{3}ma^2$ in your head without any of the integration that we did in Section 2.7!

By way of illustration, consider three spheres, each of radius a and mass M, but the density between center and surface varies as

$$\rho = \rho_0 (1 - \tfrac{kr}{a}), \quad \rho = \rho_0 (1 - \tfrac{kr^2}{a^2}), \quad \rho = \rho_0 \sqrt{1 - \tfrac{kr^2}{a^2}}$$

for the three spheres.

Example 2.19.1

Calculate for each the moment of inertia about an axis through the center of the sphere. Express the answer in the form $\frac{2}{5}Ma^2 \times f(k)$.

Solution

The mass of a sphere is

$$M = 4\pi \int_0^a \rho(r) r^2 dr$$

and so

$$\frac{2}{5}Ma^2 = \frac{8\pi a^2}{5} \int_0^a \rho(r) r^2 dr$$

The moment of inertia about the center is

$$\iota = 4\pi \int_0^a \rho(r) r^4 dr.$$

and so the moment of inertia about an axis through the center is

$$I = \frac{8}{3} \int_0^a \rho(r) r^4 dr.$$

Therefore

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{5\int_o^a \rho(r) r^4 dr}{3a^2 \int_o^a \rho(r) r^2 dr}$$

For the first two spheres the integrations are straightforward. I make it

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{12 - 10k}{12 - 9k}$$

for the first sphere, and

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{35 - 25k}{35 - 21k}$$

for the second sphere. The integrations for the third sphere need a little more patience, but I make the answer

$$\frac{I}{\frac{2}{5}Ma^2} = \frac{5(12\alpha - 3\sin 2\alpha - 3\sin 4\alpha + \sin 6\alpha)}{18\sin^2\alpha(4\alpha - \sin 4\alpha)}$$

where $\sin\alpha = \sqrt{k}$.

Example 2.19.1 should be enough to convince that the concept of ι is useful – but it is not its only use. We shall meet it again in Chapter 3 on the dynamics of systems of particles; in particular, it will play a role in what we shall become familiar with as the virial theorem.

2.20: Ellipses and Ellipsoids

Here are some problems concerning ellipses and ellipsoids that might be of interest.

Determine the principal moments of inertia of the following:

1. A uniform plane lamina of mass m in the form of an ellipse of semi axes a and b.
2. A uniform plane ring of mass m in the form of an ellipse of semi axes a and b.
3. A uniform solid triaxial ellipsoid of mass m and semi axes a, b and c.
4. A uniform hollow triaxial ellipsoid of mass m and semi axes a, b and c.

1. By integration, an elliptical lamina is slightly difficult, but by physical insight it is very easy!

The distribution of mass around the minor axis is the same as for a circular lamina of radius a, and therefore the moment B is the same as for the circular lamina, namely $B=\frac{1}{4} m a^2$. Similarly, $A=\frac{1}{4} m b^2$, and hence, by the perpendicular axes theorem, $C=\frac{1}{4} m\left(a^2+b^2\right)$.

I think you will find that the shape of the momental ellipse is the same as the shape of the original elliptical lamina.

2. An elliptical ring (hoop) is remarkably difficult. It cannot be expressed in terms of elementary functions, and it has to be calculated numerically. It can be expressed in terms of elliptic integrals (no surprise there), but most of us aren't sure what elliptic integrals are and they hardly count as elementary functions, and they have to be calculated numerically anyway. We take the ellipse to be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with $b \leq a$.

Even calculating the circumference of an ellipse isn't all that easy. The circumference is

$$\oint d s=4 \int_0^a\left[1+\left(\frac{d y}{d x}\right)^2\right] d x \text{ , with } y=b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} .$$

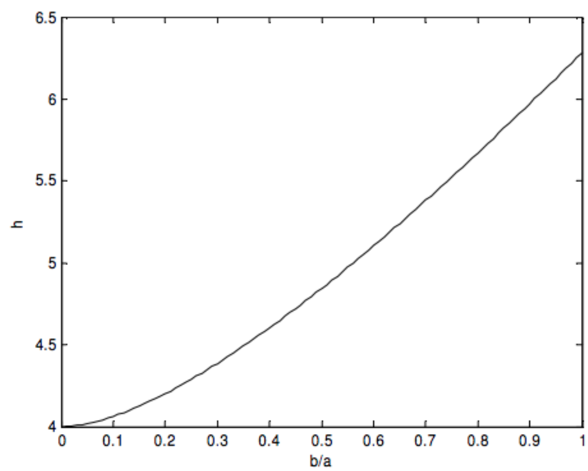
After a bit of algebra, this can be written as

$$\frac{4 a}{x} \int_0^a \sqrt{\frac{c^2-x^2}{a^2-z^2}} d x, \text{ where } c^2=\frac{a^4}{a^2-b^2} .$$

At first this looks easy, but I do not think you can do it in terms of elementary functions. No problem, then – just integrate it numerically. Unfortunately the integrand becomes infinite at the upper limit, so there is still a bit of a problem. However, a change of variable to $x=a \sin \theta$ solves that problem. The expression for the circumference becomes simply

$$4 a \int_0^{\pi / 2}\left[1-\left(\frac{a^2-b^2}{a^2}\right) \sin ^2 \theta\right]^{\frac{1}{2}} d \theta ,$$

which can be integrated numerically without infinity problems at the limits. According to my calculations, the circumference of the ellipse is $h a$, where h is a function of b / a as follows:



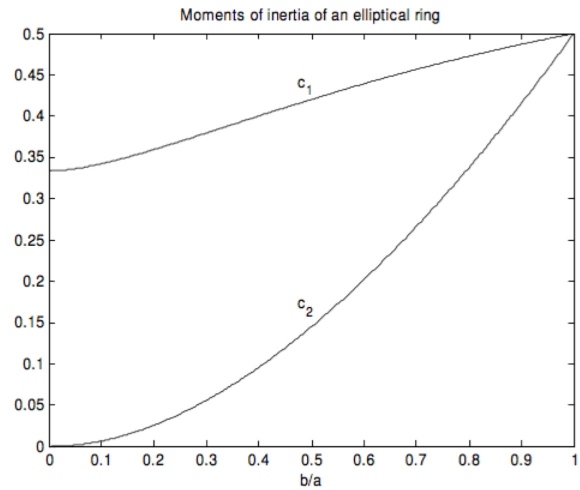
To find the moment of inertia (or the second moment of length) about the minor axis, we have to multiple the integrand by x^2, or $a^2 \sin ^2 \theta$, and integrate. Thus the moment of inertia of the elliptical hoop about its minor axis is $c_1 m a^2$, where

$$c_1 = \frac{\int_0^{\pi/2}[1-(\frac{a^2-b^2}{a^2})\sin^2\theta]^{1/2}\sin^2\theta d\theta}{\int_0^{\pi/2}[1-(\frac{a^2-b^2}{a^2})\sin^2\theta]^{1/2}d\theta}$$

The moment of inertia about the major axis is $c_2 ma^2$, where

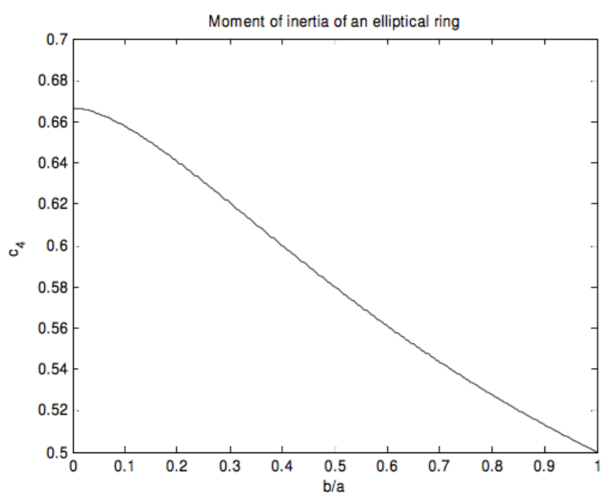
$$c_2 = \frac{\frac{b^2}{a^2}\int_0^{\pi/2}[1-(\frac{a^2-b^2}{a^2})cos^2]^{1/2}\sin^2\theta d\theta}{\int_0^{\pi/2}[1-(\frac{a^2-b^2}{a^2})\sin^2\theta]^{1/2}d\theta}$$

These two coefficients of ma^2 are shown below as a function of b/a.



The moments of inertia of an elliptical ring of mass m and semi major and semi minor axes a and b are $c_1 ma^2$ about the minor axis and $c_2 ma^2$ about the major axis, where c_1 and c_2 are shown as functions of b/a.

The moment of inertia about the major axis can also be conveniently expressed in terms of b rather than a. If we write the moment of inertia about the major axis as $c_4 mb^2$, then c_4 as a function of b/a is shown below.



The moment of inertia of an elliptical ring of mass m and semi major and semi minor axes a and b is $c_4 mb^2$ about the major axis, where c_4 is shown as a function of b/a.

The moment of inertia about an axis perpendicular to the plane of the ellipse and passing through its centre is $c_3 ma^2$, where, of course (by the perpendicular axes theorem), $c_3 = c_1 + c_2$.

It is also equal to $c_1 ma^2 + c_4 mb^2$.

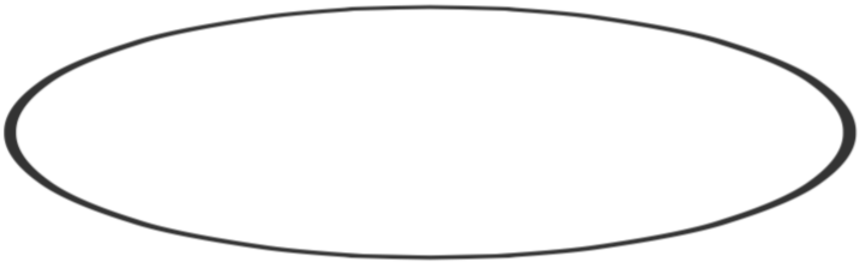
3. For a uniform solid triaxial ellipsoid, the moments of inertia are

$$A=\frac{1}{5}m(b^2+c^2) \qquad B=\frac{1}{5}m(c^2+a^2) \qquad C=\frac{1}{5}m(c^2+a^2)$$

The momental ellipsoid is not of the same shape. Its axes are in the ratio

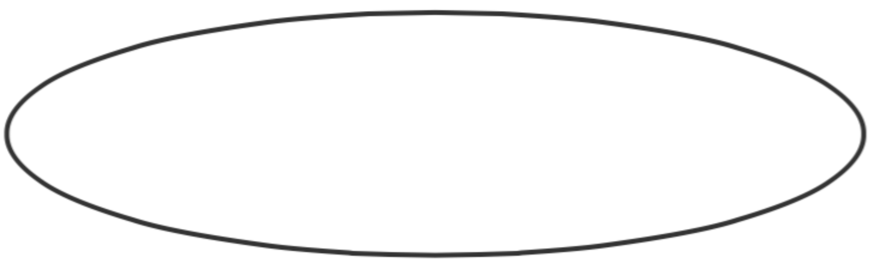
For example, if the axial ratios of the original ellipsoid are 1 : 2 : 3, the axial ratios of the corresponding momental ellipsoid is $1:\sqrt{\frac{13}{10}}:\sqrt{\frac{13}{5}}=1:1.140:1.612$, which is slightly more spherical than the original ellipsoid.

4. Triaxial elliptical shell. We have to think carefully about what a triaxial elliptical shell is. If we imagine the inner surface of the shell to be an ellipsoid, and the outer surface to be a similar ellipsoid, but with all linear dimensions increased by the same small fractional increment, then we obtain a figure like this:



In this drawing the linear size of the outer surface is 3 percent larger than that of the inner surface. E. J. Routh correctly shows in his treatise on rigid bodies that the principal moments of inertia of such a figure are $\frac{1}{3}m(b^2+c^2)$, $\frac{1}{3}m(c^2+a^2)$, $\frac{1}{3}m(a^2+b^2)$.

But it can be seen that such a figure is not (as presumably a rugger ball is) of uniform thickness. I draw below a shell of uniform thickness. In such a case the inner and outer surfaces are not exactly similar.



In attempting to calculate the moment of inertia of such a figure I shall restrict myself to the case of a *spheroidal* shell of uniform thickness. That is to say, an ellipsoid with two equal axes, represented by the equation, in cylindrical coordinates

$$\frac{\rho^2}{a^2}+\frac{z^2}{c^2}=1,$$

where $\rho^2=x^2+y^2$. Further, if I put $c=\chi a$, the equation to the spheroid can be written

$$\rho^2+\frac{z^2}{\chi^2}=a^2,$$

If $\chi<1$, the spheroid is oblate. If $\chi>1$, the spheroid is prolate.

We'll first need to calculate its surface area, which is

$$A=4\pi\int_0^c \rho[1+(\frac{d\rho}{dz})^2]^{\frac{1}{2}}dz$$

After some algebra, this comes to

$$A=4\pi a^2 f(\chi),$$

where

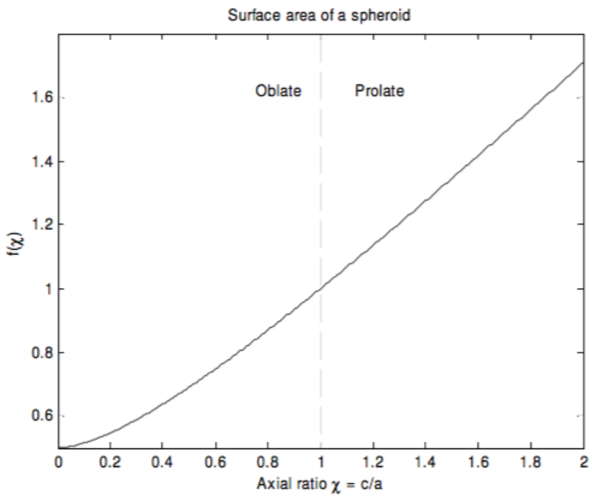
$$f(\chi)=\frac{1}{2}\left[\frac{\chi^2}{\sqrt{1-\chi^2}}\ln\left(\frac{1+\sqrt{1-\chi^2}}{\chi}\right)+1\right] \text{ for } \chi\le 1$$

and

$$f(\chi)=\frac{1}{2}\left[\frac{\chi^2}{\sqrt{\chi^2-1}}\sin^{-1}\left(\frac{\sqrt{\chi^2-1}}{\chi}\right)+1\right]]\text{ for }\chi\geq 1$$

This function is shown below as far as $\chi=2$. For $\chi=0$, the figure is a disc whose total area

(upper and lower surface) is $2\pi a^2$, and $f=\frac{1}{2}$. For $\chi=1$, the figure is a sphere whose area is $4\pi a^2$, and $f=1$. The function goes to infinity as χ goes to infinity.



The moment of inertia about the z-axis is

$$I=\frac{4\pi m}{A}\int_0^c\rho^3[1+(\frac{d\rho}{dz})^2]^{1/2}dz.$$

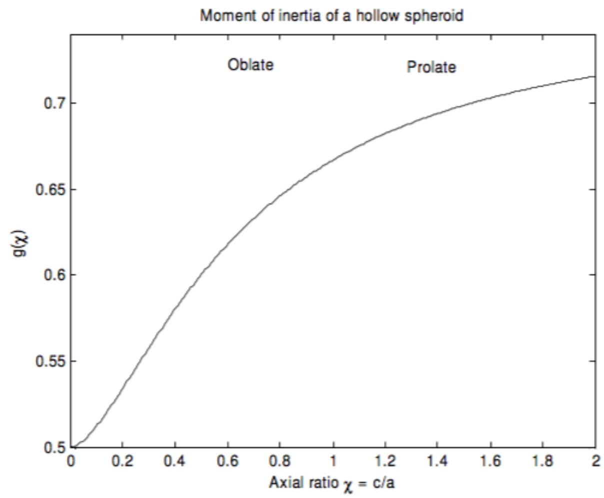
After some algebra this becomes

$$I=ma^2g(\chi)$$

$$g(\chi)=\frac{(2-\chi^2)(1-\chi^2)-\chi^4\ln\left[(1+\sqrt{1-\chi^2})/\chi\right]}{4\left\{(1-\chi^2)^{3/2}+\chi^2(1-\chi^2)\ln\left[(1+\sqrt{1-\chi^2})/\chi\right]\right\}}\text{ for }\chi\leq 1$$

$$g(\chi)=1-\frac{\frac{\chi^4}{(\chi-1)^{3/2}}sin^{-1}\left(\frac{\sqrt{\chi^2-1}}{\chi}\right)+\frac{\chi^2-2}{\chi^2-1}}{4\left\{\frac{\chi^2}{\sqrt{\chi^2-1}}sin^{-1}\left(\frac{\sqrt{\chi^2-1}}{\chi}\right)+1\right\}}\text{ for }\chi\geq 1$$

This function is shown below as far as $\chi=2$ For $\chi=0$, the figure is a disc whose moment of inertia is $\frac{1}{2}\pi a^2$, and $f=\frac{1}{2}$. For $\chi=1$, the figure is a hollow sphere whose moment of inertia is $\frac{2}{3}\pi a^2$, and $f=\frac{2}{3}$. The function goes to 1 as χ goes to infinity; the moment of inertia then approaches that of a hollow cylinder.



Moment of inertia of a hollow spheroid
Oblate
Prolate
g(χ)
0.7
0.65
0.6
0.55
0.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Axial ratio χ = c/a

2.21: Tetrahedra

? Exercise 2.21.1

Show that the moment of inertia about an axis through the centre of mass of a uniform solid regular tetrahedron of mass m and edge length a is $\frac{1}{20}ma^2$

? Exercise 2.21.2

Show that the moment of inertia of a methane molecule about an axis through the carbon atom is $\frac{8}{3}ml^2$, where l is the bond length and m is the mass of a hydrogen atom.

And, in case you are wondering that I haven't specified the *orientation* of the axis in either case, the solid regular tetrahedron and the methane molecule are both spherical tops, and the moment of inertia is the same about *any* axis through the centre of mass.

CHAPTER OVERVIEW

3: Systems of Particles

3.1: Introduction to Systems of Particles

By systems of particles I mean such things as a swarm of bees, a star cluster, a cloud of gas, an atom, a brick. A brick is indeed composed of a system of particles – atoms - which are constrained so that there is very little motion (apart from small amplitude vibrations) of the particles relative to each other. In a system of particles, there may be very little or no interaction between the particles (as in a loose association of stars separated from each other by large distances) or there may be (as in the brick) strong forces between the particles. Most (perhaps all) of the results to be derived in this chapter for a system of particles apply equally to an apparently solid body such as a brick. Even if scientists are wrong and a brick is not composed of atoms but is a genuine continuous solid, we can in our imagination suppose the brick to be made up of an infinite number of infinitesimal mass and volume elements, and the same results will apply.

What sort of properties shall we be discussing? Perhaps the simplest one is this: *The total linear momentum of a system of particles is equal to the total mass times the velocity of the center of mass.* This is true, and it may be "obvious" - but it still requires proof. It may be equally "obvious" to some that "the total kinetic energy of a system of particles is equal to $\frac{1}{2}M\bar{v}^2$ where M is the total mass and $\bar{v}$ is the velocity of the center of mass" - but this one, however "obvious", is <u>not</u> true!

Before we get round to properties of systems of particles, I want to clarify what I mean by the *moment* of a vector such as a force or a momentum. You are already familiar, from Chapters 1 and 2, with the moments of *mass*, which is a scalar quantity.

3.2: Moment of Force

First, let's look at a familiar two-dimensional situation. In Figure III.1 I draw a force $\mathbf{F}$ and a point O. The moment of the force with respect to O can be defined as

Force times perpendicular distance from O to the line of action of $\mathbf{F}$.

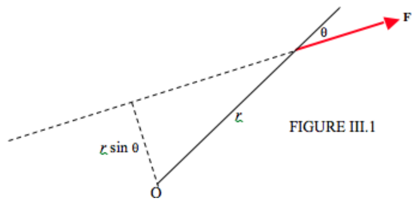


FIGURE III.1

Alternatively, (Figure III.2) the moment can be defined equally well by

Transverse component of force times distance from O to the point of application of the force.

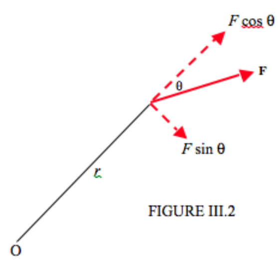


FIGURE III.2

Either way, the magnitude of the moment of the force, also known as the *torque*, is $rF\sin\theta$ We can regard it as a vector, $\boldsymbol{\tau}$, perpendicular to the plane of the paper:

$$\boldsymbol{\tau}=\mathbf{r}\times\mathbf{F} \tag{3.2.1}$$

Now let me ask a question. Is it correct to say the moment of a force with respect to (or "about") a *point* or with respect to (or "about") an *axis*?

In the above two-dimensional example, it does not matter, but now let me move on to three dimensions, and I shall try to clarify.

In Figure III.3, I draw a set of rectangular axes, and a force $\mathbf{F}$, whose position vector with respect to the origin is $\mathbf{r}$.

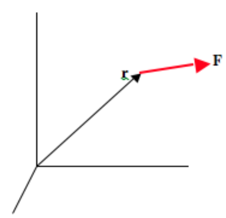


FIGURE III.3

The moment, or *torque*, of $\mathbf{F}$ *with respect to the origin* is the vector

$$\boldsymbol{\tau}=\mathbf{r}\times\mathbf{F} \tag{3.2.2}$$

The $x-, y-$ and z *-components* of $\boldsymbol{\tau}$ are the moments of $\mathbf{F}$ *with respect to the* $x-, y-$ *and z-axes.* You can easily find the components of $\boldsymbol{\tau}$ by expanding the cross product 3.2.2:

$$\boldsymbol{\tau}=\hat{\mathbf{x}}(yF_z-zF_y)+\hat{\mathbf{y}}(yF_x-xF_z)+\hat{\mathbf{z}}(xF_y-yF_x) \tag{3.2.3}$$

where $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ are the unit vectors along the x, y, z axes. In Figure III.4, we are looking down the x-axis, and I have drawn the components F_y and F_z, and you can see that, indeed, $\tau_x=yF_z-zF_y$.

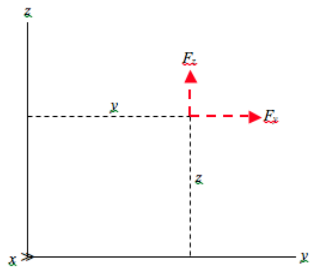


FIGURE III.4

The dimensions of moment of a force, or torque, are ML^2T^{-2}, and the SI units are N m. (It is best to leave the units as N m rather than to express torque in joules.)

3.3: Moment of Momentum

In a similar way, if a particle at position $\mathbf{r}$ has linear momentum $\mathbf{p} = m\mathbf{v}$, its *moment of momentum with respect to the origin is the vector* $\mathbf{l}$ *defined by*

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} \tag{3.3.1}$$

and its *components* are the moments of momentum *with respect to the axes.* Moment of momentum plays a role in rotational motion analogous to the role played by linear momentum in linear motion, and is also called *angular momentum.* The dimensions of angular momentum are ML^2T^{-1}. Several choices for expressing angular momentum in SI units are possible; the usual choice is J s (joule seconds).

3.4: Notation

In this section I am going to suppose that we n particles scattered through three-dimensional space. We shall be deriving some general properties and theorems – and, to the extent that a solid body can be considered to be made up of a system of particles, these properties and theorems will apply equally to a solid body.

In the Figure III.5, I have drawn just two of the particles, (the rest of them are left to your imagination) and the centre of mass C of the system.

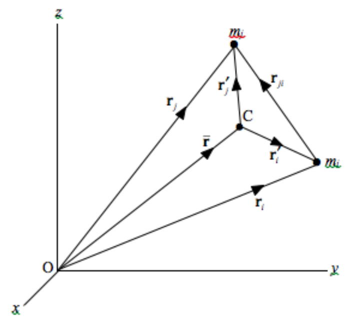


FIGURE III.5

A given particle may have an *external force* $\mathbf{F_i}$ acting upon it. (It may, of course, have *several* external forces acting on it, but I mean by $\mathbf{F_i}$ the vector sum of all the external forces acting on the *i* th particle.) It may also interact with the other particles in the system, and consequently it may have *internal forces* $\mathbf{F_{ij}}$ acting upon it, where j goes from 1 to n except for i. I define the vector sum $\mathbf{F} = \sum \mathbf{F_i}$ as the total external force acting upon the *system*.

I am going to establish the following notation for the purposes of this chapter.

- Mass of the i th particle = m_i
- Total mass of the system $M = \sum m_i$
- Position vector of the i th particle referred to a fixed point O: $\mathbf{r}_i = x_i\hat{\mathbf{x}} + y_i\hat{\mathbf{y}} + z_i\hat{\mathbf{z}}$
- Velocity of the i th particle referred to a fixed point O: $\mathbf{r}_i$ or $\mathbf{v}_i$ (Speed = v_i)
- Linear momentum of the i th particle referred to a fixed point O: $\mathbf{p}_i = m_i\mathbf{v}_i$
- Linear momentum of the *system*: $\mathbf{P} = \sum \mathbf{P}_i = \sum m_i\mathbf{v}_i$
- External force on the i th particle: $\mathbf{F}_i$
- Total external force on the system: $\mathbf{F} = \sum \mathbf{F}_i$
- Angular momentum (moment of momentum) of the i th particle referred to a fixed point O:

$$\mathbf{l}_i = \mathbf{r}_i \times \mathbf{p}_i$$

- Angular momentum of the system: $\mathbf{L} = \sum \mathbf{l}_i = \sum \mathbf{r}_i \times \mathbf{p}_i$
- Torque on the *i* th particle referred to a fixed point O: $\boldsymbol{\tau}_i = \mathbf{r}_i \times \mathbf{F}_i$
- Total external torque on the system with respect to the origin:

$$\boldsymbol{\tau} = \sum \boldsymbol{\tau}_i = \sum \mathbf{r}_i \times \mathbf{F}_i$$

Kinetic energy of the system: (We are dealing with a system of *particles* – so we are dealing only with *translational* kinetic energy – no rotation or vibration):

$$T = \sum \tfrac{1}{2} m_i v_i^2$$

Position vector of the *centre of mass* referred to a fixed point O: $\bar{\mathbf{r}}_i = \bar{x}\hat{\mathbf{x}} + \bar{y}\hat{\mathbf{y}} + \bar{z}\hat{\mathbf{z}}$

The centre of mass is defined by $M\bar{\mathbf{r}} = \sum m_i\mathbf{r}_i$

Velocity of the centre of mass referred to a fixed point O: $\bar{\mathbf{r}} = \bar{\mathbf{v}}$ (Speed = $\bar{v}$)

For position vectors, unprimed single-subscript symbols will refer to O. Primed single-subscript symbols will refer to C. This will be clear, I hope, from Figure III.5.

Position vector of the i th particle *referred to the centre of mass* C: $\mathbf{r}_i' = \mathbf{r}_i - \overline{\mathbf{r}}_i$

Position vector of particle j with respect to particle i: $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$

(Internal) force exerted on particle i by particle j: $\mathbf{F}_{ij}$

(Internal) force exerted on particle j by particle i: $\mathbf{F}_{ji}$

If the force between two particles is *repulsive* (e.g. between electrically-charged particles of the same sign), then $\mathbf{F}_{ji}$ and $\mathbf{r}_{ji}$ are in the same direction. But if the force is an *attractive* force, $\mathbf{F}_{ji}$ and $\mathbf{r}_{ji}$ are in opposite directions.

According to Newton's Third Law of Motion (Lex III), $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$

Total angular momentum of system referred to the centre of mass C: $\mathbf{L}_C$

Total external torque on system referred to the centre of mass C: $\boldsymbol{\tau}_C$

For the velocity of the centre of mass I may use either $\dot{\overline{\mathbf{r}}}$ or $\overline{\mathbf{v}}$

O is an arbitrary origin of coordinates. C is the centre of mass.

Note that

$$\mathbf{r}_i = \overline{\mathbf{r}} + \mathbf{r}_i' \tag{3.4.1}$$

and therefore

$$\dot{\overline{\mathbf{r}}}_i = \dot{\overline{\mathbf{r}}} + \dot{\mathbf{r}}_i'; \tag{3.4.2}$$

that is to say

$$\mathbf{v}_i = \overline{\mathbf{v}} + \mathbf{v}_i' \tag{3.4.3}$$

Note also that

$$\sum m_i \mathbf{r}_i' = 0 \tag{3.4.4}$$

Note further that

$$\sum m_i \mathbf{v}_i' = \sum m_i (\mathbf{v}_i - \overline{\mathbf{v}}) = \sum m_i \mathbf{v}_i - \overline{\mathbf{v}} \sum m_i = M\overline{\mathbf{v}} - \overline{\mathbf{v}} M = 0 \tag{3.4.5}$$

That is, *the total linear momentum with respect to the centre of mass is zero.*

Having established our notation, we now move on to some theorems concerning systems of particles. It may be more useful for you to conjure up a physical picture in your mind what the following theorems mean than to memorize the details of the derivations.

3.5: Linear Momentum

Theorem:

The total momentum of a system of particles equals the total mass times the velocity of the centre of mass.

Thus:

$$\mathbf{P} = \sum m_i \mathbf{v}_i = \sum m_i (\overline{\mathbf{v}} + \mathbf{v}_i') = M\overline{\mathbf{v}} + 0. \tag{3.5.1}$$

3.6: Force and Rate of Change of Momentum

Theorem:

The rate of change of the total momentum of a system of particles is equal to the sum of the external forces on the system.

Thus, consider a single particle. By Newton's second law of motion, the rate of change of momentum of the particle is equal to the sum of the forces acting upon it:

$$\dot{\mathbf{P}}_i = \mathbf{F}_i + \sum_i \mathbf{F}_{ij} \qquad (j \neq i) \tag{3.6.1}$$

Now sum over all the particles:

$$\begin{aligned} \dot{\mathbf{P}}_i = \sum_i \mathbf{F}_i + \sum_i \sum_j \mathbf{F}_{ij} \qquad (j \neq i) \\ \mathbf{F} + \frac{1}{2} \sum_i \sum_j \mathbf{F}_{ij} + \frac{1}{2} \sum_j \sum_i \mathbf{F}_{ij} \\ \mathbf{F} + \frac{1}{2} \sum_i \sum_j \mathbf{F}_{ji} + \mathbf{F}_{ij} \end{aligned} \tag{3.6.2}$$

But, by Newton's third law of motion, $\mathbf{F}_{ji} + \mathbf{F}_{ij} = 0$, so the theorem is proved.

Corollary:

If the sum of the external forces on a system is zero, the linear momentum is constant. (Law of Conservation of Linear Momentum.)

3.7: Angular Momentum

Notation:

- $\mathbf{L}_C$= angular momentum of system with respect to centre of mass C.
- $\mathbf{L}$ = angular momentum of system relative to some other origin O.
- $\overline{\mathbf{r}}$ = position vector of C with respect to O.
- $\mathbf{P}$ = linear momentum of system with respect to O.
- (The linear momentum with respect to C is, of course, zero.)

Theorem:

$$\mathbf{L}=\mathbf{L}_C+\overline{\mathbf{r}}\times\mathbf{P} \tag{3.7.1}$$

Thus:

$$\begin{aligned}\mathbf{L} &=\sum \mathbf{r}_i\times\mathbf{p}_i=\sum m_i(\mathbf{r}_i\times\mathbf{v}_i)=\sum m_i(\overline{\mathbf{r}}+\mathbf{r}_i')\times(\overline{\mathbf{v}}+\mathbf{v}_i')\\ &=(\overline{\mathbf{r}}\times\overline{\mathbf{v}})\sum m_i+\overline{\mathbf{r}}\times\sum m_i\mathbf{v}_i'+\left(\sum m_i\mathbf{r}_i'\right)\times\overline{\mathbf{v}}+\sum \mathbf{r}_i'\times\mathbf{p}_i'\\ &=M(\overline{\mathbf{r}}\times\overline{\mathbf{v}})+\overline{\mathbf{r}}\times 0+0\times\overline{\mathbf{v}}+\mathbf{L}_C\end{aligned}$$

therefore

$$\mathbf{L}=\mathbf{L}_C+\overline{\mathbf{r}}\times\mathbf{P}$$

Example 3.7.1

A hoop of radius *a* rolling along the ground (Figure III.6):

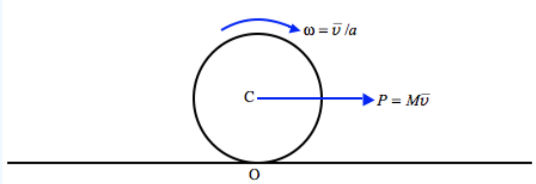


FIGURE III.6

The angular momentum with respect to C is $L_C = I_{C\omega}$ where I_C is the rotational inertia about C. The angular momentum about O is therefore

$$I=I_C\omega+M\overline{v}a=I_C\omega+Ma^2\omega=(I_C+Ma^2)=I\omega$$

where

$$I=I_C+Ma^2$$

is the *rotational inertia* about O.

3.8: Torque

Notation:

- $\boldsymbol{\tau}_C$ = vector sum of all the torques about C.
- $\boldsymbol{\tau}$ = vector sum of all the torques about the origin O.
- $\mathbf{F}$ = vector sum of all the external forces.

Theorem

$$\boldsymbol{\tau} = \boldsymbol{\tau}_C + \bar{\mathbf{r}} \times \mathbf{F}$$

Thus:

$$\begin{aligned} \boldsymbol{\tau} &= \sum \mathbf{r}_i \times \mathbf{F}_i = \sum (\mathbf{r}'_i + \bar{\mathbf{r}}) \times \mathbf{F}_i \qquad (3.8.1) \\ &= \sum \mathbf{r}'_i \times \mathbf{F}_i + \bar{\mathbf{r}} \sum \mathbf{F}_i \qquad (3.8.2) \end{aligned}$$

therefore

$$\boldsymbol{\tau} = \boldsymbol{\tau}_C + \bar{\mathbf{r}} \times \mathbf{F}$$

3.9: Comparison

At this stage I compare some somewhat similar formulas.

$\mathbf{L}=\mathbf{L}_C+\overline{\mathbf{r}}\times\mathbf{P}$	$\boldsymbol{\tau}=\boldsymbol{\tau}_C+\overline{\mathbf{r}}\times\mathbf{F}$
$\mathbf{L}=\sum m_i\mathbf{r}_i\times\mathbf{v}_i$	$\boldsymbol{\tau}=\sum m_i\mathbf{r}_i\times\dot{\mathbf{v}}_i$
$\mathbf{L}_C=\sum m_i\mathbf{r}_i'\times\mathbf{v}_i'$	$\boldsymbol{\tau}'=\sum m_i\mathbf{r}_i'\times\dot{\mathbf{v}}_i$
$\mathbf{P}=\sum m_i\mathbf{v}_i$	$\mathbf{F}=\sum m_i\dot{\mathbf{v}}_i$

3.10: Kinetic energy

We remind ourselves that we are discussing *particles*, and that all kinetic energy is translational kinetic energy.

Notation:

- *[Math Processing Error]* = kinetic energy with respect to the centre of mass C.
- *[Math Processing Error]* = kinetic energy with respect to the origin O.

Theorem:

[Math Processing Error]

Thus:

[Math Processing Error]

[Math Processing Error].

[Math Processing Error].

Corollary:

If *[Math Processing Error]*. (Think about what this means.)

Corollary:

Corollary: For a non-rotating rigid body, *[Math Processing Error]* = 0, and therefore *[Math Processing Error]*

(Think about what this means.)

3.11: Torque and Rate of Change of Angular Momentum

Theorem:

The rate of change of the total angular momentum of a system of particles is equal to the sum of the external torques on the system.

Thus:

$$L = \sum_i \mathbf{r}_i \times p_i \tag{3.11.1}$$

$$\therefore \qquad \dot{\mathbf{L}} = \sum_i \dot{\mathbf{r}}_i \times \dot{\mathbf{p}}_i \tag{3.11.2}$$

But the first term is zero, because $\dot{\mathbf{r}}$ and $\mathbf{p}_i$ are parallel.

Also

$$\dot{\mathbf{r}}_i = \mathbf{F}_i + \sum \mathbf{F}_{ij} \tag{3.11.3}$$

$$\dot{\mathbf{L}}_i = \sum_i \mathbf{r}_i \times (\mathbf{r}_i + \sum_j \mathbf{F}_{ij}) = \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_i \mathbf{r}_i \times \sum_j \mathbf{F}_{ii}$$

$$\therefore \qquad \sum_i r_i \times F_i + \sum_i r_i \times \sum_j F_{ii}$$

But $\sum_i \sum_j \mathbf{F}_{ij} = 0$ by Newton's third law of motion, and so $\sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij} = 0$.

Also $\sum_i \mathbf{r}_i \times \mathbf{F}_i = \boldsymbol{\tau}$, and so we arrive at

$$\dot{\mathbf{L}} = \boldsymbol{\tau} \tag{3.11.4}$$

which was to be demonstrated.

Corollary: Law of Conservation of Angular Momentum

If the sum of the external torques on a system is zero, the angular momentum is constant.

3.12: Torque, Angular Momentum and a Moving Point

In Figure III.7 I draw the particle m_i, which is just one of n particles, $n-1$ of which I haven't drawn and are scattered around in 3-space. I draw an arbitrary origin O, the centre of mass C of the system, and another point Q, which may (or may not) be moving with respect to O. The question I am going to ask is: Does the equation $\dot{\mathbf{L}} = \boldsymbol{\tau}$ apply to the point Q? It obviously does if Q is stationary, just as it applies to O. But what if Q is moving? If it does not apply, just what *is* the appropriate relation?

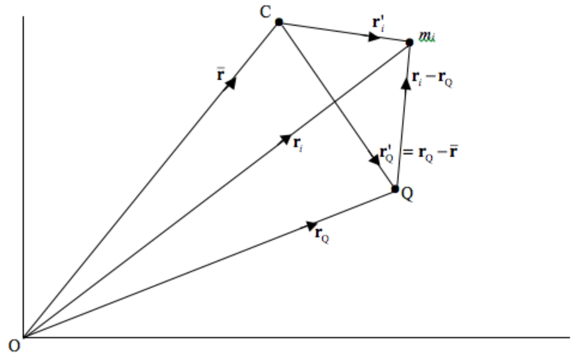


FIGURE III.7

The theorem that we shall prove – and interpret - is

$$\dot{\mathbf{L}}_Q = \boldsymbol{\tau}_Q + M\mathbf{r}'_Q \times \ddot{\mathbf{r}}_Q. \tag{3.12.1}$$

We start:

$$\mathbf{L}_Q = \sum (\mathbf{r}_i - \mathbf{r}_Q) \times [m_i(\mathbf{v}_i - \mathbf{v}_Q)] \tag{3.12.2}$$

$$\therefore \quad \dot{\mathbf{L}}_Q = \sum (\mathbf{r}_i - \mathbf{r}_Q) \times m_i(\dot{\mathbf{v}}_i - \dot{\mathbf{v}}_Q) + \sum (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_Q) \times m_i(\mathbf{v}_i - \mathbf{v}_Q). \tag{3.12.3}$$

The second term is zero, because $\dot{\mathbf{r}} = \mathbf{v}$

Continue:

$$\dot{\mathbf{L}} = \sum (\mathbf{r}_i - \mathbf{r}_Q) \times m_i \dot{\mathbf{v}}_i - \sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_Q + \sum m_i \mathbf{r}_Q \times \dot{\mathbf{v}}_Q \tag{3.12.4}$$

Now $m_i \dot{\mathbf{v}}_i = \mathbf{F}_i$, so that the first term is just $\boldsymbol{\tau}_Q$

Continue:

$$\begin{aligned} \dot{\mathbf{L}} &= \boldsymbol{\tau}_Q - \sum m_i \mathbf{r}_i \times dot \mathbf{v}_Q + \sum M_i \mathbf{r}_Q \times dot \mathbf{v}_Q \\ &= \boldsymbol{\tau}_Q - M\bar{\mathbf{r}} \times \ddot{\mathbf{r}}_Q + M\mathbf{r}_Q \times \ddot{\mathbf{r}}_Q \\ &= \boldsymbol{\tau}_Q + M(\mathbf{r}_Q - \bar{\mathbf{r}}) \times \ddot{\mathbf{r}}_Q \\ \therefore \dot{\mathbf{L}}_Q &= \boldsymbol{\tau}_Q + M\mathbf{r}'_Q \times \ddot{r}_Q \qquad Q.E.D \end{aligned} \tag{3.12.5}$$

Thus in general, $\dot{\mathbf{L}}_Q \neq \boldsymbol{\tau}_Q$, but $\dot{\mathbf{L}}_Q = \boldsymbol{\tau}_Q$ under any of the following three circumstances:

i. $\mathbf{r}'_Q = 0$ - that is, Q coincides with C.
ii. $\ddot{\mathbf{r}}_Q = 0$ - that is, Q is not accelerating.
iii. $\ddot{\mathbf{r}}_Q$ and $\mathbf{r}'_Q$ are parallel, which would happen, for example, if O were a centre of attraction or repulsion and Q were accelerating towards or away from O.

3.13: The Virial Theorem

First, let me say that I am not sure how this theorem got its name, other than that my Latin dictionary tells me that *vis*, *viris* means *force*, and its plural form, *vires*, *virium* is generally translated as *strength*. The term was apparently introduced by Rudolph Clausius of thermodynamics fame. We do not use the word *strength* in any particular technical sense in classical mechanics, although we do talk about the *tensile strength* of a wire, which is the force that it can summon up before it snaps. We use the word *energy* to mean *the ability to do work*; perhaps we could use the word *strength* to mean *the ability to exert a force*. But enough of these idle speculations.

Before proceeding, I define the quantity

$$\boldsymbol{\iota}=\sum_i m_i r_i^2 \tag{3.13.1}$$

as the second moment of mass of a system of particles with respect to the origin. As discussed in Chapter 2, mass is (apart from some niceties in general relativity) synonymous with inertia, and the second moment of mass is used so often that it is nearly always called simply "the" moment of inertia, as though there were only one moment, the second, worth considering. Note carefully, however, that you are probably much more used to thinking about the moment of inertia with respect to an *axis* rather than with respect to a *point*. This distinction is discussed in Section 2.19. Note also that, since the symbol I tends to be heavily used in any discussion of moments of inertia, for moment of inertia with respect to a point I am using the symbol $\boldsymbol{\iota}$

I can also write Equation 3.13.1 as

$$\boldsymbol{\iota}=\sum_i m_i(\mathbf{r}_i\times\mathbf{r}_i) \tag{3.13.2}$$

Differentiate twice with respect to time:

$$\dot{\boldsymbol{\iota}}=2\sum_i m_i(\mathbf{r}_i\dot{\mathbf{r}}_i), \tag{3.13.3}$$

and

$$\ddot{\boldsymbol{\iota}}=2\sum_i m_i(\dot{\mathbf{r}}_i^2+\mathbf{r}_i\ddot{\mathbf{r}}_i), \tag{3.13.4}$$

or

$$\ddot{\boldsymbol{\iota}}=4T+2\sum_i \mathbf{r}_i m_i\ddot{\mathbf{r}}_i, \tag{3.13.5}$$

where T is the *kinetic energy* of the system of particles. The sums are understood to be over all particles - i.e. i from 1 to n.

$m_i\ddot{\mathbf{r}}_i$ is the force on the ith particle. I am now going to suppose that there are no *external* forces on any of the particles in the system, but the particles interact with each other with conservative forces, $\mathbf{F}_{ij}$ being the force exerted on particle i by particle j. I am also going to introduce the notation \({\bf r} _{ij} = {\bf r} _{i} -{\bf r} _{i}\), which is a vector directed from particle i to particle j. The relation between these three vectors in shown in Figure III.8.

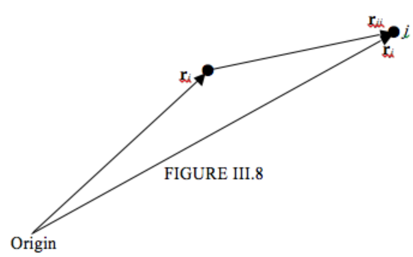


FIGURE III.8

I have not drawn the force $\mathbf{F}_{ij}$, but it will be in the opposite direction to $\mathbf{r}_{ji}$ if it is a repulsive force and in the same direction as $\mathbf{r}_{ji}$ if it is an attractive force.

The total force on particle *i* is $\sum_{j\neq i}\mathbf{F}_{ij}$ and this is equal to $m_i\ddot{\mathbf{r}}_i$ Therefore, Equation 3.13.5 becomes

$$\ddot{I} = 4T + 2\sum_i \mathbf{r}_i \sum_{j\neq i} \mathbf{F}_{ij} \tag{3.13.6}$$

Now it is clear that

$$\sum_i \mathbf{r}_i \sum_{j\neq i} \mathbf{F}_{ij} = \sum_i \sum_{j<i} \mathbf{r}_{ij}\mathbf{F}_{ij} \tag{3.13.7}$$

However, in case, like me, you find double subscripts and summations confusing and you have really no idea what Equation 3.13.7 means, and it is by no means at all clear, I write it out in full in the case where there are five particles. Thus:

$$\begin{aligned}
\textstyle\sum_i \mathbf{r}_i \sum_{j\neq i} \mathbf{F}_{ij} = \mathbf{r}_1(\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14} + \mathbf{F}_{15}) \\
+\mathbf{r}_2(\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_{24} + \mathbf{F}_{25}) \\
+\mathbf{r}_3(\mathbf{F}_{31} + \mathbf{F}_{33} + \mathbf{F}_{34} + \mathbf{F}_{35}) \\
+\mathbf{r}_4(\mathbf{F}_{41} + \mathbf{F}_{43} + \mathbf{F}_{44} + \mathbf{F}_{45}) \\
+\mathbf{r}_5(\mathbf{F}_{51} + \mathbf{F}_{53} + \mathbf{F}_{54} + \mathbf{F}_{55})
\end{aligned}$$

Now apply Newton's third law of motion:

$$\begin{aligned}
\textstyle\sum_i r_i \sum_{j\neq i} \mathbf{F}_{ij} = r_1(-\mathbf{F}_{21} - \mathbf{F}_{31} - \mathbf{F}_{41} + \mathbf{F}_{51}) \\
+\mathbf{r}_2(\mathbf{F}_{21} - \mathbf{F}_{32} - \mathbf{F}_{42} - \mathbf{F}_{52}) \\
+\mathbf{r}_3(\mathbf{F}_{31} - \mathbf{F}_{32} - \mathbf{F}_{43} - \mathbf{F}_{53}) \\
+\mathbf{r}_4(\mathbf{F}_{41} - \mathbf{F}_{42} - \mathbf{F}_{43} - \mathbf{F}_{54}) \\
+\mathbf{r}_5(\mathbf{F}_{51} - \mathbf{F}_{52} - \mathbf{F}_{53} + \mathbf{F}_{54})
\end{aligned}$$

Now bear in mind that $\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_{21}$, and we see that this becomes

$$\begin{aligned}
\textstyle\sum_i r_i \sum j \neq i\mathbf{F}_{ij} = \mathbf{F}_{21} * r_{21} + \mathbf{F}_{31} * r_{31} + \mathbf{F}_{41} * r_{41} + \mathbf{F}_{51} * r_{51} \\
\mathbf{F}_{32} * \mathbf{r}_{32} + \mathbf{F}_{42} * \mathbf{r}_{42} + \mathbf{F}_{52} * \mathbf{r}_{52} \\
+\mathbf{F}_{43} * \mathbf{r}_{43} + \mathbf{F}_{53} * \mathbf{r}_{53} \\
+\mathbf{F}_{54} * \mathbf{r}_{54}
\end{aligned}$$

and we have arrived at Equation 3.13.7. Equation 3.13.6 then becomes

$$\ddot{I} = 4T + 2\sum_i \sum_{j<i} \mathbf{r}_{ij}\mathbf{F}_{ij} \tag{3.13.8}$$

This is the most general form of the virial Equation. It tells us whether the cluster is going to disperse ($\ddot{I}$ positive) or collapse ($\ddot{I}$ negative) – though this will evidently depend on the nature of the force law $\mathbf{F}_{ij}$.

Now suppose that the particles attract each other with a force that is inversely proportional to the nth power of their distance apart. For gravitating particles, of course, n = 2. The force between two particles can then be written in various forms, such as

$$\mathbf{F}_{ij} = -\mathbf{F}_{ij}\hat{\mathbf{r}}_{ij} = -\frac{k}{\mathbf{r}^{\mathbf{n}}_{\mathbf{ij}}}\hat{\mathbf{r}}_{ij} = -\frac{k}{r^{n+1}_{ij}}\mathbf{r}_{ij} \tag{3.13.9}$$

and the mutual potential energy between two particles is minus the integral of $\mathbf{F}_{ij}dr$, or

$$U_{ij} = -\frac{k}{(n-1)\mathbf{r}^{n-1}} \tag{3.13.10}$$

I now suppose that the forces between the particles are gravitational forces, such that

$$\mathbf{F}_{ij} = -\frac{Gm_im_j}{\mathbf{r}^{\mathbf{3}}_{\mathbf{ij}}}\mathbf{r}_{ij} \tag{3.13.11}$$

Now return to the term $\mathbf{r}_{ij}\mathbf{F}_{ij}$ which occurs in Equation 3.13.8:

$$\mathbf{r}_{ij}\mathbf{F}_{ij} = -\frac{k}{\mathbf{r}_{\mathbf{ij}}^{\mathbf{n+1}}}\mathbf{r}_{ij}\mathbf{r}_{ij} = -\frac{k}{r_{ij}^{n-1}} = (n-1)U_{ij} \tag{3.13.12}$$

Thus Equation 3.13.8 becomes

$$\ddot{\imath} = 4T + 2(n-1)U, \tag{3.13.13}$$

where T and U are the kinetic and potential energies of the system. Note that for gravitational interaction (or any attractive) forces, the quantity U is *negative*. **Equation 3.13.13 is the virial theorem for a system of particles with an r^{-2} attractive force between them.** The system will disperse or collapse according the sign of $\ddot{\imath}$ For a system of **gravitationally-interacting** particles, $n = 2$, and so the virial theorem takes the form

$$\ddot{\imath} = 4T + 2U \tag{3.13.14}$$

changing from moment to moment, but always in such a manner that Equation 3.13.13is satisfied.

In a *stable, bound* system, by which I mean that, over a long period of time, there is no long-term change in the moment of inertia of the system, and the system is neither irreversibly dispersing or contracting, that is to say in a system in which the average value of $\ddot{\imath}$ over a long period of time is zero (I'll define "long" soon), the virial theorem for a stable, bound system of r^{-n} particles takes the form

$$2\langle t\rangle + (n-1) < u >= 0, \tag{3.13.15}$$

and **for a <u>stable</u> system of gravitationally-interacting particles,**

$$2\langle t\rangle + < u >= 0, \tag{3.13.16}$$

Here the angular brackets are understood to mean the average values of the kinetic and potential energies over a long period of time. By a "long" period we mean, for example, long compared with the time that a particle takes to cross from one side of the system to the other, or long compared with the time that a particle takes to move in an orbit around the centre of mass of the system. (In the absence of external forces, of course, the centre of mass does not move, or it moves with a constant velocity.)

For example, if a bound cluster of stars occupies a spherical volume of uniform density, the potential energy is $\frac{3GM^2}{5a}$ (Equation 5.9.1 of Celestial Mechanics), so the virial theorem (Equation 3.13.16) will enable you to work out the mean kinetic energy and hence speed of the stars. A globular cluster has roughly spherical symmetry, but it is not of uniform density, being centrally condensed. If you assume some functional form for the density distribution, this will give a slightly different formula for the potential energy, and you can then still use the virial theorem to calculate the mean kinetic energy.

Example 3.13.1

Consider a planet of mass m moving in a circular orbit of radius a around a Sun of mass M, such that $m << M$ and the Sun does not move.

The potential energy of the system is $U = -GM\frac{m}{a}$.
The speed of the planet is given by equating $\frac{mv^2}{a}$ to $\frac{GMm}{a^2}$, from which $T = GM\frac{m}{(2a)}$,

so we easily see in this case that $2T + U = 0$.

CHAPTER OVERVIEW

4: Rigid Body Rotation

No real solid body is perfectly rigid. A rotating nonrigid body will be distorted by centrifugal force* or by interactions with other bodies. Nevertheless most people will allow that in practice some solids are fairly rigid, are rotating at only a modest speed, and any distortion is small compared with the overall size of the body. No excuses, therefore, are needed or offered for analyzing, to begin with the rotation of a rigid body.

4.1: Introduction to Rigid Body Rotation

No real solid body is perfectly rigid. A rotating nonrigid body will be distorted by centrifugal force* or by interactions with other bodies. Nevertheless most people will allow that in practice some solids are fairly rigid, are rotating at only a modest speed, and any distortion is small compared with the overall size of the body. No excuses, therefore, are needed or offered for analyzing, to begin with the rotation of a rigid body.

*I do not in this chapter delve deeply into whether there really is "such thing" as "centrifugal force". Some would try to persuade us that there is no such thing. But is there "such thing" as a "gravitational force"? And is one any more or less "real" than the other? These are deep questions best left to the philosophers. In physics we use the concept of "force" – or indeed any other concept – according to whether it enables us to supply a description of how physical bodies behave. Many of us would, I think, be challenged if we were faced with an examination question: "Explain, without using the term *centrifugal force*, why Earth bulges at its equator."

We have already discussed some aspects of solid body rotation in Chapter 2 on Moment of Inertia, and indeed the present Chapter 4 should not be plunged into without a good understanding of what is meant by "moment of inertia". One of the things that we found was that, while the comfortable relation $L = I\omega$ we are familiar with from elementary physics is adequate for problems in two dimensions, in three dimensions the relation becomes $\mathbf{L} = \mathbf{l}\boldsymbol{\omega}$, where $\mathbf{l}$ is the *inertia tensor*, whose properties were discussed at some length in Chapter 2. We also learned in Chapter 2 about the concepts of *principal moments of inertia*, and we introduced the notion that, unless a body is rotating about one of its principal axes, the equation $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ implies that the angular momentum and angular velocity vectors *are not in the same direction.* We shall discuss this in more detail in this chapter.

A full treatment of the rotation of an *asymmetric top* (whose three principal moments of inertia are unequal and which has as its momental ellipsoid a triaxial ellipsoid) is very lengthy, since there are so many cases to consider. I shall restrict consideration of the motion of an asymmetric top to a qualitative argument that shows that rotation about the principal axis of greatest moment of inertia or about the axis of least moment of inertia is stable, whereas rotation about the intermediate axis is unstable.

I shall treat in more detail the free rotation of a *symmetric top* (which has two equal principal moments of inertia) and we shall see how it is that the angular velocity vector precesses while the angular momentum vector (in the absence of external torques) remains fixed in magnitude and direction.

I shall also discuss the situation in which a symmetric top is subjected to an external torque (in which case $\mathbf{L}$ is certainly not fixed), such as the motion of a top. A similar situation, in which Earth is subject to external torques from the Sun and Moon, causes Earth's axis to precess with a period of 26,000 years, and this will be dealt with in a chapter of the notes on Celestial Mechanics.

Before discussing these particular problems, there are a few preparatory topics, namely, angular velocity and Eulerian angles, kinetic energy, Lagrange's equations of motion, and Euler's equations of motion.

4.2: Angular Velocity and Eulerian Angles

Let $Oxyz$ be a set of space-fixed axis, and let $Ox_0y_0z_0$ be the body-fixed principal axes of a rigid body.

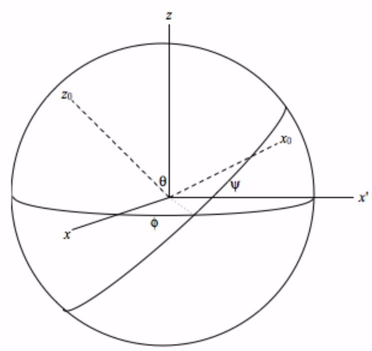


FIGURE IV.1a

The orientation of the body-fixed principal axes $Ox_0y_0z_0$ with respect to the space-fixed axes $Oxyz$ can be described by the three Euler angles: θ, ϕ, and ψ. These are illustrated in Figure IV.1a. Those who are not familiar with Euler angles or who would like a reminder can refer to their detailed description in Chapter 3 of my notes on Celestial Mechanics.

We are going to examine the motion of a body that is rotating about a non-principal axis. If the body is freely rotating in space with no external torques acting upon it, its angular momentum $\mathbf{L}$ will be constant in magnitude and direction. The angular velocity vector ω, however, will not be constant, but will wander with respect to both the space-fixed and body-fixed axes, and we shall be examining this motion. I am going to call the instantaneous components of ω relative to the body-fixed axes $\omega_1, \omega_2, \omega_3$, and its magnitude ω. As the body tumbles over and over, its Euler angles will be changing continuously. We are going to establish a geometrical relation between the instantaneous rates of change of the Euler angles and the instantaneous components of ω. That is, we are going to find how ω_1, ω_2 and ω_3 are related to $\dot{\theta}, \dot{\phi}$ and $\dot{\psi}$.

I have indicated, in Figure VI.2a, the angular velocities $\dot{\boldsymbol{\theta}}, \dot{\boldsymbol{\phi}}$ and $\dot{\boldsymbol{\psi}}$ as vectors in what I hope will be agreed are the appropriate directions.

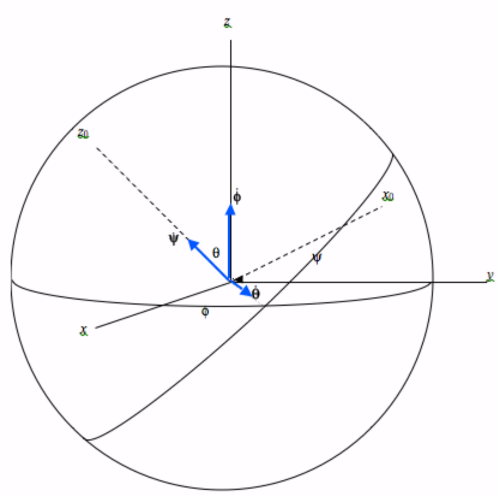
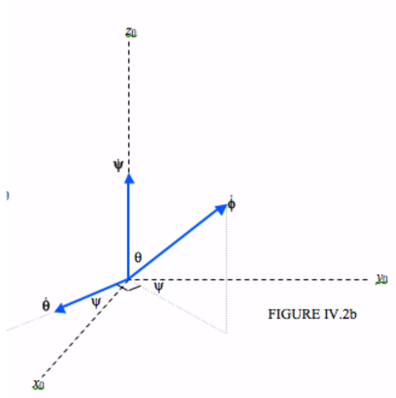


FIGURE IV.2a

It should be clear that ω_1 is equal to the x_0-component of $\dot{\boldsymbol{\phi}}$ plus the x_0-component of $\dot{\boldsymbol{\theta}}$ and that ω_2 is equal to the y_0-component of $\dot{\boldsymbol{\phi}}$ plus the y_0-component of $\dot{\boldsymbol{\theta}}$ and that ω_3 is equal to the z_0-component of $\dot{\boldsymbol{\phi}}$ plus $\dot{\boldsymbol{\psi}}$

Let us look at Figure IV.2b



We see that the x_0 and y_0 components of $\dot{\boldsymbol{\theta}}$ are $\dot{\theta}\cos\psi$ and $-\dot{\theta}\sin\psi$ respectively. The x_0, y_0 and z_0 components of $\dot{\boldsymbol{\phi}}$ are, respectively:

- $\dot{\phi}\sin\theta\sin\psi$,
- $\dot{\phi}\sin\theta\sin\psi$, and
- $\dot{\phi}\cos\theta$.

Hence we arrive at

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi. \tag{4.2.1}$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi. \tag{4.2.2}$$

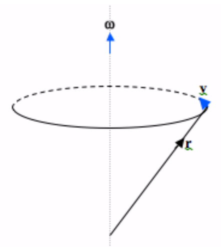
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \tag{4.2.3}$$

4.3: Kinetic Energy of Rigid Body Rotation

Most of us are familiar with the formula $\frac{1}{2}I\boldsymbol{\omega}^2$ for the rotational kinetic energy of a rotating solid body. This formula is adequate for simple situations in which a body is rotating about a principal axis, but is not adequate for a body rotating about a non-principal axis.

I am going to think of a rotating solid body as a collection of point masses, fixed relative to each other, but all revolving with the same angular velocity about a common axis – and those who believe in atoms assure me that this is indeed the case. (If you believe that a solid is a continuum, you can still divide it in your imagination into lots of small mass elements.)

FIGURE IV.3



In Figure IV.3, I show just one particle of the rotating body. The position vector of the particle is {\bf r}. The body is rotating at angular velocity $\boldsymbol{\omega}$. I hope you'll agree that the linear velocity {\bf v} of the particle is (now think about this carefully) \(v = \boldsymbol \omega \times \bf{r} \).

The rotational kinetic energy of the solid body is

$$\mathbf{T}_{rot} = \frac{1}{2}\sum \mathbf{m}\nu^2 = \sum \mathbf{v}\cdot\mathbf{m}\mathbf{v} = \frac{1}{2}\sum(\boldsymbol{\omega}\times\mathbf{r})\cdot\mathbf{p}$$

The triple scalar product is the volume of a parallelepiped, which justifies the next step:

$$T_{rot} = \frac{1}{2}\sum \boldsymbol{\omega}\cdot(r\times p)$$

All particles have the same angular velocity, so:

$$T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\sum(\mathbf{r}\times\mathbf{p}) = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{l}\boldsymbol{\omega}$$

Thus we arrive at the following expressions for the rotational kinetic energy:

$$T_{rot} = \frac{1}{2}\boldsymbol{\omega}\mathbf{L} = \frac{1}{2}\boldsymbol{\omega}\mathbf{I}\boldsymbol{\omega} \tag{4.3.1}$$

If the body is rotating about a nonprincipal axis, the vectors $\boldsymbol{\omega}$ and $\mathbf{L}$ are not parallel (we shall discuss this in more detail in later sections). If it is rotating about a principal axis, they *are* parallel, and the expression reduces to the familiar $\frac{1}{2}I\omega^2$

In matrix notation, this can be written

$$T_{rot} = \frac{1}{2}\tilde{\boldsymbol{\omega}}\mathbf{I}\omega \tag{4.3.2}$$

Here $\mathbf{I}$ is the inertia tensor, ω is a column vector containing the rectangular components of the angular velocity and $\tilde{\boldsymbol{\omega}}$ is its transpose, namely a row vector.

That is:

$$T_{rot} = (\omega_x\omega_y\omega_z)\begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{pmatrix}\begin{pmatrix}\omega_x \\ \omega_y \\ \omega_z\end{pmatrix} \tag{4.3.3}$$
$$\tag{4.3.4}$$
$$= \frac{1}{2}(A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2F\omega_y\omega_z - 2G\omega_z\omega_x - 2H\omega_x\omega_y \tag{4.3.1}$$

This expression gives the rotational kinetic energy when the components of the inertia tensor and the angular velocity vector are referred to an arbitrary set of axes. If we refer them to the *principal* axes, the off-diagonal elements are zero. I am going to call the principal moments of inertia I_1, I_2 and I_3. (I could call them A, B and C, but I shall often use the convention that $A < B < C$, and I do not want to specify at the present which of the three moments is the greatest and which is the greatest, so I'll call them I_1, I_2 and I_3, with $I_1 = \sum m(y^2 + z^2$, etc.). I'll also call the angular velocity components referred to the principal axes ω_1, ω_2, ω_3. Referred, then, to the principal axes, the rotational kinetic energy is

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2). \tag{4.3.5}$$

I have now dropped the subscript "rot", because in this chapter I am dealing entirely with rotational motion, and so T can safely be understood to mean rotational kinetic energy.

We can also now write the kinetic energy in terms of the rates of change of the Eulerian angles, and the expression we obtain will be useful later when we derive Euler's equations of motion:

$$T = \frac{1}{2}I_1(\dot{\phi}\sin\psi + \dot{\theta}\cos\psi)^2 + \frac{1}{2}I_2(\dot{\phi}\sin\theta cos\psi - \dot{\theta}\sin\psi)^2 + I_1(\dot{\phi}\cos\theta + \dot{\psi})^2 \tag{4.3.6}$$

You will probably want a concrete example in order to understand this properly,

Example 4.3.1

Let us imagine that we have a concrete brick of dimensions 10 cm x 15 cm x 20 cm, and of density 4 g cm^{-3}, and that it is rotating about a body diameter (the ends of which are fixed) at an angular speed of 6 rad s^{-1}.

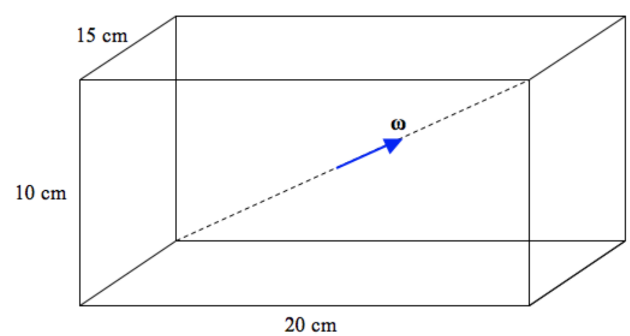


FIGURE IV.4

I hope you'll agree that the mass is 12000 g = 12 kg.

The principal moment of inertia about the vertical axis is

$$I_3 = \frac{1}{3} \times 12000 \times (10^2 + 7.5^2) \quad = 625{,}000 \text{ g cm}^2 = 0.0625 \text{ kg m}^2.$$

Similarly the other principal moments are

$$I_1 = 0.0500 \text{ kg m}^2 \text{ and } I_2 = 0.0325 \text{ kg m}^2.$$

The direction cosines of the vector ω are

$$\frac{15}{\sqrt{15^2+20^2+10^2}}, \frac{20}{\sqrt{15^2+20^2+10^2}}, \frac{10}{\sqrt{15^2+20^2+10^2}}$$

Therefore ω_1 = 3.34252 rad s^{-1}, ω_2 = 4.45669 rad s^{-1}, ω_3 = 2.22834 rad s^{-1}.

Hence $\underline{T = 0.02103 \text{ J}}$

4.4: Lagrange's Equations of Motion

In Section 4.5 I want to derive Euler's equations of motion, which describe how the angular velocity components of a body change when a torque acts upon it. In deriving Euler's equations, I find it convenient to make use of Lagrange's equations of motion. This will cause no difficulty to anyone who is already familiar with Lagrangian mechanics. Those who are not familiar with Lagrangian mechanics may wish just to understand what it is that Euler's equations are dealing with and may wish to skip over their derivation at this stage. Later in this series, I hope to add a longer chapter on Lagrangian mechanics, when all will be made clear (maybe). In the meantime, for those who are not content just to accept Euler's equations but must also understand their derivation, this section gives a five-minute course in Lagrangian mechanics.

To begin with, I have to introduce the idea of *generalized coordinates* and *generalized forces.*

The geometrical description of a mechanical system at some instant of time can be given by specifying a number of *coordinates.* For example, if the system consists of just a single particle, you could specify its rectangular coordinates xyz or its cylindrical coordinates $\rho\phi z$, or its spherical coordinates $r\theta\phi$. Certain theorems to be developed will be equally applicable to any of these, so we can think of *generalized coordinates* $q_1q_2q_3$, which could mean any one of the rectangular, cylindrical of spherical set.

In a more complicated system, for example a polyatomic molecule, you might describe the geometry of the molecule at some instant by a set of interatomic distances plus a set of angles between bonds. A fairly large number of distances and angles may be necessary. These distances and angles can be called the *generalized coordinates.* Notice that generalized coordinates need not always be of dimension L. Some generalized coordinates, for example, may have the dimensions of angle.

[See Appendix of this Chapter for a brief discussion as to whether angle is a dimensioned or a dimensionless quantity.]

While the generalized coordinates at an instant of time describe the geometry of a system at an instant of time, they alone do not predict the future behaviour of the system.

I now introduce the idea of *generalized forces*. With each of the generalized coordinates there is associated a *generalized force.* With the generalized coordinate q_i there is associated a corresponding generalized force P_i. It is defined as follows. If, when the generalized coordinate q_i increases by δq_i, the work done on the system is $P_i\delta q_i$ then P_i is the generalized force associated with the generalized coordinate q_i. For example, in our simple example of a single particle, if one of the generalized coordinates is merely the x-coordinate, the generalized force associated with x is the x-component of the force acting on the particle.

Note, however, that often one of the generalized coordinates might be an *angle*. In that case the generalized force associated with it is a *torque* rather than a force. In other words, a generalized force need not necessarily have the dimensions MLT^{-2}.

Before going on to describe Lagrange's equations of motion, let us remind ourselves how we solve problems in mechanics using Newton's law of motion. We may have a ladder leaning against a smooth wall and smooth floor, or a cylinder rolling down a wedge, the hypotenuse of which is rough (so that the cylinder does not slip) and the smooth base of which is free to obey Newton's third law of motion on a smooth horizontal table, or any of a number of similar problems in mechanics that are visited upon us by our teachers. The way we solve these problems is as follows. We draw a large diagram using a pencil , ruler and compass. Then we mark in red all the *forces*, and we mark in green all the *accelerations*. If the problem is a two-dimensional problem, we write $F=ma$ in any two directions; if it is a three-dimensional problem, we write $F=ma$ in any three directions. Usually, this is easy and straightforward. Sometimes it does not seem to be as easy as it sounds, and we may prefer to solve the problem by Lagrangian methods.

To do this, as before, we draw a large diagram using a pencil , ruler and compass. But this time we mark in blue all the *velocities* (including angular velocities).

> *Lagrange, in the Introduction to his book La méchanique analytique (modern French spelling omits the h) pointed out that there were no diagrams at all in his book, since all of mechanics could be done analytically – hence the title of the book. Not all of us, however, are as mathematically gifted as Lagrange, and we cannot bypass the step of drawing a large, neat and clear diagram.*

Having drawn in the velocities (including angular velocities), we now calculate the *kinetic energy*, which in advanced texts is often given the symbol T, presumably because potential energy is traditionally written U or V. There would be no harm done if you prefer to write E_k, E_p and E for kinetic, potential and total energy. I shall stick to T, U or V, and E.

Now, instead of writing $F = ma$, we write, for each generalized coordinate, the Lagrangian equation (whose proof awaits a later chapter):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial \dot{q}_i} = P_i \tag{4.4.1}$$

The only further intellectual effort on our part is to determine what is the generalized force associated with that coordinate. Apart from that, the procedure goes quite automatically. We shall use it in use in the next section.

That ends our five-minute course on Lagrangian mechanics.

4.5: Euler's Equations of Motion

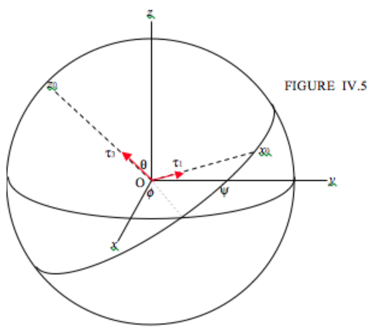
In our first introduction to classical mechanics, we learn that when an external torque acts on a body its angular momentum changes (and if no external torques act on a body its angular momentum does not change.) We learn that the rate of change of angular momentum is equal to the applied torque. In the first simple examples that we typically meet, a symmetrical body is rotating about an axis of symmetry, and the torque is also applied about this same axis. The angular momentum is just $I\omega$, and so the statement that torque equals rate of change of angular momentum is merely $\tau = I\dot{\omega}$ and that's all there is to it.

Later, we learn that $\mathbf{L}$ = $I\boldsymbol{\omega}$, where $\mathbf{l}$ is a tensor, and $\mathbf{L}$ and $\boldsymbol{\omega}$ are not parallel. There are three principal moments of inertia, and $\mathbf{L}$, $\boldsymbol{\omega}$ and the applied torque $\boldsymbol{\tau}$ each have three components, and the statement "torque equals rate of change of angular momentum" somehow becomes much less easy.

Euler's Equations sort this out, and give us a relation between the components of the $\boldsymbol{\tau}$, $\mathbf{l}$ and $\boldsymbol{\omega}$.

For Figure IV.5, I have just reproduced, with some small modifications, Figure III.19 from my notes on this Web site on Celestial Mechanics, where I defined *Eulerian angles*. Again it is suggested that those who are unfamiliar with Eulerian angles consult Chapter III of Celestial Mechanics.

In Figure IV.5, $Oxyz$ are space-fixed axes, and $Ox_0y_0z_0$ are the *body-fixed principal axes*. The axis Oy_0 is behind the plane of your screen; you will have to look inside your monitor to find it.



I suppose an external torque $\boldsymbol{\tau}$ acts on the body, and I have drawn the components τ_1 and τ_3. Now let's suppose that the body rotates in such a manner that the Eulerian angle ψ were to increase by $\delta\psi$. I think it will be readily agreed that the work done on the body is $\tau_3\delta\psi$. This means, following our definition of generalized force in Section 4.4, that τ_3 *is the generalized force associated with the generalized coordinate* ψ. Having established that, we can now apply the Lagrangian Equation 4.4.1:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = \tau_3 \tag{4.5.1}$$

Here the kinetic energy is the expression that we have already established in Equation 4.3.6. In spite of the somewhat fearsome aspect of Equation 4.3.6, it is quite easy to apply Equation 4.5.1 to it. Thus

$$\frac{\partial T}{\partial \dot{\psi}} = I_3(\dot{\phi} cos\theta + \psi) = I_3\omega \tag{4.5.2}$$

where I have made use of Equation 4.2.3.

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = I_3\dot{\omega}_3 \tag{4.5.3}$$

And, if we make use of Equations 4.2.1,2,3, it is easy to obtain

$$\frac{\partial T}{\partial \psi}) = I_1\omega_1\omega_2 - I_2\omega_2\omega_2 = \omega_1\omega_2(I_1 - I_2) \tag{4.5.4}$$

Thus Equation 4.5.1 becomes:

$$I_3\dot{\omega_3} - (I_1 - I_2)\omega_1\omega_1 = \tau_3 \tag{4.5.5}$$

This is one of the *Eulerian Equations of motion*.

Now, although we saw that τ_3 is the generalized force associated with the coordinate y, it will we equally clear that τ_1 is *not* the generalized force associated with q, nor is τ_2 the generalized force associated with ϕ. However, we do not have to think about what the generalized forces associated with these two coordinates are; it is much easier than that. To obtain the remaining two Eulerian Equations, all that is necessary is to carry out a cyclic permutation of the subscripts in Equation 4.5.5. Thus the three Eulerian Equation are:

$$I_1\dot{\omega_1} - (I_2 - I_2)\omega_2\omega_3 = \tau_1, \tag{4.5.6}$$

$$I_2\dot{\omega_2} - (I_3 - I_1)\omega_3\omega_1 = \tau_2, \tag{4.5.7}$$

$$I_3\dot{\omega_3} - (I_1 - I_2)\omega_1\omega_2 = \tau_3. \tag{4.5.8}$$

These take the place of $\tau = I\dot{\omega}$ which we are more familiar with in elementary problems in which a body is rotating about a principal axis and a torque is applied around that principal axis.

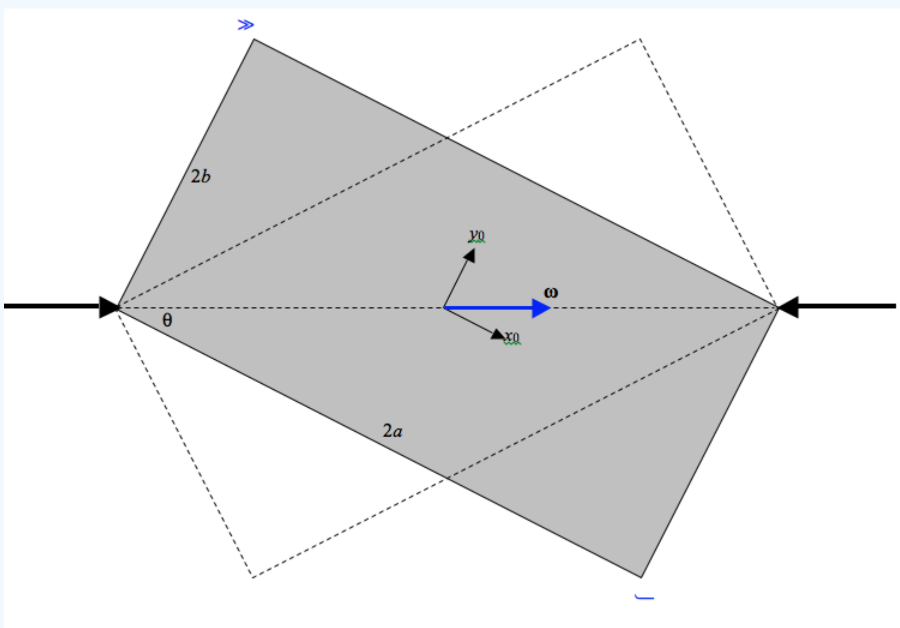
If there are no external torques acting on the body, then we have Euler's Equations of free rotation of a rigid body:

$$I_1\dot{\omega_1} = (I_2 - I_3)\omega_2\omega_3, \tag{4.5.9}$$

$$I_1\dot{\omega_2} = (I_3 - I_1)\omega_3\omega_1, \tag{4.5.10}$$

$$I_3\dot{\omega_3} = (I_1 - I_2)\omega_1\omega_2. \tag{4.5.11}$$

Example 4.5.1



In the above drawing, a rectangular lamina is spinning with constant angular velocity $\boldsymbol{\omega}$ between two frictionless bearings. We are going to apply Euler's Equations of motion to it. We shall find that the bearings are exerting a torque on the rectangle, and the rectangle is exerting a torque on the bearings. The angular momentum of the rectangle is not constant – at least it is not constant in *direction*. We shall calculate the torque (its magnitude and its direction) and see what is happening to the angular momentum.

We note that the principal (second) moments of inertia are

$$I_1 = \tfrac{1}{3}mb^2 \qquad I_2 = \tfrac{1}{3}ma^2 \qquad I_3 = \tfrac{1}{3}m(a^2 + b^2)$$

and that the components of angular velocity are

$$\omega_1 = \omega\cos\theta \qquad \omega_2 = \omega\sin\theta \qquad \omega_3 = 0.$$

Also, $\dot{\boldsymbol{\omega}}$ and all of its components are zero. We immediately obtain, from Euler's Equations, that τ_1 and τ_2 are zero, and that the torque exerted **on** the rectangle **by** the bearings is

$\tau_3 = (I_2 - I_1)\omega_1\omega_2 = \frac{1}{3}m(a^2 - b^2)\omega^2 sin\theta \cos\theta$

And since

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}} \quad and \quad \cos\theta = \frac{b}{\sqrt{a^2+b^2}},$$

we obtain

$$\tau_3 = \frac{m(a^2-b^2)ab}{3(a^2+b^2)}\omega^2$$

Thus $\boldsymbol{\tau}$, the torque exerted **on** the rectangle **by** the bearings is directed normal to the plane of the rectangle (out of the plane of the paper in the instantaneous snapshot above).

The angular momentum is given by $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$. That is to say:

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \frac{1}{3}m \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \begin{pmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

$$L_1 = \frac{1}{3}mb^2\omega\cos\theta = \frac{1}{3}m\frac{ab^2}{\sqrt{a^2+b^2}}\omega$$

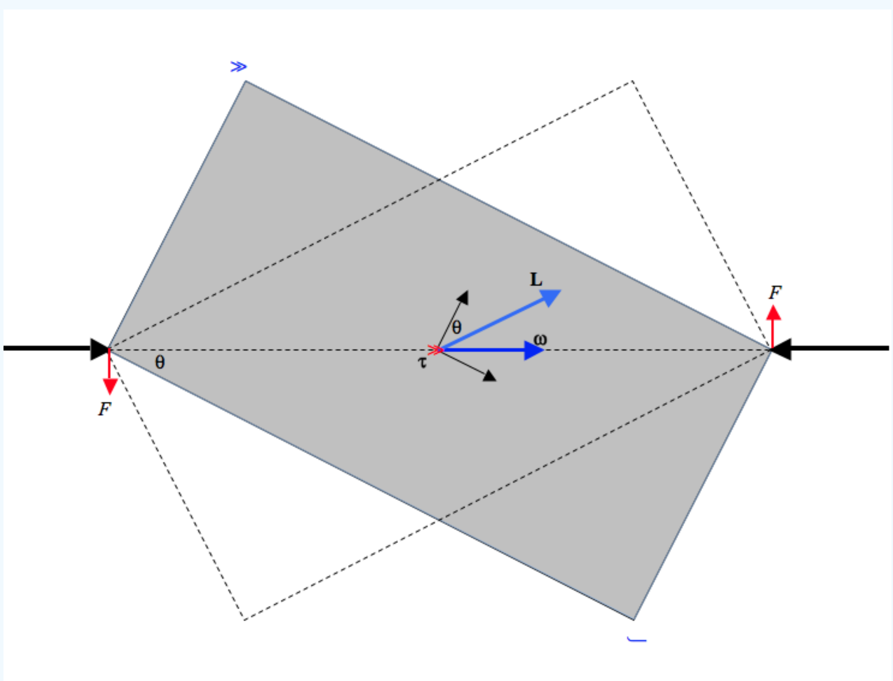
$$L_2 = \frac{1}{3}mb^2\omega\sin\theta = \frac{1}{3}m\frac{ab^2}{\sqrt{a^2+b^2}}\omega$$

$$L_3 = 0$$

$$L = \frac{1}{3}mab\omega$$

$$L_2/L_1 = \frac{a^2 sin\theta}{b^2 cos\theta} = \cot\theta = tan(90^\circ - \theta)$$

This tells us that $\mathbf{L}$ is in the plane of the rectangle, and makes an angle 90° - θ with the x-axis, or q with the y-axis, and it rotates around the vector $\boldsymbol{\tau}$. $\boldsymbol{\tau}$ is perpendicular to the plane of the rectangle, and of course the change in $\mathbf{L}$ takes place in that direction. The torque does no work, and $\boldsymbol{\omega}$ and T are constant. The reader might find an analogy in the situation of a planet in orbit around the Sun in a circular orbit.. The planet experiences a force that is always perpendicular to its velocity. The force does no work, and the speed and kinetic energy remain constant.



The torque on the plate can be represented as a *couple* of forces exerted by the bearings on the plate, each of magnitude $\frac{\tau_3}{2\sqrt{a^2+b^2}}$, or $\frac{m(a^2-b^2)}{6(a^2-b^2)^{\frac{3}{2}}}\omega^2$ Forces exerted by the plate on the bearings are, of course, in the opposite direction.

Example 4.5.2

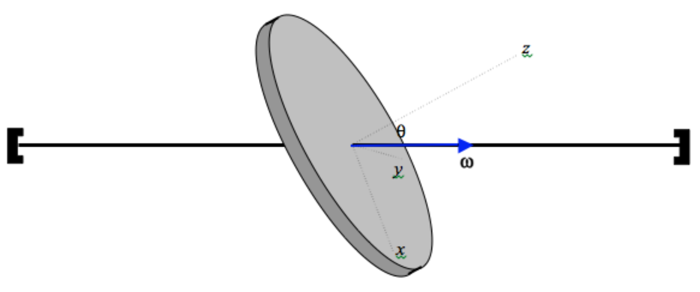


FIGURE IV.6

Figure IV.6 shows a disc of mass m, radius a, spinning at a constant angular speed ω about at axle that is inclined at an angle θ to the normal to the disc. I have drawn three body-fixed principal axes. The x- and y- axes are in the plane of the disc\boldsymbol; the direction of the x-axis is chosen so that the axle (and hence the vector $\boldsymbol{\omega}$) is in the zx-plane. The disc is evidently unbalanced and there must be a torque on it to maintain the motion.

Since $\boldsymbol{\omega}$ is constant, all components of $\dot{\boldsymbol{\omega}}$ are zero, so that Euler's Equations are

$$\tau_1 = (I_3 - I_2)\omega_3\omega_2,$$

$$\tau_2 = (I_1 - I_3)\omega_1\omega_3,$$

$$\tau_3 = (I_2 - I_1)\omega_2\omega_1,$$

Now $\omega_1 = \omega\sin\theta, \omega_2 = \omega\cos\theta, I_1 = \frac{1}{4}ma^2, I_2 = \frac{1}{4}ma^2, I_3 = \frac{1}{1}ma^2$

Therefore $\tau_1 = \tau_3 = 0, and \tau_2 = -\frac{1}{4}ma^2\omega^2 sin\theta cos\theta = -\frac{1}{8}ma^2\omega^2 sin2\theta$

(Check, as always, that this expression is dimensionally correct.) Thus the torque acting on the disc is in the negative y-direction.

Can you reconcile the fact that there is a torque acting on the disc with the fact that is it moving with constant angular velocity? Yes, most decidedly! What is *not* constant is the *angular momentum* $\mathbf{L}$, which is moving around the axle in a cone such that $\dot{\mathbf{L}} = -\tau_2\mathbf{j}$, where $\mathbf{j}$ is the unit vector along the y-axis.

4.6: Force-free Motion of a Rigid Asymmetric Top

By "asymmetric top" I mean a body whose three principal moments of inertia are unequal. While we often think of a "top" as a symmetric body spinning on a table, in this section the "top" will not necessarily be symmetric, and it will not be in contact with any table, nor indeed subjected to any external forces or torques.

A complete description of the motion of an asymmetric top is quite complicated, and therefore all that we shall attempt in this chapter is a qualitative description of certain aspects of the motion. That our description is going to be "qualitative" does not by any means imply that this section is not going to be replete with equations or that we can give our poor brains a rest.

The first point that we can make is that, *provided that no external torques act on the body*, its angular momentum $\mathbf{L}$ is constant in magnitude and direction. A second point is that, *provided the body is rigid and has no internal degrees of freedom*, the rotational kinetic energy T is constant. I deal briefly with nonrigid bodies in Section 4.7. Although the angular velocity vector $\boldsymbol{\omega}$ is by no means fixed in either magnitude and direction, and the body can tumble over and over, these two conditions impose some constraints of the magnitude and direction of $\boldsymbol{\omega}$.

We are going to examine these two conditions to see what constraints are imposed on $\boldsymbol{\omega}$. One of the things we shall find is that rotation of a body about a principal axis of greatest or of least moment of inertia is stable against small displacements, whereas rotation about the principal axis of intermediate moment of inertia is unstable.

Absence of an external torque means that the angular momentum is constant:

$$L^2 = L_1^2 + L_2^2 + L_3^2 = constant, \tag{4.6.1}$$

so that, at all times,

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L^2 \tag{4.6.2}$$

Thus, for a given L. the angular velocity components always satisfy

$$\frac{\omega_1^2}{(L/I_1)^2} + \frac{\omega_2^2}{(L/I_2)^2} + \frac{\omega_3^2}{(L/I_3)^2} = 1. \tag{4.6.3}$$

That is to say, the angular velocity vector is constrained such that the tip of the vector $\boldsymbol{\omega}$ is always on the surface of an ellipsoid of semi axes $\frac{L}{I_1}$, $\frac{L}{I_2}$, $\frac{L}{I_3}$,

In addition to the constancy of angular momentum, the kinetic energy is also constant:

$$\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 = T \tag{4.6.4}$$

Thus the tip of the angular velocity vector must also be on the surface of the ellipsoid

$$\frac{\omega_1^2}{(\sqrt{2T/I_1})^2} + \frac{\omega_2^2}{(\sqrt{2T/I_2})^2} + \frac{\omega_3^2}{(\sqrt{2T/I_3})^2} = 1. \tag{4.6.5}$$

This ellipsoid (which is similar in shape to the momental ellipsoid) has semi axes $\sqrt{2T/I_1}$, $\sqrt{2T/I_2}$, $\sqrt{2T/I_3}$.

Thus, how ever the body tumbles over and over, $\boldsymbol{\omega}$ is constrained in magnitude and direction so that its tip is on the curve where these two ellipses intersect.

Example 4.6.1

Suppose, that we have a rigid body with

- I_1 = 0.2 kgm^2
- I_2 = 0.3 kgm^2
- I_3 = 0.5 kgm^2

and that we set it in motion such that the angular momentum and kinetic energy are L = 4 J s and T = 20 J.

(The angular momentum and kinetic energy will be determined by the magnitude and direction of the initial velocity vector by which it is set in motion.)

The tip of $\boldsymbol{\omega}$ is constrained to be on the curve of intersection of the two ellipsoids

$$\frac{\omega_1^2}{20^2}+\frac{\omega_2^2}{13.3^2}+\frac{\omega_3^2}{8^2}=1 \tag{4.6.6}$$

and

$$\frac{\omega_1^2}{14.14^2}+\frac{\omega_2^2}{11.55^2}+\frac{\omega_3^2}{8.94^2}=1 \tag{4.6.7}$$

It is not easy (or I do not find it so) to imagine what this curve of intersection looks like in three-dimensional space, but one of my students, Leif Petersen, prepared the drawing below, and I am grateful to him for permission to reproduce it here. You can see that the curve of intersection is not a plane curve.

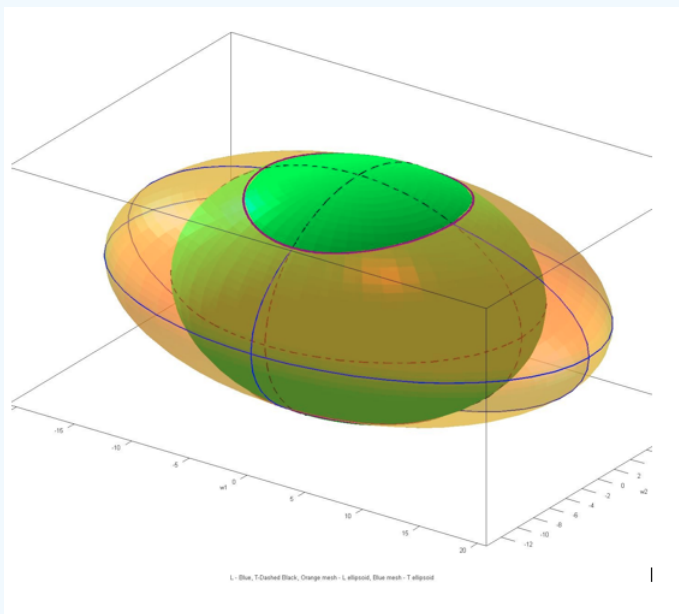
In case it's of any help, you might want to note that equations 4.6.6 and 4.6.7 can be written

$$4\omega_1^2+9\omega_2^2+25\omega_3^2=1600 \tag{4.6.8}$$

and

$$2\omega_1^2+3\omega_2^2+5\omega_3^2=400 \tag{4.6.9}$$

but I'm going to leave the equations in the form 4.6.6 and 4.6.7, and in figure IV.7, I'll sketch one octant of the two ellipsoidal surfaces.



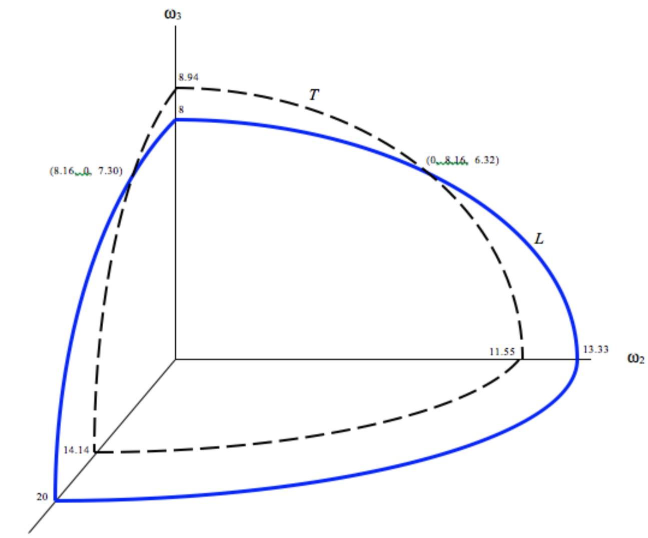


FIGURE IV.7

The continuous blue curve shows an octant of the ellipsoid L = constant, and the dashed black curve shows an octant of the ellipsoid T = constant. The angular momentum vector can end only on the curve (not drawn) where the two ellipsoids intersect. Two points on the curve are indicated in Figure IV.7. If, for example, $\boldsymbol{\omega}$ is oriented so that ω_1 = 0, the other two components must be ω_2 = 8.16 and ω_3 = 6.32. If it is oriented so that ω_3 = 0, the other two components must be ω_3 = 7.30 and ω_1 = 8.16. If ω_3 = 0, there are no real solutions for ω_1 and ω_2. This means that, for the given values of L and T, ω_3 cannot be zero.

Now I'm going to address myself to the stability of rotation when a symmetric top is initially set to spin about one of its principal axes, which I'll take to be the z-axis. We'll suppose that initially ω_1 = ω_2 = 0, and ω_3 = Ω. In that case the angular momentum and the kinetic energy are $L = I_3\Omega$ In any subsequent motion, the tip of $T = I_3\Omega^2$ is restricted to move along the curve of intersection of the ellipsoids given by equations 4.6.3 and 4.6.5. That is to say, along the curve of intersection of the ellipsoids

$$\frac{\omega_1^2}{(\frac{I_3}{I_1}\Omega)^2}+\frac{\omega_2^2}{(\frac{I_3}{I_2}\Omega)^2}+\frac{\omega_3^2}{\Omega^2}=1 \tag{4.6.10}$$

and

$$\frac{\omega_1^2}{(\sqrt{\frac{I_3}{I_1}}\Omega)^2}+\frac{\omega_2^2}{(\sqrt{\frac{I_3}{I_2}}\Omega)^2}+\frac{\omega_3^2}{\Omega^2}=1 \tag{4.6.11}$$

For a specific example, I'll suppose that the moments of inertia are in the ratio $2:3:5$, and we'll consider three cases in turn.

Case I. Rotation about the axis of *least* moment of inertia. That is, we'll take $I_3=2$, $I_1=3$, $I_2=5$. Since I_3 is the smallest moment of inertia, each of the ratios $\frac{I_3}{I_1}$ and $\frac{I_3}{I_2}$ are less than 1, and $\sqrt{\frac{I_3}{I_1}}>\frac{I_3}{I_1} \quad and \sqrt{\frac{I_3}{I_2}}>\frac{I_3}{I_2}$ The two ellipsoids are

$$\frac{\omega_1^2}{(0.667\Omega)^2}+\frac{\omega_2^2}{(0.400\Omega)^2}+\frac{\omega_3^2}{\Omega^2}=1 \tag{4.6.12}$$

and

$$\frac{\omega_1^2}{(0.816\Omega)^2}+\frac{\omega_2^2}{(0.632\Omega)^2}+\frac{\omega_3^2}{\Omega^2}=1 \tag{4.6.13}$$

I’ll try and sketch these:

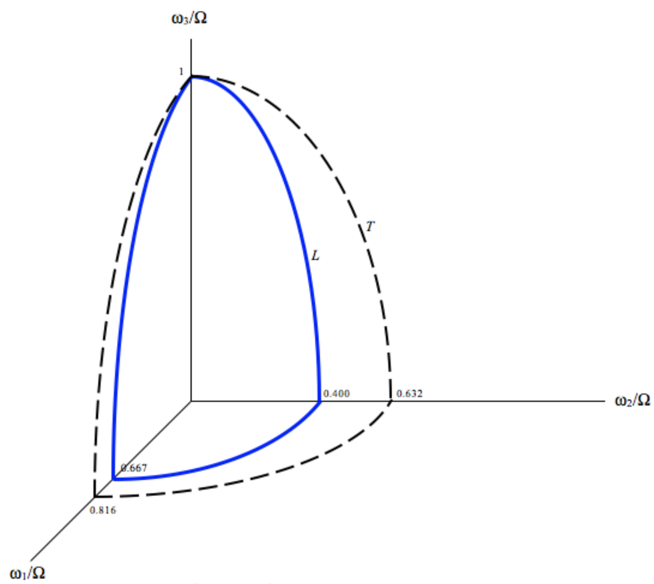


FIGURE IV.8

Initially, we suppose, the body was set in motion rotating about the z-axis with angular speed Ω, which determines the values of L and T, which will remain constant. The tip of the vector ω is constrained to remain on the surface of the ellipsoid L = 0 and on the ellipsoid T = 0, and hence on the intersection of these two surfaces. But these two surface touch only at one point, namely $(\omega_1, \omega_2, \omega_3)$ = (0 , 0 , Ω). Thus there the vector ω remains, and the rotation is stable.

Case II. Rotation about the axis of *greatest* moment of inertia. That is, we’ll take $I_3 = 5$, $I_1 = 2$, $I_2 = 3$. Since I_3 is the greatest moment of inertia, each of the ratios $\frac{I_3}{I_1}$ and $\frac{I_3}{I_2}$ are greater than 1, and $\sqrt{\frac{I_3}{I_1}} < \frac{I_3}{I_1}$ and $\sqrt{\frac{I_3}{I_2}} < \frac{I_3}{I_2}$ The two ellipsoids are

$$\frac{\omega_1^2}{(2.50\Omega)^2} + \frac{\omega_2^2}{(1.67\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1 \tag{4.6.14}$$

and

$$\frac{\omega_1^2}{(1.58\Omega)^2} + \frac{\omega_2^2}{(1.29\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1 \tag{4.6.15}$$

I’ll try and sketch these:

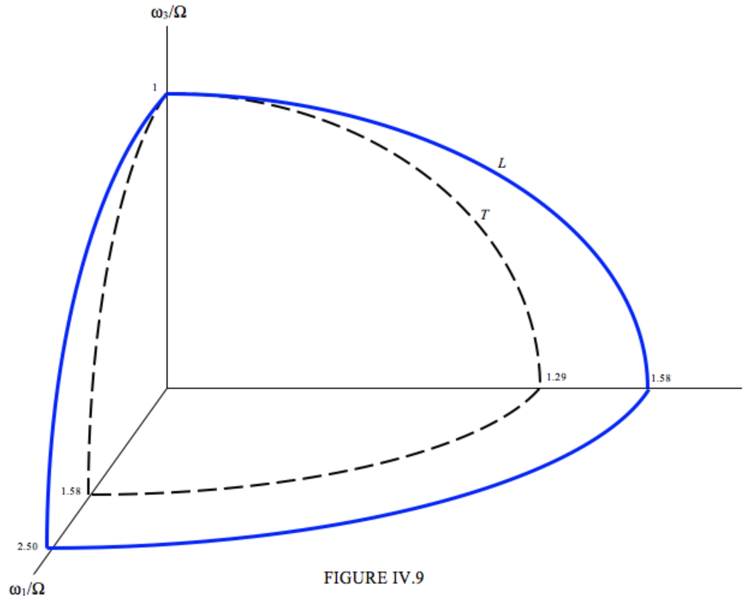


FIGURE IV.9

Again, and for the same reason as for Case I, we see that this motion is stable.

Case III. Rotation about the *intermediate* axis. That is, we'll take $I_3 = 3$, $I_1 = 5$, $I_2 = 2$. This time $\frac{I_3}{I_1}$ is less than 1 and $\frac{I_3}{I_2}$ is less than 1, and $\sqrt{\frac{I_3}{I_1}} < \frac{I_3}{I_1}$ and $\sqrt{\frac{I_3}{I_2}} < \frac{I_3}{I_2}$ The two ellipsoids are

$$\frac{\omega_1^2}{(0.60\Omega)} + \frac{\omega_2^2}{(1.50\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1 \tag{4.6.16}$$

and

$$\frac{\omega_1^2}{(0.77\Omega)^2} + \frac{\omega_2^2}{(1.22\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1 \tag{4.6.17}$$

I'll try and sketch these:

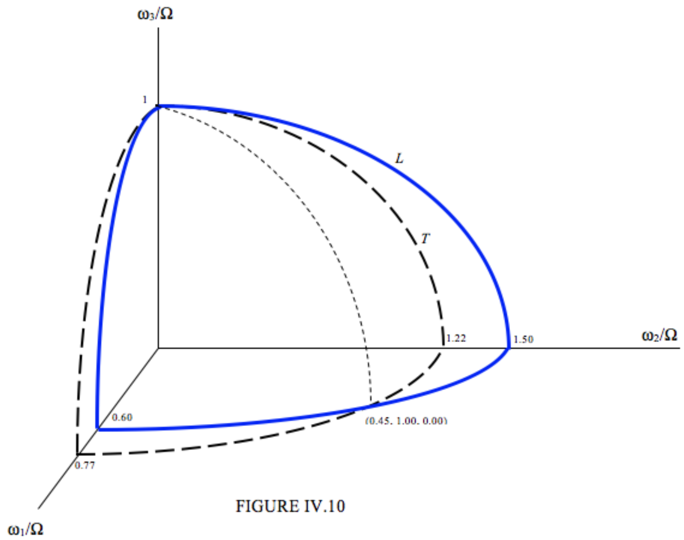


FIGURE IV.10

Unlike the situation for Cases I and II, in which the two ellipsoids touch at only a single point, the two ellipses for Case III intersect in the curve shown as a dotted line in figure IV.10. Thus ω is not restricted to lying along the z-axis, but it can move anywhere along the dotted line. The motion, therefore, is not stable.

You should experiment by throwing a body in the air in such a manner as to let it spin around one of its principal axes. A rectangular block will do, though the effect is particularly noticeable with something like a table-tennis bat.

Here is another approach to reach the same result. We imagine an asymmetric top spinning about one of its principal axes with angular velocity \(\boldsymbol\omega \ = \omega \hat{\bf z}). It is then given a small perturbation, so that its angular velocity is now.

$$\omega = \epsilon\hat{\mathbf{x}} + \eta\hat{\mathbf{y}} + \omega_z\hat{\mathbf{z}} \tag{4.6.18}$$

Here the "hatted" quantities are the unit orthogonal vectors; ϵ and η are supposed small compared with ω_z. Euler's equations are :

$$I_1\dot{\epsilon} = \eta\omega_Z(I_2 - I_3), \tag{4.6.19}$$

$$I_1\dot{\eta} = \omega_Z\epsilon(I_3 - I_1), \tag{4.6.20}$$

$$I_1\dot{\omega_Z} = \epsilon\eta(I_1 - I_2). \tag{4.6.21}$$

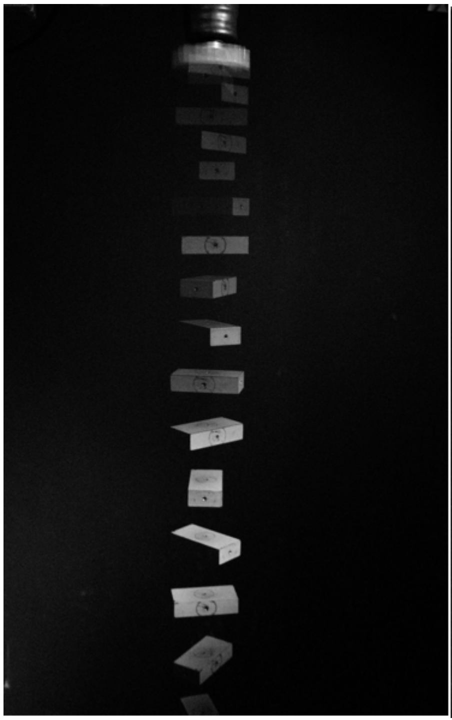
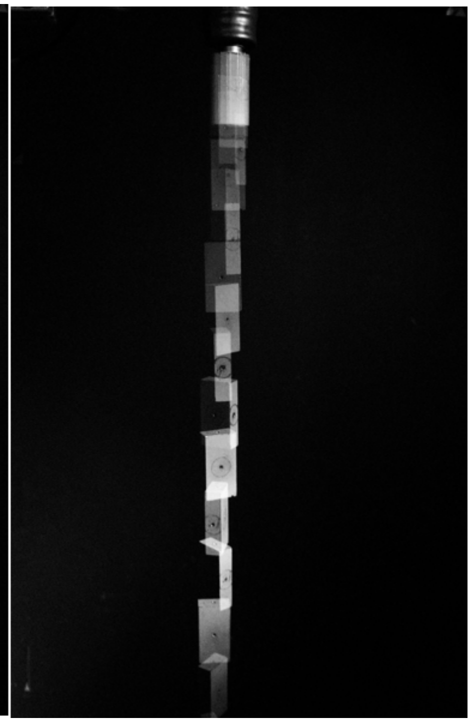
If $\epsilon\eta << \dot{\omega_Z}$, then ω_Z is approximately constant. Elimination of η from the first two equations yields

$$\ddot{\epsilon} = -[\frac{(I_2 - I_3)(I_1 - I_3)\omega_z^2}{I_1 I_2}]\epsilon \tag{4.6.22}$$

Elimination of ϵ instead results in a similar equation in η.

If I_3 is either the largest or the smallest of the three moments of inertia, the two parentheses in the denominator have the same sign, so the expression in the brackets is positive. Equation 4.6.22 is then the equation for simple harmonic motion, and the motion is stable. If, however, I_3 is intermediate between the other two, the two parentheses have opposite sign, and the expression in brackets in negative. In that case ϵ and η increase exponentially, and the motion is unstable.

Mr Neil Honkanen of the University of Victoria conducted an experiment to illustrate the stability of rotation about the three principal axes. The body in question was a small "brick" of mild steel (density 7.83 g/cm^3) of dimensions 3/8 inch × 3/4 inch × 1 1/2 inch, mass 54.1 g. In round figures, this corresponds to principal moments of inertia A_0 = 2 ×10^{-6} kg m^2, B_0 = 7 × 10^{-6} kg m^2, C_0 = 8 × 10^{-6} kg m^2. He suspended it from an electromagnet, which he set in rotation at about 25 revolutions per second, and then let it fall, while photographing it stroboscopically. He did three experiments rotation respectively about the three principle axes. You can see from the photographs below that the rotation is stable when the rotation is about the axes of greatest or least moment of inertia, but is unstable when the rotation is about the axis of intermediate moment of inertia.


4.7: Nonrigid Rotator

The rotational kinetic energy of a body rotating about a principal axis is $\frac{1}{2}I\omega^3$, where I is the moment of inertia about that principal axis, and the angular momentum is L = Iω. (For rotation about a nonprincipal axis, see Section 4.3.) Thus the rotational kinetic energy can be written as

$$KE_{rot} = \frac{L^2}{2I}. \tag{4.7.1}$$

When an asymmetric top is rotating about a nonprincipal axes, the body experiences internal stresses, which, if the body is nonrigid, result in periodic strains which periodically distort the shape of the body. As a result of this, rotational kinetic energy becomes degraded into heat; the rotational kinetic energy of the body gradually decreases. In the absence of external torques, however, the angular momentum is constant. Equation 4.7.1 shows that the kinetic energy is least for a given angular momentum when the moment of inertia is greatest. Thus eventually the body rotates about its principal axis of greatest moment of inertia. After that, it no longer loses kinetic energy to heat, because, when the body is rotating about a principal axis, it is no longer subject to internal stresses.

The time taken (the "relaxation time") for a body to reach its final state of rotation about its principal axis of greatest moment of inertia depends, among other things, on how fast the body is rotating. A fast rotator will reach its final state relatively soon, whereas it takes a long time for a slow rotator to reach its final state. Thus it is not surprising to find that, among the asteroids, most of the fast rotators are principal axis rotators, whereas many slow rotators are also nonprincipal axis rotators. There are, however, a few fast rotators that are still rotating about a nonprincipal axis. It is assumed that such asteroids may have suffered a collision in the recent past.

4.8: Force-free Motion of a Rigid Symmetric Top

Notation:

- I_1, I_2, I_3 are the principal moments of inertia. I_3 is the unique moment. If it is the largest of the three, the body is an *oblate symmetric top*; if it is the smallest, it is a *prolate spherical top*.
- Ox_0, Oy_0, Oz_0 are the corresponding body-fixed principal axes.
- $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity vector $\boldsymbol{\omega}$ with respect to the principal axes.

In the analysis that follows, we are going to have to think about three vectors. There will be the angular momentum vector $\mathbf{L}$, which, in the absence of external torques, is fixed in magnitude and in the direction in laboratory space. There will be the direction of the axis of symmetry, the Oz_0 axis, which is fixed in the body, but not necessarily in space, unless the body happens to be rotating about its axis of symmetry; we'll denote a unit vector in this direction by $\hat{\mathbf{z}}_0$. And there will be the instantaneous angular velocity vector ω which is neither space- nor body-fixed.

What we are going to find is the following. We shall find that $\boldsymbol{\omega}$ precesses in the body about the body-fixed symmetry axis in a cone called the *body cone*. The angle between $\boldsymbol{\omega}$ and $\hat{\mathbf{z}}_{\mathbf{0}}$ is constant (we'll be calling this angle α), and the magnitude ω of $\boldsymbol{\omega}$ of is constant. We shall find that the sense of the precession is the same as the sense of the spin if the body is oblate, but opposite if it is prolate. The direction of the symmetry axis, however, is not fixed in space, but it precesses about the space-fixed angular momentum vector $\mathbf{L}$ in another cone. This cone is narrower than the body cone if the body is oblate, but broader than the body cone if the body is prolate. The net result of these two precessional motions is that precesses ω in space about the space-fixed angular momentum vector in a cone called the space cone. For a prolate top, the semi vertical angle of the space cone can be anything from 0° to 90°; for an oblate top, however, the semi vertical angle of the space cone cannot exceed 19° 28' . That's quite a lot to take in in one breath!

We can start with Euler's equations of motion for force-free rotation of a symmetric top:

$$I_1\dot{\omega_1} = -\omega_2\omega_3(I_3 - I_1), \tag{4.8.1}$$

$$I_1\dot{\omega_2} = -\omega_1\omega_3(I_3 - I_1), \tag{4.8.2}$$

$$I_3\dot{\omega_3} = 0. \tag{4.8.3}$$

From the first of these we obtain the result

$$\omega_3 = \text{constant} \tag{4.8.4}$$

For brevity, I am going to let

$$\frac{(I_3 - I_1)}{I_1} = \Omega, \tag{4.8.5}$$

although in a moment Ω will have a physical meaning.

Equations 4.8.1 and 4.8.2 become:

$$\dot{\omega_1} = \Omega\omega_2 \tag{4.8.6}$$

and

$$\dot{\omega_2} = \Omega\omega_1 \tag{4.8.7}$$

Eliminate ω_2 from these to obtain

$$\dot{\omega_1} = -\Omega^2\omega_1 \tag{4.8.8}$$

This is the Equation for simple harmonic motion and its solution is

$$\omega_1 = \omega_0\cos(\Omega t + \epsilon) \tag{4.8.9}$$

in which ω_0 and ϵ, the two constants of integration, whose values depend on the initial conditions in the usual fashion, are the amplitude and initial phase angle. On combining this with Equation 4.8.6, we obtain

$$\omega_2 = \omega_0\sin(\Omega t + \epsilon) \tag{4.8.10}$$

From these we see that $(\omega_1^2+\omega_2^2)^{1/2}$, which is the magnitude of the component of ω in the x_0y_0-plane, is constant, equal to ω_0; and since ω_3 is also constant, it follows that $(\omega_1^2+\omega_2^2+\omega_3^2)^{1/2}$, which is the magnitude of ω, is also constant. The cosine of the angle α between $\hat{z_0}$ and $\boldsymbol{\omega}$ is $\omega_3/(\omega_1^2+\omega_2^2+\omega_3^2)^{1/2}$, and its sine is $\omega_0/(\omega_1^2+\omega_2^2+\omega_3^2)^{1/2}$, so that α is constant. Equations 4.8.9 and 4.8.10 tell us, then, that the vector ω is precessing around the symmetry axis at an angular speed Ω. Making use of Equation 4.8.5, we find that

$$\cos\alpha = \frac{\omega_3}{\omega} = \frac{I_1\Omega}{(I_3-I_1)\omega} \tag{4.8.11}$$

If we take the direction of the z_0 axis to be the direction of the component of ω along the symmetry axis, then $\mathbf{\Omega}$ is in the same direction as $\mathbf{z}_0$ if $I_3 > I_1$ (that is, if the top is oblate) and it is in the opposite direction if the top is prolate. The situation for oblate and prolate tops is shown in Figure IV.11.

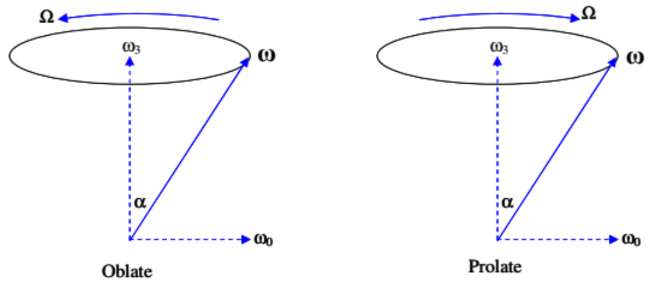


FIGURE IV.11

We have just dealt with how the instantaneous axis of rotation precesses about the body-fixed symmetry axis, describing the body cone of semi vertical angle α.

Now we are going to consider the precession of the body-fixed symmetry axis about the space-fixed angular momentum vector $\mathbf{L}$. I am going to make use of the idea of Eulerian angles for expressing the orientation of one three-dimensional set of axes with respect to another. If you are not already familiar with Eulerian angles or would like a refresher, you can go to to Chapter 3 of Celestial Mechanics especially Section 3.7.

Recall that we are using $Ox_0y_0z_0$ for *body-fixed* coordinates, referred to the principal axes. I shall use $Oxyz$ for *space-fixed* coordinates, and there is no loss of generality if I choose the Oz axis to coincide with the angular momentum vector $\mathbf{L}$. Let me try to draw the situation in Figure IV.12a. The axes $Oxyz$ are the space-fixed axes. The axes $Ox_0y_0z_0$ are the body-fixed principal axes. The angular momentum vector $\mathbf{L}$ is directed along the axis Oz. The symmetry axis of the body is directed along the axis Oz_0. The Eulerian angles of the body-fixed axes relative to the space fixed axes are (ϕ, θ, ψ).

Recall, with the aid of Figure IV.12b, how these Euler angles are formed:

First, a rotation by ϕ about Oz. Second, a rotation by θ about the dashed line Ox' to form an intermediate set of axes $Ox'y'z'$. Third, a rotation by ψ about Oz' to form the body- fixed principal axes $Ox_0y_0z_0$.

Spend a little time trying to visualize these three sets of axes. Please also convince yourself, from the way the Euler angles were formed through three rotations, that the vector $\mathbf{L}$ is in the $y_\prime z_\prime$ plane and has no x' component. It is also in the y_0z_0 plane and has no x_0 component.

You will then agree that

$$L_{x'} = 0, \quad L_{y'} = L\sin\theta \quad L_{z'} = L\cos\theta. \tag{4.8.12}$$

Now if $L_{x'} = 0$, then $\omega_{x'}$ is also zero, which means that $\boldsymbol{\omega}$, like $\mathbf{L}$, is in the $y'z'$ plane.

We have seen that $\boldsymbol{\omega}$ makes an angle α with the symmetry axis Oz_0, where α is given by Equation 4.8.11.

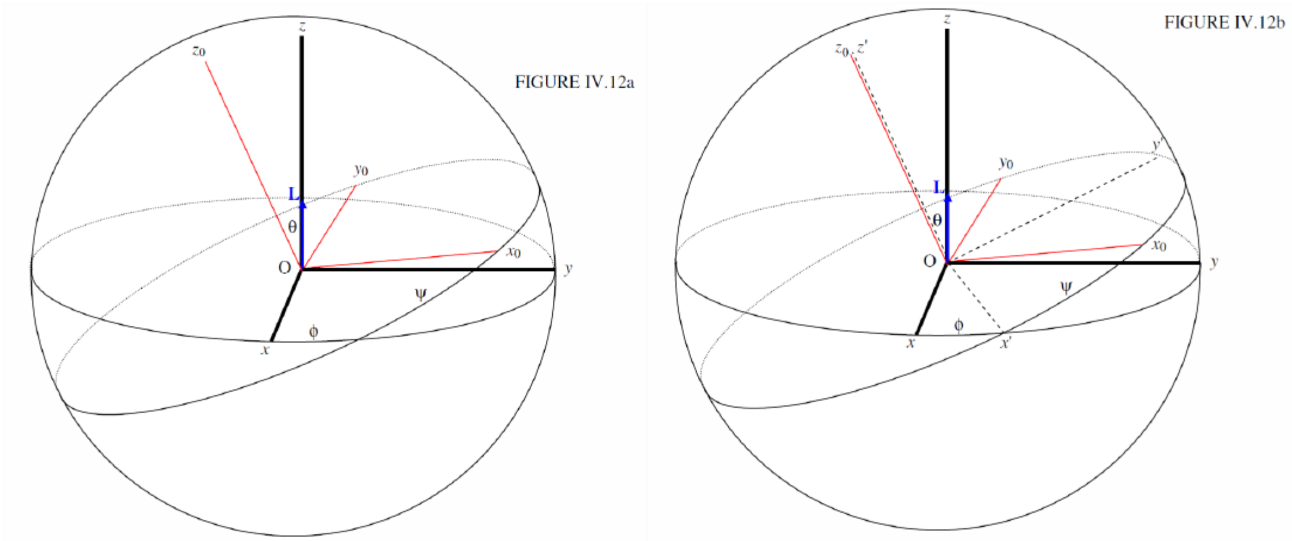


Figure 4.8.1: Paste Caption Here

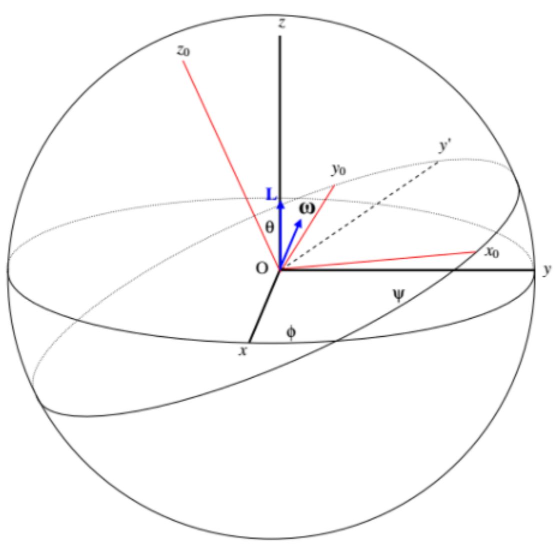
I'll now add $\boldsymbol{\omega}$ to the drawing to make Figure IV.13. Like $\mathbf{L}$, it is in the y'z' plane and has no x' component. I haven't marked in the angle α. I leave it to your imagination. It is the angle between $\boldsymbol{\omega}$ and z_0. You should easily agree that

$$\omega_{x'} = 0, \quad \omega_{y'} + \omega \sin\alpha, \quad \omega_{z'} = \omega cos\alpha. \tag{4.8.13}$$

From these, together with $L_{y'} = I_1\omega_{y'}$ and $L_{z'} = I_1\omega_{z'}$ we obtain

$$I_1 = \tan\alpha = I_3\tan\theta \tag{4.8.14}$$

- For an oblate symmetric top, $I_3 > I_1$, $\alpha > \theta$.
- For a prolate symmetric top, $I_3 < I_1$, $\alpha < \theta$.



Now $\boldsymbol{\omega}$ can be written as the vector sum of the rates of change of the three Euler angles:

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\psi}} \tag{4.8.15}$$

The components of $\dot{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{\psi}}$ along Oy' are each zero, and therefore the component of ω along Oy' is equal to the component of $\dot{\boldsymbol{\psi}}$ along Oy' .

$$\therefore \qquad \omega\sin\alpha = \dot{\phi}\sin\theta \tag{4.8.16}$$

In summary, then:

1. The instantaneous axis of rotation, which makes an angle α with the symmetry axis, precesses around it at angular speed

$$\Omega=\frac{I_3-I_1}{I_1}\omega\cos\alpha \tag{4.8.17}$$

   which is in the same sense as $\boldsymbol{\omega}$ if the top is oblate and opposite if it is prolate.
2. The symmetry axis makes an angle θ with the space-fixed angular momentum vector $\mathbf{L}$, where

$$\tan\theta=\frac{I_1}{I_3}\tan\alpha. \tag{4.8.18}$$

   For an oblate top, $\theta<\alpha$. For a prolate top, $\theta>\alpha$.
3. The speed of precession of the symmetry axis about $\mathbf{L}$ is

$$\dot{\phi}=\frac{\sin\alpha}{\sin\theta}\omega, \tag{4.8.19}$$

   or, by elimination of θ between 4.8.18 and 4.8.19,

$$\dot{\phi}=[1+\frac{I_3^2-I_1^2}{I_3^1}\cos^2\alpha]^{1/2}\omega. \tag{4.8.20}$$

The net result of this is that $\boldsymbol{\omega}$ preceses about $\mathbf{L}$ at a rate $\dot{\phi}$ in the *space cone*, which has a semi-vertical angle $\alpha-\theta$ for an oblate rotator, and $\theta-\alpha$ for a prolate rotator. The space cone is fixed in space, while the body cone rolls around it, always in contact, $\boldsymbol{\omega}$ being a mutual generator of both cones. If the rotator is oblate, the space cone is smaller than the body cone and is inside it. If the rotator is prolate, the body cone is outside the space cone and can be larger or smaller than it.

Write

$$c=I_3/I_1 \tag{4.8.21}$$

for the ratio of the principal moments of inertia. Note that for a pencil, $c=0$; for a sphere, $c=1$; for a plane disc or any regular plane lamina, $c=2$. (The last of these follows from the perpendicular axes theorem.) The range of c, then, is from 0 to 2, 0 to 1 being prolate, 1 to 2 being oblate.

Equations 4.8.17 and 4.8.20 can be written

$$\frac{\Omega}{\omega}=(c-1)\cos\alpha \tag{4.8.22}$$

and

$$\frac{\dot{\phi}}{\omega}=[1+(c^2-1)\cos^2\alpha]^{1/2} \tag{4.8.23}$$

Figures IV.15 and IV.16 show, for an oblate and a prolate rotator respectively, the instantaneouss rotation vector $\boldsymbol{\omega}$ precessing around the body-fixed symmetry axis at a rate Ω in the body cone of semi vertical angle α; the symmetry axis precessing about the space-fixed angular momentum vector $\mathbf{L}$ at a rate $\dot{\phi}$ in a cone of semi vertical angle θ (which is less than α for an oblate rotator, and greater than α for a prolate rotator; and consequently the instantaneous rotation vector $\boldsymbol{\omega}$ precessing around the space-fixed angular momentum vector $\mathbf{L}$ at a rate $\dot{\phi}$ in the space cone of semi vertical angle $\alpha-\theta$ (oblate rotator) or $\theta-\alpha$ (prolate rotator).

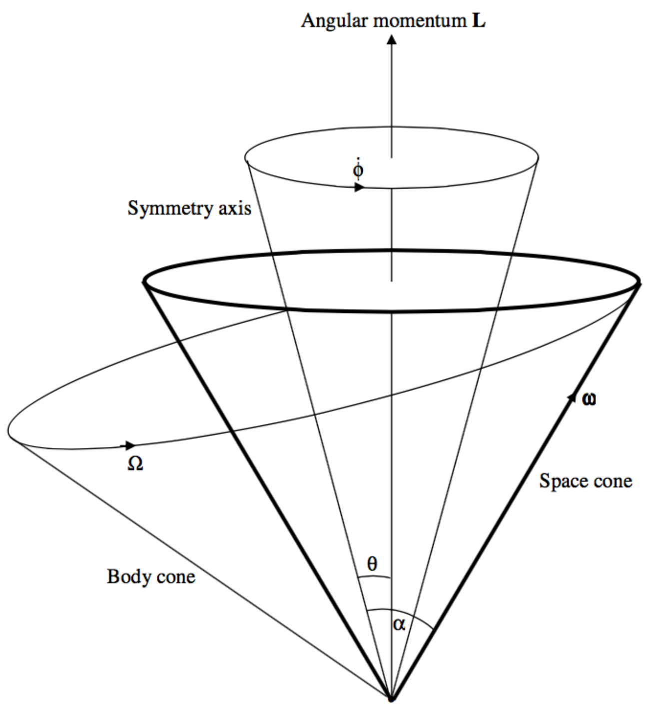


FIGURE IV.15

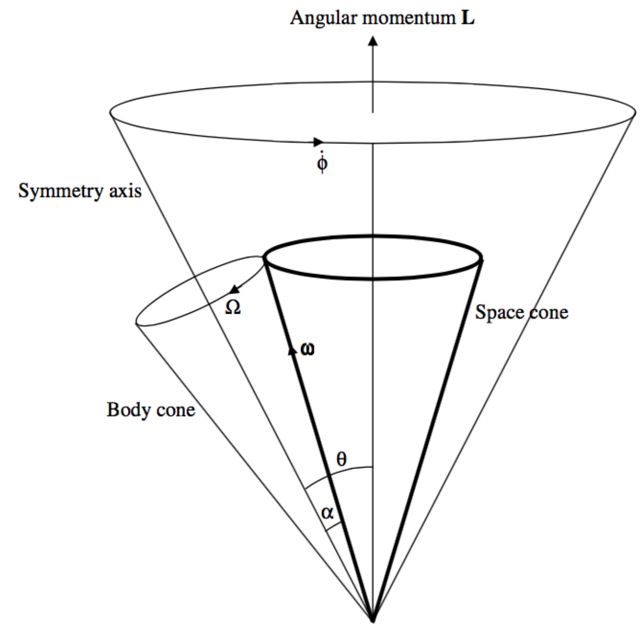


FIGURE IV.16

One can see from figures IV.15 and 16 that the angle between $\mathbf{L}$ and $\boldsymbol{\omega}$ is limited for an oblate rotator, but it can be as large as 90° for a prolate rotator. The angle between $\mathbf{L}$ and $\boldsymbol{\omega}$ is $\theta-\alpha$ (which is negative for an oblate rotator). We have

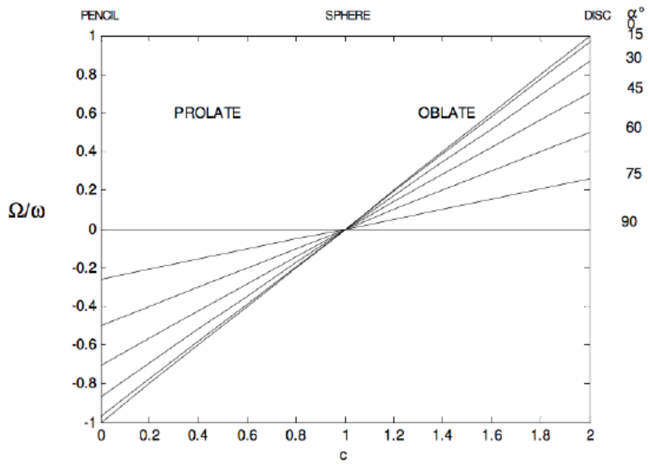
$$\tan(\theta-\alpha)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}=\frac{(1-c)\tan\alpha}{c+\tan^2\alpha} \tag{4.8.24}$$

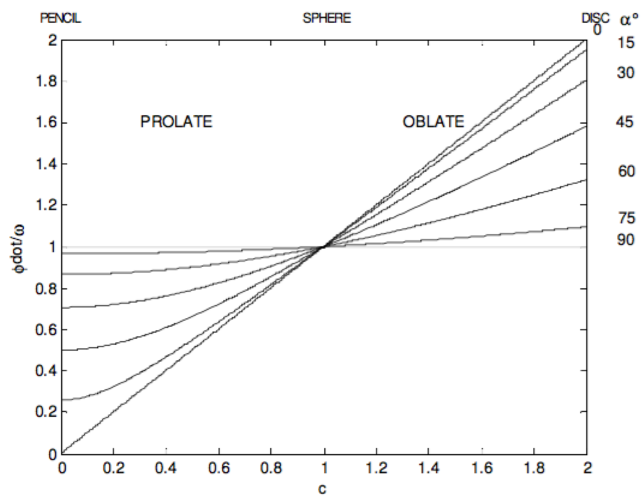
By calculus this reaches a maximum value of $\frac{1-c}{2\sqrt{c}}$ for $\tan\alpha=\sqrt{c}$

For a rod or pencil (prolate), in which $c=0$, the angle between $\mathbf{L}$ and $\boldsymbol{\omega}$ can be as large as 90°. Recalling exactly what are meant by the vectors $\mathbf{L}$ and $\boldsymbol{\omega}$, the reader should try now and imagine in his or her mind's eye a pencil rotating so that $\mathbf{L}$ and $\boldsymbol{\omega}$ are at right angles. The spin vector $\boldsymbol{\omega}$ is along the length of the pencil and the angular momentum vector $\mathbf{L}$ is at right angles to the length of the pencil.

For an oblate rotator, the angle between $\mathbf{L}$ and $\boldsymbol{\omega}$ is limited. The most oblate rotator is a flat disc or any regular flat lamina. The parallel axis theorem shows that for such a body, $c=2$. The greatest angle between $\mathbf{L}$ and $\boldsymbol{\omega}$ for a disc occurs when $\tan\alpha=\sqrt{2}\alpha$ = 54° 44'),and then $\tan\alpha-\theta=\frac{1}{\sqrt{8}}$, $\alpha-\theta$ =19° 28'.

In the following figures I illustrate some of these results graphically. The ratio $\frac{I_3}{I_1}$ goes from 0 for a pencil through 1 for a sphere to 2 for a disc.


PENCIL
SPHERE
DISC
α°
0
15
30
45
60
75
90
PROLATE
OBLATE
Ω/ω
c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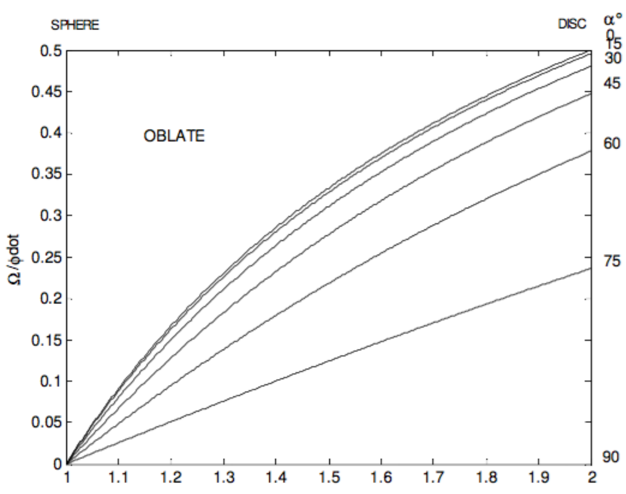
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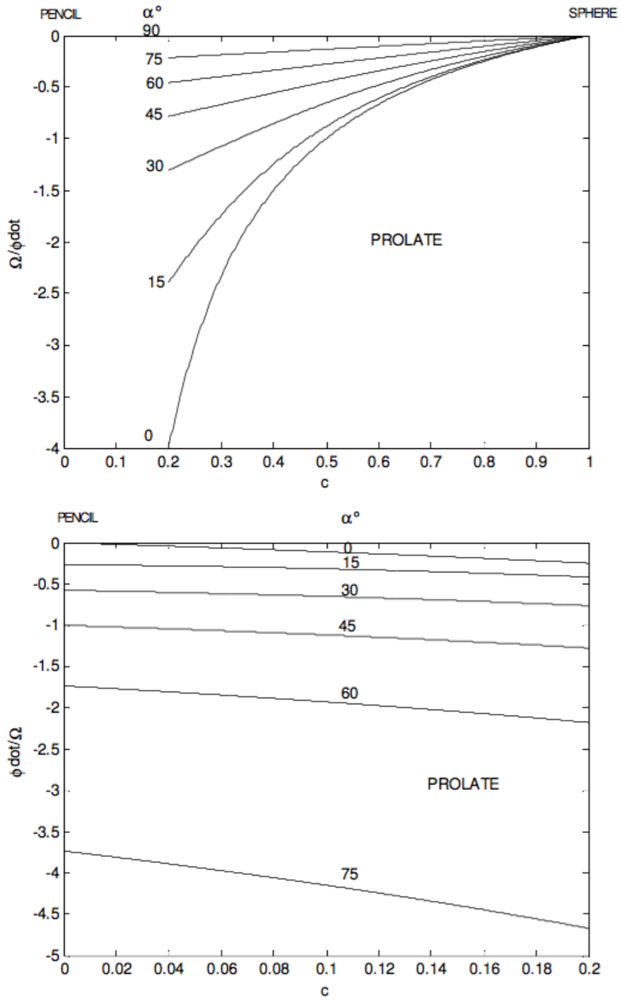
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PROLATE
φdot/Ω
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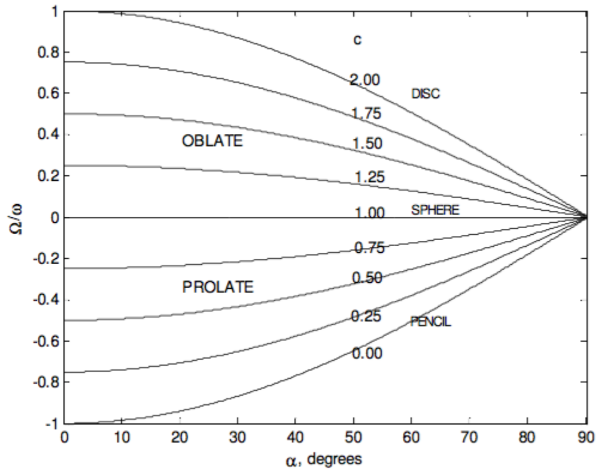
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DISC
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PENCIL
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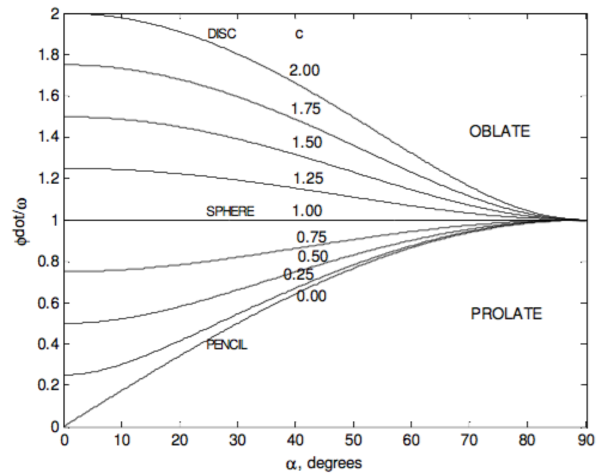
Ω/ω
α, degrees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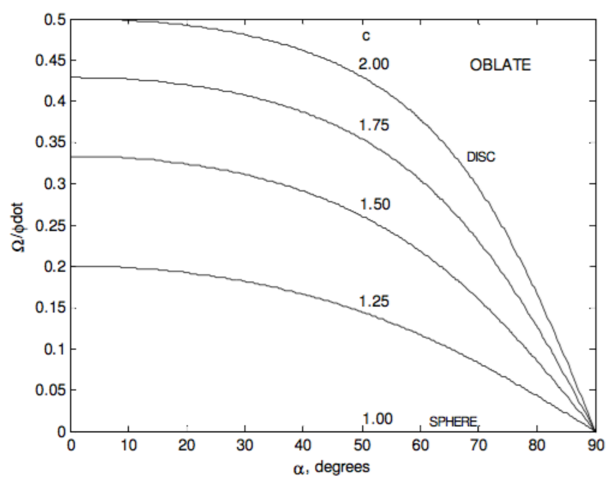
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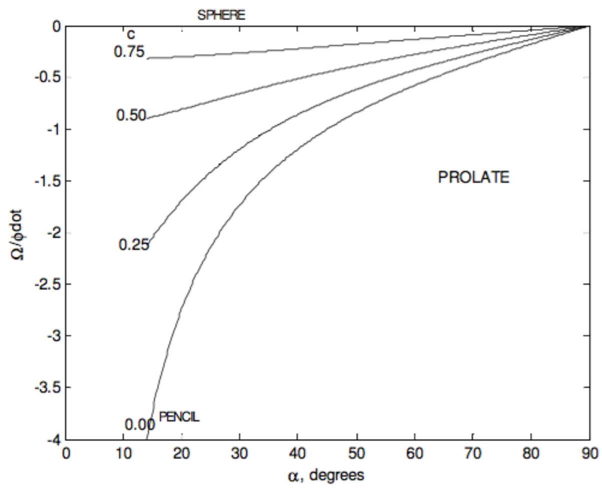
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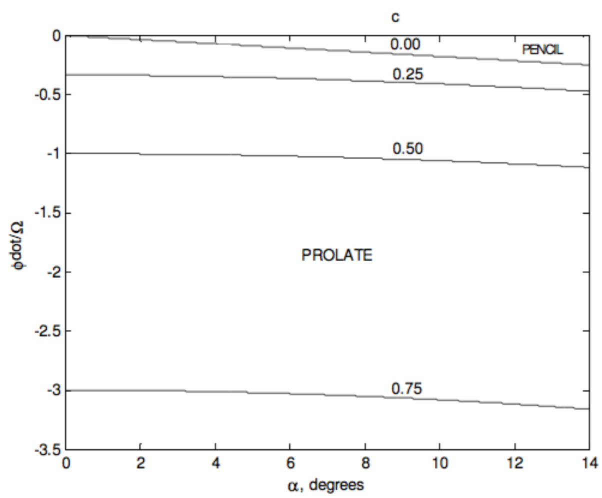
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φdot/Ω
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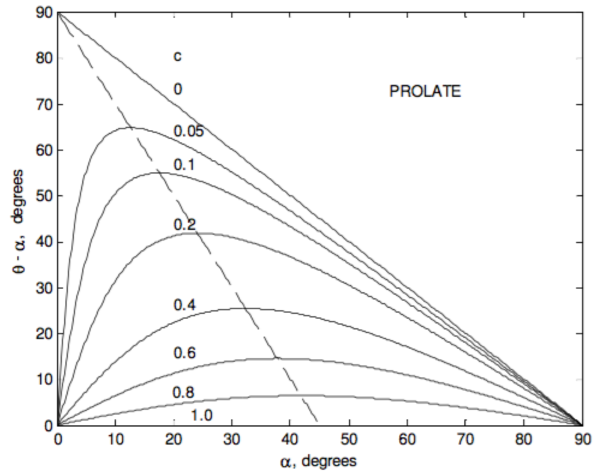
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0.2
0.4
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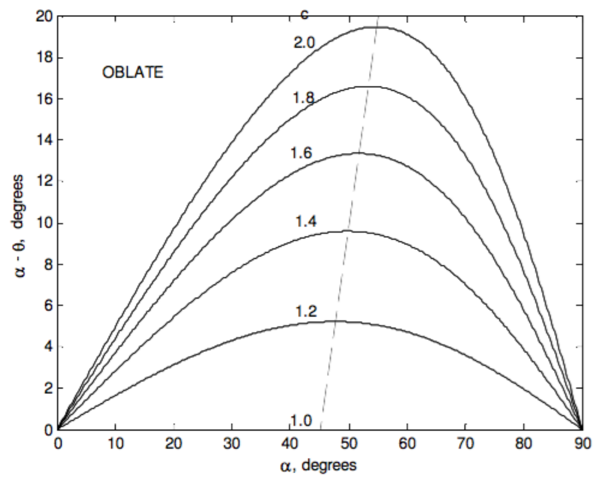
0.8
1.0
θ - α, degrees
α, degrees


OBLATE
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1.6
1.4
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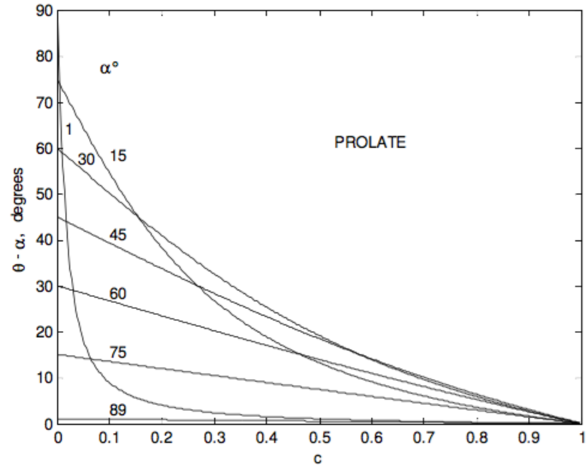
1.0
α - θ, degrees
α, degrees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PROLATE
θ - α, degrees
c
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15
45
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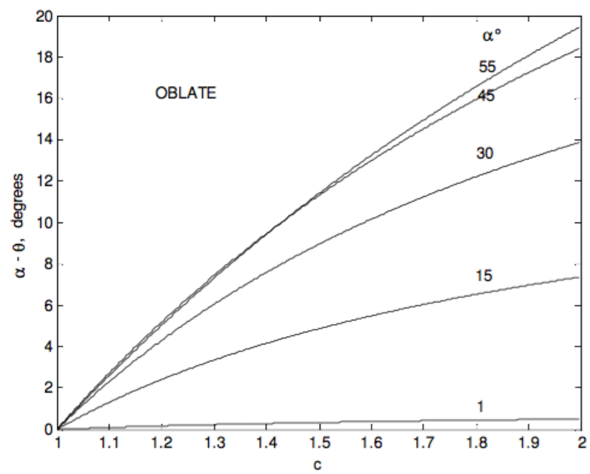
75
89


α°
OBLATE
α - θ, degrees
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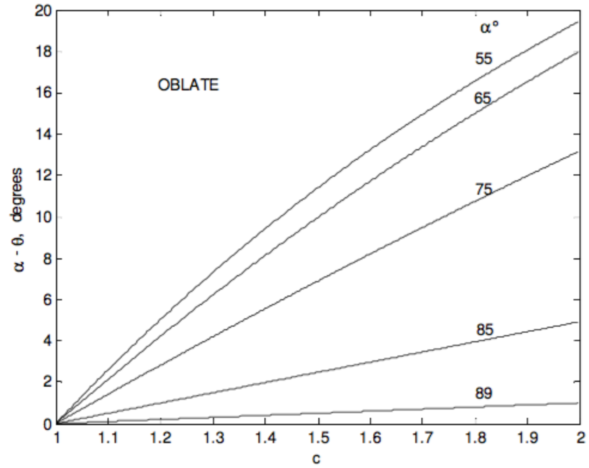
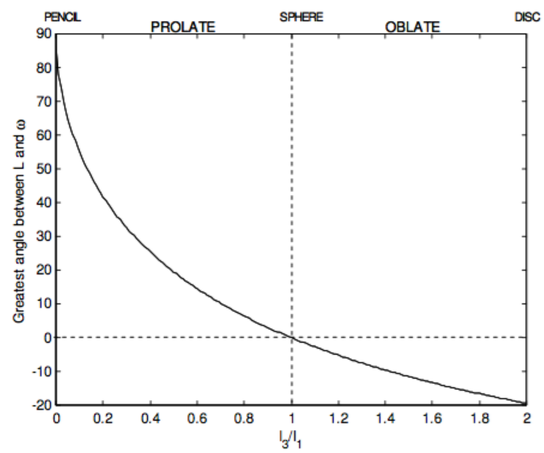




Our planet Earth is approximately an oblate spheroid, its dynamical ellipticity $\frac{(I_3 - I_1)}{I_1}$ being about 3.285 × 10^{-3}. It is not rotating exactly abut its symmetry axis; the angle α between $\boldsymbol{\omega}$ and the symmetry axis being about one fifth of an arcsecond, which is about six metres on the surface. The rotation period is one sidereal day (which is a few minutes shorter than 24 solar hours.) Equation 4.8.17 tells us that the spin axis precesses about the symmetry axis in a period of about 304 days, all within the area of a tennis court. The actual motion is a little more complicated than this. The period is closer to 432 days because of the nonrigidity of Earth, and superimposed on this is an annual component caused by the annual movement of air masses. This precessional motion of a symmetric body spinning freely about an axis inclined to the symmetry axis gives rise to variations of latitude of amplitude about a fifth of an arcsecond. It is not to be confused with the 26,000 year period of the *precession of the equinoxes*, which is caused by external torques from the Moon and the Sun.

4.9: Centrifugal and Coriolis Forces

We are usually told in elementary books that there is "no such thing" as centrifugal force. When a satellite orbits around Earth, it is not held in equilibrium between two equal and opposite forces, namely gravity acting towards Earth and centrifugal force acting outwards. In reality, we are told, the satellite is accelerating (the centripetal acceleration); there is only one force, namely the gravitational force, which is equal to the mass times the centripetal acceleration.

Yet when we drive round a corner too fast and we feel ourselves flung away from the centre of curvature of our path, the "centrifugal force" certainly feels real enough, and indeed we can solve problems referred to rotating coordinate systems as if there "really" were such a thing as "centrifugal force".

Let's look at an even simpler example, not even involving rotation. A car is accelerating at a rate a towards the right. See Figure IV.20 – but forgive my limited artistic abilities. Drawing a motor car is somewhat beyond my skills.

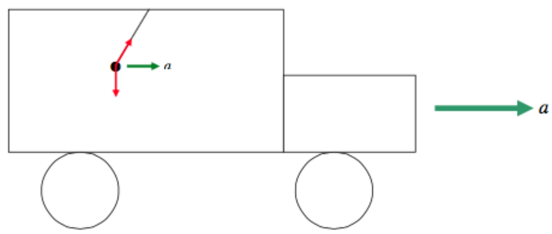


FIGURE IV.20

There is a plumb-bob hanging from the roof of the car, but, because of the acceleration of the car, it is not hanging vertically. Some would say that there are but two forces on the plumb-bob – its weight and the tension in the string – and, as a result of these, the bob is accelerating towards the right. By application of $F = ma$ it is easily possible to find the tension in the string and the angle that the string makes with the vertical.

The passenger in the car, however, sees things rather differently (Figure IV.21.)

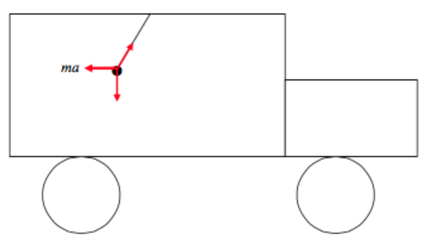


FIGURE IV.21

To the passenger in the car, nothing is accelerating. The plumb-bob is merely in static equilibrium under the action of three forces, one of which is a force ma towards the left. To the question "But what is the agent that is causing this so-called force?" I counter with the question "What is the agent that is causing the downward force that you attribute to some mysterious 'gravity' "?

It seems that, when referred to the reference frame of Figure IV.20, there are only two forces, but when referred to the accelerating reference frame of Figure IV.21, the system can be described perfectly well by postulating the existence of a force ma pulling towards the left. This is in fact a principle in classical mechanics, known as **d'Alembert's principle**, whereby, if one refers the description of a system to an accelerating reference frame, one can replace an acceleration with a force in the opposite direction. It results in a perfectly valid description of the behavior of a system, and will accurately predict how the system will behave. So who's to say which forces are "real" and which are "fictitious", and which reference frame is better than another?

The situation is similar with respect to centrifugal force. If you consider a satellite in orbit around Earth, some would say that there is only one force acting on the satellite, namely the gravitational force towards Earth. The satellite, being in a circular orbit, is accelerating towards the centre of the circle, and all is as expected - $F = ma$. The acceleration is the centripetal acceleration (peto – I desire). An astronaut on board the satellite may have a different point of view. He is at a constant distance from Earth, not accelerating; he is in static equilibrium, and he feels no net force on him whatever – he feels weightless. If he has been taught that Earth exerts a gravitational force, then this must be balanced by a force away from Earth. This force, which becomes apparent when referred to a corotating reference frame, is the centrifugal force (fugo – I flee, like a fugitive). It would need a good lawyer to argue that the invisible gravitational force towards Earth is a real force, while the equally invisible force acting away from Earth is imaginary. In truth, all forces are "imaginary" – in that they are only devices or concepts used in physics to describe and predict the behavior of matter.

I mentioned earlier a possible awful examination question: Explain why Earth bulges at the equator, without using the term "centrifugal force". Just thank yourself lucky if you are not asked such a question! People who have tried to answer it have come up with some interesting ideas. I have heard (I do not know whether it is true) that someone once offered a prize of $1000 to anyone who could prove that Earth is rotating, and that the prize has never been claimed! Some have tried to imagine how you would determine whether Earth is rotating if it were the only body in the universe. There would be no external reference points against which one could measure the orientation of Earth. It has been concluded (by some) that even to think of Earth rotating in the absence of any external reference points is meaningless, so that one certainly could not determine how fast, or even whether and about what axis, Earth was rotating. Since "rotation" would then be meaningless, there would be no centrifugal force, Earth would not bulge, nor would the Foucault pendulum rotate, nor would naval shells deviate from their paths, nor would cyclones and anticyclones exist in the atmosphere.

Centrifugal force comes into existence only when there is an external universe. It is the external universe, then, revolving around the stationary Earth, that causes centrifugal force and all the other effects that we have mentioned. These are deep waters indeed, and I do not pursue this aspect further here. We shall merely take the pragmatic view that problems in mechanics can often be solved by referring motions to a *corotating reference frame*, and that the behavior of mechanical systems can successfully and accurately be described and predicted by postulating the "existence" of "inertial" forces such as **centrifugal** and **Coriolis forces**, which make themselves apparent only when referred to a rotating frame. Thus, rather than involving ourselves in difficult questions about whether such forces are real, we shall take things easy with just a few simple Equations.

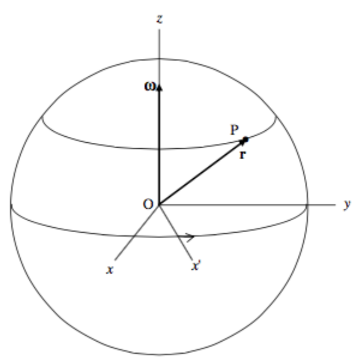


FIGURE IV.22

$\Sigma = Oxyz$ is an inertial reference frame (i.e. not accelerating or rotating).

$\Sigma = Ox'y'z'$ is a frame that is rotating about the z-axis with angular velocity $\omega = \omega\hat{z}$.

Three questions:

1. If P is a point that is fixed with respect to Σ', what is its velocity $\mathbf{v}$ with respect to Σ?
2. If P is a point that is moving with velocity $\mathbf{v}'$ with respect to Σ', what is its velocity $\mathbf{v}$ with respect to Σ?
3. If P is a point that has an acceleration $\mathbf{a}'$ with respect to Σ', what is its acceleration a with respect to Σ?

1. The answer to the first question is, I think, fairly easy. Just by inspection of Figure XV.22, I hope you will agree that it is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{4.9.1}$$

In case this is not clear, try the following argument. At some instant the position vector of P with respect to Σ is $\mathbf{r}$. At a time δt later its position vector is $\mathbf{r} + \delta\mathbf{r}$, where $\delta\mathbf{r} = \mathbf{r}\sin\theta\omega\delta\mathbf{t}$ and $\delta\mathbf{r}$ is at right angles to $\mathbf{r}$, and is directed along the small circle

whose zenith angle is θ, in the direction of motion of P with respect to Σ. Expressed

alternatively,

$$\delta \mathbf{r}=\mathbf{r}\sin\theta\omega\delta\mathbf{t}\hat{\phi}.$$

Draw the vector $\delta\mathbf{r}$ on the Figure if it helps. Divide

both sides by δt and take the limit as $\delta t\to 0$ to get $\dot{\mathbf{r}}$. The magnitude and direction of $\dot{\mathbf{r}}$ are then expressed by the single vector Equation $\dot{\mathbf{r}}=\boldsymbol{\omega}\times\mathbf{r}$, or $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$.

The only thing to look out for is this. In two-dimensional problems we are often used to expressing the relation between linear and angular speed by $v=r\omega$, and it doesn't matter which way round we write r and ω. When we are doing a three-dimensional problem using vector notation, it is important to remember that it is $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$ .and not the other way round.

2. If P is moving with velocity $\mathbf{v}'$ with respect to Σ', then its velocity $\mathbf{v}$ with respect to Σ must be

$$\mathbf{v}=\mathbf{v}'+\boldsymbol{\omega}\times\mathbf{r} \tag{4.9.2}$$

This shows that

$$(\frac{d}{dt})_{\Sigma}=(\frac{d}{dt})_{\Sigma'}+\boldsymbol{\omega}\times \tag{4.9.3}$$

What this Equation is intended to convey is that the operation of differentiating with respect to time when referred to the inertial frame Σ has the same result as differentiating with respect to time when referred to the rotating frame Σ', plus the operation $\boldsymbol{\omega}\times$. We shall understand this a little better in the next paragraph.

3. If P is accelerating with respect to Σ', we can apply the operation 4.9.3 to the Equation 4.9.2:

$$\begin{aligned}\mathbf{a}&=(\frac{d}{dt})_{\Sigma'}(\mathbf{v}'+\omega\times\mathbf{r})+\boldsymbol{\omega}\times(\mathbf{v}'+\boldsymbol{\omega}\times\mathbf{r})\\&=\mathbf{a}'+\boldsymbol{\omega}\times\mathbf{v}'+\boldsymbol{\omega}\times\mathbf{v}'+\omega\times(\boldsymbol{\omega}\times\mathbf{r}).\\ \therefore\mathbf{a}&=\mathbf{a}'+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+2\boldsymbol{\omega}\times\mathbf{v}'\end{aligned} \tag{4.9.4}$$

This, then, anwers the third question we posed. All we have to do now is understand what it means.

To start with, let us return to the case where P is neither moving nor accelerating with respect to Σ'. In that case, Equation 4.9.4 is just

$$\mathbf{a}=\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{4.9.5}$$

which we could easily have obtained by applying the operator 4.9.3 to 4.9.1. Let us try and understand what this means. In what follows, a “hat” (^) denotes a unit vector.

We have

$$\boldsymbol{\omega}\times\mathbf{r}=\boldsymbol{\omega}\mathbf{r}\sin\theta\hat{\phi}$$

and hence

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=\boldsymbol{\omega}\hat{z}\times\boldsymbol{\omega}\mathbf{r}\sin\theta\hat{\phi}=\mathbf{r}\boldsymbol{\omega}\sin\theta\, b\hat{f}z\times\hat{\boldsymbol{\phi}}$$

and $\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}}$ is a unit vector directed towards the z-axis. Notice that the point P is moving at angular speed ω in a small circle of radius $r\sin\theta$. The expression $a=\omega\times(\omega\times r)=r\omega^2\sin\theta\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}}$, then, is just the familiar centripetal acceleration, of magnitude r $r\omega^2\sin\theta$, directed towards the axis of rotation.

We could also think of $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ as a *triple vector product.*

We recall that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

so that

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=(w\cdot r)\boldsymbol{\omega}-\omega^2\mathbf{r}.$$

That is $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=\ bfr\omega^2 cos\theta\hat{\mathbf{z}}-r\omega^2\hat{\mathbf{r}}$.

This can be illustrated by the vector diagram shown in Figure IV.23. The vectors are drawn in green, in accordance with my convention of red, blue and green for force, velocity and acceleration respectively.

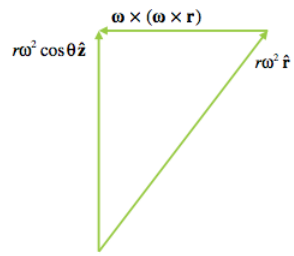


FIGURE IV.23

However, Equation 4.9.4 also tells us that, if a particle is moving with velocity $\mathbf{v}'$ with respect to Σ', it has an additional acceleration with respect to Σ of $2\omega \times v'$, which is at right angles to $\mathbf{v}'$ and to $\boldsymbol{\omega}$. This is the Coriolis acceleration.

The converse of Equation 4.9.4 is

$$\mathbf{a}' = \mathbf{a} + \boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}) = 2\mathbf{v}^{\mathbf{prime}} \times \boldsymbol{\omega} \tag{4.9.6}$$

$$\mathbf{F}' = \mathbf{F} + m\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}) + 2m\mathbf{v}' \times \boldsymbol{\omega} \tag{4.9.7}$$

It is worth now spending a few moments thinking about the direction of the Coriolis force

$$2m\mathbf{v}' \times \boldsymbol{\omega}.$$

Earth is spinning on its axis with a period of 24 sidereal hours (23h and 56m of solar time.). The vector ω is directed upwards through the north pole. Now go to somewhere on Earth at latitude 45° N. Fire a naval shell to the north. To the east. To the south. To the west. Now go to the equator and repeat the experiment. Go to the north pole. There you can fire only due south. Repeat the experiment at 45° south, and at the south pole. Each time, think about the direction of the vector $2mv' \times \omega$. If your thoughts are to my thoughts, your mind to my mind, you should conclude that the shell veers to the right in the northern hemisphere and to the left in the southern hemisphere, and that the Coriolis force is zero at the equator. As air rushes out of a high pressure system in the northern hemisphere, it will swirl clockwise around the pressure centre. As it rushes in to a low pressure system, it will swirl counterclockwise. The opposite situation will happen in the southern hemisphere.

You can think of the Coriolis force on a naval shell as being a consequence of conservation of angular momentum. Go to 45° N and point your naval gun to the north. Your shell, while waiting in the breech, is moving around Earth's axis at a linear speed of $\omega R \sin$ 45°, where R is the radius of Earth. Now fire the shell to the north. By the time it reaches latitude 50° N, it is being carried around Earth's axis in a small circle of radius only $R \sin$ 40°. In order for angular momentum to be conserved, its angular speed around the axis must speed up – it will be deviated towards the east.

Now try another thought experiment (*Gedanken Prüfung*.). Go to the equator and build a tall tower. Drop a stone from the top of the tower. Think now about the direction of the vector $2mv' \times \omega$. I really mean it – think hard. Or again, think about conservation of angular momentum. The stone drops closer to Earth's axis of rotation. It must conserve angular momentum. It falls to the east of the tower (*not* to the west!).

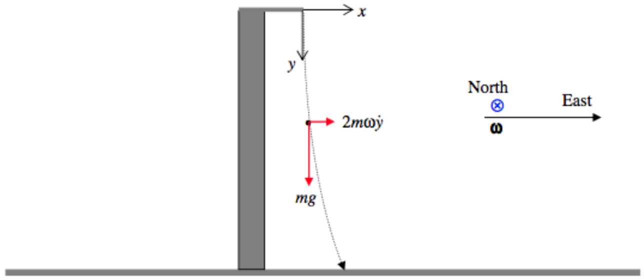


FIGURE IV.24

The two forces on the stone are its weight mg and the Coriolis force. Earth's spin vector $\boldsymbol{\omega}$ is to the north. The Coriolis force is at right angles to the stone's velocity. If we resolve the stone's velocity into a vertically down component $\dot{y}$ and a horizontal east component $\dot{z}$, the corresponding components of the Coriolis force will be $2m\omega\dot{y}$ to the east and $2m\omega\dot{x}$ upward. However, I'm going to assume that $\dot{z}$ << $\dot{y}$ and the only significant Coriolis force is the eastward component $2m\omega\dot{y}$, which I have drawn. Another way of stating the approximation is to say that the upward component of the Coriolis force is negligible compared with the weight mg of the stone.

After dropping for a time t, the y-coordinate of the stone is found in the usual way from

$$\ddot{y} = g, \quad \dot{y} = gt, \quad y = \frac{1}{2}gt^2,$$

and the x- coordinate is found from

$$\ddot{x} = 2\omega\dot{y} = 2\omega gt, \quad \dot{x} = \omega gt^2, \quad x = \frac{1}{3}\omega gt^3$$

Thus you can find out how far to the east it has fallen after two seconds, or how far to the east it has fallen if the height of the tower is 100 metres. The Equation to the trajectory would be the t-eliminant, which is

$$x^2 = \frac{8\omega^2}{9g}y^3 \tag{4.9.8}$$

For Earth, ω = 7.292 × 10^{-5} rad s^{-1}, and at the equator g = 9.780 m s^{-2}, so that

$$\frac{8\omega^2}{9g} = 4.788 \times 10^{-10} m^{-1}.$$

The path is graphed in Figure IV.25 for a 100-metre tower. The horizontal scale is exaggerated by a factor of about 6000.

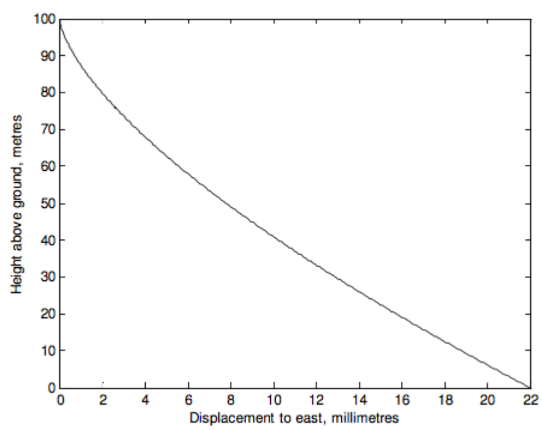


FIGURE IV.25

I once asked myself the question whether a migrating bird could navigate by using the Coriolis force. After all, if it were flying north in the northern hemisphere, it would experience a Coriolis force to its right; might this give it navigational information? I published an article on this in *The Auk* **97**, 99 (1980). Let me know what you think!

You may have noticed the similarity between the Equation for the Coriolis force

$$\mathbf{F} = \mathbf{2mv'} \times \omega$$

and the Equation for the Lorentz force on an electric charge moving in a magnetic field:

$$\mathbf{F} = \mathbf{qv} \times \mathbf{B}.$$

The analogy can be pursued a bit further. If you rotate a coil in an electric field, a current will flow in the coil. That's electromagnetic induction, and it is the principle of an electric generator. Sometime early in the twentieth century, the American physicist Arthur Compton (that's not Denis Compton, of whom only a few of my readers will have heard, and very few indeed in North America) successfully tried an interesting experiment. He made some toroidal glass tubes, filled with water coloured with $KMnO^4$, so that he could see the water, and he rotated these tubes about a horizontal or vertical diameter, and, lo, the water flowed around the tubes, just as a current flows in a coil when it is rotated in a magnetic field. Imagine a toroidal tube set up in an east-west vertical plane at the equator. The top part of the tube is slightly further from Earth's rotation axis than the bottom part, and consequently the water near the top of tube has more angular momentum per unit mass, around Earth's axis, than the water near the bottom. Now rotate the tune through 180° about its east-west horizontal diameter. The high angular momentum fluid moves closer to Earth's rotation axis, and the low angular momentum fluid moves further from Earth's axis. Therefore, in order to conserve angular momentum, the fluid must flow around the tube. By carrying out a series of such experiments, Compton was able, at least in principle, to measure the speed of Earth's rotation, and even his latitude, without looking out of the window, and indeed without even being aware that there was an external universe out there. You may think that this was a very difficult experiment to do, but you do it yourself every day. There are three mutually orthogonal semicircular canals inside your ear, and, every time you move your head, fluid inside these semicircular canals flows in response to the Coriolis force, and this fluid flow is detected by little nerves, which send a message to your brain to tell you of your movements and to help you to keep your balance. You have a wonderful brain, which is why understanding physics is so easy.

Going back to the Lorentz force, we recall that a moving charge in a magnetic field experiences a force at right angles to its velocity. But what is the origin of a magnetic field? Well, a magnetic field exists, for example, in the interior of a solenoid in which there is a current of moving electrons in the coil windings, and it is these circulating electrons that ultimately cause the Lorentz force on a charge in the interior of the solenoid – just as it is the galaxies in the universe revolving around our stationary Earth which are the ultimate cause of the Coriolis force on a particle moving with respect to our Earth. But there I seem to be getting into deep waters again, so perhaps it is time to move on to something easier.

4.10: The Top

We have classified solid bodies technically as symmetric, asymmetric, spherical and linear tops, according to the relative sizes of their principal moments of inertia. In this section, or at least in the title of this section, I mean "top" in the nontechnical sense of the child's toy – that is to say, a symmetric body, pointed at one end, spinning around its axis of symmetry, with the pointed end on the ground or on a table. Technically, it is a "heavy symmetric top with one point fixed."

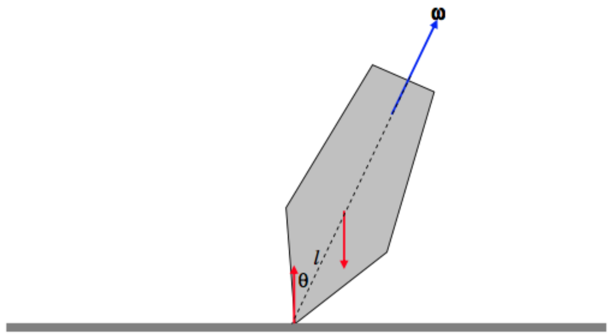


FIGURE IV.26

I have drawn it is Figure IV.26, spinning about its symmetry axis, which makes an angle θ with the vertical. The distance between the centre of mass and the point of contact with the table is l. It has a *couple* of forces acting on it – its weight and the equal, opposite reaction of the table. In Figure IV.27, I replace these two forces by a torque, $\boldsymbol{\tau}$, which is of magnitude $Mgl\sin\theta$.

Note that, since there is an *external torque* acting on the system, *the angular momentum vector is not fixed.*

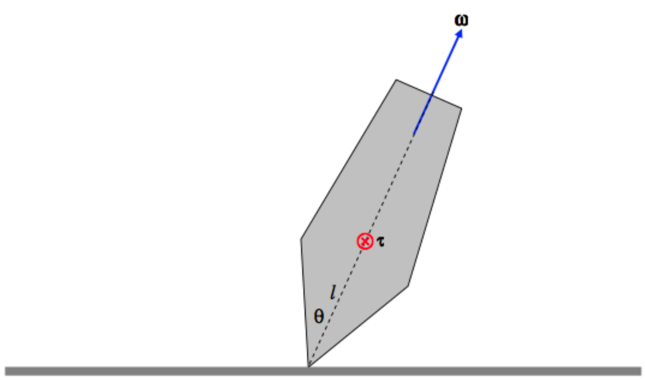


FIGURE IV.27

Before getting too involved with numerous Equations, let's spend a little while describing qualitatively the motion of a top, and also describing the various coordinate systems and angles we shall be discussing. First, we shall be making use of a set of *space-fixed coordinates*. We'll let the origin O of the coordinates be at the (fixed) point where the tip of the top touches the table. The axis Oz points vertically up to the zenith. The Ox and Oy axes are in the (horizontal) plane of the table. Their exact orientation is not very important, but let's suppose that Ox points due south, and Oy points due east. $Oxyz$ then constitutes a right-handed set. We'll also make use of a set of *body-fixed axes*, which I'll just refer to for the moment as 1, 2 and 3. The 3-axis is the symmetry axis of the top. The 1- and 2-axes are perpendicular to this. Their exact positions are not very important, but let's suppose that the 31-plane passes through a small ink-dot which you have marked on the side of the top, and that the 123 system constitutes a right-handed set.

We are going to describe the orientation of the top at some instant by means of the three Eulerian angles θ, ϕ and ψ (see Figure IV.28). The symmetry axis of the top is represented by the heavy arrow, and it is tipped at an angle θ to the z-axis. I'll refer to a plane normal to the axis of symmetry as the "equator" of the top, and it is inclined at θ to the xy-plane. The ascending node of the equator on the xy-plane has an azimuth ϕ, and ψ is the angular distance of the 1-axis from the node. The azimuth of the symmetry axis of the top is ϕ - 90 ° = ϕ + 270°.

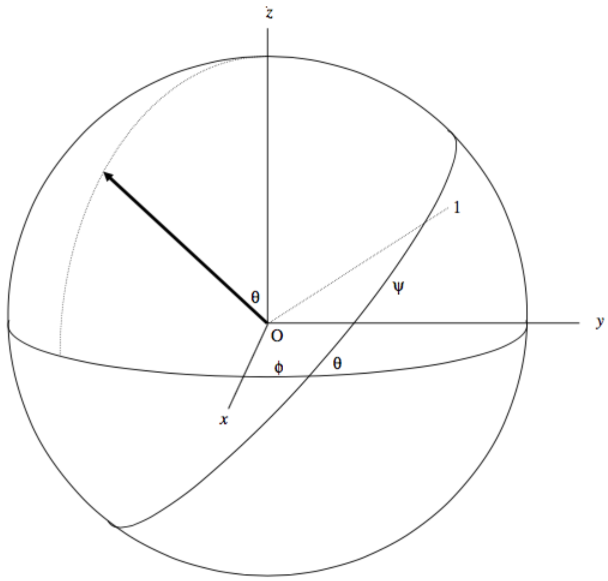


FIGURE IV.28

Now let me anticipate a bit and describe the motion of the top while it is spinning and subject to the torque described above.

The symmetry axis of the top is going to precess around the z-axis, at a rate that will be described as $\dot{\phi}$. Except under some conditions (which I shall eventually describe) this precessional motion is *secular*. That means that ϕ increases all the time – it does not oscillate to and fro. However, the symmetry axis does not remain at a constant angle with the z-axis. It oscillates, or nods, up and down between two limits. This motion is called *nutation* (Latin: nutare, to nod). One of our aims will be to try to find the rate of nutation $\dot{\phi}$ and to find the period and amplitude of the nutation.

It may look as though the top is spinning about its axis of symmetry, but this isn't quite so. If the angular velocity vector were exactly along the axis of symmetry, it would stay there, and there would be no precession or nutation, and this cannot be while there is a torque acting on the top. An exception would be if the top were spinning vertically (θ = 0), when there would be no torque acting on it. The top can in fact do that, except that, unless the top is spinning quite fast, this situation is unstable, and the top will tip away from its vertical position at the slightest perturbation. At high spin speeds, however, such motion is stable, and indeed one of our aims must be to determine the least angular speed about the symmetry axis such motion is stable.

However, as mentioned, unless the top is spinning vertically, the vector ω is not directed along the symmetry axis. We'll call the three components of ω along the three body- fixed axes ω_1, ω_2 and ω_3, the last of these being the component of ω along the symmetry axis. One of the things we shall discover when we proceed with the analysis is that ω_3 remains constant throughout the motion. Also, you should be able to distinguish between ω_3 and $\dot{\psi}$. These are not the same, because of the motion of the node. In fact you will probably understand that $\psi = \omega_3 - \dot{\phi}\cos\theta$. Indeed, we have already derived the relations between the component of the angular velocity vector and the rate of change of the Eulerian angles – see Equations 4.2.1 ,2 and 3. We shall be making use of these relations in what follows.

To analyse the motion of the top, I am going to make use of Lagrange's Equations of motion for a conservative system. If you are familiar with Lagrange's Equations, this will be straightforward. If you are not, you might prefer to skip this section until you have become more familiar with Lagrangian mechanics in Chapter 13. However, I introduced Lagrange's Equation briefly in Section 4.4, in which Lagrange's Equation of motion was given as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial \dot{q}_j}\right) = P_j. \tag{4.10.1}$$

Here T is the kinetic energy of the system. P_j is the generalized force associated with the generalized coordinate q_j. If the force is a conservative force, then P_j can be expressed as the negative of the derivative of a potential energy function:

$$P_j = -\left(\frac{\partial V}{\partial q_j}\right) \tag{4.10.2}$$

Thus we have Lagrange's Equation of motion for a **system of conservative forces**

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial V}{\partial q_j} = -\frac{\partial V}{\partial q_j} \tag{4.10.3}$$

Thus, in solving problem in Lagrangian dynamics, the first line in our calculation is to write down an expression for the kinetic energy. The first line begins: "$T = \dots$".

In the present problem, the kinetic energy is

$$T = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 \tag{4.10.4}$$

Here the subscripts refer to the principal axes, 3 being the symmetry axis. The Eulerian angles θ and ϕ are zenith distance and azimuth respectively of the symmetry axis with respect to laboratory fixed (space fixed) axes. The Eulerian angle ψ is measured around the symmetry axis. The components of the angular velocity are related to the rates of change of the Eulerian angles by previously derived formulas (Equations 4.2.1,2,3), so the $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$.

$$T = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 \tag{4.10.5}$$

The potential energy is

$$V = Mgl\cos\theta + constant. \tag{4.10.6}$$

Having written down the expressions for the kinetic and potential energies in terms of the Eulerian angles, we are now in a position to apply the Lagrangian Equations of motion 4.10.3 for each of the three coordinates. We'll start with the coordinate ϕ. The Lagrangian Equation is

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{\phi}}) - \frac{\partial T}{\partial \phi} = -\frac{\partial V}{\partial \phi} \tag{4.10.7}$$

We see that $\frac{\partial T}{\partial \phi}$ and $\frac{\partial V}{\partial \phi}$ are each zero, so that $\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = 0$, or $\frac{\partial T}{\partial \dot{\phi}}$ = constant. This has the dimensions of angular momentum, so I'll call the constant L$_1$. On evaluating the derivative $\frac{\partial T}{\partial \dot{\phi}}$, we obtain for the Lagrangian Equation in ϕ:

$$I_1\dot{\phi}sin^2\theta + I_3\dot{\phi}\cos^2\theta + I_3\dot{\psi}\cos\theta = L_1 \tag{4.10.8}$$

I'll leave the reader to carry out exactly the same procedure with the coordinate ψ. You'll quickly conclude that $\frac{\partial T}{\partial \dot{\psi}}$= constant, which has the dimensions of angular momentum, so call it L_3, and you will then arrive at the following for the Lagrangian Equation in ψ:

$$I_3(\psi + \dot{\phi}\cos\theta) = L_3 \tag{4.10.9}$$

But the expression in parentheses is equal to ω_3 (see Equation 4.2.3, although we have already used it in Equation 4.10.5), so we obtain the result that ω_3, the component of the angular velocity about the symmetry axis, is constant during the motion of the top. It would probably be worth the reader's time at this point to distinguish again carefully in his or her mind the difference between ω_3 and $\dot{\psi}$.

Eliminate $\dot{\psi}$ from Equations 4.10.8 and 4.10.9:

$$\dot{\phi} = \frac{L_1 - L_3 cos\theta}{I_1\sin^2\theta} \tag{4.10.10}$$

This Equation tells us how the rate of precession varies with θ as the top nods or nutates up and down.

We could also eliminate $\dot{\phi}$ from Equations 4.10.8 and 4.10.9:

$$\dot{\psi}=\frac{L_3}{I_3}-\frac{(L_1-L_3\cos\theta)\cos\theta}{I_1\sin^2\theta} \tag{4.10.11}$$

The Lagrangian Equation in θ is a little more complicated, but we can obtain a third Equation of motion from the constancy of the total energy:

$$\frac{1}{2}I_1(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)+\frac{1}{2}I_3(\dot{\psi}+\dot{\phi}\cos\theta)^2+Mgl\cos\theta=E. \tag{4.10.12}$$

We can eliminate $\dot{\phi}$ and $\dot{\psi}$ from this, using Equations 4.10.10and 4.10.11, to obtain an Equation in θ and the time only. After a little algebra, we obtain

$$\dot{\theta}^2=A-B\cos\theta-\left(\frac{L_1-L_3\cos\theta}{I_1\sin\theta}\right)^2, \tag{4.10.13}$$

where

$$A=\frac{1}{I_1}\left(2E-\frac{L_3^2}{I_3}\right) \tag{4.10.14}$$

and

$$B=\frac{2Mgl}{I_1} \tag{4.10.15}$$

The turning points in the θ-motion (i.e. the nutation) occur where $\dot{\theta}=0$. This results (after some algebra! – but quite straightforward all the same) in a cubic Equation in $c=\cos\theta$:

$$a_0=a_1c+a_2c^2+Bc^3=0 \tag{4.10.16}$$

where

$$a_0=A-\left(\frac{L_1}{I_1}\right)^2=\frac{2E}{I_1}-\frac{L_3^2}{I_1I_3}-\frac{L_1^2}{I_1^2}, \tag{4.10.17}$$

$$a_1=\frac{2L_1L_3}{I_1^2}-B=\frac{2L_1L_3}{I_1^2}-\frac{2Mgl}{I_1} \tag{4.10.18}$$

and

$$a_2=-A-\left(\frac{L_3}{I_1}\right)^2=-\left[\frac{2E}{I_1}-\frac{L_3^2}{I_1I_3}-\left(\frac{L_3}{I_1}\right)^2\right] \tag{4.10.19}$$

Now Equation 4.10.16is a cubic Equation in $\cos\theta$ and it has either one real root or three real roots, and in the latter case two of them or all three might be equal. We must also bear in mind that θ is real only if $\cos\theta$ is in the range −1 to+1. We are trying to find the nutation limits, so we are hoping that we will find two and only two real values of θ. (If the tip of the top were poised on top of a point – e.g. if it were poised on top of the Eiffel Tower, rather than on a horizontal table − you could have θ > 90°.)

To try and understand this better, I constructed in my mind a top somewhat similar in shape to the one depicted in Figures IV.25 and 26, about 4 cm diameter, 7 cm high, made of brass. For the particular shape and dimensions that I imagined, it worked out to have the following parameters, rounded off to two significant Figures:

M=0.53 kg l= 0.044 m I_1 = 1.7×10^{-4} kg m^2 I_3 = 9.8×10^{-5} kg m^2

I thought I'd spin the top so that ω_3 (which, as we have seen, remains constant throughout the motion) is 250 rad s^{-1}, and I'd start the top ($\dot{\phi}=\dot{\theta}=0$) at θ = 30°. and then let go. Presumably it would then immediately start to fall, and 30° would then be the upper bound to the nutation. We want to see how far it will fall before nodding upwards again. With ω_3 = 250 rad s^{-1} we find, from Equation 4.10.9that

L_3 = 2.45 x 10^{-2} Js

Also, I am assuming that $\dot{\phi}$= 0 when θ = 30°, and Equation 4.10.10tells us that

L_1 = 2.121762 x 10^{-2} Js

Then with g = 9.8 m s^{-2}, we have, from Equation 4.10.15,

B = 2.688659 x 10^3 s^{-2}.

My initial conditions are that $\dot{\phi}$ and $\dot{\theta}$ are each zero when θ = 30°, and Equations 4.10.10 and 4.10.13 between them tell us that $A = B\cos$ 30°, so that

A = 2.328447 x 10^3 s^{-2}.

From Equations 4.12.17, 18 and 19 we now have

a_0= -1.324989 x 10^3 s^{-2}

a1= +3.328586 x 10^3 s^{-2}

a2= -2.309834 x 10^3 s^{-2}

and we already have

B = 2.688659 x 10^3 s^{-2}.

The "sign rule" for polynomial Equations, if you are familiar with it, tells us that there are no negative real roots (i.e. no solutions with θ > 90°), and indeed if we solve the cubic Equation 4.10.16 we obtain

$$c = 0.824596, 0.866025, 6.9000406$$

The second of these corresponds to θ = 30°, which we already knew must be a solution. Indeed we could have divided Equation 4.10.16 by $c - \cos 30°$ to obtain a quadratic Equation for the remaining two roots, but it is perhaps best to solve the cubic Equation as it is, in order to verify that cos 30° is indeed a solution, thus providing a check on the arithmetic. The third solution does not give us a real θ (we were rather hoping this would happen). The second solution is the lower limit of the nutation, corresponding to θ = 34° 27'.

Generally, however, the top will nutate between two values of θ. Let us call these two values α and β, α being the smaller (more vertical) of the two. I'll refer to $\theta = \alpha$ as the "upper bound" of the motion, even though $\alpha < \beta$, since this corresponds to the more vertical position of the top. We have looked a little at the motion in θ; now let's look at the motion in ϕ, starting with Equation 4.10.10:

$$\dot{\phi} = \frac{L_1 - L_3\cos\theta}{I_1\sin^2\theta} \tag{4.10.10.}$$

If the initial conditions are such that $L_1 > L_3\cos\alpha$ (and therefore always greater than $L_3\cos\theta$) $\hat{\phi}$ is always positive. The motion is then something like I try to illustrate in Figure IV.29

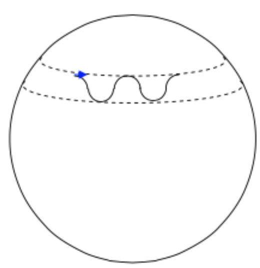


FIGURE IV.29

This motion corresponds to an initial condition in which you give the top an initial push in the forward direction as indicated by the little blue arrow. If the initial conditions are such that $\cos\alpha > \frac{L_1}{L_3} > \cos\beta$, the sign of $\dot{\phi}$ is different at the upper and lower bounds. Thus is illustrated in Figure IV.30

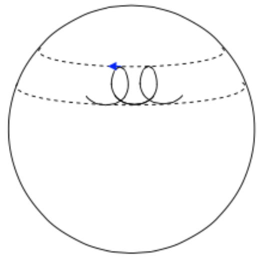


FIGURE IV.30

This motion would arise if you were initially to give a little backward push before letting go of the top, as indicated by the little blue arrow.

If the initial conditions are such that $L_1 = L_3 \cos\alpha$, then $\dot{\theta}$ and $\dot{\phi}$ are each zero at the upper bound of the nutation, and this was the situation in our numerical example. It corresponds to just letting the top drop when you let it go, without giving it either a forward or a backward push. This is illustrated in Figure IV.31.

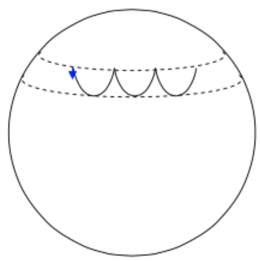


FIGURE IV.31

As we discovered while doing our numerical example, the initial conditions $\dot{\theta} = \dot{\phi} = 0$ when $\theta = \alpha$ lead, in this third type of motion, to

$$L_1 = L_3 \cos\alpha \tag{4.10.20}$$

and

$$A = B\cos\alpha \tag{4.10.21}$$

In the case Equation 4.10.13becomes

$$\dot{\theta}^2 = B(\cos\alpha - \cos\theta) - [\frac{L_3(\cos\alpha - \cos\theta)}{I_1 \sin\theta}]^2 \tag{4.10.22}$$

The lower bound to the nutation(i.e. how far the top falls) is found by putting $\theta = \beta$ when $\dot{\theta}$=0. This gives the following quadratic Equation for β:

$$\cos^2\beta - \frac{L_3^2}{I_1^2 B}\cos\beta + \frac{L_3^2 \cos\alpha}{I_1 \sin\theta} \tag{4.10.23}$$

In our numerical example, this is

$$\cos^2\beta - 7.725002\cos\beta + 5.690048 = 0, \tag{4.10.24}$$

which, naturally, has the same two solutions as we obtained when we solved the cubic Equation, namely 0.824 596 and 6.900 406.

Recalling the definition of B (Equation 4.10.15), we see that Equation 4.10.23can be written

$$\cos\alpha - \cos\beta = \frac{2MglI_1}{L_3^2}\sin^2\beta, \tag{4.10.25}$$

from which we see that the greater L_3, the smaller the difference between α and β - i.e. the smaller the amplitute of the nutation.

Equation 4.10.12, with the help of Equations 4.10.10and 4.10.11, can be written:

$$E-\frac{L_3^2}{2I_3}-\frac{1}{2}I_1\dot{\theta}^2=\frac{1}{2I_1}(L_1\csc\theta-L_3\cot\theta)^2+Mglcos\theta. \tag{4.10.26}$$

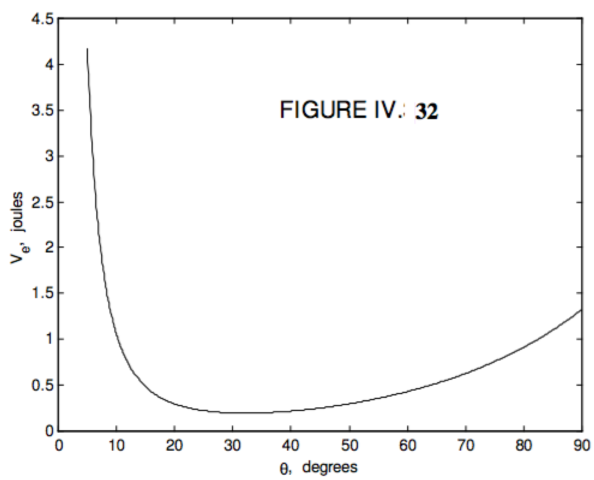
The left hand side is the total energy minus the spin and nutation kinetic energies. Thus the right hand side represents the effective potential energy $V_e(\theta)$ referred to a reference frame that is co-rotating with the precession. The term $Mgl\sin\theta$ needs no explanation. The negative of the derivative of the first term on the right hand side would be the “fictitious” force that “exists” in the corotating reference frame. The effective potential energy $V_e(\theta)$ is given by

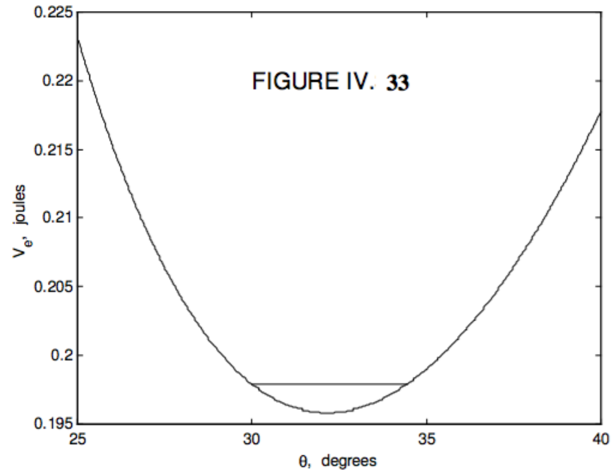
$$\frac{V_e(\theta)}{L_1^2/(2I_1)}=[\csc\theta-(L_3/L_1)\cot\theta]^2+\frac{2I_1Mgl\cos\theta}{L_1^2}. \tag{4.10.27}$$

I draw $V_e(\theta)$ in Figures IV.32 and 33 using the values that we used in our numerical example – that is:

$$V_e(\theta)=1.32081(\csc\theta-1.154701\cot\theta)^2+0.229536\cos\theta \qquad \text{joules.} \tag{4.10.28}$$

Figure 32 is plotted up to 90° (although as mentioned earlier one could go further than this if the top were not spinning on a horizontal table), and Figure 33 is a close look close 2 to the minimum. One can see that if $E-L_3^3/(2I_2)=0.1979$ the effective potential energy (which cannot go higher than this, and reaches this value only when) $\dot{\theta}$ = 0 , the nutation limits are between 30° and 34° 24' . For a given L_3, for a larger total energy, the nutation limits are correspondingly wider. But for a given total energy, the larger the component L_3 of the angular momentum is, the lower will be the horizontal line and the narrower the nutation limits. If the top loses energy (e.g. because of air resistance, or friction at the point of contact with the table), the E = constant line will become lower 2 and lower, and the amplitude of the nutation will become less and less. If $E-L_3^2/(2I_3)$ is equal to the minimum value of $V_e(\theta)$ there is only one solution for θ, and there is no nutation. For energy less than this, there is no stationary value of θ and the top falls over.





We can find the rate of true regular precession quite simply as follows – and this is often done in introductory books.

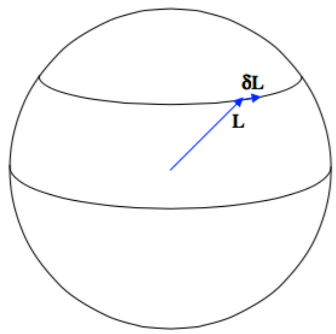


FIGURE IV.34

In Figure IV.34, the vector $\mathbf{L}$ represents the angular momentum at some time, and in a time interval δt later the change in the angular momentum is $\boldsymbol{\delta}\mathbf{L}$. The angular momentum is changing because of the external torque, which is a horizontal vector of magnitude $Mgl\sin\theta$ (remind yourself from Figure XV .26 and 27). The rate of change of angular momentum is given by $\mathbf{L} = \tau$. In time δt the tip of the vector $\mathbf{L}$ moves through a "distance" $\tau\delta t$. Denote by $\boldsymbol{\Omega}$ precessional angular velocity (the magnitude of which we have hitherto called $\dot{\phi}$). The tip of the angular momentum vector is moving in a small circle of radius $L\sin\theta$. We therefore see that $\tau = \Omega L\sin\theta$. Further, τ is perpendicular to both $\boldsymbol{\Omega}$ and $\mathbf{L}$. Therefore, in vector notation,

$$\boldsymbol{\tau} = \boldsymbol{\Omega}\times\mathbf{L} \tag{4.10.29}$$

Note that the magnitude of $\boldsymbol{\tau}$is $Mgl\sin\theta$ and the magnitude of $\boldsymbol{\Omega}\times\mathbf{L}$ is $\Omega L\sin\theta$, so that the rate of precession is

$$\Omega = \frac{Mgl}{L} \tag{4.10.30}$$

and is independent of θ.

One can continue to analyse the motion of a top almost indefinitely, but there are two special cases that are perhaps worth noting and which I shall describe.

Special Case I. $L_1 = L_3$.

In this case, Equation 4.10.27becomes

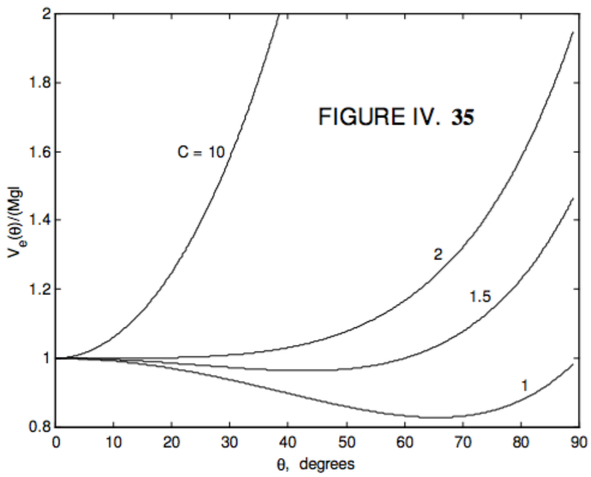
$$\frac{V_e(\theta)}{Mgl} = C(csc\theta - cot\theta)^2 + \cos\theta \tag{4.10.31}$$

where

$$C = \frac{L_1^2}{2MglI_1} \tag{4.10.32}$$

It may be rather unlikely that $L_1 = L_3$ exactly, but this case is of interest partly because it is exceptional in that $V_e(0)$ does not go to infinity; in fact $V_e(0) = Mgl$ whatever the value of C. Try substituting θ = 0 in Equation 4.10.31 and see what you get! The right hand side is indeed 1, but you may have to work a little to get there. The other reason why this case is of interest is that it makes a useful introduction to case II, which is not impossibly unlikely, namely that L_1 is *approximately* equal to L_3, which leads to motion of some interest.

In Figure IV.35, I draw $\frac{V_e(0)}{Mgl}$ for several different C.



From the graphs, it looks as though, if $C \geq 2$, there is one equilibrium position, it is at θ = 0°(i.e. the top is vertical), and the equilibrium is stable, If $C < 2$, there are two equilibrium positions: the vertical position is unstable, and the other equilibrium position is stable. Thus if the top is spinning fast (large C) it can spin in the vertical position only (a "sleeping top"), but, as the top slows down owing to friction and air resistance, the vertical position will become unstable, and the top will fall down to a positive value of θ.

These deductions are correct, for $\frac{dV_e}{d\theta} = 0$ results in

$$2C(1 - cos\theta)^2 = \sin^4\theta \tag{4.10.33}$$

One solution is $\theta = 0$, and a second differentiation will show that this is stable or unstable according to whether C is greater than or less than 2, although the second differentiation is slightly tedious, and it can be avoided. We can also note that $1 - \cos\theta$ is a common factor of the two sides of Equation 4.10.33, and it can be divided out to yield a cubic Equation in $\cos\theta$:

$$2C - 1 - (2C + 1)\cos\theta - \cos^2\theta - \cos^3\theta = 0, \tag{4.10.34}$$

which could be solved to find the second equilibrium point – but that again is slightly tedious. A less tedious way might be to take the square root of each side of Equation 4.10.33:

$$\sqrt{2C}(1 - \cos\theta) = 1 - \cos^2\theta \tag{4.10.35}$$

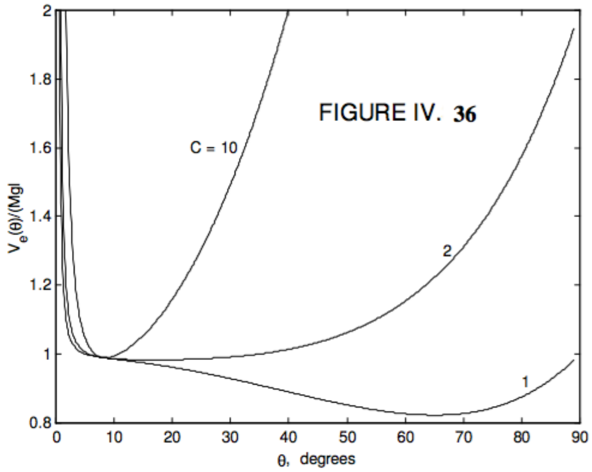
and then divide by the common factor (1 − \cos θ) to obtain

$$\cos\theta = \sqrt{2C} - 1, \tag{4.10.36}$$

which gives a real θ only if $C \geq 2$. Note also,if $C = \frac{1}{2}$, θ = 90° .

Special Case II. $L_3 \approx L_1$.

In other words, L_1 and L_3 are not very different. In Figure IX.36 I draw $\frac{V_e(0)}{Mgl}$ for several different C, for $L_3 = 1.01L_1$.



We see that for quite a large range of C greater than 2 the stable equilibrium position is close to vertical. Even though the curve for $C = 2$ has a very broad minimum, the actual minimum is a little less than 17°. (I haven't worked out the exact position – I'll leave that to the reader.)

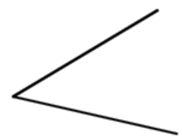
4.11: Appendix

In Section 4.4 we raised the question as to whether angle is a dimensioned or a dimensionless quantity, and in Section 4.8 we raised the question as to whether angle is a vector quantity.

I can present two arguments. One of them will prove incontrovertibly that angle is dimensionless. The other will prove, equally incontrovertibly, and equally convincingly that angle has dimensions. Angle, as you know, is defined as the ratio of arc length to radius. It is the ratio of two lengths, and is therefore incontrovertibly dimensionless. Q.E.D. On the other hand, it is necessary to state the units in which angle is expressed. You cannot merely talk about an angle of 1. You must state whether that is 1 degree or 1 degree. Angle therefore has dimensions. Q.E.D. So – you may take your pick. In many contexts, I like to think of angle as a dimensioned quantity, having dimensions Θ. That is to say, not a combination of mass, length and time, but having its own dimensions in its own right. I find I can carry on with dimensional analysis successfully like this.

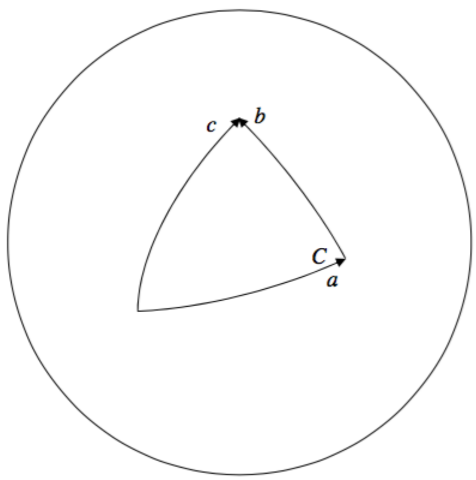
Now for the question: Is angle a vector?

An angle certainly has both magnitude and a direction associated with it. Thus the direction associated with the angle



is at right angles to the plane of the screen, or the paper.
However, this evidently isn't enough for it to be a vector in the sense that we know it.

For example, if you turn through an angle a, and then through an angle b, you cannot say that the net resultant of these is to turn through an angle c, where $c^2 = a^2 + b^2 - 2ab\cos C$.



Thus, although angle has both magnitude and direction, and could be thought of thus far as a vector, angles do not obey the ordinary triangle law of vector addition. For this reason, angles are sometimes called "pseudo-vectors".

In fact, as any astronomy student will tell you, the correct relation between the angles is

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \ .$$

If the angles a, b, c (not C) are very small, then the triangle becomes almost plane. The angles add more and more like the usual plane triangle rule for vector addition. This is probably obvious when thinking about the geometry, but you can also convince yourself of it by expanding the sines and cosines (except for $\cos C$) as series, and, to the second order of small quantities $(\cos\theta \approx 1 - \frac{1}{2}\theta^2, \sin\theta \approx 1)$, you'll find that the equation $\cos c = \cos a \cos b + \sin a \sin b \cos C$ reduces to $c^2 = a^2 + b^2 - 2ab\cos C$. For this reason it is sometimes said that an "infinitesimal rotation" can

be regarded as a true vector. Also for this reason, the time rate of change of an angle, $\frac{d\theta}{dt}$, that is to say an angular velocity, can quite safely be treated as a true vector, since the numerator and denominator of the derivatives are both infinitesimals.

Thus, although angle has direction associated with it, angle is not a true vector in that angles do not follow the usual rules for vector addition. However, very small angles do approximately follow the addition rules, so that, in the infinitesimal limit, angles can be treated as vectors. And hence angular velocity, being a ratio of infinitesimals ($d\theta$ and dt), can correctly be treated as vectors.

CHAPTER OVERVIEW

5: Collisions

In this chapter on collisions, we shall have occasion to distinguish between elastic and inelastic collisions. An elastic collision is one in which there is no loss of translational kinetic energy. That is, not only must no translational kinetic energy be degraded into heat, but none of it may be converted to vibrational or rotational kinetic energy. In laying out the principles involved in collisions between particles, we need not suppose that the particles actually "bang into" – i.e. touch – each other. For example most of the principles that we shall be describing apply equally to collisions between balls that "bang into" each other and to phenomena such as Rutherford scattering, in which an alpha particle is deviated from its path by a gold nucleus without actually "touching" it. Of course, if you think about it at an atomic level, when two billiard balls collide, the atoms don't actually "touch" each other; they are repelled from each other by electromagnetic forces, just as the alpha particle and the gold nucleus repelled each other in the Rutherford-Geiger-Marsden experiment.

Topic hierarchy

5.1: Introduction

In this chapter on collisions, we shall have occasion to distinguish between elastic and inelastic collisions. An elastic collision is one in which there is no loss of translational kinetic energy. That is, not only must no translational kinetic energy be degraded into heat, but none of it may be converted to vibrational or rotational kinetic energy. It is well known, for example, that if a ball makes a glancing (i.e. not head-on) elastic collision with another ball of the same mass, initially stationary, then after collision the two balls will move off at right angles to reach other. But this is so only if the balls are smooth. If they are rough, after collision the balls will be spinning, so this result – and any other results that assume no loss of translational kinetic energy − will not be valid. When molecules collide, they may be set into rotational and vibrational motion, and in that case the collision will not be elastic in the sense in which we are using the term. If two atoms collide, one (or both) may be raised to an excited electronic level. Some of the translational kinetic energy has then been converted to potential energy. If the excited atom subsequently drops down to a lower level, that energy is radiated away and lost from the system. Superelastic collisions are also possible. If one atom, before collision, is in an excited electronic state, on collision it may make a radiationless downwards transition, and the potential energy released is then converted to translational kinetic energy, so the collision is superelastic. None of this is intended to mean that elastic collisions are impossible or even rare. In the case of collisions involving macroscopic bodies, such as smooth, hard billiard balls, collisions may not be 100% elastic, but they may be close to it. In the case of low-energy (low temperature) collisions between atoms, there need be no excitation to excited levels, in which case the collision will be elastic. Some subatomic particles, in particular leptons (of which the electron is the best-known example), are believed to have no internal degrees of freedom, and therefore collisions between them are necessarily elastic.

In laying out the principles involved in collisions between particles, we need not suppose that the particles actually "bang into" – i.e. touch – each other. For example most of the principles that we shall be describing apply equally to collisions between balls that "bang into" each other and to phenomena such as Rutherford scattering, in which an alpha particle is deviated from its path by a gold nucleus without actually "touching" it. Of course, if you think about it at an atomic level, when two billiard balls collide, the atoms don't actually "touch" each other; they are repelled from each other by electromagnetic forces, just as the alpha particle and the gold nucleus repelled each other in the Rutherford-Geiger-Marsden experiment.

The theory of collisions is used a great deal, of course, in the study of high-energy collisions between particles in particle physics. Bear in mind, however, that in "atom-smashing" experiments with modern huge particle accelerators, or even in relatively mild collisions such as Compton scattering of x-rays, the particles involved are moving at speeds that are not negligible compared with the speed of light, and therefore relativistic mechanics is needed for a proper analysis. In this chapter, collisions are treated entirely from a nonrelativistic point of view.

5.2: Bouncing Balls

When a ball is dropped to the ground, one of four things may happen:

1. It may rebound with exactly the same speed as the speed at which it hit the ground. This is an *elastic collision*.
2. It may come to a complete rest, for example if it were a ball of soft putty. I shall call this a completely *inelastic collision*.
3. It may bounce back, but with a reduced speed. For want of a better term I shall refer to this as a somewhat *inelastic collision*.
4. If there happens to be a little heap of gunpowder lying on the table where the ball hits it, it may bounce back with a faster speed than it had immediately before collision. That would be a *superelastic collision*.

The ratio

$$\frac{\text{speed after collision}}{\text{speed before collision}}$$

is called the *coefficient of restitution*, for which I shall use the speed before collision symbol e. The coefficient is 1 for an elastic collision, less than 1 for an inelastic collision, zero for a completely inelastic collision, and greater than 1 for a **superelastic** collision. The ratio of kinetic energy (after) to kinetic energy (before) is evidently, *in this situation*, e^2.

If a ball falls on to a table from a height h_0, it will take a time $t_0 = \sqrt{2H_0 lg}$ to fall. If the collision is somewhat inelastic it will then rise to a height $h_1 = e^2 h_0$ and it will take a time et to reach height h_1. Then it will fall again, and bounce again, this time to a lesser height. And, if the coefficient of restitution remains the same, it will continue to do this for an infinite number of bounces. After a billion bounces, there is still an infinite number of bounces yet to come. The total distance travelled is

$$h = h_0 + 2h_0(e^2 + e^4 + e^6 + \dots) \tag{5.2.1}$$

and the time taken is

$$t = t_0 + 2t_0(e + e^2 + e^3 + \dots). \tag{5.2.2}$$

These are geometric series, and their sums are

$$h = h_0\left(\frac{1+e^2}{1-e^2}\right), \tag{5.2.3}$$

which is independent of g (i.e. of the planet on which this experiment is performed), and

$$t = t_0\left(\frac{1+e}{1-e}\right) \tag{5.2.4}$$

For example, suppose h_0 = 1 m, e = 0.5, g = 9.8 m s^{-2}, then the ball comes to rest in 1.36 s after having travelled 1.67 m after an infinite number of bounces.

Discuss

Does the ball ever stop bouncing, given that, after every bounce, there is still an infinite number yet to come; yet after 1.36 seconds it is no longer bouncing...?

5.3: Head-on Collision of a Moving Sphere with an Initially Stationary Sphere

The coefficient of restitution is

$$e = \frac{\text{relative speed of recession after collision}}{\text{relative speed of approach before collision}}. \tag{5.3.1}$$

We suppose that the two masses m_1 and m_2, the initial speed u, and the coefficient of restitution e are known; we wish to find v_1 and v_2.

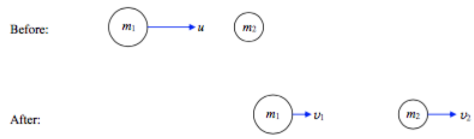


FIGURE V.1

We evidently need two equations. Since there are no *external* forces on the *system*, the linear momentum of the *system* is conserved:

$$m_1 u = m_1 v_1 + m_2 v_2. \tag{5.3.2}$$

The second equation will be the restitution equation (Equation 5.3.1):

$$v_2 - v_1 = eu. \tag{5.3.3}$$

These two equations can be solved to yield

$$v_1 = \left(\frac{m_1 - m_2 e}{m_1 + m_2}\right) u \tag{5.3.4}$$

and

$$v_2 = \left(\frac{m_1(1+e)}{m_1 + m_2}\right) u. \tag{5.3.5}$$

The relation between the kinetic energy loss and the coefficient of restitution isn't quite as simple as in Section 5.2.

Exercise 5.3.1

Show that

$$\frac{\text{kinetic energy (after)}}{\text{kinetic energy (before)}} = \frac{m_1 v_1^2 + m_1 v_2^2}{m_1 u^2} = \frac{m_1 + m_2 e^2}{m_1 + m_2}. \tag{5.3.6}$$

- If $m_2 = \infty$ (as in Section 5.2), this becomes just e^2. If $e = 1$, it becomes unity, so all is well.
- If $m_1 << m_2$ (Ping-pong ball collides with cannon ball), $v_1 = -u, v_2 = 0$.
- If $m_1 = m_2$ (Ping-pong ball collides with ping-pong ball), $v_1 = 0, v_2 = u$.
- If $m_1 >> m_2$ (Cannon ball collides with ping-pong ball), $v_1 = u, v_2 = 2u$.

Example 5.3.1

A moving sphere has a head-on elastic collision with an initially stationary sphere. After collision the kinetic energies of the two spheres are equal. Show that the mass ratio of the two spheres is 0.1716.

Which of the two spheres is the more massive? (I guarantee that your answer to this will be correct.)

5.4: Oblique Collisions

In Figure V.2 I show two balls just before collision, and just after collision. The horizontal line is the line joining the centres – for short, the "line of centres". We suppose that we know the velocity (speed and direction) of each ball before collision, and the coefficient of restitution. The direction of motion is to be described by the angle that the velocity vector makes with the line of centres. We want to find the velocities (speed and direction) of each ball after collision. That is, we want to find four quantities, and therefore we need four equations. These equations are as follows.

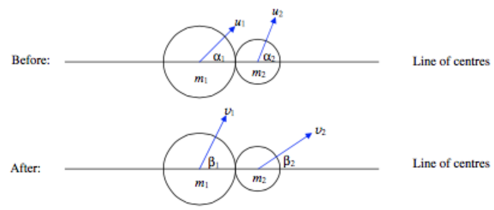


FIGURE V.2

There are no external forces on the system along the line of centres. Therefore the component of momentum of the system along the line of centres is conserved:

$$m_1 v_1 \cos \beta_1 + m_2 v_2 \cos \beta_2 = m_1 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2. \tag{5.4.1}$$

If we assume that the balls are smooth - i.e. that there are no forces perpendicular to the line of centres and the balls are not set into rotation, then the component of the momentum of each ball separately perpendicular to the line of centres is conserved:

$$v_1 \sin \beta_1 = u_1 \sin \alpha_1 \tag{5.4.2}$$

and

$$v_2 \sin \beta_2 = u_2 \sin \alpha_2. \tag{5.4.3}$$

The last of the four equations is the restitution equation

$$e = \frac{\text{elative speed of recession along the line of centres after collision}}{\text{relative speed of approach along the line of centres before collision}}.$$

That is,

$$v_2 \cos \beta_2 - v_1 \cos \beta_1 = e(u_1 \cos \alpha_1 - u_2 \cos \alpha_2). \tag{5.4.4}$$

Example 5.4.1A

Suppose m_1 =3kg, m_2 = 2kg, u_1 = 40ms^{-1} u_2 = 15ms^{-1}

α_1 = 10° , α_2 = 70°, e = 0.8

Find v_1, v_2, β_1, β_2.

Solution

v_1 = 16.28 m s−1 v_2 = 44.43 m s−1

β_1 = 25°15' β_2 = 18°30'

Example 5.4.1B

Suppose m_1 = 3kg, m_2 = 3kg, u_1 = 12ms^{-1} u_2 = 15ms^{-1}

α_1 = 20° , α_2 = 50°, β_2 = 47°

Find v_1, v_2, β_1, e.

Solution

v_1 = 10.50 m s^{-1} v_2 = 15.71 m s^{-1}

β_1= 23°00' e= 0.6418

Exercise 5.4.1

If $u_2 = 0$, and if $e = 1$ and if $m_1 = m_2$, show that β_1 = 90° and β_2 = 0°.

5.5: Oblique (Glancing) Elastic Collisions, Alternative Treatment

In Figure V.3, unlike Figure V.2, the horizontal line is not intended to represent the line of centers. Rather, it is the direction of the initial velocity of m_1, and m_2 is initially at rest. The second mass m_2 is slightly off the line of the velocity of m_1. I am assuming that the collision is elastic, so that $e = 1$. In the "before" part of the Figure, I have indicated, as well as the two masses, the position and velocity $\mathbf{V}$ of the center of mass C. The velocity of C remains constant, because there are no external forces on the system. I have not drawn C in the "after" part of the Figure, because it would get a little in the way. Think about where it is.

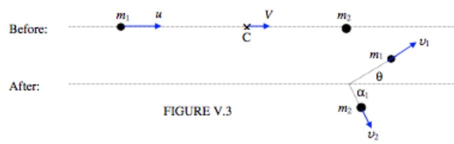


FIGURE V.3

Figure V.3 shows the situation in "laboratory space". (Later, we'll look at the situation referred to a reference frame in which C is at rest – "center of mass space".) The angle θ is the angle through which m_1 has been scattered (the "scattering angle"). I have indicated in the Figure how it is related to the α_1 and β_1 of Section 5.3. Note that m_2 (initially stationary) scoots off along the line of centers.

The following two equations express the constancy of linear momentum of the system.

$$(m_1 + m_2)V = m_1 u = m_1 v_1 \cos\theta + m_2 v_2 \cos\alpha_1. \tag{5.5.1}$$

I'm going to draw, in Figure V.4, the situation "close-up", so that you can see the geometry more clearly. Note that the distance b is called the *impact parameter*. It is the distance by which the two centers would have missed each other had the first particle not been scattered.

In Figure V.5, I draw the situation in center of mass space, in which the center of mass C is stationary. In this reference frame, I just have to subtract $\mathbf{V}$ from all the velocities. Note that in center of mass space the *speeds* of the particles are unaltered by the collision. In center of mass space, m_1 is scattered through an angle θ', and I am going to find a relation between θ', θ and the mass ratio $\frac{m_2}{m_1}$.

I shall start with the profound statement that

$$\tan\theta = \frac{v_1 \sin\theta}{v_1 \cos\theta} \tag{5.5.2}$$

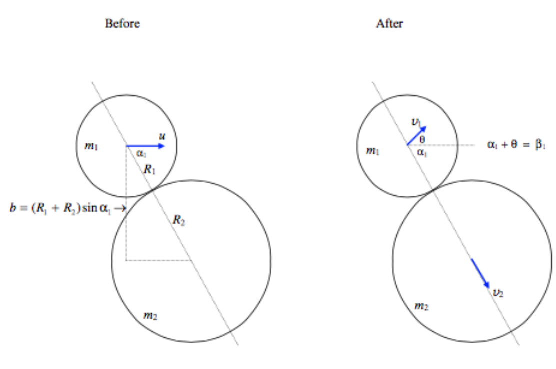


FIGURE V.4

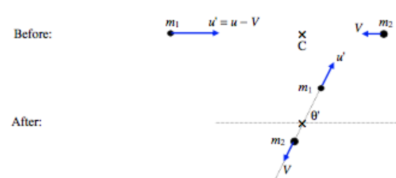


FIGURE V.5

Now $v_1 \sin\theta$ is the y-component of the final velocity of m_1 in laboratory space. The y-component of the final velocity of m_1 in center of mass space is $u' \sin\theta'$, and these two are equal, since the y-component of the motion is unaffected by the change of reference frame. Therefore

$$\tan\theta = \frac{u' \sin\theta'}{v_1 \cos\theta}. \tag{5.5.3}$$

Therefore

$$v_1 \sin\theta = u' \sin\theta'. \tag{5.5.4}$$

The x-components of the "before" and "after" velocities of m_1 are related by

$$v_1 \cos\theta = u' \sin\theta' + V. \tag{5.5.6}$$

Substitute Equations 5.5.4 and 5.5.6 into Equation 5.5.2 to obtain

$$\tan\theta = \frac{\sin\theta'}{\cos\theta' + V/u'} \tag{5.5.7}$$

But

$$(m_1 + m_2)V = m_1 u = m_1(u' + V), \tag{5.5.8}$$

from which

$$\frac{V}{u'} = \frac{m_1}{m_2}. \tag{5.5.9}$$

On substituting this into Equation 5.5.7, we obtain the relation we sought:

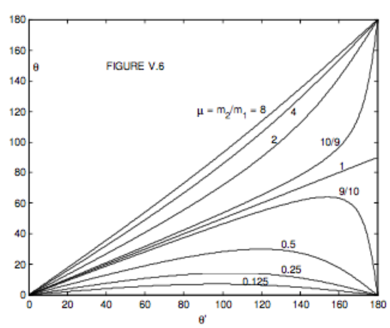
$$\tan\theta = \frac{\sin\theta'}{cos\theta' + m_1/m_2}. \tag{5.5.10}$$

This relation is illustrated in Figure V.6 for several mass ratios.

Let us try to interpret the Figure. For $m_2 > m_1$, any scattering angle, forward or backward, is possible, but for $m_2 < m_1$, backward scattering is not possible, and forward scattering is possible only up to a maximum. This is only to be expected. Thus for an impact parameter of zero or of $R_1 + R_2$, and $m_2 < m_1$, the scattering angle θ must be zero, and therefore for intermediate impact parameters it must go through a maximum. This would be clearer if we could plot the scattering angle versus the impact parameter, and indeed that is something that we shall try to do. In the meantime it is easy to show, by differentiation of Equation 5.5.10 (do it!), that the maximum scattering angle is $\sin^{-1}\mu$, where

$$\mu = \frac{m_2}{m_1}.$$

That is, if the scattered particle is very massive compared with the scattering particle, the maximum scattering angle is small – just to be expected.



I want to do two things now - <u>one</u>, to calculate the scattering angle θ as a function of impact parameter, and <u>two</u>, to calculate $\frac{v_1}{u}$ as a function of scattering angle. I'm going to start with Equations 5.4.1, 5.4.2 and 5.4.4, except for the following. I'll assume $e = 1$ (elastic collision), and $u_2 = 0$ (m_2 is initially stationary), and $\beta_2 = 0$ (since m_2 is initially stationary, it must move along the line of centers after collision). Since I want to try to calculate the scattering angle, I'll write $\theta + \alpha_1$ for β_1 (see Figure V.4). I'm also going to write r_1, r_2 and μ for the dimensionless ratios $\frac{v_1}{u}$, $\frac{v_2}{u}$ and $\frac{m_2}{m_1}$ respectively. With those small changes, Equations 5.4.1, 5.4.2 and 5.4.4 become

$$r_1 \cos(\theta + \alpha_1) + \mu r_2 = \cos\alpha_1, \tag{5.5.11}$$

$$r_1 sin(\theta + \alpha_1) = \sin\alpha_1, \tag{5.5.12}$$

$$r_2 - r_1 \cos(\theta + \alpha_1) = \cos\alpha_1. \tag{5.5.13}$$

Eliminate r_2 from Equations 5.5.11 and 5.5.13 to obtain

$$r_1 \cos(\theta + \alpha_1) = M \cos\alpha_1, \tag{5.5.14}$$

where

$$M = \frac{1-\mu}{1+\mu} = \frac{m_1 - m_2}{m_1 + m_2}. \tag{5.5.15}$$

If we now eliminate α_1 from Equations 5.5.12 and 5.5.14, we obtain the relation between $\frac{v_1}{u}$ and the scattering angle, which was the second of our two aims above. The elimination is easily done as follows. Expand \sin and cos of $\theta + \alpha_1$ in the two equations, divide both sides of each equation by $\cos\alpha_1$ and eliminate $\tan\alpha_1$ between the two equations. The result is

$$r_1^2(1+M)\cos\theta + M = 0. \tag{5.5.16}$$

We'll have a look at this equation in a moment, but in the meantime, instead of eliminating α_1 from Equations 5.5.12 and 5.5.14, let's eliminate r_1. This will give us a relation between the scattering angle θ and α_1, and, since α_1 is closely relation to the impact parameter (see Figure V.4) this will achieve the first of our aims, namely to find the scattering angle as a function of the impact parameter. If you do the algebra, you should find that the relation between θ and α_1 is

$$t = \frac{a(1-M)}{a^2 + M}, \tag{5.5.17}$$

where

$$t = tan\theta \quad and \quad a = tan\alpha_1. \tag{5.5.17a,b}$$

Now let

$$b' = \frac{b}{R_1 + R_2} \tag{5.5.18}$$

and from Figure V.5 we see that

$$b' = \sin\alpha_1. \tag{5.5.19}$$

On elimination of α_1 from Equations 5.5.17 and 5.5.19, we obtain the required relation between scattering angle θ and (dimensionless) impact parameter b':

$$\tan\theta = \frac{s\mu b'\sqrt{1-b'^2}}{1-\mu+2\mu b'^2}. \tag{5.5.20}$$

This relation is shown in Figure V .7. The values of the mass ratio μ ($= fracm_2 m_1$) are (from the

lowest up) $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{9}{10}, 1, \frac{10}{9}, 2, 4, 8$and (dashed) ∞. This Figure is perhaps slightly easier to interpret than Figure V.6. One can see that for $\mu > 1$, any scattering angle is possible, but for $\mu < 1$, the scattering angle has a maximum possible value, less than 90 °, and the scattering angle is zero for b' = 0 or 1.

Exercise 5.5.1

We saw, by differentiation of Equation 5.5.10, that the maximum scattering angle was $\sin^{-1}\mu$. Now show the same thing by differentiation of Equation 5.5.20. (This is not so easy, is it?)

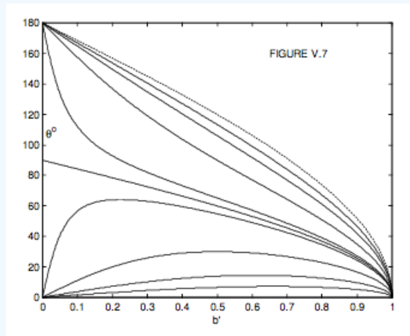
Show that the scattering angle is greatest for an impact parameter of

$$b' = \sqrt{\frac{1-\mu}{2}}. \tag{5.5.21}$$

Solution

You will notice that, for /(b' = 0/) (head-on collision) the scattering angle changes abruptly from 0 to 180° as the mass ratio changes from less than 1 to more than 1. No problem there. But if the mass ratio is *exactly* 1 (not the tiniest bit less or the tiniest bit more) the scattering angle is apparently 90°. This may cause some puzzlement until it is realized that for a head-on collision with $\mu = 1$ the first sphere comes to a dead halt.

The case of $mu = \infty$ (second sphere immovable) is of some interest. It is easy in that case to calculate how the scattering angle varies with impact parameter for an elastic collision, merely by requiring the scattered sphere to obey the law of reflection, and without any reference to Equation 5.5.20.



Exercise 5.5.2

Easy Exercise.

Without any reference to Equation 5.5.20, show that, if the second ball is immovable, the scattering angle is related to the impact parameter by

$$\theta = 180^\circ - 2sin^{-1}b'. \tag{5.5.22}$$

Not-so-easy exercise. Show that, in the limit as $\mu \to \infty$, Equation 5.5.20 approaches Equation 5.5.22.

In any case, the limiting case as the second sphere becomes immovable is shown as a dashed curve in Figure V.7.

Exercise of Intermediate Difficulty. The mass ratio $\dfrac{m_2}{m_1}$ is 0.9, and the scattering angle is 50°. What was the impact parameter?

Answers

b' = 0.07270 or 0.58540.

We have now dealt with the direction of motion of m_1 after scattering as a function of impact parameter. We should now look at the speed of m_1 after collision, and this takes us back to Equation 5.5.16, which gives is the speed ($r_1 = \dfrac{v_1}{u}$) as a function of scattering angle θ. It is 11 quadratic in r_1, so, for a given scattering angle there are two possible speeds – which is not surprising, because a given scattering angle can arise from two different impact parameters, as we have just found out. We can conveniently show the relation between r_1 and θ simply by plotting the equation in polar coordinates. I'll re-write the equation here for easy reference:

$$r_1^2 = r_1(1+M)\cos\theta + M = 0. \tag{5.5.16.}$$

Here, $M = \dfrac{1-\mu}{1+\mu} = \dfrac{1}{1+\mu}$, but I want to write the equation in terms of the *mass fractions*

$$q_1 = \frac{m_1}{m_1+m_2} = \frac{1}{1+\mu} \quad and \quad q_2 = \frac{m_2}{m_1+m_2} = \frac{\mu}{1+\mu}. \tag{5.5.23a,b}$$

If you work at this for a short while, you will find that Equation 5.5.16 becomes

$$r_1^2 + q_1^2 - 2r_1q_1\cos\theta = q_2^2. \tag{5.5.24}$$

and one is then overcome with an overwhelming desire to draw a triangle:

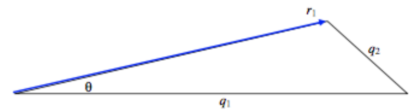


FIGURE V.8

For a given mass ratio, the locus of r_1 (the speed) versus θ (the scattering angle) is such that q_1 and q_2 are constant – in other words, it is a circle:

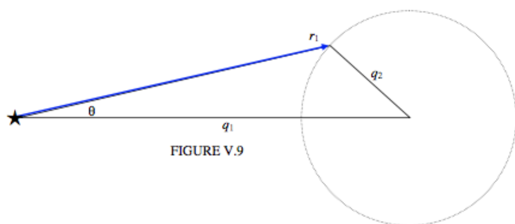


FIGURE V.9

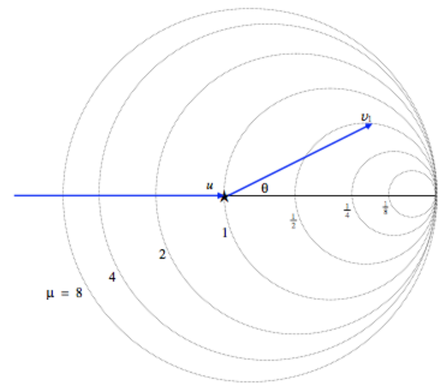


FIGURE V.10

One can imagine that the first particle comes in from the left at speed u and the collision takes place at the asterisk, and, after collision, it is moving at a speed r_1 times u in a direction θ, the magnitude of its velocity vector being determined by where the vector intersects the circle (in two possible places) given by Equation 5.5.24. The maximum scattering angle corresponds to a velocity vector that is tangent to the circle. If the asterisk is the pole (origin) of the polar coordinates, the centre of the circle is at a distance q_1 from the pole, and its radius is q_2. Figure V.10 shows the circles corresponding to several mass ratios. The Figure graphically illustrates the relation between u, v_1, θ and μ. You can see, for example, that if $\mu > 1$, scattering through any angle is possible, and the relation between v_1 and θ is unique; but if $\mu < 1$, only forward scattering is possible, up to a maximum θ, and, for a given θ, there are two solutions for v_1.

This deals with what happens to the sphere m_1. We can now turn our attention to m_2. Starting from Equations 5.5.11, 5.5.12 and 5.5.13, we are going to want to eliminate r_1 and θ - indeed anything that pertains to the sphere m_1.

If you refer to Figure V.4 you will see that, after collision, m_2 scoots off at an angle α_1 to the original direction of motion of m_1. Therefore I think it is of interest to find a relation between r_2 (\dfrac{v_{2}}{u})and α_1. If we succeed in doing this, it means that we can also find a relation, if we want it, between r_2 and the impact parameter, since $b' = \sin\alpha_1$. It is easy to eliminate r_1 from Equations 5.5.12 and 5.5.13, and then you can get $\tan(\theta+\alpha_1)$ from Equation 5.5.14, and hence get the required relation:

$$r_2 = \frac{2\cos\alpha_1}{1+\mu}. \tag{5.5.25}$$

I'll draw this relation as a polar graph, r_2 versus α_1, in Figure V.11. I'll leave the reader to work out and draw the relation between r_2 and b' if he or she wishes. Equation V.11 is the polar equation to a circle of radius $\frac{1}{1+\mu}$.

Example 5.5.1

Suppose the mass ratio $\mu = \frac{m_2}{m_1} = 0.5$ and the scattering angle is $\theta = 20°$. Equation 5.5.16 or Figure V.10 will show that r_1 = 0.8696 or 0.3833. Equation 5.5.17 will show that α_1 = 58° .4 or 11° .6 . And Equation 5.5.25 or Figure V.11 will show that r_2 = 0.6983 or 1.306. I'll leave it to the reader to determine which alternative values of r_1, r_2 and α_1 go together.

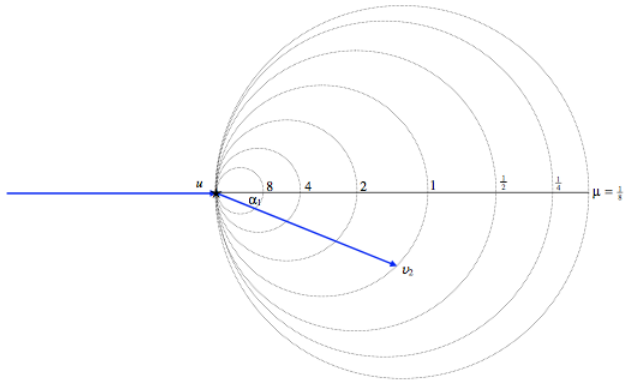


FIGURE V.11

CHAPTER OVERVIEW

6: Motion in a Resisting Medium

In studying the motion of a body in a resisting medium, we assume that the resistive force on a body, and hence its deceleration, is some function of its speed. Such resistive forces are not generally conservative, and kinetic energy is usually dissipated as heat. For simple theoretical studies one can assume a simple force law, such as the resistive force is proportional to the speed, or to the square of the speed, or to some function that we can conveniently handle mathematically. For slow, laminar, nonturbulent motion through a viscous fluid, the resistance is indeed simply proportional to the speed, as can be shown at least by dimensional arguments. One thinks, for example, of Stokes's Law for the motion of a sphere through a viscous fluid. For faster motion, when laminar flow breaks up and the flow becomes turbulent, a resistive force that is proportional to the square of the speed may represent the actual physical situation better.

6.1: Introduction

In studying the motion of a body in a resisting medium, we assume that the resistive force on a body, and hence its deceleration, is some function of its speed. Such resistive forces are not generally conservative, and kinetic energy is usually dissipated as heat. For simple theoretical studies one can assume a simple force law, such as the resistive force is proportional to the speed, or to the square of the speed, or to some function that we can conveniently handle mathematically. For slow, laminar, nonturbulent motion through a viscous fluid, the resistance is indeed simply proportional to the speed, as can be shown at least by dimensional arguments. One thinks, for example, of Stokes's Law for the motion of a sphere through a viscous fluid. For faster motion, when laminar flow breaks up and the flow becomes turbulent, a resistive force that is proportional to the square of the speed may represent the actual physical situation better.

6.2: Uniformly Accelerated Motion

Before studying motion in a resisting medium, a brief review of uniformly accelerating motion might be in order. That is, motion in which the resistance is zero. Any formulas that we develop for motion in a resisting medium must go to the formulas for uniformly accelerated motion as the resistance approaches zero.

One may imagine a situation in which a body starts with speed v_0 and then accelerates at a rate a. One may ask three questions:

How fast is it moving after time t ?
How far has it moved in time t ?
How fast is it moving after it has covered a distance x ?

The answers to these questions are well known to any student of physics:

$$v = v_0 + at, \tag{6.2.1}$$

$$v = v_0 t + \frac{1}{2}at^2, \tag{6.2.2}$$

$$v^2 = v_0^2 + 2ax. \tag{6.2.3}$$

Since the acceleration is uniform, there is no need to use calculus to derive these. The first follows immediately from the meaning of acceleration. Distance travelled is the area under a speed : time graph. Figure VI.1 shows a speed : time graph for constant acceleration, and Equation 6.2.2 is obvious from a glance at the graph. Equation 6.2.3 can be obtained by elimination of t between Equations 6.2.1 and 6.2.2. (It can also be deduced from energy considerations, though that is rather putting the cart before the horse.)

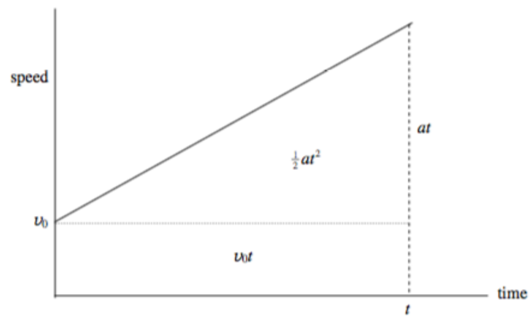


FIGURE IV.1

Nevertheless, although calculus is not necessary, it is instructive to see how calculus can be used to analyse uniformly accelerated motion, since calculus will be necessary in less simple situations. We shall be using calculus to answer the three questions posed earlier in the section.

For uniformly accelerated motion, the *Equation of motion* is

$$\ddot{x} = a. \tag{6.2.4}$$

To answer the first question, we write $\ddot{x}$ as $\frac{dv}{dt}$, and then the integral (with initial condition $x = 0$ when $t = 0$) is

$$v = v_0 + at. \tag{6.2.5}$$

This is the *first time* integral.
Next, we write v as $\frac{dv}{dt}$ and integrate again with respect to time, to get

$$x = v_0 t + \frac{1}{2}at^2. \tag{6.2.6}$$

This is the *second time integral.*
To obtain the answer to the third question, which will be called the *space integral*, we must remember to write $\ddot{x}$ as $v\frac{dv}{dt}$.
Thus the Equation of motion (Equation 6.2.4) is

$$v\frac{dv}{dx} = a. \tag{6.2.7}$$

When this is integrated with respect to x (with initial condition $v = v_0$ when $x = 0$) we obtain

$$v^2 = v_0^2 + 2ax. \tag{6.2.8}$$

This is the space integral.

Examples.

Here are a few quick examples of problems in uniformly accelerated motion. It is probably a good idea to work in *algebra* and obtain *algebraic* solutions to each problem. That is, even if you are told that the initial speed is 15 ms^{-1}, call it v_0, or, if you are told that the height is 900 feet, call it h. You will probably find it helpful to sketch graphs either of distance versus time or speed versus time in most of the problems. One last little hint: Remember that the two solutions of a quadratic Equation are equal if $b^2 = 4ac$.

Example 6.2.1

A body is dropped from rest. The last third of the distance before it hits the ground is covered in time T. Show that the time taken for the entire fall to the ground is 5.45T.

Example 6.2.2

The Lady is 8 metres from the bus stop, when the Bus, starting from rest at the bus stop, starts to move off with an acceleration of 0.4 m s^{-2}. What is the least speed at which the Lady must run in order to catch the Bus?

Answer: 2.53ms^{-1}.

Example 6.2.3

A parachutist is descending at a constant speed of 10 feet per second. When she is at a height of 900 feet, her friend, directly below her, throws an apple up to her. What is the least speed at which he must throw the apple in order for it to reach her? How long does it take to reach her, what height is she at then, and what is the relative speed of parachutist and apple? Assume g = 32 ft s^{-2}. Neglect air resistance for the apple (but not for the parachutist!)

Answer:230fts^{-1}, 7.5s, 825 ft, 0fts^{-1}.

Example 6.2.4

A lunar explorer performs the following experiment on the Moon in order to determine the gravitational acceleration g there. He tosses a lunar rock upwards at an initial speed of 15 m s^{-1}. Eight seconds later he tosses another rock upwards at an initial speed of 10 m s^{-1}. He observes that the rocks collide 16.32 seconds after the launch of the first rock. Calculate g and also the height of the collision.

Answer: 1.64ms^{-2}, 26.4m

Example 6.2.5

Mr A and Mr B are discussing the merits of their cars. Mr A can go from 0 to 50 mph in ten seconds, and Mr B can go from 0 to 60 mph in 20 seconds. Mr B gives Mr A a start of one second. Assuming that each driver first accelerates uniformly to his maximum speed and thereafter travels at each uniform speed, how long does it take Mr B to catch Mr A, and how far have the cars travelled by then?

Answer: 41 s, half a mile.

I make the answers as follows. Let me know (jtatum@uvic.ca) if you think I have got any of them wrong.

SECTION OVERVIEW

6.3: Uniformly Accelerated Motion

If the only force on a body is a resistive force that is proportional to its speed, the equation of motion is

$$m\ddot{x} = -b\dot{x}.$$

One thinks, for example, of Stokes's equation for the laminar motion of a sphere through a viscous fluid, in which the resistive force is $6\pi\eta av$, where η is the coefficient of dynamic viscosity. If we divide both sides of the equation by the mass m, we obtain

$$m\ddot{x} = -\gamma\dot{x},$$

where $\gamma = \frac{b}{m}$ is the damping constant. It has dimension T^{-1} and SI units s^{-1}.

Topic hierarchy

6.3A: Resistive Force Only

6.3B: Body falling under gravity in a resisting medium, resistive force proportional to the speed

6.3C: Body thrown vertically upwards with initial speed v_0

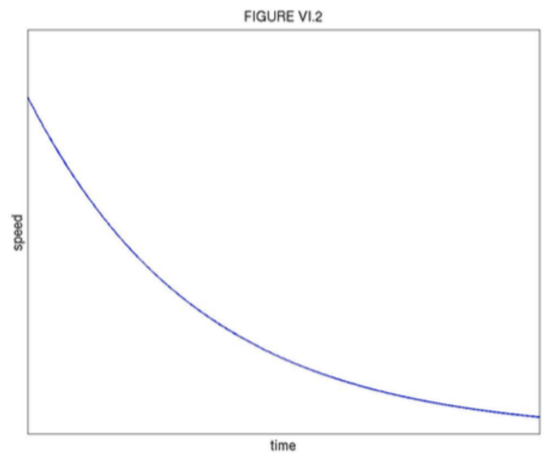
6.3A: Resistive Force Only

It is difficult to imagine a real situation in which the one and only force is a resistive force proportional to the speed. A body falling through the air won't do, because, in addition to the resistive force, there is the acceleration due to gravity. Perhaps we could imagine a puck sliding across the ice. The ice would have to be presumed to be completely frictionless, and the only force on the puck would be the resistance of the air. It is a slightly artificial situation, because we want the puck to be going so fast that the frictional force is negligible compared with the air resistance, but not so fast that the airflow is turbulent - but we need to start somewhere. The frictional force is, at least to a very good approximation, not a function of speed, but is constant, and we shall start by assuming that it is negligible and that the only horizontal force on the puck is air resistance and that the air resistance is proportional to the speed.

In this case, the Equation of motion is indeed Equation 6.3.2. To obtain the *first time integral*, we write $\dot{x}$ as v and the first time integral is readily found to be

$$v = v_0 e^{-\gamma t}. \tag{6.3.3}$$

Here v_0 is the initial speed. This is illustrated in Figure VI.2



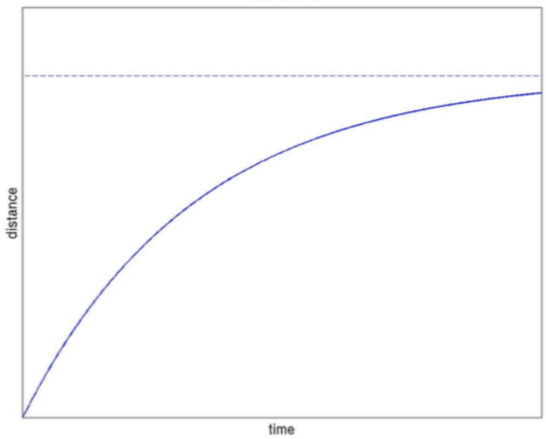
The speed is reduced to half of the initial speed in a time

$$t_{\frac{1}{2}} = \frac{\ln 2}{\gamma} = \frac{0.693}{\gamma}. \tag{6.3.4}$$

The second time integral is found by writing v in Equation 6.3.3 as $\frac{dx}{dt}$. Integration, with initial condition $x = 0$ when $t = 0$, gives

$$x = x_\infty (1 - e^{-\gamma t}), \tag{6.3.5}$$

where $x_\infty = \frac{v_0}{\gamma}$. This is illustrated in Figure VI.3. It is seen that the puck travels an eventual distance of x_∞, but only after an infinite time.



We can obtain the *space integral* either by eliminating t from between the two time integrals, or by writing the Equation of motion as

$$v\frac{dv}{dx} = -\gamma v. \tag{6.3.6}$$

With initial condition $v = v_0$ when $x = 0$, this becomes

$$v = v_0 - \gamma x, \tag{6.3.7}$$

which is illustrated in Figure VI.4. The speed drops linearly with distance (but exponentially with time) reaching zero after having travelled a finite distance $x_\infty = \frac{v_0}{\gamma}$ in an infinite time.

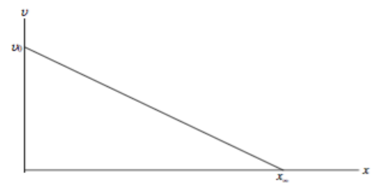


FIGURE VI.4

This analysis has assumed that the only force was the resistive force proportional to the speed. In the case of our imaginary ice puck, we were assuming that the resistive force was that of the air, the friction being negligible. Of course, as the puck slows down and the resistive force becomes less, there will come a point when the frictional force is no longer negligible compared with the ever-decreasing air resistance, so that the above Equations no longer accurately describe the motion. We shall come back to this point in subsection 3c.

6.3B: Body falling under gravity in a resisting medium, resistive force proportional to the speed

We are here probably considering a small sphere falling slowly through a viscous liquid, with laminar flow around the sphere, rather than a skydiver hurtling through the air. In the latter case, the airflow is likely to be highly turbulent and the resistance proportional to a higher power of the speed than the first.

We'll use the symbol y for the distance fallen. That is to say, we measure y downwards from the starting point. The equation of motion is

$$\ddot{y} = g - \gamma v, \tag{6.3.8}$$

where g is the gravitational acceleration.

The body reaches a constant speed when $\ddot{y}$ becomes zero. This occurs at a speed $\hat{v} = \frac{g}{\gamma}$, which is called the *terminal speed*.

To obtain the *first time integral*, we write the equation of motion as

$$\frac{dv}{dt} = \gamma(\hat{v} - v) \tag{6.3.9}$$

or

$$\frac{dv}{\hat{v} - v} = \gamma dt. \tag{6.3.10}$$

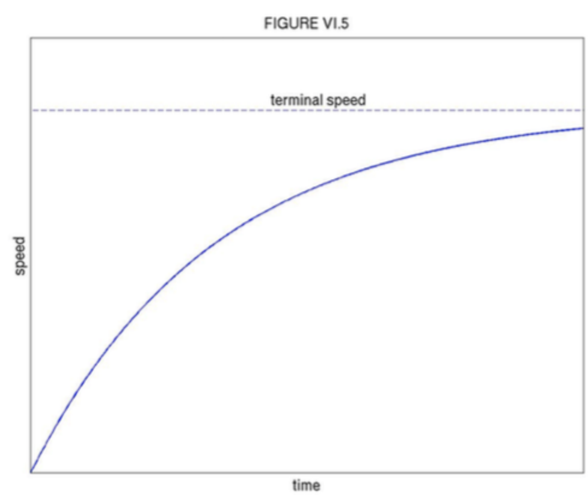
$$\frac{dv}{v - \hat{v}} = -\gamma dt. \tag{6.3.11}$$

DON'T! In the middle of an exam, while covering this derivation that you know so well, you can suddenly find yourself in inextricable difficulties. The thing to note is this. If you look at the left hand side of the equation, you will anticipate that a logarithm will appear when you integrate it. Keep the denominator positive! Some mathematicians may know the meaning of the logarithm of a negative number, but most of us ordinary mortals do not - so keep the denominator positive!

With initial condition $v = 0$ when $t = 0$, the *first time integral* becomes

$$v = \hat{v}(1 - e^{\gamma t}). \tag{6.3.12}$$

This is illustrated in Figure VI.5.



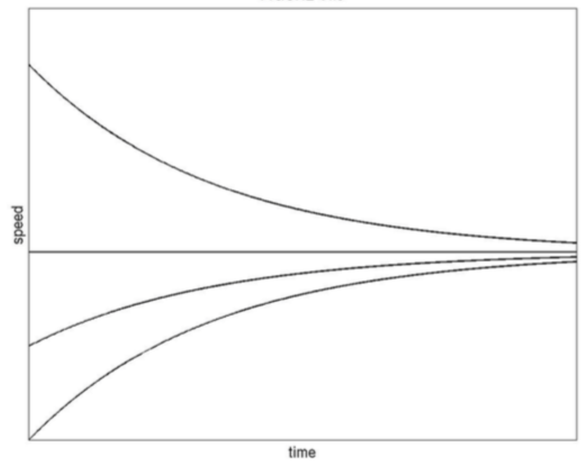
Students will have seen equations similar to this before in other branches of physics - e.g. growth of charge in a capacitor or growth of current in an inductor. That is why learning physics becomes easier all the time, because you have seen it all before in quite different contexts. Perhaps you have already noticed that third-year physics is easier than second-year physics; just think how much easier fourth-year is going to be! At any rate, v approaches the terminal speed asymptotically, never quite reaching it, but reaching half of the terminal speed in time $\frac{\ln 2}{\gamma} = \frac{.693}{\gamma}$ (you have seen that before while studying radioactive decay), and reaching $(1 - e^{-1})$ = 63% of the terminal speed in time $\frac{1}{\gamma}$.

If the body is thrown downwards, so that its initial speed is not zero but is $v = v_0$ when $t = 0$, you will write the equation of motion either as Equation 6.3.10 or as Equation 6.3.11, depending on whether the initial speed is slower than or faster than the terminal speed, thus ensuring that the denominator is kept firmly positive. In either case, the result is

$$v = \hat{v} + (v_0 - \hat{v})e^{-\gamma t} \tag{6.3.13}$$

Figure VI.6 shows v as a function of t for initial conditions $v_0 = 0, \frac{1}{2}\hat{v}, \hat{v}, 2\hat{v}$.

FIGURE VI.6

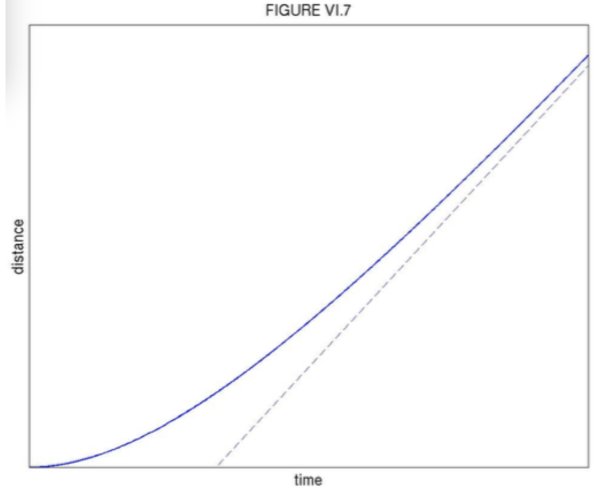


Returning to the initial condition $v = 0$ when $t = 0$, we readily find the second time integral to be

$$y = \hat{v}t - \frac{\hat{v}}{\gamma}(1 - e^{-\gamma t}). \tag{6.3.14}$$

You should check whether this equation is what is expected for when $t = 0$ and when t approaches infinity. The second time integral is shown in Figure VI.7.

FIGURE VI.7



The space integral is found either by eliminating t between the first and second time integrals, or by writing $\ddot{y}$ as $v\frac{dv}{dy}$ in the equation of motion:

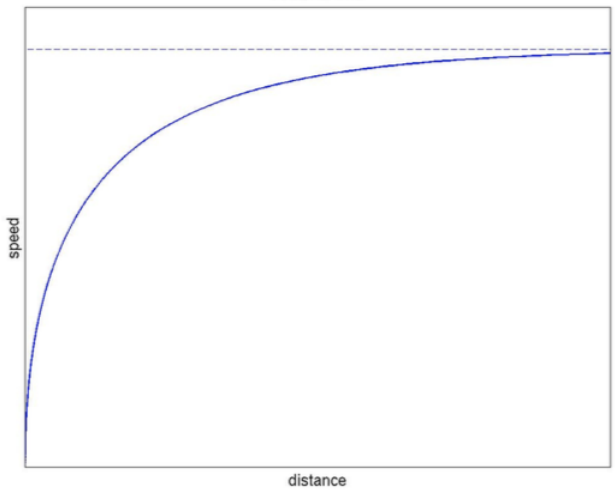
$$v\frac{dv}{dy} = \gamma(\hat{v} - v), \tag{6.3.15}$$

whence

$$y=\frac{\hat{v}}{\gamma}\ln(1-\frac{v}{\hat{v}})-\frac{v}{\gamma}. \tag{6.3.16}$$

This is illustrated in Figure VI.8. Notice that the equation gives y as a function of v, but only numerical calculation will give v for a given y.

FIGURE VI.8



Exercise 6.3B.1

Assume g = 9.8 m s^{-2}. A particle, starting from rest, is dropped through a medium such that the terminal speed is 9.8 m s^{-1}. How long will it take to fall through 9.8 m?

Solution

We are asked for t, given y, and we know the equation relating t and y- it is the *second time integral*, Equation 6.3.14 - so what could be easier? We have $\gamma=\frac{g}{\hat{v}}$ = 1s^{-1}, so Equation 6.3.14 becomes

$$9.8=9.8t-9.8(1-e^{-t}) \tag{6.3.17}$$

and suddenly we find that it is not as easy as expected!

The equation can be written

$$f(t)=t+e^{-t}-2=0. \tag{6.3.18}$$

For Newton-Raphson iteration we need

$$f'(t)=1-e^{-t}. \tag{6.3.19}$$

and, after some rearrangement, the Newton-Raphson iteration $(t\rightarrow t-\frac{f}{f'})$ becomes

$$t=\frac{1-t}{e^t-1}+2. \tag{6.3.20}$$

(It may be noticed that 6.3.20, which derives from the Newton-Raphson process, is merely a rearrangement of Equation 6.3.18)

Starting with an exceedingly stupid first guess of t = 100 s, the iterations proceed as follows:

t = 100.000 000 000

2.000 000 000

1.843 482 357

1.841 406 066

1.841 405 661

1.841 405 660 s

Exercise $6.3B.2$

Assume g = 9.8 m s^{-2} . A particle, starting from rest, falls through a resisting medium, the damping constant being γ= 1.96 s^{-1} (i.e. $\hat{v}$ = 5 m s^{-1}). How fast is it moving after it has fallen 0.3 m?

Solution

We are asked for v, given y. We want the space integral, Equation 6.3.16. On substituting the data, we obtain

$$f(v) = 5\ln(1-0.2v)+v+0.588 = 0. \tag{6.3.21}$$

From this,

$$f'(v) = v/(v-5) \tag{6.3.22}$$

The Newton-Raphson process $(t \rightarrow t - \frac{f}{f'})$, after some algebra, arrives at

$$v = \frac{u(5\ln(0.2u)+0.588)}{v}+5 = \frac{u(5\ln u - 7.459189560)}{v}+5, \tag{6.3.23}$$

where $u = 5-v$.

This time Newton-Raphson does not allow us the luxury of an exceedingly stupid first guess, but we know that the answer must lie between 0 and 5 m s^{-1} , so our moderately intelligent first guess can be v=2.5ms^{-1} .

Newton-Raphson iterations:

v = 2.500 000 000

2.122 264 100

2.051 880 531

2.049 766 247

2.049 764 400 m s^{-1}

Problems

Here are four problems concerning a body falling from rest such that the resistance is proportional to the speed. Assume that v = 9.8 m s^{-2} . Answers to questions 6.3.3 - 6.3.6 are to be given to a precision of 0.0001 seconds.

Exercise $6.3B.3$

A particle falls from rest in a medium such that the damping constant is γ = 1.0 s^{-1}. How long will it take to fall through 10 m?

Exercise $6.3B.4$

It takes t seconds to fall through y metres. Construct a table showing t for 201 values of y going from 0 to 20 metres in steps of 0.1 metre, assuming that γ = 1.0 s^{-1}.

Exercise $6.3B.5$

Construct a table showing t for 201 values of y going from 0 to 20 metres in steps of 0.1 metres for γ = 0.0, 0.5, 1.0, 1,5, 2.0 s . The table is to have six columns. The first column gives the distance fallen to a precision of 0.1 metres. The remaining five columns will give the time, to a precision of 0.0001 seconds, that the body takes to fall a given distance, to a precision of 0.0001 seconds

Exercise $6.3B.6$

Draw, by computer, a graph showing t (the dependent variable, plotted vertically) versus y (plotted horizontally) for the five values of γ in question 3.

These four problems are in order of increasing difficulty. The first is merely an exercise in solving an implicit equation (Equation 6.3.14) numerically, and might serve as an introductory example of how, for example, to solve an equation by Newton-Raphson iteration (I make the answer 1.8656 s.) The last two, if started from scratch, could well take up an entire afternoon before it is solved to one's complete satisfaction. It might be observed that the graphs of question 4 could be drawn fairly easily by calculating y explicitly as a function of t, thus obviating the necessity of Newton-Raphson iteration. No such short cuts, however, can be made for constructing the table of question 3.

In fact I solved questions 3 and 4 in just a few minutes − but I did not start from scratch. As you progress through your scientific career, you will become aware that there are certain operations that you encounter time and time again. To do questions 3 and 4, for example, you need to be able to solve an equation by Newton-Raphson iteration; you need to be able to construct a table of a function $y = f(x \quad ; \quad a)$, or in this case $t = f(y \quad ; \quad \gamma)$; and you need to be able to instruct a computer to draw graphs of tabulated values. I learned long ago that all of these are problems that crop up frequently, and I therefore long ago wrote short programs (only a few lines of Fortran each) for doing each of them. All I had to do on this occasion was to marry these existing programs together, tailored to the particular functions needed. Likewise a student will recognize similar problems for which he or she frequently needs a solution. You should accumulate and keep a set of these small programs for use in the future whenever you may need them. For example, this is by no means the last time you will need Newton-Raphson iteration to solve an equation. Write a Newton-Raphson program now and keep it for future occasions!

6.3C: Body thrown vertically upwards with initial speed v_0

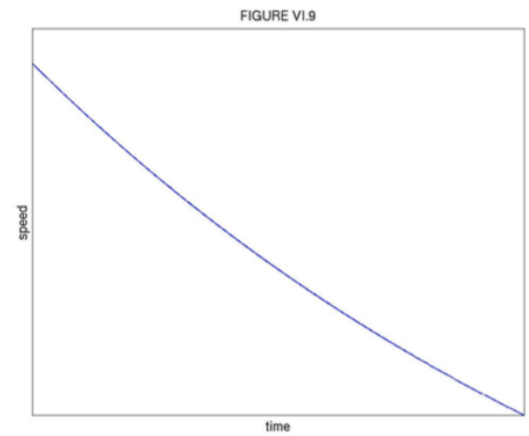
If we measure y upwards from the ground, the equation of motion is

$$\ddot{y} = -g - \gamma v = -\gamma(\hat{v} + v). \tag{6.3.24}$$

The *first time integral* is

$$v = -\hat{v} + (v_0 + \hat{v})e^{-\gamma t} \tag{6.3.25}$$

and this is shown in Figure VI.9.



It reaches a maximum height after time T, when $v = 0$ (at which time the acceleration is just $-g$):

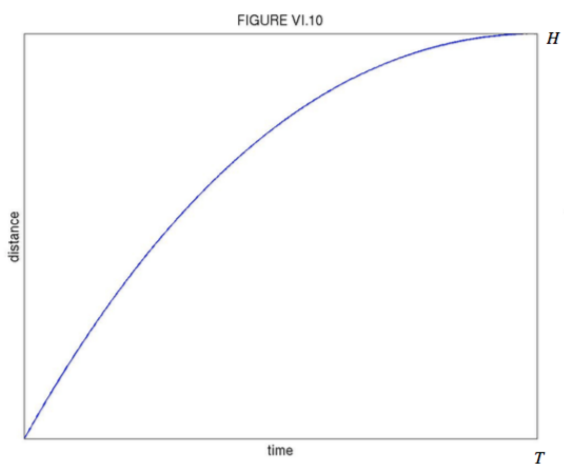
$$t + \frac{1}{\gamma}\ln(1 + \frac{v_0}{\hat{v}}). \tag{6.3.26}$$

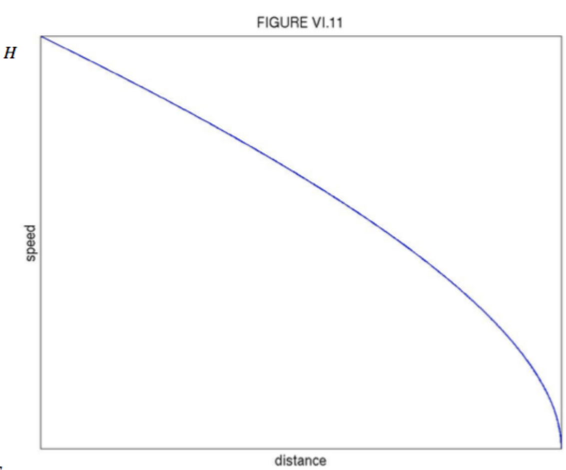
The *second time integral* (obtained by writing v as $\frac{dy}{dt}$ in Equation 6.3.25) and the *space integral* (obtained by writing $\ddot{y}$ as $v\frac{dv}{dy}$ in the equation of motion) require some patience, but the results are

$$y = \frac{(v_0 + \hat{v}}{\gamma}(1 - e^{-\gamma t} - \hat{v}t, \tag{6.3.27}$$

$$v = v_0 - \gamma y - \hat{v}\ln\left(\frac{\hat{v} + v_0}{\hat{v} + v}\right). \tag{6.3.28}$$

These are illustrated in Figures VI.10 and VI.11.





SECTION OVERVIEW

6.4: Motion in which the Resistance is Proportional to the Square of the Speed

There are not really any new principles; it is just a matter of practice with slightly more difficult integrals. We assume that the resistive force per unit mass is $k\dot{x}^2$. Here, although k plays a somewhat similar role to the γ of Section 3, it is not exactly the same thing as γ, and indeed it is not dimensionally the same as γ. What are the dimensions, and the SI units, of k?

Topic hierarchy

6.4A: Resistive Force Only

6.4A: Resistive Force Only

We'll imagine a puck sliding along a frictionless surface against turbulent air resistance. The Equation of motion is:

$$\ddot{x} = -kv^2. \tag{6.4A.1}$$

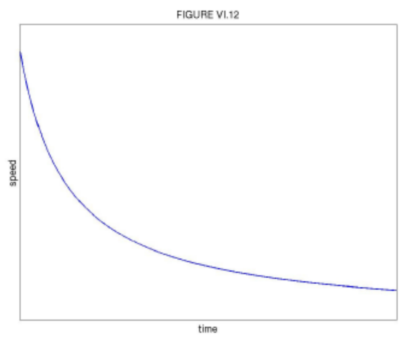
By this time we assume that the student knows how to obtain the first and second time integrals and the space integral. The actual integrations may be slightly more difficult, but we leave it to the reader to obtain the results

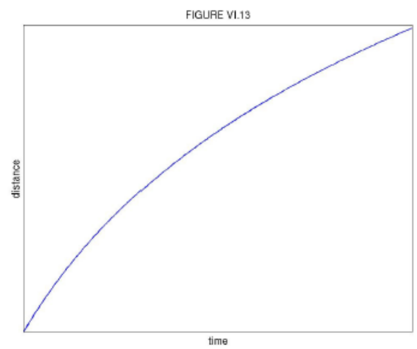
$$v = \frac{v_0}{1 + kv_0 t}$$

$$x = \frac{\ln(1 + kv_0 t)}{k}$$

$$v = v_0 e^{-kx}.$$

These are illustrated in Figures VI.12,13,14. Note that, provided that Equation 6.4A.1 accurately describes the entire motion (which may not be the case in a practical situation), there is no finite limit to x, nor does the speed drop to zero in any finite time.





CHAPTER OVERVIEW

7: Projectiles

Topic hierarchy

7.1: No Air Resistance

We suppose that a particle is projected from a point O at the origin of a coordinate system, the y-axis being vertical and the x-axis directed along the ground. The particle is projected in the xy-plane, with initial speed V_0 at an angle α to the horizon. At any subsequent time in its motion its speed is V and the angle that its motion makes with the horizontal is ψ.

The initial horizontal component if the velocity is $V_0\cos\alpha$, and, in the absence of air resistance, this horizontal component remains constant throughout the motion. I shall also refer to this constant horizontal component of the velocity as u. I.e. $u = V_0\cos\alpha =$ constant throughout the motion.

The initial vertical component of the velocity is $V_0\cos\alpha$, but the vertical component of the motion is decelerated at a constant rate g. At a later time during the motion, the vertical component of the velocity is $V_0\cos\psi$, which I shall also refer to as v.

In the following, I write in the left hand column the horizontal component of the equation of motion and the first and second time integrals; in the right hand column I do the same for the vertical component.

Horizontal	Vertical	
$\ddot{x} = 0$	$\ddot{y} = -g$	$7.1.1a,b$
$\dot{x} = u = V_0\cos\alpha$	$\dot{x} = u = V_0\cos\alpha$	$7.1.2a,b$
$\dot{x} = u = V_0 t\cos\alpha$	$\dot{x} = u = V_0 t\cos\alpha - \frac{1}{2}gt^2$	$7.1.3a,b$

Equations $7.1.3a,b$ are the parametric equations to the trajectory. In vector form, these two equations could be written as a single vector equation:

$$r = V_0 t + \frac{1}{2}gt^2 \tag{7.1.4}$$

Note the + sign on the right hand side of Equation 7.1.4. The vector **g** is directed downwards.

The xy-equation to the trajectory is found by eliminating t between Equations $7.1.3a$ and $7.1.3b$ to yield:

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2 cos^2\alpha} \tag{7.1.5}$$

Now, re-write this in the form

$$x^2 - ()x = -()y$$

Add to each side (half the coefficient of x)^2 in order to "complete the square" on the left hand side, and, after some algebra, it will be found that the equation to the trajectory can be written as:

$$(x-A)^2 = -4a(y-B), \tag{7.1.6}$$

where

$$A = \frac{V_0^2\sin\alpha\cos\alpha}{g} = \frac{V_0\sin 2\alpha}{2g} \tag{7.1.7}$$

$$B = \frac{V_0^2\sin^2\alpha}{2g} \tag{7.1.8}$$

and

$$a = \frac{V_0^2\cos^2\alpha}{2g} \tag{7.1.9}$$

Having re-arranged Equation 7.1.5 in the form 7.1.6, we see that the trajectory is a *parabola* whose vertex is at (A , B). The *range on the horizontal plane* is 2A, or $\frac{V_0^2\sin^2 2\alpha}{g}$ The greatest range on the horizontal plane is obtained when $\sin 2\alpha$ = 1, or α = 45°. The greatest range on the horizontal plane is therefore $\frac{V_0^2}{g}$ The *maximum height* reached is B, or $\frac{V_0^2\sin^2\alpha}{2g}$ The *distance between vertex* *and focus* is *a*, or $\frac{V_0^2\cos^2\alpha}{2g}$.The focus is above ground if this is less than the maximum height, and below ground if it is greater than the maximum height. That is, the focus is above ground if $\cos^2\alpha < \cos^2\alpha$. That is to say, the focus is above ground if α > 45^o and below ground if α < 45^o.

The radius of curvature ρ anywhere along the trajectory can be found using the usual formula $\rho = \frac{(1+y\prime^2)^{\frac{3}{2}}}{y\prime\prime}$. At the top of the trajectory, $y\prime = 0$, so that $rho = \frac{1}{y\prime}$ Alternatively (in case one has forgotten or is unfamiliar with the "usual formula"), we note that the speed at the top of the path is just equal to the (constant) horizontal component of the velocity $V_0\cos\alpha$. We can then equate the centripetal acceleration $V_0^2\cos^2\frac{\alpha}{\rho}$ to g and hence obtain:

$$\rho = \frac{V_0^2\cos^2\alpha}{g}. \tag{7.1.10}$$

By subtracting this from our expression for the maximum height of the projectile, we find that the height of the center of curvature above the ground is $\frac{V_0^2(1-3\cos^2\alpha)}{2g}$ The center of curvature is above ground if α > 54^o

44'.

The *range r on a plane inclined at an angle θ to the horizontal* can be found by substituting $x = r\cos\theta$ and $y = rsin\theta$ in the Equation 7.1.5 to the trajectory. This results, after some algebra, in

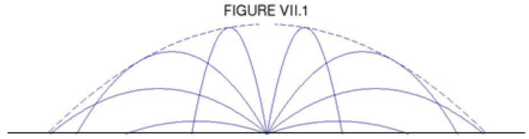
$$r = \frac{V_0^2}{gcos^2\theta}[\sin(2\alpha-\theta)-\sin\theta]. \tag{7.1.11}$$

This is greatest when $2\alpha-\theta$ = 90^o; *i.e.* when the angle of projection bisects the angle between the inclined plane and the vertical. The maximum range is

$$r = \frac{V_0^2}{g(1+\sin\theta)}. \tag{7.1.12}$$

This is the equation, in polar coordinates, of a *parabola*, and this parabola, when rotated about its vertical axis, describes a *paraboloid*, known as the *paraboloid of safety*. It is the envelope of all possible trajectories with an initial speed V_0. If a gun is firing shells with initial speed V_0, or a lawn sprinkler is ejecting water at initial speed V_0 , you are safe as long as you are outside the paraboloid of safety. Figure VII.1 shows trajectories for a = 20, 40, 60, 80, 100, 120, 140 and 160 degrees, and, as a dashed line, the paraboloid of safety. Notice how the range changes with a and that it is greatest for a = 45^o.

FIGURE VII.1



? Exercise 7.1.1

A gun projects a shell, in the absence of air resistance, at an initial angle α to the horizontal. The speed of projection varies with angle of projection and is given by

$$\text{Initial speed} = V_0\cos\frac{1}{2}\alpha$$

Show that, in order to achieve the greatest range on the horizontal plane, the shell should be projected at an angle to the horizontal whose cosine *c* is given by the solution of the equation

$$3c^3+2c^2-2c-1=0$$

Find the optimum angle to a precision of one arcminute.

7.2: Air Resistance Proportional to the Speed

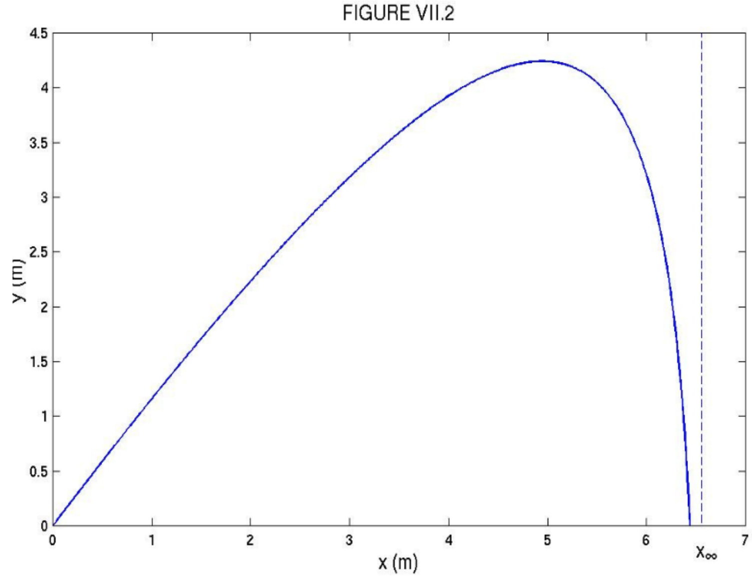
As in the previous section, I shall write the x-component of the equation of motion, and of the first and second time integrals, in the left hand column, and the y-component in the right-hand column. The x-component of the air resistance per unit mass is $\gamma\dot{x}$ and the y-component is $\gamma\dot{y}$. Here γ is the damping constant, defined in Section 6.3. The x- and y-components of the initial velocity are, respectively, $V_0\cos\alpha$ and $V_0\sin\alpha$. It should be readily seen that the equations of motion and their time integrals are as follows:

Horizontal	Vertical	
$\ddot{x} = -\gamma\dot{x}$	$\ddot{y} = -g - \gamma\dot{y}$	$7.2.1a, b$
$\dot{x} = u = V_0\cos\alpha.\, e^{-\gamma t}$	$\dot{y} = v = V_0\sin\alpha e^{-\gamma t} - \hat{v}(1 - e^{\gamma t})$ where $\hat{v} = g/\gamma$	$7.2.2a, b$
$x = x_\infty\,(1 - e^{-\gamma t})$ where $x_\infty = \frac{V_0\cos\alpha}{\gamma}$	$y = \frac{1}{\gamma}(V_0\sin\alpha + \hat{v})(1 - e^{-\gamma t}) - \hat{v}t$	$7.2.3a, b$

(In case it is not "readily seen", for the horizontal motion refer to Chapter 6, Section 3, especially Equations 6.3.2, 6.3.3 and 6.3.5, and for the vertical motion refer to Chapter 6, Section 3b, especially Equations 6.3.24, 6.3.25 and 6.3.27.) It will be seen that, as $t \to \infty$, $u \to 0$, $v \to -\hat{v}$, $x \to x_\infty$. The *xy-equation to the trajectory* is the t-eliminant of Equations $6.2.3a$ and $6.2.3b$. After a small amount of algebra this is found to be:

$$y = \frac{x(V_0\sin\alpha + \hat{v})}{V_0\cos\alpha} + \frac{\hat{v}}{\gamma}\ln\left(1 - \frac{x}{x_\infty}\right). \tag{7.2.4}$$

FIGURE VII.2



The *range on a horizontal plane* is found by setting y = 0, to obtain either

$$x = -Aln(1 - \frac{x}{x_\infty}) \tag{7.2.5}$$

or

$$x = x_\infty(1 - e^{-\frac{x}{A}}), \tag{7.2.6}$$

where

$$A = \frac{\hat{v}V_0 cos\alpha}{\gamma(V_0 sin\alpha + \hat{v})}, x_\infty = \frac{V_0 cos\alpha}{\gamma}$$

and

$$\hat{v} = \frac{g}{\gamma}.$$

Example 7.2.1

Suppose

- V_0 = 20 ms^{-1}
- α = 50°
- g =9.8 ms^{-2}
- γ =1.96s^{-1} (∴ $\hat{v}$=5 ms^{-1})

Then A =1.613 870 65 m

and x_∞ = 6.55905724 m.

Try to find the range on the horizontal plane, using either Equation 7.2.5 or 7.2.6, to nine significant figures. Which equation works best? Newton-Raphson may fail with a stupid first guess - but it should not be difficult to make a fairly intelligent first guess. I should not tell you, but figure VII.2 was calculated using the data of this example.

I make the answer 6.437 584 2 m.

Here's a more difficult problem.

Example 7.2.2

It is well known that, in the absence of air resistance, the maximum range on the horizontal plane is effected by choosing the initial launch elevation to be α = 45°. What if there is air resistance, with damping constant γ? What, then, should be the angle of launch to achieve the greatest range on the horizontal plane? Given Equation 7.2.6, $x = x_\infty(1 - e^{\frac{-x}{A}})$, for what value of α is x greatest?

Solution

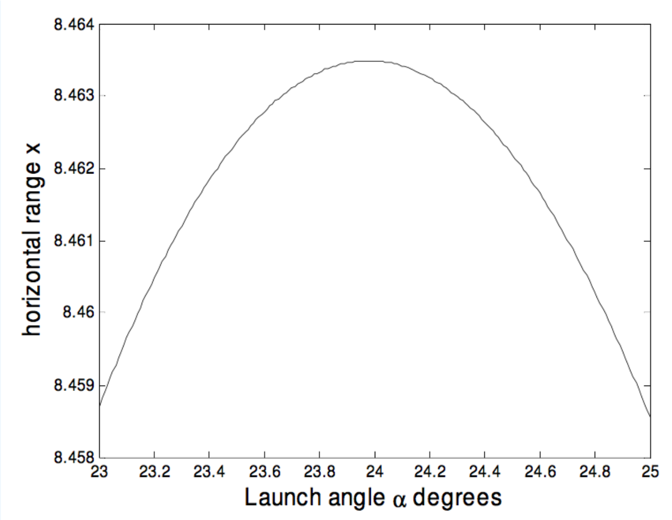
Equation 7.2.6, written in full, is

$$x = \frac{V_0 cos\alpha}{\gamma}\left[1 - exp\left(\frac{-\gamma(V_0\sin\alpha + \hat{v})x}{\hat{v}V_0\cos\alpha}\right)\right]. \tag{7.2.7}$$

This can be written

$$x = acos\alpha\left[1 - exp\left(-\frac{(b\sin\alpha + 1)x}{a\cos\alpha}\right)\right], \tag{7.2.8}$$

where $a = \frac{V_0}{\gamma}$ and $b = \frac{V_0}{\hat{v}} = \frac{\gamma V_0}{g}$. We have to find for what value of α is x greatest. It seems a simple enough problem, but at the moment I can't find a good way of solving it. If anyone has a clue, let me know (jtatum@uvic.ca). In the meantime, the best I can offer is, for our particular numerical example, to calculate the range, x, for several values of α and see where it goes through a maximum. For our particular numerical example, a = 10.204 081 63 m and b = 4. Here is a graph of range versus launch angle, for an initial speed of 20 ms^{-1}. A launch angle of about 23° 59′ gives a range of about 8.4635 m. For a given γ and g, the optimum launch angle depends on the launch speed V_0. Is this intuitively obvious?


8.464
8.463
8.462
8.461
8.46
8.459
8.458
horizontal range x
23
23.2
23.4
23.6
23.8
24
24.2
24.4
24.6
24.8
25
Launch angle α degrees

7.3: Air Resistance Proportional to the Square of the Speed

Notation: $\mathbf{V}$ is the velocity, V is the speed. The horizontal and vertical components of the velocity are, respectively, $u = \dot{x} = V\cos\psi$ and $v = \dot{y} = V sin\psi$. Here ψ is the angle that the instantaneous velocity $\mathbf{V}$ makes with the horizontal. The resistive force per unit mass is kV^2. The horizontal and vertical components of the resistive force per unit mass are $kV^2 cos\psi$ and $kV^2 sin\psi$ respectively. The launch speed is V_0 and the launch angle (i.e. the initial value of ψ) is α. Distance traveled from the launch point, measured along the trajectory, is s and speed $V = \dot{s}$. The Equations of motion are:

Horizontal:

$$\ddot{x} = -kV^2\cos\psi \tag{7.3.1}$$

Vertical:

$$\ddot{y} = -g - kV^2\sin\psi \tag{7.3.2}$$

These cannot be integrated as conveniently as in the previous cases, but we can get a simple relation between the horizontal component u of the speed and the intrinsic coordinate s. Thus, when we make use of $\ddot{x} = \dot{u}$, $V = \dot{s}$ and $V cos\psi = u$, Equation 7.3.1 takes the form

$$\dot{u} = -ku\dot{s} \tag{7.3.3}$$

Integration, with initial condition $u = V_0 cos\theta$, yields

$$u = V_0\cos\alpha . e^{-ks}. \tag{7.3.4}$$

We can also obtain an exact explicit *intrinsic Equation* to the trajectory by consideration of the *normal* Equation of motion.

The *intrinsic Equation* to any curve is a relation between the *intrinsic coordinates* (s, ψ). The rate at which the slope angle ψ changes as you move along the curve, i.e. $\frac{d\psi}{ds}$, is called the *curvature* at a point along the curve. If the slope is increasing with s, the curvature is positive. The reciprocal of the curvature at a point, $\frac{ds}{d\psi}$, is the radius of curvature at the point, denoted here by ρ.

The *normal* Equation of motion is the Equation $F = ma$ applied in a direction normal to the curve. The acceleration appropriate here is the *centripetal acceleration* $\frac{V^2}{\rho}$ or $V^2\frac{d\psi}{ds}$.

In a direction normal to the motion, the air resistance has no component, and gravity has a component $-g\cos\theta$. (It is *minus* because the curvature is clearly negative.) The normal Equation of motion is therefore

$$V^2\frac{d\psi}{ds} = -g\cos\theta. \tag{7.3.5}$$

But

$$V = \frac{u}{\cos\psi} = \frac{V_0\cos\alpha . e^{-ks}}{\cos\psi} \tag{7.3.6}$$

Therefore

$$V_0^2\cos^2\alpha . e^{-2ks}\frac{d\psi}{ds} = -g\cos^3\psi. \tag{7.3.7}$$

Separate the variables, and integrate, with appropriate initial conditions:

$$\int_\alpha^\psi \sec^3\psi d\psi = -\frac{g}{V_0^2\cos^2\alpha}\int_0^s e^{2ks}ds \tag{7.3.8}$$

From here it is good integration practice to show that the intrinsic Equation is

$$\sec\psi\tan\psi - \sec\alpha\tan\alpha + \ln\left(\frac{\sec\psi + \tan\psi}{\sec\alpha + \tan\alpha}\right) = \frac{g}{kV_0^2\cos^2\alpha}(1 - e^{2ks}) \tag{7.3.9}$$

This Equation is of the form

$$\sec \psi \tan \psi+\ln (\sec \psi+\tan \psi)=A-B e^{2 k s} . \tag{7.3.10}$$

While it would be straightforward now to compute s as a function of ψ and hence to plot a graph of s versus ψ, we really want to show y as a function of x, and x and y as a function of time. I am indebted to Dario Bruni of Italy for the following analysis.

Let (x_1, y_1) be a point on the trajectory. When the projectile moves a short distance Δs, the new coordinates will be (x_2, x_2), where

$$x_2=x_1+\Delta s \cos \psi_1 \tag{7.3.11}$$

and

$$y_2=y_1+\Delta s \sin \psi_1, \tag{7.3.12}$$

provided that Δs is taken to be sufficiently small that the path between the two points is approximately a straight line. The calculation starts with $x_1 = y_1 = 0$ and $\psi = \alpha$. At each stage of the calculation, the new value of ψ can be calculated from Equation 7.3.10. This can be done easily, for example, by Newton-Raphson iteration, since the derivative of the left hand side of this equation with respect to ψ is just $2 \sec^3 \psi$. Thus, with a sufficiently small interval Δs, the shape of the trajectory can be built up point by point.

While this gives us the shape of the trajectory, it tells us nothing about the time. To do this, we can write the Equations of motion, Equations 7.3.1 and 7.3.2 in the forms

$$\ddot{x}=-k \dot{x} \sqrt{\dot{x}^2+\dot{y}^2} \tag{7.3.13}$$

and

$$\ddot{y}=-g-k \dot{y} \sqrt{\dot{x}^2+\dot{y}^2} . \tag{7.3.14}$$

Let (x_1, y_1) be a point on the trajectory. After a short time Δt, the new coordinates will be (x_2, y_2), where

$$x_2=x_1+\dot{x}_1 \Delta t+\frac{1}{2} \ddot{x}_1(\Delta t)^2 \tag{7.3.15}$$

and

$$y_2=y_1+\dot{y}_1 \Delta t+\frac{1}{2} \ddot{y}_1(\Delta t)^2, \tag{7.3.16}$$

provided that Δt is taken to be sufficiently small that the acceleration between the two instants of time is approximately constant. Also, the new velocity components are given by

$$\dot{x}_2=\dot{x}_1+\ddot{x}_1 \Delta t \tag{7.3.17}$$

and

$$\dot{y}_2=\dot{y}_1+\ddot{y}_1 \Delta t . \tag{7.3.18}$$

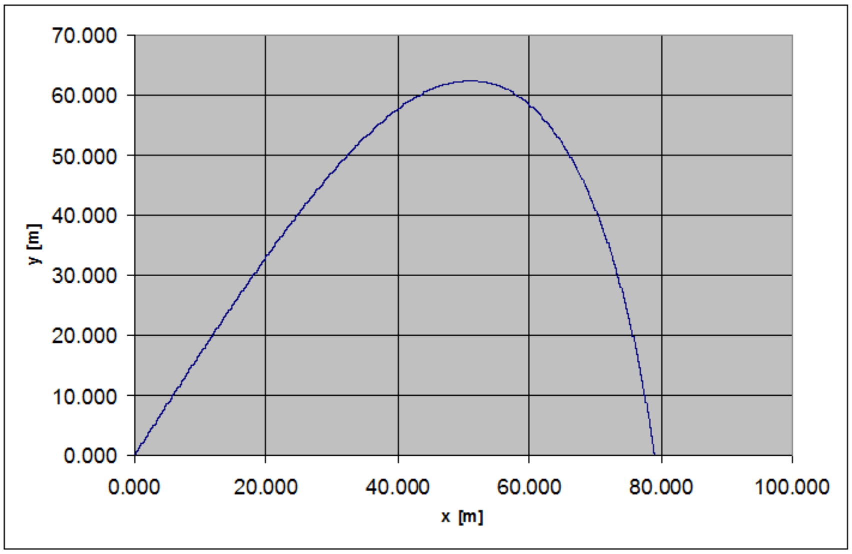
The calculation starts with

$$\dot{x}=V_0 \cos \alpha, \quad \dot{y}=V_0 \sin \alpha, \quad \ddot{x}=-k V_0^2 \cos \alpha, \quad \ddot{y}=-g-k V_0^2 \sin \alpha$$

and after each increment Δt the new coordinates and velocity and acceleration components are calculated. The results of Sr Bruni's calculations are shown in Figure VII.3 for

$$k = 0.0177\text{m}^{-1},\ V_0 = 90.5\text{ms}^{-1},\ \alpha = 60^\circ,\ g = 9.8\text{ms}^{-2}$$

FIGURE VII.3



Plotted with step by step method from intrinsic Equation with Δs= 0.025 m. Horizontal range 79.0 m; maximum height 62.4 m. Total flight duration 7.1 seconds. The time taken to reach the maximum height is 2.8 seconds, so the descent time is longer than the ascent time.

An alternative approach has been given by Ambrose Okune, of Uganda. In Okune's analysis, he obtains explicit expressions for t, x and $y$in terms of the angle ψ. (In Equation 7.3.10we already have a relation between s and ψ.)

We start with Equation 7.3.1, the horizontal Equation of motion

$$\ddot{x}=-kV^{2}\cos\psi=-kVV\cos\psi. \tag{7.3.19}$$

Now $\ddot{x}=\dot{u}$, $V=\sqrt{u^2+v^2}$, and $V\cos\psi=u$ so that

$$\dot{u}=-ku\sqrt{u^{2}+v^{2}}. \tag{7.3.20}$$

Similarly, Equation 7.3.2, the vertical Equation of motion, is

$$\ddot{y}=-g-kV^{2}\sin\psi=-g-kVV\sin\psi, \tag{7.3.21}$$

and, with $\ddot{y}=\dot{v}$, $V=\sqrt{u^2+v^2}$, and $V\sin\psi=v$,this becomes

$$\dot{v}=-g-kv\sqrt{u^{2}+v^{2}}. \tag{7.3.22}$$

Now

$$\frac{\dot{v}}{\dot{u}}=\frac{dv}{du}=\frac{v}{u}+\frac{g}{ku\sqrt{u^2+v^2}}. \tag{7.3.23}$$

Also v = $u\tan\psi$ so that

$$\frac{dv}{du}=\tan\psi+usec^{2}\frac{d\psi}{du}. \tag{7.3.24}$$

On comparison of Equations 7.3.23and 7.3.24, we see that

$$\frac{g}{ku\sqrt{u^2+v^2}}=u\sec^{3}\psi d\psi. \tag{7.3.25}$$

Upon substitution of v = $u\tan\psi$ this becomes

$$\frac{g}{ku^{3}}=\sec^{3}\psi\frac{d\psi}{du}. \tag{7.3.26}$$

and hence

$$\frac{g}{k}\int u^{-3}du=\int\sec^3\psi d\psi. \tag{7.3.27}$$

Upon integration, we obtain

$$\frac{g}{ku^2}+\ln(\sec\psi+\tan\psi)+\sec\psi\tan\psi=A=\frac{g}{ku_0^2}+\ln(\sec\alpha+\tan\alpha)+\sec\alpha\tan\alpha. \tag{7.3.28}$$

From this, we obtain

$$u=\sqrt{\frac{g}{k}}\frac{1}{\sqrt{A-\ln(sec\psi+\tan\psi)-sec\psi\tan\psi}}, \tag{7.3.29}$$

and hence

$$v=\sqrt{\frac{g}{k}}\frac{\tan\psi}{\sqrt{A-\ln(sec\psi+tan\psi)-\sec\psi\tan\psi}}. \tag{7.3.30}$$

Thus we now have the velocity components explicitly in terms of the angle y. For simplicity, let us write

$$\lambda=A-\ln(\sec\psi+\tan\psi)-\sec\psi\tan\psi. \tag{7.3.31}$$

Then the Equations for the velocity components are

$$u=\sqrt{\frac{g}{k}}\frac{1}{\sqrt{\lambda}} \tag{7.3.32}$$

and

$$v=\sqrt{\frac{g}{k}}\frac{\tan\psi}{\sqrt{\lambda}}. \tag{7.3.33}$$

In the limit, as $u\to 0$, $\psi\to -90^\circ$, $y\to-\infty$, the motion approaches a vertical asymptote. As $\psi\to-90^\circ$, $\lambda\to -sec\psi tan\psi$ and hence $\lim_{\psi\to-90^\circ}\frac{tan\psi}{\sqrt{\lambda}}=-1$. Thus the limiting value of the vertical component of the velocity is $-\sqrt{\frac{g}{k}}$. This agrees precisely with what one would expect for a body falling vertically at terminal speed, with resistance proportional to the square of the speed (see Equation 6.4.5).

We now aim to find an expression relating ψ to t, which we do by noting that

$$\frac{d\psi}{dt}=\frac{\frac{du}{dt}}{\frac{du}{d\psi}}=\frac{\frac{du}{dt}}{\frac{du}{d\lambda}\frac{d\lambda}{d\psi}}. \tag{7.3.34}$$

The derivative $\frac{du}{dt}$ can be found from the horizontal Equation of motion $\ddot{x}=-kV^2cos\psi$, which can be written (because $u=V$ and $\ddot{x}=\dot{u}$) as $\dot{u}=-ku^2sec\psi$. Then, making use of Equation 7.3.32, we obtain

$$\frac{du}{dt}=-\frac{g}{\lambda}\sec\psi \tag{7.3.35}$$

The derivative $\frac{du}{d\lambda}$ can be found from Equation 7.3.32 and is

$$\frac{du}{d\lambda}=-\frac{1}{2}\sqrt{\frac{g}{k}}\frac{1}{\lambda^{\frac{3}{2}}}. \tag{7.3.36}$$

The derivative $\frac{d\lambda}{d\psi}$ can be found from Equation 7.3.31 and is

$$\frac{d\lambda}{d\psi}=-2\sec^3\psi. \tag{7.3.37}$$

Thus the relation we seek is

$$\frac{d\psi}{dt} = -\sqrt{gk}\sqrt{\lambda}\cos^2\psi. \tag{7.3.38}$$

If the initial motion of the projectile at time zero makes an angle α with the horizontal, then integration of Equation 7.3.38 gives the following expression for the subsequent time t when the motion makes an angle ψ with the horizontal.

$$t = \frac{1}{\sqrt{gk}}\int_\psi^\alpha \frac{d\psi}{\sqrt{\lambda}\cos^2\psi}. \tag{7.3.39}$$

Also $u = \frac{dx}{dt} = \frac{dx}{d\psi}\frac{d\psi}{dt}$. With u and $\frac{d\psi}{dt}$ given respectively by Equations 7.3.32 and 7.3.38, we obtain

$$\frac{dx}{d\psi} = -\frac{1}{k\lambda\cos^2\psi}, \tag{7.3.40}$$

from which we can calculate x as a function of ψ:

$$x = \frac{1}{k}\int_\psi^\alpha \frac{d\psi}{\lambda\cos^2\psi}. \tag{7.3.41}$$

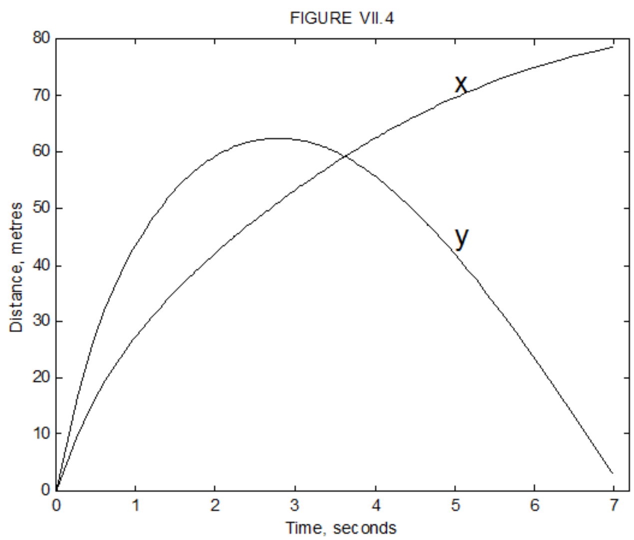
Further, $v = \frac{dy}{dt} = \frac{dy}{d\psi}\frac{d\psi}{dt}$. With v and $\frac{d\psi}{dt}$ given respectively by Equations 7.3.33 and 7.3.38, we obtain

$$\frac{dy}{d\psi} = -\frac{\tan\psi}{k\lambda\cos^2\psi}, \tag{7.3.42}$$

from which we can calculate y as a function of ψ:

$$y = \frac{1}{k}\int_\psi^\alpha \frac{\tan\psi d\psi}{\lambda\cos^2\psi}. \tag{7.3.43}$$

Equations 7.3.39, 7.3.41 and 7.3.43 enable us to calculate t, x and y as a function of ψ, and hence to calculate any one of them in terms of any of the others. In each case a numerical integration is required, such as by Simpson's rule or by Gaussian quadrature, or other integration algorithm, and, as is always the case, sufficient points must be sampled to obtain adequate precision. Numerical integration of these Equations, using the data of Dario Bruno's example above, produced the same $x : y$ trajectory as calculated for Figure VII.3 by Bruno, and the $x : t$ and $y : t$ relations shown in Figure VII.4.



I am greatly indebted to Dario Bruni and to Ambrose Okune for their interesting and instructive contributions to this section – an inspirational example of international scientific cooperation between, Italy, Uganda and Canada!

CHAPTER OVERVIEW

8: Impulsive Forces

Topic hierarchy

8.1: Introduction

As it goes about its business, a particle may experience many different sorts of forces. In Chapter 7, we looked at the effect of forces that depend only on the *speed* of the particle. In a later chapter we shall look at forces that depend only on the *position* of the particle. (Such forces will be called *conservative* forces.) In this chapter we shall look at the effect of forces that vary with *time*. Of course, it is quite possible that an unfortunate particle may be buffeted by forces that depend on its speed, on its position, and on the time - but, as far as this chapter is concerned, we shall be looking at forces that depend only on the time.

Everyone knows that Newton's second law of motion states that when a force acts on a body, the momentum of the body changes, and the rate of change of momentum is equal to the applied force. That is, $F = \frac{dp}{dt}$. If a force that varies with time, $F(t)$, acts on a body for a time T, the integral of the force over the time, $\int_0^T F(t)dt$ is called the *impulse* of the force, and it results in a *change of momentum* ΔP which is equal to the impulse. I shall use the symbol J to represent impulse, or the time integral of a force. Its SI units would be N s, and its dimensions MLT^{-1}, which is the same as the dimensions of momentum.

Thus, Newton's second law of motion is

$$F = \dot{p}.$$

When integrated over time, this becomes

$$J = \Delta p.$$

Likewise, in rotational motion, the *angular momentum* L of a body changes when a *torque* τ acts on it, such that the rate of change of angular momentum is equal to the applied torque:

$$\tau = \dot{L}.$$

If the torque, which may vary with time, acts over a time T, the integral of the torque over the time, $\int_0^T \tau dt$ is the *angular impulse*, which I shall denote by the symbol K, and it results in a change of the angular momentum:

$$K = \Delta L.$$

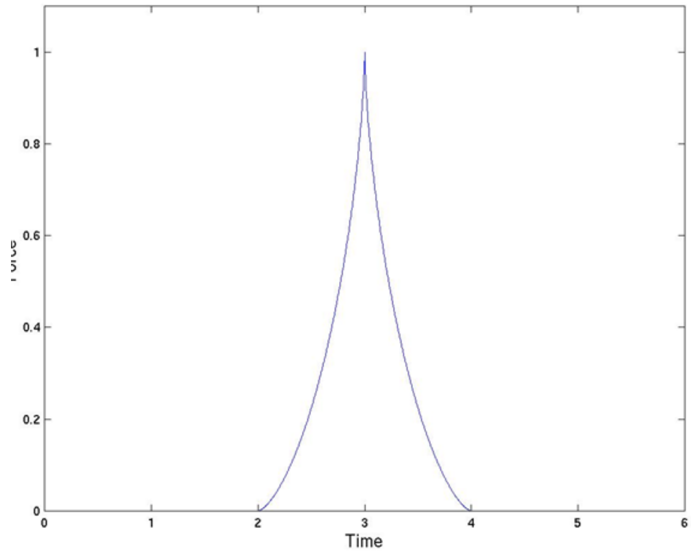
The SI units of angular impulse are N m s, and the dimensions are ML^{2}T^{-1}, which are the same as those of angular momentum.

For example, suppose that a golf ball is struck by a force varies with time as

$$F = \hat{F}\left[1 - \left(\frac{|t - t_0|}{\tau}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}$$

This may look like a highly-contrived and unlikely function, but in Figure VIII.1 I have drawn it for $\hat{F} = 1$, $t_0 = 3$, $\tau = 1$ and you will see that it is a reasonably plausible function. The club is in contact with the ball from time $t_0 - \tau$ to $t_0 + \tau$.

FIGURE VIII.1



If the ball, of mass m, starts from rest, what will be its speed V immediately after it leaves the club? The answer is that its new momentum, mV, will equal the impulse (or the time integral) of the above force:

$$mV = \hat{F}\int_{t_0-\tau}^{t_0+\tau}\left[1-\left(\frac{|t-t_0|}{\tau}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}} dt.$$

This is very easy to understand; if there is any difficulty it might be in the mechanics of working out this integral. It is good integration practice, but, if you can't do it after a reasonable effort, and you want a hint, ask me (jtatum@uvic.ca) and I'll see what I can do. You should get

$$mV = \frac{3\pi}{16}\hat{F}\tau = 0.589\hat{F}\tau$$

Example 8.1.1

Here is a very similar example, except that the integration is rather easier. A ball of mass 500 g, initially at rest, is struck with a force that varies with time as

$$F = \hat{F}\left[1-\left(\frac{t-t_0}{\tau}\right)^2\right]^{\frac{1}{2}},$$

where $\hat{F}$ = 4000N, t_0 = 10 ms, t = 3 ms. Draw (accurately, by computer) a graph of F versus time (it doesn't look quite like Figure VIII.1). How fast is the ball moving immediately after impact?

(I make it 37.7 m s^{-1}.)

8.2: Problem

In this section I offer a set of miscellaneous problems. In a typical problem it is assumed that an impulsive force or torque acts for only a very short time. By "a very short" time, I mean that the time during which the force or torque acts is very small or is negligible compared with other times that might be involved in the problem. For example, if a golf club hits a ball, the club is in contact with the ball for a time that is negligible compared with the time in which the ball is in the air. Or if a pendulum is subjected to an impulsive torque, the time during which the impulsive torque is applied is negligible compared with the period of the pendulum.

In many problems, you will be told that a body is subjected to an impulse J. The easiest way to interpret this is to say that the linear momentum of the body suddenly changes by J. Or you may be told that the body is subjected to an impulsive torque K. The easiest way to interpret this is to say that the angular momentum of the body suddenly changes by K.

In some of the problems, for example the first one, the body concerned is freely hinged about a fixed point; that is, it can freely rotate about that point.

Before giving the first problem, here is a little story. One of the most inspiring lectures I remember going to was one given by a science educator. She complained that a professor, instead of inspiring his students with a love and appreciation of the great and profound ideas of science and civilization, "infantalized" the class with a tiresome insistence that the class use blue pencils for velocity vectors, green for acceleration, and red for forces. I recognized immediately that this was a great way of imparting to students an appreciation of the profound ideas of physics, and I insisted on it with my own students ever since. In some of the following drawings I have used this colour convention, though I don't know whether your computer will reproduce the colours that I have used. In any case, I strongly recommend that you use the colour convention so deprecated by the educator if you want to understand the great ideas of civilization, such as the ideas of impulsive forces.

Exercise 8.2.1

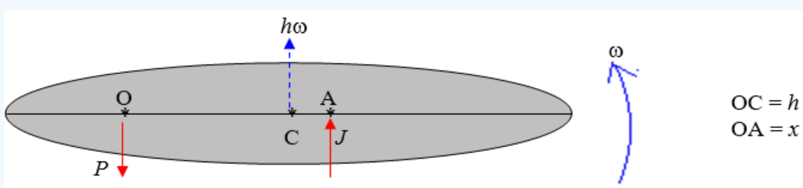


FIGURE VIII.2

In Figure VIII.2, a body is free to rotate about a fixed axis O. The centre of mass of the body is at C. The distance OC is *h*. The body is struck with a force of impulse J at A, such that OA = x. The mass of the body is m. Its rotational inertia about C is mk^2, and its rotational inertia about O is $m(k^2+h^2)$.

As a result of the blow, the body will rotate with angular speed w and the centre of mass will move forward with linear speed *h*w. *One of the questions* in this problem is to *calculate* ω

The body will also push with an impulsive force against the axis at O. It is not immediately obvious whether the body will push upwards against the axis in the same direction as J, or whether the left hand end of the body will swing downwards and the body will push downwards on the axis. You will probably agree that if A is very close to O, the body will push upwards on the axis, but if A is near the right hand end, the body will push downward on the axis. In the Figure, I am assuming that the *body* pushes *upwards* on the *axis*; the *axis* therefore pushes *downwards* on the *body*, with a force of impulse P, and what the Figure shows is the two impulsive forces that act *on the body*. *The second question* to be asked in this problem is to find P in terms of J and x.

If we are right in our intuitive feeling that P acts upwards or downwards according to the position of A - i.e. on where the body is struck - there is presumably some position of A such that the reactive impulse of the axis on the body is zero. Indeed there is, and the position of A that gives rise to zero reactive impulse at A is called the *centre of percussion*, and a *third question* in this problem is to find the position of the centre of percussion. Where on the bat should you hit the baseball if you want zero impulsive reaction on your wrists? Where should you position a doorstop so as to result in zero reaction on the door hinges? Never let it be said that theoretical physics does not have important practical applications. The very positioning of a door-stop depends on a thorough understanding of the principles of classical mechanics.

The net upward impulse is $J-P$, and this results in a change in linear momentum $mh\omega$:

$$J-P=mh\omega \tag{8.2.1}$$

The impulsive torque about O is Jx, and this results in a change in angular momentum $I\omega$; that is to say $m(k^2+h^2)\omega$:

$$Jx=m(k^2+h^2)\omega. \tag{8.2.2}$$

These two equations enable us to solve for the two unknowns ω and P. Indeed, Equation 8.2.2 gives us ω immediately, and elimination of ω between the two equations gives us P:

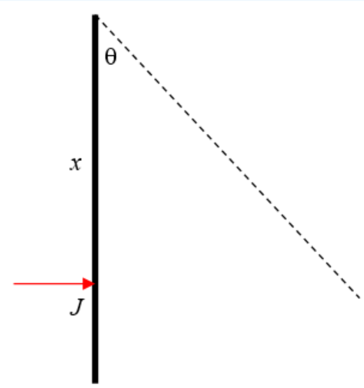
$$P=J\left(1-\frac{xh}{k^2+h^2}\right). \tag{8.2.3}$$

If the right hand side of Equation 8.2.3 is positive, then Figure VIII.2 is correct: the axis pushes down on the body, and the body pushes upwards on the axis. That is, P acts downwards if $x<\frac{k^2+h^2}{h}$, and upwards if $x>\frac{k^2+h^2}{h}$. The position of the centre of percussion is $x=\frac{k^2+h^2}{h}$.

If the body is a uniform rod of length l, O is at one end of the rod, then $k^2=\frac{1}{12}l^2$ so that, in this case, $x=\frac{2}{3}l$. This is where you should position a door-stop. It is also where you should hit a baseball with the bat - if the bat is a uniform rod. However, I admit to not knowing a great deal about baseball bats, and if such a bat is not a uniform rod, but is, for example, thicker and heavier at the distal than the proximal end, the centre of percussion will be further towards the far end.

? Exercise 8.2.2

FIGURE VIII.3



A heavy rod, of mass m and length $2l$, hangs freely from one end. It is given an impulse J as shown at a point at a distance x from the upper end. Calculate the maximum angular height through which the rod rises.

We can use Equation 8.2.2 to find the angular speed ω immediately after impact. In this equation, $m(k^2+h^2)$ is the rotational inertia of the rod about its end, which is $\frac{4ml^2}{3}$, so that

$$\omega=\frac{3Jx}{4ml^2}.$$

The kinetic energy immediately after impact is $\frac{1}{2}\cdot\frac{4}{3}ml^2\cdot\omega^2$ and we have to equate this to the subsequent gain in potential energy $mgl(1-\cos\theta)$.

Thus

$$\cos\theta=1-\frac{2l\omega^2}{3g}=1-\frac{2J^2x^2}{8gm^2l^3}.$$

To get the rod to swing through 180°, the angular impulse applied must be

$$Jx=4ml\sqrt{\frac{gl}{3}}.$$

Exercise 8.2.3

FIGURE VIII.4

A uniform rod of mass m and length $2l$ is freely hinged at one end O. A mass cm (where c is a constant) is attached to the rod at a distance x from O. An impulse J is applied to the other end of the rod from O. Where should the mass cm be positioned if the linear speed of the mass cm immediately after the application of the impulse is to be greatest?

The angular impulse about O is $2lJ$. The rotational inertia about O is $\frac{4}{3}ml^2 + cmx^2$. If w is the angular speed immediately after the blow, the angular momentum is $(\frac{4}{3}ml^2 + cmx^2)\omega$. If we equate this to the impulse, we find

$$\omega = \frac{6lJ}{m(4l^2+3cx^2)}.$$

The linear speed of the mass cm is x times this, or $\frac{6lJ}{m(4l^2+3cx^2)}$. By calculus, this is greatest when $x = \frac{2l}{\sqrt{3c}}$.

Exercise 8.2.4

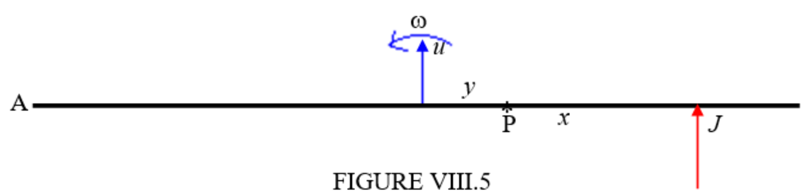


FIGURE VIII.5

A uniform rod is of mass m and length $2l$. An impulse J is applied as shown at a distance x from the mid-point of the rod. P is a point at a distance y from the mid-point of the rod. Does P move forward or backward? Which way does A move?

The first thing we can do is to find the linear speed u of the centre of mass of the rod and the angular speed ω of the rod. We do this by equating the impulse to the increase in linear momentum and the moment of the impulse (i.e. the angular impulse) to the increase in angular momentum:

$$J = mu$$

and

$$Jx = \frac{1}{3}ml^2\omega.$$

The forward velocity of P is $u + y\omega$. That is to say $\frac{J}{m} + \frac{3Jxy}{ml^2}$. This is positive if $y > -\frac{l^2}{3x}$ but negative otherwise. For the point A, $y = -l$, so that A will move forward if $x < \frac{l}{3}$, and it will move backwards otherwise.

Exercise 8.2.5

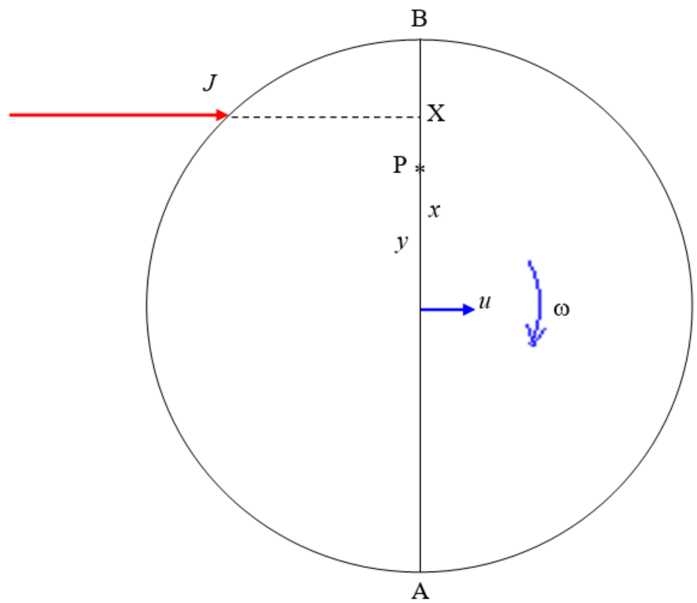


FIGURE VIII.6

A spherical planet, mass m, radius a, is struck by an asteroid with an impulse J as shown, the impact parameter being x. P is a point on the diameter, a distance y from the centre of the planet. Does P move forward or backward? Which way does A move?

As in the previous problem, we can easily find u and ω:

$$J = mu$$

and

$$Jx = \frac{2}{5}ma^2\omega.$$

The forward velocity of P is $u + y\omega$. That is to say $\frac{J}{m}(1 + \frac{5xy}{2a^2})$. This is positive if $y > -\frac{2a^2}{5x}$ but negative otherwise. For the point A, $y = -a$, so that A will move forward if $x < \frac{2a}{5}$, and it will move backward otherwise. That is to say A will move backwards if the blow is more that 70% of the way from A to B.

? Exercise 8.2.6

A hoop, radius a, mass m, moving at speed v, hits a kerb of height h as shown. Will it mount the kerb, or will it fall back?

The only impulse acts at the point A. The moment of the impulse about A is therefore zero, and therefore there is no change in angular momentum with respect to the point A.

Before impact, the angular momentum with respect to the point A is

$$mva + mv(a-h) = mv(2a-h) \quad .$$

Let ω be the angular speed about A after impact. The angular momentum about A is then $2ma^2\omega$. These are equal, so that

$$\omega = \frac{(2a-h)v}{2a^2}$$

If it is to mount the kerb, the new kinetic energy $\frac{1}{2}(2ma^2)\omega^2$ must be greater than the potential energy that is to be overcome, mgh.

Therefore

$$v > \frac{2a\sqrt{gh}}{2a-h} \, .$$

Exercise 8.2.7

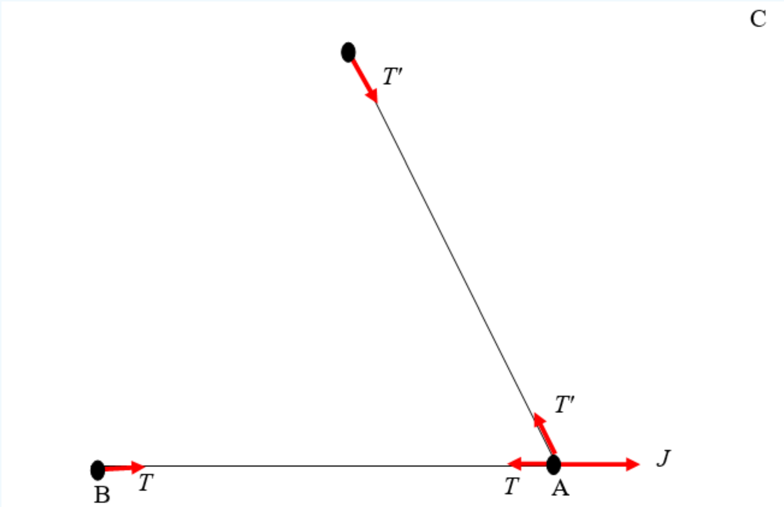


FIGURE VIII.8

Three equal particles, A, B and C, each of mass *m*, are joined by light inextensible strings as shown, the angle BAC being 60^{o}. A is given a sharp tug of impulse J as shown. Find the initial velocities of the particles and the initial tensions in the strings.

Let the initial tensions in BA and AC be T and T' respectively.

Let the initial velocity of A be $u\mathbf{i}+\mathbf{v}\mathbf{j}$.

Then the initial velocity of B is $u\mathbf{i}$

and the initial velocity of C is $\frac{1}{2}(u-\sqrt{3v})$ towards A.

The equations of impulsive motion are:

For B:

$$mu = T$$

For C:

$$\frac{1}{2}(u-\sqrt{3v}) = T'$$

For A:

$$mu = J - T - \frac{1}{2}T'$$

$$mv = \frac{1}{2}\sqrt{3}T'$$

The solutions of these equations are:

$$u = \frac{7J}{15m}, v = \frac{\sqrt{3}J}{15m},$$

$$T = \frac{7J}{15}, T' = \frac{2J}{15}.$$

Exercise 8.2.8

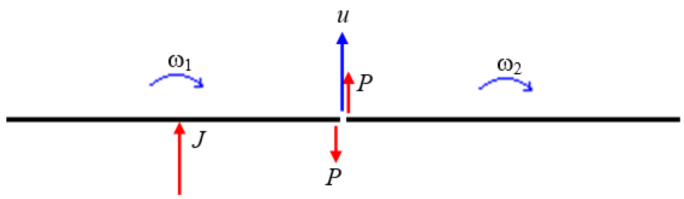


FIGURE VIII.9

Two rods, each of mass m and length $2l$, are freely jointed as shown. One of them is given an impulse J as shown. What happens then is that the end of one rod gives the end of the other an impulsive kick P, and the other gives the one an equal kick in the opposite direction. The centre of mass of the system moves forward with speed u and the two rods rotate with angular speeds ω_1 and ω_2. The problem is to determine P, u, ω_1 and ω_2.

The equations of impulsive motion are:

LH rod, translation:

$$J-P=m(u+l\omega_1)\ ,$$

RH rod, translation:

$$P=m(u+l\omega_2)$$

LH rod, rotation:

$$Pl=\tfrac{1}{3}ml^2\omega_1$$

RH rod, rotation:

$$Pl=\tfrac{1}{3}ml^2\omega_1$$

The solutions to these equations are:

$$u=\tfrac{J}{2m}\,,$$

$$\omega_1=\omega_2=\tfrac{3J}{8ml}\ ,$$

$$P=\tfrac{J}{8}\,.$$

Exercise 8.2.9

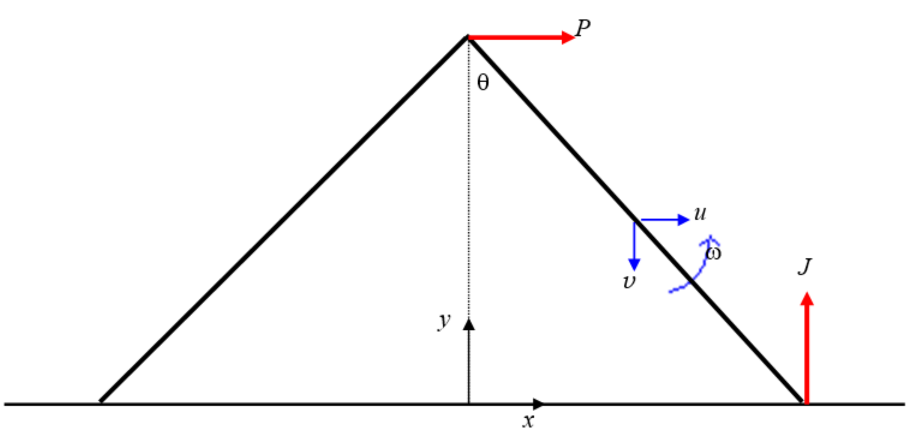


FIGURE VIII.10

Two rods, each of mass m and length $2l$, are freely jointed initially at right angles. They are dropped on to a smooth horizontal table and strike it with speed V. Find the rate $\dot{\theta}$ at which the rods splay apart immediately after impact.

We consider the dynamics of the right hand rod. On impact, it experiences a vertical impulse J at its lower end, and it experiences a horizontal impulse P (from the other rod) at its upper end. Immediately after impact, let the components of the velocity of the centre of mass of the right hand rod be u and v, and the angular speed of the rod is the required quantity $\dot{\theta}$.

From geometry, we have

$$x = l\sin\theta \text{ and } y = l\cos\theta$$

and hence

$$u = \dot{x} = l\cos\theta\dot{\theta} = \frac{l\dot{\theta}}{\sqrt{2}} \text{ and } v = -\dot{y} = l\sin\theta\dot{\theta} = \frac{l\theta}{\sqrt{2}} \ .$$

The impulsive equations of motion are

$$P = mu$$

and

$$J = m(V - v) \ .$$

After that, some algebra results in

$$\dot{\theta} = \frac{\sqrt{18}V}{8l}.$$

Exercise 8.2.10

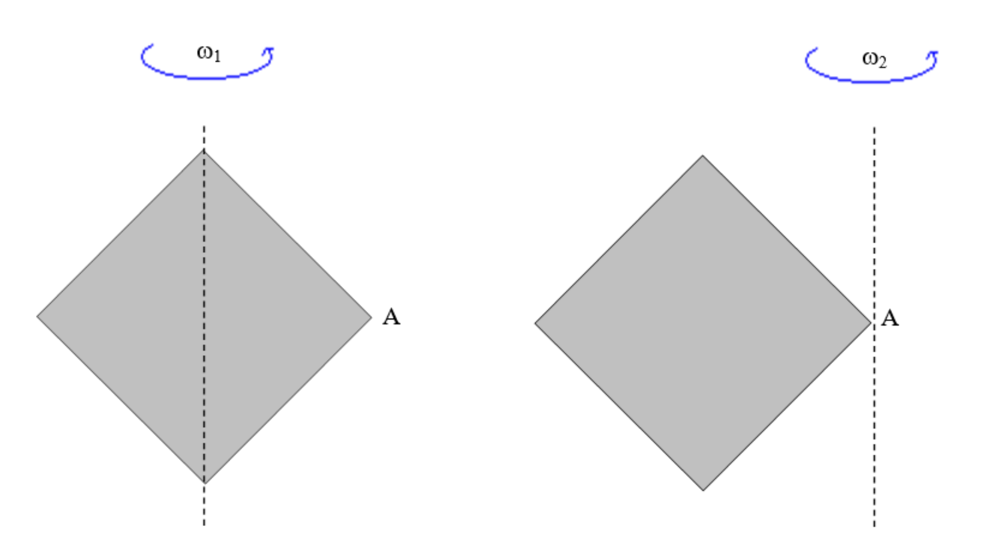


FIGURE VIII.11

A square plate is spinning about a vertical diameter at angular speed ω_1. It strikes a fixed obstacle at the corner A, so that it subsequently spins about a vertical axis through A at angular speed ω_2. Find ω_2.

Since the impulse is at A, the moment of the impulse about A is zero, so that angular momentum about A is conserved. The relevant moments of inertia can be calculated from the information in Chapter 2, and hence we obtain

$$\frac{1}{3}ma^2\omega_1 = \frac{7}{3}ma^2\omega_2 .$$
$$\omega_2 = \frac{\omega_1}{7} .$$

CHAPTER OVERVIEW

9: Conservative Forces

A conservative force is a force with the property that the work done in moving a particle between two points is *independent* of the taken path.

Topic hierarchy

9.1: Introduction

In Chapter 7 we dealt with forces on a particle that depend on the speed of the particle. In Chapter 8 we dealt with forces that depend on the time. In this chapter, we deal with forces that depend only on the position of a particle. Such forces are called *conservative forces*. While only conservative forces act, the sum of potential and kinetic energies is conserved.

Conservative forces have a number of properties. One is that the work done by a conservative force (or, what amounts to the same thing, the *line integral* of a conservative force) as it moves from one point to another is route-independent. The work done depends only on the coordinates of the beginning and end points, and not on the path taken to get from one to the other. It follows from this that the work done by a conservative force, or its line integral, round a closed path is zero. (If you are reminded here of the properties of a *function of state* in thermodynamics, all to the good.) Another property of a conservative force is that it can be derived from a potential energy function. Thus for any conservative force, there exists a scalar function $V(x, y, z)$ such that the force is equal to -**grad** V, or $-\nabla V$. In a one-dimensional situation, a sufficient condition for a force to be conservative is that it is a function of its position alone. In two- and three-dimensional situations, this is a necessary condition, but it is not a sufficient one. That a conservative force must be derivable from the gradient of a potential energy function and that its line integral around a closed path must be zero implies that the **curl** of a conservative force must be zero, and indeed a zero **curl** is a necessary and a sufficient condition for a force to be conservative.

This is all very well, but suppose you are stuck in the middle of an exam and your mind goes blank and you can't think what a line integral or a **grad** or a **curl** are, or you never did understand them in the first place, how can you tell if a force is conservative or not? Here is a rule of thumb that will almost never fail you: If the force is the tension in a stretched elastic string or spring, or the thrust in a compressed spring, or if the force is gravity or if it is an electrostatic force, the force is conservative. If it is not one of these, it is not conservative.

Example $9.1.1$

A man lifts up a basket of groceries from a table. Is the force that he exerts a conservative force?

Solution

No, it is not. The force is not the tension in a string or a spring, nor is it electrostatic. And, although he may be fighting against gravity, the force that he exerts with his muscles is not a gravitational force. Therefore it is not a conservative force. You see, he may be accelerating as he moves the basket up, in which case the force that he is exerting is greater than the weight of the groceries. If he is moving at constant speed, the force he exerts is equal to the weight of the groceries. Thus the force he exerts depends on whether he is accelerating or not; the force does not depend only on the position.

Example $9.1.2$

But you are not in an exam now, and you have ample time to remind yourself what a **curl** is. Each of the following two forces are functions of position only - a necessary condition for them to be conservative. But it is not a sufficient condition. In fact one of them is conservative and the other isn't. You will have to find out by evaluating the **curl** of each. The one that has zero **curl** is the conservative one. When you have identified it, work out the potential energy function from which it can be derived. In other words, find $V(x, y, z)$ such that $\mathbf{F} = -\mathbf{\nabla V}$.

i. $\mathbf{F} = (\mathbf{3x^2z - 3y^2z})\mathbf{i} - \mathbf{6xyzj} + (\mathbf{x^3 - xy^2})\mathbf{k}$

ii. $\mathbf{F} = \mathbf{ax^2yzi} - \mathbf{bxy^2zj} + \mathbf{cxyz^2k}$

When you have identified which of these forces is irrotational (i.e. has zero **curl**), you can find the potential function by calculating the work done when the force moves from the origin to (x,y,z) along any route you choose. Indeed, you might try more than one route to convince yourself that the line integral is route-independent.

One could devise many exercises in determining whether various force functions are conservative, and, if so, what the corresponding potential energy functions are, but I am going to restrict this chapter to just one more topic, namely

9.2: The Time and Energy Equation

Consider a one-dimensional situation in which there is a force $F(x)$ that depends on the one coordinate only and is therefore a conservative force. If a particle moves under this force, its equation of motion is

$$m\ddot{x} = F(x)$$

and we can obtain the space integral in the usual fashion by writing $\ddot{x}$ as $v\frac{dv}{dx}$.

Thus

$$mv\frac{dv}{dx} = F(x)$$

Integration yields

$$\frac{1}{2}mv^2 = \int F(x)dx + T_0$$

Here $\frac{1}{2}mv^2$ is called the *kinetic energy* and the integration constant T_0 can be interpreted as the initial kinetic energy. Thus the gain in kinetic energy is

$$T - T_0 = \int F(x)dx \tag{9.2.4}$$

the right hand side merely being the work done by the force.

Since F is a function of x alone, we can find a V such that $F = -\frac{dV}{dx}$. [It is true that we could also find a function V such that $F = +\frac{dV}{dx}$, but we shall shortly find that the choice of the minus sign gives V a desirable property that we can make use of.] If we integrate this equation, we find

$$V = -\int F(x)dx + V_0 \tag{9.2.5}$$

Here V is the *potential energy* and V_0 is the initial potential energy. From Equations 9.2.4 and 9.2.5 we obtain

$$V + T = V_0 + T_0 \tag{9.2.6}$$

Thus the quantity $V+T$ is conserved under the action of a conservative force. (This would not have been the case if we had chosen the + sign in our definition of V.) We may call the sum of the two energies E, the total energy, and we have

$$T = E - V(x) \tag{9.2.7}$$

or

$$\frac{1}{2}mv^2 = E - V(x) \tag{9.2.8}$$

With $v = \frac{dx}{dt}$, we obtain, by integrating Equation 9.2.8,

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E - V(x)}} \tag{9.2.9a}$$

This may at first appear to be a very formal and laborious way of arriving at something very obvious and something we have known since we first studied physics, but we shall see that it can often be a quite useful equation. You might, by the way, check that this equation is dimensionally correct.

The choice of the sign in Equation 9.2.9a may require some care, as will be evident in the examples that follow in the next section. If the particle is moving away from the origin, then its speed is $v = \frac{dx}{dt}$, and we choose the positive sign. If the particle is moving towards from the origin, then its speed is $v = -\frac{dx}{dt}$, and we choose the negative sign. However, I believe the following to be true: If the particle is moving away from the origin, then the initial value of x is smaller than the final value. If the particle is moving toward the origin, then the initial value of x is larger than the final value. It would seem to be safe, then, always to use the positive sign, but then the lower limit of integration is the smaller value of x (not necessarily the initial value), and the upper limit of integration is the larger value of x (not necessarily the final value). It may therefore be easier to write the equation in the form

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_{smaller}}^{x_{larger}}\frac{dx}{\sqrt{E-V(x)}} \tag{9.2.9b}$$

All that this means is that, for a conservative force, the time taken for a "return" journey is just equal to the time taken for the outbound journey, so one might as well always calculate the time for the outbound journey.

In some classes of problem such as pendulums, or rods falling over, the potential energy can be written as a function of an angle, and the kinetic energy is rotational kinetic energy written in the form $\frac{1}{2}I\omega^2$ where $\omega = \frac{d\theta}{dt}$. In that case, Equation 9.2.9b takes the form

$$t = \pm\sqrt{\frac{I}{2}}\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{E-V(\theta)}}. \tag{9.2.10}$$

You should check that this, too, is dimensionally correct.

9.3: Virtual Work

We have seen that a mechanical system subject to conservative forces is in equilibrium when the derivatives of the potential energy with respect to the coordinates are zero. A method of solving such problems, therefore, is to write down an expression for the potential energy and put the derivatives equal to zero.

A very similar method is to use the principle of virtual work. In this method, we imagine that we act upon the system in such a manner as to increase one of the coordinates. We imagine, for example, what would happen if we were to stretch one of the springs, or to increase the angle between two jointed rods, or the angle that the ladder makes as it leans against the wall. We ask ourselves how much work we have to do on the system in order to increase this coordinate by a small amount. If the system starts from equilibrium, this work will be very small, and, in the limit of an infinitesimally small displacement, this "virtual work" will be zero. This method is very little different from setting the derivative of the potential energy to zero. I mention it here, however, because the concept might be useful in Chapter 13 in describing Hamilton's variational principle.

Let's start by doing a simple ladder problem by the method of virtual work. The usual uniform ladder of high school physics, of length $2l$ and weight mg, is leaning in limiting static equilibrium against the usual smooth vertical wall and the rough horizontal floor whose coefficient of limiting static friction is μ. What is the angle θ that the ladder makes with the vertical wall?

I have drawn the four forces on the ladder, namely: its weight mg; the normal reaction of the floor on the ladder, which must also be mg; the frictional force, which is μmg; and the normal (and only) reaction of the wall on the ladder, which must also be μmg.

There are several ways of doing this, which will be familiar to many readers. The only small reminder that I will give is to point out that, if you wish to combine the two forces at the foot of the ladder into a single force acting upwards and somewhat to the left, so that there are then just three forces acting on the ladder, the three forces must act through a single point, which will be above the middle of the ladder and to the right of the point of contact with the wall. But we are interested now in solving this problem by the principle of virtual work.

Before starting, I should warn that it is important in using the principle of virtual work to be meticulously careful about signs, and in that respect I remind readers that in the differential calculus the symbols δ and d in front of a scalar quantity x do not mean "a small change in" or "an infinitesimal change" in x. Such language is vague. The symbols stand for "a small increase in" and "an infinitesimal increase in".

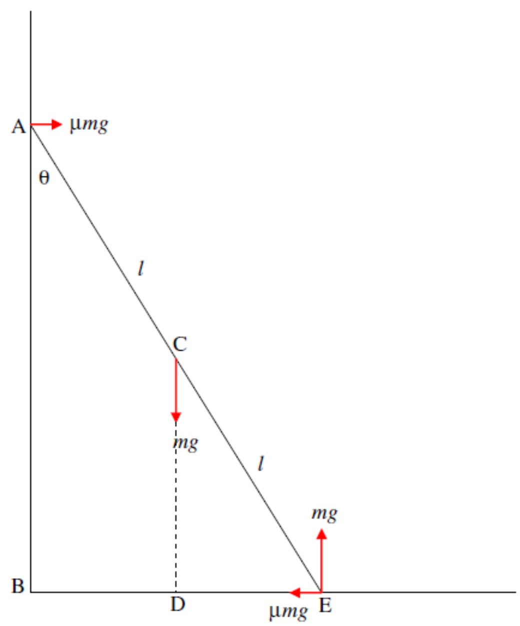


FIGURE IX.4

Let us take note of the following distances:

$$CD = l\cos\theta$$

and

$$BE = 2l\sin\theta.$$

If we were to **increase** θ by $\delta\theta$, keeping the ladder in contact with wall and floor, the **increases** in these distances would be

$$\delta(CD) = -l\sin\theta d\theta.$$

and

$$\delta(BE) = 2l\cos\theta d\theta.$$

Further, if were to **increase** θ by $\delta\theta$, the work done **by** the force at C would be mg times the decrease of the distance CD, and the work done **by** the frictional force at E would be *minus* μmg times the increase of the distance BE. The other two forces do no work. Thus the "virtual work" done **by** the external forces **on** the ladder is

$$mg.\,l\sin\theta\delta\theta - \mu mg.2l\cos\theta\delta\theta.$$

On putting the expression for the virtual work to zero, we obtain

$$\tan\theta = 2\mu.$$

You should verify that this is the same answer as you get from other methods – the easiest of which is probably to take moments about E.

There is something about virtual work which reminds me of thermodynamics. The first law of thermodynamics, for example is $\Delta U = \Delta q + \Delta w$, where ΔU is the **increase** of the internal energy of the system, Δq is the heat added **to** the system, and Δw is the work done **on** the system. Prepositions play an important part in thermodynamics. It is always mandatory to state clearly and without ambiguity whether work is done **by** the piston **on** the gas, or **by** the gas **on** the system; or whether heat is gained **by** the system or lost **from** it. Without these prepositions, all discussion is meaningless. Likewise in solving a problem by the principle of virtual work, it is always essential to say whether you are describing the work done **by** a force **on** what part of the system (on the ladder or on the floor?) and whether you are describing an increase or a decrease of some length or angle.

Let us move now to a slightly more difficult problem, which we'll try by three different methods – including that of virtual work.

In Figure IX.5, a uniform rod AB of weight Mg and length $2a$ is freely hinged at A. The end B carries a smooth ring of negligible mass. A light inextensible string of length l has one end attached to a fixed point C at the same level as A and distant $2a$ from it. It passes through the ring and carries at its other end a weight $\frac{1}{10}Mg$ hanging freely. (The "smooth" ring means that the tension in the string is the same on both sides of the ring.) Find the angle CAB when the system is in equilibrium.

I have marked in various angles and lengths, which can easily be determined from the geometry of the system, and I have also marked the four forces on the rod.

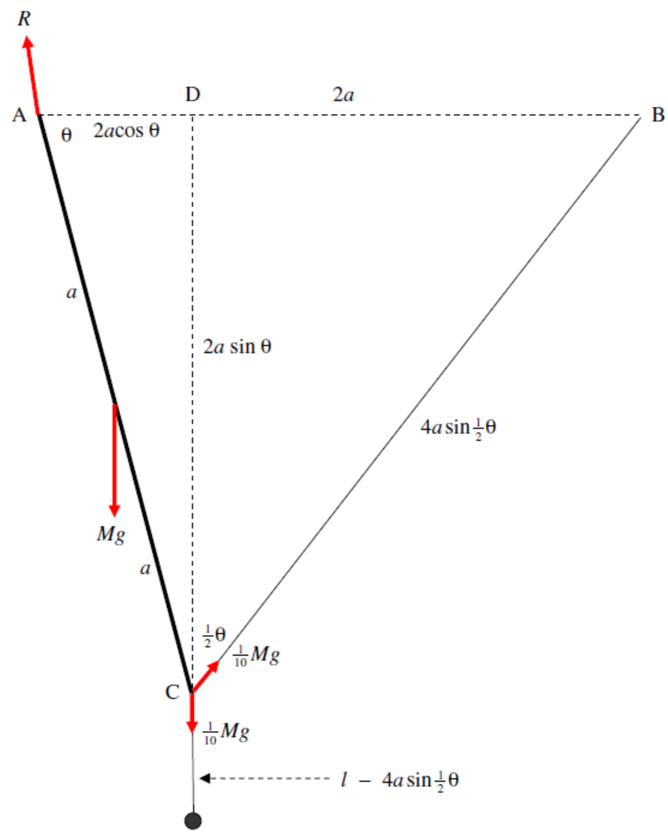


FIGURE IX.5

Let us first try a very conventional method. We know rather little about the force **R** of the hinge on the rod (though see below), and therefore this is a good reason for taking moments about the point A. We immediately obtain

$$Mga\cos\theta + \frac{1}{10}Mg.2a\cos\theta = \frac{1}{10}Mg.2a\cos\frac{1}{2}\theta.$$

Divide by Mga and set $\cos\theta = 2c^2 - 1$, where $c = \cos\frac{1}{2}\theta$. After a little algebra, we obtain $12c^2 - c - 6 = 0$ and hence we find for the equilibrium condition that θ = 82° 49' or 263° 37'. The latter, by the way, is a physically valid solution – you might want to sketch it.

Now let's try the same problem using energy conditions. We'll take the zero of potential energy when the rod is horizontal – at which time the small mass is at a distance l below the level AC.

When the angle CAB = θ, the distance of the centre of mass of the rod below AC is $a\sin\theta$ and the distance of the small mass below AC is $l - 4a\sin\frac{1}{2}\theta + 2a\sin\theta$ so that the potential energy is

$$V = -Mga\sin\theta + \frac{1}{10}Mg[l - (l - 4a\sin\frac{1}{2}\theta + 2a\sin\theta)] = -\frac{2}{3}Mga(3\sin\theta - \sin\frac{1}{2}\theta)$$

The derivative is

$$\frac{dV}{d\theta} = -\frac{2}{3}Mga(3\cos\theta - \frac{1}{2}cos\frac{1}{2}\theta),$$

and setting this to zero will produce the same results as before. Further differentiation (do it), or a graph of $V : \theta$ (do it), will show that the 82° 49' solution is stable and the 263° 37' solution is unstable.

Now let's try it by virtual work. We are going to increase θ by $\delta\theta$ and see how much work is done.

The distance of the centre of mass of the rod below AC is $a \sin \theta$, and if θ increases by $\delta\theta$, this will increase by $a \cos \theta \delta\theta$, and the work done by Mg will be $Mga \cos \theta \delta\theta$.

The distance of the ring below AC is $2a \sin \theta$, and if θ increases by $\delta\theta$, this will increase by $2a \cos \theta \delta\theta$, and the work done by the downward force will be $\frac{1}{10} Mg.2a \cos \theta \delta\theta$.

The distance BC is $4a \sin \frac{1}{2}\theta$, and if θ increases by $\delta\theta$, this will increase by $2a \cos \frac{1}{2}\theta \delta\theta$ and the work done by the sloping force will be ***MINUS*** $\frac{1}{10} Mg.2a \cos \frac{1}{2}\theta \delta\theta$.

Thus the virtual work is

$$Mg.\, a \cos \theta \delta\theta + \frac{1}{10} Mg.2a \cos \theta \delta\theta - \frac{1}{10} Mg.2a \cos \frac{1}{2}\theta \delta\theta.$$

If we put this equal to zero, we obtain the same result as before.

9.E: Conservative Forces (Exercises)

Exercise $9.E.1$

How long does it take a particle to fall to the ground from a height h, with a uniform acceleration g? From elementary methods you know that the answer is $\sqrt{\frac{2h}{g}}$, but we are going to use this simple example to illustrate the use of Equation 9.2.9. The use of the equation to solve such a simple problem might seem a rather tedious way of solving an elementary problem, but the method is useful in more difficult cases, and the use of the method is best introduced with a simple example.

Let us take the origin to be the ground, and we shall measure distances y upward from the ground. The speed is then given by $v=-\frac{dy}{dt}$, so we use the negative sign in Equation 9.2.9a. The initial and final values of y are h and 0 respectively. Alternatively, you can use Equation 9.2.9b, with the positive sign, in which case the loser and upper limits of integration are 0 and h respectively.

The total energy E is the initial potential energy mgh (since the initial kinetic energy is zero), and the potential energy at height y is $V(y)=mgy$ Equation 9.2.9 therefore takes the form

$$t=-\sqrt{\frac{m}{2}}\int_h^0\frac{dy}{\sqrt{mgh-mgy}}.$$

Exercise $9.E.2$

How long does it take for a particle, thrown vertically upwards with initial speed v_0, to reach a height h? Again, by elementary methods, you will easily find (do it!) that the answer is $t=\frac{v_0-\sqrt{v_0^2-2gh}}{g}$, but let's see if we can do it from Equation 9.2.9.

Let us take the origin to be the ground, and we shall measure distances y upward from the ground. The speed is then given by $v=+\frac{dy}{dt}$, so we use the positive sign in Equation 9.2.9. The initial and final values of y are 0 and h respectively.

The total energy E is the initial kinetic energy $\frac{1}{2}mv_0^2$(since the initial potential energy is zero), and the potential energy at height y is $V(y)=mgy$. Equation 9.2.9 therefore takes the form

$$t=+\sqrt{\frac{m}{2}}\int_0^h\frac{dy}{\sqrt{\frac{1}{2}mv_0^2-mgy}}.$$

This yields the expected answer

$$t=\frac{v_0-\sqrt{v_0^2-2gh}}{g}.$$

Exercise $9.E.3$

In this example, we shall have a stone falling from a height that is not negligible compared with the radius of the Earth, so that the acceleration is not constant. We shall suppose that we drop a stone from a point at a distance $r=b$ from the centre of the Earth, and we ask how long it will take to reach the surface of the Earth, radius a.

Let us take the origin to be the centre of the Earth, and we shall measure distances r radially outward from the centre. The speed is then given by $v=-\frac{dy}{dt}$, so we use the negative sign in Equation 9.2.9. The initial and final values of r are b and a respectively.

The total energy E is the initial potential energy $-\frac{GMm}{b}$(since the initial kinetic energy is zero), and the potential energy at distance r from the centre is $V(r)=-\frac{GMm}{r}$. Equation 9.2.9 therefore takes the form

$$t=-\sqrt{\frac{m}{2}}\int_b^a\frac{dr}{\sqrt{\frac{GMm}{r}-\frac{GMm}{b}}}\tag{9.E.1}$$

The gravity on the surface of Earth is $g_0 = \frac{GM}{a^2}$ so that equation 9.E.1 can be written

$$t = -\frac{1}{a}\sqrt{\frac{b}{2g_0}}\int_b^a \frac{dr}{\sqrt{\frac{b}{r}-1}} \tag{9.E.2}$$

This can be integrated analytically to give

$$t = \frac{b}{a}\sqrt{\frac{b}{2g_0}}\int_b^a (\alpha + \frac{1}{2}\sin 2\alpha), \tag{9.E.3}$$

where

$$\cos^2\alpha = \frac{a}{b}. \tag{9.E.4}$$

Here’s a numerical example: How long would it take for a stone to fall to Earth from an initial height of 240,000 miles? In case the above method is too difficult, here’s another way to do it - in your head in a few seconds!

240,000 miles is the radius of the Moon's orbit. The stone is falling in a highly elliptical orbit of major axis equal to the distance to the Moon - i.e. its semi major axis is half that of the Moon. Therefore by Kepler's third law, its period is equal to the Moon's period (which is 28 days) divided by $2\sqrt{2}$ which is 2.8. The orbital period of the stone is therefore 10 days. The time taken to drop to Earth is half of this, or five days. You might want to calculate it from Equations 9.E.3 and 9.E.4 and see if you get the same answer!

? Exercise $9.E.4$

The answer is that it is four times the time that it takes to rise from the vertical position to an angle α from the vertical. Thus we shall work out this time from Equation 9.2.10 and multiply by four.

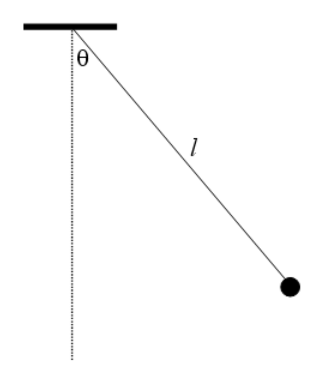


FIGURE IX.1

Let us adopt the upper end of the string as our level for zero potential energy. The potential energy $V(\theta)$ is then $-mgl\cos\theta$. The total energy E is equal to the potential energy when $\theta = \alpha$; that is to say, $E = -mgl\cos\alpha$. If we take the initial angle to be 0 and the final angle to be α, then the upward motion is such that $\omega = +\frac{d\theta}{dt}$, and we choose the positive sign for Equation 9.2.10. The rotational inertia I is ml^2. Thus Equation 9.2.10 gives

$$P = 4t = \sqrt{\frac{8l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}. \tag{9.E.5}$$

This can doubtless be expressed in terms of special functions that most of us are unfamiliar with, so you will probably opt to evaluate this numerically as a function of α. There is a small difficulty at the upper limit of the integration when the integrand becomes infinite. Indeed, in all cases in which the system is at rest at either the start or the finish, the denominator of Equation 9.2.9 or 9.2.10 will necessarily be zero and hence the integrand will be infinite. In some cases, as in examples **(i)** to **(iii)**, the integral can be done analytically and there is no problem. If, however, as in the present example, the integration has to be done numerically, there is a potential problem and some ingenuity (often by making a change of variable) will have to be exercised.

Using a trigonometric identity, you can write Equation 9.E.5 as

$$P=\sqrt{\frac{4l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\frac{1}{2}\alpha-\sin^2\frac{1}{2}\theta}}.$$

This doesn't get rid of the infinity, but now make a change of variable by letting

$$\sin\frac{1}{2}\theta=\sin\frac{1}{2}\alpha\sin\phi$$

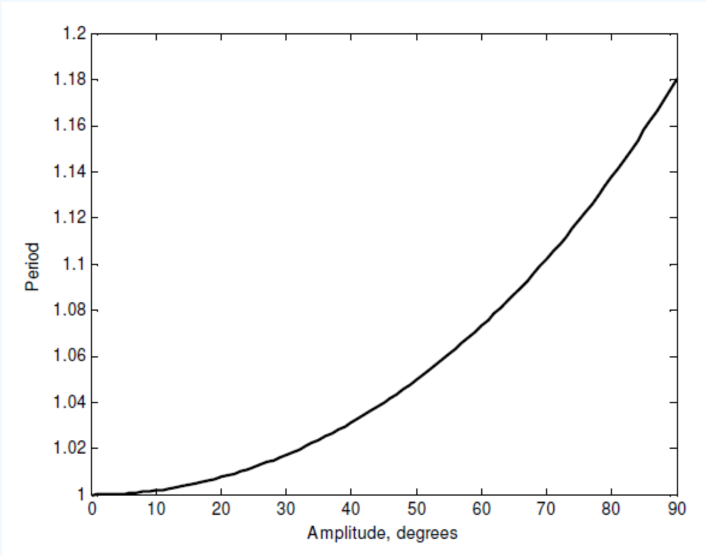
and the difficulty will disappear. In particular, the expression for the period becomes

$$P=\sqrt{\frac{l}{g}}\times\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\sin^2\frac{1}{2}\alpha.\sin^2\phi}}.$$

Below is a graph of the period, in units of $2\pi\sqrt{\frac{l}{g}}$ versus α.

A further question might be asked. For example: What is the amplitude of the pendulum swing such that the period is 10 percent more than the small angle limit of $2\pi\sqrt{\frac{l}{g}}$? In other words, solve the equation $\frac{\sqrt{2}}{\pi}\int_0^\alpha\frac{d\theta}{\sqrt{\cos\theta-\cos\alpha}}=1.1$ for α. This will be quite a challenge.

I make it α = 69º.325146



Exercise 9.E.5

A Semicircle, a Ring and a String

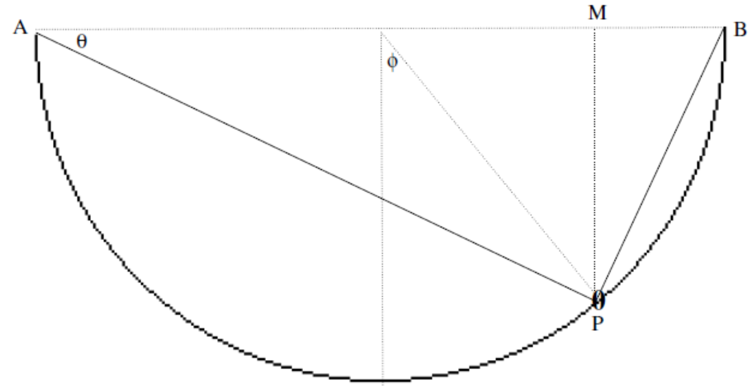


FIGURE IX.2

In this example, we have a smooth semicircular wire in a vertical plane. The radius of the semicircle is a. A ring of mass m at P can slide smoothly around the ring. An elastic string of natural length $2a$ is attached to the ends of the wire at A and B and is threaded through the ring. The force constant of the string is k. The ring is subjected to three conservative forces - gravity and the tensions in the two parts of the string. The ring is smooth, so there is no nonconservative friction.

By geometry the lengths of the following are

AP: $2a\cos\theta$

BP: $2a\sin\theta$

PM: $2a\sin\theta\cos\theta = a\sin 2\theta$

We are going to find the equilibrium position(s) of the ring, and see how long it takes to slide from one position on the semicircle to another.

Before doing any calculations, let's think about the physics qualitatively. Suppose that it is a very heavy ring and a weak string. In that case the ring will surely slide down to the bottom of the semicircle and stay there. On the other hand suppose that the ring is not very heavy but the string is quite strong. In that case we may well imagine that the ring may rest in stable equilibrium farther up the semicircle; indeed, if the string is very strong, the stable equilibrium position of the ring might be quite near the top. Of course, by the symmetry of the situation, there will always be an equilibrium position at the bottom of the semicircle, but, if the string is very strong, this position will be unstable, and, upon the slightest displacement, the ring will snap up to a higher position. Whether the position at the bottom of the semicircle will be stable or unstable depends on the relative strengths of the weight of the ring and the tension in the string. Let us then, in anticipation, refer to the ratio $\frac{mg}{ka}$ by the symbol λ. In fact we shall find that if $\lambda > 0.586$, the position at the bottom of the semicircle will be stable, but if λ is less than this the positions of stable equilibrium will be higher up.

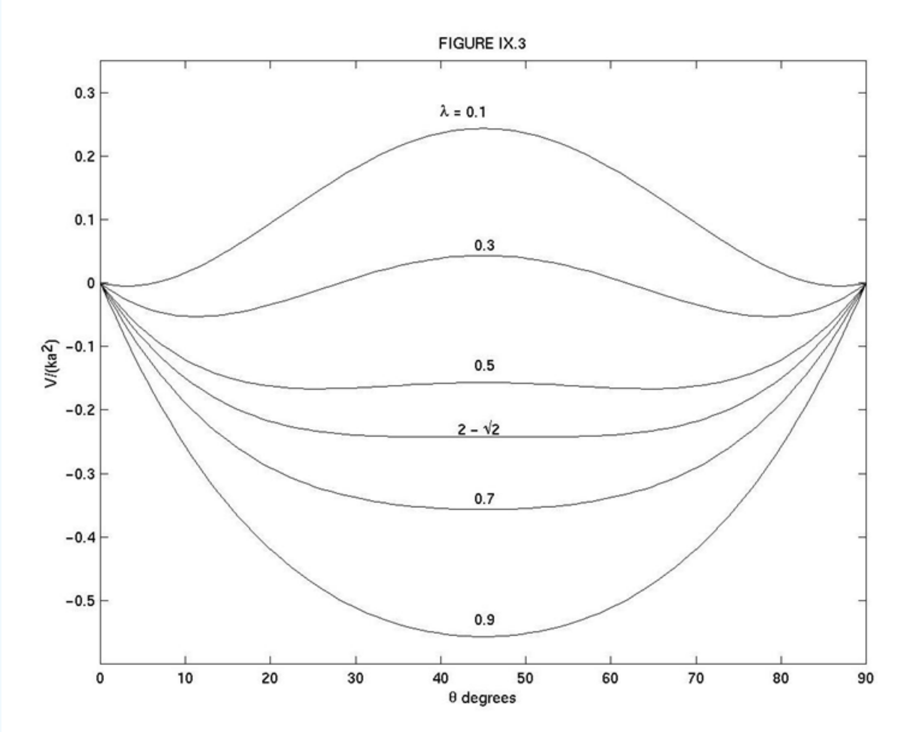
The extension of the string above its natural length is $2a(\cos\theta+\sin\theta-1)$ and the depth of the ring below AB is $a\sin 2\theta$.Therefore, if we take AB to be the level for zero gravitational potential energy, the potential energy (elastic plus gravitational)) is

$$\begin{aligned} V(\theta = & \frac{1}{2}\lfloor 2a(\cos\theta+\sin\theta-1)^2\rfloor - mga\sin 2\theta \\ & = ka^2\lfloor 2(\cos\theta+\sin\theta-1)^2 - \lambda\sin 2\theta\rfloor . \end{aligned} \tag{9.E.6}$$

Figure IX.3 shows this potential energy as a function of θ for several values of λ, including 0.586. We can see that, for λ greater than this, the position at the bottom of the ring (θ = 45°) is the only equilibrium position and it is stable. For small values of λ, the bottom, while an equilibrium position, is unstable, and there are two stable positions higher up. To calculate these equilibrium positions exactly, we need to determine where the derivative of V is zero, and to find whether these positions are stable or unstable, we need to examine the sign of the second derivative.

I leave it to the reader to work through the first derivative and to show that one condition for the derivative to be zero is for $\cos\theta$ to equal $\sin\theta$; that is, θ = 45^0 , which corresponds to the bottom of the semicircle. Another condition for the first derivative to be zero is slightly more challenging to find, but you should find that the derivative is zero if

$$\sin(\theta+\frac{\pi}{4})=\frac{\sqrt{2}}{2-\lambda}.$$



This corresponds to a real value of θ only if $\lambda\leq 2-\sqrt{2}=0.586$. The second derivatives are necessary to determine whether the equilibria are stable (V a minimum) or unstable (V a maximum) or a glance at figure IX.3 will be easier.

Now let's express the potential energy as a function U of the angle ϕ, so that $U(\phi)=V(\theta)$. From Figure IX.2, we see that $\phi=90^\circ-2\theta$. After some algebra and trigonometry, I find that the potential energy as a function of ϕ is given by

$$\frac{U(\phi)}{ka^2}=4\cos^2\frac{1}{2}\phi-4\sqrt{2}\cos\frac{1}{2}\phi-\lambda\cos\phi-\lambda-4+4\sqrt{2}.$$

Now expand this carefully by Taylor's theorem as far as ϕ^2:

$$\frac{U(\phi)}{ka^2}=2(\lambda+\sqrt{2}-2)\phi^2.$$

What we have done is to approximate the potential energy function $U(\phi)$ by a parabola for small ϕ. Now a parabolic potential well is characteristic of simple harmonic motion. For example, in linear simple harmonic motion obeying the equation $m\ddot{x}=-kx$ the potential energy per unit mass is $\frac{1}{2}kx^2$ and the period is $2\pi\sqrt{\frac{m}{k}}$. In rotational simple harmonic motion obeying the equation $I\ddot{\theta}=-c\theta$ the potential energy per unit rotational inertia is $\frac{1}{2}c\theta^2$ and the period is $2\pi\sqrt{\frac{I}{c}}$. The rotational inertia here is just ma^2, so we find that the period of small oscillations is

$$P=\pi\sqrt{\frac{\lambda a}{(\lambda+\sqrt{2}-2)g}}, \tag{9.E.7}$$

provided, of course, that $\lambda>2-\sqrt{2}$.

If you want to find how long the ring takes to slide from an initial position $\phi=\alpha$ to the bottom you can use Equations 9.2.10 and 9.E.6, with $E=U(\alpha)$. You will find the usual difficulty that the integrand is zero at the start. I haven't actually tried the problem, because it looks slightly tedious, but I am fairly certain that it can be integrated analytically. If you do it and get an answer, make sure that, in the limit of small ϕ, you get the same as Equation 9.E.7

Exercise $9.E.6$

A rod of length $2l$ and mass m has one end freely pivoted on a horizontal floor. The rod is held at an initial angle of 45º to the vertical and then released. How long does it take for the rod to hit the floor? I'll leave you to work this one out. By the way, if I had started with the rod vertical, you would find that it takes an infinite time to fall - because the vertical position, although unstable, is an equilibrium position, and it would never get going unless given an infinitesimal displacement.

CHAPTER OVERVIEW

10: Rocket Motion

Topic hierarchy

10.1: Introduction

If you are asked to state Newton's Second Law of Motion, I hope you will not reply: "Force equals mass times acceleration" - because that is not Newton's Second Law of Motion. Newton's Second Law of Motion is:

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

In short: Force equals Rate of Change of Momentum, or, in symbols, $F = \dot{p}$. On differentiating the right hand side, we obtain $F = m\dot{v} + \dot{m}v$. In other words, if the mass is constant then indeed force equals mass times acceleration - but *only* if the mass is constant. In a rocket, a very appreciable fraction of the mass of the rocket is fuel, which is burned and ejected at a very high rate, so that the mass of the rocket is rapidly diminishing during the motion. It is one of the great problems of rocket design that such a high proportion of the initial mass must be fuel. For this reason, other possible methods of driving spacecraft are being investigated by many groups. For example, in the ion propulsion system of the Deep Space One spacecraft, electrically accelerated ions are ejected at high speed from the spacecraft. The force produced and the acceleration are minute, but, because it can be kept up for a very long time, very high speeds can eventually be reached. "Solar sail" systems similarly rely on the very tiny force that can be exerted by the solar wind, but this tiny force can be exerted during most of the lifetime of a spacecraft's flight, and hence again high speeds can be reached.

This chapter, however, concerns just conventional rocket motion. In the next section I consider the motion of a rocket in space subject only to the one force from the high-speed ejection of burned fuel in the absence of any other forces. At a later date, if I can find the time and energy, I may add further sections on rocket motion against gravity, which might be uniform or might fall off with distance from Earth, and we might include air resistance or not. But to begin with, we deal solely with a rocket isolated in space and subject to no additional forces.

10.2: An Integral

So that we don't get bogged down later with an integral that is going to crop up, see if you can do the following integration:

$$\int \ln(a - bt)dt$$

You should get $-t - \frac{1}{b}(a - bt)\ln(a - bt) +$ constant.

10.3: The Rocket Equation

Initially at time t = 0, the mass of the rocket, including fuel, is m_0.

We suppose that the rocket is burning fuel at a rate of b kg s^{-1} so that, at time t, the mass of the rocket-plus-remaining-fuel is $m = m_0 - bt$. The rate of increase of mass with time is $\frac{dm}{dt} = -b$ and is supposed constant with time. (The rate of "increase" is, of course, negative.)

We suppose that the speed of the ejected fuel, relative to the rocket, is V. The thrust of the ejected fuel on the rocket is therefore Vb, or $-V\frac{dm}{dt}$. This is equal to the instantaneous mass times acceleration of the rocket:

$$Vb = m\frac{dv}{dt} = (m_0 - bt)\frac{dv}{dt}. \tag{10.3.1}$$

Thus

$$\int_0^v dv = Vb\int_0^t \frac{dt}{m_0 - bt}. \tag{10.3.2}$$

(Don't be tempted to write the right hand side as $-Vb\int_0^t \frac{dt}{bt-m_0}$. You are anticipating a logarithm, so keep the denominator positive. We have met this before in Chapter 6.) On integration, we obtain

$$v = V\ln\frac{m_0}{m_0 - bt}. \tag{10.3.3}$$

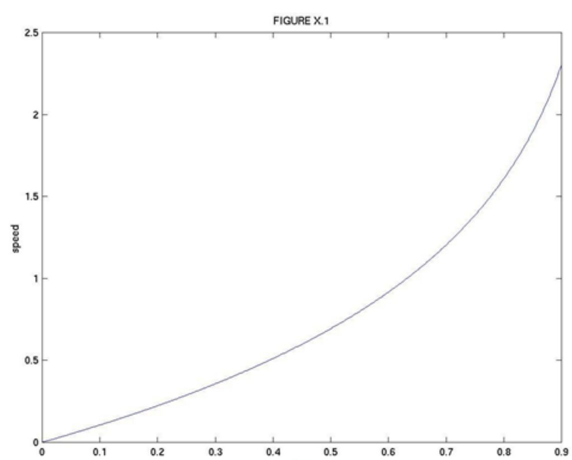
The acceleration is

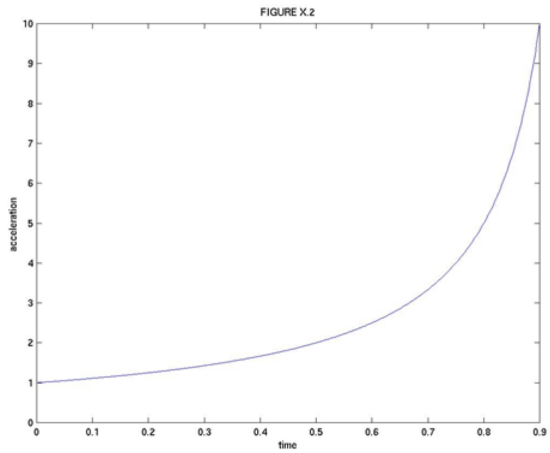
$$\frac{dv}{dt} = \frac{Vb}{m_0 - bt}. \tag{10.3.4}$$

At t = 0, the speed is zero and the acceleration is $\frac{Vb}{m_0}$.

At time $t = \frac{m_0}{b}$, the remaining mass is zero and the speed and acceleration are both infinite. However, this is so only if the initial mass is 100% fuel and nothing else. This is not realistic. If the fraction of the total mass was initially f, the fuel will be completely expended after a time $\frac{fm_0}{b}$ at which time the speed will be (which is, of course, positive), and the speed will remain constant thereafter. For example, if f = 99%, the final speed will be 4.6 V.

Equations 10.3.3 and 10.3.4 are shown in Figures X.1 and X.2. In Figure X.1, the speed of the rocket is plotted in units of V, the ejection speed of the burnt fuel. The time is plotted in units of $\frac{m_0}{b}$. The fuel initially comprised 90% of the rocket, so that the rocket runs out of fuel in time 0.9 $\frac{m_0}{b}$, at which time its speed is 2.3V. In Figure X.2, the acceleration is plotted in units of the initial acceleration, which is $\frac{Vb}{m_0}$. When the fuel is exhausted, the acceleration is ten times this.





In Equation 10.3.3, v is of course $\frac{dx}{dt}$, so the equation can be integrated to obtain the distance:time relation:

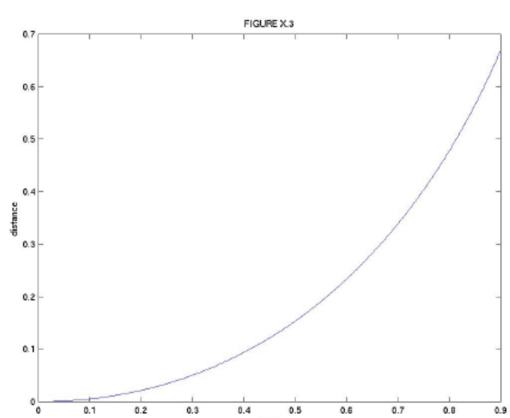
$$x = V[t + (\frac{m_0}{b} - t)\ln(1 - \frac{bt}{m_0})]. \tag{10.3.5}$$

Elimination of t between Equations 10.3.3 and 10.3.5 gives the relation between speed and distance:

$$x = \frac{Vm_0}{b}[1 - (1 + \frac{v}{V})e^{\frac{-v}{V}}]. \tag{10.3.6}$$

If f is the fraction of the initial mass that is fuel, the fuel supply will be exhausted after a time $\frac{fm_0}{b}$, at which time its speed will be $-V\ln(1-f)$ (this is positive, because 1 $-f$ is less than 1), its acceleration will be $\frac{1}{(1-f)}$ and it will have travelled a distance $\frac{Vm_0}{b}[f + (1-f)\ln(1-f)]$. If the entire initial mass is fuel, so that f = 1, the fuel will burn for a time $\frac{m_{0i}}{b}$, at which time its speed and acceleration will be infinite, it will have travelled a finite distance $\frac{Vm_0}{b}$ and the mass will have been reduced to zero, This remarkable result is not very believable, for two reasons. In the first place it is not very realistic. More importantly, when the speed becomes comparable to the speed of light, the equations which we have developed for nonrelativistic speeds are no longer approximately valid, and the correct relativistic equations must be used. The speed cannot then reach the speed of light as long as the remaining mass is non-zero.

Equations 10.3.5 and 10.3.6 are illustrated in Figures X.3 and X.4, in which , the fraction of the initial mass that is fuel, is 0.9. The units for distance, time and speed in these graphs are, respectively, $\frac{Vm_0}{b}$, $\frac{m_0}{b}$ and V.



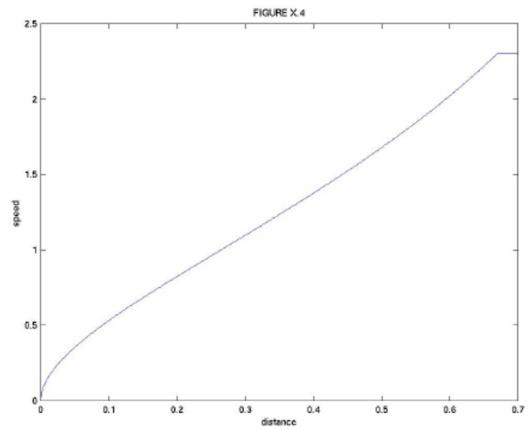


FIGURE X.4
speed
distance
0
0.5
1
1.5
2
2.5
0.1
0.2
0.3
0.4
0.5
0.6
0.7

10.E: Rocket Motion (Exercises)

Exercise $10.E.1$

Derive the integral in section 2.

Exercise $10.E.2$

Integrate equation 10.3.2 to obtain equation 10.3.3

Exercise $10.E.3$

Integrate equation 10.3.3 to obtain equation 10.3.5

Exercise $10.E.4$

Obtain equation 10.3.6

In the following problems, (numbers 5 - 8) assume:

$$V = 2 \text{ km s}^{-1}.$$
$$m0 = 2000 \text{ kg}$$
$$b = 0.5 \text{ kg s}^{-1}$$
$$f = 90\%$$

Exercise $10.E.5$

What is the maximum speed, and how long does it take to attain it?

Exercise $10.E.6$

How long does it take to reach a speed of 3 km s^{-1}?

Exercise $10.E.7$

How long does it take for the rocket to travel 600 km?

Exercise $10.E.8$

How fast is it moving when it has travelled 300 km?

CHAPTER OVERVIEW

11: Simple and Damped Oscillatory Motion

Topic hierarchy

11.1: Simple Harmonic Motion

I am assuming that this is by no means the first occasion on which the reader has met simple harmonic motion, and hence in this section I merely summarize the familiar formulas without spending time on numerous elementary examples

Simple harmonic motion can be defined as follows: It a point P moves in a circle of radius a at constant angular speed ω (and hence period $\frac{2\pi}{\omega}$) its projection Q on a diameter moves with simple harmonic motion. This is illustrated in Figure XI.1, in which the velocity and acceleration of P and of Q are shown as coloured arrows. The velocity of P is just $a\omega$ and its acceleration is the centripetal acceleration $a\omega^2$. As in Chapter 8 and elsewhere, I use blue arrows for velocity vectors and green for acceleration.

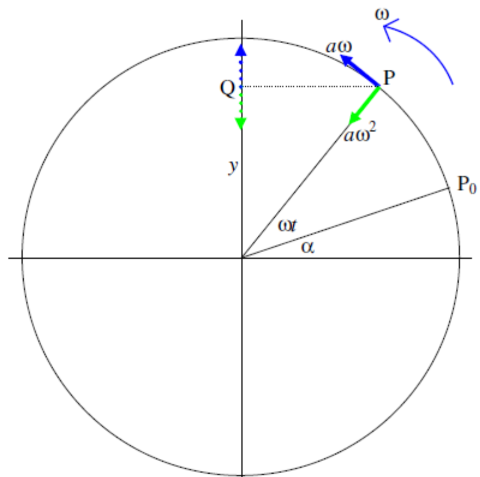


FIGURE XI.1

P_0 is the initial position of P - i.e. the position of P at time $t=0$ - and a is the *initial phase angle.* At time t later, the phase angle is $\omega t+\alpha$. The projection of P upon a diameter is Q. The displacement of Q from the origin, and its velocity and acceleration, are

$$y=a\sin(\omega t+\alpha) \tag{11.1.1}$$

$$v=\dot{y}=a\omega\cos(\omega t+\alpha) \tag{11.1.2}$$

$$\ddot{y}=-a\omega^2\sin(\omega t+\alpha). \tag{11.1.3}$$

Equations 11.1.2 and 11.1.3 can be obtained immediately either by inspection of Figure XI.1 or by differentiation of Equation 11.1.1. Elimination of the time from Equations 11.1.1 and 11.1.2 and from Equations 11.1.1 and 11.1.3 leads to

$$v=\dot{y}=\omega(a^2-y^2)^{\frac{1}{2}} \tag{11.1.4}$$

and

$$\ddot{y}=-\omega^2 y \tag{11.1.5}$$

An alternative definition of simple harmonic motion is to define as simple harmonic motion any motion that obeys the differential Equation 11.1.5. We then have the problem of solving this differential Equation. We can make no progress with this unless we remember to write $\ddot{y}$ as $v\frac{dv}{dy}$ (recall that we did this often in Chapter 6.) Equation 11.1.5 then immediately integrates to

$$v^2=\omega^2(a^2-y^2)$$

A further integration, with $v=\frac{dy}{dt}$, leads to

$$y=a\sin(\omega t+\alpha)$$

provided we remember to use the appropriate initial conditions. Differentiation with respect to time then leads to Equation 11.1.2, and all the other Equations follow.

? Exercise 11.1.1

Important Problem.

Show that $y = a\sin(\omega t + \alpha)$ can be written

$$y = A\sin\omega t + B\cos\omega t \tag{11.1.6}$$

where $A = a\cos\alpha$ and $B = a\sin\alpha$. The converse of these are $a = \sqrt{A^2+B^2}$, $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2+B^2}}$. It is important to note that, if A and B are known, in order to calculate a without ambiguity of quadrant it is entirely necessary to calculate both $\cos\alpha$ and $\sin\alpha$. It will not do, for example, to calculate α solely from $\alpha = \tan^{-1}(\frac{y}{x})$ because this will give two possible solutions for a differing by 180°.

Show also that Equation 11.1.6 can also be written

$$y = Me^{i\omega t} + Ne^{-i\omega t}, \tag{11.1.7}$$

where $M = \frac{1}{2}(B - iA)$ and $N = \frac{1}{2}(B + iA)$ show that the right hand side of Equation 11.1.7 is real.

The four large satellites of Jupiter furnish a beautiful demonstration of simple harmonic motion. Earth is almost in the plane of their orbits, so we see the motion of satellites projected on a diameter. They move to and fro in simple harmonic motion, each with different amplitude (radius of the orbit), period (and hence angular speed ω) and initial phase angle α.

11.2: Mass Attached to an Elastic Spring

I am thinking of a mass m resting on a smooth horizontal table, rather than hanging downwards, because I want to avoid the unimportant distraction of the gravitational force (weight) acting on the mass. The mass is attached to one end of a spring of force constant k, the other end of the spring being fixed, and the motion is restricted to one dimension.

I suppose that the force required to stretch or compress the spring through a distance x is proportional to x and to no higher powers; that is, the spring obeys *Hooke's Law*. When the spring is stretched by an amount x there is a *tension* kx in the spring; when the spring is compressed by x there is a *thrust* kx in the spring. The constant k is the *force constant* of the spring.

When the spring is stretched by an distance x, its acceleration $\ddot{x}$ is given by

$$m\ddot{x} = -kx. \tag{11.2.1}$$

This is an Equation of the type 11.1.5, with $\omega^2 = \frac{k}{m}$, and the motion is therefore simple harmonic motion of period

$$P = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \tag{11.2.2}$$

At this stage you should ask yourself two things: Does this expression have dimensions T? Physically, would you expect the oscillations to be slow for a heavy mass and a weak spring? The reader might be interested to know (and this is literally true) that when I first typed Equation 11.2.2, I inadvertently typed $\sqrt{\frac{k}{m}}$ and I immediately spotted my mistake by automatically asking myself these two questions. The reader might also like to note that you can deduce that $P\infty\sqrt{\frac{m}{k}}$ by the method of dimensions, although you cannot deduce the proportionality constant 2π. Try it.

Energy Considerations. The work required to stretch (or compress) a Hooke's law spring by x is $\frac{1}{2}kx^2$, and this can be described as the potential energy or the elastic energy stored in the spring. I shall not pause to derive this result here. It is probably already known by the reader, or s/he can derive it by calculus. Failing that, just consider that, in stretching the spring, the force increases linearly from 0 to kx, so the average force used over the distance x is $\frac{1}{2}kx$ and so the work done is $\frac{1}{2}kx^2$.

If we assume that, while the mass is oscillating, no mechanical energy is dissipated as heat, the total energy of the system at any time is the sum of the elastic energy $\frac{1}{2}kx^2$ stored in the spring and the kinetic energy $\frac{1}{2}mv^2$ of the mass. (I am assuming that the mass of the spring is negligible compared with m.)

Thus

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \tag{11.2.3}$$

and there is a continual exchange of energy between elastic and kinetic. When the spring is fully extended, the kinetic energy is zero and the total energy is equal to the elastic energy then, $\frac{1}{2}ka^2$ when the spring is unstretched and uncompressed, the energy is entirely kinetic; the mass is then moving at its maximum speed $a\omega$ and the total energy is equal to the kinetic energy then, $\frac{1}{2}ma^2\omega^2$. Any of these expressions is equal to the total energy:

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}ka^2 = \frac{1}{2}ma^2\omega^2 \tag{11.2.4}$$

11.3: Torsion Pendulum

A torsion pendulum consists of a mass of rotational inertia I hanging by a thin wire from a fixed point. If we assume that the torque required to twist the wire through an angle θ is proportional to θ and to no higher powers, then the ratio of the torque to the angle is called the torsion constant c. It depends on the shear modulus of the material of which the wire is made, is inversely proportional to its length, and, for a wire of circular cross-section, is proportional to the fourth power of its diameter. A thick wire is much harder to twist than a thin wire. Ribbonlike wires have comparatively small torsion constants. The work required to twist a wire through an angle θ is $\frac{1}{2}c\theta^2$.

When a torsion pendulum is oscillating, its Equation of motion is

$$I\ddot{\theta} = -c\theta. \tag{11.3.1}$$

This is an Equation of the form 11.1.5 and is therefore simple harmonic motion in which $\omega = \sqrt{\frac{c}{I}}$. This example, incidentally, shows that our second definition of simple harmonic motion (i.e. motion that obeys a differential Equation of the form of Equation 11.1.5) is a more general definition than our introductory description as the projection upon a diameter of uniform motion in a circle. In particular, do not imagine that ω here is the same thing as $\dot{\theta}$!

Exercise 11.3.1

Write down the torsional analogues of all the Equations given for linear motion in Sections 11.1 and 11.2.

11.4: Ordinary Homogeneous Second-order Differential Equations

This is not a full mathematical course on differential Equations, but it may be useful as a reminder for those who have already studied differential Equations, and may even be just enough for our purposes for those who have not.

We suppose that $y=y(x)$ and $y^{'}$ denotes $\frac{dy}{dx}$. An ordinary homogenous second-order differential equation is an Equation of the form

$$ay^{''}+by^{'}+cy=0, \tag{11.4.1}$$

and we have to find a function $y(x)$ which satisfies this. It turns out that it is quite easy to do this, although the nature of the solutions depends on whether b^2 is less than, equal to or greater than $4ac$.

A first point to notice is that, if $y=f(x)$ is a solution, so is $Af(x)$ - just try substituting this in the Equation 11.4.1. If $y=g(x)$ is another solution, the same is true of g - i.e. $Bg(x)$ is also a solution. And you can also easily verify that any linear combination, such as

$$y=Af(x)+Bg(x), \tag{11.4.2}$$

is also a solution.

Now Equation 11.4.1 is a second-order Equation - i.e. the highest derivative is a second derivative - and therefore there can be only two arbitrary constants of integration in the solution - and we already have two in Equation 11.4.1, and consequently there are no further solutions. All we have to do, then, is to find two functions that satisfy the differential Equation.

It will not take long to discover that solutions of the form $y=e^{kx}$ satisfy the Equation, because then $y^{'}=ky$ and $y^{''}=k^2y$, and, if you substitute these in Equation 11.4.1, you obtain

$$(ak^2+bk+c)y=0. \tag{11.4.3}$$

You can always find two values of k that satisfy this, although these may be complex, which is why the nature of the solutions depends on whether b^2 is less than or greater than $4ac$. Thus the general solution is

$$y=Ae^{k_1x}+Be^{k_2x} \tag{11.4.4}$$

where k_1 and k_2 are the solutions of the Equation

$$ax^2+bx+c=0. \tag{11.4.5}$$

There is one complication, however, if $b^2=4ac$ because then the two solutions of Equation 11.4.5 are each equal to $(\frac{-b}{(2a)}$. The solution of the differential Equation is then

$$y=(A+B)\exp\left[\frac{-bx}{(2a)}\right] \tag{11.4.6}$$

and the two constants can be combined into a single constant $C=A+B$ so that Equation 11.4.6 can be written

$$y=C\exp\left[\frac{-bx}{(2a)}\right]. \tag{11.4.7}$$

This solution has only one independent arbitrary constant, and so an additional solution must be possible. Let us try and see whether a function of the form

$$y=xe^{mx} \tag{11.4.8}$$

might be a solution of Equation 11.4.1. From Equation 11.4.8 we obtain $y^{'}=(1+mx)e^{mx}$ and $y^{''}=m(2+mx)e^{mx}$. On substituting these into Equation 11.4.1, remembering that $c=\frac{b^2}{(4a)}$, we obtain for the left hand side of Equation 11.4.1, after some algebra,

$$\frac{e^{mx}}{4a}\left[(2am+b)^2x+4a(2am+b)\right]. \tag{11.4.9}$$

This is identically zero if $m=\dfrac{-b}{(2a)}$, and hence

$$y=x\exp\left[\frac{-bx}{(2a)}\right] \tag{11.4.10}$$

is a solution of Equation 11.4.1 the general solution of Equation 11.4.1, if $b^2=4ac$, is therefore

$$y=(C=Dx)\exp\left[\frac{-bx}{(2a)}\right]. \tag{11.4.11}$$

We shall discover what these solutions actually look like in the next section.

SECTION OVERVIEW

11.5: Damped Oscillatory Motion

As pointed out in Section 11.2, the equation of motion for a mass m vibrating on the end of a spring of force constant k, in the absence of any damping, is

$$m\ddot{x} = -kx. \tag{11.5.1}$$

Here, I am assuming that the displacement x is a function of time, and a dot denotes $\frac{d}{dt}$.

However, in most real situations, there is some damping, or loss of mechanical energy, which is dissipated as heat. In the case of our example of a mass oscillating on a horizontal table, damping may be caused by friction between the mass and the table. For a mass hanging vertically from a spring, we might imagine the mass to be immersed in a viscous fluid. These are obvious examples. Slightly less obvious, it may be that the constant bending and stretching of the spring produces heat, and the motion is damped from this cause. In any case, in this analysis we shall assume that, in addition to the restoring force kx, there is also a damping force that is proportional to the speed at which the particle is moving. I shall denote the damping force by $b\dot{x}$. The equation of motion is then

$$m\ddot{x} = -kx - b\dot{x}. \tag{11.5.2}$$

If I divide by m and write ω_0^2 for $\frac{k}{m}$ and γ for $\frac{b}{m}$, we obtain the equation of motion in its usual form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0. \tag{11.5.3}$$

Here γ is the *damping constant*, which we have already met in Chapter 10, and, from Section 11.4, we are ready to solve the differential Equation 11.5.3. Indeed, we know that the general solution is

$$x = Ae^{k_1 t} + Be^{k_2 t}, \tag{11.5.4}$$

where k_1 and k_2 are the solutions of the quadratic equation

$$k^2 + \gamma k + \omega_0^2 = 0. \tag{11.5.5}$$

An exception occurs if $k_1 = k_2$, and we shall deal with that exceptional case in due course (subsection 11.5iii). Otherwise, k_1 and k_2 are given by

$$k_1 = -\frac{1}{2}\gamma + \sqrt{\frac{1}{4}\gamma^2 - \omega_0^2}, \quad k_2 = -\frac{1}{2}\gamma - \sqrt{\frac{1}{4}\gamma - \omega_0^2}. \tag{11.5.6}$$

In Section 11.5.3, we pointed out that the nature of the solution depends on whether b^2 is less than, equal to or greater than $4ac$, or, in the present case, upon whether γ is less than, equal to or greater than. These cases are referred to, respectively, as lightly damped, critically damped and heavily damped. We shall start by considering light damping.

Topic hierarchy

11.5i: Light damping- γ<2 ω 0 γ<2ω0

Since $\gamma < 2\omega_0$, we have to write Equations 11.5.6 as

$$k_1 = -\frac{1}{2}\gamma + i\sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}, \quad k_2 = -\frac{1}{2}\gamma - i\sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}. \tag{11.5.7}$$

Further, I shall write

$$\omega' = \sqrt{w_0^2 - \frac{1}{4}\gamma^2}. \tag{11.5.8}$$

Equation 11.5.4 is therefore

$$x = Ae^{-\frac{1}{2}\gamma t + i\omega' t} + Be^{-\frac{1}{2}\gamma t - i\omega' t} = e^{-\frac{1}{2}\gamma t}(Ae^{+i\omega' t} + Be^{-i\omega' t}). \tag{11.5.9}$$

If x is to be real, A and B must be complex. Also, since $e^{-i\omega' t} = (e^{i\omega' t})*$ must equal $A*$, where the asterisk denotes the complex conjugate.

Let $A = \frac{1}{2}(a - ib)$ and $B = \frac{1}{2}(a + ib)$ where a and b are real. Then the reader should be able to show that Equation 11.5.9 can be written as

$$x = e^{-\frac{1}{2}\gamma t}(a\cos\omega' t + b\sin\omega' t). \tag{11.5.10}$$

And if $C = \sqrt{a^2 + b^2}$, $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$, the equation can be written

$$x = Ce^{-\frac{1}{2}\gamma t}\sin(\omega' t + \alpha). \tag{11.5.11}$$

Equations 11.5.9, 11.5.10or 11.5.11are three equivalent ways of writing the solution. Each has two arbitrary integration constants (A, B), (a, b) or (C, α), whose values depend on the *initial conditions* - i.e. on the values of x and $\dot{x}$ when $t = 0$. Equation 11.5.11shows that the motion is a sinusoidal oscillation of period a little less than ω_0, with an exponentially decreasing amplitude.

To find C and α in terms of the initial conditions, differentiate Equation 11.5.11 with respect to the time in order to obtain an equation showing the speed as a function of the time:

$$\dot{x} = Ce^{-\frac{1}{2}\gamma t}[\omega' \cos(\omega' t + \alpha) - \frac{1}{2}\gamma\sin(\omega' t + \alpha)]. \tag{11.5.12}$$

By putting $t = 0$ in Equations 11.5.11and 11.5.12we obtain

$$x_0 = C\sin\alpha \tag{11.5.13}$$

and

$$(\dot{x})_0 = C(\omega' \cos\alpha - \frac{1}{2}\gamma\sin\alpha). \tag{11.5.14}$$

From these we easily obtain

$$\cot\alpha = \frac{1}{\omega'}[\frac{(\dot{x})_0}{\dot{x}_0} + \frac{\gamma}{2}] \tag{11.5.15}$$

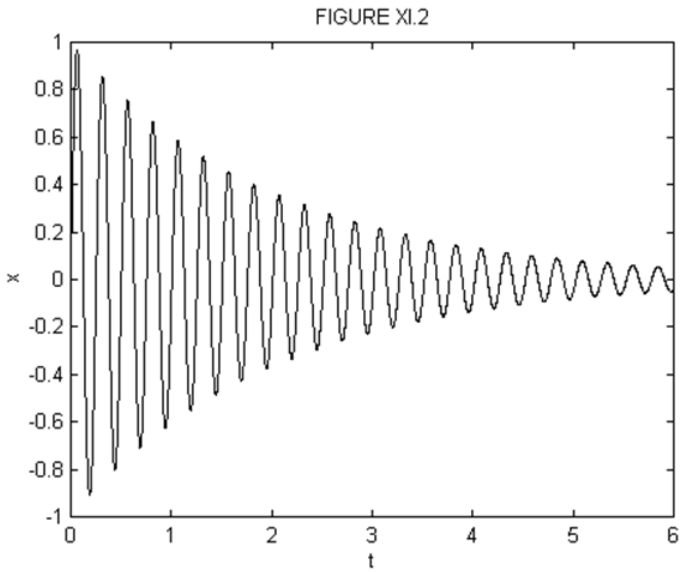
and

$$C = x_0\csc\alpha. \tag{11.5.16}$$

The quadrant of α can be determined from the signs of $\cot\alpha$ and $\csc\alpha$, C always being positive.

Note that the *amplitude* of the motion falls off with time as $e^{-\frac{1}{2}\gamma t}$, but the mechanical *energy*, which is proportional to the square of the amplitude, falls off as $e^{-\gamma t}$.

Figures XI.2 and XI.3 are drawn for $C = 1, \alpha = 0, \gamma = 1$. Figure XI.2 has $\omega_0 = 25\gamma$ and hence $x = e^{-\frac{1}{2}t}\sin 0.24.9949995t$, and figure XI.3 has $\omega_0 = 4\gamma$ and hence $x = e^{-\frac{1}{2}t}\sin 3.968626967t$



> **? Exercise $11.5i.1$**
>
> Draw displacement : time graphs for an oscillator with m = 0.02 kg, k = 0.08 N m^{-1}, g = 1.5 s^{-1}, t = 0 to 15 s, for the following initial conditions:
>
> i. $x_0 = 0, \quad (\dot{x})_0 = 4 \text{ ms}^{-1}$
> ii. $(\dot{x})_0 = 0, \quad x_0 = 3 \text{ m}$
> iii. $(\dot{x}) = -2 \text{ ms}^{-1}$, $x_0 = 2$
>
> Although the motion of a damped oscillator is not strictly "periodic", in that the motion does not repeat itself exactly, we could define a "period" $P = \frac{2\pi}{\omega'}$ as the interval between two consecutive ascending zeroes. Extrema do not occur exactly halfway between consecutive zeroes, and the reader should have no difficulty in showing, by differentiation of Equation 11.5.11, that extrema occur at times given by $\tan(\omega' t + \alpha) = \frac{2\omega'}{\gamma}$. However, provided that the damping is not very large, consecutive extrema occur approximately at intervals of $\frac{P}{2}$. The ratio of consecutive maximum displacements is, then,
>
> $$\frac{|x_n|}{|x_{n+1}|} = \frac{e^{-\frac{1}{2}\gamma t}}{e^{-\frac{1}{2}\gamma(t+\frac{1}{2}P)}}. \tag{11.5.17}$$
>
> From this, we find that the *logarithmic decrement* is
>
> $$ln(\frac{|x_n|}{|x_{n+1}|}) = \frac{P\gamma}{4}, \tag{11.5.18}$$
>
> from which the damping constant can be determined.

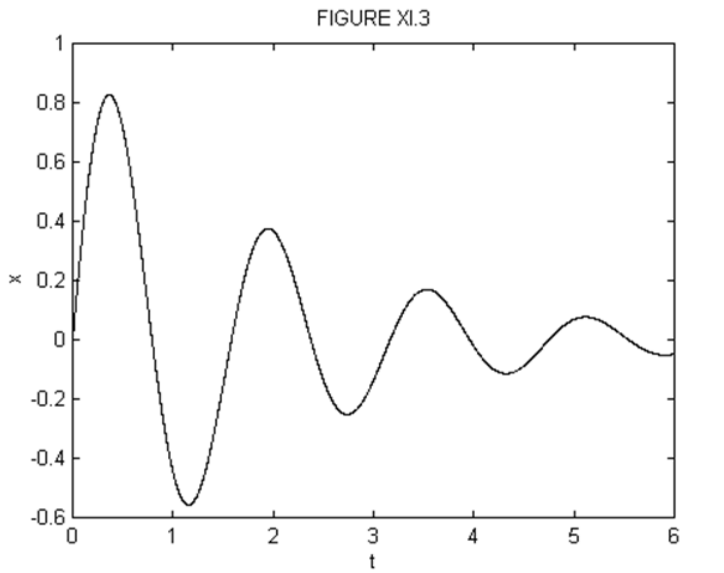


FIGURE XI.3
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
x
0
1
2
3
4
5
6
t

11.5ii: Heavy damping- γ>2 ω 0 γ>2ω0

The motion is given by Equations 11.5.4 and 11.5.6 where, this time, k_1 and k_2 are each real and negative. For convenience, I am going to write $\lambda_1 = -k_1$ and $\lambda_2 = -k_2$. λ_1 are λ_2 both real and positive, with $\lambda_2 > \lambda_1$ given by

$$\lambda_1 = \frac{1}{2}\gamma - \sqrt{(\frac{1}{2}\gamma)^2 - \omega_0^2}, \quad \lambda_2 = \frac{1}{2}\gamma + \sqrt{(\frac{1}{2}\gamma)^2 - \omega_0^2} \tag{11.5.19}$$

The general solution for the displacement as a function of time is

$$x = Ae^{-\lambda_1 t} + Be^{-\lambda_2 t}. \tag{11.5.20}$$

The speed is given by

$$\dot{x} = -A\lambda_1 e^{-\lambda_1 t} - B\lambda_2 e^{-\lambda_2 t}. \tag{11.5.21}$$

The constants A and B depend on the initial conditions. Thus:

$$x_0 = A + B \tag{11.5.22}$$

and

$$(\dot{x})_0 = -(A\lambda_1 + B\lambda_2). \tag{11.5.23}$$

From these, we obtain

$$A = \frac{(\dot{x})_0 + \lambda_2 x_0}{\lambda_2 - \lambda_1}, \qquad B = -[\frac{(\dot{x})_0 + \lambda_1 x_0}{\lambda_2 - \lambda_1}]. \tag{11.5.24}$$

Example $11.5ii.1$

$x_0 \neq 0, \quad (\dot{x})_0 = 0.$

$$x = \frac{x_0}{\lambda_2 - \lambda_1}(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}). \tag{11.5.25}$$

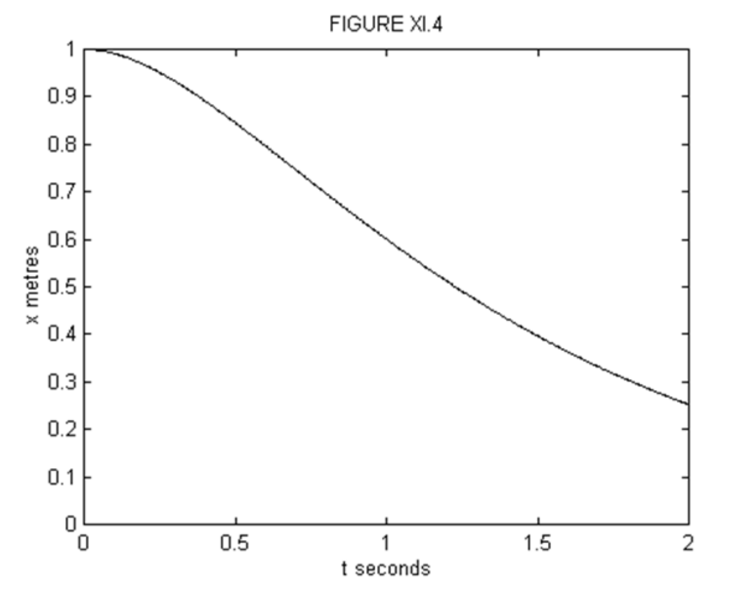
Figure XI.4 shows $x \quad : \quad t$ for x_0 = 1 m, λ_1 = 1 s^{-1}, λ_2 = 2 s^{-1}.

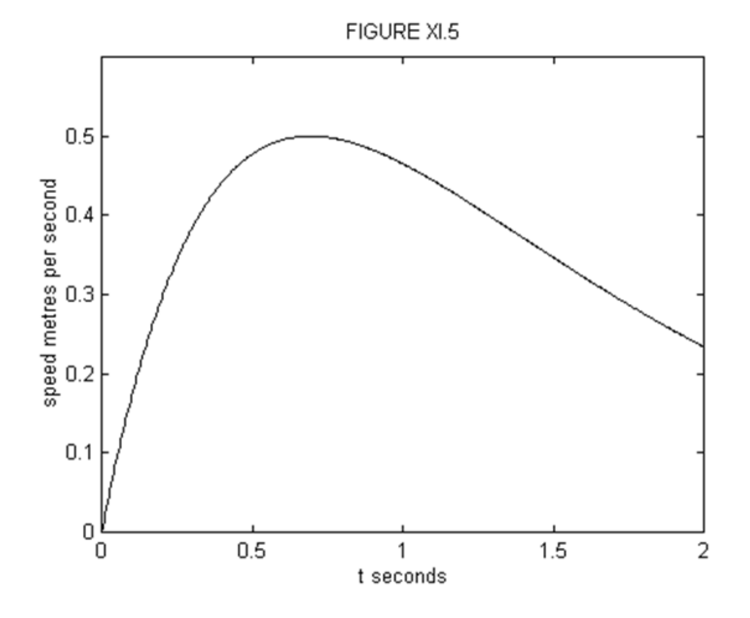
The displacement will fall to half of its initial value at a time given by putting $\frac{x}{x_0} = \frac{1}{2}$ in Equation 11.5.25. This will in general require a numerical solution. In our example, however, the equation reduces to $\frac{1}{2} = 2e^{-t} - e^{-2t}$ and if we let $u = e^{-t}$, this becomes $u^2 - 2u + \frac{1}{2} = 0$. The two solutions of this are $u = 1.707107$ or 0.292893 The first of these gives a negative *t*, so we want the second solution, which corresponds to $t = 1.228$ seconds.

The velocity as a function of time is given by

$$\dot{x} = -\frac{\lambda_1\lambda_2 x_0}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{\lambda_2 t}). \tag{11.5.26a}$$

This is always negative. In figure XI.5, is shown the speed, which is $|\dot{x}|$ as a function of time, for our numerical example. Those who enjoy differentiating can show that the maximum speed is reached in a time $-\dot{x}$ and that the maximum speed is $\frac{\lambda_1\lambda_2 x_0}{\lambda_2-\lambda_1}[(\frac{\lambda_1}{\lambda_2})^{\frac{\lambda_2}{\lambda_2-\lambda_1}} - (\frac{\lambda_1}{\lambda_2})^{\frac{\lambda_1}{\lambda_2-\lambda_1}}]$. (Are these dimensionally correct?) In our example, the maximum speed, reached at $t = \ln 2 = 0.6931$ seconds, is 0.5 m s^{-1}.





Example $11.5ii.2$

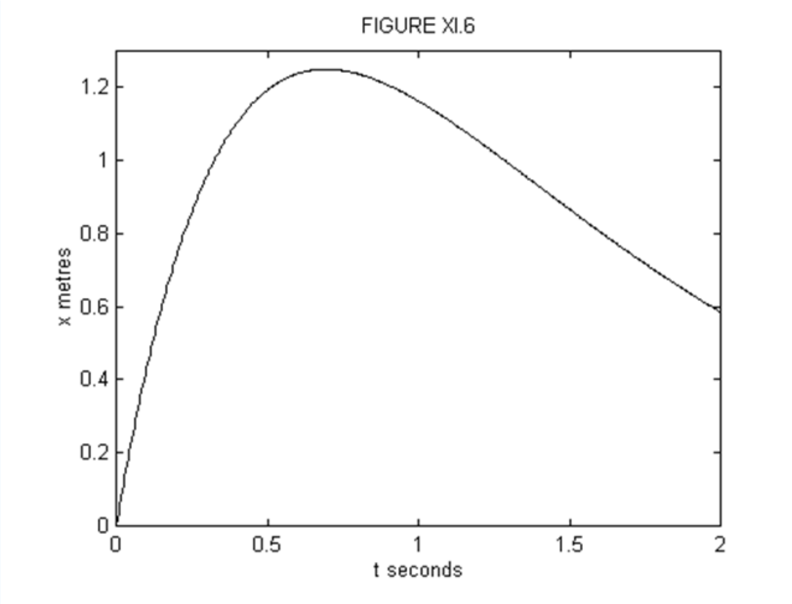
$x_0 = 0, \quad (\dot{x})_0 = V(>0)$.

In this case it is easy to show that

$$x = \frac{V}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}). \tag{11.5.26b}$$

It is left as an exercise to show that x reaches a maximum value of $\frac{V}{\lambda_2}(\frac{\lambda_1}{\lambda_2})^{\frac{\lambda_1}{\lambda_2 - \lambda_1}}$ when $t = \frac{\ln(\frac{\lambda_2}{\lambda_1})}{\lambda_2 - \lambda_1}$. Figure XI.6 illustrates Equation 11.5.26a for λ_1 = 1 s^{-1}, λ_2 = 2 s^{-1}, V = 5 m s^{-1}. The maximum displacement of 1.25 m is reached when $t = \ln 2 = 0.6831$ s. It is also left as an exercise to show that equation 11.5.26acan be written

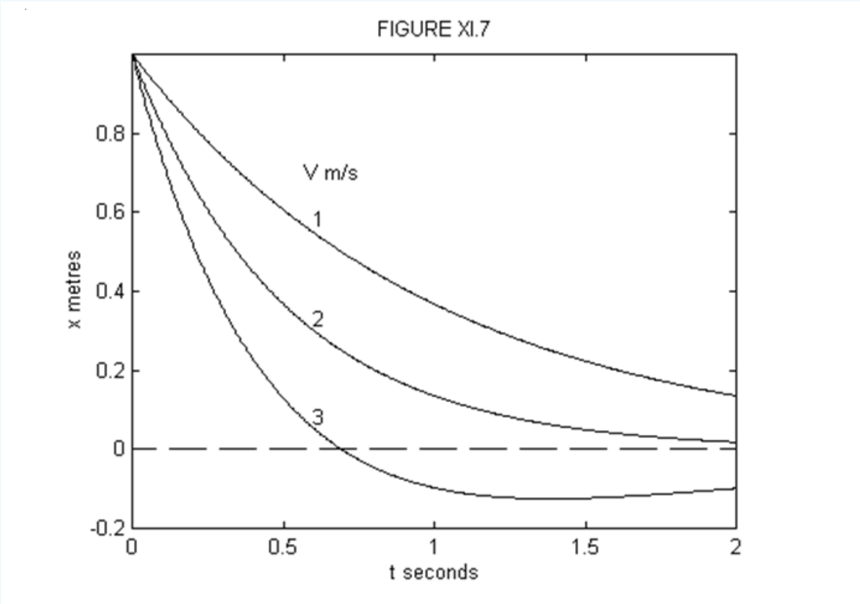
$$x = \frac{2Ve^{-\frac{1}{2}\lambda t}}{\lambda_2 - \lambda_1}\sinh(\frac{1}{4}\gamma^2 - \omega_0^2). \tag{11.5.27}$$



Example 11.5ii.3

$x_0 \neq 0, \quad (\dot{x})_0 = -V$.

This is the really exciting example, because the suspense-filled question is whether the particle will shoot past the origin at some finite time and then fall back to the origin; or whether it will merely tamely fall down asymptotically to the origin without ever crossing it. The tension will be almost unbearable as we find out. In fact, I cannot wait; I am going to plot x versus t in figure XI.7 for λ_1 = 1 s^{-1}, λ_2 = 2 s^{-1}, x_0 = 1 m, and three different values of V, namely 1, 2 and 3 m s^{-1}.



We see that if V = 3 m s^{-1} the particle overshoots the origin after about 0.7 seconds. If V = 1 m s^{-1}, it does not look as though it will ever reach the origin. And if V = 2 m s^{-1}, I'm not sure. Let's see what we can do. We can find out when it crosses the origin by putting x= 0 in Equation 11.5.20, where A and B are found from Equations 11.5.24with $(\dot{x})_0 = -V$. This gives, for the time when it crosses the origin,

$$t = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{V - \lambda_1 x_0}{V - \lambda_2 x_0}\right). \tag{11.5.28}$$

Since $\lambda_2 > \lambda_1$, this implies that the particle will overshoot the origin if $V > \lambda_2 x_0$, and this in turn implies that, for a given V, it will overshoot only if

$$\gamma < \frac{\frac{V^2}{x_0^2} + \omega_0^2}{\frac{V}{x_0}}. \tag{11.5.29}$$

For our example, $\lambda_2 x_0$= 2 m s^{-1}, so that it just fails to overshoot the origin if V= 2 m s^{-1}. For V = 3 m s^{-1}, it crosses the origin at $t = \ln 2 = 0.6931$ s. In order to find out how far past the origin it goes, and when, we can do this just as in

I make it that it reaches its maximum negative displacement of -0.125 m at $t = \ln 4 = 1.386$ s.

11.5iii: Critical damping- [Math Processing Error] γ=2ω0

Before embarking on this section, you might just want to refresh your memory of differential equations as described in Section 11.4.

In this case, *[Math Processing Error]* and *[Math Processing Error]* are each equal to *[Math Processing Error]*. As discussed in Section 4, the general solution is of the form

[Math Processing Error]

which can also be written in the form

[Math Processing Error]

Either way, there are two arbitrary constants, which can be determined by the initial values of the displacement and speed. It is easy to show that

[Math Processing Error]

The particle will not go through zero unless

[Math Processing Error]

I'll leave it to the reader to draw a graph of Equation 11.5.19.

Ideally the hydraulic door closer that you see near the tops of doors in public buildings should be critically damped. This will cause the door to close fastest without slamming. And we have already used the physics of impulsive forces in Problem 2.1 of Chapter 8 to work out where to place a door stop for minimum reaction on the hinges. Truly an understanding of physics is of enormous importance in achieving the task of closing a door!

A more subtle example is in the design of a moving-coil ammeter. In this instrument, the electric current is passed through a coil between the poles of a magnet, and the coil then swings around against the restoring force of a little spiral spring. The coil is wound on a light aluminium frame called a former, and, as the coil (and hence the former) moves in the magnetic field, a little current is induced in the former, and this damps the motion of the coil. In order that the coil and the pointer should move to the equilibrium position in the fastest possible time without oscillating, the system should be critically damped - which means that the rotational inertia and the electrical resistance of the little aluminium former has to be carefully designed to achieve this.

11.6: Electrical Analogues

A charged capacitor of capacitance C is connected in series with a switch and an inductor of inductance L. The switch is closed, and charge flows out of the capacitor and hence a current flows through the inductor. Thus while the electric field in the capacitor diminishes, the magnetic field in the inductor grows, and a back electromotive force (EMF) is induced in the inductor. Let Q be the charge in the capacitor at some time. The current I flowing from the positive plate is equal to $-\dot{Q}$. The potential difference across the capacitor is $\frac{Q}{C}$ and the back EMF across the inductor is $L\dot{I} = -L\ddot{Q}$. The potential drop around the whole circuit is zero, so that $\frac{Q}{C} = -L\ddot{Q}$. The charge on the capacitor is therefore governed by the differential equation

$$\ddot{Q} = -\frac{Q}{LC}, \tag{11.6.1}$$

which is simple harmonic motion with $\omega_0 = \frac{1}{\sqrt{LC}}$. You should verify that this has dimensions T^{-1}.

If there is a resistor of resistance R in the circuit, while a current flows through the resistor there is a potential drop $RI = -R\dot{Q}$ across it, and the differential equation governing the charge on the capacitor is then

$$LC\ddot{Q} + RC\dot{Q} + Q = 0. \tag{11.6.2}$$

This is damped oscillatory motion, the condition for critical damping being

$$R^2 = \frac{4L}{C}.$$

In fact, it is not necessary actually to have a physical resistor in the circuit. Even if the capacitor and inductor were connected by superconducting wires of zero resistance, while the charge in the circuit is slopping around between the capacitor and the inductor, it will be radiating electromagnetic energy into space and hence losing energy. The effect is just as if a resistance were in the circuit.

If a battery of EMF E were in the circuit, the differential equation for Q would be

$$LC\ddot{Q} + RC\dot{Q} + Q = EC. \tag{11.6.3}$$

This is not quite an equation of the form 11.4.1, and I shan't spend time on it here. However, if we are interested in the current as a function of time, we just differentiate Equation 11.6.3 with respect to time:

$$LC\ddot{I} + RC\dot{I} + I = 0. \tag{11.6.4}$$

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CHAPTER OVERVIEW

12: Forced Oscillations

Topic hierarchy

12.1: More on Differential Equations

In Section 11.4 we argued that the most general solution of the differential equation

$$ay'' + by' + cy = 0 \tag{11.4.1}$$

is of the form

$$y = Af(x) + Bg(x). \tag{11.4.2}$$

In this chapter we shall be looking at equations of the form

$$ay'' + by' + cy = h(x). \tag{12.1.1}$$

If you look back at the arguments that led to the conclusion that Equation 11.4.2 is the most general solution of Equation 11.4.1, you will be able to conclude that 11.4.2 is still a solution of Equation 11.4.1, but it is not the only solution. There is another function that is a solution, so that the most general solution to equation 12.1.1 is of the form

$$y = Af(x) + Bg(x) + H(x). \tag{12.1.2}$$

The solution $H(x)$ is called the *particular integral*, while the part $Af(x) + Bg(x)$ is the *complementary function.* I shall be dealing in this chapter mainly with the particular integral, though we shall not entirely forget the complementary function.

This is a book on classical mechanics rather than on differential equations, so I am not going into how to obtain the particular integral $H(x)$ for a given $h(x)$. There are several ways of doing it; for those who know what they are and are in practice with them, Laplace transforms are among the more attractive methods. Some readers will already know how to do it. They will doubtless want to go back to Equation 11.6.3 in the previous chapter and try their hand at finding the particular integral for that. Those who do not may be happy and content to take my word for the particular integral in the sections that follow, or perhaps at least to differentiate it to verify that it is indeed a solution.

12.2: Forced Oscillatory Motion

We are thinking of a mass *[Math Processing Error]* attached to a spring of force constant *[Math Processing Error]* and subject to a damping force *[Math Processing Error]*, but also subject to a periodic sinusoidal force *[Math Processing Error]*. The equation of motion is

[Math Processing Error]

or, if we divide by *[Math Processing Error]*:

[Math Processing Error]

Here *[Math Processing Error]*. *[Math Processing Error]* is the forcing angular frequency and *[Math Processing Error]* is the natural frequency of mass and spring in the absence of damping. One part of the general solution of Equation *[Math Processing Error]* is the complementary function, which we have dealt with at length in Chapter 11. In this section I shall be interested in the particular integral. I shall not derive it here (those who are familiar with differential equations will be able to do so), but you should at least verify by differentiation and substitution that the following is a solution, and it is indeed the particular integral:

[Math Processing Error]

This can also be written

[Math Processing Error]

where

[Math Processing Error]

The response frequency is the same as the forcing frequency, but there is a phase lag between *[Math Processing Error]* and *[Math Processing Error]*. Figure XII.1 shows *[Math Processing Error]* as a function of *[Math Processing Error]* for several different values of *[Math Processing Error]*.

The particular integral can also be written

[Math Processing Error]

where the *displacement amplitude* *[Math Processing Error]* varies with forcing frequency *[Math Processing Error]* as

[Math Processing Error]

If we now introduce dimensionless quantities

[Math Processing Error]

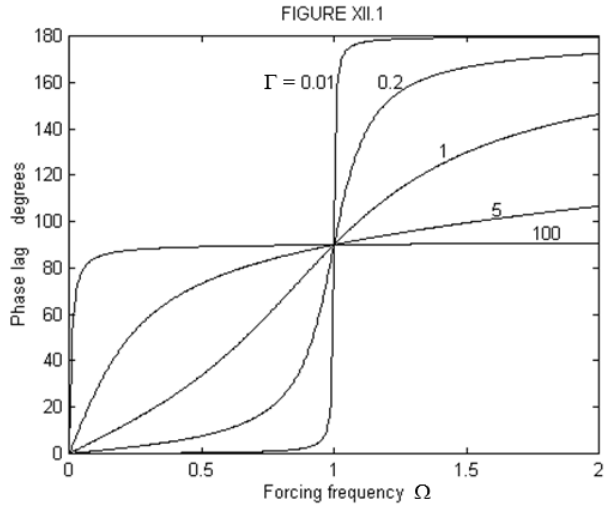
Equations *[Math Processing Error]* and *[Math Processing Error]* become

[Math Processing Error]

and

[Math Processing Error]

The phase lag *[Math Processing Error]* and the displacement amplitude *[Math Processing Error]* are shown as a function of forcing frequency for various values of the damping constant in figures XII.1 and 2.

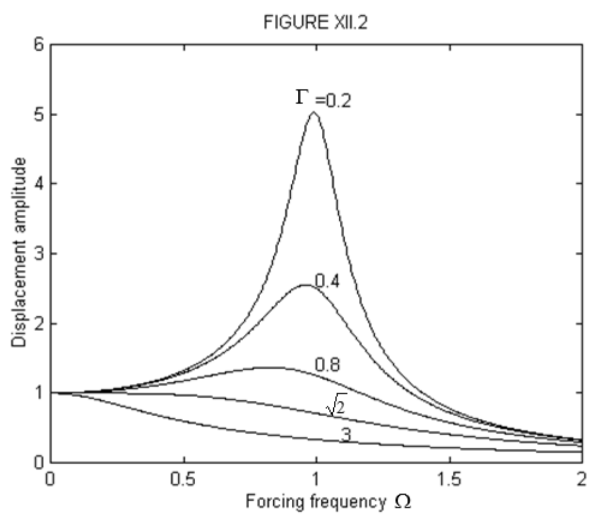


A common misunderstanding is that the displacement amplitude is greatest when the forcing frequency is equal to the undamped frequency *[Math Processing Error]*. That this is far from the case is immediately obvious from a glance at figure XII.2. We can find the frequency that results in the greatest displacement amplitude by maximizing Equation *[Math Processing Error]* or *[Math Processing Error]*. This is most easily achieved by minimizing the square of the denominator. Let *[Math Processing Error]* be the square of the denominator of equation *[Math Processing Error]*, and let *[Math Processing Error]* and *[Math Processing Error]*. Then *[Math Processing Error]*, which is greatest for *[Math Processing Error]* or, provided *[Math Processing Error]*

[Math Processing Error]

This is less not only than *[Math Processing Error]*, but also less than *[Math Processing Error]*. For the frequency given by equation *[Math Processing Error]*, the displacement amplitude will be

[Math Processing Error]



The locus of the maxima in figure XII.2 is found by eliminating *[Math Processing Error]* from equation 12.2.7 and *[Math Processing Error]* which gives

[Math Processing Error]

The solution given by Equations *[Math Processing Error]* and *[Math Processing Error]* is the *particular integral*. As pointed out in Section 1 of this chapter, the complete solution is the sum of the particular integral and the *complementary function*, the latter being the unforced solutions of Chapter 11. The particular integral represents the *steady state* solution, whereas the complementary function, which dies out with time, is a *transient solution.* When a mechanical oscillation is started, or when an alternating current electric circuit is first switched on, the solution is the sum of transient and steady state parts, the former more or less rapidly dying away. Often when an electric fuse blows, the overload is caused by the large, but temporary, amplitude of the transient part of the solution.

Equations *[Math Processing Error]* and *[Math Processing Error]* give the displacement of the system as a function of time. Differentiation with respect to time gives the velocity as a function of time. Thus:

[Math Processing Error]

where

[Math Processing Error]

is the *velocity amplitude.* In dimensionless units, this can be written

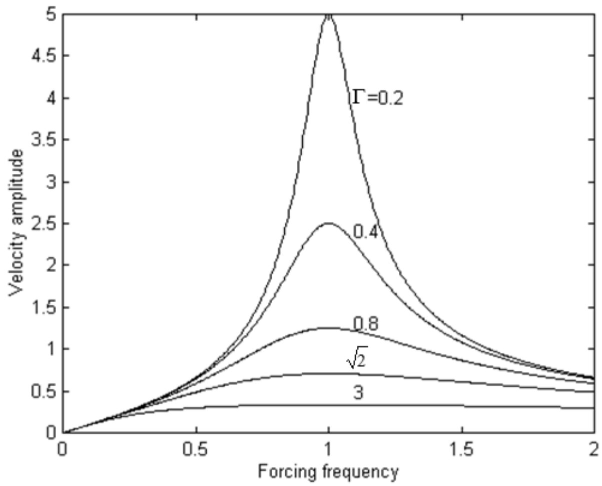
[Math Processing Error]

where

[Math Processing Error]

This is illustrated in figure XII.3.

FIGURE XII.3



It is left to the reader to show that the velocity amplitude is greatest and equal to *[Math Processing Error]* when *[Math Processing Error]*.

We have now found the phase lag and the displacement and velocity amplitudes as a function of forcing frequency, but I must now try the reader's patience one step further for the most important part of the analysis, which really must not be skipped. Damping of oscillatory motion implies that some of the mechanical energy (which, in an undamped system, alternates between kinetic and potential energy) is lost - or, rather, that it is dissipated as heat. This happens if the damping is caused by the oscillator being immersed in a viscous fluid, or if it is caused by the repeated expansion and compression of a spring, or, in an electric circuit, by the dissipation of heat in the resistive part of the circuit. We aim now to find the rate at which the mechanical energy is dissipated as heat.

We return to the equation of motion, Equation *[Math Processing Error]*:

[Math Processing Error]

and multiply each side by *[Math Processing Error]*:

[Math Processing Error]

Introduce the total mechanical energy:

[Math Processing Error]

The instantaneous rate of change of [Math Processing Error] is [Math Processing Error] while the instantaneous rate at which [Math Processing Error] does work is [Math Processing Error]. The difference (see Equation [Math Processing Error]), [Math Processing Error], is therefore the rate at which work is being dissipated as heat, which, of course, is zero if [Math Processing Error].

The average of [Math Processing Error] over a complete period is

[Math Processing Error]

where the bars denote the average value over a period. But [Math Processing Error], so the average rate at which work is being dissipated as heat, for which I shall use the symbol [Math Processing Error] is

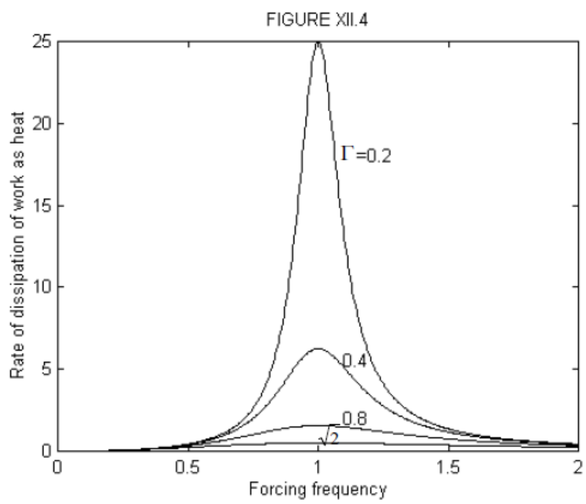
[Math Processing Error]

The reader should check that the right hand side has the dimensions of rate of dissipation of energy and hence the SI unit of watts.

In dimensionless units, in which [Math Processing Error] this can be written

[Math Processing Error]

This is illustrated in figure XII.4. The reader can easily prove that the rate at which work is dissipated as heat is greatest when the forcing frequency is equal to [Math Processing Error].



Summary

Phase lag: Equation [Math Processing Error]

[Math Processing Error].

Displacement amplitude: Equation [Math Processing Error]

[Math Processing Error]

Velocity amplitude: Equation [Math Processing Error]

[Math Processing Error]

Rate of dissipation of work as heat: Equation [Math Processing Error]

[Math Processing Error].

In terms of dimensionless variables,

Phase lag: Equation [Math Processing Error]

[Math Processing Error]

Displacement amplitude: Equation [Math Processing Error]

[Math Processing Error]

Velocity amplitude: [Math Processing Error]

[Math Processing Error]

Rate of dissipation of work as heat: [Math Processing Error]

[Math Processing Error]

12.3: Electrical Analogue

Suppose that an alternating potential difference $E = \hat{E} \sin \omega t$ is applied across an LCR circuit. We refer to Equation 11.6.3, and we see that the equation that governs the charge on the capacitor is

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \hat{E} \sin \omega t. \tag{12.3.1}$$

We can differentiate both sides with respect to time, and divide by L, and hence see that the current is given by

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = \frac{\hat{E}\omega}{L} \cos \omega t. \tag{12.3.2}$$

We can compare this directly with Equation 12.2.2, so that we have

$$\gamma = \frac{R}{L}, \quad \omega_0^2 = \frac{1}{\sqrt{LC}}, \quad \hat{f} = \frac{\hat{E}\omega}{L}. \tag{12.3.3}$$

Then, by comparison with Equation 12.2.5, we see that I will lag behind E by α, where

$$\tan \alpha = \frac{\frac{R\omega}{L}}{\frac{1}{LC} - \omega^2} = \frac{R}{\frac{1}{C\omega} - L\omega}. \tag{12.3.4}$$

This is just what we obtain from the more familiar complex number approach to alternating current circuits.

CHAPTER OVERVIEW

13: Lagrangian Mechanics

Sometimes it is not all that easy to find the equations of motion and there is an alternative approach known as **lagrangian mechanics** which enables us to find the equations of motion when the newtonian method is proving difficult. In lagrangian mechanics we start, as usual, by drawing a large, clear diagram of the system, using a ruler and a compass. But, rather than drawing the forces and accelerations with red and green arrows, we draw the *velocity* vectors (including angular velocities) with blue arrows, and, from these we write down the *kinetic energy* of the system. If the forces are *conservative* forces (gravity, springs and stretched strings), we write down also the *potential energy*. That done, the next step is to write down the *lagrangian equations of motion* for each coordinate. These equations involve the kinetic and potential energies, and are a little bit more involved than $F = ma$, though they do arrive at the same results.

13.1: Introduction to Lagrangian Mechanics

The usual way of using newtonian mechanics to solve a problem in dynamics is first of all to draw a large, clear diagram of the system, using a ruler and a compass. Then mark in the forces on the various parts of the system with red arrows and the accelerations of the various parts with green arrows. Then apply the equation $F = ma$ in two different directions if it is a two-dimensional problem or in three directions if it is a three-dimensional problem, or $\tau = I\ddot{\theta}$ if torques are involved. More correctly, if a mass or a moment of inertia is not constant, the equations are $F = \dot{p}$ and $\tau = \dot{L}$. In any case, we arrive at one or more *equations of motion*, which are differential equations which we integrate with respect to space or time to find the desired solution. Most of us will have done many, many problems of that sort.

Sometimes it is not all that easy to find the equations of motion as described above. There is an alternative approach known as lagrangian mechanics which enables us to find the equations of motion when the newtonian method is proving difficult. In lagrangian mechanics we start, as usual, by drawing a large, clear diagram of the system, using a ruler and a compass. But, rather than drawing the forces and accelerations with red and green arrows, we draw the *velocity* vectors (including angular velocities) with blue arrows, and, from these we write down the *kinetic energy* of the system. If the forces are *conservative* forces (gravity, springs and stretched strings), we write down also the *potential energy*. That done, the next step is to write down the *lagrangian equations of motion* for each coordinate. These equations involve the kinetic and potential energies, and are a little bit more involved than $F = ma$, though they do arrive at the same results.

I shall derive the lagrangian equations of motion, and while I am doing so, you will think that the going is very heavy, and you will be discouraged. At the end of the derivation you will see that the lagrangian equations of motion are indeed rather more involved than $F = ma$, and you will begin to despair – but do not do so! In a very short time after that you will be able to solve difficult problems in mechanics that you would not be able to start using the familiar newtonian methods, and the speed at which you do so will be limited solely by the speed at which you can write. Indeed, you scarcely have to stop and think. You know straight away what you have to do. Draw the diagram. Mark the velocity vectors. Write down expressions for the kinetic and potential energies, and apply the lagrangian equations. It is automatic, fast, and enjoyable.

Incidentally, when Lagrange first published his great work *La méchanique analytique* (the modern French spelling would be *mécanique*), he pointed out with some pride in his introduction that there were no drawings or diagrams in the book – because all of mechanics could be done *analytically* – i.e. with algebra and calculus. Not all of us, however, are as gifted as Lagrange, and we cannot omit the first and very important step of drawing a large and clear diagram with ruler and compass and marking all the velocity vectors.

13.2: Generalized Coordinates and Generalized Forces

In two-dimensions the positions of a point can be specified either by its *rectangular coordinates* (x, y) or by its *polar coordinates*. There are other possibilities such as confocal conical coordinates that might be less familiar. In three dimensions there are the options of *rectangular coordinates* (x, y, z), or *cylindrical coordinates* ρ, ϕ, z or *spherical coordinates* r, ω, ϕ – or again there may be others that may be of use for specialized purposes (inclined coordinates in crystallography, for example, come to mind). The state of a molecule might be described by a number of parameters, such as the bond lengths and the angles between the bonds, and these may be varying periodically with time as the molecule vibrates and twists, and these bonds lengths and bond angles constitute a set of *coordinates* which describe the molecule. We are not going to think about any particular sort of coordinate system or set of coordinates. Rather, we are going to think about *generalized coordinates*, which may be lengths or angles or various combinations of them. We shall call these coordinates $(q_1, q_2, q_3, \ldots)$. If we are thinking of a single particle in three-dimensional space, there will be three of them, which could be rectangular, or cylindrical, or spherical. If there were N particles, we would need $3N$ coordinates to describe the system – unless there were some constraints on the system.

With each generalized coordinate q_j is associated a *generalized force* P_j, which is defined as follows. If the work required to increase the coordinate q_j by δq_j is $P_j \delta q_j$, then P_j is the generalized force associated with the coordinate q_j.

> A generalized force need not always be dimensionally equivalent to a force. For example, if a generalized coordinate is an angle, the corresponding generalized force will be a torque.

One of the things that we shall want to do is to identify the generalized force associated with a given generalized coordinate.

13.3: Holonomic Constraints

The complete description of a system of *[Math Processing Error]* unconstrained particles requires *[Math Processing Error]* coordinates. You can think of the state of the system at any time as being represented by a single point in *[Math Processing Error]*-dimensional space. If the system consists of molecules in a gas, or a cluster of stars, or a swarm of bees, the coordinates will be continually changing, and the point that describes the system will be moving, perhaps completely unconstrained, in its *[Math Processing Error]*-dimensional space.

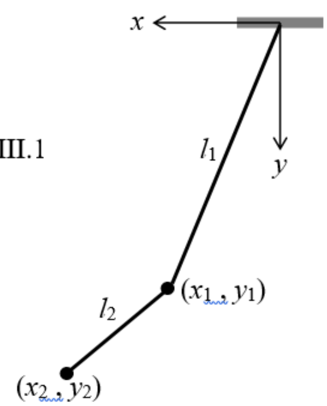
However, in many systems, the particles may not be free to wander anywhere at will; they may be subject to various *constraints*. A constraint that can be described by an equation relating the coordinates (and perhaps also the time) is called a *holonomic constraint*, and the equation that describes the constraint is a *holonomic equation*. If a system of *[Math Processing Error]* particles is subject to *[Math Processing Error]* holonomic constraints, the point in *[Math Processing Error]*-dimensional space that describes the system at any time is not free to move anywhere in *[Math Processing Error]*-dimensional space, but it is constrained to move over a surface of dimension *[Math Processing Error]*. In effect only *[Math Processing Error]* coordinates are needed to describe the system, given that the coordinates are connected by *[Math Processing Error]* holonomic equations.

Incidentally, I looked up the word "holonomic" in *The Oxford English Dictionary* and it said that the word was from the Greek ὅ*[Math Processing Error]*, meaning "whole" or "entire" and *[Math Processing Error]*, meaning "law". It also said "applied to a constrained system in which the equations defining the constraints are integrable or already free of differentials, so that each equation effectively reduces the number of coordinates by one; also applied to the constraints themselves."

As an example, consider a bar of wet soap slithering around in a hemispherical basin of radius *[Math Processing Error]*. You can describe its position in the basin by means of the usual two spherical angles *[Math Processing Error]*; the motion is otherwise constrained by its remaining in contact with the basin; that is to say it is subject to the holonomic constraint *[Math Processing Error]*. Thus instead of needing three coordinates to describe the position of a totally unconstrained particle, we need only two coordinates.

Or again, consider the double pendulum shown in Figure XIII.1, and suppose that the pendulum is constrained to swing only in the plane of the paper – or of the screen of your computer monitor.

FIGURE XIII.1



Two unconstrained particles would require six coordinates to specify their positions but this system is subject to four holonomic constraints. The holonomic equations *[Math Processing Error]* and *[Math Processing Error]* constrain the particles to be moving in a plane, and, if the strings are kept taut, we have the additional holonomic constraints *[Math Processing Error]* and *[Math Processing Error]*. Thus only two coordinates are needed to describe the system, and they could conveniently be the angles that the two strings make with the vertical.

13.4: The Lagrangian Equations of Motion

This section might be tough – but do not be put off by it. I promise that, after we have got over this section, things will be easy. But in this section I do not like all these summations and subscripts any more than you do.

Suppose that we have a system of N particles, and that the force on the ith particle ($i = 1$ to N) is $\mathbf{F_i}$. If the ith particle undergoes a displacement $\delta\mathbf{r_i}$, the total work done on the system is $\sum_i \mathbf{F_i}\cdot\partial\mathbf{r_i}$. The position vector $\mathbf{r}$ of a particle can be written as a function of its generalized coordinates; and a change in $\mathbf{r}$ can be expressed in terms of the changes in the generalized coordinates. Thus the total work done on the system is

$$\sum_i \mathbf{F_i}\cdot\sum_{\mathbf{j}}\frac{\partial\mathbf{r_i}}{\partial\mathbf{q_j}}\delta\mathbf{q_j} \tag{13.4.1}$$

which can be written

$$\sum_j\sum_i \mathbf{F_i}\cdot\frac{\partial\mathbf{r_i}}{\partial\mathbf{q_j}}\delta\mathbf{q_j}. \tag{13.4.2}$$

But by definition of the generalized force, the work done on the system is also

$$\sum_j P_j\cdot\delta q_j. \tag{13.4.3}$$

Thus the generalized force P_j associated with generalized coordinate q_j is given by

$$P_j = \sum_i \mathbf{F_i}\cdot\frac{\partial\mathbf{r_i}}{\partial\mathbf{q_j}}. \tag{13.4.4}$$

Now $\mathbf{F_i} = \mathbf{m_i}\ddot{\mathbf{r}}_\mathbf{i}$, so that

$$P_j = \sum_i m_i\ddot{r}_i\cdot\frac{\partial\mathbf{r_i}}{\partial q_j}. \tag{13.4.5}$$

Also

$$\frac{d}{dt}\left(\dot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r_i}}{\partial q_j}\right) = \ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r_i}}{\partial q_j} + \dot{\mathbf{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{r_i}}{\partial q_j}\right). \tag{13.4.6}$$

Substitute for $\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r_i}}{\partial q_j}$ from Equation 13.4.6 into Equation 13.4.5 to obtain

$$P_j = \sum_i m_i\left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r_i}}{\partial q_j}\right) - \dot{\mathbf{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial\mathbf{r_i}}{\partial q_j}\right)\right]. \tag{13.4.7}$$

Now

$$\frac{\partial\mathbf{r}_i}{\partial q_j} = \frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j} \tag{13.4.8}$$

and

$$\frac{d}{dt}\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right) = \frac{\partial\dot{\mathbf{r}}_i}{\partial q_j}. \tag{13.4.9}$$

Therefore

$$P_j = \sum_i m_i\left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i\cdot\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j}\right) - \dot{\mathbf{r}}_i\cdot\left(\frac{\partial\dot{\mathbf{r}}_i}{\partial q_j}\right)\right] \tag{13.4.10}$$

> You may not be immediately comfortable with the assertions in Equations 13.4.8 and 13.4.9 so I'll interrupt the flow briefly here with an example to try to justify these assertions and to understand what they mean.

Consider the relation between the coordinate x and the spherical coordinates r, θ, ϕ:

$$x = r\sin\theta\cos\phi \tag{A1}$$

In this example, x would correspond to one of the components of $\mathbf{r_i}$, and r, θ, ϕ are the q_1, q_2, q_3.

From Equation A1, we easily derive

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \tag{A2.1}$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \tag{A2.2}$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi \tag{A2.3}$$

and differentiating Equation A1 with respect to time, we obtain

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\dot{\theta}\cos\phi - r\sin\theta\sin\phi\dot{\phi} \tag{A3}$$

And from this we see that

$$\frac{\partial \dot{x}}{\partial \dot{r}} = \sin\theta\cos\phi \tag{A4.1}$$

$$\frac{\partial \dot{x}}{\partial \dot{\theta}} = r\cos\theta\cos\phi \tag{A4.2}$$

$$\frac{\partial \dot{x}}{\partial \dot{\phi}} = -r\sin\theta\sin\phi \tag{A4.3}$$

Thus the first assertion is justified in this example, and I think you'll see that it will always be true no matter what the functional dependence of $\mathbf{r_i}$ on the q_j.

For the second assertion, consider

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \tag{A5}$$

and hence

$$\frac{d}{dt}\frac{\partial x}{\partial r} = \cos\theta\dot{\theta}\cos\phi - \sin\theta\sin\phi\dot{\phi}. \tag{A6}$$

From Equation A3 we find that

$$\frac{\partial \dot{x}}{\partial r} = \cos\theta\dot{\theta}\cos\phi - \sin\theta\sin\phi\dot{\phi}, \tag{A7}$$

and the second assertion is justified. Again, I think you'll see that it will always be true no matter what the functional dependence of $\mathbf{r_i}$ on the q_j.

The kinetic energy T is

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2 = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \tag{13.4.11}$$

Therefore

$$\frac{\partial T}{\partial q_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \tag{13.4.12}$$

and

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}. \tag{13.4.13}$$

On substituting these in Equation 13.4.10we obtain

$$P_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}. \tag{13.4.14}$$

This is one form of Lagrange's equation of motion, and it often helps us to answer the question posed in the last sentence of Section 13.2 – namely to determine the generalized force associated with a given generalized coordinate.

Conservative Forces

If the various forces in a particular problem are **conservative** (gravity, springs and stretched strings, including valence bonds in a molecule) then the generalized force can be obtained by the negative of the gradient of a potential energy function – i.e. $P_j = -\frac{\partial V}{\partial q_j}$. In that case, Lagrange's equation takes the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}. \tag{13.4.15}$$

In my experience, this is the most useful and most often encountered version of Lagrange's equation.

The quantity $L = T - V$ is known as the **lagrangian** for the system, and Lagrange's equation can then be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0. \tag{13.4.16}$$

This form of the equation is seen more often in theoretical discussions than in the practical solution of problems. It does enable us to see one important result. If, for one of the generalized coordinates, $\frac{\partial L}{\partial q_j} = 0$ (this could happen if neither T nor V depends on q_j – but of course it could also happen if $\frac{\partial T}{\partial q_j}$ and $\frac{\partial V}{\partial q_j}$ were nonzero but equal and opposite in sign), then that generalized coordinate is called an *ignorable coordinate* – presumably because one can ignore it in setting up the lagrangian. However, it does not really mean that it should be ignored altogether, because it *immediately reveals a constant* of the motion. In particular, if $\frac{\partial L}{\partial q_j}$ = 0, then $\frac{\partial L}{\partial \dot{q}_j}$ *is constant*. It will be seen that if q_j has the dimensions of length, $\frac{\partial L}{\partial \dot{q}_j}$ has the dimensions of linear momentum. And if q_i is an angle, $\frac{\partial L}{\partial \dot{q}_j}$ has the dimensions of angular momentum. The derivative $\frac{\partial L}{\partial \dot{q}_j}$ is usually given the symbol p_j and is called *the generalized momentum conjugate to the generalized coordinate* q_j. If q_j is an "ignorable coordinate", then p_j is a constant of the motion.

In each of Equations 13.4.14, 13.4.15and 13.4.16one of the qs has a dot over it. You can see which one it is by thinking about the *dimensions* of the various terms. Dot has dimension T^{-1}.

So, we have now derived Lagrange's equation of motion. It was a hard struggle, and in the end we obtained three versions of an equation which at present look quite useless. But from this point, things become easier and we rapidly see how to use the equations and find that they are indeed very useful.

13.5: Acceleration Components

In Section 3.4 of the Celestial Mechanics "book", I derived the radial and transverse components of velocity and acceleration in two-dimensional coordinates. The radial and transverse velocity components are fairly obvious and scarcely need derivation; they are just $\dot{\rho}$ and $\rho\dot{\phi}$. For the acceleration components I reproduce here an extract from that chapter:

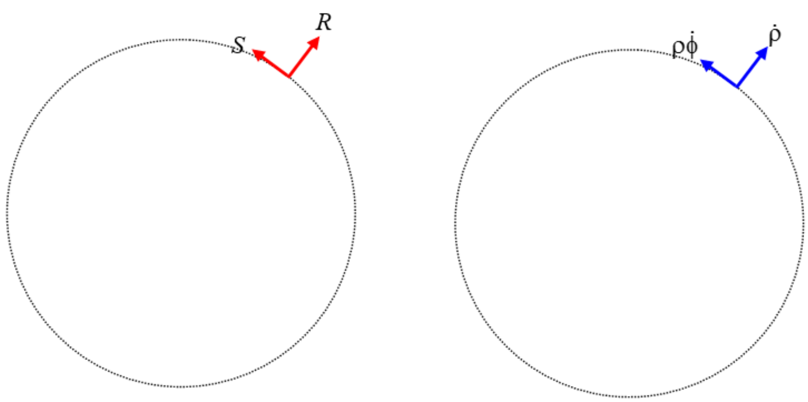
"The radial and transverse components of acceleration are therefore $(\ddot{\rho}-\rho\dot{\phi}^2)$ and $(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi})$ respectively."

I also derived the radial, meridional and azimuthal components of velocity and acceleration in three-dimensional spherical coordinates. Again the velocity components are rather obvious; they are $\dot{r}$, $r\dot{\theta}$ and $r\sin\theta\dot{\phi}$ while for the acceleration components I reproduce here the relevant extract from that chapter.

"On gathering together the coefficients of $\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}$ we find that the components of acceleration are:

- Radial: $\ddot{r}-r\dot{\theta}^2-r\sin^2\theta\dot{\phi}^2$
- Meridional: $r\ddot{\theta}+2\dot{r}\dot{\theta}-r\sin\theta\cos\theta\dot{\phi}^2$
- Azimuthal: $2\dot{r}\dot{\phi}\sin\theta+2r\dot{\theta}\dot{\phi}\cos\theta+r\sin\theta\ddot{\phi}$. "

You might like to look back at these derivations now. However, I am now going to derive them by a different method, using Lagrange's equation of motion. You can decide for yourself which you prefer.



We'll start in two dimensions. Let R and S be the radial and transverse components of a force acting on a particle. ("Radial" means in the direction of increasing ρ; "transverse" means in the direction of increasing ϕ.) If the radial coordinate were to increase by $\delta\rho$, the work done by the force would be just $R\delta\rho$. Thus the generalized force associated with the coordinate ρ is just $P_\rho=R$. If the azimuthal angle were to increase by $\delta\phi$, the work done by the force would be $S\rho\delta\phi$. Thus the generalized force associated with the coordinate ϕ is $P_\phi=S\rho$. Now we do not have to think about how to start; in Lagrangian mechanics, the first line is always "T= ...", and I hope you'll agree that

$$T=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\phi}^2). \tag{13.5.1}$$

If you now apply Equation 13.4.12 in turn to the coordinates ρ and ϕ, you obtain

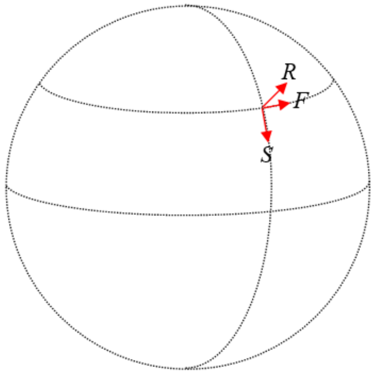
$$P_\rho=m(\ddot{\rho}-\rho\dot{\phi}^2) \quad and \quad P_\phi=m\rho(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi}), \tag{13.5.2a,b}$$

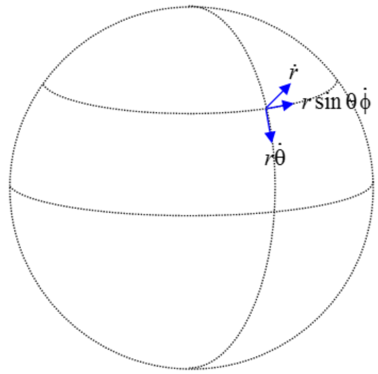
and so

$$R=m(\ddot{\rho}-\rho\dot{\phi}^2) \quad and \quad S=m(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi}). \tag{13.5.3a,b}$$

Therefore the radial and transverse components of the acceleration are $(\ddot{\rho}-\rho\dot{\phi}^2)$ and $(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi})$ respectively.

We can do exactly the same thing to find the acceleration components in three-dimensional spherical coordinates. Let R, S and F be the radial, meridional and azimuthal (i.e. in direction of increasing r, θ and ϕ) components of a force on a particle.





- If r increases by δr, the work on the particle done is $R\delta r$.
- If θ increases by $\delta\theta$, the work done on the particle is $Sr\delta\theta$.
- If ϕ increases by $\delta\phi$, the work done on the particle is $Fr\sin\theta\delta\phi$.

Therefore $P_r = R, \quad P_\theta = Sr$ and $P_\phi = Fr\sin\theta$.

Start:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \tag{13.5.4}$$

If you now apply Equation 13.4.12 in turn to the coordinates r, θ and ϕ, you obtain

$$P_r = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2), \tag{13.5.5}$$

$$P_\theta = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2) \tag{13.5.6}$$

and

$$P_\phi = m(r^2\sin^2\theta\ddot{\phi} + 2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta + 2r\dot{r}\dot{\phi}\sin^2\theta). \tag{13.5.7}$$

Therefore

$$R = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2), \tag{13.5.8}$$

$$S = m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2) \tag{13.5.9}$$

and

$$F = m(r\sin\theta\ddot{\phi} + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta). \tag{13.5.10}$$

Thus the acceleration components are

- Radial: $\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2$
- Meridional: $r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2$
- Azimuthal: $2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}$.

Be sure to check the dimensions. Since dot has dimension T^{-1}, and these expressions must have the dimensions of acceleration, there must be an r and two dots in each term.

13.6: Slithering Soap in Conical Basin

We imagine a slippery (no friction) bar of soap slithering around in a conical basin. An isolated bar of soap in intergalactic space would require three coordinates to specify its position at any time, but, if it is subject to the holonomic constraint that it is to be in contact at all times with a conical basin, its position at any time can be specified with just two coordinates. I shall, first of all, analyse the problem with a newtonian approach, and then, for comparison, I shall analyse it using lagrangian methods. Either way, we start with a large diagram. In the newtonian approach we mark in the forces in red and the accelerations in green. See Figure XIII.2. The semi vertical angle of the cone is α.

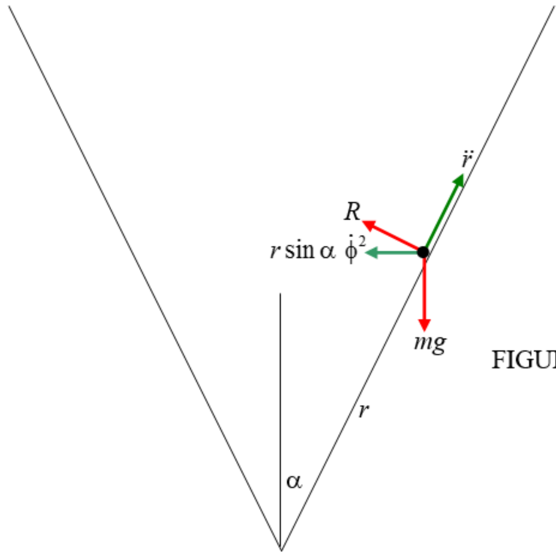


FIGURE XIII.2

The two coordinates that we need are r, the distance from the vertex, and the azimuthal angle ϕ, which I'll ask you to imagine, measured around the vertical axis from some arbitrary origin. The two forces are the weight mg and the normal reaction R of the basin on the soap. The accelerations are $\ddot{r}$ and the centripetal acceleration as the soap moves at angular speed $\dot{\phi}$ in a circle of radius $r\sin\alpha$ is $r\sin\alpha\dot{\phi}^2$.

We can write the newtonian equation of motion in various directions:

Horizontal: $R\cos\alpha = m(r\sin\alpha\dot{\phi}^2 - \ddot{r}\sin\alpha)$

i.e.

$$R = m\tan\alpha(r\dot{\phi}^2 - \ddot{r}). \tag{13.6.1}$$

Vertical:

$$R\sin\alpha - mg = m\ddot{r}\cos\alpha. \tag{13.6.2}$$

Perpendicular to surface:

$$R - mg\sin\alpha = mr\sin\alpha\cos\alpha\dot{\phi}^2. \tag{13.6.3}$$

Parallel to surface:

$$g\cos\alpha = r\sin^2\alpha\dot{\phi}^2 - \ddot{r}. \tag{13.6.4}$$

Only two of these are independent, and we can choose to use whichever two we want to at our convenience. There are, however, three quantities that we may wish to determine, namely the two coordinates r and ϕ, and the normal reaction R. Thus we need another equation. We note that, since there are no azimuthal forces, the angular momentum per unit mass, which is $r^2\sin^2\alpha\dot{\phi}$, is conserved, and therefore $r^2\dot{\phi}$ is constant and equal to its initial value, which I'll call $l^2\Omega$. That is, we start off at a distance l from the vertex with an initial angular speed Ω. Thus we have as our third independent equation

$$r^2\dot{\phi} = l^2\Omega \tag{13.6.5}$$

This last equation shows that $\dot{\phi} \to \infty$ as $r \to 0$.

One possible type of motion is circular motion at constant height (put $\ddot{r} = 0$). From Equations 13.6.1 and 13.6.2 it is easily found that the condition for this is that

$$r\dot{\phi}^2 = \frac{g}{\sin\alpha\tan\alpha}. \tag{13.6.6}$$

In other words, if the particle is projected initially horizontally ($\dot{r} = 0$) at $r = l$ and $\dot{\phi} = \Omega$, it will describe a horizontal circle (for ever) if

$$\Omega = \left(\frac{g}{l\sin\alpha\tan\alpha}\right)^{\frac{1}{2}} = \Omega_C, \quad say. \tag{13.6.7}$$

If the initial speed is less than this, the particle will describe an elliptical orbit with a minimum $r < l$; if the initial speed is greater than this, the particle will describe an elliptical orbit with a maximum $r > l$.

Now let's do the same problem in a lagrangian formulation. This time we draw the same diagram, but we mark in the velocity components in blue. See Figure XIII.3. We are dealing with conservative forces, so we are going to use Equation 13.4.13, the most useful form of Lagrange's equation.

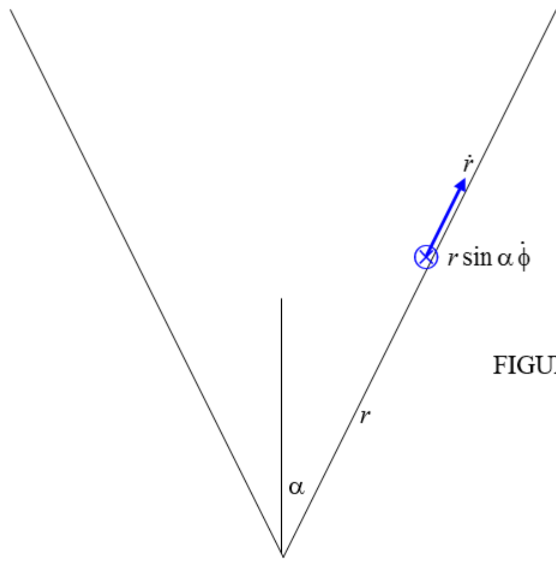


FIGURE XIII.3

We need not spend time wondering what to do next. The first and second things we always have to do are to find the kinetic energy T and the potential energy V, in order that we can use Equation 13.4.13.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\dot{\phi}^2) \tag{13.6.8}$$

and

$$V = mgr\cos\alpha + constant. \tag{13.6.9}$$

Now go to Equation 13.4.13, with $q_i = r$, and work out all the derivatives, and you should get, when you apply the lagrangian equation to the coordinate r:

$$\ddot{r} - r\sin^2\alpha\dot{\phi}^2 = -g\cos\alpha. \tag{13.6.10}$$

Now do the same thing with the coordinate ϕ. You see immediately that $\frac{\partial T}{\partial \phi}$ and $\frac{\partial V}{\partial \phi}$ are both zero. Therefore $\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}}$ is zero and therefore $\frac{\partial T}{\partial \dot{\phi}}$ is constant. That is, $mr^2\sin^2\alpha\dot{\phi}$ is constant and so $r^2\dot{\phi}$ is constant and equal to its initial value $l^2\Omega$. Thus the second lagrangian equation is

$$r^2\dot{\phi} = l^2\Omega. \tag{13.6.11}$$

Since the lagrangian is independent of ϕ, ϕ is called, in this connection, an “ignorable coordinate” – and the momentum associated with it, namely $mr^2\dot{\phi}$ is constant.

Now it is true that we arrived at both of these equations also by the newtonian method, and you may not feel we have gained much. But this is a simple, introductory example, and we shall soon appreciate the power of the lagrangian method,

Having got these two equations, whether by newtonian or lagrangian methods, let’s explore them further. For example, let’s eliminate $\dot{\phi}$ between them and hence get a single equation in r:

$$\ddot{r} - \frac{l^4\Omega^2\sin^2\alpha}{r^3} = -g\cos\alpha. \tag{13.6.12}$$

We know enough by now (see Chapter 6) to write $\ddot{r}$ as $v\frac{dv}{dr}$, where $v = \dot{r}$ and if we let the constants $l^4\Omega^2\sin^2\alpha$ and $g\cos\alpha$ equal A and B respectively, Equation 13.6.12becomes

$$\nu\frac{d\nu}{dr} = \frac{A}{r^3} - B. \tag{13.6.13}$$

(It may just be useful to note that the dimensions of A and B are L^4T^{-2} and LT^{-2} respectively. This will enable us to keep track of dimensional analysis as we go.)

If we start the soap moving horizontally ($v = 0$) when $r = l$, this integrates, with these initial conditions, to

$$\nu^2 = A(\frac{1}{l^2} - \frac{1}{r^2}) + 2B(l-r). \tag{13.6.14}$$

Again, so that we can see what we are doing, let $\frac{A}{l^2} + 2Bl = C$ (note that $[C] = L^2T^{-2}$), and Equation 13.6.14becomes

$$\nu^2 = C - \frac{A}{r^2} - 2Br. \tag{13.6.15}$$

This gives $\nu(=\dot{r})$ as a function of r. The particle reaches is maximum or minimum height when $\nu = 0$; that is where

$$2Br^3 - Cr^2 + A = 0. \tag{13.6.16}$$

One solution of this is obviously $r = l$. Of the other two solutions, one is positive (which we want) and the other is negative (which we do not want).

If we go back to the original meanings of A, B and C, and write $x = \frac{r}{l}$ equation (16) becomes, after a little tidying up

$$x^3 - (\frac{l\Omega^2\sin\alpha\tan\alpha}{2g} + 1)x^2 + \frac{l\Omega^2\sin\alpha\tan\alpha}{2g} = 0. \tag{13.6.17}$$

Recall from Equation 13.6.7that $\Omega_c = (\frac{g}{l\sin\alpha\tan\alpha})^{\frac{1}{2}}$, and the equation becomes

$$x^3 - (\frac{\Omega^2}{2\Omega_c^2} + 1)x^2 + \frac{\Omega^2}{2\Omega_c^2}, \tag{13.6.18}$$

or, with $a = \frac{\Omega^2}{2\Omega_c^2}$,

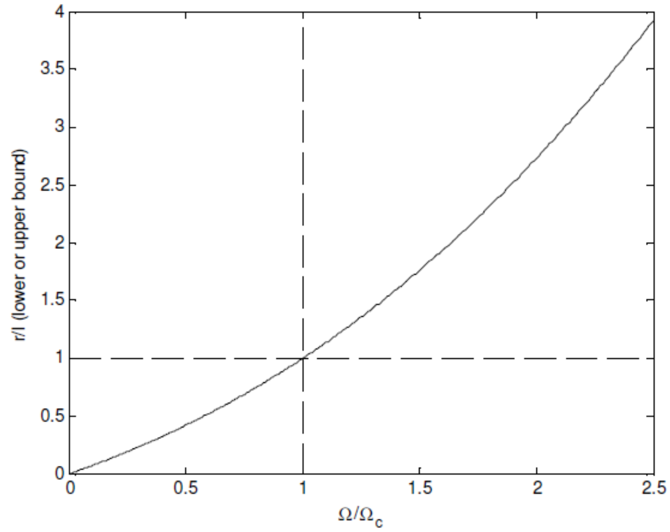
$$x^3 - (a+1)x^2 + a = 0. \tag{13.6.19}$$

This factorizes to

$$(x-1)(x^2 - ax - a) = 0. \tag{13.6.20}$$

The solution we are interested in is

$$x = \frac{1}{2}(a + \sqrt{a(a+4)}). \tag{13.6.21}$$


r/l (lower or upper bound)
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
Ω/Ω$_c$

13.7: Slithering Soap in Hemispherical Basin

Suppose that the basin is of radius a and the soap is subject to the holonomic constraint $r=a$ - i.e. that it remains in contact with the basin at all times. Note also that this is just the same constraint of a pendulum free to swing in three-dimensional space except that it is subject to the holonomic constraint that the string be taut at all times. Thus any conclusions that we reach about our soap will also be valid for a pendulum.

We'll start with the newtonian approach, and I'll draw in red the two forces on the soap, namely its weight and the normal reaction of the basin on the soap. Figure XIII.4

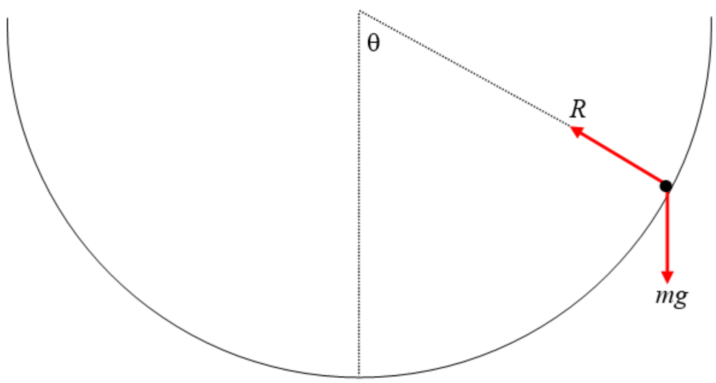


FIGURE XIII.4

We'll make use of the expressions for the radial, meridional and azimuthal accelerations from Section 13.5 and we'll write down the equations of motion in these directions:

Radial:

$$mg\cos\theta - R = m(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2). \tag{13.7.1}$$

Meridional:

$$-mg\sin\theta = m(r\ddot{\theta} - 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2), \tag{13.7.2}$$

Azimuthal:

$$0 = m(2\dot{r}\dot{\phi}\sin\theta + 2r\dot{r}\dot{\phi}\cos\theta + r\sin\theta\ddot{\phi}) \tag{13.7.3}$$

We also have the constraint that $r=a$ and hence that $\dot{r}=\ddot{r}=0$, after which these equations become

$$mg\sin\theta - R = -ma(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2), \tag{13.7.4}$$

$$-g\sin\theta = a(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2), \tag{13.7.5}$$

$$0 = \sin\theta\ddot{\phi} + 2\dot{\theta}\dot{\phi}\cos\theta. \tag{13.7.6}$$

These, then, are the newtonian equations of motion. If you still prefer the newtonian method to the lagrangian method, and you wish to integrate these and find expressions θ, ϕ and R separately, by all means go ahead and do so – but I'm now going to try the lagrangian approach.

Although Lagrange himself would not have drawn a diagram, we shall not omit that step – but instead of marking in the forces, we'll mark in the velocity components, and then we'll immediately write down expressions for the kinetic and potential energies. Indeed the first line of a lagrangian calculation is always "$T=$...".

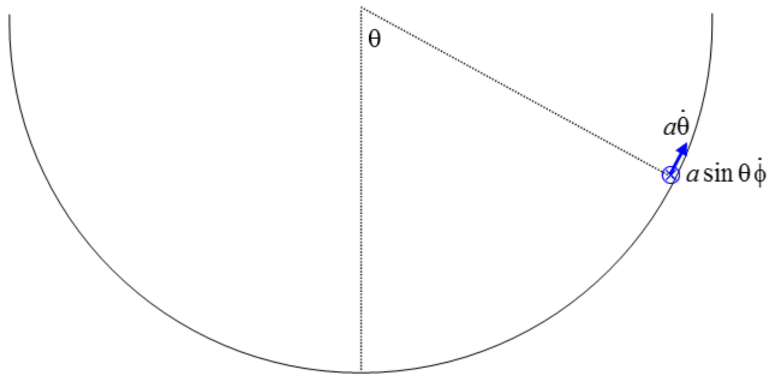


FIGURE XIII.5

$$T=\frac{1}{2}ma^2(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2) \tag{13.7.7}$$

$$V=-mga\cos\theta+constant \tag{13.7.8}$$

Now apply Equation 13.4.13 in turn to the coordinates θ and ϕ.

θ:

$$a\ddot{\theta}-a\sin\theta\cos\theta\dot{\phi}^2=-g\sin\theta. \tag{13.7.9}$$

ϕ:

As for the conical basin, we see that $\frac{\partial T}{\partial \phi}$ and $\frac{\partial V}{\partial \phi}$ are both zero (ϕ is an "ignorable coordinate") and therefore $\frac{\partial T}{\partial \dot{\phi}}$ is constant and equal to its initial value. If the initial values of $\dot{\phi}$ and θ are Ω and α respectively, then

$$\sin^2\theta\dot{\phi}=\sin^2\alpha\Omega. \tag{13.7.10}$$

This is merely stating that angular momentum is conserved.

We can easily eliminate $\dot{\phi}$ from Equations 13.4.9 and 13.4.10 to obtain

$$\ddot{\theta}=k\cot\theta\csc^2\theta-\frac{g\sin\theta}{a}, \tag{13.7.11}$$

where

$$k=\sin^4\alpha\Omega^2. \tag{13.7.12}$$

Write $\ddot{\theta}$ as $\dot{\theta}\frac{d\dot{\theta}}{d\theta}$ in the usual way and integrate to obtain the first space integral:

$$\dot{\theta}^2=\frac{2g}{a}(\cos\theta-\cos\alpha)-k(\csc^2\theta-\csc^2\alpha). \tag{13.7.13}$$

The upper and lower bounds for θ occur when $\dot{\theta}=0$

Example. Suppose that the initial value of θ is α = 45° and that we start by pushing the soap horizontally ($\dot{\theta}=0$) at an initial angular speed Ω = 3 rad s^{-1}, so that k = 2.25 rad^2 s^{-2}. Suppose that the radius of the basin is a = 1.96 m and that g = 9.8 m s^{-2}. You can then put $\dot{\theta}=0$ in Equation 13.7.13 and solve it for θ. One solution, of course, is $\theta=\alpha$. We could find the other solution by Newton-Raphson iteration, or by putting $\csc^2\theta=\frac{1}{(1-\cos^2\theta)}$ and solving it as a cubic equation in $\cos\theta$. Alternatively, try this:

Let $\frac{ak}{2g}=n, \quad \cos\theta=x, \quad \cos\alpha=c,$

so that $\csc^2\theta=\frac{1}{(1-x^2)}$ and $\csc^2\alpha=\frac{1}{(1-c^2)}$.

The equation then becomes a quadratic equation in x, with solution

That is, $x = \frac{-n+\sqrt{n^2-4(1-c^2)[c(n+c)-1]}}{2(1-c^2)}$

I'll leave you to re-write this in terms of what these quantities originally meant.

13.8: More Lagrangian Mechanics Examples

Example 13.8.1

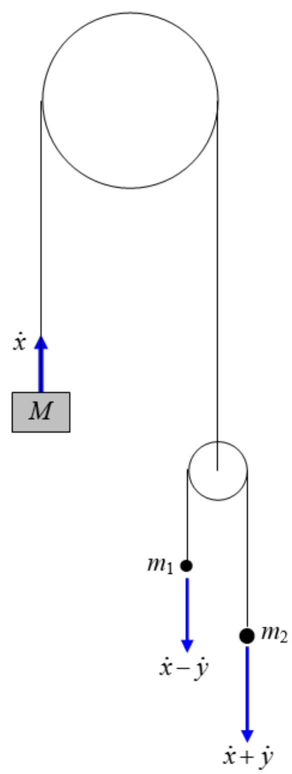


FIGURE XIII.6

The upper pulley is fixed in position. Both pulleys rotate freely without friction about their axles. Both pulleys are "light" in the sense that their rotational inertias are small and their rotation contributes negligibly to the kinetic energy of the system. The rims of the pulleys are rough, and the ropes do not slip on the pulleys. The gravitational acceleration is g.

The mass M moves upwards at a rate $\dot{x}$ with respect to the upper, fixed, pulley, and the smaller pulley moves downwards at the same rate. The mass m_1 moves upwards at a rate $\dot{y}$ with respect to the small pulley, and consequently its speed in laboratory space is $\dot{x}-\dot{y}$. The speed of the mass m_2 is therefore $\dot{x}+\dot{y}$ in laboratory space. The object is to find $\ddot{x}$ and $\ddot{y}$ in terms of g.

The kinetic energy is

$$T=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m_1(\dot{x}-\dot{y})^2+\frac{1}{2}m_2(\dot{x}+\dot{y})^2. \tag{13.8.1}$$

The potential energy is

$$V=g[Mx-m_1(x-y)-m_2(x+y)]+constant. \tag{13.8.2}$$

Apply Lagrange's equation (13.4.13) in turn to the coordinates x and y:

x:

$$M\ddot{x}+m_1(\ddot{x}-\ddot{y})+m_2(\ddot{x}+\ddot{y})=-g(M-m_1-m_2). \tag{13.8.3}$$

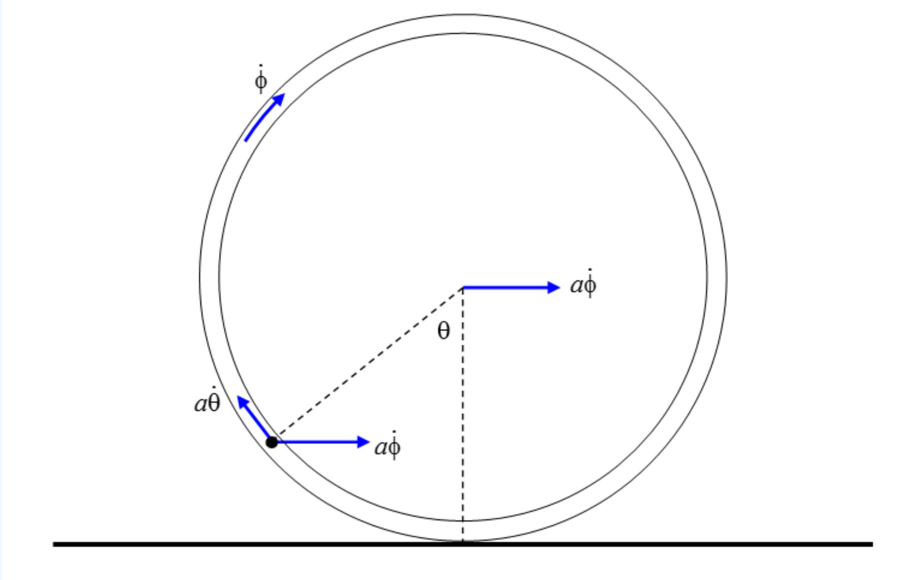
y:

$$-m_1(\ddot{x}-\ddot{y})+m_2(\ddot{x}+\ddot{y})=-g(m_1-m_2) \tag{13.8.4}$$

These two equations can be solved at one's leisure for $\ddot{x}$ and $\ddot{y}$.

Example 13.8.2

A torus of mass M and radius a rolls without slipping on a horizontal plane. A pearl of mass m slides smoothly around inside the torus. Describe the motion.



I have marked in the several velocity vectors. The torus is rolling at angular speed $\dot{\phi}$ Consequently the linear speed of the centre of mass of the hoop is $a\dot{\phi}$ and the pearl also shares this velocity. In addition, the pearl is sliding relative to the torus at an angular speed $\dot{\theta}$ and consequently has a component to its velocity of $a\dot{\theta}$ tangential to the torus. We are now ready to start.

The kinetic energy of the torus is the sum of its translational and rotational kinetic energies:

$$\tfrac{1}{2}M(a\dot{\phi})^2+\tfrac{1}{2}(Ma^2)\dot{\phi}^2=Ma^2\dot{\phi}^2$$

The kinetic energy of the pearl is

$$\tfrac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2-2\dot{\theta}\dot{\phi}\cos\theta)$$

Therefore

$$T=Ma^2\dot{\phi}^2+\frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2-2\dot{\theta}\dot{\phi}\cos\theta). \tag{13.8.5}$$

The potential energy is

$$V=constant-mga\cos\theta. \tag{13.8.6}$$

The lagrangian equation in θ becomes

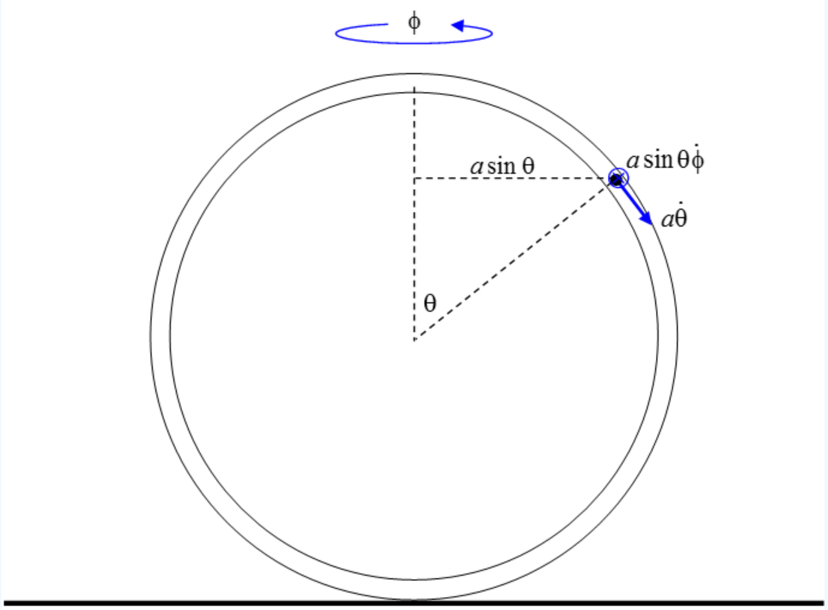
$$a(\ddot{\theta}-\ddot{\phi}\cos\theta+\dot{\phi}sin\theta\dot{\theta})+g\sin\theta=0. \tag{13.8.7}$$

The lagrangian equation in ϕ becomes

$$(2M+m)\ddot{\phi}=m(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta) \tag{13.8.8}$$

These, then, are two differential equations in the two variables. The lagrangian part of the analysis is over.

Example 13.8.3



As in example ii, we have a torus of radius a and mass M, and a pearl of mass m which can slide freely and without friction around the torus. This time, however, the torus is not rolling along the table, but is spinning about a vertical axis at an angular speed $\dot{\phi}$. The pearl has a velocity component $a\dot{\theta}$ because it is sliding around the torus, and a component $a\sin\theta\dot{\phi}$ because the torus is spinning. The resultant speed is the orthogonal sum of these. The kinetic energy of the system is the sum of the translational kinetic energy of the pearl and the rotational kinetic energy of the torus:

$$T = \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{2}(\frac{1}{2}Ma^2)\dot{\phi}^2. \tag{13.8.9}$$

If we refer potential energy to the centre of the torus:

$$V = mga\cos\theta. \tag{13.8.10}$$

The lagrangian equations with respect to the two variables are:

θ:

$$a(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2) - g\sin\theta = 0. \tag{13.8.11}$$

ϕ:

$$m\sin^2\theta\dot{\phi} + \frac{1}{2}M\dot{\phi} = constant. \tag{13.8.12}$$

The constant is equal to whatever the initial value of the left hand side was. E.g., maybe the initial values of θ and $\dot{\phi}$ were α and ω. This finishes the lagrangian part of the analysis. The rest is up to you. For example, it would be easy to eliminate $\dot{\phi}$ between these two equations to obtain a differential equation between θ and the time. If you then write $\ddot{\theta}$ as $\dot{\theta}\frac{d\dot{\theta}}{d\theta}$ in the usual way, I think it wouldn't be too difficult to obtain the first space integral and hence get $\dot{\theta}$ as a function of θ. I haven't tried it, but I'm sure it'll work.

Example 13.8.4

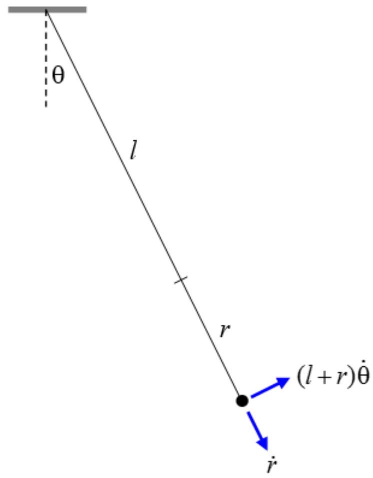


FIGURE XIII.9

Figure XIII.10 shows a pendulum. The mass at the end is m. It is at the end not of the usual inflexible string, but of an elastic spring obeying Hooke's law, of force constant k. The spring is sufficiently stiff at right angles to its length that it remains straight during the motion, and all the motion is restricted to a plane. The unstretched natural length of the spring is l, and, as shown, its extension is r. The spring itself is "light" in the sense that it does not contribute to the kinetic or potential energies. (You can give the spring a finite mass if you want to make the problem more difficult.) The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{r}^2 + (l+r)^2\dot{\theta}^2) \tag{13.8.13}$$

and

$$V = constant - mg(l+r)\cos\theta + \frac{1}{2}kr^2. \tag{13.8.14}$$

Apply Lagrange's equation in turn to r and to θ and see where it leads you.

Example 13.8.5

Another example suitable for lagrangian methods is given as problem number 11 in Appendix A of these notes.

Lagrangian methods are particularly applicable to vibrating systems, and examples of these will be discussed in Chapter 17. These chapters are being written in more or less random order as the spirit moves me, rather than in logical order, so that vibrating systems appear after the unlikely sequence of relativity and hydrostatics.

13.9: Hamilton's Variational Principle

Hamilton's variational principle in dynamics is slightly reminiscent of the principle of virtual work in statics, discussed in Section 9.4 of Chapter 9. When using the principle of virtual work in statics we imagine starting from an equilibrium position, and then increasing one of the coordinates infinitesimally. We calculate the virtual work done and set it to zero. I am slightly reminded of this when discussing Hamilton's principle in dynamics

Imagine some mechanical system – some contraption including in its construction various wheels, jointed rods, springs, elastic strings, pendulums, inclined planes, hemispherical bowls, and ladders leaning against smooth vertical walls and smooth horizontal floors. It may require N generalized coordinates to describe its configuration at any time. Its configuration could be described by the position of a point in N-dimensional space. Or perhaps it is subject to k holonomic constraints – in which case the point that describes its configuration in N-dimensional space is not free to move anywhere in that space, but is constrained to slither around on a surface of dimension $N-k$.

The system is not static, but it is evolving. It is changing from some initial state at time t_1 to some final state at time t_2. The generalized coordinates that describe it are changing with time – and the point in N-space is slithering round on its surface of dimension $N-k$. One can imagine that at any instant of time one can calculate its kinetic energy T and its potential energy V, and hence its lagrangian $L=T-V$. You can multiply L at some moment by a small time interval δt and then add up all of these products between t_1 and t_2 to form the integral

$$\int_{t_1}^{t_2} L\,dt.$$

This quantity – of dimension ML^2T^{-1} and SI unit J s – is sometimes called the "action". There are many different ways in which we can imagine the system to evolve from its initial state to its final state – and there are many different routes that we can imagine might be taken by our point in N-space as its moves from its initial position to its final position, as long as it moves over its surface of dimension $N-k$. But, although we can *imagine* many such routes, the manner in which the system will *actually* evolve, and the route that the point will actually take is determined by Hamilton's principle; and the route, according to this principle, is such that the integral $\int_{t_1}^{t_2} L dt$ is a minimum, or a maximum, or an inflection point, when compared with other imaginable routes. Stated otherwise, let us suppose that we calculate $\int_{t_1}^{t_2} L dt$ over the actual route taken and then calculate the *variation* in $\int_{t_1}^{t_2} L dt$ if the system were to move over a slightly different adjacent path. Then (and here is the analogy with the principle of virtual work in a statics problem) this *variation*

$$\delta\int_{t_1}^{t_2} L\,dt$$

from what $\int_{t_1}^{t_2} L dt$ would have been over the actual route is zero. And this is *Hamilton's variational principle*.

The next questions will surely be: Can I use this principle for solving problems in mechanics? Can I prove this bald assertion? Let me try to use the principle to solve two simple and familiar problems, and then move on to a more general problem.

Example 13.9.1

Imagine that we have a particle than can move in one dimension (i.e. one coordinate – for example its height y above a table - suffices to describe its position), and that when its coordinate is y its potential energy is

$$V=mgy. \tag{13.9.1}$$

Its kinetic energy is, of course,

$$T=\frac{1}{2}m\dot{y}^2. \tag{13.9.2}$$

We are going to use the variational principle to find the equation of motion – i.e we are going to find an expression for its acceleration. I imagine at the moment you have no idea what its acceleration could possibly be – but do not worry, for we know that the lagrangian is

$$L=\frac{1}{2}m\dot{y}^2-mgy, \tag{13.9.3}$$

and we'll make short work of it with Hamilton's variational principle and soon find the acceleration. According to this principle, y must vary with t in such a manner that

$$m\delta\int_{t_1}^{t_2}(\frac{1}{2}\dot{y}^2-gy)dt=0. \tag{13.9.4}$$

Let us vary $\dot{y}$ by $\delta\dot{y}$ and y by δy see how the integral varies.

The integral is then

$$m\int_{t_1}^{t_2}(\dot{y}\delta\dot{y}-g\delta y)dt, \tag{13.9.5}$$

which I'll call I_1-I_2 .

Now $\dot{y}=\frac{dy}{dt}$ and if y varies by δy, the resulting variation in $\dot{y}$ will be $\delta\dot{y}=\frac{d}{dt}\delta y$, or $\delta\dot{y}dt=d\delta y$.

Therefore

$$I_1=m\int_{t_1}^{t_2}\dot{y}d\delta y. \tag{13.9.6}$$

(If unconvinced of this, consider $\int e^t\cos tdt=\int e^t\frac{d}{dt}\sin tdt=\int e^td\sin t$.)

By integration by parts:

$$I_1=[m\dot{y}\delta y]_{t_1}^{t_2}-m\int_{t_1}^{t_2}\delta yd\dot{y}. \tag{13.9.7}$$

The first term is zero because the variation is zero at the beginning and end points. In the second term, $d\dot{y}=\ddot{y}dt$ and therefore

$$I_1=-m\int_{t_1}^{t_2}\ddot{y}\delta ydt \tag{13.9.8}$$

$$\delta\int_{t_1}^{t_2}Ldt=-m\int_{t_1}^{t_2}(\ddot{y}+g)\delta ydt, \tag{13.9.9}$$

and, for this to be zero, we must have

$$\ddot{y}=-g. \tag{13.9.10}$$

This is the equation of motion that we sought. You would never have guessed this, would you?

Now let's do another one-dimensional problem.

Example 13.9.2

Only one coordinate, x, describes the particle's position, and, when its coordinate is x we'll suppose that its potential energy is $V=\frac{1}{2}m\omega^2x^2$ and its kinetic energy is, of course, $T=\frac{1}{2}m\dot{x}^2$. The equation of motion, or the way in which the acceleration varies with position, must be such as to satisfy

$$\frac{1}{2}m\delta\int_{t_1}^{t_2}(\dot{x}^2-\omega^2x^2)dt=0. \tag{13.9.11}$$

If we vary $\dot{x}$ by $\delta\dot{x}$ and x by δx the variation in the integral will be

$$m\int_{t_1}^{t_2}(\dot{x}\delta\dot{x}-\omega^2x\delta x)dt=I_1-I_2. \tag{13.9.12}$$

By precisely the same argument as before, the first integral is found to be $-m\int_{t_1}^{t_2}\ddot{x}\delta x\,dt$

Therefore

$$\delta \int_{t_1}^{t_2} L\,dt = -m\int_{t_1}^{t_2} \ddot{x}\delta x\,dt - m\omega^2 \int_{t_1}^{t_2} x\delta x\,dt, \tag{13.9.13}$$

and, for this to be zero, we must have

$$\ddot{x} = -\omega^2 x. \tag{13.9.14}$$

These two examples must have given the impression that we are doing something very difficult in order to derive something that is immediately obvious – but the examples were just intended to show the direction of a more general argument we are about to make.

This time, we'll consider a very general system, in which we write the lagrangian as a function of the (several) generalized coordinates and their time rates of change - i.e. $L = L(q_i, \dot{q}_i)$ - without specifying any particular form of the function – and we'll carry out the same sort of argument to derive a very general equation of motion.

We have

$$\delta \int_{t_1}^{t_2} Ldt = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right) dt = 0. \tag{13.9.15}$$

As before, $\delta \dot{q}_i = \frac{d}{dt}\delta q_i$ so that

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} d\delta q_i = \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} dt \tag{13.9.16}$$

$$\delta \int_{t_1}^{t_2} Ldt = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt = 0. \tag{13.9.17}$$

Thus we arrive at the general equation of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0. \tag{13.9.18}$$

Thus we have derived Lagrange's equation of motion from Hamilton's variational principle, and this is indeed the way it is often derived. However, in this chapter, I derived Lagrange's equation quite independently, and hence I would regard this derivation not so much as a proof of Lagrange's equation, but as a vindication of the correctness of Hamilton's variational principle.

CHAPTER OVERVIEW

14: Hamiltonian Mechanics

Hamiltonian mechanics can be used to describe simple systems such as a bouncing ball, a pendulum or an oscillating spring in which energy changes from kinetic to potential and back again over time, its strength is shown in more complex dynamic systems, such as planetary orbits in celestial mechanics. The more degrees of freedom the system has, the more complicated its time evolution.

14.1: Introduction to Hamiltonian Mechanics

The hamiltonian equations of motion are of deep theoretical interest. Having established that, I am bound to say that I have not been able to think of a problem in *classical* mechanics that I can solve more easily by hamiltonian methods than by newtonian or lagrangian methods. That is not to say that real problems cannot be solved by hamiltonian methods. What I have been looking for is a problem which I can solve easily by hamiltonian methods but which is more difficult to solve by other methods. So far, I have not found one. Having said that, doubt not that hamiltonian mechanics is of deep theoretical significance.

Having expressed that mild degree of cynicism, let it be admitted that Hamilton theory – or more particularly its extension the Hamilton-Jacobi equations - does have applications in celestial mechanics, and of course hamiltonian operators play a major part in quantum mechanics, although it is doubtful whether Sir William would have recognized his authorship in that connection.

14.2: A Thermodynamics Analogy

Page ID 7017

Readers may have noticed from time to time – particularly in Chapter 9 - that I have perceived some connection between parts of classical mechanics and thermodynamics. I perceive such an analogy in developing hamiltonian dynamics. Those who are familiar with thermodynamics may also recognize the analogy. Those who are not can skip this section without seriously prejudicing their understanding of subsequent sections.

Please do not misunderstand: The hamiltonian in mechanics is not at all the same thing as enthalpy in thermodynamics, even though we use the same symbol, *[Math Processing Error]*. Yet there are similarities in the way we can introduce these concepts.

In thermodynamics we can describe the state of the system by its internal energy, defined in such a way that when heat is supplied **to** a system and the system does external work, the **increase** in internal energy of the system is equal to the heat supplied **to** the system minus the work done **by** the system:

[Math Processing Error]

From this point of view we are describing the state of the system by specifying its internal energy as a function of the entropy and the volume:

[Math Processing Error]

so that

[Math Processing Error]

from which we see that

[Math Processing Error]

and

[Math Processing Error]

However, it is sometimes convenient to change the basis of the description of the state of a system from *[Math Processing Error]* and *[Math Processing Error]* to *[Math Processing Error]* and *[Math Processing Error]* by defining a quantity called the enthalpy *[Math Processing Error]* defined by

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In that case, if the state of the system changes, then

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I.e.

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Thus we see that, if heat is added to a system held at constant *volume*, the increase in the *internal energy* is equal to the heat added; whereas if heat is added to a system held at constant *pressure*, the increase in the *enthalpy* is equal to the heat added.

From this point of view we are describing the state of the system by specifying its enthalpy as a function of the entropy and the pressure:

[Math Processing Error]

so that

[Math Processing Error]

from which we see that

[Math Processing Error]

and

[Math Processing Error]

None of this has anything to do with hamiltonian dynamics, so let's move on.

14.3: Hamilton's Equations of Motion

In classical mechanics we can describe the state of a system by specifying its Lagrangian as a function of the coordinates and their time rates of change:

$$L = L(q_i, \dot{q}) \tag{14.3.1}$$

If the coordinates and the velocities increase, the corresponding increment in the Lagrangian is

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i. \tag{14.3.2}$$

Definition: generalized momenta

The *generalized momentum* p_i associated with the generalized coordinate q_i is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{14.3.3}$$

[You have seen this before, in Section 13.4. Remember "ignorable coordinate"?]

It follows from the Lagrangian equation of motion (Equation 13.4.14)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

that

$$\dot{p}_i = \frac{\partial L}{\partial q_i}. \tag{14.3.4}$$

Thus

$$dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i. \tag{14.3.5}$$

(I am deliberately numbering this Equation 14.3.5, to maintain an analogy between this section and Section 14.2.)

However, it is sometimes convenient to change the basis of the description of the state of a system from q_i and $\dot{q}_i$ to q_i and $\dot{p}_i$ by defining a quantity called the hamiltonian H defined by

$$H = \sum_i p_i \dot{q}_i - L. \tag{14.3.6}$$

Definition: hamiltonian

In that case, if the state of the system changes, then

$$\begin{aligned} dH &= \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - dL \\ &= \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i - \sum_i p_i d\dot{q}_i \end{aligned}$$

That is

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \dot{p}_i dq_i. \tag{14.3.7}$$

We are regarding the hamiltonian as a function of the generalized coordinates and generalized momenta:

$$H = H(q_i, p_i) \tag{14.3.8}$$

so that

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i, \tag{14.3.9}$$

from which we see that

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \tag{14.3.10}$$

and

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{14.3.11}$$

In summary, then, Equations 14.3.3, 14.3.4, 14.3.10and 14.3.11:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{14.3.12}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{14.3.13}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \tag{14.3.14}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{14.3.15}$$

which I personally find impossible to commit accurately to memory (although note that there is one dot in each equation) except when using them frequently, may be regarded as Hamilton's equations of motion. I'll refer to these equations as **A**, **B**, **C** and **D**.

Note that, in Equation 14.3.13, if the Lagrangian is independent of the coordinate q_i the coordinate q_i is referred to as an "ignorable coordinate". I suppose it is called "ignorable" because you can ignore it when calculating the lagrangian, but in fact a so-called "ignorable" coordinate is usually a very interesting coordinate indeed, because it means (look at the second equation) that the corresponding generalized momentum is conserved.

Now the kinetic energy of a system is given by $T = \frac{1}{2}\sum_i p_i \dot{q}_i$ (for example, $\frac{1}{2} m\nu\nu$), and the hamiltonian (Equation 14.3.6) is defined as $H = \sum_i p_i \dot{q}_i - L$. For a *conservative system*, $L = T - V$, and hence, for a conservative system, $H = T + V$. If you are asked in an examination to explain what is meant by the hamiltonian, by all means say it is $T + V$. That's fine for a conservative system, and you'll probably get half marks. That's 50% - a D grade, and you've passed. If you want an A+, however, I recommend Equation 14.3.6.

14.4: Hamiltonian Mechanics Examples

I'll do two examples by hamiltonian methods – the simple harmonic oscillator and the soap slithering in a conical basin. Both are conservative systems, and we can write the hamiltonian as $T+V$, but we need to remember that we are regarding the hamiltonian as a function of the generalized coordinates and *momenta*. Thus we shall generally write translational kinetic energy as $\frac{p^2}{(2m)}$ rather than as $\frac{1}{2}m\nu^2$, and rotational kinetic energy as $\frac{L^2}{(2I)}$ rather than as $\frac{1}{2}I\omega^2$

Simple harmonic oscillator

The potential energy is $\frac{1}{2}kx^2$, so the hamiltonian is

$$H=\frac{p^2}{2m}+\frac{1}{2}kx^2.$$

From equation **D**, we find that $\dot{x}=\frac{p}{m}$, from which, by differentiation with respect to the time, $\dot{p}=m\ddot{x}$. And from equation **C**, we find that $\dot{p}=-kx$. Hence we obtain the equation of motion $m\ddot{x}=-kx$.

Conical basin

We refer to Section 13.6:

$$T=\frac{1}{2}m(\dot{r}^2+r^2\sin^2\alpha\dot{\phi}^2)$$

$$V=mgr\cos\alpha$$

$$L=\frac{1}{2}m(\dot{r}^2+r^2\sin^2\alpha\dot{\phi}^2)-mgr\cos\alpha$$

$$L=\frac{1}{2}m(\dot{r}^2+r^2\sin^2\alpha\dot{\phi}^2)+mgr\cos\alpha$$

But, in the hamiltonian formulation, we have to write the hamiltonian in terms of the generalized momenta, and we need to know what they are. We can get them from the lagrangian and equation **A** applied to each coordinate in turn. Thus

$$P_r=\frac{\partial L}{\partial\dot{r}}=m\dot{r} \tag{14.4.1}$$

and

$$P_\phi=\frac{\partial L}{\partial\dot{\phi}}=mr^2\sin^2\alpha\dot{\phi}. \tag{14.4.2}$$

Thus the hamiltonian is

$$H=\frac{P_r^2}{2m}+\frac{p_\phi^2}{2mr^2\sin^2\alpha}+mgr\cos\alpha. \tag{14.4.3}$$

Now we can obtain the equations of motion by applying equation **D** in turn to r and ϕ and then equation **C** in turn to r and ϕ:

$$\dot{r}=\frac{\partial H}{\partial p_r}=\frac{p_r}{m}, \tag{14.4.4}$$

$$\dot{\phi}=\frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{mr^2\sin^2\alpha}, \tag{14.4.5}$$

$$\dot{p}_r=-\frac{\partial H}{\partial r}=\frac{p_\phi^2}{mr^3\sin^2\alpha}-mg\cos\alpha, \tag{14.4.6}$$

$$\dot{p_\phi}=\frac{\partial H}{\partial\phi}=0. \tag{14.4.7}$$

Equations 14.4.2 and 14.4.7 tell us that $mr^2\sin^2\alpha\dot{\phi}$ is constant and therefore that

$$r^2\dot{\phi} \quad is \quad constant, = h, \quad say. \tag{14.4.8}$$

This is one of the equations that we arrived at from the lagrangian formulation, and it expresses constancy of angular momentum.

By differentiation of Equation 14.4.1 with respect to time, we see that the left hand side of Equation 14.4.6 is $m\ddot{r}$. On the right hand side of Equation 14.4.6, we have p_ϕ, which is constant and equal to $mh\sin^2\alpha$. Equation 14.4.6 therefore becomes

$$\ddot{r} = \frac{h^2\sin^2\alpha}{r^3} - g\cos\alpha, \tag{14.4.9}$$

which we also derived from the lagrangian formulation.

14.5: Poisson Brackets

Let f and g be functions of the generalized coordinates and momenta. Think first of all of one coordinate, say q_i, and its conjugate momentum p_i(defined, you may remember, as $\frac{\partial L}{\partial \dot{q}_i}$). I now ask the question: Is $\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$ the same thing as $\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$?

After thinking about it you will probably say something like: Well, I dare say that you *might be able to find* two functions such that that is so, but I do not see why it should be so for *any* two arbitrary functions. If that is what you thought, you thought right. Pairs of functions such that these two expressions are equal are of special significance. And pairs of functions such that these two expressions are *not* equal are also of special significance

The *Poisson bracket* of two functions of the coordinates and momenta is defined as

$$[f,g] \quad = \quad \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \tag{14.5.1}$$

(Poisson brackets are sometimes written as *braces* - i.e. {}. I'm not sure whether braces {} or brackets [] are the commoner. I have chosen brackets here, so that I do not have to call them Poisson braces.)

Poisson brackets have important applications in celestial mechanics and in quantum mechanics. In celestial mechanics, they are used in the developments of *Lagrange's planetary equations*, which are used to calculate the perturbations of the elements of the planetary orbits under small deviations from ideal two-body point-source orbits. See, for example, Chapter 14 of the Celestial Mechanics set of these notes. Readers who have had an introductory course in quantum mechanics may have come across the *commutator* of two operators, and will (or should!) understand the significance of two operators that commute. (It means that a function can be found that is simultaneously an eigenfunction of both operators.) You may not have thought of the commutator as being a *Poisson bracket*, but you soon will.

Let's suppose (because it does not make any essential difference) that there is just a single generalized coordinate and its conjugate generalized momentum, so that the Poisson bracket is just

$$[f,g] \quad = \quad \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{14.5.2}$$

Example 14.5.1

Now let's suppose that f is just q, the coordinate, and that g is the Hamiltonian, H, which is defined, you will recall, as $p\dot{q} - L$, and is a function of the coordinate and the momentum. What, then is the Poisson bracket $[q, H]$?

Solution

$$[q,H] \quad = \quad \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial H}{\partial q}. \tag{14.5.3}$$

The coordinate and the momentum are independent variables, so that $\frac{\partial q}{\partial p}$ is zero, so the second term on the right hand side of Equation 14.5.3is zero. In the first term on the right hand side, $\frac{\partial q}{\partial q}$ is of course 1, and $\frac{\partial H}{\partial p}$, by Hamilton's equations of motion, is $\dot{q}$. Thus, the answer is

$$[q,H] = \dot{q}. \tag{14.5.4}$$

In a similar vein, you will find (DO IT!!) that

$$[p,H] = \dot{p} \tag{14.5.5}$$

Thus neither the generalized coordinate nor the generalized momentum commutes with the Hamiltonian.

Now go a little further, and suppose that there are more than one coordinate and more than one momentum. Two will do, so that

$$[f,g] = \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} + \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2} \tag{14.5.6}$$

? Exercise 14.5.1

Can you show that:

$$[p_1, p_2] = [q_1, q_2] = [p_1, q_2] = [q_1, p_2] = 0; \quad [q_1, p_1] = 1.? \tag{14.5.7}$$

I shan't go any further than that here, because it would take us too far into quantum mechanics. However, those readers who have done some introductory quantum mechanics may recall that there are various pairs of operators that do or do not commute, and may now begin to appreciate the relation between the Poisson brackets of certain pairs of observable quantities and the commutator of the operators representing these quantities. For example, consider the last of these. It shows that a coordinate such as x does not commute with its corresponding momentum p_x. There is nothing more certain that this. So certain is it that it ought to be called Heisenberg's Certainty Principle. But for some reason people often seem to present quantum mechanics as something uncertain or mysterious, whereas in reality there is nothing uncertain or mysterious about it at all.

CHAPTER OVERVIEW

15: Special Relativity

The phrase "special" relativity deals with the transformations between reference frames that are moving with respect to each other at constant relative velocities. Reference frames that are accelerating or rotating or moving in any manner other than at constant speed in a straight line are included as part of general relativity and are not considered in this chapter.

15.1: Introduction to Special Relativity

Why a chapter on relativity in a book on "classical mechanics"? A first excuse might be that the phrase "classical mechanics" is used by different authors to mean different things. To some, it means "pre-relativity"; to others it means "pre-quantum mechanics". For the purposes of this chapter, then, I mean the latter, so that special relativity may fairly be included in "classical" mechanics. A second excuse is that, apart from one brief foray into an electromagnetic problem, this chapter deals only with mechanical, kinematic and dynamical problems, and therefore deals with only a rather restricted part of relativity that can be dealt with conveniently in a single chapter of classical mechanics rather than in a separate book. This is in fact a quite substantial restriction, because electromagnetic theory plays a major role in special relativity. It was in fact difficulties with electromagnetic theory that led Einstein to the special theory of relativity. Indeed, Einstein's theory of relativity was introduced to the world in a paper with the title *Zur Elektrodynamik bewegter Körper* (*On the Electrodynamics of Moving Bodies*), *Annalen der Physik*, **17**, 891 (1905).

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15.2: Preparation

The speed of light is, by definition, exactly 2.997 924 58 % 10^8 m s^{-1}, and is the same relative to all observers. This seemingly simple sentence invites several comments.

First: Note that I have used the word “speed”. Some writers use the word “velocity” as if it were merely a more impressive and scientific-sounding synonym for “speed”. I trust that all readers of these notes know the difference and will use the word “speed” when they mean “speed”, and the word “velocity” when they mean “velocity – surely not an unreasonable demand. To say that the “velocity” of light is the same for all observers means that the direction of travel of light is the same relative to all observers. This is doubtless not at all what a writer who uses the word “velocity” intends to convey – but it is the literal (and of course quite erroneous) meaning of the assertion.

Second: How can we possibly *define* the speed of light to have a certain *exact* value? Surely the speed of light is what we find it to be, and we are not free to *define* its value. But in fact we *are* allowed to do this, and the explanation, briefly, is as follows.

Over the course of history, the *metre* has been defined in several different ways. At one time it was a specified fraction of the circumference of Earth. Later, it was the distance between two scratches on a bar of platinum-iridium alloy held in Paris. Later still it was a specified number of wavelengths of a particular line in the spectrum of mercury, or cadmium, or argon or krypton. In our present state of technology it is far easier to measure and reproduce precise standards of *frequency* than it is to measure and reproduce standards of *length*. Because of that, the current SI (Système International) unit of time is the SI second, which is based on the frequency of a particular transition in the spectrum of caesium, and from there, the metre is *defined* as the distance travelled by light *in vacuo* in a defined fraction of an SI second, the speed of light being assigned the exact value quoted above.

Detailed discussion of the exact definitions of the units of time, distance and speed is part of the subject of *metrology*. That is an important and interesting subject, but it is only marginally relevant to the topic of relativity, and consequently, having quoted the exact value of the speed of light, we leave further discussion of metrology here.

Third: How can the speed of light be the same relative to *all observers*? This assertion is absolutely central to the theory of special relativity, and it may be regarded as its fundamental and most important principle. We shall discuss it further in the remainder of the chapter.

15.3: Preparation

The ratio of the speed ν of a body (or a particle, or a reference frame) is often given the symbol β:

$$\beta=\frac{\nu}{c}. \tag{15.3.1}$$

For reasons that will become apparent (I hope!) later, the range of β is usually restricted to between 0 and 1. In our study of special relativity, we shall find that we have to make frequent use of a number of functions of β. The most common of these are

$$\gamma=(1-\beta^2)^{-\frac{1}{2}}, \tag{15.3.2}$$

$$k=\sqrt{\frac{(1+\beta)}{(1-\beta)}}, \tag{15.3.3}$$

$$z=k-1, \tag{15.3.4}$$

$$\phi=\frac{1}{2}\ln\left[\frac{(1+\beta)}{(1-\beta)}\right]=\tanh^{-1}\beta=\ln k. \tag{15.3.5}$$

$$\theta=\cos^{-1}\gamma=\sin^{-1}(i\beta\gamma). \tag{15.3.6}$$

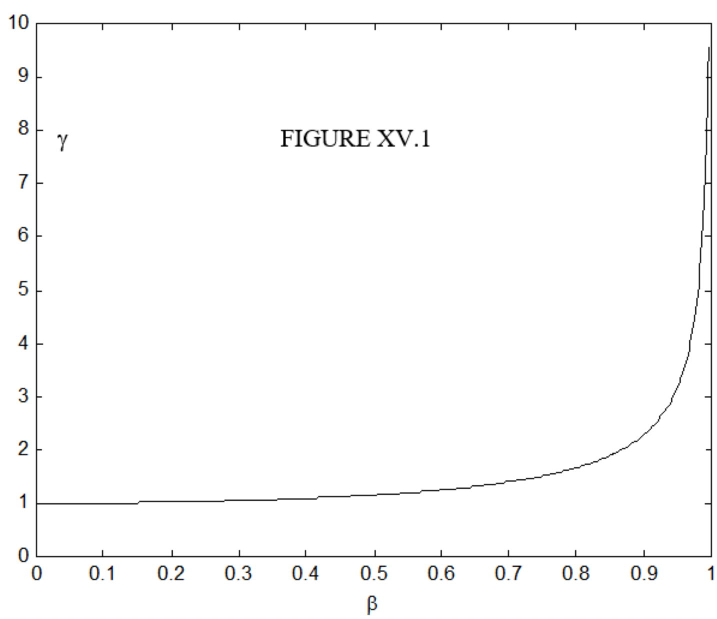
In Figures XV.1-3 I draw γ, k and ϕ as functions of β. The functions γ and k go from 1 to ∞ as b goes from 0 to 1; z, K and ϕ go from 0 to ∞. The function θ is imaginary.

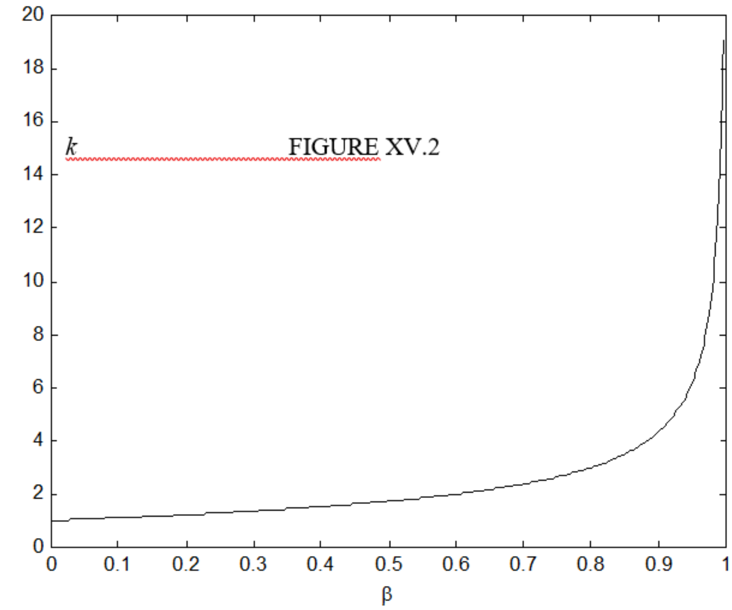
Redundancy

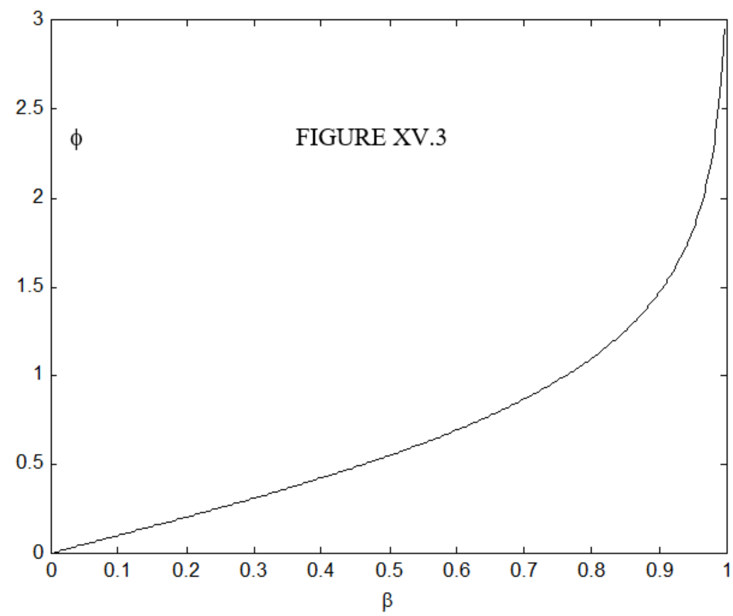
Many – one might even say most – problems in special relativity (including examination and homework questions!) amount, when stripped of their verbiage, to the following:

"Given one of the quantities β, γ, k, z, K, ϕ, θ, calculate one of the others."

Thus I would suggest that, even before you have any idea what these quantities mean, you might write a program for your computer (or programmable calculator) such that, when you enter any one of the real quantities, the computer will instantly return all six of them. This will save you, on future occasions, from having to remember the exact formulas or having to bother with tedious arithmetic, so that you can concentrate your mind on understanding the relativity.







Just for future reference, I tabulate here the relations between these various quantities. This has involved some algebra and typesetting; I do not think there are any mistakes, but I hope some reader might check through them all carefully and will let me know (jtatum@uvic.ca) if he or she finds any.

$$\beta=\sqrt{1-\frac{1}{\gamma^2}}=\frac{k^2-1}{k^2+1}=\frac{z(z+2)}{(z+1)^2+1}=\frac{\sqrt{K(K+2)}}{K+1}=\tanh\phi \quad \text{or } \frac{e^{2\phi}-1}{e^{2\phi}+1}=-i\tan\theta$$

$$\gamma=\frac{1}{\sqrt{1-\beta^2}}=\frac{k^2+1}{2k}=\frac{(z+1)^2+1}{2(z+1)}=K+1=\cosh\phi \quad \text{or } \frac{1}{2}(e^{\phi}+e^{-\phi})=\cos\theta$$

$$k=\sqrt{\frac{1+\beta}{1-\beta}}=\gamma+\sqrt{\gamma^2-1}=z+1=K+1+\sqrt{K(K+2)}=e^{\phi}=e^{-i\theta}$$

$$z=\sqrt{\frac{1+\beta}{1-\beta}}-1=\gamma-1+\sqrt{\gamma^2-1}=k-1=K+\sqrt{K(K+2)}=e^{\phi}-1=e^{-i\theta}-1$$

$$K=\frac{1}{\sqrt{1-\beta^2}}-1=\gamma-1=\frac{(k-1)^2}{2k}=\frac{z^2}{2(z+1)}=\frac{(e^{\phi}-1)^2}{2e^{\phi}}=\cos\theta-1$$

$$\phi = \tanh^{-1}\beta \text{ or } \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) = \cosh^{-1}\gamma \text{ or} \ln(\gamma + \sqrt{\gamma^2 - 1}) = \ln k = \ln(z+1) = \ln\left(K + 1 + \sqrt{K(K+2)}\right) = -i\theta$$

$$\theta = \frac{i}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) = i\ln(\gamma + \sqrt{\gamma^2 - 1}) = i\ln k = i\ln(z+1) = i\ln\left[K + 1 + \sqrt{K(K+2)}\right] = i\phi$$

15.4: Speed is Relative - The Fundamental Postulate of Special Relativity

You are sitting in a railway carriage (or a railroad car, if you prefer the term). The windows and curtains are closed and you cannot see outside. You are asked to measure the constant speed of the carriage along its tracks. You try a number of experiments. You measure the period of a simple pendulum. You slide a puck and roll a ball down an inclined plane. You throw a ball vertically up in the air and catch it as it comes down. You throw it up at an angle and you watch it describe a graceful parabola. You cause billiard balls to collide on the billiards table thoughtfully provided in your carriage. You experiment with a torsion pendulum. You stand a pencil on its end and you watch it as it falls to a horizontal position.

All your careful work is to no avail. None of them tells you what speed you are moving at, or even if you are moving at all. After exhausting all mechanical experiments you can think of, you are led to the conclusion:

> *It is impossible to determine the speed of motion of a uniformly-moving reference frame by means of any mechanical experiment performed within that frame.*

Frustrated, you open a curtain on one side of the carriage. You look out and you see that there is another train on the line next to you. It appears to be moving backwards. Or are you moving forwards? Or are you both moving in the same direction but at different speeds? You still can't tell.

You move to the other side of the carriage and open the curtain there. This time you see the station platform, and the station platform is moving backwards. Or are you moving forwards? (Those of you who have not done much travel by train may not appreciate just how very strong the impression can be that the platform is moving.) What does it mean, anyway, to say that it is you that is moving rather than the platform?

The following story is not true, but it ought to be. (It is an "apocryphal" story.) Einstein was travelling by train across Canada. Halfway across the Prairies he leant across and tapped on the knee of his fellow passenger and asked: "Excuse me, mein Herr, bitte, but does Regina stop at this train?"

You are about to conclude that it is not possible by any means, whether by experiment or by observation, to determine the speed of your reference frame, or even whether it is moving or stationary.

But not so hasty! I am about to invent a speedometer, which I intend to patent and to use to make myself rich. I am going to use my invention to measure the speed of our train – without even looking out of the window!

We shall set up two long parallel glass rods in the middle of the corridor, parallel to the railway lines and to the velocity of the train. We shall suspend the rods horizontally, side by side from a common support, and we shall rub each of them with a silken handkerchief, so that each of them bears an electrostatic charge of λCm^{-1}. They will repel each other with an electrostatic force per unit length of

$$F_e = \frac{\lambda^2}{4\pi\epsilon_0 r} Nm^{-1}, \tag{15.4.1}$$

where r is their distance apart, and consequently they will hang out of the vertical – see figure XV.4.

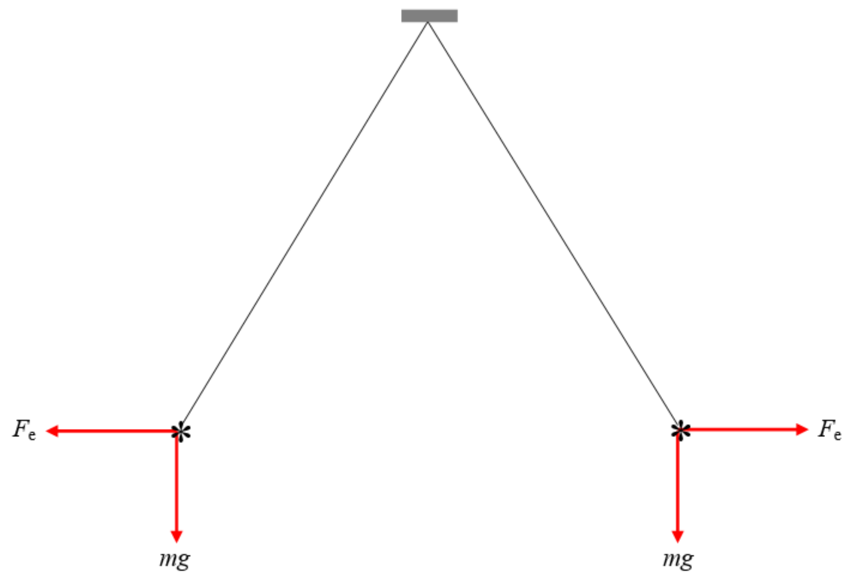


FIGURE XV.4

Now see what happens when the train moves forward at speed ν. Each rod, bearing a charge λ per unit length, is now moving forward at speed ν, and therefore each rod constitutes an electric current $\lambda\nu A$. Therefore, by Ampère's law, in addition to the Coulomb repulsion, they will experience a magnetic attraction per unit length equal to

$$F_m = \frac{\mu_0\lambda^2\nu^2}{4\pi r} N m^{-1} \tag{15.4.2}$$

The net repulsive force per unit length is now

$$\frac{\lambda^2}{4\pi\epsilon_0 r^2}(1-\mu_0\epsilon_0\nu^2). \tag{15.4.3}$$

This is a little less than it was when the train was stationary, so the angle between the suspending strings is a little less, as shown in figure XV.5. It might be noted that the force between the strings is reduced to zero (and the angle also becomes zero) when the train is travelling at a speed $\frac{1}{\sqrt{\mu_0\epsilon_0}}$. We remember from electromagnetic theory that the permeability of free space is $\mu_0 = 4\pi$ % 10^{-7} H m^{-1} and that the permittivity ϵ_0 is 8.8542 % 10^{-12} F m^{-1}; consequently the force and the angle drop to zero and the strings hang vertically, when the train is moving at a speed of 2.998 % 10^8 m s^{-1}.

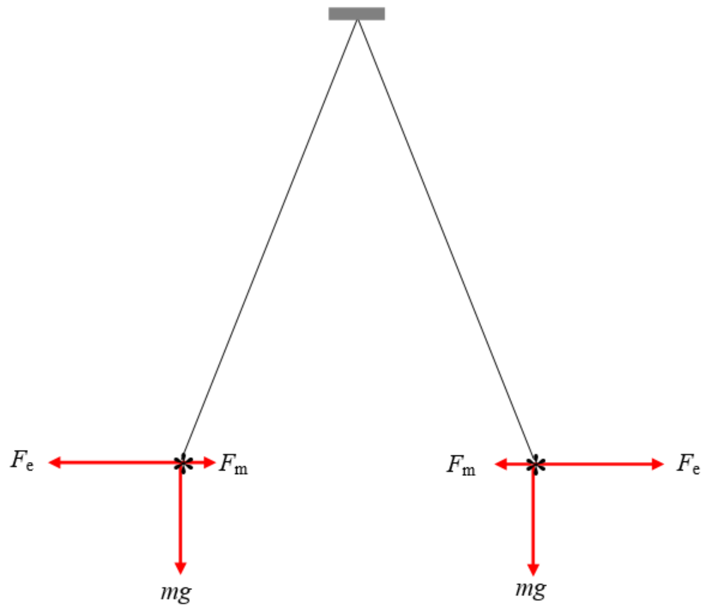


FIGURE XV.5

To complete my invention, I am now going to attach a protractor to the instrument, but instead of marking the protractor in degrees, I am going to calibrate it in miles per hour, and my speedometer is now ready for use (figure XV.6).

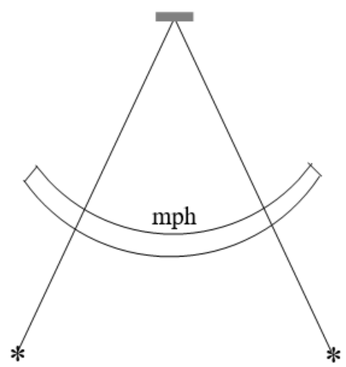


FIGURE XV.6

You now have a choice. Either:

i. You can choose to believe that the speedometer will work and you can accompany me to the patent office to see if they will grant a patent for this invention, which will measure the speed of a train without reference to any external reference frame. If you choose to believe this, there is no need for you to read the remainder of the chapter on special relativity.

or

ii. You can say that it defies common sense to believe that it is possible to determine whether a given reference frame is moving or stationary, let alone to determine its speed. Common sense dictates that

It is impossible to determine the speed of motion of a uniformly-moving reference frame by any means whatever, whether by a mechanical or electrical or indeed any experiment performed entirely or partially within that frame, or even by reference to another frame.

Your common sense, then, leads you – as it should – to the fundamental principle of special relativity. Whereas some people protest that relativity "defies common sense", in fact relativity *is* common sense, and its predictions (such as your prediction that my speedometer will not work) are exactly what common sense would lead you to expect.

15.5: The Lorentz Transformations

For the remainder of this chapter I am taking, as a fundamental postulate, that

> *It is impossible to determine the speed of motion of a uniformly-moving reference frame by any means whatever, whether by a mechanical or electrical or indeed any experiment performed entirely or partially within that frame, or even by reference to another frame*

and consequently I am choosing to believe that my speedometer will not work. If it is impossible by any electrical experiment to determine our speed, we must assume that all the electromagnetic equations that we know, not just the ones that we have quoted, but indeed Maxwell's equations, which embrace all electromagnetic phenomena, are the same in all uniformly-moving reference frames.

One of the many predictions of Maxwell's equations is that electromagnetic radiation (which includes light) travels at a speed

$$c=\frac{1}{\sqrt{\mu_0\epsilon_0}} \tag{15.5.1}$$

Presumably neither the permeability nor the permittivity of space changes merely because we believe that we are travelling through space – indeed it would defy common sense to suppose that they would. Consequently, our acceptance of the fundamental principle of special relativity is equivalent to accepting as a fundamental postulate that the speed of light *in vacuo* is the same for all observers in uniform relative motion. We shall take anything other than this to be an outrage against common sense – though acceptance of the principle will require a careful examination of our ideas concerning the relations between time and space.

Let us imagine two reference frames, $\sum$ and $\sum'$. $\sum'$is moving to the right (positive x-direction) at speed ν relative to $\sum$. (For brevity, I shall from time to time refer to S as the "stationary" frame, in the hope that this liberty will not lead to misunderstanding.) At time $t=t'=0$ the two frames coincide, and at that instant someone strikes a match at the common origin of the two frames. At a later time, which I shall call t if referred to the frame $\sum$, and t' if referred to $\sum'$, the light from the match forms a spherical wavefront travelling radially outward at speed c from the origin O of $\sum$, and the equation to this wavefront, when referred to the frame $\sum$, is

$$x^2+y^2+z^2-c^2t^2=0. \tag{15.5.2}$$

Referred to $\sum'$, it also travels outward at speed c from the origin O' of S', and the equation to this wavefront, when referred to the frame $\sum'$, is

$$x'^2+y'^2+z'^2-c^2t'^2=0. \tag{15.5.3}$$

Most readers will accept, I think, that $y=y'$ and $z=z'$. Some formal algebra may be needed for a rigorous proof, but that would distract from our main purpose of finding a transformation between the primed and unprimed coordinates such that

$$x'^2-c^2t'^2=x^2-c^2t^2. \tag{15.5.4}$$

It is easy to show that the "Galilean" transformation $x'=x-ct$, $t'=t$ does not satisfy this equality, so we shall have to try harder.

Let us seek linear transformations of the form

$$x'=Ax+Bt, \tag{15.5.5}$$

$$t'=Cx+Dt, \tag{15.5.6}$$

which satisfy Equation 15.5.4.

We have

$$\frac{x'}{t'}=\frac{Ax+Bt}{Cx+Dt}, \tag{15.5.7}$$

and, by inversion,

$$\frac{x}{t} = \frac{Dx' - Ct'}{Ax' - Ct'}. \tag{15.5.8}$$

Consider the motion of O' relative to $\sum$ and to $\sum'$. We have $\frac{x}{t} = \nu$ and $x' = 0$.

$$\nu = -\frac{B}{A}. \tag{15.5.9}$$

Consider the motion of O relative to $\sum'$ and to $\sum$. We have $\frac{x'}{t'} = -\nu$ and $x = 0$.

$$-\nu = \frac{B}{D} \tag{15.5.10}$$

From these we find that $D = A$ and $B = -A\nu$, so we arrive at

$$x' = A(x - \nu t) \tag{15.5.11}$$

and

$$t' = Cx + At. \tag{15.5.12}$$

On substitution of Equations 15.5.11and 15.5.12into Equation 15.5.4, we obtain

$$A^2(x-\nu t)^2 - c^2(Cx + At)^2 = x^2 - c^2t^2. \tag{15.5.13}$$

Equate powers of t^2 to obtain

$$A = \frac{1}{\sqrt{\frac{1-\nu^2}{c^2}}} = \gamma. \tag{15.5.14}$$

Equate powers of xt to obtain

$$C = -\frac{\nu\gamma}{c}. \tag{15.5.15}$$

Equating powers of x^2 produces no new information.

We have now determined A, B, C and D, and we can substitute them into Equations 15.5.5and 15.5.6, and hence we arrive at

$$x' = \gamma(x - \nu t) \tag{15.5.16}$$

and

$$t' = \gamma\left(\frac{t - \nu x}{c^2}\right). \tag{15.5.17}$$

These, together with $y = y'$ and $z = z'$, constitute the *Lorentz transformations*, which, by suitable choice of axes, guarantee the invariance of the speed of light in all reference frames moving at constant velocities relative to one another.

To express x and t in terms of x' and t', you may, if you are good at algebra, solve Equations 15.5.16and 15.5.17simultaneously for x' and t', or, if instead, you have good physical insight, you will merely reverse the sign of ν and interchange the primed and unprimed quantities. Either way, you should obtain

$$x = \gamma(x' + \nu t') \tag{15.5.18}$$

and

$$t = \gamma\left(\frac{t' + \nu x'}{c^2}\right) \tag{15.5.19}$$

15.6: But This Defies Common Sense

At this stage one may hear the protest: “But this defies common sense!”. One may hear it again as we encounter several predictions of the invariance of the speed of light and of the Lorentz transformations. But, if you have read this far, it is too late to make such protest. You have already, at the end of Section 15.4, made your choice, and you then decided that it defies common sense to suppose that one can somehow determine the speed of a reference frame by some experiment or observation. You rejected that notion, and it was the *application* of common sense, not its abandonment, that led us into the Lorentz transformations and the invariance of the speed of light.

There may be other occasions when we are tempted to protest “But this defies common sense!”, and it is therefore always salutary to recall this. For example, we shall later learn that if a train is moving at speed V relative to the station platform, and a passenger is walking towards the front of the train at a speed ν relative to the train, then, relative to the platform, he is moving at a speed just a little bit less that $V+\nu$. When we protest, we are often presented with an “explanation” along the following lines:

In every day life, trains do not move at speeds comparable to the speed of light, nor do walking passengers. Therefore, we do not notice that the combined speed is a little bit less than $V+\nu$. After all, if V= 60 mph and ν= 4 mph, the combined speed is 0.999 999 999 999 999 5 % 64 mph. The formula $V+\nu$ is just an *approximation*, we are told, and we have the erroneous impression that the combined speed is exactly $V+\nu$ only because we are accustomed, in daily life, to experiencing speeds that are small compared with the speed of light.

This explanation somehow does not seem to be satisfactory – and nor should it, for it is *not* a correct explanation. It seems to be an explanation invented for the benefit of the nonscientific layman – but nothing is ever made easy to understand by giving an incorrect explanation under the pretence of “simplifying” something. It is *not* correct merely to say that the Galilean transformations are just an “approximation” to the “real” transformations.

The problem is that it is exceedingly difficult – perhaps impossible – to describe exactly what is meant by “distance” and “time interval”. It is almost as difficult as describing colours to a blind person, or even describing your sensation of the colour red to another seeing person. We have no guarantee that every person’s perception of colour is the same. The best that can be done to describe what we mean by distance and time interval is to *define* how distances and times *transform* between reference frames. The Lorentz transformations, which we have adopted in order to make it meaningless to discuss the absolute velocity of a reference frame, amount to a useful *working definition* of the meanings of space and time. Once we have adopted this definition, “common sense” no longer comes into the matter. There is no longer a *mystery* which our minds cannot quite grasp; from this point on it merely becomes a matter of algebra as to how a measurement of length or of time interval, or of speed, or of mass, as appropriately defined, transforms when referred to one reference or to another. There is no impossible feat of imagination to be done.

15.7: The Lorentz Transformation as a Rotation

The Lorentz transformation can be written

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{15.7.1}$$

where $x_1 = x,\ x_2 = y,\ x_3 = z$ and $x_4 = -ict$, and similarly for primed quantities. Please do not just take my word for this; multiply the matrices, and verify that this Equation does indeed represent the Lorentz transformation. You could, if you wish, also write this for short:

$$\mathbf{x}' = \lambda\mathbf{x}. \tag{15.7.2}$$

Another way of writing the Lorentz transformation is

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_0' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix}, \tag{15.7.3}$$

where $x_1 = x,\ x_2 = y,\ x_3 = z$ and $x_0 = ct$, and similarly for primed quantities.

Some people prefer one version; others prefer the other. In any case, a set of four quantities that transforms like this is called a 4-*vector*. Those who dislike version 15.7.1 dislike it because of the introduction of imaginary quantities. Those who like version 15.7.1 point out that the expression

$$\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + (\Delta x_4)^2}$$

(the “interval” between two events) is invariant in four-space – that is, it has the same value in all uniformly-moving reference frames, just as the distance between two points in three-space, $[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{\frac{1}{2}}$, is independent of the position or orientation of any reference frame. In version 15.7.3, the invariant interval is

$$\sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + (\Delta x_0)^2}\ .$$

Those who prefer version 15.7.1 dislike the minus sign in the expression for the interval. Those who prefer version 15.7.3 dislike the imaginary quantities of version 15.7.1.

For the time being, I am going to omit y and z, so that I can concentrate my attention on the relations between x and t. Thus I am going to write 15.7.1 as

$$\begin{pmatrix} x_1' \\ x_4' \end{pmatrix} = \begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ict \end{pmatrix} \tag{15.7.4}$$

and Equation 15.7.3 as

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{15.7.5}$$

Readers may notice how closely Equation 15.7.4 resembles the Equation for the transformation of coordinates between two reference frames that are inclined to each other at an angle. (See Celestial Mechanics Section 3.6.) Indeed, if we let $\cos\theta = \gamma$ and $\sin\theta = i\beta\gamma$, Equation 15.7.4 becomes

$$\begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{15.7.6}$$

The matrices in Equations 15.7.1, 15.7.4 and 15.7.6 are orthogonal matrices and they satisfy each of the criteria for orthogonality described, for example, in Celestial Mechanics Section 3.7. We can obtain the converse relations (i.e. we can express x and t in terms of x' and t') by interchanging the primed and unprimed quantities and either reversing the sign of β or of θ or by interchanging the rows and columns of the matrix.

There is a difficulty in making the analogy between the Lorentz transformation as expressed by Equation 15.7.4 and rotation of axes as expressed by Equation 15.7.6 in that, since $\gamma > 1$, θ is an imaginary angle. (At this point you may want to reach for your ancient, brittle, yellowed notes on complex numbers and hyperbolic functions.) Thus

$\theta = \cos^{-1}\gamma$ and for $\gamma > 1$, this means that $\theta = i\cosh^{-1}\gamma = i\ln\left(\gamma + \sqrt{\gamma^2 - 1}\right)$. And $\theta = \sin^{-1}(i\beta\gamma) = i\sinh^{-1}(\beta\gamma) = i\ln\left(\beta\gamma + \sqrt{\beta^2\gamma^2 + 1}\right)$. Either of these expressions reduces to $\theta = i\ln[\gamma(1+\beta)]$. Perhaps a yet more convenient way of expressing this is

$$\theta = i\tanh^{-1}\beta = \frac{1}{2} i \ln\left(\frac{1+\beta}{1-\beta}\right). \tag{15.7.7}$$

For example, if β = 0.8, θ = 1.0986i, which might be written (not necessarily particularly usefully) as $i62^\circ\ 57'$.

At this stage, you are probably thinking that you much prefer the version of Equation 15.7.5, in which all quantities are real, and the expression for the interval between two events is $[(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - (\Delta x_0)^2]^{\frac{1}{2}}$. The minus sign in the expression is a small price to pay for the realness of all quantities. Equation 15.7.5 can be written

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \tag{15.7.8}$$

where $\cosh\phi = \gamma,\ \sinh\phi = \beta\gamma,\ \tanh\phi = \beta$. On the face of it, this looks much simpler.

No messing around with imaginary angles. Yet this formulation is not without its own set of difficulties. For example, neither the matrix of Equation 15.7.5 nor the matrix of Equation 15.7.8 is orthogonal. You cannot invert the Equation to find x and t in terms of x' and t' merely by interchanging the primed and unprimed symbols and interchanging the rows and columns. The converse of Equation 15.7.8 is in fact

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{15.7.9}$$

which can also (understandably!) be written

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh(-\phi) & \sinh(-\phi) \\ \sinh(-\phi) & \cosh(-\phi) \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{15.7.10}$$

which demands as much skill in handling hyperbolic functions as the other formulation did in handling complex numbers. A further problem is that the formulation 15.7.5 does not allow the analogy between the Lorenz transformation and the rotation of axes. You take your choice.

It may be noticed that the determinants of the matrices of Equations 15.7.5 and 15.7.8 are each unity, and it may therefore be thought that each matrix is orthogonal and that its reciprocal is its transpose. But this is not the case, for the condition that the determinant is inity is not a sufficient condition for a matrix to be orthogonal. The necessary tests are summarized in Celestial Mechanics, Section 3.7, and it will be found that several of the conditions are not satisfied.

15.8: Timelike and Spacelike 4-Vectors

I am going to refer some events to a coordinate system whose origin is here and now and which is moving at the same velocity as you happen to be moving. In other words, you are sitting at the origin of the coordinate system, and you are stationary with respect to it. Let us suppose that an event A occurs at the following coordinates referred to this reference frame, in which the distances $x_1,\ y_1,\ z_1$ are expressed in light-years (lyr) the time t_1 is expressed in years (yr).

$$x_1\ =2\quad y_1\ =3\quad z_1\ =7\quad t_1\ =\ -1$$

A "light-year" is a unit of distance used when describing astronomical distances to the layperson, and it is also useful in describing some aspects of relativity theory. It is the distance travelled by light in a year, and is approximately 9.46 % 10^{15} m or 0.307 parsec (pc). Event A, then, occurred a year ago at a distance of $\sqrt{62}\ =\ 7.87$ lyr, when referred to this reference frame. Note that, if referred to a reference frame that coincides with this one at $t=0$, but is moving with respect to it, all four coordinates might be different, and the distance $\sqrt{x^2+y^2+z^2}$ and the time of occurrence would be different, but, according to the way in which we have defined space and time by the Lorentz transformation, the quantity $\sqrt{x^2+y^2+z^2-c^2t^2}$ would be the same.

Imagine now a second event, B, which occurs at the following coordinates:

$$x_2\ =5\quad y_2\ =8\quad z_2\ =10\quad t_2\ =\ +2$$

That is to say, when referred to the same reference frame, it will occur in two years' time at a distance of $\sqrt{189}=13.75$ lyr.

The 4-vector **s** = **B** - **A** connects these two events, and the magnitude s of **s** is the *interval* between the two events. Note that the *distance* between the two events, *when referred to our reference frame*, is $\sqrt{(5-2)^2+(8-3)^2+(10-7)^2}\ =6.56$ lyr. The *interval* between the two events is $\sqrt{(5-2)^2+(8-3)^2+(10-7)^2-(2+1)^2}\ =5.83$ lyr, and this is independent of the velocity of the reference frame. That is, if we "rotate" the reference frame, it obviously makes no difference to the *interval* between the two events, which is *invariant*.

As another example, consider two events A and B whose coordinates are

$$x_1\ =2\quad y_1\ =5\quad z_1\ =3\quad t_1\ =\ -2$$

$$x_2\ =3\quad y_2\ =7\quad z_2\ =4\quad t_2\ =\ +6$$

with distances, as before, expressed in lyr, and times in yr. Calculate the interval between these two events – i.e. the magnitude of the 4-vector connecting them. If you carry out this calculation, you will find that s^2 = -58, so that the interval s is *imaginary* and equal to 7.62i.

So we see that some pairs of events are connected by a 4-vector whose magnitude is real, and other pairs are connected by a 4-vector whose magnitude is imaginary. There are differences in character between real and imaginary intervals, but, in order to strip away distractions, I am going to consider events for which $y\quad=\quad z\quad=0$. We can now concentrate on the essentials without being distracted by unimportant details.

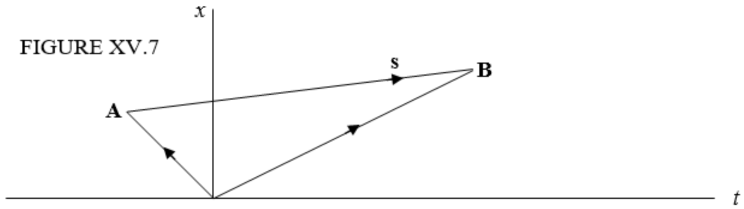
Let us therefore consider two events A and B whose coordinates are

$$x_1\ =2\ \text{lyr}\ t_1=-2\ \text{yr}$$

$$x_2\ =3\ \text{lyr}\ t_2=+6\ \text{yr}$$

These events and the 4-vector connecting them are shown in Figure XV.7.Event A happened two years ago (referred to our reference frame); event B will occur (also referred to our reference frame) in six years' time. The square of the *interval* between the two events (which is invariant) is -63 lyr^2, and the interval is imaginary. If someone wanted to experience both events, he would have to travel only 1 lyr (referred to our reference frame), and he could take his time, for he would have eight years (referred to our reference frame) in which to make the journey to get to event B in time. He couldn't totally dawdle, however; he would have to travel at a speed of at least $\frac{1}{8}$ times the speed of light, but that's not extremely fast for anyone well versed in relativity.

FIGURE XV.7



Let's look at it another way. Let's suppose that event A is the *cause* of event B. This means that some agent must be capable of conveying some information from A to B at a speed at least equal to $\frac{1}{8}$ times the speed of light. That may present some technical problems, but it presents no problems to our imagination.

You'll notice that, in this case, the interval between the two events – i.e. the magnitude of the 4-vector connecting them - is *imaginary*. A 4-vector whose magnitude is *imaginary* is called a *timelike* 4-vector.There is quite a long time between events A and B, but not much distance.

Now consider two events A and B whose coordinates are

$$x_1 = 2 \text{ lyr } t_1 = -1 \text{ yr}$$

$$x_2 = 7 \text{ lyr } t_2 = +3 \text{ yr}$$

The square of the magnitude of the interval between these two events is +9 lyr^2, and the interval is real. A 4-vector whose magnitude is *real* is called a *spacelike* 4-vector. It is shown in Figure XV.8.

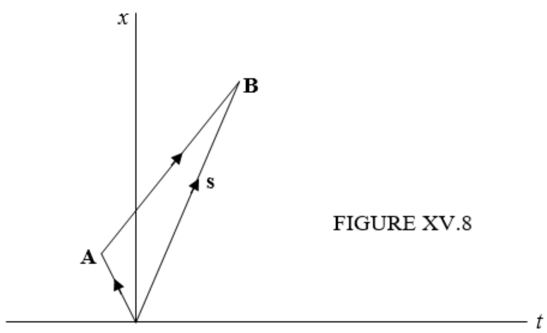


FIGURE XV.8

Perhaps I could now ask how fast you would have to travel if you wanted to experience both events. They are quite a long way apart, and you haven't much time to get from one to the other. Or, if event A is the *cause* of event B, how fast would an information-carrying agent have to move to convey the necessary information from A in order to instigate event B? Maybe you have already worked it out, but I'm not going to ask the question, because in a later section we'll find that *two events A and B cannot be mutually causally connected if the interval between them is real.* Note that I have said "mutually"; this means that A cannot cause B, and B cannot cause A. A and B must be quite independent events; there simply is too much space in the interval between them for one to be the cause of the other. It does not mean that the two events cannot have a common cause. Thus, Figure XV.9 shows two events A and B with a spacelike interval between them (very steep) and a third event C such the intervals CA and CB (very shallow) are timelike. C could easily be the cause of both A and B; that is, A and B could have a common cause.But there can be no *mutual* causal connection between A and B. (It might be noted parenthetically that Charles Dickens temporarily nodded when he chose the title of his novel *Our Mutual Friend*. He really meant our common friend. C was a friend common to A and to B. A and B were friends mutually to each other.)

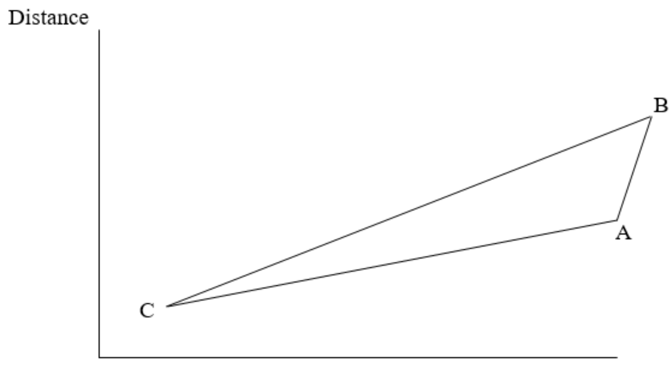


FIGURE XV.9

Exercise 15.8.1

The distance of the Sun from Earth is 1.496 % 10^{11} m. The speed of light is 2.998 % 10^{8} m s^{-1}.

How long does it take for a photon to reach Earth from the Sun?

Event A: A photon leaves the Sun on its way to Earth. Event B: The photon arrives at Earth.

What is the interval (i.e. *s* in 4-space) between these two events?

15.9: The FitzGerald-Lorentz Contraction

This is sometimes described in words something like the following:

If a measuring-rod is moving with respect to a "stationary" observer, it "appears" to be shorter than it "really" is.

This is not a very precise statement, and the words that I have placed in inverted commas call for some clarification.

We have seen that, while the interval between two events is invariant between reference frames, the distance between two points (and hence the length of a rod) depends on the coordinate frame to which the points are referred. Let us now define what we mean by the *length* of a rod. Figure XV.10 shows a reference frame, and a rod lying parallel to the x-axis. For the moment I am not specifying whether the rod is moving with respect to the reference frame, or whether it is stationary.

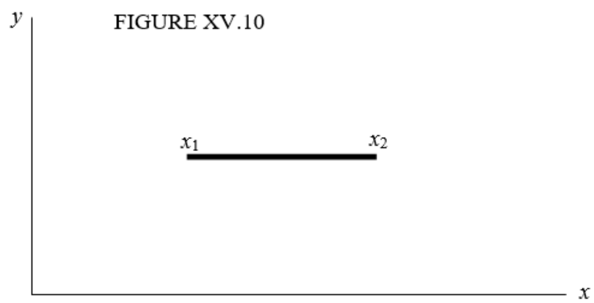


FIGURE XV.10

Let us suppose that the *x*-coordinate of the left-hand end of the rod is x_1, and that, *at the same time referred to this reference frame*, the x-coordinate of the right-hand end is x_2. The length l of the rod is defined as $l = x_2 - x_1$. That could scarcely be a simpler statement – but note the little phrase "at the same time referred to this reference frame". That simple phrase is important.

Now let's look at the FitzGerald-Lorentz contraction. See Figure XV.11.

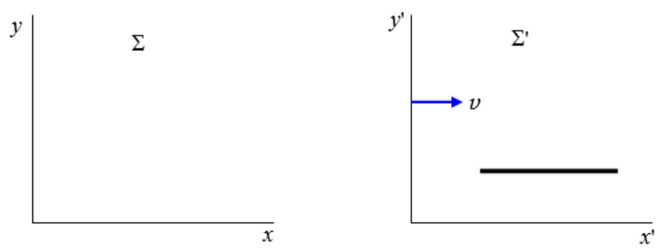


FIGURE XV.11

The are two reference frames, $\sum$ and sum'. The frame $\sum'$ is moving to the right with respect to $\sum$ with speed ν. A rod *is at rest with respect to the frame* $\sum'$, and is therefore moving to the right with respect to $\sum$ at speed ν.

In my younger days I often used to travel by train, and I still like to think of railway trains whenever I discuss relativity. Modern students usually like to think of spacecraft, presumably because they are more accustomed to this mode of travel. In the very early days of railways, it was customary for the stationmaster to wear top hat and tails. Those days are long gone, but, when thinking about the FitzGerald-Lorentz contraction, I like to think of $\sum$ as being a railway station in which there resides a stationmaster in top hat and tails, while $\sum'$ is a railway train.

The length of the rod, referred to the frame $\sum'$, is $l' = x'_2 - x'_1$, in what I hope is obvious notation, and of course these two coordinates are determined at the same time referred to $\sum'$.

The length of the rod *referred to a frame in which it is at rest* is called its *proper length.* Thus l' is the proper length of the rod.

Now it should be noted that, according to the way in which we have defined distance and time by means of the Lorentz transformation, although x'_2 and x'_1 are measured simultaneously with respect to $\sum'$, these two events (the determination of the coordinates of the two ends of the rod) are not simultaneous when referred to the frame $\sum$ (a point to which we shall return in a later section dealing with simultaneity). The length of the rod referred to the frame $\sum$ is given by $l = x_2 - x_1$, where these two coordinates are to be determined at the same time when referred to $\sum$. Now Equation 15.5.16 tells us that $x_2 = \frac{x_2'}{\gamma} + \nu t$ and $x_1 = \frac{x_1'}{\gamma} + \nu t$. (Readers should note this derivation very carefully, for it is easy to go wrong. In particular, be very clear what is meant in these two equations by the symbol t. It is the single instant of time, referred to $\sum$, when the coordinates of the two ends are determined simultaneously with respect to $\sum$.) From these we reach the result:

$$l = \frac{l'}{\gamma}. \tag{15.9.1}$$

This is the FitzGerald-Lorentz contraction.

It is sometimes described thus: A railway train of proper length 100 yards is moving past a railway station at 95% of the speed of light (γ = 3.2026.) To the stationmaster the train "appears" to be of length 31.22 yards; or the stationmaster "thinks" the length of the train is 31.22 yards; or, "according to" the stationmaster the length of the train is 31.22 yards. This gives a false impression, as though the stationmaster is under some sort of misapprehension concerning the length of the train, or as if he is labouring under some sort of illusion, and it introduces some sort of unnecessary "mystery" into what is nothing more than simple algebra. In fact what the stationmaster "thinks" or "asserts" is entirely irrelevant. Two correct statements are: 1. The length of the train, referred to a reference frame in which it is at rest – i.e. the proper length of the train – is 100 yards. 2. The length of the train when referred to a frame with respect to which it is moving at a speed of 0.95c is 31.22 yards. And that is all there is to it. Any phrase such as "this observer thinks that" or "according to this observer" should always be interpreted in this manner. It is not a matter of what an observer "thinks". It is a matter of which frame a measurement is referred to. Nothing more, nothing less.

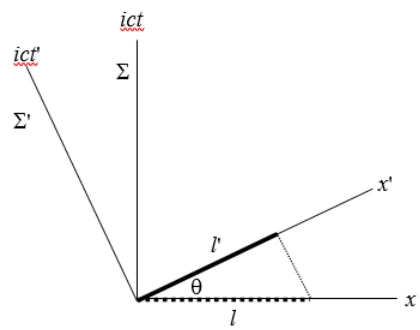


FIGURE XV.12

It is possible to describe the Lorentz-FitzGerald contraction by interpreting the Lorentz transformations as a rotation in 4-space. Whether it is helpful to do so only you can decide. Thus Figure XV.12 shows $\sum$ and $\sum'$ related by a rotation in the manner described in Section 15.7. The thick continuous line shows a rod oriented so that its two ends are drawn at the same time with respect to $\sum'$. Its length is, referred to sum', l', and this is its proper length. The thick dotted line shows the two ends at the same time with respect to $\sum$. Its length referred to $\sum$ is $l = \frac{l'}{\cos\theta}$. And, since $\cos\theta = \gamma$, which is greater than 1, this means that, in spite of appearances in the Figure, $l < l'$. The Figure is deceptive because, as discussed in Section 15.7, θ is imaginary. As I say, only you can decide whether this way of looking at the contraction is helpful or merely confusing. It is, however, at least worth looking at, because I shall be using this concept of rotation in a forthcoming section on simultaneity and order of events. Illustrating the Lorentz transformations as a rotation like this is called a *Minkowski diagram.*

15.10: Time Dilation

We imagine the same railway train $\sum'$ and the same railway station $\sum$ as in the previous section except that, rather than measuring a length referred to the two reference frames, we measure the time interval between two events. We'll suppose that a passenger in the railway train $\sum'$ claps his hands twice. These are two events which, when referred to the reference frame $\sum'$, take place *at the same place when referred to this reference frame.* Let the instants of time when the two events occur, referred to $\sum'$, be t'_1 and t'_2. The time interval T' is defined as $t'_2 - t'_1$. But the Lorentz transformation is

$$t = \gamma(t' + \frac{\nu x'}{c^2})$$

and so the time interval when referred to $\sum$ is

$$T = \gamma T'. \tag{15.10.1}$$

This is the *dilation of time.* The situation is illustrated by a Minkowski diagram in Figure XV.13. While it is clear from the figure that $T = T' \cos\theta$ and therefore that $T = \gamma T'$ it is not so clear from the figure that this means that T is greater than T' – because $\cos\theta > 1$ and θ is imaginary.

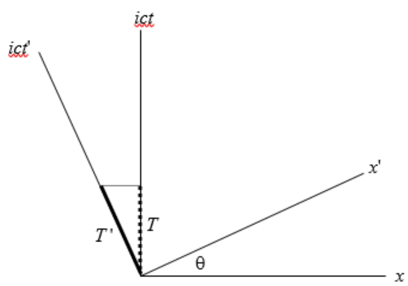


FIGURE XV.13

Thus, let us suppose that a passenger on the train holds a 1-metre measuring rod (its length in the direction of motion of the train) and he claps his hands at an interval of one second apart. Let's suppose that the train is moving at 98% of the speed of light (γ = 5.025). In that case the stationmaster thinks that the length of the rod is only 19.9 cm and that the time interval between the claps is 5.025 seconds.

I deliberately did not word that last sentence very well. It is not a matter of what the stationmaster or anyone else "thinks" or "asserts". It is not a matter that the stationmaster is somehow deceived into erroneously believing that the rod is 19.9 cm long and the claps 5.025 seconds apart, whereas they are "really" 1 metre long and 1 second apart. It is a matter of how length and time are defined (by subtracting two space coordinates determined at the same time, or two time coordinates at the same place) and how space-time coordinates are defined by means of the Lorentz transformations. The length *is* 19.9 cm, and the time interval *is* 5.025 seconds when referred to the frame $\sum$. It is true that the *proper length* and the *proper time interval* are the length and the time interval referred to a frame in which the rod and the clapper are at rest. In that sense one could loosely say that they are "really" 1 metre long and 1 second apart. But the Lorentz contraction and the time dilation are not determined by what the stationmaster or anyone else "thinks".

Another way of looking at it is this. The interval s between two events is clearly independent of the orientation any reference frames, and is the same when referred to two reference frames that may be inclined to each other. But the components of the vector joining two events, or their projections on to the time axis or a space axis are not at all expected to be equal.

By the way, in Section 15.3 I urged you to write a computer or calculator programme for the instant conversion between the several factors commonly encountered in relativity. I still urge it. As soon as I typed that the train was travelling at 98% of the speed of light, I was instantly able to generate γ. You need to be able to do that, too.

15.11: The Twins Paradox

During the late 1950s and early 1960s there was great controversy over a problem known as the "Twins Paradox". The controversy was not confined to within scientific circles, but was argued, by scientists and others, in the newspapers, magazines and many serious journals. It goes something like this:

There are two 20-year-old twins, Albert and Betty. Albert is a sedentary type who likes nothing better than to stay at home tending the family vineyards. His twin sister Betty is a more adventurous type, and has trained to become an astronaut. On their twentieth birthday, Betty waves a cheery *au revoir* to her brother and takes off on what she intends to be a brief spaceflight, at which she travels at 99.98 % of the speed of light (γ = 50). After six months by her calendar she turns back and on her 21st birthday she arrives back home to greet her brother, only to find that he is now old and sere and has laboured, by his calendar for 50 years and is now an aged man of 71 years. If we accept what we have derived in the previous section about the dilation of time, there would seem to be no particular problem with that. It has even been argued that travel between the stars may not be an impossibility. Whereas to an Earthbound observer it may take many decades for a spacecraft to travel to a star and back, for the astronauts on board much less time has elapsed.

And yet a paradox was pointed out. According to the principles of the relativity of motion, it was argued, one could refer everything to Betty's reference frame, and from that point of view one could regard Betty as being the stationary twin and Albert as the one who travelled off into the distance and returned later. Thus, it could be argued, it would be Albert who had aged only one year, while Betty would have aged fifty years. Thus we have a *paradox*, which is a problem which apparently gives rise to opposite conclusions depending on how it is argued. And the only way that the paradox could be resolved was to suppose that both twins were the same age when they were re-united.

A second argument in favour of this interpretation that the twins were the same age when re-united points out that dilation of time arises because two events that may occur in the same place when referred to one reference frame do not occur in the same place when referred to another. But in this case, the two events (Betty's departure and re-arrival) occur at the same place when referred to both reference frames.

The argument over this point raged quite furiously for some years, and a particularly plausible tool that was used was something referred to as the "k-calculus" – an argument that is, however, fatally flawed because the "rules" of the k-calculus inherently incorporate the desired conclusion. Two of the principal leaders of the very public scientific debate were Professors Fred Hoyle and Herbert Dingle, and this inspired the following letter to a weekly magazine, *The Listener*, in 1960:

Sir:

The ears of a Hoyle may tingle;
The blood of a Hoyle may boil
When Hoyle pours hot oil upon Dingle,
And Dingle cold water on Hoyle.

But the dust of the wrangle will settle.
Old stars will look down on new soil.
The pot will lie down with the kettle,
And Dingle will mingle with Hoyle.

So what are you, the reader, expected to believe? Let us say this: If you are a student who has examinations to pass, or if you are an untenured professor who has to hold on to a job, be in no doubt whatever: The original conclusion is the canonically-accepted correct conclusion, namely that Albert has aged 50 years while his astronaut sister has aged but one. This is now firmly accepted truth. Indeed it has even been claimed that it has been "proved" experimentally by a scientist who took a clock on commercial airline flights around the world, and compared it on his return with a stay-at-home clock. For myself I have neither examinations to pass nor, alas, a job to hold on to, so I am not bound to believe one thing or the other, and I elect to hold my peace.

I do say this, however – that what anyone "believes" is not an essential point. It is not a matter of what Albert or Betty or Hoyle or Dingle or your professor or your employer "believes". The real question is this: What is it that is predicted by the special theory of relativity? From this point of view it does not matter whether the theory of relativity is "true" or not, or whether it represents a correct description of the real physical world. Starting from the basic precepts of relativity, whether "true" or not, it must be only a matter of algebra (and simple algebra at that) to decide what is predicted by relativity.

A difficulty with this is that it is not, strictly speaking, a problem in special relativity, for special relativity deals with transformations between reference frames that are in uniform motion relative to one another. It is pointed out that Albert and Betty are not in uniform motion relative to one another, since one or the other of them has to change the direction of motion – i.e. has to accelerate. It could still be argued that, since motion is relative, one can regard either Albert or Betty as the one who accelerates – but the response to this is that only *uniform* motion is relative. Thus there is no symmetry between Albert and Betty. Betty either accelerates or experiences a gravitational field (depending on whether her experience is referred to Albert's or her own reference frame). And, since there is no symmetry, there is no paradox. This argument, however, admits that the age difference between Albert and Betty on Betty's return is not an effect of special relativity, but of general relativity, and is an effect caused by the acceleration (or gravitational field) experienced by Betty.

If this is so, there are some severe difficulties is describing the effect under general relativity. For example, whether the general theory allows for an instantaneous change in direction by Betty (and infinite deceleration), or whether the final result depends on how she decelerates – at what rate and for how long – must be determined by those who would tackle this problem. Further, the alleged age difference is supposed to depend upon the time during which Betty has been travelling and the length of her journey – yet the portion of her journey during which she is accelerating or decelerating can be made arbitrarily short compared with the time during which she is travelling at constant speed.

If the effect were to occur solely during the time when she was accelerating or decelerating, then the total length and duration of the constant speed part of her journey should not affect the age difference at all.

Since this chapter deals only with special relativity, and this is evidently not a problem restricted to special relativity, I leave the problem, as originally stated, here, without resolution, for readers to argue over as they will

15.12: A, B and C

A, B and C were three characters in the Canadian humorist Stephen Leacock's essay on *The Human Element in Mathematics*. "A, B and C are employed to dig a ditch. A can dig as much in one hour as B can dig in two..."

We can ask A, B and C to come to our aid in a *modified version* of the twins' problem, for we can arrange all three of them to be moving with constant velocities relative to each other. It goes like this (figure XV.14):

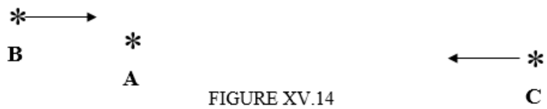


FIGURE XV.14

The scenario is probably obvious from the figure. There are three events:

1. B passes A
2. B meets C
3. C meets A

At event 1, B and A synchronize their watches so that each reads zero. At event 2, C sets his watch so that it reads the same as $B's$. At event 3, C and A compare watches. I shall leave the reader to cogitate over this. The only thing I shall point out is that this problem differs from the problem described as the Twins Paradox in two ways. In the first place, unlike in the Twins Paradox, all three characters, A, B and C are moving at constant velocities with respect to each other. Also, the first and third events occur at the same place relative to A but at different places referred to B or to C. In the twin paradox problem, the two events occur at the same place relative to both frames.

15.13: Simultaneity

If the time interval referred to one reference frame can be different when referred to another reference frame (and since time interval is merely one component of a four-vector, the magnitude of the component surely depends on the orientation in four space of the four axes) this raises the possibility that there might be a time interval of zero relative to one frame (i.e. two events are simultaneous) but are not simultaneous relative to another. This is indeed the case, provided that the two events do not occur in the same place as well as at the same time. Look at Figure XV.15.

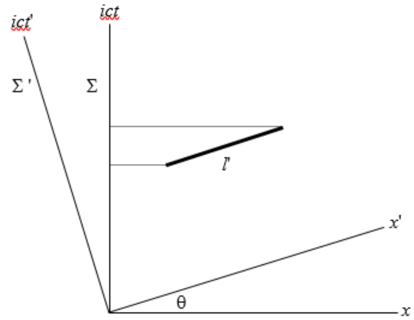


FIGURE XV.15

I have drawn two reference frames at an (imaginary) angle θ to each other. Think of $\sum$ as the railway station and of $\sum'$ as the railway train, and that the speed of the railway train is $c\tan\theta$ (You may have to go back to Section 15.3 or 15.7 to recall the relation of θ to the speed.) The thick line represents the interval between two events that are simultaneous when referred to $\sum'$, but are separated in space (one occurs near the front of the train; the other occurs near the rear). (Note also in this text that I am using the phrase "time interval" to denote the time-component of the "interval". For two simultaneous events, the *time interval* is zero, and the *interval* is then merely the distance between the two events.)

While the thick line has zero component along the ict' axis, its component along the ict axis is $l'\sin\theta$. That is, $ic(t_2-t_1)=l'\sin\theta=l'\times i\beta\gamma$.

Hence:

$$t_2-t_1=\frac{\beta\gamma l'}{c}. \tag{15.13.1}$$

For example, if the events took place simultaneously 100,000 km apart in the train (it is a long train) and if the train were travelling at 95% of the speed of light (γ = 3.203; it is a fast train), the two events would be separated when referred to the railway station by 1.01 seconds. The event near the rear of the train occurred first.

15.14: Order of Events, Causality and the Transmission of Information

Maybe it is even possible that if one event precedes another in one reference frame, in another reference frame the other precedes the one. In other words, the order of occurrence of events may be different in two frames. This indeed can be the case, and Minkowski diagrams (Figure XV.16) can help us to see why and in what circumstances.

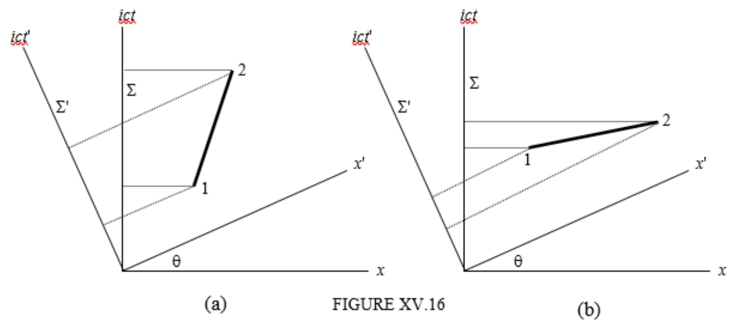


FIGURE XV.16

In part (a), of the two events 1 and 2, 1 occurs before 2 in either $\sum$ or $\sum'$. (from this point on I shall use a short phrase such as "in $\sum$" rather than the more cumbersome "when referred to the reference frame $\sum$". But in part (b), event 1 occurs before event 2 in $\sum$, but after event 2 in $\sum'$. One can see that there is reversal of order of events if the slope of the line joining to two events is less than the angle θ. The angle θ, it may be recalled, is an imaginary angle such than $\tan\theta = i\beta = \frac{i\nu}{c}$, where ν is the relative speed of the two frames. In Figure XV.17, for simplicity I am going to suppose that event 1 occurs at the origin of both frames, and that event 2 occurs at coordinates $(\nu\, t\ ,\ ict)$ in S. The condition for no reversal of events is then evidently

$$\frac{ict}{\nu t} \geq \tan\theta = i\beta = \frac{i\nu}{c};$$

or

$$\nu \leq c \tag{15.14.1}$$

This means, in effect, that neither mass nor energy can be transmitted faster than the speed of light. That is not quite the same thing as saying that "nothing" can be transmitted faster than the speed of light. For example a Moiré pattern formed by two combs with slightly different tooth spacings can move faster than light if one of the combs is moved relative to the other; but then I suppose it has to be admitted that in that case "nothing" is actually being transmitted – and certainly nothing that can transmit information or that can cause an event. An almost identical example would be the modulation envelope of the sum of two waves of slightly different frequencies. A well-known example from wave mechanics is that of the wave representation of a moving particle. The wave group (which is the integral of a continuous distribution of wavelengths whose extent is governed by Heisenberg's principle) moves with the particle at a sub-luminal speed, but there is nothing to prevent the wavelets within the group moving through the group at any speed. These wavelets may start at the beginning of the group and rapidly move through the group and extinguish themselves at the end. No "information" is transmitted from A to B at a speed any faster than the particle itself is moving.

15.15: Derivatives

We'll pause here and establish a few derivatives just for reference and in case we need them later.

We recall that the Lorentz relations are

$$x=\gamma(x'+\nu t') \tag{15.15.1}$$

and

$$t=\gamma\left(t'+\frac{\beta x'}{c}\right) \tag{15.15.2}$$

From these we immediately find that

$$\left(\frac{\partial x}{\partial x'}\right)_{t'}=\gamma;\quad \left(\frac{\partial x}{\partial t'}\right)_{x'}=\gamma\nu;\quad \left(\frac{\partial t}{\partial x'}\right)_{t'}=\frac{\beta\gamma}{c};\quad \left(\frac{\partial t}{\partial t'}\right)_{x'}=\gamma. \tag{15.15.3a,b,c,d}$$

We shall need these in future sections.

Caution

It is not impossible to make a mistake with some of these derivatives if one allows one's attention to wander. For example, one might suppose that, since $\frac{\partial x}{\partial x'}=\gamma$ then "obviously" $\frac{\partial x'}{\partial x}=\frac{1}{\gamma}$ - and indeed this is correct if t' is being held constant. However, we have to be sure that this is really what we want. The difficulty is likely to arise if, when writing a partial derivative, we neglect to specify what variables are being held constant, and no great harm would be done by insisting that these always be specified when writing a partial derivative. If you want the *inverses* rather than the *reciprocals* of Equations 15.15.3a,b,c,dthe rule, as ever, is: Interchange the primed and unprimed symbols and change the sign of ν or β. For example, the reciprocal of $\left(\frac{\partial x}{\partial x'}\right)_{t'}$ is $\left(\frac{\partial x'}{\partial x}\right)_{t'}$, while its inverse is $\left(\frac{\partial x'}{\partial x}\right)_{t}$. For completeness, and reference, then, I write down all the possibilities:

$$\left(\frac{\partial x'}{\partial x}\right)_{t'}=\frac{1}{\gamma};\quad \left(\frac{\partial t'}{\partial x}\right)_{x'}=\frac{1}{\gamma\nu};\quad \left(\frac{\partial x'}{\partial t}\right)_{t'}=\frac{c}{\beta\gamma};\quad \left(\frac{\partial t'}{\partial t}\right)_{x'}=\frac{1}{\gamma}. \tag{5.15.3e,f,g,h}$$

$$\left(\frac{\partial x'}{\partial x}\right)_{t}=\gamma;\quad \left(\frac{\partial x'}{\partial t}\right)_{x}=-\gamma\nu;\quad \left(\frac{\partial t'}{\partial x}\right)_{t}=-\frac{\beta\gamma}{c};\quad \left(\frac{\partial t'}{\partial t}\right)_{x}=\gamma. \tag{15.15.3i,j,k,l}$$

$$\left(\frac{\partial x}{\partial x'}\right)_{t}=\frac{1}{\gamma};\quad \left(\frac{\partial t}{\partial x'}\right)_{x}=-\frac{1}{\gamma\nu};\quad \left(\frac{\partial x}{\partial t'}\right)_{t}=-\frac{c}{\beta\gamma};\quad \left(\frac{\partial t}{\partial t'}\right)_{x}=\frac{1}{\gamma}. \tag{15.15.3m,n,o,p}$$

Now let's suppose that $\psi=\psi(x,t)$ where x and t are in turn functions (Equations 15.15.1and 15.15.2) of x' and t'. Then

$$\frac{\partial\psi}{\partial x'}=\frac{\partial x}{\partial x'}\frac{\partial\psi}{\partial t}+\frac{\partial t}{\partial x'}\frac{\partial\psi}{\partial t}=\gamma\frac{\partial\psi}{\partial x}+\frac{\beta\gamma}{c}\frac{\partial\psi}{\partial t} \tag{15.15.3}$$

and

$$\frac{\partial\psi}{\partial t'}=\frac{\partial x}{\partial t'}\frac{\partial\psi}{\partial x}+\frac{\partial t}{\partial t'}\frac{\partial\psi}{\partial t}=\gamma\nu\frac{\partial\psi}{\partial x}+\gamma\frac{\partial\psi}{\partial t}. \tag{15.15.4}$$

The reader will doubtless notice that I have here ignored my own advice and I have not indicated which variables are to be held constant. It would be worth spending a moment here thinking about this.

We can write Equations 15.15.3and 15.15.4as equivalent operators:

$$\frac{\partial}{\partial x'}=\gamma\left(\frac{\partial}{\partial x}+\frac{\beta}{c}\frac{\partial}{\partial t}\right) \tag{15.15.5}$$

and

$$\frac{\partial}{\partial t'}=\gamma\left(\nu\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right). \tag{15.15.6}$$

We can also, if we wish, find the second derivatives. Thus

$$\frac{\partial^2 \psi}{\partial x'^2} = \gamma^2 \left(\frac{\partial^2}{\partial x^2} + \frac{2\beta}{c} \frac{\partial^2}{\partial x \partial t} + \frac{\beta^2}{c^2} \frac{\partial^2}{\partial t^2} \right). \tag{15.15.7}$$

In a similar manner we obtain

$$\frac{\partial^2}{\partial x' \partial t'} = \gamma^2 \left(\nu \frac{\partial^2}{\partial x^2} + (1+\beta^2) \frac{\partial^2}{\partial x \partial t} + \frac{\beta}{c} \frac{\partial^2}{\partial t^2} \right) \tag{15.15.8}$$

and

$$\frac{\partial^2}{\partial t'^2} = \gamma^2 \left(\nu^2 \frac{\partial^2}{\partial x^2} + 2\nu \frac{\partial^2}{\partial x \partial t} + \frac{\partial^2}{\partial t^2} \right). \tag{15.15.9}$$

The inverses of all of these relations are to be found by interchanging the primed and unprimed coordinates and changing the signs of ν and β.

15.16: Addition of Velocities

A railway train trundles towards the east at speed ν_1, and a passenger strolls towards the front at speed ν_2. What is the speed of the passenger relative to the railway station? We might at first be tempted to reply: "Why, $\nu_1+\nu_2$ of course." In this section we shall show that the answer as predicted from the Lorentz transformations is a little less than this, and we shall develop a formula for calculating it. We have already discussed (in Section 15.6) our answer to the objection that this defies common sense. We pointed out there that the answer (to the perfectly reasonable objection) that "at the speeds we are accustomed to we would hardly notice the difference" is not a satisfactory response. The reason that the resultant speed is a little less than $\nu_1+\nu_2$ results from the way in which we have defined the Lorentz transformations between references frames and the way in which distances and time intervals are defined with reference to reference frames in uniform relative motion

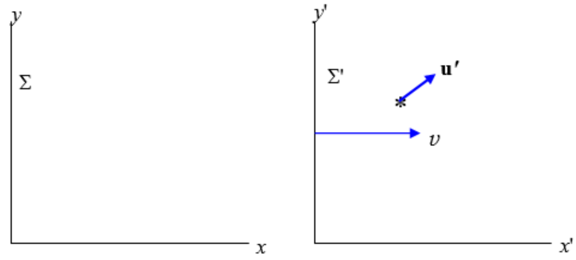


FIGURE XV.17

Figure XV.17 shows two references frames, Σ and Σ', the latter moving at speed ν with respect to the former. A particle is moving with velocity $\mathbf{u}'$ in Σ', with components $u'_{x'}$ and $u'_{y'}$. (" in Σ' " = "referred to the reference frame Σ'".)

What is the velocity of the particle in Σ?

Let us start with the x-component.

We have:

$$u=\frac{dx}{dt}=\frac{\left(\frac{\partial x}{\partial x'}\right)_{t'}dx'+\left(\frac{\partial x}{\partial t'}\right)_{x'}dt'}{\left(\frac{\partial t}{\partial x'}\right)_{t'}dx'+\left(\frac{\partial t}{\partial t'}\right)_{x'}dt'}=\frac{\left(\frac{\partial x}{\partial x'}\right)_{t'}u'+\left(\frac{\partial x}{\partial t'}\right)_{x'}}{\left(\frac{\partial t}{\partial x'}\right)_{t'}u'+\left(\frac{\partial t}{\partial t'}\right)_{x'}} \tag{15.16.1}$$

We take the derivatives from Equations 15.15.3a,b,c,d, and, writing $\frac{\nu}{c}$ for β, we obtain

$$u_x=\frac{u'_x+\nu}{1+u'_{x'}\frac{\nu}{c^2}}. \tag{15.16.2}$$

The inverse is obtained by interchanging the primed and unprimed symbols and reversing the sign of ν.

The y-component is found in an exactly similar manner, and I leave its derivation to the reader. The result is

$$u_y=\frac{u'+\nu}{1+u'\frac{\nu}{c^2}} \tag{15.16.3}$$

Special cases:

I. If $u'_{x'}=u'$ and $u'_{y'}=0$, then

$$u_x=\frac{u'+\nu}{1+u'\frac{\nu}{c^2}} \tag{15.16.4a}$$

$$u_y=0 \tag{15.16.4b}$$

II. If $u'_{x'} = 0$ and $u'_{y'} = u'$ then

$$u_x = \nu \tag{15.16.5a}$$

$$u_y = \frac{u'}{\gamma} \tag{15.16.5b}$$

Equation 15.16.4a as written is not easy to commit to memory, though it is rather easier if we write $\beta_1 = \frac{\nu}{c}$, $\beta_2 = \frac{u'}{c}$ and $\beta = \frac{u_x}{c}$. Then the equation becomes

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{15.16.4}$$

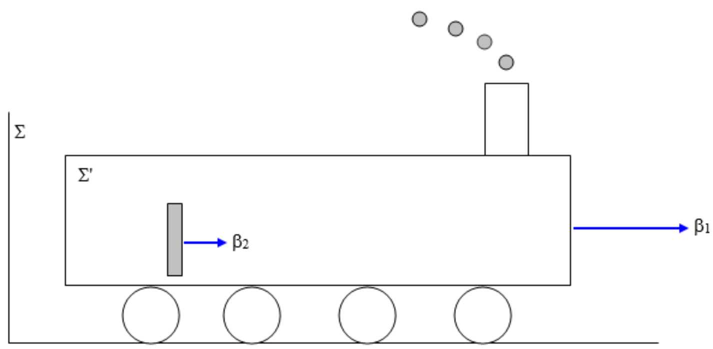
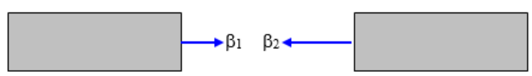


FIGURE XV.18

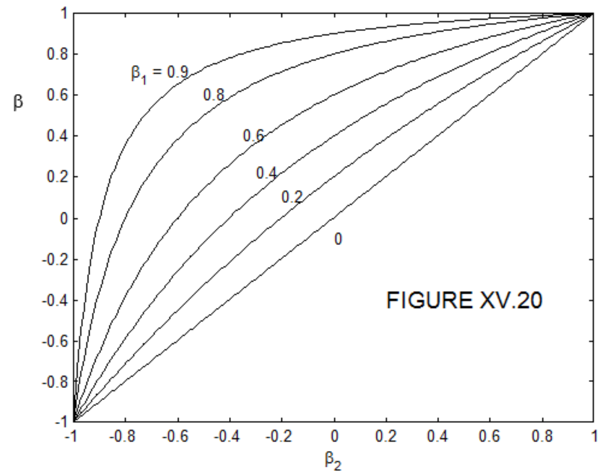


In Figure XV.18, a train Σ' is trundling with speed β_1 (times the speed of light) towards the right, and a passenger is strolling towards the front at speed β_2. The speed β of the passenger relative to the station Σ is then given by Equation 15.16.4. In Figure XV.19, two trains, one moving at speed β_1 and the other moving at speed β_2, are moving towards each other. (If you prefer to think of protons rather than trains, that is fine.) Again, the relative speed b of one train relative to the other is given by Equation 15.16.4.

Example 15.16.1

A train trundles to the right at 90% of the speed of light relative to Σ, and a passenger strolls to the right at 15% of the speed of light relative to Σ'. The speed of the passenger relative to Σ is 92.5% of the speed of light.

The relation between β_1, β_2 and β is shown graphically in Figure XV.20.



If I use the notation $\frac{\beta_1}{\beta_2}$ to mean "combining β_1 with β_2", I can write Equation 15.16.4as

$$\beta_1 \oplus \beta_2 = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \tag{15.16.5}$$

You may notice the similarity of Equation 15.16.4$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_1}$ to the hyperbolic function identity

$$\tanh(\phi_1 + \phi_2) = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2} \tag{15.16.6}$$

Thus I can represent the speed of an object by giving the value of ϕ, where

$$\beta = \tanh \phi \tag{15.16.7}$$

or

$$\phi = \tanh^{-1} \beta = \frac{1}{2} \ln \left(\frac{1 + \beta}{1 - \beta} \right) \tag{15.16.8}$$

The factor ϕ combines simply as

$$\frac{\phi_2}{\phi_2} = \phi_1 + \phi_2 \tag{15.16.9}$$

If you did what I suggested in Section 15.3 and programmed your calculator or computer to convert instantly from one relativity factor to another, you now have a quick way of adding speeds.

Example 15.16.2

A train trundles to the right at 90% of the speed of light (ϕ_1 = 1.47222) relative to S, and a passenger strolls to the right at 15% of the speed of light (ϕ_2 = 0.15114) relative to Σ'. The speed of the passenger relative to Σ is ϕ = 1.62336, or 92.5% of the speed of light.

Example 15.16.3

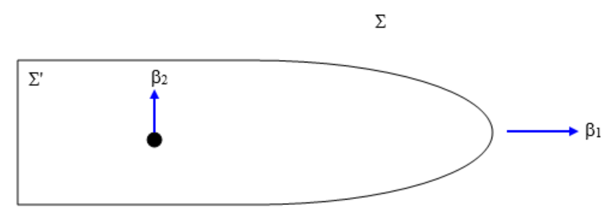


FIGURE XV.22

(Sorry – there is no Figure XV.21.)

An ocean liner Σ' sails serenely eastwards at a speed β_1 = 0.9c (g_1 = 2.29416) relative to the ocean Σ. A passenger ambles athwartships at a speed β_2 = 0.5c relative to the ship. What is the velocity of the passenger relative to the ocean?

The northerly component of her velocity is given by Equation 15.16.5b and is 0.21794c. Her easterly component is just 0.9c. Her velocity relative to the ocean is therefore 0.92601c in a direction 13° 37' north of east.

? Exercise 15.16.1

Show that, if the speed of the ocean liner is β_1 and the athwartships speed of the passenger is β_2, the resultant speed β of the passenger relative to the ocean is given by

$$\beta^2 = \beta_1^2 + \beta_2^2 - \beta_1^2\beta_2^2 \tag{15.16.10}$$

and that her velocity makes and angle α with the velocity of the ship given by

$$\tan\alpha = \beta_2\sqrt{1 - \frac{\beta_1^2}{\beta_1}}. \tag{15.16.11}$$

✓ Example 15.16.4

A railway train Σ' of proper length L_0 = 100 yards thunders past a railway station Σ at such a speed that the stationmaster thinks its length is only 40 yards. (Correction: It is not a matter of what he "thinks". What I should have said is that the length of the train, referred to a reference frame Σ in which the stationmaster is at rest, is 40 yards.) A dachshund waddles along the corridor towards the front of the train. (A dachshund, or badger hound, is a cylindrical dog whose proper length is normally several times its diameter.) The proper length l_0 of the dachshund is 24 inches, but to a seated passenger, it appears to be... no, sorry, I mean that its length, referred to the reference frame Σ', is 15 inches. What is the length of the dachshund referred to the reference frame Σ in which the stationmaster is at rest?

We are told, in effect, that the speed of the train relative to the station is given by γ_1 = 2.5, and that the speed of the dachshund relative to the train is given by γ_2 = 1.6. So how do these two gammas combine to make the factor γ for the dachshund relative to the station?

There are several ways in which you could do this problem. One is to develop a general algebraic method of combining two gamma factors. Thus:

? Exercise 15.16.2

Show that two gamma factors combine according to

$$\gamma_1 \oplus \gamma_2 = \gamma_1\gamma_2 + \sqrt{(\gamma_1^2 - 1)(\gamma_2^2 - 1)}. \tag{15.16.12}$$

I'll leave you to try that. The other way is to take advantage of the programme you wrote when you read Section 15.3, by which you can instantaneously convert one relativity factor to another. Thus you instantly convert the gammas to phis.

Thus $\gamma_1 = 2.5 \;\Rightarrow\; \phi_1 = 1.56680$

and $\gamma_1 = 1.6 \;\Rightarrow\; \phi_1 = 1.04697$

$$\therefore \qquad \phi = 2.61377 \ \Rightarrow \ \gamma = 6.86182$$

Is this what Equation 15.16.12gets?

Therefore, referred to the railway station, the length of the dachshund is $\frac{24}{\gamma}$ = 3.5 inches.

15.17: Aberration of Light

The direction of Earth's velocity on any particular date is called the *Apex of the Earth's Way*. In part (a) of Figure XV.23 I show Earth moving towards the apex at speed ν, and light coming from a star at speed c from an angle χ from the apex.

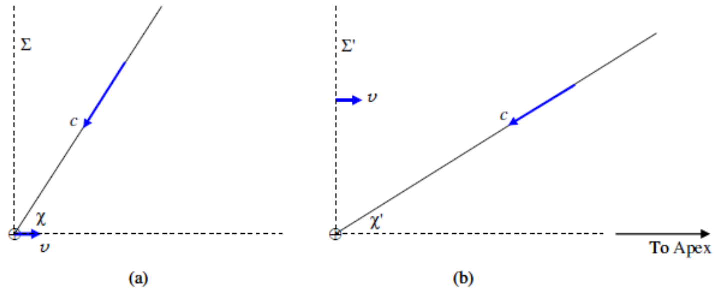


FIGURE XV.23

The x- and y- components of the velocity of light are respectively $-c\cos\chi$ and $-c\sin\chi$. Relative to Earth (part (b)), the x'- and y'- components are, by Equations 15.16.2 and 15.16.3 (or rather their inverses)

$$-\frac{c\cos\chi+\nu}{1+\left(\frac{\nu}{c}\right)\cos\chi}$$

and

$$-\frac{c\sin\chi}{\gamma\left(1+\left(\frac{\nu}{c}\right)\cos\chi\right)}.$$

You can verify that the orthogonal sum of these two components is c, as it should be according to our fundamental assumption that the speed of light is the same referred to all reference frames in uniform relative motion.

The apparent direction of the star is therefore given by

$$\sin\chi' = -\frac{\sin\chi}{\gamma\left(1+\left(\frac{\nu}{c}\right)\cos\chi\right)} \tag{15.17.1}$$

It is left as an exercise to show that, for small $\frac{\nu}{c}$, this becomes

$$\chi-\chi' = \frac{\nu\sin\chi}{c}. \tag{15.17.2}$$

with ν = 29.8 km s^{-1}, $\frac{\nu}{c}$ is about 20".5. More details about aberration of light, including the derivation of Equation 15.17.2, can be found in Celestial Mechanics, Section 11.3.

15.18: Doppler Effect

It is well known that the formula for the Doppler effect in sound is different according to whether it is the source or the observer that is in motion. An answer to the question “Why should this be?” to the effect that “Oh, that’s just the way the algebra works out” is obviously unsatisfactory, so I shall try to explain why, physically, there is a difference. Then, when you have thoroughly understood that observer in motion is an entirely different situation from source in motion, and the formulas must be different, we shall look at the Doppler effect in light, and we’ll return to square one when we find that the formulas for source in motion and observer in motion are the same!

This section on the Doppler effect will probably be rather longer than it need be, just because some aspects interested me – but if you find it too long, just skip the parts that aren’t of special interest to you. These will quite likely include the parts on the *ballistic* Doppler effect.

First, we’ll deal with the Doppler effect in sound. All speeds are supposed to be very small compared with the speed of light, so that we need not trouble ourselves with Lorentz transformations. First, let’s deal with observer in motion (Figure XV. 24).

When the source is at rest, it emits *concentric* equally-spaced spherical wavefronts at some frequency. When an observer moves towards the source, he will pass these wavefronts at a higher frequency that the frequency at which they were emitted, and that is the cause of the Doppler effect with a stationary source and moving observer.

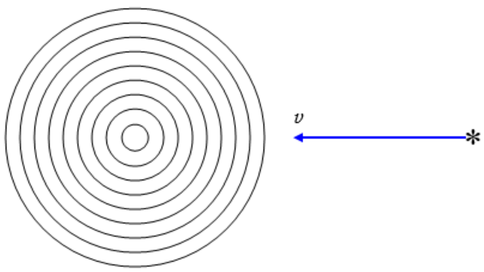


FIGURE XV.24

Now, we’ll look at the source-in-motion situation. (Figure XV.25).

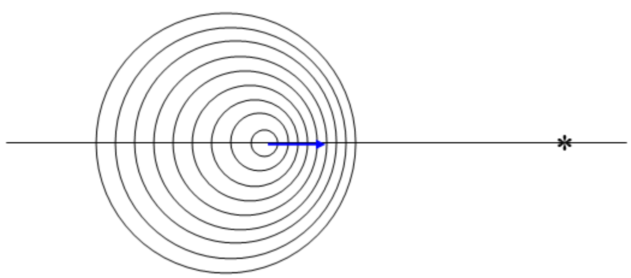


FIGURE XV.25

Here we see that the wavefronts are not equally spaced, but are compressed ahead of the motion of the source, and for that reason they will pass a stationary observer at a higher frequency than the frequency at which they were emitted. Thus the nature of the effect is a little different according to whether it is the source or the observer that is in motion, and thus one would not expect identical equations to describe the two situations.

We shall move on shortly to discuss the effect quantitatively and develop the relevant equations. I shall assume that the reader is familiar with the usual relation connecting wavelength, frequency and speed of a wave. Nevertheless I shall write down the relation in large print, three times, just to make sure:

SPEED = FREQUENCY **% WAVELENGTH**

FREQUENCY = SPEED ÷ WAVELENGTH

WAVELENGTH = SPEED ÷ FREQUENCY

I am going to start with the Doppler effect in sound, where the speed of the signal is constant with respect to the medium than transmits the sound – usually air. I shall give the necessary formulas for source *and* observer each in motion. If you want the formulas for one or the other stationary, you just put one of the speeds equal to zero. The speeds of the source S and of the observer O relative to the air will be denoted respectively by ν_1 and ν_2 and the speed of sound in air will be denoted by c. The situation is shown in Figure XV.26.

FIGURE XV.26

The relevant formulas are shown below:

	Source	Observer
Frequency	ν_0	$\nu_0\left(\dfrac{c-v_2}{c-v_1}\right)$
	$\downarrow$	$\uparrow$
Speed	$c - v_1$	$c - v_2$
	$\downarrow$	$\uparrow$
Wavelength	$(c - v_1)/\nu_0 \longrightarrow$	$(c - v_1)/\nu_0$

The way we work this table is just to follow the arrows. Starting at the top left, we suppose that the source emits a signal of frequency ν_0. The speed of the signal *relative to the source* is $c-v_1$, and so the wavelength is $\frac{(c-v_1)}{\nu_0}$. The wavelength is the same for the observer (we are supposing that all speeds are very much less than the speed of light, so the Lorentz factor is effectively 1.) The speed of sound relative to the observer is $c-v_2$, and so the frequency heard by the observer is the last (upper right) entry of the table.

Two special cases:

a. Observer in motion and approaching a stationary source at speed v. $v_1=0$ and $v_2=-v$. In that case the frequency heard by the observer is

$$\nu=\nu_0(1+\frac{v}{c}). \tag{15.18.1}$$

b. Source in motion and approaching a stationary observer at speed v. $v_1=v$ and $v_2=0$. In that case the frequency heard by the observer is

$$\nu=\frac{\nu_0}{(1-\frac{v}{c})}\approx\nu_0(1+(\frac{v}{c})+(\frac{v}{c})^2+\ldots). \tag{15.18.2}$$

We might now consider *reflection*. Thus, suppose you approach a brick wall at speed v while whistling a note of frequency ν_0. What will be the frequency of the echo that you hear? Let's make the question a little more general. A source S, emitting a whistle of frequency ν_0, approaches a brick wall M at speed v_1. A separate observer O approaches the wall (from the same side) at speed v_2. And, for good measure, let's have the brick wall moving at speed v_3. (The reader may notice at this point that theoretical physics is rather easier than experimental physics.) The situation is shown in Figure XV.27.

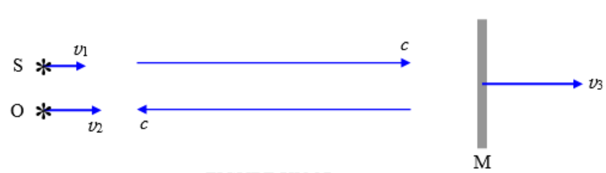


FIGURE XV.27

We construct a table similar to the previous one.

	Source	Mirror before reflection	Mirror after reflection	Observer
Frequency	ν_0	$\nu_0\left(\frac{c-v_3}{c-v_1}\right)$	$\nu_0\left(\frac{c-v_3}{c-v_1}\right)$	$\nu_0\left(\frac{(c+v_2)(c-v_3)}{(c-v_1)(c+v_3)}\right)$
Speed	$c-v_1$	$c-v_3$	$c+v_3$	$c+v_2$
Wavelength	$\frac{c-v_1}{\nu_0}$	$\frac{c-v_1}{\nu_0}$	$\frac{(c-v_1)(c+v_3)}{\nu_0(c-v_3)}$	$\frac{(c-v_1)(c+v_3)}{\nu_0(c-v_3)}$

At all times, the speed relative to the air is c.

The answer to our initial question, in which the source and the observer were one and the same, and the mirror (wall) was stationary is found by putting $v_1 = v_2 = v$ and $v_3 = 0$ in the last (top right) formula in the table. This results in

$$\nu = \nu_0\left(\frac{c+v}{c-v}\right) \approx \nu_0(1+2(\frac{v}{c})+2(\frac{v}{c})^2+2(\frac{v}{c})^3\ldots). \tag{15.18.3}$$

So much for the Doppler effect in sound. Before moving on to light, I want to look at what I shall call the Doppler effect in ballistics, or "cops and robbers". An impatient reader may safely skip this discussion of ballistic Doppler effect. A police ("cop") car is chasing a stolen car driven by robbers. The cop car is the "source" and the robber's car (or, rather the car that they have stolen, for it is not theirs) are the "observers". The cop car ("source") is travelling at speed v_1 and the robbers ("observer") is travelling at speed v_2. The cops are firing bullets (the "signal") towards the robbers. (No one gets hurt in this thought experiment, which is all make-believe.) The bullets leave the muzzle of the revolver at speed c(that is the speed of the *bullets*, and is nothing to do with *light*) relative to the revolver, and hence they travel (relative to the lamp-posts at the side of the road) at speed $c+v_1$ and relative to the robbers at speed $c+v_1-v_2$. The cops fire bullets at frequency ν_0, and our task is to find the frequency with which the bullets are "received" by the robbers. The distance between the bullets is the "wavelength".

This may not be a very important exercise, but it is not entirely pointless, for fairness dictates that, when we are considering (even if only to discard) possible plausible mechanisms for the propagation of light, we might consider, at least briefly, the so-called "ballistic" theory of light propagation, in which the speed of light through space is equal to the speed at which it leaves the source plus the speed of the source. Some readers may be aware of the Michelson-Morley experiment. That experiment demonstrated that light was not propagated at a speed that was constant with respect to some all-pervading "luminiferous aether" – but it must be noted that it did nothing to prove or disprove the "ballistic" theory of light propagation, since it did not measure the speed of light from moving sources. In the intervening years, some attempts have indeed been made to measure the speed of light from moving sources, though their interpretation has not been free from ambiguity.

I now construct a table showing the "frequency", "speed" and "wavelength" for ballistic propagation in exactly the same way as I did for sound.

	Source	Observer
Frequency	ν_0	$\nu_0\left(\frac{c+v_1-v_2}{c}\right)$
Speed	c	$c+v_1-v_2$
Wavelength	c/ν_0	c/ν_0

In order not to spend longer on "ballistic" propagation than is warranted by its importance, I'll just let the reader spend as much or as little time pondering over this table as he or she wishes. Just one small point might be noted, namely that the formulas for "observer in motion" and "source in motion" are the same.

For completeness rather than for any important application, I shall also construct here the table for "reflection". A source of bullets is approaching a mirror at speed v_1. An observer is also approaching the mirror, from the same side, at speed v_2. And the mirror is moving at speed v_3, and reflection is elastic (the coefficient of restitution is 1.) You are free to put as many of these speeds equal to zero as you wish.

The entries for "speed" give the speed relative to the source or mirror or observer. The speed relative to stationary lampposts at the side of the road is $c+v_1$ before reflection and $c+v_1-2v_3$ after reflection.

	Source	Mirror before reflection	Mirror after reflection	Observer
Frequency	ν_0	$\nu_0\left(\frac{c+v_1-v_3}{c}\right)$	$\nu_0\left(\frac{c+v_1-v_3}{c}\right)$	$\nu_0\left(\frac{c+v_1+v_2-2v_3}{c}\right)$
Speed	c	$c+v_1-v_3$	$c+v_1-v_3$	$c+v_1+v_2-2v_3$
Wavelength	$\frac{c}{\nu_0}$	$\frac{c}{\nu_0}$	$\frac{c}{\nu_0}$	$\frac{c}{\nu_0}$

We now move on to the only aspect of the Doppler effect that is really relevant to this chapter, namely the Doppler effect in light. In the previous two situations I have been able to assume that all speeds were negligible compared with the speed of light, and we have not had to concern ourselves with relativistic effects. Here, however, the signal *is* light and is propagated at the speed of light, and this speed is the same whether referred to the reference frame in which the source is stationary or the observer is stationary. Further, the Doppler effect is noticeable only if source or observer are moving at speeds comparable to that of light. We shall see that the difference between the frequency of a signal relative to an observer and the frequency relative to the source is the result of *two* effects, which, while they may be treated separately, are both operative and in that sense inseparable. These two effects are the *Doppler effect proper*, which is a result of the changing distance between source and observer, and the relativistic *dilation of time*.

I am going to use the symbol T to denote the time interval between passage of consecutive crests of an electromagnetic wave. I'll call this the *period*. This is merely the reciprocal of the frequency ν. I am going to start by considering a situation in which a source and an observer a receding from each other at a speed v. I have drawn this in Figure XV.27, which is referred to a frame in which the observer is at rest. The speed of light is c.

FIGURE XV.27

Let us suppose that S emits an electromagnetic wave of period $T_0=\frac{1}{\nu_0}$ *referred to the frame in which S is at rest.* We are going to have to think about *four* distinct periods or frequencies:

1. The time interval between the emission of consecutive crests by S *referred to the reference frame in which S is at rest.* This is the period T_0 and the frequency ν_0 that we have just mentioned.

2. The time interval between the emission of consecutive crests by S *referred to the reference frame in which O is at rest.* By the relativistic formula for the dilation of time this is

$$\gamma T_0 \quad or \quad \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{15.18.4}$$

3. The time interval between the reception of consecutive crests by O as a result of the increasing distance between O and S (the "true" Doppler effect, as distinct from time dilation) *referred to the reference frame in which S is at rest.* This is

$$T_0(1+\frac{v}{c}). \tag{15.18.5}$$

4. The time interval between the reception of consecutive crests by O as a result of the increasing distance between O and S (the "true" Doppler effect, as distinct from time dilation) *referred to the reference frame in which O is at rest.* This is

$$\gamma \quad \text{times} \quad T_0\left(1+\frac{v}{c}\right). \tag{15.18.6}$$

This, of course, is what O "observes", and, when you do the trivial algebra, you find that this is

$$T=T_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}, \tag{15.18.7}$$

or, in terms of frequency,

$$\nu=\nu_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}. \tag{15.18.8}$$

If source and observer *approach* each other at speed v, the result is

$$\nu=\nu_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}. \tag{15.18.9}$$

The factor $\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$ is often denoted by the symbol k, and indeed that was the symbol I used in Section 15.3 (see Equation 15.3.3).

? Exercise 15.18.1

Expand Equation 15.18.9by the binomial theorem as far as $(\frac{v}{c})^2$ and compare the result with Equations 15.18.1and 15.18.2.

I make it

$$\nu=\nu_0(1+(\frac{v}{c})+\frac{1}{2}(\frac{v}{c})^2\ldots). \tag{15.18.10}$$

? Exercise 15.18.2

An observer O sends an electromagnetic signal of frequency ν_0 at speed c to a mirror that is receding at speed v. When the reflected signal arrives back at the observer, what is its frequency (to first order in $\frac{v}{c}$)? Is it $\nu_0(1-\frac{v}{c})$ or is it $\nu_0(1-\frac{2v}{c})$?

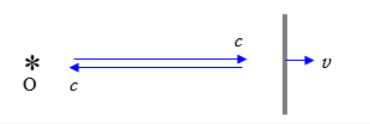


FIGURE XV.28

I can think offhand of two applications of this. If you examine the solar Fraunhofer spectrum reflected of the equatorial limb of a rotating planet, and you observe the fractional change $\frac{\Delta\nu}{\nu_0}$ in the frequency of a spectrum line, will this tell you $\frac{v}{c}$ or $\frac{2v}{c}$, where v is the equatorial speed of the planet's surface? And if a policeman directs a radar beam at your car, does the frequency of the returning beam tell him the speed of your car, or twice its speed? You could try arguing this case in court – or, better, stick to the speed limit so there is no need to do so. The answer, by the way, is $\nu_0(1-\frac{2v}{c})$.

Redshift. When a galaxy is moving away from us, a spectrum line of laboratory wavelength λ_0 will appear to have a frequency for the observer of $\lambda=k\lambda_0$. The fractional increase in wavelength $\frac{\lambda-\lambda_0}{\lambda_0}$ is generally given the symbol z, which is evidently equal to $k-1$. (Only to first order in β is it approximately equal to β. It is important to note that the definition of z is $\frac{\lambda-\lambda_0}{\lambda_0}$, and not $\frac{v}{c}$.

A note on terminology: If a source is receding from the observer the light is observed to be shifted towards longer wavelengths, and if it is approaching the observer the light is shifted towards shorter wavelengths. Traditionally a shift to longer wavelengths is called a “redshift”, and a shift towards shorter wavelengths is called a “blueshift”. Note, however, that if an infrared source is approaching an observer, its light is shifted towards the red, and if an ultraviolet source is receding from an observer, its light is shifted towards the blue! Nevertheless, I shall continue in this chapter to refer to shifts to longer and shorter wavelengths as redshifts and blueshifts respectively.

Example 15.18.1

A red galaxy R of wavelength 680.0 nm and a green galaxy G of wavelength 520.0 nm are on opposite sides of an observer X, both receding from him/her. To the observer, the wavelength of the red galaxy appears to be 820.0 nm, and the wavelength of the green galaxy appears to be 640.0 nm. What is the wavelength of the green galaxy as seen from the red galaxy?

FIGURE XV.29

Solution

We are told that k for the red galaxy is 82/68 = 1.20588, or z = 0.20588, and that k for the green galaxy k is 64/52 = 1.23077, or z = 0.23077. Because of the preparation we did in Section 15.3, we can instantly convert these to ϕ. Thus for the red galaxy ϕ= 0.187212 and for the green galaxy ϕ = 0.207639. The sum of these is 0.394851. We can instantly convert this to $k = 1.48416$ or $z = 0.48416$.. Thus, as seen from R, the wavelength of G is 771.8 nm.

Alternatively.

Show that the factor k combines as

$$k_1 \oplus k_2 = k_1 k_2 \tag{15.18.11}$$

and verify that $\frac{82}{68} \times \frac{64}{52} = 1.48416$. Show also that the redshift factor z combines as

$$z_1 \oplus z_2 = z_1 z_2 + z_1 + z_2. \tag{15.18.12}$$

15.19: The Transverse and Oblique Doppler Effects

I pointed out in Section 15.18 that the observed Doppler effect, when the transmitted signal is electromagnetic radiation and observer or source or both are travelling at speed comparable to that of light, is a combination of two effects – the "true" Doppler effect, caused by the changing distance between source and observer, and the effect of time dilation. This raises the following questions.

Transverse Doppler effect

If a source of light is moving at right angles to (transverse to) the line joining observer to source, will the observer see a change in frequency or wavelength, even though the distance between observer and source at that instant is not changing? The answer is yes, certainly, and the effect is sometimes called the "transverse Doppler effect", although it is the effect of relativistic time dilation rather than of a true Doppler effect.

Thus let us suppose that a source is moving transverse to the line of sight at a speed described by its parameter β or γ, and that the period of the radiation referred to the reference frame in which the source is at rest is T_0 and the frequency is ν_0. The time interval between emission of consecutive wavecrests when referred to the frame in which the observer is at rest is longer by the gamma-factor, and the frequency is correspondingly less. That is, the frequency, referred to the observer's reference frame, is

$$\nu=\frac{\nu_0}{\gamma}=\nu_0\sqrt{1-\beta^2} \tag{15.19.1}$$

The light from the source is therefore seen by the observer to be redshifted, even though there is no radial velocity component.

Oblique Doppler Effect

This raises a further question. Suppose a source is moving almost but not quite at right angles to the line of sight, so that it has a large transverse velocity component, and a small velocity component towards the observer. In that case, its "redshift" resulting from the time dilation might be appreciable, while its "blueshift" resulting from "true" Doppler effect (the decreasing distance between source and observer) is still very small. Therefore, even though the distance between source and observer is slightly decreasing, there is a net redshift of the spectrum. This is in fact correct, and is the "oblique Doppler effect".

In Figure XV.30, a source S is moving at speed β times the speed of light in a direction that makes an angle θ with the line of sight. It is emitting a signal of frequency ν_0 in S. (I am here using the frame "in S" as earlier in the chapter to mean "referred to a reference frame in which S is at rest.) The signal arrives at the observer O at a slightly greater frequency as a result of the decreasing distance of S from O, and at a slightly lesser frequency as a result of the time dilation, the two effects opposing each other.

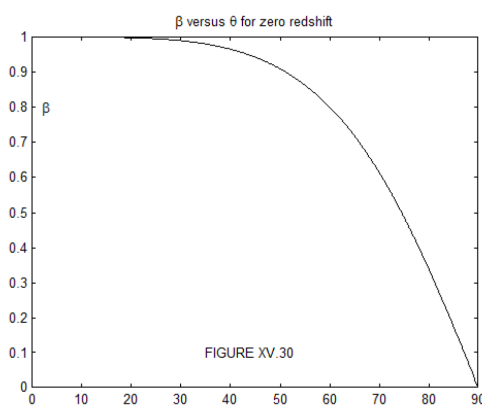


FIGURE XV.30

The frequency of the received signal at O, in O, is

$$\nu=\frac{\nu_0\sqrt{1-\beta^2}}{1-\beta\cos\theta}. \tag{15.19.2}$$

For a given angle θ the redshift is zero for a speed of

$$\beta=\frac{2\cos\theta}{1+\cos^2\theta} \tag{15.19.3}$$

or, for a given speed, the direction of motion resulting in a zero redshift is given by

$$\beta\cos^2\theta-2\cos\theta+\beta=0. \tag{15.19.4}$$

This relation is shown in Figure XV.30. (Although Equation 15.19.4 is quadratic in $\cos\theta$ there is only one real solution θ for β between 0 and 1. Prove this assertion.) It might be noted that if the speed of the source is 99.99% of the speed of light the observer will see a redshift unless the direction of motion of S is no further than 9º 36' from the line from S to O. That is worth repeating: S is moving very close to the speed of light in a direction that is close to being directly towards the observer; the observer will see a redshift.

Equation 15.19.2, which gives ν as a function of θ for a given β, will readily be recognized at the equation of an ellipse of eccentricity β, semi minor axis ν_0 and semi major axis $\gamma\nu_0$. This relation is shown in Figure XV.31 for several β. The curves are red where there is a redshift and blue where there is a blueshift. There is no redshift or blueshift for $\beta=0$, and the ellipse for that case is a circle and is drawn in black.

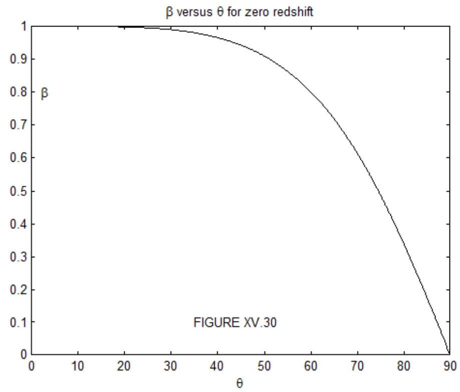


FIGURE XV.30

An alternative and perhaps more useful way of looking at Equation 15.19.2 is to regard it as an equation that gives β as a function of θ for a given Doppler ratio $\frac{\nu}{\nu_0}$. For example, if the Doppler ratio of a galaxy is observed to be 0.75, the velocity vector of the galaxy could be any arrow starting at the black dot and ending on the curve marked 0.75. The curves are ellipses with semi major axis equal to $\frac{1}{\sqrt{1-(\frac{\nu}{\nu_0})^2}}$ and semi minor axis $\frac{1}{(1-(\frac{\nu}{\nu_0})^2)}$.

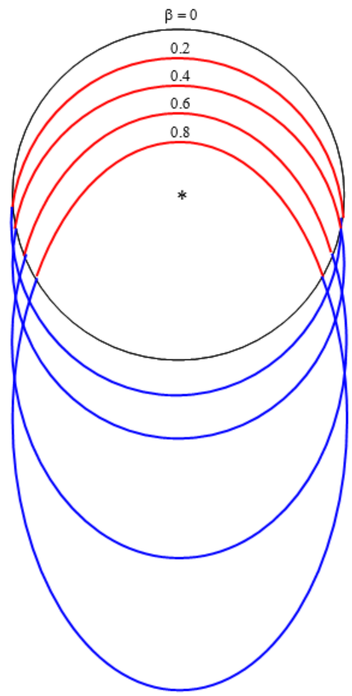


FIGURE XV.31

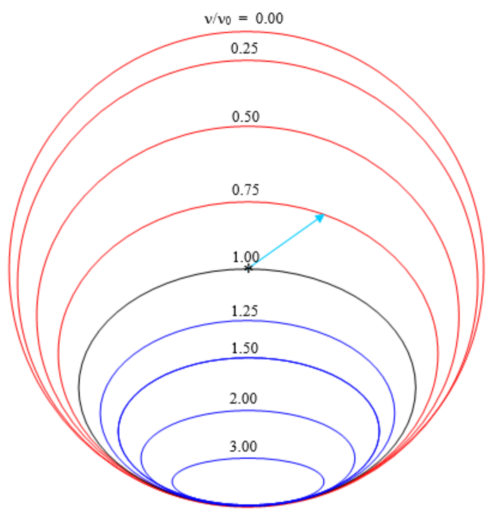


FIGURE XV.32

15.20: Acceleration

Figure XV.33 shows two references frames, Σ and Σ', the latter moving at speed v with respect to the former.

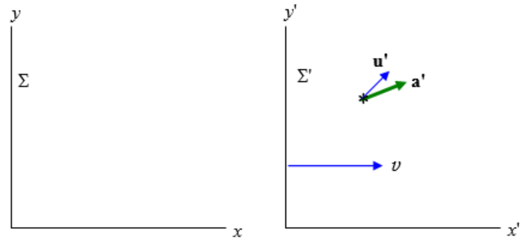


FIGURE XV.33

A particle is moving with acceleration $\mathbf{a}'$ in Σ'. (" in Σ' " = "referred to the reference frame Σ' ".) The velocity is not necessarily, of course, in the same direction as the acceleration, and we'll suppose that its velocity in Σ' is $\mathbf{u}'$. The acceleration and velocity components in Σ' are $a'_{x'}, a'_{y'}, u'_{x'}, u'_{y'}$.

What is the acceleration of the particle in Σ? We shall start with the x-component.

The x-component of its acceleration in Σ is given by

$$a_x = \frac{du_x}{dt}, \tag{15.20.1}$$

where

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \tag{15.20.2}$$

and

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{15.5.19}$$

Equations 15.16.2 and 15.5.19 give us

$$du_x = \frac{du_x}{du'_x} du'_x = \frac{du'_x}{\gamma^2 (1 + \frac{u'_x v}{c^2})^2} \tag{15.20.3}$$

and

$$dt \ = \frac{\partial t}{\partial t'} dt' + \frac{\partial t}{\partial x'} dx' = \gamma dt' \ + \ \frac{\gamma v}{c^2} dx' \tag{15.20.4}$$

On substitution of these into Equation 15.20.1and a very little algebra, we obtain

$$a_x = \frac{a'}{\gamma^3 (1 + \frac{u'_x v}{c^2})^3} \tag{15.20.5}$$

The y-component of its acceleration in Σ is given by

$$a_y = \frac{du_y}{dt}, \tag{15.20.6}$$

We have already worked out the denominator dt(Equation 15.20.4). We know that

$$u_y = \frac{u'_{y'}}{\gamma (1 + \frac{u'_{x'} v}{c^2})} \tag{15.16.3}$$

from which

$$du_y = \frac{\partial u_y}{\partial u'_{x'}} + \frac{\partial u_y}{\partial u'_{y'}} \partial u'_{y'} = \frac{1}{\gamma}\left(-\frac{vu'_{y'}}{c^2\left(1+\frac{vu'_{x'}}{c^2}\right)^2} du'_{x'} + \frac{1}{1+\frac{vu'_{x'}}{c^2}} du'_{y'} \right). \tag{15.20.7}$$

Divide Equation 15.20.7by Equation 15.20.4to obtain

$$a_y = \frac{1}{\gamma^2}\left(-\frac{vu'_{y'}}{c^2\left(1+\frac{vu'_{x'}}{c^2}\right)^2} a'_{x'} + \frac{1}{1+\frac{vu'_{x'}}{c^2}} a'_{y'} \right). \tag{15.20.8}$$

15.21: Mass

It is well known that "in relativity" the mass of an object increases as its speed increases. This may be well known, but I am not certain that it is a very precise statement of the true situation. Or at least it is no more precise than to say that the length of a rod decreases as its speed increases. The length of a rod when referred to a frame in which it is at rest is called its *proper length* l_0, and the mass of a body when referred to a frame in which it is at rest is called its *rest mass* m_0, and both of these things are invariant. The length of a rod when referred to a reference frame that is moving with respect to it (i.e., in Minkowski language, its component along an inclined axis) and the mass of a body referred to a frame that is moving with respect to it may indeed be different from the proper length of the rod or the rest mass of the body.

In order to derive the FitzGerald-Lorentz contraction, we had to think about what we mean by "length" and how to measure it. Likewise, in order to derive the "relativistic increase of mass" (which may be a misnomer) we have to think about what we mean by mass and how to measure it.

The fundamental unit of mass used at present in science is the International Prototype Kilogram, a platinum-iridium alloy, held in Sèvres, Paris, France. In order to determine the mass, or inertia, of another body, we need to carry out an experiment to compare its reluctance to accelerate when a force is applied to it with the reluctance of the standard kilogram when the same force is applied. We might, for example, attach the body to a spring, stretch the spring, let go, and see how fast the body accelerates. Then we carry out the same experiment with the International Prototype Kilogram. Or we might apply an impulse ($\int I dt$ - see Chapter 8) to the body and to the Kilogram, and measure the speed immediately after applying the impulse. This might be done, for example, but striking the body and the Kilogram with a golf club, or, for a more controlled experiment, one could press each body up against a compressed spring, release the spring, and measure the resulting speed imparted to the body and to the Kilogram. (It is probable that the International Prototype Kilogram is kept under some sort of guard, and its curators may not altogether appreciate such experiments, so perhaps these experiments had better remain Thought Experiments.) Yet another method would be to cause the body and the Kilogram to collide with each other, and to assume that the collision is elastic (no internal degrees of freedom) and that momentum (defined as the product of mass and velocity) are conserved.

All of these experiments measure the reluctance to accelerate under a force; in other words the *inertia* or the *inertial mass* or just the *mass* of the body.

Another possible experiment to determine the mass of the body would be to place it and the Kilogram at a measured distance from another mass (such as the Earth) and measure the gravitational force (weight) of each. One has an uneasy feeling that this sort of measurement is somehow a little different from the others, in that it isn't a measure of *inertia*. Some indeed would differentiate between the *inertial mass* and the *gravitational mass* of a body, although the two are in fact observed to be strictly proportional to one another. Some would not find the proportionality between inertial and gravitational mass particularly remarkable; to others, the proportionality is a surprising fact of the profoundest significance.

In this chapter we do not deal with general relativity or with gravity, so we shall think of mass in terms of its inertia. I am going to measure the ratio of two masses (one of which might be the International Prototype Kilogram) by allowing them to collide, and their masses are to be defined by assuming that the momentum of the system is conserved in all uniformly moving reference frames.

Figure XV.34 shows two references frames, Σ and Σ', the latter moving to the right at speed v relative to the former. Two bodies, of identical masses in Σ' (i.e. referred to the frame Σ'), are moving at speed u' in Σ', one of them to the right, the other to the left. Their mutual centre of mass is stationary in Σ'.

FIGURE XV.34

Now let us refer the situation to the frame Σ (see figure XV.35).

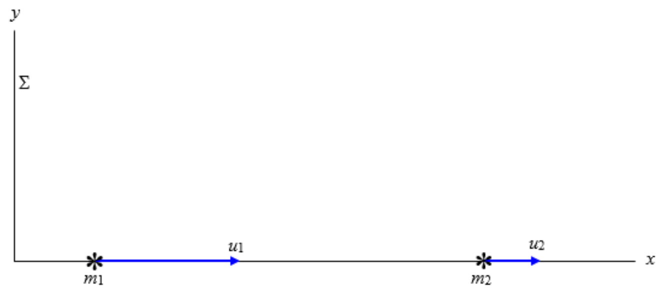


FIGURE XV.35

The total momentum of the system in Σ is $m_1u_1+m_2u_2$. But the centre of mass (which is stationary in Σ') is moving to the right in Σ with speed v. Therefore the momentum is also $(m_1+m_2)v$. If they stick together upon collision, we are left with a single particle of mass m_1+m_2 moving at speed v, and, because there are no external forces, the momentum is conserved. In any case, whether the collision is elastic or not, we have

$$m_1u_1+m_2u_2=(m_1+m_2). \tag{15.21.1}$$

But

$$u_1=\frac{u'+v}{1+\frac{u'v}{c^2}} \tag{15.21.2a}$$

and

$$u_2=\frac{-u'+v}{1-\frac{u'v}{c^2}} \tag{15.21.2b}$$

Our aim is to try to find a relation between the masses and speeds referred to Σ. Therefore we must eliminate v and u' from Equations 15.21.1, 15.21.2a and 15.21.2b. This can be a bit fiddly, but the algebra is straightforward, and I leave it to the reader to show that the result is

$$\frac{m_1}{m_2}=\sqrt{\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}} \tag{15.21.2}$$

This tells us that the mass m of a body referred to Σ is proportional to $\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$, where u is its speed referred to Σ. If we call the proportionality constant m_0, then

$$m=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \tag{15.21.3}$$

If $u=0$, then $m=m_0$, and m_0 is called the *rest mass*, and it is the mass when referred to a frame in which the body is at rest. The mass m is generally called the *relativistic mass*, and it is the mass when referred to a frame in which the speed of the body is u.

Equation 15.21.3 gives the mass referred to Σ assuming that the mass is at rest in Σ'. But what if the mass is not at rest in Σ'?

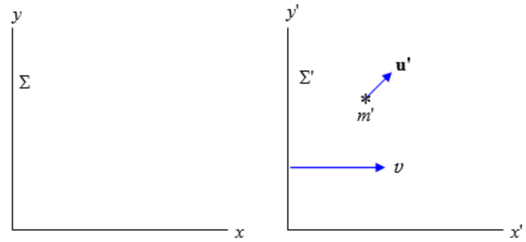


FIGURE XV.36

In figure XV.36 we see a mass m' moving with velocity $\mathbf{u}'$ in Σ'. Referred to Σ its mass will be m, where

$$\frac{m}{m'}=\sqrt{\frac{1-\frac{u'^2}{c^2}}{1-\frac{u^2}{c^2}}}. \tag{15.21.4}$$

Its velocity $\mathbf{u}$ will be in a different direction (referred to Σ) from the direction of $\mathbf{u}'$ in Σ', and the speed will be given by

$$u^2=u_x^2+u_y^2 \tag{15.21.5}$$

where u_x and u_y are given by Equations 15.16.2 and 15.16.3. Substitute Equations 15.21.5, 15.16.2 and 15.16.3 into Equation 15.21.4. The objective is to replace u entirely by primed quantities. The algebra is slightly boring, but it is worth persisting. You will find that $u_{y'}^{\prime 2}$ appears when you use Equation 15.16.3. Replace that by $u'^2-u_{x'}^{\prime 2}$. Also write $\frac{1}{\left(1-\frac{v^2}{c^2}\right)}$ for γ^2. After a little while you should arrive at

$$\frac{m}{m'}=\gamma\left(1+\frac{vu'_{x'}}{c^2}\right). \tag{15.21.6}$$

The transformation for mass between the two frames depends only on the x' component of its velocity in Σ'. It would have made no difference, other than to increase the tedium of the algebra, if I had added $+u_z^2$ to the right hand side of Equation 15.21.5.

The inverse of Equation 15.21.6 is found in the usual way by interchanging the primed and unprimed quantities and changing the sign of v:

$$\frac{m'}{m}=\gamma\left(1-\frac{vu_x}{c^2}\right). \tag{15.21.7}$$

Example.

Example 15.21.1

Let's return to the problem of the dachshund that we met in Section 15.16. A railway train Σ' is trundling along at a speed $\frac{v}{c}$ = 0.9 (γ = 2.294). The dachshund is waddling towards the front of the train at a speed $\frac{u'_{x'}}{c}$. In the reference frame of the train Σ' the mass of the dog is m' = 8 kg. In the reference frame of the railway station, the mass of the dog is given by Equation 15.21.6 and is 31.6 kg. (Its length is also much compressed, so it is very dense when referred to Σ and is disc-shaped.)

I leave it to the reader to show that the rest mass of the dog is 4.8 kg.

15.22: Momentum

The linear momentum $\mathbf{p}$ of a body, referred to a frame Σ, is defined as

$$\mathbf{p}=\mathbf{m}\mathbf{u}. \tag{15.22.1}$$

Here m and $\mathbf{u}$ are its mass and velocity referred to Σ. Note that m is not the rest mass.

Example 15.22.1

The rest mass of a proton is 1.67 % 10^{-27} kg. What is its momentum referred to a frame in which it is moving at 99% of the speed of light?

Answer = 3.51 % 10^{-18} kg m s^{-1}.

15.23: Some Mathematical Results

Before proceeding with the next section, I just want to establish few mathematical results, so that we do not get bogged down in heavy algebra later on when we should be concentrating on understanding physics.

First, if

$$\gamma = \left(1 - \frac{u^2}{c^2}\right), \tag{15.23.1}$$

Then, by trivial differentiation,

$$\frac{d\gamma}{du} = \frac{\gamma^3 u}{c^2}. \tag{15.23.2}$$

$$\dot{\gamma} = \frac{\gamma^3 u \dot{u}}{c^2}. \tag{15.23.3}$$

From this, we quickly find that

$$\frac{\gamma u \dot{u}}{\dot{\gamma}} = c^2 - u^2. \tag{15.23.4}$$

Now for a small result concerning a scalar (dot) product.

Let **A** be a vector such that **A** * **A** = A^2.

Then

$$\frac{d}{dt}(A^2) = 2A\dot{A} \text{ and } \frac{d}{dt}(\mathbf{A} \cdot \mathbf{A}) = \mathbf{2A} \cdot \dot{\mathbf{A}}$$

$$A \cdot \dot{A} = A\dot{A} \tag{15.23.5}$$

We can now safely proceed to the next section.

15.24: Kinetic Energy

If a force $\mathbf{F}$ acts on a particle moving with velocity $\mathbf{u}$, the rate of doing work – i.e. the rate of increase of kinetic energy T is $\dot{T} = \mathbf{F} \cdot \mathbf{u}$. But $\mathbf{F = \dot{p}}$ where $\mathbf{p = mu} = \gamma \mathbf{m_0 u}$.

(A point about notation may be in order here. I have been using the symbol $\mathbf{v}$ and v for the velocity and speed of a frame Σ' relative to a frame Σ, and my choice of axes without significant loss of generality has been such that $\mathbf{v}$ has been directed parallel to the x-axis. I have been using the symbol $\mathbf{u}$ for the velocity (speed = u) of a particle relative to the frame Σ. Usually the symbol γ has meant $\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$, but here I am using it to mean $\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$. I hope that this does not cause too much confusion and that the context will make it clear. I toyed with the idea of using a different symbol, but I thought that this might make matters worse. Just be on your guard, anyway.)

We have, then

$$\mathbf{F = m_0(\dot{\gamma} u + \gamma \dot{u})} \tag{15.24.1}$$

and therefore

$$\dot{T} = m_0(\dot{\gamma} u^2 + \gamma \mathbf{\dot{u} \cdot u}). \tag{15.24.2}$$

Making use of Equations 15.23.5 and 15.23.6 we obtain

$$\dot{T} = \dot{\gamma} m_0 c^2 \tag{15.24.3}$$

Integrate with respect to time, with the condition that when γ = 1, T = 0, and we obtain the following expression for the kinetic energy:

$$T = (\gamma - 1) m_0 c^2. \tag{15.24.4}$$

Exercise. Expand γ by the binomial theorem as far as $\frac{u^2}{c^2}$, and show that, to this order, $T = \frac{1}{2} m u^2$.

I here introduce the dimensionless symbol

$$K = \frac{T}{m_0 c^2} = \gamma - 1 \tag{15.24.5}$$

to mean the kinetic energy in units of $m_0 c^2$. The second half of this was already given as Equation 15.3.5.

15.25: Addition of Kinetic Energies

I want now to consider two particles moving at nonrelativistic speeds – by which I mean that the kinetic energy is given to a sufficient approximation by the expression $\frac{1}{2}mu^2$ and so that parallel velocities add linearly.

Consider the particles in figure XV.37, in which the velocities are shown relative to laboratory space.

FIGURE XV.37

Referred to laboratory space, the kinetic energy is $\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2$. However, the centre of mass is moving to the right with speed $V=\frac{(m_1u_1+m_2u_2)}{(m_1+m_2)}$, and, referred to centre of mass space, the kinetic energy is $\frac{1}{2}m_1(u_1-V)^2+\frac{1}{2}m(u_2+V)^2$. On the other hand, if we refer the situation to a frame in which m_1 is at rest, the kinetic energy is $\frac{1}{2}m_2(u_1+u_2)^2$, and, if we refer the situation to a frame in which m_2 is at rest, the kinetic energy is $\frac{1}{2}m_1(u_1+u_2)^2$.

All we are saying is that the kinetic energy depends on the frame to which speed are referred – and this is not something that crops up only for relativistic speeds.

Let us put some numbers in. Let us suppose, for example that

$$m_1=3\,\text{kg}\;u_1=4\,\text{m s}^{-1}$$

$$m_2=2\,\text{kg}\;u_3=4\,\text{m s}^{-1}$$

so that

$$V=1.2\,\text{m s}^{-1}.$$

In that case, the kinetic energy

referred to laboratory space is 33 J,

referred to centre of mass space is 29.4 J,

referred to m_1 is 49 J,

referred to m_2 is 73.5 J.

In this case the kinetic energy is least when referred to centre of mass space, and is greatest when referred to the lesser mass.

Exercise. Is this always so, whatever the values of m_1, m_2 , u_1 and u_2?

It may be worthwhile to look at the special case in which the two masses are equal (m) and the two speeds(whether in laboratory or centre of mass space) are equal (u).

In that case the kinetic energy in laboratory or centre of mass space is mu^2, while referred to either of the masses it is $2mu^2$.

We shall now look at the same problem for particles travelling at relativistic speeds, and we shall see that the kinetic energy referred to a frame in which one of the particles is at rest is very much greater than (not merely twice) the energy referred to a centre of mass frame.

If two particles are moving towards each other with "speeds" given by g_1 and g_2 in centre of mass space, the g of one relative to the other is given by equation 15.16.14, and, since K = g - 1, it follows that if the two particles have kinetic energies K_1 and K_2 in centre of mass space (in units of the m_0c^2 of each), then the kinetic energy of one relative to the other is

$$K=K_1\oplus K_2=K_1+K_2+K_1K_2+\sqrt{K_1K_2(K_1+2)(K_2+2)}. \tag{15.25.1}$$

If two identical particles, each of kinetic energy K_1 times m_0c^2, approach each other, the kinetic energy of one relative to the other is

$$K=2K_1(K_1+2). \tag{15.25.2}$$

For nonrelativistic speeds as $K_1\to 0$, this tends to $K=4K_1$, as expected.

Let us suppose that two protons are approaching each other at 99% of the speed of light in centre of mass space (K_1 = 6.08881). Referred to a frame in which one proton is at rest, the kinetic energy of the other will be K = 98.5025, the relative speeds being 0.99995 times the speed of light. Thus $K = 16K_1$ rather than merely $4K_1$ as in the nonrelativistic calculation. For more energetic particles, the ratio $\frac{K}{K_1}$ is even more. These calculations are greatly facilitated if you wrote, as suggested in Section 15.3, a program that instantly connects all the relativity factors given there.

Exercise 15.25.1

Two protons approach each other, each having a kinetic energy of 500 GeV in laboratory or centre of mass space. (Since the two rest masses are equal, these TWO spaces are identical.) What is the kinetic energy of one proton in a frame in which the other is at rest?

(Answer: I make it 535 TeV.)

The factor K (the kinetic energy in units of m_0c^2) is the last of several factors used in this chapter to describe the speed at which a particle is moving, and I take the opportunity here of summarising the formulas that have been derived in the chapter for combining these several measures of speed. These are

$$\beta_1 \oplus \beta_2 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}. \tag{15.16.7}$$

$$\gamma_1 \oplus \gamma_2 = \gamma_1\gamma_2 + \sqrt{(\gamma_1^2 - 1)(\gamma_2^2 - 1)}. \tag{15.16.14}$$

$$k_1 \oplus k_2 = k_1k_2 \tag{15.18.11}$$

$$z_1 \oplus z_2 = z_1z_2 + z_1 + z_2. \tag{15.18.12}$$

$$K = K_1 \oplus K_2 = K_1 + K_2 + K_1K_2 + \sqrt{K_1K_2(K_1+2)(K_2+2)} \quad .$$

$$\frac{\phi_1}{\phi_2} = \phi_1 + \phi_2 \tag{15.16.11}$$

If the two speeds to be combined are equal, these become

$$\beta_1 \oplus \beta_1 = \frac{2\beta_1}{1 + \beta_1^2}. \tag{15.25.3}$$

$$\gamma_1 \oplus \gamma_1 = 2\gamma_1^2 - 1 \tag{15.25.4}$$

$$\frac{k_1}{k_1} = k_1^2 \tag{15.25.5}$$

$$z_1 \oplus z_1 = z_1(z_1 + 2) \tag{15.25.6}$$

$$K_1 \oplus K_1 = 2K_1(K_1 + 2). \tag{15.25.7}$$

$$\phi_1 \oplus \phi_1 = 2\phi. \tag{15.25.8}$$

These formulas are useful, but for numerical examples, if you already have a program for interconverting between all of these factors, the easiest and quickest way of combinng two “speeds” is to convert them to ϕ. We have seen examples of how this works in Sections 15.16 and 15.18. We can do the same thing with our example from the present section when combining two kinetic energies. Thus we were combining two kinetic energies in laboratory space, each of magnitude K_1 = 6.08881 (ϕ_1 = 2.64665). From this, ϕ = 5.29330, which corresponds to K = 98.5025.

15.26: Energy and Mass

The nonrelativistic expression for kinetic energy $T=\frac{1}{2}mu^2$ has just one term in it, a term which depends on the speed. The relativistic expression which approximates to the nonrelativistic expression at low speeds) can be written $T=mc^2-m_0c^2$ that is, a speed-dependent term minus a constant term. The kinetic energy can be thought of as the excess over the energy over the constant term m_0c^2. The expression m_0c^2 is known as the *rest-mass energy*. The sum of the kinetic energy and the rest-mass energy is the "total energy", or just the "energy" E:

$$E=T+m_0c^2=mc^2 \tag{15.26.1}$$

This means that, if the kinetic energy of a particle is zero, the total energy of the particle is not zero – it still has its rest-mass energy m_0c^2.

Of course, giving the name "rest-mass energy" to the constant term m_0c^2, and calling the speed-dependent term mc^2 the "total energy" and writing the famous equation $E=mc^2$, does not by itself immediately and directly tell us that "matter" can be converted to "energy" or the other way round. Whether such conversion can in fact take place is a matter for experiment and observation to determine. The equation by itself merely tells us how much mass is held by a given quantity of energy, or how much energy is held by a given quantity of mass. That entities that we traditionally think of as "matter" can be converted into entities that we traditionally think of as "energy" is well established with, for example, the "annihilation" of an electron and a positron ("matter" and "antimatter") to form photons ("energy") as is the inverse process of pair production (production of an electron-positron pair from a gamma ray in the presence of a third body).

It is unfortunate that the main (almost the only) example of application of the equation $E=mc^2$ persistently presented to the nonscientific public is the atom bomb, whose operation actually has nothing at all to do with the equation $E=mc^2$, nor, contrary to the popular mind, is any "matter" converted to energy.

I have heard it said that you can find out on the Web how to build an atom bomb, so here goes – here is how an atom bomb works. A uranium-235 nucleus is held together by strong attractive forces between the nucleons, which, at short femtometre ranges are much stronger than the Coulomb repulsive forces between the protons. When the nucleus absorbs an additional neutron, the resulting ^{236}U nucleus is unstable and breaks up into two intermediate-mass nuclei plus two or three neutrons. The two intermediate-mass nuclei are generally not of exactly equal mass; one is usually a bit less than half of the uranium nucleus and the other a bit more than half, but that's a detail. The potential energy required to bind the nucleons together in the uranium nucleus is rather greater than the binding energy of the two resulting intermediate-mass nuclei; the difference is of order 200 MeV, and that potential energy is converted into kinetic energy of the two resulting nuclei and, to a lesser extent, the two or three neutrons released. That is all. It is merely the familiar conversion of potential binding energy (admittedly a great deal of energy) into kinetic energy. No matter, no protons, no neutrons, are "destroyed" or "converted into energy", and $E=mc^2$ simply does not enter into it anywhere! The rest-mass energy of a proton or a neutron is about 1 GeV, and that much energy would be released if a proton were miraculously and for no cause converted into energy. Let us hope that no one invents a bomb that will do that – though we may rest assured that that is rather unlikely.

Where the equation $E=mc^2$ does come in is in the familiar observation that the mass of any nucleus other than hydrogen is a little less than the sum of the masses of the constituent nucleons. It is for that reason that nuclear masses, even for pure isotopes, are not integral. The mass of a nucleus is equal to the sum of the masses of the constituent nuclei plus the mass of the binding energy, the latter being a negative quantity since the inter-nucleon forces are attractive forces. The equation $E=mc^2$ tells us that energy (such as, for example, the binding energy between nucleons) has mass.

15.27: Energy and Momentum

A moving particle has energy arising from its momentum and also from its rest mass, and we need to find an expression relating energy to rest mass and momentum. It is fairly easy and it goes like this:

[Math Processing Error]

[Math Processing Error]

Thus we obtain for the energy in terms of rest mass and momentum

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If the speed (and hence momentum) is zero, the energy is merely *[Math Processing Error]*. If the rest mass is zero (as, for example, a photon) and the energy is not zero, then *[Math Processing Error]*. But also *[Math Processing Error]*, so that, if the rest mass of a particle is zero and the energy is not, the particle must be moving at the speed of light. This could be regarded as the reason why photons, which have zero rest mass, travel at the speed of photons. If neutrinos have zero rest mass, they, too, will travel at the speed of light; if they are not massless, they won't.

In addition to Equation *[Math Processing Error]*, which relates the energy to the magnitude of the momentum, it will be of interest to see how the *components* of momentum transform between reference frames. As usual, we are considering frame *[Math Processing Error]* to be moving with respect to *[Math Processing Error]* at a speed *[Math Processing Error]* with respect to *[Math Processing Error]*. There is no difficulty with the *[Math Processing Error]*- and *[Math Processing Error]*- components. We have merely *[Math Processing Error]* and *[Math Processing Error]*. However:

[Math Processing Error] and *[Math Processing Error]*.

Also *[Math Processing Error]*, from which *[Math Processing Error]*.

After a little algebra, we obtain

[Math Processing Error]

And this is

[Math Processing Error]

The inverse is found in the usual way:

[Math Processing Error]

If we eliminate *[Math Processing Error]* from Equations *[Math Processing Error]* and *[Math Processing Error]*, we'll find *[Math Processing Error]* in terms of *[Math Processing Error]* and *[Math Processing Error]*:

[Math Processing Error]

Thus the transformations between energy and the three spatial components of momentum is similar to the transformation between time and the three space coordinates, and are described by a similar 4-vector:

[Math Processing Error]

The reader should multiply this out to verify that it does reproduce Equations *[Math Processing Error]* and *[Math Processing Error]*.

15.28: Units

It is customary in the field of particle physics to express energy (whether total, kinetic or rest-mass energy) in electron volts (eV) or in keV, MeV, GeV or TeV (10^3, 10^6, 10^9, or 10^{12} eV respectively). A electron volt is the kinetic energy gained by an electron if it is accelerated through an electrical potential of 1 volt; alternatively it is the work required to move an electron through one volt. Either way, since the charge on an electron is 1.602 % 10^{-19}) C, 1eV = 1.602 % 10^{-19} J.

The use of such a unit may understandably dismay those who would insist always on expressing any physical quantity in SI units, and I am much in sympathy with this view. Yet, to those who deal daily with particles whose charge is equal to or is a small multiple or rational fraction of the electronic charge, the eV has its attractions. Thus if you accelerate a particle through so many volts, you do not have to remember the exact value of the electronic charge or carry out a long multiplication every time you do so. One might also think of a hypothetical question such as: An electron is accelerated through 3426.7189628471 volts. What is its gain in kinetic energy? You *cannot answer this in joules* unless you know the value of the electronic charge to a comparable precision; but of course you do know the answer in eV.

One situation that does require care is this. An a-particle is accelerated through 1000 V. What is the gain in kinetic energy? Because the charge on an a-particle is twice that of an electron, the answer is 2000 eV.

Very often you know the energy of a particle (because you have accelerated it through so many volts) and you want to know its momentum; or you know its momentum (because you have measured the curvature of its path in a magnetic field) and you want to know its energy. Thus you will frequent occasion to make use of Equation 15.27.1:

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You have to be careful to remember how many *[Math Processing Error]*s there are, and what is the exact value of *[Math Processing Error]*. Particle physicists prefer to make life easier for themselves (not necessarily for the rest of us!) by preferring not to state what the momentum of a particle is, or its rest mass, but rather to give the values of *[Math Processing Error]* or of *[Math Processing Error]* – and to express *[Math Processing Error]*,*[Math Processing Error]* and *[Math Processing Error]* all in eV (or keV, MeV or GeV). Thus one may hear that

[Math Processing Error] = 6.2 GeV

[Math Processing Error] = 0.938 GeV.

More often this is expressed, somewhat idiosyncratically and in somewhat doubtful use of English, as

[Math Processing Error] = 6.2 GeV/c

[Math Processing Error] = 0.938 GeV/c^2

or in informal casual conversation (one hopes not for publication) merely as

[Math Processing Error] = 6.2 GeV

[Math Processing Error] = 0.938 GeV.

While this may puzzle some and raise the ire of others, it is not entirely without merit, because, provided one uses these units, the relation between energy, momentum and rest mass is then simply

[Math Processing Error].

The practice is not confined to energy, momentum and rest mass. For example, the SI unit of magnetic dipole moment is N m T^{-1} (newton metre per tesla). Now N m (unit of torque) is not quite the same as a joule (unit of energy), although dimensionally similar. Yet it is common practice to express the magnetic moments of subatomic particles in eV T^{-1}. Thus the Bohr magneton is a unit of magnetic dipole moment equal to 9.27 % 10^{-24} N m T^{-1}, and this may be expressed as 5.77 % 10^{-5} eV T^{-1}.

One small detail to be on guard for is this. One may hear talk of "a 500 MeV proton". Does this mean that the *kinetic* energy is 500 MeV or that its *total* energy is 500 MeV? In this case the answer is fairly clear (although it would have been completely clear if the speaker had been explicit). The rest-mass energy of a proton is 938 MeV, so he must have been referring to the kinetic energy. If, however, he had said "a 3 GeV proton", there would be no way of deducing whether he was referring to the kinetic or the total energy. And if he had said "a 3 GeV particle", there would be no way of telling whether he was referring to its total energy, its kinetic energy or its rest-mass energy. It is incumbent on all of us – or at least those of us who wish to be understood by others – always to make ourselves explicitly clear and not to suppose that others will correctly guess what we mean.

15.29: Force

Force is defined as rate of change of momentum, and we wish to find the transformation between forces referred to frames in uniform relative motion such that this relation holds on all such frames.

Suppose that, in Σ', a mass has instantaneous mass m' and velocity whose instantaneous components are $u'_{x'}$ and $u'_{y'}$. If a force acts on it, then the velocity *and hence also the mass* are functions of time. The x-component of the force is given by

$$F'_{x'} = \frac{d}{dt}(m'u'_{x'}). \tag{15.29.1}$$

We want to express everything on the right hand side in terms of unprimed quantities. Thus from Equation 15.21.8 and the inverse of Equation 15.16.2, we obtain

$$m'u'_{x'} = m\gamma(u_x - v). \tag{15.29.2}$$

Also

$$\frac{d}{dt'} = \frac{dt}{dt'}\frac{d}{dt} \tag{15.29.3}$$

Let us first evaluate $\frac{d}{dt}(m\gamma u_x - m\gamma v)$. In this expression, v and γ are independent of time (the frame Σ' is moving at constant velocity relative to Σ), and $\frac{d}{dt}$ of mu_x is the x-component of the force in Σ, that is F_x. Thus

$$\frac{d}{dt}(m\gamma u_x - m\gamma v) = \gamma\left(F_x - v\frac{dm}{dt}\right). \tag{15.29.4}$$

Now we need to evaluate $\frac{d}{dt'}$ *in terms of unprimed quantities*. If we start with

$$dt' = \left(\frac{\partial t'}{\partial x}\right)_t dx + \left(\frac{\partial t'}{\partial t}\right)_x dt \tag{15.29.5}$$

and we'll evaluate $\frac{dt'}{dt}$ which, being a total derivative, is the reciprocal of $\frac{dt}{dt'}$. The partial derivatives are given by Equations 15.15.3j,k and l, while $\frac{dx}{dt} = u_x$. Hence we obtain

$$\frac{dt}{dt'} = \frac{1}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}. \tag{15.29.6}$$

Thus we arrive at

$$F'_{x'} = \frac{F_x - v\left(\frac{dm}{dt}\right)}{1 - \frac{u_x v}{c^2}} \tag{15.29.7}$$

The mass is not constant (i.e. $\frac{dm}{dt}$ is not zero) because there is a force acting on the body, and we have to relate the term $\frac{dm}{dt}$ to the force. At some instant when the force and velocity (in Σ) are $\mathbf{F}$ and $\mathbf{u}$, the rate at which $\mathbf{F}$ is doing work on the body is $\mathbf{F} * \mathbf{u}$ $= F_x u_x + F_y u_y + F_z u_z$ and this is equal to the rate of increase of energy of the body, which is $\dot{m}c^2$. (In Section 15.24, in deriving the expression for kinetic energy, I wrote that the rate of doing work was equal to the rate of increase of *kinetic* energy. Now I have just written that it is equal to the rate of increase of (total) energy. Which is right?)

$$\frac{dm}{dt} = \frac{1}{c^2}(F_x u_x + F_y u_y + F_z u_z). \tag{15.29.8}$$

Substitute this into Equation 15.29.7and, after a very little more algebra, we finally obtain the transformation for $F'_{x'}$:

$$F'_{x'} = F_x - \frac{v}{c^2 - u_x v}(u_y F_y + u_z F_z). \tag{15.29.9}$$

The $y'-$ and $z'-$ components are a little easier, and I leave it as an exercise to show that

$$F'_{y'} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{u_x v}{c}} F_y \tag{15.29.10}$$

$$F'_{z'} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{u_x v}{c}} F_z. \tag{15.29.11}$$

As usual, the inverse transformations are found by interchanging the primed and unprimed quantities and changing the sign of v.

The force on a particle and its resultant acceleration are not in general in the same direction, because the mass is not constant. (Newton's second law is not $\mathbf{F} = m\mathbf{a}$; it is $\mathbf{F} = \dot{\mathbf{p}}$) Thus

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{u}) = m\mathbf{a} + \dot{m}\mathbf{u}. \tag{15.29.12}$$

Here

$$m = \frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{1}{2}}} \tag{15.29.13}$$

and so

$$\dot{m} = \frac{m_0 u a}{c^2\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}. \tag{15.29.14}$$

Thus

$$\mathbf{F} = \frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{\frac{1}{2}}}\left(\mathbf{a} + \frac{ua}{c^2-u^2}\mathbf{u}\right). \tag{15.29.15}$$

15.30: The Speed of Light

The speed of light is, by definition, exactly 2.997 924 58 x 10^8 ms^{-1}, and is the same relative to all observers.

This seemingly simple sentence invites several comments.

First: Note that I have used the word “speed”. Some writers use the word “velocity” as if it were merely a more impressive and scientific-sounding synonym for “speed”. I trust that all readers of these notes know the difference and will use the word “speed” when they mean “speed”, and the word “velocity” when they mean “velocity – surely not an unreasonable demand. To say that the “velocity” of light is the same for all observers means that the direction of travel of light is the same relative to all observers. This is doubtless not at all what a writer who uses the word “velocity” intends to convey – but it is the literal (and of course quite erroneous) meaning of the assertion.

Second: How can we possibly *define* the speed of light to have a certain *exact* value? Surely the speed of light is what we find it to be, and we are not free to *define* its value. But in fact we *are* allowed to do this, and the explanation, briefly, is as follows.

Over the course of history, the *metre* has been defined in several different ways. At one time it was a specified fraction of the circumference of Earth. Later, it was the distance between two scratches on a bar of platinum-iridium alloy held in Paris. Later still it was a specified number of wavelengths of a particular line in the spectrum of mercury, or cadmium, or argon or krypton. In our present state of technology it is far easier to measure and reproduce precise standards of *frequency* than it is to measure and reproduce standards of length. Because of that, the current SI (Système International) unit of time is the SI second, which is based on the frequency of a particular transition in the spectrum of caesium, and from there, the metre is *defined* as the distance travelled by light in vacuo in a defined fraction of an SI second, the speed of light being assigned the exact value quoted above.

Detailed discussion of the exact definitions of the units of time, distance and speed is part of the subject of *metrology*. That is an important and interesting subject, but it is only marginally relevant to the topic of relativity, and consequently, having quoted the exact value of the speed of light, we leave further discussion of metrology here.

Third: How can the speed of light be the same relative to *all observers*? This assertion is absolutely central to the theory of special relativity, and it may be regarded as its fundamental and most important principle. We shall discuss it further in the remainder of the chapter.

15.31: Electromagnetism

These notes are intended to cover only mechanics, and therefore I resist the temptation to cover here special relativity and electromagnetism. I point out only that in many ways this misses many of the most exciting parts of special relativity, and indeed it was some puzzles with electromagnetism that led Einstein to formulate the theory of special relativity. One proceeds as we have done with mechanical quantities; that is, we have to define carefully what is meant by each quantity and how in principle it is possible to measure it, and then see how it transforms between frames in such a manner that the laws of physics – in particular Maxwell's equations - are the same in each. One such transformation that is found, for example, is $\mathbf{E'} = \gamma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$ so that what appears in one frame as an electric field appears in another at least in part as a magnetic field. The Coulomb force transforms to a Lorentz force; Coulomb's law transforms to Ampères law.

Although I do no more than mention this topic here, I owe it to the reader to say just a little bit more about the speedometer that I designed in Section 15.4. It is indeed true that, as the train moves forward, the net repulsive force between the two rods does diminish, although not quite as I have indicated, for one has to make the correct transformations between frames for force, current, electric field, magnetic field, and so on. But it turns out that the weights of the rods – i.e. the downward forces on them – also diminish in exactly the same ratio, and the angle between the strings remains stubbornly the same. Our trip to the patent office will be in vain. The speedometer will not work, and it remains impossible to determine the absolute motion of the train.

CHAPTER OVERVIEW

16: Hydrostatics

This relatively short chapter deals with the pressure under the surface of an incompressible fluid, which in practice means a liquid, which, compared with a gas, is nearly, if not quite, incompressible. It also deals with Archimedes' principle and the equilibrium of floating bodies. The chapter is perhaps a little less demanding than some of the other chapters, though it will assume a familiarity with the concepts of centroids and radius of gyration, which are dealt with in Chapters 1 and 2.

16.1: Introduction to Hydrostatics

This relatively short chapter deals with the pressure under the surface of an incompressible fluid, which in practice means a liquid, which, compared with a gas, is nearly, if not quite, incompressible. It also deals with Archimedes' principle and the equilibrium of floating bodies. The chapter is perhaps a little less demanding than some of the other chapters, though it will assume a familiarity with the concepts of centroids and radius of gyration, which are dealt with in Chapters 1 and 2.

16.2: Density

There is little to be said about density other than to define it as mass per unit volume. However, this expression does not literally mean the mass of a cubic metre, for after all a cubic metre is a large volume, and the density may well vary from point to point throughout the volume. Density is an intensive quantity in the thermodynamical sense, and is defined at every *point*. A more exact definition of density, for which I shall usually use the symbol ρ, is

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V}. \tag{16.2.1}$$

The awful term “specific gravity” was formerly used, and is still regrettably often heard, as either a synonym for density, or the dimensionless ratio of the density of a substance to the density of water. It should be avoided. The only concession I shall make is that I shall use the symbol *s* to mean the ratio of the density of a body to the density of a fluid in which it may be immersed or on which it may be floating,

The density of water varies with temperature, but at 4 ºC is 1 g cm^{-3} or 1000 kg m^{-3}, or 10 lb gal^{-1}. The original gallon was the volume of 10 pounds (lb) of water. These are Imperial (UK) gallons, and avoirdupois pounds - not the gallons (wet or dry) used in the U.S., and not the pounds (troy) used in the jewellery trade.

16.3: Pressure

Pressure is force per unit area, or, more precisely,

$$\rho = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}. \tag{16.3.1}$$

There is no particular direction associated with pressure – it acts in all directions – and it is a scalar quantity. The SI unit is the *pascal* (Pa), which is a pressure of one newton per square metre (N m^{-2}). Blaise Pascal (1623-1662) was a French mathematician and philosopher who contributed greatly to the theory of conic sections and to hydrostatics. He showed that the barometric pressure decreases with height – hence the famous examination question: "Explain how you would use a barometer to measure the height of a tall building" – to which the most accurate answer is said to be: "I would drop it out of the window and time how long it took to reach the ground."

The CGS unit of pressure is dyne cm^{-2}, and 1 Pa = 10 dyne cm^{-2}.

Some other silly units for pressure are often seen, such as psi, bar, Torr or mm Hg, and atm.

A psi or "pound per square inch" is all right for those who define a "pound" as a unit of force (US usage) but is less so for those who define a pound as a unit of mass (UK usage). A psi is about 6894.76 Pa and a bar is 10^5 Pa or 100 kPa.

> *[The "British Engineering System", as far as I know, is used exclusively in the U.S. and is not and never has been used in Britain, where it would probably be unrecognized. In the "British" Engineering System, the pound is defined as a unit of force, whereas in Britain a pound is a unit of mass.]*

A Torr is a pressure under a column of mercury 760 mm high. This may be convenient for casual conversational use where extreme precision is not expected in laboratory experiments in which pressure is actually indicated by a mercury barometer or manometer. To find out exactly what the pressure in Pa under 760 mm Hg is, one would have to know the exact value of the local gravitational acceleration and also the exact density of mercury, which varies with temperature and with isotopic constitution. A Torr is usually given as 133.322 Pa. Evangelista Torricelli (1608 – 1647) is regarded as the inventor of the mercury barometer. He succeeded Galileo as professor of mathematics at the University of Florence.

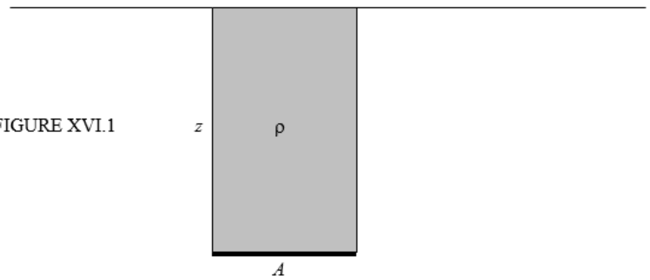
An atm is 760 torr or about 14.7 psi or 101 325 Pa. That is to say, 1.013 25 bar

As usual, the use of a variety of different units, and knowing the exact definitions and conversion factors between all of them and carrying out all the tedious multiplications, is an unnecessary chore that is inflicted upon all of us in all branches of physics.

16.4: Pressure on a Horizontal Surface. Pressure at Depth

Figure XVI.1 shows a horizontal surface of area A immersed at a depth z under the surface of a fluid at depth z. The force F on the area A is equal to the weight of the superincumbent fluid.

FIGURE XVI.1



This gives us occasion to use a gloriously pompous word. "Incumbent" means "lying down", so that "superincumbent" is lying down above the area. It is incumbent upon all of us to understand this. The weight of the superincumbent fluid is evidently its volume Az times its density ρ times the gravitational acceleration g. Thus

$$F = \rho g z A, \tag{16.4.1}$$

and, since pressure is force per unit area, we find that the pressure at a depth z is

$$P = \rho g z. \tag{16.4.2}$$

This is, of course, in addition to the atmospheric pressure that may exist above the surface of the liquid.

The pressure is the same at all points at the same horizontal level within a **homogeneous incompressible fluid**. This seemingly trivial statement may sometimes be worth remembering under the stress of examination conditions. Thus, let's look at an example.

Example 16.4.1

In Figure XVI.2, the vessel at the left is partly filled with a liquid of density 0.8 g cm^{-3}, the upper part of the vessel being filled with air. The liquid also fills the tube along to the point C. From C to B, the tube is filled with mercury of density 13.6 g cm^{-3}. Above that, from B to A, is water of density 1.0 g cm^{-3}, and above that is the atmospheric pressure of 101 kPa.

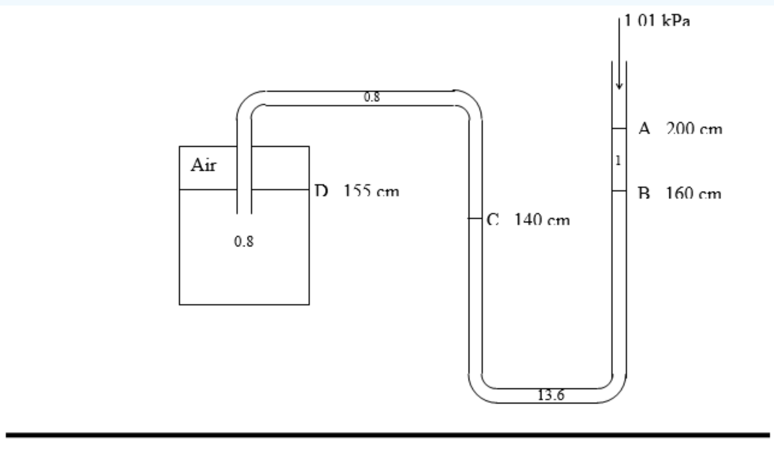


FIGURE XVI.2

The height of the four interfaces above the thick black line are

- A: 200 cm
- B: 160 cm
- C: 140 cm
- D: 155 cm

With g = 9.8 m s^{-2}, what is the pressure of the air in the closed vessel?

Solution

I'll do the calculation in SI units.

- Pressure at A = 101000 Pa
- Pressure at B = 101000 + 1000 % 9.8 % 0.4 = 104920 Pa
- Pressure at C = 104920 + 13600 % 9.8 % 0.20 = 131576 Pa
- Pressure at D = 131576 - 800 % 9.8 % 0.15 = 130400 Pa,

and this is the pressure of the air in the vessel.

Rather boring so far, and the next problem will also be boring, but the problem after that should keep you occupied arguing about it over lunch.

Exercise 16.4.1

This problem is purely geometrical and nothing to do with hydrostatics – but the result will help you with the next problem after this. If you do not want to do it, just use the result in the next problem.

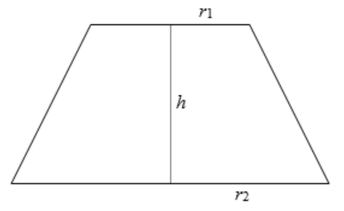


FIGURE XV1.3

Show that the volume of the frustrum of a cone, whose upper and lower circular faces are of radii r_1 and r_2, and whose height is h, is $\frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$

Exercise 16.4.2: Pascal's Paradox

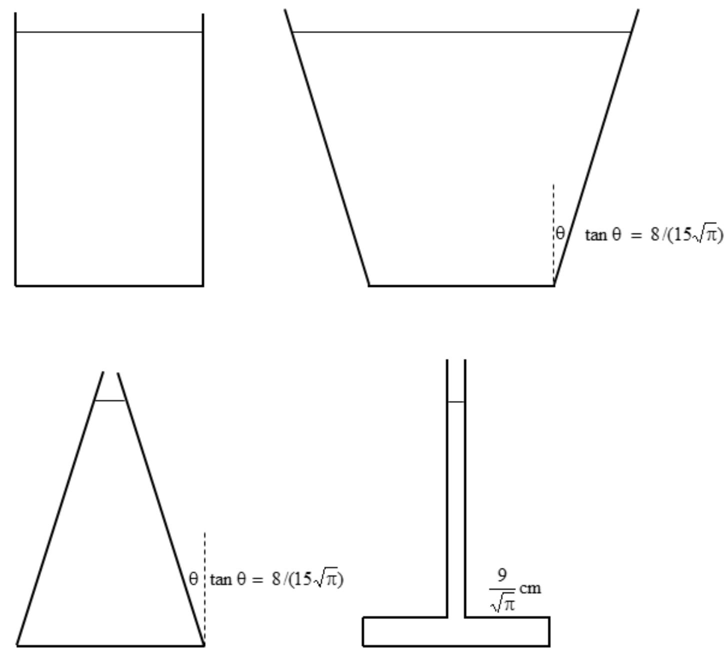


FIGURE XVI.4

Figure XVI.4 shows four vessels. The base of each is circular with the same radius, $\frac{10}{\sqrt{\pi}}$ cm, so the area is 100 cm^2. Each is filled with water (density = 1 g cm^{-3}) to a depth of 15 cm.

Calculate

1. The mass of water in each.
2. The pressure at the bottom of each vessel.
3. The force on the bottom of each vessel.

If the bottom of each vessel were made of glass, which cracked under a certain pressure, which would crack first if the vessels were slowly filled up? If the bottom of each vessel were welded to the scale of a weighing machine, what weight would be recorded?

I'll leave you to argue about this for as long as you wish.

16.5: Pressure on a Vertical Surface

Figure XVI.5 shows a vertical surface from the side and face-on. The pressure increases at greater depths. I show a strip of the surface at depth z.

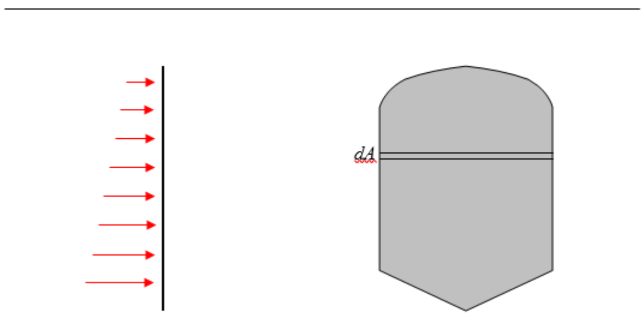


FIGURE XVI.5

Suppose the area of that strip is dA. The pressure at depth z is $\rho g z$, so the force on the strip is $\rho g z dA$. The force on the entire area is $\rho g \int z dA$, and that, by definition of the centroid (see Chapter 1), is $\rho g \bar{z} A$ where $\bar{z}$ is the depth of the centroid. The same result will be obtained for an inclined surface.

Therefore:

> *The total force on a submerged vertical or inclined plane surface is equal to the area of the surface times the depth of the centroid.*

Example 16.5.1

Figure XVI.6 shows a triangular area. The uppermost side of the triangle is parallel to the surface at a depth z. The depth of the centroid is $z+\frac{1}{3}h$, so the pressure at the centroid is $\rho g\left(z+\frac{1}{3}h\right)$. The area of the triangle is $\frac{1}{2}bh$ so the total force on the triangle is $\frac{1}{2}\rho g b h\left(z+\frac{1}{3}h\right)$.

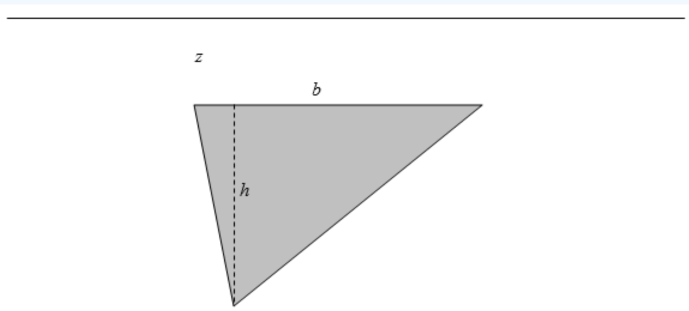


FIGURE XVI.6

16.6: Centre of Pressure

"The centre of pressure is the point at which the pressure may be considered to act." This is a fairly meaningless sentence, yet it is not entirely devoid of all meaning. If you refer to the left hand side of Figure XVI.5, you will see an infinite number (I've drawn only eight) of forces. If you were to replace all of these forces by a single force, where would you put it? Or, more precisely, if you were to replace all of these forces by a single force such that the (first) *moment of this force* about a line through the surface of the fluid is the same as the (first) *moment of all the actual forces*, where would you place this single force? You would place it at the *centre of pressure*. The depth of the centre of pressure is a depth such that the moment of the total force on a vertical surface about a line in the surface of the fluid is the same as the moment of all the hydrostatic forces about a line in the surface of the fluid. I shall use the Greek letter ζ to indicate the depth of the centre of pressure. We can continue to use Figure XVI.5.

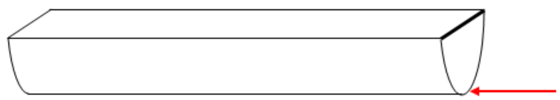


FIGURE XVI.7

The force on the strip of area dA at depth z is, as we have seen, $\rho gz dA$, so the first moment of that force is $\rho gz^2 dA$. The total moment is therefore $\rho g \int z^2 dA$ which is, by definition of radius of gyration k, (see Chapter 2), $\rho gk^2 A$. The total force, as we have seen, is $pg\bar{z}A$ and the total moment is to be this times ζ. Thus the depth of the centre of pressure is

$$\zeta = \frac{k^2}{\bar{z}} \tag{16.6.1}$$

Example 16.6.1

A semicircular trough of radius a is filled with water, density ρ. One semicircular end of the trough is freely hinged at its diameter (the thick line in the Figure). What force must be exerted at the bottom of the trough to prevent the end from swinging open?

Solution

The area of the semicircle is $\frac{1}{2}\pi a^2$. The depth of the centroid is $\frac{4a}{3\pi}$ so the total hydrostatic force is $\frac{2}{3}\rho ga^2$. The square of the radius of gyration is $\frac{1}{4}a^2$, so the depth of the centre of pressure is $\zeta = \frac{3\pi}{16}a$. The moment of the hydrostatic forces is therefore $\frac{1}{8}\pi\rho ga^3$. If the required force is F, this must equal Fa, and therefore $F = \frac{1}{8}\pi\rho ga^2$.

16.7: Archimedes' Principle

The most important thing about Archimedes’ principle is to get the apostrophe in the right place and to spell principle correctly.

Archimedes was a Greek scientist who lived in Syracuse, Sicily. He was born about 287 BC and died about 212 BC. He made many contributions to mechanics. He invented the Archimedean screw, he is reputed to have said “Give me a fulcrum and I shall move the world”, and he probably did not set the Roman invading fleet on fire by focussing sunlight on them with concave mirrors – though it makes a good story. The most famous story about him is that he was commissioned by King Hiero of Sicily to determine whether the king’s crown was contaminated with base metal. Archimedes realized that he would need to know the density of the crown. Measuring its weight was no problem, but – how to measure the volume of such an irregularly-shaped object? One day, he went to take a bath, and he had filled the bath full right to the rim. When he stepped into the bath he was much surprised that some of the water slopped over the edge of the bath on to the floor. Suddenly, he realized that he had the solution to his problem, so straightway he raced out of the house and ran absolutely starkers through the streets of Syracuse shouting “ $\epsilon \upsilon \rho \eta \kappa \alpha$! $\epsilon \upsilon \rho \eta \kappa \alpha$!”, which is Greek for "Eureka, Eureka" meaning "I found it, I found it."

When a body is totally or partially immersed in a fluid, it experiences a hydrostatic upthrust equal to the **weight** of fluid displaced.

Figure XVI.8 is a drawing of some water or other fluid. I have outlined with a dashed curve an arbitrary portion of the fluid. It is subject to hydrostatic pressure from the rest of the fluid. The small pressure of the fluid above it is pushing it down; the larger pressure of the fluid below it is pushing it up. Therefore there is a net upthrust. The portion of the fluid outlined is in equilibrium between its own weight and the hydrostatic upthrust. If we were to replace this portion of the fluid with a lump of iron, we wouldn’t have changed the hydrostatic forces. Therefore the upthrust is equal to the **weight** of fluid displaced.

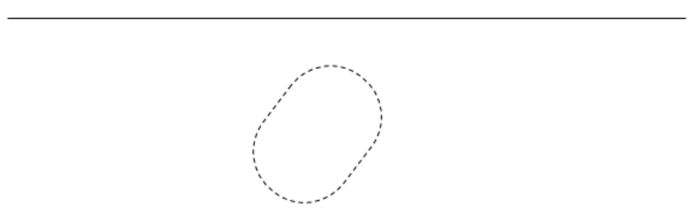


FIGURE XVI.8

If a body is *floating* on the surface, the hydrostatic upthrust, as well as being equal to the **weight** of fluid displaced, is also equal to the weight of the body.

16.8: Some Simple Examples

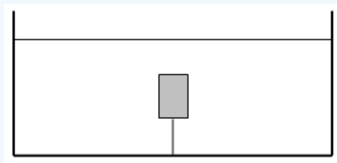
As we pointed out in the introduction to this chapter, this chapter is less demanding than some of the others, and indeed it has been quite trivial so far. Just to show how easy the topic is, here are a few quick examples.

Example 16.8.1

A cylindrical vessel of cross-sectional area A is partially filled with water. A mass m of ice floats on the surface. The density of water is ρ_0 and the density of ice is ρ. Calculate the change in the level of the water when the ice melts, and state whether the water level rises or falls.

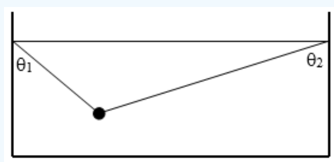
Example 16.8.2

A cork of mass m, density ρ, is held under water (density ρ_0) by a string. Calculate the tension in the string. Calculate the initial acceleration if the string is cut.



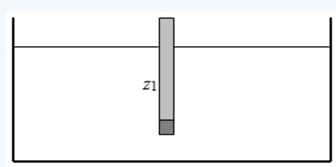
Example 16.8.3

A lump of lead (mass m, density ρ) is held hanging in water (density ρ_0) by two strings as shown. Calculate the tension in the strings.



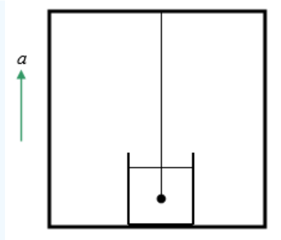
Example 16.8.4

A hydrometer (for our purposes a hydrometer is a wooden rod weighted at the bottom for stability when it floats vertically) floats in equilibrium to a depth z_1 in water of density ρ_1. If salt is added to the water so that the new density is ρ_2, what is the new depth z_2?



Example 16.8.5

A mass m, density ρ, hangs in a fluid of density ρ_0 from the ceiling of an elevator (lift). The elevator accelerates upwards at a rate a. Calculate the tension in the string.

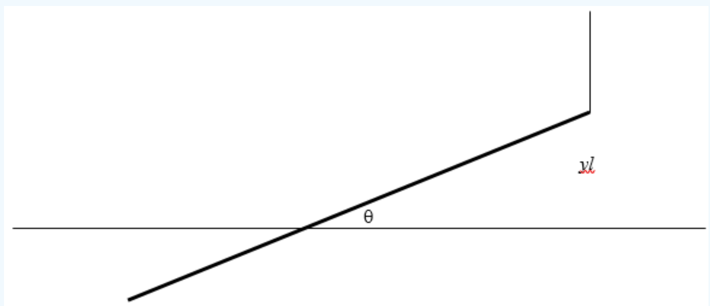


Example 16.8.6

A hydrometer of mass m and cross-sectional area A floats in equilibrium do a depth h in a liquid of density ρ. The hydrometer is then gently pushed down and released. Determine the period of oscillation.

Example 16.8.7

A rod of length l and density $s\rho(s<1)$ floats in a liquid of density ρ. One end of the rod is lifted up through a height yl so that a length xl remains immersed. I have drawn it with the rope vertical. Must it be?)



i. Find x as a function of s.

ii. Find θ as a function of y and s.

iii. Find the tension T in the rope as a function of $m,\ g$ and s.

Draw the following graphs:

a. x and $\frac{T}{(mg)}$ versus s.

b. θ versus y for several s.

c. θ versus s for several y.

d. x versus y for several s.

e. $\frac{T}{(mg)}$ versus y for several s.

Answers

1. No, it does not.

2. $T=\left(\frac{\rho_0-\rho}{\rho}\right)mg$

3. $T_1=\frac{\left(\frac{\rho-\rho_0}{\rho}\right)mg}{\cos\theta_1+\frac{\sin\theta_1}{\tan\theta_2}}\ T_2=\frac{\left(\frac{\rho-\rho_0}{\rho}\right)mg}{\cos\theta_2+\frac{\sin\theta_2}{\tan\theta_1}}$

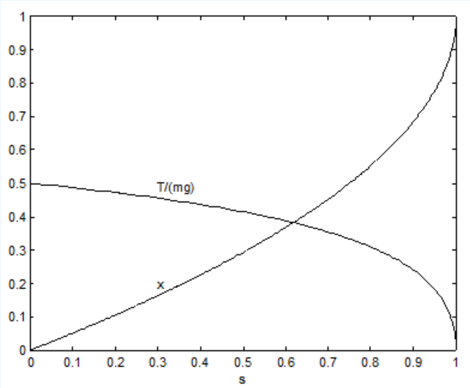
4. $z_2=\frac{\rho_1}{\rho_2}z_1$

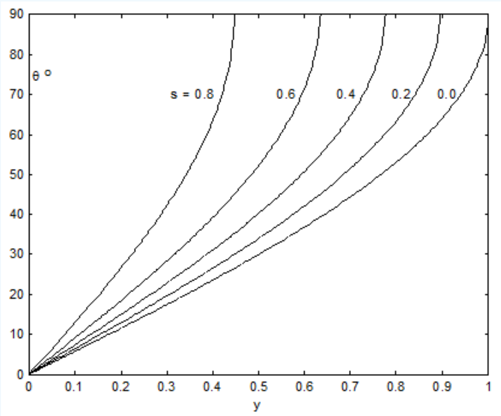
5. $T=m\left[a+g\left(\frac{\rho-\rho_0}{\rho}\right)\right]$

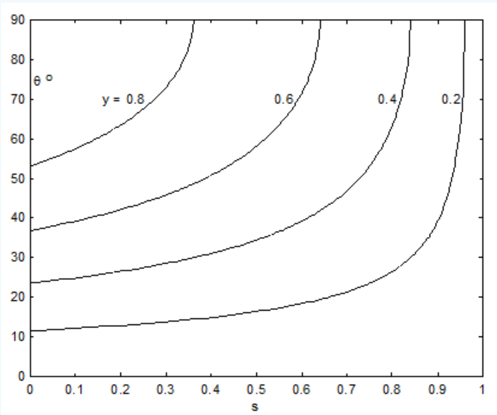
6. $P = 2\pi\sqrt{\frac{m}{\rho A g}}$

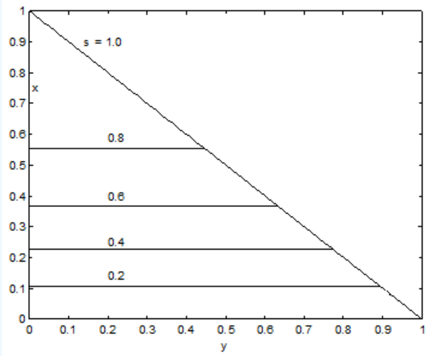
7.

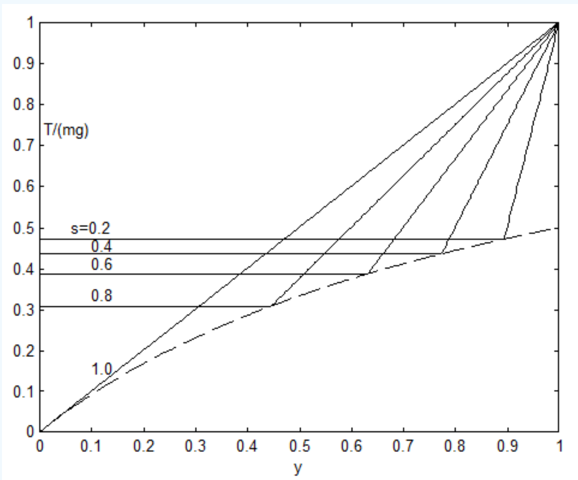
i. $x = 1 - \sqrt{1-s}$
ii. $\sin\theta = \frac{y}{\sqrt{1-s}}$
iii. $T = mg\left(\frac{\sqrt{1-s}-(1-s)}{s}\right)$

a. 

b. 

c. 

d. 

e. 

16.9: Floating Bodies

This is the most grisly topic in hydrostatics.

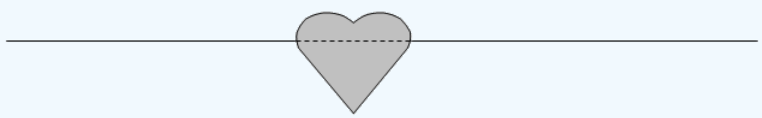
We can start with an observation that we have already made in Section 16.7, namely that, if a body is freely floating, the hydrostatic upthrust is equal to the weight of the body.

I also introduce here the term *centre of buoyancy*, which is the centre of mass of the displaced fluid. In a freely-floating body in equilibrium, the centre of buoyancy is vertically below the centre of mass of the floating body. As far as calculating the *moment* about some axis of the hydrostatic upthrust is concerned, the upthrust can be considered to act through the centre of buoyancy, just as the weight of an object can be considered to act through its centre of mass. See Section 1.1 of Chapter 1, for example, for a discussion of this point.

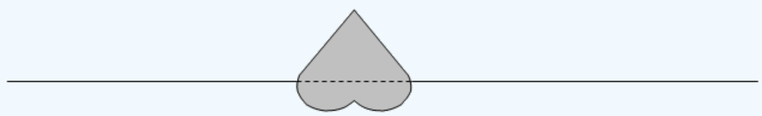
Also, before we get going, here is another small problem.

Problem 8

The drawing shows a body, whose relative density (i.e. its density relative to that of the fluid that it is floating in) is s_1. The dashed line is the *water-line section*.



Now, in the next drawing, a body of exactly the same size and shape (but not necessarily the same density) is floating upside down, with the same water-line section.



What is the relative density of this second body?

Answer

Let us establish some notation.

V = total volume of each body

fV = volume of liquid displaced by the first body (i.e. volume below the waterline in the first drawing)

$(1-f)V$ = volume of liquid displaced by the second body (i.e. volume below the waterline in the second drawing)

ρ_0 = density of liquid

ρ_1 = density of first body = $s_1\rho_0$

ρ_2 = density of first body = $s_2\rho_0$

g = gravitational acceleration

Now:

Weight of first body = weight of liquid displaced: $V\rho_1\mathrm{g} = fV\rho_0\mathrm{g}$ I.e. $s_1 = f$

Weight of second body = weight of liquid displaced: $V\rho_2\mathrm{g} = (1-f)V\rho_0\mathrm{g}$ I.e. $s_2 = f$

Hence $\underline{\underline{s_2 = f}}$.

I want to look now at the *stability* of equilibrium of a freely-floating body. While at first sight this may not be a very interesting topic, if you ever happen to be a passenger on an ocean liner, you might then find it to be quite interesting, for you will be interested to know, if the liner is given a small angular displacement from the vertical position, whether it will capsize and throw you into the sea, or whether it will right itself. Under such circumstances it becomes a very interesting subject indeed.

Before I start, I just want to establish one small geometric result.

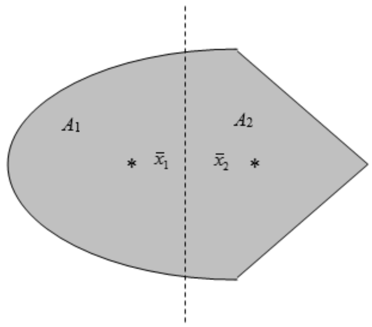


FIGURE XVI.9

Figure XVI.9 shows a plane bilaterally-symmetric area. I have drawn a dashed line through the centroid of the area. The areas to the left and right of this line are A_1 and A_2, and I have indicated the positions of the centroids of these two areas. (I haven't calculated the positions of the three centroids accurately – I just drew them approximately where I thought they would be.) Note that, since the dashed line goes through the centroid of the whole area, $A_1\bar{x}_1 = A_2\bar{x}_2$. Now rotate the area about the dashed line through an angle θ. By the theorem of Pappus (see Chapter 1, Section 1.6), the volume swept out by A_1 is $A_1 \times \bar{x}_1\theta$ and the volume swept out by A_2 is $A_2 \times \bar{x}_2\theta$. Thus we have established the geometrical result that I wanted, namely, that when a bilaterally symmetric area is rotated about an axis perpendicular to its axis of symmetry and passing through its centroid, the areas to left and right of the axis of rotation sweep out equal volumes.

We can now return to floating bodies, and I am going to consider the stability of equilibrium of a bilaterally symmetric floating body to a rotational displacement about an axis lying in the water line section and perpendicular to the axis of symmetry.

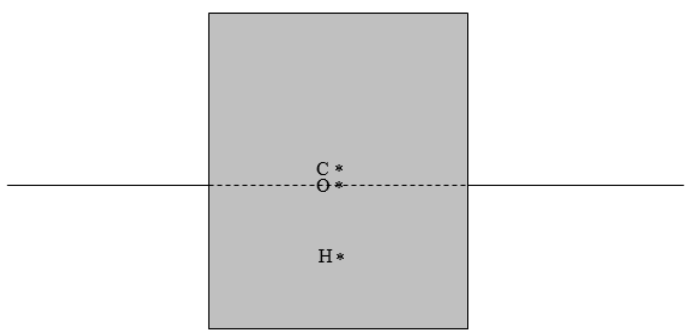


FIGURE XVI.10

I have drawn in figure XVI.10 the centre of mass C of the whole body, the centre of buoyancy H, and the centroid of the water-line section. The body is bilaterally symmetric about the plane of the paper, and we are going to rotate the body about an axis through O perpendicular to the plane of the paper, and we want to know whether the equilibrium is stable against such an angular displacement. We are going to rotate it in such a manner that the volume submerged is unaltered by the rotation – which means that the hydrostatic upthrust will remain equal to the weight of the body, and there will be no vertical acceleration. The geometrical theorem that we have just established shows that, if we rotate the body about an axis through the centroid of the water-line section, the volume submerged will be constant; conversely, our condition that the volume submerged is constant implies that the rotation is about an axis through the centroid of the water-line section.

I am going to establish a set of rectangular axes, origin O, with the x-axis to the right, the y-axis towards you, and the z-axis downwards. I'm going to call the depth of the centre H of buoyancy $\bar{z}$. Now let's carry out the rotation about O through an angle θ.

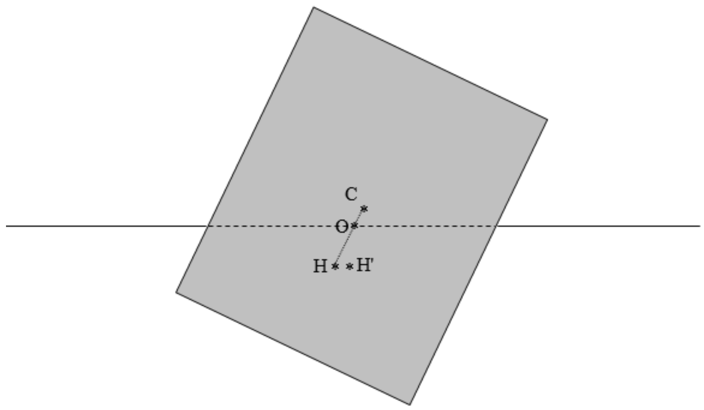


FIGURE XVI.11

I have drawn the position of the new centre of buoyancy H' and I wish to find its coordinates $(\overline{x}', \overline{z}')$ relative to O. We shall find that it has moved a little horizontally compared with the original position of H, but its depth is almost unchanged. Indeed, for small θ, we shall find that $\overline{z}' - \overline{z}$ is of order θ^2, while $\overline{x}' - \overline{x}$ is of order θ. Thus, to first order in θ, I shall assume that the depth of the centre of buoyancy has remained unchanged.

However, the coordinate $\overline{x}'$ of the new centre of buoyancy will be of interest for the following reason. The weight of the body acts at its centre of mass C while the hydrostatic upthrust acts at the new centre of buoyancy H' and these two forces form a couple and exert a torque. You will understand from figure XVI.11 that if H' is to the left of C, the torque will topple the body over, whereas if H' is to the right of C, the torque will stabilize the body. Indeed, the horizontal distance between C and H' is known as the *righting lever*. The point on the line COH vertically above H' is called the *metacentre*. I haven't drawn it on the diagram, in order to minimise clutter, but I shall use the symbol M to indicate the metacentre. We can see that the condition for stability of equilibrium is that HM > HC. This is why we are interested in finding the exact position of the new centre of buoyancy H'.

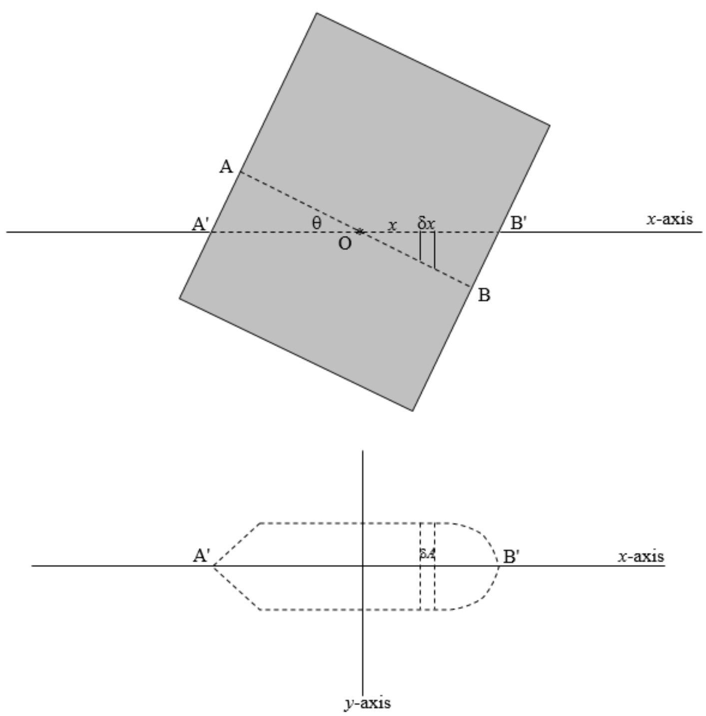


FIGURE XVI.12

In the upper part of figure XVI.12 I have drawn the old and new water-line sections as seen from the side, and in the lower part I have drawn the new water-line section seen from above. I have indicated an elemental volume of width δx of the displaced fluid at a distance x from the centroid O of the water-line section. For small θ the depth of this element is $x\theta$. Let's call its area in the water-line section δA, so that the volume element is $x\theta\delta A$. We'll call the total volume of the displaced fluid (which is unaltered by the rotation) V.

Consider the moments of volume about the x-axis. We have

$$V\bar{z}' = V\bar{z} - \int_O^{A'} \tfrac{1}{2}x\theta .\, x\theta\delta A + \int_O^{B'} \tfrac{1}{2}x\theta .\, x\theta\delta A$$

$$V(\bar{x}' - \bar{x}) = \theta \int_{A'}^{B'} x^2 dA. \tag{16.9.1}$$

Thus, as previously asserted, the vertical displacement of the centre of buoyancy is of order θ^2, and, to first order in θ may be neglected.

Now consider the moments of volume about the y-axis. We have

$$V\bar{x}' = V\bar{x} - \int_O^{A'} x\theta\, dA\, x + \int_O^{A'} x\theta\, dA\, x$$

$$V\,(\bar{x}' - \bar{x}) = \theta \int_{A'}^{B'} x^2 dA. \tag{16.9.2}$$

But the integral on the right hand side of Equation 16.9.2 is Ak^2, where A is the area of the water-line section, and k is its radius of gyration.

Thus

$$HH' = \frac{Ak^2\theta}{V}. \tag{16.9.3}$$

Now $HH' = HM \sin\theta$, where M is the metacentre, or, to first order in θ, HH' = HM % θ.

$$\mathrm{HM} = \frac{Ak^2}{V}. \tag{16.9.4}$$

Therefore the *condition for stability of equilibrium* is that

$$\frac{Ak^2}{V} > \mathrm{HC}. \tag{16.9.5}$$

Here, A and k^2 refer to the water-line section, V is the volume submerged, and HC is the distance between centre of buoyancy and centre of mass.

Example. Suppose that the body is a cube of side $2a$ and of relative density s. The water-line section is a square, and $A = 4a^2$ and $k^2 = \frac{a^2}{3}$. The volume submerged is $8a^3s$. The distance between the centres of mass and buoyancy is $a(1-s)$, and so the condition for stability is

$$\frac{a}{6s} > a(1-s) \tag{16.9.6}$$

The equilibrium is unstable if

$$6s^2 - 6s + 1 < 0. \tag{16.9.7}$$

That is, the equilibrium is unstable if s is between 0.2113 and 0.7887. The cube will float vertically only if the density is less than 0.2113 or if it is greater than 0.7887.

? Problem 9

Here in British Columbia there is a large logging industry, and many logs float horizontally in the water. They gradually become waterlogged, and, when the density of a log is nearly as dense as the water, the vertical position become stable and the log tips to the vertical position, nearly all of it submerged, with only an inch or so above the surface. It then becomes a danger to boats. If the length of the log is $2l$ and its radius is a, what is the least relative density for which the vertical position is stable?

Answer

The *condition for stability of equilibrium* is that

$$\frac{Ak^2}{V} > \mathrm{HC}$$

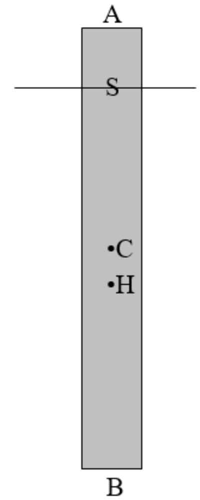
Here, A and k^2 refer to the water-line section, V is the volume submerged, and HC is the distance between centre of mass and centre of buoyancy.

In the present case we have a log of radius a and length $2l$. In this case

$$A = \pi a^2, \quad k^2 = \frac{1}{2}a^2, \quad V = 2\pi a^2 l\ .$$

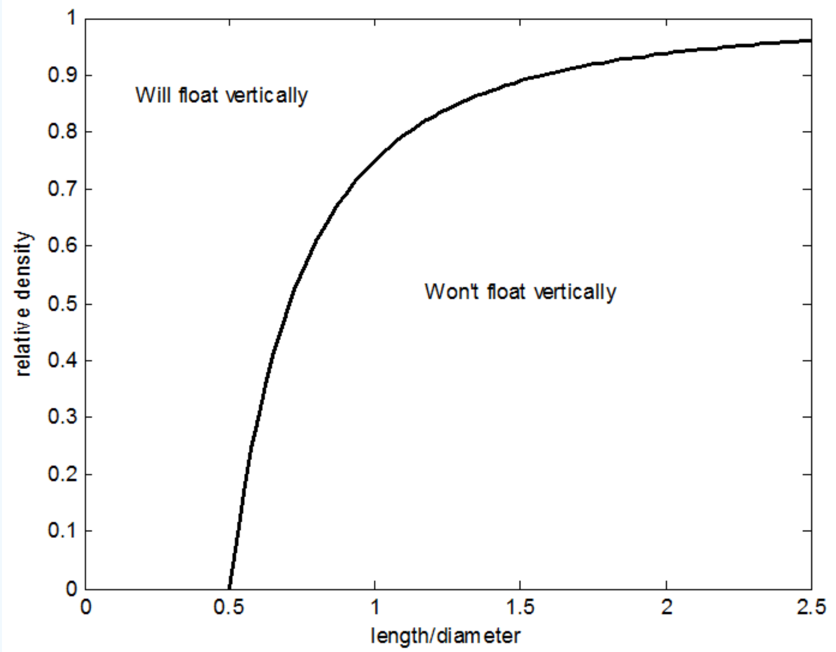
$$\frac{Ak^2}{V} = \frac{a^2}{4l}$$

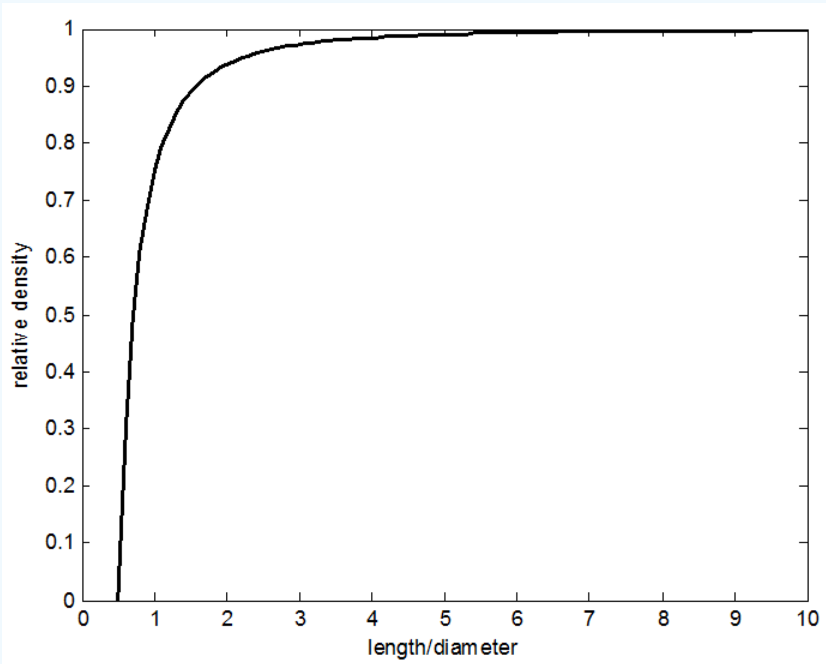
Density of log = ρ
Density of water = ρ_0
Relative density $s = \frac{\rho}{\rho_0}$
Some distances:
AB = $2l$
AC = l
SB = $2ls$
AS = $2l(1-s)$
SH = ls
SC = AC - AS = $2ls - l$ 2ls-l\)
HC = SH - SC = $l(1-s)$
The condition for stability is that $\frac{a^2}{4l} > l(1-s)$
That is: $s > 1 - \frac{1}{4}\left(\frac{\text{diameter}}{\text{length}}\right)^2$.



length/diameter =	0.5	0.71	1	2	10	40
relative density >	0	0.50	0.75	0.9375	0.9975	0.9988

A flat log, whose length is less than half its diameter, floats with its axis vertical, whatever its density (provided, of course, that it is less than that of water, when it will sink). If its length is equal to its diameter, it will float vertically provided that its density is at least 0.75 that of water. A very long log floats horizontally until it is almost completely saturated with water, and then it will tip over to a vertical position, almost completely submerged, when it is not readily visible and it is then a danger to boats. The condition for vertically-floating stable equilibrium is illustrated in the two graphs below.


Will float vertically
Won't float vertically
relative density
length/diameter


relative density
length/diameter

CHAPTER OVERVIEW

17: Vibrating Systems

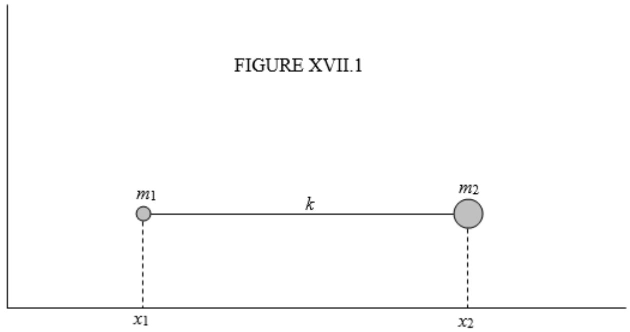
Topic hierarchy

17.1: Introduction

A mass m is attached to an elastic spring of force constant k, the other end of which is attached to a fixed point. The spring is supposed to obey Hooke's law, namely that, when it is extended (or compressed) by a distance x from its natural length, the tension (or thrust) in the spring is kx, and the equation of motion is $m\ddot{x} = -kx$. This is simple harmonic motion of period $\frac{2\pi}{\omega}$, where $\omega^2 = \frac{k}{m}$. Most readers will have no difficulty with that problem. But now suppose that, instead of one end of the spring being attached to a fixed point, we have two masses, m_1 and m_2, one at either end of the spring. A diatomic molecule is much the same thing. Can you calculate the period of simple harmonic oscillations? It looks like an easy problem, but it somehow seems difficult to get a hand on it by conventional newtonian methods. In fact it can be done quite readily by newtonian methods, but this problem, as well as more complicated problems where you have several masses connected by several springs and several possible modes of vibration, is particularly suitable by lagrangian methods, and this chapter will give several examples of vibrating systems tackled by lagrangian methods.

17.2: The Diatomic Molecule

Two particles, of masses m_1 and m_2 are connected by an elastic spring of force constant k. What is the period of oscillation?



Let's suppose that the equilibrium separation of the masses – i.e. the natural, unstretched, uncompressed length of the spring – is a. At some time suppose that the x-coordinates of the two masses are x_1 and x_2. The extension q of the spring from its natural length at that moment is $q = x_2 - x_1 - a$. We'll also suppose that the velocities of the two masses at that instant are $\dot{x}_1$ and $\dot{x}_2$. We know from Chapter 13 how to start any calculation in lagrangian mechanics. We do not have to think about it. We always start with T= ... and V= ... :

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2, \tag{17.2.1}$$

$$V = \frac{1}{2}kq^2. \tag{17.2.2}$$

We want to be able to express the equations in terms of the internal coordinate q. V is already expressed in terms of q. Now we need to express T(and therefore $\dot{x}_1$ and $\dot{x}_2$) in terms of q. Since $q = x_2 - x_1 - a$ we have, by differentiation with respect to time,

$$\dot{q} = \dot{x}_2 - \dot{x}_1. \tag{17.2.3}$$

We need one more equation. The linear momentum is constant and there is no loss in generality in choosing a coordinate system such that the linear momentum is zero:

$$0 = m_1\dot{x}_1 + m_2\dot{x}_2. \tag{17.2.4}$$

From these two equations, we find that

$$\dot{x}_1 = \frac{m_2}{m_1 + m_2}\dot{q} \tag{17.2.5a}$$

and

$$\dot{x}_2 = \frac{m_1}{m_1 + m_2}\dot{q}. \tag{17.2.5b}$$

Thus we obtain

$$T = \frac{1}{2}m\dot{q}^2 \tag{17.2.5}$$

and

$V = \frac{1}{2}kq^2$

where

$$m = \frac{m_1 m_2}{m_1 + m_2}. \tag{17.2.6}$$

Now apply Lagrange's equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = \frac{\partial V}{\partial q_j}. \tag{13.4.13}$$

to the single coordinate q in the fashion to which we became accustomed in Chapter 13, and the equation of motion becomes

$$m\ddot{q} = -kq, \tag{17.2.7}$$

which is simple harmonic motion of period $2\pi\sqrt{\frac{m}{k}}$ where m is given by Equation 17.2.6. The frequency is the reciprocal of this, and the "angular frequency" ω, also sometimes called the "pulsatance", is 2π times the frequency, or $\sqrt{\frac{k}{m}}$.

The quantity $\frac{m_1m_2}{(m_1+m_2)}$ is usually called the "reduced mass" and one may wonder is what sense it is "reduced". I believe the origin of this term may come from an elementary treatment of the Bohr atom of hydrogen, in which one at first assumes that there is an electron moving around an immovable nucleus – i.e. a nucleus of "infinite mass". One develops formulas for various properties of the atom, such as, for example, the Rydberg constant, which is the energy required to ionize the atom from its ground state. This and similar formulas include the mass m of the electron. Later, in a more sophisticated model, one takes account of the finite mass of the nucleus, with nucleus and electron moving around their mutual centre of mass. One arrives at the same formula, except that m is replaced by $\frac{mM}{(m+M)}$, where M is the mass of the nucleus. This is slightly less (by about 0.05%) than the mass of the electron, and the idea is that you can do the calculation with a fixed nucleus provided that you use this "reduced mass of the electron" rather than its true mass. Whether this is the appropriate term to use in our present context is debatable, but in practice it is the term almost universally used.

It may also be remarked upon by readers with some familiarity with quantum mechanics that I have named this section "The Diatomic Molecule" – yet I have ignored the quantum mechanical aspects of molecular vibration. This is true – in this series of notes on *Classical Mechanics* I have adopted an entirely classical treatment. It would be wrong, however, to assume that classical mechanics does not apply to a molecule, or that quantum mechanics would not apply to a system consisting of a cricket ball and a baseball connected by a metal spring. In fact both classical mechanics and quantum mechanics apply to both. The formula derived for the frequency of vibration in terms of the reduced mass and the force constant ("bond strength") applies as accurately for the molecule as for the cricket ball and baseball. Quantum mechanics, however, predicts that the total *energy* (the eigenvalue of the hamiltonian operator) can take only certain discrete values, and also that the lowest possible value is not zero. It predicts this not only for the molecule, but also for the cricket ball and baseball – although in the latter case the energy levels are so closely spaced together as to form a quasi continuum, and the zero point vibrational energy is so close to zero as to be unmeasurable. Quantum mechanics makes its effects *evident* at the molecular level, but this does not mean that it does not *apply* at macroscopic levels. One might also take note that one is not likely to understand why wave mechanics predicts only discrete energy levels unless one has had a good background in the classical mechanics of waves. In other words, one must not assume that classical mechanics does not apply to microscopic systems, or that quantum mechanics does not apply to macroscopic systems.

Below leaving this section, in case you tried solving this problem by newtonian methods and ran into difficulties, here's a hint. Keep the centre of mass fixed. When the length of the spring is x, the lengths of the portions on either side of the centre of mass are $\frac{m_2x}{m_1+m_2}$ and $\frac{m_1x}{m_1+m_2}$. The force constants of the two portions of the spring are inversely proportional to their lengths. Take it from there.

17.3: Two Masses, Two Springs and a Brick Wall

The system is illustrated in Figure XVII.2, first in its equilibrium (unstretched) position, and then at some instant when it is not in equilibrium and the springs are stretched. You can imagine that the masses are resting upon and can slide upon a smooth, horizontal table. I could also have them hanging under gravity, but this would introduce a distracting complication without illustrating any further principles. I also want to assume that all the motion is linear, so we could have them sliding on a smooth horizontal rail, or have them confined in the inside of a smooth, fixed drinking-straw. For the present, I do not want the system to bend.

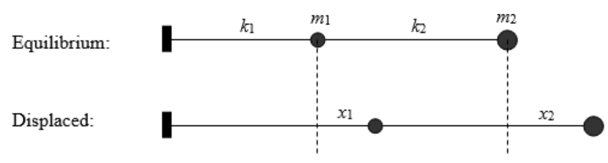


FIGURE XVII.2

The displacements from the equilibrium positions are x_1 and x_2, so that the two springs are stretched by x_1 and $x_2 - x_1$ respectively. The velocities of the two masses are $\dot{x}_1$ and $\dot{x}_2$. We now start the lagrangian calculation in the usual manner:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2, \tag{17.3.1}$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2. \tag{17.3.2}$$

Apply Lagrange's equation to each coordinate in turn, to obtain the following equations of motion:

$$m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2 \tag{17.3.3}$$

and

$$m_2\ddot{x}_2 = k_2x_1 - k_2x_2. \tag{17.3.4}$$

Now we seek solutions in which the system is vibrating in simple harmonic motion at angular frequency ω; that is, we seek solutions of the form $\ddot{x}_1 = -\omega^2 x_1$ and $\ddot{x}_2 = -\omega^2 x_2$.

When we substitute these in Equations 17.3.3and 17.3.4, we obtain

$$(k_1 + k_2 - m_1\omega^2)x_1 - k_2x_2 = 0 \tag{17.3.5}$$

an\

$$k_2x_1 - (k_2 - m_2\omega^2)x_2 = 0. \tag{17.3.6}$$

Either of these gives us the displacement ratio $\frac{x_2}{x_1}$ (and hence amplitude ratio). The first gives us

$$\frac{x_2}{x_1} = \frac{-m_1\omega^2 + k_1 + k_2}{k_2} \tag{17.3.7}$$

and the second gives us

$$\frac{x_2}{x_1} = \frac{k_2}{k_2 - m_2\omega^2}. \tag{17.3.8}$$

These are equal, and, by equating the right hand sides, we obtain the following equation for the *angular frequencies of the normal modes*:

$$m_1m_2\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2 \ = \ 0. \tag{17.3.9}$$

This equation can also be derived by noting, from the theory of equations, that Equations 17.3.5 and 17.3.6 are consistent only if the determinant of the coefficients is zero.

The meaning of these equations and of the expression "normal modes" can perhaps be best illustrated with a numerical example. Let us suppose, for example, that $k_1 = k_2 = 1$ and $m_1 = 3$ and $m_2 = 2$. In that case Equation 17.3.9 is

$6\omega^4 - 7\omega^2 + 1 = 0$. This is a quartic equation in ω, but it is also a quadratic equation in ω^2, and there are just two positive solutions for ω. These are $\frac{1}{\sqrt{6}} = 0.4082$ (slow, low frequency) and 1 (fast, high frequency). If you put the low frequency ω into either of Equations 17.3.7 or 17.3.8 (or in both, to check for arithmetic or algebraic mistakes) you find a displacement ratio of +1.5; but if you put the high frequency ω into either equation, you find a displacement ratio of -1.0. The first of these *normal modes* is a low-frequency slow oscillation in which the two masses oscillate in phase, with m_2 having an amplitude 50% larger than m_1. The second normal mode is a high-frequency fast oscillation in which the two masses oscillate out of phase but with equal amplitudes.

So, how does the system actually oscillate? This depends on the *initial conditions.* For example, if you displace the first mass by one inch to the right and the second mass by 1.5 inches to the right (this implies stretching the first spring by 1 inch and the second by 0.5 inches), and then let go, the system will oscillate in the slow, in-phase mode. But if you start by displacing the first mass by one inch to the right and the second mass by one inch to the left (this implies stretching the first spring by 1 inch and compressing the second by 2 inches), the system will oscillate in the fast, out-of-phase mode. For other initial conditions, the system will oscillate in a *linear combination of the normal modes.*

Thus, m_1 might oscillate with an amplitude A in the slow mode, and an amplitude B in the fast mode:

$$x_1 = A\cos(\omega_1 t + \alpha_1) \, + \, B\cos(\omega_2 t + \alpha_2) \, , \tag{17.3.10}$$

in which case the oscillation of m_2 is given by

$$x_2 = 1.5A\cos(\omega_1 t + \alpha_1) \, - \, B\cos(\omega_2 t + \alpha_2). \tag{17.3.11}$$

In our example, ω_1 and ω_2 are $\frac{1}{\sqrt{6}}$ and 1 respectively.

Let's suppose that the initial conditions are that, at $t = 0$, $\dot{x}_1$ and $\dot{x}_2$ are both zero. This means that α_1 and α_2 are both zero or π (I'll take them to be zero), so that

$$x_1 = A\cos\omega_1 t \, + \, B\cos\omega_2 t \tag{17.3.12}$$

and

$$x_2 = 1.5A\cos\omega_1 t \, - \, B\cos\omega_2 t. \tag{17.3.13}$$

Suppose further that at $t = 0$, x_1 and x_2 are both +1, which means that we start by stretching both springs equally. Equations 17.3.12 and 17.3.13 then become $1 = A + B$ and $1 = 1.5A - B$. That is, $A = 0.8$ and $B = 0.2$. I'll leave you to draw graphs of x_1 and x_2 versus time.

Here's an exercise that might be useful if, perhaps, you wanted to construct a real system with two equal masses m and two equal springs, each of constant k, to demonstrate the vibrations. Show that in that case, the angular frequency (which is, of course, 2π times the actual frequency) of the slow, in phase, mode is

$\omega_1 = \frac{1}{2}\left(\sqrt{5} - 1\right)\sqrt{\frac{k}{m}} \, = 0.6180\sqrt{\frac{k}{m}}$

with a displacement ratio $\frac{x_2}{x_1} = \frac{1}{2}(\sqrt{5} + 1) \, = \, 1.6180$;

and the angular frequency of the fast, out of phase, mode is

$\omega_2 = \frac{1}{2} = (\sqrt{5} + 1)\sqrt{\frac{k}{m}} = 1.6180\sqrt{\frac{k}{m}}$

with a displacement ratio, $x_2/x_1 = -\frac{1}{2}(\sqrt{5} - 1) = -0.6180$.

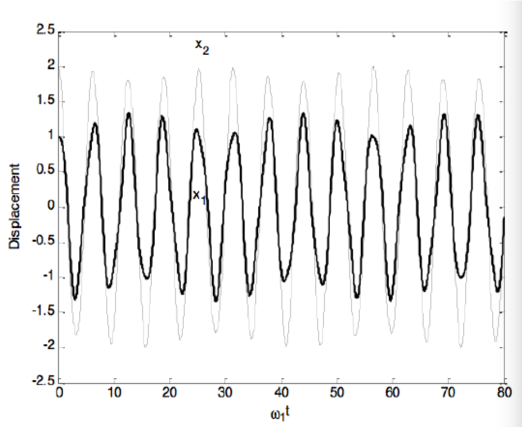
Knowing these displacements ratios will enable you to start with the appropriate initial conditions for each normal mode.

If you were to start at $t = 0$ with a displacements $x_1 = 1$ and $x_2 = 2$ which isn't right for either normal mode, you can show that the subsequent displacements would be

$$x_1 = 1.170820\cos\omega_1 t - 0.170820\cos\omega_2 t$$

$$x_2 = 1.894427\cos\omega_1 t - 0.105572\cos\omega_2 t$$

That looks like

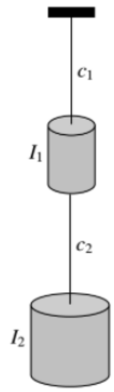


Although at first it looks like fast in-phase mode for both of them, you can see the influence of the slow mode, which has about 2.6 times the period of the last mode, in the slow amplitude modulation. If you look carefully at the modulation amplitudes of both displacements, you will see that the amplitude of the x_1 displacement is out of phase with the amplitude of the x_2 displacement.

17.4: Double Torsion Pendulum

Here we have two cylinders of rotational inertias I_1 and I_2 hanging from two wires of torsion constants c_1 and c_2.

FIGURE XVII.3



At any instant, the top cylinder is turned through an angle θ_1 from the equilibrium position and the lower cylinder by an angle θ_2 from the equilibrium position (so that, relative to the upper cylinder, it is turned by). The equations and the description of the motion are just the same as in the previous example, except that $x_1, x_2, m_1, m_2, k_1, k_2$ are replaced by $\theta_1, \theta_2, I_1, I_2, c_1, c_2$. The kinetic and potential energies are

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2, \tag{17.4.1}$$

$$T = \frac{1}{2} c_1 \theta_1^2 + \frac{1}{2} c_2 (\theta_2 - \theta_1)^2. \tag{17.4.2}$$

The equations for ω and the displacement ratios are just the same, and there is an in-phase and an out-of-phase mode.

17.5: Double Pendulum

This is another similar problem, though, instead of assuming Hooke's law, we shall assume that angles are small ($\sin\theta\approx\theta, \cos\theta\approx 1-\frac{1}{2}\theta^2$). For clarity of drawing, however, I have drawn large angles in Figure XVIII.4.

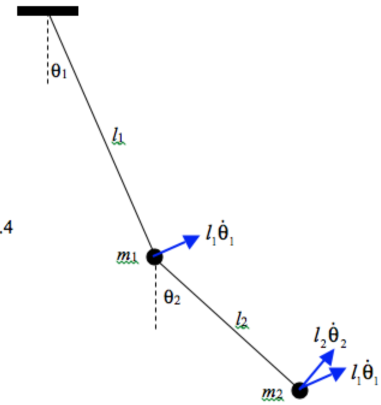


FIGURE XVII.4

Because I am going to use the lagrangian equations of motion, I have not marked in the forces and accelerations; rather, I have marked in the velocities. I hope that the two components of the velocity of m_2 that I have marked are self-explanatory; the speed of m_2 is given by $v_2^2=l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)$. The kinetic and potential energies are

$$T=\frac{1}{2}m_1l_1^2\dot{\theta}_1^2+\frac{1}{2}m_1[l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)], \tag{17.5.1}$$

$$V=constant-m_1gl_1\cos\theta_1-m_2g(l_1\cos\theta_1+l_2\cos\theta_2). \tag{17.5.2}$$

If we now make the small angle approximation, these become

$$T=\frac{1}{2}m_1l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2(l_1\dot{\theta}_1+l_2\dot{\theta}_2)^2 \tag{17.5.3}$$

and

$$V=contant+\frac{1}{2}m_1gl_1\theta_1^2+\frac{1}{2}m_1g(l_1\theta_1^2+l_2\theta_2^2)-m_1gl_1-m_2gl_2. \tag{17.5.4}$$

Apply the lagrangian equation in turn to θ_1 and θ_2:

$$(m_1+m_2)l_1^2\ddot{\theta}_1+m_2l_1l_2\ddot{\theta}=-(m_1+m_2)gl_1\theta_1 \tag{17.5.5}$$

and

$$m_2l_1l_2\ddot{\theta}_1+m_2l_2^2\ddot{\theta}_2=-m_2gl_2\theta_2. \tag{17.5.6}$$

Seek solutions in the form of $\dot{\theta}_1=-\omega\theta_1$ and $\dot{\theta}_2=-\omega^2\theta_2$.

Then

$$(m_1+m_2)(l_2\omega^2-g)\theta_1+m_2l_1l_2\omega^2\theta_2=0 \tag{17.5.7}$$

and

$$l_1\omega^2\theta_1+(l_2\omega^2-g)\theta_2=0. \tag{17.5.8}$$

Either of these gives the displacement ratio θ_2/θ_1. Equating the two expressions for the ratio θ_2/θ_1, or putting the determinant of the coefficients to zero, gives the following equation for the frequencies of the normal modes:

$$m_1l_1l_2\omega^4-(m_1+m_2)g(l_1+l_2)\omega^2+(m_1+m_2)g^2=0. \tag{17.5.9}$$

As in the previous examples, there is a slow in-phase mode, and fast out-of-phase mode.

For example, suppose m_1 = 0.01 kg, m_2 = 0.02 kg, l_1 = 0.3 m, l_2 = 0.6 m, g = 9.8 m s^{-2}.

Then $0.0018\omega^4 - 0.02446\omega^2 = 0$. The slow solution is ω = 3.441 rad s^{-1} (P = 1.826 s), and the fast solution is ω = 11.626 rad s^{-1} (P =0.540 s). If we put the first of these (the slow solution) in either of equations 17.5.7 or 8 (or both, as a check against mistakes) we obtain the displacement ratio θ_2/θ_1 = 1.319, which is an in-phase mode. If we put the second (the fast solution) in either equation, we obtain θ_2/θ_1 = −0.5689 , which is an out-of-phase mode. If you were to start withθ_2/θ_1 = 1.319 and let go, the pendulum would swing in the slow in-phase mode. If you were to start with θ_2/θ_1 = −0.5689 and let go, the pendulum would swing in the fast out-of-phase mode. Otherwise the motion would be a linear combination of the normal modes, with the fraction of each determined by the initial conditions, as in the example in Section 17.3.

17.6: Linear Triatomic Molecule

In Chapter 2, Section 2.9, we discussed a *rigid* triatomic molecule. Now we are going to discuss three masses held together by springs, of force constants k_1 and k_2. We are going to allow it to vibrate, but not to rotate. Also, for the time being, I do not want the molecule to bend, so we'll put it inside a drinking straw to that all the vibrations are linear. By the way, for real triatomic molecules, the force constants and rotational inertias are such that molecules vibrate much faster than they rotate. To see their vibrations you look in the near infra-red spectrum; to see their rotation, you have to go to the far infrared or the microwave spectrum.

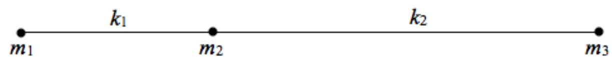


FIGURE XVII.5

Suppose that the equilibrium separations of the atoms are a_1 and a_2. Suppose that at some instant of time, the x-coordinates (distances from the left hand edge of the page) of the three atoms are x_1, x_2, x_3. The extensions from the equilibrium distances are then $q_1 = x_2 - x_1 - a_1, q_2 = x_3 - x_2 - a_2$. We are now ready to start:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_2\dot{x}_3^2, \tag{17.6.1}$$

$$V = \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_2q_2^2. \tag{17.6.2}$$

We need to express the kinetic energy in terms of the internal coordinate, and, just as for the diatomic molecule (Section 17.2), the relevant equations are

$$\dot{q}_1 = \dot{x}_2 - \dot{x}_1, \tag{17.6.3}$$

$$\dot{q}_2 = \dot{x}_3 - \dot{x}_2, \tag{17.6.4}$$

and

$$0 = m_1\dot{x}_1 + m_2\dot{x}_2 + m_3\dot{x}_3. \tag{17.6.5}$$

These can conveniently be written

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \tag{17.6.6}$$

By one dexterous flick of the fingers (!) we invert the matrix to obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\frac{m_2+m_3}{M} & -\frac{m_3}{M} & \frac{1}{M} \\ \frac{m_1}{M} & -\frac{m_3}{M} & \frac{1}{M} \\ \frac{m_1}{M} & \frac{m_1+m_2}{M} & \frac{1}{M} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ 0 \end{pmatrix}, \tag{17.6.7}$$

where $M = m_1 + m_2 + m_3$. On putting these into Equation 17.6.1, we now have

$$T = \frac{1}{2}(a\dot{q}_1^2 + 2h\dot{q}_1\dot{q}_2 + b\dot{q}_2^2) \tag{17.6.8}$$

and

$$V = \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_2q_2^2 \tag{17.6.9}$$

where

$$a = m_1(m_2 + m_3)/M, \tag{17.6.10}$$

$$h=m_3m_1/M, \tag{17.6.11}$$

$$b=m_3(m_1+m_2)/M \tag{17.6.12}$$

and for future reference,

$$ab-h^2=m_1m_2m_2/M=m^2h. \tag{17.6.13}$$

On application of Lagrange's equation in turn to the two internal coordinates we obtain

$$a\ddot{q}_1+h\ddot{q}_2+k_1q_1=0 \tag{17.6.14}$$

and

$$b\ddot{q}_2+h\ddot{q}_2+k_2q_2=0 \tag{17.6.15}$$

Seek solutions of the form $\ddot{q}_1=-\omega^2q_1$ and $\ddot{q}_2=-\omega^2q_2$ and we obtain the following two expressions for the extension ratios:

$$\frac{q_1}{q_2}=\frac{h\omega^2}{k_1-a\omega^2}=\frac{k_2-b\omega^2}{h\omega^2}. \tag{17.6.16}$$

Equating them gives the equation for the normal mode frequencies:

$$(ab-h^2)\omega^4-(ak_2+bk_1)\omega^2+k_1k_2=0. \tag{17.6.17}$$

For example, if $k_1=k_2=l$ and $m_1=m_2=m_3=m$, we obtain, for the slow symmetric ("breathing") mode, $q_1/q_2=+1$ and $\omega^2=k/m$. For the fast asymmetric mode, $q_1/q_2=-1$ and $\omega^2=3k/m$.

Example.

Consider the linear OCS molecule whose atoms have masses 16, 12 and 32. Suppose that the angular frequencies of the normal modes, as determined from infrared spectroscopy, are 0.905 and 0.413. (I just made these numbers up, in unstated units, just for the purpose of illustrating the calculation. Without searching the literature, I can't say what they are in the real OCS molecule.) Determine the force constants.

In Chapter 2 we considered a rigid triatomic molecule. We were given the moment of inertia, and we were asked to find the two internuclear distances. We couldn't do this with just one moment of inertia, so we made an isotopic substitution (^{18}O instead of ^{16}O) to get a second equation, and so we could then solve for the two internuclear distances. This time, we are dealing with vibration, and we are going to use Equation 17.6.17 to find the two force constants. This time, however, we are given two frequencies (of the normal modes), and so we have no need to make an isotopic substitution − we already have two equations.

Here are the necessary data.

^{16}OCS

Fast ω 0.905

Slow ω 0.413

$m_1m_2m_3$ 16 12 32

M 60

a 11.73

h 8.53

b 14.93

$ab-h^2$ 102.4

Use equation 17.6.16 for each of the frequencies, and you'll get two equations, in k_1 and k_2. As in the rotational case, they are quadratic equations, but they are a bit easier to solve than in the rotational case. You'll get two equations, each of the form of $A-Bk_1-Ck_2+k_1k_2=0$, where the coefficients are functions of a,b,h,ω. You'll have to work out the values of these coefficients, but, before you substitute the numbers in, you might want to give a bit of thought to how you would go about solving two simultaneous equations of the form $A-Bk_1-Ck_2+k_1k_2=0$.

You will find that there are two possible solutions:

$$k_1 = 2.8715 \qquad k_2 = 4.9818$$

and

$$k_1 = 3.9143 \qquad k_2 = 3.6547$$

Both of these will result in the same frequencies. You would need some additional information to determine which obtains for the actual molecule, perhaps with measurements on an isotopomer, such as ^{18}OCS.

Note that in this section we considered a linear triatomic molecule that was not allowed either to rotate or to bend, whereas in Chapter 2 we considered a rigid triatomic molecule that was not allowed either to vibrate or to bend. If all of these restrictions are removed, the situation becomes rather more complicated. If a rotating molecule vibrates, the moving atoms, in a co-rotating reference frame, are subject to the Coriolis force, and hence they do not move in a straight line. Further, as it vibrates, the rotational inertia changes periodically, so the rotation is not uniform. If we allow the molecule to bend, the middle atom can oscillate up and down in the plane of the paper (so to speak) or back and forth at right angles to the plane of the paper. These two motions will not necessarily have either the same amplitude or the same phase. Consequently the middle atom will whirl around in a Lissajous ellipse, giving rise to what has been called "vibrational angular momentum". In a real triatomic molecule, the vibrations are usually much faster than the relatively slow, ponderous rotation, so that vibration-rotation interaction is small – but is by no means negligible and is readily observed in the spectrum of the molecule.

17.7: Two Masses, Three Springs, Two brick Walls

The three masses are equal, and the two outer springs are identical. Figure XVII.6 shows the equilibrium position.

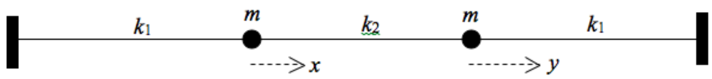


FIGURE XVII.6

Suppose that at some instant the first mass is displaced a distance x to the right and the second mass is displaced a distance y to the right. The extensions of the first two springs are x and $y-x$ respectively, and the compression of the third spring is y. If the speeds of the masses are $\dot{x}$ and $\dot{y}$, we have for the kinetic and potential energies:

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2 \tag{17.7.1}$$

and

$$V=\frac{1}{2}k_1x^2+\frac{1}{2}k_2(y-x)^2+\frac{1}{2}k_1y^2. \tag{17.7.2}$$

Apply Lagrange's equation in turn to x and to y.

$$m\ddot{x}+(k_1+k_2)x-k_2y=0 \tag{17.7.3}$$

and

$$m\ddot{y}+(k_1+k_2)y-k_2x=0. \tag{17.7.4}$$

Seek solutions of the form $\ddot{x}=-\omega^2x$ and $\ddot{y}=-\omega^2y$.

$$(-m\omega^2+k_1+k_2)-k_2y=0 \tag{17.7.5}$$

and

$$-k_2x+(-m\omega^2+k_1+k_2)y=0. \tag{17.7.6}$$

On putting the determinant of the coefficients to zero, we find for the frequencies of the normal modes

$$\omega^2=\frac{k_1}{m} \qquad \text{and} \qquad \omega^2=\frac{k_1+2k_2}{m}, \tag{17.7.7}$$

corresponding to displacement ratios

$$\frac{x}{y}=1 \qquad \text{and} \qquad \frac{x}{y}=-1. \tag{17.7.8}$$

In the first, slow, mode, the masses move in phase and there is no extension or compression of the connecting spring. In the second, fast, mode, the masses move in antiphase and the compression or extension of the coupling spring is twice the extension or compression of the outer springs.

The general motion is a linear combination of the normal modes:

$$x=A\cos(\omega_1t+\alpha_1)+B\cos(\omega_2t+\alpha_2), \tag{17.7.9}$$

$$y=A\cos(\omega_1t+\alpha_1)-B\cos(\omega_2t+\alpha_2), \tag{17.7.10}$$

$$\dot{x}=-A\sin(\omega_1t+\alpha_1)-B\sin(\omega_2t+\alpha_2), \tag{17.7.11}$$

$$\dot{y}=-A\sin(\omega_1t+\alpha_1)+B\sin(\omega_2t+\alpha_2), \tag{17.7.12}$$

Suppose that the initial condition is at $t=0, y=\dot{y}=0, x=x_0, \dot{x}=0$. That is, we pull the first mass a little to the right (keeping the second mass fixed) and then we let go. The second two equations establish that $\alpha_1=\alpha_2=0$, and the first two equations tell us that $A = B = x_0/2$. The displacements are then given by

$$x = \frac{1}{2}x_0(\cos\omega_1 t + \cos\omega_2 t) = x_0 \cos\frac{1}{2}(\omega_1 - \omega_2)t \cos\frac{1}{2}(\omega_1 + \omega_2)t \tag{17.7.13}$$

and

$$y = \frac{1}{2}x_0(\cos\omega_1 t + \cos\omega_2 t) = -x_0 \sin\frac{1}{2}(\omega_1 - \omega_2)t \sin\frac{1}{2}(\omega_1 + \omega_2)t. \tag{17.7.14}$$

Let us imagine, for example, that k_2 is much less than k_1 (but not negligible), so that we have two *weakly-coupled oscillators*. In that case equations 17.7.7 tell us that the frequencies of the two normal modes are nearly equal. What equation 17.7.13 describes, then, is a rapid oscillation of the first mass with angular frequency $\frac{1}{2}(\omega_1 + \omega_2$ whose amplitude is modulated with a slow angular frequency $\frac{1}{2}(\omega_1 - \omega_2)$. Equation 17.7.14 describes the same sort of motion for the second mass, except that the modulation is out of phase by 90º with the modulation of the motion of the first mass. For a while the first mass will oscillate with a large amplitude. This will gradually decrease, while the amplitude of the motion of the second mass increases until the motion of the first mass momentarily ceases. After that, the amplitude of the motion of the second mass starts to decrease, while the first mass starts up again. And so the motion continues, with the first mass and the second mass alternately taking up the motion.

17.8: Transverse Oscillations of Masses on a Taut String

A light string of length $4a$ is held taut, under tension F between two fixed points.

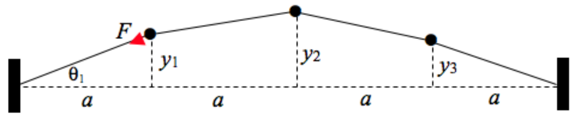


FIGURE XVII.7

Three equal masses m are attached at equidistant points along the string. They are set into transverse oscillation of small amplitudes, the transverse displacements of the three masses at some time being y_1, y_2 and y_3.

The kinetic energy is easy. It is just

$$T = \frac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2). \tag{17.8.1}$$

The potential energy is slightly more difficult.

In the undisplaced position, the length of each portion of the string is a.

In the displaced position, the lengths of the four portions of the string are, respectively,

$$\sqrt{y^2+1+a^2} \qquad \sqrt{(y_2-y_1)^2+a^2} \qquad \sqrt{(y_2-y_3)^2+a^2} \qquad \sqrt{y_3^2+a^2}$$

For small displacements (i.e. the ys much smaller than a), these are, approximately (by binomial expansion),

$$a+\frac{y_1^2}{2a} \qquad a+\frac{(y_2-y_1)^2}{2a} \qquad a+\frac{(y_2-y_3)^2}{2a} \qquad a+\frac{y_1^2}{2a}$$

so the extensions are

$$\frac{y_1^2}{2a} \qquad \frac{(y_2-y_1)^2}{2a} \qquad \frac{(y_2-y_3)^2}{2a} \qquad \frac{y_3^2}{2a}$$

It is also supposed that the tension in the string is F and that the displacements are sufficiently small that this is constant. The work done in displacing the masses, which is the elastic energy stored in the string as a result of the displacements, is therefore

$$V = \frac{F}{2a}[y_1^2 + (y_2-y_1)^2 + (y_2-y_3)^2 + y_3^2] = \frac{F}{a}(y_1^2 + y_2^2 + y_3^2 - y_1y_2 - y_2y_3) \quad .$$

We note with mild irritation the presence of the cross-terms y_1y_2, y_2y_3.

Apply Lagrange's equation in turn to the three coordinates:

$$am\ddot{y}_1 + F(2y_1 - y_2) = 0, \tag{17.8.2}$$

$$am\ddot{y}_2 + F(-y_1 + 2y_2 - y_3) = 0, \tag{17.8.3}$$

$$am\ddot{y}_3 + F(-y_2 + 2y_3) = 0. \tag{17.8.4}$$

Seek solutions of the form $\ddot{y}_1 = -\omega^2 y_1, \ddot{y}_2 = -\omega^2 y_2, \ddot{y}_2 = -\omega^2 y_3$.

Then

$$(2F - am\omega^2)y_1 - Fy_2 = 0, \tag{17.8.5}$$

$$-Fy_1 + (2F - am\omega^2)y_2 - Fy_3 = 0, \tag{17.8.6}$$

$$-Fy_2 + (2F - am\omega^2)y_3 = 0. \tag{17.8.7}$$

Putting the determinant of the coefficients to zero gives an equation for the frequencies of the normal modes. The solutions are:

Slow	Medium	Fast
$\omega_1^2 = \frac{(2-\sqrt{2}F}{am}$	$\omega_2^2 = \frac{2F}{am}$	$\omega_3^2 = \frac{(2-\sqrt{2}F}{am}$

Substitution of these into equations 17.8.5 to 17.8.7 gives the following displacement ratios for these three modes:

$$y_1 : y_2 : y_3 = 1 : \sqrt{2} : 1 \qquad 1 : 0 : -1 \qquad 1 : \sqrt{2} : 1$$

These are illustrated in Figure XVII.8.

As usual, the general motion is a linear combination of the normal modes, the relative amplitudes and phases of the modes depending upon the initial conditions.

If the motion of the first mass is a combination of the three modes with relative amplitudes in the proportion $\hat{q}_1 : \hat{q}_2 : \hat{q}_3$ and $\alpha_1 : \alpha_2 : \alpha_3$ with initial phases its motion is described by

$$y_1 = \hat{q}_1 \sin(\omega_1 t + \alpha_1) + \hat{q}_2 \sin(\omega_2 t + \alpha_2) + \hat{q}_3 \sin(\omega_3 t + \alpha_3) \tag{17.8.8}$$

The motions of the second and third masses are then described by

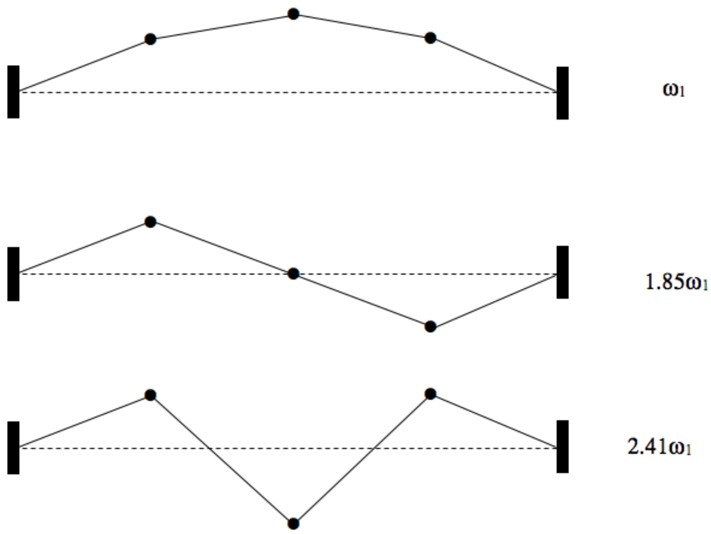


FIGURE XVII.8

$$\sqrt{2}\hat{q}_1 \sin(\omega_1 t + \alpha_1) \tag{17.8.9}$$

and

$$y_3 = \hat{q_1} \sin(\omega_1 t + \alpha_1) - \hat{q_2} \sin(\omega_2 t + \alpha_2) + \hat{q_3} \sin(\omega_3 t + \alpha_3). \tag{17.8.10}$$

These can be written

$$y_1 = q_1 + q_2 + q_3, \tag{17.8.11}$$

$$y_1 = \sqrt{2}q_1 - \sqrt{2}q_3 \tag{17.8.12}$$

and

$$y_1 = q_1 - q_2 + q_3, \tag{17.8.13}$$

where the q_i, like the y_i, are time-dependent coordinates.

We could, if we wish, express the q_i in terms of the y_i, by solving these equations:

$$q_1 = \frac{1}{4}(y_1 + \sqrt{2}y_2 + y_3), \tag{17.8.14}$$

$$q_2 = \frac{1}{2}(y_1 - y_3) \tag{17.8.15}$$

and

$$q_3 = \frac{1}{4}(y_1 - \sqrt{2}y_2 + y_3). \tag{17.8.16}$$

We have hitherto described the state of the system as a function of time by giving the values of the coordinates y_1, y_2 and y_3. We could equally well, if we wished, describe the state of the system by giving, instead, the values of the coordinates q_1, q_2 and q_3. Indeed it turns out that it is very useful to do so, and these coordinates are called the *normal coordinates*, and we shall see that they have some special properties. Thus, if you express the kinetic and potential energies in terms of the normal coordinates, you get

$$T = \frac{1}{2}m(4\dot{q}_1^2 + 2\dot{q}_2^2 + 4\dot{q}_3^2) \tag{17.8.17}$$

and

$$V = \frac{2F}{a}[(2-\sqrt{2})q_1^2 + q_2^2 + (2+\sqrt{2})q_3^2]. \tag{17.8.18}$$

Note that there are *no cross terms*. When you apply Lagrange's equation in turn to the three normal coordinates, you obtain

$$am\ddot{q}_1 = -(2-\sqrt{2})Fq_1, \tag{17.8.19}$$

$$am\ddot{q}_2 = -2Fq_2, \tag{17.8.20}$$

and

$$am\ddot{q}_3 = -(2+\sqrt{2})Fq_3. \tag{17.8.21}$$

Notice that the normal coordinates have become completely separated into three independent equations and that each is of the form $\ddot{q} = -\omega^2 q$ and that each of the normal coordinates oscillates with one of the frequencies of the normal modes. Much of the art of solving problems involving vibrating systems concerns identifying the normal coordinates.

17.9: Vibrating String

It is possible that the three modes of vibration of the three masses in Section 17.8 reminded you of the fundamental and first two harmonic vibrations of a stretched string – and it is quite proper that it did. If you were to imagine ten masses attached to a stretched string and to carry out the same sort of analysis, you would find ten normal modes, of which one would be quite like the fundamental mode of a stretched string, and the remainder would remind you of the first nine harmonics. You could continue with the same analysis but with a very large number of masses, and eventually you would be analysing the vibrations of a continuous heavy string. We do that now, and we assume that we have a heavy, taut string of mass μ per unit length, and under a tension F.

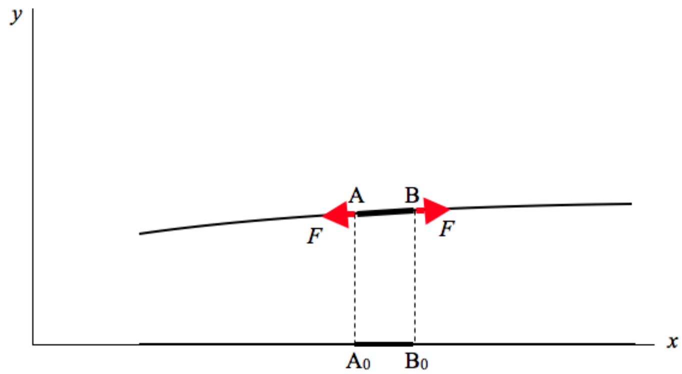


FIGURE XVII.9

I show in Figure XVII.9 a portion of length δx of a vibrating rope, represented by A_0B_0 in its equilibrium position and by AB in a displaced position. The rope makes an angle ψA with the horizontal at A and an angle ψ B with the horizontal at B. The tension in the rope is F. The vertical equation of motion is

$$F(\sin\psi_B - \sin\psi_A) = \mu\delta x \frac{\partial^2 y}{\partial t^2}. \tag{17.9.1}$$

If the angles are small, then $\sin\psi \cong \frac{\partial y}{\partial x}$, so the expression in parenthesis is $\frac{\partial^2 y}{\partial x^2}\delta x$. The equation of motion is therefore

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{17.9.2}$$

where

$$c = \sqrt{\frac{T}{\mu}} \tag{17.9.3}$$

As can be verified by substitution, the general solution to this is of the form

$$y = f(x - ct) + g(x + ct) \tag{17.9.4}$$

This represents a function that can travel in either direction along the rope at a speed c given by Equation 17.9.3. Should the disturbance be a periodic disturbance, then a wave will travel along the rope at that speed. Further analysis of waves in ropes and strings is generally done in chapters concerned with wave motion. This section, however, at least establishes the speed at which a disturbance (periodic or otherwise) travels along a stretched strong or rope.

17.10: Water

Water consists of a mass M ("oxygen") connected to two smaller equal masses m ("hydrogen") by two equal springs of force constants k, the angle between the springs being 2θ. The equilibrium length of each spring is r. The torque needed to increase the angle between the springs by $2\delta\theta$ is $2c\delta\theta$. See Figure XVII.10. (θ is about 52°.)

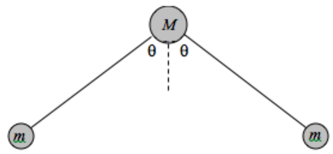


FIGURE XVII.10

At any time, let the coordinates of the three masses (from left to right) be

$$(x_1, y_1), \qquad (x_2, y_2), \qquad (x_3, y_3)$$

and let the equilibrium positions be

$$(x_{10}, y_{10}), \qquad (x_{20}, y_{20}), \qquad (x_{30}, y_{30}), \text{ where} y_{30} = y_{10}$$

We suppose that these coordinates are referred to a frame in which the centre of mass of the system is stationary.

Let us try and imagine, in Figure XVII.11, the vibrational modes. We can easily imagine a mode in which the angle opens and closes symmetrically. Let is resolve this mode into an x-component and a y-component. In the x-component of this motion, one hydrogen atom moves to the right by a distance q_1 while the other moves to the left by and equal distance q_1. In the y-component of this symmetric motion, both hydrogens move upwards by a distance q_2, while, in order to keep the centre of mass of the system unmoved, the oxygen necessarily moves down by a distance $2mq_2/M$. We can also imagine an asymmetric mode in which one spring expands while the other contracts. One hydrogen moves down to the left by a distance q_3, while the other moves up to the left by the same distance. In the meantime, the oxygen must move to the right by a distance $(2mq_3 \sin\theta)/M$, in order to keep the centre of mass unmoved.

We are going to try to write down the kinetic and potential energies in terms of the internal coordinates q_1, q_2 and q_3.

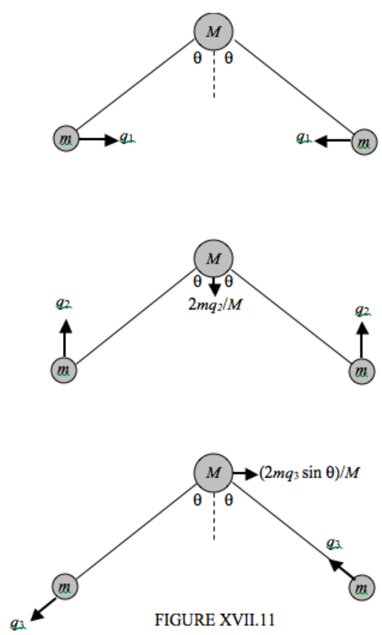


FIGURE XVII.11

It is easy to write down the kinetic energy in terms of the (x, y) coordinates:

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}M(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}m(\dot{x}_3^2 + \dot{y}_3^2). \tag{17.10.1}$$

From geometry we have:

$$\dot{x}_1 = \dot{q}_1 - \dot{q}_3 \sin\theta \tag{17.10.2}$$

$$\dot{y}_1 = \dot{q}_2 - \dot{q}_3 \cos\theta \tag{17.10.3}$$

$$\dot{x}_2 = \frac{2m\dot{q}_3 \sin\theta}{M} \tag{17.10.4}$$

$$\dot{y}_2 = -\frac{2m\dot{q}_2}{M} \tag{17.10.5}$$

$$\dot{x}_3 = -\dot{q}_1 - \dot{q}_3 \sin\theta \tag{17.10.6}$$

$$\dot{y}_3 = \dot{q}_2 + \dot{q}_3 \cos\theta \tag{17.10.7}$$

On putting these into equation 17.10.1we obtain

$$T = m\dot{q}_1^2 + m\left(1 + \frac{2m}{M}\right)\dot{q}_2^2 + m\left(1 + \frac{(2m\sin^2\theta)}{M}\right)\dot{q}_3^2 \tag{17.10.8}$$

For short, I am going to write this as

$$T = a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + a_{33}\dot{q}_3^2 \tag{17.10.9}$$

Now for the potential energy.

The extension of the left hand spring is

$$\begin{aligned} \delta r_1 &= -q_1 \sin\theta - q_2\cos\theta - \frac{2mq_2\cos\theta}{M} + q_3 + \frac{2mq_3\sin\theta\cos\theta}{M} \\ &= -q_1\sin\theta - q_2\left(\frac{1+2m}{M}\right)\cos\theta + q_3\left(1 + \frac{(2m\sin^2\theta)}{M}\right) \end{aligned} \tag{17.10.10}$$

The extension of the right hand spring is

$$\begin{aligned} \delta r_2 &= -q_1 \sin\theta - q_2\cos\theta - \frac{2mq_2\cos\theta}{M} - q_3 - \frac{2mq_3\sin^2\theta}{M} \\ &= -q_1\sin\theta - q_2\left(\frac{1+2m}{M}\right)\cos\theta - q_3\left(1 + \frac{(2m\sin^2\theta)}{M}\right). \end{aligned} \tag{17.10.11}$$

The increase in the angle between the springs is

$$2\delta\theta = -\frac{2q_1\cos\theta}{r} + \frac{2(1+\frac{2m}{M})q_2\sin\theta}{r}. \tag{17.10.12}$$

The potential energy (above the equilibrium position) is

$$V = \frac{1}{2}k(\delta r_1)^2 + \frac{1}{2}k(\delta r_2)^2 + \frac{1}{2}c(2\delta\theta)^2. \tag{17.10.13}$$

On substituting Equations 17.10.10, 17.10.11and 17.10.12into this, we obtain an equation of the form

$$V = b_{11}q_1^2 + 2bq_{12}q_1q_2 + b_{22}q_2^2 + b_{33}q_3^2, \tag{17.10.14}$$

where I leave it to the reader, if s/he wishes, to work out the detailed expressions for the coefficients. We still have a cross term, so we can't completely separate the coordinates, but we can easily apply Lagrange's equation to Equations 17.10.9and 17.10.14, and then seek simple harmonic solutions in the usual way. Setting the determinant of the coefficients to zero leads to the following equation for the angular frequencies of the normal modes:

$$\begin{bmatrix} b_{11} - \omega^2 a_{11} & b_{12} & 0 \\ b_{12} & b_{22} - \omega^2 a_{22} & 0 \\ 0 & 0 & b_{33} - \omega^2 a_{33} \end{bmatrix} = 0. \tag{17.10.15}$$

Thus, given the masses and r, θ, k and c, one can predict the frequencies of the normal modes. Can one calculate k and c given the frequencies? I do not know, to tell the truth. Can I leave it to the reader to investigate further?

17.11: A General Vibrating System

For convenience, I'll refer to a collection of masses connected by springs as a "molecule", and the individual masses as "atoms". In a molecule with N atoms, the number of degrees of vibrational freedom (the number of normal modes of vibration) $n = 3N - 6$ for nonlinear molecules, or $n = 3N - 5$ for linear molecules. Three equations are needed to express zero translational motion, and three (or two) are needed to express zero rotational motion.

> While reading this Section, it might be worthwhile for the reader to follow at the same time the treatment given to the OCS molecule in Section 17.6. Bear in mind, however, that in that section we did not consider the possibility of the molecule bending. Indeed we treated the molecule as if it were constrained inside a drinking straw, and it remained linear at all times. That being the case only N coordinates (rather than $3N$) suffice to describe the state of the molecule. Only one equation is needed to express zero translational motion, and none are needed to express zero rotational motion. Thus there are $N - 1$ internal coordinates, and hence $N - 1$ normal vibrational modes. In the case of OCS, $N = 3$, so there are two normal vibrational modes.

A molecule with n degrees of vibrational freedom can be described at some instant of time by n internal coordinates q_i. A typical such coordinate may be related to the external coordinates of two atoms, for example, by some expression of the form $q = x_2 - x_1 - a$, as we saw in our example of the molecule OCS. Its potential energy can be written in the form

$$\begin{aligned} 2V =& \kappa_{11}q_1^2 + \kappa_{12}q_1q_2 + \ldots \quad + \kappa_{1n}q_1q_n \\ &+ \kappa_{21}q_2q_1 + \kappa_{22}q_2^2 \ldots \quad + \kappa_{2n}q_2q_n \\ &+ \ldots \\ &+ \kappa_{n1}q_nq_1 + \kappa_{12}q_nq_2 \ldots \quad + \kappa_{nm}q_nq_n. \end{aligned} \tag{17.11.1}$$

Unless the q are the judiciously chosen "normal coordinates" (see our example of the transverse vibrations of three masses on an elastic string), there will in general be cross terms, such as q_1q_2. If both qs of a term are linear displacements, the corresponding κ is a force constant (dimensions MT^{-2}). If both qs are angles, κ is a torsion constant (dimensions ML^2T^{-2}. If one is a linear displacement and he other is an angular displacement, κ will be a coefficient of dimensions MLT^{-2}.

The matrix is symmetric, so that Equation 17.11.1could also be written

$$\begin{aligned} 2V =& \kappa_{11}q_1^2 + \kappa_{12}q_1q_2 + \ldots \quad + 2\kappa_{1n}q_1q_n \\ &+ \kappa_{22}q_2^2 \quad + \ldots \quad + 2\kappa_{2n}q_2q_n \\ &+ \ldots \\ &\qquad\qquad + \kappa_{nm}q_nq_n. \end{aligned} \tag{17.11.2}$$

In matrix notation, the Equation (i.e. Equations 17.11.1or 17.11.2) could be written:

$$2V = \tilde{\mathbf{q}}\kappa\mathbf{q}. \tag{17.11.3}$$

or in vector/tensor notation,

$$2V = \mathbf{q} \cdot \kappa\mathbf{q}. \tag{17.11.4}$$

The kinetic energy can be written in terms of the time rates of change of the external coordinates x_i:

$$2T = m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + \ldots + m_{3N}\dot{x}_{3N}^2. \tag{17.11.5}$$

To make use of the Lagrangian equations of motion, we need to express V and T in terms of the same coordinates, and it is usually advantageous if these be the n internal coordinates rather than the $3N$ external coordinates – so that we have to deal with only n rather than $3N$ lagrangian equations. (Recall that $n = 3N - 6$ or 5.) The relations between the external and internal coordinates are given as a set of equations that express a choice of coordinates such that there is no pure translation and no pure rotation of the molecule. These equations are of the form

$$\mathbf{q} = \mathbf{A}\mathbf{x}. \tag{17.11.6}$$

Here $\mathbf{q}$ is an $n \times 1$ column matrix, x a $3N \times 1$ column matrix, and $\mathbf{A}$ is a matrix with n rows and $3N$ columns, and it may need a little trouble to set up. We could then use this to express V in terms of the external coordinates, so we would then have both V and

T in terms of the external coordinates. We could then apply Lagrange's equation to each of the $3N$ external coordinates and arrive at $3N$ simultaneous differential equations of motion.

A better approach is usually to set up the equations connecting $\dot{q}$ and $\dot{x}$:

$$\dot{\mathbf{q}} = \mathbf{B}\dot{\mathbf{x}}. \tag{17.11.7}$$

(These correspond to Equations 17.6.3 and 17.6.4 in our example of the linear triatomic molecule in Section 17.6.) We then want to invert Equations 17.11.7 in order to express $\dot{x}$ in terms of $\dot{q}$. But we can't do this, because $\mathbf{B}$ is not a square matrix. $\dot{x}$ has $3N$ elements while $\dot{q}$ has only n. We have to add an additional six (or five for linear molecules) equations to express zero pure translational and zero pure rotational motion. This adds a further 6 or 5 rows to $\mathbf{B}$, so that $\mathbf{B}$ is now square (this corresponds to Equation 17.6.6), and we can then invert Equation 17.11.7:

$$\dot{\mathbf{x}} = \mathbf{B}^{-1}\dot{\mathbf{q}} \tag{17.11.8}$$

(This corresponds to Equation 17.6.7.)

By this means we can express the kinetic energy in terms of the time rates of change of only the n internal coordinates:

$$\begin{aligned} 2T = & \mu_{11}\dot{q}_1^2 + \mu_{12}\dot{q}_1\dot{q}_2 + \ldots \quad + \mu_{1n}\dot{q}_1\dot{q}_n \\ & + \mu_{21}\dot{q}_2\dot{q}_1 + \mu_{22}\dot{q}_2^2 \ldots \quad + \mu_{2n}\dot{q}_2\dot{q}_n \\ & + \ldots \\ & + \mu_{n1}\dot{q}_n\dot{q}_1 + \mu_{12}\dot{q}_n\dot{q}_2 \ldots \quad + \mu_{nm}\dot{q}_n\dot{q}_n \end{aligned} \tag{17.11.9}$$

Since the matrix is symmetric, the equation could also be written in a form analogous to Equation 17.11.2. The equation can also be written in matrix notation as

$$2T = \tilde{\dot{\mathbf{q}}}\mu\dot{\mathbf{q}}. \tag{17.11.10}$$

or in vector/tensor notation,

$$2T = \dot{\mathbf{q}} \cdot \mu\dot{\mathbf{q}}. \tag{17.11.11}$$

Here the μ_{ij} are functions of the masses. If both qs in a particular term have the dimensions of a length, the corresponding μ and κ will have dimensions of mass and force constant. If both qs are angles, the corresponding μ and κ will have dimensions of rotational inertia and torsion constant.. If one q is a length and the other is an angle, the corresponding μ and κ will have dimensions ML and MLT^{-2}.

Apply Lagrange's equation successively to $q_1, \ldots, q_n$ to obtain n equations of the form

$$\mu_{11}\ddot{q}_1 + \ldots + \mu_{1n}\ddot{q}_n + \kappa_{11}q_1 + \ldots + \kappa_{1n}q_n = 0. \tag{17.11.12}$$

That is to say

$$\mu\ddot{\mathbf{q}} = -\kappa\mathbf{q}. \tag{17.11.13}$$

Seek simple harmonic solutions of the form $\ddot{q} = -\omega^2 q$

and we obtain n equations of the form

$$(\kappa_{11} - \mu_{11}\omega^2)cq_1 + \ldots + (\kappa_{1n} - \mu_{1n}\omega^2)q_n = 0. \tag{17.11.14}$$

The frequencies of the normal modes can be obtained by equating the determinant of the coefficients to zero, and hence the displacement ratios can be determined.

If N is large, this could be a formidable task. The work can be very much reduced by making use of symmetry relations of the molecule, in which case the determinant of the coefficients may be factored into a number of much smaller subdeterminants. Further, if the configuration of the molecule could be expressed in terms of *normal coordinates* (combinations of the internal coordinates) such that the potential energy contained no cross terms, the equations of motion for each normal coordinate would be in the form $\ddot{q} = -\omega^2 q$.

17.12: A Driven System

It would probably be useful before reading this and the next section to review Chapters 11 and 12.

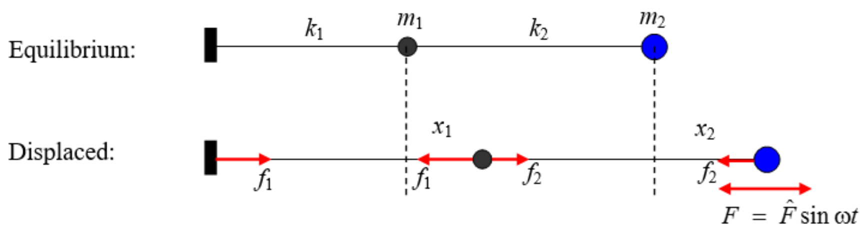


FIGURE XVII.12

Figure XVII.12 shows the same system as figure XVII.2, except that, instead of being left to vibrate on its own, the second mass is subject to a periodic force $F = \hat{F}\sin\omega t$. For the time being, we'll suppose that there is no damping. Either way, it is not a conservative force, and Lagrange's equation will be used in the form of Equation 13.4.12. As in Section 17.2, the kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{17.12.1}$$

Lagrange's equations are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_1} - \frac{\partial T}{\partial x_1} = P_1 \tag{17.12.2}$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_2} - \frac{\partial T}{\partial x_2} = P_2. \tag{17.12.3}$$

We have to identify the generalized forces P_1 and P_2.

In the nonequilibrium position, the extension of the left hand spring is x_1 and so the tension in that spring is $f_1 = k_1x_1$. The extension of the right hand spring is $x_2 - x_2$ and so the tension in that spring is $f_2 = k_2(x_2 - x_1)$. If x_1 were to increase by δx_1, the work done on m_1 would be $(f_2 - f_1)\delta x_1$ and therefore the generalized force associated with the coordinate x_1 is $P_1 = k_2(x_2 - x_1) - k_1x_1$. If x_2 were to increase by δx_2, the work done on m_2 would be $(F - f_2)\delta x_2$ and therefore the generalized force associated with the coordinate x_2 is $P_2 = \hat{F}\sin\omega t - k_2(x_2 - x_1)$. The lagrangian equations of motion therefore become

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \tag{17.12.4}$$

and

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = \hat{F}\sin\omega t. \tag{17.12.5}$$

Seek solutions of the form $\ddot{x}_1 = -\omega^2x_1$ and $\ddot{x}_2 = -\omega^2x_2$. The equations become

$$(k_1 + k_2 - m_1\omega^2)x_1 - k_2x_2 = 0 \tag{17.12.6}$$

and

$$-k_2x_1 + (k_2 - m_2\omega^2)x_2 = \hat{F}\sin\omega t. \tag{17.12.7}$$

We do <u>not</u>, of course, now equate the determinants of the coefficients to zero (why not?!), but we can solve these equations to obtain

$$x_1 = \frac{k_2\hat{F}\sin\omega t}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2} \tag{17.12.8}$$

and

$$x_2 = \frac{(k_1+k_2-m_1\omega^2)\hat{F}\sin\omega t}{(k_1+k_2-m_1\omega^2)(k_2-m_2\omega^2)-k_2^2}. \tag{17.12.9}$$

The amplitudes of these motions (and how they vary with the forcing frequency ω) are

$$\hat{x}_1 = \frac{k_2\hat{F}}{m_1m_2\omega^4 - (m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} \tag{17.12.10}$$

and

$$\hat{x}_2 = \frac{(k_1+k_2-m_1\omega^2)\hat{F}}{m_1m_2\omega^4 - (m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} \tag{17.12.11}$$

where I have re-written the denominators in the form of a quadratic expression in ω^2.

For illustration I draw, in figure XVII.13, the amplitudes of the motion of m_1(continuous curve, in black) and of m_2(dashed curve, in blue) for the following data:

$$\hat{F}=1,\ k_1=k_2=1,\ m_1=3,\ m_2=2,$$

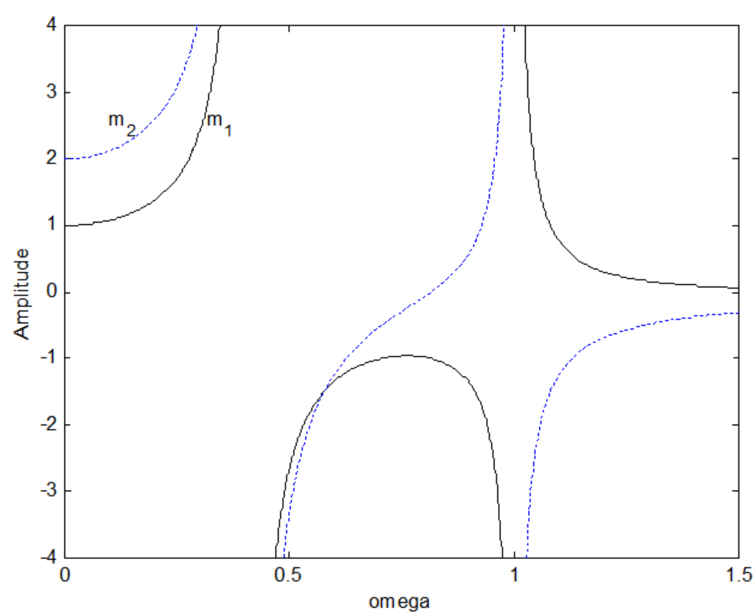
when the equations become

$$\hat{x}_1 = \frac{1}{6\omega^4-7\omega^2+1} = \frac{1}{(6\omega^2-1)(\omega^2-1)} \tag{17.12.12}$$

and

$$\hat{x}_1 = \frac{2-3\omega^2}{6\omega^4-7\omega^2+1} = \frac{2-3\omega^2}{(6\omega^2-1)(\omega^2-1)} \tag{17.12.13}$$

FIGURE XVII.13



Where the amplitude is negative, the oscillations are out of phase with the force F. The amplitudes go to infinity (remember we are assuming here zero damping) at the two frequencies where the denominators of Equations 17.12.10 and 17.12.11 are zero. The amplitude of the motion of m_2 is zero when the numerator of Equation 17.12.11 is zero. This is at an angular frequency of $\sqrt{\frac{(k_1+k_2)}{m_1}}$, which is just the angular frequency of the motion of m_1 held by the two springs between two fixed points. In our numerical example, this is $\omega = \sqrt{\frac{2}{3}} = 0.8165$. This is an example of *antiresonance*.

17.13: A Damped Driven System

I'll leave the reader to add some damping to the system described in Section 17.12. Let us here try it with the system described in Section 17.7. We'll apply a periodic force to the left hand mass, and we'll suppose that the damping constant for each mass is $\gamma = \frac{b}{m}$. We could write the periodic force as $F = \hat{F}\sin\omega t$, but the algebra will be easier if we write it as $F = \hat{F}e^{i\omega t}$. If the initial condition is such that $F = 0$ when $t = 0$, then we choose just the imaginary part of this in subsequent expressions.

The equations of motion are

$m\ddot{x}$ = - the damping force $b\dot{x}$

- the tension in the left hand spring k_1x

+ the force F

+ the tension in the middle spring $k_2(y-x)$

(this last is a thrust whenever $y < x$)

and

$m\ddot{y}$ - the damping force $b\dot{y}$

- the thrust in the right hand spring k_1y

- the tension in the middle spring $k_2(y-x)$

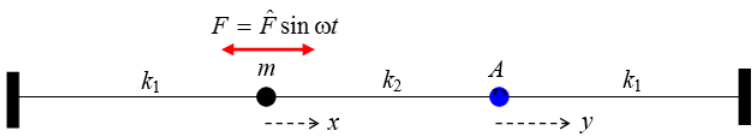


FIGURE XVII.14

That is,

$$m\ddot{x} + b\dot{x} + (k_1+k_2)x - k_2y = \hat{F}e^{i\omega t} \tag{17.13.1}$$

and

$$m\ddot{y} + b\dot{y} + (k_1+k_2)y - k_2y = 0. \tag{17.13.2}$$

For the steady-state motion, seek solutions of the form

$$\ddot{x} = -\omega^2 x,\ \ddot{y} = -\omega^2 y, \text{ so that } \dot{x} = i\omega x \text{ and } \dot{y} = i\omega y.$$

The equations then become

$$(k_1 + k_2 - m\omega^2 + ib\omega)x - k_2y = \hat{F}e^{i\omega t} \tag{17.13.3}$$

and

$$-k_2x + (k_1 + k_1 - m\omega^2 + ib\omega)y = 0. \tag{17.13.4}$$

There is now a little algebra to be carried out. Solve these equations for x and y, and when, in doing so, there is a complex number in the denominator, multiply top and bottom by the conjugate in the usual way, so as to get x and y in the forms $x' + ix''$ and $y' + iy''$. Then find expressions for the amplitudes $\hat{x}$ and $\hat{y}$. After some algebra, the amount of which depends on one's skill, experience and luck (it is not always obvious how to gather terms in the most economical way, and you need some luck in this) you eventually get, for the amplitudes of the motion

$$\hat{x}^2 = \frac{((k_1 + k_2 - m\omega^2) + b^2\omega^2)\hat{F}^2}{((k_1 - m\omega^2)^2 + b^2\omega^2)((k_1 + 2k_2 - m\omega^2)^2 + b^2\omega^2)} \tag{17.13.5}$$

and

$$\hat{y}^2 = \frac{k_2^2 \hat{F}^2}{((k_1 - m\omega^2)^2 + b^2\omega^2)((k_1 + 2k_2 - m\omega^2)^2 + b^2\omega^2)}. \tag{17.13.6}$$

There are many variables in these expressions, but in order to see qualitatively what the steady state motion is like, I'm going to put $\hat{F}$, m and $k_1 = 1$. I think if I also put $b = 1$, this will give light damping in the sense described in Chapter 11. As for k_2, I am going to introduce a coupling coefficient α defined by $\alpha = \frac{k_2}{k_1 + k_2}$ or $k_2 = \left(\frac{\alpha}{1-\alpha}\right) k_1$. This coupling constant will be close to zero if the middle spring is very weak, and 1 if the middle connector is a rigid rod. The equations now become

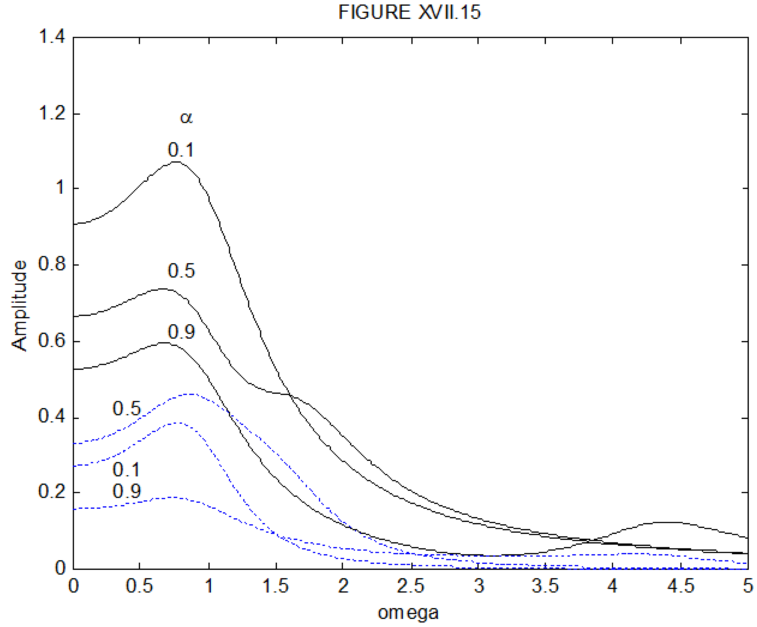
$$\hat{x}^2 = \frac{((\frac{1}{1-\alpha} - \omega^2)^2 + \omega^2)}{((1-\omega^2)^2 + \omega^2)(\frac{1+\alpha}{1-\alpha} - \omega^2)^2 + \omega^2)}. \tag{17.13.7}$$

and

$$\hat{y}^2 = \frac{\frac{\alpha}{(1-\alpha)}}{((1-\omega^2)^2 + \omega^2)(\frac{1+\alpha}{1-\alpha} - \omega^2)^2 + \omega^2)} \tag{17.13.8}$$

For computational efficiency you might want to rewrite these equations a little. For example you could write $(1-\omega^2)^2 + \omega^2$ as $1 - \Omega(1-\Omega)$, where $\Omega = \omega^2$. In any case, figure XVII.15 shows the amplitudes of the motions of the two masses as a function of frequency, for $\alpha = 0.1,\ 0.5$ and 0.9. The continuous black curves are for the left hand mass; the dashed blue curve is for the right hand mass.

FIGURE XVII.15



CHAPTER OVERVIEW

18: The Catenary

If a flexible chain or rope is loosely hung between two fixed points, it hangs in a curve that looks a little like a parabola, but in fact is not quite a parabola; it is a curve called a *catenary*, which is a word derived from the Latin *catena*, a chain.

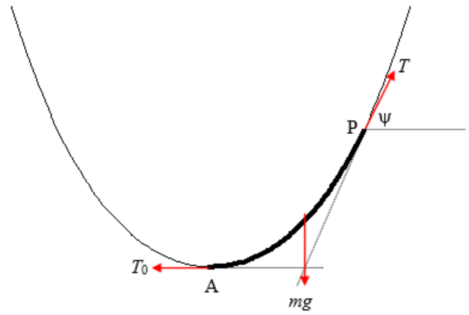
18.1: Introduction

If a flexible chain or rope is loosely hung between two fixed points, it hangs in a curve that looks a little like a parabola, but in fact is not quite a parabola; it is a curve called a *catenary*, which is a word derived from the Latin *catena*, a chain.

18.2: The Intrinsic Equation to the Catenary

We consider the equilibrium of the portion AP of the chain, A being the lowest point of the chain (Figure XVIII.1).

FIGURE XVIII.1



It is in equilibrium under the action of three forces: The horizontal tension T_0 at A; the tension T at P, which makes an angle ψ with the horizontal; and the weight of the portion AP. If the mass per unit length of the chain is μ and the length of the portion AP is s, the weight is $\mu s\mathrm{g}$. It may be noted than these three forces act through a single point.

Clearly,

$$T_0 = T\cos\psi \tag{18.2.1}$$

and

$$\mu s\mathrm{g} = T\sin\psi, \tag{18.2.2}$$

from which

$$(\mu s\mathrm{g})^2 + T_0^2 = T^2 \tag{18.2.3}$$

and

$$\tan\psi = \frac{\mu\mathrm{g}s}{T_0} \tag{18.2.4}$$

Introduce a constant a having the dimensions of length defined by

$$a = \frac{T_0}{\mu\mathrm{g}}. \tag{18.2.5}$$

Then Equations 18.2.3and 18.2.4become

$$T = \mu\mathrm{g}\sqrt{s^2 + a^2} \tag{18.2.6}$$

and

$$s = a\tan\psi. \tag{18.2.7}$$

Equation 18.2.7is the *intrinsic equation* of the catenary.

18.3: Equation of the Catenary in Rectangular Coordinates, and Other Simple Relations

The slope at some point is $y' = \frac{dy}{dx} = \tan\psi = \frac{s}{a}$ from which $\frac{ds}{dx} = a\frac{d^2y}{dx^2}$. But, from the usual pythagorean relation between intrinsic and rectangular coordinates $ds = (1+y'^2)^{\frac{1}{2}}dx$, this becomes

$$(1+y'^2)^{\frac{1}{2}} = a\frac{dy'}{dx}. \tag{18.3.1}$$

On integration, with the condition that $y' = 0$ where $x = 0$, this becomes

$$y' = \sinh\left(\frac{x}{a}\right), \tag{18.3.2}$$

and, on further integration,

$$y = a\cosh\left(\frac{x}{a}\right) + C \tag{18.3.3}$$

If we fix the origin of coordinates so that the lowest point of the catenary is at a height a above the x-axis, this becomes

$$y = a\cosh\left(\frac{x}{a}\right) \tag{18.3.4}$$

This, then, is the x, y Equation to the catenary. The x-axis is the *directrix* of this catenary.

The following additional simple relations are easily derived and are left to the reader:

$$s = a\sinh\left(\frac{x}{a}\right) \tag{18.3.5}$$

$$y^2 = a^2 + s^2 , \tag{18.3.6}$$

$$y = a\sec\psi. \tag{18.3.7}$$

$$x = a\ln\,(\sec\psi + \tan\psi \tag{18.3.8}$$

$$T = \mu g y \tag{18.3.9}$$

Equations 18.3.7 and 18.3.8 may be regarded as parametric Equations to the catenary.

If one end of the chain is fixed, and the other is looped over a smooth peg, Equation 18.3.9 shows that the loosely hanging vertical portion of the chain just reaches the directrix of the catenary, and the tension at the peg is equal to the weight of the vertical portion.

? Exercise 18.3.1

By expanding Equation 18.3.4 as far as x^2, show that, near the bottom of the catenary, or for a tightly stretched catenary with a small sag, the curve is approximately a parabola. Actually, it does not matter what Equation 18.3.4 is – if you expand it as far as x^2, provided the x^2 term is not zero, you'll get a parabola – so, in order not to let you off so lightly, show that the semi latus rectum of the parabola is a.

? Exercise 18.3.2

Expand Equation 18.3.5 as far as x^3.

Now: let $2s$ = total length of chain, $2k$ = total span, and d = sag. Show that for a shallow catenary $s-k = k^3/(6a^3)$ and $k^2 = 2ad$ hence that length − span = $\frac{8}{3}$ sag^2/span.

✓ Example 18.3.1

A cord is stretched between points on the same horizontal level. How large a force must be applied so that the cord is no longer a catenary, but is accurately a straight line?

Answer:

There is no force however great

Can stretch a string however fine

Into a horizontal line

That shall be accurately straight.

I am indebted to Hamilton Carter of Texas A & M University for drawing my attention to a note by C. A. Chant in J. Roy. Astron. Soc. Canada 33, 72, (1939), where this doggerel is attributed to the early nineteenth century Cambridge mathematician William Whewell.

Exercise 18.3.3

And here's something for engineers. We, the general public, expect engineers to built safe bridges for us. The suspension chain of a suspension bridge, though scarcely shallow, is closer to a parabola than to a catenary. There is a reason for this. Discuss.

18.4: Area of a Catenoid

A theorem from the branch of mathematics known as the calculus of variations is as follows. Let $y = y(x)$ with $y' = dy/dx$ and let $f(y, y', x)$ be some function of y, y' and x. Consider the line integral of f from A to B along the route $y = y(x)$.

$$\int_A^B f(y, y', x) dx \tag{18.4.1}$$

In general, and unless f is a function of x and y alone, and not of y', the value of this integral will depend on the *route* (i.e. $y = y(x)$) over which this line integral is calculated. The theorem states that the integral is an extremum for a route that satisfies

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y} \tag{18.4.2}$$

By "extremum" we mean either a minimum or a maximum, or an inflection, though in many – perhaps most – cases of physical interest, it is a minimum. It can be difficult for a newcomer to this theorem to try to grasp exactly what this theorem means, so perhaps the best way to convey its meaning is to start by giving a simple example. Following that, I give an example involving the catenary. There will be another example, involving a famous problem in dynamics, in Chapter 19, and in fact we have already encountered an application of it in Chapter 14 in the discussion of Hamilton's variational principle.

Let us consider, for example, the problem of calculating the distance, measured along some route $y(x)$ between two points; that is, we want to calculate the arc length $\int ds$ From the usual pythagorean relation between ds, dx and dy, this is $\int (1 + y')^{1/2} dx$. The variational principle says that this distance – measured along $y(x)$ – is least for a route $y(x)$ that satisfies Equation 18.4.2, in which in this case $f = (1 + y')^{1/2}$

For this case, we have $\frac{df}{dy} = 0$ and $\frac{df}{dy} = \frac{y'}{(1+y')^{1/2}}$. Thus integration of Equation 18.4.2 gives

$$y' = c(1 + y'^2)^{1/2}, \tag{18.4.3}$$

where c is the integration constant. If we solve this for y', we obtain which is just another constant, which I'll write as a, so that $y' = a$. Integrate this to find

$$y = ax + b \tag{18.4.4}$$

This probably seems rather a long way to prove that the shortest distance between two points is a straight line – but that wasn't the point of the exercise. The aim was merely to understand the meaning of the variational principle.

Let's try another example, in which the answer will not be so obvious.

Consider some curve $y = y(x)$, and let us rotate the curve through an angle ϕ (which need not necessarily be a full (\2 \pi \) radians) about the y-axis. An element ds of the curve can be written as $\sqrt{1 + y'^2 dx}$ and the distance moved by the element ds (which is at a distance x from the y-axis) during the rotation is ϕx. Thus the area swept out by the curve is

$$A = \phi \int x\sqrt{1 + y'^2} dx. \tag{18.4.5}$$

For what shape of curve, $y = y(x)$, is this area least? The answer is – a curve that satisfies Equation 18.4.2, where $f = x\sqrt{1 = y'^2}$. For this function, we have $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial y} = \frac{xy'}{\sqrt{1+y'^2}}$.

Therefore the required curve satisfies

$$\frac{xy'}{\sqrt{1 + y'^2}} = a. \tag{18.4.6}$$

That is,

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}. \tag{18.4.7}$$

On substitution of $x = a\cosh\theta$ and looking up everything we have forgotten about hyperbolic functions, and integrating, we obtain

$$y = a\cosh(x/a). \tag{18.4.8}$$

Thus the required curve is a catenary.

If a soap bubble is formed between two identical horizontal rings, one beneath the other, it will take up the shape of least area, namely a catenoid.

CHAPTER OVERVIEW

19: The Cycloid

A cycloid generated by a rolling circle. (CC BY-SA 3.0; Zorgit).

19.1: Introduction to Cycloids

Let us set up a coordinate system Oxy, and a horizontal straight line $y = 2a$. We imagine a circle of diameter $2a$ between the x-axis and the line $y = 2a$, and initially the lowest point on the circle, P, coincides with the origin of coordinates O. We now allow the circle to roll counterclockwise without slipping on the line $y = 2a$, so that the centre of the circle moves to the right. As the circle rolls on the line, the point P describes a curve, which is known as a *cycloid*.

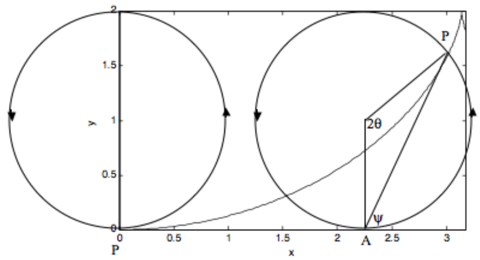


FIGURE XIX.1

When the circle has rolled through an angle 2θ, the centre of the circle has moved to the right by a horizontal distance $2a\theta$, while the horizontal distance of the point P from the centre of the circle is $a\sin 2\theta$ and the vertical distance of the point P below the centre of the circle is $a\cos 2\theta$. Thus the coordinates of the point P are

$$x = a(2\theta + \sin 2\theta) \tag{19.1.1}$$

and

$$y = a(1 - \cos 2\theta). \tag{19.1.2}$$

Equations 19.1.1 and 19.1.2 are the parametric equations of the cycloid. Using a simple trigonometric identity, Equation 19.1.2 can also be written

$$y = 2a\sin^2\theta. \tag{19.1.3}$$

✔ Example 19.1.1

When the x-coordinate of P is $2.500a$, what (to four significant figures) is its y-coordinate?

Solution

We have to find 2θ by solution of $2\theta + \sin 2\theta$. By Newton-Raphson iteration or otherwise, we find 2θ = 0.931 599 201 radians, and hence y = 0.9316a.

19.2: Tangent to the Cycloid

The slope of the tangent to the cycloid at P is dy/dx, which is equal to $dy/d\theta$, and these can be obtained from Equations 19.1.1 and 19.1.2.

? Exercise 19.2.1

Show that the slope of the tangent at P is tan θ. That is to say, the tangent at P makes an angle θ with the horizontal.

Having done that, now consider the following:

Let A be the lowest point of the circle. The angle ψ that AP makes with the horizontal is given by $\tan\psi = \frac{y}{x-2a\theta}$

? Exercise 19.2.2

Show that $\psi = \theta$. Therefore the line AP is the tangent to the cycloid at P; or the tangent at P is the line AP.

19.3: The Intrinsic Equation to the Cycloid

An element ds of arc length, in terms of dx and dy, is given by the theorem of Pythagoras: $ds = ((dx)^2 + (dy)^2))^{1/2}$ or, since x and y are given by the parametric Equations 19.1.1 and 19.1.2, by And of course we have just shown that the intrinsic coordinate ψ (i.e. the angle that the tangent to the cycloid makes with the horizontal) is equal to θ.

Exercise 19.3.1

Integrate ds (with initial condition s = 0, θ = 0) to show that the intrinsic equation to the cycloid is

$$s = 4a\sin\psi \tag{19.3.1}$$

Also, eliminate ψ (or θ) from Equations 19.3.1 and 19.1.2 to show that the following relation holds between arc length and height on the cycloid:

$$s^2 = 4ay. \tag{19.3.2}$$

19.4: Variations

In Sections 19.1,2,3, we imagined that the cycloid was generated by a circle that was rolling counterclockwise along the line $y = 2a$. We can also imagine variations such as the circle rolling clockwise along $y = 0$, or we can start with P at the top of the circle rather than at the bottom. I summarise in this section four variations. The distinction between ψ and θ is as follows. The angle that the tangent to the cycloid makes with the positively-directed x-axis is ψ; that is to say, $dx/dy = tan\psi$.The circle rolls through an angle 2θ. There is a simple relation between ψ and θ, which is different for each case.

In each figure, x and y are plotted in units of a. The vertical height between vertices and cusps is $2a$, the horizontal distance between a cusp and the next vertex is πa, and the arc length between a cusp and the next vertex is $4a$.

I. Circle rolls counterclockwise along $y = 2a$. P starts at the bottom. The cusps are up. A vertex is at the origin.

$$x = a(2\theta + \sin 2\theta)$$

$$y = 2a\sin^2\theta \tag{19.4.2}$$

$$s = 4a\sin\theta \tag{19.4.3}$$

$$^2 = 8ay \tag{19.4.4}$$

$$\psi = \theta. \tag{19.4.5}$$

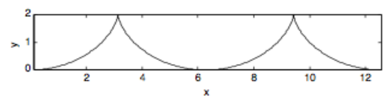


FIGURE XIX.2

II. Circle rolls clockwise along $y = 0$. P starts at the bottom. The cusps are down. A cusp is at the origin.

$$x = a(2\theta - sin2\theta) \tag{19.4.6}$$

$$y = 2asin^2\theta \tag{19.4.7}$$

$$s = 4a(1 - cos\theta) \tag{19.4.8}$$

$$s^2 = 8a(y - s) \tag{19.4.9}$$

$$\psi = 90\circ - \theta. \tag{19.4.10}$$

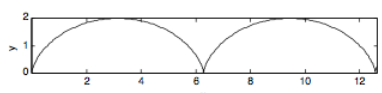


FIGURE XIX.3

III. Circle rolls clockwise along $y = 0$. P starts at the top. The cusps are down. A vertex is at $x = 0$.

$$x = a(2\theta + \sin 2\theta \tag{19.4.11}$$

$$y = 2a\cos^2\theta \tag{19.4.12}$$

$$s = 4a\sin\theta \tag{19.4.13}$$

$$s^2 = 8a(2a - y) \tag{19.4.14}$$

$$\psi = 180\circ - \theta. \tag{19.4.15}$$

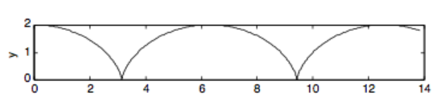


FIGURE XIX.4

IV. Circle rolls counterclockwise along $y = 2a$. P starts at the top. The cusps are up. A cusp is at $x = 0$.

$$x=a(2\theta-\sin 2\theta) \tag{19.4.16}$$

$$y=2a\cos^2\theta \tag{19.4.17}$$

$$s=4a(1-\cos\theta) \tag{19.4.18}$$

$$s^2-8as+8a(2a-y)=0 \tag{19.4.19}$$

$$\psi=90\circ+\theta \tag{19.4.20}$$

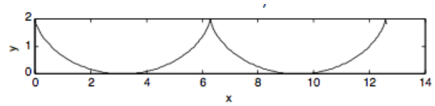


FIGURE XIX.5

19.5: Motion on a Cycloid, Cusps Up

We shall imagine either a particle sliding down the inside of a smooth cycloidal bowl, or a bead sliding down a smooth cycloidal wire, Figure XIX.6.

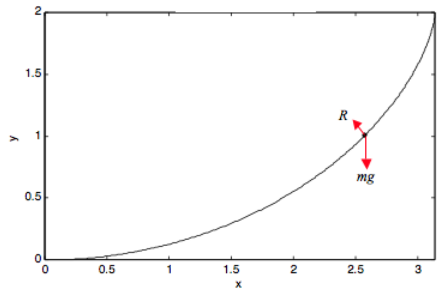


FIGURE XIX.6

We shall work in intrinsic coordinates to obtain the tangential and normal Equations of motion. These Equations are, respectively:

$$\ddot{s} = -g \sin\psi \tag{19.5.1}$$

and

$$\frac{mv^2}{\rho} = R - mg\cos\psi. \tag{19.5.2}$$

Here R is the normal (and only) reaction of the bowl or wire on the particle and ρ is the radius of curvature. The radius of curvature is $ds/d\psi$, which, from Equation 19.3.1, (or Equations 19.4.3 and 19.4.5) is

$$\rho = 4a\cos\psi \tag{19.5.3}$$

From Equations 19.3.1 and 19.5.1 we see that the tangential Equation of motion can be written, without approximation:

$$\ddot{s} = -\frac{g}{4a}s. \tag{19.5.4}$$

This is simple harmonic motion of period $4\pi\sqrt{a/g}$, independent of the amplitude of the motion. This is the *isochronous* property of the cycloid. Likewise, if the particle is released from rest, it will reach the bottom of the cycloid in a time $\pi\sqrt{a/g}$ whatever the starting position.

Let us see if we can find the value of R where the generating angle is ψ. Let us suppose that the particle is released from rest at a height y_0 above the x-axis (generating angle = ψ_0); what is its speed v when it has reached a height y (generating angle ψ)? Clearly this is given by

$$\frac{1}{2}mv^2 = mg(y_0 - y), \tag{19.5.5}$$

and, following Equation 19.3.2, and recalling that $\theta = \psi$, this is

$$v^2 = 2ga(cos2\psi - cos2\psi_0). \tag{19.5.6}$$

On substituting this and Equation 19.5.3 into Equation 19.5.2, we find for R:

$$R = \frac{mg}{2\cos\psi}(1 + 2\cos 2\psi - \cos 2\psi_0) \tag{19.5.7}$$

19.6: Motion on a Cycloid, Cusps Down

We imagine a particle sliding down the outside of an inverted smooth cycloidal bowl, or a bead sliding down a smooth cycloidal wire. We shall suppose that, at time $t = 0$, the particle was at the top of the cycloid and was projected forward with a horizontal velocity v_0. See Figure XIX.7.

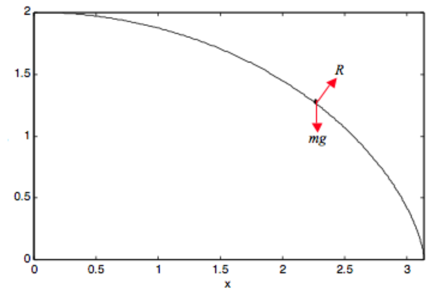


FIGURE XIX.7

This time, the equations of motion are

$$\ddot{s} = gsin\psi \tag{19.6.1}$$

and

$$\frac{mv^2}{\rho} = mg\cos\phi - R. \tag{19.6.2}$$

By arguments similar to those made in Section 19.5, we find that

$$\ddot{s} = \frac{gs}{4a} \tag{19.6.3}$$

The general solution to this is

$$s = Ae^{pt} + Be^{-pt}, \tag{19.6.4}$$

where

$$p = \sqrt{g/(2a)}. \tag{19.6.5}$$

With the initial condition given (at $t = 0$, $s = 0$, $\dot{s} = v_0$), we can find A and B and hence:

$$s = v_0\sqrt{\frac{a}{g}}(e^{pt} - e^{-pt}) \tag{19.6.6}$$

Again proceeding as in Section 19.5, we find for R:

$$R = \frac{m}{4\cos\psi}(4ga\cos 2\psi - v_0^2). \tag{19.6.7}$$

So – what happens?

If the constraint is *two-sided* (bead sliding on a wire) R becomes zero when $\cos 2\pi = v_0^2/(2/ga)$, and thereafter R is in the opposite direction.

If the constraint is *one-sided* (particle sliding down the outside of a smooth cycloidal bowl):

1. If $v_0^2 > 4ga$, the particle loses contact at the moment of projection.
2. If If $v_0^2 < 4ga$ the particle loses contact as soon as $\cos 2\pi = v_0^2/(2ga)$, is very small (i.e. very much smaller than $\sqrt(2ga)$), this will happen when $\psi = 45\circ$; for faster initial speeds, contact is lost sooner.

Example 19.6.1

A particle is projected horizontally with speed $v_0 = 1 \text{ m s}^{-1}$ from the vertex of the smooth cycloidal hill

$$x = a(2\theta + \sin 2\theta$$

$$y = 2a\cos^2\theta,$$

where $a = 2$ m. Assuming that $g = 9.8 \text{ m s}^{-2}$, how long does it take to get halfway down the hill (i.e. to $y = a$)?

Solution

We have to use Equation 19.6.6. With the numerical data given, this is

$$s = 0.451754(e^{1.565248t} - e^{-1.565248t}).$$

We can find s from Equation 19.4.12, which gives us s = 2.828427 m. If we let we now have to solve 6.26099 = $\xi - 1/\xi$, or $\xi^2 - 6.26099\xi - 1 = 0$. From this, ξ = 6.41683 and hence t = 1.19 s.

I leave it to the reader to calculate R at this time – and indeed to see whether the particle loses contact with the hill before then. Perhaps the fact that I got a positive real root for ξ means that we are all right and the particle is still in contact – but I wouldn't be sure of that. I leave it to the reader to investigate further.

19.7: The Brachystochrone Property of the Cycloid

A small point. The word is sometimes spelled brachistochrone, and I have no recommendation one way or the other. For what it's worth, the only dictionary within easy reach of my desk has brachiopod and brachycephalic. In any case, the word is derived from Greek, and means shortest time.

The famous brachystochrone problem problem is this: A smooth wire, which can be of any desired length, is to connect two points O and P; P is at a lower level than O, but is not vertically below O. The wire is to be bent to a shape, and cut to a length, such that the time taken for a bead to slide down the wire from O to P is least.

It is not easy to prove that the required curve is a cusps-up cycloid; but it is quite reasonable to speculate or to guess that this might be so. And, having speculated that it might be a cycloid, it is easy to verify that the required curve is indeed a cusps-up cycloid, the bead starting from rest at a cusp of the cycloid.

A speculation might go something like this. Generally one would expect that the further P is from O, the longer it will take for the bead to slide from O to P. But, if O and P are connected with a cycloidal wire, the time taken to go from O to P does not increase with distance. (See the isochronous property of the cycloid discussed in Section 19.5.) Thus, as you increase the distance between O and P, the time taken to travel by any route other than the cycloidal one must take longer than the cycloidal route. This argument may not sound like a rigorous proof, though it is enough to arouse our suspicions and to test whether it is correct.

Since I am going to deal with a bead sliding downwards under gravity, I am going to find it convenient to set up our coordinate axes such that x increases to the right, and y increases *downwards*. In that case, the parametric equations to a cusps-up cycloid, with the origin at a cusp, are

$$x = a(2\theta - \sin 2\theta) \tag{19.7.1}$$

and

$$y = 2a\sin^2\theta \tag{19.7.2}$$

− and these are the equations that we shall be testing.

The time taken for the bead to travel a distance ds along the wire, while it is moving at speed v is ds/v. In (x, y) coordinates, ds is $\sqrt{1+y'^2}\,dx$ where $y' = dy/dx$. Also, the speed reached is related (by equating the gain in kinetic energy to the loss of potential energy) to the vertical distance y dropped by $v = \sqrt{2gy}$.Thus the time taken to go from O to P is

$$\frac{1}{\sqrt{2g}} = \int_0^P \frac{\sqrt{1y'^2}}{\sqrt{y}}dx = \frac{1}{2g}\int_O^P f(y, y')dx. \tag{19.7.3}$$

This is least (see Chapter 18 for a discussion of this theorem from the calculus of variations) for a function $y(x)$ that satisfies

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}. \tag{19.7.4}$$

We have:

$$f = \frac{1+y'^2}{\sqrt{y}}, \tag{19.7.5}$$

$$\frac{\partial f}{\partial y} = -\frac{(1+y'^2)^{1/2}}{2y^{3/2}} \tag{19.7.6}$$

$$\frac{\partial f}{\partial y} = -\frac{y'}{y^{1/2}(1+y'^2)^{1/2}} \tag{19.7.7}$$

It is left for the reader to see whether equations 19.7.1 and 19.7.2 satisfy equation 19.7.4. You should find that both sides of the equation are equal to Thus our speculation is confirmed, and a cusps-up cycloid is indeed the curve that offers passage from O to P in the shortest time.

19.8: Contracted and Extended Cycloids

As in Section 19.1, we consider a circle of radius a rolling to the right on the line $y = 2a$. The point P is initially below the centre of the circle, but, instead of being on the rim of the circle, its distance from the centre of the circle is r. If $r < a$, the path described by P will be a **contracted cycloid**; if $r > a$, the path is an **extended cycloid**. (I think there's a case for using this nomenclature the other way round, but most authors seem to use "contracted" for $r < a$ and "extended" for $r > a$.) It should not take long to be convinced, by arguments similar to those in Section 19.1, that the parametric equations to a contracted or extended cycloid are

$$x = 2a\theta + r\sin 2\theta \tag{19.8.1}$$

and

$$y = a - r\cos 2\theta \tag{19.8.2}$$

These are illustrated in Figures XIX.8 and XIX.9 for a contracted cycloid with $r = 0.5a$ and an extended cycloid with $r = 1.5a$.

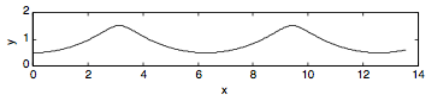


FIGURE XIX.8

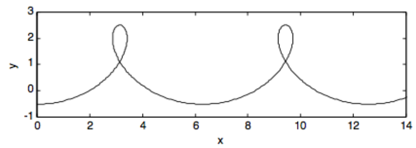


FIGURE XIX.9

19.9: The Cycloidal Pendulum

Let us imagine building a wooden construction in the shape of the cycloid shown with the thick line in Figure XIX.10.

$$x = a(2\theta - \sin 2\theta) \tag{19.9.1}$$

$$y = 2a\cos^2\theta \tag{19.9.2}$$

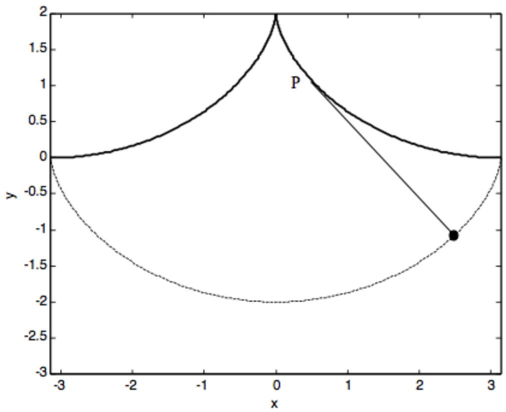


FIGURE XIX.10

Now suspend a pendulum of length $4a$ from the cusp, and allow it to swing to and fro, partially wrapping itself against the wooden frame as it does so. If the arc length from the cusp to P is s, then the length of the "free" string is $4a - s$, and so the coordinates of the bob at the end of the pendulum are

$$\begin{aligned} x &= a(2\theta - \sin 2\theta) + (4a - s)\cos(180^\circ - \psi) \\ &= a(2\theta - \sin 2\theta) + (4a - s)\sin\theta. \end{aligned} \tag{19.9.3}$$

and

$$\begin{aligned} y &= 2a\cos^2\theta - (4a - s)\sin(180^\circ - \psi) \\ &= 2a\cos^2\theta - (4a - s)\cos\theta \end{aligned} \tag{19.9.4}$$

(You will need to remind yourself of the exact meaning of ψ and also make use of Equation 19.4.20.) Now Equation 19.4.18 tells us that $s = 4a(1 - \cos\theta$, and, on substitution of this in equations 19.9.3 and 19.9.4, we find (after a very little algebra and trigonometry) for the parametric equations to the path described by the bob of the pendulum:

$$x = a(2\theta + \sin 2\theta) \tag{19.9.5}$$

and

$$y = -2a\cos^2\theta. \tag{19.9.6}$$

Thus the path of the pendulum bob (shown as a dashed line in Figure XIX.10) is a cycloid, and hence its period is independent of its amplitude. (Recall Section 19.5.) Thus the pendulum is *isochronous* or *tautochronous*. It is astonishing to learn that Huygens constructed just such a pendulum as long ago as 1673.

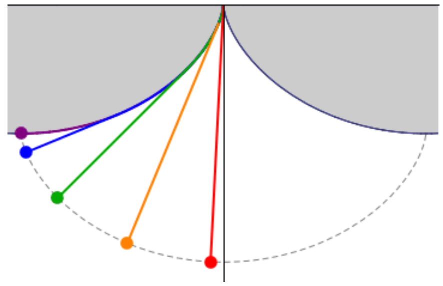


Figure XIX.10: Five isochronous cycloidal pendula with different amplitudes. All of them are isochronous, meaning they have the same frequency regardless of their amplitudes. Notice the two upper cycloidal arcs which make the bobs describe cycloidal trajectories. (CC BY-SA 4.0; Rem088roy via Wikipedia).

19.10: Examples of Cycloidal Motion in Physics

Several examples of cycloidal motion in physics come to mind. One is the nutation of a top, which is described in Section 4.10 of Chapter 10. Earth's axis nutates in a similar fashion. Another well known example is the motion of an electron in crossed electric and magnetic fields. This is described in Chapter 8 of the Electricity and Magnetism section of these notes. In cosmology, if the mean density of the Universe is low, the Universe expands indefinitely, but, if the density is higher than a certain critical density, the (dimensionless) scale factor R of the Universe expands and contracts with time t according to the following parametric cycloidal equations:

$$R=\frac{\Omega_0}{2(\Omega_0-1)}(1-\cos 2\theta), \tag{19.10.1}$$

$$t=\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}(2\theta-\sin 2\theta). \tag{19.10.2}$$

Here t is expressed in units of the reciprocal of the present Hubble constant, and Ω_0 is the ratio of the present density of the Universe to the density required to "close" the Universe.

A less well known example concerns the propagation of sound in the atmosphere. In the troposphere, which is the lower part of the atmosphere up to about 11 km, the temperature decreases roughly linearly with height. In that case sound travels through the troposphere in a cycloidal path. The speed of sound in a gas is proportional to the square root of the temperature. (If you are wondering how it depends on pressure P and density ρ, the answer is that it depends on the ratio P/ρ - and this ratio is proportional to the temperature.) In any case, if the temperature decreases linearly with height, the sound speed v varies with height y as

$$v=v_0\sqrt{1-cy} \tag{19.10.3}$$

where c is a constant, equal to about 0.023 km^{-1}.Now to trace a sound ray through the atmosphere, we have to understand how the direction of propagation changes as the sound passes through layers of air of different temperature. This is governed, as with light, by Snell's law (see figure XIX.11):

$$\frac{dv}{v}=-\tan\psi d\psi \tag{19.10.4}$$

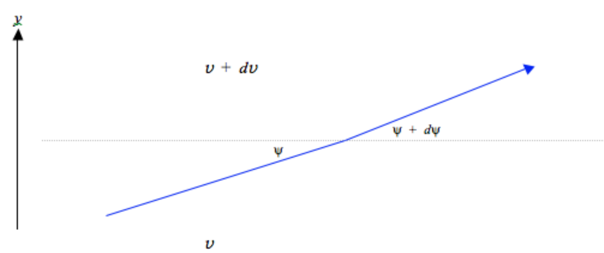


FIGURE XIX.11

Snell's law states that when sound (or light) enters a slower medium (i.e. one in which the speed of propagation is slower) it is bent towards the normal. I have drawn figure XIX.11 to represent the situation in the troposphere where the temperature (and hence the sound speed v) decreases with height. That is, dv/dy is negative. In other words dv in figure XIX is negative, and Equation 19.10.4indicates that $d\psi$ is positive, as drawn. In case you do not recognize this differential form of Snell's law, try integrating it from v_1 to v_2 and from ψ_1 to ψ_2, and it should assume its more familiar integral form. If you now eliminate v between Equations 19.10.3and 19.10.4, you will get a differential relation between y and ψ, which, upon integration, becomes

$$cy=1-\frac{\cos^2\psi}{\cos^2\psi_0} \tag{19.10.5}$$

where ψ_0 is the ground-level value of ψ. If we introduce

$$a=\frac{1}{2c\cos^2\psi_0}, \tag{19.10.6}$$

equation 19.10.5can be conveniently re-written

$$y=2a(\sin^2\psi-s\in^2\psi_0)=2a(\cos^2\psi_0-\cos^2\psi \tag{19.10.7}$$

Now $\tan\psi=dy/dx$, and elimination of y between this and Equation 19.10.7will give a differential relation between x and ψ, which, upon integration, becomes

$$x=a[2(\psi-\psi_0)+\sin 2\psi-\sin 2\psi_0]. \tag{19.10.8}$$

Equations 19.10.7and 19.10.8are the parametric equations of the sound path through the troposphere, and describe a cycloid.

Exercise 19.10.1: Problem for a Rainy Day

If x = 2.0 and y = 1.6, what are ψ and ψ_0?

Solution

I make it $\psi=69^\circ,\psi_0=15^\circ 52'$.

CHAPTER OVERVIEW

20: Miscellaneous

This chapter is a miscellany of diverse and unrelated topics – namely surface tension, shear modulus and viscosity – discussed only for the purpose of presenting a few more examples of elementary problems in mechanics. It is not intended in any way to substitute for a comprehensive course in any of the vast and interesting fields of surface chemistry, elasticity or hydrodynamics. All of these subjects have a huge and specialized literature, each worthy of a full-length course, and I am not remotely competent to offer one. Nevertheless, the few simple problems chosen in this chapter are suitable for a bit more practice in classical mechanics.

Topic hierarchy

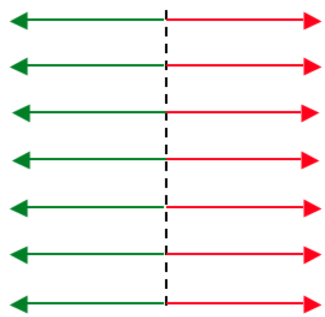
20.1: Introduction

This chapter is a miscellany of diverse and unrelated topics – namely surface tension, shear modulus and viscosity – discussed only for the purpose of presenting a few more examples of elementary problems in mechanics. It is not intended in any way to substitute for a comprehensive course in any of the vast and interesting fields of surface chemistry, elasticity or hydrodynamics. All of these subjects have a huge and specialized literature, each worthy of a full-length course, and I am not remotely competent to offer one. Nevertheless, the few simple problems chosen in this chapter are suitable for a bit more practice in classical mechanics.

20.2: Surface Tension

The cause of surface tension is often explained roughly as follows. Molecules within a liquid are subject to intermolecular forces whose exact nature and origin need not concern us other than to say that they are principally van der Waals forces and they hold the liquid together and prevent it from evaporating. A molecule deep within the liquid is surrounded in all directions by other molecules, and so the net force on it averages zero. But a molecule on the surface experiences forces from beneath the surface, and consequently it tends to get dragged beneath the surface. This results in as few molecules as possible remaining on the surface; i.e. it results in the surface contracting to as small an area as possible consistent with whatever other geometrical constraints may exist. That is, the surface appears to be in a state of tension causing it to contract to the least possible area.

FIGURE XX.1



This tension can be described qualitatively thus. In Figure XX.1, the dashed line is an imaginary line drawn in the surface of a liquid. The liquid to the left of the line is being pulled to the right as indicated by the red arrows; the liquid to the right of the line is being pulled equally to the left as indicated by the green arrows. The force per unit length perpendicular to a line drawn in the surface of the liquid is the *surface tension*. Its SI unit is newtons per metre, and its CGS unit is dynes per centimetre. The dimensions are MT^{-2}.

I have seen various symbols, such as T, S and γ used for surface tension. The first two of these symbols are already heavily worked in thermodynamics, so I shall use the symbol γ (although, it must be admitted, γ is heavily worked in thermodynamics, too.) Not everyone is comfortable with a definition involving forces perpendicular to an imaginary line drawn in the surface, and an alternative approach may be more palatable to some. The idea of a molecule beneath the surface being surrounded on all sides by other molecules and hence experiencing zero net average force, while a molecule on the surface is pulled asymmetrically by the molecules beneath it, remains. But instead of drawing an imaginary line on the surface, we reason that it requires *work* to move a molecule from within the liquid to the surface, and it requires a lot of work to move many molecules from beneath to the surface. That is, it requires work to create new surface. Thus we can define surface tension as the *work required to create unit area of new surface*. The conditions under which this work is done have to be carefully defined in any precise definition, and, from a thermodynamical point of view, the strict definition is the increase in the Gibbs free energy per unit area of new surface created under conditions of constant temperature and pressure. That is:

$$\gamma = \left(\frac{\partial G}{\partial A}\right)_{T,P} \tag{20.2.1}$$

This is consistent with the definition of the Gibbs free energy as a quantity whose increase is equal to the work , other than PdV work, done on a system in a reversible, isothermal isobaric process.

Such a nicety will be of interest to those versed in thermodynamics (and I have added a bit about the thermodynamics of surface energy in Chapter 12 of Thermodynamics), but for those not so versed, you may, without any serious prejudice to understanding most of the matter in this section, think of surface tension either as the force per unit length perpendicular to an imaginary line in the surface, or as the work required to create unit area of new surface. You may express surface tension either in newtons per metre or in joules per square metre (or, if you are of CGS persuasion, dynes per centimetre or ergs per square centimetre). These are dimensionally equivalent.metre or in joules per square metre (or, if you are of CGS persuasion, dynes per centimetre or ergs per square centimetre). These are dimensionally equivalent.

Topic hierarchy

20.2.1: Excess Pressure Inside Drops and Bubbles

The pressure inside a spherical drop is greater than the pressure outside. The way in which the excess pressure P depends on the radius a of the drop, and the surface tension γ and density ρ of the liquid is amenable to dimensional analysis. One can suppose that $P \propto a^\alpha \gamma^\beta \rho^\delta$, after which I leave it to the reader to show that $\alpha = -1, \beta = 1, \delta = 0$, and therefore $P \propto \gamma/a$.

However, it is also quite easy to calculate the excess pressure (other than as a mere proportionality) in terms of the surface tension and the radius of the drop. In Figure XX.2 I have divided a spherical drop of radius a into two hemispheres, and we are going to consider the equilibrium of the upper hemisphere.

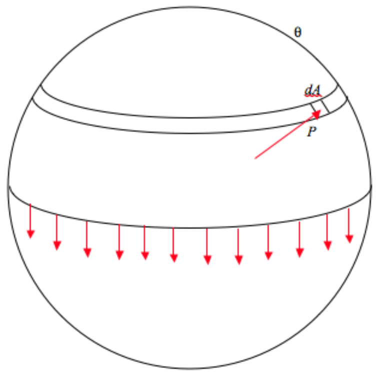


FIGURE XX.2

The upper hemisphere is being pulled down by surface tension all round the base of the hemisphere, and this downward force is equal to the circumference of the base times the surface tension, or $2\pi\gamma a$. If the excess pressure inside the drop is P, the upward component of the force due to this pressure is equal to P times the area of the base, πa^2. In case this is not obvious, consider an elemental area dA as shown, at a spherical angle θ from the top of the drop. The force on this element is equal to PdA. The upward component of this force is $P\cos\theta dA$, and this is equal to P times the horizontal projection of dA. Now you are welcome to do a nice double integration over the hemisphere, but since this (i.e " this is equal to P times the horizontal projection of dA ") is true for every elemental area over the surface of the hemisphere, the total upward force must be equal to P times the area of the base. Thus $2\pi\gamma a = \pi a 2P$, and so the excess pressure inside the drop is

$$P = \frac{2\gamma}{a} \tag{20.2.2}$$

The smaller the drop, the greater the excess pressure. You may regard this as an explanation as to why droplets cannot form from a vapor unless there is a dust nucleus of finite size for them to condense upon. Of course, two molecules colliding with each other cannot in any case coalesce unless there is something to remove or absorb the kinetic energy.

The case of a nonspherical drop might be mentioned in passing. It is a well known result in geometry (or at least it is well known to those who already know it) that if $z = z(x, y)$ is a nonspherical surface, and you take two vertical planes at right angles to each other, and if a_1 and a_2 are the radii of curvature of the intersections of the two planes with the surface, then $\frac{1}{a_1} + \frac{1}{a_2}$ is independent of the orientations of the two planes, as long as they remain perpendicular to each other. In other words, a_1 and a_2 do not have to be the maximum and minimum radii of curvature. The excess pressure inside a nonspherical drop is

$$P = \gamma\left(\frac{1}{a_1} + \frac{1}{a_2}\right) \tag{20.2.3}$$

What about the pressure inside a spherical bubble of air (or other gas) under water (or other liquid)? If we are hasty, we might suggest that, since this is the opposite situation to a liquid drop in air, maybe the pressure is less inside an underwater bubble. This would be a very hasty conclusion, and quite wrong. If you go through exactly the same argument as we did for a drop, considering the equilibrium of one hemisphere, you will see immediately that there is (as for the drop) an excess pressure inside the bubble given again by Equation 20.2.2. And exactly the same would apply to a spherical drop of one liquid under the surface of a second liquid, if the two liquid are immiscible. But, rather than just repeat the identical derivation, let's try a different approach.

Let us imagine that we have a bubble of radius a in a liquid of surface tension γ, and suppose that we are able, by means of a fine syringe, to inject some more air inside so as to increase the radius of the bubble by da at constant pressure and temperature. The surface area of a sphere of radius a is $A = 4\pi a^2$, so, if we increase the radius by da we increase the surface area by $8\pi a da$, and we increase the volume by $4\pi a2da$. The work done against the surface tension is $8\pi\gamma a da$, and this must also be equal to $4\pi Pa^2 da$, where P is the excess pressure inside the bubble. Equating these two expressions leads again to Equation 20.2.2.

What about a hollow spherical soap bubble in air? Here the soap has two surfaces – inside and out. If you repeat either of the above derivations to this case, you will see that the excess pressure inside a hollow spherical soap bubble is

$$P = \frac{4\gamma}{a} \tag{20.2.4}$$

20.2.2: Angle of Contact

When a static quantity of liquid is in contact with an impermeable solid surface, it generally rests so that there is a characteristic angle (measured in the liquid) between the surface of the liquid and the surface of the solid. This angle is the angle of contact, and is shown as the angle θ is Figure XX.3. Figure XX.3(a) shows an acute angle of contact, in which the liquid spreads out a little and "wets" the surface. Figure XX.3(b) shows an obtuse angle of contact, in which the liquid "bunches up", and does not wet the surface, rather like drops of mercury on most surfaces, or drops of water on the surface of a car that has been freshly waxed. In many cases the angle of contact is close to either 0° or 180°, although it will be appreciated that if θ were exactly zero, the liquid would spread

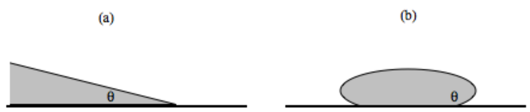


FIGURE XX.3

out in an infinitesimally thin layer to cover or "wet" the entire surface; and if it were exactly 180°, the liquid, in the absence of other forces (such as its weight!), would form a spherical globule in contact with the surface only at a single point. The angle of contact is determined by the nature of both surfaces, and is very sensitive to any surface contamination. In order to wet a surface, water may need to be helped by a small amount of wetting agent or detergent; only a small amount is necessary, because only the surface, and not the bulk, of the liquid is involved. The chemical nature of wetting agents and detergents is beyond the scope of these notes (i.e. it is beyond my scope!), but how the angle of contact depends on the surface tension provides a useful example of the technique of virtual work (see Section 9.4 of Chapter 9).

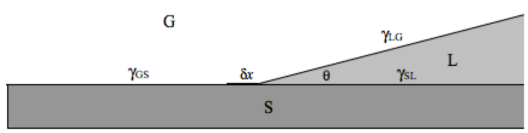


FIGURE XX.4

Figure XX.4 represents a liquid, L, (e.g. water) in contact with a solid, S, (e.g. glass) and a gas, G, (e.g. air). The angle of contact is θ, and the surface tensions (energy per unit area, or, for those who are versed in thermodynamics, the Gibbs free energy per unit area) of the three interfaces are as shown. We'll suppose that the three media extend for a distance l at right angles to the plane of the paper (or computer screen). The three phases are in equilibrium. Now, if we move the SLG boundary to the left by a distance δx, we create a new area $l\delta x$ of SL interface and a new area $l\cos\theta\delta x$ of LG interface, while we lose an area $l\delta x$ of GS interface. The work done on the system is therefore $\gamma_{SL}l\delta x+\gamma_{LG}l\cos\theta\delta x-\gamma_{GS}l\delta x$. By the principle of virtual work, this is zero, and therefore

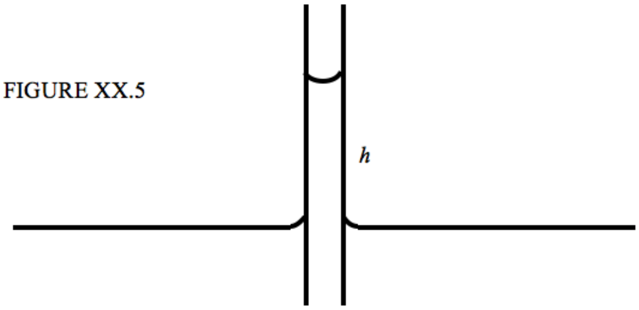
$$\cos\theta=\frac{\gamma_{GS}-\gamma_{SL}}{\gamma_{LG}}. \tag{20.2.5}$$

The angle of contact is acute or obtuse, according to whether γ_{GS} is greater than or less than γ_{SL}.

20.2.3: Capillary Rise

When the lower end of a narrow capillary tube is immersed in a liquid, the liquid inside the tube rises a little above the level of the liquid outside. If is then very simple to calculate how far the liquid rises in terms of the surface tension, the angle of contact and the inside radius of the tube. See Figure XX.5.

FIGURE XX.5



The upward force due to surface tension is $2\pi a\gamma\cos\theta$ where a is the inside radius of the tube, and, if we neglect the very small mass of the liquid in the meniscus (the curved surface at the top of the liquid column), the weight of the liquid column is $\pi a^2 h\rho g$, and therefore

$$h = \frac{2\gamma\cos\theta}{\rho g a}. \tag{20.2.6}$$

Of course if θ is *obtuse* (as with mercury in contact with glass), h will be *negative*, and the level of the mercury in the tube will be *below* the outside level.

20.3: Shear Modulus and Torsion Constant

Imagine that we have a rectangular block of solid material, as shown on the left hand side of Figure XX.6. We now apply a couple of tangential forces F as shown on the right hand side. (I have not decided to go all chatty and informal by saying "a couple" of forces; far from it – I am using the word "couple" in its formal sense in mechanics.) The material will undergo an angular deformation, and the ratio of the tangential force per unit area to the resulting angular deformation is called the *shear modulus* or the *rigidity modulus*. Its SI unit is N m^{-2} rad^{-1} and its dimensions are ML^{-1}T^{-2}θ^{-1}. (I'd advise against using "pascals" per radian. The unit "pascal" is best restricted to pressure, which is normal force per unit area, and is not quite the same thing as the tangential force per unit area that we are discussing here.) You should convince yourself that the definition must specify the force F, not the torque provided by the couple. If the block were twice as thick, and the forces were the same, you'd still get the same angular deformation.

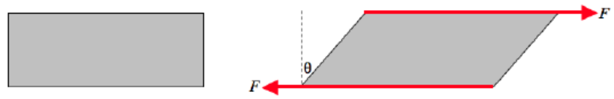


FIGURE XX.6

If you hold one end of a wire or rod fixed and apply a torque to the other end, this end will twist through an angle, and the ratio of the applied torque to the angle through which the wire twists is the *torsion constant*, *c*, of the wire. You can see how the torsion constant depends on the shear modulus η of the metal and the radius a and length l of the wire by the method of dimensions. You can start by supposing that

$$c \propto \eta^{\alpha} a^{\beta} l^{\gamma},$$

but you will soon find yourself in difficulty because a and l are each of dimension L. However, you will probably have no difficulty with making the assumption that $\gamma = -1$ (the longer the wire, the easier it is to twist), and dimensional analysis will soon show that $\alpha = 1$ and $\beta = 4$- which, being interpreted, means that it is much more difficult to twist a thick wire than a thin wire. But can we do better and get an expression other than a mere proportionality for the torsion constant? Can we find the proportionality constant? Let's try some simpler problems first, and see how things go.

Let us consider a long, thin strip or ribbon of metal. By long and thin I mean that its length is much greater than its width, and its width is much greater than its thickness. I can use any symbol I like to represent any quantity I like, so I could, if I wished, use Ξ for the length, m_{α} for the width, and G_2 for the thickness. Instead, the symbols that I shall choose to represent the length, width and thickness of the strip are going to be, respectively, $l, 2\pi r$ and δr. This seems silly at the moment, but in the end you'll be glad that I made this choice. The strip is shown at the left hand side of Figure XX.7.

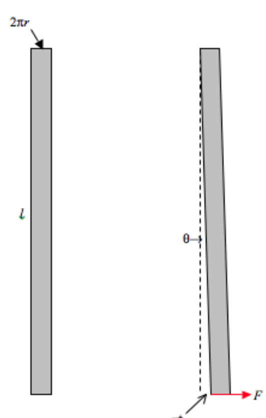


FIGURE XX.7

I am now going to fix the upper end of the strip and apply a force F to the lower end, as shown in the right hand side of Figure XX.7, and I can use any symbol I like to represent the displacement of the lower end, and I choose the symbol $r\phi$. This means that the angular displacement θ is equal to $r\phi/l$.The tangential force per unit area is $F/(2\pi r\delta r)$, and therefore

$$\eta=\frac{Fl}{2\pi\phi r^2\delta r}, \tag{20.3.1}$$

or

$$F=\frac{2\pi\eta\phi r^2\delta r}{l}. \tag{20.3.2}$$

Now I'm going to restore the strip to its original shape, and then I'm going to roll it into a hollow cylindrical tube, so that it now looks like a metal drinking straw. The circumference of the straw is $2\pi r$, its radius is r and its thickness is δr (Figure XX.8). (Now my notation is beginning to make some sense!)

FIGURE XX.8

I shall hold the upper end of the tube fixed and I shall apply a torque $\tau=Fr$ to the lower end. The tube will evidently twist through an azimuthal angle ϕ given by

$$\tau=\frac{2\pi\eta^3\delta r}{l}\phi. \tag{20.3.3}$$

The torsion constant of the hollow tube is therefore

$$c=\frac{2\pi\eta r^3\delta r}{l}. \tag{20.3.4}$$

The torsion constant of a long solid cylinder (a wire) of radius a is the integral of this from 0 to a, which is

$$c=\frac{\pi\eta a^4}{2l} \tag{20.3.5}$$

SECTION OVERVIEW

20.4: Viscosity

Consider a river flowing over a smooth bed, as in Figure XX.9.

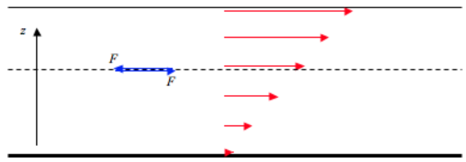


FIGURE XX.9

There will be a *transverse velocity gradient* dv/dz, with the liquid stationary at the river bottom, and the speed becoming faster as we ascend from the bottom. As a consequence of the transverse velocity gradient, the liquid below the dashed line will be dragged forward by the tangential force of the faster liquid above it, and the liquid above the dashed line will be dragged backward by the tangential force of the more sluggish liquid below it. The ratio of the tangential force per unit area to the transverse velocity gradient is called the coefficient of *dynamic viscosity*, for which the usual symbol is η. The dimensions of dynamic viscosity are $ML^{-1}T^{-1}$. The CGS unit of dynamic viscosity is the *poise*. The abbreviation for the unit is P – though it would be well to define it if you use it, since not everyone will recognize it. The unit is named after a nineteenth century French doctor, Jean Poiseuille, who was interested in blood pressure and hence in the rate of flow of liquids through tubes. That is to say, if, for a transverse velocity gradient of 1 cm s^{-1} per cm, the tangential force per unit area is 1 dyne cm^{-2}, the dynamic viscosity is one poise. The SI (MKS) unit is the decapoise (also spelled dekapoise), though the SI unit the pascal second (Pa s), which is dimensionally correct, is also seen. If, for a transverse velocity gradient of 1 m s^{-1} per cm, the tangential force per unit area is 1 N m^{-2}, the dynamic viscosity is one decapoise. The dynamic viscosity of water varies from about 1.8 centipoise at 0○ C to about 0.3 centipoise at 100○C.

The ratio of the coefficient of dynamic viscosity to the density is the coefficient of *kinematic viscosity*, for which the usual symbol is the Greek letter ν. The dimensions of kinematic viscosity are L^2T^{-1}. The CGS unit of kinematic viscosity is *the stokes* (abbreviation St). It is named after nineteenth century British physicist, Sir George Stokes, who made major contributions to diverse areas of physics. The SI unit of kinematic viscosity is usually given simply as m^2 s^{-1}, and 1 m^2 s^{-1} = 104 stokes. The kinematic viscosity of water varies from about 1.8 centistokes (1.8 - 10^{-6} m^2 s^{-1}) at 0○ C to about 0.3 centistokes (3 - 10^{-7} m^2 s^{-1}) at 100○C.

Hydrodynamics is a huge and very difficult subject (at least I think it is), but there are a couple of simple problems that, if nothing else, make good homework problems. These are Poiseuille’s law and the Couette viscometer.

Topic hierarchy

20.4.1: Poiseuille's Law

Poiseuille's law tells you how the rate of nonturbulent flow of a liquid through a cylindrical pipe depends on the viscosity of the liquid, the radius of the pipe, and the pressure gradient. If all else fails, you can at least try dimensional analysis. Assume that the rate of flow of liquid (in cubic metres per second) is proportional to $\eta^\alpha a^\beta \left(\frac{dP}{dx}\right)^\gamma$, and show by dimensional analysis that $\alpha = -1, \beta = -4$ and $\gamma = 1$, which shows that the rate of flow is very sensitive to the radius of the pipe. That $\beta = -4$ tells you that if your arteries are at all constricted, even by a little bit, you had better watch out. Gas flow is more complicated because gases are compressible, (so are liquids, but not by much), but $\beta = -4$ tells you that the rate at which you can pump out gas from a system depends a lot on the size of the smallest tube you have between the volume that you are trying to evacuate and the pump. Now let's try and analyse it further.

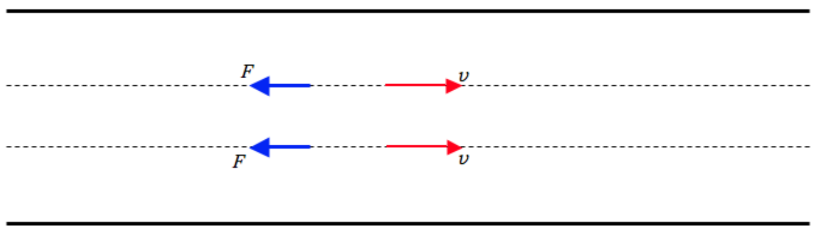


FIGURE XX.10

Figure XX.10 represents a pipe of radius a with liquid flowing to the right. At a distance r from the axis of the pipe the speed of the liquid is v. The length of the pipe is l, and there is a pressure gradient along the length of the pipe, the pressure at the left end being higher than the pressure at the right by P. There is a velocity gradient in the pipe. The speed of the liquid along the axis of the pipe is v0, and the speed at the circumference of the pipe is zero. That is, the speed decreases from axis to circumference, so that the velocity gradient (dv/dr) is negative.

Now consider the equilibrium of the liquid inside radius r. (It is in equilibrium because it is moving at constant speed.) It is being pushed forward by the pressure gradient. This rightward force is $\pi r^2 P$. It is being dragged back by the viscous force acting on the area $2\pi rl$. This leftward force is $-2\pi\eta lr(dv/dr)$, this expression for the leftward force being positive.

Therefore

$$-2\eta l \frac{dv}{dr} = Pr. \tag{20.4.1}$$

Integrate from the axis $(r = 0, v = v_0)$ to r:

$$v = v_0 - \frac{Pr^2}{4\eta l}. \tag{20.4.2}$$

Thus the speed decreases quadratically (parabolically) as you move away from the axis. The speed is zero at the circumference, and hence the speed on the axis is

$$v_0 = \frac{Pr^2}{4\eta l}. \tag{20.4.3}$$

Verify the dimensions.

Now the volume flow through a cylindrical shell of radii r and $r + dr$ is the speed times the area $2\pi r dr$,, which is $\frac{\pi r^3 dr}{2\eta l}$, and if you integrate that through the whole pipe, from 0 to a, you find that the rate of flow of liquid through the pipe (cubic metres per second) is

$$\frac{\pi a^4 P}{8\eta l}. \tag{20.4.4}$$

This is *Poiseuille's Law.*

20.4.2: The Couette Viscometer

A cylindrical vessel of radius b contains the liquid whose viscosity is to be measured. A smaller, solid cylinder of radius a and length l is suspended from a torsion wire, whose torsion constant c is known, and is immersed in the liquid in the vessel, the two cylinders being coaxial. The vessel containing the liquid is spun about its axis at an angular speed Ω, thus setting the liquid in motion. This causes a viscous torque on the inner cylinder, which is therefore pulled round through an angle ϕ. When the restoring torque of the torsion wire $c\phi$ is equal to the viscous torque, the system will be in equilibrium, and one can then calculate the viscosity η of the liquid. We shall refer to Figure XX.11. In the simple analysis given below, we suppose that the angular and linear speed and gradients are sufficiently small that the flow is nonturbulent. We also neglect the effects of viscous drag on the flat ends of the cylinder. Thus the diameter of the cylinder, in our analysis, must be much less than its length.

> *Incidentally, for a long time I thought that the word "couette" must be French for something. It is – it's French for "feather bed" or for "pigtail". But the Couette viscometer is actually named after a little-known nineteenth century French scientist, Maurice Couette.*

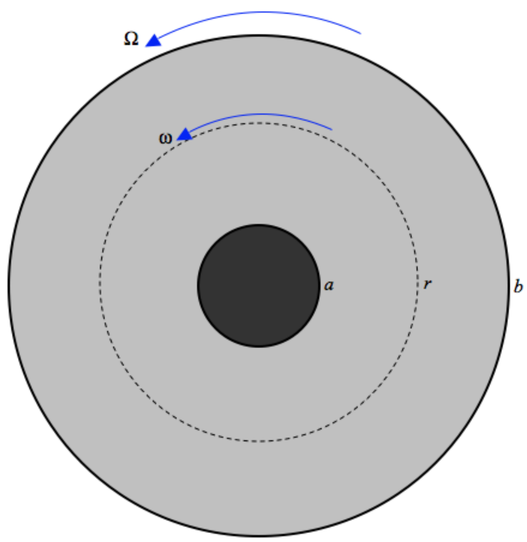


FIGURE XX.11

Let us calculate the viscous torque on the liquid within radius r. Notice that, since we have a steady-state situation, this torque is *independent* of r; in particular the torque on the liquid within radius r is the same as the torque (which we can measure with the torsion wire) on the inner cylinder. The area of the curved surface of the liquid within radius r is $2\pi lr$. The viscous torque on this surface is r times η times the area times the transverse velocity gradient. But we have to be careful about this last term. If the whole body of the liquid were rotating as a solid body with angular speed ω, the speed at radius r would be $r\omega$ and hence there would be a transverse velocity gradient equal to ω − but no viscous drag! But the liquid is not, of course, rotating as a solid, and ω (as well as v) is a function of r. Since $v = r\omega$, the velocity gradient is $\frac{dv}{dr} = r\frac{d\omega}{dr} + \omega$ and the only part of this that goes into the expression for the viscous torque is the part $r\frac{d\omega}{dr}$.Thus the expression for the torque on the liquid within radius r (and hence also on the inner cylinder) is

$$\tau = r.\eta.2\pi rl.r\frac{d\omega}{dr} \tag{20.4.5}$$

That is,

$$\frac{d\omega}{dr} = \frac{\tau}{2\pi\eta lr^3}. \tag{20.4.6}$$

Integration from $r = a, \omega = 0$ to $r = b, \omega = \Omega$ gives

$$\tau = \frac{4\pi\eta l\Omega a^2 b^2}{b^2 - a^2} \tag{20.4.7}$$

In equilibrium, this is equal to $c\phi$, where c is the torsion constant of the suspension and ϕ is the angle through which the inner cylinder has turned, and hence the viscosity can be determined. You should, as usual, check the dimensions of Equation 20.4.7.

CHAPTER OVERVIEW

21: Central Forces and Equivalent Potential

21.1: Introduction to Central Forces

When a particle is in orbit around a point under the influence of a central attractive force (i.e. a force $F(r)$ which is directed towards a central point, with no transverse component) it experiences, *when referred to an inertial reference frame*, a centripetal acceleration. If, however, the system is described *with respect to a co-rotating reference frame*, there is no centripetal acceleration; rather, it appears as though an additional force, the centrifugal force, is pushing it away from the centre of attraction. In the co-rotating frame, this force depends only on the distance of the particle from the centre of attraction, and it is therefore a conservative force – and, like any conservative force, it can be described by the negative of the derivative of a potential energy function. When describing the motion with respect to the co-rotating frame, we must add this potential to any additional "real" potentials (such as originate from the gravitational fields of other bodies), to form an equivalent potential which constrains the motion of the particle. An excellent example of this method is the analysis of the restricted three-body problem given in some detail in Chapter 16 of the Celestial Mechanics notes. But I deal first, by way of example, with some simpler problems involving central forces, in which we shall be able, by simple arguments, to deduce some basic characteristics of the motion.

21.2: Motion Under a Central Force

I consider the two-dimensional motion of a particle of mass m under the influence of a conservative central force $F(r)$, which can be either attractive or repulsive, but depends only on the radial coordinate r. Recalling the formula $\ddot{r} - r\dot{\theta}^2$ \)for acceleration in polar coordinates (the second term being the centripetal acceleration), we see that the equation of motion is

$$m\ddot{r} - mr\dot{\theta}^2 = F(r). \tag{21.2.1}$$

This describes, in polar coordinates, two-dimensional motion in a plane. But since there are no transverse forces, the angular momentum $m^2\dot{\theta}^2$ is constant and equal to L, say. Thus we can write Equation 21.2.1 as

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3}. \tag{21.2.2}$$

This has reduced it to a one-dimensional equation; that is, we are describing, relative to a co-rotating frame, how the distance of the particle from the centre of attraction (or repulsion) varies with time. In this co-rotating frame it is as if the particle were subject not only to the force $F(r)$, but also to an additional force $\frac{L^2}{mr^3}$. In other words the total force on the particle (referred to the co-rotating frame) is

$$F'(r) = F(r) + \frac{L^2}{mr^3}. \tag{21.2.3}$$

Now $F(r)$, being a conservative force, can be written as minus the derivative of a potential energy function, $F = -\frac{dV}{dr}$. Likewise, $\frac{L^2}{m^3}$ is minus the derivative of $\frac{L^2}{2mr^2}$. Thus, in the co-rotating frame, the motion of the particle can be described as constrained by the potential energy function V', where

$$V' = V + \frac{L^2}{2mr^2}. \tag{21.2.4}$$

This is the *equivalent potential energy*. If we divide both sides by the mass m of the orbiting particle, this becomes

$$\Phi' = \Phi + \frac{h^2}{2r^2}. \tag{21.2.5}$$

Here h is the angular momentum per unit mass of the orbiting particle, Φ is the potential in the inertial frame, and $\Phi'$is the *equivalent potential* in the corotating frame.

21.3: Inverse Square Attractive Force

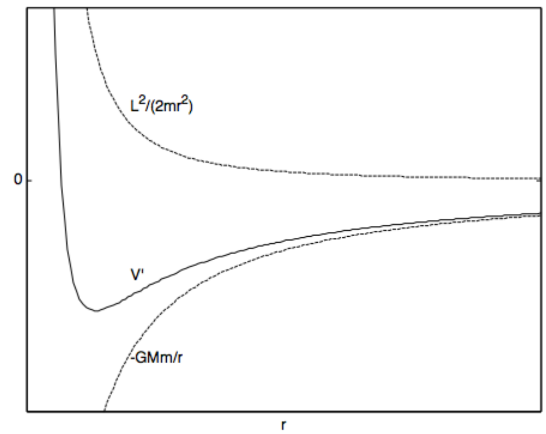
This is dealt with in detail in Chapter 9 of Celestial Mechanics. Here we investigate some general properties of the motion.

If $F=-\frac{GMn}{r^2}$ then $V=\frac{GMn}{r}$, and hence

$$V'=-\frac{GMn}{r}+\frac{L^2}{2mr^2}. \tag{21.3.1}$$

I sketch this in Figure XXI.1. The total energy (potential + kinetic) is constant (independent of r) and is greater than (or equal to) the potential energy. If the total energy is less than zero, you can see from the graph that r has a lower (perihelion) and upper (aphelion) limit; this corresponds to an elliptic orbit. But if the total energy is positive, r has a lower limit, but no upper limit; this corresponds to a hyperbolic orbit. If the total energy is equal to the minimum of V', only one value of r is possible, and the orbit is a circle.

FIGURE XXI.1



21.4: Hooke's Law

We imagine a particle whirling around on the end of a spring, oscillating in and out as it does so. The force constant of the spring is k, the force on the particle is $-kr$ and the potential (elastic) energy is $V=\frac{1}{2}kr^2$. This is akin to a non-rigid rotor.

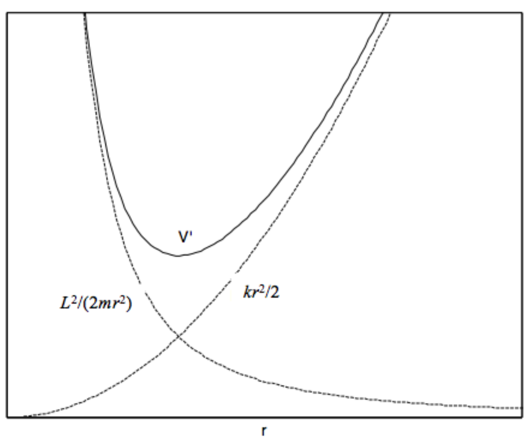
Figure 21.4.1: In the absence of the spring, the particles would fly apart. However, the force exerted by the extended spring pulls the particles onto a periodic, oscillatory path. (CC BY-SA 3.0; Cleonis).

The effective potential energy is therefore

$$V'=\frac{1}{2}kr^2+\frac{L^2}{2mr^2}. \tag{21.4.1}$$

and is sketched in Figure XVI.2. The total energy (potential + kinetic) is constant (independent of r) and is greater than (or equal to) the potential energy. The distance of the particle from the centre of attraction is bounded above and below. The motion is a **Lissajous ellipse**, with the centre of attraction at the centre (not the focus) of the ellipse. The lower bound is the semi minor axis and the upper bound is the semi major axis.

FIGURE XXI.2



An inverse square force (e.g. a gravitational force, or a Coulomb's law electrostatic force) and a Hooke's law force (kx) are obvious examples of real forces in nature. In what follows we shall investigate the behavior of a particle under the influence of other force laws, such as inverse fourth power and inverse cube forces. It is difficult to imagine whether such forces actually exist in nature (the field of an electric dipole falls off as the cube of the distance - but the field is not radial, and the force is not a central force), and to that extent much of what follows is an exercise in mathematics more than in physics. But inverse square and Hooke's law forces are certainly not the only forces to operate in nature. What is the force law, for example, for the residual strong interactions between nucleons in an atomic nucleus, or the force law between the quarks within a nucleon? It will be worthwhile investigating the simpler hypothetical forces to be discussed here in order to understand the principles and methods that may be applicable to a more difficult problem.

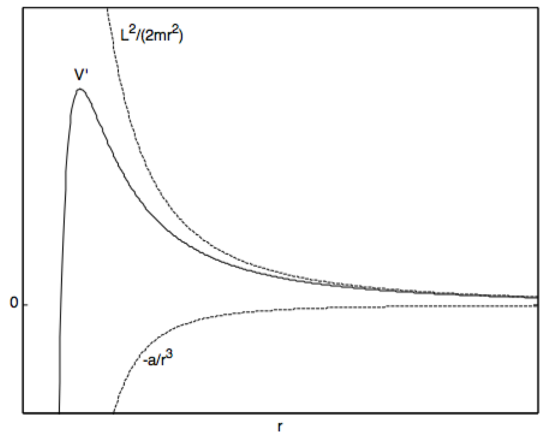
21.5: Inverse Fourth Power Attractive Force

If $F=-\frac{3a}{r^4}$ then $V=-\frac{a}{r^3}$, and hence

$$V'=-\frac{a}{r^3}+\frac{L^2}{2mr^2} \tag{21.5.1}$$

I sketch this in Figure XXI.3. The total energy (potential + kinetic) is constant (independent of r) and is greater than (or equal to) the potential energy. If the total energy is negative, the distance r has an upper limit, but the only lower limit is the origin, or the centre of attraction, and particle will eventually end there. If the total energy is greater than the maximum of V', the motion is completely unbounded. If the total energy is positive but less than V'_{max}, the motion depends on the initial value of r. For small r the motion is bounded above, and the particle will eventually end at the origin. For large r, there is a minimum distance to which the particle can approach the origin, and the particle will eventually wander off to infinity. For total energy in this range, there is a range of r that is not possible.

FIGURE XXI.3



21.6: A General Central Force

Let us suppose that we have a particle that is moving under the influence of a central force $F(r)$. The equations of motion are

Radial:

$$m(\ddot{r} - r\dot{\theta}^2) = F(r) \tag{21.6.2}$$

Transverse:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{21.6.3}$$

These can also be written

$$\ddot{r} - \dot{\theta}^2 = a(r) \tag{21.6.4}$$

$$r^2\dot{\theta} = h. \tag{21.6.5}$$

Here a is the radial force per unit mass (i.e. the radial acceleration) and h is the (constant) angular momentum per unit mass. [If you are unsure of why Equations 21.6.3 and 21.6.5 are the same, differentiate equation 21.6.5 with respect to time.]

These are two simultaneous equations in r, θ, t. In principle, if we could eliminate t between them, we would obtain a relation between r and θ, which would tell us the shape of the path pursued by the particle. In Chapter 9 of my Celestial Mechanics notes we do this for the gravitational case, and we find that the path is an ellipse of the form $r = \frac{l}{1+e\cos\theta}$. Or perhaps we could eliminate r and hence find out how the angle θ changes with time. Or again we might be able to eliminate θ and hence get a relation telling us how r varies with the time. Yet again we might be told the shape of the path $r(\theta)$, and asked to find the force law $F(r)$. Or again, rather than the force, we might be given the form of the potential energy $V(r)$, which is related to the force by $F = -dV/dr$. The potential Φ is the potential energy per unit mass, and $-d\Phi/dr$ is the radial force per unit mass - i.e. it is the radial acceleration $a(r)$ of the orbiting particle. The angular momentum of the particle, which is constant, is $L - mr^2\dot{\theta}$, and the angular momentum per unit mass is $h = r^2\dot{\theta}$, which is twice the rate at which the radius vector sweeps out area.

We might also remember that, if we are given the potential energy V or the potential Φ in an inertial frame, we might also want to work in a co-rotating frame, making use of the equivalent potential energy $V' = V + \frac{L^2}{2mr^2}$ or the equivalent potential $\Omega' = \Omega + \frac{h^2}{2^2}$.

One last thing to bear in mind before starting any problems of this class. It turns out that, very often, a change of variable $u = 1/r$ turns out to be useful. Conservation of angular momentum then takes the form $\dot{\theta}/u^2 = h$ Also

$$\dot{r} = \frac{dr}{du}\frac{du}{dt} = \frac{dr}{d\theta}\frac{dr}{dt}\frac{du}{d\theta} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta} = -h\frac{du}{d\theta} \tag{21.6.6}$$

and

$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{dt}\frac{du}{d\theta} = -h\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta} = -h.\,hu^2.\,\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}. \tag{21.6.7}$$

Equations 21.5.4 and 21.6.5 now become

$$h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = -a(r). \tag{21.6.8}$$

and

$$\dot{\theta} = hu^2. \tag{21.6.9}$$

We can now easily eliminate the time which was one of our aims:

$$h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = -a(r). \tag{21.6.10}$$

[As ever, check the dimensions.] This equation, which does not contain the time, when integrated will give us the equation to the path.

With these remarks in mind, let us try a few problems. For example:

21.7: Inverse Cube Attractive Force

A particle moves in a field such that the attractive force on it varies inversely as the cube of the distance from a centre of attraction. What is the shape of the path? How does the angle θ vary with time?

Let's suppose that the radial acceleration is $a(r) = -k^3/r^3 = -k^3u^3$. (I want the coefficient of $1/r^3$ to be negative, so that the force is attractive, which is why I have written the coefficient as $-k^2$. Besides, the dimensions of k are then L^2T^{-1}, which are the same as those of h, the angular momentum per unit mass, which helps to make the algebra simple.) The differential equation to the path (Equation 21.6.10) is then $h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = k^2u^3$ or

$$h^2\frac{d^2u}{d\theta^2} + h^2u = k^3u. \tag{21.7.1}$$

That is,

$$\frac{d^2u}{d\theta^2} = \frac{k^2 - h^2}{h^2}u. \tag{21.7.2}$$

The form of the motion evidently depends on whether $k^2 > h^2$ (a strongly attractive force, or a small angular momentum), or if $k^2 < h^2$ (a weak force, or a large angular momentum.) If we start the particle rolling with just the right amount of angular momentum $(k^2 = h^2)$, there will evidently be zero radial acceleration, and the particle will move in a circle.

Before integrating Equation 21.7.2, let us look at the *equivalent potential*. For $a(r) = -k^2/r^3$, the potential in the inertial frame is $\Omega = -\frac{1}{2}k^2/r^2$ provided we take the potential at infinity to be zero. The equivalent potential is then (see equation 21.2.5)

$$\Omega' = -\frac{k^2}{2r^2} + \frac{h^2}{2r^2}. \tag{21.7.3}$$

We see that, if $k^2 = h^2$, the potential is zero and independent of distance. If $h^2 < k^2$, the equivalent potential is negative, increasing to zero as $r \to \infty$, and the particle accelerates towards the centre of attraction. If $h^2 > k^2$, the potential is positive, decreasing to zero as $r \to \infty$, and the particle accelerates away from the centre of attraction. This sounds like a contradiction, but what is happening is $h^2 > k^2$, that means that the particle has initially been given a large angular momentum, and, in the corotating frame, the centrifugal force is larger than the attractive force.

If $h^2 < k^2$, the equation of motion (Equation 21.7.2) is

$$\frac{d^2u}{d\theta^2} = c^2u, \tag{21.7.14}$$

where

$$c^2 = \frac{k^2 - h^2}{h^2}. \tag{21.7.15}$$

The general solution is

$$u = Ae^{c\theta} + Be^{-c\theta} \tag{21.7.16}$$

If the initial conditions are that at $t = 0, r = r_0, u = u_0, \frac{du}{d\theta} = 0$ (this last condition means that the particle was launched in a direction at right angles to the radius vector, this solution becomes

$$u = y_0 \cos hc\theta. \tag{21.7.17}$$

That is,

$$r = r_0 \sec hc\theta \tag{21.7.18}$$

I have drawn this below for $c = 0.1$; that is, for $k \approx 1.05h$ And for $c = 0.5$; that is $k \approx 1.22h$, for a smaller angular momentum.

FIGURE XXI.4

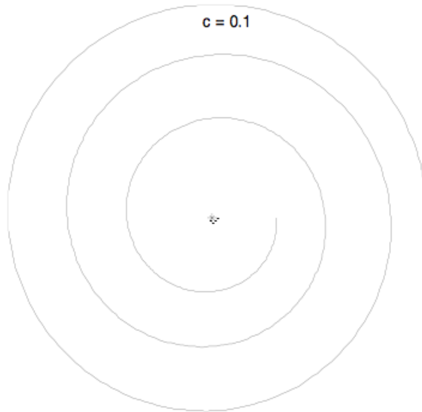
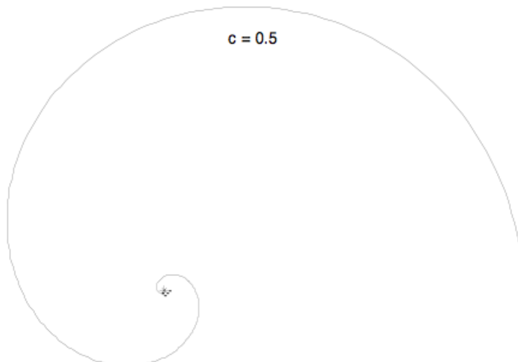


FIGURE XXI.5



We also need to consider the case $h^2 > k^2$, in which case the general solution is of the form $u = A\cos c\theta + B\sin c\theta$. Alas, I haven't had the energy to do this yet. Perhaps some viewer can beat me to it, and let me know.

CHAPTER OVERVIEW

22: Dimensions

22.1: Mass, Length and Time

Any mechanical quantity can be expressed in terms of three fundamental quantities, *mass, length* and *time.* For example, speed is a length divided by time. Force is mass times acceleration, and is therefore a mass times a distance divided by the square of a time.

We therefore say that [Force] = MLT^{-2}. The square brackets mean: "The dimensions of the quantity within". The equations indicate how force depends on mass, length and time. We use the symbols MLT (not in *italics*) to indicate the fundamental dimensions of mass, length and time. In the above equation, MLT^{-2} are *not* enclosed within square brackets; it would make no sense to do so.

We distinguish between the *dimensions* of a physical quantity and the units in which it is expressed. In the case of MKS units (which are a subset of SI units), the units of mass, length and time are the kg, the m and the s. Thus we could say that the *units* in which force is expressed are kg m s^{-2}, while its dimensions are MLT^{-2}.

For electromagnetic quantities we need a fourth fundamental quantity. We could choose, for example, quantity of electricity Q, in which case the dimensions of current are QT^{-1}. We do not deal further with the dimensions of electromagnetic quantities here. Further details are to be found in my notes on Electricity and Magnetism, http://orca.phys.uvic.ca/~tatum/elmag.html

To determine the dimensions of a physical quantity, the easiest way is usually to look at the definition of that quantity. Most readers will have no difficulty in understanding that, since work is force times distance, the dimensions of work (and hence also of energy) are $\mathrm{ML}^2\mathrm{T}^{-2}$. A more challenging one would be to find [dynamic viscosity]. One would have to refer to its definition (see Chapter 20) as tangential force per unit area per unit transverse velocity gradient.

Thus [dynamic viscosity] = $\left[\frac{\text{force}}{\text{area}}\frac{\text{distance}}{\text{velocity}}\right] = \frac{\mathrm{MLT}^{-2}}{\mathrm{L}^{-2}}\frac{\mathrm{L}}{\mathrm{LT}^{-1}} = \mathrm{MLT}^{-1}\mathrm{L}^{-1}$.

22.2: Table of Dimensions

I supply here a table of dimensions and MKS units of some mechanical quantities. Some are obvious and trivial. Others might be less so, and readers to whom this topic is new are encouraged to derive some of them from the definitions of the quantities concerned. Let me know (jtatum at uvic.ca) if you detect any mistakes.

I do not know whether angle is a dimensionless or a dimensioned quantity. I can convince you that it is dimensionless by reminding you that it is defined as a ratio of two lengths. I can convince you that it is dimensioned by pointing out that it is necessary to state the units (e.g. radians or degrees) in which it is expressed. This might make for an interesting lunchtime conversation

Mass	M	kg	
Length	L	m	
Time	T	s	
Density	ML^{-3}	$kg\ m^{-3}$	
Speed	LT^{-1}	$m\ s^{-1}$	
Acceleration	LT^{-2}	$m\ s^{-2}$	
Force	MLT^{-2}	$kg\ m\ s^{-2}$	N
Work, Energy, Torque	ML^2T^{-2}	$kg\ m^2\ s^{-2}$	J, N m
Action	ML^2T^{-1}	$kg\ m^2\ s^{-1}$	J s
Rotational inertia	ML^2	$kg\ m^2$	
Angular speed	T^{-1}	s^{-1}	$rad\ s^{-1}$
Angular acceleration	T^{-2}	s^{-2}	$rad\ s^{-2}$
Angular momentum	ML^2T^{-1}	$kg\ m^2\ s^{-1}$	J s
Pressure, elastic modulus	$ML^{-1}T^{-2}$	$kg\ m^{-1}\ s^{-2}$	Pa
Gravitational constant	$M^{-1}L^3T^{-2}$	$kg^{-1}\ m^3\ s^{-2}$	$N\ m^2\ kg^{-2}$
Dynamic viscosity	$ML^{-1}T^{-1}$	$kg\ m^{-1}\ s^{-1}$	dekapoise
Kinematic viscosity	L^2T^{-1}	$m^2\ s^{-1}$	
Force constant	MT^{-2}	$kg\ s^{-2}$	$N\ m^{-1}$
Torsion constant	ML^2T^{-2}	$kg\ m^2\ s^{-2}$	$N\ m\ rad^{-1}$
Surface tension	MT^{-2}	$kg\ s^{-2}$	$N\ m^{-1}$
Schrödinger wavefunction Ψ	$L^{-3/2}T^{-1/2}$	$m^{-3/2}s^{-1/2}$	
Schrödinger wavefunction ψ	$L^{-3/2}$	$m^{-3/2}$	

22.3: Checking Equations

When you are doing a complicated calculation involving difficult equations connecting several physical quantities, you must, routinely, check the dimensions of every line in your calculation. If the equation does not balance dimensionally, you know immediately that you have made a mistake, and the dimensional imbalance may even give you a hint as to what the mistake is. If the equation does balance dimensionally, this, of course, does not guarantee that it is correct - you may, for example, have missed a dimensionless constant in the equation.

Suppose that you have deduced (or have read in a book) that the period of oscillations of a torsion pendulum is $P = 2\pi\sqrt{\frac{I}{C}}$, where I is the rotational inertia and c is the torsion constant. You have to check to see whether the dimensions of the right hand side are indeed that of time. We have

$$\left[\sqrt{\frac{I}{C}}\right] = \sqrt{\frac{\mathrm{ML}^2}{\mathrm{ML}^2\mathrm{T}^{-2}}}.$$

This does indeed come to T, and so the equation balances dimensionally.

22.4: Deducing Relationships

i. We may suppose that the period P of a simple pendulum depends upon its mass m, its length l, and the gravitational acceleration g. In particular we suppose that the period is proportional to some power α of the mass, some power β of the length, and some power γ of the gravitational acceleration. That is

$$p \propto m^{\alpha} l^{\beta} g^{\gamma}.$$

Both sides must have the same dimension - namely T.

That is

$$[m^{\alpha} l^{\beta} g^{\gamma}] = T$$

That is

$$\mathrm{M}^{\alpha}\mathrm{L}^{\beta}(\mathrm{LT}^{-2})^{\gamma} = T$$

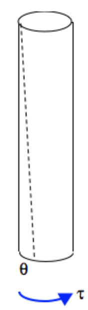
We equate powers of M, L and T to get three equations in α, β, γ:

$$\alpha = 0, \beta + \gamma = 0, -2\gamma = 1,$$

with solutions $\alpha = 0, \beta = \frac{1}{2}, \gamma = -\frac{1}{2}$, which shows that

$$P \propto m^{0} l^{\frac{1}{2}} g^{-\frac{1}{2}} \text{, or } P \propto \sqrt{\frac{l}{g}}$$

ii. Here's another: The torque τ required to twist a solid metallic cylinder through an angle θ is proportional to $\theta : \tau = c\theta$.



c is the torsion constant. How does c depend upon the length l and radius a of the cylinder, its density ρ and its shear modulus η ? There is an immediate difficulty, in that we have four quantities to consider − l, a, ρ and η, yet we have only three dimensions L, M , T to deal with. Hence we shall have three equations in four unknowns. Further, two of the quantities, l and a have similar dimensions, which adds to the difficulties.

In cases like this we may have to make a sensible assumption about one of the quantities. We may, for example, find it easy to accept that, the longer the cylinder, the easier it is to twist, and we may make the assumption that the torsion constant is inversely proportional to the first power of its length. Then we can suppose that

$$cl \propto a^{\alpha} \rho^{\beta} \eta^{\gamma}$$

in which case

$$[cl] \quad h \quad [a^{\alpha} \rho^{\beta} \eta^{\gamma}]$$

That is

$$\mathrm{ML}^{2}\mathrm{T}^{-2}L \quad h \quad \mathrm{L}^{\alpha}(\mathrm{ML}^{-3})^{\beta}(\mathrm{ML}^{-1}\mathrm{T}^{-2})^{\gamma}$$

Equate the powers of M, L and T:

$$1 = \beta + \gamma; \qquad 3 = \alpha - 3\beta - \gamma; \qquad -2 = -2\gamma$$

This gives $\alpha = 4, \beta = 0, \gamma = 1,$ and hence $c \propto \frac{\eta a^{4}}{l}$.

iii. How does the orbital period P of a planet depend on the radius of its orbit, the mass M of the Sun, and the gravitational constant G?

Assume

$$P \propto G^{\alpha} M^{\beta} a^{\gamma}$$

It is left to the reader to show that $P \propto \sqrt{\frac{a^{3}}{GM}}$.

iv. A sphere of radius a moves slowly at a speed v through a fluid of density ρ and dynamic viscosity η. How does the viscous drag F depend upon these four variables?

Four variables, but only three dimensions, and hence three equations! What to do? If you have better insight than I have, or if you already know the answer, you can assume that it does not depend upon the density. I haven't got such clear insight, but I'd be willing to suppose that the viscous drag is proportional to the first power of the dynamic viscosity. In which case I'd be happy to assume that

$$\frac{F}{\eta} \propto a^{\alpha} \rho^{\beta} v^{\gamma}$$

Then

$$\frac{\mathrm{MLT^{-2}}}{\mathrm{ML^{-1}T^{-1}}} = \mathrm{L}^{\alpha} (\mathrm{ML}^{-3})^{\beta} (\mathrm{LT}^{-1})^{\gamma}$$

Equate the powers of M, L and T:

$$0 = \beta; \qquad 2 = \alpha - 3\beta + \gamma' \qquad -1 = -\gamma$$

This gives $\alpha = 1, \beta = 0, \gamma = 1$, and hence $F \propto \eta a v$.

22.5: Dimensionless Quantities

Dimensionless Quantities are used extensively in fluid dynamics. For example, if a body of some difficult shape, such as an aircraft, is moving through a fluid at speed V, it will experience all sorts of forces, external and internal. The ratio of the internal forces to the external forces will depend upon its speed, and the viscosity of the fluid, and the size of the body. By “size” of a body of “difficult” shape we could take the distance between two defined points on the body, such as its top and bottom, or its front and back, or its greatest width, or whatever. Call that distance l. But the ratio of the internal to the viscous forces is dimensionless, so it must depend on some combination of the viscosity, speed V and linear size l that is dimensionless. Since V and l do not contain M in their dimensions, the viscosity concerned must be the *kinematic* viscosity ν, which is the ratio of dynamic viscosity to density and does not have M in its dimensions. So, what combination of ν, V and l is dimensionless?

It is easy to see that $\frac{VI}{V}$ - or any power of it, positive, negative, zero, integral, nonintegral - is dimensionless. $\frac{VI}{V}$ is called the Reynolds number, and is usually given the symbol Re. It is supposed that if you make a small model of the aircraft (or whatever the body is) and move it through some fluid and some speed, the ratio of internal to viscous forces in the model will be the same as in the real thing provided that the Reynolds numbers in the model and in the real thing are the same.

There are oodles of similar dimensionless numbers used in fluid dynamics, such as **Froude’s number** and **Mach number**, but this example of Reynolds number should give the general idea.

22.6: Different Fundamental Quantities

We stated at the beginning of this chapter that any mechanical quantity could be expressed in terms of three fundamental quantities, *mass*, *length* and *time*. But there is nothing particularly magic about these quantities. For example, we might decide that we could express any mechanical quantity in terms of, say, *energy* E, *speed* V and *angular momentum* J. We might then say that the dimensions of area could be expressed as $E^{-2}V^2J^2$.(Verify this!)

While agreeing that such a system might be possible, you might feel that it would be totally absurd and there is no point in reading further.

But stop! Such a system is not only possible, but it is *normally and routinely used* in the field of *high-energy particle physics*. That, perhaps, is a surprise, but, if you are thinking of taking an interest in particle physics, read on.

The *units* generally used in particle physics to express the fundamental quantities energy, speed and angular momentum are GeV (or MeV, or TeV , etc) for energy, the speed of light c for speed, and the modified Planck constant $\hbar$ for angular momentum. There are often referred to as "natural" units, the speed of light being a "natural" unit of speed and ħ being a "natural" unit for angular momentum, whereas metre, kilogram and second are not so "natural" in this sense as they are "man-made". It is true that a GeV is not particularly "natural", but at least a system with GeV, c and $\hbar$ as fundamental quantities is certainly more "natural" than metre-kilogram-second.

In any case, the dimensions of *mass* in this system are EV^{-2}. (You can see this immediately, for example from Einstein's famous equation $E = mc^2$.) The units used in this system are GeV/c^2. Thus the rest mass of a proton is 0.9383 GeV/c^2, and the rest mass of an electron is 0.5110 MeV/c^2. One way to interpret this, if you like, is to say that the rest-mass energy of a proton (i.e. its m_0c^2) is 0.9383 GeV.

Likewise the dimensions of linear momentum are EV^{-1}, and units in which it is expressed are GeV/c. (You can see this, for example, if you look at the energy and momentum of a photon: E = hν, p = h/γ, from which $\frac{P}{E}=\frac{1}{v\gamma}=\frac{1}{c}$)

Torque (which has the same dimensions as energy) is equal to rate of change of angular momentum, from which we see that time has dimensions E−1J and could be expressed in units of ħ/GeV. Alternatively you can see that [time] = ℏ/GeV immediately from Planck's equation . $E=\hbar\omega$ And speed is distance over time, so that we see that distance, or length, has dimensions $E^{-1}VJ$, and hence units ħc/GeV.

Using data from the 2010 Particle Physics Booklet, I calculate as follows.

Mass:	1 GeV/c^2 = 1.782 661 76 ~ 10^{-27} kg
Length:	1 ℏc/GeV = 1.973 269 63 ~ 10^{-16} m
Time:	1 ℏ/GeV = 6.582 118 99 ~ 10^{-26} s
Energy:	1 GeV = 1.602 176 49 ~ 10^{-10} J
Linear Momentum:	1 GeV/c = 5.344 285 50 ~ 10^{-19} kg m s^{-1}

I give here a table of the dimensions (in terms of EVJ) of the same quantities as in the table of page 2. I dare say some of them are never likely to be needed, but some certainly will be needed, and, rather than predict which will be useful and which not, I might as well give them all. The dynamic viscosity of water at room temperature is about 10^{-3} kg m^{-1} s^{-1}, or 10^{-3} dekapoise. I cannot imagine anyone needing to know that the dynamic viscosity of water at room temperature is about 7.3 ~ 10^{-18} $(GeV)^3/(c^3\hbar^2)$, or that its surface tension is so many $(GeV)^3/(c\hbar)^2$ − but you never know

Mass	EV^{-2}	GeV/c^2
Length	$E^{-1}VJ$	$\hbar c$/GeV
Time	$E^{-1}J$	$\hbar$/GeV
Density	$E^2V^{-5}J^{-3}$	$(\text{GeV})^2/(c^5\hbar^3)$
Speed	V	c
Acceleration	EVJ^{-1}	GeV $c/\hbar$
Force	$E^2V^{-1}J^{-1}$	$(\text{GeV})^2/(c\hbar)$
Work, Energy, Torque	E	GeV
Action	J	$\hbar$
Rotational inertia	$E^{-1}J^2$	$\hbar^2$/GeV
Angular speed	EJ	GeV $\hbar$
Angular acceleration	E^2J^2	$(\text{GeV})^2\hbar^2$
Angular momentum	J	$\hbar$
Pressure, elastic modulus	$E^4V^{-3}J^{-3}$	$(\text{GeV})^4/(c\hbar)^3$
Gravitational constant	$E^{-2}V^5J$	$c^5\hbar/(\text{GeV})^2$
Dynamic viscosity	$E^3V^{-3}J^{-2}$	$(\text{GeV})^3/(c^3\hbar^2)$
Kinematic viscosity	$E^{-1}V^2J$	$c^2\hbar/(\text{GeV})$
Force constant	$E^3V^{-2}J^{-2}$	$(\text{GeV})^3/(c\hbar)^2$
Torsion constant	E	GeV
Surface tension	$E^3V^{-2}J^{-2}$	$(\text{GeV})^3/(c\hbar)^2$
Schrödinger wavefunction Ψ	$E^2V^{-3/2}J^{-2}$	$(\text{GeV})^2/(c^{3/2}\hbar^2)$
Schrödinger wavefunction ψ	$E^{3/2}V^{-3/2}J^{-3/2}$	$(\text{GeV})^{3/2}/(c^{3/2}\hbar^{3/2})$

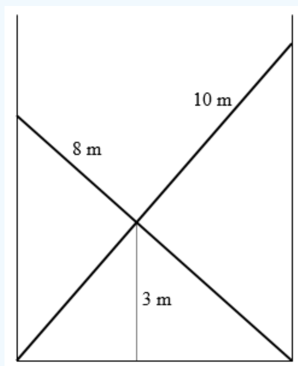
22.7: Appendix A

Miscellaneous Problems

In this Appendix I offer a number of random problems in classical mechanics. They are not in any particular order – they come just as I happen to think of them, and they are not necessarily related to any of the topics discussed in any of the chapters. They are intended just to occupy you on those dull, rainy days when you have nothing better to do. Solutions will be in Appendix B – except that whenever I add any new problems to Appendix A, which I shall from time to time, I shall wait a few days before posting the solutions in Appendix B.

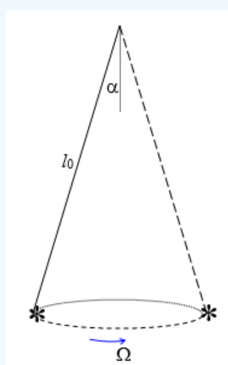
Exercise 22.7.1

No book on classical mechanics is complete without a problem of a ladder leaning against a wall. Here, then, is a ladder problem – except that it is nothing whatever to do with mechanics, and it is put here just for fun. It is a problem only in geometry, yet it is one which some people at first find difficult. It even seems difficult to try to find an approximate solution by trying to draw it accurately to scale, and I have deliberately *not* drawn it to scale, so you can't find the answer merely by taking a ruler and measuring it!



Two ladders, of lengths 8 m and 10 m, are leaning against two walls as shown. Their point of intersection is 3 m above the ground. What is the distance between the walls?

Exercise 22.7.2



A pendulum of length l_0, is set into motion so that it describes a cone as shown of semi-vertical angle α, the bob describing a horizontal circle at angular speed Ω

Show that

$$\cos\alpha = \frac{g}{l_0\Omega^2}.$$

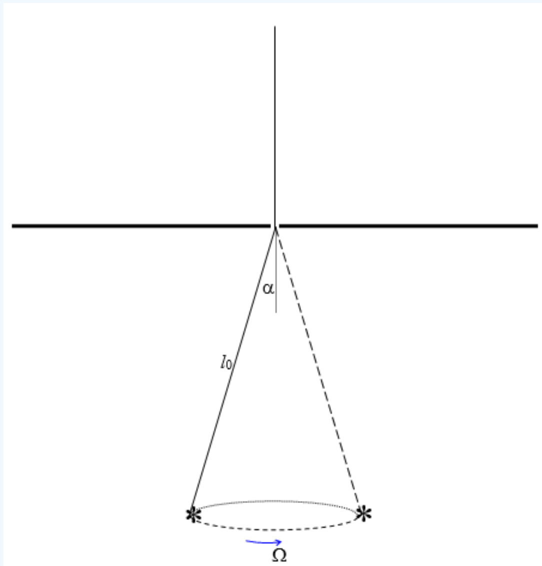
This, of course, is a very trivial problem not worthy of your mettle. It is given only as an introduction to the next problem.

Exercise 22.7.3a

A string of a pendulum passes through a board as shown in the figure below, in such a manner that, by lowering or raising the board, the length of the string below the board can be varied. The part below the board is initially of length l_0, and it is set into motion as a conical pendulum so that the angular speed and the semi vertical angle are related by

$$\cos\alpha = \frac{g}{l_0\Omega^2}.$$

As the board is raised or lowered (or alternatively the pendulum is lowered or raised) and consequently the length l below the board is varied, the semivertical angle θ will change and so will the angular speed ω. (The symbols l_0, α and Ω represent the initial values of these quantities.)



Show that

i. $l^3 \sin^3\theta \tan\theta$ is constant,
ii. $\omega^3 \cot^2\theta$ is constant,
iii. $\omega^3(\omega l^2 - \Omega l_0^2 \sin^2\alpha)$ is constant.

Exercise 22.7.3b

Start with the following initial conditions:

$$l_0 = 50 \text{ cm} \qquad \Omega = 5 \text{ rad s}^{-1}$$

and assume that $g = 9.8 \text{ m s}^{-2}$, so that $\alpha = 38^\circ\ 22'$.

i. Plot a graph of θ (vertically) versus l (horizontally), for $l = 0$ to 1 m. When l = 40 cm, what is θ correct to one arcmin?
ii. Plot a graph of ω (vertically) versus θ (horizontally), for θ = 0 to 70°.
iii. Plot a graph of ω (vertically) versus l (horizontally) for l = 16 cm to 1 m. When l = 60 cm, what is ω correct to four significant figures?

The next few problems involve a rod with its lower end in contact with a horizontal table and the rod falling over from an initial vertical (or inclined) position. There are several versions of this problem. The table could be smooth, so that the rod freely slips over the table. Or the lower end could be freely hinged at the table, so that the lower end does not move as the rod falls over. Or the table might be rough, so that the rod might or might not slip.

Exercise 22.7.4

A uniform rod of mass m and length $2l$ is initially vertical with its lower end in contact with a smooth horizontal table. It is given an infinitesimal angular displacement from its initial position, so that it falls over. When the rod makes an angle θ with the vertical, find:

The angular speed of the rod;

The speed at which the centre of the rod is falling;

The speed at which the lower end of the rod is moving;

Show that the speed of the lower end is greatest when $\theta = 37^\circ\ 50'$.

If the length of the rod is 1 metre, and g = 9.8 m s^{-2}, what is the angle θ when the speed of the lower end is 1 m s^{-1}?

Exercise 22.7.5

A uniform rod is initially vertical with its lower end smoothly hinged to a horizontal table. Show that, when the rod falls over, the reaction of the hinge upon the rod is vertical when the rod makes an angle 48^{o} 11' with the vertical, and is horizontal when the rod makes an angle 70^{o} 31' with the vertical.

Exercise 22.7.6

A uniform rod of length 1 metre, with its lower end smoothly hinged to a horizontal table, is initially held at rest making an angle of 40^{o} with the vertical. It is then released. If g = 9.8 m s^{-1}, calculate its angular speed when it hits the table in a horizontal position (easy) and how long it takes to get there (not so easy).

Exercise 22.7.7

A uniform rod is initially vertical with its lower end in contact with a rough horizontal table, the coefficient of friction being μ.

Show that:

Exercise 22.7.7a

If μ < 0.3706, the lower end of the rod must slip before the rod makes an angle θ with the vertical of 35^{o} 05'.

Exercise 22.7.7b

If μ > 0.3706, the rod will not slip before θ = 51^{o} 15', but it will certainly slip before θ = 70^{o} 31' .

Exercise 22.7.7c

If μ = 0.25, at what angle θ will the lower end slip? If μ = 0.75?

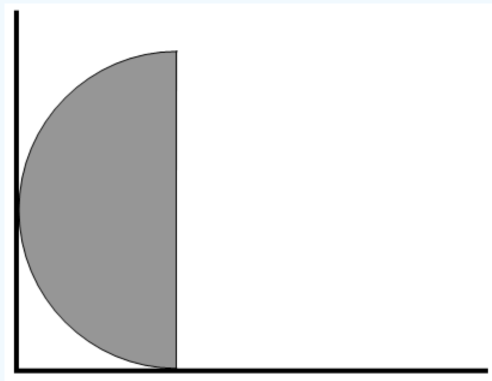
Exercise 22.7.8

It is time for another ladder problem. Most ladders in elementary mechanics problems rest on a rough horizontal floor and lean against a smooth vertical wall. In this problem, both floor and wall are smooth. The ladder starts making an angle of a with the vertical, and then it is released. It immediately starts to slip, of course. After a while it will cease contact with the smooth vertical wall. Show that, at the moment when the upper end of the ladder loses contact with the wall, the angle q that the ladder makes with the vertical is given by $\cos\theta = \frac{2}{3}\cos\alpha$.

Exercise 22.7.9

If you managed that one all right, this one, which is somewhat similar, should be easy. Maybe.

A uniform solid semicylinder of radius a and mass m is placed with its curved surface against a smooth vertical wall and a smooth horizontal floor, its base initially being vertical.



It is then released. Find the reaction N_1 of the floor on the semicylinder and the reaction N_2 of the wall on the semicylinder when its base makes an angle θ with the vertical.

Show that the semicylinder loses contact with the wall when $\theta = 90^{\circ}$, and that it then continues to rotate until its base makes an angle of $39^{\circ}\ 46'$ with the vertical before it starts to fall back.

Many problems in elementary mechanics involve a body resting upon or sliding upon an inclined plane. It is time to try a few of these. The first one is very easy, just to get us started. The two following that might be more interesting.

Exercise 22.7.10

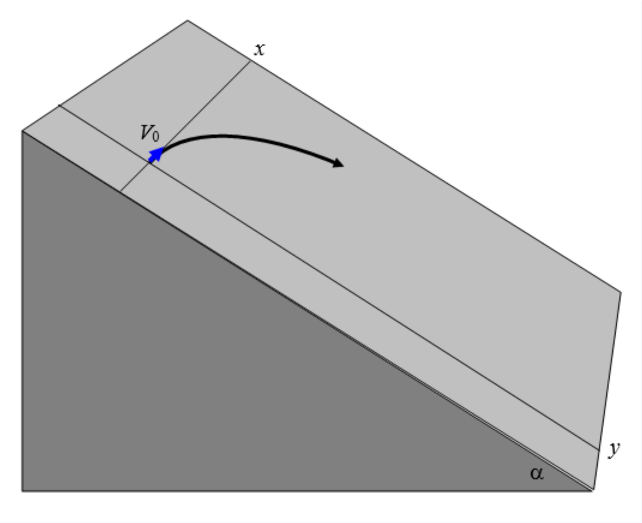
A particle of mass m is placed on a plane which is inclined to the horizontal at an angle α that is greater than $\tan^{-1}\mu$, where μ is the coefficient of limiting static friction. What is the least force required to prevent the particle from sliding down the plane?

Exercise 22.7.11

A cylinder or mass m, radius a, and rotational inertia ka^2 rolls without slipping down the rough hypotenuse of a wedge on mass M, the smooth base of which is in contact with a smooth horizontal table. The hypotenuse makes an angle α with the horizontal, and the gravitational acceleration is g. Find the linear acceleration of the wedge as it slips along the surface of the table, in terms of m, M, g, a, k and α.

[Note that by saying that the rotational inertia is ka^2, I am letting the question apply to a hollow cylinder, or a solid cylinder, or even a hollow or solid sphere.]

Exercise 22.7.12

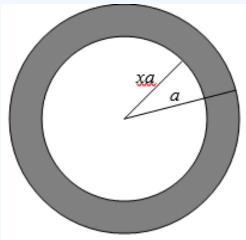


A particle is placed on a rough plane inclined at an angle a to the horizontal. It is initially in limiting static equilibrium. It is given an initial velocity V_0 along the x-axis. Ignoring the small difference between the coefficients of moving and limiting static friction, show that at a point on the subsequent trajectory where the tangent to the trajectory makes an angle ψ with the x-axis, the speed V is given by

$V = \frac{V_0}{1+\cos\psi}$

What is the limiting speed reached by the particle after a long time?

Exercise 22.7.13

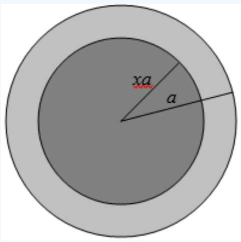


Calculate the moment of inertia of a hollow sphere, mass M, outer radius a, inner radius xa. Express your answer in the form

$$I = \frac{2}{5}Ma^2 \times f(x) .$$

What does your expression become if $x = 0$? And if $x \to 1$?

Exercise 22.7.14



Calculate the moment of inertia of a spherical planet of outer radius a, consisting of a dense core of radius xa surrounded by a mantle of density s times the density of the core. Express your answer in the form

$$I = \frac{2}{5}Ma^2 \times f(x, s) .$$

Make sure that, if the density of the core is zero, your expression reduces to the answer you got for Exercise 13.

Draw graphs of $\frac{I}{(\frac{2}{5}Ma^2)}$ versus x (x going from 0 to 1), for s = 0.2, 0.4, 0.6 and 0.8.

Show that, for a given mass M and density ratio s, the moment of inertia is least for a core size give by the solution of

$$2(I-s)x^5+15x^2-9=0$$

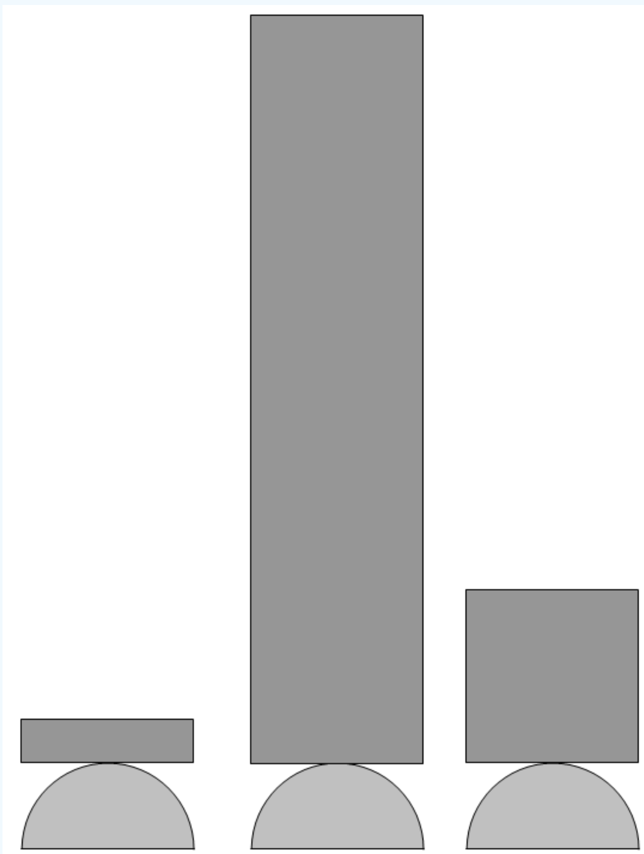
For a mantle-to-density ratio of 0.6, calculate the core size for which the moment of inertia is least and calculate (in units of $\frac{2}{5}Ma^2$) the moment of inertia for that core.

Now let's see if we can determine the core size from a knowledge of the moment of inertia. It is sometimes asserted that one can determine the moment of inertia (and hence the core size) of a planet from the rate of precession of the orbit of a satellite. I am not sure how this would work with a planet such as Mercury, which has never had a satellite in orbit around it. (Mariner 10, while in orbit around the Sun, made three fly-bys past Mercury). Unless a planet departs from spherical symmetry, the orbit of a satellite will not precess, since the gravitational planet is then identical with that from a point mass. And, even if a planet were dynamically oblate, the rate of precession allows us to determine the dynamical ellipticity $\frac{(C-A)}{C}$, but not either moment of inertia separately.

Nevertheless, let's suppose that the moment of inertia of a planet is (0.92 ± 1%) % $\frac{2}{5}Ma^2$; specifically, let's suppose that the moment of inertia has been determined to be between 0.911 and 0.929 % $\frac{2}{5}Ma^2$, and that the mantle-to-core density ratio is known (how?) to be 0.6. Calculate the possible range in the value of the core radius x.

Exercise 22.7.15

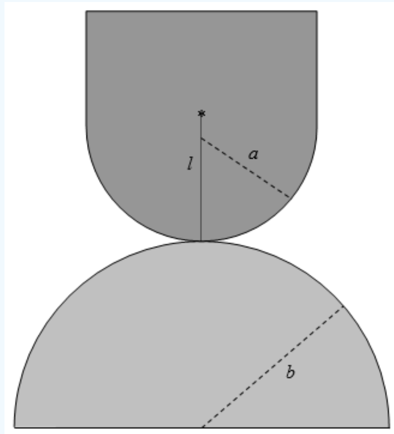
A rectangular brick of length $2l$ rests (with the sides of length $2l$ vertically) on a rough semicylindrical log of radius R. The drawing below shows three such bricks. In the first one, $2l$ is quite short, and it looks is if it is stable. In the second one, $2l$ is rather long, and the equilibrium looks decidedly wobbly. In the third one, we're not quite sure whether the equilibrium is stable or not. What is the longest brick that is stable against small angular displacements from the vertical?



Exercise 22.7.16

A Thing with a semicylindrical (or hemispherical) base of radius a is balanced on top of a rough semicylinder (or hemisphere) of radius b as shown. The distance of the centre of mass of the Thing from the line (or point) of contact is l. Show that the equilibrium is stable if

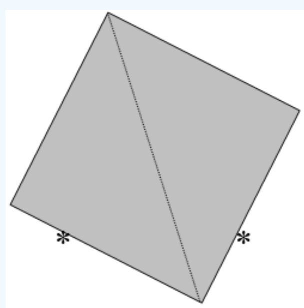
$$\frac{1}{l} > \frac{1}{a} + \frac{1}{b} \,.$$



If $a = b$, is the equilibrium stable if the Thing is

1. A hollow semicylinder?
2. A hollow hemisphere?
3. A uniform solid semicylinder?
4. A uniform solid hemisphere?

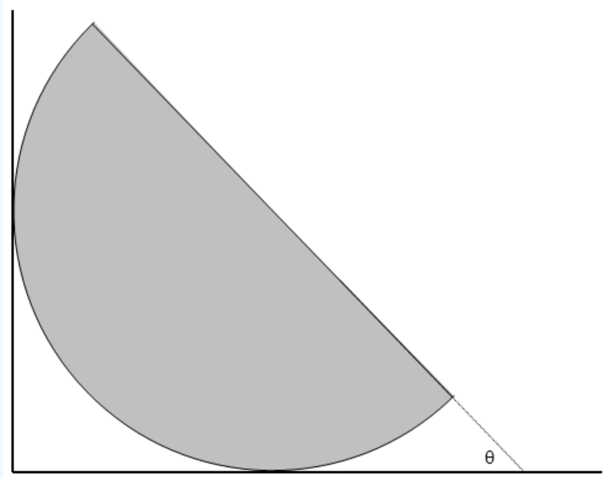
Exercise 22.7.17



A log of square cross-section, sides $2a$, rests on two smooth pegs a distance $2ka$ apart, one of the diagonals making an angle θ with the vertical.

Show that, if $k < \frac{1}{\sqrt{8}} = 0.354$ the only equilibrium position possible is θ = 90º, but that this position is unstable; consequently, following a small displacement, the log will fall out of the pegs. Show that if the pegs are farther apart, with $0.354 < k < 0.500$, three equilibrium positions are possible. Which of them are stable, and which are unstable? If $k = 0.45$, what are the possible equilibrium values of θ? Show that, if $0.500 < k < 1.414$, only one equilibrium position is possible, and that it is stable.

Exercise 22.7.18



A uniform solid hemisphere of radius a rests in limiting static equilibrium with its curved surface in contact with a smooth vertical wall and a rough horizontal floor (coefficient of limiting static friction μ). Show that the base of the hemisphere makes an angle θ with the floor, where

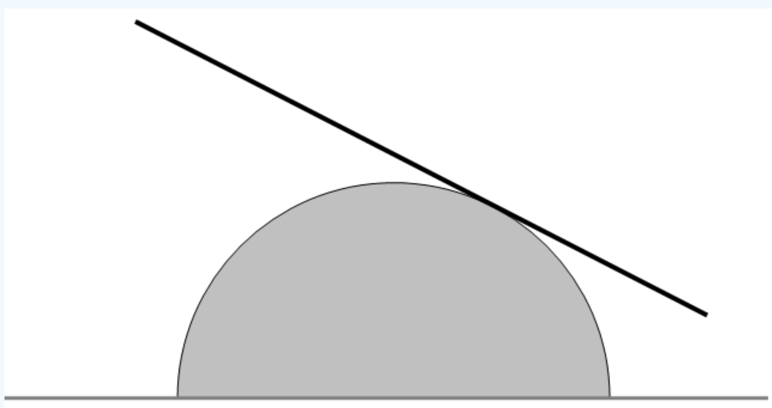
$$\sin\theta = \frac{8\mu}{3}.$$

Calculate the value of θ if (a) $\mu = \frac{1}{4}$ and $\mu = \frac{3}{8}$.

What happens if $\mu > \frac{3}{8}$?

Exercise 22.7.19

A uniform rod of length $2l$ rocks to and fro on the top of a rough semicircular cylinder of radius a. Calculate the period of small oscillations.



A uniform solid hemisphere of radius a with its curved surface in contact with a rough horizontal table rocks through a small angle. Show that the period of small oscillations is

$$P = 2\pi\sqrt{\frac{26a}{15\mathrm{g}}}.$$

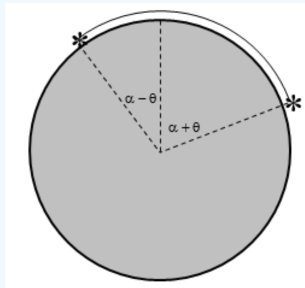
Exercise 22.7.21

The density ρ of a solid sphere of mass M and radius a varies with distance r from the centre as

$$\rho = \rho_0\left(1 - \frac{r}{a}\right).$$

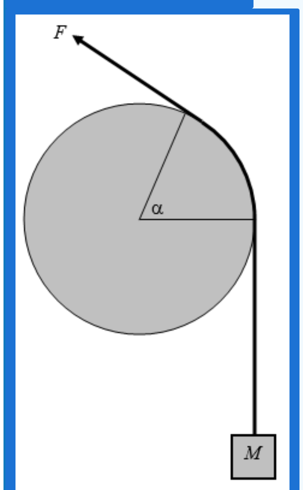
Calculate the (second) moment of inertia about an axis through the centre of the sphere. Express your answer in the form of constant % Ma^2.

Exercise 22.7.22



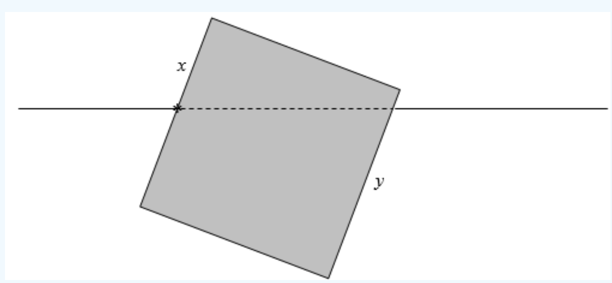
Two identical particles are connected by a light string of length $2a\alpha$. The system is draped over a cylinder of radius a as shown, the coefficient of limiting static friction being μ. Determine the angle θ when the system is in limiting equilibrium and just about to slide.

Exercise 22.7.23



A mass M hangs from a light rope which passes over a rough cylinder, the coefficient of friction being μ and the angle of lap being α. What is the least value of F, the tension in the upper part of the rope, required to prevent the mass from falling?

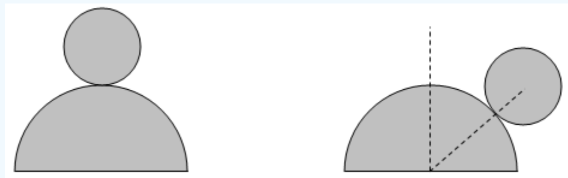
Exercise 22.7.24



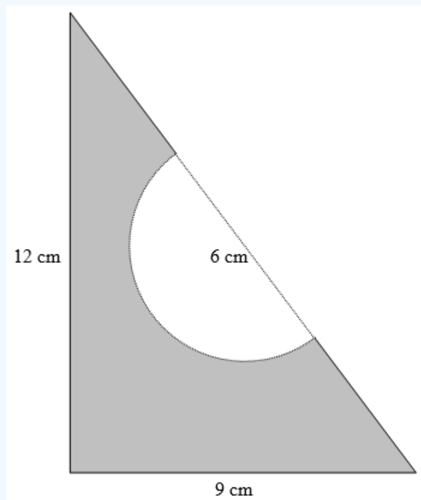
A wooden cube floats on water. One of its faces is freely hinged to an axis fixed in the surface of the water. The hinge is fixed at a distance from the top of the face equal to x times the length of a side. The opposite face is submerged to a distance y times the length of a side. Find the relative density s (that is, relative to the density of the water) of the wood in terms of x and y.

Exercise 22.7.25

A uniform solid sphere sits on top of a rough semicircular cylinder. It is given a small displacement so that it rolls down the side of the cylinder. Show that the sphere and cylinder part company when the line joining their centres makes an angle 53º 58' with the vertical.



Exercise 22.7.26



A student has a triangular sandwich of sides 9 cm, 12 cm, 15 cm. She takes a semicircular bite of radius 3 cm out of the middle of the hypotenuse. Where is the centre of mass of the remainder? Is it inside or outside the bite?

Exercise 22.7.27

An rubber elastic band is of length $2\pi a$ and mass m; the force constant of the rubber is k. The band is thrown in the air, spinning, so that it takes the form of a circle, stretched by the centrifugal force. (This takes much practice, skill and manual dexterity.) Find a relation between its radius and angular speed, in terms of a, m and k.

Exercise 22.7.28

Most of us have done simple problems on friction at high school or in first year at college or university. You know the sort – a body lies on a rough horizontal table. A force is applied to it. What happens? Try this one.

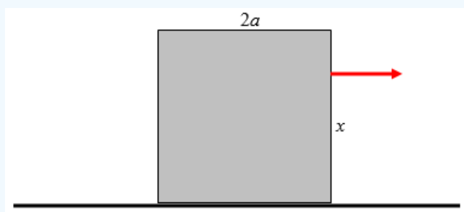
Find a uniform rod AB. A ruler will do as long as it is straight and not warped. Or a pencil of hexagonal (not circular) cross-section, provided that it is uniform and does not have an eraser at the end. Place it on a rough horizontal table. Gradually apply a horizontal force perpendicular to the rod at the end A until the rod starts to move. The end A will, of course, move forward. Look at the end B – it moves backward. There is a point C somewhere along the rod that is stationary. I.e., the initial motion of the rod is a rotation about the point C. Calculate – and measure – the ratio AC/AB. What is the force you are exerting on A when the rod is just about to move, in terms of its weight and the coefficient of friction?

Exercise 22.7.29

Some of the more dreaded friction problems are of the “Does it tip or does it slip?” type. This and the following four are examples of this type.

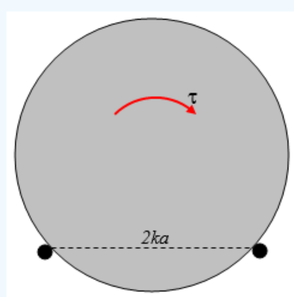
A uniform solid right circular cone of height h and basal radius a is placed on an inclined plane whose inclination to the horizontal is gradually increased. The coefficient of limiting static friction is μ. Does the cone slip, or does it tip?

Exercise 22.7.30



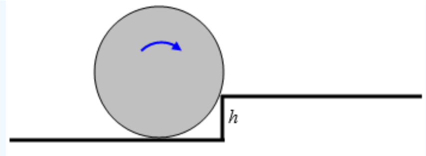
A cubical block of side $2a$ rests on a rough horizontal table, the coefficient of limiting static friction being μ. A gradually increasing horizontal force is applied as shown at a distance x above the table. Will the block slip or will it tip? Show that, if $\mu < \frac{1}{2}$, the block will slip whatever the value of x.

Exercise 22.7.31

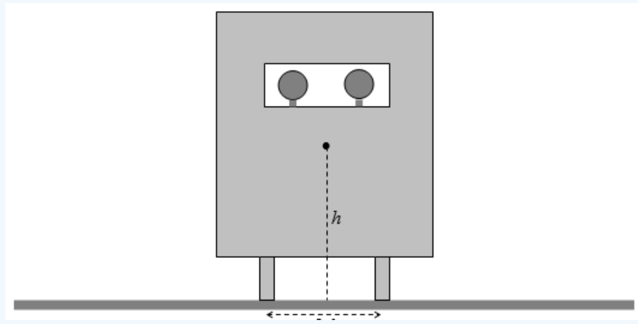


A cylindrical log of diameter $2a$ and mass m rests on two rough pegs (coefficient of limiting static friction μ) a distance $2ka$ apart. A gradually increasing torque τ is applied as shown. Does the log slip (i.e. rotate about its axis) or does it tip (about the right hand peg)?

When you’ve done that one, you can try a variant (which I haven’t worked out and haven’t posted a solution) in which a cylinder of radius a is resting against a kerb (or curb, if you prefer that spelling) of height h, and a torque is applied. Will it tip or will is slip?

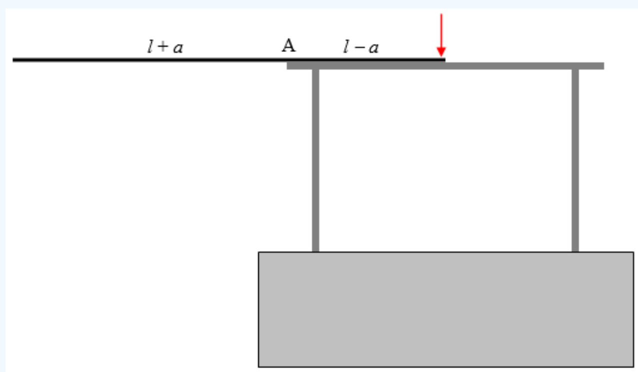


Exercise 22.7.32



This problem reveals the severe limitations of my artistic abilities, but the drawing above, believe it or not, represents a motor car seen from behind. You can see the driver and passenger. The height of the centre of mass is h and the distance between the wheels is $2d$. The car is travelling on a horizontal road surface, coefficient of friction μ, and is steering to the left in a circle of radius R, the centre of curvature being way off to the left of the drawing. As they gradually increase their speed, will the car slip to the right, or will it tip over the right hand wheel, and at what speed will this disaster take place? Fortunately, driver and passenger were both wearing their seat belts and neither of them was badly hurt, and never again did they drive too fast round a corner.

Exercise 22.7.33



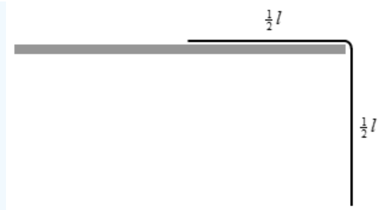
A uniform rod of length $2l$ rests on a table, with a length $l-a$ in contact with the table, and the remainder, $l+a$ sticking over the edge – that is, a is the distance from the edge of the table to the middle of the rod. It is initially prevented from falling by a force as shown. When the force is removed, the rod turns about A. Show that the rod slips when it makes an angle θ with the horizontal, where

$$\tan\,\theta\, =\, \frac{2\mu}{2+9(\frac{a}{l})^2}$$

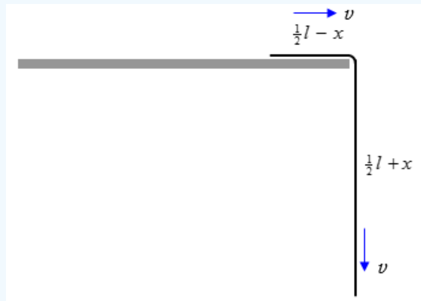
Here m is the coefficient of limiting static friction at A.

Exercise 22.7.34

A flexible chain of mass m and length l is initially at rest with one half of it resting on a smooth horizontal table, and the other half dangling over the edge:



It is released, so that it starts to slide off the table. At a subsequent time t, a length $\frac{1}{2}l - x$ remains in contact with the table, the remaining length $\frac{1}{2}l + x$ hanging vertically, and the speed of the chain is v.



Show that

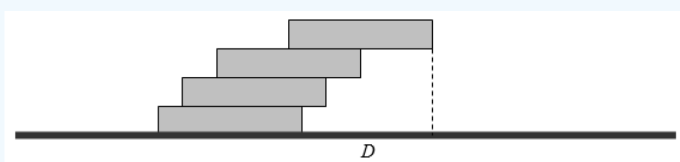
$$v^2 \;=\; gx \;+\; \frac{g}{l}x^2 \;,$$

$$x \;=\; \frac{l(e^{\sqrt{\frac{g}{lt}}} - 1)^2}{4e^{\sqrt{\frac{g}{lt}}}}$$

$$v \;=\; \frac{\sqrt{gl}(e^{\sqrt{\frac{4g}{lt}}} - 1)}{4e^{\sqrt{\frac{4g}{lt}}}}$$

Exercise 22.7.35a

Four books, each of width $2w$, are stacked on top of each other in a heap, thus:



What is the maximum possible overhang, D?

Exercise 22.7.35b

How many books would be needed to achieve an overhang of $10w$?

Exercise 22.7.35c

Given an unlimited supply of books, what is the maximum overhang achievable?

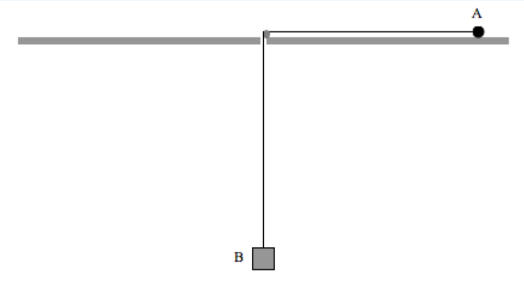
Exercise 22.7.36

The Man and the Dog.

At time $t = 0$, the Man is at the origin of coordinates, and he starts to walk up the y-axis at constant speed v. The Dog starts at $(a, 0)$ and runs at constant speed $Av(v > 1)$ towards the Man. The velocity of the Dog is always directed straight towards the

Man. Find an equation for the path pursued by the Dog, and draw a graph of this path. How far has the Man walked when the Dog reaches the Man, and how long does this take?

Exercise 22.7.37



A particle A of mass m is attached by a light string to a second particle, B, also of mass m. A rests on a smooth horizontal table, while B hangs vertically through a hole in the table. At time zero, the length of the horizontal portion of the string (i.e. the distance of A from the hole) is a, and A is moving on the table in a horizontal circle of radius a with initial angular speed ω_0.

At some subsequent time the length of the horizontal portion of the string is r and the angular speed of A is ω. Let us denote by the rate of increase of r with time, which will evidently be negative if B is falling.

Exercise 22.7.37a

Show that $\dot{r}$ is given by

$$\frac{\dot{r}^2}{ga} = 1 + \frac{a\omega_0^2}{2g}\left(1 - \frac{\omega}{\omega_0}\right) - \sqrt{\frac{\omega}{\omega_0}}. \tag{22.7.1}$$

Exercise 22.7.37b

Show that, if $a\omega_0^2 = g$, where $\Omega = \omega\,\omega_0$.

$$\frac{\dot{r}^2}{ga} = \frac{3}{2} - \frac{1}{2}\Omega - 1/\sqrt{\Omega}, \tag{22.7.2}$$

Exercise 22.7.37c

Show that there is only one value of Ω, namely 1, for which there is a real solution for $\dot{r}$, namely $\dot{r} = 0$. This implies that the system remains in equilibrium, with the radius of the circle, the angular speed of A and the height of B remaining constant, with the centrifugal force on A remaining equal to the weight of B.

Exercise 22.7.37d

Show that if $a\omega_0^2 = 2g$,

$$\frac{\dot{r}^2}{ga} = 2 - \Omega - 1/\sqrt{\Omega}$$

and that A moves outwards (its angular speed decreasing) and B moves upwards,

reaching a maximum speed of $\dot{r} = 0.331841\sqrt{ga}$

where $\dot{r} = 1.259921a$

when $\omega = 0.629961\omega_0$,

and it reaches an equilibrium when $\dot{r} = 0$

where $\dot{r} = 1.618034a$

when $\omega = 0.381966\omega_0$.

Exercise 22.7.37e

Show that if $a\omega_0^2 = \frac{1}{2}g$,

$$\frac{\dot{r}^2}{ga} = \frac{5}{4} - \frac{1}{4}\Omega - 1/\sqrt{\Omega},$$

and that A moves inwards (its angular speed increasing) and B moves downwards,

reaching a maximum speed of $\dot{r} = -0.243822\sqrt{ga}$

where $r = 0.793701a$

when $\omega = 1.587401\omega_0$,

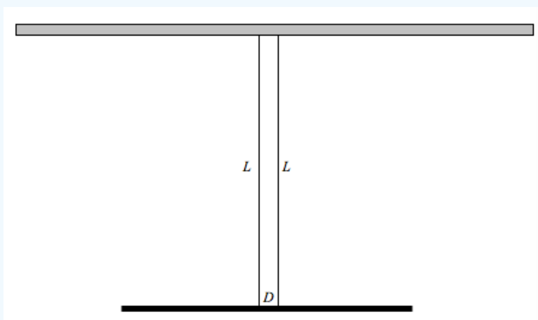
and it reaches an equilibrium when $\dot{r} = 0$

where $r = 0.640338a$

when $\omega = 2.438447\omega_0$.

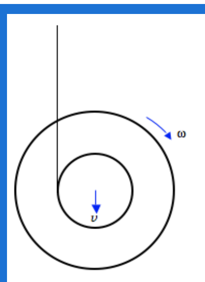
Exercise 22.7.38

This problem – the bifilar torsion pendulum − was suggested to me by Claude Plathey, who used the method in a practical application to determine the rotational inertia (moment of inertia) of a real nonuniform rod. He also drew my attention to an interesting paper on the determination of the moments of inertia of bodies (such as aircraft!) by the method: naca.larc.nasa.gov/digidoc/re...ACA-TR-467.PDF



A symmetric but not necessarily uniform rod of mass m and moment of inertia I is suspended from the ceiling by two light threads each of length L a distance D apart $(D << L)$.The rod is twisted about a vertical axis through its midpoint through a small angle and then released. Find the period of small oscillations in the horizontal plane.

Exercise 22.7.39



A yo-yo is of mass M and rotational inertia I. The radius of its axle is a, and it falls in the usual way with a length of string wrapped around the axle.

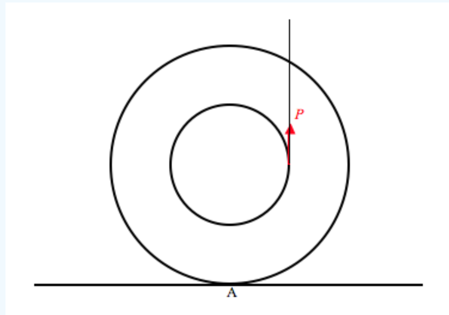
How that its linear acceleration downwards is

$$\frac{Ma^2}{Ma^2+I} \times g$$

and that the tension in the string i

$$\frac{I}{Ma^2+1} \times Mg$$

Exercise 22.7.40a



A yo-yo, mass M, axle radius a, outer radius b, rests on a horizontal table as shown. The string, wrapped around the axle, is held vertically as shown, and a force P is applied. The coefficient of friction between yo-yo and table is μ. Show that, if

$$\mu > \frac{MabP}{(Mg-P)(I+Mb^2}\text{,}$$

the initial motion of the yo-yo will be to roll to the left without slipping, with an initial linear acceleration

$$\frac{abP}{I+Mab}\text{;}$$

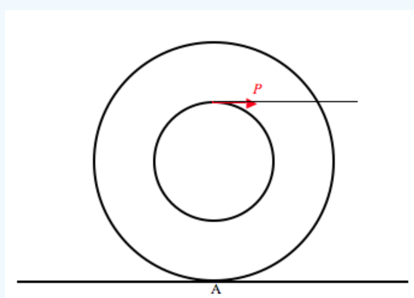
but that if

$$\mu < \frac{MabP}{(Mg-P)(I+Mb^2}\text{,}$$

the yo-yo will rotate counterclockwise without rolling, with an initial angular acceleration about C of

$$\frac{P(a-\mu)-\mu Mg}{I}.$$

Exercise 22.7.40b



A yo-yo, mass M, axle radius a, outer radius b, rests on a horizontal table as shown. The string, wrapped around the axle, is held horizontally as shown, and a force P is applied. The coefficient of friction between yo-yo and table is μ.

(i) Show that, ***if $I > Mab$*:**

If

$$\mu > \left(\frac{I-Mab}{I+Mb^2}\right)\left(\frac{P}{\text{Mg}}\right)$$

the initial motion of the yo-yo will be to roll to the right without slipping, with an initial linear acceleration

$$\frac{Pb(a+b)}{I+Mb^2}\text{;}$$

but that, if

$$\mu < \left(\frac{I-Mab}{I+Mb^2}\right)\left(\frac{P}{Mg}\right),$$

the yo-yo simultaneously accelerates to the right with a linear acceleration of

$$\frac{P-\mu Mg}{M}$$

while undergoing a clockwise angular acceleration about C of

$$\frac{Pa+\mu Mgb}{I}.$$

(ii) Show that, ***if*** $\boldsymbol{I > Mab}$***:***

If

$$\mu > \left(\frac{Mab-I}{Mb^2+I}\right)\left(\frac{P}{Mg}\right),$$

the initial motion of the yo-yo will be to roll to the right without slipping, with an initial linear acceleration

$$\frac{Pb(a+b)}{I+Mb^2};$$

but that, if

$$\mu < \left(\frac{Mab-I}{Mb^2+I}\right)\left(\frac{P}{Mg}\right),$$

the yo-yo simultaneously accelerates to the right with a linear acceleration of

$$\frac{P+\mu Mg}{M}$$

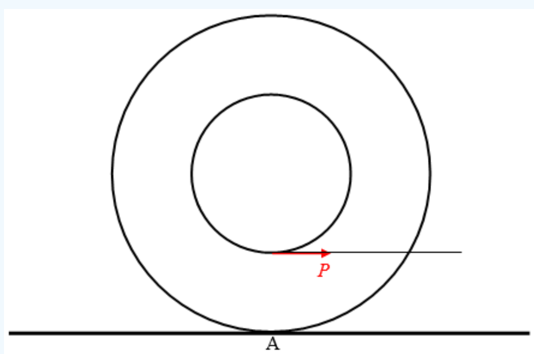
while undergoing a clockwise angular acceleration about C of

$$\frac{Pa-\mu Mgb}{I}.$$

(iii) Show that, ***if*** $\boldsymbol{I > Mab}$***:***

? Exercise $22.7.40c$

(After Problem 40(b) this one is a good deal easier and a welcome relief.)



A yo-yo, mass M, axle radius a, outer radius b, rests on a horizontal table as shown. The string, wrapped around the axle, is held horizontally as shown, and a force P is applied. The coefficient of friction between yo-yo and table is m.

Show that, if

$$\mu > \left(\frac{I+Mab}{I+Mb^2}\right)\left(\frac{P}{Mg}\right),$$

the initial motion of the yo-yo will be to roll to the right without slipping, with an initial linear acceleration

$$\frac{Pb(b-a)}{I+Mb^2}$$

and angular acceleration

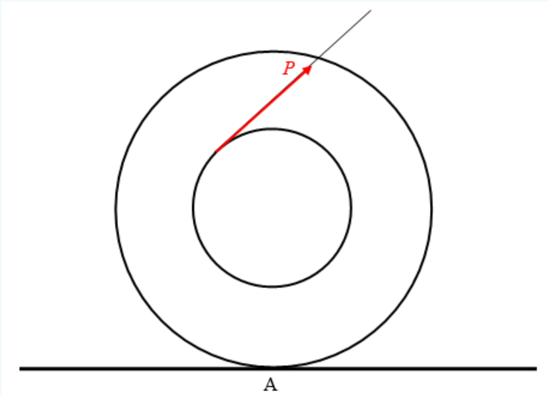
$$\frac{P(b-a)}{I+Mb^2}$$

but that, if

$$\mu < \left(\frac{I+Mab}{I+Mb^2}\right)\left(\frac{P}{Mg}\right)$$

the yo-yo slips at A. C accelerates to the right at a rate of $\frac{P-\mu Mg}{M}$, while the yo-yo spins around C with a counterclockwise angular acceleration of $\frac{Pa-\mu Mgb}{I}$.

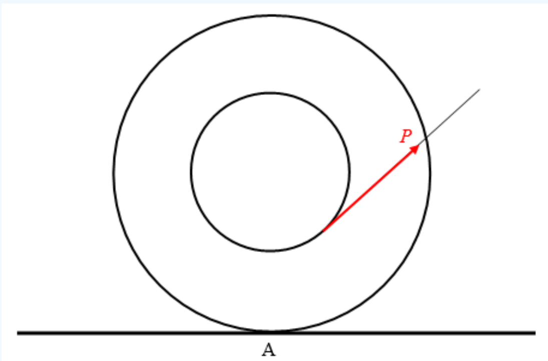
? Exercise 22.7.40d



A yo-yo, mass M, axle radius a, outer radius b, rests on a horizontal table as shown. The string, wrapped around the axle, is held at an angle θ to the horizontal as shown, and a force P is applied. The coefficient of friction between yo-yo and table is μ.

The complete analysis of this problem is similar to that of Problem 40(b), except that a factor of $\cos\theta$ appears in many of the equations. No new phenomena appear, and the analysis is tedious without any new points of interest. For that reason I limit this problem to asking you to show that the direction of the frictional force of the table on the yo-yo at A depends upon whether the $\cos\theta$ is less than or greater than $\frac{Mab}{I}$.

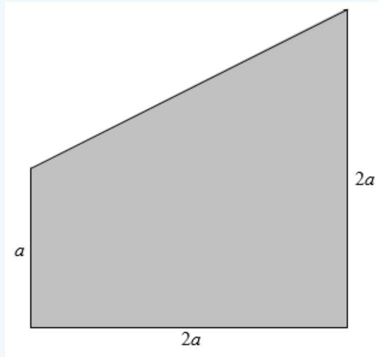
? Exercise 22.7.40e



A yo-yo, mass M, axle radius a, outer radius b, rests on a horizontal table as shown. The string, wrapped around the axle, is held at an angle θ to the horizontal as shown, and a force P is applied. The coefficient of friction between yo-yo and table is μ.

Show that, provided there is no slipping, the yo-yo rolls to the right if $\cos\theta > \frac{a}{b}$ and to the left if $\cos\theta < \frac{a}{b}$. Describe what happens if $\cos\theta = \frac{a}{b}$.

Exercise 22.7.41

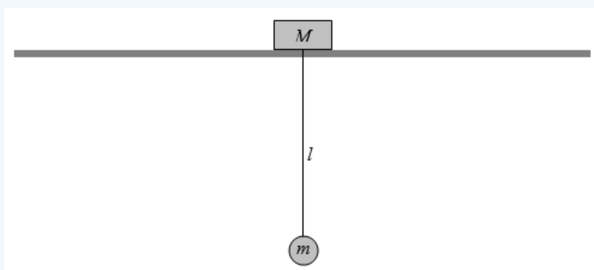


A uniform plane lamina of mass $3m$ is in the form of a truncated square like the one above.

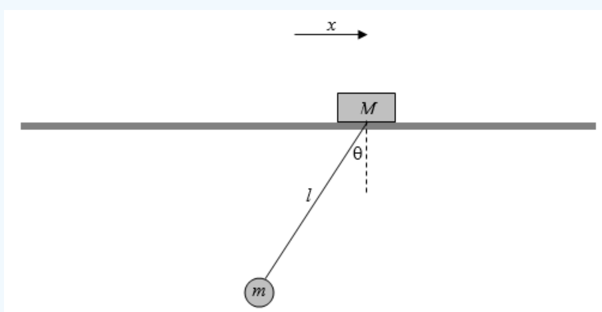
Find the position of the centre of mass, the principal moments of inertia with respect to the centre of mass, and the eccentricity and inclination of the momental ellipse.

Exercise 22.7.42

A mass M sits on a smooth horizontal table. A second mass, m, hangs from the first by a light inextensible string. A slot in the table allows m and the string to swing as a pendulum.



The system is then set in motion with the pendulum swinging, and the mass M sliding back and forth on the table. At some instant when the horizontal displacement of M from its equilibrium position is x the string makes an angle θ with the vertical.



Show that the equations of motion are

$$(M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0 \quad ,$$

$$I\ddot{\theta} + \ddot{x}\cos\theta - \dot{x}\sin\theta\dot{\theta} = -\mathrm{g}\sin\theta \quad .$$

Show that for small oscillations $(\cos\theta \approx 1,\ \sin\theta \approx \theta,\ \dot{\theta}^2\theta << \ddot{\theta},\ \dot{x}\theta\dot{\theta} << \mathrm{g})$ the period the motion is approximately $2\pi\sqrt{\frac{Ml}{(M+m)\mathrm{g}}}$. Note that, if $m << M$, this reduces to $2\pi\sqrt{\frac{l}{\mathrm{g}}}$ as expected.

? Exercise 22.7.43

A gun projects a shell, in the absence of air resistance, at an initial angle to the horizontal. The speed of projection varies with the angle α of projection and is given by

$$\text{initial speed} = V_0 \cos \tfrac{1}{2}\alpha.$$

Show that, in order to achieve the greatest range on the horizontal plane, the shell should be projected at an angle to the horizontal whose cosine c is given by the solution of the equation

$$3c^3 + 2c^2 - 2c - 1 = 0$$

and determine this angle to the nearest arcminute.

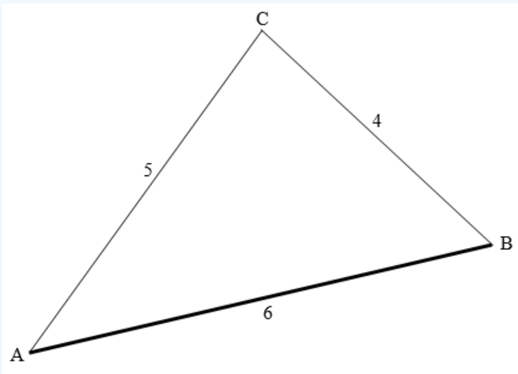
? Exercise 22.7.44

The length of a cylindrical log is L times its diameter, and its density is s times that of water $(0 < s < 1)$. Show that the log can float vertically in stable equilibrium, whatever its density, provided that $L < 0.707$. and that, if its length is greater than this, it can float vertically in stable equilibrium only if

$$L < \frac{1}{\sqrt{8s(1-s)}}.$$

Show that, if the length is equal to the diameter, it can float in stable equilibrium with its cylindrical axis vertical only if its density is less than 0.146 or greater than 0.854 times that of water.

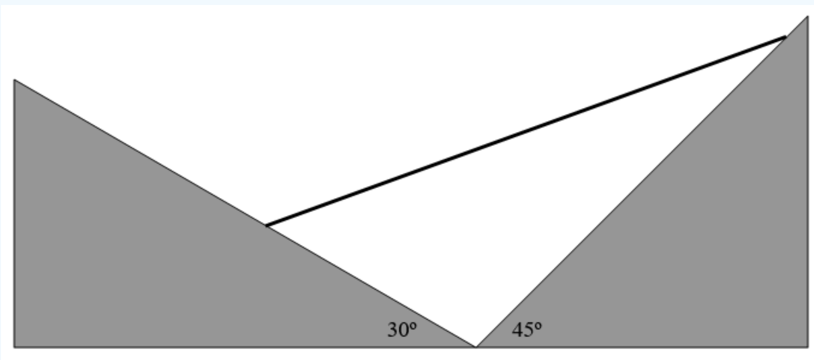
? Exercise 22.7.45



A uniform heavy rod of length 6 hangs from a fixed point C by means of two light strings of lengths 4 and 5. What angle does the rod make with the horizontal?

Incidentally, a (4, 5, 6) triangle has the interesting property that one of its angles is exactly twice one of the other ones.

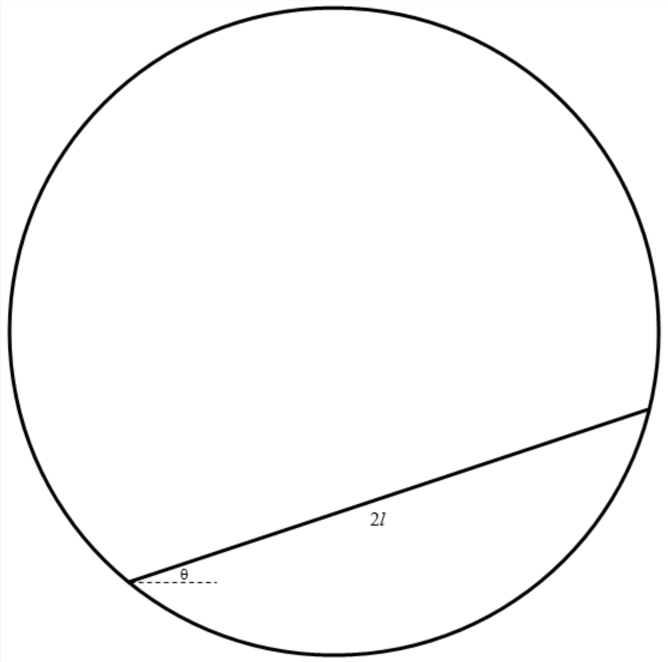
? Exercise 22.7.46



A uniform rod rests on two smooth (frictionless) planes inclined at 30º and 45º to the horizontal. What angle does the rod make with the horizontal?

❓ Exercise 22.7.47

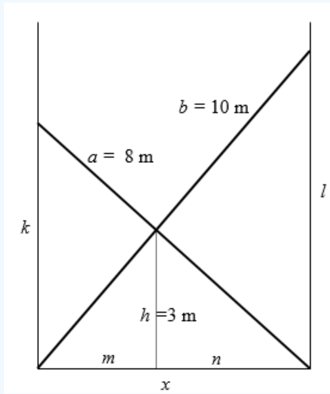
A uniform rod of length $2l$ rests on the inside of a circular cylindrical pipe of radius a. The coefficient of limiting static friction (often known for short, if with less precision, as "the"coefficient of friction) μ. What is the maximum angle θ that the rod can make with the horizontal in static equilibrium?



22.8: Appendix B

Solutions to Miscellaneous Problems

Exercise 22.8.1



By proportions, $\frac{h}{k} = \frac{n}{x}$ and $\frac{h}{l} = \frac{m}{x}$ and therefore $\frac{h}{k} + \frac{h}{l} = 1$.

Therefore by Pythagoras:

$$h\left(\frac{1}{\sqrt{a^2-x^2}} + \frac{1}{\sqrt{b^2-x^2}}\right) = 1..$$

Everything but x is known in this equation, which can therefore be solved for x. There are several ways of solving it; here's a suggestion. If we put in the numbers, the equation becomes

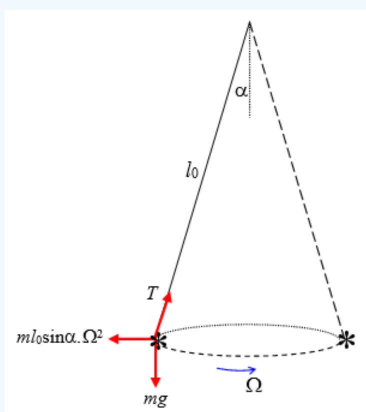
$3\left(\frac{1}{\sqrt{64-x^2}} + \frac{1}{\sqrt{100-x^2}}\right) - 1 = 0.$

Put $X = 100 - x^2$, and the equation becomes

$$3\left(\frac{1}{\sqrt{X-36}} + \frac{1}{\sqrt{X}}\right) - 1 = 0.$$

This can be written $f(X) = 3(A+B) - 1 = 0$, where A and B are obvious functions of X. Differentiation with respect to X gives $f'(X) = -\frac{3}{2}(A^3 + B^3)$ and Newton-Raphson iteration $(X = X - \frac{f}{f'})$ soon gives X, from which it is then found that x= 6.326 182 m.

Exercise 22.8.2



In the corotating frame the bob is in equilibrium under the action of three forces – its weight, the tension in the string and the centrifugal force. (If you do not like rotating reference frames and centrifugal force, it will be easy for you to do it "properly".)

Resolve the forces perpendicular to the string: $ml_0 \sin\alpha\Omega^2 . \cos\alpha \;=\; mg\sin\alpha$ and the problem is finished.

Exercise 22.8.3a

Raising or lowering the board does not apply any torques to the system, so the angular momentum L is conserved. That is,

$$L = ml^2 \sin^2\theta . \omega \qquad \text{is constant.} \tag{1}$$

We also have that

$$\mathrm{g} \;=\; l\cos\theta . \omega . \tag{2}$$

i. Eliminate w from these equations. This gives:

$$l^3 \sin^3\theta \tan\theta = \frac{L^2}{\mathrm{g}m^2} \tag{3}$$

which is constant.

ii. Eliminate l from equations (1) and (2). This gives:

$$\omega^3 \cot^2\theta = \frac{m\mathrm{g}^2}{L}, \tag{4}$$

which is constant.

iii. Eliminate θ from equations (1) and (2). This gives:

$$\omega^3\left(\omega l^2 - \frac{L}{m}\right) = \mathrm{g}^2 . \tag{5}$$

(Check the dimensions of all the equations.) Then we can get $\frac{L}{m}$ from equation (1) and hence

$$\omega^3(\omega l^2 - \Omega l_0^2 \sin^2\alpha) = \mathrm{g}^2 \;,$$

which is constant.

Exercise 22.8.3b

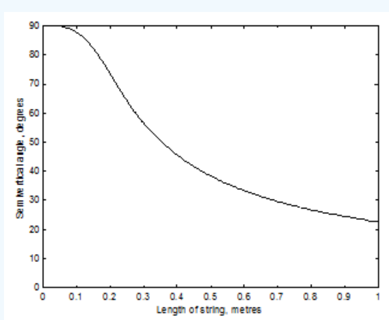
i.

$$l^3 \sin^3\theta \tan\theta = l_0^3 \sin^3\alpha \tan\alpha = 0.023675 \text{ m}^3 .$$

Although we are asked to plot θ vertically versus l horizontally, it is easier, when working out numerical values, to calculate l as a function of θ. That is,

$$l \;=\; \frac{0.287142}{\sin\theta\sqrt[3]{\tan\theta}} \;.$$

(The number in the numerator is the cube root of 0.023675.)



For l = 40 cm = 0.4 m, the semivertical angle is given by

$$\sin^3\theta \tan\theta = 0.369\,923 .$$

The solution to this is

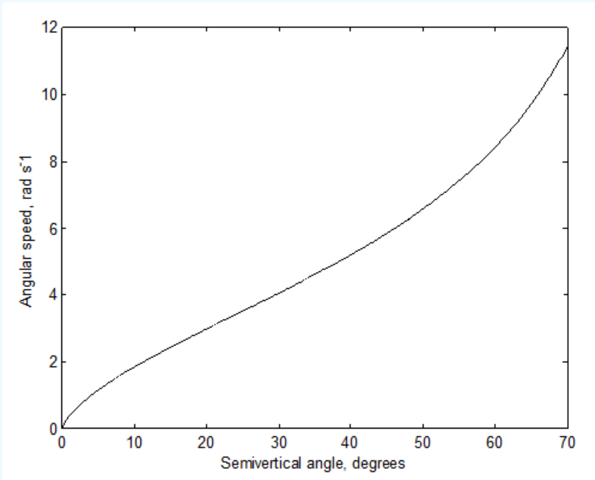
$$\theta = 45^\circ\ 31'\ .$$

(See Section 1.4 of Celestial Mechanics if you need to know how to solve the equation $f(x) = 0$.)

ii.

$$\omega^3 \cot^2\theta = \Omega^3 \cot^2\alpha$$

With the given data, this is $\omega^3 = 199.385 \tan^2\theta$.



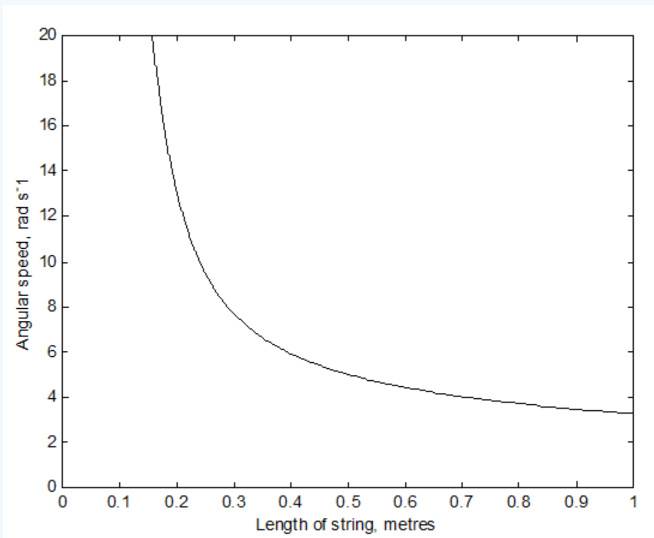
iii. $\omega^3(\omega l^2 - \Omega l_0^2 \sin^2\alpha) = \ \Omega^3(\Omega l_0^2\ -\ \Omega l_0^2 \sin^2\alpha) = \Omega^4 l_0^2 \cos^2\alpha\quad.$

That is, $\omega^3(\omega l^2\ - a) = \ \mathrm{g}^2$ where, with the given initial data,

$$a = 0.48168\ \mathrm{m}^2\mathrm{s}^{-1} \text{ and } \mathrm{g}^2\ =\ 96.04\ \mathrm{m}^2\mathrm{s}^{-4}.$$

Although we are asked to plot ω vertically versus l horizontally, it is easier, when working out numerical values, to calculate l as a function of ω. That is,

$$l^2\ =\ \frac{\mathrm{g}^2}{\omega^4} + \frac{a}{\omega}\ .$$



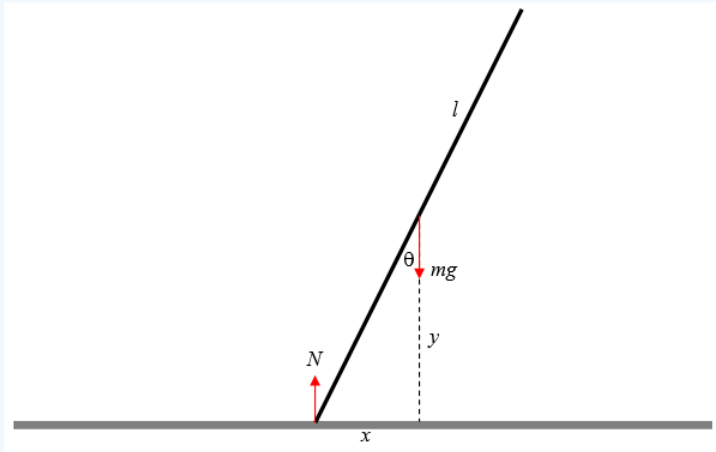
To solve the above equation for ω might be slightly easier with the substitution of u for $\frac{1}{\omega}$:

$$\mathrm{g}^2 u^4\ +\ au\ - l^2 = 0\quad.$$

With l = 0.6 m, this gives u= 0.226121 rad^{-1} s, and hence ω = 4.422 rad s^{-1}. As in part (b) i, it is necessary to know how to solve the equation $f(x) = 0$. See Section 1.4 of Celestial Mechanics if you need to know how.

Exercise 22.8.4

There are no horizontal forces, because the table is smooth. Therefore the centre of mass of the rod falls vertically.centre of mass of the rod falls vertically.



From energy considerations

$$\frac{1}{2}m\dot{y}^2 + \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 + mgy = \text{constant.} \tag{1}$$

But $y = l\cos\theta$ and therefore $\dot{y} = -l\sin\theta.\dot{\theta}$.

$$(3\sin^2\theta + 1)l\dot{\theta}^2 + 6\text{g}\sin\theta = C. \tag{2}$$

Initially $\theta = \dot{\theta} = 0, \quad \therefore C = 6\text{g}$.

$$\underline{\underline{\dot{\theta}^2 = \frac{6\text{g}(1-\cos\theta)}{l(3\sin^2\theta+1)}}} \tag{3}$$

Also, since $\dot{y}^2 = l^2\sin^2\theta\dot{\theta}^2$ and $\dot{x}^2 = l^2\cos^2\theta\dot{\theta}^2$, we obtain

$$\underline{\underline{\dot{y}^2 = \frac{6\text{g}l\sin^2\theta(1-\cos\theta)}{3\sin^2\theta+1}}} \tag{4}$$

and

$$\underline{\underline{\dot{x}^2 = \frac{6\text{g}l\cos^2\theta(1-\cos\theta)}{3\sin^2\theta+1}}} \tag{5}$$

Of course $\dot{\theta}$ and $\dot{y}$ increase monotonically with θ; but $\dot{x}$ starts and finishes at zero, and must go through a maximum. With $c = \cos\theta$, Equation (5) can be written

$$\dot{x}^2 = \frac{6\text{g}lc(1-c)}{4-3c^2}, \tag{6}$$

and by differentiating $\dot{x}^2$ with respect to c, we see that $\dot{x}^2$ is greatest at an angle θ given by

$$3c^2 - 12c + 8 = 0, \tag{7}$$

the solution of which is $\underline{\underline{\theta = 37^\circ\ 50'}}$.

If the length of the rod is 1 m (l = 0.5 m) and $\dot{x}$ = 1 m s^{-1}, Equation 4 becomes

$$26.4c^2 - 29.4c + 4 = 0, \tag{8}$$

and the two solutions are $\underline{\underline{\theta = 17^\circ\ 15'}}$ and $\underline{\underline{80^\circ\ 52'}}$.

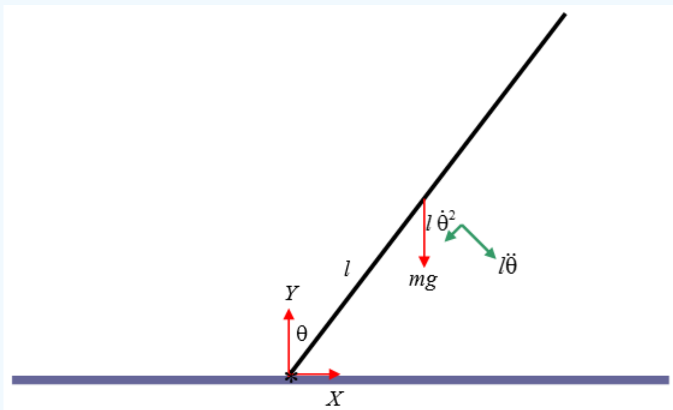
The reader who has done all the problems so far will be aware of the importance of being able instantly to solve the equation $f(x) = 0$. If you have not already done so, you should write a computer or calculator program that enables you to do this instantly and at a moment's notice. See Section 1.4 of Celestial Mechanics if you need to know how.

If you want to find the normal reaction N of the table on the lower end of the rod, you could maybe start with the vertical equation of motion $m\ddot{y} = N - mg$. Differentiate Equation (4): $2\dot{y}\ddot{y} =$ whatever, and the use equation

4 again for $\dot{y}$. This looks like rather heavy and uninteresting algebra to me, so I shan't pursue it. There may be a better way...

Exercise 22.8.5

In the figure below I have marked in red the forces on the rod, namely its weight mg and the horizontal and vertical components X and Y of the reaction of the hinge on the rod. I have also marked, in green, the transverse and radial components of the acceleration of the centre of mass. The transverse component is $l\ddot{\theta}$ and the radial component is the centripetal acceleration $l\dot{\theta}^2$.centre of mass. The transverse component is $l\ddot{\theta}$ and the radial component is the centripetal acceleration $l\dot{\theta}^2$.



From consideration of the moment of the force mg about the lower end of the rod, it is evident that the angular acceleration is

$$\ddot{\theta} = \frac{3g\sin\theta}{4l} \tag{1}$$

and by writing $\ddot{\theta}$ as $\frac{\dot{\theta}\, d\dot{\theta}}{d\theta}$ and integrating (with initial conditions $\theta = \dot{\theta} = 0$), or from energy considerations, we obtain the angular speed:

$$\dot{\theta}^2 = \frac{3g(1-\cos\theta)}{2l}. \tag{2}$$

The horizontal and vertical equations of motion are:

$$X = ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \tag{3}$$

and

$$mg - Y = ml(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta). \tag{4}$$

(As ever, check the dimensions - and count the dots!)

After substitution for $\ddot{\theta}$ and $\dot{\theta}^2$ we find

$$X = \frac{3}{4}mg\sin\theta(3\sin\theta - 2) \tag{5}$$

and

$$Y = \frac{1}{4}mg(1-3\cos\theta)^2. \tag{6}$$

The results follow immediately.

? Exercise 22.8.6

Call the length of the rod $2l$. Initially the height above the table of its centre of mass is $l\cos 40^{\circ}$, and its gravitational potential energy is $mg\ l\ \cos 40^{\circ}$. When it hits the table at angular speed w, its kinetic energy is $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{4}{3}ml^2\right)\omega^2 = \frac{2}{3}ml^2\omega^2$. Therefore,

$$\omega = \sqrt{\frac{3g\cos 40^{\circ}}{2l}} = \underline{\underline{4.746 \text{ rad s}^{-1} = 271.9 \text{ deg s}^{-1}}}\ .$$

To find the time taken, you can use Equation 9.2.10:

$$t = \sqrt{\frac{I}{2}}\int_{40^{\circ}}^{90^{\circ}}\frac{d\theta}{\sqrt{E-V(\theta)}}. \tag{7}$$

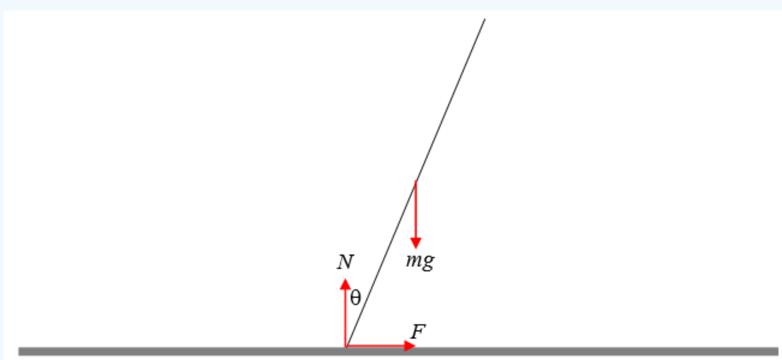
Here, $I = \frac{3}{4}ml^2,\ E = mgl\cos 40^{\circ},\ V(\theta) = mgl\cos\theta$ and therefore

$$t = \sqrt{\frac{2l}{3g}}\int_{40}^{90}\frac{d\theta}{\sqrt{\cos 40^{\circ}-\cos\theta}}. \tag{8}$$

The magnitude of the quantity before the integral sign is 0.184428 s. To find the value of the integral requires either that you be an expert in elliptic integrals or (more likely and more useful) that you know how to integrate numerically (see Celestial Mechanics 1.2.) I make the value of the integral 2.187314, so that the time taken is 0.4034 seconds. When integrating, note that the value of the integrand is infinite at the lower limit. How to deal with this difficulty is dealt with in Celestial Mechanics 1.2. It cannot be glossed over.

? Exercise 22.8.7

Here is the diagram. The forces are the weight mg of the rod, and the force of the table on the rod. However, I have resolved the latter into two components – the *normal* reaction N of the table on the rod, and the frictional force F, which may be either to the left or the right, depending on whether rod is tending to slip towards the right or the left. The magnitude of F is less than μN as long as the rod is not jus about the slip. When the rod is just about to slip, $F = \mu N,\ \mu$ being the coefficient of limiting static friction.



Just as in Problem 5, the equations of motion, as long as the rod does not slip, are

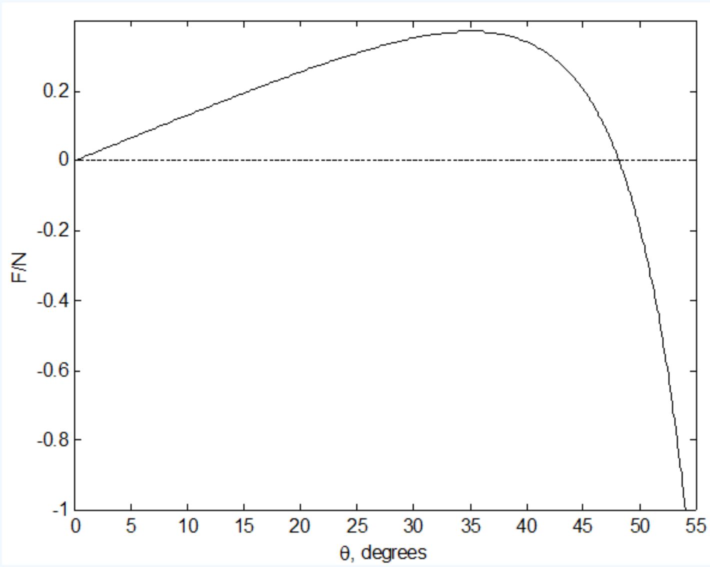
$$F = \frac{3}{4}mg\sin\theta(3\cos\theta-2) \tag{1}$$

and

$$N = \frac{1}{4}mg(1-3\cos\theta)^2. \tag{2}$$

$$\frac{F}{N}=\frac{3\sin\theta(3\cos\theta-2)}{(1-3\cos\theta)^2}. \tag{3}$$

The figure below shows $\frac{F}{N}$ as a function of θ. One sees that, as the rod falls over, $\frac{F}{N}$ increases, and, as soon as it attains a value of μ, the rod will slip. We see, however, that $\frac{F}{N}$ reaches a maximum value, and by calculus we can determine that it reaches a maximum value of $\frac{15\sqrt{10}}{128}=0.3706$ when $\theta=\theta^{-1}\left(\frac{9}{11}\right)=35^\circ 06'$. If μ < 0.3706, the bottom of the rod will slip before $\theta=35^\circ 06'$. If, however, μ > 0.3706, the rod will not have slipped by the time $\theta=35^\circ 06'$, and it is safe for a while as $\frac{F}{N}$ starts to decrease. When θ reaches $\cos^{-1}\left(\frac{2}{3}\right)=48^\circ 11'$, the frictional force changes sign and thereafter acts to the left. (The frictional force of the table on the rod acts to the left; the frictional force of the rod on the table acts to the right.) We know by now (since the rod survived slipping before $\theta=35^\circ 06'$ that the magnitude of $\frac{F}{N}$ can be at least as large



as 0.3706, and it does not reach this until $\theta=51^\circ 15'$. Therefore, if the rod hasn't slipped by $\theta=51^\circ 15'$ it won't slip before $\theta=51^\circ 15'$. But after that it is in danger again of slipping. $\frac{F}{N}$ becomes infinite ($N=0$) when $\theta=\cos^{-1}\left(\frac{1}{3}\right)=70^\circ 32'$, so it will certainly slip (to the right) before then.

If $\mu=0.25$, the rod will slip to the left when

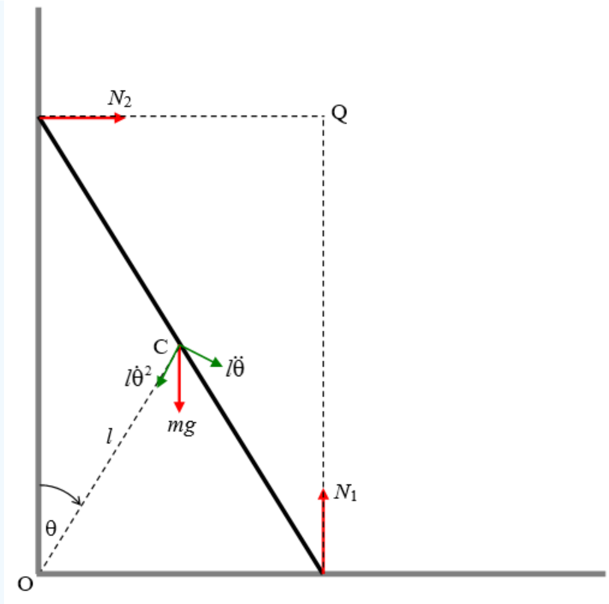
$$\frac{3\sin\theta(3\cos\theta-2)}{(1-3\cos\theta)^2}=\frac{1}{4}, \text{ or } \underline{\underline{\theta=19^\circ 39'}}\ .$$

If $\mu=0.75$, the rod will slip to the right when

$$\frac{3\sin\theta(3\cos\theta-2)}{(1-3\cos\theta)^2}=-\frac{3}{4}, \text{ or } \underline{\underline{\theta=53^\circ 07'}}\ .$$

Again, it is very necessary that you prepare for yourself a program that will instantly solve the equation $f(x)=0$.

? Exercise 22.8.8



Let the length of the ladder be $2l$. By geometry, the distance OC remains equal to l throughout the motion; therefore C describes a circle of radius l, centre O. I have marked in, in green, the radial and transverse components of the acceleration of C, namely $l\dot{\theta}^2$ and $l\ddot{\theta}$. The angular speed of the ladder is $\dot{\theta}$ and the linear speed of the centre of mass C is $l\ddot{\theta}$. I have also marked, in red, the three forces acting on the ladder, namely its weight and the reactions of the floor and the wall on the ladder.centre O. I have marked in, in green, the radial and transverse components of the acceleration of C, namely $l\dot{\theta}^2$ and $l\ddot{\theta}$. The angular speed of the ladder is $\dot{\theta}$ and the linear speed of the centre of mass C is $l\ddot{\theta}$. I have also marked, in red, the three forces acting on the ladder, namely its weight and the reactions of the floor and the wall on the ladder.

The angular speed $\dot{\theta}$ can be obtained from energy considerations. That is, the loss of potential energy in going from angle a to the vertical to angle θ is equal to the gain in translational and rotational kinetic energies:

$$mgl(\cos\alpha - \cos\theta) = \tfrac{1}{2}m(l\dot{\theta})^2 + \tfrac{1}{2}\left(\tfrac{1}{2}ml^2\right)\dot{\theta}^2 .$$

$$\dot{\theta}^2 = \frac{3g}{2l}(\cos\alpha - \cos\theta). \tag{1}$$

The angular acceleration $\ddot{\theta}$ can be obtained from the following equation:

$$mgl\sin\theta = \frac{4}{3}ml^2\ddot{\theta} \tag{2}$$

The derivation of Equation (2) raises some points of interest, and I discuss it in an Appendix at the end of the problem.

The vertical and horizontal equations of motion are:

$$N_2 = m(l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) \tag{3}$$

and

$$mg - N_1 = m(l\ddot{\theta}\sin\theta + l\dot{\theta}\cos\theta), \tag{4}$$

although we need only the first of these, because we wish to find out when $N_2 = 0$.

On substitution for $\ddot{\theta}$ and $\dot{\theta}^2$ we find that

$$N_2 = \frac{3}{4}mg\sin\theta(3\cos\theta - 2\cos\alpha) \tag{5}$$

and

$$N_1 = \frac{1}{4} mg(1 - 6\cos\alpha\cos\theta + 9\cos^2\theta) \tag{6}$$

We need only the first of these to see that N_2 becomes zero (and hence the upper end loses contact with the wall) when $\cos\theta = \frac{2}{3}\cos\alpha$.

Appendix: Derivation of equation (2).

In my original posting of this solution I had derived Equation (2) by considering that the total moment of all forces about Q is $mgl\sin\theta$, and the rotational inertia with respect to Q is $\frac{4}{3}ml^2$. I then equated $mgl\sin\theta$ to $\frac{4}{3}ml^2\ddot{\theta}$. I am indebted to correspondent Amin Rezaee Zadeh for pointing out a flaw in this argument, and for supplying a correct derivation. The flaw is that I am applying the equation $\tau = \dot{\mathbf{L}}$ to a *moving point* Q. In Section 3.12 of Chapter 3 of these notes it is pointed out that $\tau = \dot{\mathbf{L}}$ can be applied to a moving point only if the moving point satisfies one or more of three conditions, and it is evident in this problem that Q satisfies none of these conditions. I present Mr Rezaee's correct derivation of Equation (2) below.

I shall be making use of Equations 3.12.1 and 3.12.2:

$$\dot{L}_Q = \tau_Q + Mr'_q \times \ddot{r}_Q. \tag{3.12.1}$$

$$L_Q = \Sigma(r_i - r_Q) \times [m_i(v_i - v_Q)]. \tag{3.12.2}$$

I shall also be making use of the notation used in Section 3.12, and I reproduce here Figure III.7 from that Section, and I also draw the relevant vectors appropriate to this ladder problem.

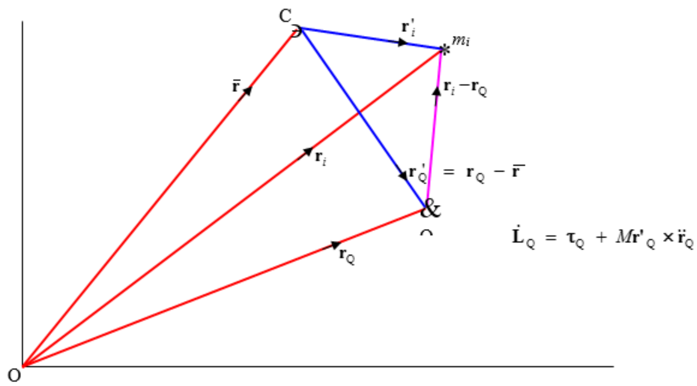
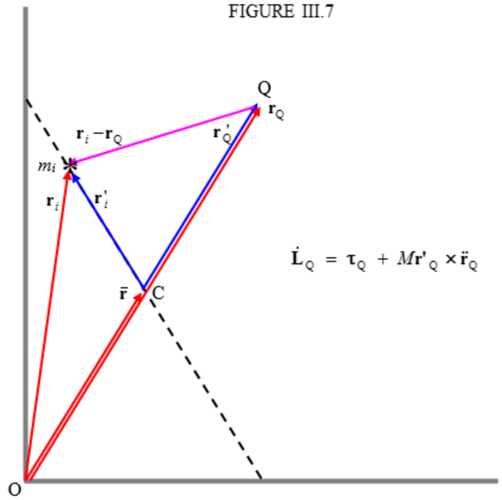


FIGURE III.7



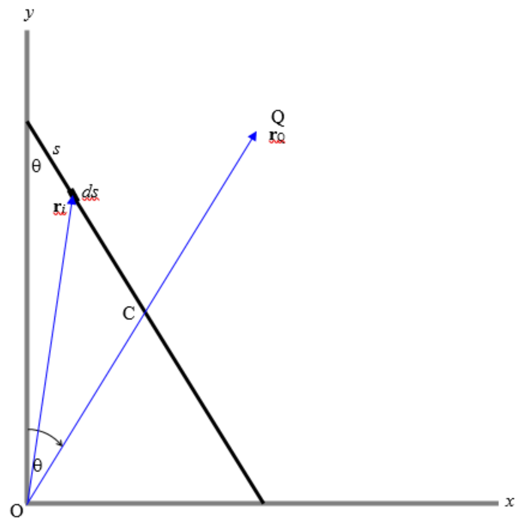
In the figure below, I have indicated an elemental portion ds of the ladder at a distance s from the upper end of the ladder. Its mass is evidently $dm = \frac{mds}{2l}$. I have drawn the position vectors $\mathbf{r_i}$ and $\mathbf{r_O}$ of ds and of Q. This notation corresponds to the same notation used in Section 3.12. From the geometry of the figure, we can determine that

$$\mathbf{r}_i = s\sin\theta\mathbf{i} + (2l - s)\cos\theta\mathbf{j} \tag{A1}$$

and

$$\mathbf{r}_Q = 2l\sin\theta\mathbf{i} + 2l\cos\theta\mathbf{j}, \tag{A2}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the $x-$ and $y-$ directions respectively.



On differentiation with respect to time, we find the following expressions for the velocities of the element ds and the point Q, in which I again retain the notation used in Section 3.12:

$$\mathbf{v}_i = s\dot{\theta}\cos\theta\mathbf{i} - (2l-s)\dot{\theta}\sin\theta\mathbf{j} \tag{A3}$$

and

$$\mathbf{v}_Q = 2l\dot{\theta}\cos\theta\mathbf{i} - 2l\dot{\theta}\sin\theta\mathbf{j} \tag{A4}$$

On making use of Equation 3.12.2, we obtain for the angular momentum of the element ds with respect to Q:

$$d\mathbf{L}_Q = \frac{m}{2l}(\mathbf{r}_i - \mathbf{r}_Q)\times[(\mathbf{v}_i - \mathbf{v}_Q)]ds. \tag{A5}$$

The instantaneous angular momentum of the entire ladder about Q is therefore

$$\mathbf{L}_Q = \frac{m}{2l}\int_0^{2l}(\mathbf{r}_i - \mathbf{r}_Q)\times[(\mathbf{v}_i - \mathbf{v}_Q)]ds. \tag{A6}$$

On substitution of Equations (A1) – (A4) into equation (A6) and a modest amount of algebra, we obtain

$$\mathbf{L}_Q = \frac{m\theta\mathbf{k}}{2l}\int_0^{2l} s(s-2l)ds = -\frac{2}{3}ml^2\dot{\theta}\mathbf{k}, \tag{A7}$$

where $\mathbf{k}$ is the unit vector in the $z-$direction. (The $z-$direction is *out of* the plane of the "paper", and therefore $\mathbf{L_Q}$ is *into* the plane of the "paper". It is worth spending a moment or two trying to imagine this. The ladder is rotating counterclockwise about C, while C and Q are moving in clockwise trajectories. It may not be immediately obvious to decide whether one would expect $\mathbf{L_Q}$ to be directed into or out of the plane of the "paper". Equation (A7) answers this question.)

We now make use of Equation 3.12.1:

$$\dot{\mathbf{L}}_Q = \tau_Q + m\mathbf{r'_Q}\times\ddot{\mathbf{r}}_\mathbf{Q}. \tag{A8}$$

Let us find expressions for the four vector quantities in this equation.

By differentiation of Equation (A7) with respect to time, we obtain

$$\dot{\mathbf{L}}_Q = -\frac{2}{3}ml^2\ddot{\theta}\mathbf{k}. \tag{A9}$$

The torque about Q is

$$\tau_Q = mgl\sin\theta\mathbf{k} \tag{A10}$$

We can see from the geometry of the figure (see especially the second of our figures, in which we see that $\mathbf{r}'_Q$ and $\overline{\mathbf{r}}$ are the same in magnitude and direction) that

$$\mathbf{r}'_Q = l\sin\theta\mathbf{i} + l\cos\theta\mathbf{j}. \tag{A11}$$

Finally, by differentiation of Equation (A4) (in which), we obtain

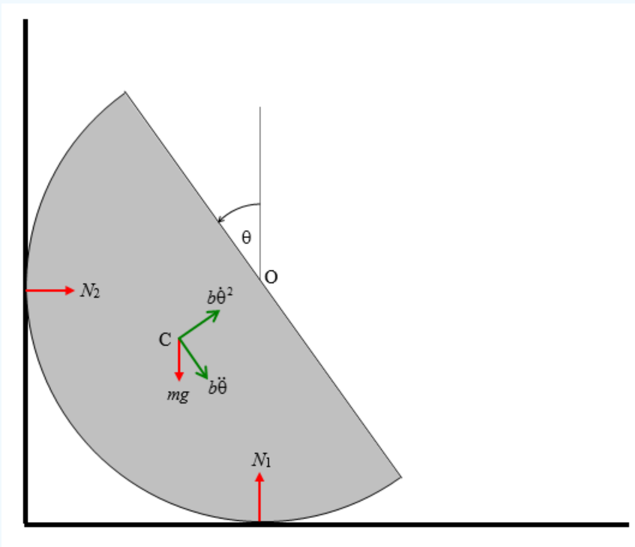
$$\ddot{\mathbf{r}}_Q = 2l[(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\mathbf{i} - (\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta)\mathbf{j}]. \tag{A12}$$

Substitution of Equations (A9) to (A12) into Equation (A8) gives, after some algebra,

$$mgl\sin\theta = \frac{4}{3}ml^2\ddot{\theta}. \tag{A13}$$

This is Equation (2), *quod erat demonstrandum*.

Exercise 22.8.9



It will, I think, be agreed that the point O remains fixed in space as long as the semicylinder remains in contact with wall and floor. Therefore the centre of mass C moves in a circle around O. We'll call the radius of the circle, which is the distance between O and C, b, which, for a semicylinder, equals centre of mass C moves in a circle around O. We'll call the radius of the circle, which is the distance between O and C, b, which, for a semicylinder, equals $\frac{4a}{(3\pi)}$ (see Chapter 1), where a is the radius of the semicylinder. I have marked, in red, the three forces on the semicylinder, and also, in green, the radial and transverse components of the acceleration.

The angular speed $\dot{\theta}$ can be obtained from energy considerations. The gain in kinetic energy in going from rest to an angular speed $\dot{\theta}$ is $\frac{1}{2}(mk^2)\dot{\theta}^2$ and the gain in potential energy when the centre of mass drops through a vertical distance $b\ \sin\theta$ is $mgb\sin\theta$. Here k is the radius of gyration about O, which, for a semicylinder, is given by $k^2 = \frac{1}{2}a^2$.

[I have left b and k as they are in the equations, so that the analysis could easily be adapted, if needed, for a hollow semicylinder, or a solid hemisphere, or a hollow hemisphere. From Chapters 1 and 2 we recall:

Solid semicylinder:	$b = \frac{4a}{3\pi}$	$k^2 = \frac{1}{2}a$	$\frac{b^2}{k^2} = \frac{32}{9\pi^2}$
Hollow semicylinder:	$b = \frac{2a}{\pi}$	$k^2 = a^2$	$\frac{b^2}{k^2} = \frac{4}{\pi^2}$
Solid hemisphere:	$b = \frac{3a}{8}$	$k^2 = \frac{2}{5}a^2$	$\frac{b^2}{k^2} = \frac{45}{128}$

Hollow hemisphere:	$b = \frac{1}{2}a$	$k^2 = \frac{2}{3}a^2$	$\frac{b^2}{k^2} = \frac{3}{8}$

On equating the gain in kinetic energy to the loss in potential energy, we obtain

$$\dot{\theta}^2 = \frac{2bg}{k^2}\sin\theta. \tag{1}$$

The angular acceleration $\ddot{\theta}$ can be obtained from applying $\tau = I\ddot{\theta}$ about O:

$$mgb\cos\theta = mk^2\ddot{\theta},$$

from which

$$\ddot{\theta} = \frac{bg}{k^2}\cos\theta \tag{2}$$

The horizontal and vertical equations of motion are

$$N_2 = mb(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta) \tag{3}$$

and

$$N_2 = mb(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta) \tag{4}$$

We do not really need Equation (4), because we are trying to determine when $N_2 = 0$.

On substitution from Equations (1) and (2), Equation (3) becomes

$$N_1 = \frac{6mb^2\text{g}}{a^2}\sin\theta\cos\theta. \tag{5}$$

This is zero when $\theta = 0^\circ$ (which was the initial condition) or when $\theta = 0^\circ$, at which point contact with the wall is lost, which it was required to show.

At this instant, the rotational velocity is $\sqrt{\frac{2bg}{k^2}}$ counterclockwise.

and the linear velocity of C is $b\sqrt{\frac{2bg}{k^2}}$ horizontally to the right.

The rotational kinetic energy is $\frac{1}{2}I\omega^2$ where $\omega = \sqrt{\frac{2bg}{k^2}}$, and I is the rotational inertia *about the centre of masscentre of mass*, which is $m(k^2 - b^2)$.

$$K_{\text{rot}} = \frac{mbg(k^2-b^2)}{k^2}.$$

The translational kinetic energy is $\frac{1}{2}mv^2$ where $v = \sqrt{\frac{2bg}{k^2}}$.

$$K_{\text{tr}} = \frac{mb^3\text{g}}{k^2}$$

The sum of these is mbg, which is just equal to the loss of the original potential energy, which serves as a check on the correctness of our algebra.

There are now no horizontal forces, so the horizontal component of the velocity of C remains constant. The semicylinder continues to rotate, however, until the rotational kinetic energy is converted to potential energy and C rises to its maximum height. If the base then makes an angle ϕ with the vertical, the gain in potential energy is $mbg\sin\phi$, and equating this to the rotational kinetic energy gives

$$\sin\phi = 1 - \frac{b^2}{k^2}.$$

This gives the following results:

Solid semicylinder:	$\phi = 39^\circ\ 46'$
Hollow semicylinder:	$\phi = 36^\circ\ 30'$

Solid hemisphere:	$\phi = 40^\circ\ 25'$
Hollow hemisphere:	$\phi = 38^\circ\ 41'$

? Exercise 22.8.10

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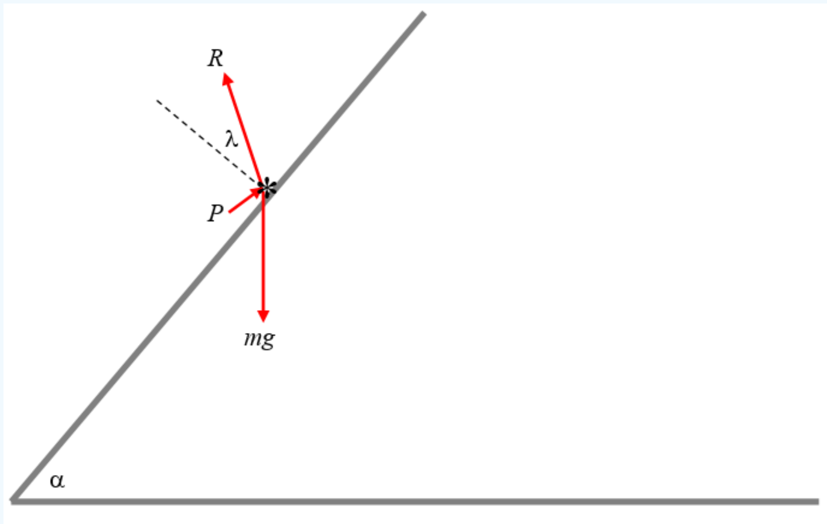
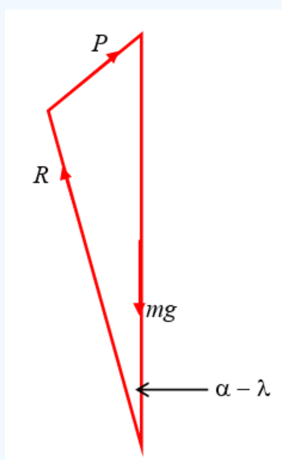
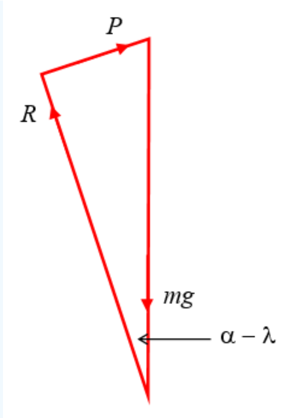


It is well known that if $\alpha > \tan^{-1}\mu$ the particle will slide down the plane unless helped by an extra force. I have drawn the three forces acting on the particle. Its weight mg. The reaction R of the plane on the particle; if the particle is in limiting static equilibrium, this reaction will make an angle λ ("the angle of friction") with the plane such that $\tan\lambda = \mu$. It therefore makes an angle $\alpha - \theta$ with the vertical. Finally, the additional force P needed; we do not initially know the direction of this force.

When three (or more) coplanar forces are in equilibrium and are drawn head-to-tail, they form a closed triangle (polygon). I draw the triangle of forces below.



It will be clear from the triangle that P is least when the angle between $\mathbf{P}$ and $\mathbf{R}$ is 90°:



The least value of P is therefore $mg(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda)$. But $\tan\lambda = \mu$ and therefore $\sin\lambda = \frac{\mu}{\sqrt{1+\mu^2}}$ and $\cos\lambda = \frac{1}{\sqrt{1+\mu^2}}$.

$$P_{\text{min}} = \frac{mg(\sin\alpha - \mu\cos\alpha)}{\sqrt{1-\mu^2}}$$

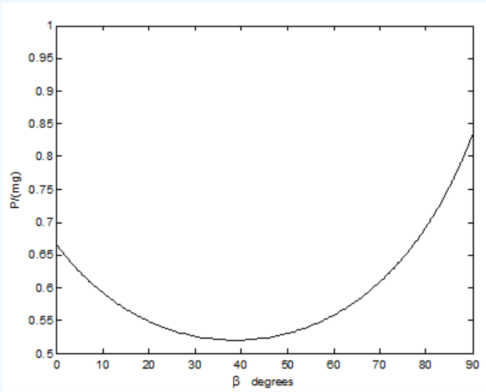
and P then makes an angle \(\lambda\) with the plane.

You may, if you wish, go further, and show that when $\mathbf{P}$ makes an angle β with the plane, it must have magnitude

$$P = \frac{\sin\alpha - \mu\cos\alpha}{\mu\sin\beta + \cos\beta} mg.$$

You can then differentiate this with respect to β (you need only differentiate the denominator) and show that this is a minimum when $\beta = \lambda$. That is just a harder way of finding what we already found by using the triangle of forces.

For $\alpha = 70^\circ$ and $\mu = 0.8$, P varies with β like this:



This goes through a minimum of $P_{\text{min}} = 0.520mg$ at $\beta = \tan^{-1} 0.8 = 38^\circ.7$.

? Exercise 22.8.11

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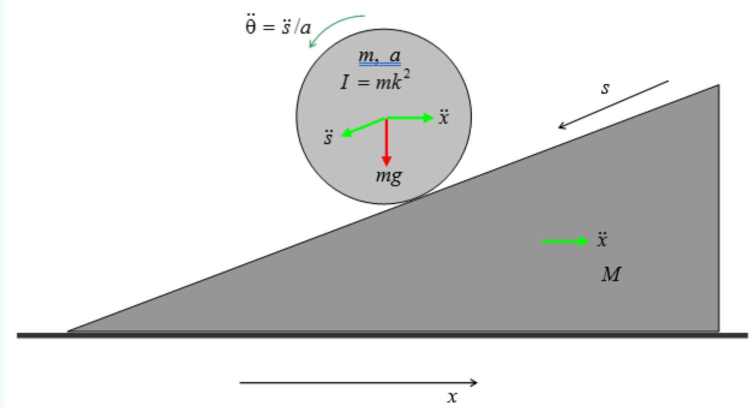


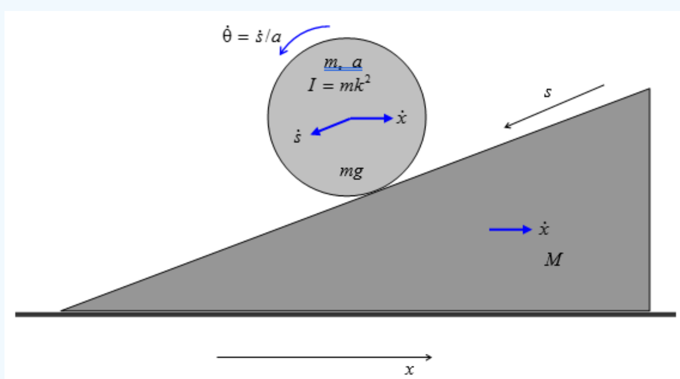
As the cylinder rolls down the plane, the wedge, because its base is smooth, will slide towards the left. Since there are no external horizontal forces on the system, the centre of mass of the system will not move horizontally (or, rather, it won't accelerate horizontally.)centre of mass of the system will not move horizontally (or, rather, it won't accelerate horizontally.)

As usual, we draw a large diagram, using a ruler , and we mark in the forces in red and the accelerations in green, after which we'll apply $F = ma$ to the cylinder, or to the wedge, or to the system as a whole, in two directions. It should be easy and straightforward.

I have drawn the linear acceleration $\ddot{s}$ of the cylinder down the slope, and its angular acceleration $\ddot{\theta}$. I have drawn the linear acceleration $\ddot{x}$ of the wedge, which is also shared with the cylinder. I have drawn the gravitational force mg on the cylinder. There is one more force on the cylinder, namely the reaction of the wedge on the cylinder. But I'm not sure in which direction to draw it. Is it normal to the plane? That would mean there is no frictional force between the cylinder and the plane. Is that correct (remembering that both the cylinder and the wedge are accelerating)? Of course I could calculate the moment of the force mg about the point of contact of the cylinder with the plane, and then I wouldn't need to concern myself with any forces at that point of contact.

But then that point of contact is not fixed. Oh, dear, I'm getting rather muddled and unsure of myself.

This problem, in fact, is ideally suited to a lagrangian rather than a newtonian treatment, and that is what we shall do. Lagrange proudly asserted that it was not necessary to draw any diagrams in mechanics, because it could all be done analytically. We are not quite so talented as Lagrange, however, so we still need a large diagram drawn with a ruler. But, instead of marking in the forces and accelerations in red and green, we mark in the *velocities* in *blue*.



No frictional or other nonconservative forces do any work, so we can use Lagrange's equations of motion for a conservative holonomic system; $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \left(\frac{\partial T}{\partial q}\right) = -\left(\frac{\partial V}{\partial q}\right)$.

The speed of the wedge is $\dot{x}$ and the speed of the centre of mass of the cylinder is centre of mass of the cylinder is $\sqrt{\dot{s}^2 + \dot{x}^2 - 2\dot{s}\dot{x}\cos\alpha}$ and the angular speed of the cylinder is $\frac{\dot{s}}{a}$.

The kinetic energy of the system is

$$T = \frac{1}{2}m(\dot{s}^2 + \dot{x}^2 - 2\dot{s}\dot{x}\cos\alpha) + \frac{1}{2}(mk^2)\left(\frac{\dot{s}}{a}\right) + \frac{1}{2}M\dot{x}^2 \quad ,$$

or

$$T = \tfrac{1}{2}m\left(1+\tfrac{k^2}{a^2}\right)\dot{s}^2 - m\dot{s}\dot{x}\cos\alpha + \tfrac{1}{2}(m+M)\dot{x}^2 \quad ,$$

and the potential energy is

$$V = \text{constant} - mgs\sin\alpha$$

Application of Lagrange's equation to the coordinate x gives us

$$\left(1+\tfrac{k^2}{a^2}\right)\ddot{s} = \ddot{x}\cos\alpha + g\sin\alpha \ .$$

and application of Lagrange's equation to the coordinate s gives us

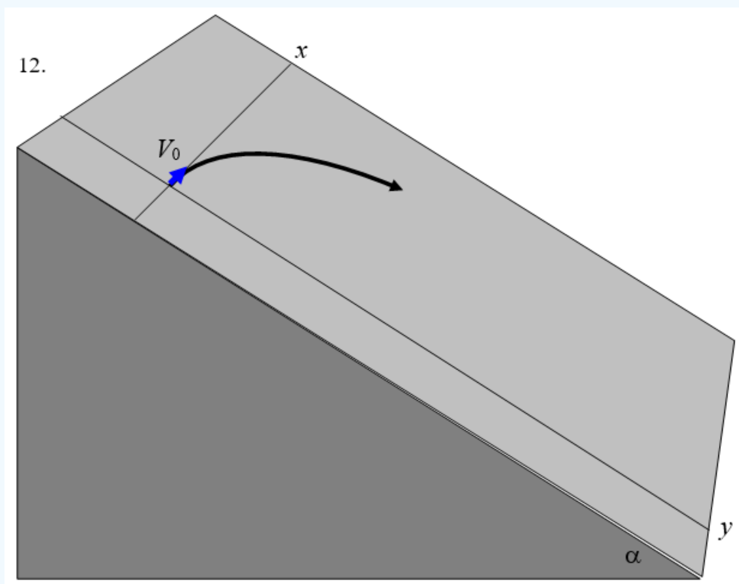
$$m\ddot{s}\cos\alpha = (m+M)\ddot{x}$$

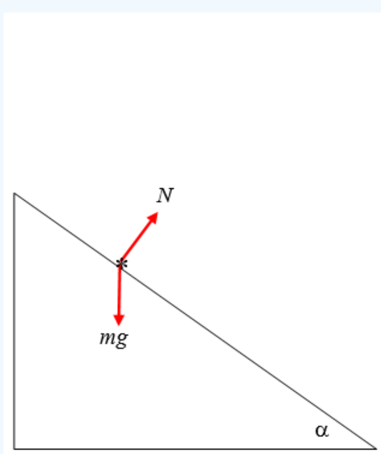
Elimination of $\ddot{s}$ from these two equations gives us

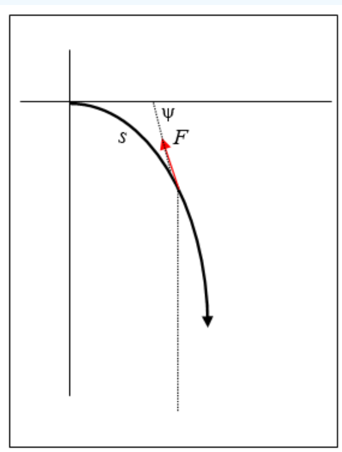
$$\ddot{x} = \frac{mg\sin\alpha\cos\alpha}{(m+M)\left(1+\frac{k^2}{a^2}\right)-m\cos^2\alpha}$$

You can also easily find an expression for $\ddot{s}$ is you wish.

Exercise 22.8.12







There is no acceleration normal to the plane, and therefore $N = mg\cos\alpha$. The frictional force F acts along the tangent to the path and is equal to μN, or $\mu mg\cos\alpha$, where μ is the coefficient of moving friction. We are told to ignore the difference between the coefficients of moving and limiting static friction. Since the particle was originally at rest in limiting static friction, we must have $\mu = \tan\alpha$. Therefore $F = mg\sin\alpha$. The tangential equation of motion is

$m\ddot{s} = -F +$ whatever the component of $m\mathbf{g}$ is in the tangential direction in the sloping plane.

The component of $m\mathbf{g}$ down the plane would be (look at the left hand drawing) $mg\sin\alpha$, and so its tangential component (look at the right hand drawing) is $mg\sin\alpha\sin\psi$. So we have, for the tangential equation of motion,

$$m\ddot{s} = -mg\sin\alpha + mg\sin\alpha\sin\psi\ ,$$

or

$$\ddot{s} = -\mathrm{g}\sin\alpha(1-\sin\psi)\ .$$

We are seeking a relation between V and ψ, so, in the now familiar fashion, we write $V\frac{dV}{ds}$ for $\ddot{s}$, so the tangential equation of motion is

$$V\frac{dV}{ds} = -\mathrm{g}\sin\alpha(1-\sin\psi). \tag{1}$$

We also need the equation of motion normal to the trajectory. The component of $m\mathbf{g}$ in that direction is $mg\sin\alpha\cos\psi$, and so the normal equation of motion is

$$\frac{mV^2}{\rho} = mg\sin\alpha\cos\psi\,.$$

Here ρ is the radius of curvature of the path, which is the reciprocal of the curvature $\frac{ds}{d\psi}$. The normal equation of motion is therefore

$$V^2\frac{d\psi}{ds} = \mathrm{g}\sin\alpha\cos\psi. \tag{2}$$

Divide Equation (1) by Equation (2) to eliminate s and thus get a desired differential equation between V and ψ:

$$\frac{1}{V}\frac{dV}{d\psi} = -\frac{(1-\sin\psi)}{\cos\psi}. \tag{3}$$

This is easily integrated; a convenient (not the only) way is to multiply top and bottom by $1 + \sin\psi$. In any case we soon arrive at

$$\ln V = -\ln(1+\sin\psi) + \text{constant}, \tag{4}$$

and with the initial condition $V = V_0$ when $\psi = 0$, this becomes

$$\underline{\underline{V = \frac{V_0}{1+\sin\psi}}}. \tag{5}$$

In the limit, as $\psi \to 90^\circ$, $V \to \frac{1}{2}V_0$. The particle is then moving at constant velocity and is in equilibrium under the forces acting upon it just when it was initially at rest.

Exercise 22.8.13

13.

M_1 = mass of complete sphere of radius a.

M_1 = mass of missing inner sphere of radius xa.

M = mass of given hollow sphere.

We have $M = M_2 - M_1$ and $\frac{M_2}{M_1} = x^3$ and therefore

$$M_1 = \frac{M}{1-x^3} \text{ and } M_2 = \frac{Mx^3}{1-x^3}\ .$$

Also $I = \frac{2}{5}M_1a^2 - \frac{2}{5}M_2x^2a^2 = \frac{2}{5}a^2(M_1 - M_2x^2)$.

Hence $I = \frac{2}{5}Ma^2 \times \frac{1-x^5}{1-x^3}$.

If $x = 0$, $I = \frac{2}{5}Ma^2$, as expected. If $x \to 1$, you may have to use de l'Hôpital's rule to show that $I \to \frac{2}{5}Ma^2$ as expected.

Exercise 22.8.14

M_1 = mass of mantle.

M_2 = mass of core.

M = mass of entire planet.

We have $M = M_1 + M_2$ and $\frac{M_1}{M_2} = \frac{s(1-x^3)}{x^3}$ and therefore

$$M_2 = M \times \frac{x^3}{x^3 + s(1-x^3)} \quad \text{and } M_1 = M \times \frac{s(1-x^3)}{x^3 + s(1-x^3)}$$

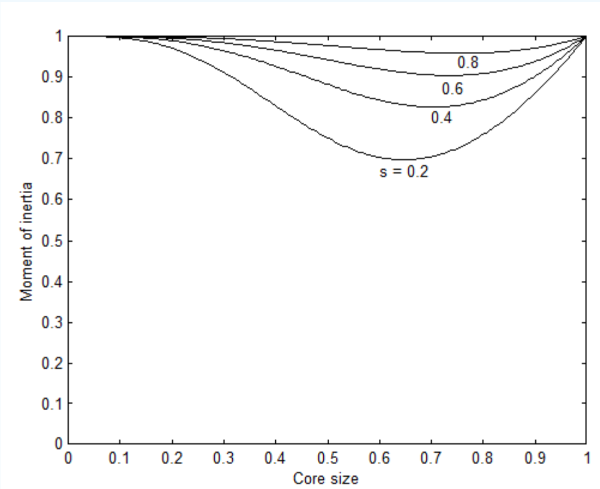
Also

$$I = I_{\text{core}} + I_{\text{mantle}} = \frac{2}{5}M_2x^2a^2 + \frac{2}{5}M_1a^2 \times \frac{1-x^5}{1-x^3} ,$$

where I have made use of the result from the previous problem. On substitution of the expressions for M_1 and M_2, we quickly obtain

$$I = \frac{2}{5}Ma^2 \times \frac{s + (1-s)x^5}{s + (1-s)x^3} \tag{1}$$

A hollow planet would correspond to $\frac{1}{s} = 0$. Divide top and bottom by s and it is immediately seen that the expression for a hollow planet would be identical to the expression obtained for the previous problem.



Note that both $x = 0$ and $x = 1$ correspond to a uniform sphere, so that in either case, $I = \frac{2}{5}Ma^2$ for all other cases, the moment of inertia is less than $\frac{2}{5}Ma^2$.

The core size for minimum moment of inertia is easily found by differentiation of the above expression for I, and the required expression follows after some algebra. For $s = 0.6$, the equation becomes $9 - 15x^2 - 4x^5 = 0$, of which the only positive real root is $x = 0.73682$, which corresponds to a moment of inertia of 0.90376 % $\frac{2}{5}Ma^2$. Note that. for $s = 0.6$, the moment of inertia, expressed in units of $\frac{2}{5}Ma^2$ varies very little as the core size goes from 0 to 1, so that measurement of the moment of inertia places very little restriction on the possible core size.

The inverse of Equation (1) is

$$(1-s)x^5 - I(1-s)x^3 + (1-I)s = 0, \tag{2}$$

where I is expressed in units of $\frac{2}{5}Ma^2$. For $I = 0.911$, there are two positive real roots (look at the graph); they are $x = 0.64753$ and 0.81523. For $I = 0.929$, the roots are 0.55589 and 0.87863. Thus the core size could be anything between 0.55589 and 0.64753 or between 0.81523 and 0.87863 a rather large range of uncertainty. Even if I were known exactly (which does not happen in science), there would be two solutions for x.

Exercise 22.8.15

This is just a matter of geometry. If, when you make a small angular displacement, you raise the centre of mass of the brick the equilibrium is stable. For, while the brick is in its vertical position, it is evidently at a potential minimum, and you have to do work to raise the centre of mass. If, on the other hand, your action in making a small angular displacement results in a lowering of the centre of mass, the equilibrium is unstable.centre of mass of the brick the equilibrium is stable. For, while the brick is in its vertical position, it is evidently at a potential minimum, and you have to do work to raise the centre of mass. If, on the other hand, your action in making a small angular displacement results in a lowering of the centre of mass, the equilibrium is unstable.

When the brick is in its vertical position, the height h_0 of its centre of mass above the base of the semicylinder is justcentre of mass above the base of the semicylinder is just

$$h_0 = R + l \ .$$

When it is displaced from the vertical by an angle θ, the point of contact between brick and semicylinder is displaced by a distance $R\theta$, and, by inspection of the drawing, the new height h is

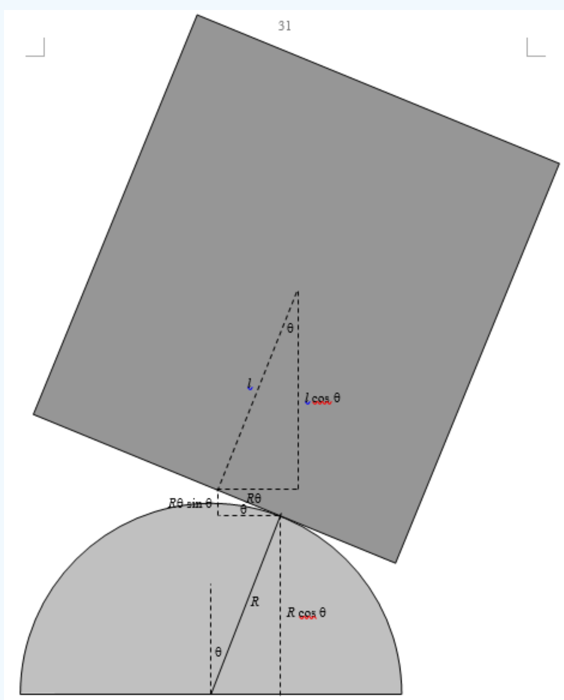
$$h = R\cos\theta + R\theta\sin\theta + l\cos\theta \quad .$$

$$h - h_0 = R\theta\sin\theta - (R+l)(1-\cos\theta) \quad .$$

If you Maclaurin expand this as far as θ^2, you arrive at

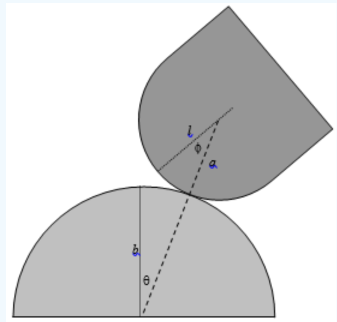
$$h - h_0 \approx \frac{1}{2}(R-l)\theta^2 \ .$$

This is positive, and therefore the equilibrium is stable, if $l < R$, or $2l < 2R$, i.e. if the length of the brick is less than the diameter of the semicylinder.



Exercise 22.8.16

As in the previous question, it is just a matter of geometry. If rolling the Thing results in raising its centre of mass, the equilibrium is stable. Initially, the height of the centre of mass is $H_0 = b + l$ .centre of mass, the equilibrium is stable. Initially, the height of the centre of mass is $H_0 = b + l$.



After rolling, the dashed line, which joins the centres and is of length $a + b$, makes an angle θ with the vertical. The short line joining the centre of mass of the Thing to the centre of curvature of its bottom is of length $l - a$ and it makes an angle $\theta +$ *phi* with the vertical. The height of the centre of mass is therefore nowcentre of mass of the Thing to the centre of curvature of its bottom is of length $l - a$ and it makes an angle $\theta +$ *phi* with the vertical. The height of the centre of mass is therefore now

$$h = (a+b)\cos\theta + (l-a)\cos(\theta+\phi)$$

The centre of mass has therefore rise through a heightcentre of mass has therefore rise through a height

$$h - h_0 = (a+b)\cos\theta + (l-a)\cos(\theta+\phi) - b - l \quad .$$

Also, the two angles are related by $a\phi = b\theta$, so that

$$h - h_0 = (a+b)\cos\theta + (l-a)\cos\left[\left\{1+\left(\frac{b}{a}\right)\right\}\theta\right] - b - l$$

$$h - h_0 = -\frac{1}{2}\theta^2[a+b + (l-a)(1+\frac{b}{a})^2] \quad .$$

For stability this must be positive, and hence $\frac{1}{l} > \frac{1}{a} + \frac{1}{b}$.

If $a = b$, this becomes $l < \frac{1}{2}a$.

For a hollow semicylinder,	$l = (1-\frac{2}{\pi})a = 0.363a$	☐ Stable
For a hollow hemisphere,	$l = 0.5a$	☐ Borderline stable
For a solid semicylinder,	$l = [1-\frac{4}{(3\pi)}]a = 0.576a$	☐ Unstable
For a solid hemisphere,	$l = \frac{5}{8}a = 0.625a$	☐ Unstable

Exercise 22.8.17

We need to find the height h of the centre of mass above the level of the pegs as a function of θ. See drawing on next page.centre of mass above the level of the pegs as a function of θ. See drawing on next page.

Angles:

$$\mathrm{BAC} = 45^\circ - \theta$$

$$\mathrm{ABX} = 45^\circ + \theta$$

Distances:

$$\mathrm{AB} = 2ka$$

$\mathrm{AC} = 2ka \cos(45^\circ - \theta)$

$\mathrm{EF} = 2ka \cos(45^\circ - \theta)\cos(45^\circ + \theta) = ak\cos 2\theta$

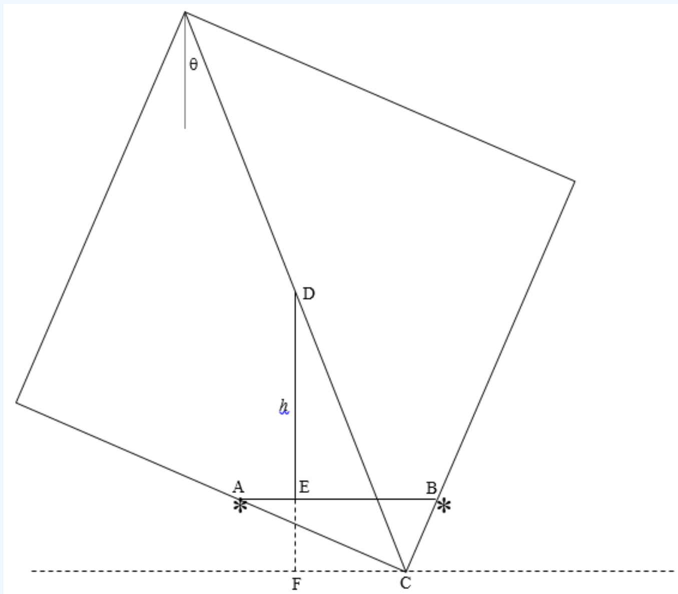
$\mathrm{DC} = a\sqrt{2}$

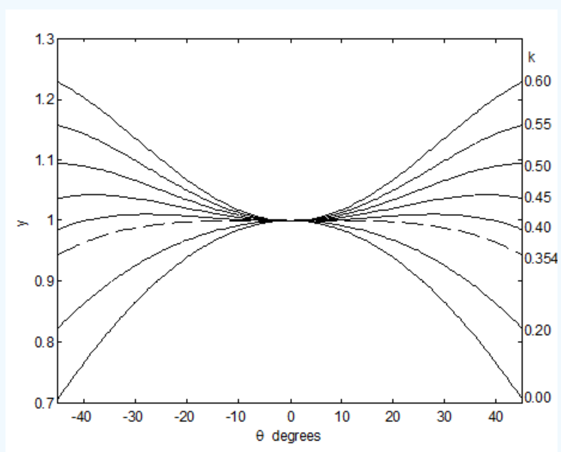
$\mathrm{DF} = a\sqrt{2}\cos\theta$

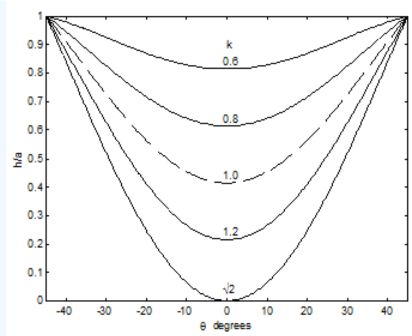
$h = \mathrm{DF} - \mathrm{EF} = a(\sqrt{2}\cos\theta - k\cos 2\theta)$

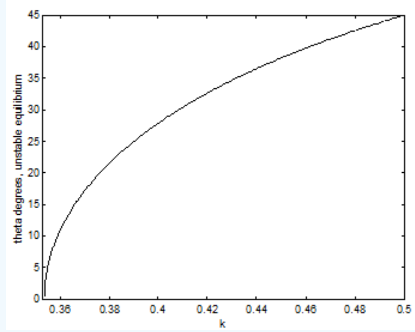
$h_0 =$ height of centre of mass above pegs when $\theta = 0$ centre of mass above pegs when $\theta = 0^\circ = a(\sqrt{2} - k)$

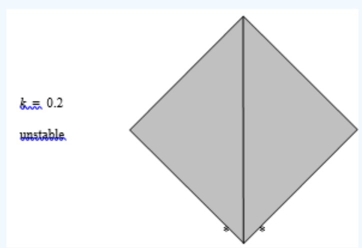
$y = \frac{h}{h_0} = \frac{\sqrt{2}\cos\theta - k\cos 2\theta}{\sqrt{2} - k}$

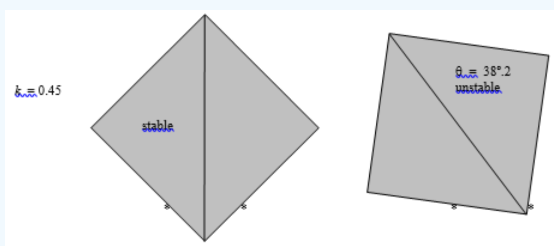




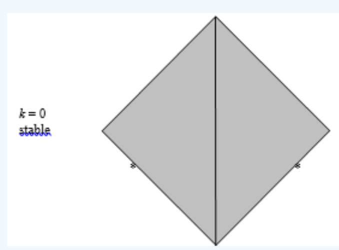




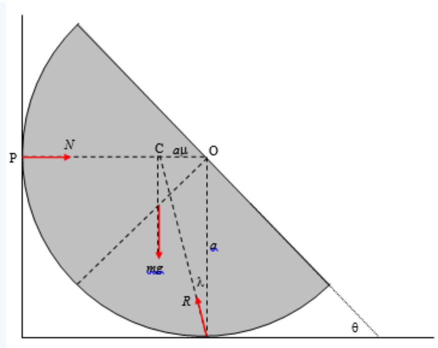




The right-hand drawing is a position of equilibrium, but it is unstable. What happens if it tips counterclockwise? What happens if it tips clockwise?



? Exercise 22.8.18



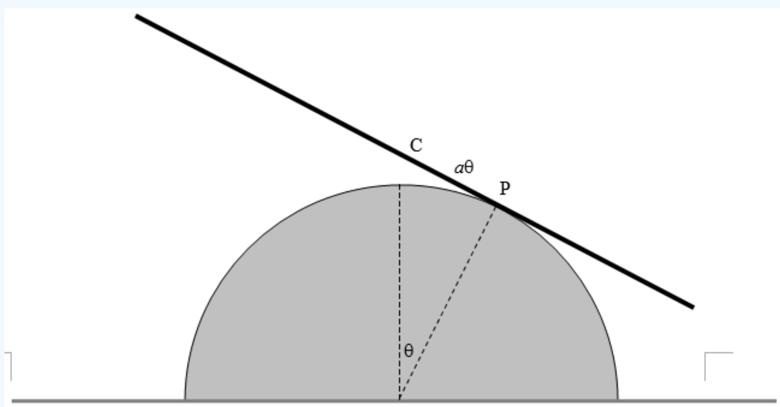
There are three forces acting on the hemisphere: Its weight mg. The reaction N of the wall, which is perpendicular to the wall since the wall is smooth. The reaction R of the floor, which acts at an angle λ to the floor, where $\mu = \tan\lambda$. Three forces in equilibrium must act through a point; therefore all three forces act through the point P. It is thus clear that

$$\sin\,\theta = \frac{\text{OP}}{\text{OC}} = \frac{a\mu}{\frac{3}{8}a} = \frac{8\mu}{3} \quad .$$

If $\mu = \frac{1}{4}$, $\theta = 41^\circ 48'$. If $\mu = \frac{3}{8}$, $\theta = 90^\circ$. If $\mu > \frac{3}{8}$ the hemisphere can rest in any position, the equilibrium not being *limiting* static equilibrium.

? Exercise 22.8.19

This solution uses the same method that Professor Marsh (Warwick University) showed me for Problem 20. I believe it to be clearer than an earlier solution that I had posted.



At an instant when the rod is tilted at angle θ, the coordinates of C with respect to the fixed point O are:

$$\bar{x} = a(\sin\theta - \theta\cos\theta), \tag{1}$$

$$\bar{y} = a(\cos\theta + \theta\sin\theta) \tag{2}$$

and so its velocity components are

$$\dot{\bar{x}} = a\theta\sin\theta\dot{\theta} \tag{3}$$

and

$$\dot{\bar{y}} = a\theta\cos\theta\dot{\theta}. \tag{4}$$

The moment of inertia of the rod about the centre of mass is $\frac{1}{3}ml^2$.

The kinetic energy T is the sum of the translational kinetic energy and the rotational kinetic energy about the centre of mass:centre of mass:

$$T = \left(\frac{1}{2}a^2\theta^2 + \frac{1}{6}l^2\right)m\dot{\theta}^2, \tag{5}$$

and the potential energy V is

$$V = mga(\cos\theta + \theta\sin\theta). \tag{6}$$

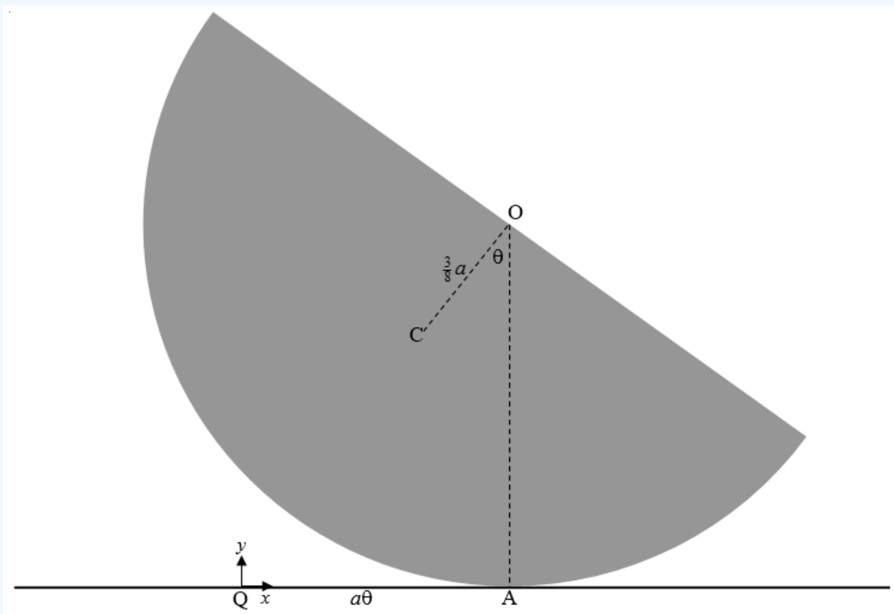
One can now get the equation of motion either by Lagrangian means or by equating the derivative with respect to θ of the total energy to zero, since there are no nonconservative forces and hence the total energy is independent of θ. In carrying out the differentiation, note that $\frac{d}{d\theta}\dot{\theta}^2 = 2\dot{\theta}\frac{d\dot{\theta}}{d\theta} = 2\ddot{\theta}$. We obtain, for the equation of motion:

$$a^2\theta\dot{\theta}^2 + (a^2\theta^2 + \frac{2}{3}l^2)\ddot{\theta} + ga\theta\cos\theta = 0 \tag{7}$$

For small θ (neglecting second and higher powers of θ), $\cos\theta \to 1$ and $a^2\theta^2$ is negligible compared with l^2, so the equation of motion becomes, approximately, $\ddot{\theta} = -\frac{3ga}{l^2}$, and so the period is $\underline{\underline{P = \frac{2\pi l}{\sqrt{3ga}}}}$.

Exercise 22.8.20

I am much indebted to Professor T. R. Marsh of Warwick University not only for finding a mistake in an earlier posted solution to this problem, but for providing the following solution.



$$I = ma^2\left(\frac{7}{5} - \frac{3}{4}\cos\theta\right). \tag{2}$$

$$x = \frac{3a\theta}{8}$$

We are going to refer the motion to a fixed point Q, which is the point of contact between hemisphere and table when the hemisphere is in its equilibrium position.

At an instant when the hemisphere is tilted at an angle θ, the distance between A and Q is $a\theta$, and the coordinates of C relative to Q are

$$\bar{x} = a\theta - \frac{3}{8}a\sin\theta, \tag{1}$$

$$\bar{y} = a - \frac{3}{8}a\cos\theta. \tag{2}$$

Therefore the velocity components of C are

$$\dot{x} = a\left(1 - \frac{3}{8}\cos\theta\right)\dot{\theta}, \tag{3}$$

$$\dot{y} = \frac{3}{8}a\sin\theta\dot{\theta}. \tag{4}$$

By the parallel axes theorem, the moment of inertia around the centre of mass iscentre of mass is

$$I = \frac{2}{5}ma^2 - m\left(\frac{3}{8}a\right)^2 = \frac{83}{320}ma^2 \tag{5}$$

The kinetic energy T is the sum of the translational kinetic energy and the rotational kinetic energy about the centre of mass:centre of mass:

$$T = \frac{1}{2}ma^2\left[\left(1 - \frac{3}{8}\cos\theta\right)^2 + \left(\frac{3}{8}\sin\theta\right)^2 + \frac{83}{230}\right]\dot{\theta}^2 = ma^2\left(\frac{7}{10} - \frac{3}{8}\cos\theta\right)\dot{\theta}^2 \tag{6}$$

The potential energy V is

$$V = mga(1 - \frac{3}{8}\cos\theta). \tag{7}$$

We can the get the equation of motion either by using the Lagrangian equations, or by calculating the derivative with respect to θ of the total energy $T + V$. The derivative is zero, because there are no nonconservative forces and total energy is constant. Note that (as in Problem 19) the derivative of $\dot{\theta}^2$ with respect to θ is $2\dot{\theta}\frac{d\dot{\theta}}{d\theta}$, which is $2\ddot{\theta}$. Either method results in the equation of motion:

$$\left(\frac{7}{5} - \frac{3}{4}\cos\theta\right)\ddot{\theta} + \frac{3}{8}\sin\theta\dot{\theta}^2 + \frac{3}{8}\mathrm{g}\sin\theta = 0. \tag{8}$$

In the small angle limit, $\cos\theta \to 1$, and $\sin\theta \to \theta$ is negligible compared with g, so the equation of motion becomes

$$\ddot{\theta} = -\frac{15\mathrm{g}}{26a}\theta \tag{9}$$

$$\underline{\underline{P = 2\pi\sqrt{\frac{26a}{15\mathrm{g}}}}}. \tag{10}$$

? Exercise 22.8.21

The (second) moment of inertia with respect to the centre (see centre (see Section 2.19 of Chapter 2) is

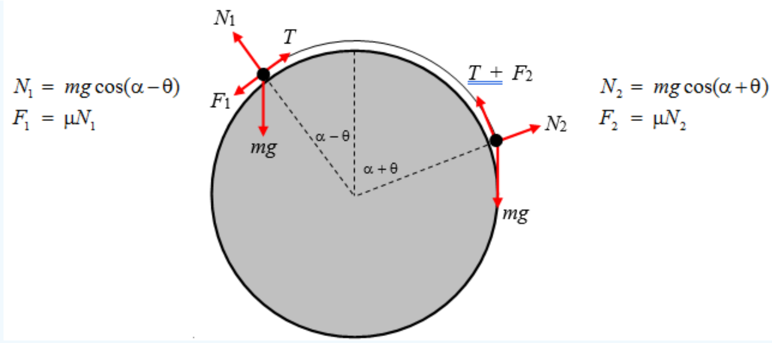
$$I_{\text{centre}} = 4\pi\rho_0\int_0^a(r^4 - r^5/a)dr = \tfrac{2}{15}\pi\rho_0a^5..$$

The moment of inertia with respect to an axis through the centre is 2/3 of this:centre is 2/3 of this:

$$I_{\text{axis}} = \tfrac{4}{45}\pi\rho_0a^5.$$

$$\therefore \qquad \underline{\underline{I_{\text{axis}} = \tfrac{4}{15}Ma^2}}.$$

? Exercise 22.8.22



Left-hand particle: $T = mg[\mu\cos(\alpha - \theta) + \sin(\alpha - \theta)]$.

Right-hand particle: $T = mg[\sin(\alpha + \theta) - \mu\cos(\alpha + \theta)]$.

$\therefore \quad \mu[\cos(\alpha - \theta) + \cos(\alpha + \theta)] = \sin(\alpha + \theta) - \sin(\alpha - \theta)$,

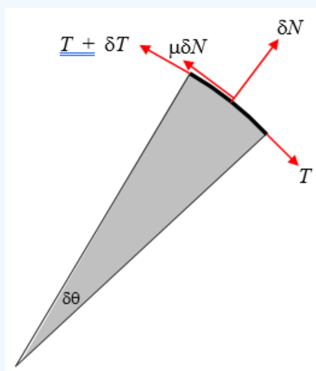
and, by the "sum and difference" trigonometrical formulae, we obtain

$$2\mu\cos\alpha\cos\theta = 2\cos\alpha\sin\theta,$$

from which

$$\underline{\underline{\tan\,\theta = \mu.}}$$

Exercise 22.8.23



Consider a portion of the rope between θ and $\delta\theta$. There are four forces on this portion. The tension T at θ. The tension $T + \delta T$ at $\theta + \delta\theta$ (δT is negative). The normal reaction δN of the cylinder on the rope. The frictional force $\mu\delta N$ of the cylinder on the rope. Note that the rope is about to slip downwards, so the friction force is upwards as shown.

We have

$$\delta N = (2T + \delta T)\sin\left(\tfrac{1}{2}\theta\right)$$

and

$$(T + \delta T)\cos\left(\tfrac{1}{2}\delta\theta\right) + \mu\delta N = T\cos\left(\tfrac{1}{2}\delta\theta\right) \; .$$

To first order, these become

$$\delta N = T\delta\theta$$
$$\delta T = -\mu\delta N$$

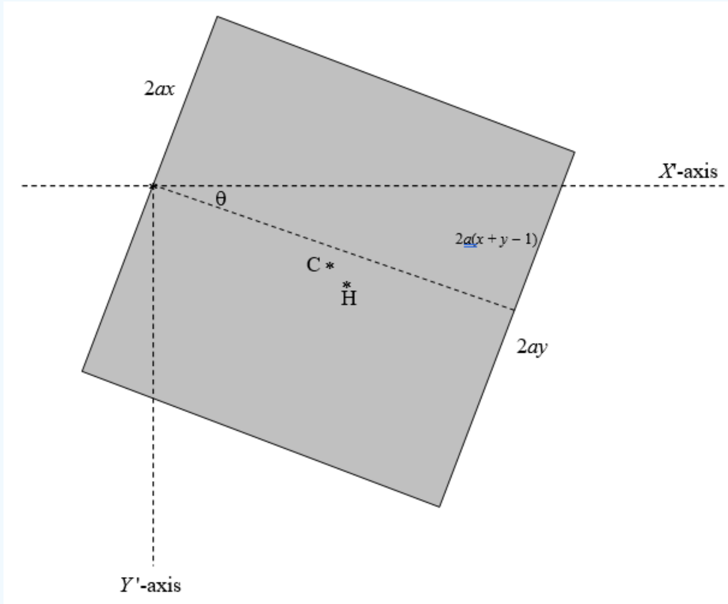
and

$$\delta T = -\mu T\delta\theta$$

and hence by integration

$$\underline{\underline{F = Mge^{-\mu\alpha}}}.$$

Exercise 22.8.24



Area of square = $4a^2$

Area of rectangle = $4a^2(1-x)$

Area of triangle = $2a^2(x+y-1)$

Area of trapezoid = $2a^2(1-x+y)$

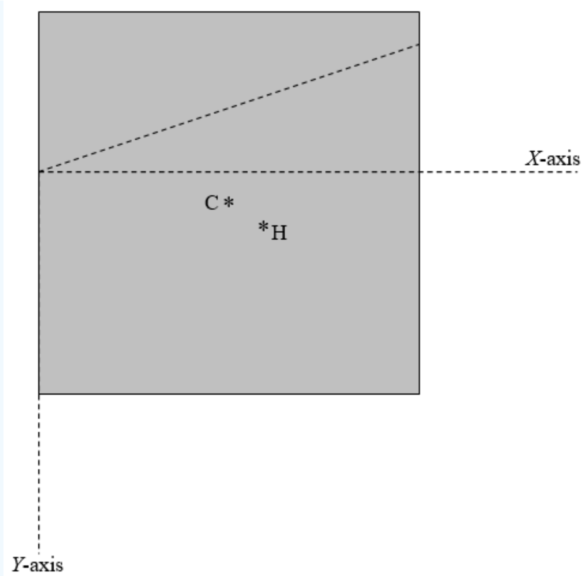
The weight of the cube is $8a^3\rho s$g, and it acts downward through C, the centre of mass. The hydrostatic upthrust is $4a^3(1-x+y)\rho$g and it acts upward through the centre of buoyancy H. Here ρ is the density of the fluid, and ρs is the density of the wood. We evidently must find the centre of mass. The hydrostatic upthrust is $4a^3(1-x+y)\rho$g and it acts upward through the centre of buoyancy H. Here ρ is the density of the fluid, and ρs is the density of the wood. We evidently must find the $X'-$ coordinate of C and of H. Let's first of all find the $X-$ and $Y-$ coordinates (see the next figure).

The $X-$ and $Y-$coordinates of C are trivial and quite easy respectively:

$$X_C = a \qquad Y_C = a(1-2x)$$

You are going to have to work quite hard at it to find the $X-$ and $Y-$ coordinates of H, the centre of buoyancy, which is the centroid of the trapezoid. "After some algebra" you should findcentre of buoyancy, which is the centroid of the trapezoid. "After some algebra" you should find

$$X_H = \frac{2(1-x+2y)a}{3(1-x+y)} \qquad Y_H = \frac{2(2-4x+2y+2x^2-2xy-y^2)a}{3(1-x+y)}$$



To find the $X'-$ coordinates of C and of H, we use the usual formulas for rotation of axes, being sure to get it the right way round:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

together with

$$\tan\,\theta = x + y - 1 \quad .$$

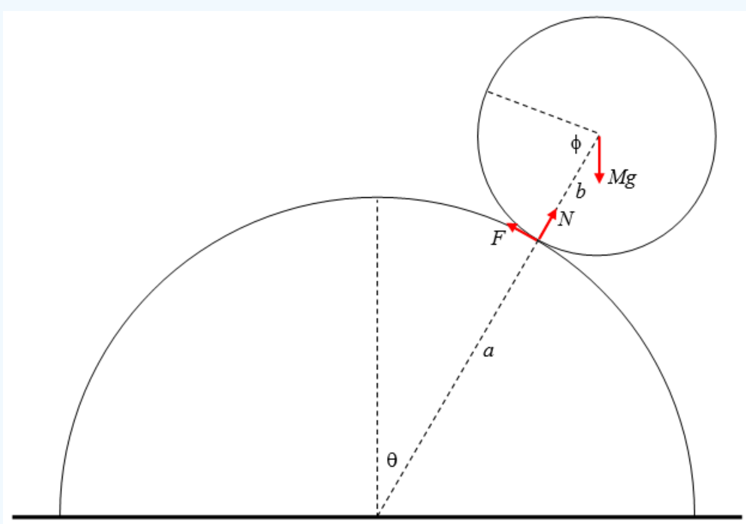
Take moments about the axle (origin):

$$8a^3\rho s\mathrm{g}X'_C = 4a^3\rho\mathrm{g}(1 - x + y)\ .$$

After a little more algebra, you should eventually arrive at

$$\underline{\underline{s = \frac{3-7x+2y+6x^2-3y^2-2x^3+y^3+3xy^2}{3(2-3x-y+2x^2+2xy)}}}$$

Exercise 22.8.25



Let the radii of the cylinder and sphere be a and b respectively, and the mass of the sphere be M. The angles θ and ϕ are related by $a\theta = b\phi$. I have drawn the three forces on the sphere, namely its weight, the normal reaction of the cylinder on the sphere, and the frictional force on the sphere. The transverse acceleration of the centre of the sphere is $(a+b)\ddot{\theta}$ and the centripetal acceleration is $(a+b)\dot{\theta}^2$. The equations of motion are:centre of the sphere is $(a+b)\ddot{\theta}$ and the centripetal acceleration is $(a+b)\dot{\theta}^2$. The equations of motion are:

$$Mg\sin\theta - F = M(a+b)\ddot{\theta} \tag{1}$$

and

$$Mg\cos\theta - N = M(a+b)\dot{\theta}^2 \tag{2}$$

The angular acceleration of the sphere about its centre is $\ddot{\theta} + \ddot{\phi} = \left(1+\frac{a}{b}\right)\ddot{\theta}$, and its rotational inertia is $\frac{2Mb^2}{5}$. The torque that is causing this angular acceleration is Fb, and therefore the rotational equation of motion is

$$Fb = \frac{2}{5}Mb^2\left(1+\frac{a}{b}\right)\ddot{\theta} \tag{3}$$

Elimination of F between Equations (1) and (3) yields

\ddot{\theta} = \frac{5\text{g}}{7(a+b)}\sin\theta. \tag{4}\label{20.4}

Write $\ddot{\theta}$ as $\dot{\theta}\frac{d\dot{\theta}}{d\theta}$ in the usual way and integrate with initial conditions $\theta = \dot{\theta} = 0$ or from energy considerations:

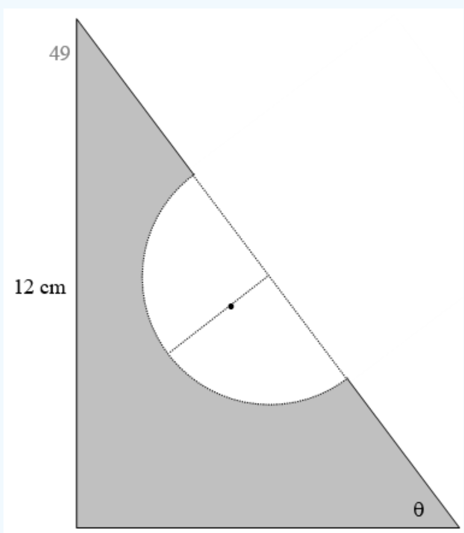
\dot{\theta}^{2} =\frac{10\text{g}}{7(a+b)}(1-\cos\theta) \tag{5}\label{20.5}

Substitute for $\ddot{\theta}$ and $\dot{\theta}^2$ into Equation (2) to obtain

N =\ M\text{g}(17\cos\theta-10). \tag{6}\label{20.6}

This is zero, and the sphere leaves the cylinder, when $\cos\theta = \frac{10}{17}$, $\theta = 53^\circ 58'$.

Exercise 22.8.26



Surface density = σ g cm^{-2}

Original sandwich:

Mass = 54σ g

x-coordinate of centre of mass = 3 cm centre of mass = 3 cm

y-coordinate of centre of mass = 4 cm centre of mass = 4 cm

Bite:

Mass = $\frac{1}{2}\pi 3^2 \sigma$ =14.137 166 94σ g

Distance of centre of mass from hypotenuse = centre of mass from hypotenuse= $\frac{4}{3\pi} \times 3 = \frac{4}{\pi}$ = 1.273 239 545 cm

x-coordinate of centre of mass = 4.5 -centre of mass = $4.5 - \frac{4}{\pi}\sin\theta = 4.5 - \frac{16}{5\pi}$ = 3.481 408 364 cm

y-coordinate of centre of mass = 6- centre of mass = $6 - \frac{4}{\pi}\cos\theta = 6 - \frac{12}{5\pi}$ = 5.236 056 273 cm

Remainder:

Mass = (54 - 14.137 166 94)σ = 39.862 833 06σ g

x-coordinate of centre of mass = $\bar{x}$centre of mass = $\bar{x}$

y-coordinate of centre of mass = $\bar{y}$centre of mass = $\bar{y}$

Moments:

39.862 833 06$\bar{x}$ + 14.137 166 94 × 3.481 408 364 = 54 × 3. $\underline{\underline{\bar{x} = 2.829\ 270\ 780 \text{ cm}}}$

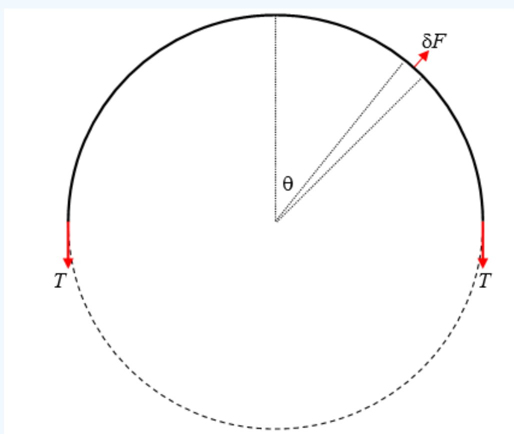
39.862 833 06$\bar{y}$ + 14.137 166 94 × 5.236 056 273 = 53 × 4. $\underline{\underline{\bar{y} = 3.561\ 638\ 436 \text{ cm}}}$

This point is very close to the edge of the bite. The centre of the bite is at (4.5, 6), and its radius is 3. Its equation is thereforecentre of the bite is at (4.5, 6), and its radius is 3. Its equation is therefore

$$(x - 4.5)^2 + (y - 6)^2 = 9, \quad \text{or } x^2 + y^2 - 9x - 12y + 47.25 = 0 \quad .$$

The line x = 2.829 270 780 cuts the circle where $y^2 - 12y + 29.791\ 336\ 13 = 0$. The lower of the two points of intersection is at y = 3.508 280 941 cm. The centre of mass is slightly higher than this and is therefore just inside the bite.centre of mass is slightly higher than this and is therefore just inside the bite.

Exercise 22.8.27

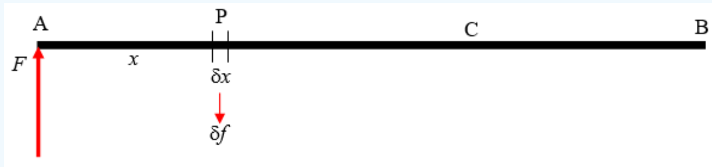


Consider a portion of the band within the angle $\delta\theta$. Its mass is $\frac{m\delta\theta}{2\pi}$ When the band is spinning at angular speed ω and its radius is r, the centrifugal force on that portion is $\delta F = \frac{mr\omega^2\delta\theta}{2\pi}$. (I leave it to the philosophers and the schoolteachers to debate as to whether there "really" is "such thing" as centrifugal force – I want to get this problem done, and I'm referring to a co-rotating frame.) The y-component of this force is $\frac{mr\omega^2\delta\theta}{2\pi}$. Also, the tension in the band when its radius is r is $T = 2\pi k(r - a)$.

Consider the equilibrium of half of the band. The y-component of the centrifugal force on it is $\frac{mr\omega^2}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\theta d\theta = \frac{mr\omega^2}{\pi}$. The opposing force is $2T = 4\pi k(r - a)$. Equating these gives $\underline{\underline{\omega^2 = \frac{4\pi^2 k(r-a)}{mr}}}$.

Exercise 22.8.28

Let the distance AB be l and the distance AC be c. Let the mass of the rod be m.



Consider an elemental portion δx of the rod at P at a distance x from A. Its weight is $\frac{m\delta x}{l}$. When the rod is about to move, it will experience a frictional force $\delta f = \frac{\mu m g \delta x}{l}$, which will be in the direction shown if P is to the left of C, and in the opposite direction if P is to the right of C. When the rod is just about to move (but has not yet done so) it is still in equilibrium. Consider the moment about A of the frictional forces on the rod. The clockwise moment of the frictional forces on AC must equal the counterclockwise moment of the frictional forces on CB. Thus

$$\frac{\mu m g}{l}\int_0^c x\,dx = \frac{\mu m g}{l}\int_c^l x\,dx\,.$$

$$\therefore \qquad \underline{\underline{c = \frac{l}{\sqrt{2}}}}\,.$$

The net force on the rod is

$$F - \frac{\mu m g}{l}\int_0^c dx + \frac{\mu m g}{l}\int_c^l dx \;\;,$$

and this is zero, and therefore

$$\underline{\underline{F = \frac{\mu m g(2c-l)}{l} = (\sqrt{2}-1)\mu m g.}}$$

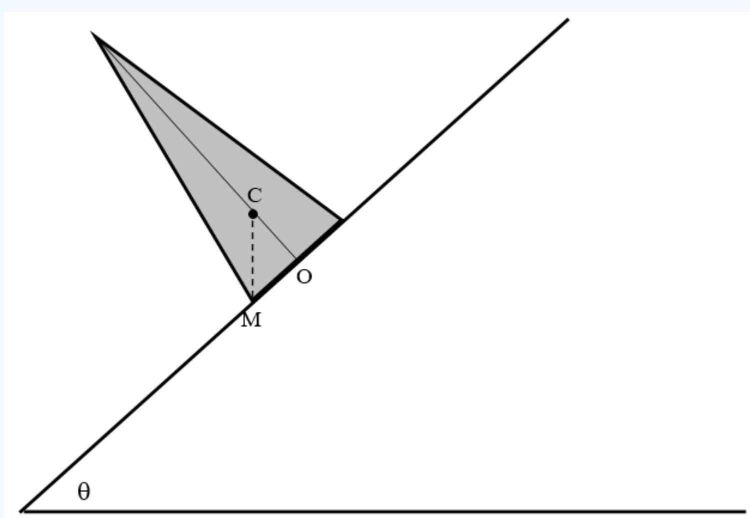
Exercise 22.8.29

The cone slips when $\tan\theta > \mu$.

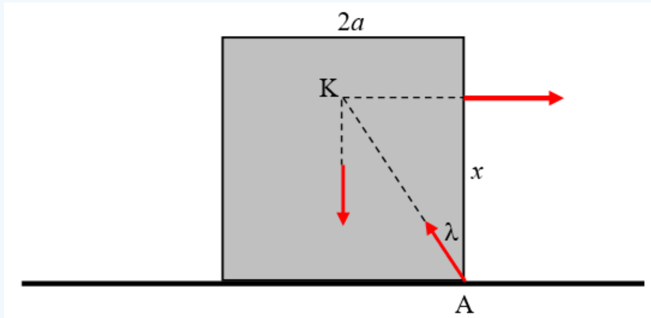
It tips when C (the centre of mass) is to the left of M. centre of mass) is to the left of M.

The distance OC is $\frac{h}{4}$. (See Chapter 1, Section 1.7). Therefore it tips when $\tan\theta > 4\frac{a}{h}$.

Thus it slips if $\mu < 4\frac{a}{h}$ and it tips if $\mu < 4\frac{a}{h}$.

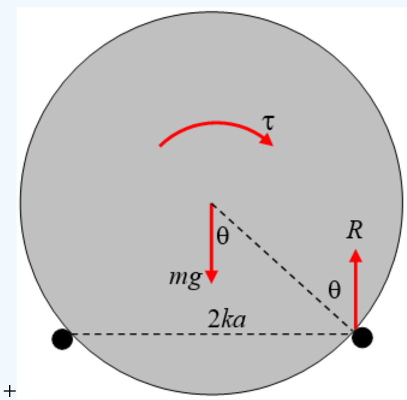


Exercise 22.8.30



When the block is just about to tip, the reaction of the table on the block acts at A and it is directed towards the point K, because, when three coplanar forces are in equilibrium they must act through a single point. The angle λ is given by $\tan\lambda = \frac{a}{x}$. However, by the usual laws of friction, the block will slip as soon as $\tan\lambda = \mu$. Thus the block will slip if $\mu < \frac{a}{x}$, and it will tip if $\mu > \frac{a}{x}$. Expressed otherwise, it will slip if $x < \frac{a}{\mu}$ and it will tip if $x > \frac{a}{\mu}$. The greatest possible value of x is $2a$; therefore the block will inevitably slip if $\mu < \frac{1}{2}$

Exercise 22.8.31



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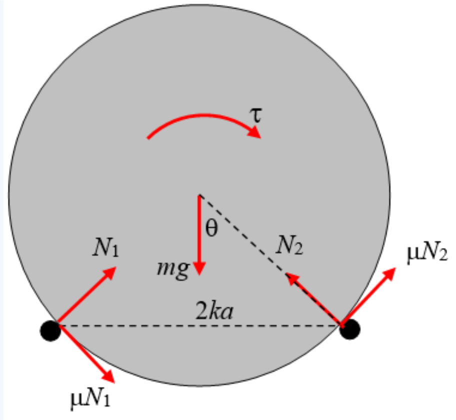
When or if the cylinder is just about to tip, it is about to lose contact with the left hand peg. The only forces on the cylinder are the torque, the weight, and the reaction R of the right hand peg on the cylinder, which must be vertical and equal to mg. But the greatest possible angle that the reaction R can make with the surface of the cylinder is the angle of friction λ given by $\tan\,\lambda\,=\,\mu$. From geometry, we see that $\sin\,\theta\,=\,k$, or $\tan\,\theta\,=\,\frac{k}{\sqrt{1-k^2}}$. Thus the cylinder will slip before it tips if $\mu\,<\,\frac{k}{\sqrt{1-k^2}}$ and it will tip before it slips if $\mu\,>\,\frac{k}{\sqrt{1-k^2}}$.

If the cylinder tips (which it will do if $\mu\,>\,\frac{k}{\sqrt{1-k^2}}$), the clockwise torque t at that moment will equal the counterclockwise torque of the couple (R and mg), which is $mgka$. Thus the torque when the cylinder tips is

TIP:

$$\tau\,=\,mgak. \tag{1}$$



When or if the cylinder is just about to slip, the forces are as shown above, in which I have resolved the reactions of the pegs on the cylinder into a normal reaction (towards the axis of the cylinder) and a frictional force, which, when slipping is about to occur, is equal to m times the normal reaction. The equilibrium conditions are

$$\mu(N_1 + N_2)\cos\theta + (N_1 - N_2)\sin\theta = 0 \quad ,$$

$$\mu(N_1 - N_2)\sin\theta - (N_1 + N_2)\cos\theta + mg = 0$$

and

$$\mu(N_1 + N_2)a = \tau \ .$$

We can find $N_1 + N_2$ by eliminating $N_1 - N_2$ from the first two equations, and then, writing $\sqrt{1-k^2}$ for $\cos\theta$, we find that, when slipping is about to occur,

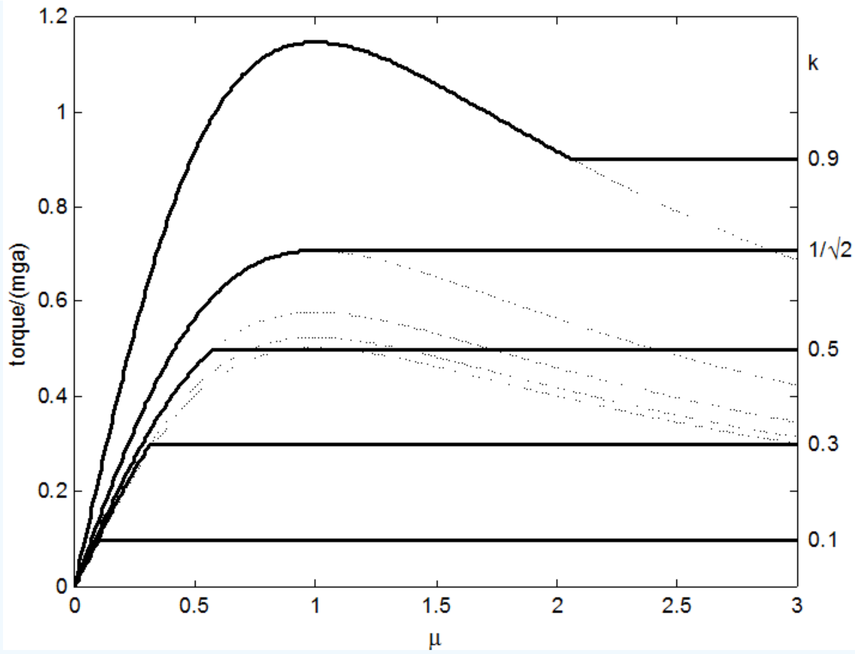
SLIP:

$$\tau = mga \times \frac{\mu}{1+\mu^2} \times \frac{1}{\sqrt{1-k^2}}. \tag{2}$$

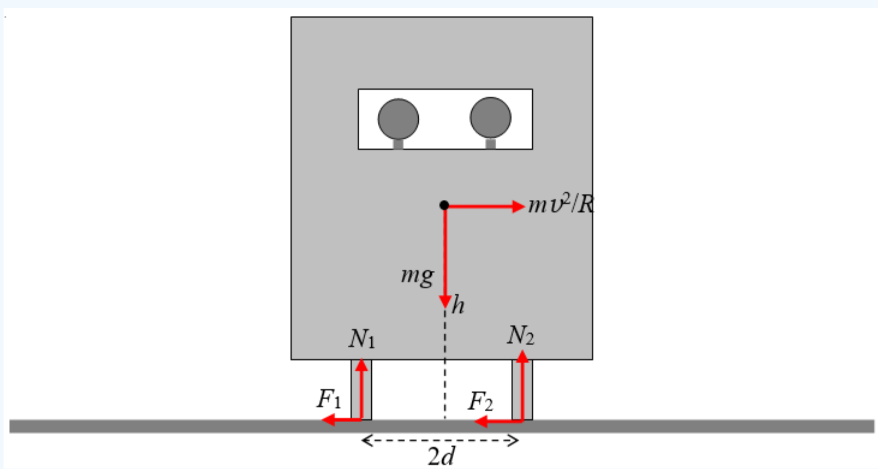
I have drawn below the functions

$$\frac{\tau}{mga} = k \text{ (tip) and } \frac{\tau}{mga} = \frac{\mu}{1+\mu^2} \times \frac{1}{\sqrt{1-k^2}} \text{ (slip)}$$

for $k = 0.1, 0.3, 0.5, \frac{1}{\sqrt{2}}$ and 0.9. The horizontal lines are the tip functions, and the curves are the slip functions. As long as $\mu < \frac{k}{\sqrt{1-k^2}}$ the cylinder will slip. As soon as $\mu < \frac{k}{\sqrt{1-k^2}}$ the cylinder will tip.



Exercise 22.8.32



We'll leave to the philosophers the question as to whether centrifugal force "really exists", and we'll work in a co-rotating reference frame, so that the car, when referred to that frame, is in static equilibrium under the six forces shown. Clearly, N_1 and $N_2 = mg$ and $F_1 + F_2 = \frac{mv^2}{R}$.

The car slips when $F_1 + F_2 = \mu(N_1 + N_2)$; that is, when $v = \sqrt{\mu g R}$.

The car tips when $\frac{mv^2h}{R} = mgd$; that is, when $v = \sqrt{\frac{dgR}{h}}$.

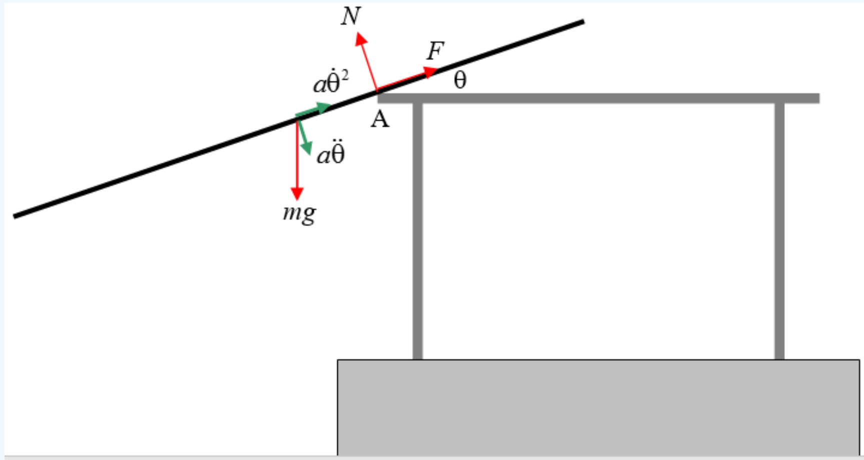
That is, it will slip or tip according as to whether $\mu < \frac{d}{h}$ or $> \frac{d}{h}$.

For example suppose $d = 60 \text{ cm}$, $h = 60 \text{ cm}$, $g = 9.8 \text{ m s}^{-2}$, $R = 30 \text{ m}$, $\mu = 0.8$.

In that case, $\frac{d}{h} = 0.75$, so it will tip at $v = 14.8 \text{ m s}^{-1} = 53.5 \text{ km hr}^{-1}$.

But if it rains, reducing μ to 0.7, it will slip at $v = 14.3 \text{ m s}^{-1} = 51.6 \text{ km hr}^{-1}$.

? Exercise 22.8.33



I have drawn in green the radial and transverse components of the acceleration of the centre of mass $a\dot{\theta}^2$ and $a\ddot{\theta}$ respectively. I have drawn in red the weight of the rod and the normal and frictional components of the force of the table on the rod at A, N and F respectively.centre of mass $a\dot{\theta}^2$ and $a\ddot{\theta}$ respectively. I have drawn in red the weight of the rod and the normal and frictional components of the force of the table on the rod at A, N and F respectively.

The following are the equations of motion:

Normal:

$$ma\ddot{\theta} = mg\cos\theta - N. \tag{1}$$

Lengthwise:

$$ma\dot{\theta}^2 = -mg\sin\theta + F. \tag{2}$$

Rotation:

$$k^2\ddot{\theta} = ga\cos\theta \tag{3}$$

Here k is the radius of gyration about A, given by

$$k^2 = \frac{1}{3}l^2 + a^2. \tag{4}$$

From Equations (1), (3) and (4), we obtain

$$N = mg\cos\theta.\left(\frac{l^2}{l^2 + 3a^2}\right). \tag{5}$$

The space integral (see Chapter 6, Section 6.2) of Equation (3), with initial condition $\dot{\theta} = 0$ when $\theta = 0$, results in

$$\dot{\theta}^2 = \frac{2ga}{k^2}\sin\theta. \tag{6}$$

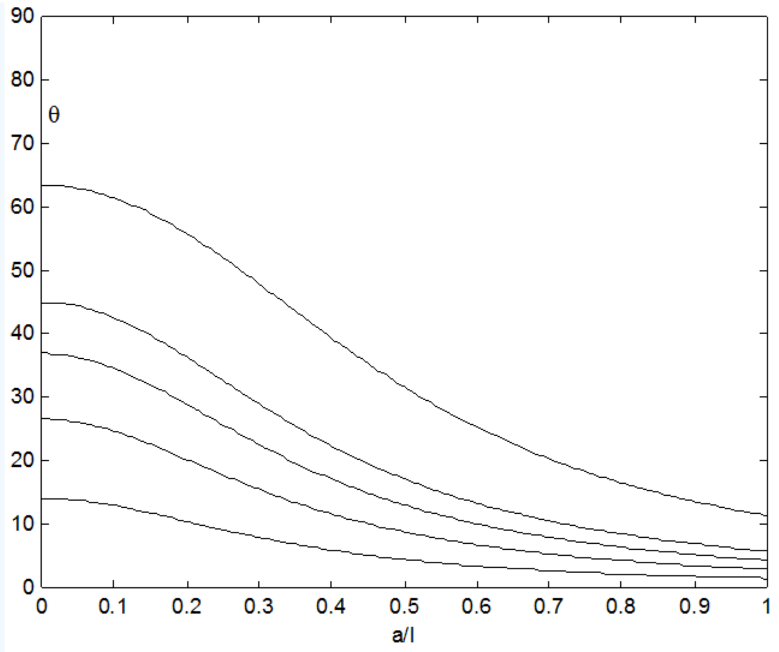
This can also be obtained by equation the loss of potential energy, $mga\sin\theta$ to the gain in kinetic energy, $\frac{1}{2}mk^2\dot{\theta}^2$.

Combining this with Equations (2) and (4) leads to

$$F = mg\sin\theta.\left(\frac{l^2 + 9a^2}{l^2 + 3a^2}\right). \tag{7}$$

At the instant of slipping, $F = \mu N$, and hence, from Equations (5) and (7) we find

$$\tan\theta = \frac{\mu}{1+9\left(\frac{a}{l}\right)^2}.$$



Exercise 22.8.34

I derive $v^2 = \mathrm{g}x + \frac{\mathrm{g}}{l}x^2$ by two different methods – one from energy considerations, the other from angular momentum considerations. First, energy.

If the table top is taken to be the zero level for potential energy, the initial potential energy was $-\frac{1}{2}m.\mathrm{g}.\frac{1}{4}l = -\frac{1}{8}mgl$.

When the length of the dangling portion is $\frac{1}{2}l + x$ the potential energy is

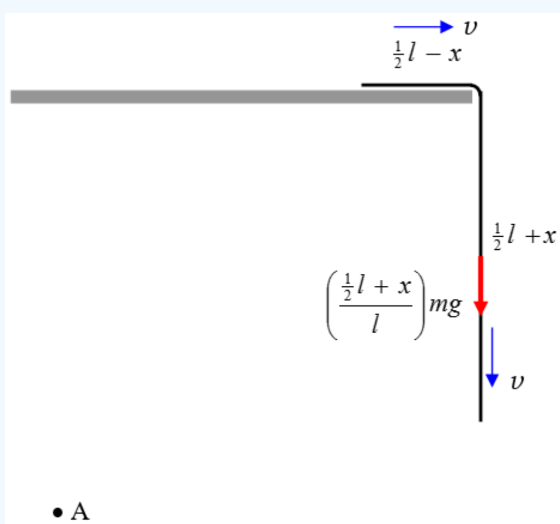
$$-\left(\frac{\frac{1}{2}l+x}{l}\right)m.\mathrm{g}.\frac{1}{2}\left(\frac{1}{2}l + x\right) = -\frac{m\mathrm{g}}{2l}\left(\frac{1}{2}l + x\right)^2 = -\frac{1}{8}mgl - \frac{1}{2}mgl - \frac{mgx^2}{2l} .$$

The loss of potential energy is therefore $\frac{1}{2}mgx + \frac{mgx^2}{2l}$.

This is equal to the gain in kinetic energy $\frac{1}{2}mv^2$, and therefore

$$v^2 = \mathrm{g}x + \frac{\mathrm{g}}{l}x^2 .$$

Another method:



Consider a point A. Anywhere will do, but I have chosen it to be a distance l below the level of the table and l to the left of the table edge. The moment of momentum (= angular momentum) of the chain about this point is $mlv = ml\dot{x}$ and its rate of change is therefore $mlv = ml\dot{x}$. The torque about A is \left(\mlv\ =\ ml\dot{x} and its rate of change is therefore $mlv = ml\dot{x}$. The torque about A is $\left(\frac{\frac{1}{2}l+x}{l}\right)mgl = \left(\frac{1}{2}l + x\right)mg$. These are equal, and so $l\ddot{x} = \mathrm{g}\left(\frac{1}{2}l + x\right)$. Write $\ddot{x} = v\frac{dv}{dx}$ in the usual way, and integrate (with $v = 0$ when $x = 0$) and the result $v^2 = \mathrm{g}x + \frac{g}{l}x^2$ follows.

To find the relation between x and t we can use the energy Equation 9.2.9 for conservative systems

$$t = \sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E-V(x)}}.$$

Here $x_0 = 0$ and we have already seen that $E - V(x) = \frac{mg}{2l}x^2 + \frac{mgx}{2}$. Upon integrating this expression, we obtain, after a little algebra and calculus,

(1)

The converse of this is the required expression

(2)

Differentiation of this with respect to time produces the third required expression:

(3)

You may verify from these last two equations, if you wish, that $v^2 = \mathrm{g}x + \frac{\mathrm{g}}{l}x^2$.

The chain falls completely off the table when $x = \frac{1}{2}l$. That is (by using Equation (1)), at time $\sqrt{\frac{l}{g}}\ln(2+\sqrt{3}) = 1.317\sqrt{\frac{l}{\mathrm{g}}}$.

If we express distances in units of l, time in units of $\sqrt{\frac{l}{\mathrm{g}}}$ and therefore necessarily speeds in units of $\sqrt{\mathrm{g}l}$, Equations (2) and (3) become

$$x = \frac{(e^t-1)^2}{4e^t} = \frac{1}{4}(e^t + e^{-t} - 2) = \frac{1}{2}(\cosh t - 1) \tag{4}$$

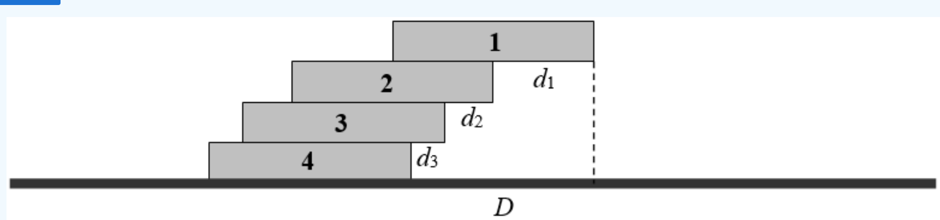
$$v = \frac{e^{2t}-1}{4e^t} = \frac{1}{2}\sinh t \tag{5}$$

and we can get the acceleration by a further differentiation:

$$a = \frac{1}{4}(e^t + e^{-t}) = \frac{1}{2}\cosh t. \tag{6}$$

We are pleased to note that, by the time that $x = \frac{1}{2}l$ [i.e. when the chain completely leaves the table at time $t = \ln(2+\sqrt{3})\sqrt{\frac{l}{\mathrm{g}}}$], the acceleration is g. The speed is then $\sqrt{\frac{3}{4}l\mathrm{g}} = 0.866\sqrt{l\mathrm{g}}$.

Exercise $22.8.35a$



The maximum overhang of book **1** is $d_1 = w$.

The centre mass of $\mathbf{1} + \mathbf{2}$ centre mass of $\mathbf{1} + \mathbf{2}$ is at $\frac{3w}{2}$ from the left hand side (LHS) of **2**, so $d3 = \frac{w}{3}$.

The distance of the centre of mass of $\mathbf{1}+\mathbf{2}+\mathbf{3}$ centre of mass of $\mathbf{1}+\mathbf{2}+\mathbf{3}$ is at $\frac{5\mathrm{w}}{2}$ from the LHS of **3**, so $d3 = \frac{w}{3}$.

Thus $D = d_1 + d_2 + d_3 = \left(1 + \frac{1}{2} + \frac{1}{3}\right) \mathrm{w} = 1.8\dot{3}\mathrm{w}$.

Exercise 22.8.35b

In a similar manner we find that, given $n + 1$ books, the maximum overhang is

$$D = \left(1+\frac{1}{2}+\frac{1}{3}\ldots\ \ldots+\frac{1}{n}\right) \mathrm{w}\ .$$

I do not know if there is a simple expression for the sum to n terms of this harmonic series. Please let me know if you know of one or can find one. Therefore I used a computer to solve

$$1+\frac{1}{2}+\frac{1}{3}\ldots\ \ldots+\frac{1}{n} = 10$$

by brute force. I got $n = 12367$, so you would need 12368 books.

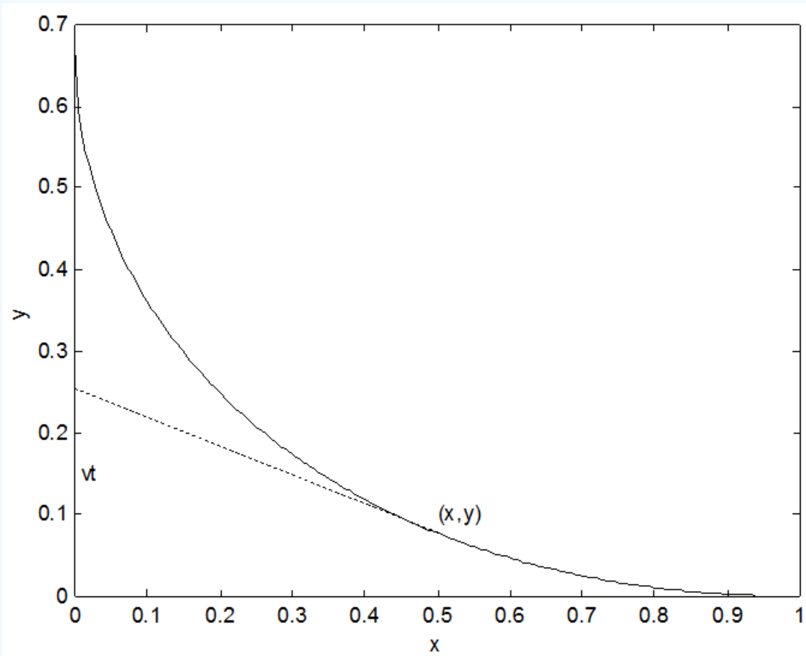
Exercise 22.8.35c

The harmonic series is divergent and has no finite limit, so there is no finite limit to the possible overhang.

You might wish to speculate on any practical limitations on constructing such a pile of books. For example, we have been assuming a uniform gravitational field – but this will no longer be valid once the overhang becomes comparable to the radius of Earth. This will, however, need quite a large number of books.

Exercise 22.8.36

In the solution that follows, a prime will be used to denote differentiation with respect to x, and $p = y' = \frac{dy}{dx}$. I shall also make use of an auxiliary variable $\phi = \sinh^{-1} p$. The initial conditions are $y=0,\ x=a,\ p=0,\ \phi=0$. The final conditions are $x=0,\ p=-\infty, \phi=-\infty, y$ to be determined.



At time t, the $y-$coordinate of the Man is $v\,t$. If $(x,\ y)$ are the coordinates of the Dog at that time, the slope of the path taken by the Dog is

$$p = -\frac{vt-y}{x}, \tag{1}$$

so that

$$v\,t = y - px. \tag{2}$$

The speed of the Dog is

$$Av = -\frac{ds}{dt} = -\sqrt{1+p^2}\frac{dx}{dt}. \tag{3}$$

[This comes from $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$. The minus sign is necessary because $\left(\frac{dx}{dt}\right)$ is negative, and Av, the speed (not *velocity*!) of the Dog is necessarily positive.]

Now $\left(\frac{dx}{dt}\right) = -\frac{1}{t'}$ so Equation (???) can be written

$$Avt' = -\sqrt{1+p^2} \tag{4}$$

If we can eliminate t between Equations (2) and (4), we will obtain a relation between the slope p and x, and hence potentially a relation between y and x.

Differentiate Equation (??? with respect to x (recalling that $y' = p$):

$$vt' = -p'x \tag{5}$$

It is now simple to eliminate t' from Equations (4) and (5):

$$Ap'x = \sqrt{1+p^2} \tag{6}$$

On separating the variables and integrating, we obtain

$$A\int\frac{dp}{\sqrt{1+p^2}} = \int\frac{dx}{x}. \tag{7}$$

With initial conditions $p = 0$ when $x = a$, this gives us

$$A\sinh^{-1}p = \ln\left(\frac{x}{a}\right), \tag{8}$$

or

$$x = ae^{A\phi}, \tag{9}$$

where

$$\phi = \sinh^{-1}p. \tag{10}$$

Equation (9), with (10), gives us the relation between x and the slope, p. Note that p and hence f are negative, so that equation says that $x < a$.

Our next task will be to find a relation between y and p (or between y and ϕ).

From Equation (10) we have

$$dy = \sinh\phi\, dx, \tag{11}$$

and from Equation (9) we have

$$dx = aAe^{A\phi}d\phi. \tag{12}$$

From these we obtain the differential relation between y and ϕ:

$$dy = aAe^{A\phi}\sinh\phi d\phi, \tag{13}$$

or

$$dy = \frac{1}{2}aA(e^{(A+1)\phi} - e^{(A-1)\phi})d\phi. \tag{14}$$

Integrate this, with initial condition $\phi = 0$ when $y = 0$, to obtain

$$y = \frac{1}{2}aA\left(\frac{\left(\frac{x}{a}\right)^{1+1/A}}{A+1} - \frac{e^{(A-1)\phi}}{A-1} + \frac{2}{A^2-1}\right). \tag{15}$$

Equation (9) and (15) are parametric equations to the path of the Dog, though it is easy to eliminate ϕ and write y explicitly as a function of x:

$$y = \frac{1}{2}aA\left(\frac{e^{(A+1)\phi}}{A+1} - \frac{e^{(A-1)\phi}}{A-1} + \frac{2}{A^2-1}\right). \tag{16}$$

The figure was drawn for $a = 1, \; A = 2,$ for which Equation (16) reduces to

$$y = \frac{1}{3}[x^{\frac{1}{2}}(x-3) + 2]. \tag{17}$$

The distance walked by the Man is found by putting $\phi = -\infty$ in Equation 15. Thus

$$y = \frac{aA}{A^2-1}, \tag{18}$$

and the time taken is

$$t = \frac{aA}{v(A^2-1)} \tag{19}$$

Exercise 22.8.37

Let l be the length of the string.

Exercise 22.8.37a

Kinetic energy of the upper mass = $\frac{1}{2}(mr^2)\omega^2 + \frac{1}{2}m\dot{r}^2$.

Kinetic energy of the lower mass = $\frac{1}{2}m\dot{r}^2$

Potential energy of the lower mass = $mg(l-r)$.

Total energy of the system =

$$\frac{1}{2}(mr^2)\omega^2 + m\dot{r}^2 - mg(l-r). \tag{1}$$

Initial total energy of the system =

$$\frac{1}{2}(ma^2)\omega_0^2 + m\dot{r}^2 - mg(l-a). \tag{2}$$

Energy is conserved and therefore, by equating (1) and (2), we obtain

$$\dot{r}^2 = g(a-r) + \frac{1}{2}a^2\omega_0^2 - \frac{1}{2}r^2\omega^2. \tag{3}$$

Angular momentum is also conserved, and therefore

$$r^2\omega = a^2\omega_0 \tag{4}$$

On elimination of r between Equations (3) and (4) we obtain, after some algebra,

$$\frac{\dot{r}^2}{ga} = 1 + \frac{a\omega_0^2}{2g}\left(1 - \frac{\omega}{\omega_0}\right) - \sqrt{\frac{\omega_0}{\omega}}. \tag{5}$$

? Exercise 22.8.37b

If $a\omega_0^2 = \text{g}$ it is trivial to show that

$$\frac{\dot{r}^2}{ga} = \frac{3}{2} - \frac{1}{2}\Omega - \frac{1}{\sqrt{\Omega}}. \tag{6}$$

? Exercise 22.8.37c

Algebra and calculus show that $\frac{3}{2} - \frac{1}{2}\Omega - \frac{1}{\sqrt{\Omega}}$ is negative for all positive Ω except for $\Omega = 1$, when it reaches a maximum value of zero.

? Exercise 22.8.37d

If $a\omega_0^2 = 2\text{g}$ and $\Omega = \frac{\omega}{\omega_0}$ it is trivial to show that

$$\frac{\dot{r}^2}{ga} = 2 - \Omega - \frac{1}{\sqrt{\Omega}} \tag{7}$$

Algebra and calculus show that $2 - \Omega - \frac{1}{\sqrt{\Omega}}$ reaches a maximum value for $\Omega = \frac{\omega}{\omega_0} = \frac{1}{2^{\frac{2}{3}}} = 0.629\,961$ at which time $\frac{\dot{r}^2}{(\text{ga})} = 0.110\,118$. That is, when $\dot{r} = 0.331\,841\sqrt{ga}$ Equation (4) (conservation of angular momentum) shows that $r = \frac{a}{\sqrt{\Omega}} = \sqrt[3]{2a} = 1.259\,921a$.

Solution of $2 - \Omega - \frac{1}{\sqrt{\Omega}} = 0$ shows that the speed is zero when $\Omega = 1$ (the initial condition) and when (the equilibrium value). Equation (4) (conservation of angular momentum) shows that $r = \frac{a}{\sqrt{\Omega}} = 1.618\,034a$.

? Exercise 22.8.37e

If $a\omega_0^2 = \frac{1}{2}\text{g}$ and $\Omega = \frac{\omega}{\omega_0}$ it is trivial to show that

$$\frac{\dot{r}^2}{ga} = \frac{5}{4} - \frac{1}{4}\Omega - \frac{1}{\sqrt{\Omega}} \tag{8}$$

Algebra and calculus show that $\frac{5}{4} - \frac{1}{4}\Omega - \frac{1}{\sqrt{\Omega}}$ reaches a maximum value for $\Omega = \frac{\omega}{\omega_0} = 2^{\frac{2}{3}} = 1.587\,401$ at which time $\frac{\dot{r}^2}{(\text{ga})} = 0.059449$. That is, when $\dot{r} = -0.243\,822\sqrt{ga}$ Equation (4) (conservation of angular momentum) shows that $r = \frac{a}{\sqrt{\Omega}} = \frac{a}{\sqrt[3]{2}} = 0.793\,701a$.

Solution of $\frac{5}{4} - \frac{1}{4}\Omega - \frac{1}{\sqrt{\Omega}}$ shows that the speed is zero when $\Omega = 1$ (the initial condition) and when $\Omega = \frac{\omega}{\omega_0} = 2.438\,447$ (the equilibrium value). Equation (4) (conservation of angular momentum) shows that $r = \frac{a}{\sqrt{\Omega}} = 0.640\,388a$.

How much further can we go with this question? By elimination of r between Equations (3) and (4) we obtained a relation between $\dot{r}$ and ω. By elimination of ω between Equations (3) and (4) we can get a relation between $\dot{r}$ and r. It will be of the form

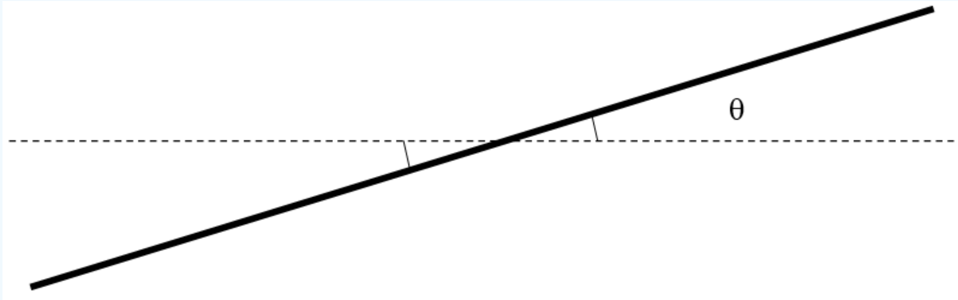
$$\dot{r} = \sqrt{A - gr - \frac{B}{r^2}}, \tag{9}$$

where $A = ga + \frac{1}{2}a^2\omega_0^2$ and $B = a^4\omega_0^4$. If you can integrate this, you then get a relation between r and t. I haven't given much though as to whether you can get integrate Equation (9) analytically (if anyone manages it, please let me know), but at least a numerical integration will certainly be possible.

In another variation of the question, you can start with an equilibrium situation in which $a\omega_0^2 = \text{g}$ and then add an extra mass m (or M, if you want to make it more general) and then follow the motion from there. I leave that to you.

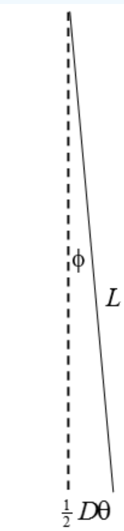
? Exercise 22.8.38

Let's look at the rod from above when it is twisted in the horizontal plane through a small angle θ.



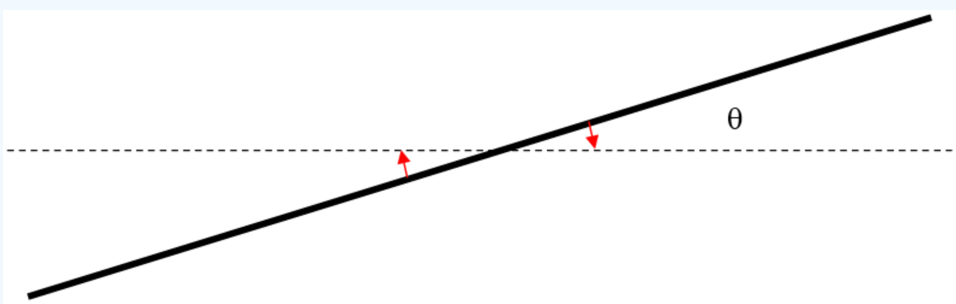
Each of the points where the threads are attached to the rod is displaced horizontally through a distance $\frac{1}{2}D\theta$ (Since θ is small and $D << L$ we can neglect the slight vertical rise in the position of the rod.) Each thread is now displaced from the vertical by an angle ϕ given by

$$\phi = \frac{\frac{1}{2}D\theta}{L}.$$



The tension T in each thread is $\frac{1}{2}mg\cos\phi$ which, to first order in ϕ, is just $\frac{1}{2}mg$.

The horizontal component of each of these forces is $\frac{1}{2}mg\sin\phi$ which, to first order in ϕ, is $\frac{1}{2}mg\phi$.



Therefore the rod experiences a restoring torque equal to $\frac{1}{2}mgD\phi$. But $\phi = \frac{\frac{1}{2}D\theta}{L}$ and therefore the restoring torque is $\frac{mgD^2\theta}{4L}$.

The equation of motion is therefore

$$I\ddot{\theta} = -\frac{mgD^2}{4L}\theta$$

and consequently the period P of small oscillations is

$P = 2\pi\sqrt{\frac{4LI}{mgD^2}} = \frac{4\pi}{D}\sqrt{\frac{LI}{mg}}$.

If the rod is uniform and of length $2l$, its moment of inertia is $\frac{1}{3}ml^2$ and in that case the period of small oscillations is

$P = \frac{4\pi l}{D}\sqrt{\frac{L}{3g}}$.

There is no need to remind the reader to check the dimensions of these equations.

Exercise 22.8.39

When the yo-yo has fallen through a distance x, it has lost potential energy Mgh, and it has gained translational kinetic energy $\frac{1}{2}mv^2$ and gained rotational kinetic energy $\frac{1}{2}I\omega^2$ where $\omega = \frac{v}{a}$. Therefore $Mgx = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v}{a}\right)^2$ from which

$$v^2 = 2 \cdot \frac{Ma^2 g}{Ma^2 + I} \cdot x \ .$$

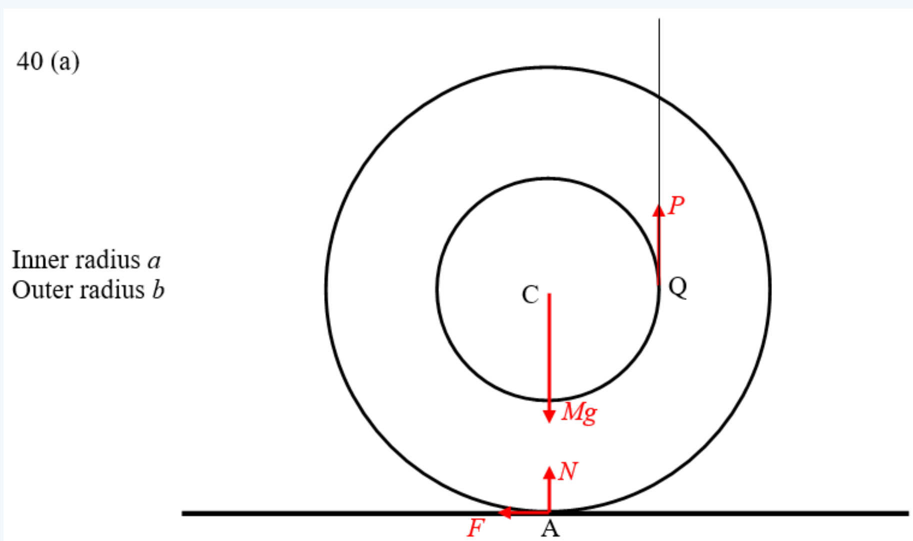
Thus, from the usual equations for constant linear acceleration, the acceleration is

$$\frac{Ma^2}{Ma^2 + I} \times g \, .$$

The net downward force is $Mg - P$ where P is the tension in the string. This is equal to M times the acceleration, from which we obtain

$$P = \frac{1}{Ma^2+1} \times Mg.$$

Exercise 22.8.40a



I have drawn four forces on the yo-yo. Its weight Mg. The tension P in the string. The normal reaction N of the table on the yo-yo. And the frictional force F of the table on the yo-yo. As long as the yo-yo is in contact with the table and there is no vertical acceleration, we must have $P + N = Mg$.

Let us suppose that there is no slipping between the yo-yo and the table, so that the yo-yo rolls to the left. We note that there is a net force F to the left, and a net counterclockwise torque about C equal to $Pa - Fb$. Thus the yo-yo accelerates to the left at a rate $\frac{F}{M}$ it and experiences a counterclockwise angular acceleration $(Pa - Fb)\frac{a}{I}$. If there is no slipping, these must be related by $\frac{F}{M} = b \times \frac{Pa-Fb}{I}$. Thus, if there is no slipping,

$$F = \left(\frac{Mab}{I + Mb^2}\right)P. \tag{1}$$

The linear acceleration to the left must be $\frac{F}{M}$, or

$$\underline{\underline{\frac{abP}{I+Mb^2}}} \tag{2}$$

Alternative derivation:

There is a net counterclockwise torque about A equal to Pa. The moment of inertia with respect to A is $I + Mb^2$. Therefore there is an angular acceleration about A equal to $\frac{Pa}{I+Mb^2}$. Therefore the linear acceleration of C to the left is $\underline{\underline{\frac{abP}{I+Mb^2}}}$ and the frictional force F is M times this, or $F = \left(\frac{Mab}{I+Mb^2}\right)P$.

End of Alternative Derivation.

However, if the yo-yo is just about to slip, $F = \mu N = \mu(Mg - P)$. Upon substitution of this into Equation (1), we see that the yo-yo will just slip if

$$P = \frac{\mu Mg(I + Mb^2)}{\mu(I + Mb^2) + Mab}. \tag{3}$$

That is, the yo-yo will roll to the left without slipping if

$$\underline{\underline{\mu > \frac{MabP}{(Mg-P)(I+Mb^2)}}} \tag{4}$$

Its linear acceleration is then given by Equation (2), namely

$$\underline{\underline{\frac{abP}{I+Mb^2}}} \tag{2}$$

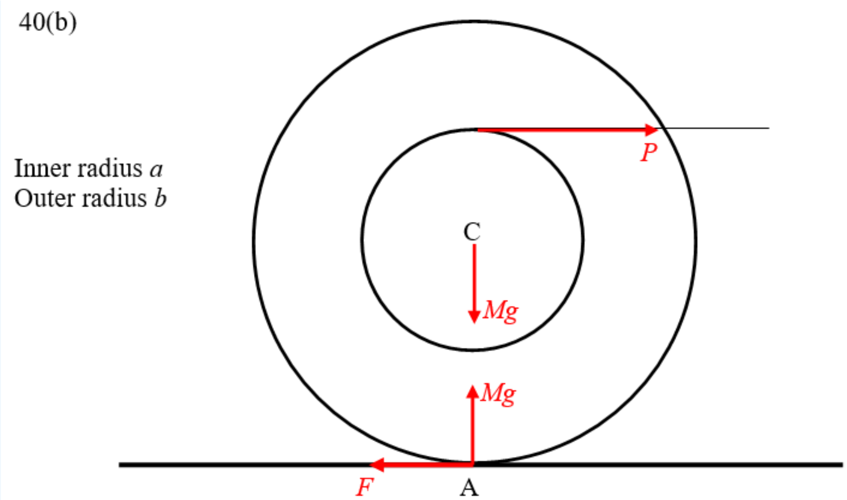
On the other hand, the yo-yo will rotate counterclockwise with no rolling if

$$\underline{\underline{\mu < \frac{MabP}{(Mg-P)(I+Mb^2)}}} \tag{5}$$

The sum of the counterclockwise moments of the forces about C is then $Pa - Fb$ where $F = \mu N = \mu(Mg - P)$. The counterclockwise angular acceleration about C is

$$\frac{Pa-Fb}{I} = \underline{\underline{\frac{P(a-\mu b)-\mu Mg}{I}}}. \tag{6}$$

? Exercise $22.8.40b$



I have drawn the four forces on the yo-yo. Its weight Mg. The normal reaction of the table on the yo-yo, which is also of magnitude Mg. The tension P in the string. And the frictional force F of the table on the yo-yo.

At this point it may not be immediately obvious whether F acts to the left or the right. For example, let us suppose that the coefficient of friction is zero. The force P will result in a translation of the yo-yo to the right together with a clockwise rotation of the yo-yo. So, in which direction does the point A on the circumference of the yo-yo move - to the left or the right? It is hard to say, but one might suppose, qualitatively, that, if the moment of inertia is large, the induced rotation will be sluggish, so that A moves to the right. Whereas if I is small, the induced rotation will be rapid, and A will move to the left in spite of the translational motion of the centre of mass to the right. From this we might conclude that, if $\mu \neq 0$, F will act to the right if I is small, and F will act to the left if I is large. The following analysis shows that this qualitative expectation is correct.centre of mass to the right. From this we might conclude that, if $\mu \neq 0$, F will act to the right if I is small, and F will act to the left if I is large. The following analysis shows that this qualitative expectation is correct.

(The reader might find some of the Problems in Section 8.2 of Chapter 8 to be helpful at this point, particularly Problem 2.5.)

For the time being, I have drawn F as if acting towards the left.

Let us suppose there is no slipping and that the yo-yo rolls.

The sum of the clockwise moments of the forces about A is $P(a+b)$ and the moment of inertia about A is $I+Mb^2$. The yo-yo therefore undergoes an initial clockwise angular acceleration about A equal to $\frac{P(a+b)}{I+Mb^2}$ and, therefore (if there is no slipping), an initial linear acceleration of C to the right equal to

$$\underline{\underline{\frac{Pb(a+b)}{I+Mb^2}}}. \tag{1}$$

The above linear acceleration must equal $\frac{(P-F)}{M}$ from which we obtain

$$F=\left(\frac{I-Mab}{I+Mb^2}\right)P \tag{2}$$

This shows that the frictional force F acts to the left, as shown, if $I>Mab$ but if $I<Mab$ the frictional force F acts to the right. This is in agreement with our qualitative expectations, namely that F will act to the left if I is large, and to the right if I is small.

Let is consider three cases in turn: $I>Mab$, $I<Mab$ and $I=Mab$.

(i) $I>Mab$

In this case, F acts to the left, as drawn. Provided $F<\mu Mg$ there will be no slipping at A, and the yo-yo will roll to the right without slipping. On recalling Equation (2), we see that the yo-yo will roll to the right without slipping, with a linear acceleration given by Equation (1) if

$$\underline{\underline{\mu > \left(\frac{I - Mab}{I + Mb^2}\right)\left(\frac{P}{Mg}\right)}}. \tag{3}$$

The linear acceleration to the right is given by Equation (1), namely

$$\underline{\underline{\frac{Pb(a+b)}{I+Mb^2}}}. \tag{1}$$

However, if

$$\underline{\underline{\mu < \left(\frac{I - Mab}{I + Mb^2}\right)\left(\frac{P}{Mg}\right)}}. \tag{4}$$

slipping occurs at A. The frictional force is the no longer given by Equation (2), but is given by

$$F = \mu Mg, \tag{5}$$

and it acts to the left.

(We are concerned in this problem with the *initial* motion. Once motion is underway, μ has to be replaced with the smaller coefficient of kinetic friction.)

The net force to the right is then $P - \mu Mg$ so the linear acceleration of C to the right is

$$\underline{\underline{\frac{P - \mu Mg}{M}}}. \tag{6}$$

Because of condition (4), this is necessarily positive.

The net clockwise moment of the forces about the centre of mass C is $Pa + \mu Mgb$. The yo-yo therefore undergoes a clockwise angular acceleration about C ofcentre of mass C is $Pa + \mu Mgb$. The yo-yo therefore undergoes a clockwise angular acceleration about C of

$$\underline{\underline{\frac{Pa + \mu Mgb}{I}}}. \tag{7}$$

The linear acceleration to the right of the point A on the circumference of the yo-yo is $\frac{P-\mu Mg}{M} - b \times \frac{Pa+\mu Mgb}{I}$, and, because of condition (4), some algebra will show that this is necessarily positive, as expected.

(ii) $I < Mab$

In this case, F acts to the right, and the linear acceleration is $\frac{(P+F)}{M}$. Provided that

$$\underline{\underline{\mu > \left(\frac{Mab - I}{Mb^2 + I}\right)\left(\frac{P}{Mg}\right)}}, \tag{8}$$

the yo-yo will roll to the right with a linear acceleration given by Equation (1), namely

$$\underline{\underline{\frac{Pb(a+b)}{I+Mb^2}}}. \tag{1}$$

However, if

$$\underline{\underline{\mu > \left(\frac{Mab - I}{Mb^2 + I}\right)\left(\frac{P}{Mg}\right)}}, \tag{9}$$

slipping occurs at A. The frictional force is then given by

$$F = \mu Mg, \tag{10}$$

and it acts to the right.

The net force to the right is then $P + \mu Mg$, so the linear acceleration to the right is

$$\frac{P + \mu M g}{M}. \tag{11}$$

The net clockwise moment of the forces about the centre of mass C is . The yo-yo therefore undergoes a clockwise angular acceleration about C ofcentre of mass C is . The yo-yo therefore undergoes a clockwise angular acceleration about C of

$$\frac{Pa - \mu M g b}{I}. \tag{12}$$

The linear acceleration to the left of the point A on the circumference of the yo-yo is $b \times \frac{Pa - \mu M g b}{I} - \frac{P + \mu M g}{M}$, and, because of condition (8), some algebra will show that this is necessarily positive, as expected.

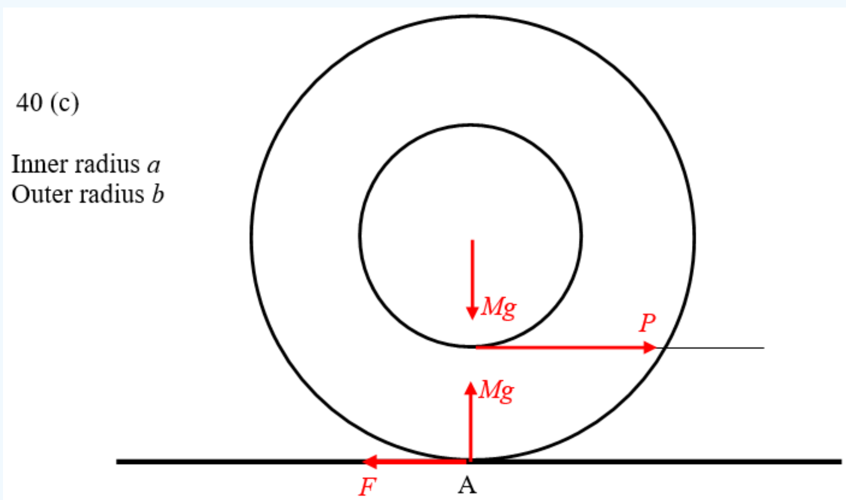
$\boldsymbol{I = Mab}$

In this case, F is zero. Whatever the coefficient of friction, even zero, the yo-yo will undergo a linear acceleration $\frac{P}{M}$ to the right (Verify that this is consistent with Equation (1)), and a clockwise angular acceleration about C equal to $\frac{Pa}{I}$. The linear acceleration to the right of the point A on the circumference of the yo-yo is

$\frac{P}{M} - b \times \frac{Pa}{I}$

which is zero. The initial linear velocity of the point A is therefore zero.

Exercise 22.8.40c



I have drawn the four forces on the yo-yo. Its weight Mg. The normal reaction of the table on the yo-yo, which is also of magnitude Mg. The tension P in the string. And the frictional force F of the table on the yo-yo. On this occasion (unlike in Problem 40 (b)) there is no question about the direction of F, which is towards the left.

Let us suppose there is no slipping.

The sum of the clockwise moments of the forces about A is $\frac{P(b-a)}{I + Mb^2}$ and the moment of inertia about A is $I + Mb^2$. The yo-yo therefore undergoes an initial clockwise angular acceleration about A equal to $\frac{P(b-a)}{I + Mb^2}$ and therefore (if there is no slipping) an initial linear acceleration to the right equal to

$$\frac{Pb(b-a)}{I + Mb^2}. \tag{1}$$

Additional string therefore becomes wrapped around the axle. (Yes, it really does! I tried it!)

The above linear acceleration must equal $\frac{(P-F)}{M}$ from which we obtain

$$F = \left(\frac{I + Mab}{I + Mb^2}\right) P. \tag{2}$$

Provided $F < \mu Mg$,there will be no slipping at A, and the yo-yo will roll to the right without slipping. Thus the yo-yo will roll to the right without slipping, with a linear acceleration given by equation (1) if

$$\mu > \frac{\left(\frac{I+Mab}{I+Mb^2}\right)}{\left(\frac{P}{Mg}\right)}. \tag{2}$$

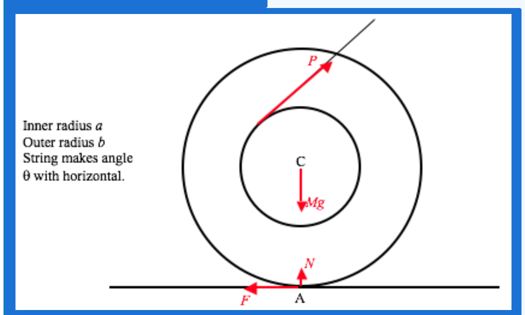
$$\mu < \frac{\left(\frac{I+Mab}{I+Mb^2}\right)}{\left(\frac{P}{Mg}\right)}, \tag{22.8.1}$$

$$F = \mu Mg, \tag{2}$$

$$\frac{P - \mu Mg}{M}. \tag{22.8.2}$$

$$\frac{Pa - \mu Mgb}{I} \tag{22.8.3}$$

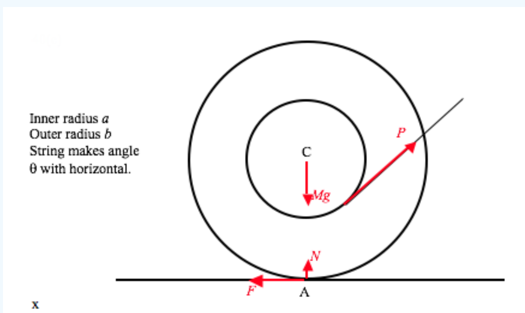
Exercise $22.8.40d$



$$\frac{Pb(a + b\cos\theta}{I + Mb^2} \tag{1}$$

$$F = \left(\frac{I\cos\theta - Mab}{I + Mb^2}\right) P. \tag{2}$$

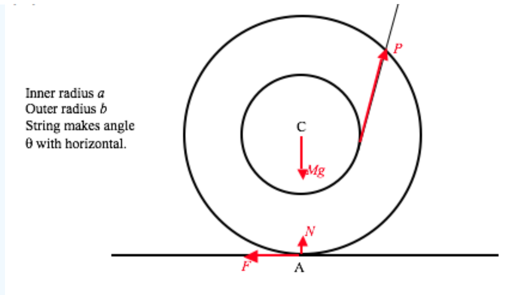
Exercise $22.8.40E$



$$\frac{P(b\cos\theta - a)}{I + Mb^2}. \tag{1}$$

$$\frac{P(b - \cos\theta - a)}{I + Mb^2}. \tag{2}$$

$$F = \left(\frac{I\cos\theta + Mab}{I + Mb^2}\right) P, \tag{3}$$

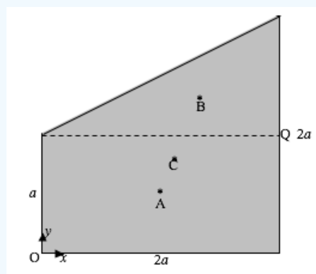


$$\frac{P(a - b\cos\theta}{I + Mb^2}. \tag{4}$$

$$\frac{Pb(a - b\cos\theta}{I + Mb^2} \tag{5}$$

F = \left(\frac{I\cos \theta + Mab}{I + Mb^2}\right)P, } \label{40e.6}

Exercise 22.8.41



$$3m\bar{x} = 2ma + m \times \tfrac{4}{3}a$$

$$3m\bar{y} = 2m \times \tfrac{1}{2}a + m \times 43a.$$

$$\bar{x} = \tfrac{10}{9}a \qquad \bar{y} = \tfrac{7}{9}a.$$

$$\frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\tfrac{1}{3}(2m)(\tfrac{1}{2}a)^2 = \tfrac{1}{6}ma^2.$$

$$= \tfrac{1}{3}(2m)a^2 = \tfrac{2}{3}ma^2.$$

$$= 0.$$

$$A_{rect} = \tfrac{1}{6}ma^2 + 2m(\tfrac{5}{18}a)^2 = \tfrac{28}{81}ma^2.$$

$$B_{rect} = \tfrac{2}{3}ma^2 + 2m(\tfrac{1}{9}a)^2 = \tfrac{65}{81}ma^2$$

$$H_{rect} = 0 + 2m(\tfrac{1}{9}a)(\tfrac{5}{18}a) = \tfrac{5}{81}ma^2.$$

$$= \tfrac{1}{6}ma^2 - m(\tfrac{1}{3}a)^2 = \tfrac{1}{18}ma^2.$$

$$= \tfrac{1}{6}(m)(2a)^2 - m(\tfrac{1}{3} \times 2a)^2 = \tfrac{2}{9}ma^2.$$

$$+\tfrac{1}{36}m(2a)(a) = +\tfrac{1}{18}ma^2.$$

$$A_{tria} = \tfrac{1}{18}ma^2 + m(\tfrac{5}{8}a)^2 = \tfrac{59}{162}ma^2.$$

$$B_{tria} = \tfrac{2}{9}ma^2 + m(\tfrac{2}{9}a)^2 = \tfrac{22}{81}ma^2.$$

$$H_{tria} = \tfrac{1}{18}ma^2 + m(-\tfrac{2}{9}a)(-\tfrac{5}{9}a) = \tfrac{29}{162}ma^2.$$

$$A=\frac{26}{81}ma^2+\frac{59}{162}ma^2=\frac{37}{51}ma^2.$$

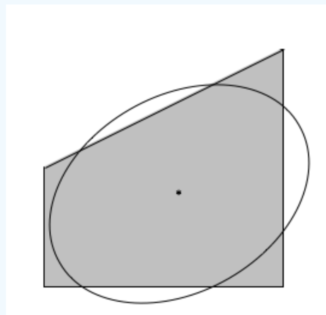
$$B=\frac{56}{81}ma^2+\frac{22}{81}ma^2=\frac{26}{27}ma^2.$$

$$H=\frac{5}{81}ma^2+\frac{29}{162}m=\frac{13}{54}ma^2.$$

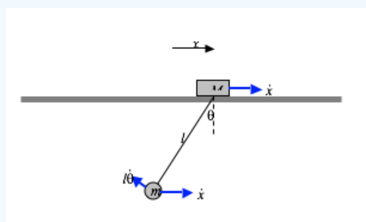
$$\theta = 30^{\circ}.009\ 180 \text{ and } 120^{\circ}.009\ 180$$

$$A\cos^2\theta-2H\sin\theta\cos\theta+b\sin^2\theta.$$

$$A_0 = 0.546142ma^2\ ,\ B_0 = 1.102006ma^2.$$



? Exercise 22.8.42



$$T=\frac{1}{2}Mx^2+\frac{1}{2}m(\dot{x}^2+l^2\dot{\theta}^2+2l\dot{\theta}\dot{x}\cos\theta$$

$$V=\text{constant}-mgl\cos\theta.$$

$$(M+m)\ddot{x}+ml(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)=0,$$

$$l\ddot{\theta}+\ddot{x}+g\theta=0.$$

$$\ddot{\theta}=-\frac{(M+m)g}{Ml}\theta$$

$$\underline{\underline{2\pi=\sqrt{\frac{Ml}{M+m}g}.}}$$

? Exercise 22.8.43

$$\frac{1}{k}\frac{dr}{d\alpha}=-\frac{1}{2}\sin\alpha\sin2\alpha+2\cos^2\frac{1}{2}\alpha\cos2\alpha$$

$$=-\sin^2\alpha\cos\alpha+(1+\cos\alpha)(2\cos^2\alpha-1)=0.$$

$$\underline{\underline{3c^2+2c^2-2c-1=0}}$$

? Exercise 22.8.44

The condition for stability, from Chapter 16 Section 16.9, Equation 16.9.5 is that

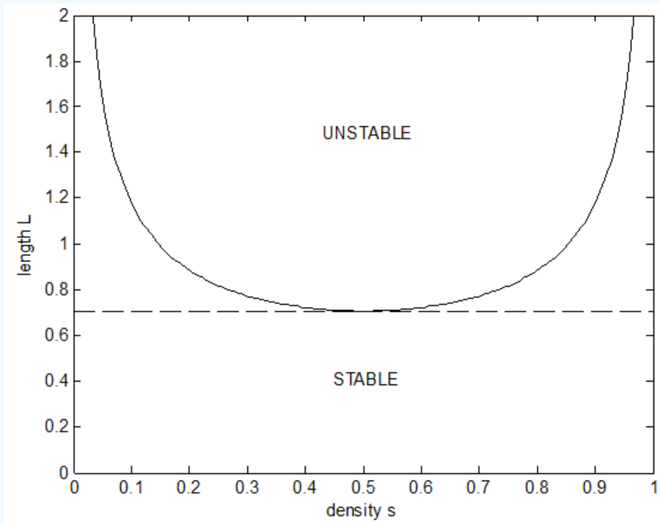
$$\frac{Ak^2}{V}>\text{HC}.$$

k^2 for a filled circle of radius a is $\frac{1}{4}a^2$. If the length of the cylinder is l, the volume immersed is Asl, so the left hand side of the inequality is $\frac{a^2}{4sl}$.

The depth of the centre of mass is $l(s-\frac{1}{2})$ and the depth of the centre of buoyancy is $\frac{1}{2}ls$, so that HC $= \frac{1}{2}l(1-s)$. The condition for stability is, then, $\frac{a^2}{4sl} > \frac{1}{2}l(1-s)$.

With $L = \frac{l}{(2a)}$, this gives, for the condition for stability,

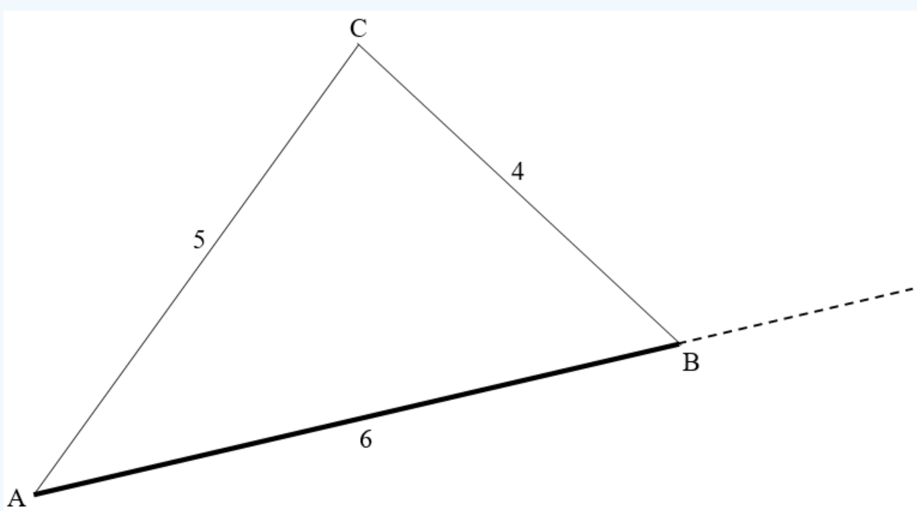
$$\underline{\underline{L < \frac{1}{\sqrt{8s(1-s)}}}}$$



This function is least for $s = \frac{1}{2}$, $L < \frac{1}{\sqrt{2}} = 0.707$. For any length less than this, the system is stable for any density. With $L = 1$, the inequality can be written $8s^2 - 8s + 1 > 0$, so that s must be less than 0.146 or greater than 0.854.

Exercise 22.8.45

Before doing the problem, let's just have a look at the "interesting" property of a (4, 5, 6) triangle



Calculate A by the cosine rule: $16 = 25 + 36 - 60\cos A$, hence $\cos A = \frac{3}{4}$.

Calculate C by the cosine rule: $36 = 16 + 25 - 40\cos C$, hence $\cos C = \frac{1}{8}$.

But $\cos 2A = 2\cos^2 A - 1 = \frac{1}{8}$. Therefore $C = 2A$.

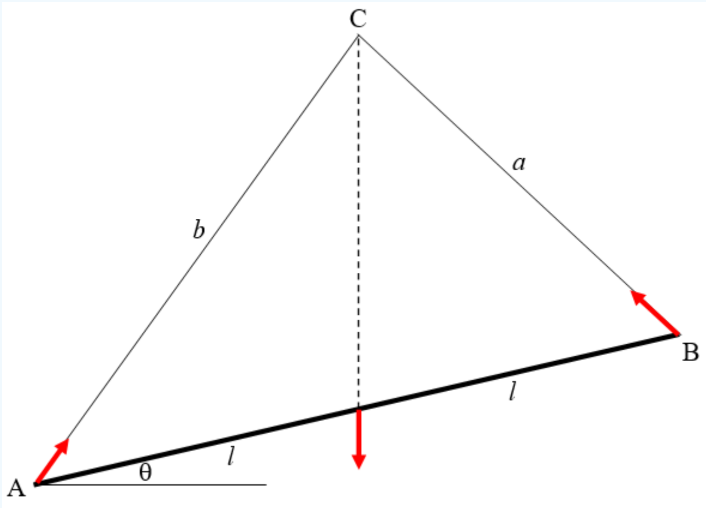
The external angle at B is $3A$.

The angles are $A = 41^\circ.4096 \quad C = 82^\circ.8192 \quad B = 55^\circ.7711 \quad \left(\cos B = \frac{9}{16}\right)$

Supplement of $B = 124^\circ.2289$

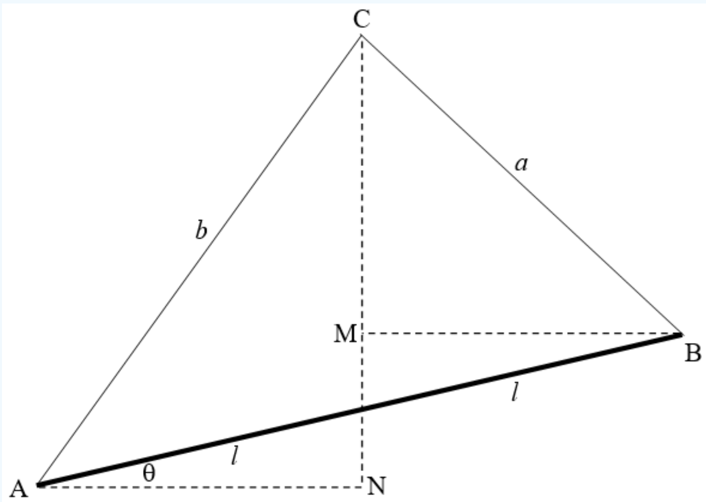
It is not the case that a triangle with one angle equal to twice another one is necessarily a (4, 5, 6) triangle.

After that diversion, let's move on to the given problem - except that we'll generalize it to make the length of the rod $2l$, and the lengths of the strings a and b.



The only physics involved is to recall that, if three coplanar forces are in equilibrium, they must be concurrent at a point - in this case the point C. This means that C must be vertically above the mid-point of the rod.

After that, there is no more physics; the rest is "just" geometry. All we have to do is to find θ in terms of a and b.



If C is to be directly above the mid-point, then AN = MB. That is:

$b\cos(A+\theta) = a\cos(B-\theta)$.

This quickly results in

$\tan\theta = \frac{b\cos A - a\cos B}{b\sin A + a\sin B}$.

In our particular example, $\frac{a}{b} = \frac{4}{5} = 0.8, \; B = 180^\circ - 3A$, and $A = 41^\circ.4096$.

Thus

$$\tan\theta = \frac{\cos A + 0.8\cos 3A}{\sin A + 0.8\sin 3A} ,$$

and

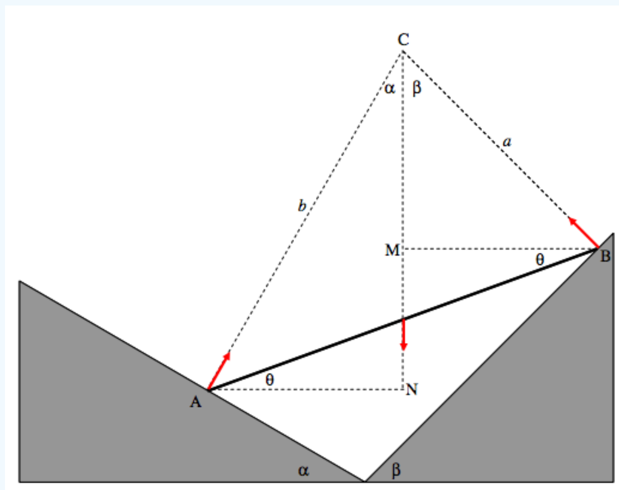
$$\underline{\theta = 12^{\circ}.78}$$

If the weight of the rod is mg, I'll leave it to you to work out the tensions in the strings.

Exercise 22.8.46

The only physics involved is to recall that, if three coplanar forces are in equilibrium, they must be concurrent at a point - in this case the point C. This means that C must be vertically above the mid-point. Also, since the planes are smooth, the forces at A and B are perpendicular to the planes.

The rest is geometry - almost the same as in Problem 45, except that in this problem we are given the angles α and β rather than the lengths a and b. Start by convincing yourself that the two angles at C are indeed α and β, as marked. Now all that is required is to express θ in terms of α and β.



Since the mid-point of the rod must be vertically below C, we must have AN = MB. That is:

$$b \sin \alpha = a \sin \beta.$$

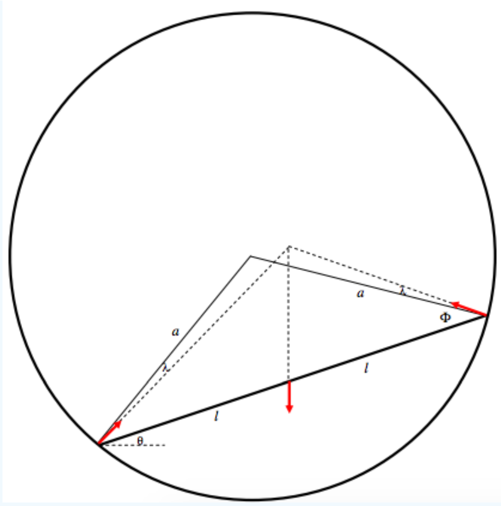
By the Sine Rule, $\frac{b}{a} = \frac{\sin B}{\sin A}$, so that $\sin \alpha \sin B = \sin \beta \sin A$

But $A = 90^{\circ} - \alpha + \theta$ and $B = 90^{\circ} - \beta - \theta$, so

$$\sin \alpha = \cos(\beta - \theta) = \sin \beta \cos(\alpha + \theta) \ ,$$

which quickly yields $\underline{\underline{\tan \theta = \frac{1}{2}(\cot \alpha - \cot \beta)}}$. In our particular example, this is $\tan \theta = \frac{1}{2}(\sqrt{3} - 1)$, $\underline{\underline{\theta =}}$ 20.1°. If you wish, you could work out the forces at A and B in terms of the weight of the rod.

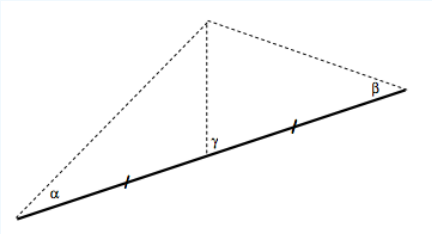
Exercise 22.8.47



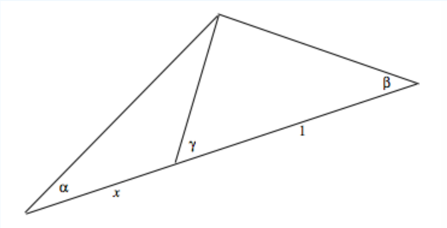
I have drawn above the three forces on the rod. The forces at the ends of the rod each make an angle λ to the normal to the surface, where $\tan\lambda = \mu$, and the three coplanar forces, being in static equilibrium, are concurrent at a point. I have also introduced the angle ϕ, given by $\cos\phi = l/a$. All we have to do is to find θ in terms of λ and ϕ - that is to say, in terms of μ and l/a

Fortunately I found the following formula for a triangle in an old geometry book:

$$\cot\gamma = \frac{1}{2}(\cos\alpha - \cot\beta)$$



I'll leave you to see if you can derive it. The book actually gave a formula for a more general case in which the base of the triangle isn't divided equally. For the case



the formula is $(1+x)\cot\gamma = \cot\alpha - x\cot\beta$.

You can use this in various problems in geometric optics, where you are trying to find relations between object distance, image distance and radius of curvature or focal length. However, for this problem, we need only the simpler formula, where the base of the triangle is equally divided.

On applying the simpler formula to our present problem we obtain

$$\underline{\underline{\tan\theta = \frac{1}{2}[\cot(\phi-\lambda) - \cot(\phi+\lambda)]}} \;,$$

and the problem is solved.

Below, I illustrate some examples. Going from left to right we have short $l/a = 0.2$, medium $l/a = 0.4$, long $l/a = 0.6$ rod. Going from top to bottom we have a slippery $\mu = 0.5$, a medium $/mu = 1.0$, and a sticky $/mu = 1.5$ surface.

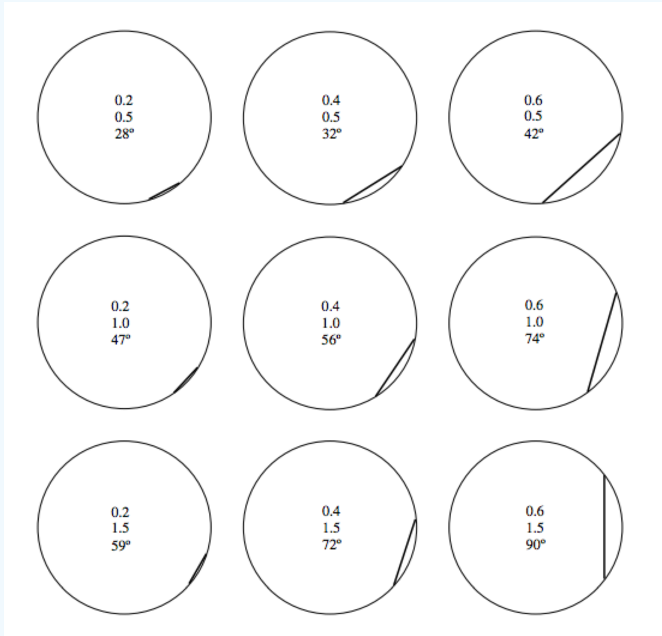
Inside each drawing, I tabulate

$$l/a$$

$$/mu$$

$$\theta°$$

The long $(l/a = 0.6)$ rod will rest vertically for any $/mu > 1.33$. But, while it will not slip, the equilibrium is no longer stable, and the rod will tip after an infinitesimal counterclockwise displacement.



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